Local Stabilization of Non-Ergodic Jackson Networks with Unreliable Nodes

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Preface

Over the past four years, I have dedicated myself to the research of general Jackson networks which cannot settle down in a global equilibrium. My research results on the local stabilization of such stochastic networks where breakdowns of nodes are incorporated are collected in this thesis.
At this point I would like to thank a number of people:

First of all, I am indebted to my supervisor Prof. Dr. Hans Daduna who introduced me to the topic of non-ergodic Jackson networks. After my graduation it was him who asked and encouraged me to continue my research and to extend my former findings to a PhD thesis. During the time of my research, I strongly benefited from his valuable advice and helpful comments.

Secondly, I thank Prof. Dr. Bernd Heidergott for reviewing my PhD thesis as a second supervisor.

Moreover, it was pleasant to work at the Center of Mathematical Statistics and Stochastic Processes at the University of Hamburg. I thank all its members for the kind atmosphere.

Finally, I am most grateful to my family for the continuous support. Special thanks go to my parents and my fiancé:

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Ich danke meinen Eltern und Pascal von ganzem Herzen und widme ihnen diese Arbeit.

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Jennifer Mylosz

Contents

Introduction

Motivation

This[1] thesis contributes to the field of queueing networks (i.e., stochastic networks).

> "Queueing networks have obtained their place in both theory and practice. New technological developments such as the Internet and wireless communications, but also advancements in existing applications such as manufacturing and production systems, public transportation, logistics, and health care, have triggered many theoretical and practical results." [BD11, p. v]

Queueing networks have been standard models in the field of telecommunication networks since the very beginning of mathematically based planning of telephone networks. In early days, such networks were decomposed into single node components (e.g., $M/M/s/0$, $M/M/s/\infty$) for which results on their performance already existed. A heuristically motivated composition of the results for these single node components has been successfully used to assess the expected performance of these networks. The idea behind this was the existence of a globally stable network, which was thought to be a system in a steady state.
The steady state analysis for such stochastic networks experienced a breakthrough when J.R. Jackson [Jac57] and W.J. Gordon and G.F. Newell [GN67] found their celebrated steady state results. At least since the publication of [Kle64], these models are well established in the area of performance analysis of communication and computer networks. As can be seen for example in [BGMT98], it is common practice now to use the results of Jackson or Gordon and Newell and the algorithms derived on the base of the explicit steady state probabilities for network analysis and planning. A consequence of this traditional approach is that strict requirements for the existence of a steady state in such networks need to hold. [MD11, pp.99-100]

The motivation for this thesis is - in contrary of the steady state situation - the following: In large and complex networks one may often observe that unstable subnetworks occur due to local overload (i.e., the nodes experience a higher input rate than their service capacity) or due to temporary non-availability of nodes. Nevertheless, at the same time one may observe that co-existing regions seem in an obvious way to stabilize locally in the long run. [MD09, p.1249] [MD11, p.100]

[1]Parts of this motivation cite the introductions of our papers [MD09] (pp.1249-1251) and [MD11] (pp.99-100). The corresponding parts will be shortly marked at the end of their paragraphs.

As it is common practice in performance analysis, [BGMT98], we describe the evolution of such networks over time by a Markov process. Within the framework of Markov process theory, such a Markovian description of the state evolution cannot lead to settling down in an equilibrium, if the network is not globally stable. In technical terms we will say that the Markov process is *not ergodic*. [MD11, p.100]

Best to our knowledge, models for globally unstable networks were successfully investigated the first time by J.B. Goodman and W.A. Massey in [GM84]. From the parameter settings of such a network only, they precisely identified the regions of instability. Moreover, they proved that in these unstable regions the local queue lengths "degenerate" out of the state space in the long run, i.e., the state distribution of unstable nodes converges in a precise meaning to a one-point distribution concentrated at infinity. For the regions of stability, they proved that the joint marginal queue lengths converge to a classical proper product form distribution. This means that the limiting queue lengths of stable nodes stay finite with probability one. [MD09, p.1250] [MD11, p.100]

This thesis is concerned with the combination of two topics:

1. the investigation of globally unstable networks with unreliable nodes, i.e., with breakdowns of nodes, and the analysis of their long-time behavior towards a local stabilization and

2. the analysis of the local stabilization of globally unstable networks.

Ad 1. The analysis of globally unstable networks with breakdowns of nodes can be motivated the following way:

There are many problems that require an integrated model to study the interplay of performance and availability. Examples are network control and routing protocols in case of link failures in IP networks, [MKC03], [NSB$^+$03], the handling of catastrophic events in mobile cellular networks, [KN03], or the investigation of unreliable machines in flexible manufacturing systems (FMS) and the evaluation of system availability, [BSY94]. Most recent applications where nodes or links may be not available for some time are mobile and ad-hoc networks or sensor networks. [MD09, pp.1249-1250]
As an example, wireless ad-hoc networks are built of varying sets of mobile users with wireless communication capabilities without relying on a pre-existing infrastructure. The emergence of wireless ad-hoc networks introduces problems concerning the availability of transmission nodes next to a user. Furthermore, in such networks, overload arises from too many users entering a region and applying for transmission. Thus, in these networks, the required high quality of service for speech and data transmission has to be guaranteed on the basis of not necessarily reliable network nodes. [SV03]
For a survey on analytical models for sensor networks see [WDW07]. A non-stationary framework in wireless sensor networks which will lead in analytical studies to evanescent networks is investigated in [WPTGG10].

Performability Theory was developed over twenty years to answer questions that arise with respect to combined reliability and performance assessment which was motivated by the needs of fault tolerant computer and communication systems, see [HMRT01].

Our aim is to contribute to performability modeling and evaluation in the framework of generalized Jackson networks, with an emphasis on situations where the global network is not stable in the classical sense. We study the interaction of reliability and performance in these networks and contribute to a better understanding of the behavior of networks that cannot settle down in a classical equilibrium state. [MD09, p.1250]

Classical product form theory for stochastic networks is concerned with stations (nodes) which have completely reliable servers. Usually, when unreliable nodes were to be considered, the method of adjusted service rates was used:

(i) The reliability of the nodes was computed using reliability theoretical methods and yielded the percentage of time the nodes are broken down.

(ii) The service speeds of the nodes were then decreased by this factor and the performance evaluation methods for product form networks could be applied, see [CM96].

The disadvantage of this method is that the interaction of performance and reliability cannot be studied in a unified model. The predominant methods developed to study the interaction of reliability and performance of complex systems in integrated models are simulation and numerical evaluation. Many different models and methods for integrated investigation of reliability and performance of systems are collected under the heading of performability in the survey [HMRT01]. [MD09, p.1250]

Our approach is different from these methods: We start from results by C. Sauer and H. Daduna in [SD03] on Jackson networks with unreliable nodes which have product form equilibrium. But we do not assume that the global network is stable, which may be due to breakdowns or overload of nodes. We describe the evolution of the network over time by a Markov process. The states of this Markov process encompass the necessary information about the availability of the nodes (up or down) and the queue lengths at the nodes. If the network is not globally stable, its Markov process is not ergodic. We continue the investigation of Goodman and Massey in [GM84] in the framework of Jackson networks with unreliable nodes. Our main result is similar to that of Goodman and Massey:

(i) Their method applies in our framework including unreliability of (all) nodes as well and allows to determine the regions of stability and instability.

(ii) The queue-length distributions of unstable nodes degenerate in the long run to one-point distributions concentrated at infinity.

(iii) In stability regions, the marginal distributions converge to a product form distribution. The limiting product form distribution is a product with respect to the availability component and the queue-length vector. Furthermore, the queue-length vector exhibits internal product form over the nodes as classical Jackson networks do.

Although we have obtained the limiting availability queue-length distribution (for the general case that all nodes may be unreliable), no stationary distribution for the network exists. Therefore, classical steady-state availability and performance evaluation, even for the stable subnetwork is not possible. The results obtained suggest to assess the quality of service inside the stabilizing part of the network in the long run by using the limiting

distributions. We discuss this in detail for the limiting throughput, but a generalization to other performance measures is obvious. Additionally, we prove that time averages of cumulative rewards associated to the queue-length process of the stable subnetwork can be approximated by state-space averages. In globally stable networks the evolution of which is described by a Markov process, this is an immediate consequence of the ergodic theorem for Markov processes. In the framework of our non-ergodic (not even Markovian) process, we have to prove this from scratch. [MD09, p.1250]
Fortunately enough, this can be derived directly from the proofs of the main theorems in Chapter 5. The results presented in this thesis provide performance indices in explicit form for non-ergodic networks which look exactly like those derived by classical product form steady-state distributions.

The product form steady states and the algorithms derived from them have been proven to be an indispensable tool set for performance evaluation. Two procedures relying on product form calculus are standard:

- If the necessary modeling assumptions for product form models can be validated, the performance indices are directly obtained;

- if the necessary assumptions for product form models are violated, the algorithms are used to obtain quick answers about main performance characteristics like the throughput via approximation in a decomposition-aggregation procedure.

More details are given in [BGMT98, Chapter 7 and 8] for the case of exact models and in [BGMT98, Chapter 10] for the case of approximating non-product form models by product form algorithms. Nevertheless, there are many situations and models where additionally (or instead) simulations or straight forward numerical algorithms have to be used.
It is common practice first to obtain some raw performance indices from (approximative) product form algorithms and then additionally to perform fine tuning via simulation studies. [MD09, pp.1250-1251]

We expect that our results open a path to justifying similar methods for our class of models: Either using the results directly when the assumptions of exponential times and sufficient independence properties can be verified from the model description, or using the results in approximation procedures for decomposition-aggregation algorithms when we cannot justify such assumptions but need raw results. [MD09, p.1251]
At least in the steady-state situation, the product form results turned out to be astonishingly robust and insensitive to violations of the assumptions, as shown in the recent paper [MAG06] where the steady-state results of [SD03] are used to analyze the performance of a wireless sensor network.

Ad 2. The analysis of the local stabilization is motivated as follows:

As shown in [MD09] and in this thesis, assessment of the quality of service inside the stabilizing part of a non-ergodic network can be done by using the obtained limiting (product form) distributions, e.g., the limiting throughput as a performance index in the long run. The asymptotic local performance indices for non-ergodic Jackson networks look exactly like the steady-state performance indices obtained from the classical product

form distributions as in [Kle76] or [BGMT98]. In the classical framework of ergodic network processes these product form steady states and the algorithms derived from them have been proven to be an indispensable tool set for performance evaluation, see [BGMT98, Chapter 7 and 8].
The observed structural similarities of the asymptotic results for ergodic network processes and for non-ergodic network processes (on the stable parts) naturally pose the question whether there exist similar features for the stable subnetwork of the globally unstable network which resemble the steady state properties of the ergodic network processes which are expressed as product form calculus. [MD11, p.100]

In this context, we will analyze a possible invariance of the marginal probabilities of the stable subnetwork process in finite time. We do this with different approaches such as considering the classical invariance of the queue length probabilities of the stable subnetwork over time (which will not succeed for more than one time step) and analyzing some kind of quasi-stationary behavior of the stable part of the network process.

The term quasi-stationarity refers to properties and tools from epidemic models, which describe in Markovian settings the evolution of, e.g., an epidemic development of infectious disease [Bai75]. Such a disease is known to die out with probability one, but in many cases seems to settle down in a state which strongly resembles stationary behavior, because the time to extinction is of a much greater order of magnitude than the observation horizon. We will reinterpret the definitions of quasi-stationarity used in the literature to fit to our model. [MD11, p.100]

We will extend the analysis of the conditional probabilities by the use of stopping times. At the end of the analysis, we will introduce an invariance-type called "stochastically sub-invariant distribution" to describe the behavior of the marginal distribution of the stable subnetwork process in finite time.

Outline

This thesis has the following structure:

First of all, we define the classical Jackson network, which is the basis of our research, in Chapter 1. The Jackson network considered here has single servers at each node. The well known steady state results of Jackson [Jac57] are reproduced and the asymptotics for non-ergodic Jackson networks proved by Goodman and Massey [GM84] are delivered. The therefore needed standard traffic equations as well as the general traffic equations are deduced and an intuitive interpretation of their solutions as input rates in the systems is given. For applications, we present an algorithm to decide which traffic equation is appropriate to calculate the input rates in a system.
In Section 1.4, we try to obtain assumptions for the non-ergodic Jackson network to classify its states. We do this in a framework of a two-nodes system. In the first approach we show how matrix geometrical methods may be used to obtain interesting facts about the behavior of non-ergodic Jackson networks. For the second approach martingale methods are used which yield assumptions in concrete terms for the non-ergodic network

to be transient.

In Chapter 2, Jackson networks are generalized by incorporating breakdowns of nodes. The model description follows [SD03]. The steady state results known from [SD03] are of product form.

Chapter 3 deals with stabilization or destabilization of the network process due to breakdowns. The results show that there is a need for side constraints in order to maintain the stability situation in a network with breakdowns. A possible stabilization of a globally unstable network by removing even stable nodes is discussed.

Chapter 4 is devoted to the study of Jackson network with buffers of an infinite amount of jobs at some nodes, as defined in [Wei05]. We prove steady-state results in case of ergodic network processes and analyze the structure of these results. Then we state and prove the asymptotics as well as partly steady-state results in case of non-ergodic network processes. After incorporating breakdowns of nodes in an integrated model, we prove similar results. The performance of such networks is presented.

In Chapter 5, we use the results of Chapter 2 and the previously worked out results of Chapter 4 to prove the asymptotics of non-ergodic Jackson networks with breakdowns of nodes. Since steady-state performance measures may not be used in such networks, an alternative method using limiting performance measurements is presented and proved.

After comparing the similar asymptotics for ergodic and for non-ergodic Jackson network processes on the stable parts, we analyze the local stability in finite time in Chapter 6. We analyze the non-ergodic network process by proceeding to its uniformization.
Different possible invariance types of the marginal distribution for the stable subnetwork (and methods to obtain them) are discussed and shown for one step. Difficulties in deriving invariance of the stable marginal distributions for more than one step are described and we point out open questions for possible further research. We conclude with a geometrically distributed upper bound for the process of the stable subnetwork of an overall unstable Jackson network established in two different ways in Section 6.5 and, in this context, describe the resulting behavior of the marginal distribution calling it "stochastically sub-invariant distribution".

Notation

Throughout this thesis the following notation is used:

All random variables and stochastic processes are defined on an underlying probability space $(\Omega, \mathcal{A}, P)$. We denote all stochastic processes X by $(X(t) : t \geq 0)$. Unless otherwise expressly agreed all processes are defined on a continuous-time scale.

We assume all homogeneous Markov processes to be regular jump processes which means:

- the row sums of their Q-matrices is zero,
- all diagonal entries of their Q-matrices are finite,

- the processes are non-explosive (the sequence of jump times diverges almost surely), and therefore
- they have cadlag (French "**c**ontinue **à** **d**roite, **l**imitée **à** **g**auche") paths, i.e., each path of a process is right-continuous and has left limits everywhere.

We denote by $\mathbb{N} := \{0, 1, 2, ...\}$ the set of all non-negative integers. We set $\mathbb{N}_+ := \mathbb{N} \setminus \{0\}$ for all positive integers. The set of all real numbers is $\mathbb{R}$, we write $\mathbb{R}_+ := [0, \infty)$ for all non-negative real numbers.

$\exists$ stands for "(there) exist(s)", $\exists!$ means "(there) exists exactly one".

The Kronecker Delta δ_{xy} is defined by

$$\delta_{xy} := \begin{cases} 1 & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases}$$

and with a set A the indicator function $1_A(\cdot)$ is defined by

$$1_A(x) := \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Empty sums are 0, empty products are 1. $\inf \emptyset = \infty$ and $\sup \emptyset = -\infty$.

The notation $\subseteq$ between sets stands for "subset or equal", $\subset$ means "proper subset". For a set A we denote by $\mathcal{P}(A)$ the set of all subsets of A and by $|A|$ the number of elements in A.

The matrix $\mathbf{I}$ denotes the identity matrix of appropriate size, i.e., a square matrix with ones on the main diagonal and zeros elsewhere. The vector e is a column vector of appropriate size with all its components equal to one.

For a vector $x = (x_i : i \in C)$, $C \subseteq \mathbb{N}$, and a subset $A \subseteq C$ we define $x_A := (x_i : i \in A)$ as restriction of the vector x to components from the set A. The components of x_A are ordered in the corresponding natural linear order. Similarly for subsets $A, B \subseteq C$, $X_{AB} := (x(i,j) : i \in A, j \in B)$ is a restriction of the matrix $X = (x(i,j) : i, j \in C)$ to the components of $A \times B$.

Applying the operator $\wedge$ on vectors $w, z \in \mathbb{N}^J$ yields the component-wise minimum of these vectors.

Chapter 1

Classical Jackson networks

1.1 Introduction

Jackson networks were introduced by J.R. Jackson ([Jac57]) and are now well established in the literature on queueing systems, e.g., in [Kle76], [Kel79], [Ser99], [CY01], [BB05], [BD11]. We consider only Jackson networks with single-server stations, i.e., at each service station there is one server.

Jackson networks are a special class of open stochastic networks.

Stochastic networks or networks of queues consist of a finite number of nodes which are service stations. These service stations are models for machines, workers, or computing servers. Customers which may be, e.g., individuals, machines, data, messages, or orders, move between those nodes. A customer who arrives at some node, waits there until he gets service at that station and after some random service time he leaves the node and randomly chooses where to go next.

A stochastic network is *open*, if customers enter the network from a source outside of the network and after being served in the network they leave the network to a sink with positive probability.

Definition 1.1 (Jackson network)**.** [Jac57] *A* Jackson network *with J nodes, numbered $j \in \{1, 2, ..., J\} =: \tilde{J}$, is an open stochastic network with J $\cdot/M/1/\infty$ nodes, i.e., for each node j we have*

- *an external Poisson(λ_j)-arrival stream ($\sum_{j\in\tilde{J}} \lambda_j = \lambda > 0$),*
- *a single server with exponential(μ_j) distributed service time,*
- *an infinite waiting room,*
- *the first-come-first-served regime.*

Customers are indistinguishable. All interarrival and service times constitute a set of independent random variables.
Routing is Markovian: Given the departure node i the selection of the next node is independent of the previous history. A customer departing from node i immediately proceeds

to node j with probability $r(i,j) \geq 0$ and departs from the network with probability $r(i,0)$. The artificial node 0 represents the outside, source and sink, of the network, $r(0,0) := 0$, $r(0,i) := \lambda_i/\lambda$. The routing matrix $R = (r(i,j) : i,j \in \{0,1,...,J\})$ is stochastic and irreducible.

Let $X_j(t)$ be a random variable which indicates that $X_j(t)$ customers are present at node j at time t, either in service or waiting. Then $X(t) = (X_1(t),...,X_J(t))$ is the vector of queue lengths in the network at time t and $X = (X(t) : t \in \mathbb{R}_+)$ the corresponding vector process called queue length process. Because of the assumptions put on the system X is a continuous-time homogeneous strong Markov process on $(\Omega, \mathcal{A}, P)$ with discrete state space $(\mathbb{N}^J, \mathcal{P}(\mathbb{N}^J))$ and with transition rates matrix $Q = (q(z,z') : z,z' \in \mathbb{N}^J)$ defined by: $\forall i,j \in \tilde{J}$, $i \neq j$

$$
\begin{aligned}
&q(n_1,...,n_i,...,n_J; n_1,...,n_i+1,...,n_J) = \lambda_i,\\
&q(n_1,...,n_i,...,n_J; n_1,...,n_i-1,...,n_J) = \mu_i r(i,0) 1_{\mathbb{N}_+}(n_i),\\
&q(n_1,...,n_i,...,n_j,...,n_J; n_1,...,n_i-1,...,n_j+1,...,n_J) = \mu_i r(i,j) 1_{\mathbb{N}_+}(n_i),\\
&q(n_1,...,n_J; n_1,...,n_J) = -\sum_{i\in\tilde{J}} \lambda_i - \sum_{i\in\tilde{J}} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i),
\end{aligned}
$$

and $q(z,z') = 0$ otherwise.

The following lemma holds for the routing matrix R of Jackson networks as defined in Definition 1.1.

Lemma 1.2. *For any subset $K \subseteq \tilde{J}$ the inverse $(\mathbf{I} - R_{KK})^{-1}$ exists and is positive.*

Proof. For the existence of $(\mathbf{I} - R_{KK})^{-1}$ it is sufficient to show that the spectral radius of R_{KK} is less than one: If $\sigma(R_{KK}) < 1$ holds, then the Neumann series $\sum_{k=0}^{\infty} R_{KK}^k$ with $R_{KK}^0 := \mathbf{I}$ converges component-wise to $(I - R_{KK})^{-1}$ (see [Heu06, Satz 12.4 and p.127]). From the definition of the routing matrix we have $\sum_{k=0}^{\infty} R_{KK}^k > 0$ component-wise, so if the inverse exists, it is positive.

Since the routing matrix R is stochastic (so $\sigma(R) = 1$), R_{KK} is a (sub-)stochastic matrix and therefore for the spectral radius holds $\sigma(R_{KK}) \leq 1$, see [Bre99, p.198].
To show $\sigma(R_{KK}) < 1$, suppose that $\sigma(R_{KK}) = 1$. Then R_{KK} has the maximal eigenvalue 1. From the Perron-Frobenius theorem (see [Sen81, Theorem 1.1, p.3]) it is known, that for this eigenvalue 1 there are positive left and right eigenvectors which are unique to constant multiples. Denote by $L \subseteq K$ the indices where the left eigenvector x has non-zero, positive entries. Then R_{LL} is stochastic, because

$$
\begin{aligned}
xR_{KK} = vx \text{ with } v = 1 \quad &\Rightarrow \; x_L R_{LL} = x_L \;\Leftrightarrow\; \sum_{i\in L} x_i r(i,j) = x_j \;\forall\, j \in L\\
\Rightarrow \sum_{j\in L}\sum_{i\in L} x_i r(i,j) = \sum_{j\in L} x_j \;&\Leftrightarrow\; \sum_{i\in L} x_i \sum_{j\in L} r(i,j) = \sum_{j\in L} x_j,
\end{aligned}
$$

which yields $\sum_{j\in L} r(i,j) = 1$ for all $i \in L$. But if R_{LL} is stochastic and $L \subseteq K \subseteq \tilde{J}$, the routing matrix R is not irreducible because even in the case where $L = K = \tilde{J}$, the source and sink denoted by the node 0 is not accessible from nodes in L. R not being irreducible is a contradiction to Definition 1.1 of a Jackson network, so $\sigma(R_{KK}) < 1$ holds for all $K \subseteq \tilde{J}$. □

1.2 The ergodic case

A Markov process, as it is the queue lengths process X, may - under certain conditions - settle down in an equilibrium state. In this context, two terms to be considered are limiting distributions and stationary distributions.

Definition 1.3. *Let X be a homogeneous Markov process with discrete state space E. X has a* limiting distribution *if*

$$\lim_{t\to\infty} P(X(t)=i) \text{ exists for all } i \in E$$

and $\sum_{i\in E} \lim_{t\to\infty} P(X(t)=i) = 1$.

A stationary process is sometimes said to be a process *in equilibrium.* [Ser09, p.34] The process is *stationary*, if the initial distribution is a stationary (also called: steady-state) distribution. If the process is started with this distribution, it remains the same over time.

Definition and Remark 1.4. (e.g., [Bre99, pp.342-343])

- *A homogeneous Markov process $X = (X(t) : t \in \mathbb{R}_+)$ on $(\Omega, \mathcal{A}, P)$ with discrete state space E is* stationary, *if for all $m \in \mathbb{N}_+$, $0 \le t_1 < t_2 < ... < t_m$, $t_i \in \mathbb{R}_+$, and for all states $i_1, ..., i_m$ holds*

  $$P(X(t_1) = i_1, ..., X(t_m) = i_m) = P(X(t_1+s) = i_1, ..., X(t_m+s) = i_m) \quad \forall s > 0,$$

  i.e., $(X(t) : t \in \mathbb{R}_+)$ and its time-shifted version $(X(t+s) : t \in \mathbb{R}_+)$ have the same distribution.

- *A non-negative vector $\pi = (\pi(i) : i \in E)$ is an* invariant measure for the continuous-time *X with transition matrix function $p^{(t)} = (p^{(t)}(i,j) : i,j \in E)$, $t \in \mathbb{R}_+$, if*

  $$\pi = \pi \cdot p^{(t)} \quad \forall t \ge 0.$$

- *If an invariant measure is a probability measure, then it is called a* stationary distribution *for X. If there exists a stationary distribution π for X and if the process is started with this distribution $P^{X_0} = \pi$, then X is stationary.*

- *Every stationary distribution π for X is an asymptotic distribution for X conditional on $P^{X_0} = \pi$.*

To avoid the computational effort of solving infinitely many equations $\pi = \pi \cdot p^{(t)}$ (for all $t \in \mathbb{R}_+$) to derive a stationary distribution, one may skip to an alternative formulation with the Q-matrix if the Q-matrix exists: (Recall that all homogeneous Markov processes in this thesis are assumed to be regular jump processes.)

Lemma 1.5. [Asm03, Theorems 4.2 and 4.3, pp.51-52] *Let $X = (X(t) : t \in \mathbb{R}_+)$ be a homogeneous Markov process on a discrete state space E with transition rates matrix $Q = (q(i,j) : i,j \in E)$ which is irreducible and positive recurrent, i.e., ergodic. A probability measure π is a stationary distribution for X if and only if it solves the global balance equation*

$$\pi \cdot Q = 0.$$

If a Markov process is ergodic, then its stationary distribution is the limiting distribution, which is positive. (e.g., [Ser09, p.40])
In this section, conditions for the process X of a Jackson network as in Definition 1.1 being ergodic are given. Before doing this, the traffic equations the solution of which is utilized for Jackson's theorem later on must be defined first.

1.2.1 The traffic equations

Definition 1.6. [Jac57] *The* traffic equations *of a Jackson network are defined as*

$$\eta_j = \lambda_j + \sum_{i=1}^{J} \eta_i r(i,j), \quad j \in \tilde{J}. \tag{1.1}$$

The classical traffic equations of a Jackson network were derived by the global balance equation of its (discrete-time) routing chain which is

$$\eta \cdot R = \eta \;\Leftrightarrow\; \eta_i = \sum_{j=0}^{J} \eta_j r(j,i) \quad \forall i = 0, 1, ..., J. \tag{1.2}$$

Because of the irreducibility of the routing matrix R, (1.2) have a solution $\eta = (\eta_0, ..., \eta_J)$ which is unique up to a factor. After normalizing the solution $\eta = (\eta_0, ..., \eta_J)$, it is a stochastic vector where η_i is the probability of a customer being located at station i, if the routing process is in equilibrium.
Prescribing

$$\eta_0 := \sum_{i\in\tilde{J}} \lambda_i = \lambda$$

in (1.2), we get

$$\eta_0 = \eta_0 \underbrace{r(0,0)}_{=0} + \sum_{i\in\tilde{J}} \eta_i r(i,0) \;\Rightarrow\; \sum_{i\in\tilde{J}} \lambda_i = \sum_{i\in\tilde{J}} \eta_i r(i,0) \tag{1.3}$$

and

$$\eta_i = \underbrace{\lambda r(0,i)}_{=\lambda_i} + \sum_{j\in\tilde{J}} \eta_j r(j,i) \quad \forall i = 1, ..., J.$$

Because of the irreducibility of R, for the immediate feedback at nodes $i \in \tilde{J}$ holds $r(i,i) < 1$, so these remaining equations have a unique solution determined by

$$\eta_i = \Big(\lambda_i + \sum_{j\in\tilde{J}\setminus\{i\}} \eta_j r(j,i)\Big) \cdot (1 - r(i,i))^{-1} \quad \forall i = 1, ..., J. \tag{1.4}$$

In general, $\eta = (\eta_1, ..., \eta_J)$ given by (1.4) is not a stochastic vector.

Remark 1.7. *Summing* (1.1) *on both sides over all $j \in \tilde{J}$ yields*

$$\sum_{j\in\tilde{J}} \eta_j = \sum_{j\in\tilde{J}} \lambda_j + \sum_{i\in\tilde{J}} \eta_i \sum_{j\in\tilde{J}} r(i,j) \;\Leftrightarrow\; \sum_{j\in\tilde{J}} \lambda_j = \sum_{j\in\tilde{J}} \eta_j r(j,0)$$

which coincides with (1.3).

Definition 1.8 (Overall arrival rate)**.** [Ser99, p.136] *Denote by* $N_i(t)$ *the number of arrivals at node* $i \in \tilde{J}$ *in the time interval* $[0, t]$*, then* $t^{-1}N_i(t)$ *is the average number of arrivals per time unit in the time interval* $[0, t]$ *at node* i*. Letting* t *tend to infinity yields* $\lim_{t\to\infty} t^{-1}N_i(t)$*, the asymptotical average number of arrivals per time unit at node* i *which is also called the (asymptotical)* overall arrival rate *at that node.*

The overall arrival rate at a node $i \in \tilde{J}$ can be calculated from the sum of its external arrival rate and of the expected number of arrivals coming from the other nodes. The following remark shows when the traffic equations (1.1) can be used to calculate the overall arrival rates in the network.

Remark 1.9. *In Theorem 1.10, we will see, that ergodicity of the queue length process* X *depends on the solution of the traffic equations* (1.1)*:* X *is ergodic if and only if* $\eta_i < \mu_i$ *holds for all nodes* $i \in \tilde{J}$*. The intuition behind* (1.1) *under equilibrium conditions of* X *is a rate balancing argument:* η_j *is in equilibrium the overall arrival rate at station* j *as well as the overall departure rate there, [Ser99, Remark 1.16, p.16].*
Note that due to R *being irreducible, the traffic equations have a unique solution independent of whether* X *is ergodic or not.*

1.2.2 Steady-state results

The following theorem was proved by J.R. Jackson (1957).

Theorem 1.10 (Jackson's theorem)**.** [Jac57] *If* X *from Definition 1.1 is ergodic, then the unique stationary and limiting distribution* π *on* $\mathbb{N}^J$ *is*

$$\pi(n_1, ..., n_J) = \lim_{t\to\infty} P(X_j(t) = n_j : j \in \tilde{J}) = \prod_{j=1}^{J} \left(1 - \frac{\eta_j}{\mu_j}\right) \left(\frac{\eta_j}{\mu_j}\right)^{n_j}. \quad (1.5)$$

X *is ergodic if and only if* $\eta_j < \mu_j$ *holds for all* $j \in \tilde{J}$*, where* $\eta = (\eta_j : j \in \tilde{J})$ *is the solution of* (1.1)*.*

Remark 1.11. *The stationary distribution* (1.5) *is a so-called* product form distribution*, which means that the joint distribution for the network process* $\pi(n_1, ..., n_J)$ *is a product of all one-dimensional marginal distributions*

$$\pi_i(n_i) = \sum_{(n_j : j\in\tilde{J}\setminus\{i\})\in\mathbb{N}^{J-1}} \pi(n_1, ..., n_J).$$

This product structure implies that the one-dimensional marginals of X *in time have independent coordinates in space. But, of course, the queue lengths at the nodes cannot be independent of each other, since the queue lengths rely on the customers moving between those nodes.*
The product form also suggests that, in equilibrium, at a given fixed time the network states behave like J *independent* $(M/M/1/\infty)$*-systems with Poisson(*η_i*) arrival streams and exponential(*μ_i*) service times which have the (geometric) distribution*

$$\pi_i(n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \quad n_i \in \mathbb{N},$$

as stationary distribution of the queue length process at node i*. But the suggestion of Poisson streams between the nodes is, in general, not true, as can be seen in [Mel79a].*

Such a product form of the stationary distribution is clearly desirable because common performance measures of the network are then easy to compute, as can be seen in the following subsection.

1.2.3 Performance measures

Definition 1.12. *Consider an ergodic Jackson network. Then the stationary throughput* TH_j *of a node is*

$$TH_j = \sum_{(n_1,...,n_J)\in\mathbb{N}^J} \pi(n_1,...,n_J)\cdot\mu_j\cdot 1_{\mathbb{N}_+}(n_j),$$

and the stationary throughput TH *of the network is*

$$TH = \sum_{(n_1,...,n_J)\in\mathbb{N}^J} \pi(n_1,...,n_J)\cdot\sum_{j\in\tilde{J}} 1_{\mathbb{N}_+}(n_j)\mu_j r(j,0) = \sum_{j\in\tilde{J}} TH_j r(j,0).$$

The followingg result is obtained by direct computations.

Proposition 1.13. *Consider an ergodic Jackson network. Then the stationary node-j throughput is* $TH_j = \eta_j$*, and the stationary throughput of the network is* $TH = \lambda$.

This result shows that, in equilibrium, the performance of Jackson networks is maximal: At any time, the departure rate of the network to the sink equals the external arrival rate into the network. Loosely speaking, what comes in comes out in average at a time unit if the system is in equilibrium.

The next theorem exploits when empirical time averages converge to space averages under the stationary and limiting distribution π.

Theorem 1.14 (Ergodic theorem)**.** (e.g., [Bre99, Theorem 6.2]) *Let* $X = (X(t) : t \in \mathbb{R}_+)$ *be an ergodic homogeneous Markov process (with state space* $(E, \mathcal{P}(E))$*) with stationary distribution* π*, and let* $f : E \to \mathbb{R}$ *be such that* $\sum_{i\in E} |f(i)|\pi(i) < \infty$*. Then for any initial distribution* $P^{(X_0)} =: p_0$ *holds* P_{p_0}*-a.s.*

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))dt = \sum_{i\in E} f(i)\pi(i).$$

Thus using the ergodic theorem, it is possible to estimate time averages of rewards or costs.

Example 1.15. *Let* $g : \mathbb{N}^J \to \mathbb{R}$ *denote a non-decreasing cost function which determines costs associated with queue lengths. To assess the quality of a system it is often desirable to predict the average accumulated cost over a time horizon* $[0, T]$,

$$d(T) = \frac{1}{T}\int_0^T g(X_i(t), i \in \tilde{J})\ dt.$$

Note, that this is an empirical measure which depends on the realized path of the system. If X *is ergodic with stationary distribution* π *as in Jackson's theorem (see Theorem 1.10)* *and if* $\sum_{(n_i:i\in\tilde{J})\in\mathbb{N}^J} |g(n_1,...,n_J)|\pi(n_1,...,n_J) < \infty$, *then the ergodic theorem (see Theorem 1.14) yields that for any initial distribution* $P^{X(0)}$ *holds* $P_{P^{X(0)}}$*-a.s.*

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g(X_1(t),...,X_J(t))dt = \sum_{(n_i:i\in\tilde{J})\in\mathbb{N}^J} g(n_1,...,n_J)\pi(n_1,...,n_J).$$

This means that the path-wise evaluated time average converges to the state space average, so path-wise evaluated time averages for a time horizon $[0,T]$ *with large* T *can be estimated by state space averages in ergodic Jackson networks:*

$$\frac{1}{T}\int_0^T g(X_1(t),...,X_J(t))dt \approx \sum_{(n_i:i\in\tilde{J})\in\mathbb{N}^J} g(n_1,...,n_J)\prod_{j=1}^{J}\left(1-\frac{\eta_j}{\mu_j}\right)\left(\frac{\eta_j}{\mu_j}\right)^{n_j}.$$

1.3 The non-ergodic case

In this section, Jackson networks are considered the state process of which is not ergodic. This means that the ergodicity condition $\eta_i < \mu_i \ \forall i \in \tilde{J}$ (see Theorem 1.10) is *not* fulfilled. The asymptotic results of J.B. Goodman and W.A. Massey ([GM84]) will be presented for which traffic equations different to those in Section 1.2 are valid.

1.3.1 The traffic equations

Definition 1.16. [GM84]

$$\eta_j = \lambda_j + \sum_{i=1}^{J}\min(\eta_i,\mu_i)r(i,j), \quad j\in\tilde{J}, \tag{1.6}$$

are the general traffic equations *for Jackson networks.*

These general traffic equations are necessary to obtain the asymptotic results of Goodman and Massey on non-ergodic Jackson networks which will be presented in the following subsection. Similar to the interpretation of the solution of (1.1) as overall arrival rates, an intuitive explanation of (1.6) can be given the following way:

The general traffic equation arises from a more careful examination of Burke's theorem for the $M/M/1$ system without immediate feedback (which means that the probability to directly leave the system after service to the sink is 1) in [Bur56]: When the external arrival rate λ is greater or equal to the service rate μ at the station, then the output of the system is still Poisson for large time, not with rate λ (as in the ergodic case) but with rate μ. To make an unconditional statement, the output process of an $M/M/1$ system for large time is Poisson with rate $\min(\lambda,\mu)$. [GM84, p.861]

Whenever the ergodicity condition is not fullfilled, the network is said to have unstable nodes.

Definition 1.17. *A node* i *is said to be* stable *if* η_i *determined by* (1.6) *is strictly less than* μ_i, *otherwise the node is called* unstable.

For unstable nodes j the input rate is different from the output rate which can be at most μ_j. So if there is at least one unstable node in the network, in determining the input rates we are no longer allowed to insert the same quantities in the traffic equations (1.1) on the right-hand side as on the left-hand side, the traffic equations need to be reformulated to (1.6).

Whenever there is more than one node for which $\eta_i < \mu_i$ does not hold (where η_i is determined by the classical traffic equation (1.1)), this does not mean, that all of these nodes where the condition does not hold are actually unstable. This is due to the interaction of those nodes and the fact that too large departure rates were taken into account by deriving the η_i's by (1.1). So the analysis of the stability of nodes is more complicated and consists, in general, of more than just one step. This will be seen in the following algorithms.

Algorithm 1.18. [GM84] *To determine for a Jackson network with J nodes which nodes are stable and which are unstable in the sense of Definition 1.17.*

(i) Assume that all nodes are unstable.
Based on this assumption, let $(\eta_i(1) : i \in \tilde{J})$ be the first estimate for the solution $(\eta_i : i \in \tilde{J})$ of (1.6), i.e., $(\eta_i(1) : i \in \tilde{J})$ is the solution of the traffic equations:

$$\eta_i(1) = \lambda_i + \sum_{j=1}^{J} \mu_j r(j,i) \quad \forall\, i \in \tilde{J}, \tag{1.7}$$

which trivially exists and is unique, because all parameters at the right-hand side of the equations are given. Comparing (1.6) and (1.7) and noticing $\min(\eta_i, \mu_i) \le \mu_i$ implies $\eta_i(1) \ge \eta_i$ $\forall i \in \tilde{J}$.

- *If $\eta_i(1) \ge \mu_i$ holds for all $i \in \tilde{J}$, all nodes are unstable and for the first estimate holds $\eta_i = \eta_i(1)$ $\forall\, i \in \tilde{J}$. Stop here.*
- *If for at least one node $i_* \in \tilde{J}$ holds $\eta_{i_*}(1) < \mu_{i_*}$, then $\mu_{i_*} > \eta_{i_*}$ holds due to $\eta_i(1) \ge \eta_i$ $\forall\, i \in \tilde{J}$, so this node i_* is stable. But since $\eta_{i_*}(1)$ is derived under the assumption that all nodes are unstable, it holds $\eta_{i_*}(1) \ge \eta_{i_*}$. Set $S(1) := \{i : \eta_i(1) < \mu_i\}$ and proceed to the next step.*

(ii) All nodes $i \in S(1)$ will be eventually stable. Assume that all other nodes $i \in \tilde{J} \setminus S(1)$ are unstable.
Based on this assumption, let $(\eta_i(2) : i \in \tilde{J})$ be the second estimate for the solution $(\eta_i : i \in \tilde{J})$ of (1.6), i.e., $(\eta_i(2) : i \in \tilde{J})$ is the solution of the traffic equations

$$\eta_i(2) = \lambda_i + \sum_{j \in \tilde{J}\setminus S(1)} \mu_j r(j,i) + \sum_{j \in S(1)} \eta_j(2) r(j,i) \quad \forall\, i \in \tilde{J}, \tag{1.8}$$

which exists and is unique, see Lemma 1.19. Again, comparing (1.6) and (1.8) and noticing $\min(\eta_i, \mu_i) \le \mu_i$ implies $\eta_i(2) \ge \eta_i$ $\forall i \in \tilde{J}$, but the assumptions are more conservative than those for $(\eta_1(1), ..., \eta_J(1))$. It holds: $\eta_i \le \eta_i(2) \le \eta_i(1)$ $\forall\, i \in \tilde{J}$ and $\eta_i(2) < \mu_i$ $\forall\, i \in S(1)$.

- *If* $S(1) = S(2) := \{i : \eta_i(2) < \mu_i\}$*, then* $\eta_i(2) = \eta_i$ *holds* $\forall\, i \in \tilde{J}$*. Stop here.*
- *If* $S(1) \neq S(2)$*, then* $S(1) \subset S(2)$*. In this case* $\eta_{i_*}(2) > \eta_{i_*}$ *holds for at least one node* $i_* \in \tilde{J}$*. Iterate the second step with* $S(2)$ *as new set of stable nodes.*

Lemma 1.19. [GM84] *The general traffic equations* (1.6) *have a unique solution which we denote by* $\eta = (\eta_j : j \in \tilde{J})$*.*

Proof. The proof consists of three steps: First it is shown that the traffic equations (1.6) are solved by an algorithm which recursively builds a sequence of vectors $\eta(n) = (\eta_1(n), ..., \eta_J(n))$, $n \in \mathbb{N}_+$, together with a sequence of sets $S(n) := \{i \in \tilde{J} : \eta_i(n) < \mu_i\}$ for which holds:

(i) $S(0) := \emptyset$,

(ii) $S(n-1) \subseteq S(n)\ \forall n \geq 1$,

(iii) $\exists! 0 < n^* \leq J : \quad S(n^*-1) \subset S(n^*) = S(n^*+1)$,

(iv) $\eta(n+1)$ solves the $S(n)$-partition of the traffic equations

$$\eta(n+1)_{S(n)} = \lambda_{S(n)} + \eta(n+1)_{S(n)} R_{S(n)S(n)} + \mu_{U(n)} R_{U(n)S(n)}, \tag{1.9}$$
$$\eta(n+1)_{U(n)} = \lambda_{U(n)} + \eta(n+1)_{S(n)} R_{S(n)U(n)} + \mu_{U(n)} R_{U(n)U(n)}, \tag{1.10}$$

where $U(n) := \tilde{J} \setminus S(n)$ and $S(n)$ is the detected set of stable nodes in the nth step of the algorithm.

It is shown that the sequence of vectors $\eta(n)$, $n \in \mathbb{N}_+$, of such an algorithm converges to the unique solution of the traffic equations (1.6) in at most J iterations, if a unique solution exists.
In the second step, the existence of a solution of the general traffic equations is shown. This is strongly related to the before mentioned algorithm: It has to be shown that for every $n \in \tilde{J}$ the $S(n)$-partition of the traffic equations, (1.9) and (1.10), has a solution. Transforming (1.9) into

$$\eta(n+1)_{S(n)} = (\lambda_{S(n)} + \mu_{U(n)} R_{U(n)S(n)})(\mathbf{I} - R_{S(n)S(n)})^{-1}$$

yields a solution of (1.9) and inserting this solution into (1.10) yields the solution of the latter equation, but the transformation is only possible, if $(\mathbf{I} - R_{S(n)S(n)})^{-1}$ exists and is positive. Because of the irreducibility of R, $(\mathbf{I} - R_{S(n)S(n)})^{-1}$ exists and is positive for all $n \in \tilde{J}$, see Lemma 1.2.
In the third step the uniqueness of a solution of (1.6) is proven straight forward. For more details see the proof of Lemma 4.10 with $V = \emptyset$, pp.56-59. □

Whenever a Jackson network is to be analyzed, it is important to know which nodes are stable and which are unstable and therefore which traffic equations are appropriate. The following algorithm avoids the computational effort of Algorithm 1.18 which arises if there is no or only one unstable node in the network. These two cases will be detected after one step already. For all other cases, the following algorithm refers to Algorithm 1.18 which is then worth the effort.

Algorithm 1.20. *To determine the set of stable nodes in a Jackson network.*

(i) Solve the standard traffic equations (1.1)*. Check if $\eta_i < \mu_i$ holds for all nodes $i \in \tilde{J}$.*

- *If the condition holds for all nodes, then all nodes are stable and* (1.1) *are the appropriate traffic equations. Stop here.*
- *If there is only one node, i_*, for which the condition does not hold, then this node is the only unstable node and the appropriate traffic equations are*

$$\eta_j = \lambda_j + \sum_{i \in \tilde{J} \setminus \{i_*\}} \eta_i r(i,j) + \mu_{i_*} r(i_*, j), \quad j \in \tilde{J}.$$

  Stop here.
- *If there are at least two nodes for which the condition does not hold, proceed to the following step.*

(ii) Run Algorithm 1.18 to solve the general traffic equations (1.6)*. With the detected set $S := \{i : \eta_i < \mu_i\}$ of stable nodes and $U := \tilde{J} \setminus S$ of unstable nodes the appropriate traffic equations are then*

$$\eta_j = \lambda_j + \sum_{i \in S} \eta_i r(i,j) + \sum_{i \in U} \mu_i r(i,j), \quad j \in \tilde{J}.$$

Remark 1.21. *Let $\eta = (\eta_j : j \in \tilde{J})$ be the unique solution of* (1.6)*. Nodes in $S := \{i \in \tilde{J} : \eta_i < \mu_i\}$ are stable, $U := \tilde{J} \setminus S$ is the set of unstable nodes. Summing* (1.6) *on both sides over all $j \in \tilde{J}$ yields*

$$\begin{aligned}
& \sum_{j \in \tilde{J}} \eta_j = \sum_{j \in \tilde{J}} \lambda_j + \sum_{j \in \tilde{J}} \sum_{i=1}^{J} \min(\eta_i, \mu_i) r(i,j) \\
\Leftrightarrow\ & \sum_{j \in \tilde{J}} \eta_j = \sum_{j \in \tilde{J}} \lambda_j + \sum_{j \in \tilde{J}} \sum_{i \in S} \eta_i r(i,j) + \sum_{j \in \tilde{J}} \sum_{i \in U} \mu_i r(i,j) \\
\Leftrightarrow\ & \sum_{j \in \tilde{J}} \lambda_j = \sum_{j \in S} \eta_j r(j,0) + \sum_{j \in U} (\eta_j - \mu_j(1 - r(j,0))) \\
\Leftrightarrow\ & \sum_{j \in \tilde{J}} \lambda_j = \sum_{j \in S} \eta_j r(j,0) + \sum_{j \in U} \mu_j r(j,0) + \sum_{j \in U} (\eta_j - \mu_j) \\
\Leftrightarrow\ & \sum_{j \in \tilde{J}} \lambda_j = \sum_{j \in \tilde{J}} \min(\eta_j, \mu_j) r(j,0) + \sum_{j \in U} (\eta_j - \mu_j)
\end{aligned}$$

so the sum of the departure rates of the system into the sink is smaller than the external arrival rate λ into the system if not all unstable nodes operate at full capacity ($\eta_j = \mu_j$), but at least one unstable node is overloaded ($\eta_j > \mu_j$). This reflects that the maximal departure rate at each node is limited by the maximal capacity of its server.

1.3.2 Long-time behavior

In the following theorem we assume that the ergodicity condition from Jackson's theorem (Theorem 1.10) does not hold which means that there is at least one node which is unstable. Although we know that the network process X will not settle down in an equilibrium, Goodman and Massey showed that parts of the network process - on the subset of stable nodes - do stabilize asymptotically.

Theorem 1.22. [GM84] *Let $X = (X(t) : t \in \mathbb{R}_+)$ denote the queue length process of a Jackson network. X is a Markov process with state space $\mathbb{N}^J$. Let $\eta = (\eta_j : j \in \tilde{J})$ be the unique solution of the general traffic equations* (1.6)*. Denote by $S = \{i : \eta_i < \mu_i\} \subseteq \tilde{J}$ the set of stable nodes in the network and by $U := \tilde{J} \setminus S$ the set of unstable nodes in the network.*
Then we have independent of the initial distribution for all $n_j \in \mathbb{N}$:

$$\lim_{t\to\infty} P(X_j(t) = n_j : j \in S) = \prod_{j\in S} \left(1 - \frac{\eta_j}{\mu_j}\right) \left(\frac{\eta_j}{\mu_j}\right)^{n_j}, \tag{1.11}$$

$$\lim_{t\to\infty} P(X_j(t) = n_j) = 0 \quad \forall j \in U. \tag{1.12}$$

The proof of Theorem 1.22 is left out here because it is a simplified version ($D = \emptyset$) of the proof of Theorem 5.10 in Chapter 5.

Remark 1.23. *It is noteworthy that the marginal probability* (1.11) *has the same structure as* (1.5)*, but the marginal process on S is not Markovian. Also one should notice that here the vector $\eta = (\eta_1, ..., \eta_J)$ solves equations which are different to those in the previous section,* (1.1)*, if there is at least one unstable node in the network.*
Moreover, the limiting joint marginal distribution (1.11) *of the queue lengths process on the subnetwork S is of classical product form as it already occurred in Jackson's theorem. This means that the limiting joint marginal distribution on S is the product of all limiting one-dimensional marginal distributions of the nodes in S:*

$$\lim_{t\to\infty} P(X_j(t) = n_j) = \left(1 - \frac{\eta_j}{\mu_j}\right) \left(\frac{\eta_j}{\mu_j}\right)^{n_j} \quad \forall\ j \in S.$$

The limiting marginal queue length probability (1.12) *being zero for all states implies that the limiting queue length distribution for every unstable node is the degenerated one-point distribution at $\{\infty\}$. The result* (1.12) *also implies*

$$\lim_{t\to\infty} P(X_j(t) = n_j, j \in U) = 0 \quad \forall n_j \in \mathbb{N},$$

which can be proved similarly as (1.12)*, but this is a weaker result.*

The asymptotic product form result (1.11) on S makes it easier to compute performance measures, at least for the stable parts of the network, when one cannot take the classical way and access steady state probabilities. This will be shown in Chapter 5, Section 5.4, with $D = \emptyset$ for the classical Jackson network considered here.

1.4 Classification of states

In Section 1.2.2, conditions for the queue length process of a Jackson network to be ergodic were presented: A Jackson network process is ergodic if and only if all of its nodes are stable. If this ergodicity condition is not fulfilled for at least one node, the global network process is not ergodic and the according Jackson network is called non-ergodic, too. Ergodicity of a network process is strongly related to the classification of its states.

Definition 1.24 (Classification of states)**.** [Chu67, pp.12-21, pp.182-187] *Let $X = (X(t) : t \in T)$ be a homogeneous Markov process with discrete state space E and time parameter set $T \subseteq \mathbb{R}$.*

- *State $j \in E$ is* accessible *from state $i \in E$ if $p^{(t)}(i,j) > 0$ for some $t > 0$.*
- *State $j \in E$* communicates *with state $i \in E$ if j is accessible from i and vice versa.*
- *A state $i \in E$ is* transient*, if*

  $$P(X(t) = i \text{ for some } t > 0 | X(0) = i) < 1,$$

  i.e., the probability to return to the state in finite time is less than one. Otherwise i is recurrent*.*
- *A recurrent state $i \in E$ is* positive recurrent*, if the mean time to return to state i for the first time is finite. Otherwise i is* null-recurrent*.*

A homogeneous Markov process is called *irreducible* if all states of its state space communicate with each other. If state i is recurrent and another state j is accessible from i, then j is recurrent as well. So if a process is irreducible and its state i is recurrent, then all other states of the state space are recurrent and the process is called recurrent. The same holds for transience, positive recurrence, and null-recurrence.

Definition 1.25. (e.g., [Ser09, p.259]) *A discrete-time homogeneous Markov process is ergodic if and only if it is irreducible, positive recurrent, and aperiodic.*
A continuous-time homogeneous Markov process is ergodic if and only if it is irreducible and positive recurrent.

The queue length process X of a Jackson network is an irreducible continuous-time homogeneous Markov process. Thus, X is ergodic if and only if all states are positive recurrent. Whenever X is not ergodic, the process cannot be positive recurrent. As a consequence the non-ergodic queue length process of a Jackson network must be either null-recurrent or transient.

In the literature, a non-ergodic network process is often called transient (see, e.g., [Wei05]), this does not automatically mean that the process actually is transient according to the above definition. Best to our knowledge it is still an open question under which conditions, in case of non-ergodic Jackson networks, the network process is transient or null-recurrent.

Throughout this chapter we will refer to a two nodes system which is defined as follows.

Definition 1.26 (The two-node model)**.** *Consider a Jackson network with two nodes where node* 1 *is stable and node* 2 *is unstable in the sense of Definition 1.17. Denote by* $\eta = (\eta_1, \eta_2)$ *the unique solution of the general traffic equations* (1.6) *which in this case are*

$$\eta_1 = \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1),$$
$$\eta_2 = \lambda_2 + \eta_1 r(1,2) + \mu_2 r(2,2).$$

Denote by $X = (X(t) = (X_1(t), X_2(t)) : t \geq 0)$ *the queue length process of the network on the state space* $\mathbb{N}^2$.

1.4.1 Matrix-geometrical approach

Matrix-geometric methods (see [Neu81]) are useful to analyze each level of a state space. The set of $\{(i, n_2) : i \in \mathbb{N}\}$ is called *level* n_2 of a two-dimensional state space. Clearly, considering the queue length process of a Jackson network as in Definition 1.26, it is only possible to move from some state (i, n_2) to some state $(j, n_2 + k)$ of the state space $\mathbb{N}^2$ $(-n_2 \leq k < \infty)$ by visiting all intermediate levels at least once. This property is also referred to as *skipfree to the left and to the right* ([Kei65]) for levels. Within a level, this property also holds in a Jackson network.

We consider the two-node model from Definition 1.26. The generator $\tilde{\mathbf{Q}}$ for level n_2 is then given by

n_2	0	1	2	3	$\cdots$
0	$\mathbf{B}$	$\mathbf{A}_0$	0	0	$\cdots$
1	$\mathbf{A}_2$	$\mathbf{A}_1$	$\mathbf{A}_0$	0	$\cdots$
2	0	$\mathbf{A}_2$	$\mathbf{A}_1$	$\mathbf{A}_0$	$\ddots$
$\vdots$	$\vdots$	$\ddots$	$\ddots$	$\ddots$	$\ddots$

This tridiagonal matrix has the structure of an infinitesimal generator of a *quasi-birth-death process*, see [Neu81, p.82], because state changes occur only between adjacent levels. $\tilde{\mathbf{Q}}$ is *level independent* because the rates at which transitions occur do not depend on the level. An overview on level-independent quasi-birth-death process is given in [Lat10]. The difference to the process presented in [Neu81] is that our process X has the state space $\mathbb{N}^2$, so the square matrices $\mathbf{B}$, $\mathbf{A_0}$, $\mathbf{A_1}$, and $\mathbf{A_2}$ have infinitely many rows and columns:

$\mathbf{B} =$

	$(0,0)$	$(1,0)$	$(2,0)$	$\cdots$
$(0,0)$	$-\lambda_1 - \lambda_2$	λ_1	0	$\cdots$
$(1,0)$	$\mu_1 r(1,0)$	$-\lambda_1 - \lambda_2 - \mu_1(1 - r(1,1))$	λ_1	$\cdots$
$(2,0)$	0	$\mu_1 r(1,0)$	$-\lambda_1 - \lambda_2 - \mu_1(1 - r(1,1))$	$\ddots$
$\vdots$	$\vdots \ddots$	$\ddots$	$\ddots$	

,

$\mathbf{A_0} =$

	$(0, n_2+1)$	$(1, n_2+1)$	$(2, n_2+1)$	$\cdots$
$(0, n_2)$	λ_2	0	0	$\cdots$
$(1, n_2)$	$\mu_1 r(1,2)$	λ_2	0	$\cdots$
$(2, n_2)$	0	$\mu_1 r(1,2)$	λ_2	$\ddots$
$\vdots$	$\vdots \ddots$	$\ddots$	$\ddots$	

,

$$\mathbf{A_2} = \begin{array}{c|ccccc} & (0,n_2-1) & (1,n_2-1) & (2,n_2-1) & (3,n_2-1) & \cdots \\ \hline (0,n_2) & \mu_2 r(2,0) & \mu_2 r(2,1) & 0 & 0 & \cdots \\ (1,n_2) & 0 & \mu_2 r(2,0) & \mu_2 r(2,1) & 0 & \cdots \\ (2,n_2) & 0 & 0 & \mu_2 r(2,0) & \mu_2 r(2,1) & \ddots \\ \vdots & \vdots\ddots & \ddots & \ddots & & \ddots \end{array}$$

and, with $\gamma := \mu_2(1 - r(2,2)) + \lambda_1 + \lambda_2$, matrix $\mathbf{A_1}$ is

$$\begin{array}{c|ccccc} & (0,n_2) & (1,n_2) & (2,n_2) & (3,n_2) & \cdots \\ \hline (0,n_2) & -\gamma & \lambda_1 & 0 & 0 & \cdots \\ (1,n_2) & \mu_1 r(1,0) & -\mu_1(1-r(1,1)) - \gamma & \lambda_1 & 0 & \cdots \\ (2,n_2) & 0 & \mu_1 r(1,0) & -\mu_1(1-r(1,1)) - \gamma & \lambda_1 & \ddots \\ \vdots & \vdots\ddots & \ddots & \ddots & & \ddots \end{array}.$$

To analyze the process related to the generator $\tilde{\mathbf{Q}}$, the matrix $\mathbf{A}$ is defined by the sum of the describing matrices $\mathbf{A_i}$, $\mathbf{A} := \mathbf{A_0} + \mathbf{A_1} + \mathbf{A_2}$, which is with $\nu := \lambda_1 + \mu_2 r(2,1)$:

$$\mathbf{A} = \begin{array}{c|ccccc} n_1 & 0 & 1 & 2 & 3 & \cdots \\ \hline 0 & -\nu & \nu & 0 & 0 & \cdots \\ 1 & \mu_1(1-r(1,1)) & -\mu_1(1-r(1,1)) - \nu & \nu & 0 & \cdots \\ 2 & 0 & \mu_1(1-r(1,1)) & -\mu_1(1-r(1,1)) - \nu & \nu & \cdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots \end{array}.$$

The row sum of $\mathbf{A}$ is zero, so $\mathbf{A}$ is conservative.

An interpretation of $\mathbf{A}$ can be given the following way: $\mathbf{A}$ is the transition rates matrix of a birth-death process where a birth means an increase of the queue length at node 1 and a death means a decrease of the queue length at node 1 regardlessly whether the transition changes the queue length at node 2 (and therefore the level) or not. That is why $\mathbf{A}$ is also called the *inter-level generator matrix*.

With $\eta_1 = \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1)$ it holds $\mathbf{A} =$

$$\begin{pmatrix} -\eta_1(1-r(1,1)) & \eta_1(1-r(1,1)) & 0 & 0\cdots \\ \mu_1(1-r(1,1)) & -\mu_1(1-r(1,1)) - \eta_1(1-r(1,1)) & \eta_1(1-r(1,1)) & 0\cdots \\ 0 & \mu_1(1-r(1,1)) & -\mu_1(1-r(1,1)) - \eta_1(1-r(1,1)) & \eta_1(1-r(1,1)) \\ \vdots\ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

$\mathbf{A}$ is the Q-matrix of a birth-death process with birth rates $\eta_1(1-r(1,1))$ and death rates $\mu_1(1-r(1,1))$. Because $\eta_1 < \mu_1$, the stationary distribution of this process is

$$\pi(n) = \left(1 - \frac{\eta_1}{\mu_1}\right) \cdot \left(\frac{\eta_1}{\mu_1}\right)^n.$$

It is striking that the stationary distribution related to $\mathbf{A}$ is the same as the marginal limiting distribution (1.11) for the stable node related to the Q-matrix of X.

If $\mathbf{A}$ is in equilibrium, $\pi \cdot \mathbf{A_0} \cdot e$ is the rate for a transition from some state (i, n_2) to the higher level $n_2 + 1$ which is given by

$$\begin{aligned}\pi \cdot \mathbf{A_0} \cdot e &= \sum_{n=0}^{\infty} \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^n \lambda_2 + \sum_{n=0}^{\infty} \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n+1} \mu_1 r(1,2) \\ &= \sum_{n=0}^{\infty} \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^n (\lambda_2 + \eta_1 r(1,2))) \\ &= \eta_2 - \mu_2 r(2,2).\end{aligned}$$

Analogously, if $\mathbf{A}$ is in equilibrium, $\pi \cdot \mathbf{A_2} \cdot e$ is the rate for a transition from some state (i, n_2) to the lower level $n_2 - 1$ (for $n_2 > 0$) which is

$$\begin{aligned}\pi \cdot \mathbf{A_2} \cdot e &= \sum_{n=0}^{\infty} \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^n \mu_2 r(2,0) + \sum_{n=1}^{\infty} \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n-1} \mu_2 r(2,1) \\ &= \sum_{n=0}^{\infty} \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^n \mu_2(1 - r(2,2)) \\ &= \mu_2(1 - r(2,2)).\end{aligned}$$

The birth and death rates of the process generated by $\mathbf{A}$ are strictly positive because $r(1,1) < 1$ is implied by the irreducibility of the routing process, therefore the birth and death process is irreducible.

In case that the matrices A_i would have a finite amount of rows and columns, with the irreducibility of $\mathbf{A}$ we would know from [Neu81, Theorem 1.7.1, p.32] that the process generated by $\tilde{\mathbf{Q}}$ is positive recurrent if and only if

$$\pi \cdot \mathbf{A_2} \cdot e > \pi \cdot \mathbf{A_0} \cdot e \quad \Leftrightarrow \quad \mu_2(1 - r(2,2)) > \eta_2 - \mu_2 r(2,2) \quad \Leftrightarrow \quad \mu_2 > \eta_2, \tag{1.13}$$

which would be a contradiction to the assumption that node 2 is unstable in the two-node model. Unfortunately enough, in our setting the matrices A_i have infinitely many rows and columns. According to G. Latouche and V. Ramaswami the "situation is more involved, and not satisfactorily settled yet, if [the number of rows and columns of the matrices A_i is] $m = \infty$"[LR99, p.147]. An approach for the case $m = \infty$ is given with [LR99, Theorem 16.2.2, p.310] which is based on [Twe82, Theorem 5]: If $\mathbf{A}$ is irreducible and if $\mathbf{B} = \mathbf{A}_1 + \mathbf{A}_2$ holds, then (1.13) is a sufficient condition for the process generated by $\tilde{\mathbf{Q}}$ being positive recurrent. Unfortunately $\mathbf{B} = \mathbf{A}_1 + \mathbf{A}_2$ does not hold in our setting. Anyways, the process generated by $\tilde{\mathbf{Q}}$ is not positive recurrent for the following reason: Notice that the generator $\tilde{\mathbf{Q}}$ is build only by restructuring the Q-matrix of the queue length process X of the two-node model which is done only by structuring its state space by the use of levels. But from the Definition 1.26 of the two-node model we know that its queue length process X is non-ergodic and therefore it is not positive recurrent.

Concluding this analysis via matrix-geometrical methods, there are interesting and striking remarks concerning analogies to our queue length process X:

- The generator $\tilde{\mathbf{Q}}$ is build only by restructuring the Q-matrix of X using levels.
- Although X is not ergodic and therefore does not have a stationary distribution, we get a stationary distribution for the $\mathbb{N}$-valued Markov process generated by $\mathbf{A}$, the inter-level generator matrix.
- This stationary distribution for the Markov process generated by $\mathbf{A}$, which is a limiting distribution for that process as well, equals the marginal limiting distribution (1.11) for the stable part of non-ergodic networks and, even more, the stationary distribution is obtained despite the assumption that node 2 is unstable. This is remarkable and we do not know yet what it means.

1.4.2 Analysis via martingale criteria

Foster's theorem gives a sufficient condition with *Lyapunov functions h* for a homogeneous Markov chain to be positive recurrent:

Theorem 1.27 (Foster's theorem)**.** [Bre99, Chapter 5, Theorem 1.1] *Let the transition probability matrix $P := (p(i,j) : i,j \in E)$ on the countable state space E be irreducible and suppose that there exists a function $h : E \to \mathbb{R}$ such that $\inf_i h(i) > -\infty$ and for some finite set $F \subseteq E$*

$$\sum_{k\in E} p(i,k)h(k) < \infty \quad \text{for all } i \in F,$$

$$\sum_{k\in E} p(i,k)h(k) \le h(i) - \varepsilon \quad \text{for all } i \notin F,$$

for some $\varepsilon > 0$. Then the corresponding time-homogeneous Markov chain is positive recurrent.

Similar criteria can be given for transience and null-recurrence. All of the following criteria are referred to as martingale criteria, because they are proved with martingale theory, see [Bre99, Chapter 5, Section 3.2]. Note that $\sum_{k\in E} p(i,k)h(k) = \mathbb{E}(h(X(n+1))|X(n) = i)$.

Theorem 1.28. [Bre99, Chapter 5, Theorem 3.4] *A necessary and sufficient condition for an irreducible time-homogeneous Markov chain to be* transient *is the existence of some state conventionally called 0 and of a bounded function $h : E \to \mathbb{R}$, not identically null and satisfying*

$$h(j) = \sum_{k\neq 0} p(j,k)h(k) \quad \text{for all } j \neq 0.$$

If we consider a non-ergodic Jackson network, the following recurrence criterion can be used as a criterion for null-recurrence:

Theorem 1.29. [Bre99, Chapter 5, Theorem 3.5] *Let the time-homogeneous Markov chain with transition probability matrix P be irreducible, and suppose that there exists a function $h : E \to \mathbb{R}$ such that $\{i : h(i) < K\}$ is finite for all finite K, and such that*

$$\sum_{k\in E} p(i,k)h(k) \le h(i) \text{ for all } i \notin F,$$

for some finite subset $F \subset E$. Then the process is recurrent.

The following theorem is an alternative to Theorem 1.28 as a criterion for transience.

Theorem 1.30. [Bre99, Chapter 5, Theorem 3.6] *Let the time-homogeneous Markov chain with transition probability matrix P be irreducible and let $h : E \to \mathbb{R}$ be a bounded function such that*

$$\sum_{k \in E} p(i,k)h(k) \leq h(i) \text{ for all } i \notin F, \tag{1.14}$$

for some subset $F \subset E$, not necessarily finite. Suppose, moreover, that there exists $i \notin F$ such that

$$h(i) < h(j) \quad \text{for all } j \in F. \tag{1.15}$$

Then the process is transient.

All these criteria are for Markov chains, i.e., discrete-time Markov processes, while our queue length process is a continuous-time Markov process. This inconformity can be remedied by the method of uniformization, as defined in, e.g., [Kij97, p.195].

Definition 1.31 (The two-node model with uniformized queue length process)**.** *Consider the two-node model from Definition 1.26. Since all entries in the main diagonal of the Q-matrix are bounded, the queue length process X is uniformizable with some uniformization constant $\xi \geq \sup_z -q(z,z)$, $z \in \mathbb{N}^2$. With*

$$\xi \geq \sum_{i=1}^{2} (\lambda_i + \mu_i)$$

as the uniformization constant, a Poisson-clock with intensity ξ generates the jump times of the uniformized process. The jump kernel of the uniformization is then given by

$$p^u = \mathbf{I} + \frac{1}{\xi} \cdot Q.$$

The relationship of the original continuous-time process and the process after uniformization can be expressed as follows: Let $(N_\xi(t) : t \geq 0)$ denote a Poisson process with rate ξ, then the processes $(X^u(N_\xi(t)) : t \geq 0)$ and $(X(t) : t \geq 0)$ are equal in distribution, see [Kij97, Theorem 4.19]. The advantage of the uniformization is that an analysis of the continuous-time process X is possible with discrete time techniques. Thus, we can check whether one of the above discrete-time criteria for recurrence or transience applies for the non-ergodic X or not.

The following proposition shows how the criteria can be used to establish conditions under which the non-ergodic process is - in this case - transient. Other conditions obtained in the same manner are conceivable.

Proposition 1.32. *Consider the two-node model with uniformized queue length process X on the state space $\mathbb{N}^2$ and jump kernel p^u from Definition 1.31. Let for the parameters of the unstable node 2 hold*

$$\mu_2(1 - r(2,2)) \leq \frac{1}{3} \cdot \lambda_2, \tag{1.16}$$

then the Markov chain is transient.

Proof. We define the function $h : \mathbb{N}^2 \to \mathbb{R}$ by

$$h(n_1, n_2) := \frac{1}{1+n_2}.$$

h is bounded, because for all possible states $(n_1, n_2) \in \mathbb{N}^2$ holds $0 < h(n_1, n_2) \leq 1$. Furthermore we define $F := \mathbb{N} \times \{0\}$ and show that with these definitions the requirements (1.14) and (1.15) of Theorem 1.30 are fulfilled:

For all $(i_1, i_2) \in \mathbb{N}_+ \times \mathbb{N}_+ \subset F^c$ holds

$$\begin{aligned}
&\sum_{(k_1,k_2)\in\mathbb{N}^2} p^u(i_1,i_2;k_1,k_2)h(k_1,k_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} p^u(i_1,i_2;k_1,k_2)\frac{1}{1+k_2} \\
&= \sum_{k_1=i_1-1}^{i_1+1}\sum_{k_2=i_2-1}^{i_2+1} p^u(i_1,i_2;k_1,k_2)\frac{1}{1+k_2} \\
&= \frac{1}{1+i_2-1}\cdot\left(\frac{\mu_2 r(2,0)}{\xi} + \frac{\mu_2 r(2,1)}{\xi}\right) + \frac{1}{1+i_2+1}\cdot\left(\frac{\lambda_2}{\xi} + \frac{\mu_1 r(1,2)}{\xi}\right) + \\
&\ + \frac{1}{1+i_2}\cdot\left(\frac{\lambda_1}{\xi} + \frac{\mu_1 r(1,0)}{\xi} + \frac{\xi - \lambda_1 - \lambda_2 - \mu_1(1-r(1,1)) - \mu_2(1-r(2,2))}{\xi}\right) \\
&= \frac{1}{1+i_2-1}\cdot\frac{\mu_2(1-r(2,2))}{\xi} + \frac{1}{1+i_2+1}\cdot\frac{\lambda_2+\mu_1 r(1,2)}{\xi} + \\
&\ + \frac{1}{1+i_2}\cdot\left(1 - \frac{\lambda_2+\mu_1 r(1,2)}{\xi} - \frac{\mu_2(1-r(2,2))}{\xi}\right) \\
&= \frac{1}{1+i_2} + \frac{\lambda_2+\mu_1 r(1,2)}{\xi}\left(\frac{1}{2+i_2} - \frac{1}{1+i_2}\right) + \frac{\mu_2(1-r(2,2))}{\xi}\left(\frac{1}{i_2} - \frac{1}{1+i_2}\right) \\
&= \frac{1}{1+i_2} + \frac{\lambda_2+\mu_1 r(1,2)}{\xi}\cdot\frac{1+i_2-2-i_2}{(1+i_2)(2+i_2)} + \frac{\mu_2(1-r(2,2))}{\xi}\cdot\frac{1+i_2-i_2}{i_2(1+i_2)} \\
&= \frac{1}{1+i_2} + \frac{\mu_2(1-r(2,2))}{\xi}\cdot\frac{1}{i_2(1+i_2)} - \frac{\lambda_2+\mu_1 r(1,2)}{\xi}\cdot\frac{1}{(1+i_2)(2+i_2)} \\
&= \frac{1}{1+i_2} + \frac{1}{\xi i_2(1+i_2)}\left(\mu_2(1-r(2,2)) - (\lambda_2+\mu_1 r(1,2))\cdot\frac{i_2}{2+i_2}\right) \\
&\overset{(1.16)}{\leq} \frac{1}{1+i_2} + \frac{1}{\xi i_2(1+i_2)}\underbrace{\left(\frac{1}{3}\lambda_2 - (\lambda_2+\mu_1 r(1,2))\cdot\frac{i_2}{2+i_2}\right)}_{\leq 0,\text{ since } \frac{i_2}{2+i_2}\in[\frac{1}{3},1]} \\
&\leq \frac{1}{1+i_2} = h(i_1,i_2).
\end{aligned}$$

For all $(0, i_2) \in \{0\} \times \mathbb{N}_+ \subset F^c$ holds

$$\sum_{(k_1,k_2)\in\mathbb{N}^2} p^u(0, i_2; k_1, k_2) h(k_1, k_2) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} p^u(0, i_2; k_1, k_2) \frac{1}{1+k_2}$$

$$= \sum_{k_1=0}^{1} \sum_{k_2=i_2-1}^{i_2+1} p^u(0, i_2; k_1, k_2) \frac{1}{1+k_2}$$

$$= \frac{1}{1+i_2-1} \cdot \left(\frac{\mu_2 r(2,0)}{\xi} + \frac{\mu_2 r(2,1)}{\xi} \right) + \frac{1}{1+i_2+1} \cdot \frac{\lambda_2}{\xi} +$$

$$+ \frac{1}{1+i_2} \cdot \left(\frac{\lambda_1}{\xi} + \frac{\xi - \lambda_1 - \lambda_2 - \mu_2(1 - r(2,2))}{\xi} \right)$$

$$= \frac{1}{1+i_2-1} \cdot \frac{\mu_2(1-r(2,2))}{\xi} + \frac{1}{1+i_2+1} \cdot \frac{\lambda_2}{\xi} + \frac{1}{1+i_2} \cdot \left(1 - \frac{\lambda_2}{\xi} - \frac{\mu_2(1-r(2,2))}{\xi} \right)$$

$$= \frac{1}{1+i_2} + \frac{\lambda_2}{\xi} \left(\frac{1}{2+i_2} - \frac{1}{1+i_2} \right) + \frac{\mu_2(1-r(2,2))}{\xi} \left(\frac{1}{i_2} - \frac{1}{1+i_2} \right)$$

$$= \frac{1}{1+i_2} + \frac{\mu_2(1-r(2,2))}{\xi} \cdot \frac{1}{i_2(1+i_2)} - \frac{\lambda_2}{\xi} \cdot \frac{1}{(1+i_2)(2+i_2)}$$

$$= \frac{1}{1+i_2} + \frac{1}{\xi i_2(1+i_2)} \left(\mu_2(1-r(2,2)) - \lambda_2 \cdot \frac{i_2}{2+i_2} \right)$$

$$\overset{(1.16)}{\leq} \frac{1}{1+i_2} + \frac{1}{\xi i_2(1+i_2)} \underbrace{\left(\frac{1}{3}\lambda_2 - \lambda_2 \cdot \frac{i_2}{2+i_2} \right)}_{\leq 0, \text{ since } \frac{i_2}{2+i_2} \in [\frac{1}{3}, 1]}$$

$$\leq \frac{1}{1+i_2} = h(0, i_2).$$

Thus (1.14) is valid. It remains to show (1.15): For all $(i_1, i_2) \in F^c = \mathbb{N} \times \mathbb{N}_+$ holds

$$h(i_1, i_2) = \frac{1}{1+i_2} < 1 = \frac{1}{1+0} = h(j_1, 0) = h(j_1, j_2) \quad \forall (j_1, j_2) \in F = \mathbb{N} \times \{0\}.$$

□

Chapter 2

Jackson networks with breakdowns: The ergodic case

2.1 Introduction

In[2] complex networks, nodes may be not accessible for some time because, e.g.,

- the server is broken down and under repair,
- the (mobile) station has left the network and is replaced sometime later by some other station,
- the server is serving customers from another network, etc.

Such nodes are called *unreliable*. The theory of such networks with unreliable nodes is part of performability theory which provides us with a tool set of well established methods and algorithms for an integrated approach to performance analysis and reliability with availability. For an overview see [HMRT01]. Our description here is with reliability arguments: We describe an integrated model for performance analysis and reliability/availability of Jackson networks with unreliable nodes, the description follows [SD03], more details can be found in [Sau06]. [MD09, p.1252]

We start from a classical Jackson network as described in Definition 1.1. We supplement the state space $\mathbb{N}^J$ for the joint queue lengths by an additional coordinate which describes the availability of the stations. If stations in $D \subseteq \tilde{J}$ can break down, the availability information is of a generic form

$$I \subseteq D \subseteq \tilde{J}$$

which indicates that the stations in I are not available (under repair), while stations in $\tilde{J} \backslash I$ are available and can serve customers. The joint queue-lengths vector combined with this knowledge will provide us with sufficient information for a Markovian description of the system. Let $\mathcal{P}(D)$ denote the set of all subsets of $D \subseteq \tilde{J}$. Then the states of our Markov process are of the form

$$(I; n_1, n_2, ..., n_J) \in \mathcal{P}(D) \times \mathbb{N}^J,$$

[2]Section 2.1 with its subsections 2.1.1-2.1.2 quote the pages 1252-1254 of our paper [MD09] which will be shortly marked after the corresponding paragraphs. The only difference to the paper is the definition of the set D of unreliable nodes.

and $\mathcal{P}(D) \times \mathbb{N}^J$ will serve as state space for the joint availability–queue-length process (Y, X). The process Y on $\mathcal{P}(D)$ indicates the availability status of the network and X on $\mathbb{N}^J$ the queue lengths in the network. [MD09, p.1252]

Remark 2.1. *Unlike the description in, e.g., [SD03] where all nodes may be unreliable, we denote the set of unreliable nodes by D which may be also equal to the node set $\tilde{J}$. For this chapter it does not make any difference but in the following Chapters 4 and 5 this definition of the set D of unreliable nodes in the network enables us to be more precise with side constraints which, as can be seen later on, must hold for unreliable nodes only.*

2.1.1 Breakdown and repair rates

Incorporating the availability status of the network into the Markovian description we have to specify transition rates for jumps of Y out of a state I into a successor state I'. Two cases have to be considered for generic transitions $I \to I'$:

- I increases by breakdown of nodes outside of I, and
- I decreases by successful repair of nodes inside of I.

Increase and decrease of I are with respect to set-inclusion. In general, the transition rates for increase and decrease of I will depend on the load ($\equiv$ queue lengths) of the nodes in I, possibly even of neighboring nodes.
As an introductory example we describe a rather general situation where breakdown and repair are independent of the load. A first approach to specify the rates for the set-valued process Y is suggested by noticing that we can encode the sets $I \in \mathcal{P}(D)$ by vectors from $\{0,1\}^D$. The bijection is given by

$$[(x_j : j \in D) \in \{0,1\}^D \Leftrightarrow I \in \mathcal{P}(D)] \Longleftrightarrow [x_i = 1 \Leftrightarrow i \in I].$$

Having this in mind we can represent the process Y as a multi-dimensional birth-and-death process on $\{0,1\}^D$ with concurrent births and concurrent deaths. The theory of multi-dimensional birth-and-death processes will provide us with intensities for changing the I-coordinate of the network process. Because we are interested in closed-form solutions for the equilibrium probabilities, we seek for birth and death intensities with steady states that can be given in closed form. An astonishing simple recipe from birth-and-death processes (transformed to our situation) can be described in terms of the set-valued process Y as follows. [MD09, pp.1252-1253]

Example 2.2. [SD03, Example 8(a)] *Take any pair of non-negative functions*

$$A : \mathcal{P}(D) \to [0, \infty) \text{ and } B : \mathcal{P}(D) \to [0, \infty),$$

subject to $A(\emptyset) = 1$ and $B(\emptyset) = 1$ and for all $K \subseteq D \subseteq \tilde{J}$ (recall $0/0 = 0$)

$$\frac{A(K \cup G)}{A(K)} < \infty \quad \forall K \cap G = \emptyset \quad \text{and} \quad \frac{B(K)}{B(K \setminus H)} < \infty \quad \forall H \subseteq K.$$

With these functions we can define breakdown rates (death rates) $\alpha(\cdot,\cdot)$ and repair rates (birth rates) $\beta(\cdot,\cdot)$ as follows:

For all subsets of down nodes $K \subseteq D$ we set

$$\alpha(K, K \cup G) = \frac{A(K \cup G)}{A(K)}, \quad K \cap G = \emptyset, \quad \text{and} \quad \beta(K, K \setminus H) = \frac{B(K)}{B(K \setminus H)}, \quad H \subseteq K.$$

By inspection we see that

$$\pi := \left(\pi(K) := \frac{A(K)}{B(K)},\ K \in \mathcal{P}(D)\right)$$

fulfills

$$\pi(K) \cdot \alpha(K, K \cup G) = \pi(K \cup G) \cdot \beta(K \cup G, K)$$

for all $K, G \in \mathcal{P}(D)$ which implies that, after normalization, π is the steady state of the breakdown and repair process. Even more, we have proved that the process Y is reversible.
A statistical procedure to derive the rates $\alpha(\cdot,\cdot), \beta(\cdot,\cdot)$ and to verify whether they are of the functional form is as follows:
$\alpha(\cdot), \beta(\cdot)$ are intensities for the process Y which is in this form a Markov process for its own. Therefore we can estimate the rates with standard methods of statistics for stochastic processes.
By estimating for any $K \subseteq D$

$$\alpha(\emptyset, K) = \frac{A(K)}{A(\emptyset)} = \frac{A(K)}{1}, \quad \beta(K, \emptyset) = \frac{B(K)}{B(\emptyset)} = \frac{B(K)}{1},$$

we obtain the candidates for the functions $A(\cdot), B(\cdot)$, and are then in a position by estimating the intensities $\alpha(K, K \cup G)$ and $\beta(K, K \setminus H)$ for all combinations K, G, H to test whether the functional form of this example is relevant for the system under investigation. (A similar approach is possible for the more complicated functions of Definition 2.3.)

Note that the (simple) structure of the intensities in this example already allows very general correlations between nodes in the breakdown and repair process.
Further special intensities for multi-dimensional birth-and-death processes are provided by Serfozo [Ser93, Table 1]. These can be incorporated into our breakdown and repair processes in a similar procedure.

Breakdown and repair intensities in Example 2.2 may depend on the interaction of nodes but not on their load.
Breakdown and repair rates which depend on the load of the nodes can be constructed following the advice given by the recipe which is behind Example 2.2 in [SD03].

Definition 2.3. *Assume that stations in the set $K \in \mathcal{P}(D)$ are down and under repair. Then the breakdown intensity (rate) of a non-void subset $G \subseteq D \setminus K$ is*

$$\alpha(K, K \cup G, n_i : i \in \tilde{J}) := \frac{A(K \cup G, n_i : i \in K \cup G)}{A(K, n_i : i \in K)}.$$

In the same availability status K of the network the repair intensity of a non-void subset $H \subseteq K$ is

$$\beta(K, K \setminus H, n_i : i \in \tilde{J}) := \frac{B(K, n_i : i \in K)}{B(K \setminus H, n_i : i \in K \setminus H)}.$$

Here A and B are non-negative functions

$$A, B : \bigcup_{I \subseteq D} (\{I\} \times \mathbb{N}^{|I|}) \to [0, \infty),$$

with $A(\emptyset, n_i : i \in \emptyset) := B(\emptyset, n_i : i \in \emptyset) := 1$.
We require all intensities $\alpha(K, K \cup G, n_i : i \in \tilde{J})$ *and* $\beta(K, K \setminus H, n_i : i \in \tilde{J})$ *to be finite.*

Some further special cases:

Example 2.4 (Locally determined breakdown and repair rates)**.** [SD03, Example 7] *This is an example where breakdowns and repairs are locally determined and the local intensities depend on the local queue length of a station only.*
For any node $i \in \tilde{J}$ *we specify non-negative functions*

$$a_i : \mathbb{N} \to [0, \infty) \text{ and } b_i : \mathbb{N} \to [0, \infty).$$

We assume intensities are of product form and set for all combinations of nodes $K \subseteq D \subseteq \tilde{J}$

$$\alpha(K, K \cup G, n_i : i \in \tilde{J}) = \prod_{i \in G} a_i(n_i), \quad K \cap G = \emptyset, G \subseteq D,$$
$$\beta(K, K \setminus H, n_i : i \in \tilde{J}) = \prod_{i \in H} b_i(n_i), \quad H \subseteq K.$$

In the setting of Definition 2.3 we take

$$A(I; n_1, \ldots, n_J) = \prod_{i \in I} a_i(n_i) \text{ and } B(I; n_1, \ldots, n_J) = \prod_{i \in I} b_i(n_i),$$

which leads to the given $\alpha(\cdot, \cdot)$ *and* $\beta(\cdot, \cdot)$. *The necessary restrictions are automatically fulfilled.*

Example 2.5 (Vacation system)**.** [SD03, Example 8(b)] *A breakdown at a station only occurs if the node is empty, i.e., there is no customer waiting or in service at this station. The breakdown and repair rates in such so-called vacation systems are for all* $G, K \subseteq D$, $K \cap G = \emptyset$ *and* $H \subseteq K$

$$\alpha(K, K \cup G, n_i : i \in \tilde{J}) = \frac{A(K \cup G) \prod_{i \in K \cup G} \delta_{0 n_i}}{A(K) \prod_{i \in K} \delta_{0 n_i}},$$
$$\beta(K, K \setminus H, n_i : i \in \tilde{J}) = \frac{B(K) \prod_{i \in K} \delta_{0 n_i}}{B(K \setminus H) \prod_{i \in K \setminus H} \delta_{0 n_i}}.$$

2.1.2 Rerouting in case of breakdowns

In case of broken down nodes we have to impose additional regulations on the movement of customers. We assume that at down nodes

- no service is provided,

- customers which are present at a node when this node breaks down are frozen and stay there, waiting for service until the node is repaired,
- no new customers are admitted, customers who want to visit a down node are rerouted according to one of the following rules,
- when a broken down node is repaired, service of the first customer in line is resumed with the residual service time the customer had reached when being interrupted.

Definition 2.6 (Stalling)**.** [SD03, p.180] *Whenever a node breaks down the system is frozen, i.e., all arrival processes are interrupted and the service anywhere in the network is stopped. Thus every movement of customers inside the network and arrivals to the network from the outside are stopped until all broken down nodes are repaired again, i.e., if nodes in* $I \neq \emptyset$ *are broken down then for all* $i \in \tilde{J}$ *holds* $\lambda_i^I = \mu_i^I = 0$*. We assume that the stopped nodes which are in up status are waiting in warm standby, i.e., they can break down although they are stalled.*

Stalling as a reaction on detected failure is implemented in many complex production systems to guarantee high quality of production, e.g., in automotive industry. A successful adaption of the stalling regime in case of failures can be found in the Toyota Production System where workers on the assembly line are able to interrupt the whole production process whenever a defective or low-quality product is found, see [Shi89, p.74].

Definition 2.7 (Skipping)**.** [SD03, p.183] *Customers are not allowed to enter down nodes and have to skip these nodes. I.e., if the next destination of a customer is a down node, the customer jumps to this node spending no time there and immediately performs the next jump according to his routing regime until he arrives at a node in up status or leaves the network. Whenever a breakdown of a subset* $I \subseteq D$ *occurs, customers are rerouted according to the following routing matrix* $R^I = (r^I(i,j) : i,j \in \{0\} \cup \tilde{J} \setminus I)$*:*

$$r^I(j,k) = r(j,k) + \sum_{i \in I} r(j,i) r^I(i,k) \quad \text{for } k,j \in \{0\} \cup \tilde{J} \setminus I \tag{2.1}$$

with

$$r^I(i,k) = r(i,k) + \sum_{l \in I} r(i,l) r^I(l,k) \quad \text{for } i \in I, k \in \{0\} \cup \tilde{J} \setminus I. \tag{2.2}$$

The external arrival rates during a breakdown of I *are*

$$\lambda_j^I = \lambda_j + \sum_{i \in I} \lambda_i r^I(i,j) \quad \text{for } j \in \tilde{J} \setminus I \tag{2.3}$$

and $\lambda_k^I = 0$ *for* $k \in I$*. The service intensities are*

$$\mu_i^I = \begin{cases} \mu_i, & i \in \tilde{J} \setminus I, \\ 0, & \text{otherwise.} \end{cases}$$

Skipping is a typical reaction in routing processes on graphs when a vertex has disappeared. In Markov chain theory it emerges when taboo sets occur in the state space of the chain. [MD09, p.1254]

Definition 2.8 (Blocking rs-rd)**.** [SD03, p.176] *Broken down stations are blocked. A customer whose next destination is a down node stays at his present node to obtain another service there. After the repeated service (rs) the customer chooses his next destination anew according to his routing matrix (random destination (rd)). Whenever a breakdown of a subset $I \subseteq D$ occurs, customers are rerouted according to the following routing matrix $R^I = (r^I(i,j) : i,j \in \{0\} \cup \tilde{J} \setminus I)$ with*

$$r^I(i,j) = \begin{cases} r(i,j), & i,j \in \{0\} \cup \tilde{J} \setminus I,\ i \neq j, \\ r(i,j) + \sum_{k \in I} r(i,k), & i \in \{0\} \cup \tilde{J} \setminus I, \quad i = j. \end{cases} \tag{2.4}$$

The external arrival rates during a breakdown of I are $\lambda_j^I = \lambda_j$ for $j \in \tilde{J} \setminus I$ and $\lambda_j^I = 0$ otherwise as well as the service intensities are

$$\mu_i^I = \begin{cases} \mu_i, & i \in \tilde{J} \setminus I, \\ 0, & \textit{otherwise}. \end{cases}$$

Repeated service in case of a blocked departure from some node due to full buffer at the destination node is used in models for telecommunications systems. There are several other blocking regimes in the literature like rejection blocking, retransmission, repeat blocking. More details can be found in [BDO01, Section 2.2].
When blocking rs-rd was used in modeling a protocol for resolving blocking, it was observed that in the Jackson networks under consideration the routing had to be reversible when a product form modeling was required. The same structural requirement turned out to be essential in the investigations of unreliable Jackson networks described in [SD03]. [MD09, p.1254]

2.1.3 The model

Definition 2.9 (Jackson network with unreliable nodes)**.** *Consider a Jackson network with node set $\tilde{J} = \{1, ..., J\}$, i.e., for each node j we have*

- *external Poisson(λ_j)-arrival streams ($\sum_{j \in \tilde{J}} \lambda_j = \lambda > 0$),*
- *single servers with exponential(μ_j) distributed service time,*
- *infinite waiting room,*
- *first-come-first-served regime.*

Customers are indistinguishable. All interarrival and service times constitute a set of independent random variables.
Routing is Markovian: Given the departure node i the selection of the next node is independent of the previous history. A customer departing from node i immediately proceeds to node j with probability $r(i,j) \geq 0$ and departs from the network with probability $r(i,0)$ (the artificial node 0 represents the outside, source and sink, of the network, $r(0,0) := 0$, $r(0,i) := \lambda_i/\lambda$). The routing matrix $R = (r(i,j) : i,j \in \{0,1,...,J\})$ is stochastic and irreducible.
Nodes from the set $D \subseteq \tilde{J}$ are unreliable and break down randomly with intensities prescribed by Definition 2.3 and are repaired with the intensities defined there. Customers

are rerouted according to either stalling, skipping, or blocking rs-rd as in Definitions 2.6 - 2.8.
Then the joint availability-queue lengths process is described by the Markov process

$$(Y, X) = ((Y(t); X_1(t), ..., X_J(t)) : t \in \mathbb{R}_+) \text{ on the state space } \mathcal{P}(D) \times \mathbb{N}^J$$

with transition rates matrix $Q = (q(z, z') : z, z' \in \mathcal{P}(D) \times \mathbb{N}^J)$ which is defined in dependence on the according rerouting regime by:
For all $i, j \in \tilde{J}, i \neq j$:

$$\begin{aligned}
&q(\emptyset, n_1, ..., n_i, ..., n_J; \emptyset, n_1, ..., n_i + 1, ..., n_J) = \lambda_i,\\
&q(\emptyset, n_1, ..., n_i, ..., n_J; \emptyset, n_1, ..., n_i - 1, ..., n_J) = \mu_i r(i, 0) 1_{\mathbb{N}_+}(n_i),\\
&q(\emptyset, n_1, ..., n_i, ..., n_j, ..., n_J; \emptyset, n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) = \mu_i r(i, j) 1_{\mathbb{N}_+}(n_i),\\
&q(\emptyset, n_1, ..., n_J; \emptyset, n_1, ..., n_J) = -\sum_{i \in \tilde{J}} \lambda_i - \sum_{i \in \tilde{J}} \mu_i (1 - r(i, i)) 1_{\mathbb{N}_+}(n_i)\\
&\qquad\qquad - \sum_{I \subseteq D} \alpha(\emptyset, n_1, ..., n_J; I, n_1, ..., n_J),\\
&q(\emptyset, n_1, ..., n_J; I, n_1, ..., n_J) = \alpha(\emptyset, n_1, ..., n_J; I, n_1, ..., n_J), \quad I \subseteq D,
\end{aligned}$$

and for all $\emptyset \neq I \subseteq D, i, j \in \tilde{J} \setminus I, i \neq j$:

$$\begin{aligned}
&q(I, n_1, ..., n_i, ..., n_J; I, n_1, ..., n_i + 1, ..., n_J) = \lambda_i^I,\\
&q(I, n_1, ..., n_i, ..., n_J; I, n_1, ..., n_i - 1, ..., n_J) = \mu_i^I r^I(i, 0) 1_{\mathbb{N}_+}(n_i),\\
&q(I, n_1, ..., n_i, ..., n_j, ..., n_J; I, n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) = \mu_i^I r^I(i, j) 1_{\mathbb{N}_+}(n_i),\\
&q(I, n_1, ..., n_J; I, n_1, ..., n_J) = -\sum_{i \in \tilde{J} \setminus I} \lambda_i^I - \sum_{i \in \tilde{J} \setminus I} \mu_i^I (1 - r^I(i, i)) 1_{\mathbb{N}_+}(n_i)\\
&\qquad\qquad - \sum_{I \subset H \subseteq D} \alpha(I, n_1, ..., n_J; H, n_1, ..., n_J)\\
&\qquad\qquad - \sum_{K \subset I \subseteq D} \beta(I, n_1, ..., n_J; K, n_1, ..., n_J),\\
&q(I, n_1, ..., n_J; H, n_1, ..., n_J) = \alpha(I, n_1, ..., n_J; H, n_1, ..., n_J), \quad I \subset H \subseteq D,\\
&q(I, n_1, ..., n_J; K, n_1, ..., n_J) = \beta(I, n_1, ..., n_J; K, n_1, ..., n_J), \quad K \subset I \subseteq D,
\end{aligned}$$

and $q(z, z') = 0$ otherwise.

2.2 The traffic equations

Definition 2.10. *The traffic equations for unreliable Jackson networks are:*

- *In case of stalling:*

$$\eta_i = \lambda_i + \sum_{j \in \tilde{J}} \eta_j r(j, i), \ i \in \tilde{J}, \tag{2.5}$$

as long as all nodes are in up status ($I = \emptyset$). Otherwise $\eta_i^I = 0$ for all $i \in \tilde{J}$. (Note that (2.5) equals (1.1).)

- *In case of rerouting according to blocking rs-rd or skipping:*

$$\eta_i^I = \lambda_i^I + \sum_{j\in\tilde{J}\setminus I} \eta_j^I r^I(j,i), \ \ i \in \tilde{J}\setminus I, \tag{2.6}$$

for all $I \subseteq D$. If $I = \emptyset$ the traffic equations (2.6) *are equal to* (1.1).

In general, a solution of such traffic equations, if it exists, does not reveal the overall arrival rates in the system since the traffic equations for some I remain in force only as long as the availability status is unchanged. Whenever the availability status of the system changes, the traffic equations are adapted according to the new set of broken down nodes. Thus each traffic equation (2.6) may have different solutions for different I. The following two lemmata show when the solution of the traffic equation (2.6) remains the same for all $I \subseteq D$.

Lemma 2.11. [SD03, Proof of Theorem 5.2, p.179] *Consider a Jackson network where nodes in $D \subseteq \tilde{J}$ are unreliable. Let for all nodes hold $\eta_i < \mu_i$ where $(\eta_i : i \in \tilde{J})$ is the unique solution of* (1.1). *In case of breakdowns of nodes we assume that customers are rerouted according to the* **blocking rs-rd** *regime. If the following reversibility constraints hold:*

$$\eta_i r(i,j) = \eta_j r(j,i) \quad \forall i,j \in \tilde{J}, \tag{2.7}$$

then for all nodes $i \in \tilde{J}\setminus I$ holds that the solution η_i^I of the traffic equation (2.6) *for all $I \subseteq D$, $I \neq \emptyset$, equals η_i.*

Proof. We make the ansatz $\eta_i = \eta_i^I$ for all $i \in \tilde{J}\setminus I$ and all $I \subseteq D$ in (2.6). We then obtain with the solution η_i of the traffic equations (1.1) for any $I \subseteq D$: $\forall i \in \tilde{J}\setminus I$

$$\begin{aligned}\eta_i &= \lambda_i^I + \sum_{j\in\tilde{J}\setminus I} \eta_j r^I(j,i) \overset{(2.4)}{=} \lambda_i + \eta_i\Big(r(i,i) + \sum_{k\in I} r(i,k)\Big) + \sum_{j\in\tilde{J}\setminus I, j\neq i} \eta_j r(j,i)\\ &= \lambda_i + \sum_{k\in I} \underbrace{\eta_i r(i,k)}_{\overset{(2.7)}{=}\eta_k r(k,i)} + \sum_{j\in\tilde{J}\setminus I} \eta_j r(j,i) = \lambda_i + \sum_{j\in\tilde{J}} \eta_j r(j,i) = (1.1).\end{aligned}$$

□

Remark 2.12. *The reversibility constraints* (2.7) *are the local balance equations of the routing process.*

Lemma 2.13. [SD03, Proof of Theorem 6.4, p.184] *Consider a Jackson network where nodes in $D \subseteq \tilde{J}$ are unreliable. Let for all nodes $i \in \tilde{J}$ hold $\eta_i < \mu_i$ where $(\eta_i : i \in \tilde{J})$ is the unique solution of* (1.1). *In case of breakdowns of nodes we assume that customers are rerouted according to the* **skipping** *regime. Then for all nodes $i \in \tilde{J}\setminus I$ holds that the solution η_i^I of the traffic equation* (2.6) *for all $I \subseteq D$, $I \neq \emptyset$, equals η_i.*

Proof. We make the ansatz $\eta_i = \eta_i^I$ for all $i \in \tilde{J}\setminus I$ and all $I \subseteq D$ in (2.6). We then obtain with the solution η_i of the traffic equations (1.1) for any $I \subseteq D$: $\forall i \in \tilde{J} \setminus I$

$$\begin{aligned}
\lambda_i^I + \sum_{j \in \tilde{J} \setminus I} \eta_j r^I(j,i) &\overset{\text{(Def. 2.7)}}{=} \lambda_i + \sum_{k \in I} \lambda_k r^I(k,i) + \sum_{j \in \tilde{J} \setminus I} \eta_j \Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big) \\
&= \underbrace{\lambda_i + \sum_{j \in \tilde{J} \setminus I} \eta_j r(j,i)}_{\overset{(1.1)}{=} \eta_i - \sum_{j \in I} \eta_j r(j,i)} + \sum_{k \in I} r^I(k,i) \underbrace{\Big(\lambda_k + \sum_{j \in \tilde{J} \setminus I} \eta_j r(j,k) \Big)}_{\overset{(1.1)}{=} \eta_k - \sum_{j \in I} \eta_j r(j,k)} \\
&= \eta_i - \sum_{j \in I} \eta_j r(j,i) + \sum_{k \in I} \eta_k r^I(k,i) - \sum_{k \in I} r^I(k,i) \sum_{j \in I} \eta_j r(j,k) \\
&= \eta_i - \sum_{j \in I} \eta_j \underbrace{\Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big)}_{\overset{(2.1)}{=} r^I(j,i)} + \sum_{k \in I} \eta_k r^I(k,i) = \eta_i.
\end{aligned}$$

□

2.3 Steady-state results

Theorem 2.14. [SD03, Theorems 5.2, 5.5, and 6.4] *Consider a Jackson network with unreliable nodes as in Definition 2.9, where the set with unreliable nodes is denoted by $D \subseteq \tilde{J}$. Let $\eta = (\eta_1, ..., \eta_J)$ be the solution of the traffic equation* (1.1).
In case of breakdown of nodes we assume either that stalling is applied or that the skipping regime is in force or that the blocking rs-rd regime is in force. In case of blocking rs-rd, we require that routing fulfills the reversibility condition (2.7).
If the availability–queue-lengths process (Y, X) is ergodic, it has a unique stationary and limiting distribution:
For all $(I; n_1, ..., n_J) \in \mathcal{P}(D) \times \mathbb{N}^J$

$$\pi(I; n_1, ..., n_J) = \; C^{-1} \, \frac{A(I, n_i : i \in I)}{B(I, n_i : i \in I)} \, \prod_{j=1}^{J} \left(\frac{\eta_j}{\mu_j} \right)^{n_j}. \tag{2.8}$$

(Y, X) is ergodic if and only if for the normalization constant holds

$$C = \sum_{(n_1,...,n_J) \in \mathbb{N}^J} \left(\prod_{j=1}^{J} \left(\frac{\eta_j}{\mu_j} \right)^{n_j} \sum_{I \subseteq D} \frac{A(I, n_i : i \in I)}{B(I, n_i : i \in I)} \right) < \infty. \tag{2.9}$$

Remark 2.15. *The stationary distribution* (2.8) *is of product form similar to the classical Jackson's result. This is even more explicit for load-independent breakdown and repair rates as in Example 2.2: For all $I \subseteq D$ and $(n_1, ..., n_J) \in \mathbb{N}^J$ holds*

$$\pi(I; n_1, ..., n_J) = \pi(I) \cdot \pi(n_1, ..., n_J) = \pi(I) \cdot \prod_{i=1}^{J} \pi_i(n_i),$$

where with a little abuse of notation (π has different meanings which are clear from the context)

$$\pi(I) := \sum_{(n_1,...,n_J) \in \mathbb{N}^J} \pi(I; n_1, ..., n_J) = \left(\sum_{K \subseteq D} \frac{A(K)}{B(K)} \right)^{-1} \cdot \frac{A(I)}{B(I)},$$

and

$$\pi(n_1, ..., n_J) := \sum_{I \subseteq D} \pi(I; n_1, ..., n_J) = \prod_{i=1}^{J} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i} = \prod_{i=1}^{J} \pi_i(n_i),$$

where

$$\pi_i(n_i) := \sum_{I \subseteq D} \sum_{(n_j : j \in \tilde{J} \setminus \{i\}) \in \mathbb{N}^{J-1}} \pi(I; n_1, ..., n_J) = \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}.$$

Hence the components for the performance and the reliability factorize and, even more, the queue-lengths distribution is of the well known Jackson network structure.
It should be noted that, in general, neither X nor Y is Markovian for its own. This is because, in general, the breakdown and repair rates depend on the load (queue length) of the nodes, and service to customers can be delivered only when a node is up.

Definition 2.16. *The condition $\eta_j < \mu_j$ is called* local stability criterion *for node j.*

2.4 Computation of availability and performance measures

When the breakdown and repair rates depend on the interaction of nodes but not on their load, as in Example 2.2, the availability process Y is an ergodic Markov process for its own, which has the unique limiting and stationary distribution

$$\pi(I) = \left(\sum_{K \subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \cdot \frac{A(I)}{B(I)} \quad \forall I \subseteq D, \tag{2.10}$$

see Example 2.2.

Remark 2.17. *As mentioned after Definition 2.10, the determination of the overall arrival rates in a system with breakdowns is slightly different to an ergodic system without breakdowns (see Remark 1.9). Assuming load-independent breakdown and repair rates as in Example 2.2 we provide the following approach:*
If nodes in I were not available and never be repaired and if no more nodes were unreliable, then we would have an ergodic Jackson network process with $J - |I|$ nodes and from [Ser99, Remark 1.16, p.16] the solutions $(\eta_i^I : i \in \tilde{J} \setminus I)$ of the traffic equations (2.6) would be the overall arrival rates in the system in equilibrium. Since the process in Theorem 2.14 jumps from one availability status to another, we need to consider the probability to be in each availability status to calculate the overall arrival rate.

- *The rerouting regimes skipping and blocking rs-rd lead to an identical result, if we require the reversibility constraint (2.7) to hold in case of blocking rs-rd. If the system is in equilibrium, the overall arrival rate at a node j is with the solution η_j^I of the traffic equation (2.6) and with $\pi(I)$ from (2.10)*

$$\sum_{I \subseteq D, j \notin I} \underbrace{\eta_j^I}_{\overset{(*)}{=} \eta_j} \cdot \pi(I) = \eta_j \cdot \sum_{I \subseteq D, j \notin I} \pi(I) \leq \eta_j,$$

where $(*)$ *holds with Lemma 2.11 in case of blocking rs-rd and Lemma 2.13 in case of skipping. Note that for all* $i \notin D$ *the overall arrival rate is*

$$\sum_{I \subseteq D} \underbrace{\eta_i^I}_{\stackrel{(*)}{=}\eta_i} \cdot \pi(I) = \eta_i \cdot \underbrace{\sum_{I \subseteq D} \pi(I)}_{=1} = \eta_i.$$

- *In case of stalling, the overall arrival rate at a node* j *is with the solution* η_j *of the traffic equation* (2.5) *and with* $\pi(I)$ *from* (2.10)

$$\eta_j \cdot \pi(\emptyset) = \eta_j \left(\sum_{I \subseteq D} \frac{A(I)}{B(I)} \right)^{-1}$$

if the system is in equilibrium, because customers arrive at a node only if all nodes are in up status. Thus in case of the stalling regime, the overall arrival rates are for all nodes less than in a reliable ergodic system which is intuitively obvious.

The following proposition shows how the point availability of a Jackson network with unreliable nodes may be computed:

Proposition 2.18. [SD03, p.185] *Consider a Jackson network with unreliable nodes as in Theorem 2.14 with load-independent breakdown and repair rates as in Example 2.2. Then the stationary joint* point availability *at time* $t \geq 0$ *for the subnetwork* $H \subseteq D$ *is*

$$PA(H)(t) := \sum_{K \subseteq D \setminus H} \pi(K),$$

where $\pi(I)$ *is the probability that exactly the nodes in set* $I \subseteq D$ *are under repair, given by* (2.10).

The following example shows how the stationary distribution of the availability process may be used to estimate costs involved with breakdowns of nodes.

Example 2.19. *Let*

$$f : \mathcal{P}(D) \to \mathbb{R}$$

denote a cost function which determines costs associated with shortages in service due to breakdown and with repair. To assess the quality of a system with unreliable components one often uses the time average of the accumulated cost over a time horizon $[0, T]$

$$c(T) = \frac{1}{T} \int_0^T f(Y(t)) \, dt.$$

Note, that this is an empirical measure which depends on the realized path of the system. Since Y *is an ergodic Markov process of its own, we can apply the ergodic theorem for Markov processes (see Theorem 1.14) to approximate for large* T

$$c(T) \approx \sum_{I \subseteq D} f(I) \cdot \pi(I) = \sum_{I \subseteq D} f(I) \, \frac{A(I)}{B(I)} \left(\sum_{K \subseteq D} \frac{A(K)}{B(K)} \right)^{-1}.$$

This approximation holds almost surely for any path of the system.

The probably most important performance measure for open networks is the throughput. The throughput of a network is the effective departure rate to the sink. The performance of a network - with fixed parameters - is maximal if the throughput equals the external arrival rate from the source into the network, as in case of ergodic Jackson networks with completely reliable nodes, see Proposition 1.13.

Definition 2.20. *Consider an ergodic Jackson network with unreliable nodes as in Definition 2.9, where the set with unreliable nodes is denoted by $D \subseteq \tilde{J}$. If nodes in $I \subseteq D$ are broken down, let μ_j^I denote the service rate at nodes $j \in \tilde{J} \setminus I$ and let $R^I = (r^I(i,j) : i,j \in \{0\} \cup \tilde{J} \setminus I)$ be the rerouting matrix which are determined by some rerouting strategy to be specified.*
Then the stationary throughput TH_j of a node $j \in \tilde{J}$ is

$$TH_j = \sum_{(I;n_1,\ldots,n_J)\in\mathcal{P}(D)\times\mathbb{N}^J} \pi(I;n_1,\ldots,n_J)\cdot\mu_j^I\cdot 1_{\mathbb{N}_+}(n_j)\cdot 1_{\tilde{J}\setminus I}(j), \tag{2.11}$$

and the stationary throughput TH of the network is

$$\begin{aligned} TH &= \sum_{(I;n_1,\ldots,n_J)\in\mathcal{P}(D)\times\mathbb{N}^J} \pi(I;n_1,\ldots,n_J)\cdot \sum_{j\in\tilde{J}\setminus I} 1_{\mathbb{N}_+}(n_j)\mu_j^I r^I(j,0) \\ &= \sum_{j\in\tilde{J}\setminus I} TH_j\cdot r^I(j,0). \end{aligned} \tag{2.12}$$

Proposition 2.21. [SD03, Proposition 7.3, p.189] *Consider an ergodic Jackson network with unreliable nodes as in Definition 2.9, where the set with unreliable nodes is denoted by $D \subseteq \tilde{J}$ and the breakdown and repair rates are load-independent as in Example 2.2.*

(i) If the rerouting regime stalling is in force, the stationary throughput at a node j is $TH_j = \eta_j \cdot \pi(\emptyset)$ and the stationary throughput of the network is $TH = \pi(\emptyset)\cdot\lambda$.

(ii) If the rerouting is according to blocking rs-rd and the reversibility constraints (2.7) *hold, the stationary throughput at a node j is $TH_j = \eta_j \cdot \sum_{I\subseteq D, j\notin I}\pi(I)$ and the stationary throughput of the network is*

$$TH = \sum_{I\subseteq D}\pi(I)\cdot\sum_{j\in\tilde{J}\setminus I}\lambda_j.$$

(iii) If the rerouting is according to skipping, the stationary throughput at a node j is $TH_j = \eta_j \cdot \sum_{I\subseteq D, j\notin I}\pi(I)$ and the stationary throughput of the network is

$$TH = \sum_{I\subseteq D}\pi(I)\cdot\lambda\sum_{j\in\tilde{J}\setminus I} r^I(0,j).$$

Example 2.22. *Let*

$$g : \mathbb{N}^J \to \mathbb{R},$$

denote a non-decreasing cost function which determines costs associated with storage. To assess the quality of a system with unreliable components it is often desirable to predict the average accumulated cost over a time horizon $[0,T]$,

$$d(T) = \frac{1}{T}\int_0^T g(X_i(t), i\in\tilde{J})\ dt.$$

Note, that this is an empirical measure which depends on the realized path of the system. Since X is not a Markov process of its own, we cannot apply the ergodic theorem for Markov processes (see Theorem 1.14) directly.
Now consider a cost function

$$g : \mathcal{P}(D) \times \mathbb{N}^J \to \mathbb{R}.$$

Then for $T \to \infty$ from the ergodic theorem for Markov processes follows for (Y, X)

$$\frac{1}{T}\int_0^T g(Y(t), X_i(t) : i \in \tilde{J})\ dt \to \sum_{(I, n_i : i \in \tilde{J}) \in \mathcal{P}(D) \times \mathbb{N}^J} g(I, n_i : i \in \tilde{J}) \pi(I, n_i : i \in \tilde{J}).$$

This holds almost surely for all paths of the process. So we have for almost all paths of (Y, X) for large T

$$\frac{1}{T}\int_0^T g(Y(t), X_i(t) : i \in \tilde{J})\ dt \approx \sum_{(I, n_i : i \in \tilde{J}) \in \mathcal{P}(D) \times \mathbb{N}^J} g(I, n_i : i \in \tilde{J}) \pi(I, n_i : i \in \tilde{J}).$$

This means that we can estimate the path-wise evaluated time average $d(T)$ by a state-space average (the phase-space average) for almost all paths.

Chapter 3

Change of the stability behavior due to breakdowns

3.1 Introduction

This chapter is dedicated to the study of the stability in networks with breakdowns. Starting with an ergodic system of reliable nodes (no breakdown and repair) we investigate the problem of maintaining stability when (some of) the nodes become unreliable and have to be repaired. First we analyze the ergodicity criterion (2.9) presented in Theorem 2.14. In a system which would be ergodic in case that all nodes are reliable, we answer the question whether the ergodicity criterion (2.9) may be not fulfilled in a system where all nodes are locally stable according to Definition 2.16, which means they experience a traffic intensity less than one. While the results in Section 3.2 deal with the form and structure of breakdown and repair intensities, we study possible changes of traffic intensities at nodes due to rerouting in Section 3.3.

3.2 Destabilization of the network process due to breakdown and repair

For[3] classical Jackson networks the ergodicity criterion $\eta_j < \mu_j \ \forall j \in \tilde{J}$ has a rate interpretation which guarantees the normalization constant of the stationary distribution to be finite. The refinement by Goodman and Massey, Theorem 1.22, towards stabilization criteria for subnets in non-ergodic networks exploits again a locally determined rate criterion. Such a local interpretation is, in general, no longer possible for the ergodicity condition (2.9) in networks with unreliable stations. Due to the rather general model structure in Theorem 2.14 a direct simple ergodicity criterion seems to be out of reach. Intuition, nevertheless, suggests that a stable Jackson network may lose stability if some nodes become unreliable. We describe an example. [MD09, p.1255]

Proposition 3.1. [MD09, Proposition 12, p.1255] *Consider an ergodic Jackson network of reliable nodes with* $\eta = (\eta_1, ..., \eta_J)$ *as the solution of the standard traffic equation* (1.1) *such that* $\eta_j < \mu_j$ *holds for all* j. *Assume that the reliable stations become unreliable with*

[3] This section covers parts of our paper [MD09, p.1255].

locally determined breakdown and repair rates according to Example 2.4. The rerouting regime follows one of the requirements in Theorem 2.14.
Then the network process (Y, X) *is ergodic if and only if in the original network with reliable stations for all* $j \in \tilde{J}$ *the stationary mean value of the function*

$$f_j : \mathbb{N} \to [0, \infty), \qquad n \mapsto \frac{a_j(n)}{b_j(n)},$$

is finite, i.e., for all $j = 1, \ldots, J$ *holds*

$$\sum_{n=0}^{\infty} \left(1 - \frac{\eta_j}{\mu_j}\right) \left(\frac{\eta_j}{\mu_j}\right)^n f_j(n) < \infty. \tag{3.1}$$

Proof. The joint availability–queue-length process is ergodic if $C < \infty$, see (2.9). We have

$$\begin{aligned} C &= \sum_{(n_1,\ldots,n_J)\in\mathbb{N}^J} \left(\prod_{j=1}^{J} \left(\frac{\eta_j}{\mu_j}\right)^{n_j} \left(\prod_{i=1}^{J} \left[1 + \frac{a_i(n_i)}{b_i(n_i)}\right]\right)\right) \\ &= \prod_{j=1}^{J} \left(\sum_{n=0}^{\infty} \left(\frac{\eta_j}{\mu_j}\right)^n \left[1 + \frac{a_j(n)}{b_j(n)}\right]\right), \end{aligned}$$

and this is finite, if and only if (3.1) holds for all $j \in \tilde{J}$. □

Note that in the situation of Proposition 3.1 the availability process Y is a Markov process for its own, but X is not Markovian.

Example 3.2. [MD09, Example 13, p.1255] *Under the conditions of Proposition 3.1 let for some node* i *and* $a_i, b_i > 0$ *the function* f_i *be*

$$f_i(n) = \frac{a_i^n}{b_i^n}, \quad n \in \mathbb{N}.$$

Then (3.1) *holds if and only if* $\eta_i \cdot a_i < \mu_i \cdot b_i$, *which seems to be a natural condition, because the factors* a_i *and* b_i *scale the arrival rate and the service rate in a way that the arriving load* $\eta_i \cdot (a_i/(a_i+b_i))$ *at node* i *remains below the maximal service capacity* $\mu_i \cdot (b_i/(a_i+b_i))$ *delivered by that node.*

Adding random noise to systems may stabilize the system under certain conditions. An example from population dynamics is discussed in [MMR02]. It is therefore an interesting question whether under special parameter choices for the breakdown and repair rates a non-ergodic network may become ergodic. The reason would be that the additional randomness due to breakdown and repair can smooth the system's behavior. The answer to this question is still open. But in the situation of Proposition 3.1, it is visible that the answer is negative, because, e.g., for $\eta_j \geq \mu_j$ and any choice of the $a_j(n), b_j(n) > 0$ the sum

$$\sum_{n=0}^{\infty} \left(\frac{\eta_j}{\mu_j}\right)^n \left[1 + \frac{a_j(n)}{b_j(n)}\right]$$

diverges. [MD09, p.1255]

In Chapter 4 and Chapter 5, we shall utilize the following fact: Load-independent breakdown and repair rates cannot destabilize a stable network. This becomes clear by the following

Example 3.3. [MD09, Example 14, p.1255] *Consider an unreliable Jackson network with stable nodes according to Definition 1.17 and load-independent breakdown and repair rates as in Example 2.2. The ergodicity criterion for the joint availability–queue-length process is*

$$C = \sum_{(n_1,\dots,n_J)\in\mathbb{N}^J} \prod_{j=1}^{J} \left(\frac{\eta_j}{\mu_j}\right)^{n_j} \sum_{I\subseteq\tilde{J}} \frac{A(I)}{B(I)} < \infty. \tag{3.2}$$

C is finite irrespectively of the choice of the breakdown and repair rate parameters, as long as the associated network with reliable nodes is ergodic.

It is remarkable that convergence of the normalization constant C in (3.2) does not depend on the breakdown and repair rates: For unstable nodes i with $\eta_i \geq \mu_i$ their instability cannot be compensated by such random smoothing. [MD09, p.1255]

3.3 Possible stabilization and destabilization of nodes due to rerouting

The[4] following proposition shows that a stable node may become unstable if some other node breaks down and is never repaired and if rerouting is according to the blocking rs-rd regime.

Proposition 3.4. [Myl08, pp.54-55] *Consider a reliable Jackson network with two nodes where node 2 is stable in the sense of Definition 1.17. Assume that node 1 may break down. Once broken down it will never be repaired, i.e., the repair time is infinite. Rerouting is according to blocking rs-rd.*
If $r(2,0) > 0$ and

$$\lambda_2 \geq \mu_2 r(2,0), \tag{3.3}$$

then for node 2 holds

$$\eta_2^{\{1\}} \geq \mu_2.$$

This result holds independent of node 1 being stable or unstable.

Proof. When node 1 breaks down, i.e., $I = \{1\}$, routing of customers is as follows:

$$r^I(0,2) = r(0,2), \quad r^I(2,0) = r(2,0), \quad r^I(2,2) = r(2,2) + r(2,1), \quad r^I(0,1) = 0,$$

and all other routing probabilities inside of the network equal zero. Thus the standard traffic equation is

$$\eta_2^I = \lambda_2^I + \eta_2^I r^I(2,2) = \lambda_2 + \eta_2^I(r(2,2) + r(2,1)),$$

which has the solution

$$\eta_2^I = \frac{\lambda_2}{r(2,0)}.$$

[4]This section covers parts of the author's diploma thesis [Myl08, Chapter 5, Section 5.2].

According to Algorithm 1.20(i), we have to check whether $\eta_2^I < \mu_2$ holds:

$$\eta_2^I < \mu_2 \Leftrightarrow \lambda_2 < \mu_2 r(2,0).$$

This contradicts assumption (3.3). The general traffic equation (1.6) reduces in the present situation to:

$$\eta_2^I = \lambda_2^I + \mu_2^I r^I(2,2) = \lambda_2 + \mu_2(r(2,2) + r(2,1)),$$

and it holds $\eta_2^I \geq \mu_2$, because

$$\lambda_2 + \mu_2(r(2,2) + r(2,1)) \geq \mu_2 \Leftrightarrow \lambda_2 \geq \mu_2 r(2,0).$$

□

A breakdown of some node may also lead to a stabilization of an unstable node, if rerouting is according to the blocking rs-rd regime:

Proposition 3.5. [Myl08, pp.51-53] *Consider a reliable Jackson network with two nodes where node 2 is unstable in the sense of Definition 1.17. Assume that node 1 may break down. Once broken down it will never be repaired, i.e., the repair time is infinite. Rerouting is according to blocking rs-rd.*
If $r(2,0) > 0$ *and*

$$\lambda_2 < \mu_2 r(2,0),$$

then for node 2 holds

$$\eta_2^{\{1\}} < \mu_2.$$

This result holds independent of node 1 being stable or unstable.

Proof. When node 1 breaks down, i.e., $I = \{1\}$, routing of customers is as follows:

$$r^I(0,2) = r(0,2), \quad r^I(2,0) = r(2,0), \quad r^I(2,2) = r(2,2) + r(2,1), \quad r^I(0,1) = 0,$$

and all other routing probabilities inside of the network equal zero. Thus the standard traffic equation is

$$\eta_2^I = \lambda_2^I + \eta_2^I r^I(2,2) = \lambda_2 + \eta_2^I(r(2,2) + r(2,1)),$$

which has the solution

$$\eta_2^I = \frac{\lambda_2}{r(2,0)}.$$

According to Algorithm 1.20(i) applied to the network after breakdown of node 1, we need to check whether $\eta_2^I < \mu_2$ holds:

$$\eta_2^I < \mu_2 \Leftrightarrow \lambda_2 < \mu_2 r(2,0),$$

which was assumed. □

Remark 3.6. *Consider the framework of Proposition 3.4. In the situation that both nodes are stable in a reliable network, we have the traffic equations*

$$\eta_1 = \lambda_1 + \eta_1 r(1,1) + \eta_2 r(2,1),$$
$$\eta_2 = \lambda_2 + \eta_1 r(1,2) + \eta_2 r(2,2),$$

which have the unique solutions (recall $\lambda_i = \lambda r(0,i)$):

$$\eta_1 = \lambda \frac{r(2,1) + r(2,0)r(0,1)}{r(1,0)[r(2,0) + r(2,1)] + r(1,2)r(2,0)},$$
$$\eta_2 = \lambda \frac{r(1,2) + r(1,0)r(0,2)}{r(1,0)[r(2,0) + r(2,1)] + r(1,2)r(2,0)}. \quad (3.4)$$

Equation (3.4) *with $\eta_2 < \mu_2$ and the assumption $\lambda_2 \geq \mu_2 r(2,0)$ yield*

$$\lambda \frac{r(0,2)}{r(2,0)} \geq \mu_2 > \lambda \frac{r(1,2) + r(1,0)r(0,2)}{r(1,0)[r(2,0) + r(2,1)] + r(1,2)r(2,0)}$$
$$\Rightarrow \frac{r(0,2)}{r(2,0)} > \frac{r(1,2) + r(1,0)r(0,2)}{r(1,0)[r(2,0) + r(2,1)] + r(1,2)r(2,0)}$$
$$\Leftrightarrow r(0,2)r(1,0)[r(2,0) + r(2,1)] + r(0,2)r(1,2)r(2,0) > r(1,2)r(2,0) + r(1,0)r(0,2)r(2,0)$$
$$\Leftrightarrow r(0,2)r(2,1)r(1,0) > (1 - r(0,2))r(1,2)r(2,0)$$
$$\Leftrightarrow r(0,1)r(1,2)r(2,0) < r(0,2)r(2,1)r(1,0),$$

i.e., the probability of a customer route from the outside entering node 2, transition to node 1 and leaving the network thereafter must be greater than the route the other way around.
In case of Proposition 3.5, the assumption $\lambda_2 < \mu_2 r(2,0)$ yields with similar computations the opposite inequality:

$$r(0,1)r(1,2)r(2,0) > r(0,2)r(2,1)r(1,0).$$

More details concerning these implications can be found in [Myl08, Chapter 5].
One should note, that

$$r(0,1)r(1,2)r(2,0) = r(0,2)r(2,1)r(1,0)$$

means that the routing process is reversible.

Proposition 3.5 offers ways to optimize networks by removing nodes causing global instability. In the light of Proposition 3.5, it might be profitable to remove a node, even a stable one, in order to obtain a globally stable network, if customers are rerouted then according to the blocking rs-rd regime.
A similar result of a stabilization of an unstable node due to a breakdown and rerouting according to the skipping regime is not possible.

Proposition 3.7. [Myl08, pp.59-61] *Consider a reliable Jackson network with two nodes where node 2 is unstable in the sense of Definition 1.17. Assume that node 1 may break down. Once broken down it will never be repaired, i.e., the repair time is infinite. Rerouting is according to skipping. Then it holds*

$$\eta_2^{\{1\}} \geq \mu_2.$$

This result holds independent of node 1 being stable or unstable.

Proof. If node 1 is unstable, we know that

$$\eta_1 = \lambda_1 + \mu_1 r(1,1) + \mu_2 r(2,1) \geq \mu_1 \;\Rightarrow\; \mu_2 \geq \frac{\mu_1(1-r(1,1)) - \lambda_1}{r(2,1)} \tag{3.5}$$

$$\eta_2 = \lambda_2 + \mu_1 r(1,2) + \mu_2 r(2,2) \geq \mu_2 \;\Rightarrow\; \frac{\lambda_2 + \mu_1 r(1,2)}{1-r(2,2)} \geq \mu_2. \tag{3.6}$$

(3.5) with (3.6) yields

$$\begin{aligned}
&\frac{\lambda r(0,2) + \mu_1 r(1,2)}{1-r(2,2)} \geq \frac{\mu_1(1-r(1,1)) - \lambda r(0,1)}{r(2,1)}\\
&\Leftrightarrow\; \lambda r(0,2) r(2,1) + \mu_1 r(1,2) r(2,1) \geq \mu_1(1-r(1,1))(1-r(2,2)) - \lambda r(0,1)(1-r(2,2))\\
&\Leftrightarrow\; \lambda(r(0,2)r(2,1) + r(0,1)(1-r(2,2))) \geq \mu_1((1-r(1,1))(1-r(2,2)) - r(1,2)r(2,1))\\
&\Leftrightarrow\; \frac{\lambda}{\mu_1} \geq \frac{(1-r(1,1))(1-r(2,2)) - r(1,2)r(2,1)}{r(0,2)r(2,1) + r(0,1)(1-r(2,2))}.
\end{aligned} \tag{3.7}$$

If node 1 is stable, we have

$$\eta_1 = \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1) \;\Leftrightarrow\; \eta_1 = \frac{\lambda_1 + \mu_2 r(2,1)}{1-r(1,1)}, \tag{3.8}$$

$$\begin{aligned}
&\eta_2 = \lambda_2 + \eta_1 r(1,2) + \mu_2 r(2,2) \overset{(3.8)}{=} \lambda_2 + \frac{\lambda_1 + \mu_2 r(2,1)}{1-r(1,1)} r(1,2) + \mu_2 r(2,2) \geq \mu_2\\
&\Rightarrow\; \lambda r(0,2) + \lambda \frac{r(0,1)r(1,2)}{1-r(1,1)} + \mu_2 \frac{r(2,1)r(1,2)}{1-r(1,1)} \geq \mu_2(1-r(2,2))\\
&\Leftrightarrow\; \lambda \frac{r(0,2)(1-r(1,1)) + r(0,1)r(1,2)}{1-r(1,1)} \geq \mu_2 \frac{(1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)}{1-r(1,1)}\\
&\Leftrightarrow\; \lambda \frac{r(0,2)r(1,0) + r(1,2)}{r(2,0)r(1,0) + r(2,0)r(1,2) + r(2,1)r(1,0)} \geq \mu_2.
\end{aligned} \tag{3.9}$$

When node 1 breaks down, i.e., $I = \{1\}$, routing of customers is as follows:

$$r^I(0,2) = r(0,2) + \frac{r(0,1)r(1,2)}{1-r(1,1)}, \quad r^I(2,0) = r(2,0) + \frac{r(2,1)r(1,0)}{1-r(1,1)},$$
$$r^I(2,2) = r(2,2) + \frac{r(2,1)r(1,2)}{1-r(1,1)},$$

and all other routing probabilities inside of the network equal zero. Thus the standard traffic equation is given by

$$\begin{aligned}
\eta_2^I &= \lambda_2^I + \eta_2^I r^I(2,2)\\
&= \lambda\Big(r(0,2) + \frac{r(0,1)r(1,2)}{1-r(1,1)}\Big) + \eta_2^I\Big(r(2,2) + \frac{r(2,1)r(1,2)}{1-r(1,1)}\Big),
\end{aligned}$$

which has the solution

$$\begin{aligned}
\eta_2^I &= \lambda \frac{r(0,2)(1-r(1,1)) + r(0,1)r(1,2)}{(1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)}\\
&= \lambda \frac{r(0,2)r(1,0) + r(1,2)}{r(2,0)r(1,0) + r(2,0)r(1,2) + r(2,1)r(1,0)}.
\end{aligned} \tag{3.10}$$

Supposing $\eta_2^I < \mu_2$ leads to an obvious contradiction to (3.9). Assumption (3.6) and (3.10) with $\eta_2^I < \mu_2$ yield:

$$\begin{aligned}
&\frac{\lambda_2 + \mu_1 r(1,2)}{1 - r(2,2)} \geq \mu_2 > \lambda \frac{r(0,2)(1-r(1,1)) + r(0,1)r(1,2)}{(1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)} \\
\Rightarrow &\frac{\lambda r(0,2) + \mu_1 r(1,2)}{1 - r(2,2)} > \lambda \frac{r(0,2)(1-r(1,1)) + r(0,1)r(1,2)}{(1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)} \\
\Leftrightarrow &\Big(\lambda r(0,2) + \mu_1 r(1,2)\Big)\Big((1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)\Big) > \\
&\quad > \lambda\Big(r(0,2)(1-r(1,1)) + r(0,1)r(1,2)\Big)(1-r(2,2)) \\
\Leftrightarrow &\mu_1 r(1,2)\Big((1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)\Big) > \\
&\quad > \lambda\Big((r(0,2)(1-r(1,1)) + r(0,1)r(1,2))(1-r(2,2)) \\
&\qquad - r(0,2)((1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2))\Big) \\
\Leftrightarrow &\mu_1\Big((1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)\Big) > \lambda\Big(r(0,1)(1-r(2,2)) + r(0,2)r(2,1)\Big) \\
\Leftrightarrow &\frac{(1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)}{r(0,1)(1-r(2,2)) + r(0,2)r(2,1)} > \frac{\lambda}{\mu_1},
\end{aligned} \tag{3.11}$$

which is a contradiction to (3.7). Thus, whether node 1 is stable or unstable, it holds $\eta_2^I \geq \mu_2$. □

Destabilization of a stable node due to a breakdown and rerouting according to skipping is not possible, if the traffic intensity at the unreliable node is less than or equal to one.

Proposition 3.8. [Myl08, pp.62-63] *Consider a reliable Jackson network with two nodes where node 1 is either stable, i.e., $\eta_1 < \mu_1$, or unstable with $\eta_1 = \mu_1$, and node 2 is stable, i.e., $\eta_2 < \mu_2$.*
Assume that the network is modified by node 1 being unreliable. Once broken down it will never be repaired, i.e., the repair time is infinite. Rerouting is according to skipping. Then it holds

$$\eta_2^{\{1\}} < \mu_2.$$

Proof. In case that node 1 is stable which means that both nodes are stable, $\eta_2^{\{1\}} < \mu_2$ follows from Lemma 2.13.
If node 1 is unstable, we know that

$$\eta_1 = \lambda_1 + \mu_1 r(1,1) + \eta_2 r(2,1) \geq \mu_1 \;\Rightarrow\; \mu_1(1 - r(1,1)) \leq \lambda_1 + \eta_2 r(2,1) \tag{3.12}$$

$$\eta_2 = \lambda_2 + \mu_1 r(1,2) + \eta_2 r(2,2) < \mu_2 \;\Leftrightarrow\; \eta_2 = \frac{\lambda_2 + \mu_1 r(1,2)}{1 - r(2,2)} < \mu_2. \tag{3.13}$$

(3.13) in (3.12) yields

$$\mu_1(1-r(1,1)) \le \lambda r(0,1) + \frac{\lambda r(0,2)+\mu_1 r(1,2)}{1-r(2,2)} r(2,1)$$
$$\Leftrightarrow \mu_1(1-r(1,1))(1-r(2,2)) \le \lambda\Big(r(0,1)(1-r(2,2)) + r(0,2)r(2,1)\Big) + \mu_1 r(1,2)r(2,1)$$
$$\Leftrightarrow \mu_1\Big((1-r(1,1))(1-r(2,2)) - r(1,2)r(2,1)\Big) \le \lambda\Big(r(0,1)(1-r(2,2)) + r(0,2)r(2,1)\Big)$$
$$\Leftrightarrow \frac{(1-r(1,1))(1-r(2,2)) - r(1,2)r(2,1)}{r(0,1)(1-r(2,2)) + r(0,2)r(2,1)} \le \frac{\lambda}{\mu_1}. \tag{3.14}$$

Note that equality in (3.14) holds if and only if equality holds in (3.12), i.e., $\eta_1 = \mu_1$. When node 1 breaks down, i.e., $I = \{1\}$, routing of customers is as follows:

$$r^I(0,2) = r(0,2) + \frac{r(0,1)r(1,2)}{1-r(1,1)}, \quad r^I(2,0) = r(2,0) + \frac{r(2,1)r(1,0)}{1-r(1,1)},$$
$$r^I(2,2) = r(2,2) + \frac{r(2,1)r(1,2)}{1-r(1,1)},$$

and all other routing probabilities inside of the network equal zero. Thus the standard traffic equation is

$$\eta_2^I = \lambda_2^I + \eta_2^I r^I(2,2)$$
$$= \lambda\Big(r(0,2) + \frac{r(0,1)r(1,2)}{1-r(1,1)}\Big) + \eta_2^I\Big(r(2,2) + \frac{r(2,1)r(1,2)}{1-r(1,1)}\Big),$$

which has the solution

$$\eta_2^I = \lambda\frac{r(0,2)(1-r(1,1)) + r(0,1)r(1,2)}{(1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)} \tag{3.15}$$
$$= \lambda\frac{r(0,2)r(1,0) + r(1,2)}{r(2,0)r(1,0) + r(2,0)r(1,2) + r(2,1)r(1,0)}.$$

Assumption (3.13) and (3.15) with $\eta_2^I \ge \mu_2$ yield

$$\lambda\frac{r(0,2)(1-r(1,1)) + r(0,1)r(1,2)}{(1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)} > \frac{\lambda_2 + \mu_1 r(1,2)}{1-r(2,2)}$$

which is (3.11) with an opposite inequality. The same transformations as done for (3.11) yield

$$\frac{(1-r(2,2))(1-r(1,1)) - r(2,1)r(1,2)}{r(0,1)(1-r(2,2)) + r(0,2)r(2,1)} < \frac{\lambda}{\mu_1}$$

which is a contradiction to (3.14) if $\eta_1 = \mu_1$. □

As a consequence of the proof of Proposition 3.8, it follows directly

Proposition 3.9. [Myl08, p.62] *Consider a reliable Jackson network with two nodes where node 1 is unstable with $\eta_1 > \mu_1$, and node 2 is stable, i.e., $\eta_2 < \mu_2$.*
Assume that node 1 may break down. Once broken down it will never be repaired, i.e., the repair time is infinite. Rerouting is according to skipping. Then node 2 may become unstable in the sense of

$$\eta_2^{\{1\}} \ge \mu_2.$$

The results concerning rerouting according to the skipping regime show that this rerouting concept may cause a destabilization but not a stabilization of nodes.

Chapter 4

Jackson networks with infinite supply

4.1 Introduction

Jackson networks with infinite supply of work as defined by G. Weiss in [Wei05] are a special class of multi-class queueing networks with virtual infinite buffers, introduced in [KW02] and [AW06]. A multi-class queueing network consists of K classes of jobs (customers) and J stations. Jobs of class $k \in \{1, ..., K\}$ queue up in buffer k and are served by a server $s_k(i)$ at station $i \in \{1, ..., J\}$. In general, a station can serve several classes of jobs that is why such networks are called multi-class networks.

For Jackson networks with infinite supply only jobs of two classes, a class of lower priority and a class of higher priority jobs, are considered and there is only one server at each station which can serve both classes of jobs. Jobs moving between the stations are of one class (higher priority), the infinite buffer at some stations is filled only with jobs of lower priority, the other class of jobs. Once completely served at their first station the lower priority jobs turn into higher priority jobs on their path through the network.

The idea of an infinite supply of lower-priority work is used frequently, e.g., in [LY75] in an $(M/G/1)$ queueing system to utilize idle times. Recent works using this concept of infinite supply are, e.g., [Guo08] where generalized Jackson networks are considered or [KNW09] where a push-pull network is provided with infinite supply.

Implementing an infinite supply at some nodes has the aim to utilize capacities to the fullest, avoid idle times completely and therefore enhance productivity.

The principle of an infinite supply of work at a node is the following:

- Whenever all jobs queued at the node have departed and the node is idle, a job from the infinite supply depot is served there. After completed service, the job departs and is routed according to the routing matrix of the network.

- If during the service of a job from the infinite supply a regular job (not coming directly from the infinite supply depot) arrives at that node, this new job has preemptive priority and the job from the infinite supply depot is sent back to the depot immediately.

- Thus jobs from the infinite supply storage have lower priority. But after its initial service, a low priority job turns into a high priority job as all others in the system.
- Nodes with infinite supply are busy all the time, hence their service capacity is fully utilized.
- As long as a job has low priority, it is not counted in the state space as a queued job, so the state description of the node does not change with its arrival.

The principle of not counting the extra arrivals (here from the infinite supply) at some node i to its state but the arrivals of these at (other) nodes after departing from i is also used in [CHT01].

The terms "preemptive"/"high" and "low" priority indicate possible utilization of a network with infinite supply to model for example communication networks where messages in transit have preemptive priority over newly generated messages at a station: If only messages in transit are counted as congestion, this model fits. A particular computer communication system that works in this way is MAN (metropolitan area network) Ethernet RPR (resilient packet ring), in which ring traffic has priority over the traffic generated at nodes, [Wei05].

Definition 4.1 (Jackson network with infinite supply)**.** [Wei05] *Consider a Jackson network with node set $\tilde{J} = \{1, ..., J\}$, i.e., for each node j we have*

- *external Poisson(λ_j)-arrival streams ($\sum_{j \in \tilde{J}} \lambda_j = \lambda > 0$),*
- *single servers with exponential(μ_j) distributed service time,*
- *infinite waiting room,*
- *first-come-first-served (FCFS) regime.*

Customers (jobs) are indistinguishable. All interarrival and service times constitute a set of independent random variables.
Routing is Markovian: Given the departure node i the selection of the next node is independent of the previous history. A customer departing from node i immediately proceeds to node j with probability $r(i, j) \geq 0$ and departs from the network with probability $r(i, 0)$ (the artificial node 0 represents the outside, source and sink, of the network, $r(0, 0) := 0$, $r(0, i) := \lambda_i/\lambda$). The routing matrix $R = (r(i, j) : i, j \in \{0, 1, ..., J\})$ is stochastic and irreducible.
Let $V := \{i \in \tilde{J} : i$ has infinite supply$\}$ denote the subset of nodes with an infinite supply of work and $W := \tilde{J} \setminus V$ the subset of nodes without an infinite supply of work.
Then the queue length process $X = ((X_1(t), ..., X_J(t)) : t \in \mathbb{R}_+)$ is a Markov process on $\mathbb{N}^J$ with transition rates matrix $Q = (q(z, z') : z, z' \in \mathbb{N}^J)$ which is defined by:
For all $i, j \in \tilde{J}, i \neq j$:

$$q(n_1, ..., n_i, ..., n_J; n_1, ..., n_i + 1, ..., n_J) = \lambda_i + \sum_{j \in V} \mu_j r(j, i) 1_{\{0\}}(n_j),$$

$$q(n_1, ..., n_i, ..., n_J; n_1, ..., n_i - 1, ..., n_J) = \mu_i r(i, 0) 1_{\mathbb{N}_+}(n_i),$$

$$q(n_1, ..., n_i, ..., n_j, ..., n_J; n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) = \mu_i r(i, j) 1_{\mathbb{N}_+}(n_i),$$

$$q(n_1, ..., n_J; n_1, ..., n_J) = -\sum_{i \in \tilde{J}} \lambda_i - \sum_{i \in \tilde{J}} \sum_{j \in V} \mu_j r(j, i) 1_{\{0\}}(n_j) - \sum_{i \in \tilde{J}} \mu_i (1 - r(i, i)) 1_{\mathbb{N}_+}(n_i),$$

and $q(z, z') = 0$ *otherwise.*

Remark 4.2. *Since the routing matrix* R *in Definition 4.1 is stochastic and irreducible as in Definition 1.1, Lemma 1.2 applies here as well.*

Theorem 4.3. [Wei05, Proposition 1(iii)] *Consider a Jackson network where nodes in* $V \subseteq \tilde{J}$ *have an infinite supply of work. Then the departure streams from nodes* $j \in V$ *with infinite supply are independent Poisson streams with rates* μ_j *and therefore the departure stream from* $j \in V$ *to* $i \in \tilde{J}$ *is Poisson with rate* $\mu_j r(j, i)$.

Proof. All departure times from node $j \in V$

- are independent of the state of the node due to the infinite supply,
- and therefore the independent times are identically exponentially distributed with rate μ_j like the service times of the node.

Thus, the departure stream of node $j \in V$ is a Poisson process with rate μ_j. Hence for node $i \in \tilde{J}$, the arrival stream from node $j \in V$ is a Poisson process with rate $\mu_j r(j, i)$, because a portion of $r(j, i)$ of the departure stream is directed to node $i \in \tilde{J}$ (decomposition of a Poisson process by some probability independent of the process and of its times of events, see [Ser09, Proposition 41, p.197]). □

4.2 The traffic equations

Definition 4.4. [Wei05] *The traffic equations of a Jackson network with infinite supply as defined in Definition 4.1 are*

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j, i) + \sum_{j \in V} \mu_j r(j, i), \quad i \in \tilde{J}. \tag{4.1}$$

This definition is different from the classical Definition 1.6 due to the infinite supply: A node $i \in V$ has an infinite supply of work which is activated whenever this node is empty. The additional customers from the infinite supply depot are not counted in the state space as regular customers to the queue length until they leave the node after completed service. Assuming that, on average, all nodes are neither fully loaded nor overloaded, the input rate of high priority customers at a node with infinite supply is less than its output rate of high priority customers. From Theorem 4.3 we know that node $i \in V$ with infinite supply generates a Poisson departure stream with rate μ_i. Therefore the output rate in the traffic equation (4.1) has to be μ_i for all nodes with infinite supply instead of η_i which in this case is the input rate.

Remark 4.5. *Summing* (4.1) *on both sides over all* $i \in \tilde{J}$ *yields:*

$$\begin{aligned}
&\sum_{i \in \tilde{J}} \eta_i = \sum_{i \in \tilde{J}} \lambda_i + \sum_{i \in \tilde{J}} \sum_{j \in W} \eta_j r(j, i) + \sum_{i \in \tilde{J}} \sum_{j \in V} \mu_j r(j, i) \\
\Leftrightarrow \quad &\sum_{i \in \tilde{J}} \lambda_i = \sum_{i \in W} \eta_i r(i, 0) + \underbrace{\sum_{i \in V} \mu_i r(i, 0)}_{(*)} + \sum_{i \in V} (\eta_i - \mu_i),
\end{aligned}$$

so the sum of the departure rates to the sink is greater than the sum of the external arrival rates, λ, if $\sum_{i\in V}(\eta_i - \mu_i) < 0$ which holds for example if $\eta_i < \mu_i \ \forall\ i \in V$. This reflects the additional customer input from the infinite supply depot. Note, that in (∗) customers from the infinite supply which immediately depart after completed service are counted as proper customers.

Lemma 4.6. [Wei05] *The traffic equations* (4.1) *have a unique solution* $\eta = (\eta_1, ..., \eta_J)$.

Proof. In order to solve (4.1), consider the traffic equations in matrix notation partitioned according to the sets V and W:

$$\eta_W = \lambda_W + \eta_W R_{WW} + \mu_V R_{VW}, \tag{4.2}$$
$$\eta_V = \lambda_V + \eta_W R_{WV} + \mu_V R_{VV}. \tag{4.3}$$

Because of the irreducibility of R, $(\mathbf{I} - R_{WW})^{-1}$ exists and is positive (see Lemma 1.2). Therefore (4.2) may be transformed into

$$\eta_W = (\lambda_W + \mu_V R_{VW})(\mathbf{I} - R_{WW})^{-1}, \tag{4.4}$$

which is the unique solution of (4.2) and yields after inserting into (4.3) the unique solution of this equation, too. □

As in the classical Jackson network, we need to define general traffic equations for the case that there are nodes which, on average, are fully loaded or overloaded due to the arrival intensity of high priority customers.

Definition 4.7.

$$\eta_i = \lambda_i + \sum_{j\in W} \min(\eta_j, \mu_j) r(j,i) + \sum_{j\in V} \mu_j r(j,i), \quad i \in \tilde{J}, \tag{4.5}$$

are the general traffic equations *for Jackson networks with infinite supply.*

These traffic equations are motivated as follows:

Definition 4.8. *Consider a Jackson network with infinite supply. A node i is* stable *if η_i determined by* (4.5) *is strictly less than its service rate μ_i, otherwise the node is* unstable.

For a node $j \in V$ with infinite supply the input rate is already different from its output rate: The low priority customers arriving from the infinite supply depot are not counted as arriving customers at that node but they lead to an output rate of proper customers equal to the service rate there. Thus, if only nodes with infinite supply are unstable, the input rates are still determined by (4.1). The only difference is, that an unstable node with infinite supply is less often feeded with low priority customers (from the infinite supply depot) than if it is stable.
For an unstable node $j \in W$ without infinite supply the input rate η_j is different from its overall maximal departure rate which can be at most μ_j. So in determining the input rates we are no longer allowed to insert the same quantities in the traffic equations (4.1) on the right-hand side as on the left-hand side, the traffic equations need to be reformulated to (4.5).

Remark 4.9. *Let $S := \{i \in \tilde{J} : \eta_i < \mu_i\}$ denote the set of stable nodes and $U := \tilde{J} \setminus S$ the set of unstable nodes according to Definition 4.8. Summing (4.5) on both sides over all $i \in \tilde{J}$ yields:*

$$\begin{aligned} \sum_{i\in\tilde{J}} \eta_i &= \sum_{i\in\tilde{J}} \lambda_i + \sum_{i\in\tilde{J}}\sum_{j\in W} \min(\eta_j,\mu_j)r(j,i) + \sum_{i\in\tilde{J}}\sum_{j\in V} \mu_j r(j,i) \\ \Leftrightarrow \sum_{i\in\tilde{J}} \lambda_i &= \sum_{i\in W} \min(\eta_i,\mu_i)r(i,0) + \sum_{i\in V}\mu_i r(i,0) + \sum_{i\in U\cup V}(\eta_i-\mu_i), \end{aligned}$$

so the sum of the departure rates to the sink is less than the sum of the external arrival rates λ, if all nodes with infinite supply (in V) are unstable. Otherwise the sum of the departure rates to the sink may be also equal to or even greater than λ, depending on the sizes of η_i and μ_i of nodes $i \in U \cup V$. This reflects the additional customer input from the infinite supply depot as well as the bottlenecks arising at unstable nodes.

Lemma 4.10. *The general traffic equations (4.5) have a unique solution which we denote by $\eta = (\eta_1, ..., \eta_J)$.*

In the proof of Lemma 4.10, the main argument is the existence of an algorithm with which the unique solution of (4.5) may be determined in at most J steps. This algorithm is presented here:

Algorithm 4.11. *Consider a Jackson network with J nodes where nodes in V have an infinite supply of work. Nodes in $W := \tilde{J}\setminus V$ work without infinite supply. It is not known which nodes are stable and which are unstable.*

(i) Assume that all nodes are unstable.
Based on this assumption, let $(\eta_i(1) : i \in \tilde{J})$ be the first estimate for the solution $(\eta_i : i \in \tilde{J})$ of the traffic equations (4.5), i.e., $(\eta_i(1) : i \in \tilde{J})$ is the solution of the traffic equations:

$$\eta_i(1) = \lambda_i + \sum_{j=1}^{J} \mu_j r(j,i) \quad \forall\ i \in \tilde{J},$$

which trivially exists and is unique, because all parameters at the right-hand side of the equations are given. Since the departure rate at each node $i \in W$ without infinite supply is $\min(\eta_i, \mu_i)$, the estimate $\eta_i(1)$ is at most overestimated, so $\eta_i(1) \geq \eta_i$ holds for all $i \in \tilde{J}$.

- *If $\eta_i(1) \geq \mu_i$ holds for all $i \in \tilde{J}$, all nodes are unstable and for the first estimate holds $\eta_i = \eta_i(1)\ \forall\ i \in \tilde{J}$. Stop here.*
- *If $\eta_i(1) \geq \mu_i$ holds for all $i \in W$, then all nodes in W are unstable. If $\eta_{i_*}(1) < \mu_{i_*}$ holds for some nodes $i_* \in V$, then $\mu_{i_*} > \eta_{i_*}$ holds due to $\eta_i(1) \geq \eta_i\ \forall\ i \in \tilde{J}$, so these nodes i_* are stable. But due to the infinite supply at these nodes, the traffic equations do not change with this information, so $\eta_i = \eta_i(1)$ holds $\forall\ i \in \tilde{J}$ and the set of stable nodes is identified as $S(1) := \{i : \eta_i(1) < \mu_i\} \subseteq V$. Stop here.*

- *If for at least one node* $i_* \in W$ *holds* $\eta_{i_*}(1) < \mu_{i_*}$, *then* $\mu_{i_*} > \eta_{i_*}$ *holds due to* $\eta_i(1) \geq \eta_i \ \forall \ i \in \tilde{J}$, *so this node* i_* *is stable. But since* $\eta_{i_*}(1)$ *is derived under the assumption that all nodes are unstable, it holds* $\eta_{i_*}(1) \geq \eta_{i_*}$. *Set* $S(1) := \{i : \eta_i(1) < \mu_i\}$ *and proceed to the next step.*

(ii) All nodes $i \in S(1)$ *will eventually be stable. Assume that all other nodes* $i \in \tilde{J} \setminus S(1)$ *are unstable.*
Based on this assumption, let $(\eta_i(2) : i \in \tilde{J})$ *be the second estimate for* $(\eta_i : i \in \tilde{J})$, *i.e.,* $(\eta_i(2) : i \in \tilde{J})$ *is the solution of the traffic equations (with* $U(1) = \tilde{J} \setminus S(1)$*):*

$$\eta_i(2) = \lambda_i + \sum_{j \in U(1) \cup V} \mu_j r(j,i) + \sum_{j \in S(1) \cap W} \eta_j(2) r(j,i) \quad \forall \ i \in \tilde{J}$$

which exists and is unique, see Proof of Lemma 4.10. Again, $(\eta_1(2), ..., \eta_J(2))$ *is at most an overestimation, but the assumptions are more conservative than those for* $(\eta_i(1) : i \in \tilde{J})$. *It holds:* $\eta_i \leq \eta_i(2) \leq \eta_i(1) \ \forall \ i \in \tilde{J}$ *and* $\eta_i(2) < \mu_i \ \forall \ i \in S(1)$.

- *If* $S(1) = S(2) := \{i : \eta_i(2) < \mu_i\}$, *then* $\eta_i(2) = \eta_i$ *holds* $\forall \ i \in \tilde{J}$. *Stop here.*
- *If* $S(1) \neq S(2)$ *(so* $S(1) \subset S(2)$*) and* $(S(2) \setminus S(1)) \cap W = \emptyset$, *then* $\eta_i(2) = \eta_i$ *holds* $\forall \ i \in \tilde{J}$, *but* $S(2)$ *is the true set of stable nodes. Stop here.*
- *If* $S(1) \neq S(2)$ *(so* $S(1) \subset S(2)$*) and* $(S(2) \setminus S(1)) \cap W \neq \emptyset$, *then* $\eta_{i_*}(2) > \eta_{i_*}$ *holds for at least one node* $i_* \in \tilde{J}$. *Iterate (ii) with* $S(2)$ *as new set of stable nodes.*

Remark 4.12. *Setting* $V = \emptyset$ *in Algorithm 4.11 yields Algorithm 1.18.*

Proof of Lemma 4.10. The traffic equations (4.5) are solved by an algorithm which recursively builds a sequence of vectors $\eta(n) = (\eta_1(n), ..., \eta_J(n))$, $n \in \mathbb{N}_+$, together with a sequence of sets $S(n) := \{i : \eta_i(n) < \mu_i\}$ of nodes, which are detected within the first n steps as being stable, for which holds:

(i) $S(0) := \emptyset$,

(ii) $S(n-1) \subseteq S(n) \ \forall n \geq 1$,

(iii) $\exists! 0 < n^* \leq J : \quad S(n^*-1) \subset S(n^*) = S(n^*+1)$,

(iv) $\eta(n+1)$ solves the following $S(n) \cap W$-partition of traffic equations (recall $U(n) := \tilde{J} \setminus S(n)$)

$$\eta(n+1)_{S(n)\cap W} = \lambda_{S(n)\cap W} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W \ S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V \ S(n)\cap W}, \tag{4.6}$$

$$\eta(n+1)_{U(n)\cup V} = \lambda_{U(n)\cup V} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W \ U(n)\cup V} + \mu_{U(n)\cup V} R_{U(n)\cup V \ U(n)\cup V}. \tag{4.7}$$

We show that the sequence $\eta(n)$ delivered by that algorithm converges to the unique solution η of the traffic equations (4.5) in at most J iterations, if a unique solution exists (which will be shown in this proof later on):

If $S(n) \subseteq S(n+1)$ holds for all $n \in \tilde{J}$, then there exists $n^* \leq J$ with $S(n^*) = S(n^*+1)$, so the set of stable nodes will be found in at most J iterations and $\eta(n^*) = \eta$ will be the solution of the traffic equations.

We therefore show:

a) $\forall n \in \mathbb{N}$: $\eta_i(n) \geq \eta_i(n+1) \quad \forall\, i \in \tilde{J} \quad \Rightarrow \quad S(n) \subseteq S(n+1)$,

b) $\eta_i(n) \geq \eta_i(n+1)$ holds for all $i \in \tilde{J}, n \in \mathbb{N}_+$.

Proof of a): For all $i \in S(n)$ holds by definition $\eta_i(n) < \mu_i$. From $\mu_i > \eta_i(n) \geq \eta_i(n+1)$ follows $i \in S(n+1)$ and therefore $S(n) \subseteq S(n+1)$ holds for all $n \in \mathbb{N}$.

Proof of b): By induction over n.
1. Basis ($n = 1$):

$$\eta(1) = \lambda + \mu R_{\tilde{J}\tilde{J}} = \lambda + \mu_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} + \mu_{U(1)\cup V} R_{U(1)\cup V\ \tilde{J}}, \tag{4.8}$$

$$\eta(2) = \lambda + \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} + \mu_{U(1)\cup V} R_{U(1)\cup V\ \tilde{J}}, \tag{4.9}$$

so $\eta(1) \geq \eta(2)$ (component-wise) is equivalent to

$$\mu_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} \geq \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}}.$$

Note that if $S(1) = \emptyset$ then $\eta(1) = \eta(2)$ follows directly. We therefore consider $S(1) \neq \emptyset$ for the remainder of the induction basis.
With (4.8) and (4.9) we have

$$\eta(1) = \eta(2) + \mu_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} - \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}}$$

and from definition $\mu_{S(1)} > \eta(1)_{S(1)}$ holds component-wise, so $\mu_{S(1)\cap W} > \eta(1)_{S(1)\cap W}$ and

$$\begin{aligned} &\mu_{S(1)\cap W} > \eta(2)_{S(1)\cap W} + \mu_{S(1)\cap W} R_{S(1)\cap W\ S(1)\cap W} - \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ S(1)\cap W} \\ \Leftrightarrow\ &\mu_{S(1)\cap W}(\mathbf{I} - R_{S(1)\cap W\ S(1)\cap W}) > \eta(2)_{S(1)}(\mathbf{I} - R_{S(1)\cap W\ S(1)\cap W}) \end{aligned}$$

Multiplying both sides of the last inequality from the right side with $(\mathbf{I} - R_{S(1)\cap W\ S(1)\cap W})^{-1}$ (which exists and is positive, see Lemma 1.2) yields

$$\mu_{S(1)\cap W} > \eta(2)_{S(1)\cap W} \ \Rightarrow\ \mu_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}} \geq \eta(2)_{S(1)\cap W} R_{S(1)\cap W\ \tilde{J}}.$$

2. Inductive step ($n \curvearrowright n+1$):
Induction hypothesis: For some $n \in \mathbb{N}_+$ holds $\eta(n-1) \geq \eta(n)$ ($\Rightarrow S(n-1) \subseteq S(n)$).
We show that $\eta(n) \geq \eta(n+1)$ holds under the induction hypothesis.
With $S' := S(n) \setminus S(n-1)$ we have

$$\begin{aligned} \eta(n) &= \lambda + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} + \mu_{U(n-1)\cup V} R_{U(n-1)\cup V\ \tilde{J}} \\ &= \lambda + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} + \mu_{S'\cap W} R_{S'\cap W\ \tilde{J}} + \mu_{U(n)\cup V} R_{U(n)\cup V\ \tilde{J}} \end{aligned} \tag{4.10}$$

$$\begin{aligned} \eta(n+1) &= \lambda + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W\ \tilde{J}} + \mu_{U(n)\cup V} R_{U(n)\cup V\ \tilde{J}} \\ &= \lambda + \eta(n+1)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{J}} + \eta(n+1)_{S'\cap W} R_{S'\cap W\ \tilde{J}} + \mu_{U(n)\cup V} R_{U(n)\cup V\ \tilde{J}} \end{aligned} \tag{4.11}$$

so $\eta(n) \geq \eta(n+1)$ (component-wise) is equivalent to

$$\eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{\jmath}} + \mu_{S'\cap W} R_{S'\cap W\ \tilde{\jmath}} \geq \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W\ \tilde{\jmath}}. \tag{4.12}$$

Note that if $S' = \emptyset$ (i.e., $S(n-1) = S(n)$) then $\eta(n) = \eta(n+1)$ follows directly. We therefore consider the case $S' \neq \emptyset$ for the remainder of the induction step.
From (4.10) we have

$$\begin{aligned}\eta(n)_{S(n-1)\cap W} = \lambda_{S(n-1)\cap W} + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ S(n-1)\cap W} +\\ + \mu_{S'\cap W} R_{S'\cap W\ S(n-1)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n-1)\cap W},\end{aligned}$$

and

$$\begin{aligned}\eta(n)_{S'\cap W} &= \lambda_{S'\cap W} + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ S'\cap W} +\\ &\quad + \mu_{S'\cap W} R_{S'\cap W\ S'\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S'\cap W}\\ \Leftrightarrow\ \mu_{S'\cap W} &= \lambda_{S'\cap W} + \eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ S'\cap W} +\\ &\quad + \mu_{S'\cap W} R_{S'\cap W\ S'\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S'\cap W} + \mu_{S'\cap W} - \eta(n)_{S'\cap W}.\end{aligned}$$

With $\eta^*(n)_{S(n)\cap W} := (\eta(n)_{S(n-1)\cap W}, \mu_{S'\cap W})$ and

$$\lambda^*_{S(n)\cap W} := (\lambda_{S(n-1)\cap W}, \lambda_{S'\cap W} + \mu_{S'\cap W} - \eta(n)_{S'\cap W})$$

we have

$$\eta^*(n)_{S(n)\cap W} = \lambda^*_{S(n)\cap W} + \eta^*(n)_{S(n)\cap W} R_{S(n)\cap W\ S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W}$$

and with the existence and positivity of $(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}$ (see Lemma 1.2) we get

$$\eta^*(n)_{S(n)\cap W} = (\lambda^*_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W})(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}.$$

Similarly we get the solution of

$$\eta(n+1)_{S(n)\cap W} \stackrel{(4.11)}{=} \lambda_{S(n)\cap W} + \eta(n+1)_{S(n)\cap W} R_{S(n)\cap W\ S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W}$$

as

$$\eta(n+1)_{S(n)\cap W} = (\lambda_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W})(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}.$$

From definition it holds $\mu_{S'} > \eta(n)_{S'}$, so $\mu_{S'\cap W} > \eta(n)_{S'\cap W}$ and therefore $\lambda^*_{S(n)\cap W} \geq \lambda_{S(n)\cap W}$ holds. Thus

$$\begin{aligned}&\eta^*(n)_{S(n)\cap W} \geq \eta(n+1)_{S(n)\cap W}\\ \Leftrightarrow\ &(\eta(n)_{S(n-1)\cap W}, \mu_{S'\cap W}) \geq (\eta(n+1)_{S(n-1)\cap W}, \eta(n+1)_{S'\cap W})\\ \Leftrightarrow\ &\eta(n)_{S(n-1)\cap W} \geq \eta(n+1)_{S(n-1)\cap W}\ \wedge\ \mu_{S'\cap W} \geq \eta(n+1)_{S'\cap W}\\ \Rightarrow\ &\eta(n)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{\jmath}} \geq \eta(n+1)_{S(n-1)\cap W} R_{S(n-1)\cap W\ \tilde{\jmath}}\\ &\wedge\ \mu_{S'\cap W} R_{S'\cap W\ \tilde{\jmath}} \geq \eta(n+1)_{S'\cap W} R_{S'\cap W\ \tilde{\jmath}}\end{aligned}$$

which yields (4.12).

Existence of a solution of (4.5):
For the existence of a solution of the general traffic equations we need to show for all $n \in \tilde{J}$ that the $S(n) \cap W$-partition of the traffic equations, (4.6) and (4.7), has a solution. Transforming (4.6) into

$$\eta(n+1)_{S(n)\cap W} = (\lambda_{S(n)\cap W} + \mu_{U(n)\cup V} R_{U(n)\cup V\ S(n)\cap W})(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}$$

yields the unique solution of (4.6) and inserting this solution into equation (4.7) yields the unique solution of (4.7), but the transformation is possible if and only if $(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}$ exists and is positive. Because of the irreducibility of R, $(\mathbf{I} - R_{S(n)\cap W\ S(n)\cap W})^{-1}$ exists and is positive for all $n \in \tilde{J}$, see Lemma 1.2.

Uniqueness of a solution of (4.5):
Suppose η and $\hat{\eta}$ are both solutions of (4.5). Then for all nodes $i \in \tilde{J}$ holds

$$\eta_i - \hat{\eta}_i = \lambda_i - \lambda_i + \sum_{j\in W}(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))r(j,i) + \sum_{j\in V}(\mu_j r(j,i) - \mu_j r(j,i))$$

$$\Rightarrow |\eta_i - \hat{\eta}_i| = \left|\sum_{j\in W}(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))r(j,i)\right|.$$

Summing over all $i \in W$ yields:

$$\begin{aligned}\sum_{i\in W}|\eta_i - \hat{\eta}_i| &= \sum_{i\in W}\left|\sum_{j\in W}(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))r(j,i)\right| \\ &\overset{(*1)}{\leq} \sum_{i\in W}\sum_{j\in W}|(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))| \cdot |r(j,i)| \\ &= \sum_{j\in W}|(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))| \cdot \underbrace{\sum_{i\in W}|r(j,i)|}_{=1-r(j,0)-\sum_{i\in V} r(j,i)} \\ &\leq \sum_{j\in W}|(\min(\eta_j, \mu_j) - \min(\hat{\eta}_j, \mu_j))| \\ &\leq \sum_{j\in W}|\eta_j - \hat{\eta}_j|, \end{aligned} \tag{4.13}$$

where $(*1)$ holds because of the triangle inequality. (4.13) yields

$$\begin{aligned}\sum_{i\in W}|\eta_i - \hat{\eta}_i| &= \sum_{i\in W}|(\min(\eta_i, \mu_i) - \min(\hat{\eta}_i, \mu_i))| \\ \Leftrightarrow |\eta_i - \hat{\eta}_i| &= |(\min(\eta_i, \mu_i) - \min(\hat{\eta}_i, \mu_i))| \ \forall\ i \in W,\end{aligned}$$

because in any case $|\eta_i - \hat{\eta}_i| \geq |(\min(\eta_i, \mu_i) - \min(\hat{\eta}_i, \mu_i))| \ \forall\ i \in W$.
So $\{i \in W : \eta_i < \mu_i\} = \{i \in W : \hat{\eta}_i < \mu_i\} =: S \cap W$ and therefore η and $\hat{\eta}$ are the solutions of the same $S \cap W$-partition of the traffic equation (which has a unique solution, see above), which means $\eta = \hat{\eta}$. □

Whenever analyzing a Jackson network with infinite supply, it is important to know which nodes are stable and which are unstable and therefore which traffic equations are appropriate.

Algorithm 4.13. *To determine which nodes are stable and which nodes are unstable and the appropriate traffic equations in a Jackson network with infinite supply at nodes in* V*. Let* $W := \tilde{J} \setminus V$ *denote the set of nodes without infinite supply.*

(i) Solve the standard traffic equations (4.1)*. Check if* $\eta_i < \mu_i$ *holds for all nodes* $i \in \tilde{J}$*.*

- *If* $\eta_i < \mu_i$ *holds for all nodes* $i \in \tilde{J}$*, then all nodes are stable and* (4.1) *are the appropriate traffic equations.*
- *If the condition holds for all nodes in* W *and if the condition does not hold for at least one node with infinite supply (*$\in V$*), then all nodes in* W *are stable, but those nodes in* V *for which the condition does not hold are unstable. Nevertheless* (4.1) *are the appropriate traffic equations.*
- *If there is only one node in* W*, say* i_**, for which the condition does not hold, this node is unstable and the appropriate traffic equations are given by:*

$$\eta_j = \lambda_j + \sum_{i \in W \setminus \{i_*\}} \eta_i r(i,j) + \sum_{i \in V \cup \{i_*\}} \mu_i r(i,j), \quad j \in \tilde{J}.$$

- *If there is more than only one node in* W *for which the condition does not hold, proceed to the following step.*

(ii) Run Algorithm 4.11 to solve the general traffic equations (4.5)*. With the detected set* $S := \{i : \eta_i < \mu_i\}$ *of stable nodes and* $U := \tilde{J} \setminus S$ *of unstable nodes the appropriate traffic equations are then given by*

$$\eta_j = \lambda_j + \sum_{i \in S \cap W} \eta_i r(i,j) + \sum_{i \in U \cup V} \mu_i r(i,j), \quad j \in \tilde{J}.$$

Remark 4.14. *If the network is expected to be on average overloaded at almost all nodes, one may skip the first task of Algorithm 4.13 and start with Algorithm 4.11 right away. But in general, Algorithm 4.13 reduces the computational effort to determine the appropriate traffic equation, because in many cases running Algorithm 4.11 is avoided.*

4.3 The ergodic case

Main parts of the next theorem were proved in [Wei05] for ergodic Jackson networks with infinite supply where nodes show no immediate feedback. The proof is sketched there only. To fit to our later needs, we generalize and prove similar statements for Jackson networks with possible immediate feedback at all nodes.

Theorem 4.15. *Consider a Jackson network where nodes in* $V \subseteq \tilde{J}$ *have an infinite supply of work as in Definition 4.1. Nodes in* $W := \tilde{J} \setminus V$ *operate without infinite supply. Assume that* $\eta_i < \mu_i$ *holds for all nodes* $i \in \tilde{J}$ *where* $\eta = (\eta_1, ..., \eta_J)$ *is the unique solution of the traffic equations* (4.1)*. Denote by* $X = ((X_1(t), ..., X_J(t)) : t \geq 0)$ *the queue-length process on* $\mathbb{N}^J$*.*

(i) *For nodes without infinite supply, the joint marginal limiting distribution is of product form:*

$$\lim_{t\to\infty} P(X_i(t) = n_i : i \in W) = \prod_{i\in W} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{4.14}$$

for all $(n_i : i \in W) \in \mathbb{N}^{|W|}$ *and this is a stationary distribution for the subset* W *as well.*

(ii) *If the system is started with an initial distribution which has* (4.14) *as marginal joint queue lengths distribution on* W*, the arrival stream at* $i \in V$ *from* $j \in W$ *is a Poisson stream with rate* $\eta_j r(j,i)$.

(iii) *If the system is started with an initial distribution which has* (4.14) *as marginal joint queue lengths distribution on* W*, then the marginal limiting distribution for a node* $i \in V$ *with infinite supply which has no immediate feedback, i.e.,* $r(i,i) = 0$*, is*

$$\lim_{t\to\infty} P(X_i(t) = n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{4.15}$$

for all $n_i \in \mathbb{N}$ *and this is a one-dimensional stationary distribution as well.*

Remark 4.16. *The main difference of Theorem 4.15 to Proposition 1 of Weiss in [Wei05] is the explicite condition "If the system is started with an initial distribution which has* (4.14) *as marginal joint queue lengths distribution on* W*" which is needed in (ii) as well as in (iii). Weiss implicitly uses this condition in his sketch of the proof. The point is that without this assumption customer streams from the subset* W *into the subset* V*, in general, are not Poisson. This will be evident in the following proof.*

Proof of Theorem 4.15. (i): Consider the subset W of nodes without infinite supply. We have the following information about the subnetwork W:

- All service times are exponentially distributed and the service discipline at all nodes is FCFS.
- Routing of customers is Markovian: A customer completing service at node $i \in W$ will either move to some node $j \in W$ with probability $r(i,j)$ or leave the subnetwork with probability $1 - \sum_{j\in W} r(i,j)$, which is non-zero for at least one $i \in W$ because of the routing matrix being irreducible for the global network on $\tilde{J}$.
- At each node $i \in W$, we have external Poisson arrival streams with rate $\lambda_i \geq 0$. Furthermore all streams from nodes $j \in V$ with infinite supply into nodes $i \in W$ are Poisson streams with rate $\mu_j r(j,i)$, see Theorem 4.3.
  All (inter-)arrival times from the source and from nodes in V into node $i \in W$ constitute a set of independent random variables. Thus all arrival streams from the outside of the subnetwork W into each node $i \in W$ constitute independent Poisson processes with rate $\lambda_i + \sum_{j\in V} \mu_j r(j,i)$.
- All service and inter-arrival times constitute a set of independent random variables.

All these properties guarantee that the subnetwork W develops as a Jackson network with $|W|$ nodes where the source and sink is represented by $\{0\} \cup V$, see Definition 1.1.
The corresponding queueing process

$$\tilde{X} := ((\tilde{X}_i(t) : i \in W) : t \in \mathbb{R}_+)$$

is a Markov process of its own. The traffic equations of the described subnetwork W are given by

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j \in W} \tilde{\eta}_j r(j,i), \quad i \in W,$$

where

$$\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i),$$

so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$.
According to Jackson's theorem (see Theorem 1.10), $\tilde{X}$ has the unique stationary and limiting distribution (4.14) because $\eta_i < \mu_i$ for all $i \in W$ holds by assumption.
Thus, even if the subnetwork V of nodes with infinite supply is not in equilibrium, the equilibrium on the subnetwork W of nodes without infinite supply is preserved, if the initial distribution has the joint marginal (4.14).

An interesting alternative way to prove (i) is as follows: If the network process is ergodic, there is a unique solution of the global balance equations $\pi Q = 0$, i.e., $\forall (n_1, ..., n_J) \in \mathbb{N}^J$:

$$\begin{aligned}
\pi(n_1, ..., n_J) \cdot \Big(&\sum_{i \in \tilde{J}} \lambda_i + \sum_{i \in \tilde{J}} \sum_{j \in V} \mu_j r(j,i) \cdot 1_{\{0\}}(n_j) + \sum_{i \in \tilde{J}} \mu_i r(i,0) \cdot 1_{\mathbb{N}_+}(n_i) + \\
&+ \sum_{i \in \tilde{J}} \sum_{j \in \tilde{J} \setminus \{i\}} \mu_i r(i,j) \cdot 1_{\mathbb{N}_+}(n_i)\Big) \\
= \sum_{i \in W} &\pi(n_1, ..., n_i - 1, ..., n_J) \cdot \Big(\lambda_i + \sum_{j \in V} \mu_j r(j,i) \cdot 1_{\{0\}}(n_j)\Big) \cdot 1_{\mathbb{N}_+}(n_i) + \\
+ \sum_{i \in V} &\pi(n_1, ..., n_i - 1, ..., n_J) \cdot \\
&\cdot \Big(\lambda_i + \sum_{j \in V \setminus \{i\}} \mu_j r(j,i) \cdot 1_{\{0\}}(n_j) + \mu_i r(i,i) 1_{\{0\}}(n_i - 1)\Big) \cdot 1_{\mathbb{N}_+}(n_i) + \\
+ \sum_{i \in \tilde{J}} &\pi(n_1, ..., n_i + 1, ..., n_J) \cdot \mu_i r(i,0) + \\
+ \sum_{i \in \tilde{J}} &\sum_{j \in \tilde{J} \setminus \{i\}} \pi(n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J) \cdot \mu_i r(i,j) \cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned} \tag{4.16}$$

Summing over all $(n_k : k \in V) \in \mathbb{N}^{|V|}$ yields:

$$\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \pi(n_1,...,n_J)\cdot\Big(\sum_{i\in\tilde{J}}\lambda_i+\sum_{i\in\tilde{J}}\sum_{j\in V}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)+\sum_{i\in\tilde{J}}\mu_i r(i,0)\cdot 1_{\mathbb{N}_+}(n_i)+$$
$$+\sum_{i\in\tilde{J}}\sum_{j\in\tilde{J}\setminus\{i\}}\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_i)\Big)$$
$$=\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\pi(n_1,...,n_i-1,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in V}\pi(n_1,...,n_i-1,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V\setminus\{i\}}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in V}\pi(n_1,...,n_i-1,...,n_J)\cdot\mu_i r(i,i)\cdot\underbrace{1_{\{0\}}(n_i-1)\cdot 1_{\mathbb{N}_+}(n_i)}_{=1_{\{1\}\cap\mathbb{N}_+}(n_i)=1_{\{1\}}(n_i)=1_{\{0\}}(n_i-1)}+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in\tilde{J}}\pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i r(i,0)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in\tilde{J}}\sum_{j\in\tilde{J}\setminus\{i\}}\pi(n_1,...,n_i+1,...,n_j-1,...,n_J)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_j)$$
$$\Leftrightarrow\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\pi(n_1,...,n_J)\cdot\Big(\sum_{i\in\tilde{J}}\lambda_i+\sum_{i\in\tilde{J}}\sum_{j\in V}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)+$$
$$+\sum_{i\in\tilde{J}}\mu_i r(i,0)\cdot 1_{\mathbb{N}_+}(n_i)+\sum_{i\in\tilde{J}}\sum_{j\in\tilde{J}\setminus\{i\}}\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_i)\Big)$$
$$=\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\pi(n_1,...,n_i-1,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in V}\pi(n_1,...,n_i,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V\setminus\{i\}}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)\Big)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in V}\pi(n_1,...,n_i,...,n_J)\cdot\mu_i r(i,i)\cdot 1_{\{0\}}(n_i)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i r(i,0)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in V}\pi(n_1,...,n_i,...,n_J)\cdot\mu_i r(i,0)\cdot 1_{\mathbb{N}_+}(n_i)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\sum_{j\in W\setminus\{i\}}\pi(n_1,...,n_i+1,...,n_j-1,...,n_J)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_j)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\sum_{j\in V}\pi(n_1,...,n_i+1,...,n_j,...,n_J)\cdot\mu_i r(i,j)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in V}\sum_{j\in W}\pi(n_1,...,n_i,...,n_j-1,...,n_J)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_j)\cdot 1_{\mathbb{N}_+}(n_i)+$$
$$+\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in V}\sum_{j\in V\setminus\{i\}}\pi(n_1,...,n_i,...,n_j,...,n_J)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_i)$$

$$\begin{aligned}
\Leftrightarrow & \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \pi(n_1,...,n_J)\cdot \\
& \Big(\sum_{i\in\tilde{J}}\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)\Big)-\sum_{i\in V}\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)\Big)+ \\
& +\sum_{i\in\tilde{J}}\mu_i r(i,0)\cdot 1_{\mathbb{N}_+}(n_i)-\sum_{i\in V}\mu_i r(i,0)\cdot 1_{\mathbb{N}_+}(n_i)+ \\
& +\sum_{i\in\tilde{J}}\sum_{j\in\tilde{J}\setminus\{i\}}\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_i)-\sum_{i\in V}\sum_{j\in V\setminus\{i\}}\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_i)\Big) \\
= & \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\pi(n_1,...,n_i-1,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)+ \\
+ & \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i r(i,0)+ \\
+ & \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\sum_{j\in W\setminus\{i\}}\pi(n_1,...,n_i+1,...,n_j-1,...,n_J)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_j)+ \\
+ & \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in W}\sum_{j\in V}\pi(n_1,...,n_i+1,...,n_j,...,n_J)\cdot\mu_i r(i,j)+ \\
+ & \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\sum_{i\in V}\sum_{j\in W}\pi(n_1,...,n_i,...,n_j-1,...,n_J)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_j)\cdot 1_{\mathbb{N}_+}(n_i)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow & \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \pi(n_1,...,n_J)\cdot \\
& \Big(\sum_{i\in W}\lambda_i+\underbrace{\sum_{i\in W}\sum_{j\in V}\mu_j r(j,i)\cdot 1_{\{0\}}(n_j)+\sum_{i\in V}\sum_{j\in W}\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_i)}_{=\sum_{i\in W}\sum_{j\in V}\mu_j r(j,i)}+ \\
& \underbrace{+\sum_{i\in W}\mu_i r(i,0)\cdot 1_{\mathbb{N}_+}(n_i)+\sum_{i\in W}\sum_{j\in\tilde{J}\setminus\{i\}}\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_i)}_{=\sum_{i\in W}\mu_i(1-r(i,i))\cdot 1_{\mathbb{N}_+}(n_i)}\Big) \\
= & \sum_{i\in W}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\pi(n_1,...,n_i-1,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)\Big)\cdot 1_{\mathbb{N}_+}(n_i)+ \\
+ & \sum_{i\in W}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i\underbrace{\Big(r(i,0)+\sum_{j\in V}r(i,j)\Big)}_{=1-\sum_{j\in W}r(i,j)}+ \\
+ & \sum_{i\in W}\sum_{j\in W\setminus\{i\}}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\pi(n_1,...,n_i+1,...,n_j-1,...,n_J)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned}$$

With $\pi_W(n_k : k\in W) := \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}}\pi(n_1,...,n_J)$, these equations are equivalent to:

$\forall (n_i : i \in W) \in \mathbb{N}^{|W|}$:

$$\pi_W(n_k : k \in W) \cdot \Big(\sum_{i \in W}\Big(\lambda_i + \sum_{j \in V} \mu_j r(j,i)\Big) + \sum_{i \in W} \mu_i(1 - r(i,i)) \cdot 1_{\mathbb{N}_+}(n_i)\Big)$$
$$= \sum_{i \in W} \pi_W(n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\lambda_i + \sum_{j \in V} \mu_j r(j,i)\Big) \cdot 1_{\mathbb{N}_+}(n_i) +$$
$$+ \sum_{i \in W} \pi_W(n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i\Big(1 - \sum_{j \in W} r(i,j)\Big) +$$
$$+ \sum_{i \in W} \sum_{j \in W \setminus \{i\}} \pi_W(n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r(i,j) \cdot 1_{\mathbb{N}_+}(n_j).$$

These equations are the global balance equations of a Jackson network with $|W|$ stable nodes and with the following characteristics: External arrival streams at nodes $i \in W$ are Poisson processes with rate $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i)$ and the service times at nodes $i \in W$ are exponentially distributed with rate μ_i. All inter-arrival and service times constitute a set of independent random variables. The routing matrix is the same as the original one where transitions to the subset V are handled as transitions to the sink. In this situation these equations are known to have a unique probability solution, which is the stationary distribution (4.14). So if the global network process is started with its stationary distribution, its marginal distribution can only be (4.14) which is the stationary distribution of the subnet on W.

(ii): It is well known that ergodic Jackson networks with Poisson arrival streams from the source to node i with rate $\tilde{\lambda}_i$ have, in equilibrium, Poisson departure streams from node i to the sink with some rate $\tilde{\eta}_i \tilde{r}(i,0)$, see [Mel79b, Example 7.1]. From the proof of (i), we know that the subset W behaves like an ergodic Jackson network for its own with $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i)$ and

$$\tilde{\eta}_i \tilde{r}(i,0) = \eta_i\Big(1 - \sum_{j \in W} r(i,j)\Big) = \eta_i\Big(r(i,0) + \sum_{j \in V} r(i,j)\Big).$$

Hence, if the global network process is started with an initial distribution which has the marginal (4.14) on W, departures to the sink from nodes $i \in W$ are Poisson streams with rate $\eta_i r(i,0)$ and departures to any node $j \in V$ are also Poisson streams with rate $\eta_i r(i,j)$, because a portion $\frac{r(i,j)}{r(i,0) + \sum_{j \in V} r(i,j)}$ of the departure stream $\tilde{\eta}_i \tilde{r}(i,0)$ from node $i \in W$ is directed to $j \in V$. This holds even if the subnetwork V is not in equilibrium.

(iii): Consider a node $i \in V$ with infinite supply and without immediate feedback (i.e., $r(i,i) = 0$):

- The node has exponential-μ_i distributed service, the service discipline is FCFS.
- Routing of customers is Markovian: A customer arriving at node i, waits in the line until he gets service, after the service he proceeds to another node $j \in \tilde{J} \setminus \{i\}$ with probability $r(i,j)$ or leaves the network with probability $r(i,0)$. Thus, the customer leaves node i after service is completed with probability

$$r(i,0) + \sum_{j \in \tilde{J} \setminus \{i\}} r(i,j) = 1 - r(i,i) = 1.$$

- The external arrival stream is Poisson with rate $\lambda_i \geq 0$. From (ii) it follows directly that, if the global network process is started with an initial distribution which has the marginal (4.14) on W, the arrival streams at node $i \in V$ from nodes $j \in W$ are Poisson with rate $\eta_j r(j,i)$. Arrival streams from nodes $j \in V \setminus \{i\}$ are Poisson streams with rate $\mu_j r(j,i)$, see Theorem 4.3. All these Poisson streams are independent of each other. Thus the arrival stream at node $i \in V$ is a Poisson process with rate

$$\hat{\lambda}_i := \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V \setminus \{i\}} \mu_j r(j,i).$$

- All service and inter-arrival times constitute a set of independent random variables.

Thus, if the subnetwork W is in equilibrium and if $r(i,i) = 0$ holds, node $i \in V$ behaves as an $(M/M/1)-$ system of its own. The corresponding queue length process $\hat{X}$ is a Markov process on the state space $\mathbb{N}$. Customers who arrive from the infinite supply storage of this node are not counted as waiting customers. The traffic equation is then given by

$$\hat{\eta}_i = \hat{\lambda}_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V \setminus \{i\}} \mu_j r(j,i),$$

thus $\hat{\eta}_i = \eta_i$ holds, see (4.1) with $r(i,i) = 0$.
If the balance equations of $\hat{X}$ have a probability solution, then this solution is the unique stationary and limiting distribution of $\hat{X}$. The balance equations are for all $n \in \mathbb{N}$

$$\begin{aligned} \pi_i(n)(\hat{\lambda}_i + \mu_i 1_{\mathbb{N}_+}(n)) &= \pi_i(n-1)\hat{\lambda}_i 1_{\mathbb{N}_+}(n) + \pi_i(n+1)\mu_i \\ \Leftrightarrow \pi_i(n)(\eta_i + \mu_i 1_{\mathbb{N}_+}(n)) &= \pi_i(n-1)\eta_i 1_{\mathbb{N}_+}(n) + \pi_i(n+1)\mu_i, \end{aligned}$$

which are solved by $\pi_i(n) = \left(\frac{\eta_i}{\mu_i}\right)^n$. Normalizing the solution to

$$\pi_i(n) = \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^n$$

yields a probability measure if and only if $\eta_i < \mu_i$ holds, which was assumed. □

The proof of Theorem 4.15(i) verifies the following

Corollary 4.17. *In the setting of Theorem 4.15, the process $X_W := (X_i : i \in W)$ is an ergodic homogeneous Markov process of its own.*

Remark 4.18. *Note that in Theorem 4.15 in (i) and (ii) immediate feedback is allowed at all nodes. Only in (iii) we required that nodes with infinite supply have no immediate feedback. This is due to the arising balance equations, as can be seen as follows:*
Consider a node $i \in V$ as in Theorem 4.15, but allow immediate feedback at all nodes. Then all facts of the proof of (iii) hold except for:

- *If the subnetwork W is in equilibrium, node $i \in V$ behaves as an $(M/M/1)-$ system with infinite supply and with immediate feedback of its own.*

- *The traffic equation is then given by*

$$\hat{\eta}_i = \lambda_i + \sum_{j\in W} \eta_j r(j,i) + \sum_{j\in V} \mu_j r(j,i), \tag{4.17}$$

thus $\hat{\eta}_i = \eta_i$ *holds, see* (4.1).

- *The balance equations of* $\hat{X}$ *are for all* $n \in \mathbb{N}$

$$\begin{aligned}
&\pi_i(n)\Big(\lambda_i + \sum_{j\in W}\eta_j r(j,i) + \sum_{j\in V\setminus\{i\}}\mu_j r(j,i) + \mu_i r(i,i)1_{\{0\}}(n) + \mu_i(1-r(i,i))1_{\mathbb{N}_+}(n)\Big)\\
&= \pi_i(n-1)\Big(\lambda_i + \sum_{j\in W}\eta_j r(j,i) + \sum_{j\in V\setminus\{i\}}\mu_j r(j,i)\Big)1_{\mathbb{N}_+}(n) +\\
&+ \pi_i(n-1)\mu_i r(i,i)1_{\{1\}}(n) + \pi_i(n+1)\mu_i(1-r(i,i)).
\end{aligned} \tag{4.18}$$

Plugging (4.17) *into* (4.18) *yields*

$$\begin{aligned}
&\pi_i(n)\Big(\eta_i - \mu_i r(i,i) + \mu_i r(i,i)1_{\{0\}}(n) + \mu_i(1-r(i,i))1_{\mathbb{N}_+}(n)\Big)\\
&= \pi_i(n-1)(\eta_i - \mu_i r(i,i))1_{\mathbb{N}_+}(n) + \pi_i(n-1)\mu_i r(i,i)1_{\{1\}}(n) + \pi_i(n+1)\mu_i(1-r(i,i)).
\end{aligned}$$

With $\pi_i(n) = \left(\frac{\eta_i}{\mu_i}\right)^n$ *this is equivalent to*

$$\begin{aligned}
&\eta_i - \mu_i r(i,i) + \mu_i r(i,i)1_{\{0\}}(n) + \mu_i(1-r(i,i))1_{\mathbb{N}_+}(n)\\
&= \frac{\mu_i}{\eta_i}(\eta_i - \mu_i r(i,i))1_{\mathbb{N}_+}(n) + \frac{\mu_i}{\eta_i}\mu_i r(i,i)1_{\{1\}}(n) + \frac{\eta_i}{\mu_i}\mu_i(1-r(i,i))
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow &-\mu_i r(i,i) + \mu_i r(i,i)1_{\{0\}}(n) - \mu_i r(i,i)1_{\mathbb{N}_+}(n)\\
&= -\frac{\mu_i}{\eta_i}\mu_i r(i,i)1_{\mathbb{N}_+}(n) + \frac{\mu_i}{\eta_i}\mu_i r(i,i)1_{\{1\}}(n) - \eta_i r(i,i)
\end{aligned}$$

$$\Leftrightarrow (\eta_i - \mu_i)r(i,i) + \mu_i r(i,i)\Big(1_{\{0\}}(n) - \frac{\mu_i}{\eta_i}1_{\{1\}}(n)\Big) = \Big(1 - \frac{\mu_i}{\eta_i}\Big)\mu_i r(i,i)1_{\mathbb{N}_+}(n).$$

With $r(i,i) > 0$ *the last equation holds if and only if*

$$\eta_i - \mu_i + \mu_i\Big(1_{\{0\}}(n) - \frac{\mu_i}{\eta_i}1_{\{1\}}(n)\Big) = \Big(1 - \frac{\mu_i}{\eta_i}\Big)\mu_i 1_{\mathbb{N}_+}(n). \tag{4.19}$$

- *In case of* $n = 0$ *equation* (4.19) *is reduced to* $\eta_i - \mu_i + \mu_i = 0 \;\Leftrightarrow\; \eta_i = 0$.
- *In case of* $n = 1$ *equation* (4.19) *reduces to* $\eta_i - \mu_i - \mu_i\frac{\mu_i}{\eta_i} = \Big(1 - \frac{\mu_i}{\eta_i}\Big)\mu_i \;\Leftrightarrow\; \eta_i = 2\mu_i$.
- *In case of* $n \geq 2$ *equation* (4.19) *is reduced to* $\eta_i - \mu_i = \Big(1 - \frac{\mu_i}{\eta_i}\Big)\mu_i \;\Leftrightarrow\; \eta_i = \mu_i$.

Thus, equation (4.19) *holds if and only if* $\mu_i = 0$ *holds for* $i \in V$ *which is a contradiction to the assumptions in Definition 4.1 of a Jackson network with infinite supply.*

Remark 4.19. *In general, except for a special network described below in Corollary 4.20, an ergodic Jackson network with infinite supply of work (without immediate feedback at nodes with infinite supply) does* **not** *have a stationary distribution* **of product form***. This means that in general, in equilibrium, the queue lengths of the nodes with an infinite supply of work ($i \in V$) are at a fixed time instant neither independent of each other nor independent of the queue lengths of the nodes without infinite supply, although all flows between the nodes with infinite supply are Poisson. In contrast, the product form of* (4.14) *implies that in equilibrium the queue length processes of the subnetwork W at a fixed time instant are independent of each other, although the flows between these nodes are, in general, not Poisson.*

The statement of Remark 4.19 can be seen by plugging

$$\pi(n_1, ..., n_J) = \prod_{i \in \tilde{J}} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

into the global balance equations (4.16) of the network process with $r(i,i) = 0 \; \forall i \in V$, which yields:

$$\begin{aligned} &\sum_{i \in \tilde{J}} \lambda_i + \sum_{i \in \tilde{J}} \sum_{j \in V} \mu_j r(j,i) 1_{\{0\}}(n_j) + \sum_{i \in \tilde{J}} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i) \\ &= \sum_{i \in \tilde{J}} \frac{\mu_i}{\eta_i} \Big(\lambda_i + \sum_{j \in V} \mu_j r(j,i) 1_{\{0\}}(n_j) \Big) 1_{\mathbb{N}_+}(n_i) + \sum_{i \in \tilde{J}} \frac{\eta_i}{\mu_i} \mu_i r(i,0) + \\ &+ \sum_{i \in \tilde{J}} \sum_{j \in \tilde{J} \setminus \{i\}} \frac{\mu_j}{\eta_j} \frac{\eta_i}{\mu_i} \mu_i r(i,j) 1_{\mathbb{N}_+}(n_j) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \; &\sum_{i \in \tilde{J}} \lambda_i + \sum_{i \in \tilde{J}} \sum_{j \in V} \mu_j r(j,i) 1_{\{0\}}(n_j) + \sum_{i \in \tilde{J}} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i) \\ &= \sum_{i \in \tilde{J}} \frac{\mu_i}{\eta_i} \underbrace{\Big(\lambda_i + \sum_{j \in V} \mu_j r(j,i) 1_{\{0\}}(n_j) + \sum_{j \in \tilde{J} \setminus \{i\}} \eta_j r(j,i) \Big)}_{\overset{(*2)}{=} \eta_i (1 - r(i,i)) - \sum_{j \in V} \mu_j r(j,i) 1_{\mathbb{N}_+}(n_j) + \sum_{j \in V} \eta_j r(j,i)} 1_{\mathbb{N}_+}(n_i) + \sum_{i \in \tilde{J}} \eta_i r(i,0) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow \; &\sum_{i \in \tilde{J}} \lambda_i + \sum_{i \in \tilde{J}} \sum_{j \in V} \mu_j r(j,i) 1_{\{0\}}(n_j) \\ &= \sum_{i \in \tilde{J}} \frac{\mu_i}{\eta_i} \sum_{j \in V} r(j,i)(\eta_j - \mu_j 1_{\mathbb{N}_+}(n_j)) 1_{\mathbb{N}_+}(n_i) + \sum_{i \in \tilde{J}} \eta_i r(i,0), \end{aligned} \tag{4.20}$$

where (∗2) holds because of (4.1) and $r(i,i) = 0 \; \forall i \in V$. With

$$\sum_{i \in \tilde{J}} \lambda_i = \sum_{i \in W} \eta_i r(i,0) + \sum_{i \in V} \mu_i r(i,0) + \sum_{i \in V} (\eta_i - \mu_i)$$

(see Remark 4.5), (4.20) is equivalent to

$$
\begin{aligned}
&\sum_{i\in V}\mu_i r(i,0)+\sum_{i\in V}(\eta_i-\mu_i)+\sum_{i\in\tilde J}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)\\
&=\sum_{i\in\tilde J}\frac{\mu_i}{\eta_i}\sum_{j\in V}r(j,i)(\eta_j-\mu_j 1_{\mathbb{N}_+}(n_j))1_{\mathbb{N}_+}(n_i)+\sum_{i\in V}\eta_i r(i,0)\\
\Leftrightarrow\ &\sum_{i\in V}(\eta_i-\mu_i)\underbrace{(1-r(i,0))}_{=\sum_{j\in\tilde J}r(i,j)}+\sum_{i\in\tilde J}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)\\
&=\sum_{i\in\tilde J}\frac{\mu_i}{\eta_i}\sum_{j\in V}r(j,i)(\eta_j-\mu_j 1_{\mathbb{N}_+}(n_j))1_{\mathbb{N}_+}(n_i)\\
\Leftrightarrow\ &\sum_{i\in V}(\eta_i-\mu_i 1_{\mathbb{N}_+}(n_i))\sum_{j\in\tilde J}r(i,j)=\sum_{i\in\tilde J}\frac{\mu_i}{\eta_i}\sum_{j\in V}r(j,i)(\eta_j-\mu_j 1_{\mathbb{N}_+}(n_j))1_{\mathbb{N}_+}(n_i)\\
\Leftrightarrow\ &\sum_{i\in V}(\eta_i-\mu_i 1_{\mathbb{N}_+}(n_i))\sum_{j\in\tilde J}r(i,j)\left(1-\frac{\mu_j}{\eta_j}1_{\mathbb{N}_+}(n_j)\right)=0.
\end{aligned}
$$

The last equation is valid only if $r(i,j)=0$ holds for all $i\in V$ and $j\in\tilde J$. This justifies the following

Corollary 4.20. *The* **only** *class of Jackson networks with infinite supply where the stationary queue lengths distribution is* **of product form** *is characterized by the following property: Customers departing from a node $i\in V$ with infinite supply leave the network directly to the sink with probability 1, i.e., $r(i,0)=1\ \forall i\in V$.*

In other words, independence of the queue lengths in the system at a fixed time instant in equilibrium is maintained only if the nodes with infinite supply do not interact with each other at all and if there are only streams from the subset W without infinite supply to nodes in V with infinite supply but not the other way around. It is intriguing that all these interrupted departures from nodes with infinite supply are Poisson streams and exactly these seem to be the source of the dependence structure in equilibrium. The low priority customers from the infinite supply depot are then directed to the sink immediately after their first service and therefore they do not influence any arrival rate in the network.

The loss of independence due to the interaction of nodes with infinite supply can be observed in the following example analyzed by I.J.B.F. Adan and G. Weiss in [AW05]:

Example 4.21. [AW05] *Consider a two-node Jackson network with infinite supply as in Definition 4.1. Let $W=\emptyset$, so both nodes have infinite supply. Furthermore let $r(i,i)=0$ for $i=1,2$ hold, thus there is no immediate feedback in the system. Now - unlike a Jackson network - assume that there is no exogeneous input, i.e., $\lambda=0$. The traffic equations are then given by*

$$
\begin{aligned}
\eta_1&=\mu_2 r(2,1),\\
\eta_2&=\mu_1 r(1,2).
\end{aligned}
$$

If $\eta_i < \mu_i$ holds for $i = 1, 2$, then the queue length process $X = ((X_1(t), X_2(t)) : t \in \mathbb{R}_+)$ on the state space $\mathbb{N}^2$ has the unique stationary and limiting distribution

$$\pi(n_1, n_2) = \sum_{k=1}^{\infty}(-1)^{k+1}[(1-\alpha_k)\alpha_k^{n_1}(1-\beta_{k+1})\beta_{k+1}^{n_2} + (1-\alpha_{k+1})\alpha_{k+1}^{n_1}(1-\beta_k)\beta_k^{n_2}]$$

where for $k \geq 1$

$$\alpha_{k+1}^{-1} = \frac{\mu_1+\mu_2}{\mu_2 r(2,1)}\beta_k^{-1} - \alpha_{k-1}^{-1} - \frac{r(2,0)}{r(2,1)}, \quad \beta_{k+1}^{-1} = \frac{\mu_1+\mu_2}{\mu_1 r(1,2)}\alpha_k^{-1} - \beta_{k-1}^{-1} - \frac{r(1,0)}{r(1,2)},$$

with initially $\alpha_0 = \beta_0 = 1$, $\alpha_1 = \frac{\eta_1}{\mu_1}$, $\beta_1 = \frac{\eta_2}{\mu_2}$.
The joint stationary distribution is not of product form, it holds

$$\pi(n_1, n_2) \neq \pi_1(n_1) \cdot \pi_2(n_2) \ \forall (n_1, n_2) \in \mathbb{N}^2,$$

where $\pi_i(n_i) = (1 - \frac{\eta_i}{\mu_i})(\frac{\eta_i}{\mu_i})^{n_i}$, $i = 1, 2$.
Moreover, Adan and Weiss proved that there is a negative correlation between the queue lengths at the nodes, if $\eta_i < \mu_i$ holds for $i = 1, 2$, so even if the system is in equilibrium:

$$Corr(X_1, X_2) < 0.$$

In the following example of a two-node Jackson network with infinite supply of the general type, the events $\{X_1(t) = 0\}$ and $\{X_2(t) = n\}$ with any $n \in \mathbb{N}$ are, in equilibrium, at all times $t \geq 0$ independent of each other, if the first coordinate represents the queue length of a node without infinite supply and the second is the queue length of a node with infinite supply.

Example 4.22. *Consider a Jackson network of two nodes, where node 2 has infinite supply but no immediate feedback. Let $r(1, 2) > 0$ and $r(2, 1) > 0$, so we do not have a product form network with infinite supply from Corollary 4.20. Then the traffic equations are given by*

$$\eta_1 = \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1) \tag{4.21}$$
$$\eta_2 = \lambda_2 + \eta_1 r(1,2). \tag{4.22}$$

Let $\eta_i < \mu_i$ hold for $i = 1, 2$. The queue length process $X = ((X_1(t), X_2(t)) : t \in \mathbb{R}_+)$ is a Markov process on the state space $\mathbb{N}^2$. From Theorem 4.15 we know that for all $n_1 \in \mathbb{N}$ holds

$$\pi_1(n_1) = \sum_{n_2 \in \mathbb{N}} \pi(n_1, n_2) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}, \tag{4.23}$$

we additionally assume that we start the system with an initial distribution which has the marginal (4.23), so for all $n_2 \in \mathbb{N}$ holds under this condition for the limiting and stationary distribution

$$\pi_2(n_2) = \sum_{n_1 \in \mathbb{N}} \pi(n_1, n_2) = \left(1 - \frac{\eta_2}{\mu_2}\right)\left(\frac{\eta_2}{\mu_2}\right)^{n_2}. \tag{4.24}$$

The global balance equations are $\forall (n_1, n_2) \in \mathbb{N}^2$*:*

$$\begin{aligned}
&\pi(n_1,n_2)\cdot(\lambda_1+\lambda_2+\mu_2 r(2,1)1_{\{0\}}(n_2)+\mu_1(1-r(1,1))1_{\mathbb{N}_+}(n_1)+\mu_2 1_{\mathbb{N}_+}(n_2))\\
&=\pi(n_1-1,n_2)\cdot(\lambda_1+\mu_2 r(2,1)1_{\{0\}}(n_2))1_{\mathbb{N}_+}(n_1)+\pi(n_1,n_2-1)\cdot\lambda_2 1_{\mathbb{N}_+}(n_2)+\\
&+\pi(n_1+1,n_2)\cdot\mu_1 r(1,0)+\pi(n_1,n_2+1)\cdot\mu_2 r(2,0)+\\
&+\pi(n_1+1,n_2-1)\cdot\mu_1 r(1,2)1_{\mathbb{N}_+}(n_2)+\pi(n_1-1,n_2+1)\cdot\mu_2 r(2,1)1_{\mathbb{N}_+}(n_1).
\end{aligned}$$

Summing over all $n_1 \in \mathbb{N}$ *on both sides yields*

$$\begin{aligned}
&\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2)\cdot(\lambda_1+\lambda_2+\mu_2 r(2,1)1_{\{0\}}(n_2)+\mu_1(1-r(1,1))1_{\mathbb{N}_+}(n_1)+\mu_2 1_{\mathbb{N}_+}(n_2))\\
&=\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2)\cdot(\lambda_1+\mu_2 r(2,1)1_{\{0\}}(n_2))+\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2-1)\cdot\lambda_2 1_{\mathbb{N}_+}(n_2)+\\
&+\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2)\cdot\mu_1 r(1,0)1_{\mathbb{N}_+}(n_1)+\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2+1)\cdot\mu_2 r(2,0)+\\
&+\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2-1)\cdot\mu_1 r(1,2)1_{\mathbb{N}_+}(n_1)1_{\mathbb{N}_+}(n_2)+\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2+1)\cdot\mu_2 r(2,1)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\ &\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2)(\lambda_2+\mu_1 r(1,2)+\mu_2 1_{\mathbb{N}_+}(n_2))-\pi(0,n_2)\mu_1 r(1,2)\\
&=\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2-1)(\lambda_2+\mu_1 r(1,2))1_{\mathbb{N}_+}(n_2)+\sum_{n_1\in\mathbb{N}}\pi(n_1,n_2+1)\mu_2\\
&-\pi(0,n_2-1)\mu_1 r(1,2)1_{\mathbb{N}_+}(n_2)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\ &\pi_2(n_2)(\lambda_2+\mu_1 r(1,2)+\mu_2 1_{\mathbb{N}_+}(n_2))-\pi(0,n_2)\mu_1 r(1,2)\\
&=\pi_2(n_2-1)(\lambda_2+\mu_1 r(1,2))1_{\mathbb{N}_+}(n_2)+\pi_2(n_2+1)\mu_2-\pi(0,n_2-1)\mu_1 r(1,2)1_{\mathbb{N}_+}(n_2)
\end{aligned}$$

$$\begin{aligned}
\overset{(4.24)}{\Rightarrow}\ &\pi_2(n_2)\Big(\lambda_2+\mu_1 r(1,2)+\mu_2 1_{\mathbb{N}_+}(n_2)-\frac{\mu_2}{\eta_2}(\lambda_2+\mu_1 r(1,2))1_{\mathbb{N}_+}(n_2)-\frac{\eta_2}{\mu_2}\mu_2\Big)\\
&=\pi(0,n_2)\mu_1 r(1,2)-\pi(0,n_2-1)\mu_1 r(1,2)1_{\mathbb{N}_+}(n_2)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\ &\pi_2(n_2)\Big(\underbrace{\lambda_2-\eta_2}_{\overset{(4.22)}{=}-\eta_1 r(1,2)}+\mu_1 r(1,2)+\mu_2 1_{\mathbb{N}_+}(n_2)-\frac{\mu_2}{\eta_2}(\underbrace{\lambda_2}_{\overset{(4.22)}{=}\eta_2-\eta_1 r(1,2)}+\mu_1 r(1,2))1_{\mathbb{N}_+}(n_2)\Big)\\
&=\pi(0,n_2)\mu_1 r(1,2)-\pi(0,n_2-1)\mu_1 r(1,2)1_{\mathbb{N}_+}(n_2)
\end{aligned}$$

$$\begin{aligned}
\Leftrightarrow\ &\pi_2(n_2)\Big(-\eta_1 r(1,2)+\mu_1 r(1,2)-\frac{\mu_2}{\eta_2}(-\eta_1 r(1,2)+\mu_1 r(1,2))1_{\mathbb{N}_+}(n_2)\Big)\\
&=\pi(0,n_2)\mu_1 r(1,2)-\pi(0,n_2-1)\mu_1 r(1,2)1_{\mathbb{N}_+}(n_2)
\end{aligned}$$

$$\begin{aligned}
\overset{(4.23)}{\Rightarrow}\ &\pi_2(n_2)\Big(\pi_1(0)\mu_1 r(1,2)-\frac{\mu_2}{\eta_2}\pi_1(0)\mu_1 r(1,2)1_{\mathbb{N}_+}(n_2)\Big)\\
&=\pi(0,n_2)\mu_1 r(1,2)-\pi(0,n_2-1)\mu_1 r(1,2)1_{\mathbb{N}_+}(n_2)
\end{aligned}$$

$$\Leftrightarrow \pi_2(n_2)\pi_1(0)\Big(1 - \frac{\mu_2}{\eta_2} 1_{\mathbb{N}_+}(n_2)\Big) = \pi(0, n_2) - \pi(0, n_2 - 1)1_{\mathbb{N}_+}(n_2). \quad (4.25)$$

It follows directly from (4.25) *that* $\pi(0, 0) = \pi_1(0) \cdot \pi_2(0)$ *holds and, since* (4.25) *is a recursive equation, by induction*

$$\pi(0, n_2) = \pi_1(0) \cdot \pi_2(n_2) \ \forall n_2 \in \mathbb{N}.$$

Thus, in equilibrium, the events $\{X_1(t) = 0\}$ *and* $\{X_2(t) = n_2\}$ *are independent of each other for all* $n_2 \in \mathbb{N}$.

This examples motivates:

Proposition 4.23. *Consider a Jackson network where nodes in* $V \subseteq \tilde{J}$ *have an infinite supply of work as in Definition 4.1. Nodes in* $W := \tilde{J} \setminus V$ *operate without infinite supply. Let* $|W| = 1$*, so there is only one node, say node* 1*, without infinite supply. We assume that there is no immediate feedback at nodes with infinite supply, i.e.,* $r(i, i) = 0$ *for all* $i \in V$*, and that all nodes are immediately accessible, i.e.,* $r(i, j) > 0$ *for all* $i \neq j$*. Let* $\eta_i < \mu_i$ *hold for all nodes* $i \in \tilde{J}$ *where* $\eta = (\eta_1, ..., \eta_J)$ *is the unique solution of the traffic equations* (4.1)*:*

$$\eta_i = \lambda_i + \eta_1 r(1, i) + \sum_{j \in V} \mu_j r(j, i), \ i \in \tilde{J}. \quad (4.26)$$

Denote by $X = ((X_1(t), ..., X_J(t)) : t \geq 0)$ *the queue-length process on* $\mathbb{N}^J$ *and by* $\pi(n_1, ..., n_J)$ *the stationary distribution of* X*. If the system is started with an initial distribution which has the marginal*

$$P(X_1(0) = n_1) := \pi_1(n_1) = \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n_1} \ \forall n_1 \in \mathbb{N}, \quad (4.27)$$

then for the joint marginal $\pi_{\{1,i\}}(0, n_i) := \sum_{(n_k : k \in \tilde{J} \setminus \{1,i\}) \in \mathbb{N}^{J-2}} \pi(0, ..., n_i, ..., n_J)$ *of the stationary distribution of* X *holds*

$$\pi_{\{1,i\}}(0, n_i) = \pi_1(0) \cdot \pi_i(n_i) \ \forall i \in V \ \forall n_i \in \mathbb{N}, \quad (4.28)$$

where $\pi_i(n_i)$ *is the one-dimensional marginal queue length distribution for node* $i \in V$

$$\pi_i(n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i} \ \forall n_i \in \mathbb{N}. \quad (4.29)$$

Proposition 4.23 states that there is a class of Jackson networks with infinite supply and with an arbitrary finite number, say J, of nodes for which, in equilibrium, the following events are independent of each other:

- The sole node without infinite supply ($i \in W$) is empty at time t.
- A node with infinite supply ($j \in V$) has a queue length n at time t.

Proof of Proposition 4.23. From Theorem 4.15 we know that if we start the system with an initial distribution which has the marginal (4.27), the marginal for any node $i \in V$ is (4.29). The global balance equations of X are for all $(n_1, ..., n_J) \in \mathbb{N}^J$:

$$\begin{aligned}
&\pi(n_1, ..., n_J) \cdot \Big(\sum_{i\in\tilde{J}} \lambda_i + \sum_{i\in\tilde{J}} \sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j) + \sum_{i\in\tilde{J}} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i)\Big) \\
&= \sum_{i\in\tilde{J}} \pi(n_1, ..., n_i - 1, ..., n_J) \cdot \Big(\lambda_i + \sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j)\Big) \cdot 1_{\mathbb{N}_+}(n_i) \\
&+ \sum_{i\in\tilde{J}} \pi(n_1, ..., n_i + 1, ..., n_J) \cdot \mu_i r(i, 0) \\
&+ \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}\setminus\{i\}} \pi(n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) \mu_j r(j,i) 1_{\mathbb{N}_+}(n_i).
\end{aligned}$$

Summing over all $n_1 \in \mathbb{N}$ yields (recall $\tilde{J} = \{1\} \cup V$)

$$\begin{aligned}
&\sum_{n_1\in\mathbb{N}} \pi(n_1, ..., n_J) \cdot \Big(\lambda_1 + \sum_{i\in V} \lambda_i + \sum_{j\in V} \mu_j r(j,1) 1_{\{0\}}(n_j) + \sum_{i\in V} \sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j) + \\
&\qquad + \mu_1 (1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) + \sum_{i\in V} \mu_i 1_{\mathbb{N}_+}(n_i)\Big) \\
&= \sum_{n_1\in\mathbb{N}} \pi(n_1 - 1, n_2, ..., n_J) \cdot \Big(\lambda_1 + \sum_{j\in V} \mu_j r(j,1) 1_{\{0\}}(n_j)\Big) \cdot 1_{\mathbb{N}_+}(n_1) \\
&+ \sum_{n_1\in\mathbb{N}} \sum_{i\in V} \pi(n_1, ..., n_i - 1, ..., n_J) \cdot \Big(\lambda_i + \sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j)\Big) \cdot 1_{\mathbb{N}_+}(n_i) \\
&+ \sum_{n_1\in\mathbb{N}} \pi(n_1 + 1, n_2, ..., n_J) \cdot \mu_1 r(1, 0) \\
&+ \sum_{n_1\in\mathbb{N}} \sum_{i\in V} \pi(n_1, ..., n_i + 1, ..., n_J) \cdot \mu_i r(i, 0) \\
&+ \sum_{n_1\in\mathbb{N}} \sum_{j\in V} \pi(n_1 - 1, ..., n_j + 1, ..., n_J) \mu_j r(j,1) 1_{\mathbb{N}_+}(n_1) \\
&+ \sum_{n_1\in\mathbb{N}} \sum_{i\in V} \pi(n_1 + 1, ..., n_i - 1, ..., n_J) \mu_1 r(1,i) 1_{\mathbb{N}_+}(n_i) \\
&+ \sum_{n_1\in\mathbb{N}} \sum_{i\in V} \sum_{j\in V\setminus\{i\}} \pi(n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) \mu_j r(j,i) 1_{\mathbb{N}_+}(n_i)
\end{aligned}$$

$\Leftrightarrow$

$$\sum_{n_1\in\mathbb{N}} \pi(n_1,...,n_J)\cdot\Big(\lambda_1+\sum_{i\in V}\lambda_i+\sum_{j\in V}\mu_j r(j,1)1_{\{0\}}(n_j)+\sum_{i\in V}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+$$
$$+\mu_1(1-r(1,1))1_{\mathbb{N}_+}(n_1)+\sum_{i\in V}\mu_i 1_{\mathbb{N}_+}(n_i)\Big)$$
$$=\sum_{n_1\in\mathbb{N}} \pi(n_1,...,n_J)\cdot\Big(\lambda_1+\sum_{j\in V}\mu_j r(j,1)1_{\{0\}}(n_j)\Big)$$
$$+\sum_{n_1\in\mathbb{N}}\sum_{i\in V}\pi(n_1,...,n_i-1,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)$$
$$+\sum_{n_1\in\mathbb{N}}\pi(n_1,...,n_J)\cdot\mu_1 r(1,0)1_{\mathbb{N}_+}(n_1)$$
$$+\sum_{n_1\in\mathbb{N}}\sum_{i\in V}\pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i r(i,0)$$
$$+\sum_{n_1\in\mathbb{N}}\sum_{j\in V}\pi(n_1,...,n_j+1,...,n_J)\mu_j r(j,1)$$
$$+\sum_{n_1\in\mathbb{N}}\sum_{i\in V}\pi(n_1,...,n_i-1,...,n_J)\mu_1 r(1,i)1_{\mathbb{N}_+}(n_i)1_{\mathbb{N}_+}(n_1)$$
$$+\sum_{n_1\in\mathbb{N}}\sum_{i\in V}\sum_{j\in V\setminus\{i\}}\pi(n_1,...,n_i-1,...,n_j+1,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_i)$$

$\Leftrightarrow$

$$\sum_{n_1\in\mathbb{N}} \pi(n_1,...,n_J)\cdot\Big(\sum_{i\in V}\lambda_i+\sum_{i\in V}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+$$
$$+\mu_1\underbrace{(1-r(1,1)-r(1,0))}_{=\sum_{i\in V} r(1,i)}1_{\mathbb{N}_+}(n_1)+\sum_{i\in V}\mu_i 1_{\mathbb{N}_+}(n_i)\Big)$$
$$=\sum_{n_1\in\mathbb{N}}\sum_{i\in V}\pi(n_1,...,n_i-1,...,n_J)\cdot$$
$$\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+\mu_1 r(1,i)1_{\mathbb{N}_+}(n_1)\Big)\cdot 1_{\mathbb{N}_+}(n_i)$$
$$+\sum_{n_1\in\mathbb{N}}\sum_{i\in V}\pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i(r(i,0)+r(i,1))$$
$$+\sum_{n_1\in\mathbb{N}}\sum_{i\in V}\sum_{j\in V\setminus\{i\}}\pi(n_1,...,n_i-1,...,n_j+1,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_i). \tag{4.30}$$

Summing over all $(n_2,...,n_{J-1})\in\mathbb{N}^{J-2}$ yields for fixed $n_J\in\mathbb{N}$:

$$\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J)\cdot\Big(\lambda_J+\sum_{i\in V\setminus\{J\}}\lambda_i+\sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j)+\sum_{i\in V\setminus\{J\}}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)$$
$$+\mu_1\sum_{i\in V\setminus\{J\}} r(1,i)1_{\mathbb{N}_+}(n_1)+\mu_1 r(1,J)1_{\mathbb{N}_+}(n_1)+\mu_J 1_{\mathbb{N}_+}(n_J)+\sum_{i\in V\setminus\{J\}}\mu_i 1_{\mathbb{N}_+}(n_i)\Big)$$
$$=\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J-1)\cdot\Big(\lambda_J+\sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j)+\mu_1 r(1,J)1_{\mathbb{N}_+}(n_1)\Big)\cdot 1_{\mathbb{N}_+}(n_J)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_i-1,...,n_J)\cdot$$
$$\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+\mu_1 r(1,i)1_{\mathbb{N}_+}(n_1)\Big)\cdot 1_{\mathbb{N}_+}(n_i)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J+1)\cdot\mu_J(r(J,0)+r(J,1))$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i(r(i,0)+r(i,1))$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{j\in V\setminus\{J\}} \pi(n_1...,n_j+1,...,n_J-1)\mu_j r(j,J)1_{\mathbb{N}_+}(n_J)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{j\in V\setminus\{J\}} \pi(n_1,...,n_j-1,...,n_J+1)\mu_J r(J,j)1_{\mathbb{N}_+}(n_j)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}}\sum_{j\in V\setminus\{J,i\}} \pi(n_1,...,n_i-1,...,n_j+1,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_i)$$

$$\Leftrightarrow\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J)\cdot\Big[\lambda_J+\sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j)+\sum_{i\in V\setminus\{J\}}\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)\Big)$$
$$+\mu_1\sum_{i\in V\setminus\{J\}} r(1,i)1_{\mathbb{N}_+}(n_1)+\mu_1 r(1,J)1_{\mathbb{N}_+}(n_1)+\mu_J 1_{\mathbb{N}_+}(n_J)+\sum_{i\in V\setminus\{J\}}\mu_i 1_{\mathbb{N}_+}(n_i)\Big]$$
$$=\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J-1)\cdot\Big(\lambda_J+\sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j)+\mu_1 r(1,J)1_{\mathbb{N}_+}(n_1)\Big)\cdot 1_{\mathbb{N}_+}(n_J)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+\mu_1 r(1,i)1_{\mathbb{N}_+}(n_1)\Big)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J+1)\cdot\mu_J(r(J,0)+r(J,1))$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_J)\cdot\mu_i(r(i,0)+r(i,1))1_{\mathbb{N}_+}(n_i)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{j\in V\setminus\{J\}} \pi(n_1,...,n_J-1)\mu_j r(j,J)1_{\mathbb{N}_+}(n_J)1_{\mathbb{N}_+}(n_j)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{j\in V\setminus\{J\}} \pi(n_1,...,n_J+1)\mu_J r(J,j)$$
$$+\sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}}\sum_{j\in V\setminus\{J,i\}} \pi(n_1,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_j)$$

$$\Leftrightarrow \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J)\cdot$$
$$\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j) + \mu_1 r(1,J)1_{\mathbb{N}_+}(n_1) + \mu_J 1_{\mathbb{N}_+}(n_J) + \sum_{i\in V\setminus\{J\}}\mu_i 1_{\mathbb{N}_+}(n_i)$$
$$\underbrace{-\sum_{i\in V\setminus\{J\}}\mu_i(r(i,0)+r(i,1))1_{\mathbb{N}_+}(n_i) - \sum_{i\in V\setminus\{J\}}\sum_{j\in V\setminus\{J,i\}}\mu_j r(j,i)1_{\mathbb{N}_+}(n_j)}_{=-\sum_{i\in V\setminus\{J\}}\mu_i(1-r(i,J))1_{\mathbb{N}_+}(n_i),\text{ since } r(i,i)=0 \text{ for } i\in V.}\Big)$$
$$= \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J-1)\cdot$$
$$\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J) + \mu_1 r(1,J)1_{\mathbb{N}_+}(n_1)\Big)\cdot 1_{\mathbb{N}_+}(n_J)$$
$$+ \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J+1)\cdot\mu_J\underbrace{\Big(r(J,0)+r(J,1)+\sum_{j\in V\setminus\{J\}}r(J,j)\Big)}_{=1,\text{ since } r(J,J)=0.}$$

$$\Leftrightarrow \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J)\cdot\Big(\lambda_J + \underbrace{\sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j) + \sum_{i\in V\setminus\{J\}}\mu_i r(i,J)1_{\mathbb{N}_+}(n_i)}_{=\sum_{j\in V}\mu_j r(j,J),\text{ since } r(J,J)=0.} +$$
$$+\mu_1 r(1,J)1_{\mathbb{N}_+}(n_1) + \mu_J 1_{\mathbb{N}_+}(n_J)\Big)$$
$$= \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J-1)\cdot$$
$$\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J) + \mu_1 r(1,J)1_{\mathbb{N}_+}(n_1)\Big)\cdot 1_{\mathbb{N}_+}(n_J)$$
$$+ \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J+1)\cdot\mu_J$$

$$\Leftrightarrow \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J)\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J) + \mu_1 r(1,J) + \mu_J 1_{\mathbb{N}_+}(n_J)\Big)$$
$$- \sum_{(n_2,...,n_{J-1})\in\mathbb{N}^{J-2}} \pi(0,n_2,...,n_J)\mu_1 r(1,J)$$
$$= \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J-1)\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J) + \mu_1 r(1,J)\Big)\cdot 1_{\mathbb{N}_+}(n_J)$$
$$- \sum_{(n_2,...,n_{J-1})\in\mathbb{N}^{J-2}} \pi(0,n_2,...,n_J-1)\mu_1 r(1,J)1_{\mathbb{N}_+}(n_J)$$
$$+ \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J+1)\cdot\mu_J.$$

With $\pi_{\{1,J\}}(0,n_J) := \sum_{(n_2,...,n_{J-1})\in\mathbb{N}^{J-2}}\pi(0,n_2,...,n_J)$ and

$\pi_J(n_J) := \sum_{(n_1,...,n_{J-1})\in\mathbb{N}^{J-1}} \pi(n_1, ..., n_J)$ this is equivalent to

$$\begin{aligned}&\pi_J(n_J)\cdot\Big(\underbrace{\lambda_J+\sum_{j\in V}\mu_j r(j,J)}_{\overset{(4.26)}{=}\eta_J-\eta_1 r(1,J)}+\mu_1 r(1,J)+\mu_J 1_{\mathbb{N}_+}(n_J)\Big)-\pi_{\{1,J\}}(0,n_J)\mu_1 r(1,J)\\&=\pi_J(n_J-1)\cdot\Big(\lambda_J+\sum_{j\in V}\mu_j r(j,J)+\mu_1 r(1,J)\Big)\cdot 1_{\mathbb{N}_+}(n_J)+\pi_J(n_J+1)\cdot\mu_J\\&-\pi_{\{1,J\}}(0,n_J-1)\mu_1 r(1,J)1_{\mathbb{N}_+}(n_J).\end{aligned}$$

(4.29) implies

$$\begin{aligned}&\pi_J(n_J)\cdot\Big(\eta_J-\eta_1 r(1,J)+\mu_1 r(1,J)+\mu_J 1_{\mathbb{N}_+}(n_J)\\&-\frac{\mu_J}{\eta_J}\Big(\eta_J-\eta_1 r(1,J)+\mu_1 r(1,J)\Big)1_{\mathbb{N}_+}(n_J)-\frac{\eta_J}{\mu_J}\mu_J\Big)\\&=\pi_{\{1,J\}}(0,n_J)\mu_1 r(1,J)-\pi_{\{1,J\}}(0,n_J-1)\mu_1 r(1,J)1_{\mathbb{N}_+}(n_J)\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow\pi_J(n_J)\cdot\Big((\mu_1-\eta_1)r(1,J)-\frac{\mu_J}{\eta_J}(\mu_1-\eta_1)r(1,J)1_{\mathbb{N}_+}(n_J)\Big)\\&\quad=(\pi_{\{1,J\}}(0,n_J)-\pi_{\{1,J\}}(0,n_J-1)1_{\mathbb{N}_+}(n_J))\mu_1 r(1,J)\end{aligned}$$

$$\begin{aligned}&\Leftrightarrow\pi_J(n_J)\cdot\Big(\underbrace{1-\frac{\eta_1}{\mu_1}}_{\overset{(4.27)}{=}\pi_1(0)}\Big)\Big(1-\frac{\mu_J}{\eta_J}1_{\mathbb{N}_+}(n_J)\Big)\mu_1 r(1,J)\\&\quad=(\pi_{\{1,J\}}(0,n_J)-\pi_{\{1,J\}}(0,n_J-1)1_{\mathbb{N}_+}(n_J))\mu_1 r(1,J).\end{aligned}$$

Since $r(1,J)>0$ is assumed, it remains

$$\pi_J(n_J)\cdot\pi_1(0)\Big(1-\frac{\mu_J}{\eta_J}1_{\mathbb{N}_+}(n_J)\Big)=\pi_{\{1,J\}}(0,n_J)-\pi_{\{1,J\}}(0,n_J-1)1_{\mathbb{N}_+}(n_J). \tag{4.31}$$

The solution of the recursion (4.31) is

$$\pi_{\{1,J\}}(0,n_J)=\pi_1(0)\cdot\pi_J(n_J)\ \forall n_J\in\mathbb{N}, \tag{4.32}$$

as proved by induction over n_J: The basis ($n_J=0$) is trivial.
Induction hypothesis (IH): (4.32) is true for some $(n_J-1)\in\mathbb{N}_+$. For the inductive step we have to show that (4.32) also holds for n_J.

$$\begin{aligned}(4.31)\Leftrightarrow\ \pi_{\{1,J\}}(0,n_J)&=\pi_J(n_J)\cdot\pi_1(0)\Big(1-\frac{\mu_J}{\eta_J}1_{\mathbb{N}_+}(n_J)\Big)+\underbrace{\pi_{\{1,J\}}(0,n_J-1)}_{\overset{(\mathrm{IH})}{=}\pi_1(0)\cdot\pi_J(n_J-1)}1_{\mathbb{N}_+}(n_J)\\&=\pi_J(n_J)\cdot\pi_1(0)\Big(1-\frac{\mu_J}{\eta_J}1_{\mathbb{N}_+}(n_J)\Big)+\pi_1(0)\cdot\underbrace{\pi_J(n_J-1)}_{\overset{(4.29)}{=}\frac{\mu_J}{\eta_J}\pi(n_J)}1_{\mathbb{N}_+}(n_J)\\&=\pi_J(n_J)\cdot\pi_1(0).\end{aligned}$$

For deriving (4.32), $J\in V$ was selected for easy notation. The proof for arbitrary $i\in V$ is similar. Hence (4.28) follows. □

The following example shows that independence of events similar to Proposition 4.23 is not possible in general, if $|W| > 1$, i.e., there are several nodes without infinite supply.

Example 4.24. *Consider a Jackson network of three nodes, where* $W = \{1, 2\}$ *and node 3 has infinite supply but no immediate feedback. Let* $r(3, 0) < 1$, *so we are not in the framework of Corollary 4.20. Then the traffic equations are*

$$\eta_i = \lambda_i + \sum_{j=1}^{2} \eta_j r(j,i) + \mu_3 r(3,i), \quad i = 1, 2, \tag{4.33}$$

$$\eta_3 = \lambda_3 + \sum_{j=1}^{2} \eta_j r(j,i). \tag{4.34}$$

Let $\eta_i < \mu_i$ *hold for* $i = 1, 2, 3$. *For the queue length process* $X = ((X_1(t), X_2(t), X_3(t)) : t \in \mathbb{R}_+)$ *we know from Theorem 4.15 that for all* $n_1, n_2 \in \mathbb{N}$ *holds*

$$\pi_W(n_1, n_2) = \sum_{n_3 \in \mathbb{N}} \pi(n_1, n_2, n_3) = \prod_{i=1}^{2} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}. \tag{4.35}$$

We start the system with an initial distribution which has the marginal (4.35), *so for all* $n_3 \in \mathbb{N}$ *holds*

$$\pi_3(n_3) = \sum_{(n_1,n_2) \in \mathbb{N}^2} \pi(n_1, n_2, n_3) = \left(1 - \frac{\eta_3}{\mu_3}\right) \left(\frac{\eta_3}{\mu_3}\right)^{n_3}. \tag{4.36}$$

The global balance equations are $\forall (n_1, n_2, n_3) \in \mathbb{N}^3$:

$$\begin{aligned}
&\pi(n_1, n_2, n_3) \cdot \Big(\sum_{i=1}^{3} \lambda_i + \sum_{i=1}^{2} \mu_3 r(3,i) 1_{\{0\}}(n_3) + \sum_{i=1}^{2} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i) + \mu_3 1_{\mathbb{N}_+}(n_3) \Big) \\
&= \sum_{i=1}^{2} \pi(n_k : k \in W \setminus \{i\}, n_i - 1, n_3) \cdot (\lambda_i + \mu_3 r(3,i) 1_{\{0\}}(n_3)) 1_{\mathbb{N}_+}(n_i) + \\
&+ \pi(n_1, n_2, n_3 - 1) \cdot \lambda_3 1_{\mathbb{N}_+}(n_3) + \\
&+ \sum_{i=1}^{3} \pi(n_k : k \in \{1,2,3\} \setminus \{i\}, n_i + 1) \cdot \mu_i r(i, 0) + \\
&+ \pi(n_1 + 1, n_2 - 1, n_3) \cdot \mu_1 r(1,2) 1_{\mathbb{N}_+}(n_2) + \pi(n_1 - 1, n_2 + 1, n_3) \cdot \mu_2 r(2,1) 1_{\mathbb{N}_+}(n_1) + \\
&+ \pi(n_1 + 1, n_2, n_3 - 1) \cdot \mu_1 r(1,3) 1_{\mathbb{N}_+}(n_3) + \pi(n_1 - 1, n_2, n_3 + 1) \cdot \mu_3 r(3,1) 1_{\mathbb{N}_+}(n_1) + \\
&+ \pi(n_1, n_2 + 1, n_3 - 1) \cdot \mu_2 r(2,3) 1_{\mathbb{N}_+}(n_3) + \pi(n_1, n_2 - 1, n_3 + 1) \cdot \mu_3 r(3,2) 1_{\mathbb{N}_+}(n_2).
\end{aligned}$$

Summing over all $(n_1,n_2)\in\mathbb{N}^2$ on both sides and shifting indices yields for fixed $n_3\in\mathbb{N}$

$$\begin{aligned}
&\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3)\cdot\\
&\cdot\Big(\sum_{i=1}^{3}\lambda_i+\sum_{i=1}^{2}\mu_3 r(3,i)1_{\{0\}}(n_3)+\sum_{i=1}^{2}\mu_i(1-r(i,i))1_{\mathbb{N}_+}(n_i)+\mu_3 1_{\mathbb{N}_+}(n_3)\Big)\\
&=\sum_{(n_1,n_2)\in\mathbb{N}^2}\sum_{i=1}^{2}\pi(n_1,n_2,n_3)\cdot(\lambda_i+\mu_3 r(3,i)1_{\{0\}}(n_3))+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3-1)\cdot\lambda_3 1_{\mathbb{N}_+}(n_3)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\sum_{i=1}^{2}\pi(n_1,n_2,n_3)\cdot\mu_i r(i,0)1_{\mathbb{N}_+}(n_i)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3+1)\cdot\mu_3 r(3,0)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3)\cdot\mu_1 r(1,2)1_{\mathbb{N}_+}(n_1)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3)\cdot\mu_2 r(2,1)1_{\mathbb{N}_+}(n_2)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3-1)\cdot\mu_1 r(1,3)1_{\mathbb{N}_+}(n_3)1_{\mathbb{N}_+}(n_1)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3+1)\cdot\mu_3 r(3,1)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3-1)\cdot\mu_2 r(2,3)1_{\mathbb{N}_+}(n_3)1_{\mathbb{N}_+}(n_2)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3+1)\cdot\mu_3 r(3,2)
\end{aligned}$$

$\Leftrightarrow$

$$\begin{aligned}
&\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3)\cdot\Big(\lambda_3+\sum_{i=1}^{2}\mu_i r(i,3)1_{\mathbb{N}_+}(n_i)+\mu_3 1_{\mathbb{N}_+}(n_3)\Big)\\
&=\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3-1)\cdot\Big(\lambda_3+\sum_{i=1}^{2}\mu_i r(i,3)1_{\mathbb{N}_+}(n_i)\Big)1_{\mathbb{N}_+}(n_3)+\\
&+\sum_{(n_1,n_2)\in\mathbb{N}^2}\pi(n_1,n_2,n_3+1)\cdot\mu_3
\end{aligned}$$

$$\Leftrightarrow \underbrace{\sum_{(n_1,n_2)\in\mathbb{N}^2} \pi(n_1,n_2,n_3)}_{=\pi_3(n_3)} \cdot \Big(\lambda_3 + \sum_{i=1}^{2} \mu_i r(i,3) + \mu_3 1_{\mathbb{N}_+}(n_3)\Big)$$
$$- \sum_{n_2\in\mathbb{N}} \pi(0,n_2,n_3)\mu_1 r(1,3) - \sum_{n_1\in\mathbb{N}} \pi(n_1,0,n_3)\mu_2 r(2,3)$$
$$= \underbrace{\sum_{(n_1,n_2)\in\mathbb{N}^2} \pi(n_1,n_2,n_3-1)}_{=\pi_3(n_3-1)} \cdot \Big(\lambda_3 + \sum_{i=1}^{2} \mu_i r(i,3)\Big) 1_{\mathbb{N}_+}(n_3) +$$
$$- \sum_{n_2\in\mathbb{N}} \pi(0,n_2,n_3-1)\mu_1 r(1,3) 1_{\mathbb{N}_+}(n_3) - \sum_{n_1\in\mathbb{N}} \pi(n_1,0,n_3-1)\mu_2 r(2,3) 1_{\mathbb{N}_+}(n_3)$$
$$+ \underbrace{\sum_{(n_1,n_2)\in\mathbb{N}^2} \pi(n_1,n_2,n_3+1)}_{=\pi_3(n_3+1)} \cdot \mu_3.$$

With $\pi_{\{1,3\}}(0,n_3) := \sum_{n_2\in\mathbb{N}} \pi(0,n_2,n_3)$ *and* $\pi_{\{2,3\}}(0,n_3) := \sum_{n_1\in\mathbb{N}} \pi(n_1,0,n_3)$ *this is equivalent to*

$$\pi_3(n_3) \cdot \Big(\lambda_3 + \sum_{i=1}^{2} \mu_i r(i,3) + \mu_3 1_{\mathbb{N}_+}(n_3)\Big) - \pi_{\{1,3\}}(0,n_3)\mu_1 r(1,3) - \pi_{\{2,3\}}(0,n_3)\mu_2 r(2,3)$$
$$= \pi_3(n_3-1) \cdot \Big(\lambda_3 + \sum_{i=1}^{2} \mu_i r(i,3)\Big) 1_{\mathbb{N}_+}(n_3) + \pi_3(n_3+1) \cdot \mu_3$$
$$- \pi_{\{1,3\}}(0,n_3-1)\mu_1 r(1,3) 1_{\mathbb{N}_+}(n_3) - \pi_{\{2,3\}}(0,n_3-1)\mu_2 r(2,3) 1_{\mathbb{N}_+}(n_3).$$

Plugging in (4.36) *yields*

$$\pi_3(n_3) \cdot \Big(\underbrace{\lambda_3}_{=\eta_3 - \sum_{i=1}^{2} \eta_i r(i,3)} + \sum_{i=1}^{2} \mu_i r(i,3) + \mu_3 1_{\mathbb{N}_+}(n_3)$$
$$- \frac{\eta_3}{\mu_3}\mu_3 - \frac{\mu_3}{\eta_3} \cdot \Big(\lambda_3 + \sum_{i=1}^{2} \mu_i r(i,3)\Big) 1_{\mathbb{N}_+}(n_3)\Big)$$
$$= \mu_1 r(1,3)(\pi_{\{1,3\}}(0,n_3) - \pi_{\{1,3\}}(0,n_3-1)1_{\mathbb{N}_+}(n_3)) +$$
$$+ \mu_2 r(2,3)(\pi_{\{2,3\}}(0,n_3) - \pi_{\{2,3\}}(0,n_3-1)1_{\mathbb{N}_+}(n_3))$$

$$\Leftrightarrow \pi_3(n_3) \cdot \Big(\sum_{i=1}^{2} (\mu_i - \eta_i) r(i,3) - \frac{\mu_3}{\eta_3} \sum_{i=1}^{2} (\mu_i - \eta_i) r(i,3) 1_{\mathbb{N}_+}(n_3)\Big)$$
$$= \mu_1 r(1,3)(\pi_{\{1,3\}}(0,n_3) - \pi_{\{1,3\}}(0,n_3-1)1_{\mathbb{N}_+}(n_3)) +$$
$$+ \mu_2 r(2,3)(\pi_{\{2,3\}}(0,n_3) - \pi_{\{2,3\}}(0,n_3-1)1_{\mathbb{N}_+}(n_3))$$

$$\Leftrightarrow \pi_3(n_3)\cdot\Big(1-\frac{\mu_3}{\eta_3}1_{\mathbb{N}_+}(n_3)\Big)\sum_{i=1}^{2}\underbrace{\Big(1-\frac{\eta_i}{\mu_i}\Big)}_{=\pi_i(0)}\mu_i r(i,3)$$
$$=\mu_1 r(1,3)(\pi_{\{1,3\}}(0,n_3)-\pi_{\{1,3\}}(0,n_3-1)1_{\mathbb{N}_+}(n_3))+$$
$$+\mu_2 r(2,3)(\pi_{\{2,3\}}(0,n_3)-\pi_{\{2,3\}}(0,n_3-1)1_{\mathbb{N}_+}(n_3))$$

$$\Leftrightarrow \pi_3(n_3)\cdot\Big(1-\frac{\mu_3}{\eta_3}1_{\mathbb{N}_+}(n_3)\Big)(\pi_1(0)\mu_1 r(1,3)+\pi_2(0)\mu_2 r(2,3))$$
$$=\mu_1 r(1,3)(\pi_{\{1,3\}}(0,n_3)-\pi_{\{1,3\}}(0,n_3-1)1_{\mathbb{N}_+}(n_3))+$$
$$+\mu_2 r(2,3)(\pi_{\{2,3\}}(0,n_3)-\pi_{\{2,3\}}(0,n_3-1)1_{\mathbb{N}_+}(n_3)). \tag{4.37}$$

Consider the case $n_3 = 0$, then (4.37) reduces to

$$\pi_3(0)\cdot(\pi_1(0)\mu_1 r(1,3)+\pi_2(0)\mu_2 r(2,3)) = \mu_1 r(1,3)\pi_{\{1,3\}}(0,0)+\mu_2 r(2,3)\pi_{\{2,3\}}(0,0),$$

and this equation is valid if $\pi_{\{i,3\}}(0,0) = \pi_i(0)\cdot\pi_3(0)$ for $i = 1,2$.
If $r(i,3) = 0$ holds for one node $i \in \{1,2\}$, node 3 with infinite supply is only directly accessible from W over node $3-i$ and it follows analogously as in Example 4.22 that

$$\pi_{\{3-i,3\}}(0,n) = \pi_{3-i}(0)\cdot\pi_3(n)$$

holds for all $n \in \mathbb{N}$.

This example motivates:

Proposition 4.25. *Consider a Jackson network where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work as in Definition 4.1. Nodes in $W := \tilde{J}\setminus V$ operate without infinite supply. Let the subset V be directly accessible from the subset W only via one node, say node $1 \in W$, so $\sum_{j\in V} r(1,j) > 0$ and $r(i,j) = 0$ for all $i \in W\setminus\{1\}$ and $j \in V$. We assume that there is no immediate feedback at nodes with infinite supply, i.e., $r(i,i) = 0$ for all $i \in V$, and that all nodes are immediately accessible from nodes in $i \in V$, i.e., $r(i,j) > 0$ for all $j \in \tilde{J}$. Let $\eta_i < \mu_i$ hold for all nodes $i \in \tilde{J}$ where $\eta = (\eta_1, ..., \eta_J)$ is the unique solution of the traffic equations (4.1):*

$$\eta_i = \lambda_i + \sum_{j\in W}\eta_j r(j,i) + \sum_{j\in V}\mu_j r(j,i),\ \ i \in \tilde{J}. \tag{4.38}$$

Denote by $X = ((X_1(t), ..., X_J(t)) : t \geq 0)$ the queue-length process on $\mathbb{N}^J$ and by $\pi(n_1, ..., n_J)$ the stationary distribution of X. If the system is started with an initial distribution which has the marginal

$$P(X_i(0) = n_i : i \in W) := \pi_W(n_i : i \in W) = \prod_{i\in W}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}\ \ \forall (n_i : i \in W) \in \mathbb{N}^{|W|}, \tag{4.39}$$

then for the joint marginal $\pi_{\{1,i\}}(0,n) := \sum_{(n_k : k\in\tilde{J}\setminus\{1,i\})\in\mathbb{N}^{J-2}}\pi(n_1,...,n_J)$ of the stationary distribution of X holds

$$\pi_{\{1,i\}}(0,n_i) = \pi_1(0)\cdot\pi_i(n_i)\ \forall i \in V\ \forall n_i \in \mathbb{N}, \tag{4.40}$$

where $\pi_i(n_i)$ is the one-dimensional marginal queue length distribution for node $i \in V$

$$\pi_i(n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i} \quad \forall n_i \in \mathbb{N}. \tag{4.41}$$

Proof. From Theorem 4.15 we know that if we start the system with an initial distribution which has the marginal (4.39), the marginal for any node $i \in V$ is (4.41). The global balance equations of X are for all $(n_1, ..., n_J) \in \mathbb{N}^J$:

$$\begin{aligned}
&\pi(n_1, ..., n_J) \cdot \Big(\sum_{i\in\tilde{J}} \lambda_i + \sum_{i\in\tilde{J}}\sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j) + \sum_{i\in\tilde{J}} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i)\Big) \\
&= \sum_{i\in\tilde{J}} \pi(n_1, ..., n_i - 1, ..., n_J) \cdot \Big(\lambda_i + \sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j)\Big) \cdot 1_{\mathbb{N}_+}(n_i) \\
&\quad + \sum_{i\in\tilde{J}} \pi(n_1, ..., n_i + 1, ..., n_J) \cdot \mu_i r(i, 0) \\
&\quad + \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}\setminus\{i\}} \pi(n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) \mu_j r(j,i) 1_{\mathbb{N}_+}(n_i).
\end{aligned}$$

Summing over all $(n_i : i \in W) \in \mathbb{N}^{|W|}$ yields

$$\begin{aligned}
&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} \pi(n_1, ..., n_J) \cdot \Big(\sum_{i\in\tilde{J}} \lambda_i + \sum_{i\in\tilde{J}}\sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j) + \sum_{i\in\tilde{J}} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i)\Big) \\
&= \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} \sum_{i\in\tilde{J}} \pi(n_1, ..., n_i - 1, ..., n_J) \cdot \Big(\lambda_i + \sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j)\Big) \cdot 1_{\mathbb{N}_+}(n_i) \\
&+ \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} \sum_{i\in\tilde{J}} \pi(n_1, ..., n_i + 1, ..., n_J) \cdot \mu_i r(i, 0) \\
&+ \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}\setminus\{i\}} \pi(n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) \mu_j r(j,i) 1_{\mathbb{N}_+}(n_i)
\end{aligned}$$

$\Leftrightarrow$

$$
\begin{aligned}
&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} \pi(n_1,...,n_J)\cdot\Big(\sum_{i\in W}\lambda_i+\sum_{i\in V}\lambda_i+\sum_{i\in W}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+\\
&\quad+\sum_{i\in V}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+\sum_{i\in W}\mu_i(1-r(i,i))1_{\mathbb{N}_+}(n_i)+\sum_{i\in V}\mu_i 1_{\mathbb{N}_+}(n_i)\Big)\\
=&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in W}\pi(n_1,...,n_i,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)\Big)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in V}\pi(n_1,...,n_i-1,...,n_J)\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in W}\pi(n_1,...,n_i,...,n_J)\cdot\mu_i r(i,0)1_{\mathbb{N}_+}(n_i)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in V}\pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i r(i,0)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in W}\sum_{j\in W\setminus\{i\}}\pi(n_1,...,n_i,...,n_j,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_j)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in W}\sum_{j\in V}\pi(n_1,...,n_i,...,n_j+1,...,n_J)\mu_j r(j,i)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in V}\sum_{j\in W}\pi(n_1,...,n_i-1,...,n_j,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_i)1_{\mathbb{N}_+}(n_j)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in V}\sum_{j\in V\setminus\{i\}}\pi(n_1,...,n_i-1,...,n_j+1,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_i)
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} \pi(n_1,...,n_J)\cdot\Big(\sum_{i\in V}\lambda_i+\sum_{i\in V}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+\\
&\quad+\sum_{i\in W}\mu_i\sum_{j\in V}r(i,j)1_{\mathbb{N}_+}(n_i)+\sum_{i\in V}\mu_i 1_{\mathbb{N}_+}(n_i)\Big)\\
=&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in V}\pi(n_1,...,n_i-1,...,n_J)\cdot\\
&\quad\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+\sum_{j\in W}\mu_j r(j,i)1_{\mathbb{N}_+}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in V}\pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i\Big(1-\sum_{j\in V}r(i,j)\Big)\\
+&\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\sum_{i\in V}\sum_{j\in V\setminus\{i\}}\pi(n_1,...,n_i-1,...,n_j+1,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_i).
\end{aligned}
$$

Now we select one node in V, fix its state, and sum over all possible states of the remaining nodes in V. Let for readability node J have infinite supply, i.e., $J \in V$. Summing over all $(n_j : j \in V \setminus \{J\}) \in \mathbb{N}^{|V|-1}$ yields for fixed $n_J \in \mathbb{N}$:

$$\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J)\cdot$$
$$\cdot\Big(\lambda_J+\sum_{i\in V\setminus\{J\}}\lambda_i+\sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j)+\sum_{i\in V\setminus\{J\}}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+$$
$$+\sum_{i\in W}\mu_i r(i,J)1_{\mathbb{N}_+}(n_i)+\sum_{i\in W}\mu_i\sum_{j\in V\setminus\{J\}} r(i,j)1_{\mathbb{N}_+}(n_i)+\mu_J 1_{\mathbb{N}_+}(n_J)+\sum_{i\in V\setminus\{J\}}\mu_i 1_{\mathbb{N}_+}(n_i)\Big)$$
$$=\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J-1)\cdot$$
$$\cdot\Big(\lambda_J+\sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j)+\sum_{j\in W}\mu_j r(j,J)1_{\mathbb{N}_+}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_J)$$
$$+\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_i-1,...,n_J)\cdot$$
$$\cdot\Big(\lambda_i+\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+\sum_{j\in W}\mu_j r(j,i)1_{\mathbb{N}_+}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)$$
$$+\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J+1)\cdot\mu_J\Big(1-\sum_{j\in V}r(J,j)\Big)$$
$$+\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_i+1,...,n_J)\cdot\mu_i\Big(1-\sum_{j\in V}r(i,j)\Big)$$
$$+\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{j_1}}\sum_{j\in V\setminus\{J\}} \pi(n_1,...,n_j+1,...,n_J-1)\mu_j r(j,J)1_{\mathbb{N}_+}(n_J)$$
$$+\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{j_1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_i-1,...,n_J+1)\mu_J r(J,i)1_{\mathbb{N}_+}(n_i)$$
$$+\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{j_1}}\sum_{i\in V\setminus\{J\}}\sum_{j\in V\setminus\{i,J\}} \pi(n_1,...,n_i-1,...,n_j+1,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_i)$$

$$\Leftrightarrow \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J)\cdot$$
$$\cdot\Big(\lambda_J + \sum_{i\in V\setminus\{J\}}\lambda_i + \sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j) + \sum_{i\in V\setminus\{J\}}\sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j)+$$
$$+\sum_{i\in W}\mu_i r(i,J)1_{\mathbb{N}_+}(n_i) + \sum_{i\in W}\mu_i\sum_{j\in V\setminus\{J\}} r(i,j)1_{\mathbb{N}_+}(n_i) + \mu_J 1_{\mathbb{N}_+}(n_J) + \sum_{i\in V\setminus\{J\}}\mu_i 1_{\mathbb{N}_+}(n_i)\Big)$$
$$= \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J-1)\cdot$$
$$\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J)1_{\{0\}}(n_j) + \sum_{j\in W}\mu_j r(j,J)1_{\mathbb{N}_+}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_J)$$
$$+ \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_J)\cdot$$
$$\cdot\Big(\lambda_i + \sum_{j\in V}\mu_j r(j,i)1_{\{0\}}(n_j) + \sum_{j\in W}\mu_j r(j,i)1_{\mathbb{N}_+}(n_j)\Big)$$
$$+ \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J+1)\cdot\mu_J\Big(1-\sum_{j\in V} r(J,j)\Big)$$
$$+ \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_J)\cdot\mu_i\Big(1-\sum_{j\in V} r(i,j)\Big)\cdot 1_{\mathbb{N}_+}(n_i)$$
$$+ \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}}\sum_{j\in V\setminus\{J\}} \pi(n_1,...,n_J-1)\mu_j r(j,J)1_{\mathbb{N}_+}(n_J)1_{\mathbb{N}_+}(n_j)$$
$$+ \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}} \pi(n_1,...,n_J+1)\mu_J r(J,i)$$
$$+ \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}}\sum_{i\in V\setminus\{J\}}\sum_{j\in V\setminus\{i,J\}} \pi(n_1,...,n_J)\mu_j r(j,i)1_{\mathbb{N}_+}(n_j)$$

$\Leftrightarrow$

$$\sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_J)\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J) + \sum_{i\in W}\mu_i r(i,J)1_{\mathbb{N}_+}(n_i) + \mu_J 1_{\mathbb{N}_+}(n_J)\Big)$$
$$= \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J-1)\cdot$$
$$\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J) + \sum_{j\in W}\mu_j r(j,J)1_{\mathbb{N}_+}(n_j)\Big)\cdot 1_{\mathbb{N}_+}(n_J)$$
$$+ \sum_{(n_i:i\in\tilde{J}\setminus\{J\})\in\mathbb{N}^{J-1}} \pi(n_1,...,n_{J-1},n_J+1)\cdot\mu_J.$$

With $\pi_{\{i,J\}}(0,n_J) := \sum_{(n_i:i\in\tilde{J}\setminus\{J,i\})\in\mathbb{N}^{J-2}}\pi(n_i : i\in\tilde{J}\setminus\{i\},0)$ and

$\pi_j(n_j) := \sum_{(n_i: i\in\tilde{J}\setminus\{j\})\in\mathbb{N}^{J-1}} \pi(n_1, ..., n_J)$ this is equivalent to

$$\pi_J(n_J)\cdot\Big(\underbrace{\lambda_J + \sum_{j\in V}\mu_j r(j,J)}_{\overset{(4.38)}{=}\eta_J - \sum_{j\in W}\eta_j r(j,J)} + \sum_{i\in W}\mu_i r(i,J) + \mu_J 1_{\mathbb{N}_+}(n_J)\Big) - \sum_{i\in W}\pi_{\{i,J\}}(0,n_J)\mu_i r(i,J)$$
$$= \pi_J(n_J-1)\cdot\Big(\lambda_J + \sum_{j\in V}\mu_j r(j,J) + \sum_{j\in W}\mu_j r(j,J)\Big)\cdot 1_{\mathbb{N}_+}(n_J) + \pi_J(n_J+1)\cdot\mu_J$$
$$- \sum_{j\in W}\pi_{\{i,J\}}(0,n_J-1)\mu_j r(j,J)\cdot 1_{\mathbb{N}_+}(n_J).$$

(4.41) implies

$$\pi_J(n_J)\cdot\Big(\eta_J - \sum_{j\in W}\eta_j r(j,J) + \sum_{i\in W}\mu_i r(i,J) + \mu_J 1_{\mathbb{N}_+}(n_J)$$
$$- \frac{\mu_J}{\eta_J}\cdot\Big[\eta_J - \sum_{j\in W}\eta_j r(j,J) + \sum_{j\in W}\mu_j r(j,J)\Big]\cdot 1_{\mathbb{N}_+}(n_J) - \frac{\eta_J}{\mu_J}\mu_J\Big)$$
$$= \sum_{i\in W}\pi_{\{i,J\}}(0,n_J)\mu_i r(i,J) - \sum_{j\in W}\pi_{\{i,J\}}(0,n_J-1)\mu_j r(j,J)\cdot 1_{\mathbb{N}_+}(n_J)$$

$\Leftrightarrow$

$$\pi_J(n_J)\cdot\Big(-\sum_{j\in W}\eta_j r(j,J) + \sum_{j\in W}\mu_j r(j,J)$$
$$- \frac{\mu_J}{\eta_J}\cdot\Big[-\sum_{j\in W}\eta_j r(j,J) + \sum_{j\in W}\mu_j r(j,J)\Big]\cdot 1_{\mathbb{N}_+}(n_J)\Big)$$
$$= \sum_{i\in W}\pi_{\{i,J\}}(0,n_J)\mu_i r(i,J) - \sum_{j\in W}\pi_{\{i,J\}}(0,n_J-1)\mu_j r(j,J)\cdot 1_{\mathbb{N}_+}(n_J)$$

$\Leftrightarrow$

$$\pi_J(n_J)\cdot\Big(1 - \frac{\mu_J}{\eta_J}\cdot 1_{\mathbb{N}_+}(n_J)\Big)\sum_{j\in W}(\mu_j - \eta_j) r(j,J)$$
$$= \sum_{i\in W}(\pi_{\{i,J\}}(0,n_J) - \pi_{\{i,J\}}(0,n_J-1)\cdot 1_{\mathbb{N}_+}(n_J))\mu_j r(j,J).$$

With the assumptions $r(1,J) > 0$ and $\sum_{j\in W\setminus\{1\}} r(j,J) = 0$, it follows directly

$$\pi_J(n_J)\cdot\Big(1 - \frac{\mu_J}{\eta_J}\cdot 1_{\mathbb{N}_+}(n_J)\Big)(\mu_1 - \eta_1) r(1,J)$$
$$= (\pi_{\{1,J\}}(0,n_J) - \pi_{\{1,J\}}(0,n_J-1)\cdot 1_{\mathbb{N}_+}(n_J))\mu_1 r(1,J)$$

$$\Leftrightarrow \pi_J(n_J)\cdot\Big(1 - \frac{\mu_J}{\eta_J}\cdot 1_{\mathbb{N}_+}(n_J)\Big)\underbrace{\Big(1 - \frac{\eta_1}{\mu_1}\Big)}_{\overset{(4.39)}{=}\pi_1(0)} = \pi_{\{1,J\}}(0,n_J) - \pi_{\{1,J\}}(0,n_J-1)\cdot 1_{\mathbb{N}_+}(n_J).$$

The solution of the recursion is

$$\pi_{\{1,J\}}(0,n_J) = \pi_1(0)\cdot\pi_J(n_J) \quad \forall n_J\in\mathbb{N}, \tag{4.42}$$

as proved by induction over n_J, see the end of Proof of Proposition 4.23, p.77. For deriving (4.42), $J \in V$ was selected for easy notation. The proof for arbitrary $j \in V$ is similar. Hence (4.40) follows. □

The analysis of the structure of the stationary and limiting distribution which in general is not of product form led to the following observation on independence in space:
Whenever a Jackson network with infinite supply features a subset of nodes without infinite supply where there is only one node, say node i_*, from which customers are transfered directly to the subset V of nodes with infinite supply (which have no immediate feedback) with positive probability, then in equilibrium the two events

- node i_* is empty at a time t,
- node $j \in V$ has a queue length of n customers at time t,

are independent of each other, although it holds for the routing probability $r(j, i_*) > 0$ for all $j \in V$.
We do not know whether this observation is an artifact only but it is striking that in case of a product form stationary and limiting distribution the independence of these events is part of the independence structure in space at a given fixed time instant.

4.4 The non-ergodic case

It is known from Chapter 1 that classical Jackson networks which have at least one unstable node cannot be ergodic, so there cannot exist a steady-state distribution for the global network process. The next theorem shows that under certain conditions, the marginal limiting distribution for the stable nodes is a stationary distribution.

Theorem 4.26. *Consider a Jackson network where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work as in Definition 4.1. Nodes in $W := \tilde{J} \setminus V$ operate without infinite supply. Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations* (4.5)*.*
$S := \{i \in \tilde{J} : \eta_i < \mu_i\}$ is the set of stable nodes, nodes in $U := \tilde{J} \setminus S$ are unstable. We assume that all nodes without infinite supply of work are stable, i.e., $W \cap U = \emptyset$. In this situation the traffic equations (4.5) *reduce to*

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i), \quad i \in \tilde{J}. \tag{4.43}$$

Denote by $X = ((X_1(t), ..., X_J(t)) : t \geq 0)$ the queue-length process on $\mathbb{N}^J$.

(i) For nodes without infinite supply, the joint marginal limiting distribution is of product form:

$$\lim_{t \to \infty} P(X_i(t) = n_i : i \in W) = \prod_{i \in W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{4.44}$$

for all $(n_i, i \in W) \in \mathbb{N}^{|W|}$, and this is a one-dimensional stationary distribution on W as well.

(ii) *If the system is started with an initial distribution which has* (4.44) *as marginal joint queue lengths distribution on W, the arrival stream from $j \in W$ to $i \in V$ is Poisson with rate $\eta_j r(j,i)$. All these Poisson streams are independent.*

(iii) *If the system is started with an initial distribution which has* (4.44) *as marginal joint queue lengths distribution on W, then for a stable node $i \in V \cap S$ with infinite supply and without immediate feedback (i.e., $r(i,i) = 0$) the marginal limiting distribution is*

$$\lim_{t\to\infty} P(X_i(t) = n_i) = \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{4.45}$$

for all $n_i \in \mathbb{N}$, and this is a one-dimensional stationary distribution as well.

(iv) *If the system is started with an initial distribution which has* (4.44) *as marginal joint queue lengths distribution on W, then for unstable nodes with infinite supply, $i \in U \subseteq V$, and without immediate feedback the limit of the marginal queue length probability is*

$$\lim_{t\to\infty} P(X_i(t) = n_i) = 0 \tag{4.46}$$

for all $n_i \in \mathbb{N}$.

Proof. (i): Consider the subset W of nodes without infinite supply. We have the following information about the subnetwork:

- All service times are exponentially distributed and the service discipline at all nodes is FCFS.
- Routing of customers is Markovian: A customer completing service at node $i \in W$ will either move to some node $j \in W$ with probability $r(i,j)$ or leave the subnetwork with probability $1 - \sum_{j\in W} r(i,j) =: \tilde{r}(i,0)$, which is non-zero for some $i \in W$ because of the routing matrix being irreducible for the global network on $\tilde{J}$.
- At each node $i \in W$, we have an external Poisson arrival stream from the source with rate $\lambda_i \geq 0$. Furthermore all streams from nodes $j \in V$ with infinite supply into nodes $i \in W$ are Poisson streams with rate $\mu_j r(j,i)$, see Theorem 4.3. All (inter-) arrival times from the source and from nodes in V into node $i \in W$ constitute a set of independent random variables. So these Poisson processes are independent of each other. Thus all arrival streams from the outside of the subnetwork into each node $i \in W$ are a Poisson process with rate $\lambda_i + \sum_{j\in V} \mu_j r(j,i)$, and these superpositions are independent over $i \in W$.
- All service and inter-arrival times constitute a set of independent random variables.

All these properties guarantee that the subnetwork W acts as a Jackson network with $|W|$ nodes where the source and sink is denoted by $\{0\} \cup V$, see Definition 1.1. The corresponding queueing process $\tilde{X} := ((\tilde{X}_i(t) : i \in W) : t \in \mathbb{R}_+)$ is a Markov process of its own. The traffic equations of the described subnetwork W are given by

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j\in W} \tilde{\eta}_j r(j,i), \quad i \in W,$$

where $\tilde{\lambda}_i := \lambda_i + \sum_{j\in V} \mu_j r(j,i)$, so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$, see (4.43). According to Jackson's theorem (see Theorem 1.10), $\tilde{X}$ has the unique stationary and limiting distribution (4.44) if and only if $\tilde{\eta}_i < \mu_i$ holds for all $i \in W$. This condition is equivalent to $\eta_i < \mu_i$ for all $i \in W$ which was assumed.
So the subnetwork on W is in equilibrium, if and only if the global network process on $\tilde{J}$ is started with an initial distribution which has the marginal (4.44).

(ii): It is well known that ergodic Jackson networks with Poisson arrival streams from the source to node i with some rate $\tilde{\lambda}_i$ have, in equilibrium, Poisson departure streams from node i to the sink with some rate $\tilde{\eta}_i\tilde{r}(i,0)$, see [Mel79b, Example 7.1]. From the proof of (i), we know that the subset W behaves like an ergodic Jackson network of its own with $\tilde{\lambda}_i := \lambda_i + \sum_{j\in V} \mu_j r(j,i)$ and

$$\tilde{\eta}_i\tilde{r}(i,0) = \eta_i\Big(1 - \sum_{j\in W} r(i,j)\Big) = \eta_i\Big(r(i,0) + \sum_{j\in V} r(i,j)\Big).$$

Hence, if the subnetwork W is in equilibrium, departures to the sink from node $i \in W$ are a Poisson stream with rate $\eta_i r(i,0)$ and departures to any node $j \in V$ are also a Poisson stream with rate $\eta_i r(i,j)$, because a portion $\frac{r(i,j)}{r(i,0)+\sum_{j\in V} r(i,j)}$ of the departure stream $\tilde{\eta}_i\tilde{r}(i,0)$ from node $i \in W$ is directed to $j \in V$.

(iii)-(iv): Consider some fixed node $i \in V$ with infinite supply and without immediate feedback, i.e., $r(i,i) = 0$:

- The node has exponential-μ_i distributed service, the service discipline is FCFS.
- Routing of customers is Markovian: A customer arriving at node i, waits in the line until he gets service, after the service he proceeds to another node $j \in \tilde{J} \setminus \{i\}$ with probability $r(i,j)$ or leaves the network with probability $r(i,0)$. Thus, the customer leaves node i after complete service with probability

$$r(i,0) + \sum_{j\in\tilde{J}\setminus\{i\}} r(i,j) = 1 - r(i,i) = 1.$$

- The external arrival stream is Poisson with rate $\lambda_i \geq 0$. From (ii) it follows directly that, if the global network process is started with an initial distribution which has the marginal (4.44) on W, the arrival streams at node $i \in V$ from nodes $j \in W$ are Poisson streams with rate $\eta_j r(j,i)$. Arrival streams from nodes $j \in V \setminus \{i\}$ are Poisson streams with rate $\mu_j r(j,i)$, see Theorem 4.3. All these Poisson streams are independent from each other. Thus the arrival stream at node $i \in V$ is a Poisson process with rate

$$\hat{\lambda}_i := \lambda_i + \sum_{j\in W} \eta_j r(j,i) + \sum_{j\in V\setminus\{i\}} \mu_j r(j,i).$$

- All service and inter-arrival times constitute a set of independent random variables.

Thus, if the subnetwork W is in equilibrium and if $r(i,i) = 0$ holds for all $i \in V$, node $i \in V$ behaves like an $(M/M/1)-$ system of its own. The corresponding queue length process $\hat{X}$ is a Markov process on the state space $\mathbb{N}$. Customers who arrive from the infinite supply storage of this node are not counted as waiting customers. The traffic equation is then given by

$$\hat{\eta}_i = \hat{\lambda}_i = \lambda_i + \sum_{j\in W} \eta_j r(j,i) + \sum_{j\in V\setminus\{i\}} \mu_j r(j,i),$$

thus $\hat{\eta}_i = \eta_i$ holds (see (4.43) with $r(i,i) = 0\ \forall i \in V$).
If the balance equations of $\hat{X}$ have a probability solution, then this solution is the unique stationary and limiting distribution of $\hat{X}$. The balance equations are for all $n \in \mathbb{N}$

$$\begin{aligned} &\pi_i(n)(\hat{\lambda}_i + \mu_i 1_{\mathbb{N}_+}(n)) = \pi_i(n-1)\hat{\lambda}_i 1_{\mathbb{N}_+}(n) + \pi_i(n+1)\mu_i \\ \Leftrightarrow\ &\pi_i(n)(\eta_i + \mu_i 1_{\mathbb{N}_+}(n)) = \pi_i(n-1)\eta_i 1_{\mathbb{N}_+}(n) + \pi_i(n+1)\mu_i, \end{aligned}$$

which are solved by $\pi_i(n) = \left(\frac{\eta_i}{\mu_i}\right)^n$. Normalizing the solution to

$$\pi_i(n) = \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^n$$

yields a probability measure if and only if $\eta_i < \mu_i$ holds. If this condition does not hold, the node is unstable and the according network process of the $(M/M/1)-$ system is not ergodic, so the limit of the queue length probabilities are zero for all states in $\mathbb{N}$ and therefore the limiting queue length distribution degenerates to a one-point distribution in ∞. □

With the proof of Theorem 4.26(i) the following corollary is verified:

Corollary 4.27. *In the setting of Theorem 4.26, the process* $X_W := (X_i : i \in W)$ *is an ergodic homogeneous Markov process of its own.*

Theorem 4.28. *Consider a Jackson network where nodes in* V *have an infinite supply of work as in Definition 4.1. Nodes in* $W := \tilde{J} \setminus V$ *operate without infinite supply.*
Denote by $\eta = (\eta_1, ..., \eta_J)$ *the unique solution of the traffic equations* (4.5).
$S := \{i \in \tilde{J} : \eta_i < \mu_i\}$ *is the set of stable nodes, nodes in* $U := \tilde{J} \setminus S$ *are unstable. Denote by* $X = ((X_1(t), ..., X_J(t)) : t \geq 0)$ *the queue-length process on* $\mathbb{N}^J$.
Then we have independent of the initial distribution for all $n_i \in \mathbb{N}$*:*

$$\lim_{t\to\infty} P(X_i(t) = n_i : i \in S \cap W) = \prod_{i\in S\cap W} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{4.47}$$

$$\lim_{t\to\infty} P(X_i(t) = n_i) = 0 \quad \forall i \in U \cap W. \tag{4.48}$$

Proof. Consider the subset W of nodes without infinite supply. We have the following information about the subnetwork:

- All service times are exponentially distributed and the service discipline at all nodes is FCFS.

- Routing of customers is Markovian: A customer completing service at node $i \in W$ will either move to some node $j \in W$ with probability $r(i,j)$ or leave the subnetwork with probability $1 - \sum_{j \in W} r(i,j)$, which is non-zero for some subset of W because of the routing matrix being irreducible for the global network on $\tilde{J}$.

- At each node $i \in W$, we have external arrival streams from the source which are Poisson streams with rate $\lambda_i \geq 0$. Furthermore all streams from nodes $j \in V$ with infinite supply into nodes $i \in W$ are Poisson streams with rate $\mu_j r(j,i)$, see Theorem 4.3. All (inter-)arrival times from the source and from nodes in V into node $i \in W$ constitute a set of independent random variables. So the Poisson processes are independent of each other. Thus all arrival streams from the outside of the subnetwork into each node $i \in W$ are a Poisson process with rate $\lambda_i + \sum_{j \in V} \mu_j r(j,i)$.

- All service and inter-arrival times at each node are independent of each other.

All these properties guarantee that the subnetwork W acts as a Jackson network with $|W|$ nodes where the source and sink is denoted by $\{0\} \cup V$, see Definition 1.1. The corresponding queueing process $\tilde{X} := ((\tilde{X}_i(t) : i \in W) : t \in \mathbb{R}_+)$ is a Markov process of its own. The traffic equations of the described subnetwork W are given by

$$\tilde{\eta}_i = \tilde{\lambda}_i + \sum_{j \in W} \min(\tilde{\eta}_j, \mu_j) r(j,i), \quad i \in W,$$

where $\tilde{\lambda}_i := \lambda_i + \sum_{j \in V} \mu_j r(j,i)$, so $\eta_i = \tilde{\eta}_i$ holds for all $i \in W$, see (4.5). According to Theorem 1.22, $\tilde{X}$ has the limiting marginal joint queue length distribution (4.47) on $S \cap W$ and the limiting marginal queue length probabilities (4.48) for nodes in $U \cap W$. □

Remark 4.29. *In the general situation of Theorem 4.28 we cannot prove a statement about the marginal limiting distribution like* (4.45) *for stable nodes with infinite supply* ($\in S \cap V$)*. If $U \cap W \neq \emptyset$ holds, the queue length process of the subnetwork W is not ergodic, so the argument of Poisson departure streams from W into nodes in V does not apply. If $U \cap W = \emptyset$ holds, Theorem 4.26 applies.*

4.5 Performance measures

The computation of the throughput is a little bit different compared to Chapter 1, Section 1.2.3, due to the infinite supply.

Definition 4.30. *Consider an ergodic Jackson network with infinite supply. The stationary throughput TH_i of a node $i \in W$ without infinite supply is*

$$TH_i = \sum_{(n_1,\ldots,n_J) \in \mathbb{N}^J} \pi(n_1, \ldots, n_J) \mu_i 1_{\mathbb{N}_+}(n_i),$$

the stationary throughput TH_j of a node $j \in V$ with infinite supply is

$$TH_j = \sum_{(n_1,\ldots,n_J) \in \mathbb{N}^J} \pi(n_1, \ldots, n_J) \mu_j.$$

The stationary throughput TH of the network is

$$TH = \sum_{i \in W} TH_i r(i,0) + \sum_{j \in V} TH_j r(j,0).$$

Proposition 4.31. *Consider a Jackson network where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work as in Definition 4.1. Nodes in $W := \tilde{J} \setminus V$ operate without infinite supply. Let for all nodes $i \in \tilde{J}$ hold $\eta_i < \mu_i$ where $\eta = (\eta_1, ..., \eta_J)$ is the unique solution of the traffic equations* (4.1).
Then the stationary throughput of nodes $i \in W$ without infinite supply is $TH_i = \eta_i$. Let $r(j,j) = 0$ hold for all $j \in V$, then the stationary throughput of nodes $j \in V$ with infinite supply is $TH_j = \mu_j$ and the stationary throughput of the system is

$$TH = \lambda - \sum_{i \in V} (\eta_i - \mu_i) > \lambda.$$

Proof. The proof uses the results of Theorem 4.15. For $i \in W$ we get

$$\begin{aligned} TH_i = & \sum_{(n_1,...,n_J) \in \mathbb{N}^J} \pi(n_1, ..., n_J) \mu_i 1_{\mathbb{N}_+}(n_i) = \sum_{(n_j : j \in W) \in \mathbb{N}^{|W|}} \mu_i 1_{\mathbb{N}_+}(n_i) \underbrace{\sum_{(n_j : j \in V) \in \mathbb{N}^{|V|}} \pi(n_1, ..., n_J)}_{=\pi_W(n_j : j \in W)} \\ = & \sum_{(n_j : j \in W) \in \mathbb{N}^{|W|}} \mu_i 1_{\mathbb{N}_+}(n_i) \prod_{k \in W} \left(1 - \frac{\eta_k}{\mu_k}\right) \left(\frac{\eta_k}{\mu_k}\right)^{n_k} = \sum_{n_i \in \mathbb{N}} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i} \mu_i 1_{\mathbb{N}_+}(n_i) \\ = & \sum_{n_i \in \mathbb{N}} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i + 1} \mu_i = \eta_i, \end{aligned}$$

and for $j \in V$:

$$TH_j = \sum_{(n_1,...,n_J) \in \mathbb{N}^J} \pi(n_1, ..., n_J) \mu_j = \mu_j \underbrace{\sum_{(n_1,...,n_J) \in \mathbb{N}^J} \pi(n_1, ..., n_J)}_{=\sum_{n_j \in \mathbb{N}} \pi_j(n_j) = 1} = \mu_j.$$

The throughput of the system is

$$\begin{aligned} TH &= \sum_{j \in \tilde{J}} TH_j r(j,0) = \sum_{j \in W} TH_j r(j,0) + \sum_{j \in V} TH_j r(j,0) \\ &= \sum_{j \in W} \eta_j r(j,0) + \sum_{j \in V} \mu_j r(j,0) \overset{\text{Remark 4.5}}{=} \lambda - \sum_{j \in V} \underbrace{(\eta_j - \mu_j)}_{<1} > \lambda. \end{aligned}$$

□

Example 4.32. *Let $g : \mathbb{N}^J \to \mathbb{R}$ denote a non-decreasing cost function which determines costs associated with queue lengths. The quality of a system is often measured by the average accumulated cost over a time horizon $[0, T]$,*

$$d(T) = \frac{1}{T} \int_0^T g(X_i(t), i \in \tilde{J}) \, dt.$$

Note, that this is an empirical measure which depends on the realized path of the system. If X is ergodic with stationary distribution π as in Corollary 4.20 and if

$$\sum_{(n_i:i\in\tilde{J})\in\mathbb{N}^J} |g(n_1,...,n_J)|\pi(n_1,...,n_J) < \infty,$$

then the ergodic theorem (see Theorem 1.14) yields that for any initial distribution $P^{X(0)}$ holds $P_{P^{X(0)}}$-a.s.

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g(X_1(t),...,X_J(t))dt = \sum_{(n_i:i\in\tilde{J})\in\mathbb{N}^J} g(n_1,...,n_J)\pi(n_1,...,n_J).$$

This means that the path-wise evaluated time average converges to the state space average in equilibrium, so path-wise evaluated time averages for a time horizon $[0,T]$ with large T can be estimated by state space averages in ergodic Jackson networks:

$$\frac{1}{T}\int_0^T g(X_1(t),...,X_J(t))dt \approx \sum_{(n_i:i\in\tilde{J})\in\mathbb{N}^J} g(n_1,...,n_J)\prod_{j=1}^J\left(1-\frac{\eta_j}{\mu_j}\right)\left(\frac{\eta_j}{\mu_j}\right)^{n_j}.$$

But also if the Jackson network with infinite supply has no product form stationary queue length distribution, one may predict the average accumulated cost over a time horizon.

Example 4.33. *Let*

$$g_i : \mathbb{N} \to \mathbb{R}$$

denote a non-decreasing cost function which determines costs associated with a queue length at some node $i \in V$ with infinite supply and

$$g_W : \mathbb{N}^{|W|} \to \mathbb{R}$$

denote a non-decreasing cost function which determines costs associated with a queue length at nodes $i \in W$ without infinite supply. The average accumulated cost of the system over a time horizon $[0,T]$ is

$$d(T) = \sum_{i\in V}\frac{1}{T}\int_0^T g_i(X_i(t))\ dt + \frac{1}{T}\int_0^T g_W(X_i(t) : i \in W)\ dt.$$

If $\eta_i < \mu_i$ holds for all $i \in \tilde{J}$ and if $\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} |g_W(n_i : i \in W)|\pi_W(n_i : i \in W) < \infty$, then the ergodic theorem (see Theorem 1.14) yields that almost surely holds under any initial distribution

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g_W(X_i(t) : i \in W)dt = \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} g_W(n_i : i \in W)\pi_W(n_i : i \in W).$$

If $\eta_i < \mu_i$ holds for all $i \in \tilde{J}$ and $r(i,i) = 0$ for all $i \in V$, and if $\sum_{n_i\in\mathbb{N}} |g_i(n_i)|\pi_i(n_i) < \infty$ for all $i \in V$, then the ergodic theorem yields that almost surely holds under any initial distribution

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T g_i(X_i(t))\ dt = \sum_{n_i\in\mathbb{N}} g_i(n_i)\pi_i(n_i)\ \forall i \in V.$$

So the path-wise evaluated time averages for a time horizon $[0,T]$ *with large* T *can be estimated by state space averages in ergodic Jackson networks with infinite supply:*

$$d(T) \approx \sum_{i\in V}\sum_{n_i\in\mathbb{N}} g_i(n_i)\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i} + \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} g_W(n_i : i\in W)\prod_{i\in W}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}.$$

For non-ergodic Jackson networks with infinite supply, we use the properties described in Theorem 4.26 and Corollary 4.27 to define the stationary throughput the following way:

Definition 4.34. *Consider a Jackson network with infinite supply where nodes without infinite supply are stable. The stationary throughput* TH_i *of a node* $i\in W$ *without infinite supply is*

$$TH_i = \sum_{(n_j:j\in W)\in\mathbb{N}^{|W|}} \pi_W(n_j : j\in W)\mu_i 1_{\mathbb{N}_+}(n_i),$$

the stationary throughput TH_j *of a stable node* $j\in V$ *with infinite supply is*

$$TH_j = \sum_{n_j\in\mathbb{N}} \pi_j(n_j)\mu_j.$$

The stationary throughput TH_S *of the subnetwork of stable nodes is*

$$TH_S = \sum_{i\in W} TH_i r(i,0) + \sum_{j\in V\cap S} TH_j r(j,0).$$

Proposition 4.35. *Consider a Jackson network where nodes in* $V\subseteq\tilde{J}$ *have an infinite supply of work as in Definition 4.1. Nodes in* $W := \tilde{J}\setminus V$ *operate without infinite supply. Denote by* $\eta=(\eta_1,...,\eta_J)$ *the unique solution of the traffic equations* (4.5).
$S := \{i\in\tilde{J} : \eta_i<\mu_i\}$ *is the set of stable nodes, nodes in* $U := \tilde{J}\setminus S$ *are unstable. We assume that all nodes without infinite supply of work are stable, i.e.,* $W\cap U=\emptyset$. *In this situation the traffic equations* (4.5) *reduce to* (4.1).
Then the stationary throughput of nodes $i\in W$ *without infinite supply is* $TH_i=\eta_i$.
Let $r(j,j)=0$ *hold for all* $j\in V\cap S$, *then the stationary throughput of nodes* $j\in V\cap S$ *with infinite supply is* $TH_j=\mu_j$ *and the throughput of the stable subnetwork is*

$$TH_S = \lambda - \sum_{i\in V}(\eta_i-\mu_i) - \sum_{j\in V\cap U}\mu_j r(j,0).$$

Proof. The proof uses the results of Theorem 4.26. For $i\in W$ we get

$$\begin{aligned} TH_i &= \sum_{(n_j:j\in W)\in\mathbb{N}^{|W|}} \pi_W(n_j : j\in W)\mu_i 1_{\mathbb{N}_+}(n_i) = \sum_{n_i\in\mathbb{N}}\pi_i(n_i)\mu_i 1_{\mathbb{N}_+}(n_i)\\ &= \sum_{n_i\in\mathbb{N}}\pi_i(n_i+1)\mu_i = \sum_{n_i\in\mathbb{N}}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i+1}\mu_i = \eta_i, \end{aligned}$$

and for $j\in V\cap S$:

$$TH_j = \underbrace{\sum_{n_j\in\mathbb{N}}\pi_j(n_j)}_{=1}\mu_j = \mu_j.$$

The throughput of the subnetwork S is

$$TH_S = \sum_{i\in W} TH_i r(i,0) + \sum_{j\in V\cap S} TH_j r(j,0) = \sum_{i\in W} \eta_i r(i,0) + \sum_{j\in V\cap S} \mu_j r(j,0)$$
$$\stackrel{\text{Remark 4.5}}{=} \lambda - \sum_{j\in V}(\eta_j - \mu_j) - \sum_{j\in V\cap U} \mu_j r(j,0).$$

□

Example 4.36. *Let*

$$g_W : \mathbb{N}^{|W|} \to \mathbb{R}$$

denote a non-decreasing cost function which determines costs associated with a queue length at nodes $i \in W$ without infinite supply. In case of non-ergodic Jackson networks with infinite supply we cannot apply the ergodic theorem to assess average costs in general. In case $W \cap U = \emptyset$, at least the average accumulated costs of the subsystem W over a time horizon $[0,T]$ may be predicted:

$$d(T) = \frac{1}{T}\int_0^T g_W(X_i(t) : i \in W)\ dt.$$

If $\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} |g_W(n_i : i \in W)|\pi_W(n_i : i \in W) < \infty$, then the ergodic theorem (see Theorem 1.14) yields for the ergodic homogeneous Markov process X_W (see Corollary 4.27) that almost surely holds

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T g_W(X_i(t) : i \in W)dt = \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} g_W(n_i : i \in W)\pi_W(n_i : i \in W).$$

So, even in non-ergodic *Jackson networks with infinite supply, the path-wise evaluated time averages on the subnet W for a time horizon $[0,T]$ with large T can be estimated by state space averages:*

$$d(T) \approx \sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}} g_W(n_i : i \in W) \prod_{i\in W}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}.$$

4.6 Breakdowns of nodes

4.6.1 Introduction

In this section, we consider Jackson networks with infinite supply of work at some nodes where some nodes break down randomly and are repaired thereafter. Breakdowns of nodes without infinite supply of work were already investigated in Chapter 2 and parts of the techniques are adapted here. It turns out that breakdowns of nodes with infinite supply require a more specific regime to control breakdown and repair behavior.

A breakdown of a node with infinite supply is handled as follows:
Whenever a node with infinite supply breaks down, the server of this node is broken down and does not serve any customers. This means that this station serves neither customers queued in the line, nor low priority customers from the storage of the infinite

supply of work (which only appear if the node is empty). Thus as already defined in Chapter 2, there are no departures from broken down nodes. Since an infinite supply is unambiguously allocated to a node, the low priority customers are not subject to rerouting if the according node breaks down.

So, all together, in case of broken down stations we assume that at down nodes

- no service is provided,
- customers which are present at a station when this station breaks down are frozen and stay there, waiting for service until the station is repaired,
- no new customers are admitted, customers who want to visit a down node are rerouted according to the already defined rules: stalling (see Definition 2.6), skipping (see Definition 2.7) or blocking rs-rd (see Definition 2.8),
- low priority customers from an infinite supply at a down node $i \in V \cap I$ (if nodes in I are broken down) are also frozen and wait until the station is repaired, so they are not subject to rerouting.

Breakdown and repair intensities are defined as in Example 2.2, so the breakdown and repair rates may depend on the interaction of nodes but not on their queue length nor on the infinite supply. Since the low priority customers from the infinite supply at a node do not count as queued customers - even if one of them is in service -, they would not directly influence load-dependent breakdown and repair intensities.

Definition 4.37 (Jackson network with unreliable nodes and infinite supply)**.** *Consider a Jackson network with unreliable nodes with node set $\tilde{J} = \{1, ..., J\}$. Denote by $D \subseteq \tilde{J}$ the subset of unreliable nodes. $V \subseteq \tilde{J}$ is the set of nodes with an infinite supply of work, nodes in $W = \tilde{J} \setminus V$ operate without infinite supply.*
The breakdown and repair intensities may depend on the interaction of nodes but not on their load, as in Example 2.2. Rerouting regime is either stalling, skipping, or blocking rs-rd according to the Definitions 2.6 - 2.8.
Then the joint availability-queue length process is described by the Markov process $(Y, X) = ((Y(t); X_1(t), ..., X_J(t)) : t \in \mathbb{R}_+)$ on the state space $\mathcal{P}(D) \times \mathbb{N}^J$ with transition rates matrix $Q = (q(z, z') : z, z' \in \mathcal{P}(D) \times \mathbb{N}^J)$ given by:
For all $i, j \in \tilde{J}, i \neq j$:

$$
\begin{aligned}
&q(\emptyset, n_1, ..., n_i, ..., n_J; \emptyset, n_1, ..., n_i + 1, ..., n_J) = \lambda_i + \sum_{k \in V} \mu_k r(k, i) 1_{\{0\}}(n_k),\\
&q(\emptyset, n_1, ..., n_i, ..., n_J; \emptyset, n_1, ..., n_i - 1, ..., n_J) = \mu_i r(i, 0) 1_{\mathbb{N}_+}(n_i),\\
&q(\emptyset, n_1, ..., n_i, ..., n_j, ..., n_J; \emptyset, n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) = \mu_i r(i, j) 1_{\mathbb{N}_+}(n_i),\\
&q(\emptyset, n_1, ..., n_J; H, n_1, ..., n_J) = \alpha(\emptyset, H) = A(H), \emptyset \subset H \subseteq D,\\
&q(\emptyset, n_1, ..., n_J; \emptyset, n_1, ..., n_J) = -\sum_{i \in \tilde{J}} \lambda_i - \sum_{i \in \tilde{J}} \sum_{k \in V} \mu_k r(k, i) 1_{\{0\}}(n_k)\\
&\qquad - \sum_{i \in \tilde{J}} \mu_i (1 - r(i, i)) 1_{\mathbb{N}_+}(n_i) - \sum_{\emptyset \subset H \subseteq D} \alpha(\emptyset, H),
\end{aligned}
$$

and for all $I \subseteq D, I \neq \emptyset, i, j \in \tilde{J} \setminus I, i \neq j$*:*

$$q(I, n_1, ..., n_i, ..., n_J; I, n_1, ..., n_i + 1, ..., n_J) = \lambda_i^I + \sum_{k \in V \setminus I} \mu_k^I r^I(k, i) 1_{\{0\}}(n_k),$$

$$q(I, n_1, ..., n_i, ..., n_J; I, n_1, ..., n_i - 1, ..., n_J) = \mu_i^I r^I(i, 0) 1_{\mathbb{N}_+}(n_i),$$

$$q(I, n_1, ..., n_i, ..., n_j, ..., n_J; I, n_1, ..., n_i - 1, ..., n_j + 1, ..., n_J) = \mu_i^I r^I(i, j) 1_{\mathbb{N}_+}(n_i),$$

$$q(I, n_1, ..., n_J; H, n_1, ..., n_J) = \alpha(I, H) = \frac{A(H)}{A(I)}, \quad I \subset H \subseteq D,$$

$$q(I, n_1, ..., n_J; K, n_1, ..., n_J) = \beta(I, K) = \frac{B(I)}{B(K)}, \quad K \subset I \subseteq D,$$

$$q(I, n_1, ..., n_J; I, n_1, ..., n_J) = -\sum_{i \in \tilde{J} \setminus I} \lambda_i^I - \sum_{i \in \tilde{J} \setminus I} \sum_{k \in V \setminus I} \mu_k^I r^I(k, i) 1_{\{0\}}(n_k)$$
$$- \sum_{i \in \tilde{J} \setminus I} \mu_i^I (1 - r^I(i, i)) 1_{\mathbb{N}_+}(n_i) - \sum_{I \subset H \subseteq D} \alpha(I, H) - \sum_{K \subset I \subseteq D} \beta(I, K),$$

and $q(z, z') = 0$ *otherwise.*

Theorem 4.38. *Consider a Jackson network with unreliable nodes where nodes in* V *have an infinite supply of work as in Definition 4.37. Nodes in* $W := \tilde{J} \setminus V$ *operate without infinite supply. If at time* t *all nodes are in up-status, i.e.,* $Y(t) = \emptyset$*, then the departure streams from node* $j \in V$ *which has an infinite supply of work is a Poisson process with rate* μ_j*. Thus the departure stream from* $j \in V$ *to* $i \in \tilde{J}$ *is Poisson with rate* $\mu_j r(j, i)$*.*
Whenever nodes in $I \neq \emptyset$ *are broken down and either skipping or blocking rs-rd is in force, the departure stream of node* $j \in V \setminus I$ *with infinite supply in up-status is Poisson with rate* μ_j*. The departure stream from* $j \in V \setminus I$ *to* $i \in \tilde{J} \setminus I$ *is Poisson with rate* $\mu_j r^I(j, i)$*, where* $r^I(j, i)$ *is determined by the rerouting regime in force. If a node* $k \in V$ *with infinite supply is broken down, i.e.,* $k \in V \cap I$*, the Poisson departure stream of this node is interrupted until its server is repaired.*
In case of stalling, all Poisson arrival streams stop whenever a breakdown occurs ($I \neq \emptyset$)*, and the Poisson streams are reactivated when all nodes recur to the up-status.*

Proof. Consider the network in its different availability states of nodes:

- If all nodes are in up-status ($I = \emptyset$), all departure times from node $j \in V$
  - are independent of the state of the node due to the infinite supply,
  - and therefore have independent and identically exponentially distributed interdeparture times with rate μ_j, the service rate of the node.

  Thus, the departure stream of node $j \in V$ is a Poisson process with rate μ_j. Hence for node $i \in \tilde{J}$, the arrival stream from node $j \in V$ is a Poisson process with rate $\mu_j r(j, i)$, because a portion of $r(j, i)$ of the departure stream is directed to node $i \in \tilde{J}$ (decomposition of a Poisson process by some probability independent of the process and of its times of events).

- If nodes in $I \neq \emptyset$ are broken down and rerouting is according to blocking rs-rd or skipping, all departure times from node $j \in V \setminus I$

  - are independent of the state of the node due to the infinite supply,
  - and therefore have independent and identically exponentially distributed interdeparture times with rate μ_j, the service rate of the node.

  Therefore the departure stream of node $j \in V$ is a Poisson process with rate μ_j. Thus, for node $i \in \tilde{J} \setminus I$, the arrival stream from node $j \in V \setminus I$ is a Poisson process with rate $\mu_j r^I(j,i)$, because a portion of $r^I(j,i)$ of the departure stream is directed to node $i \in \tilde{J} \setminus I$ (decomposition of a Poisson process by some probability independent of all the process and of its times of events). This holds for all states $I \subseteq D$ of the availability process Y which means that whenever a node with infinite supply breaks down, its Poisson departure stream is interrupted until the node is repaired.

- If nodes in $I \neq \emptyset$ are broken down and stalling occurs, all network processes - except for the breakdown and repair processes - are frozen, i.e., no service is provided in the network and there is no arrival stream. Thus, as long as not all nodes are in up status, there are no arrival streams in the network.

□

4.6.2 The traffic equations

For the results on the long-time behavior of such networks, the following traffic equations are needed. The definition of the appropriate traffic equations is established analogously to Definition 4.4.

Definition 4.39. *The traffic equations for unreliable Jackson networks with infinite supply are:*

- *In case of stalling:*

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i), \; i \in \tilde{J}, \tag{4.49}$$

  as long as all nodes are in up status ($I = \emptyset$). Otherwise $\eta_i^I = 0$ for all $i \in \tilde{J}$. (Note that (4.49) *equals* (4.1).*)*

- *In case of rerouting according to blocking rs-rd or skipping:*

$$\eta_i^I = \lambda_i^I + \sum_{j \in W \setminus I} \eta_j^I r^I(j,i) + \sum_{j \in V \setminus I} \mu_j^I r^I(j,i), \; i \in \tilde{J} \setminus I, \tag{4.50}$$

  for all $I \subseteq D$. If $I = \emptyset$ the traffic equations (4.50) *are equal to* (4.1).

The traffic equations for some $I \subseteq D$ remain valid only as long as the availability status is unchanged. Whenever the availability status of the system changes, the traffic equations are adapted according to the new set of broken down nodes. Thus each traffic equation (4.50) may have different solutions for different I. The following two lemmata show under which constraints the solution of the traffic equation (4.50) remains the same for all $I \subseteq D$.

Lemma 4.40. *Consider a Jackson network where nodes in $D \subseteq \tilde{J}$ are unreliable and nodes in $V \subseteq \tilde{J}$ have an infinite supply of work. $W := \tilde{J} \setminus V$ is the set of nodes without infinite supply. Let for all nodes $i \in W$ without infinite supply hold $\eta_i < \mu_i$ where $(\eta_i : i \in \tilde{J})$ is the unique solution of* (4.1). *In case of breakdowns of nodes we assume that customers are rerouted according to the* **blocking rs-rd** *regime.*

(i) If the following reversibility constraints hold:

$$\eta_i r(i,j) = \eta_j r(j,i) \quad \forall i,j \in W, \tag{4.51}$$
$$\eta_i r(i,j) = \mu_j r(j,i) \quad \forall i \in W, j \in V, \tag{4.52}$$

then for all nodes $i \in W \setminus I$ holds that the solution η_i^I of the traffic equation (4.50) *for all $I \subseteq D$, $I \neq \emptyset$, equals η_i.*

(ii) Let (4.51) *and* (4.52) *hold. If we additionally require the reversibility constraint*

$$\mu_i r(i,j) = \mu_j r(j,i) \quad \forall i,j \in V, \tag{4.53}$$

then for the solution of (4.50) *holds $\eta_i^I = \eta_i$ for all $i \in V \setminus I$ and all $I \subseteq D$, as well.*

Proof. (i): We make the ansatz $\eta_i = \eta_i^I$ for all $i \in W \setminus I$ and all $I \subseteq D$. We then obtain with the solution η_i of the traffic equations (4.1) for any $I \subseteq D$: $\forall i \in W \setminus I$

$$\begin{aligned}
\eta_i &= \lambda_i^I + \sum_{j\in W\setminus I} \eta_j r^I(j,i) + \sum_{j \in V\setminus I} \mu_j r^I(j,i) \\
&\overset{\text{Def. 2.8}}{=} \lambda_i + \eta_i\Big(r(i,i) + \sum_{k\in I} r(i,k)\Big) + \sum_{j\in W\setminus I, j\neq i} \eta_j r(j,i) + \sum_{j\in V\setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{k\in I\cap W} \underbrace{\eta_i r(i,k)}_{\overset{(4.51)}{=}\eta_k r(k,i)} + \sum_{k\in I\cap V} \underbrace{\eta_i r(i,k)}_{\overset{(4.52)}{=}\mu_k r(k,i)} + \sum_{j\in W\setminus I} \eta_j r(j,i) + \sum_{j\in V\setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j\in W} \eta_j r(j,i) + \sum_{j\in V} \mu_j r(j,i) = (4.1).
\end{aligned}$$

(ii): For any $I \subseteq D$ holds $\forall i \in V \setminus I$:

$$\begin{aligned}
\eta_i^I &\overset{(4.50)}{=} \lambda_i^I + \sum_{j\in W\setminus I} \underbrace{\eta_j^I}_{\overset{(i)}{=}\eta_j} r^I(j,i) + \sum_{j\in V\setminus I} \mu_j r^I(j,i) \\
&\overset{\text{Def. 2.8}}{=} \lambda_i + \sum_{j\in W\setminus I} \eta_j r(j,i) + \mu_i\Big(r(i,i) + \sum_{k\in I} r(i,k)\Big) + \sum_{j\in V\setminus I, j\neq i} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j\in W\setminus I} \eta_j r(j,i) + \sum_{k\in I\cap W} \underbrace{\mu_i r(i,k)}_{\overset{(4.52)}{=}\eta_k r(k,i)} + \sum_{k\in I\cap V} \underbrace{\mu_i r(i,k)}_{\overset{(4.53)}{=}\mu_k r(k,i)} + \sum_{j\in V\setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j\in W} \eta_j r(j,i) + \sum_{j\in V} \mu_j r(j,i) \overset{(4.1)}{=} \eta_i.
\end{aligned}$$

□

Remark 4.41. *The reversibility constraints* (4.51), (4.52) *and* (4.53) *are different from the classical reversibility constraints* (2.7) *which are the local balance equations of the routing process. But the interpretation of* (4.51), (4.52) *and* (4.53) *is the same as for* (2.7)*: The departure rate from one node multiplied with the routing probability to another node has to be equal to the according flow rate of the opposite direction.*

In the following example the reversibility constraints (4.51), (4.52) and (4.53) are compatible with the definition of a Jackson network with infinite supply.

Example 4.42. *Consider a Jackson network with two nodes where node* 2 *has an infinite supply of work. Both nodes may break down. Let* $\eta_1 < \mu_1$ *hold. Then the traffic equations in case of all nodes in up status are*

$$\eta_1 = \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1) \tag{4.54}$$

$$\eta_2 = \lambda_2 + \eta_1 r(1,2) + \mu_2 r(2,2). \tag{4.55}$$

Since $1 - r(1,1) > 0$ *equation* (4.54) *is equivalent to*

$$\eta_1 = \frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)}. \tag{4.56}$$

Let the following reversibility constraint

$$\eta_1 r(1,2) = \mu_2 r(2,1) \tag{4.57}$$

hold. This constraint leads to the following structural consequences for the Jackson network:

- *If* $r(2,1) = 0$ *then either* $r(1,2) = 0$ *or* $\lambda_1 = 0$*, i.e., if* $r(2,1) = 0$ *holds then either the two nodes must be two separate* $M/M/1/\infty$ *systems (where one has infinite supply) or node 1 has no input at all.*
- *If* $r(1,2) = 0$ *then* $r(2,1) = 0$*, i.e., if* $r(1,2) = 0$ *holds then the two nodes must be two separate* $M/M/1/\infty$ *systems (where one has infinite supply).*
- *It remains to analyze the case* $r(1,2) \neq 0 \neq r(2,1)$*. Then the constraint* (4.57) *is equivalent to*

$$\begin{aligned} &\frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)} r(1,2) = \mu_2 r(2,1) \\ &\Leftrightarrow \; \lambda_1 r(1,2) + \mu_2 r(2,1) r(1,2) = \mu_2 r(2,1)(\underbrace{1 - r(1,1)}_{=r(1,0)+r(1,2)}) \\ &\Leftrightarrow \; \lambda_1 r(1,2) = \mu_2 r(2,1) r(1,0). \end{aligned}$$

  Thus, $r(1,0) = 0$ *if and only if* $\lambda_1 = 0$*. Let* $r(1,0) > 0$ *then we get*

$$\mu_2 r(2,1) = \lambda_1 \frac{r(1,2)}{r(1,0)} \;\Leftrightarrow\; \mu_2 = \lambda_1 \frac{r(1,2)}{r(2,1) r(1,0)} > 0.$$

  Plugging this information into (4.56) *yields*

$$\eta_1 = \frac{\lambda_1 (1 + \frac{r(1,2)}{r(1,0)})}{1 - r(1,1)} = \frac{\lambda_1}{r(1,0)} = \lambda \frac{r(0,1)}{r(1,0)} > 0.$$

  So if $r(1,2) \neq 0 \neq r(2,1)$ *and* $r(1,0) > 0$*, then* $\lambda_1 > 0$*. Thus constraint* (4.57) *has influence on the parameters of the network, but it does not lead to contradictions.*

Lemma 4.43. *Consider a Jackson network where nodes in $D \subseteq \tilde{J}$ are unreliable and nodes in $V \subseteq \tilde{J}$ have an infinite supply of work. $W := \tilde{J} \setminus V$ is the set of nodes without infinite supply. Let for all nodes $i \in W$ without infinite supply hold $\eta_i < \mu_i$ where $(\eta_i : i \in \tilde{J})$ is the unique solution of (4.1). In case of breakdowns of nodes we assume that customers are rerouted according to the* **skipping** *regime. Let the following side constraint hold:*

$$\eta_i = \mu_i \quad \forall i \in V \cap D. \tag{4.58}$$

Then for all nodes $i \in \tilde{J} \setminus I$ holds that the solution η_i^I of the traffic equation (4.50) for all $I \subseteq D$, $I \neq \emptyset$, equals η_i.

Proof. We make the ansatz $\eta_i = \eta_i^I$ for all $i \in W \setminus I$ and all $I \subseteq D$. We then obtain with the solution η_i of the traffic equations (4.1) for any $I \subseteq D$: $\forall i \in W \setminus I$

$$\begin{aligned}
&\lambda_i^I + \sum_{j \in W \setminus I} \eta_j r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \\
&\overset{(\text{Def. } 2.7)}{=} \lambda_i + \sum_{k \in I} \lambda_k r^I(k,i) + \sum_{j \in W \setminus I} \eta_j \Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big) + \\
&\quad + \sum_{j \in V \setminus I} \mu_j \Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big) \\
&= \underbrace{\lambda_i + \sum_{j \in W \setminus I} \eta_j r(j,i) + \sum_{j \in V \setminus I} \mu_j (r(j,i) +}_{\overset{(4.1)}{=} \eta_i - \sum_{j \in I \cap W} \eta_j r(j,i) - \sum_{j \in I \cap V} \mu_j r(j,i)} \\
&\quad + \sum_{k \in I} r^I(k,i) \Big(\underbrace{\lambda_k + \sum_{j \in W \setminus I} \eta_j r(j,k) + \sum_{j \in V \setminus I} \mu_j r(j,k)}_{\overset{(4.1)}{=} \eta_k - \sum_{j \in I \cap W} \eta_j r(j,k) - \sum_{j \in I \cap V} \mu_j r(j,k)} \Big) \\
&= \eta_i - \sum_{j \in I \cap W} \eta_j r(j,i) - \sum_{j \in I \cap V} \mu_j r(j,i) + \\
&\quad + \sum_{k \in I} \eta_k r^I(k,i) - \sum_{k \in I} r^I(k,i) \sum_{j \in I \cap W} \eta_j r(j,k) - \sum_{k \in I} r^I(k,i) \sum_{j \in I \cap V} \mu_j r(j,k) \\
&= \eta_i - \sum_{j \in I \cap W} \eta_j \underbrace{\Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big)}_{\overset{(2.2)}{=} r^I(j,i)} + \\
&\quad + \sum_{k \in I} \eta_k r^I(k,i) - \sum_{j \in I \cap V} \mu_j \underbrace{\Big(r(j,i) + \sum_{k \in I} r(j,k) r^I(k,i) \Big)}_{\overset{(2.2)}{=} r^I(j,i)} \\
&= \eta_i + \sum_{k \in I \cap W} \eta_k r^I(k,i) - \sum_{j \in I \cap W} \eta_j r^I(j,i) + \sum_{k \in I \cap V} \eta_k r^I(k,i) - \sum_{j \in I \cap V} \mu_j r^I(j,i) \\
&= \eta_i + \sum_{k \in I \cap V} \underbrace{(\eta_k - \mu_k)}_{\overset{(4.58)}{=} 0} r^I(k,i) = \eta_i.
\end{aligned} \tag{4.59}$$

Since $\eta_j^I = \eta_i$ holds for all $j \in W \setminus I$ and all $I \subseteq D$, it follows for all $i \in V \setminus I$ and $I \subseteq D$:

$$\begin{aligned}\eta_i^I &= \lambda_i^I + \sum_{j \in W \setminus I} \eta_j^I r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i) \\ &= \lambda_i^I + \sum_{j \in W \setminus I} \eta_j r^I(j,i) + \sum_{j \in V \setminus I} \mu_j r^I(j,i)\end{aligned}$$

which is the very left side of (4.59) with $i \in V \setminus I$. The above computations in (4.59) are valid for all $i \in \tilde{J} \setminus I$, hence it follows $\eta_i^I = \eta_i$ for all $i \in V \setminus I$ and $I \subseteq D$, too. □

Remark 4.44. *The constraint* (4.58) *means that nodes with infinite supply which may break down, on average, are fully loaded by customers of preemptive priority. So there are less low priority customers from the infinite supply served at that node than if $\eta_k < \mu_k$ holds for this node $k \in V \cap D$.*

In the following example it can be seen that constraint (4.58) is compatible with the definition of a Jackson network with infinite supply.

Example 4.45. *Consider a Jackson network with two nodes where node 2 has an infinite supply of work. Both nodes may break down. Let $\eta_1 < \mu_1$ and $\eta_2 = \mu_2$ hold. Then the traffic equations in case of all nodes in up status are*

$$\eta_1 = \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1) \tag{4.60}$$
$$\eta_2 = \lambda_2 + \eta_1 r(1,2) + \mu_2 r(2,2). \tag{4.61}$$

Since $1 - r(1,1) > 0$ equation (4.60) *is equivalent to*

$$\eta_1 = \frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)}.$$

Plugging this into (4.61) *yields*

$$\eta_2 = \lambda_2 + \frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)} r(1,2) + \mu_2 r(2,2).$$

With $\eta_2 = \mu_2$ we get

$$\begin{aligned}&\mu_2 = \lambda_2 + \frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)} r(1,2) + \mu_2 r(2,2) \\ \Leftrightarrow\ &\mu_2(1 - r(2,2)) = \lambda_2 + \frac{\lambda_1 r(1,2)}{1 - r(1,1)} + \frac{\mu_2 r(2,1) r(1,2)}{1 - r(1,1)} \\ \Leftrightarrow\ &\mu_2\Big(1 - r(2,2) - \frac{r(2,1)r(1,2)}{1 - r(1,1)}\Big) = \lambda_2 + \frac{\lambda_1 r(1,2)}{1 - r(1,1)} \\ \Leftrightarrow\ &\mu_2\Big(\underbrace{(1 - r(1,1))(1 - r(2,2)) - r(2,1)r(1,2)}_{=r(1,0)(1-r(2,2))+r(1,2)r(2,0)>0}\Big) = \lambda_2(1 - r(1,1)) + \lambda_1 r(1,2) \\ \Leftrightarrow\ &\mu_2(r(2,0)(1 - r(1,1)) + r(2,1)r(1,0)) = \lambda_2(1 - r(1,1)) + \lambda_1 r(1,2).\end{aligned}$$

So if we require $\eta_1 < \mu_1$ and $\eta_2 = \mu_2$ to hold, μ_2 is

$$\mu_2 = \lambda_2 \underbrace{\frac{1 - r(1,1)}{r(2,0)(1-r(1,1)) + r(2,1)r(1,0)}}_{>0} + \lambda_1 \underbrace{\frac{r(1,2)}{r(2,0)(1-r(1,1)) + r(2,1)r(1,0)}}_{>0}.$$

Because of the irreducibility of the routing process, $\lambda_i > 0$ holds for at least one node, so the right-hand side of the equation is greater than zero. Thus constraint (4.58) is compatible with the definition of a two-node Jackson network with infinite supply.

4.6.3 Long-time behavior

Theorem 4.46. *Consider a Jackson network with unreliable nodes where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work as described in Definition 4.37. Nodes in $W := \tilde{J} \setminus V$ operate without infinite supply. Nodes in S are stable, $U = \tilde{J} \setminus S$ is the set of unstable nodes. Let $W \cap U = \emptyset$, so all nodes without infinite supply are stable. Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations (4.1).*
Nodes of the set $D \subseteq \tilde{J}$ may break down. In case of breakdowns customers are rerouted according to the **stalling** *regime. The availability-queue lengths process (Y, X) is the global network process.*

(i) For nodes without infinite supply, the joint marginal limiting distribution is:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i : i \in W) = \\ = \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i\in W} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{4.62}$$

for all $I \subseteq D$ and all $(n_i : i \in W) \in \mathbb{N}^{|W|}$, and this is a stationary distribution on W as well.

(ii) If the global network process is started with an initial distribution which has the marginal (4.62) on W, the arrival stream from $i \in W$ to $j \in V$ is Poisson with rate $\eta_i r(i,j)$ whenever all nodes are in up status. And these streams are independent given the nodes are up.

(iii) If the global network process is started with an initial distribution which has the marginal (4.62) on W, then the marginal limiting distribution for a stable node $i \in V$ with infinite supply and without immediate feedback (i.e., $r(i,i) = 0$) is:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i) = \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \tag{4.63}$$

for all $I \subseteq D$ and all $n_i \in \mathbb{N}$, if and only if $\eta_i < \mu_i$, and this is a one-dimensional stationary distribution as well.
If $\eta_i \geq \mu_i$ holds for this node $i \in V$, then for its limiting probability holds for all $I \subseteq D$ and all $n_i \in \mathbb{N}$:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i) = 0, \tag{4.64}$$

if the global network process is started with an initial distribution which has the marginal (4.62) on W.

In the following proof of Proposition 4.46(iii), showing the limiting probability (4.64) for unstable nodes with infinite supply is skipped, because it will be validated (implicitly) by the proof of Theorem 5.10 later on in Chapter 5, see p.121 for the theorem and pp.127-131 for its proof.

Proof of Proposition 4.46. (i): Consider the subset W of nodes without infinite supply. As long as all nodes are in up status ($I = \emptyset$), we have the following information about the subnetwork:

- All service times are exponentially distributed and the service discipline at all nodes is FCFS.
- Routing of customers is Markovian: A customer completing service at node $i \in W$ will either move to some node $j \in W$ with probability $r(i,j)$ or leave the subnetwork with probability $1 - \sum_{j\in W} r(i,j)$, which is non-zero for some $i \in W$ because of the routing matrix being irreducible for the global network on $\tilde{J}$.
- At each node $i \in W$, we have external arrivals from the source which is a Poisson stream with rate $\lambda_i \geq 0$. Furthermore all arrivals from nodes $j \in V$ with infinite supply into nodes $i \in W$ are a Poisson stream with rate $\mu_j r(j,i)$, see Theorem 4.3. The sum of independent Poisson streams is a Poisson stream, hence all arrivals from the outside of the subset W into each node $i \in W$ are a Poisson arrival stream with rate $\lambda_i + \sum_{j\in V} \mu_j r(j,i)$.
- All service and inter-arrival times at all nodes are independent of each other.

All these properties guarantee that the subnetwork W acts as a Jackson network with $|W|$ nodes where the source and sink is denoted by $\{0\} \cup V$, see Definition 1.1. The queue length process for the subnetwork W is denoted by $\tilde{X}$. This queue length process is coupled with an availability process Y which only depends on the interaction of the nodes in $D \subseteq \tilde{J}$ but not on their load. Whenever a node in D breaks down, stalling occurs, so all nodes go into a warm standby and all arrivals and services are interrupted until all nodes recur to the up status.
The network process $(Y, \tilde{X})$ is a Markov process on the state space $\mathcal{P}(D) \times \mathbb{N}^{|W|}$. The balance equations for the subnetwork W are given by

$$\begin{aligned}
&\pi(\emptyset, n_k : k \in W)\left(\sum_{i\in W}\Big(\lambda_i + \sum_{j\in V}\mu_j r(j,i)\Big) + \sum_{i\in W}\mu_i(1 - r(i,i))1_{\mathbb{N}_+}(n_i) + \sum_{\emptyset\neq I\subseteq D}\alpha(\emptyset, I)\right)\\
&= \sum_{i\in W}\pi(\emptyset, n_k : k \in W\setminus\{i\}, n_i - 1)\cdot\Big(\lambda_i + \sum_{j\in V}\mu_j r(j,i)\Big)\cdot 1_{\mathbb{N}_+}(n_i) +\\
&+ \sum_{i\in W}\pi(\emptyset, n_k : k \in W\setminus\{i\}, n_i + 1)\cdot\mu_i\Big(1 - \sum_{j\in W} r(i,j)\Big) +\\
&+ \sum_{i\in W}\sum_{j\in W\setminus\{i\}}\pi(\emptyset, n_k \in W\setminus\{i,j\}, n_i + 1, n_j - 1)\cdot\mu_i r(i,j)\cdot 1_{\mathbb{N}_+}(n_j)\\
&+ \sum_{\emptyset\neq I\subseteq D}\pi(I, n_k : k \in W)\cdot\beta(I, \emptyset)
\end{aligned} \tag{4.65}$$

for all $(\emptyset, n_k : k \in W) \in \{\emptyset\} \times \mathbb{N}^{|W|}$ and

$$\pi(I, n_k : k \in W) \left(\sum_{I \subset H \subseteq D} \alpha(I, H) + \sum_{\emptyset \neq K \subset I} \beta(I, K) \right)$$
$$= \sum_{\emptyset \neq K \subset I} \pi(K, n_k : k \in W) \cdot \alpha(K, I) + \sum_{I \subset H \subseteq D} \pi(H, n_k : k \in W) \cdot \beta(H, I) \qquad (4.66)$$

for all $(I, n_k : k \in W) \in \mathcal{P}(D) \times \mathbb{N}^{|W|}$ with $I \neq \emptyset$.

We have to show, that (4.62) solves these equations. In the following we denote

$$\hat{\pi}(I, n_k : k \in W) := \frac{A(I)}{B(I)} \prod_{i \in W} \left(\frac{\eta_i}{\mu_i} \right)^{n_i}$$

for all $(I, n_k : k \in W) \in \mathcal{P}(D) \times \mathbb{N}^{|W|}$, which is (4.62) before normalization, and plug it into the above balance equations for $\pi(I, n_k : k \in W)$.

In the first equation (4.65) the term

$$\hat{\pi}(\emptyset, n_k : k \in W)\alpha(\emptyset, I) = \hat{\pi}(\emptyset, n_k : k \in W)A(I) = \hat{\pi}(I, n_k : k \in W)B(I)$$

on the left-hand side is equal to the term $\hat{\pi}(I, n_k : k \in W)\beta(I, \emptyset) = \hat{\pi}(I, n_k : k \in W)B(I)$ on the right-hand side for each $\emptyset \neq I \subseteq D$. The remainder of (4.65) is the global balance equation of a classical Jackson network which has the solution

$$\hat{\pi}(\emptyset, n_k : k \in W) := \hat{\pi}(n_k : k \in W) = \prod_{i \in W} \left(\frac{\eta_i}{\mu_i} \right)^{n_i},$$

see Theorem 1.10.

Consider the second equation (4.66) for some fixed $I \neq \emptyset$. For any $K \subset I$, $K \neq \emptyset$, the term

$$\hat{\pi}(I, n_k : k \in W)\beta(I, K) = \hat{\pi}(I, n_k : k \in W)\frac{B(I)}{B(K)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat{\pi}(K, n_k : k \in W)\alpha(K, I) = \hat{\pi}(K, n_k : k \in W)\frac{A(I)}{A(K)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(I)}{B(K)}.$$

Moreover, for any $I \subset H \subseteq D$ the term

$$\hat{\pi}(I, n_k : k \in W)\alpha(I, H) = \hat{\pi}(I, n_k : k \in W)\frac{A(H)}{A(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat{\pi}(H, n_k : k \in W)\beta(H, I) = \hat{\pi}(H, n_k : k \in W)\frac{B(H)}{B(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the right-hand side.

The last step of the proof of (i) is done by defining the normalizing constant as

$$\left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)\sum_{(n_i:i\in W)\in\mathbb{N}^{|W|}}\prod_{i\in W}\left(\frac{\eta_i}{\mu_i}\right)^{n_i},$$

which is finite, because $\eta_i<\mu_i$ holds for all $i\in W$.

(ii): It is well known that ergodic Jackson networks have, in equilibrium, Poisson departure streams from node i to the sink with some rate $\tilde{\eta}_i\tilde{r}(i,0)$, see [Mel79b, Example 7.1]. From the proof of (i), we know that the subset W behaves like an ergodic Jackson network with unreliable nodes of its own with $\tilde{\lambda}_i := \lambda_i+\sum_{j\in V}\mu_j r(j,i)$ and

$$\tilde{\eta}_i\tilde{r}(i,0)=\eta_i\Big(1-\sum_{j\in W}r(i,j)\Big)=\eta_i\Big(r(i,0)+\sum_{j\in V}r(i,j)\Big).$$

Hence, if the subnetwork W is in equilibrium, as long as all nodes are in up status, departures to the sink from nodes $i\in W$ are Poisson streams with rate $\eta_i r(i,0)$ and departures from $i\in W$ to any node $j\in V$ are also Poisson streams with rate $\eta_i r(i,j)$, because a portion of $\frac{r(i,j)}{r(i,0)+\sum_{j\in V}r(i,j)}$ of the departure stream from node $i\in W$ is directed to $j\in V$.

(iii): Consider some fixed node $i\in V$ with infinite supply and without immediate feedback (i.e., $r(i,i)=0$). As long as all nodes $j\in\tilde{J}$ are in up status, we have the following information about the node:

- Service times are exponentially distributed with rate μ_i, the service discipline is FCFS.
- Routing of customers is Markovian: A customer arriving at node i, waits in the line until he gets service, after the service he proceeds to another node $j\in\tilde{J}\setminus\{i\}$ with probability $r(i,j)$ or leaves the network with probability $r(i,0)$. Thus, the customer leaves node i after completed service with probability

$$r(i,0)+\sum_{j\in\tilde{J}\setminus\{i\}}r(i,j)=1-r(i,i)=1.$$

- External arrivals (from the artificial node 0) are Poisson with rate $\lambda_i\geq 0$. From (ii) it follows directly that, if the global network process is started with an initial distribution which has the marginal (4.62) on W, the arrival streams at node $i\in V$ from nodes $j\in W$ are Poisson with rate $\eta_j r(j,i)$. Arrival streams from nodes $j\in V\setminus\{i\}$ are Poisson streams with rate $\mu_j r(j,i)$, see Theorem 4.38. All these Poisson streams are independent from each other, so the arrival stream at node $i\in V$ is a Poisson process with rate

$$\hat{\lambda}_i := \lambda_i+\sum_{j\in W}\eta_j r(j,i)+\sum_{j\in V\setminus\{i\}}\mu_j r(j,i).$$

- All service and inter-arrival times constitute a set of independent random variables.

Thus, if the global network process is started with an initial distribution which has the marginal (4.14) on W, node $i \in V$ is an $(M/M/1)-$ system of its own. Let $\hat{X}$ denote the queue length process of $i \in V$. Then the traffic equation is

$$\hat{\eta}_i = \hat{\lambda}_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V \setminus \{i\}} \mu_j r(j,i),$$

so $\hat{\eta}_i = \eta_i$ holds, see (4.1) with $r(i,i) = 0$. The queue length process is coupled with an availability process Y on $\mathcal{P}(D)$, $D \subseteq \tilde{J}$, where breakdown and repair of nodes only depend on the interaction of the nodes but not on their queue length. Whenever a node in D breaks down, stalling occurs, so all nodes go into a warm standby and all arrivals and services are interrupted until all nodes recur to the up status.
The network process $(Y, \hat{X})$ is a Markov process on the state space $\mathcal{P}(D) \times \mathbb{N}$. The balance equations are

$$\pi_i(\emptyset, n_i)\Big(\hat{\lambda}_i + \mu_i 1_{\mathbb{N}_+}(n_i) + \sum_{\emptyset \neq I \subseteq D} \alpha(\emptyset, I)\Big)$$
$$= \pi_i(\emptyset, n_i - 1) \cdot \hat{\lambda}_i \cdot 1_{\mathbb{N}_+}(n_i) + \pi_i(\emptyset, n_i + 1) \cdot \mu_i + \sum_{\emptyset \neq I \subseteq D} \pi_i(I, n_i) \cdot \beta(I, \emptyset) \tag{4.67}$$

for all $(\emptyset, n_i) \in \{\emptyset\} \times \mathbb{N}$ and

$$\pi_i(I, n_i)\Big(\sum_{I \subset H \subseteq D} \alpha(I, H) + \sum_{\emptyset \neq K \subset I} \beta(I, K)\Big)$$
$$= \sum_{\emptyset \neq K \subset I} \pi_i(K, n_i) \cdot \alpha(K, I) + \sum_{I \subset H \subseteq D} \pi_i(H, n_i) \cdot \beta(H, I) \tag{4.68}$$

for all $(I, n_i) \in \mathcal{P}(D) \times \mathbb{N}$ with $I \neq \emptyset$.

We have to show, that (4.63) solves these equations. In the following we set

$$\hat{\pi}_i(I, n_i) := \frac{A(I)}{B(I)} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

for all $(I, n_i) \in \mathcal{P}(D) \times \mathbb{N}$ as the nonnormalized proposed solution density.

In the first equation (4.67) the term

$$\hat{\pi}_i(\emptyset, n_i)\alpha(\emptyset, I) = \hat{\pi}_i(\emptyset, n_i)A(I) = \hat{\pi}_i(I, n_i)B(I)$$

on the left-hand side is equal to the term $\hat{\pi}_i(I, n_i)\beta(I, \emptyset) = \hat{\pi}_i(I, n_i)B(I)$ on the right-hand side for each $\emptyset \neq I \subseteq D$. The remainder of (4.67) is the global balance equation of an $(M/M/1)$ system which has the solution

$$\hat{\pi}_i(\emptyset, n_i) := \hat{\pi}_i(n_i) = \left(\frac{\eta_i}{\mu_i}\right)^{n_i},$$

since $\hat{\lambda}_i = \eta_i$ holds.

Consider the second equation (4.68) for some fixed $I \neq \emptyset$. For any $K \subset I$, $K \neq \emptyset$, the term

$$\hat{\pi}_i(I, n_i)\beta(I, K) = \hat{\pi}_i(I, n_i)\frac{B(I)}{B(K)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat{\pi}_i(K, n_i)\alpha(K, I) = \hat{\pi}_i(K, n_i)\frac{A(I)}{A(K)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(I)}{B(K)}.$$

Moreover, for any $I \subset H \subseteq D$ the term

$$\hat{\pi}_i(I, n_i)\alpha(I, H) = \hat{\pi}_i(I, n_i)\frac{A(H)}{A(I)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat{\pi}_i(H, n_i)\beta(H, I) = \hat{\pi}_i(H, n_i)\frac{B(H)}{B(I)} = \hat{\pi}_i(\emptyset, n_i)\frac{A(H)}{B(I)}$$

on the right-hand side.

The last step of the proof of (iii) is done by defining the normalizing constant as

$$\left(\sum_{K \subseteq D} \frac{A(K)}{B(K)}\right) \sum_{n \in \mathbb{N}} \left(\frac{\eta_i}{\mu_i}\right)^n,$$

then the solution is a probability measure, if and only if $\eta_i < \mu_i$ holds. □

Theorem 4.47. *Consider a Jackson network with unreliable nodes where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work as described in Definition 4.37. Nodes in $W := \tilde{J} \setminus V$ operate without infinite supply. Nodes in S are stable, $U = \tilde{J} \setminus S$ is the set of unstable nodes. Let $W \cap U = \emptyset$, so all nodes without infinite supply are stable. Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations* (4.1).
Nodes of the set $D \subseteq \tilde{J}$ may break down. In case of breakdowns customers are rerouted according to the **blocking rs-rd** *regime or the* **skipping** *regime. If blocking rs-rd is in force we require the reversibility-constraints* (4.51) *and* (4.52). *If skipping is in force, let* (4.58) *hold.*
The availability-queue lengths process (Y, X) is the global network process. For nodes without infinite supply, the joint marginal limiting distribution is:

$$\lim_{t \to \infty} P(Y(t) = I; X_i(t) = n_i : i \in W) =$$
$$= \left(\sum_{K \subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i \in W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \quad (4.69)$$

for all $I \subseteq D$ and all $(n_i : i \in W) \in \mathbb{N}^{|W|}$, and this is a stationary distribution on W as well.

Proof. Consider the subset W of nodes without infinite supply. For any subset $I \subseteq D$ of broken down nodes, we have the following facts for the subset $W \setminus I$ which remain in force as long as I is unchanged:

- All service times of all up-nodes are exponentially distributed and the service discipline at all nodes is FCFS.
- Routing of customers is Markovian: A customer completing service at node $i \in W \setminus I$ will either move to some node $j \in W \setminus I$ with probability $r^I(i,j)$ or leave the subnetwork with probability $1 - \sum_{j \in W \setminus I} r^I(i,j)$.
- At each node $i \in W \setminus I$, we have external arrivals from the source which are independent Poisson streams with rate $\lambda_i^I \geq 0$. Furthermore all arrivals from nodes $j \in V \setminus I$ with infinite supply into nodes $i \in W \setminus I$ are independent Poisson streams at rate $\mu_j r^I(j,i)$, see Theorem 4.38. The sum of independent Poisson streams is a Poisson stream, hence the arrival stream from the outside of the subset $W \setminus I$ into each node $i \in W \setminus I$ is a Poisson process with rate $\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j,i)$.
- All service times and all interarrival times are independent of each other.

Let $\tilde{X} := ((\tilde{X}_i(t) : i \in W \setminus I) : t \in \mathbb{R}_+)$ be the queueing process of this subnetwork. The process is supplemented with a Markov process $Y = (Y(t) : t \in \mathbb{R}_+)$ which describes the availability status of the nodes and therefore gives information on how long the network process on the subnet $W \setminus I$ lives until it jumps to the next Markov process on some randomly chosen subnet $W \setminus K$, $K \subseteq D$. Rerouting is according to the blocking rs-rd regime (skipping, resp.).
The joint availability-queue length process $(Y, \tilde{X}_i : i \in W)$ has the balance equations

$$\begin{aligned}
&\pi(I, n_k : k \in W) \cdot \Big(\sum_{i \in W \setminus I} \Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j,i)\Big) + \\
&\quad + \sum_{i \in W \setminus I} \mu_i (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) + \sum_{I \subset H \subseteq D} \alpha(I,H) + \sum_{K \subset I \subseteq D} \beta(I,K)\Big) \\
&= \sum_{i \in W \setminus I} \pi(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j,i)\Big) \cdot 1_{\mathbb{N}_+}(n_i) \\
&\quad + \sum_{i \in W \setminus I} \pi(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i,j)\Big) + \\
&\quad + \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \pi(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j) \\
&\quad + \sum_{K \subset I \subseteq D} \pi(K, n_k : k \in W) \cdot \alpha(K,I) \\
&\quad + \sum_{I \subset H \subseteq D} \pi(H, n_k : k \in W) \cdot \beta(H,I)
\end{aligned} \tag{4.70}$$

for all $(I, n_i : i \in W) \in \mathcal{P}(D) \times \mathbb{N}^{|W|}$.

We have to show that the distribution given by (4.69) solves equation (4.70) for all $(n_i : i \in W) \in \mathbb{N}^{|W|}$ and all $I \subseteq D$. In the following we set

$$\hat{\pi}(I, n_k : k \in W) := \frac{A(I)}{B(I)} \prod_{i \in W} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

for all $(n_i : i \in W) \in \mathbb{N}^{|W|}$ and all $I \subseteq D$, and consider equation (4.70) for some fixed $I \subseteq D$.

For any $K \subset I$, $K \neq \emptyset$, the term

$$\hat{\pi}(I, n_k : k \in W)\beta(I, K) = \hat{\pi}(I, n_k : k \in W)\frac{B(I)}{B(K)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(I)}{B(K)}$$

on the left-hand side is equal to the term on the right-hand side

$$\hat{\pi}(K, n_k : k \in W)\alpha(K, I) = \hat{\pi}(K, n_k : k \in W)\frac{A(I)}{A(K)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(I)}{B(K)}.$$

Moreover, for any $I \subset H \subseteq D$ the term

$$\hat{\pi}(I, n_k : k \in W)\alpha(I, H) = \hat{\pi}(I, n_k : k \in W)\frac{A(H)}{A(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the left-hand side is equal to the term

$$\hat{\pi}(H, n_k : k \in W)\beta(H, I) = \hat{\pi}(H, n_k : k \in W)\frac{B(H)}{B(I)} = \hat{\pi}(\emptyset, n_k : k \in W)\frac{A(H)}{B(I)}$$

on the right-hand side. The remainder of (4.70) is

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i \in W \setminus I} \Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j, i)\Big) + \sum_{i \in W \setminus I} \mu_i (1 - r^I(i, i)) \cdot 1_{\mathbb{N}_+}(n_i)\Big) \\
&= \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j r^I(j, i)\Big) \cdot 1_{\mathbb{N}_+}(n_i) + \\
&+ \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i, j)\Big) + \\
&+ \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i, j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i, j) \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned} \tag{4.71}$$

With $\eta_i^I = \lambda_i^I + \sum_{j \in W \setminus I} \eta_j^I r^I(j, i) + \sum_{j \in V \setminus I} \mu_j r^I(j, i)$ (see (4.50)) this is equivalent to

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i \in W \setminus I} \Big(\eta_i^I - \sum_{j \in W \setminus I} \eta_j^I r^I(j, i)\Big) + \sum_{i \in W \setminus I} \mu_i (1 - r^I(i, i)) \cdot 1_{\mathbb{N}_+}(n_i)\Big) \\
&= \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\eta_i^I - \sum_{j \in W \setminus I} \eta_j^I r^I(j, i)\Big) \cdot 1_{\mathbb{N}_+}(n_i) + \\
&+ \sum_{i \in W \setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j \in W \setminus I} r^I(i, j)\Big) + \\
&+ \sum_{i \in W \setminus I} \sum_{j \in W \setminus I, j \neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i, j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i, j) \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned}$$

Under the required condition of either (4.51) and (4.52) in case of blocking rs-rd or (4.58) in case of skipping holds $\eta_i = \eta_i^I$ for all $i \in W \setminus I$ and all $I \subseteq D$ for the respective reduced traffic equations. Therefore from Lemma 4.40 or Lemma 4.43 respectively this is equivalent to

$$\begin{aligned}
&\hat{\pi}(I, n_k : k \in W) \cdot \Big(\sum_{i\in W\setminus I} \Big(\eta_i - \sum_{j\in W\setminus I} \eta_j r^I(j,i)\Big) + \sum_{i\in W\setminus I} \mu_i (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) \Big) \\
&= \sum_{i\in W\setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i - 1) \cdot \Big(\eta_i - \sum_{j\in W\setminus I} \eta_j r^I(j,i)\Big) \cdot 1_{\mathbb{N}_+}(n_i) + \\
&+ \sum_{i\in W\setminus I} \hat{\pi}(I, n_k : k \in W \setminus \{i\}, n_i + 1) \cdot \mu_i \Big(1 - \sum_{j\in W\setminus I} r^I(i,j)\Big) + \\
&+ \sum_{i\in W\setminus I} \sum_{j\in W\setminus I, j\neq i} \hat{\pi}(I, n_k : k \in W \setminus \{i,j\}, n_i + 1, n_j - 1) \cdot \mu_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned}$$

Plugging in $\hat{\pi}(I, n_k : k \in W) = \frac{A(I)}{B(I)} \prod_{i\in W} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$ yields

$$\begin{aligned}
&\sum_{i\in W\setminus I} \Big(\eta_i - \sum_{j\in W\setminus I} \eta_j r^I(j,i)\Big) + \sum_{i\in W\setminus I} \mu_i (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) \\
&= \sum_{i\in W\setminus I} \frac{\mu_i}{\eta_i} \cdot \Big(\eta_i - \sum_{j\in W\setminus I} \eta_j r^I(j,i)\Big) \cdot 1_{\mathbb{N}_+}(n_i) + \sum_{i\in W\setminus I} \frac{\eta_i}{\mu_i} \cdot \mu_i \Big(1 - \sum_{j\in W\setminus I} r^I(i,j)\Big) + \\
&+ \sum_{i\in W\setminus I} \sum_{j\in W\setminus I, j\neq i} \frac{\eta_i}{\mu_i} \frac{\mu_j}{\eta_j} \mu_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned}$$

$$\Leftrightarrow 0 = -\sum_{i\in W\setminus I} \frac{\mu_i}{\eta_i} \cdot \sum_{j\in W\setminus I, j\neq i} \eta_j r^I(j,i) \cdot 1_{\mathbb{N}_+}(n_i) + \sum_{i\in W\setminus I} \sum_{j\in W\setminus I, j\neq i} \frac{\mu_j}{\eta_j} \eta_i r^I(i,j) \cdot 1_{\mathbb{N}_+}(n_j).$$

Thus

$$\hat{\pi}(I, n_k : k \in W) = \frac{A(I)}{B(I)} \prod_{i\in W} \left(\frac{\eta_i}{\mu_i}\right)^{n_i}$$

solves the balance equations (4.70). The last step of proving (i) is by defining the normalizing constant as

$$\sum_{K\subseteq D} \frac{A(K)}{B(K)} \sum_{(n_i : i\in W)\in \mathbb{N}^{|W|}} \prod_{i\in W} \left(\frac{\eta_i}{\mu_i}\right)^{n_i},$$

which makes the solution a probability solution because $\eta_i < \mu_i$ holds for all $i \in W$. □

The proofs of Theorem 4.46(i) and Theorem 4.47 justify the following

Corollary 4.48. *In the settings of Theorem 4.46 and Theorem 4.47, the process*

$$(Y, X_W) := (Y, X_i : i \in W)$$

is an ergodic homogeneous Markov process of its own.

Remark 4.49. *In the setting of Theorem 4.47 a statement as in Theorem 4.46(iii) cannot be proved. This is due to the definitions of rerouting according to skipping and blocking rs-rd (Definition 2.7 and 2.8). Whenever nodes in $I \neq \emptyset$ are broken down, immediate feedback may emerge at nodes with infinite supply (where $r(i,i) = 0$) because of the rerouting regime, i.e., $r^I(i,i) > 0$ at nodes $i \in V \setminus I$, if $r(i,j) > 0$ for at least one $j \in I$. But immediate feedback at nodes with infinite supply leads to a situation discussed in Remark 4.18.*

4.6.4 Computation of availability and performance measures

Due to the load-independent breakdown and repair rates as in Example 2.2, the availability process Y in Theorem 4.46 and Theorem 4.47 is an ergodic Markov process of its own, which has the unique limiting and stationary distribution (2.10), as it was already shown in Chapter 2, Section 2.4. Here Proposition 2.18 applies to compute stationary point-availability and Example 2.19 to estimate costs involved with breakdowns of nodes, as well.

The computation of the throughput is a little bit different compared to the Chapter 2, Section 2.4, due to the infinite supply. We therefore define the stationary throughput of a subnetwork.

Definition 4.50. *Consider a Jackson network with infinite supply where nodes without infinite supply are stable and nodes in $D \subseteq \tilde{J}$ are unreliable. The stationary throughput TH_i of a node $i \in W$ without infinite supply is*

$$TH_i = \sum_{(I,n_j:j\in W)\in\mathcal{P}(D)\times\mathbb{N}^{|W|}} \pi_W(I, n_j : j \in W)\mu_i^I 1_{\mathbb{N}_+}(n_i),$$

the stationary throughput TH_j of a stable node $j \in V$ with infinite supply is

$$TH_j = \sum_{(I,n_j)\in\mathcal{P}(D)\times\mathbb{N}} \pi_j(I, n_j)\mu_i^I.$$

The stationary throughput TH_W of the subnetwork of nodes without infinite supply is

$$TH_W = \sum_{I\subseteq D}\sum_{i\in W\setminus I} TH_i r^I(i,0)$$

and the stationary throughput TH_S of the subnetwork of stable nodes is

$$TH_S = TH_W + \sum_{I\subseteq D}\sum_{j\in(V\cap S)\setminus I} TH_j r^I(j,0).$$

Proposition 4.51. *Consider a Jackson network where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work and where nodes in $D \subseteq \tilde{J}$ are unreliable as in Def. 4.37. Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations* (4.5). *$S := \{i \in \tilde{J} : \eta_i < \mu_i\}$ is the set of stable nodes, nodes in $U := \tilde{J} \setminus S$ are unstable. We assume that all nodes without infinite supply of work are stable, i.e., $W \cap U = \emptyset$. So in this situation the traffic equations* (4.5) *reduce to* (4.1).

In case of breakdowns **stalling** *occurs according to Definition 2.6. Then the stationary throughput at nodes* $i \in W$ *without infinite supply is* $TH_i = \eta_i \cdot \pi(\emptyset)$. *Let* $r(i,i) = 0$ *hold for all* $i \in V \cap S$ *then the stationary throughput at stable nodes* $i \in V \cap S$ *with infinite supply is* $TH_i = \mu_i \cdot \pi(\emptyset)$ *and the throughput of the stable subnetwork is*

$$TH_S = \pi(\emptyset) \cdot \Big(\lambda - \sum_{i \in V}(\eta_i - \mu_i) - \sum_{j \in V \cap U} \mu_j r(j,0)\Big).$$

Proof. The proof uses the results of Theorem 4.46. For $i \in W$ we get

$$\begin{aligned} TH_i &= \sum_{(I,n_j:j\in W)\in\mathcal{P}(D)\times\mathbb{N}^{|W|}} \pi_W(I, n_j : j \in W) \underbrace{\mu_i^I}_{=0 \text{ if } I\neq\emptyset} 1_{\mathbb{N}_+}(n_i) = \sum_{n_i\in\mathbb{N}} \pi_i(\emptyset, n_i)\mu_i 1_{\mathbb{N}_+}(n_i) \\ &= \sum_{n_i\in\mathbb{N}} \pi_i(\emptyset, n_i+1)\mu_i = \sum_{n_i\in\mathbb{N}} \pi(\emptyset)\left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i+1} \mu_i = \eta_i \cdot \pi(\emptyset), \end{aligned}$$

and for $j \in V \cap S$:

$$TH_j = \sum_{(I,n_j)\in\mathcal{P}(D)\times\mathbb{N}} \pi_j(I, n_j) \underbrace{\mu_i^I}_{=0 \text{ if } I\neq\emptyset} = \underbrace{\sum_{n_j\in\mathbb{N}} \pi_j(\emptyset, n_j)}_{=\pi(\emptyset)} \mu_i = \mu_i \cdot \pi(\emptyset).$$

The throughput of the stable subnetwork is

$$\begin{aligned} TH_S &= \sum_{i\in S} TH_i r(i,0) = \sum_{i\in W} \eta_i r(i,0) \cdot \pi(\emptyset) + \sum_{j\in V\cap S} \mu_j r(j,0) \cdot \pi(\emptyset) \\ &\overset{\text{(Remark 4.5)}}{=} \Big(\lambda - \sum_{i\in V}(\eta_i - \mu_i) - \sum_{j\in V\cap U} \mu_j r(j,0)\Big) \cdot \pi(\emptyset). \end{aligned}$$

□

Proposition 4.52. *Consider a Jackson network where nodes in* $V \subseteq \tilde{J}$ *have an infinite supply of work and where nodes in* $D \subseteq \tilde{J}$ *are unreliable as in Def. 4.37. Denote by* $\eta = (\eta_1, ..., \eta_J)$ *the unique solution of the traffic equations* (4.5). $S := \{i \in \tilde{J} : \eta_i < \mu_i\}$ *is the set of stable nodes, nodes in* $U := \tilde{J} \setminus S$ *are unstable. We assume that all nodes without infinite supply of work are stable, i.e.,* $W \cap U = \emptyset$. *So in this situation* (4.5) *reduces to* (4.1).

(i) If the rerouting is according to **blocking rs-rd** *and if* (4.51), (4.52) *and* (4.53) *hold, the stationary throughput at a node* $j \in W$ *is* $TH_j = \eta_j \cdot \sum_{I\subseteq D, j\notin I} \pi(I)$ *and the stationary throughput of the subnetwork* W *is*

$$TH_W = \sum_{I\subseteq D} \pi(I) \cdot \Big(\sum_{i\in\tilde{J}\setminus I} \lambda_i - \sum_{i\in V\setminus I} \mu_i r(i,0) - \sum_{i\in V\setminus I} (\eta_i - \mu_i)\Big).$$

(ii) If the rerouting is according to **skipping** *and if* (4.58) *holds, the stationary throughput at a node* $j \in W$ *is* $TH_j = \eta_j \cdot \sum_{I\subseteq D, j\notin I} \pi(I)$ *and the stationary throughput of the subnetwork* W *is*

$$TH_W = \sum_{I\subseteq D} \pi(I) \cdot \Big(\lambda(1 - r^I(0,0)) - \sum_{i\in V\setminus I} \mu_i r^I(i,0) - \sum_{i\in V\setminus I} (\eta_i - \mu_i)\Big).$$

Proof. The proof uses the results of Theorem 4.47. For nodes $i \in W$ we get

$$\begin{aligned} TH_i &= \sum_{(I,n_j:j\in W)\in\mathcal{P}(D)\times\mathbb{N}^{|W|}} \pi_W(I,n_j:j\in W)\mu_i^I 1_{\mathbb{N}_+}(n_i) \\ &= \sum_{(I,n_i)\in\mathcal{P}(D)\times\mathbb{N}} \pi_i(I,n_i) \underbrace{\mu_i^I}_{=\mu_i \text{ if } i\notin I} 1_{\mathbb{N}_+}(n_i) = \sum_{I\subseteq D, i\notin I} \sum_{n_i\in\mathbb{N}} \pi_i(I,n_i+1)\mu_i \\ &= \sum_{I\subseteq D, i\notin I} \pi(I) \sum_{n_i\in\mathbb{N}} \left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i+1} \mu_i = \eta_i \cdot \sum_{I\subseteq D, i\notin I} \pi(I), \end{aligned}$$

The throughput of the subnetwork W is

$$TH_W = \sum_{I\subseteq D} \sum_{i\in W\setminus I} TH_i r^I(i,0) = \sum_{I\subseteq D} \pi(I) \sum_{i\in W\setminus I} \eta_i r^I(i,0).$$

Since $\eta_i = \eta_i^I$ holds for all $i \in \tilde{J}\setminus I$ and $I \subseteq D$ (see Lemma 4.40 or Lemma 4.43 resp.), η_i solves

$$\eta_i = \lambda_i^I + \sum_{j\in W\setminus I} \eta_j r^I(j,i) + \sum_{j\in V\setminus I} \mu_j r^I(j,i).$$

Summing over all $i \in \tilde{J}\setminus I$ on both sides yields

$$\begin{aligned} &\sum_{i\in\tilde{J}\setminus I} \eta_i = \sum_{i\in\tilde{J}\setminus I} \lambda_i^I + \sum_{i\in\tilde{J}\setminus I} \sum_{j\in W\setminus I} \eta_j r^I(j,i) + \sum_{i\in\tilde{J}\setminus I} \sum_{j\in V\setminus I} \mu_j r^I(j,i) \\ \Leftrightarrow &\sum_{i\in\tilde{J}\setminus I} \lambda_i^I = \sum_{i\in W\setminus I} \eta_i r^I(i,0) + \sum_{i\in V\setminus I} \mu_i r^I(i,0) + \sum_{i\in V\setminus I} (\eta_i - \mu_i) \\ \Leftrightarrow &\sum_{i\in W\setminus I} \eta_i r^I(i,0) = \sum_{i\in\tilde{J}\setminus I} \lambda_i^I - \sum_{i\in V\setminus I} \mu_i r^I(i,0) - \sum_{i\in V\setminus I} (\eta_i - \mu_i). \end{aligned} \tag{4.72}$$

In case of (i) equation (4.72) is equivalent to

$$\sum_{i\in W\setminus I} \eta_i r^I(i,0) = \sum_{i\in\tilde{J}\setminus I} \lambda_i - \sum_{i\in V\setminus I} \mu_i r(i,0) - \sum_{i\in V\setminus I} (\eta_i - \mu_i).$$

In case of (ii) equation (4.72) is equivalent to

$$\sum_{i\in W\setminus I} \eta_i r^I(i,0) = \lambda \underbrace{\sum_{i\in\tilde{J}\setminus I} r^I(0,i)}_{=(1-r^I(0,0))} - \sum_{i\in V\setminus I} \mu_i r^I(i,0) - \sum_{i\in V\setminus I} (\eta_i - \mu_i).$$

□

Chapter 5

Jackson networks with breakdowns: The non-ergodic case

5.1 Introduction

Our aim is to study Jackson networks with unreliable stations which are not ergodic due to overload at some stations: Is it possible that in such a network subnetworks stabilize? We shall prove that the answer is positive.

Our setting is described in Definition 2.9. We here assume that the network process (Y, X) is not ergodic and that instability of nodes results from overload as described and investigated in Section 1.3 in the sense of [GM84].

For maintaining tractability of the model we additionally assume that breakdown and repair rates are load-independent. Although it would be desirable to weaken this assumption we note that there is a great variety of correlation structures for the breakdown and repair process included in this class of transition rates.

5.2 The traffic equations

As in the previous chapters we have to define the general traffic equations for each availability status of the availability–queue-length process first. The following definition is established analogously to (1.6).

Definition 5.1. *The general traffic equations for Jackson networks with unreliable nodes as in Definition 2.9 are:*

- *In case of stalling according to Definition 2.6:*

$$\eta_i = \lambda_i + \sum_{j \in \tilde{J}} \min(\eta_j, \mu_j) r(j, i), \ \ i \in \tilde{J}, \tag{5.1}$$

  as long as all nodes are in up status ($I = \emptyset$). Otherwise $\eta_i^I = 0$ for all $i \in \tilde{J}$. (Note that (5.1) *is* (1.6) *but is valid only when no repair is ongoing.)*

- *In case of rerouting according to skipping as in Definition 2.7 or to blocking rs-rd as in Definition 2.8:*

$$\eta_i^I = \lambda_i^I + \sum_{j\in\tilde{J}\setminus I} \min(\eta_j^I, \mu_j^I) r^I(j,i), \; i \in \tilde{J} \setminus I, \tag{5.2}$$

for all $I \subseteq D$. If no node is under repair, i.e., $I = \emptyset$, the traffic equations (5.2) *are* (1.6)*.*

The traffic equations for some $I \subseteq D$ remain valid only as long as the availability status is unchanged. Whenever the availability status of the system changes, the traffic equations are adapted according to the new set of broken down nodes. Thus each traffic equation (5.2) may have different solutions for different I. The following two lemmata show under which constraints the solution of the traffic equation (5.2) remains the same for all $I \subseteq D$. Lemma 5.2 and Lemma 5.5 resemble Lemma 4.40 and Lemma 4.43 for Jackson networks with infinite supply, this is due to the similarity of the traffic equations. Therefore the lemmata can be proved nearly the same way.

Lemma 5.2. *Consider a Jackson network where nodes in $D \subseteq \tilde{J}$ are unreliable. Let $\eta = (\eta_j : j \in \tilde{J})$ be the unique solution of the general traffic equations* (1.6)*. Denote by $S = \{i : \eta_i < \mu_i\} \subseteq \tilde{J}$ the set of stable nodes in the network and $U := \tilde{J} \setminus S$ the set of unstable nodes in the network.*
In case of breakdowns of nodes we assume that customers are rerouted according to the **blocking rs-rd** *regime. If the following reversibility constraints hold:*

$$\eta_i r(i,j) = \eta_j r(j,i) \quad \forall i,j \in S, \tag{5.3}$$
$$\eta_i r(i,j) = \mu_j r(j,i) \quad \forall i \in S, j \in U, \tag{5.4}$$
$$\mu_i r(i,j) = \mu_j r(j,i) \quad \forall i,j \in U, \tag{5.5}$$

then for the solution $\eta^I := (\eta_i^I : i \in \tilde{J} \setminus I)$ of (5.2) *holds $\eta_i^I = \eta_i$ for all $i \in \tilde{J} \setminus I$ and all $I \subseteq D$.*

Proof. (i): We make the ansatz $\eta_i = \eta_i^I$ for all $i \in S \setminus I$ and all $I \subseteq D$ in (5.2). We then obtain with the solution η_i of the traffic equations (1.6) for any $I \subseteq D$: $\forall i \in S \setminus I$

$$\begin{aligned}
\eta_i &= \lambda_i^I + \sum_{j\in\tilde{J}\setminus I} \min(\eta_j, \mu_j) r^I(j,i) \\
&= \lambda_i^I + \sum_{j\in S\setminus I} \eta_j r^I(j,i) + \sum_{j\in U\setminus I} \mu_j r^I(j,i) \\
&\overset{\text{Def. 2.8}}{=} \lambda_i + \eta_i\Big(r(i,i) + \sum_{k\in I} r(i,k)\Big) + \sum_{j\in S\setminus I, j\neq i} \eta_j r(j,i) + \sum_{j\in U\setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{k\in I\cap S} \underbrace{\eta_i r(i,k)}_{\overset{(5.3)}{=} \eta_k r(k,i)} + \sum_{k\in I\cap U} \underbrace{\eta_i r(i,k)}_{\overset{(5.4)}{=} \mu_k r(k,i)} + \sum_{j\in S\setminus I} \eta_j r(j,i) + \sum_{j\in U\setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j\in S} \eta_j r(j,i) + \sum_{j\in U} \mu_j r(j,i) = (1.6).
\end{aligned}$$

(ii): For any $I \subseteq D$ holds $\forall i \in U \setminus I$:

$$\begin{aligned}
\eta_i^I &\overset{(5.2)}{=} \lambda_i^I + \sum_{j \in S \setminus I} \underbrace{\eta_j^I}_{\overset{(i)}{=}\eta_j} r^I(j,i) + \sum_{j \in U \setminus I} \mu_j r^I(j,i) \\
&\overset{\text{Def. 2.8}}{=} \lambda_i + \sum_{j \in S \setminus I} \eta_j r(j,i) + \mu_i \Big(r(i,i) + \sum_{k \in I} r(i,k) \Big) + \sum_{j \in U \setminus I, j \neq i} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j \in S \setminus I} \eta_j r(j,i) + \sum_{k \in I \cap S} \underbrace{\mu_i r(i,k)}_{\overset{(5.4)}{=} \eta_k r(k,i)} + \sum_{k \in I \cap U} \underbrace{\mu_i r(i,k)}_{\overset{(5.5)}{=} \mu_k r(k,i)} + \sum_{j \in U \setminus I} \mu_j r(j,i) \\
&= \lambda_i + \sum_{j \in S} \eta_j r(j,i) + \sum_{j \in U} \mu_j r(j,i) \overset{(1.6)}{=} \eta_i.
\end{aligned}$$

□

Remark 5.3. *The reversibility constraints* (5.3), (5.4) *and* (5.5) *are different to the classical reversibility constraints* (2.7) *which are the local balance equations of the routing process. But the interpretation of* (5.3), (5.4) *and* (5.5) *is the same as for* (2.7)*: The departure rate from one node multiplied with the routing probability to another node has to be equal to the according flow rate of the opposite direction.*

The following example shows that the reversibility constraints (5.3), (5.4) and (5.5) are compatible with the definition of a Jackson network. Clearly, these reversibility analogues put structural restrictions on the network. But these do not lead to contradictions.

Example 5.4. *Consider a Jackson network with two nodes. Both nodes may break down. Let $\eta_1 < \mu_1$ and $\eta_2 \geq \mu_2$ hold. Then the traffic equations in case of all nodes in up status are*

$$\begin{aligned}
\eta_1 &= \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1), \\
\eta_2 &= \lambda_2 + \eta_1 r(1,2) + \mu_2 r(2,2).
\end{aligned}$$

Since routing is irreducible, $r(1,1) < 1$ holds and therefore $1 - r(1,1) > 0$. Thus the first equation is equivalent to

$$\eta_1 = \frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)}. \tag{5.6}$$

Let the reversibility constraint (5.4) *hold which is*

$$\eta_1 r(1,2) = \mu_2 r(2,1). \tag{5.7}$$

This constraint leads to the following structural consequences for the Jackson network:

- *If $r(2,1) = 0$ then either $r(1,2) = 0$ or $\lambda_1 = 0$, i.e., if $r(2,1) = 0$ holds then either the two nodes must be two separate $M/M/1/\infty$ systems or node 1 has no input at all.*
- *If $r(1,2) = 0$ then $r(2,1) = 0$, i.e., $r(1,2) = 0$ holds then the two nodes must be two separate $M/M/1/\infty$ systems.*

- *It remains to analyze the case* $r(1,2) \neq 0 \neq r(2,1)$. *Then the required constraint* (5.7) *is with* (5.6) *equivalent to*

$$\begin{aligned} &\frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)} r(1,2) = \mu_2 r(2,1) \\ \Leftrightarrow\ &\lambda_1 r(1,2) + \mu_2 r(2,1) r(1,2) = \mu_2 r(2,1)(\underbrace{1 - r(1,1)}_{=r(1,0)+r(1,2)}) \\ \Leftrightarrow\ &\lambda_1 r(1,2) = \mu_2 r(2,1) r(1,0). \end{aligned}$$

Thus, $r(1,0) = 0$ *if and only if* $\lambda_1 = 0$. *Let* $r(1,0) > 0$ *hold, then we get*

$$\mu_2 r(2,1) = \lambda_1 \frac{r(1,2)}{r(1,0)} \Leftrightarrow \mu_2 = \lambda_1 \frac{r(1,2)}{r(2,1) r(1,0)} > 0.$$

Plugging this information into (5.6) *yields*

$$\eta_1 = \frac{\lambda_1 (1 + \frac{r(1,2)}{r(1,0)})}{1 - r(1,1)} = \frac{\lambda_1}{r(1,0)} = \lambda \frac{r(0,1)}{r(1,0)} > 0.$$

So if $r(1,2) \neq 0 \neq r(2,1)$ *and* $r(1,0) > 0$, *then* $\lambda_1 > 0$. *Thus constraint* (5.7) *has influence on the parameters of the network, but it does not lead to contradictions.*

Lemma 5.5. *Consider a Jackson network where nodes in* $D \subseteq \tilde{J}$ *are unreliable. Let* $\eta = (\eta_j : j \in \tilde{J})$ *be the unique solution of the general traffic equations* (1.6). *Denote by* $S = \{i : \eta_i < \mu_i\} \subseteq \tilde{J}$ *the set of stable nodes in the network and by* $U := \tilde{J} \setminus S$ *the set of unstable nodes in the network.*
In case of breakdowns of nodes we assume that customers are rerouted according to the **skipping** *regime. Let the following side constraint hold:*

$$\eta_i = \mu_i \quad \forall i \in U \cap D. \tag{5.8}$$

Then for all nodes $i \in \tilde{J} \setminus I$ *holds that the solution* η_i^I *of the traffic equation* (5.2) *for all* $I \subseteq D$, $I \neq \emptyset$, *equals* η_i.

Proof. We make the ansatz $\eta_i = \eta_i^I$ for all $i \in S \setminus I$ and all $I \subseteq D$ in (5.2). We then obtain with the solution η_i of the traffic equations (1.6) for any $I \subseteq D$: $\forall i \in S \setminus I$

$$
\begin{aligned}
&\lambda_i^I + \sum_{j\in S\setminus I} \eta_j r^I(j,i) + \sum_{j\in U\setminus I} \mu_j r^I(j,i) \\
&\overset{(\text{Def. } 2.7)}{=} \lambda_i + \sum_{k\in I} \lambda_k r^I(k,i) + \sum_{j\in S\setminus I} \eta_j \Big(r(j,i) + \sum_{k\in I} r(j,k) r^I(k,i) \Big) + \\
&\quad + \sum_{j\in U\setminus I} \mu_j \Big(r(j,i) + \sum_{k\in I} r(j,k) r^I(k,i) \Big) \\
&= \underbrace{\lambda_i + \sum_{j\in S\setminus I} \eta_j r(j,i) + \sum_{j\in U\setminus I} \mu_j r(j,i)}_{\overset{(1.6)}{=} \eta_i - \sum_{j\in I\cap S} \eta_j r(j,i) - \sum_{j\in I\cap U} \mu_j r(j,i)} + \\
&\quad + \sum_{k\in I} r^I(k,i) \underbrace{\Big(\lambda_k + \sum_{j\in S\setminus I} \eta_j r(j,k) + \sum_{j\in U\setminus I} \mu_j r(j,k) \Big)}_{\overset{(1.6)}{=} \eta_k - \sum_{j\in I\cap S} \eta_j r(j,k) - \sum_{j\in I\cap U} \mu_j r(j,k)} \\
&= \eta_i - \sum_{j\in I\cap S} \eta_j r(j,i) - \sum_{j\in I\cap U} \mu_j r(j,i) + \\
&\quad + \sum_{k\in I} \eta_k r^I(k,i) - \sum_{k\in I} r^I(k,i) \sum_{j\in I\cap S} \eta_j r(j,k) - \sum_{k\in I} r^I(k,i) \sum_{j\in I\cap U} \mu_j r(j,k) \\
&= \eta_i - \sum_{j\in I\cap S} \eta_j \underbrace{\Big(r(j,i) + \sum_{k\in I} r(j,k) r^I(k,i) \Big)}_{\overset{(2.1)}{=} r^I(j,i)} + \\
&\quad + \sum_{k\in I} \eta_k r^I(k,i) - \sum_{j\in I\cap U} \mu_j \underbrace{\Big(r(j,i) + \sum_{k\in I} r(j,k) r^I(k,i) \Big)}_{\overset{(2.1)}{=} r^I(j,i)} \\
&= \eta_i + \sum_{k\in I\cap S} \eta_k r^I(k,i) - \sum_{j\in I\cap S} \eta_j r^I(j,i) + \sum_{k\in I\cap U} \eta_k r^I(k,i) - \sum_{j\in I\cap U} \mu_j r^I(j,i) \\
&= \eta_i + \sum_{k\in I\cap U} \underbrace{(\eta_k - \mu_k)}_{\overset{(5.8)}{=} 0} r^I(k,i) = \eta_i.
\end{aligned}
\tag{5.9}
$$

Since $\eta_j^I = \eta_j$ holds for all $j \in S \setminus I$ and all $I \subseteq D$, it follows for all $i \in U \setminus I$ and $I \subseteq D$:

$$
\begin{aligned}
\eta_i^I &= \lambda_i^I + \sum_{j\in S\setminus I} \eta_j^I r^I(j,i) + \sum_{j\in U\setminus I} \mu_j r^I(j,i) \\
&= \lambda_i^I + \sum_{j\in S\setminus I} \eta_j r^I(j,i) + \sum_{j\in U\setminus I} \mu_j r^I(j,i)
\end{aligned}
$$

which is the very left side of (5.9) with $i \in U \setminus I$. The above computations in (5.9) are valid for all $i \in \tilde{J} \setminus I$, hence it follows $\eta_i^I = \eta_i$ for all $i \in U \setminus I$ and $I \subseteq D$, too. □

Remark 5.6. *The constraint* (5.8) *means that unstable nodes which may break down, on average, are fully loaded but not overloaded.*

In the following example it can be seen that constraint (5.8) is compatible with the definition of a Jackson network.

Example 5.7. *Consider a Jackson network with two nodes. Both nodes may break down. Let $\eta_1 < \mu_1$ and $\eta_2 = \mu_2$ hold. Then the traffic equations in case of all nodes in up status are*

$$\begin{aligned}\eta_1 &= \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1),\\ \eta_2 &= \lambda_2 + \eta_1 r(1,2) + \mu_2 r(2,2).\end{aligned} \tag{5.10}$$

Since routing is irreducible, $r(1,1) < 1$ holds and therefore $1 - r(1,1) > 0$. Thus the first equation is equivalent to

$$\eta_1 = \frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)}.$$

Plugging this into (5.10) *yields*

$$\eta_2 = \lambda_2 + \frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)} r(1,2) + \mu_2 r(2,2).$$

With $\eta_2 = \mu_2$ we get

$$\begin{aligned}&\mu_2 = \lambda_2 + \frac{\lambda_1 + \mu_2 r(2,1)}{1 - r(1,1)} r(1,2) + \mu_2 r(2,2)\\
\Leftrightarrow\ &\mu_2(1 - r(2,2)) = \lambda_2 + \frac{\lambda_1 r(1,2)}{1 - r(1,1)} + \frac{\mu_2 r(2,1) r(1,2)}{1 - r(1,1)}\\
\Leftrightarrow\ &\mu_2\Big(1 - r(2,2) - \frac{r(2,1) r(1,2)}{1 - r(1,1)}\Big) = \lambda_2 + \frac{\lambda_1 r(1,2)}{1 - r(1,1)}\\
\Leftrightarrow\ &\mu_2\Big(\underbrace{(1 - r(1,1))(1 - r(2,2)) - r(2,1) r(1,2)}_{= r(1,0)(1 - r(2,2)) + r(1,2) r(2,0) > 0}\Big) = \lambda_2(1 - r(1,1)) + \lambda_1 r(1,2)\\
\Leftrightarrow\ &\mu_2(r(2,0)(1 - r(1,1)) + r(2,1) r(1,0)) = \lambda_2(1 - r(1,1)) + \lambda_1 r(1,2).\end{aligned}$$

So if we require $\eta_1 < \mu_1$ and $\eta_2 = \mu_2$ to hold, μ_2 is

$$\mu_2 = \lambda_2 \underbrace{\frac{1 - r(1,1)}{r(2,0)(1 - r(1,1)) + r(2,1) r(1,0)}}_{>0} + \lambda_1 \underbrace{\frac{r(1,2)}{r(2,0)(1 - r(1,1)) + r(2,1) r(1,0)}}_{>0}.$$

Because of the irreducibility of the routing process, $\lambda_i > 0$ holds for at least one node, so the right-hand side of the last equation is greater than zero. Thus constraint (5.8) *is compatible with the definition of a Jackson network with two nodes.*

5.3 Asymptotic results

In the theorems of this section the essential result is that the non-ergodic state process (Y, X) on $\mathcal{P}(D) \times \mathbb{N}^J$ has - under certain conditions - a product form limiting joint marginal distribution on the stable subnetwork and limiting marginal queue length probabilities for each unstable node which represent the degenerated one-point distribution at infinity.

Definition 5.8 (Framework for Chapter 5.3)**.** *Consider a Jackson network with unreliable nodes according to Definition 2.9. Nodes in $D \subseteq \tilde{J}$ are unreliabe and breakdowns and repairs of the nodes are controlled by load-independent breakdown and repair intensities as in Example 2.2. Let $\eta = (\eta_1, ..., \eta_J)$ denote the unique solution of the general traffic equations* (1.6)*. Denote by $S = \{i : \eta_i < \mu_i\} \subseteq \tilde{J}$ the set of stable nodes in the network, nodes in $U := \tilde{J} \setminus S$ are unstable. In case of broken down nodes rerouting follows one of the regimes stalling, blocking rs-rd, or skipping. Let (Y, X) denote the availability-queue lengths process (which is a Markov process) on the state space $\mathcal{P}(D) \times \mathbb{N}^J$ with Q-matrix given in Definition 2.9.*

For the process (Y, X) we will prove in the framework of Definition 5.8 that the following limiting probabilities occur:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) = n_i : i \in S) = \left(\sum_{K \subseteq D} \frac{A(K)}{B(K)} \right)^{-1} \frac{A(I)}{B(I)} \prod_{i \in S} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i} \quad (5.11)$$

for $(I, n_i : i \in S) \in \mathcal{P}(D) \times \mathbb{N}^{|S|}$, and for $(I, n_j) \in \mathcal{P}(D) \times \mathbb{N}$:

$$\lim_{t\to\infty} P(Y(t) = I; X_j(t) = n_j) = 0 \quad \forall j \in U. \quad (5.12)$$

Remark 5.9. *Whenever U is not empty, then the network process (Y, X) is not ergodic. Nevertheless the limiting joint marginal process of the queue lengths in S is of product form as it already occurred in Theorem 2.14 (see Remark 2.15).*
(5.12) *implies that the limiting marginal distribution for each unstable node is the degenerated one-point distribution at ∞. It also follows directly (and by the proof later on) that for all states $(I, n_j : j \in U) \in \mathcal{P}(D) \times \mathbb{N}^{|U|}$ holds*

$$\lim_{t\to\infty} P(Y(t) = I; X_j(t) = n_j : j \in U) = 0. \quad (5.13)$$

On the other hand, the availability process Y is an ergodic Markov process of its own and its state distribution converges to its unique steady-state distribution.

The following theorems are extensions of the results in [MD09]. The limiting probabilities (5.11) and (5.12) are proved in [MD09] under the restriction that only stable nodes are unreliable. This restriction is removed here, which obviously covers more realistic situations. We summarize the results.

Theorem 5.10. *Consider a Jackson network with unreliable stations with non-ergodic state process (Y, X) as described in Definition 5.8. We assume that nodes in $D \subseteq \tilde{J}$ are unreliable. In case of breakdown* **stalling** *is applied according to Definition 2.6 (note that servers are under warm stand-by, i.e., they can still break down and may be repaired). Then the Markov process (Y, X) has the marginal limiting probabilities* (5.11) *and* (5.12)*.*

Theorem 5.11. *Consider a Jackson network with unreliable stations with non-ergodic state process (Y, X) as described in Definition 5.8. We assume that nodes in $D \subseteq \tilde{J}$ are unreliable. In case of breakdown customers are rerouted according to the* **skipping** *regime as in Definition 2.7. Let* (5.8) *hold:*

$$\eta_i = \mu_i \quad \forall i \in U \cap D.$$

Then the Markov process (Y, X) has the marginal limiting probabilities (5.11) *and* (5.12)*.*

Theorem 5.12. *Consider a Jackson network with unreliable stations with non-ergodic state process* (Y, X) *as described in Definition 5.8. We assume that only nodes in* $D \subseteq \tilde{J}$ *are unreliable. In case of breakdown customers are rerouted according to the* **blocking rs-rd** *regime as in Definition 2.8. Let the reversibility constraints* (5.3)*,* (5.4) *and* (5.5) *hold:*

$$\eta_i r(i,j) = \eta_j r(j,i) \quad \forall i,j \in S,$$
$$\eta_i r(i,j) = \mu_j r(j,i) \quad \forall i \in S, j \in U,$$
$$\mu_i r(i,j) = \mu_j r(j,i) \quad \forall i,j \in U.$$

Then the Markov process (Y, X) *has the marginal limiting probabilities* (5.11) *and* (5.12)*.*

5.3.1 Idea and outline of the proofs

Because the proofs of the three theorems are rather long, but have the same structure, the idea and the layout used will be described first.

The problem of proving the marginal limiting probabilities is that we cannot apply Theorem 2.14 to compute the limiting probabilities of (Y, X_S), because the marginal process (Y, X_S) on $\mathcal{P}(D) \times \mathbb{N}^{|S|}$ is not Markov.

Our approach to prove (5.11) (and (5.12) as well) follows the principles used by Goodman and Massey [GM84] in case of reliable nodes. We additionally have to incorporate breakdown and repair which makes the constructions much more complicated.

The idea is to use an upper bound and sequences of lower bounds for (Y, X) in the sense of stochastic ordering and to show that these bounds converge. The bounding processes will represent ergodic Jackson network processes with unreliable nodes and Jackson networks with infinite supply at the unstable nodes and with unreliable nodes, so Theorem 2.14 and Theorem 4.46 or Theorem 4.47 (respective the considered rerouting regime) will apply to the processes and generate limiting product form distributions. This guarantees that on the subnetwork S a product form limiting distribution is established.

The order structure on $\mathcal{P}(D) \times \mathbb{N}^{|S|}$ is the product of the ordered spaces $(\mathcal{P}(D), \subseteq)$ (the inclusion order) and $(\mathbb{N}^{|S|}, \leq^{|S|})$ (the natural coordinate-wise order on $\mathbb{N}^{|S|}$). We shall denote this order by

$$\prec := (\subseteq \times \leq^{|S|}),$$

and the associated integral stochastic order on the set of probability measures on $\mathcal{P}(D) \times \mathbb{N}^{|S|}$ by

$$\prec_{st} .$$

The construction of the bounding processes will accomplish the following:

- For all bounding processes we take the breakdown and repair intensities as prescribed by the original network. Recall that the availability process Y is Markov of its own.
- The sequence of network processes that serve as lower bounds is denoted by

$$((Y, X^{-}(\varepsilon)) : \varepsilon \to 0)$$

and represents a sequence of Jackson networks with unreliable nodes.

- As upper bound we use a network process (Y, X^+) which represents a Jackson network with infinite supply at the unstable nodes in U and with unreliable nodes.
- The construction will guarantee that at any time $t \geq 0$ and for all $\varepsilon > 0$ we have for the availability status of the network and the queue lengths in the subnet S

$$\begin{aligned}(Y(t); X_i^-(\varepsilon)(t) : i \in S) \\ \prec_{st} (Y(t); X_i(t) : i \in S) \\ \prec_{st} (Y(t); X_i^+(t) : i \in S),\end{aligned} \tag{5.14}$$

as well as for the availability status of the network and the queue lengths in the subnet U

$$(Y(t); X_i^-(\varepsilon)(t) : i \in U) \prec_{st} (Y(t); X_i(t) : i \in U). \tag{5.15}$$

Outline of the proofs: We will perform the following tasks under the respective additional assumptions of the theorems:

1. Construct the upper bound process (Y, X^+).
2. Prove that $(Y, X_S^+) := (Y, X_i^+ : i \in S)$ is an ergodic Markov process of its own: Show that for the solution η_i^+ of the traffic equations of (Y, X^+) holds
   - $\eta_i^+ = \eta_i$,
   - $\eta_i^+ < \mu_i^+$ and
   - $\eta_i^+ = \eta_i^{+,I}$ (in case of skipping and blocking rs-rd),

   for all $i \in S \setminus I$ and $I \subseteq D$, where η_i is the solution of the traffic equations of (Y, X). Then with Corollary 4.48 it follows directly that (Y, X_S^+) is an ergodic homogeneous Markov process of its own.
3. Construct the lower bound processes $(Y, X^-(\varepsilon))$, $\varepsilon > 0$.
4. Prove that $(Y, X^-(\varepsilon))$ is ergodic: Show that for the solution $\eta_i^-(\varepsilon)$ of the traffic equations of $(Y, X^-(\varepsilon))$ holds
   - $\eta_i^-(\varepsilon) < \mu_i^-(\varepsilon)$,
   - $\eta_i^-(\varepsilon) = \eta_i^{-,I}(\varepsilon)$ (in case of skipping and blocking rs-rd) and
   - $\lim_{\varepsilon \to 0} \eta_i^-(\varepsilon) = \eta_i$,

   for all $i \in \tilde{J}$ and $I \subseteq D$, where η_i is the solution of the traffic equations of (Y, X). Moreover, it will hold $\lim_{\varepsilon \to 0} \mu_i^-(\varepsilon) = \mu_i \; \forall i \in S$ and $\lim_{\varepsilon \to 0} \mu_i^-(\varepsilon) = \eta_i \; \forall i \in U$.
5. Show the stochastic order (5.14) and (5.15) of the processes:
   Construct $((Y, X^-(\varepsilon)), (Y, X), (Y, X^+))$ for any $\varepsilon > 0$ on a common probability space such that if the processes are started ω-wise ordered

$$(Y, X^-(\varepsilon))(0, \omega) \prec (Y, X)(0, \omega) \prec (Y, X^+)(0, \omega),$$

then over the time horizon $[0, \infty)$ the paths of the three processes fulfill

$$(Y, X^{-}(\varepsilon))(t, \omega) \prec (Y, X)(t, \omega) \prec (Y, X^{+})(t, \omega) \text{ for all } t.$$

Thus, these versions of the processes are (path-wise) ordered on the path space of right continuous functions with left-hand limits. From Strassen's theorem follows that the complete process distributions are stochastically ordered with respect to the stochastic integral order generated by the product order $\prec^{[0,\infty)}$ on the space of functions $(\mathcal{P}(D) \times \mathbb{N}^{|S|})^{[0,\infty)}$ (restricted to the subspace of the right continuous functions with left-hand limits), see [MS02, Theorem 2.6.3] or [KKO77]. This trivially implies (5.14) and (5.15).

Remark 5.13. *An intuitive explanation for the stochastic comparison of the processes (Y, X) and (Y, X^{+}) can be visualized by coloring*

- *all jobs in (Y, X) blue,*
- *all (high priority) jobs which originate from the external Poissonian arrival streams in (Y, X^{+}) blue as well, and*
- *all additional (low priority turning into high priority) jobs (from the infinite supply) in (Y, X^{+}) red.*

Letting the processes run one may observe the following: In both processes the queue lengths at each node will have the same amount of blue jobs, but in case of (Y, X^{+}) we will additionally have red jobs in the queue lengths as well.

These tasks build the structure of the proofs. If all these tasks are done, then it follows directly:

6. Consequences for the limiting probabilities:

Recall that (Y, X^{+}) represents an unreliable Jackson network with infinite supply at the nodes in U which has according to Theorem 4.46 or Theorem 4.47, resp., a uniquely defined limiting distribution in product form on the subnet W ($\equiv S$ by definition), independent of the initial distribution. Also recall that all the $(Y, X^{-}(\varepsilon))$, $\varepsilon > 0$, are ergodic Jackson network processes with unreliable nodes which have according to Theorem 2.14 a uniquely defined limiting distribution in product form, independent of the initial distribution. Then the proof will be finished by the observation that for $\varepsilon \to 0$ the limiting marginal distributions of $((Y, X_S^{-}(\varepsilon))(t) : t \geq 0)$ converge to the limiting marginal distributions of $((Y, X_S^{+})(t) : t \geq 0)$.

For all $n_i \in \mathbb{N}$ and $I \subseteq D$ holds in case of stalling with Theorem 4.46(i), in case of skipping or blocking rs-rd with Theorem 4.47 by stochastic ordering:

$$\begin{aligned}
&\liminf_{t\to\infty} P(Y(t)=I; X_i(t)<n_i : i\in S)\\
&\geq \liminf_{t\to\infty} P(Y(t)=I; X_i^+(t)<n_i : i\in S)\\
&= \lim_{t\to\infty} P(Y(t)=I; X_i^+(t)<n_i : i\in S)\\
&= \sum_{((s_i,i\in S):0\leq s_i<n_i\forall i\in S)} \lim_{t\to\infty} P(Y(t)=I; X_i^+(t)=s_i : i\in S)\\
&= \sum_{((s_i,i\in S):0\leq s_i<n_i\forall i\in S)} \left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1}\frac{A(I)}{B(I)}\prod_{i\in S}\left(\frac{\eta_i}{\mu_i}\right)^{s_i}\left(1-\frac{\eta_i}{\mu_i}\right).
\end{aligned}$$

For all $\varepsilon>0$ and $n_i\in\mathbb{N}$ and $I\subseteq D$ holds with Theorem 2.14:

$$\begin{aligned}
&\limsup_{t\to\infty} P(Y(t)=I; X_i(t)<n_i : i\in S)\\
&\leq \limsup_{t\to\infty} P(Y(t)=I; X_i^-(\varepsilon)(t)<n_i : i\in S)\\
&= \lim_{t\to\infty} P(Y(t)=I; X_i^-(\varepsilon)(t)<n_i : i\in S)\\
&= \sum_{((s_i,i\in S):0\leq s_i<n_i\forall i\in S)} \lim_{t\to\infty} P(Y(t)=I; X_i^-(\varepsilon)(t)=s_i : i\in S)\\
&= \sum_{((s_i,i\in S):0\leq s_i<n_i\forall i\in S)} \left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1}\frac{A(I)}{B(I)}\prod_{i\in S}\left(\frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)}\right)^{s_i}\left(1-\frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)}\right).
\end{aligned}$$

Letting ε go zero yields from $\frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)} \overset{\varepsilon\searrow 0}{\longrightarrow} \frac{\eta_i}{\mu_i}$ $\forall i\in S$

$$\begin{aligned}
&\limsup_{t\to\infty} P(Y(t)=I; X_i(t)<n_i : i\in S)\\
&\leq \sum_{((s_i,i\in S):0\leq s_i<n_i\forall i\in S)} \left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1}\frac{A(I)}{B(I)}\prod_{i\in S}\left(\frac{\eta_i}{\mu_i}\right)^{s_i}\left(1-\frac{\eta_i}{\mu_i}\right).
\end{aligned}$$

Thus, for all $n_i\in\mathbb{N}$ and $I\subseteq D$

$$\begin{aligned}
&\sum_{((s_i,i\in S):0\leq s_i<n_i\forall i\in S)} \left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1}\frac{A(I)}{B(I)}\prod_{i\in S}\left(\frac{\eta_i}{\mu_i}\right)^{s_i}\left(1-\frac{\eta_i}{\mu_i}\right)\\
&\leq \liminf_{t\to\infty} P(Y(t)=I; X_i(t)<n_i : i\in S)\\
&\leq \limsup_{t\to\infty} P(Y(t)=I; X_i(t)<n_i : i\in S)\\
&\leq \sum_{((s_i,i\in S):0\leq s_i<n_i\forall i\in S)} \left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1}\frac{A(I)}{B(I)}\prod_{i\in S}\left(\frac{\eta_i}{\mu_i}\right)^{s_i}\left(1-\frac{\eta_i}{\mu_i}\right).
\end{aligned}$$

So (5.11) is proved by

$$\begin{aligned}
&\lim_{t\to\infty} P(Y(t)=I; X_i(t)<n_i : i\in S)\\
&= \sum_{((s_i,i\in S):0\leq s_i<n_i\forall i\in S)} \left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1}\frac{A(I)}{B(I)}\prod_{i\in S}\left(\frac{\eta_i}{\mu_i}\right)^{s_i}\left(1-\frac{\eta_i}{\mu_i}\right).
\end{aligned}$$

Finally, Theorem 2.14 implies for $i \in U$ and for all $n_i \in \mathbb{N}$ and $I \subseteq D$:

$$\begin{aligned}
&\limsup_{t\to\infty} P(Y(t) = I; X_i(t) < n_i : i \in U) \\
&\leq \limsup_{t\to\infty} P(Y(t) = I; X_i^-(\varepsilon)(t) < n_i : i \in U) \\
&= \sum_{((s_i, i\in U):0\leq s_i<n_i \forall i\in U)} \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i\in U} \left(\frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)}\right)^{s_i} \left(1 - \frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)}\right).
\end{aligned}$$

With $\lim_{\varepsilon\searrow 0} \frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)} = \frac{\eta_i}{\eta_i} = 1 \ \forall \ i \in U$ it follows directly for all $I \subseteq D$ and $n_i \in \mathbb{N}$:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) < n_i : i \in U) = 0,$$

hence (5.13) is valid. Even more, Theorem 2.14 also implies for $i \in U$ and for all $n_i \in \mathbb{N}$ and $I \subseteq D$:

$$\begin{aligned}
&\limsup_{t\to\infty} P(Y(t) = I; X_i(t) < n_i) \\
&\leq \limsup_{t\to\infty} P(Y(t) = I; X_i^-(\varepsilon)(t) < n_i) \\
&= \sum_{(0\leq s_i<n_i)} \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \left(\frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)}\right)^{s_i} \left(1 - \frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)}\right).
\end{aligned}$$

With $\lim_{\varepsilon\searrow 0} \frac{\eta_i^-(\varepsilon)}{\mu_i^-(\varepsilon)} = \frac{\eta_i}{\eta_i} = 1 \ \forall \ i \in U$ it follows directly for all $I \subseteq D$ and $n_i \in \mathbb{N}$:

$$\lim_{t\to\infty} P(Y(t) = I; X_i(t) < n_i) = 0,$$

$i \in U$, hence (5.12) is valid.

Remark 5.14. *As mentioned before, the result in [GM84] considers reliable non-ergodic networks only, while the results in our paper [MD09] cover the case that stable nodes are unreliable but all unstable nodes are reliable. Still the lower process networks in case that all nodes are in up status are constructed the same way in the proofs of the Theorems 5.10-5.12 as in [MD09] and in [GM84] as well. The structure of the proofs of the Theorems 5.10-5.12 resembles the structure of the proof of Theorem 15 in our paper [MD09]. Apart from other differences to the proof in [MD09] in the details, the generalization in Theorems 5.10-5.12 that all nodes may be unreliable requires the use of a different upper bound process and therefore the theory of Jackson networks with infinite supply, see Chapter 4. Furthermore, especially in case of rerouting according to skipping, some more technical computations are necessary here in order to proof the stochastic order of the three processes.*

5.3.2 Proofs

All proofs in this section will use the idea and layout described in the previous Section 5.3.1. For readability some arguments will be recapitulated.

Proof of Theorem 5.10 (Rerouting according to stalling)

1. Construction of the upper bound: We derive (Y, X^+) from (Y, X) as follows:

- The breakdown and repair processes are (stochastically) identical;
- the service rates at all nodes are the same as in the original network;
- all nodes in U are supplemented by an infinite supply of work, this leads to Poissonian departure processes from those nodes (see Theorem 4.3 and Theorem 4.38);
- the rerouting regime is selected as stalling, so whenever nodes of $I \subseteq D$ break down, the transition intensities are as in Definition 4.37 with Definition 2.6.

2. (Y, X_S^+) **is an ergodic homogeneous Markov process of its own:** (Y, X^+) is the Markov state process of a Jackson network with unreliable nodes where the nodes in U have an infinite supply of work, as in Definition 4.37 with $U = V$ and $S = W$. $(\eta_i^+ : i \in \tilde{J})$ is the unique solution of the traffic equations (4.1):

$$\begin{aligned}\eta_i^+ &= \lambda_i + \sum_{j\in W} \eta_j^+ r(j,i) + \sum_{j\in V} \mu_j r(j,i) \\ &= \lambda_i + \sum_{j\in S} \eta_j^+ r(j,i) + \sum_{j\in U} \mu_j r(j,i), \quad \forall i \in \tilde{J},\end{aligned}$$

thus $(\eta_i^+ : i \in \tilde{J})$ solves the traffic equations (1.6) of (Y, X). Therefore $\eta_i^+ = \eta_i$ for all $i \in \tilde{J}$. Recalling $S = \{i : \eta_i < \mu_i\}$, it follows directly that $\eta_i^+ < \mu_i$ holds for all $i \in S$. In case of stalling it is trivial that this local stability criterion for nodes in S holds in any availability status.

Since $U \cap W = \emptyset$ and rerouting is according to stalling, Theorem 4.46 applies and provides us with a product form stationary and limiting distribution for $(Y, X_S^+) := (Y, X_i^+ : i \in S)$ which is the restriction of the process (Y, X^+) on the stable subnetwork. From Corollary 4.48 we know that (Y, X_S^+) is an ergodic homogeneous Markov process independent of the behavior of the remaining nodes in the network. The last step of the argument is that under load-independent breakdown and repair rates, an ergodic Jackson network remains ergodic, see Example 3.3.

3. Construction of the lower bounds: We derive $(Y, X^-(\varepsilon))$ from (Y, X) for $\varepsilon > 0$ as follows:

- The breakdown and repair processes are (stochastically) identical;
- the external arrival rates at all nodes are identical to those in the original network;
- the service rates at nodes in S are the same as in the original network, the service rates at nodes in U are increased from μ_i to $\eta_i + \varepsilon$:

$$\mu_i^-(\varepsilon) = \begin{cases} \mu_i, & i \in S, \\ \eta_i + \varepsilon, & i \in U, \end{cases}$$

  which guarantees that all nodes in the network are stable now, as will be shown in the next step (4.) of the proof;

- this generates additional customer flows (at least) out of the nodes in U; this additional load on the network is then reduced by increasing the probability for customers to leave nodes in U directly to the external sink; we prescribe

$$r^-(i,j)(\varepsilon) = \begin{cases} r(i,j), & i \in S, j \in \tilde{J}, \\ \frac{\mu_i}{\eta_i+\varepsilon}\, r(i,j), & i \in U, j \in \tilde{J}, \end{cases}$$
$$r^-(i,0)(\varepsilon) = \begin{cases} r(i,0), & i \in S, \\ \frac{\mu_i r(i,0)+\eta_i-\mu_i+\varepsilon}{\eta_i+\varepsilon}, & i \in U; \end{cases}$$

- the combination of adjusted service rates and new routing probabilities yields the desired reproduction of flows on S with

$$\mu_i^-(\varepsilon) r^-(i,j)(\varepsilon) = \mu_i r(i,j) \quad \forall\, i,j \in \tilde{J},$$
$$\mu_i^-(\varepsilon) r^-(i,0)(\varepsilon) = \begin{cases} \mu_i r(i,0), & i \in S, \\ \mu_i r(i,0) + \eta_i - \mu_i + \varepsilon, & i \in U; \end{cases} \tag{5.16}$$

- in case that nodes break down, the stalling regime according to Definition 2.6 is in force.

4. $(Y, X^-(\varepsilon))$, $\varepsilon > 0$, **is ergodic:** Consider the network described above by the process $(Y, X^-(\varepsilon))$, $\varepsilon > 0$, with completely reliable nodes, which is just a classical Jackson network with queue-length process $X^-(\varepsilon)$ and traffic equations

$$\eta_i^-(\varepsilon) = \lambda_i^-(\varepsilon) + \sum_{j\in\tilde{J}} \min(\eta_j^-(\varepsilon), \mu_j^-(\varepsilon)) r^-(j,i)(\varepsilon) \;\forall i \in \tilde{J}$$
$$\Leftrightarrow\; \eta^-(\varepsilon) = \lambda^-(\varepsilon) + (\eta^-(\varepsilon) \wedge \mu^-(\varepsilon)) R^-_{\tilde{J}\tilde{J}}(\varepsilon).$$

- For the solution η_i of the traffic equation (1.6) holds $\forall i \in \tilde{J}$

$$\begin{aligned} \eta_i &= \lambda_i + \sum_{j\in S} \eta_j r(j,i) + \sum_{j \in U} \mu_j r(j,i) \\ &= \lambda_i^-(0) + \sum_{j\in S} \eta_j r^-(j,i)(0) + \sum_{j\in U} \eta_j r^-(j,i)(0) \\ &= \lambda_i^-(0) + \sum_{j\in \tilde{J}} \eta_j r^-(j,i)(0), \end{aligned}$$

thus $\eta = (\eta_i : i \in \tilde{J})$ solves the traffic equation for the process $(Y, X^-(0))$ (in matrix notation)

$$\eta^-(0) = \lambda^-(0) + \eta^-(0) R^-_{\tilde{J}\tilde{J}}(0).$$

For all $i \in S$ it holds $\eta_i^-(0) = \eta_i < \mu_i = \mu_i^-(0)$ and for all $i \in U$ we have $\eta_i^-(0) = \eta_i = \eta_i + 0 = \mu_i^-(0)$. Therefore $\eta = (\eta_i : i \in \tilde{J})$ also solves the traffic equation (in matrix notation)

$$\eta^-(0) = \lambda^-(0) + (\eta^-(0) \wedge \mu^-(0)) R^-_{\tilde{J}\tilde{J}}(0).$$

- Now we show that $\eta_i^-(\varepsilon) < \mu_i^-(\varepsilon)$ holds for all $i \in \tilde{J}$:

$$\begin{aligned} \eta^-(\varepsilon) &= \lambda + (\eta^-(\varepsilon) \wedge \mu^-(\varepsilon))R^-_{\tilde{J}\tilde{J}}(\varepsilon) \\ &\leq \lambda + (\eta^-(\varepsilon) \wedge \mu^-(\varepsilon))R^-_{\tilde{J}\tilde{J}}(0) \\ &\leq \lambda + \eta^-(\varepsilon)R^-_{\tilde{J}\tilde{J}}(0) \\ \Rightarrow \quad \eta^-(\varepsilon) &\leq \lambda\left(\mathbf{I} - R^-_{\tilde{J}\tilde{J}}(0)\right)^{-1} = \eta^-(0), \end{aligned}$$

and furthermore $\eta_U^-(0) = \mu_U^-(0) < \mu_U^-(\varepsilon)$ and $\eta_S^-(0) = \eta_S < \mu_S = \mu_S^-(\varepsilon)$ holds component-wise for $\varepsilon > 0$. So $\eta^-(\varepsilon) < \mu^-(\varepsilon)$ holds component-wise for all $\varepsilon > 0$ where $\eta^-(\varepsilon)$ is determined by

$$\eta^-(\varepsilon) = \lambda\left(\mathbf{I} - R^-_{\tilde{J}\tilde{J}}(\varepsilon)\right)^{-1},$$

hence $X^-(\varepsilon)$ is ergodic. In case of stalling it is trivial that the local stability criterion $\eta_i^-(\varepsilon) < \mu_i^-(\varepsilon)\ \forall i \in \tilde{J}$ holds in any availability status and, under load-independent breakdown and repair rates, an ergodic Jackson network remains ergodic, see Example 3.3.

- It remains to show that $\lim_{\varepsilon\to 0}\eta_i^-(\varepsilon) = \eta_i$ holds for all $i \in \tilde{J}$. Since $\eta^-(0) = \eta$, we will show that $\lim_{\varepsilon\to 0}\eta^-(\varepsilon) = \eta^-(0)$. Recalling that $X^-(\varepsilon)$ is ergodic, $|\eta^-(\varepsilon)|_1 < |\mu^-(\varepsilon)|_1$ holds for all $\varepsilon > 0$, so there is $\varepsilon' < \infty$ such that $\{\eta^-(\varepsilon) : 0 \leq \varepsilon \leq \varepsilon'\}$ is a bounded set of vectors in $\mathbb{R}^J$. Hence there is a convergent subsequence $(\eta^-(\varepsilon_l) : l \geq 0)$ for $\lim_{l\to\infty}\varepsilon_l = 0$. Furthermore, it holds component-wise

$$\lim_{l\to\infty} R^-_{\tilde{J}\tilde{J}}(\varepsilon_l) = R^-_{\tilde{J}\tilde{J}}(0)$$

and the traffic equations of $X^-(\varepsilon_l)$ are

$$\begin{aligned} \eta^-(\varepsilon_l) = \lambda + \eta^-(\varepsilon_l)R^-_{\tilde{J}\tilde{J}}(\varepsilon_l) \Rightarrow\ & \lim_{l\to\infty}\eta^-(\varepsilon_l) = \lim_{l\to\infty}\left(\lambda + \eta^-(\varepsilon_l)R^-_{\tilde{J}\tilde{J}}(\varepsilon_l)\right) \\ \overset{\text{[Kön01, p.43]}}{\Leftrightarrow}\ & \lim_{l\to\infty}\eta^-(\varepsilon_l) = \lim_{l\to\infty}\lambda + \left(\lim_{l\to\infty}\eta^-(\varepsilon_l)\right)\left(\lim_{l\to\infty}R^-_{\tilde{J}\tilde{J}}(\varepsilon_l)\right) \\ \Leftrightarrow\ & \lim_{l\to\infty}\eta^-(\varepsilon_l) = \lambda + \left(\lim_{l\to\infty}\eta^-(\varepsilon_l)\right)R^-_{\tilde{J}\tilde{J}}(0), \end{aligned}$$

which is the traffic equation of $X^-(0)$, so $\lim_{l\to\infty}\eta^-(\varepsilon_l) = \eta^-(0)$ holds for all convergent subsequences $(\eta^-(\varepsilon_l) : l \geq 0)$. Thus $\lim_{\varepsilon\to 0}\eta^-(\varepsilon) = \eta^-(0)$ is valid.

5. Stochastic ordering of the processes: The joint construction of the processes is a standard coupling construction and utilizes ideas of Goodman and Massey [GM84]. The special issue is to handle the breakdown and repair processes. We shall perform a sequence of successive constructions each for a random time during which the availability status of the networks stays invariant. Whenever the availability status of the networks changes we switch to a new, suitably adapted, construction scheme. The construction ensures that switching the construction mode for all involved processes happens at exactly the same times.

From the definition of the processes it will be immediately seen that all the Q-matrices have bounded diagonals and therefore are uniformizable, i.e., they have a compound Poisson representation: Let $0 < \zeta < \infty$ be an upper bound of the modulus of the diagonal entries of all three processes to be compared. Then in a standard coupling the processes are driven by a Poisson-ζ process, which determines the common jump times, and three (different) Markov chain jump matrices, which are coupled and determine the direction of the jumps.

Our plan is not to use a single Poisson process to determine the common jump times, but to use Poisson processes of different intensities over random time intervals, and then similarly different jump matrices.

To start the system we assume that initially all networks are empty and all nodes are up. Let us assume that at some (random) time τ a change of the availability status has happened such that nodes in I are under repair and assume that (ω-wise)

$$(Y, X^-(\varepsilon))(\tau) \prec (Y, X)(\tau) \prec (Y, X^+)(\tau)$$

holds. Now we describe the development of the processes until the next availability change occurs at some (random) time τ'. We run a Poisson process with rate

$$\zeta(I) := \sum_{i\in\tilde{J}} \lambda_i + \sum_{i\in\tilde{J}} \mu_i + \sum_{i\in U} (\eta_i - \mu_i) + \varepsilon|U| + \sum_{\emptyset\neq K\subseteq D\setminus I} \alpha(I, I\cup K) + \sum_{\emptyset\neq H\subseteq I} \beta(I, I\setminus H),$$

which generates the jump instants for the processes.

At each jump instant we select according to some probability law (to be specified below) an activity from the following set, which is performed for all the three processes jointly unless otherwise indicated:

$[A_i]$	Add a customer to node i.
$[B_i]$	Delete a customer from node i if the queue length at i is positive, otherwise do nothing.
$[C_{ij}]$	Transfer a customer from i to j if the queue length at i is positive, otherwise do nothing.
$[D]$	Do nothing.
$\left[E_{(I,I\cup K)}\right]$	τ' occurs, deactivate nodes of the non-void subset $K \subseteq D\setminus I$, stop the running Poisson process and start an independent new Poisson-$\zeta(I\cup K)$ process.
$\left[E_{(I,I\setminus H)}\right]$	τ' occurs, reactivate nodes of the non-void subset $H \subset I$, stop the running Poisson process and start an independent new Poisson-$\zeta(I\setminus H)$ process.

The breakdown and repair activities are selected for all processes the same way according to the following rules:

- With probability $\alpha(I, I\cup K)/\zeta(I)$ select $\left[E_{(I,I\cup K)}\right]$;
- with probability $\beta(I, I\setminus H)/\zeta(I)$ select $\left[E_{(I,I\setminus H)}\right]$.

Note, that by these activities a $\prec$-order of the processes is maintained.

To perform the selection of queue-length activities only the case that all nodes are in up status is interesting, because if at least one node is broken down all nodes are stalled according to Definition 2.6, which means that in any other availability status than $I = \emptyset$ no queue-length activities may be selected with positive probability. That is why we assume $Y(\tau) = \emptyset$ and recall that the processes have cadlag paths, so the transition intensities are fixed on $[\tau, \tau')$. Whenever the Poisson process indicates a jump and no availability change was selected by the randomization procedure, we utilize the intensities to select changes in the queue lengths:

- For all $i \in \tilde{J}$ select with probability $\lambda_i/\zeta(\emptyset)$:
  $[A_i]$ = Add a customer to node i;

- for all $i \in \tilde{J}$ with probability $\mu_i r(i,0)/\zeta(\emptyset)$ select:
  $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;

- for all $i \in S, j \in \tilde{J}, j \neq i$, select with probability $\mu_i r(i,j)/\zeta(\emptyset)$:
  $[C_{ij}]$ = Transfer a customer from i to j if the queue length at i is positive, otherwise do nothing;

- for all $i \in U, j \in S, j \neq i$, with probability $\mu_i r(i,j)/\zeta(\emptyset)$ select
  ◦ for (Y, X^+): $[A_j]$ = Add a customer at node j if the queue length at i is zero,
  ◦ for (Y, X) and $(Y, X^-(\varepsilon))$: $[D]$ = Do nothing if the queue length at i is zero,
  ◦ for (Y, X^+), (Y, X), and $(Y, X^-(\varepsilon))$: $[C_{ij}]$ = Transfer a customer from i to j if the queue length at i is positive;
  (*This reflects the infinite supply at the nodes in U in (Y, X^+).*)

- for all $i \in U$ select with probability $(\eta_i - \mu_i + \varepsilon)/\zeta(\emptyset)$:
  ◦ for $(Y, X^-(\varepsilon))$: $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;
  ◦ for (Y, X) and (Y, X^+): $[D]$ = Do nothing.
  (*This reflects the additional departure rates of $(Y, X^-(\varepsilon))$ which compensate the additional flows generated by making the nodes in U stable, see* (5.16).)

- For all $i \in \tilde{J}$ select with probability $\mu_i r(i,i)/\zeta(\emptyset)$:
  $[D]$ = Do nothing.

Note, that by these activities a $\prec$-order of the processes is maintained.

We conclude that by starting the processes $\prec_{st}$-ordered with the same $I \subseteq D$ this order will be maintained over time, so the equilibria of the bounding processes will still be ordered and they bound the limiting distribution of the process (Y, X).

The last step of the proof is demonstrated by the consequences for the limiting probabilities of (Y, X) which can be found in the outline of the proofs numbered as 6th task on pages 124-126. □

Remark 5.15. *As mentioned before in Chapter 4, the limiting probability* (4.64) *for unstable nodes with infinite supply in Proposition 4.46(iii) follows from the proof of Theorem 5.10: If the global state process of the Jackson network with infinite supply is started with an initial distribution which has the marginal* (4.62) *on the subset* W *of nodes without infinite supply, then the availability–queue lengths process for each node* $i \in V$ *(with infinite supply) is a Markov process on its own (see the proof of Proposition 4.46) for which a similar lower bound process* (Y, X^-) *may be constructed as in the proof above.*

Remark 5.16. *With* $D = \emptyset$*, the proof of Theorem 5.10 serves as a proof for Theorem 1.22. Comparing the proof of Theorem 1.22 given by Goodman and Massey in [GM84] with this one, our version is shorter, because the Jackson network with infinite supply for the upper bound is less complicated than Goodman and Massey's construction of an upper bound process.*

Proof of Theorem 5.11 (Rerouting according to skipping)

1. Construction of the upper bound: We derive (Y, X^+) from (Y, X) as follows:

- The breakdown and repair processes are (stochastically) identical;
- the service rates at all nodes are the same as in the original network;
- all nodes in U are supplemented by an infinite supply of work, this leads to Poissonian departure processes from those nodes (see Theorem 4.3 and Theorem 4.38);
- the rerouting regime is selected as skipping, so whenever nodes of $I \subseteq D$ break down, the transition intensities are as in Definition 4.37 with Definition 2.7.

2. (Y, X_S^+) **is an ergodic homogeneous Markov process of its own:** (Y, X^+) is the Markov state process of a Jackson network with unreliable nodes where the nodes in U have an infinite supply of work, as in Definition 4.37 with $U = V$ and $S = W$. $(\eta_i^+ : i \in \tilde{J})$ is the unique solution of the traffic equations (4.1)

$$\begin{aligned}\eta_i^+ &= \lambda_i + \sum_{j\in W} \eta_j^+ r(j,i) + \sum_{j \in V} \mu_j r(j,i) \\ &= \lambda_i + \sum_{j\in S} \eta_j^+ r(j,i) + \sum_{j \in U} \mu_j r(j,i), \quad \forall i \in \tilde{J},\end{aligned}$$

so $\eta_i^+ = \eta_i$ for all $i \in \tilde{J}$. Recalling $S = \{i : \eta_i < \mu_i\}$, it follows directly that $\eta_i^+ < \mu_i$ holds for all $i \in S$. This local stability criterion holds in any availability status, i.e., $\eta_i^{I,+} < \mu_i^I$ $\forall i \in S \setminus I, I \subseteq \tilde{J}$, because with (5.8) (which yields $\eta_i^+ = \mu_i$ $\forall i \in V \cap D$) the solution of the standard traffic equations remains the same for nodes in up status, as can be seen in Lemma 4.43.

Since $W \cap U = \emptyset$, rerouting is according to skipping, and (5.8) yields $\eta_i^+ = \mu_i$ $\forall i \in V \cap D$, Theorem 4.47 applies and provides us with a product form stationary and limiting distribution for the process $(Y, X_S^+) := (Y, X_i^+ : i \in S)$. From Corollary 4.48 we know that (Y, X_S^+) is ergodic independent of the behavior of the remaining nodes in the network. The last step of the argument is that under load-independent breakdown and repair rates, an ergodic Jackson network remains ergodic, see Example 3.3.

3. Construction of the lower bounds: We derive $(Y, X^-(\varepsilon))$ from (Y, X) for $\varepsilon > 0$ as follows:

- The breakdown and repair processes are (stochastically) identical;
- the external arrival rates at all nodes are identical to those in the original network;
- the service rates at nodes in S are the same as in the original network, the service rates at nodes in $\tilde{J} \setminus S$ are increased from μ_i to

$$\mu_i^-(\varepsilon) = \begin{cases} \mu_i, & i \in S, \\ \eta_i + \varepsilon, & i \in U, \end{cases}$$

  which guarantees that all nodes in the network are stable now, as will be shown in the next step (4.) of the proof;
- this generates additional customer flows (at least) out of the nodes in U; this additional load on the network is then reduced by increasing the probability for customers to leave nodes in U directly to the external sink; we prescribe

$$r^-(i,j)(\varepsilon) = \begin{cases} r(i,j), & i \in S, j \in \tilde{J}, \\ \frac{\mu_i}{\eta_i+\varepsilon}\, r(i,j), & i \in U, j \in \tilde{J}, \end{cases}$$
$$r^-(i,0)(\varepsilon) = \begin{cases} r(i,0), & i \in S, \\ \frac{\mu_i r(i,0)+\eta_i-\mu_i+\varepsilon}{\eta_i+\varepsilon}, & i \in U; \end{cases}$$

- the combination of adjusted service rates and new routing probabilities yields the desired reproduction of flows on S with

$$\mu_i^-(\varepsilon) r^-(i,j)(\varepsilon) = \mu_i r(i,j) \quad \forall\, i,j \in \tilde{J},$$
$$\mu_i^-(\varepsilon) r^-(i,0)(\varepsilon) = \begin{cases} \mu_i r(i,0), & i \in S, \\ \mu_i r(i,0) + \eta_i - \mu_i + \varepsilon, & i \in U; \end{cases} \tag{5.17}$$

- the rerouting regime is selected as skipping, so whenever nodes in $I \subseteq D$ break down, the parameters are constructed in the standard way according to Definition 2.7 (so $r^{I,-}(\varepsilon)$ exists and is unique):

$$r^{I,-}(i,j)(\varepsilon) = \begin{cases} r(i,j) + \sum_{k \in I} r(i,k) r^{I,-}(k,j)(\varepsilon), & i \in S \setminus I, j \in \tilde{J} \setminus I, \\ \frac{\mu_i}{\eta_i+\varepsilon}\left(r(i,j) + \sum_{k \in I} r(i,k) r^{I,-}(k,j)(\varepsilon)\right), & i \in U \setminus I, j \in \tilde{J} \setminus I, \\ 0, & \text{else}, \end{cases}$$
$$r^{I,-}(i,0)(\varepsilon) = \begin{cases} r(i,0) + \sum_{k \in I} r(i,k) r^{I,-}(k,0)(\varepsilon), & i \in S \setminus I, \\ \frac{\mu_i r(i,0)+\eta_i-\mu_i+\varepsilon}{\eta_i+\varepsilon} + \frac{\mu_i}{\eta_i+\varepsilon} \sum_{k \in I} r(i,k) r^{I,-}(k,0)(\varepsilon), & i \in U \setminus I, \\ 0, & i \in I, \end{cases}$$
$$\mu_i^{I,-}(\varepsilon) = \begin{cases} \mu_i, & i \in S \setminus I, \\ \eta_i + \varepsilon, & i \in U \setminus I, \\ 0, & i \in I, \end{cases}$$

and the transition intensities of the network process $(Y, X^-(\varepsilon))$ are

$$\lambda_i^{I,-}(\varepsilon) = \begin{cases} \lambda_i + \sum_{k\in I} \lambda_k r^{I,-}(k,i)(\varepsilon), & i \in \tilde{J} \setminus I, \\ 0, & i \in I, \end{cases}$$

$$\mu_i^{I,-}(\varepsilon) r^{I,-}(i,j)(\varepsilon) = \begin{cases} \mu_i \left(r(i,j) + \sum_{k\in I} r(i,k) r^{I,-}(k,j)(\varepsilon) \right), & i,j \in \tilde{J} \setminus I, \\ 0, & \text{else}, \end{cases}$$

$$\mu_i^{I,-}(\varepsilon) r^{I,-}(i,0)(\varepsilon) = \begin{cases} \mu_i \big(r(i,0) + \sum_{k\in I} r(i,k) r^{I,-}(k,0)(\varepsilon) \big), & i \in S \setminus I, \\ \mu_i \big(r(i,0) + \sum_{k\in I} r(i,k) r^{I,-}(k,0)(\varepsilon) \big) + \eta_i - \mu_i + \varepsilon, & i \in U \setminus I, \\ 0, & i \in I. \end{cases}$$

4. $(Y, X^-(\varepsilon))$, $\varepsilon > 0$, **is ergodic:** Consider the network described above by the process $(Y, X^-(\varepsilon))$, $\varepsilon > 0$, with completely reliable nodes, which is just a classical Jacksonian network with queue-length process $X^-(\varepsilon)$ and traffic equations

$$\eta_i^-(\varepsilon) = \lambda_i^-(\varepsilon) + \sum_{j\in\tilde{J}} \min(\eta_j^-(\varepsilon), \mu_j^-(\varepsilon)) r^-(j,i)(\varepsilon) \ \forall i \in \tilde{J}$$
$$\Leftrightarrow \ \eta^-(\varepsilon) = \lambda^-(\varepsilon) + (\eta^-(\varepsilon) \wedge \mu^-(\varepsilon)) R^-_{\tilde{J}\tilde{J}}(\varepsilon).$$

- For the solution η_i of the traffic equation (1.6) holds $\forall i \in \tilde{J}$

$$\begin{aligned} \eta_i &= \lambda_i + \sum_{j\in S} \eta_j r(j,i) + \sum_{j\in U} \mu_j r(j,i) \\ &= \lambda_i^-(0) + \sum_{j\in S} \eta_j r^-(j,i)(0) + \sum_{j\in U} \eta_j r^-(j,i)(0) \\ &= \lambda_i^-(0) + \sum_{j\in\tilde{J}} \eta_j r^-(j,i)(0), \end{aligned}$$

thus $\eta = (\eta_i : i \in \tilde{J})$ solves the traffic equation for the process $(Y, X^-(0))$ (in matrix notation)

$$\eta^-(0) = \lambda^-(0) + \eta^-(0) R^-_{\tilde{J}\tilde{J}}(0).$$

For all $i \in S$ it holds $\eta_i^-(0) = \eta_i < \mu_i = \mu_i^-(0)$ and for all $i \in U$ we have $\eta_i^-(0) = \eta_i = \eta_i + 0 = \mu_i^-(0)$. Therefore $\eta = (\eta_i : i \in \tilde{J})$ also solves the traffic equation (in matrix notation)

$$\eta^-(0) = \lambda^-(0) + (\eta^-(0) \wedge \mu^-(0)) R^-_{\tilde{J}\tilde{J}}(0).$$

- Now we show that $\eta_i^-(\varepsilon) < \mu_i^-(\varepsilon)$ holds for all $i \in \tilde{J}$:

$$\begin{aligned} \eta^-(\varepsilon) &= \lambda + (\eta^-(\varepsilon) \wedge \mu^-(\varepsilon)) R^-_{\tilde{J}\tilde{J}}(\varepsilon) \\ &\le \lambda + (\eta^-(\varepsilon) \wedge \mu^-(\varepsilon)) R^-_{\tilde{J}\tilde{J}}(0) \\ &\le \lambda + \eta^-(\varepsilon) R^-_{\tilde{J}\tilde{J}}(0) \\ \Rightarrow \ \eta^-(\varepsilon) &\le \lambda \left(\mathbf{I} - R^-_{\tilde{J}\tilde{J}}(0) \right)^{-1} = \eta^-(0), \end{aligned}$$

and furthermore $\eta^-_U(0) = \mu^-_U(0) < \mu^-_U(\varepsilon)$ and $\eta^-_S(0) = \eta_S < \mu_S = \mu^-_S(\varepsilon)$ holds component-wise for $\varepsilon > 0$. So $\eta^-(\varepsilon) < \mu^-(\varepsilon)$ holds component-wise for all $\varepsilon > 0$ where $\eta^-(\varepsilon)$ is determined by

$$\eta^-(\varepsilon) = \lambda \left(\mathbf{I} - R^-_{\tilde{J}\tilde{J}}(\varepsilon)\right)^{-1},$$

hence $X^-(\varepsilon)$ is ergodic. The local stability criterion holds in any availability status, i.e., $\eta_i^{I,-}(\varepsilon) < \mu_i^{I,-}(\varepsilon)\ \forall i \in \tilde{J}\backslash I, I \subseteq D$, see Lemma 2.13, and under load-independent breakdown and repair rates, an ergodic Jackson network remains ergodic, see Example 3.3.

- It remains to show that $\lim_{\varepsilon\to 0} \eta_i^-(\varepsilon) = \eta_i$ holds for all $i \in \tilde{J}$. Since $\eta^-(0) = \eta$, we will show that $\lim_{\varepsilon\to 0} \eta^-(\varepsilon) = \eta^-(0)$. Recalling that $X^-(\varepsilon)$ is ergodic, $|\eta^-(\varepsilon)|_1 < |\mu^-(\varepsilon)|_1$ holds for all $\varepsilon > 0$, so there is $\varepsilon' < \infty$ such that $\{\eta^-(\varepsilon) : 0 \le \varepsilon \le \varepsilon'\}$ is a bounded set of vectors in $\mathbb{R}^J$. Hence there is a convergent subsequence $(\eta^-(\varepsilon_l) : l \ge 0)$ with $\lim_{l\to\infty} \varepsilon_l = 0$. Then it holds component-wise

$$\lim_{l\to\infty} R^-_{\tilde{J}\tilde{J}}(\varepsilon_l) = R^-_{\tilde{J}\tilde{J}}(0)$$

and the traffic equations of $X^-(\varepsilon_l)$ are

$$\begin{aligned}\eta^-(\varepsilon_l) = \lambda + \eta^-(\varepsilon_l)R^-_{\tilde{J}\tilde{J}}(\varepsilon_l) \Rightarrow \lim_{l\to\infty} \eta^-(\varepsilon_l) &= \lim_{l\to\infty} \left(\lambda + \eta^-(\varepsilon_l)R^-_{\tilde{J}\tilde{J}}(\varepsilon_l)\right) \\ \overset{\text{[Kön01, p.43]}}{\Leftrightarrow} \lim_{l\to\infty} \eta^-(\varepsilon_l) &= \lim_{l\to\infty} \lambda + \left(\lim_{l\to\infty} \eta^-(\varepsilon_l)\right)\left(\lim_{l\to\infty} R^-_{\tilde{J}\tilde{J}}(\varepsilon_l)\right) \\ \Leftrightarrow \lim_{l\to\infty} \eta^-(\varepsilon_l) &= \lambda + \left(\lim_{l\to\infty} \eta^-(\varepsilon_l)\right) R^-_{\tilde{J}\tilde{J}}(0),\end{aligned}$$

which is the traffic equation of $X^-(0)$, so $\lim_{l\to\infty} \eta^-(\varepsilon_l) = \eta^-(0)$ holds for all convergent subsequences $(\eta^-(\varepsilon_l) : l \ge 0)$. Thus $\lim_{\varepsilon\to 0} \eta^-(\varepsilon) = \eta^-(0)$ is valid.

The following lemma will be used in the following task of showing the stochastic ordering of the three processes.

Lemma 5.17. *Consider the framework of the proof of Theorem 5.11.*
(i) If nodes in $I \neq \emptyset$ are broken down and $I \cap U = \emptyset$ (i.e., $I \subseteq S$), then the vectors

$$(r^I(k,j) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I), (r^{I,+}(k,j) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I),$$
$$\textit{and } (r^{I,-}(k,j)(\varepsilon) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I)$$

are component-wise equal.
(ii) If for the set I of broken down nodes holds $I \cap U \neq \emptyset$, only the vectors

$$(r^I(k,j) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I) \textit{ and } (r^{I,+}(k,j) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I)$$

are equal. For all $k \in I$ holds:

$$r^{I,-}(k,j)(\varepsilon) \le r^I(k,j)\ \forall j \in \tilde{J} \setminus I, \varepsilon > 0, \tag{5.18}$$
$$r^{I,-}(k,0)(\varepsilon) \ge r^I(k,0)\ \forall \varepsilon > 0. \tag{5.19}$$

In case $\varepsilon = 0$ the parameters are equal.

The proof of this lemma is rather technical and not essential for the further understanding of the proof. For readability one may skip the proof of Lemma 5.17 and proceed to task 5.

Proof of Lemma 5.17. (i) If $I \cap U = \emptyset$ ($I \subseteq S$) then the vectors $(r^I(k,j) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I)$, $(r^{I,+}(k,j) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I)$, and $(r^{I,-}(k,j)(\varepsilon) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I)$ are the unique solution of the equations

$$r^{I,*}(k,j) = r(k,j) + \sum_{l \in I} r(k,l) r^{I,*}(l,j), \quad k \in I, j \in \tilde{J} \setminus I, \tag{5.20}$$

$$r^{I,*}(k,0) = r(k,0) + \sum_{l \in I} r(k,l) r^{I,*}(l,0), \quad k \in I, \tag{5.21}$$

where $r^{I,*}$ is a representative for r^I, $r^{I,+}$ and $r^{I,-}(\varepsilon)$, for all $\varepsilon \geq 0$, so the vectors are component-wise equal.

(ii) If $I \cap U \neq \emptyset$, only $(r^I(k,j) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I)$ and $(r^{I,+}(k,j) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I)$ are the unique solutions of (5.20) and (5.21), so they are equal. In contrast, the vector $(r^{I,-}(k,j)(\varepsilon) : k \in I, j \in \{0\} \cup \tilde{J} \setminus I)$ is the unique solution of the equations, where we utilize (5.8) which is $\eta_i = \mu_i \ \forall i \in U \cap D$,

$$r^{I,-}(k,j)(\varepsilon) = r(k,j) + \sum_{l \in I} r(k,l) r^{I,-}(l,j)(\varepsilon), \ k \in I \cap S, j \in \tilde{J} \setminus I, \tag{5.22}$$

$$r^{I,-}(k,j)(\varepsilon) = \frac{\mu_k}{\mu_k + \varepsilon} \Big(r(k,j) + \sum_{l \in I} r(k,l) r^{I,-}(l,j)(\varepsilon) \Big), \ k \in I \cap U, j \in \tilde{J} \setminus I, \tag{5.23}$$

$$r^{I,-}(k,0)(\varepsilon) = r(k,0) + \sum_{l \in I} r(k,l) r^{I,-}(l,0)(\varepsilon), \ k \in I \cap S, \tag{5.24}$$

$$r^{I,-}(k,0)(\varepsilon) = \frac{\mu_k}{\mu_k + \varepsilon} \Big(\frac{\varepsilon}{\mu_k} + r(k,0) + \sum_{l \in I} r(k,l) r^{I,-}(l,0)(\varepsilon) \Big), \ k \in I \cap U, \tag{5.25}$$

for all $\varepsilon \geq 0$. In matrix notation we have:

$$(5.20) \ \Rightarrow \ R^I_{I \ \tilde{J} \setminus I} = R_{I \ \tilde{J} \setminus I} + R_{II} R^I_{I \ \tilde{J} \setminus I}, \tag{5.26}$$

$$(5.21) \ \Rightarrow \ R^I_{I \ \{0\}} = R_{I \ \{0\}} + R_{II} R^I_{I \ \{0\}}, \tag{5.27}$$

$$\begin{aligned}(5.22) \ \Leftrightarrow \ R^{I,-}_{I \cap S \ \tilde{J} \setminus I}(\varepsilon) &= R_{I \cap S \ \tilde{J} \setminus I} + R_{I \cap S \ I} R^{I,-}_{I \ \tilde{J} \setminus I}(\varepsilon) \\ &= R_{I \cap S \ \tilde{J} \setminus I} + R_{I \cap S \ I \cap S} R^{I,-}_{I \cap S \ \tilde{J} \setminus I}(\varepsilon) + R_{I \cap S \ I \cap U} R^{I,-}_{I \cap U \ \tilde{J} \setminus I}(\varepsilon),\end{aligned} \tag{5.28}$$

$$(5.23) \ \Leftrightarrow \ R^{I,-}_{I \cap U \ \tilde{J} \setminus I}(\varepsilon) = C(\varepsilon) \left(R_{I \cap U \ \tilde{J} \setminus I} + R_{I \cap U \ I} R^{I,-}_{I \ \tilde{J} \setminus I}(\varepsilon) \right), \tag{5.29}$$

$$\begin{aligned}(5.24) \ \Leftrightarrow \ R^{I,-}_{I \cap S \ \{0\}}(\varepsilon) &= R_{I \cap S \ \{0\}} + R_{I \cap S \ I} R^{I,-}_{I \ \{0\}}(\varepsilon) \\ &= R_{I \cap S \ \{0\}} + R_{I \cap S \ I \cap S} R^{I,-}_{I \cap S \ \{0\}}(\varepsilon) + R_{I \cap S \ I \cap U} R^{I,-}_{I \cap U \ \{0\}}(\varepsilon),\end{aligned} \tag{5.30}$$

$$(5.25) \ \Leftrightarrow \ R^{I,-}_{I \cap U \ \{0\}}(\varepsilon) = C(\varepsilon) \left(d(\varepsilon)^\top + R_{I \cap U \ \{0\}} + R_{I \cap U \ I} R^{I,-}_{I \ \{0\}}(\varepsilon) \right), \tag{5.31}$$

where $C(\varepsilon) := (\delta_{ij} \frac{\mu_i}{\mu_i + \varepsilon} : i, j \in I \cap U)$ and $d(\varepsilon) := (\frac{\varepsilon}{\mu_i} : i \in I \cap U)$. And more detailed it holds:

$$(5.26) \Rightarrow R^I_{I\cap S\ \tilde{J}\setminus I} = R_{I\cap S\ \tilde{J}\setminus I} + R_{I\cap S\ I} R^I_{I\ \tilde{J}\setminus I}, \tag{5.32}$$

$$R^I_{I\cap U\ \tilde{J}\setminus I} = R_{I\cap U\ \tilde{J}\setminus I} + R_{I\cap U\ I} R^I_{I\ \tilde{J}\setminus I}, \tag{5.33}$$

$$(5.27) \Rightarrow R^I_{I\cap S\ \{0\}} = R_{I\cap S\ \{0\}} + R_{I\cap S\ I} R^I_{I\ \{0\}}, \tag{5.34}$$

$$R^I_{I\cap U\ \{0\}} = R_{I\cap U\ \{0\}} + R_{I\cap U\ I} R^I_{I\ \{0\}}. \tag{5.35}$$

Transforming and multiplying with the existing and positive inverse $(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}$ (see Lemma 1.2) yields

$$(5.28) \Leftrightarrow R^{I,-}_{I\cap S\ \tilde{J}\setminus I}(\varepsilon) = (\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}\left(R_{I\cap S\ \tilde{J}\setminus I} + R_{I\cap S\ I\cap U} R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon)\right), \tag{5.36}$$

$$(5.30) \Leftrightarrow R^{I,-}_{I\cap S\ \{0\}}(\varepsilon) = (\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}\left(R_{I\cap S\ \{0\}} + R_{I\cap S\ I\cap U} R^{I,-}_{I\cap U\ \{0\}}(\varepsilon)\right), \tag{5.37}$$

$$(5.32) \Leftrightarrow R^I_{I\cap S\ \tilde{J}\setminus I} = (\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}\left(R_{I\cap S\ \tilde{J}\setminus I} + R_{I\cap S\ I\cap U} R^I_{I\cap U\ \tilde{J}\setminus I}\right), \tag{5.38}$$

$$(5.34) \Leftrightarrow R^I_{I\cap S\ \{0\}} = (\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}\left(R_{I\cap S\ \{0\}} + R_{I\cap S\ I\cap U} R^I_{I\cap U\ \{0\}}\right). \tag{5.39}$$

Plugging these solutions into (5.29), (5.31), (5.33) and (5.35) implies:

$$(5.36) \text{ in } (5.29) \Rightarrow R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon) = C(\varepsilon)\Big(R_{I\cap U\ \tilde{J}\setminus I} + R_{I\cap U\ I\cap U} R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon) + \\ + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}\Big[R_{I\cap S\ \tilde{J}\setminus I} + R_{I\cap S\ I\cap U} R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon)\Big]\Big), \tag{5.40}$$

$$(5.37) \text{ in } (5.31) \Rightarrow R^{I,-}_{I\cap U\ \{0\}}(\varepsilon) = C(\varepsilon)\Big(d(\varepsilon)^\top + R_{I\cap U\ \{0\}} + R_{I\cap U\ I\cap U} R^{I,-}_{I\cap U\ \{0\}}(\varepsilon) + \\ + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}\Big[R_{I\cap S\ \{0\}} + R_{I\cap S\ I\cap U} R^{I,-}_{I\cap U\ \{0\}}(\varepsilon)\Big]\Big), \tag{5.41}$$

$$(5.38) \text{ in } (5.33) \Rightarrow R^I_{I\cap U\ \tilde{J}\setminus I} = R_{I\cap U\ \tilde{J}\setminus I} + R_{I\cap U\ I\cap U} R^I_{I\cap U\ \tilde{J}\setminus I} + \\ + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}\Big(R_{I\cap S\ \tilde{J}\setminus I} + R_{I\cap S\ I\cap U} R^I_{I\cap U\ \tilde{J}\setminus I}\Big), \tag{5.42}$$

$$(5.39) \text{ in } (5.35) \Rightarrow R^I_{I\cap U\ \{0\}} = R_{I\cap U\ \{0\}} + R_{I\cap U\ I\cap U} R^I_{I\cap U\ \{0\}} + \\ + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}\Big(R_{I\cap S\ \{0\}} + R_{I\cap S\ I\cap U} R^I_{I\cap U\ \{0\}}\Big). \tag{5.43}$$

Transforming these equations yields

$$(5.40) \Leftrightarrow \left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ I\cap U}]\right) R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon) = \\ = C(\varepsilon)\left(R_{I\cap U\ \tilde{J}\setminus I} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ \tilde{J}\setminus I}\right), \tag{5.44}$$

$$(5.41) \Leftrightarrow \left(\boldsymbol{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ I\cap U}]\right) R^{I,-}_{I\cap U\ \{0\}}(\varepsilon) = \\ = C(\varepsilon)\left(d(\varepsilon)^\top + R_{I\cap U\ \{0\}} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ \{0\}}\right), \tag{5.45}$$

$$(5.42) \Leftrightarrow \left(\mathbf{I} - [R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ I\cap U}]\right) R^I_{I\cap U\ \tilde{J}\setminus I} = \\ = R_{I\cap U\ \tilde{J}\setminus I} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ \tilde{J}\setminus I}, \tag{5.46}$$

$$(5.43) \Leftrightarrow \left(\mathbf{I} - [R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ I\cap U}]\right) R^I_{I\cap U\ \{0\}} = \\ = R_{I\cap U\ \{0\}} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ \{0\}}. \tag{5.47}$$

After multiplying from the left with $C(\varepsilon)^{-1}$ on both sides of equation (5.44) and comparing this with (5.46) yields

$$\begin{aligned}
&C(\varepsilon)^{-1}\left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right) R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon) = \\
&\quad = \left(\mathbf{I} - [R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right) R^{I}_{I\cap U\ \tilde{J}\setminus I} \\
&\Leftrightarrow \left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right) R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon) = \\
&\quad = \left(C(\varepsilon)\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right) R^{I}_{I\cap U\ \tilde{J}\setminus I} \\
&\Leftrightarrow R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon) = R^{I}_{I\cap U\ \tilde{J}\setminus I} + \\
&\quad + \underbrace{\left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right)^{-1}}_{\overset{(\spadesuit)}{\geq} 0} \underbrace{(C(\varepsilon)\mathbf{I} - \mathbf{I})}_{<0} R^{I}_{I\cap U\ \tilde{J}\setminus I},
\end{aligned}$$

thus $R^{I,-}_{I\cap U\ \tilde{J}\setminus I}(\varepsilon) \leq R^{I}_{I\cap U\ \tilde{J}\setminus I}$ holds component-wise for all $\varepsilon > 0$ and therefore (5.18) is valid for all $k \in I$.
Analogously, it follows by comparison of (5.45) and (5.47):

$$\begin{aligned}
&\left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right) R^{I,-}_{I\cap U\ \{0\}}(\varepsilon) - C(\varepsilon)d(\varepsilon)^{\top} = \\
&= C(\varepsilon)\left(\mathbf{I} - [R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right) R^{I}_{I\cap U\ \{0\}} \\
&\Leftrightarrow \\
&\left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right) R^{I,-}_{I\cap U\ \{0\}}(\varepsilon) = \\
&= \left(C(\varepsilon)\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right) R^{I}_{I\cap U\ \{0\}} + C(\varepsilon)d(\varepsilon)^{\top} \\
&\Leftrightarrow \\
&R^{I,-}_{I\cap U\ \{0\}}(\varepsilon) = R^{I}_{I\cap U\ \{0\}} + \\
&\quad + \left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right)^{-1} (C(\varepsilon)\mathbf{I} - \mathbf{I})R^{I}_{I\cap U\ \{0\}} + \\
&\quad + \left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right)^{-1} C(\varepsilon)d(\varepsilon)^{\top} \\
&\Leftrightarrow \\
&R^{I,-}_{I\cap U\ \{0\}}(\varepsilon) = R^{I}_{I\cap U\ \{0\}} + \underbrace{\left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right)^{-1}}_{\overset{(\spadesuit)}{\geq} 0} \cdot \\
&\qquad\qquad \cdot (C(\varepsilon)(R^{I}_{I\cap U\ \{0\}} + d(\varepsilon)^{\top}) - R^{I}_{I\cap U\ \{0\}}),
\end{aligned}$$

hence for all $k \in I$ holds (5.19) for $\varepsilon > 0$ because $C(\varepsilon)(R^{I}_{I\cap U\ \{0\}} + d(\varepsilon)^{\top}) - R^{I}_{I\cap U\ \{0\}} > 0$ as can be seen component-wise for all $i \in I \cap U$:

$$\begin{aligned}
\frac{\mu_i}{\mu_i + \varepsilon}\left(r(i,0) + \frac{\varepsilon}{\mu_i}\right) - r(i,0) &= \frac{1}{\mu_i + \varepsilon}(\mu_i r(i,0) + \varepsilon - \mu_i r(i,0) - \varepsilon r(i,0)) \\
&= \frac{1}{\underbrace{\mu_i + \varepsilon}_{>0}} \underbrace{\varepsilon}_{\geq 0} \underbrace{(1 - r(i,0))}_{\geq 0} \geq 0.
\end{aligned}$$

It remains to show ($\spadesuit$): More precisely, we have to show that the inverse

$$\left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1}R_{I\cap S\ I\cap U}]\right)^{-1}$$

exists and is entry-wise non-negative for every $\varepsilon > 0$. Recall that $C_{ij}(\varepsilon) \in (0,1)$ for all $\varepsilon > 0$ and all entries $i, j \in I \cap U$ and $\lim_{\varepsilon \to 0} C(\varepsilon) = \mathbf{I}$.
We will therefore show that

$$\tilde{R}_{I\cap U\ I\cap U} := R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ I\cap U}$$

is a substochastic routing matrix because then $C(\varepsilon)\tilde{R}_{I\cap U\ I\cap U}$ a fortiori is a strict substochastic matrix, so its spectral radius is less than one (see [Bre99, p.198]). It follows directly that the Neumann series $\sum_{k=0}^{\infty}(C(\varepsilon)\tilde{R}_{I\cap U\ I\cap U})^k$ with $(C(\varepsilon)\tilde{R}_{I\cap U\ I\cap U})^0 := \mathbf{I}$ converges component-wise to $(\mathbf{I} - C(\varepsilon)\tilde{R}_{I\cap U\ I\cap U})^{-1}$ (see [Heu06, Satz 12.4 and p.127]).

$\tilde{R}_{I\cap U\ I\cap U}$ has the structure of a routing matrix according to skipping, if exactly the nodes in $I \cap S$ are broken down, see Definition 2.7, and the sub-matrix on $I \cap U \times I \cap U$ is considered. Since it is a routing matrix, $\tilde{R}_{I\cap U\ I\cap U}$ can be either stochastic or substochastic.

Suppose that $\tilde{R}_{I\cap U\ I\cap U}$ is stochastic. Then each row of the matrix sums to 1, i.e., $\sum_{j\in I\cap U} \tilde{r}(i,j) = 1$ holds for all $i \in I \cap U$. This implies that $\tilde{r}(i,j) = 0$ for all $i \in I \cap U$ and all $j \in \{0\} \cup \tilde{J} \setminus I$, so $I \cap U$ is a closed set whenever nodes in $I \cap S$ are broken down and rerouting is according to skipping. But $I \cap U$ being a closed set is a contradiction to the new routing matrix $\tilde{R}$ (after rerouting if nodes in $I \cap S$ are broken down) being irreducible on $\{0\} \cup \tilde{J} \setminus (I \cap S)$, see [Sau06, Remark 1.2.18]. Thus $\tilde{R}_{I\cap U\ I\cap U}$ is substochastic.

Since $\tilde{R}_{I\cap U\ I\cap U}$ is a matrix of routing probabilities, all entries are non-negative, hence $C(\varepsilon)\tilde{R}_{I\cap U\ I\cap U}$ is also entry-wise non-negative for every $\varepsilon > 0$, so $\sum_{k=0}^{\infty}(C(\varepsilon)\tilde{R}_{I\cap U\ I\cap U})^k \geq 0$ holds which implies that the inverse

$$\left(\mathbf{I} - C(\varepsilon)[R_{I\cap U\ I\cap U} + R_{I\cap U\ I\cap S}(\mathbf{I} - R_{I\cap S\ I\cap S})^{-1} R_{I\cap S\ I\cap U}]\right)^{-1}$$

is entry-wise non-negative, too. □

5. Stochastic ordering of the processes: The joint construction of the processes is a standard coupling construction and utilizes ideas of Goodman and Massey [GM84]. Again, the special issue is to handle the breakdown and repair processes. We shall perform a sequence of successive constructions each for a random time during which the availability status of the networks stays invariant. Whenever the availability status of the networks changes we switch to a new, suitably adapted, construction scheme. The construction ensures that switching the construction mode for all processes happens at exactly the same times.

From the construction of the processes it will be immediately seen that all the Q-matrices have bounded diagonals and therefore are uniformizable, i.e., they have a compound Poisson representation: Let $0 < \zeta < \infty$ be an upper bound of the modulus of the diagonal entries of all three processes to be compared. Then in a standard coupling the processes are driven by a Poisson-ζ process, which determines the common jump times, and three (different) Markov chain jump matrices, which are coupled and determine the direction of the jumps.

Our plan is not to use a single Poisson process to determine the common jump times, but to use Poisson processes of different intensities over random time intervals, and then similarly different jump matrices.

To start the system we assume that initially all networks are empty and all nodes are up. Let us assume that at some (random) time τ a change of the availability status has happened such that nodes in I are under repair and assume that (ω-wise)

$$(Y, X^{-}(\varepsilon))(\tau) \prec (Y, X)(\tau) \prec (Y, X^{+})(\tau)$$

holds. Now we describe the development of the processes until the next availability change occurs at some (random) time τ'. We run a Poisson process with rate

$$\zeta(I) := \sum_{i \in \tilde{J}} \lambda_i + \sum_{i \in \tilde{J} \setminus I} \mu_i + \sum_{j \in U \setminus I} (\eta_j - \mu_j) + \varepsilon |U \setminus I| + \\ + \sum_{\emptyset \neq K \subseteq D \setminus I} \alpha(I, I \cup K) + \sum_{\emptyset \neq H \subseteq I} \beta(I, I \setminus H),$$

which generates the jump instants for the processes. At each jump instant we select according to some probability law (to be specified below) an activity from the following set, which is performed for all the three processes jointly unless otherwise indicated:

$[A_i]$	Add a customer to node i.
$[B_i]$	Delete a customer from node i if the queue length at i is positive, otherwise do nothing.
$[C_{ij}]$	Transfer a customer from i to j if the queue length at i is positive, otherwise do nothing.
$[D]$	Do nothing.
$\left[E_{(I, I \cup K)}\right]$	τ' occurs, deactivate nodes of the non-void subset $K \subseteq D \setminus I$, stop the running Poisson process and start an independent new Poisson-$\zeta(I \cup K)$ process.
$\left[E_{(I, I \setminus H)}\right]$	τ' occurs, reactivate nodes of the non-void subset $H \subset I$, stop the running Poisson process and start an independent new Poisson-$\zeta(I \setminus H)$ process.

The breakdown and repair activities are selected for all processes the same way according to the following rules:

- With probability $\alpha(I, I \cup K)/\zeta(I)$ select $\left[E_{(I, I \cup K)}\right]$;
- with probability $\beta(I, I \setminus H)/\zeta(I)$ select $\left[E_{(I, I \setminus H)}\right]$.

Note, that by these activities a $\prec$-order of the processes is maintained.

To perform the selection of queue-length activities under the skipping regime the transition intensities after rerouting are utilized. Whenever nodes in $I \neq \emptyset$ are down, transition intensities for (Y, X) are as in Definition 2.7, for the bounding processes as defined above in the construction. These intensities are fixed on $[\tau, \tau')$.
Whenever the Poisson process indicates a jump and no availability change was selected by the randomization procedure, we utilize the intensities to select changes in the queue

lengths:
If nodes in $I \subseteq D$ are broken down and $I \cap U = \emptyset$, Lemma 5.17(i) applies:

- For all $i \in \tilde{J} \setminus I$ select with probability $(\lambda_i + \sum_{k\in I} \lambda_k r^I(k,i))/\zeta(I)$:
  $[A_i]$ = Add a customer to node i;

- for all $i \in \tilde{J} \setminus I$ with probability $\mu_i(r(i,0) + \sum_{k\in I} r(i,k) r^I(k,0))/\zeta(I)$:
  $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;

- for all $i \in \tilde{J}\setminus I, j \in \tilde{J}\setminus I, j \neq i$, with probability $\mu_i(r(i,j) + \sum_{k\in I} r(i,k) r^I(k,j))/\zeta(I)$:
  $[C_{ij}]$ = Transfer a customer from i to j if the queue length at i is positive, otherwise do nothing;

- for all $i \in U, j \in \tilde{J} \setminus I$, select with probability $\mu_i(r(i,j) + \sum_{k\in I} r(i,k) r^I(k,j))/\zeta(I)$, if the queue length at i is zero:
  ∘ for (Y, X^+): $[A_j]$ = Add a customer to node j,
  ∘ for (Y, X) and $(Y, X^-(\varepsilon))$: $[D]$ = Do nothing;
  (*This reflects the infinite supply at nodes in* U *in* (Y, X^+).)

- for all $i \in U$ with probability $(\eta_i - \mu_i + \varepsilon)/\zeta(I)$ select
  ∘ for $(Y, X^-(\varepsilon))$: $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;
  ∘ for (Y, X) and (Y, X^+): $[D]$ = Do nothing.
  (*This reflects the additional departure rates of* $(Y, X^-(\varepsilon))$ *which compensate the additional flows generated by making the nodes in* U *stable, see* (5.17).)

- For all $i \in \tilde{J} \setminus I$ select with probability $\mu_i(r(i,i) + \sum_{k\in I} r(i,k) r^I(k,i))/\zeta(I)$:
  $[D]$ = Do nothing.

If nodes in $I \subseteq D$ are broken down and $I \cap U \neq \emptyset$, for (Y, X) and (Y, X^+) all probabilities for the selection of activities are as above, whereas the rerouting probabilities for $(Y, X^-(\varepsilon))$ are now not the same as for the other processes, if $\varepsilon > 0$, because Lemma 5.17(ii) applies:

- For all $i \in \tilde{J} \setminus I$ select with probability $(\lambda_i + \sum_{k\in I} \lambda_k r^{I,-}(k,i))/\zeta(I)$:
  $[A_i]$ = Add a customer to node i;

- for all $i \in \tilde{J} \setminus I$ select with probability $\sum_{k\in I} \lambda_k (r^I(k,i) - r^{I,-}(k,i)(\varepsilon))/\zeta(I)$:
  ∘ for (Y, X) and (Y, X^+): $[A_i]$ = Add a customer to node i;
  ∘ for $(Y, X^-(\varepsilon))$: $[D]$ = Do nothing.
  (*Note that* $r^I(k,i) - r^{I,-}(k,i)(\varepsilon) \geq 0$ *holds for all* $k \in I$, $i \in \tilde{J} \setminus I$, $\varepsilon > 0$, *see Lemma 5.17(ii).*)

- For all $i \in \tilde{J} \setminus I$ with probability $\mu_i(r(i,0) + \sum_{k\in I} r(i,k) r^I(k,0))/\zeta(I)$:
  $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;

- for all $i \in \tilde{J} \setminus I$ with probability $\mu_i \sum_{k\in I} r(i,k)(r^{I,-}(k,0)(\varepsilon) - r^I(k,0))/\zeta(I)$:
  ∘ for $(Y, X^-(\varepsilon))$: $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;

∘ for (Y,X) and (Y,X^+): $[D]$ = Do nothing.
(*Note that* $r^{I,-}(k,0)(\varepsilon)-r^I(k,0)\ge 0$ *holds for all* $k\in I$, $\varepsilon>0$, *see Lemma 5.17(ii).*)

- For all $i\in\tilde{J}\setminus I, j\in\tilde{J}\setminus I, j\neq i$, with probability $\mu_i(r(i,j)+\sum_{k\in I}r(i,k)r^{I,-}(k,j))/\zeta(I)$:
  $[C_{ij}]$ = Transfer a customer from i to j if the queue length at i is positive, otherwise do nothing;

- for all $i\in\tilde{J}\setminus I, j\in\tilde{J}\setminus I, j\neq i$, select with probability $\mu_i\sum_{k\in I}r(i,k)(r^I(k,j)-r^{I,-}(k,j)(\varepsilon))/\zeta(I)$:
  ∘ for (Y,X) and (Y,X^+): $[C_{ij}]$ = Transfer a customer from i to j if the queue length at i is positive, otherwise do nothing;
  ∘ for $(Y,X^-(\varepsilon))$: $[D]$ = Do nothing.
  (*Note that* $r^I(k,j)-r^{I,-}(k,j)(\varepsilon)\ge 0$ *holds for all* $k\in I$, $j\in\tilde{J}\setminus I$, $\varepsilon>0$, *see Lemma 5.17(ii). This implies that the probability for a transition inside the network including immediate feedback at up nodes is smaller for* (Y,X^-) *than for* (Y,X) *and* (Y,X^+) *(or equal):*

$$\mu_i\Big(r(i,j)+\sum_{k\in I}r(i,k)r^{I,-}(k,j)(\varepsilon)\Big)\frac{1}{\zeta(I)}\le\mu_i\Big(r(i,j)+\sum_{k\in I}r(i,k)r^I(k,j)\Big)\frac{1}{\zeta(I)}$$

  holds for all $i,j\in\tilde{J}\setminus I$ *(for* $i\neq j$ *as well as for* $i=j$*).*)

- For all $i\in U, j\in\tilde{J}\setminus I$, select with probability $\mu_i(r(i,j)+\sum_{k\in I}r(i,k)r^I(k,j))/\zeta(I)$, if the queue length at i is zero:
  ∘ for (Y,X^+): $[A_j]$ = Add a customer to node j;
  ∘ for (Y,X) and $(Y,X^-(\varepsilon))$: $[D]$ = Do nothing.
  (*This reflects the infinite supply at nodes in* U *in* (Y,X^+).)

- For all $i\in U$ with probability $(\eta_i-\mu_i+\varepsilon)/\zeta(I)$ select
  ∘ for $(Y,X^-(\varepsilon))$: $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;
  ∘ for (Y,X) and (Y,X^+): $[D]$ = Do nothing.
  (*This reflects the additional departure rates of* $(Y,X^-(\varepsilon))$ *which compensate the additional flows generated by making the nodes in* U *stable, see* (5.17).)

Note, that by these activities a $\prec$-order of the processes is maintained.

We conclude that by starting the processes $\prec_{st}$-ordered with the same $I\subseteq D$ this order will be maintained over time, so the equilibria of the bounding processes will still be ordered and they bound the limiting distribution of the process (Y,X).

The last step of the proof is the conclusion of the limiting probabilities of (Y,X) which can be found in the outline of the proofs numbered as 6th task on pages 124-126. □

Proof of Theorem 5.12 (Rerouting according to blocking rs-rd)

1. Construction of the upper bound: We derive (Y,X^+) from (Y,X) as follows:

- The breakdown and repair processes are (stochastically) identical;

- the service rates at all nodes are the same as in the original network;
- all nodes in U are supplemented by an infinite supply of work, this leads to Poissonian departure processes from those nodes (see Theorem 4.3 and Theorem 4.38);
- the rerouting regime is selected as blocking rs-rd, so whenever nodes of $I \subseteq D$ break down, the transition intensities are as in Definition 4.37 with Definition 2.8.

2. (Y, X_S^+) is an ergodic homogeneous Markov process of its own: (Y, X^+) represents a Jackson network with unreliable nodes where the nodes in U have an infinite supply of work, as in Definition 4.37 with $U = V$ and $S = W$. $(\eta_i^+ : i \in \tilde{J})$ is the unique solution of the traffic equations (4.1)

$$\begin{aligned}\eta_i^+ &= \lambda_i + \sum_{j \in W} \eta_j^+ r(j,i) + \sum_{j \in V} \mu_j r(j,i) \\ &= \lambda_i + \sum_{j \in S} \eta_j^+ r(j,i) + \sum_{j \in U} \mu_j r(j,i), \quad \forall i \in \tilde{J},\end{aligned}$$

so $\eta_i^+ = \eta_i$ for all $i \in \tilde{J}$. Recalling $S = \{i : \eta_i < \mu_i\}$, it follows directly that $\eta_i^+ < \mu_i$ holds for all $i \in S$. This local stability criterion holds in any availability status, because with (5.3), (5.4) and (5.5) the solution of the standard traffic equations remains the same for nodes in up status, as can be seen in Lemma 4.40.

Since $W \cap U = \emptyset$, rerouting is according to blocking rs-rd and the reversibility contraints (5.3), (5.4) and (5.5) hold, Theorem 4.47 applies and provides us with a product form limiting distribution for the process $(Y, X_S^+) := (Y, X_i^+ : i \in S)$. From Corollary 4.48 we know that (Y, X_S^+) is ergodic independent of the behavior of the remaining nodes in the network. And again, the last step of the argument is that under load-independent breakdown and repair rates, an ergodic Jackson network remains ergodic, see Example 3.3.

3. Construction of the lower bounds: We derive $(Y, X^-(\varepsilon))$ from (Y, X) for $\varepsilon > 0$ as follows:

- The breakdown and repair processes are (stochastically) identical;
- the external arrival rates at all nodes are identical to those in the original network;
- the service rates at nodes in S are the same as in the original network, the service rates at nodes in U are increased from μ_i to $\eta_i + \varepsilon$:

$$\mu_i^-(\varepsilon) = \begin{cases} \mu_i, & i \in S, \\ \eta_i + \varepsilon, & i \in U, \end{cases}$$

  which guarantees that all nodes in the network are stable now, as will be shown in the next step (4.) of the proof;

- this generates additional customer flows (at least) out of the nodes in U; this additional load on the network is then reduced by increasing the probability for customers to leave nodes in U directly to the external sink; we prescribe

$$r^{-}(i,j)(\varepsilon)=\begin{cases}r(i,j), & i\in S, j\in\tilde{J},\\ \frac{\mu_i}{\eta_i+\varepsilon}\, r(i,j), & i\in U, j\in\tilde{J},\end{cases}$$
$$r^{-}(i,0)(\varepsilon)=\begin{cases}r(i,0), & i\in S,\\ \frac{\mu_i r(i,0)+\eta_i-\mu_i+\varepsilon}{\eta_i+\varepsilon}, & i\in U;\end{cases}$$

- the combination of adjusted service rates and new routing probabilities yields the desired reproduction of flows on S with

$$\mu_i^{-}(\varepsilon)r^{-}(i,j)(\varepsilon)=\mu_i r(i,j)\quad \forall\, i,j\in\tilde{J},$$
$$\mu_i^{-}(\varepsilon)r^{-}(i,0)(\varepsilon)=\begin{cases}\mu_i r(i,0), & i\in S,\\ \mu_i r(i,0)+\eta_i-\mu_i+\varepsilon, & i\in U;\end{cases}\tag{5.48}$$

- the rerouting regime is selected as blocking rs-rd, so whenever nodes in $I\subseteq D$ break down and because blocking rs-rd is in force, the parameters are

$$r^{I,-}(i,j)(\varepsilon)=\begin{cases}r(i,j), & i\in\{0\}\cup S\setminus I, j\in\tilde{J}\setminus I, i\neq j,\\ r(i,i)+\sum_{k\in I} r(i,k), & i\in\{0\}\cup S\setminus I, i=j,\\ \frac{\mu_i}{\eta_i+\varepsilon}\, r(i,j), & i\in U\setminus I, j\in\tilde{J}\setminus I, i\neq j,\\ \frac{\mu_i}{\eta_i+\varepsilon}\left(r(i,i)+\sum_{k\in I} r(i,k)\right), & i\in U\setminus I, i=j,\\ 0, & \text{else},\end{cases}$$
$$r^{I,-}(i,0)(\varepsilon)=\begin{cases}r(i,0), & i\in S\setminus I,\\ \frac{\mu_i r(i,0)+\eta_i-\mu_i+\varepsilon}{\eta_i+\varepsilon}, & i\in U\setminus I,\\ 0, & i\in I,\end{cases}$$
$$\mu_i^{I,-}(\varepsilon)=\begin{cases}\mu_i, & i\in S\setminus I,\\ \eta_i+\varepsilon, & i\in U\setminus I,\\ 0, & i\in I,\end{cases}$$

and the transition intensities of $(Y, X^{-}(\varepsilon))$ for changes in the queue lengths are

$$\lambda_i^{I,-}(\varepsilon)=\begin{cases}\lambda_i, & i\in\tilde{J}\setminus I,\\ 0, & i\in I,\end{cases}$$
$$\mu_i^{I,-}(\varepsilon)r^{I,-}(i,j)(\varepsilon)=\begin{cases}\mu_i r(i,j), & i,j\in\tilde{J}\setminus I, i\neq j\\ \mu_i\left(r(i,i)+\sum_{k\in I} r(i,k)\right), & i,j\in\tilde{J}\setminus I, i=j\\ 0, & \text{else},\end{cases}$$
$$\mu_i^{I,-}(\varepsilon)r^{I,-}(i,0)(\varepsilon)=\begin{cases}\mu_i r(i,0), & i\in S\setminus I,\\ \mu_i r(i,0)+\eta_i-\mu_i+\varepsilon, & i\in U\setminus I,\\ 0, & \text{else}.\end{cases}$$

4. $(Y, X^{-}(\varepsilon)),\ \varepsilon > 0$, **is ergodic:** Consider the network described above by the process $(Y, X^{-}(\varepsilon)),\ \varepsilon > 0$, with completely reliable nodes, which is just a classical Jacksonian network with queue-length process $X^{-}(\varepsilon)$ and traffic equations

$$\begin{aligned} &\eta_i^{-}(\varepsilon) = \lambda_i^{-}(\varepsilon) + \sum_{j \in \tilde{J}} \min(\eta_j^{-}(\varepsilon), \mu_j^{-}(\varepsilon)) r^{-}(j,i)(\varepsilon)\ \forall i \in \tilde{J} \\ \Leftrightarrow\ &\eta^{-}(\varepsilon) = \lambda^{-}(\varepsilon) + (\eta^{-}(\varepsilon) \wedge \mu^{-}(\varepsilon)) R^{-}_{\tilde{J}\tilde{J}}(\varepsilon). \end{aligned}$$

- For the solution η_i of the traffic equation (1.6) holds $\forall i \in \tilde{J}$

$$\begin{aligned} \eta_i &= \lambda_i + \sum_{j \in S} \eta_j r(j,i) + \sum_{j \in U} \mu_j r(j,i) \\ &= \lambda_i^{-}(0) + \sum_{j \in S} \eta_j r^{-}(j,i)(0) + \sum_{j \in U} \eta_j r^{-}(j,i)(0) \\ &= \lambda_i^{-}(0) + \sum_{j \in \tilde{J}} \eta_j r^{-}(j,i)(0), \end{aligned}$$

  thus $\eta = (\eta_i : i \in \tilde{J})$ solves the traffic equation for the process $(Y, X^{-}(0))$ (in matrix notation)

$$\eta^{-}(0) = \lambda^{-}(0) + \eta^{-}(0) R^{-}_{\tilde{J}\tilde{J}}(0).$$

  For all $i \in S$ it holds $\eta_i^{-}(0) = \eta_i < \mu_i = \mu_i^{-}(0)$ and for all $i \in U$ we have $\eta_i^{-}(0) = \eta_i = \eta_i + 0 = \mu_i^{-}(0)$. Therefore $\eta = (\eta_i : i \in \tilde{J})$ also solves the traffic equation (in matrix notation)

$$\eta^{-}(0) = \lambda^{-}(0) + (\eta^{-}(0) \wedge \mu^{-}(0)) R^{-}_{\tilde{J}\tilde{J}}(0).$$

- Now we show that $\eta_i^{-}(\varepsilon) < \mu_i^{-}(\varepsilon)$ holds for all $i \in \tilde{J}$:

$$\begin{aligned} \eta^{-}(\varepsilon) &= \lambda + (\eta^{-}(\varepsilon) \wedge \mu^{-}(\varepsilon)) R^{-}_{\tilde{J}\tilde{J}}(\varepsilon) \\ &\le \lambda + (\eta^{-}(\varepsilon) \wedge \mu^{-}(\varepsilon)) R^{-}_{\tilde{J}\tilde{J}}(0) \\ &\le \lambda + \eta^{-}(\varepsilon) R^{-}_{\tilde{J}\tilde{J}}(0) \\ \Rightarrow\ \eta^{-}(\varepsilon) &\le \lambda \left(\mathbf{I} - R^{-}_{\tilde{J}\tilde{J}}(0)\right)^{-1} = \eta^{-}(0), \end{aligned}$$

  and furthermore $\eta_U^{-}(0) = \mu_U^{-}(0) < \mu_U^{-}(\varepsilon)$ and $\eta_S^{-}(0) = \eta_S < \mu_S = \mu_S^{-}(\varepsilon)$ holds component-wise for $\varepsilon > 0$. So $\eta^{-}(\varepsilon) < \mu^{-}(\varepsilon)$ holds component-wise for all $\varepsilon > 0$ where $\eta^{-}(\varepsilon)$ is determined by

$$\eta^{-}(\varepsilon) = \lambda \left(\mathbf{I} - R^{-}_{\tilde{J}\tilde{J}}(\varepsilon)\right)^{-1},$$

  hence $X^{-}(\varepsilon)$ is ergodic. With (5.3), (5.4) and (5.5), which yields the reversibility constraint

$$\eta_i^{-}(\varepsilon) r^{-}(i,j)(\varepsilon) = \eta_j^{-}(\varepsilon) r^{-}(j,i)(\varepsilon) \quad \forall i,j \in \tilde{J},$$

  the local stability criterion holds in any availability status, i.e., $\eta_i^{I,-}(\varepsilon) < \mu_i^{I,-}(\varepsilon)$ $\forall i \in \tilde{J} \backslash I, I \subseteq D$, see Lemma 2.11. Furthermore under load-independent breakdown and repair rates, an ergodic Jackson network remains ergodic, see Example 3.3.

- It remains to show that $\lim_{\varepsilon\to 0}\eta_i^-(\varepsilon)=\eta_i$ holds for all $i\in\tilde{J}$. Since $\eta^-(0)=\eta$, we will show that $\lim_{\varepsilon\to 0}\eta^-(\varepsilon)=\eta^-(0)$. Recalling that $X^-(\varepsilon)$ is ergodic, $|\eta^-(\varepsilon)|_1<|\mu^-(\varepsilon)|_1$ holds for all $\varepsilon>0$, so there is $\varepsilon'<\infty$ such that $\{\eta^-(\varepsilon):0\le\varepsilon\le\varepsilon'\}$ is a bounded set of vectors in $\mathbb{R}^J$. Hence there is a convergent subsequence $(\eta^-(\varepsilon_l):l\ge 0)$ with $\lim_{l\to\infty}\varepsilon_l=0$. Then it holds component-wise

$$\lim_{l\to\infty} R^-_{\tilde{J}\tilde{J}}(\varepsilon_l)=R^-_{\tilde{J}\tilde{J}}(0)$$

and the traffic equations of $X^-(\varepsilon_l)$ are

$$\begin{aligned}\eta^-(\varepsilon_l)=\lambda+\eta^-(\varepsilon_l)R^-_{\tilde{J}\tilde{J}}(\varepsilon_l)\Rightarrow &\lim_{l\to\infty}\eta^-(\varepsilon_l)=\lim_{l\to\infty}\Big(\lambda+\eta^-(\varepsilon_l)R^-_{\tilde{J}\tilde{J}}(\varepsilon_l)\Big)\\ \overset{\text{[Kön01, p.43]}}{\Leftrightarrow}&\lim_{l\to\infty}\eta^-(\varepsilon_l)=\lim_{l\to\infty}\lambda+\Big(\lim_{l\to\infty}\eta^-(\varepsilon_l)\Big)\Big(\lim_{l\to\infty}R^-_{\tilde{J}\tilde{J}}(\varepsilon_l)\Big)\\ \Leftrightarrow&\lim_{l\to\infty}\eta^-(\varepsilon_l)=\lambda+\Big(\lim_{l\to\infty}\eta^-(\varepsilon_l)\Big)R^-_{\tilde{J}\tilde{J}}(0),\end{aligned}$$

which is the traffic equation of $X^-(0)$, so $\lim_{l\to\infty}\eta^-(\varepsilon_l)=\eta^-(0)$ holds for all convergent subsequences $(\eta^-(\varepsilon_l):l\ge 0)$. Thus $\lim_{\varepsilon\to 0}\eta^-(\varepsilon)=\eta^-(0)$ is valid.

5. Stochastic ordering of the processes: The joint construction of the processes is a standard coupling construction and utilizes ideas of Goodman and Massey [GM84]. Again, the special issue is to handle the breakdown and repair processes. We shall perform a sequence of successive constructions each for a random time during which the availability status of the networks stays invariant. Whenever the availability status of the networks changes we switch to a new, suitably adapted, construction scheme. The construction ensures that switching the construction mode for all processes happens at exactly the same times.

From the definition of the processes it is immediately seen that all the Q-matrices have bounded diagonals and therefore are uniformizable, i.e., they have a compound Poisson representation: Let $0<\zeta<\infty$ be an upper bound of the modulus of the diagonal entries of all three processes to be compared. Then in a standard coupling the processes are driven by a Poisson-ζ process, which determines the common jump times, and three (different) Markov chain jump matrices, which are coupled and determine the direction of the jumps.

Our plan is not to use a single Poisson process to determine the common jump times, but to use Poisson processes of different intensities over random time intervals, and then similarly different jump matrices.

To start the system we assume that initially all networks are empty and all nodes are up. Let us assume that at some (random) time τ a change of the availability status has happened such that nodes in I are under repair and assume that

$$(Y,X^-(\varepsilon))(\tau)\prec(Y,X)(\tau)\prec(Y,X^+)(\tau)$$

holds. Now we describe the development of the processes until the next availability change occurs at some (random) time τ'. We run a Poisson process with rate

$$\zeta(I) := \sum_{i\in\tilde{J}} \lambda_i + \sum_{i\in\tilde{J}\setminus I} \mu_i + \sum_{j\in U\setminus I} (\eta_j - \mu_j) + \varepsilon|U \setminus I| + \\ + \sum_{\emptyset\neq K\subseteq D\setminus I} \alpha(I, I\cup K) + \sum_{\emptyset\neq H\subseteq I} \beta(I, I\setminus H),$$

which generates the jump instants for the processes.
At each jump instant we select according to some probability law (to be specified below) an activity from the following set, which is performed for all the three processes jointly unless otherwise indicated:

$[A_i]$	Add a customer to node i.
$[B_i]$	Delete a customer from node i if the queue length at i is positive, otherwise do nothing.
$[C_{ij}]$	Transfer a customer from i to j if the queue length at i is positive, otherwise do nothing.
$[D]$	Do nothing.
$\left[E_{(I,I\cup K)}\right]$	τ' occurs, deactivate nodes of the non-void subset $K \subseteq D \setminus I$, stop the running Poisson process and start an independent new Poisson-$\zeta(I \cup K)$ process.
$\left[E_{(I,I\setminus H)}\right]$	τ' occurs, reactivate nodes of the non-void subset $H \subset I$, stop the running Poisson process and start an independent new Poisson-$\zeta(I \setminus H)$ process.

The breakdown and repair activities are selected for all processes the same way according to the following rules:

- With probability $\alpha(I, I \cup K)/\zeta(I)$ select $\left[E_{(I,I\cup K)}\right]$;
- with probability $\beta(I, I \setminus H)/\zeta(I)$ select $\left[E_{(I,I\setminus H)}\right]$.

Note, that by these activities a $\prec$-order of the processes is maintained.

To perform the selection of queue-length activities under the blocking rs-rd regime the transition intensities after rerouting are utilized. These intensities are fixed on $[\tau, \tau')$. Whenever the Poisson process indicates a jump and no availability change was selected by the randomization procedure, we utilize the intensities to select changes in the queue lengths:

- For all $i \in \tilde{J} \setminus I$ with probability $\lambda_i/\zeta(I)$:
  $[A_i]$ = Add a customer to node i;
- for all $i \in \tilde{J} \setminus I$ with probability $\mu_i r(i, 0)/\zeta(I)$:
  $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;
- for all $i \in \tilde{J} \setminus I, j \in \tilde{J} \setminus I, j \neq i$, select with probability $\mu_i r(i, j)/\zeta(I)$:
  $[C_{ij}]$ = Transfer a customer from i to j if the queue length at i is positive, otherwise do nothing;

- for all $i \in U \setminus I, j \in \tilde{J} \setminus I$, select with probability $\mu_i r^I(i,j)/\zeta(I)$ if the queue length at i is zero:
  ◦ for (Y, X^+): $[A_j]$ = Add a customer at node j,
  ◦ for (Y, X), and $(Y, X^-(\varepsilon))$: $[D]$ = Do nothing;
  (*This reflects the infinite supply at the nodes in* U *in* (Y, X^+).)

- for all $i \in U \setminus I$ with probability $(\eta_i - \mu_i + \varepsilon)/\zeta(I)$ select
  ◦ for $(Y, X^-(\varepsilon))$: $[B_i]$ = Delete a customer from node i if the queue length at i is positive, otherwise do nothing;
  ◦ for (Y, X) and (Y, X^+): $[D]$ = Do nothing.
  (*This reflects the additional departure rates of* $(Y, X^-(\varepsilon))$ *which compensate the additional flows generated by making the nodes in* U *stable, see* (5.48).)

- For all $i \in \tilde{J} \setminus I$ select with probability $\mu_i(r(i,i) + \sum_{k \in I} r(i,k))/\zeta(I)$:
  $[D]$ = Do nothing.

Note, that by these activities a $\prec$-order of the processes is maintained.

We conclude that by starting the processes $\prec_{st}$-ordered with the same $I \subseteq D$ this order will be maintained over time, so the equilibria of the bounding processes will still be ordered and they bound the limiting distribution of the process (Y, X).

The last step of the proof is the conclusion of the limiting probabilities of (Y, X) which can be found in the outline of the proofs numbered as 6th task on pages 124-126. □

5.4 Computation of availability and performance measures

Computing availability and performance measures of systems with non-stationary processes is often an extremely difficult task. Usually steady-state availability and performance measures are computed to assess the quality of service in such complex systems. This is not possible for networks with unstable nodes, because no steady state exists for the globally unstable system. Our results suggest to apply instead of steady-state analysis a limiting analysis - at least for those parts of the network process which stabilize over time towards a limiting distribution. [MD09, pp.1259-1260]
Theorem 5.10, Theorem 5.11 and Theorem 5.12 provide us with the required limiting marginal distributions for the queue lengths of the stable subnetwork S and for availabilities in a non-ergodic unreliable Jackson network.

Recall that under the conditions of Theorems 5.10-5.12 the availability process Y is an ergodic Markov process of its own, which has a unique limiting and stationary distribution. It is easy to see that the availability component Y of (Y, X) must converge to this steady-state distribution, which is

$$\pi(I) := \lim_{t\to\infty} P(Y(t) = I) = \sum_{(n_i:i\in S)\in\mathbb{N}^{|S|}} \lim_{t\to\infty} P(Y(t) = I; \{X_i = n_i : i \in S\})$$
$$= \left(\sum_{K\subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)}, \quad I \subseteq D,$$

where $D \subseteq \tilde{J}$ is the set of unreliable nodes. Consequently, we can even start the network process (Y, X) with the availability component Y in its steady state, and we can then consider the availability process as a stable environment process for the globally unstable network process.
In this situation, we can compute steady-state point-availability to find at time t exactly the nodes in the set K functioning. Exactly the nodes in $D \setminus K$ are down in steady-state with probability

$$\pi(D \setminus K) = \left(\sum_{I\subseteq D} \frac{A(I)}{B(I)}\right)^{-1} \frac{A(D \setminus K)}{B(D \setminus K)},$$

so this is the required probability for exactly the nodes in K being up at any time t, see [MD09].

Remark 5.18. *It is remarkable that, although the global network process is not ergodic, we can compute the stationary point-availability to find exactly the nodes in K functioning at some time t the same way as in the ergodic case, see Proposition 2.18. Example 2.19 applies here as well.*

Computing a stationary throughput is not possible, if there is no stationary distribution. The following definition and theorem propose a possible approach to compute the throughput of at least the stable subnetwork of a globally non-ergodic Jackson network with unreliable nodes.

Definition 5.19. *Consider a Jackson network where nodes in D are unreliable. Let $S := \{i : \eta_i < \mu_i\}$ denote the set of stable nodes and the breakdown and repair intensities be load-independent as in Example 2.2.*
The time-dependent throughput at time t for the stable subnetwork is

$$TH_S(t) = \sum_{(I;n_i:i\in S)\in\mathcal{P}(D)\times\mathbb{N}^{|S|}} P(Y(t) = I; \{X(t) = n_i : i \in S\}) \sum_{j\in S\setminus I} 1_{\mathbb{N}_+}(n_i)\mu_j^I r^I(j, 0).$$

Letting $t \to \infty$ yields the limiting throughput $\lim_{t\to\infty} TH_S(t)$ of the stable subnetwork.

Theorem 5.20. *Consider a Jackson network where nodes in D are unreliable. Let $\eta = (\eta_1, ..., \eta_J)$ be the unique solution of the general traffic equations* (1.6). *Let $S := \{i : \eta_i < \mu_i\}$ denote the set of stable nodes, nodes in $U := \tilde{J} \setminus S$ are unstable. Let the breakdown and repair intensities be load-independent as in Example 2.2.*

(i) In case of stalling, the limiting throughput equals

$$\lim_{t\to\infty} TH_S(t) = \pi(\emptyset)\Big(\sum_{j\in\tilde{J}} \lambda_j - \sum_{j\in U} \mu_j r(j, 0) - \sum_{j\in U}(\eta_j - \mu_j)\Big).$$

(ii) If skipping is in force and (5.8) *holds, the limiting throughput is*

$$\lim_{t\to\infty} TH_S(t) = \sum_{I\subseteq D} \pi(I)\Big(\lambda \sum_{j\in\tilde{J}\setminus I} r^I(0,j) - \sum_{j\in U\setminus I} \mu_j r^I(j,0) + \sum_{j\in U\setminus D} (\mu_j - \eta_j)\Big).$$

(iii) If rerouting is according to blocking rs-rd and if (5.3), (5.4), *and* (5.5) *hold, the limiting throughput is*

$$\lim_{t\to\infty} TH_S(t) = \sum_{I\subseteq D} \pi(I) \sum_{j\in S\setminus I} \lambda_j.$$

Proof. (i): With Theorem 5.10, we have

$$\begin{aligned}
&\lim_{t\to\infty} TH_S(t) = \\
&\overset{(\dagger 1)}{=} \sum_{(I;n_i:i\in S)\in\mathcal{P}(D)\times\mathbb{N}^{|S|}} \lim_{t\to\infty} P(Y(t)=I;\{X(t)=n_i : i\in S\}) \sum_{j\in S\setminus I} 1_{\mathbb{N}_+}(n_j) \underbrace{\mu_j^I r^I(j,0)}_{\overset{\text{Def. 2.6}}{=}\mu_j r(j,0)1_{\{\emptyset\}}(I)} \\
&= \sum_{(I;n_i:i\in S)\in\{\emptyset\}\times\mathbb{N}^{|S|}} \Big(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\Big)^{-1}\frac{A(I)}{B(I)} \prod_{i\in S}\Big(1-\frac{\eta_i}{\mu_i}\Big)\Big(\frac{\eta_i}{\mu_i}\Big)^{n_i} \sum_{j\in S} 1_{\mathbb{N}_+}(n_j)\mu_j r(j,0) \\
&= \underbrace{\Big(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\Big)^{-1}\frac{A(\emptyset)}{B(\emptyset)}}_{=\pi(\emptyset)} \sum_{j\in S} \underbrace{\sum_{(n_i:i\in S)\in\mathbb{N}^{|S|}} \prod_{i\in S}\Big(1-\frac{\eta_i}{\mu_i}\Big)\Big(\frac{\eta_i}{\mu_i}\Big)^{n_i}}_{=1} \eta_j r(j,0) \\
&\overset{\text{R. 1.21}}{=} \pi(\emptyset)\Big(\sum_{j\in\tilde{J}}\lambda_j - \sum_{j\in U}\mu_j r(j,0) - \sum_{j\in U}(\eta_j-\mu_j)\Big).
\end{aligned}$$

Here $\overset{(\dagger 1)}{=}$ follows from the fact that the convergence in (5.11) is weak convergence and that the throughput functions

$$(n_i : i\in S\setminus I) \mapsto \sum_{j\in S\setminus I} 1_{\mathbb{N}_+}(n_i)\mu_j^I r^I(j,0)$$

are bounded.

(ii)-(iii): With Theorem 5.11 and 5.12, resp., we have

$$\begin{aligned}
&\lim_{t\to\infty} TH_S(t) = \\
&\overset{(\dagger 2)}{=} \sum_{(I;n_i:i\in S)\in\mathcal{P}(D)\times\mathbb{N}^{|S|}} \lim_{t\to\infty} P(Y(t)=I;\{X(t)=n_i : i\in S\}) \sum_{j\in S\setminus I} 1_{\mathbb{N}_+}(n_j)\mu_j^I r^I(j,0) \\
&= \sum_{(I;n_i:i\in S)\in\mathcal{P}(D)\times\mathbb{N}^{|S|}} \Big(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\Big)^{-1}\frac{A(I)}{B(I)} \prod_{i\in S}\Big(1-\frac{\eta_i}{\mu_i}\Big)\Big(\frac{\eta_i}{\mu_i}\Big)^{n_i} \sum_{j\in S\setminus I} 1_{\mathbb{N}_+}(n_j)\mu_j r^I(j,0) \\
&= \sum_{I\subseteq D} \underbrace{\Big(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\Big)^{-1}\frac{A(I)}{B(I)}}_{=\pi(I)} \sum_{j\in S\setminus I} \underbrace{\sum_{(n_i:i\in S)\in\mathbb{N}^{|S|}} \prod_{i\in S}\Big(1-\frac{\eta_i}{\mu_i}\Big)\Big(\frac{\eta_i}{\mu_i}\Big)^{n_i}}_{=1} \eta_j r^I(j,0).
\end{aligned}$$

Here again $\overset{(\dagger 2)}{=}$ follows from the fact that the convergence in (5.11) is weak convergence and that the throughput functions

$$(n_i : i \in S \setminus I) \mapsto \sum_{j \in S \setminus I} 1_{\mathbb{N}_+}(n_i) \mu_j^I r^I(j,0)$$

are bounded.

In case of skipping, it holds:

$$\begin{aligned}
&\sum_{j \in \tilde{J} \setminus I} \eta_j = \sum_{j \in \tilde{J} \setminus I} \lambda_j^I + \sum_{j \in \tilde{J} \setminus I} \sum_{i \in S \setminus I} \eta_i r^I(i,j) + \sum_{j \in \tilde{J} \setminus I} \sum_{i \in U \setminus I} \mu_i r^I(i,j) \\
\Leftrightarrow &\sum_{j \in S \setminus I} \eta_j \underbrace{\Big(1 - \sum_{i \in \tilde{J} \setminus I} r^I(j,i)\Big)}_{=r^I(j,0)} + \sum_{j \in U \setminus I} \eta_j = \sum_{j \in \tilde{J} \setminus I} \lambda_j^I + \sum_{j \in \tilde{J} \setminus I} \sum_{i \in U \setminus I} \mu_i r^I(i,j) \\
\Leftrightarrow &\sum_{j \in S \setminus I} \eta_j r^I(j,0) = \lambda \sum_{j \in \tilde{J} \setminus I} r^I(0,j) - \sum_{j \in U \setminus I} \mu_j r^I(j,0) + \underbrace{\sum_{j \in U \setminus I} (\mu_j - \eta_j)}_{\overset{(5.8)}{=} \sum_{j \in U \setminus D} (\mu_j - \eta_j)} ,
\end{aligned}$$

thus (ii) follows.

In case of blocking rs-rd, for all $j \in \tilde{J} \setminus I$, $I \subseteq D$, holds $r^I(j,0) = r(j,0)$ and for all $j \in S \setminus I$ holds:

$$\begin{aligned}
\eta_j &= \lambda_j^I + \sum_{i \in S \setminus I} \eta_i r^I(i,j) + \sum_{i \in U} \mu_i r^I(i,j) \\
&\overset{\text{Def. } 2.8}{=} \lambda_j + \sum_{i \in S \setminus I} \underbrace{\eta_i r(i,j)}_{\overset{(5.3)}{=} \eta_j r(j,i)} + \sum_{i \in I} \eta_j r(j,i) + \sum_{i \in U} \underbrace{\mu_i r(i,j)}_{\overset{(5.4)}{=} \eta_j r(j,i)} \\
&= \lambda_j + \eta_j (1 - r(j,0)),
\end{aligned}$$

hence (iii) is valid. □

Remark 5.21. *It is noteworthy that, in the situation of load-independent breakdown and repair rates, the limiting throughput of the stable subnet of the non-ergodic Jackson network with unreliable nodes is under the blocking rs-rd regime the same as the steady-state throughput of an ergodic Jackson network with unreliable nodes, see Proposition 2.21.*

To compare the limiting throughput according to the three rerouting regimes, we require the same constraints for all considered rerouting regimes:

Corollary 5.22. *Consider a Jackson network with unreliable nodes. Let $\eta = (\eta_1, ..., \eta_J)$ be the unique solution of the general traffic equations* (1.6)*. Let $S := \{i : \eta_i < \mu_i\}$ denote the set of stable nodes, nodes in $U := \tilde{J} \setminus S$ are unstable. Let the breakdown and repair intensities be load-independent as in Example 2.2. Let the reversibility contraints* (5.3)*,* (5.4)*, and* (5.5) *hold, and let $\eta_i = \mu_i$ hold for all $i \in U$.*
Then the limiting throughput $\lim_{t \to \infty} TH_S(t)$ of the stable subnetwork is

(i) In case of stalling:

$$\lim_{t\to\infty} TH_S(t) = \pi(\emptyset)\Big(\sum_{j\in\tilde{J}} \lambda_j - \sum_{j\in U} \mu_j r(j,0)\Big),$$

(ii) in case of skipping:

$$\lim_{t\to\infty} TH_S(t) = \sum_{I\subseteq D} \pi(I)\Big(\lambda \sum_{j\in\tilde{J}\setminus I} r^I(0,j) - \sum_{j\in U\setminus I} \mu_j r^I(j,0)\Big),$$

(iii) in case of blocking rs-rd:

$$\lim_{t\to\infty} TH_S(t) = \sum_{I\subseteq D} \pi(I) \sum_{j\in S\setminus I} \lambda_j.$$

Example 5.23. *The problem of estimating time averages of rewards or costs, determined by a non-decreasing cost function*

$$g : \mathbb{N}^{|S|} \to \mathbb{R},$$

associated with the development of the queue lengths in the stable subnetwork over time horizon $[0,T]$,

$$d(T) = \frac{1}{T}\int_0^T g(X_i(t), i\in S)\; dt,$$

is more complicated because we have no ergodicity, in fact not even the Markov property for $((X_i(t) : i\in S) : t\geq 0)$. *So we cannot apply the ergodic theorem for Markov processes to solve the problem. Because of the lack of stationarity, Birkhoff's ergodic theorem (e.g., [Bre92, Th. 6.21]) is neither applicable.*
The solution to the problem follows from the path-wise construction in the proof of Theorem 5.10, Theorem 5.11 or Theorem 5.12, see Section 5.3.2.
We consider g *as a function*

$$g : \mathcal{P}(D)\times \mathbb{N}^{|S|} \to \mathbb{R}.$$

Then for $T\to\infty$ *from the ergodic theorem for Markov processes follows for the upper bound Markov process* (Y, X^+)

$$\frac{1}{T}\int_0^T g(Y(t), X_i^+(t) : i\in S)\; dt \;\to \tag{5.49}$$
$$\to \sum_{(I,n_i:i\in S)\in\mathcal{P}(D)\times\mathbb{N}^{|S|}} g(I, n_i : i\in S)\cdot \frac{A(I)}{B(I)}\left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1} \prod_{i\in S}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}.$$

Similarly we conclude that for $T\to\infty$ *for the lower bound Markov processes* $(Y, X^-(\varepsilon))$ *holds (for all* $\varepsilon > 0$*)*

$$\frac{1}{T}\int_0^T g(Y(t), X_i^-(\varepsilon)(t) : i\in S)\; dt \;\to \tag{5.50}$$
$$\to \sum_{(I,n_i:i\in S)\in\mathcal{P}(D)\times\mathbb{N}^{|S|}} g(I, n_i : i\in S)\cdot \frac{A(I)}{B(I)}\left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1} \prod_{i\in S}\left(1-\frac{\eta_i^-(\varepsilon)}{\mu_i}\right)\left(\frac{\eta_i^-(\varepsilon)}{\mu_i}\right)^{n_i}$$
$$\left(\xrightarrow{\varepsilon\to 0} \sum_{(I,n_i:i\in S)\in\mathcal{P}(D)\times\mathbb{N}^{|S|}} g(I, n_i : i\in S)\cdot \frac{A(I)}{B(I)}\left(\sum_{K\subseteq D}\frac{A(K)}{B(K)}\right)^{-1} \prod_{i\in S}\left(1-\frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}\right).$$

We note that (5.49) *and* (5.50) *hold almost surely for all paths of the upper bound process and lower bound processes. From the coupling construction in the proof of Theorem 5.10, Theorem 5.11 or Theorem 5.12, resp., and the non-decreasing property of* g *we see that almost surely holds*

$$\begin{aligned}\frac{1}{T}\int_0^T g(Y(t), X_i^+(t) : i \in S)\ dt \\ \geq \frac{1}{T}\int_0^T g(Y(t), X_i(t) : i \in S)\ dt \\ \geq \frac{1}{T}\int_0^T g(Y(t), X_i^-(\varepsilon)(t) : i \in S)\ dt.\end{aligned}$$

So we have for almost all paths of (Y, X) *for large* T

$$\frac{1}{T}\int_0^T g(Y(t), X_i(t) : i \in S)\ dt \approx \tag{5.51}$$

$$\approx \sum_{(I, n_i : i \in S) \in \mathcal{P}(D) \times \mathbb{N}^{|S|}} g(I, n_i : i \in S) \cdot \frac{A(I)}{B(I)} \left(\sum_{K \subseteq D} \frac{A(K)}{B(K)}\right)^{-1} \prod_{i \in S} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}. \tag{5.52}$$

This means that even in the case of the non-stationary process $((Y(t), X_i(t) : i \in S) : t \geq 0)$ *on the state space* $\mathcal{P}(D) \times \mathbb{N}^{|S|}$*, which is not a Markov process, we can estimate the path-wise evaluated time average* $d(T)$ *in* (5.51) *by the state-space average* (5.52) *(the phase-space average) for almost all paths. This is the meaning of classical ergodic theorems, see, e.g., [Bre92, pp.113-115].*

Chapter 6

Non-ergodic Jackson networks: Analysis of the local stabilization

6.1 Introduction

The analysis in this chapter is motivated by a comparison of the long-time behavior of ergodic and non-ergodic Jackson networks presented in Chapter 1.

From Theorem 1.22 on non-ergodic Jackson networks, the asymptotic joint distribution (1.11) for the subset S of stable nodes is

$$\lim_{t\to\infty} P(X_j(t) = n_j : j \in S) = \prod_{j\in S} \left(1 - \frac{\eta_j}{\mu_j}\right) \left(\frac{\eta_j}{\mu_j}\right)^{n_j},$$

whereas from Jackson's theorem (Theorem 1.10) on ergodic Jackson networks we have the asymptotic joint distribution (1.5) for the set $\tilde{J}$ (of stable nodes only!)

$$\lim_{t\to\infty} P(X_j(t) = n_j : j \in \tilde{J}) = \prod_{j=1}^{J} \left(1 - \frac{\eta_j}{\mu_j}\right) \left(\frac{\eta_j}{\mu_j}\right)^{n_j}.$$

There are several similarities of those asymptotic distributions:

- Both asymptotic joint distributions are of product form;
- all marginals are geometric distributions with parameter $1 - \frac{\eta_j}{\mu_j}$;
- the only information needed to calculate the distributions is the traffic intensity η_j/μ_j at each stable node j.

These obvious similarities motivated us to analyze possible further similarities which might be in behind this first observation.

Indeed, one may only speak of similarities of the asymptotics presented above, because η_j, $j \in \tilde{J}$, is determined differently in those two cases:
In Jackson's theorem all nodes are stable, thus $(\eta_j : j \in \tilde{J})$ is the unique solution of the (classical) traffic equations (1.1). Whereas in the situation of Theorem 1.22, if at least

one node is unstable, $(\eta_j : j \in \tilde{J})$ is the unique solution of the general traffic equations (1.6).

The main difference between the two results is that - contrary to the non-ergodic case - the asymptotic distribution (1.5) for ergodic networks is a **stationary** distribution as well. This information features that if the process is started with the asymptotic distribution as initial distribution it remains invariant over time (see Definition and Remark 1.4).

The intuitive question arising is whether, starting a non-ergodic network process ($S \subset \tilde{J}$) with an initial distribution which has the marginal (1.11), this marginal stays invariant over time as in the ergodic case. Being more specific, are there conditions on how the initial distribution must look like in order to obtain such an invariance of the stable marginals?

We start our investigation with non-ergodic Jackson networks with only two nodes where one of the nodes is stable and the other one is unstable in the sense of Definition 1.17.

Throughout this chapter we will refer to the following two-nodes system for readability and give the according results for networks of an arbitrary number of nodes if these are at hand. As in Chapter 1, Section 1.4, we will consider a uniformization of the network process.

Definition 6.1 (The two-node model with uniformized queue length process)**.** *Consider a Jackson network with two nodes where node* 1 *is stable and node* 2 *is unstable in the sense of Definition 1.17. Denote by* $\eta = (\eta_1, \eta_2)$ *the unique solution of the general traffic equations* (1.6) *which in this case reduce to*

$$\eta_1 = \lambda_1 + \eta_1 r(1,1) + \mu_2 r(2,1), \tag{6.1}$$

$$\eta_2 = \lambda_2 + \eta_1 r(1,2) + \mu_2 r(2,2). \tag{6.2}$$

Denote by $X = (X(t) = (X_1(t), X_2(t)) : t \geq 0)$ *the queue length process of the network on the state space* $\mathbb{N}^2$ *and denote by*

$$P^{(X_1(0),X_2(0))} := P(X(0) = (n_1, n_2) : (n_1, n_2) \in \mathbb{N}^2)$$

the initial distribution.

Consider the uniformization of the queue length process: Since all entries in the main diagonal of the Q-matrix are bounded, X is uniformizable. With

$$\xi \geq \sum_{i=1}^{2} (\lambda_i + \mu_i)$$

as the uniformization constant, a Poisson-clock with intensity ξ *generates the jump times of the uniformized process. The jump kernel of the uniformization is then given by*

$$p^u = \mathbf{I} + \frac{1}{\xi} \cdot Q.$$

As the continuous-time process, the uniformized process has cadlag paths.

As mentioned in Section 1.4.2, the advantage of the uniformization is that an analysis of the continuous-time process X is possible with discrete time techniques.

6.2 One-step invariance of the marginal distribution

We[5] start the non-ergodic network process of the two-node model with an initial distribution such that the following holds:

- the unstable node 2 is busy (not idle) at time 0,
- the marginal probability for the stable node 1 is according to (1.11) for all $n_1 \in \mathbb{N}$

$$P(X_1(0) = n_1) = \lim_{t\to\infty} P(X_1(t) = n_1) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}. \qquad (6.3)$$

The aim is to clarify whether the marginal distribution (6.3) can remain invariant over time. Unfortunately enough, it turned out that this is in general (seemingly) not possible. [MD11, p.103]

In course of the computations yielding this negative result we came across the following observation: As long as we can guarantee that the unstable node 2 is not idle at the beginning of the observation period, the desired invariance at the stable node 1 is maintained even right after the first jump. The astonishing fact is that this is independent of the form of the initial distribution as long as the two requirements above are valid. This leads to the following proposition. [MD11, p.103]

Proposition 6.2. [MD11, Proposition 7] *Consider the two-node model of Definition 6.1. Assume that the initial distribution fulfills the following conditions:*
- *the marginal for the stable part is the limiting probability* (1.11) *of Theorem 1.22, i.e.,*

$$P(X_1(0) = n_1) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}, \quad n_1 \in \mathbb{N}, \qquad (6.4)$$

- *the marginal for the unstable part is busy with probability* 1*, but carries no further restriction, i.e.,*

$$P(X_2(0) = 0) = 0. \qquad (6.5)$$

Denote by t_1 the (random) time of the first jump of the uniformized process X. Then for all $n_1 \in \mathbb{N}$ holds

$$P(X_1(t_1) = n_1) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}. \qquad (6.6)$$

Proof. Denote by p^u the jump kernel of the uniformization with Poisson-ξ clock, where $\xi \geq \sum_{i=1}^{2}(\lambda_i + \mu_i)$, and compute directly for all $(n_1, n_2) \in \mathbb{N}^2$

[5] Since parts of this section are published in our paper [MD11] (pp.103-104), its introduction and some of the comments cite the paper which is shortly marked at the end of the concerning paragraphs.

$$
\begin{aligned}
&P(X(t_1) = (n_1, n_2)) = \\
&= \sum_{(m_1,m_2)\in\mathbb{N}^2} P(X(0) = (m_1, m_2)) \cdot p^u(m_1, m_2; n_1, n_2) \\
&= P(X(0) = (n_1 - 1, n_2)) \cdot \frac{\lambda_1}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) + P(X(0) = (n_1, n_2 - 1)) \cdot \frac{\lambda_2}{\xi} \cdot 1_{\mathbb{N}_+}(n_2) + \\
&+ P(X(0) = (n_1 + 1, n_2)) \cdot \frac{\mu_1 r(1,0)}{\xi} + P(X(0) = (n_1, n_2 + 1)) \cdot \frac{\mu_2 r(2,0)}{\xi} + \\
&+ P(X(0) = (n_1 + 1, n_2 - 1)) \cdot \frac{\mu_1 r(1,2)}{\xi} \cdot 1_{\mathbb{N}_+}(n_2) + \\
&+ P(X(0) = (n_1 - 1, n_2 + 1)) \cdot \frac{\mu_2 r(2,1)}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) + \\
&+ P(X(0) = (n_1, n_2)) \cdot \frac{1}{\xi} \cdot \\
&\qquad \cdot (\xi - \lambda_1 - \lambda_2 - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) - \mu_2(1 - r(2,2)) 1_{\mathbb{N}_+}(n_2))
\end{aligned}
$$

Summing over all $n_2 \in \mathbb{N}$ yields

$$
\begin{aligned}
&P(X_1(t_1) = n_1) = \sum_{n_2\in\mathbb{N}} P(X(t_1) = (n_1, n_2)) = \\
&= \sum_{n_2\in\mathbb{N}} P(X(0) = (n_1 - 1, n_2)) \cdot \frac{\lambda_1}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) + \sum_{n_2\in\mathbb{N}} P(X(0) = (n_1, n_2)) \cdot \frac{\lambda_2}{\xi} + \\
&+ \sum_{n_2\in\mathbb{N}} P(X(0) = (n_1 + 1, n_2)) \cdot \frac{\mu_1 r(1,0)}{\xi} + \sum_{n_2\in\mathbb{N}_+} P(X(0) = (n_1, n_2)) \cdot \frac{\mu_2 r(2,0)}{\xi} + \\
&+ \sum_{n_2\in\mathbb{N}} P(X(0) = (n_1 + 1, n_2)) \cdot \frac{\mu_1 r(1,2)}{\xi} + \\
&+ \sum_{n_2\in\mathbb{N}_+} P(X(0) = (n_1 - 1, n_2)) \cdot \frac{\mu_2 r(2,1)}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) + \\
&+ \sum_{n_2\in\mathbb{N}} P(X(0) = (n_1, n_2)) \cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot (\xi - \lambda_1 - \lambda_2 - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) - \mu_2(1 - r(2,2)) \underbrace{1_{\mathbb{N}_+}(n_2)}_{\overset{(6.5)}{=} 1}).
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&P(X_1(t_1) = n_1) = \\
&= \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1 - 1, n_2)) \cdot \frac{\lambda_1 + \mu_2 r(2,1)}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) + \\
&+ \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1 + 1, n_2)) \cdot \frac{\mu_1(1 - r(1,1))}{\xi} + \\
&+ \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1, n_2)) \cdot \frac{1}{\xi} \cdot (\xi - \lambda_1 - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) - \mu_2 r(2,1)) \\
&\overset{(6.1)}{=} \frac{\eta_1(1 - r(1,1))}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1 - 1, n_2)) + \\
&+ \frac{\mu_1(1 - r(1,1))}{\xi} \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1 + 1, n_2)) + \\
&+ \frac{1}{\xi} \cdot (\xi - \lambda_1 - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) - \mu_2 r(2,1)) \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1, n_2)).
\end{aligned}
$$

Filling in (6.4) yields

$$
\begin{aligned}
P(X_1(t_1) = n_1) &= \frac{\eta_1(1 - r(1,1))}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) \cdot \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n_1 - 1} + \\
&+ \frac{\mu_1(1 - r(1,1))}{\xi} \cdot \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n_1 + 1} + \\
&+ \frac{1}{\xi} \cdot (\xi - \lambda_1 - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) - \mu_2 r(2,1)) \cdot \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n_1} \\
&= \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n_1} \cdot \frac{1}{\xi} \cdot \Big(\xi - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) \\
&\quad - \underbrace{(\lambda_1 + \mu_2 r(2,1))}_{\overset{(6.1)}{=} \eta_1(1 - r(1,1))} + \frac{\mu_1}{\eta_1} \eta_1 (1 - r(1,1)) \cdot 1_{\mathbb{N}_+}(n_1) + \frac{\eta_1}{\mu_1} \mu_1 (1 - r(1,1)) \Big) \\
&= \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n_1}.
\end{aligned}
$$

□

In proving the proposition we observe that the product form structure at the stable node 1 holds also at the time when the unstable node becomes idle for the first time after starting the process. Since we assume cadlag paths this property is also true at the time right after the jump of X into the state $(n_1, 0)$, and until just before the next jump at time t_2, say. [MD11, p.104]

Even more, Proposition 6.2 implies that if we know of at least $K > 0$ customers waiting at the unstable node 2 at the beginning ($t = 0$), then for $K - 1$ jumps we can guarantee that the assumption (6.5) still holds. Therefore after $K - 1$ jumps the result of the proposition applies and for at least K jumps the product form for the stable node 1 is maintained. If at time $t = 0$ a number of K customers are present at node 2, it follows from the non-idling property of the service discipline that these customers are served after an Erlang distributed time with K phases and shape parameter μ_2. [MD11, p.104]

The following proposition is a generalization of Proposition 6.2: Even in Jackson networks with J nodes which may be unreliable, the marginal distribution for the stable subnetwork reproduces after one step as long as all unstable nodes are almost surely not empty at the beginning.

Proposition 6.3. *Consider a Jackson network with J nodes. Denote by* $\eta = (\eta_1, ..., \eta_J)$ *the unique solution of the general traffic equations*

$$\eta_i = \lambda_i + \sum_{j\in\bar{J}} \min(\eta_j, \mu_j) r_{ji}, \quad i, j \in \bar{J}. \tag{6.7}$$

Nodes of the subset $S := \{i : \eta_i < \mu_i\}$ *are stable and nodes of the subset* $U := \bar{J} \setminus S$ *are unstable.*
Assume that all nodes are unreliable ($D = \tilde{J}$). *Breakdowns and repairs of the nodes are controlled by load-independent breakdown and repair intensities as in Example 2.2. In case of breakdowns, customers are rerouted according to the stalling regime, the skipping regime or the blocking rs-rd regime. If skipping is in force, we assume*

$$\eta_i = \mu_i \quad \forall i \in U. \tag{6.8}$$

In case of the blocking rs-rd regime assume that the following reversibility constraints hold:

$$\eta_i r(i,j) = \eta_j r(j,i) \quad \forall i, j \in S, \tag{6.9}$$
$$\eta_i r(i,j) = \mu_j r(j,i) \quad \forall i \in S, j \in U, \tag{6.10}$$
$$\mu_i r(i,j) = \mu_j r(j,i) \quad \forall i, j \in U. \tag{6.11}$$

Denote by (Y, X) *the Markovian availability-queue lengths process on the state space* $\mathcal{P}(\tilde{J}) \times \mathbb{N}^J$ *and assume that the initial distribution*

$$P^{(Y(0),X_1(0),...,X_J(0))} := P((Y,X)(0) = (I, n_1, ..., n_J) : (I, n_1, ..., n_J) \in \mathcal{P}(\tilde{J}) \times \mathbb{N}^J)$$

fulfills the following conditions:

$$P(Y(0) = I, X_i(0) = n_i : i \in S) = \left(\sum_{K\subseteq\tilde{J}} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i\in S} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i} \tag{6.12}$$

for all $n_i \in \mathbb{N}, I \subseteq \tilde{J}$, *and*

$$P(X_j(0) = 0) = 0 \quad \forall j \in U. \tag{6.13}$$

Consider the uniformization of (Y, X) *with Poisson-clock with intensity*

$$\xi \geq \sum_{i=1}^{J} (\lambda_i + \mu_i) + \sum_{I\subseteq\tilde{J}} \sum_{\emptyset\neq K\subseteq\tilde{J}\setminus I} \alpha(I, I\cup K) + \sum_{I\subseteq\tilde{J}} \sum_{\emptyset\neq K\subseteq I} \beta(I, I\setminus K),$$

and denote by t_1 *the (random) time of the first jump of the uniformized process* X. *Then for all* $n_i \in \mathbb{N}$ *and all* $I \subseteq \tilde{J}$ *holds*

$$P(Y(t_1) = I, X_i(t_1) = n_i : i \in S) = \left(\sum_{K\subseteq\tilde{J}} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i\in S} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}. \tag{6.14}$$

Proof. Consider the process uniformized with uniformization constant ξ, i.e., jumps of the process are controlled with a Poisson-ξ clock. Denote by $p^u := \mathbf{I} + \frac{1}{\xi}Q$ the jump kernel, i.e., the entries of the kernel are the probabilities of a jump. We consider the rerouting according to skipping and blocking rs-rd together, the case of stalling is a simplification of the following proof by setting $I = \emptyset$.
For all $I \subseteq \tilde{J}$ holds:

$$
\begin{aligned}
&P(Y(t_1) = I, X(t_1) = (n_1, ..., n_J)) = \\
&= \sum_{(m_1,...,m_J)\in\mathbb{N}^J} \sum_{H\subseteq\tilde{J}} P(Y(0) = H, X(0) = (m_1, ..., m_J)) \cdot p^u(H, m_1, ..., m_J; I, n_1, ..., n_J) \\
&= \sum_{H\subseteq\tilde{J}, H\neq I} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot p^u(H, n_1, ..., n_J; I, n_1, ..., n_J)+ \\
&\quad + P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot p^u(I, n_1, ..., n_J; I, n_1, ..., n_J)+ \\
&\quad + \sum_{j\in\tilde{J}\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot p^u(I, n_1, ..., n_j - 1, ..., n_J; I, n_1, ..., n_j, ..., n_J) \cdot 1_{\mathbb{N}_+}(n_j)+ \\
&\quad + \sum_{j\in\tilde{J}\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j + 1, ..., n_J))\cdot \\
&\qquad \cdot p^u(I, n_1, ..., n_j + 1, ..., n_J; I, n_1, ..., n_j, ..., n_J)+ \\
&\quad + \sum_{i\in\tilde{J}\setminus I} \sum_{j\in\tilde{J}\setminus I, j\neq i} P(Y(0) = I, X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot p^u(I, n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J; I, n_1, ..., n_i, ..., n_j, ..., n_J) \cdot 1_{\mathbb{N}_+}(n_j) \\
&= \sum_{H\subset I} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{\alpha(H, I)}{\xi}+ \\
&\quad + \sum_{I\subset K\subseteq\tilde{J}} P(Y(0) = K, X(0) = (n_1, ..., n_J)) \cdot \frac{\beta(K, I)}{\xi}+ \\
&\quad + P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi}\cdot \\
&\quad \cdot \Big(\xi - \sum_{i\in\tilde{J}\setminus I} \lambda_i^I - \sum_{i\in\tilde{J}\setminus I} \mu_i^I(1 - r^I(i, i)) \cdot 1_{\mathbb{N}_+}(n_i) - \sum_{I\subset K\subseteq\tilde{J}} \alpha(I, K) - \sum_{H\subset I} \beta(I, H)\Big)+ \\
&\quad + \sum_{j\in\tilde{J}\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot \frac{\lambda_j^I}{\xi} \cdot 1_{\mathbb{N}_+}(n_j)+ \\
&\quad + \sum_{j\in\tilde{J}\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \frac{\mu_j^I r^I(j, 0)}{\xi}+ \\
&\quad + \sum_{i\in\tilde{J}\setminus I} \sum_{j\in\tilde{J}\setminus I, j\neq i} P(Y(0) = I, X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{\mu_i^I r^I(i, j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned}
$$

Summing over all $(n_k : k \in U) \in \mathbb{N}^{|U|}$ yields

$$
\begin{aligned}
&P(Y(t_1)=I, X_i(t_1)=n_i : i\in S) = \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(t_1)=I, X(t_1)=(n_1,...,n_J)) = \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} \sum_{H\subset I} P(Y(0)=H, X(0)=(n_1,...,n_J))\cdot\frac{\alpha(H,I)}{\xi} + \\
&+ \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} \sum_{I\subset K\subseteq\tilde{J}} P(Y(0)=K, X(0)=(n_1,...,n_J))\cdot\frac{\beta(K,I)}{\xi} + \\
&+ \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_J))\cdot \\
&\qquad \cdot\frac{1}{\xi}\cdot\Big(\xi-\sum_{i\in\tilde{J}\setminus I}\lambda_i^I-\sum_{i\in\tilde{J}\setminus I}\mu_i^I(1-r^I(i,i))\cdot 1_{\mathbb{N}_+}(n_i) \\
&\qquad\qquad -\sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K)-\sum_{H\subset I}\beta(I,H)\Big)+ \\
&+ \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} \sum_{j\in\tilde{J}\setminus I} P(Y(0)=I, X(0)=(n_1,...,n_j-1,...,n_J))\cdot\frac{\lambda_j^I}{\xi}\cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} \sum_{j\in\tilde{J}\setminus I} P(Y(0)=I, X(0)=(n_1,...,n_j+1,...,n_J))\cdot\frac{\mu_j^I r^I(j,0)}{\xi} + \\
&+ \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} \sum_{i\in\tilde{J}\setminus I} \sum_{j\in\tilde{J}\setminus I, j\neq i} P(Y(0)=I, X(0)=(n_1,...,n_i+1,...,n_j-1,...,n_J))\cdot \\
&\qquad \cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow P(Y(t_1)=I, X_i(t_1)=n_i : i\in S)= \\
&\quad = \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} \sum_{H\subset I} P(Y(0)=H, X(0)=(n_1,...,n_J))\cdot\frac{\alpha(H,I)}{\xi}+ \\
&\quad + \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} \sum_{I\subset K\subseteq\tilde{J}} P(Y(0)=K, X(0)=(n_1,...,n_J))\cdot\frac{\beta(K,I)}{\xi}+ \\
&\quad + \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_J))\cdot\frac{1}{\xi}\cdot \\
&\qquad \cdot\Big(\xi-\sum_{i\in\tilde{J}\setminus I}\lambda_i^I-\sum_{i\in\tilde{J}\setminus I}\mu_i^I(1-r^I(i,i))\cdot 1_{\mathbb{N}_+}(n_i)-\sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K)-\sum_{H\subset I}\beta(I,H)\Big)+ \\
&\quad + \sum_{j\in S\setminus I}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_j-1,...,n_J))\cdot\frac{\lambda_j^I}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)+ \\
&\quad + \sum_{j\in U\setminus I}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_j,...,n_J))\cdot\frac{\lambda_j^I}{\xi}+ \\
&\quad + \sum_{j\in S\setminus I}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_j+1,...,n_J))\cdot\frac{\mu_j^I r^I(j,0)}{\xi}+ \\
&\quad + \sum_{j\in U\setminus I}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_j,...,n_J))\cdot\frac{\mu_j^I r^I(j,0)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)+ \\
&\quad + \sum_{i\in S\setminus I}\sum_{j\in S\setminus I, j\neq i}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_i+1,...,n_j-1,...,n_J))\cdot \\
&\qquad \cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j) \\
&\quad + \sum_{i\in S\setminus I}\sum_{j\in U\setminus I}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_i+1,...,n_j,...,n_J))\cdot \\
&\qquad \cdot\frac{\mu_i^I r^I(i,j)}{\xi} \\
&\quad + \sum_{i\in U\setminus I}\sum_{j\in S\setminus I}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_i,...,n_j-1,...,n_J))\cdot \\
&\qquad \cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)\cdot 1_{\mathbb{N}_+}(n_i) \\
&\quad + \sum_{i\in U\setminus I}\sum_{j\in U\setminus I, j\neq i}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_i,...,n_j,...,n_J))\cdot \\
&\qquad \cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_i)
\end{aligned}
$$

$$\Leftrightarrow P(Y(t_1)=I, X_i(t_1)=n_i : i\in S) =$$

$$= \sum_{H\subset I} \underbrace{\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=H, X(0)=(n_1,...,n_J))\frac{A(I)}{A(H)}}_{\overset{(6.12)}{=}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I,X(0)=(n_1,...,n_J))\frac{B(I)}{B(H)}} \cdot\frac{1}{\xi}+$$

$$+ \sum_{I\subset K\subseteq\tilde{J}} \underbrace{\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=K, X(0)=(n_1,...,n_J))\frac{B(K)}{B(I)}}_{\overset{(6.12)}{=}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I,X(0)=(n_1,...,n_J))\frac{A(K)}{A(I)}} \cdot\frac{1}{\xi}+$$

$$+ \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_J))\cdot\frac{1}{\xi}\cdot$$

$$\cdot\Big(\xi - \sum_{i\in S\setminus I}\lambda_i^I - \sum_{i\in S\setminus I}\mu_i^I(1-r^I(i,i))\cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in U\setminus I}\sum_{j\in S\setminus I}\mu_i^I r^I(i,j)\cdot\underbrace{1_{\mathbb{N}_+}(n_i)}_{\overset{(6.13)}{=}1}$$

$$- \sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K) - \sum_{H\subset I}\beta(I,H)\Big)+$$

$$+ \sum_{j\in S\setminus I} \underbrace{\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_j-1,...,n_J))}_{\overset{(6.12)}{=}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I,X(0)=(n_1,...,n_J))\frac{\mu_j}{\eta_j}} \cdot\frac{1}{\xi}\cdot$$

$$\cdot\Big(\lambda_j^I + \sum_{i\in U\setminus I}\mu_i^I r^I(i,j)\cdot\underbrace{1_{\mathbb{N}_+}(n_i)}_{\overset{(6.13)}{=}1}\Big)\cdot 1_{\mathbb{N}_+}(n_j)+$$

$$+ \sum_{j\in S\setminus I} \underbrace{\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_j+1,...,n_J))}_{\overset{(6.12)}{=}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I,X(0)=(n_1,...,n_J))\frac{\eta_j}{\mu_j}} \cdot\frac{1}{\xi}\cdot$$

$$\cdot\mu_j^I\Big(r^I(j,0) + \sum_{i\in U\setminus I} r^I(j,i)\Big)+$$

$$+ \sum_{i\in S\setminus I}\sum_{j\in S\setminus I, j\neq i} \underbrace{\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_i+1,...,n_j-1,...,n_J))}_{\overset{(6.12)}{=}\sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I,X(0)=(n_1,...,n_J))\frac{\eta_i}{\mu_i}\frac{\mu_j}{\eta_j}}\cdot$$

$$\cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)$$

$\Leftrightarrow$

$$
\begin{aligned}
&P(Y(t_1)=I, X_i(t_1)=n_i : i\in S) = \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_J))\cdot\frac{1}{\xi}\cdot \\
&\cdot\Big[\xi - \sum_{i\in S\setminus I}\lambda_i^I - \sum_{i\in S\setminus I}\mu_i^I(1-r^I(i,i))\cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in U\setminus I}\sum_{j\in S\setminus I}\mu_i^I r^I(i,j) \\
&- \sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K) - \sum_{H\subset I}\beta(I,H) + \sum_{H\subset I}\beta(I,H) + \sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K) + \\
&+ \sum_{j\in S\setminus I}\frac{\mu_j}{\eta_j}\Big(\lambda_j^I + \sum_{i\in U\setminus I}\mu_i^I r^I(i,j)\Big)\cdot 1_{\mathbb{N}_+}(n_j) + \sum_{j\in S\setminus I}\frac{\eta_j}{\mu_j}\mu_j^I\Big(r^I(j,0) + \sum_{i\in U\setminus I} r^I(j,i)\Big) + \\
&+ \sum_{i\in S\setminus I}\sum_{j\in S\setminus I, j\neq i}\frac{\eta_i}{\mu_i}\frac{\mu_j}{\eta_j}\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)\Big]
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&P(Y(t_1)=I, X_i(t_1)=n_i : i\in S) = \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_J))\cdot\frac{1}{\xi}\cdot \\
&\cdot\Big[\xi - \sum_{i\in S\setminus I}\lambda_i^I - \sum_{i\in S\setminus I}\mu_i^I(1-r^I(i,i))\cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in U\setminus I}\sum_{j\in S\setminus I}\mu_i^I r^I(i,j) \\
&+ \sum_{j\in S\setminus I}\frac{\mu_j}{\eta_j}\Big(\lambda_j^I + \sum_{i\in U\setminus I}\mu_i^I r^I(i,j)\Big)\cdot 1_{\mathbb{N}_+}(n_j) + \sum_{j\in S\setminus I}\frac{\eta_j}{\mu_j}\mu_j^I\Big(r^I(j,0) + \sum_{i\in U\setminus I} r^I(j,i)\Big) + \\
&+ \sum_{i\in S\setminus I}\sum_{j\in S\setminus I, j\neq i}\frac{\eta_i}{\mu_i}\frac{\mu_j}{\eta_j}\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)\Big].
\end{aligned}
$$

In case of skipping and blocking rs-rd $\mu_i^I = \mu_i$ holds for all $i\in\tilde{J}\setminus I$ and with Lemma 5.2 and Lemma 5.5 we have $\eta_i^I = \eta_i$ for all $i\in\tilde{J}\setminus I$. This yields

$$
\begin{aligned}
&P(Y(t_1)=I, X_i(t_1)=n_i : i\in S) = \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0)=I, X(0)=(n_1,...,n_J))\cdot\frac{1}{\xi}\cdot \\
&\cdot\Big[\xi - \sum_{i\in S\setminus I}\lambda_i^I - \sum_{i\in S\setminus I}\mu_i^I(1-r^I(i,i))\cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in U\setminus I}\sum_{j\in S\setminus I}\mu_i^I r^I(i,j) \\
&+ \sum_{j\in S\setminus I}\frac{\mu_j^I}{\eta_j^I}\underbrace{\Big(\lambda_j^I + \sum_{i\in U\setminus I}\mu_i^I r^I(i,j)\Big)}_{\overset{(6.7)}{=}\eta_j^I - \sum_{i\in S\setminus I}\eta_i^I r^I(i,j)}\cdot 1_{\mathbb{N}_+}(n_j) + \sum_{j\in S\setminus I}\eta_j^I\Big(1 - \sum_{i\in S\setminus I} r^I(j,i)\Big) + \\
&+ \sum_{i\in S\setminus I}\sum_{j\in S\setminus I, j\neq i}\frac{\mu_j^I}{\eta_j^I}\frac{\eta_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)\Big].
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&P(Y(t_1) = I, X_i(t_1) = n_i : i \in S) = \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \\
&\cdot \Big[\xi - \sum_{i\in S\setminus I} \lambda_i^I - \sum_{i\in S\setminus I} \mu_i^I (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in U\setminus I}\sum_{j\in S\setminus I} \mu_i^I r^I(i,j) \\
&+ \sum_{j\in S\setminus I} \frac{\mu_j}{\eta_j} \Big(\eta_j^I (1 - r^I(j,j)) - \sum_{i\in S\setminus I, i\neq j} \eta_i^I r^I(i,j)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{j\in S\setminus I} \eta_j^I \Big(1 - \sum_{i\in S\setminus I} r^I(j,i)\Big) + \sum_{i\in S\setminus I}\sum_{j\in S\setminus I, j\neq i} \frac{\mu_j^I}{\eta_j^I} \frac{\eta_i^I r^I(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j)\Big] \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \\
&\cdot \Big[\xi - \sum_{i\in S\setminus I} \lambda_i^I - \sum_{i\in S\setminus I} \mu_i^I (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in U\setminus I}\sum_{j\in S\setminus I} \mu_i^I r^I(i,j) \\
&+ \sum_{j\in S\setminus I} \frac{\mu_j^I}{\eta_j^I} \eta_j^I (1 - r^I(j,j)) \cdot 1_{\mathbb{N}_+}(n_j) + \sum_{j\in S\setminus I} \eta_j^I \Big(1 - \sum_{i\in S\setminus I} r^I(j,i)\Big)\Big] \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \\
&\cdot \Big[\xi + \sum_{j\in S\setminus I} \eta_j^I - \underbrace{\Big(\sum_{i\in S\setminus I} \lambda_i^I + \sum_{i\in U\setminus I}\sum_{j\in S\setminus I} \mu_i^I r^I(i,j) + \sum_{j\in S\setminus I}\sum_{i\in S\setminus I} \eta_j^I r^I(j,i)\Big)}_{\overset{(6.7)}{=} \sum_{j\in S\setminus I} \eta_j^I}\Big] \\
&= \sum_{(n_k:k\in U)\in\mathbb{N}^{|U|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)).
\end{aligned}
$$

In case of stalling, setting $I = \emptyset$ in the beginning of the proof yields the same result. For $I \neq \emptyset$ we have:

$$
\begin{aligned}
&P(Y(t_1) = I, X(t_1) = (n_1, ..., n_J)) = \\
&= \sum_{H\subset I} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{\alpha(H,I)}{\xi} + \\
&+ \sum_{I\subset K\subseteq \tilde{J}} P(Y(0) = K, X(0) = (n_1, ..., n_J)) \cdot \frac{\beta(K,I)}{\xi} + \\
&+ P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{I\subset K\subseteq \tilde{J}} \alpha(I,K) - \sum_{H\subset I} \beta(I,H)\Big).
\end{aligned}
$$

Summing over all $(n_k : k \in U) \in \mathbb{N}^{|U|}$ yields

$$
\begin{aligned}
&P(Y(t_1) = I, X_i(t_1) = n_i : i \in S) = \\
&= \sum_{H \subset I} \sum_{(n_k : k \in U) \in \mathbb{N}^{|U|}} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{\alpha(H, I)}{\xi} + \\
&+ \sum_{I \subset K \subseteq \tilde{J}} \sum_{(n_k : k \in U) \in \mathbb{N}^{|U|}} P(Y(0) = K, X(0) = (n_1, ..., n_J)) \cdot \frac{\beta(K, I)}{\xi} + \\
&+ \sum_{(n_k : k \in U) \in \mathbb{N}^{|U|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{I \subset K \subseteq \tilde{J}} \alpha(I, K) - \sum_{H \subset I} \beta(I, H)\Big) \\
&\overset{(6.12)}{=} \sum_{H \subset I} \frac{\alpha(H, I)}{\xi} \cdot P(Y(0) = I, X_i(0) = n_i : i \in S) \cdot \frac{A(H)}{A(I)} \frac{B(I)}{B(H)} + \\
&+ \sum_{I \subset K \subseteq \tilde{J}} \frac{\beta(K, I)}{\xi} \cdot P(Y(0) = I, X_i(0) = n_i : i \in S) \cdot \frac{A(K)}{A(I)} \frac{B(I)}{B(K)} + \\
&+ P(Y(0) = I, X_i(0) = n_i : i \in S) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{I \subset K \subseteq \tilde{J}} \alpha(I, K) - \sum_{H \subset I} \beta(I, H)\Big) \\
&= P(Y(0) = I, X_i(0) = n_i : i \in S) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{I \subset K \subseteq \tilde{J}} \frac{A(K)}{A(I)} - \sum_{H \subset I} \frac{B(I)}{B(H)} + \\
&\quad + \sum_{H \subset I} \frac{A(I)}{A(H)} \frac{A(H)}{A(I)} \frac{B(I)}{B(H)} + \sum_{I \subset K \subseteq \tilde{J}} \frac{B(K)}{B(I)} \frac{A(K)}{A(I)} \frac{B(I)}{B(K)}\Big) \\
&= P(Y(0) = I, X_i(0) = n_i : i \in S).
\end{aligned}
$$

□

The statement of Proposition 6.2 and its proof strongly relies on observing the uniformized process until just after the unstable node is empty the first time. The same holds for Proposition 6.3 where there can be more than one unstable node. In case of an ergodic network process, idling of nodes occurs as recurrent events and does not disturb the equilibrium state of the system. In the setting of Theorem 1.22 this seems to be different. Our preliminary explanation is that, in the framework of our two-node model, in the general traffic equations (6.1) and (6.2) node 2 functions as a Poisson-μ_2 source, similar to the Poisson-λ_i sources, as long as it is not idle. [MD11, p.104]

This intuitive explanation by Poisson-sources coming from other nodes strongly reminds of Jackson networks with infinite supply defined previously in Chapter 4. The following proposition shows that with Poisson departure streams from unstable nodes the stable marginals are indeed reproduced for all jump times.

Proposition 6.4. *Consider a Jackson network with J nodes where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work as in Definition 4.1. Nodes in $W := \tilde{J} \setminus V$ operate without infinite supply. Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations (4.5). $S := \{i \in \tilde{J} : \eta_i < \mu_i\}$ is the set of stable nodes, nodes in $U := \tilde{J} \setminus S$ are unstable. We assume that all unstable nodes have an infinite supply of work, i.e., $U \subseteq V \subseteq \tilde{J}$. In* *this situation the traffic equations* (4.5) *reduce to*

$$\eta_i = \lambda_i + \sum_{j\in W} \eta_j r(j,i) + \sum_{j\in V} \mu_j r(j,i), \quad i \in \tilde{J}. \tag{6.15}$$

Denote by $(X(t) = (X_1(t), ..., X_J(t)) : t \geq 0)$ *the queue length process of the network and assume that the initial distribution*

$$P^{(X_1(0),...,X_J(0))} := P(X(0) = (n_1, ..., n_J) : (n_1, ..., n_J) \in \mathbb{N}^J)$$

fulfills the following marginal product form property for the stable part without infinite supply:

$$P(X_i(0) = n_i : i \in W) = \prod_{i\in W} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}, \quad n_i \in \mathbb{N}. \tag{6.16}$$

Consider the uniformization of X *with Poisson-clock with intensity*

$$\xi \geq \sum_{i=1}^{J} (\lambda_i + \mu_i),$$

and denote by t_n *the (random) time of the nth jump of the uniformized process* X*. Then for all* $n_i \in \mathbb{N}$ *holds for all* $n \in \mathbb{N}$

$$P(X_i(t_n) = n_i : i \in W) = \prod_{i\in W} \left(1 - \frac{\eta_i}{\mu_i}\right)\left(\frac{\eta_i}{\mu_i}\right)^{n_i}. \tag{6.17}$$

Before proving Proposition 6.4, it should be pointed out that the resulting reproduction of the marginal distribution on the stable part of the network is exactly what we tried to achieve for non-ergodic Jackson networks without infinite supply. In contrary to non-ergodic Jackson networks without infinite supply, here the reproduction for more than one step holds without any further side constraints to be fulfilled. Indeed, the reproduction of the marginal product form property (for the uniformized process) implies that stationarity can be achieved at least on the stable part of the globally non-ergodic network process, see Theorem 4.26.

Proof of Proposition 6.4. The proof consists of showing the reproduction of the marginal after one jump (the first jump time will be denoted by t_1), i.e., (6.17) with $t_n = t_1$. Since the marginal distribution (6.16) is the only assumption on the initial distribution to hold, the reproduction of this marginal distribution after one time step is sufficient for iterated reproduction in the same manner, hence a reproduction after n jumps for any t_n will be proved.

Consider the process uniformized with uniformization constant $\xi \geq \sum_{i=1}^{J}(\lambda_i + \mu_i)$, i.e., jumps of the process are controlled by a Poisson-ξ clock. Denote by $p^u := I + \frac{1}{\xi}Q$ the jump kernel, i.e., the entries of the kernel are the probabilities of a jump, and denote by

t_1 the time of the first jump. Recall that we assume that X has cadlag paths.

$$
\begin{aligned}
&P(X(t_1) = (n_1, ..., n_J)) = \sum_{(m_1,...,m_J)\in\mathbb{N}^J} P(X(0) = (m_1, ..., m_J)) \cdot p^u(m_1, ..., m_J; n_1, ..., n_J) \\
&= P(X(0) = (n_1, ..., n_J)) \cdot p^u(n_1, ..., n_J; n_1, ..., n_J)+ \\
&+ \sum_{j\in\tilde{J}} P(X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot p^u(n_1, ..., n_j - 1, ..., n_J; n_1, ..., n_j, ..., n_J) \cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{j\in\tilde{J}} P(X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot p^u(n_1, ..., n_j + 1, ..., n_J; n_1, ..., n_j, ..., n_J) \\
&+ \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}, j\neq i} P(X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J)) \cdot \\
&\quad \cdot p^u(n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J; n_1, ..., n_i, ..., n_j, ..., n_J) \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned}
$$

Plugging in the probabilities of the jump kernel yields

$$
\begin{aligned}
&P(X(t_1) = (n_1, ..., n_J)) = \\
&= P(X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{i\in\tilde{J}} \lambda_i - \sum_{i\in\tilde{J}} \sum_{j\in V} \mu_j r(j, i) 1_{\{0\}}(n_j) + \\
&\qquad - \sum_{i\in\tilde{J}} \mu_i r(i, 0) \cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}\setminus\{i\}} \mu_i r(i, j) 1_{\mathbb{N}_+}(n_i)\Big) + \\
&+ \sum_{j\in W} P(X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \Big(\lambda_j + \sum_{i\in V} \mu_i r(i, j) 1_{\{0\}}(n_i)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{j\in V} P(X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j + \sum_{i\in V\setminus\{j\}} \mu_i r(i, j) 1_{\{0\}}(n_i) + \mu_j r(j, j) 1_{\{0\}}(n_j - 1)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{j\in\tilde{J}} P(X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \frac{\mu_j r(j, 0)}{\xi} + \\
&+ \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}\setminus\{i\}} P(X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J)) \cdot \frac{\mu_i r(i, j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned}
$$

Summing over all $(n_k : k \in V) \in \mathbb{N}^{|V|}$ yields

$$
\begin{aligned}
&P(X_i(t_1) = n_i : i \in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(t_1) = (n_1, ..., n_J)) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{i\in\tilde{J}} \lambda_i - \sum_{i\in\tilde{J}} \sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j) \\
&\qquad\qquad - \sum_{i\in\tilde{J}} \mu_i r(i,0) \cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}\setminus\{i\}} \mu_i r(i,j) 1_{\mathbb{N}_+}(n_i)\Big) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in W} P(X(0) = (n_1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j + \sum_{i\in V} \mu_i r(i,j) 1_{\{0\}}(n_i)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in V} P(X(0) = (n_1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j + \sum_{i\in V\setminus\{j\}} \mu_i r(i,j) 1_{\{0\}}(n_i) + \mu_j r(j,j) 1_{\{0\}}(n_j - 1)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in\tilde{J}} P(X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \frac{\mu_j r(j,0)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}\setminus\{i\}} P(X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J)) \cdot \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow P(X_i(t_1) = n_i : i \in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{i\in\tilde{J}} \lambda_i - \sum_{i\in\tilde{J}} \sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j) \\
&\qquad\qquad - \sum_{i\in\tilde{J}} \mu_i r(i,0)\cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in\tilde{J}} \sum_{j\in\tilde{J}\setminus\{i\}} \mu_i r(i,j) 1_{\mathbb{N}_+}(n_i)\Big) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in W} P(X(0) = (n_1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j + \sum_{i\in V} \mu_i r_{ij} 1_{\{0\}}(n_i)\Big) \cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in V} P(X(0) = (n_1, ..., n_j, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j + \sum_{i\in V} \mu_i r(i,j) \cdot 1_{\{0\}}(n_i)\Big) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in W} P(X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \frac{\mu_j r(j,0)}{\xi} \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in V} P(X(0) = (n_1, ..., n_j, ..., n_J)) \cdot \frac{\mu_j r(j,0)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in W} \sum_{j\in W\setminus\{i\}} P(X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in W} \sum_{j\in V} P(X(0) = (n_1, ..., n_i + 1, ..., n_j, ..., n_J)) \cdot \frac{\mu_i r(i,j)}{\xi} \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in V} \sum_{j\in W} P(X(0) = (n_1, ..., n_i, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \cdot 1_{\mathbb{N}_+}(n_i) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in V} \sum_{j\in V\setminus\{i\}} P(X(0) = (n_1, ..., n_i, ..., n_j, ..., n_J)) \cdot \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_i)
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow\ & P(X_i(t_1) = n_i : i \in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{i\in\tilde{J}} \lambda_i - \sum_{i\in\tilde{J}}\sum_{j\in V} \mu_j r(j,i) 1_{\{0\}}(n_j) \\
&\qquad\qquad - \sum_{i\in\tilde{J}} \mu_i r(i,0)\cdot 1_{\mathbb{N}_+}(n_i) - \sum_{i\in\tilde{J}}\sum_{j\in\tilde{J}\setminus\{i\}} \mu_i r(i,j) 1_{\mathbb{N}_+}(n_i)\Big) \\
&+ \sum_{j\in W}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j + \sum_{i\in V} \mu_i r(i,j) 1_{\{0\}}(n_i)\Big) \cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \sum_{j\in V}\Big(\lambda_j + \sum_{i\in V} \mu_i r(i,j)\cdot 1_{\{0\}}(n_i)\Big) \\
&+ \sum_{j\in W}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \frac{\mu_j r(j,0)}{\xi} \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_J)) \sum_{j\in V} \frac{\mu_j r(j,0)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{i\in W}\sum_{j\in W\setminus\{i\}}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{i\in W}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_i + 1, ..., n_J)) \sum_{j\in V} \frac{\mu_i r(i,j)}{\xi} \\
&+ \sum_{j\in W}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_j - 1, ..., n_J)) \sum_{i\in V} \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \cdot 1_{\mathbb{N}_+}(n_i) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_J)) \sum_{i\in V}\sum_{j\in V\setminus\{i\}} \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_i)
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow\ & P(X_i(t_1) = n_i : i \in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_J))\cdot \\
&\quad \cdot \frac{1}{\xi} \cdot \Big(\xi + \underbrace{\sum_{j\in V}\Big(\lambda_j + \sum_{i\in V}\mu_i r(i,j) 1_{\{0\}}(n_i)\Big) - \sum_{i\in\tilde{J}}\lambda_i - \sum_{i\in\tilde{J}}\sum_{j\in V}\mu_j r(j,i) 1_{\{0\}}(n_j)}_{=-\sum_{i\in W}\lambda_i - \sum_{i\in W}\sum_{j\in V}\mu_j r(j,i) 1_{\{0\}}(n_j)} + \\
&\qquad + \underbrace{\sum_{i\in V}\sum_{j\in V\setminus\{i\}}\mu_i r(i,j) 1_{\mathbb{N}_+}(n_i) - \sum_{i\in\tilde{J}}\sum_{j\in\tilde{J}\setminus\{i\}}\mu_i r(i,j) 1_{\mathbb{N}_+}(n_i)}_{=-\sum_{i\in W}\sum_{j\in\tilde{J}\setminus\{i\}}\mu_i r(i,j) 1_{\mathbb{N}_+}(n_i) - \sum_{i\in V}\sum_{j\in W}\mu_i r(i,j) 1_{\mathbb{N}_+}(n_i)} + \\
&\qquad + \underbrace{\sum_{j\in V}\mu_j r(j,0) 1_{\mathbb{N}_+}(n_j) - \sum_{i\in\tilde{J}}\mu_i r(i,0) 1_{\mathbb{N}_+}(n_i)}_{=-\sum_{i\in W}\mu_i r(i,0) 1_{\mathbb{N}_+}(n_i)}\Big) + \\
&\quad + \sum_{j\in W}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j + \sum_{i\in V}\mu_i r(i,j)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&\quad + \sum_{j\in W}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \mu_j \underbrace{\Big(r(j,0) + \sum_{i\in V} r(j,i)\Big)}_{=1-\sum_{i\in W} r(j,i)} + \\
&\quad + \sum_{i\in W}\sum_{j\in W\setminus\{i\}}\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J))\cdot \\
&\qquad \cdot \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_J))\cdot \\
&\quad \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{i\in W}\lambda_i \underbrace{- \sum_{i\in W}\sum_{j\in V}\mu_j r(j,i) 1_{\{0\}}(n_j) - \sum_{i\in V}\sum_{j\in W}\mu_i r(i,j) 1_{\mathbb{N}_+}(n_i)}_{=-\sum_{i\in W}\sum_{j\in V}\mu_j r(j,i)} \\
&\qquad - \sum_{i\in W}\mu_i r(i,0) 1_{\mathbb{N}_+}(n_i) - \sum_{i\in W}\sum_{j\in\tilde{J}\setminus\{i\}}\mu_i r(i,j) 1_{\mathbb{N}_+}(n_i)\Big) \\
&\quad + \sum_{j\in W}\frac{1}{\xi} \cdot \underbrace{\Big(\lambda_j + \sum_{i\in V}\mu_i r(i,j)\Big)}_{\overset{(6.15)}{=}\eta_j - \sum_{i\in W}\eta_i r(i,j)} \cdot 1_{\mathbb{N}_+}(n_j) \cdot \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_j - 1, ..., n_J)) + \\
&\quad + \sum_{j\in W}\frac{1}{\xi} \cdot \mu_j \Big(1 - \sum_{i\in W} r(j,i)\Big) \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_j + 1, ..., n_J)) + \\
&\quad + \sum_{i\in W}\sum_{j\in W\setminus\{i\}}\frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J))
\end{aligned}
$$

$$
\begin{aligned}
&\Leftrightarrow P(X_i(t_1) = n_i : i \in W) = \\
&\quad = \frac{1}{\xi} \cdot \Big(\xi \underbrace{- \sum_{i \in W} \lambda_i - \sum_{i \in W} \sum_{j \in V} \mu_j r(j,i)}_{\overset{(6.15)}{=} - \sum_{i \in W} \eta_i + \sum_{i \in W} \sum_{j \in W} \eta_j r(j,i)} + \\
&\qquad \underbrace{- \sum_{i \in W} \mu_i r(i,0) 1_{\mathbb{N}_+}(n_i) - \sum_{i \in W} \sum_{j \in \tilde{J} \setminus \{i\}} \mu_i r(i,j) 1_{\mathbb{N}_+}(n_i) \Big)}_{= - \sum_{i \in W} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i)} \cdot \\
&\qquad \cdot \sum_{(n_k : k \in V) \in \mathbb{N}^{|V|}} P(X(0) = (n_1, \ldots, n_J)) + \\
&\quad + \sum_{j \in W} \frac{1}{\xi} \cdot \Big(\eta_j - \sum_{i \in W} \eta_i r(i,j) \Big) \cdot 1_{\mathbb{N}_+}(n_j) \sum_{(n_k : k \in V) \in \mathbb{N}^{|V|}} P(X(0) = (n_1, \ldots, n_j - 1, \ldots, n_J)) + \\
&\quad + \sum_{j \in W} \frac{1}{\xi} \cdot \mu_j \Big(1 - \sum_{i \in W} r(j,i) \Big) \sum_{(n_k : k \in V) \in \mathbb{N}^{|V|}} P(X(0) = (n_1, \ldots, n_j + 1, \ldots, n_J)) + \\
&\quad + \sum_{i \in W} \sum_{j \in W \setminus \{i\}} \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \sum_{(n_k : k \in V) \in \mathbb{N}^{|V|}} P(X(0) = (n_1, \ldots, n_i + 1, \ldots, n_j - 1, \ldots, n_J)) \\
&\overset{(6.16)}{=} \frac{1}{\xi} \cdot \Big(\xi - \sum_{i \in W} \eta_i + \sum_{i \in W} \sum_{j \in W} \eta_j r(j,i) - \sum_{i \in W} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i) \Big) \cdot \\
&\qquad \cdot \prod_{k \in W} \left(1 - \frac{\eta_k}{\mu_k} \right) \left(\frac{\eta_k}{\mu_k} \right)^{n_k} \\
&\quad + \sum_{j \in W} \frac{1}{\xi} \cdot \Big(\eta_j (1 - r(j,j)) - \sum_{i \in W \setminus \{j\}} \eta_i r(i,j) \Big) \cdot 1_{\mathbb{N}_+}(n_j) \cdot \frac{\mu_j}{\eta_j} \prod_{k \in W} \left(1 - \frac{\eta_k}{\mu_k} \right) \left(\frac{\eta_k}{\mu_k} \right)^{n_k} \\
&\quad + \sum_{j \in W} \frac{1}{\xi} \cdot \mu_j \Big(1 - \sum_{i \in W} r(j,i) \Big) \frac{\eta_j}{\mu_j} \prod_{k \in W} \left(1 - \frac{\eta_k}{\mu_k} \right) \left(\frac{\eta_k}{\mu_k} \right)^{n_k} \\
&\quad + \sum_{i \in W} \sum_{j \in W \setminus \{i\}} \frac{\mu_i r(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j) \cdot \frac{\eta_i}{\mu_i} \frac{\mu_j}{\eta_j} \prod_{k \in W} \left(1 - \frac{\eta_k}{\mu_k} \right) \left(\frac{\eta_k}{\mu_k} \right)^{n_k} \\
&= \prod_{k \in W} \left(1 - \frac{\eta_k}{\mu_k} \right) \left(\frac{\eta_k}{\mu_k} \right)^{n_k} \cdot \frac{1}{\xi} \cdot \\
&\qquad \cdot \Big[\xi - \sum_{i \in W} \eta_i + \sum_{i \in W} \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in W} \eta_j \Big(1 - \sum_{i \in W} r(j,i) \Big) \\
&\qquad + \sum_{j \in W} \mu_j (1 - r(j,j)) \cdot 1_{\mathbb{N}_+}(n_j) - \sum_{i \in W} \mu_i (1 - r(i,i)) 1_{\mathbb{N}_+}(n_i) \\
&\qquad + \sum_{i \in W} \sum_{j \in W \setminus \{i\}} \eta_i r(i,j) \cdot 1_{\mathbb{N}_+}(n_j) \cdot \frac{\mu_j}{\eta_j} - \sum_{j \in W} \sum_{i \in W \setminus \{j\}} \eta_i r(i,j) \cdot 1_{\mathbb{N}_+}(n_j) \cdot \frac{\mu_j}{\eta_j} \Big] \\
&= \prod_{k \in W} \left(1 - \frac{\eta_k}{\mu_k} \right) \left(\frac{\eta_k}{\mu_k} \right)^{n_k}.
\end{aligned}
$$

□

A statement similar to Proposition 6.4 can be proven for Jackson networks with infinite supply and with unreliable nodes. Note that Proposition 6.4 where all nodes are reliable is a special case of the following proposition.

Proposition 6.5. *Consider a Jackson network with J nodes where nodes in $V \subseteq \tilde{J}$ have an infinite supply of work as in Definition 4.1. Nodes in $W := \tilde{J} \setminus V$ operate without infinite supply. Denote by $\eta = (\eta_1, ..., \eta_J)$ the unique solution of the traffic equations (4.5). $S := \{i \in \tilde{J} : \eta_i < \mu_i\}$ is the set of stable nodes, nodes in $U := \tilde{J} \setminus S$ are unstable. We assume that all unstable nodes have an infinite supply of work, i.e., $U \subseteq V \subseteq \tilde{J}$. In this situation the traffic equations (4.5) reduce to*

$$\eta_i = \lambda_i + \sum_{j \in W} \eta_j r(j,i) + \sum_{j \in V} \mu_j r(j,i), \quad i \in \tilde{J}. \tag{6.18}$$

Assume that all nodes are unreliable. Breakdowns and repairs of the nodes are controlled by load-independent breakdown and repair intensities as in Example 2.2. In case of breakdowns, customers are rerouted according to the stalling regime, the skipping regime or the blocking rs-rd regime. If skipping is in force, we assume

$$\eta_i = \mu_i \quad \forall i \in V. \tag{6.19}$$

In case of the blocking rs-rd regime assume that the following reversibility constraints hold:

$$\eta_i r(i,j) = \eta_j r(j,i) \quad \forall i,j \in W, \tag{6.20}$$

$$\eta_i r(i,j) = \mu_j r(j,i) \quad \forall i \in W, j \in V, \tag{6.21}$$

$$\mu_i r(i,j) = \mu_j r(j,i) \quad \forall i,j \in V. \tag{6.22}$$

Denote by (Y, X) the Markovian availability-queue lengths process on the state space $\mathcal{P}(\tilde{J}) \times \mathbb{N}^J$ and assume that the initial distribution

$$P^{(Y(0),X_1(0),...,X_J(0))} := P((Y,X)(0) = (I, n_1, ..., n_J) : (I, n_1, ..., n_J) \in \mathcal{P}(\tilde{J}) \times \mathbb{N}^J)$$

fulfills the marginal product form property for the stable part without infinite supply:

$$P(Y(0) = I, X_i(0) = n_i : i \in W) = \left(\sum_{K \subseteq \tilde{J}} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i \in W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i} \tag{6.23}$$

for all $n_i \in \mathbb{N}, I \subseteq \tilde{J}$.
Consider the uniformization of (Y, X) with Poisson-clock with intensity

$$\xi \geq \sum_{i=1}^{J} (\lambda_i + \mu_i) + \sum_{I \subseteq \tilde{J}} \sum_{\emptyset \neq K \subseteq \tilde{J} \setminus I} \alpha(I, I \cup K) + \sum_{I \subseteq \tilde{J}} \sum_{\emptyset \neq K \subseteq I} \beta(I, I \setminus K),$$

and denote by t_n the (random) time of the nth jump of the uniformized process X. Then for all $n_i \in \mathbb{N}$ and all $I \subseteq \tilde{J}$ holds for all $n \in \mathbb{N}$

$$P(Y(t_n) = I, X_i(t_n) = n_i : i \in W) = \left(\sum_{K \subseteq \tilde{J}} \frac{A(K)}{B(K)}\right)^{-1} \frac{A(I)}{B(I)} \prod_{i \in W} \left(1 - \frac{\eta_i}{\mu_i}\right) \left(\frac{\eta_i}{\mu_i}\right)^{n_i}. \tag{6.24}$$

Proof. The proof consists of showing the reproduction of the marginal after one jump (the first jump time will be denoted by t_1), i.e., (6.24) with $t_n = t_1$.

Since the marginal distribution (6.23) is the only assumption on the initial distribution to hold, the reproduction of this marginal distribution after one time step is sufficient for iterated reproduction in the same manner, hence a reproduction after n jumps for any t_n is proven.

Consider the process uniformized with uniformization constant ξ, i.e., jumps of the process are controlled by a Poisson-ξ clock. Denote by $p^u := I + \frac{1}{\xi}Q$ the jump kernel, i.e., the entries of the kernel are the probabilities of a jump, and denote by t_1 the time of the first jump.

We consider the rerouting according to skipping and blocking rs-rd together, the case of stalling is a simplification of the following proof by setting $I = \emptyset$.

Recall that we assume that X has cadlag paths.

For all $I \subseteq \tilde{J}$ holds:

$$
\begin{aligned}
&P(Y(t_1) = I, X(t_1) = (n_1, ..., n_J)) = \\
&= \sum_{(m_1,...,m_J)\in\mathbb{N}^J} \sum_{H\subseteq\tilde{J}} P(Y(0) = H, X(0) = (m_1, ..., m_J)) \cdot p^u(H, m_1, ..., m_J; I, n_1, ..., n_J) \\
&= \sum_{H\subseteq\tilde{J}, H\neq I} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot p^u(H, n_1, ..., n_J; I, n_1, ..., n_J) + \\
&\quad + P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot p^u(I, n_1, ..., n_J; I, n_1, ..., n_J) + \\
&\quad + \sum_{j\in\tilde{J}\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot \\
&\qquad \cdot p^u(I, n_1, ..., n_j - 1, ..., n_J; I, n_1, ..., n_j, ..., n_J) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&\quad + \sum_{j\in\tilde{J}\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \\
&\qquad \cdot p^u(I, n_1, ..., n_j + 1, ..., n_J; I, n_1, ..., n_j, ..., n_J) + \\
&\quad + \sum_{i\in\tilde{J}\setminus I} \sum_{j\in\tilde{J}\setminus I, j\neq i} P(Y(0) = I, X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J)) \cdot \\
&\qquad \cdot p^u(I, n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J; I, n_1, ..., n_i, ..., n_j, ..., n_J) \cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned}
$$

Plugging in the probabilities of the jump kernel yields

$$
\begin{aligned}
&P(Y(t_1)=I,X(t_1)=(n_1,...,n_J))=\\
&=\sum_{H\subset I}P(Y(0)=H,X(0)=(n_1,...,n_J))\cdot\frac{\alpha(H,I)}{\xi}+\\
&+\sum_{I\subset K\subseteq\tilde{J}}P(Y(0)=K,X(0)=(n_1,...,n_J))\cdot\frac{\beta(K,I)}{\xi}+\\
&+P(Y(0)=I,X(0)=(n_1,...,n_J))\cdot\frac{1}{\xi}\cdot\\
&\quad\cdot\Big(\xi-\sum_{i\in\tilde{J}\setminus I}\lambda_i^I-\sum_{i\in\tilde{J}\setminus I}\sum_{j\in V\setminus I}\mu_j^I r^I(j,i)1_{\{0\}}(n_j)-\sum_{i\in\tilde{J}\setminus I}\mu_i^I r^I(i,0)\cdot 1_{\mathbb{N}_+}(n_i)\\
&\quad-\sum_{i\in\tilde{J}\setminus I}\sum_{j\in\tilde{J}\setminus I,j\neq i}\mu_i^I r^I(i,j)1_{\mathbb{N}_+}(n_i)-\sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K)-\sum_{H\subset I}\beta(I,H)\Big)+\\
&+\sum_{j\in W\setminus I}P(Y(0)=I,X(0)=(n_1,...,n_j-1,...,n_J))\cdot\\
&\quad\cdot\frac{1}{\xi}\cdot\Big(\lambda_j^I+\sum_{i\in V\setminus I}\mu_i^I r^I(i,j)1_{\{0\}}(n_i)\Big)\cdot 1_{\mathbb{N}_+}(n_j)+\\
&+\sum_{j\in V\setminus I}P(Y(0)=I,X(0)=(n_1,...,n_j-1,...,n_J))\cdot\\
&\quad\cdot\frac{1}{\xi}\cdot\Big(\lambda_j^I+\sum_{i\in V\setminus I,i\neq j}\mu_i^I r^I(i,j)1_{\{0\}}(n_i)+\mu_j^I r^I(j,j)1_{\{0\}}(n_j-1)\Big)\cdot 1_{\mathbb{N}_+}(n_j)+\\
&+\sum_{j\in\tilde{J}\setminus I}P(Y(0)=I,X(0)=(n_1,...,n_j+1,...,n_J))\cdot\frac{\mu_j^I r^I(j,0)}{\xi}+\\
&+\sum_{i\in\tilde{J}\setminus I}\sum_{j\in\tilde{J}\setminus I,j\neq i}P(Y(0)=I,X(0)=(n_1,...,n_i+1,...,n_j-1,...,n_J))\cdot\\
&\quad\cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j).
\end{aligned}
$$

Summing over all $(n_k : k\in V)\in\mathbb{N}^{|V|}$ yields

$$
\begin{aligned}
&P(Y(t_1) = I, X_i(t_1) = n_i : i \in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(t_1) = I, X(t_1) = (n_1, ..., n_J)) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{H\subset I} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{\alpha(H,I)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{I\subset K\subseteq \tilde{J}} P(Y(0) = K, X(0) = (n_1, ..., n_J)) \cdot \frac{\beta(K,I)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \\
&\quad \cdot \Big(\xi - \sum_{i\in\tilde{J}\setminus I} \lambda_i^I - \sum_{i\in\tilde{J}\setminus I} \sum_{j\in V\setminus I} \mu_j^I r^I(j,i) 1_{\{0\}}(n_j) - \sum_{i\in\tilde{J}\setminus I} \mu_i^I r^I(i,0) \cdot 1_{\mathbb{N}_+}(n_i) \\
&\quad - \sum_{i\in\tilde{J}\setminus I} \sum_{j\in\tilde{J}\setminus I, j\neq i} \mu_i^I r^I(i,j) 1_{\mathbb{N}_+}(n_i) - \sum_{I\subset K\subseteq\tilde{J}} \alpha(I,K) - \sum_{H\subset I} \beta(I,H)\Big) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in W\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot \\
&\quad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j^I + \sum_{i\in V\setminus I} \mu_i^I r^I(i,j) 1_{\{0\}}(n_i)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in V\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot \\
&\quad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j^I + \sum_{i\in V\setminus I, i\neq j} \mu_i^I r^I(i,j) 1_{\{0\}}(n_i) + \mu_j^I r^I(j,j) 1_{\{0\}}(n_j - 1)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in\tilde{J}\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \frac{\mu_j^I r^I(j,0)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in\tilde{J}\setminus I} \sum_{j\in\tilde{J}\setminus I, j\neq i} P(Y(0) = I, X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J)) \cdot \\
&\quad \cdot \frac{\mu_i^I r^I(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow\ & P(Y(t_1)=I, X_i(t_1)=n_i : i\in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{H\subset I} P(Y(0)=H, X(0)=(n_1,...,n_J))\cdot\frac{\alpha(H,I)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{I\subset K\subseteq\tilde{J}} P(Y(0)=K, X(0)=(n_1,...,n_J))\cdot\frac{\beta(K,I)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0)=I, X(0)=(n_1,...,n_J))\cdot\frac{1}{\xi}\cdot \\
&\quad\cdot\Big(\xi - \sum_{i\in\tilde{J}\setminus I}\lambda_i^I - \sum_{i\in\tilde{J}\setminus I}\sum_{j\in V\setminus I}\mu_j^I r^I(j,i)1_{\{0\}}(n_j) - \sum_{i\in\tilde{J}\setminus I}\mu_i^I r^I(i,0)\cdot 1_{\mathbb{N}_+}(n_i) \\
&\quad - \sum_{i\in\tilde{J}\setminus I}\sum_{j\in\tilde{J}\setminus I, j\neq i}\mu_i^I r^I(i,j)1_{\mathbb{N}_+}(n_i) - \sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K) - \sum_{H\subset I}\beta(I,H)\Big) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in W\setminus I} P(Y(0)=I, X(0)=(n_1,...,n_j-1,...,n_J))\cdot \\
&\quad\cdot\frac{1}{\xi}\cdot\Big(\lambda_j^I + \sum_{i\in V\setminus I}\mu_i^I r^I(i,j)1_{\{0\}}(n_i)\Big)\cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in V\setminus I} P(Y(0)=I, X(0)=(n_1,...,n_j,...,n_J))\cdot \\
&\quad\cdot\frac{1}{\xi}\cdot\Big(\lambda_j^I + \sum_{i\in V\setminus I}\mu_i^I r^I(i,j)1_{\{0\}}(n_i)\Big) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in W\setminus I} P(Y(0)=I, X(0)=(n_1,...,n_j+1,...,n_J))\cdot\frac{\mu_j^I r^I(j,0)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in V\setminus I} P(Y(0)=I, X(0)=(n_1,...,n_j,...,n_J))\cdot\frac{\mu_j^I r^I(j,0)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in W\setminus I}\sum_{j\in W\setminus I, j\neq i} P(Y(0)=I, X(0)=(n_1,...,n_i+1,...,n_j-1,...,n_J))\cdot \\
&\quad\cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in W\setminus I}\sum_{j\in V\setminus I} P(Y(0)=I, X(0)=(n_1,...,n_i+1,...,n_j,...,n_J))\cdot \\
&\quad\cdot\frac{\mu_i^I r^I(i,j)}{\xi} \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in V\setminus I}\sum_{j\in W\setminus I} P(Y(0)=I, X(0)=(n_1,...,n_i,...,n_j-1,...,n_J))\cdot \\
&\quad\cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_j)\cdot 1_{\mathbb{N}_+}(n_i) \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in V\setminus I}\sum_{j\in V\setminus I, j\neq i} P(Y(0)=I, X(0)=(n_1,...,n_J))\cdot\frac{\mu_i^I r^I(i,j)}{\xi}\cdot 1_{\mathbb{N}_+}(n_i)
\end{aligned}
$$

$$
\begin{aligned}
\Leftrightarrow\ & P(Y(t_1) = I, X_i(t_1) = n_i : i \in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{H\subset I} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{\alpha(H, I)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{I\subset K\subseteq \tilde{J}} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{\beta(K, I)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \\
&\quad \cdot \Big(\xi - \sum_{i\in W\setminus I} \lambda_i^I - \sum_{i\in W\setminus I} \sum_{j\in V\setminus I} \mu_j^I r^I(j, i) 1_{\{0\}}(n_j) - \sum_{i\in W\setminus I} \mu_i^I r^I(i, 0) \cdot 1_{\mathbb{N}_+}(n_i) \\
&\quad - \sum_{i\in W\setminus I} \sum_{j\in \tilde{J}\setminus I, j\neq i} \mu_i^I r^I(i, j) 1_{\mathbb{N}_+}(n_i) - \sum_{i\in V\setminus I} \sum_{j\in W\setminus I} \mu_i^I r^I(i, j) 1_{\mathbb{N}_+}(n_i) \\
&\quad - \sum_{I\subset K\subseteq \tilde{J}} \alpha(I, K) - \sum_{H\subset I} \beta(I, H)\Big) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in W\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot \\
&\quad \cdot \frac{1}{\xi} \cdot \Big(\lambda_j^I + \sum_{i\in V\setminus I} \mu_i^I r^I(i, j)\Big) \cdot 1_{\mathbb{N}_+}(n_j) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{j\in W\setminus I} P(Y(0) = I, X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot \\
&\quad \cdot \frac{1}{\xi} \cdot \mu_j^I \cdot \Big(r^I(j, 0) + \sum_{i\in V\setminus I} r^I(j, i)\Big) + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} \sum_{i\in W\setminus I} \sum_{j\in W\setminus I, j\neq i} P(Y(0) = I, X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J)) \cdot \\
&\quad \cdot \frac{\mu_i^I r^I(i, j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j)
\end{aligned}
$$

$$\Leftrightarrow P(Y(t_1) = I, X_i(t_1) = n_i : i \in W) =$$

$$= \sum_{H \subset I} \underbrace{\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{A(I)}{A(H)}}_{\overset{(6.23)}{=} \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0)=I,X(0)=(n_1,...,n_J)) \frac{B(I)}{B(H)}} \cdot \frac{1}{\xi} +$$

$$+ \sum_{I \subset K \subseteq \tilde{J}} \underbrace{\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = K, X(0) = (n_1, ..., n_J)) \cdot \frac{B(K)}{B(I)}}_{\overset{(6.23)}{=} \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0)=I,X(0)=(n_1,...,n_J)) \frac{A(K)}{A(I)}} \cdot \frac{1}{\xi} +$$

$$+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot$$

$$\cdot \Big(\xi - \sum_{i\in W\setminus I} \lambda_i^I - \sum_{i\in W\setminus I} \sum_{j\in V\setminus I} \mu_j^I r^I(j,i) 1_{\{0\}}(n_j) - \sum_{i\in W\setminus I} \mu_i^I r^I(i,0) \cdot 1_{\mathbb{N}_+}(n_i)$$

$$- \sum_{i\in W\setminus I} \sum_{j\in \tilde{J}\setminus I, j\neq i} \mu_i^I r^I(i,j) 1_{\mathbb{N}_+}(n_i) - \sum_{i\in V\setminus I} \sum_{j\in W\setminus I} \mu_i^I r^I(i,j) 1_{\mathbb{N}_+}(n_i)$$

$$- \sum_{I\subset K\subseteq \tilde{J}} \alpha(I,K) - \sum_{H\subset I} \beta(I,H)\Big) +$$

$$+ \sum_{j\in W\setminus I} \underbrace{\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_j - 1, ..., n_J)) \cdot}_{\overset{(6.23)}{=} \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0)=I,X(0)=(n_1,...,n_J)) \frac{\mu_j}{\eta_j}}$$

$$\cdot \frac{1}{\xi} \cdot \Big(\lambda_j^I + \sum_{i\in V\setminus I} \mu_i^I r^I(i,j)\Big) \cdot 1_{\mathbb{N}_+}(n_j) +$$

$$+ \sum_{j\in W\setminus I} \underbrace{\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_j + 1, ..., n_J)) \cdot}_{\overset{(6.23)}{=} \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0)=I,X(0)=(n_1,...,n_J)) \frac{\eta_j}{\mu_j}}$$

$$\cdot \frac{1}{\xi} \cdot \mu_j^I \cdot \Big(r^I(j,0) + \sum_{i\in V\setminus I} r^I(j,i)\Big) +$$

$$+ \sum_{i\in W\setminus I} \sum_{j\in W\setminus I, j\neq i} \underbrace{\sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_i + 1, ..., n_j - 1, ..., n_J)) \cdot}_{\overset{(6.23)}{=} \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0)=I,X(0)=(n_1,...,n_J)) \frac{\eta_i}{\mu_i} \frac{\mu_j}{\eta_j}}$$

$$\cdot \frac{\mu_i^I r^I(i,j)}{\xi} \cdot 1_{\mathbb{N}_+}(n_j)$$

$$
\begin{aligned}
&\Leftrightarrow P(Y(t_1)=I, X_i(t_1)=n_i : i\in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0)=I, X(0)=(n_1,\dots,n_J))\cdot\frac{1}{\xi}\cdot \\
&\cdot\Big[\xi-\sum_{i\in W\setminus I}\lambda_i^I-\sum_{i\in W\setminus I}\sum_{j\in V\setminus I}\mu_j^I r^I(j,i)1_{\{0\}}(n_j)-\sum_{i\in W\setminus I}\mu_i^I r^I(i,0)\cdot 1_{\mathbb{N}_+}(n_i) \\
&-\sum_{i\in W\setminus I}\sum_{j\in\tilde{J}\setminus I, j\neq i}\mu_i^I r^I(i,j)1_{\mathbb{N}_+}(n_i)-\sum_{i\in V\setminus I}\sum_{j\in W\setminus I}\mu_i^I r^I(i,j)1_{\mathbb{N}_+}(n_i) \\
&-\sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K)-\sum_{H\subset I}\beta(I,H)+\sum_{H\subset I}\beta(I,H)+\sum_{I\subset K\subseteq\tilde{J}}\alpha(I,K)+ \\
&+\sum_{j\in W\setminus I}\frac{\mu_j}{\eta_j}\cdot\Big(\lambda_j^I+\sum_{i\in V\setminus I}\mu_i^I r^I(i,j)\Big)\cdot 1_{\mathbb{N}_+}(n_j)+ \\
&+\sum_{j\in W\setminus I}\frac{\eta_j}{\mu_j}\cdot\mu_j^I\cdot\Big(r^I(j,0)+\sum_{i\in V\setminus I}r^I(j,i)\Big)+ \\
&+\sum_{i\in W\setminus I}\sum_{j\in W\setminus I, j\neq i}\frac{\eta_i}{\mu_i}\frac{\mu_j}{\eta_j}\cdot\mu_i^I r^I(i,j)\cdot 1_{\mathbb{N}_+}(n_j)\Big].
\end{aligned}
$$

In case of skipping and blocking rs-rd $\mu_i^I=\mu_i$ holds for all $i\in\tilde{J}\setminus I$ and with Lemma 4.40 and Lemma 4.43 we have $\eta_i^I=\eta_i$ for all $i\in\tilde{J}\setminus I$. This yields

$$
\begin{aligned}
&P(Y(t_1)=I, X_i(t_1)=n_i : i\in W) = \\
&= \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0)=I, X(0)=(n_1,\dots,n_J))\cdot\frac{1}{\xi}\cdot \\
&\cdot\Big[\xi-\sum_{i\in W\setminus I}\lambda_i^I-\sum_{i\in W\setminus I}\sum_{j\in V\setminus I}\mu_j^I r^I(j,i)-\sum_{i\in W\setminus I}\mu_i^I r^I(i,0)\cdot 1_{\mathbb{N}_+}(n_i) \\
&-\sum_{i\in W\setminus I}\sum_{j\in\tilde{J}\setminus I, j\neq i}\mu_i^I r^I(i,j)1_{\mathbb{N}_+}(n_i)+\sum_{j\in W\setminus I}\frac{\eta_j^I}{\mu_j^I}\cdot\mu_j^I\cdot\Big(r^I(j,0)+\sum_{i\in V\setminus I}r^I(j,i)\Big)+ \\
&+\sum_{j\in W\setminus I}\frac{\mu_j^I}{\eta_j^I}\cdot\Big(\lambda_j^I+\sum_{i\in V\setminus I}\mu_i^I r^I(i,j)\Big)\cdot 1_{\mathbb{N}_+}(n_j)+\sum_{i\in W\setminus I}\sum_{j\in W\setminus I, j\neq i}\frac{\eta_i^I}{\mu_i^I}\frac{\mu_j^I}{\eta_j^I}\cdot\mu_i^I r^I(i,j)\cdot 1_{\mathbb{N}_+}(n_j)\Big]
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&P(Y(t_1) = I, X_i(t_1) = n_i : i \in W) = \\
&= \sum_{(n_k : k \in V) \in \mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \Big[\xi - \sum_{i \in W \setminus I} \underbrace{\Big(\lambda_i^I + \sum_{j \in V \setminus I} \mu_j^I r^I(j,i)\Big)}_{\overset{(6.18)}{=} \eta_i^I - \sum_{j \in W \setminus I} \eta_j^I r^I(j,i)} \\
&\quad - \sum_{i \in W \setminus I} \mu_i^I (1 - r^I(i,i)) \cdot 1_{\mathbb{N}_+}(n_i) + \sum_{j \in W \setminus I} \eta_j^I \cdot \Big(1 - \sum_{i \in W \setminus I} r^I(j,i)\Big) \\
&\quad + \sum_{j \in W \setminus I} \frac{\mu_j^I}{\eta_j^I} \cdot \underbrace{\Big(\lambda_j^I + \sum_{i \in V \setminus I} \mu_i^I r^I(i,j) + \sum_{i \in W \setminus I, i \neq j} \eta_i^I r^I(i,j)\Big)}_{\overset{(6.18)}{=} \eta_j^I (1 - r^I(j,j))} \cdot 1_{\mathbb{N}_+}(n_j)\Big] \\
&= \sum_{(n_k : k \in V) \in \mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)).
\end{aligned}
$$

In case of stalling, setting $I = \emptyset$ in the beginning of the proof yields the same result. For $I \neq \emptyset$ we have:

$$
\begin{aligned}
&P(Y(t_1) = I, X(t_1) = (n_1, ..., n_J)) = \\
&= \sum_{H \subset I} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{\alpha(H, I)}{\xi} + \\
&\quad + \sum_{I \subset K \subseteq \tilde{J}} P(Y(0) = K, X(0) = (n_1, ..., n_J)) \cdot \frac{\beta(K, I)}{\xi} + \\
&\quad + P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{I \subset K \subseteq \tilde{J}} \alpha(I, K) - \sum_{H \subset I} \beta(I, H)\Big).
\end{aligned}
$$

Summing over all $(n_k : k \in V) \in \mathbb{N}^{|V|}$ yields

$$
\begin{aligned}
&P(Y(t_1) = I, X_i(t_1) = n_i : i \in W) = \\
&= \sum_{H \subset I} \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = H, X(0) = (n_1, ..., n_J)) \cdot \frac{\alpha(H,I)}{\xi} + \\
&+ \sum_{I \subset K \subseteq \tilde{J}} \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = K, X(0) = (n_1, ..., n_J)) \cdot \frac{\beta(K,I)}{\xi} + \\
&+ \sum_{(n_k:k\in V)\in\mathbb{N}^{|V|}} P(Y(0) = I, X(0) = (n_1, ..., n_J)) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{I \subset K \subseteq \tilde{J}} \alpha(I,K) - \sum_{H \subset I} \beta(I,H)\Big) \\
&\overset{(6.23)}{=} \sum_{H \subset I} \frac{\alpha(H,I)}{\xi} \cdot P(Y(0) = I, X_i(0) = n_i : i \in W) \cdot \frac{A(H)}{A(I)} \frac{B(I)}{B(H)} + \\
&+ \sum_{I \subset K \subseteq \tilde{J}} \frac{\beta(K,I)}{\xi} \cdot P(Y(0) = I, X_i(0) = n_i : i \in W) \cdot \frac{A(K)}{A(I)} \frac{B(I)}{B(K)} + \\
&+ P(Y(0) = I, X_i(0) = n_i : i \in W) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{I \subset K \subseteq \tilde{J}} \alpha(I,K) - \sum_{H \subset I} \beta(I,H)\Big) \\
&= P(Y(0) = I, X_i(0) = n_i : i \in W) \cdot \frac{1}{\xi} \cdot \Big(\xi - \sum_{I \subset K \subseteq \tilde{J}} \frac{A(K)}{A(I)} - \sum_{H \subset I} \frac{B(I)}{B(H)} + \\
&\quad + \sum_{H \subset I} \frac{A(I)}{A(H)} \frac{A(H)}{A(I)} \frac{B(I)}{B(H)} + \sum_{I \subset K \subseteq \tilde{J}} \frac{B(K)}{B(I)} \frac{A(K)}{A(I)} \frac{B(I)}{B(K)}\Big) \\
&= P(Y(0) = I, X_i(0) = n_i : i \in W).
\end{aligned}
$$

□

6.3 One-step quasi-stationary marginal distribution

The[6] observation of the previous section suggests that, in the two-node model, the time instance right after the unstable node 2 becomes idle the first time is the critical time point, and, saying it the other way round, the event that node 2 becomes idle is a critical event for the system. In an obvious sense this event resembles the event of absorption in the theory of quasi-stationary distributions of absorbing Markov processes:
An absorbing Markov process is a Markov process with at least one absorbing state. A state i is *absorbing* if, once entered state i, remaining in state i has probability one. Quasi-stationary distributions are distributions for absorbing processes which remain invariant over time given that the process is not absorbed yet. [MD11, p.104]

Quasi-stationary distributions were introduced by M.S. Bartlett [Bar57], more information is given in the book [Sen81] of E. Seneta. The following definition refers to a paper of J.N. Darroch and E. Seneta in 1967.

[6] Since parts of this section are published in our paper [MD11](pp.104-106), its introduction and some of the comments cite the paper which is shortly marked at the end of the concerning paragraphs.

Definition 6.6. [DS67] *Let $Z = (Z(t) : t \geq 0)$ be an absorbing Markov process on a finite state space $E = \{0, 1, ..., n\}$ where 0 is the sole absorbing state.*
Denote by

$$d_i(t) = \frac{P(Z(t) = i)}{1 - P(Z(t) = 0)} = P(Z(t) = i | Z(t) \neq 0)$$

the probability for the process to be in state $i \in E \setminus \{0\}$ at time t under the condition that the process is not absorbed yet.
If

$$d_i(t) = d_i \qquad \forall t \geq 0$$

for all non-absorbing states $i \in E \setminus \{0\}$ holds, then $d = (d_i : i \in E \setminus \{0\})$ is called a quasi-stationary distribution for Z.

Of course, the queue length process X of our two-node model is not an absorbing process. Our intention is to construct an analogue to this quasi-stationary distribution for the marginal process on the stable part of the network as long as the critical event $\{X_2(t) = 0\}$ does not occur. That is, starting the network with the relevant limiting distribution for the stable node 1 we search for conditions which guarantee that this initial distribution is maintained over time. A first attempt is the following proposition. [MD11, p.104]

Proposition 6.7. [MD11, Proposition 9] *Consider the two-node model of Definition 6.1 with $r(2,0) > 0$ and $\mu_1 r(2,1) < \eta_1 r(2,0)$.*
Assume that the initial distribution fulfills the following conditions:

- *the marginal for the stable part is the limiting probability (1.11) of Theorem 1.22, i.e.,*

$$P(X_1(0) = n_1) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}, \quad n_1 \in \mathbb{N}; \tag{6.25}$$

- *the marginal for the unstable part is busy with probability 1, but carries no further restriction, i.e.,*

$$P(X_2(0) = 0) = 0; \tag{6.26}$$

- *the covariance structure of $P^{(X_1(0), X_2(0))}$ fulfills for all $n_1 \in \mathbb{N}$*

$$\begin{aligned} &P(X(0) = (n_1, 1)) = \\ &= P(X_1(0) = n_1) \cdot P(X_2(0) = 1) \cdot \frac{1 - r(2,2)}{r(2,0)} \sum_{k=0}^{n_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^k . \end{aligned} \tag{6.27}$$

Denote by t_1 the (random) time of the first jump of the uniformized process X. Then for all $n_1 \in \mathbb{N}$ holds

$$\sum_{n_2=1}^{\infty} P(X(t_1) = (n_1, n_2) | X_2(t_1) \neq 0) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1} . \tag{6.28}$$

Proof. Denote by p^u the jump kernel of the uniformization with Poisson-ξ clock, where $\xi \geq \sum_{i=1}^{2}(\lambda_i + \mu_i)$. The left side of (6.28) is

$$
\begin{aligned}
&\sum_{n_2=1}^{\infty} P(X(t_1) = (n_1, n_2) | X_2(t_1) \neq 0) = \\
&= \sum_{n_2=1}^{\infty} \frac{P(X(t_1) = (n_1, n_2), X_2(t_1) \neq 0)}{P(X_2(t_1) \neq 0)} = \frac{\sum_{n_2=1}^{\infty} P(X(t_1) = (n_1, n_2))}{P(X_2(t_1) \neq 0)} \\
&= \frac{\sum_{n_2=0}^{\infty} P(X(t_1) = (n_1, n_2)) - P(X(t_1) = (n_1, 0))}{P(X_2(t_1) \neq 0)} \\
&= \frac{1}{1 - P(X_2(t_1) = 0)} \cdot \Big(P(X_1(t_1) = n_1) - \Big[P(X(0) = (n_1, 1)) \cdot p^u(n_1, 1; n_1, 0) + \\
&\qquad\qquad + 1_{\mathbb{N}_+}(n_1) \cdot P(X(0) = (n_1 - 1, 1)) \cdot p^u(n_1 - 1, 1; n_1, 0) \Big] \Big) \\
&= \frac{1}{1 - P(X_2(t_1) = 0)} \cdot \Big(\underbrace{P(X_1(t_1) = n_1)}_{\stackrel{(6.6)}{=} P(X_1(0) = n_1)} - \Big[P(X(0) = (n_1, 1)) \cdot \frac{\mu_2 r(2,0)}{\xi} + \\
&\qquad\qquad + 1_{\mathbb{N}_+}(n_1) \cdot P(X(0) = (n_1 - 1, 1)) \cdot \frac{\mu_2 r(2,1)}{\xi} \Big] \Big) \\
&\stackrel{(6.27)}{=} \frac{1}{1 - P(X_2(t_1) = 0)} \cdot \Big(P(X_1(0) = n_1) - \Big[P(X_1(0) = n_1) \cdot P(X_2(0) = 1) \cdot \\
&\quad \cdot \frac{1 - r(2,2)}{r(2,0)} \cdot \sum_{k=0}^{n_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)} \right)^k \cdot \frac{\mu_2 r(2,0)}{\xi} + 1_{\mathbb{N}_+}(n_1) \cdot P(X_1(0) = n_1 - 1) \cdot \\
&\quad \cdot P(X_2(0) = 1) \cdot \frac{1 - r(2,2)}{r(2,0)} \sum_{k=0}^{n_1 - 1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)} \right)^k \cdot \frac{\mu_2 r(2,1)}{\xi} \Big] \Big) \\
&= \frac{1}{1 - P(X_2(t_1) = 0)} \cdot \Big(P(X_1(0) = n_1) - P(X_2(0) = 1) \cdot \frac{\mu_2 (1 - r(2,2))}{\xi} \cdot \\
&\quad \cdot \Big[P(X_1(0) = n_1) \sum_{k=0}^{n_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)} \right)^k + \\
&\quad + 1_{\mathbb{N}_+}(n_1) \underbrace{P(X_1(0) = n_1 - 1)}_{\stackrel{(6.25)}{=} P(X_1(0) = n_1) \cdot \frac{\mu_1}{\eta_1}} \cdot \frac{r(2,1)}{r(2,0)} \sum_{k=0}^{n_1 - 1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)} \right)^k \Big] \Big) \\
&= \frac{1}{1 - P(X_2(t_1) = 0)} P(X_1(0) = n_1) \left(1 - P(X_2(0) = 1) \cdot \frac{\mu_2 (1 - r(2,2))}{\xi} \right).
\end{aligned}
$$

The proof is completed by observing

$$
\begin{aligned}
&P(X_2(t_1) = 0) = \\
&\overset{(6.26)}{=} \sum_{n_1 \in \mathbb{N}} \sum_{m_1 \in \mathbb{N}} P(X(0) = (m_1, 1)) \cdot p^u(m_1, 1; n_1, 0) \\
&= \sum_{n_1 \in \mathbb{N}} P(X(0) = (n_1, 1)) \cdot \frac{\mu_2 r(2,0)}{\xi} + \sum_{n_1 \in \mathbb{N}_+} P(X(0) = (n_1 - 1, 1)) \cdot \frac{\mu_2 r(2,1)}{\xi} \\
&= \frac{\mu_2 r(2,0)}{\xi} \underbrace{\sum_{n_1 \in \mathbb{N}} P(X(0) = (n_1, 1))}_{=P(X_2(0)=1)} + \frac{\mu_2 r(2,1)}{\xi} \underbrace{\sum_{n_1 \in \mathbb{N}_+} P(X(0) = (n_1 - 1, 1))}_{=P(X_2(0)=1)} \\
&= P(X_2(0) = 1) \cdot \frac{\mu_2 (1 - r(2,2))}{\xi} \qquad (6.29)
\end{aligned}
$$

and from (6.25). □

Remark 6.8. [MD11, p.105]

- *The result is not obvious, because from assuming cadlag paths we observe with $X(t_1) = (n_1, n_2)$ the state of the system* **after** *the jump. Clearly, until just before the second jump the property derived here is maintained.*
- *The side constraints on the initial distribution are seemingly rather weak. Nevertheless, it can be shown that for $n_1 \in \mathbb{N}$ the probabilities $P(X(t_1) = (n_1, 1) | X_2(t_1) \neq 0)$ do not fulfill the correlation property* (6.27) *without additional requirements for the initial distribution.*
- *The marginal initial probability for the unstable node is restricted only by the requirement that the node is busy with probability* 1.
  If $P(X_2(0) = 1) = 0$, the requirement (6.27) *is always fulfilled for any joint distribution $P^{(X_1(0), X_2(0))}$. So (similar to Proposition 6.2) if we know of at least $K > 0$ customers waiting at node 2 at the beginning ($t = 0$), then for $K - 2$ jumps we can guarantee* (6.27) *still to hold. Thus after $K - 2$ jumps the result of Proposition 6.7 applies and for at least $K - 1$ jumps the product form* (6.28) *for node 1 is maintained.*

Remark 6.9. [MD11, p.105] *The correlation property* (6.27) *can be given an appealing interpretation resembling the concept of the squared* coefficient of variation *of some non-negative random variable X with $\mathbb{E}(X) > 0$:*

$$C_X^2 := \frac{Var(X)}{(\mathbb{E}(X))^2}.$$

Consider a two-dimensional random vector (X, Y) with values in $\mathbb{R}_+^2$ and with

$$\mathbb{E}(X) \cdot \mathbb{E}(Y) > 0.$$

Then the analogue is

$$\frac{Cov(X,Y)}{\mathbb{E}(X) \cdot \mathbb{E}(Y)} = \frac{\mathbb{E}(X \cdot Y)}{\mathbb{E}(X) \cdot \mathbb{E}(Y)} - 1.$$

If we apply this to the random vectors

$$(X, Y) := (1_{\{n_1\}}(X_1(0)), 1_{\{1\}}(X_2(0))),$$

we obtain

$$\frac{Cov(X,Y)}{\mathbb{E}(X) \cdot \mathbb{E}(Y)} = \frac{1 - r(2,2)}{r(2,0)} \sum_{k=0}^{n_1} \left(- \frac{\mu_1 r(2,1)}{\eta_1 r(2,0)} \right)^k - 1.$$

The following example considers the special network class of tandems, where two nodes are connected in series and there is only one way to pass through the network.

Example 6.10. *Consider a tandem Jackson network without immediate feedback, i.e.,* $r(0,1) = r(1,2) = r(2,0) = 1$. *Let node 1 be stable and node 2 be unstable, then* $\eta_1 = \eta_2 = \lambda$ *and* $\lambda \geq \mu_2$. *Let* X *be the queue length process on* $\mathbb{N}^2$. *Consider the uniformization of* X *with uniformization constant* $\xi \geq \lambda + \mu_1 + \mu_2$.
(a) *Assume that the initial distribution fulfills*

$$P(X_1(0) = n_1) = \left(1 - \frac{\lambda}{\mu_1}\right) \left(\frac{\lambda}{\mu_1}\right)^{n_1}, \quad n_1 \in \mathbb{N}.$$

Denote by t_1 *the (random) time of the first jump of the uniformized process* X. *Then for all* $n_1 \in \mathbb{N}$ *holds*

$$
\begin{aligned}
&P(X_1(t_1) = n_1) = \sum_{n_2\in\mathbb{N}} P(X(t_1) = (n_1, n_2)) \\
&= \sum_{n_2\in\mathbb{N}} \sum_{(m_1,m_2)\in\mathbb{N}^2} P(X(0) = (m_1, m_2))p^u(m_1, m_2; n_1, n_2) \\
&= \sum_{n_2\in\mathbb{N}} \Big(P(X(0) = (n_1, n_2))p^u(n_1, n_2; n_1, n_2) + \\
&\quad + P(X(0) = (n_1 - 1, n_2))p^u(n_1 - 1, n_2; n_1, n_2)1_{\mathbb{N}_+}(n_1) + \\
&\quad + P(X(0) = (n_1 + 1, n_2 - 1))p^u(n_1 + 1, n_2 - 1; n_1, n_2)1_{\mathbb{N}_+}(n_2) + \\
&\quad + P(X(0) = (n_1, n_2 + 1))p^u(n_1, n_2 + 1; n_1, n_2)\Big) \\
&= \sum_{n_2\in\mathbb{N}} \Big(P(X(0) = (n_1, n_2))\frac{1}{\xi}(\xi - \lambda - \mu_1 1_{\mathbb{N}_+}(n_1) - \mu_2 1_{\mathbb{N}_+}(n_2)) + \\
&\quad + P(X(0) = (n_1 - 1, n_2))\frac{\lambda}{\xi}1_{\mathbb{N}_+}(n_1) + \\
&\quad + P(X(0) = (n_1 + 1, n_2 - 1))\frac{\mu_1}{\xi}1_{\mathbb{N}_+}(n_2) + P(X(0) = (n_1, n_2 + 1))\frac{\mu_2}{\xi}\Big) \\
&= \sum_{n_2\in\mathbb{N}} \Big(P(X(0) = (n_1, n_2))\frac{1}{\xi}(\xi - \lambda - \mu_1 1_{\mathbb{N}_+}(n_1) - \mu_2) + \\
&\quad + P(X(0) = (n_1 - 1, n_2))\frac{\lambda}{\xi}1_{\mathbb{N}_+}(n_1) \\
&\quad + P(X(0) = (n_1 + 1, n_2))\frac{\mu_1}{\xi} + P(X(0) = (n_1, n_2))\frac{\mu_2}{\xi}\Big) \\
&\quad + P(X(0) = (n_1, 0))\frac{\mu_2}{\xi} - P(X(0) = (n_1, 0))\frac{\mu_2}{\xi} \\
&= \sum_{n_2\in\mathbb{N}} P(X(0) = (n_1, n_2))\frac{1}{\xi}\Big(\xi - \lambda - \mu_1 1_{\mathbb{N}_+}(n_1) - \mu_2 + \frac{\mu_1}{\lambda}\lambda 1_{\mathbb{N}_+}(n_1) + \frac{\lambda}{\mu_1}\mu_1 + \mu_2\Big) \\
&= \sum_{n_2\in\mathbb{N}} P(X(0) = (n_1, n_2))\frac{1}{\xi}\Big(\xi - \lambda - \mu_1 1_{\mathbb{N}_+}(n_1) - \mu_2 + \mu_1 1_{\mathbb{N}_+}(n_1) + \lambda + \mu_2\Big) \\
&= P(X_1(0) = n_1). \qquad (6.30)
\end{aligned}
$$

Iterating these computations leads to a reproduction of the marginal for all jump times of the Poisson-ξ process. In this special case node 1 acts as an $M/M/1/\infty$ system for its own, where node 2 is defined as the sink. With this approach it is easy to see that the marginal limiting distribution for the first and stable node is stationary even in continuous time.

(b) *Assume that the initial distribution fulfills the following conditions:*

$$P(X_1(0) = n_1) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}, \quad n_1 \in \mathbb{N},$$

$$P(X_2(0) = 0) = 0,$$

and that the covariance structure of $P^{(X_1(0),X_2(0))}$ *fulfills* (6.27) *for all* $n_1 \in \mathbb{N}$, *i.e., from* $r(2,1) = r(2,2) = 0$ *we have*

$$P(X(0) = (n_1, 1)) = P(X_1(0) = n_1) \cdot P(X_2(0) = 1), \tag{6.31}$$

which means that the events $\{X_1(0) = n_1\}$ *and* $\{X_2(0) = 1\}$ *are independent of each other.*
Denote by t_1 *the (random) time of the first jump of the uniformized process* X. *Then for all* $n_1 \in \mathbb{N}$ *holds by Proposition 6.7*

$$\begin{aligned}\sum_{n_2=1}^{\infty} P(X(t_1) = (n_1, n_2)|X_2(t_1) \neq 0) &= \frac{P(X_1(t_1) = n_1) - P(X(t_1) = (n_1, 0))}{1 - P(X_2(t_1) = 0)} \\ &\overset{(6.29)(6.30)}{=} \frac{P(X_1(0) = n_1) - P(X(0) = (n_1, 1)) \cdot \frac{\mu_2}{\xi}}{1 - P(X_2(0) = 1) \cdot \frac{\mu_2}{\xi}} \\ &\overset{(6.31)}{=} P(X_1(0) = n_1).\end{aligned}$$

In the next example, a stability situation the other way around than in Example 6.10 is considered. The first node is unstable and the second is stable. The first node has only external Poisson input and from this node's point of view, again, the second node can be regarded as the sink already. But for the second node, assumed to be stable, the queue length distribution strongly depends on what is happening at the first node. Therefore stationarity of the marginal for the stable node, as in the previous example, cannot be expected in general.

Example 6.11. *Consider a tandem Jackson network without immediate feedback, i.e.,* $r(0,1) = r(1,2) = r(2,0) = 1$. *Let node 1 be unstable and node 2 be stable, then* $\eta_1 = \lambda \geq \mu_1$ *and* $\eta_2 = \mu_1 < \mu_2$. *Note that with these assumptions together with the routing probabilities the assumptions of Proposition 6.7 are not fullfilled.*
Let X *be the queue length process on* $\mathbb{N}^2$. *Consider the uniformization of* X *with uniformization constant* $\xi \geq \lambda + \mu_1 + \mu_2$.
Assume that the initial distribution fulfills the following conditions:

$$P(X_2(0) = n_2) = \left(1 - \frac{\eta_2}{\mu_2}\right)\left(\frac{\eta_2}{\mu_2}\right)^{n_2}, \quad n_2 \in \mathbb{N},$$

and

$$P(X_1(0) = 0) = 0.$$

(a) *Denote by* t_1 *the (random) time of the first jump of the uniformized process* X. *Then it follows directly from Proposition 6.2 that for all* $n_2 \in \mathbb{N}$ *holds*

$$P(X_2(t_1) = n_2) = P(X_2(0) = n_2). \tag{6.32}$$

(b) *Assume that furthermore for all* $n_2 \in \mathbb{N}$ *holds*

$$P(X(0) = (1, n_2 - 1)) \cdot 1_{\mathbb{N}_+}(n_2) = P(X_1(0) = 1) \cdot P(X_2(0) = n_2), \tag{6.33}$$

i.e., by case differentiation it holds for all $n_2 > 0$

$$P(X(0) = (1, n_2 - 1)) = P(X_1(0) = 1) \cdot P(X_2(0) = n_2)$$

and for $n_2 = 0$

$$0 = P(X_1(0) = 1) \cdot P(X_2(0) = 0) = P(X_1(0) = 1) \cdot \left(1 - \frac{\eta_2}{\mu_2}\right)$$
$$\Rightarrow P(X_1(0) = 1) = 0.$$

Denote by t_1 *the time of the first jump of the uniformized process* X*. Then for all* $n_2 \in \mathbb{N}$ *holds*

$$\begin{aligned}
&\sum_{n_1=1}^{\infty} P(X(t_1) = (n_1, n_2) | X_1(t_1) \neq 0) \\
&= \frac{P(X_2(t_1) = n_2) - P(X(t_1) = (0, n_2))}{1 - P(X_1(t_1) = 0)} \\
&\overset{(6.29)(6.32)}{=} \frac{P(X_2(0) = n_2) - P(X(0) = (1, n_2 - 1)) \cdot \frac{\mu_1}{\xi} \cdot 1_{\mathbb{N}_+}(n_2)}{1 - P(X_1(0) = 1) \cdot \frac{\mu_1}{\xi}} \\
&\overset{(6.33)}{=} P(X_2(0) = n_2).
\end{aligned}$$

Both examples are extreme cases of possible non-ergodic Jackson networks with two nodes where one may be interested in the marginal limiting distribution of the stable part of the network. In both special cases we obtain (under certain assumptions) the expected results, as proven before for the general non-ergodic case, the two-node model.
In case of Example 6.10 we even get more than a one-step invariance of the marginal distribution, the desired stationarity of the marginal is achieved. This is due to the first and stable node acting like an ergodic $M/M/1/\infty$ system of its own.
In terms of the one-step quasi-stationary marginal distribution, again a covariance property is required to be fulfilled. But the assumed covariance property (6.27) in Proposition 6.7 looks slightly different to the properties assumed in (b) of both examples. This is due to the specific structure of the routing matrix in a tandem network.

6.4 One-step invariance of the marginal distribution before and after stopping times

The following approach to the problem of local stationarity is motivated by a paper of S.D. Jacka and G.O. Roberts, [JR95], where continuous-time Markov chains on a countable state space conditioned on not to hit an absorbing barrier before time T are analyzed with regard to weak convergence as T tends to ∞.
Of course, our queue length process of the two-node model is not an absorbing process, but the idea of using stopping times will be incorporated into our analysis as well.

Definition 6.12. *Let* $Y = (Y(n) : n \in \mathbb{N})$ *be a discrete-time homogeneous Markov process on a discrete state space* E*. Denote by* F *a subset of the state space. Then*

$$\tau := \inf\{n \geq 0 : Y(n) \in F\}$$

is the time of the first entrance of Y *into* F *and*

$$\sigma := \sup\{n \geq 0 : Y(n) \in F\}$$

is the time of its last exit out of F *into* $F^c := E \setminus F$.

As already mentioned, the critical event for not reproducing the marginal product form distribution for the stable node (node 1) of the two-node model seems to be that the unstable node (node 2) runs idle.
The aim of this section is to analyze whether there is an invariance of the uniformized process X (of the two-node model) conditioned on node 2 was not idle for some time or will never become idle again.
We will therefore denote the "bad" subset of the state space by $F := \mathbb{N} \times \{0\}$ and the "good" subset by $F^c := \mathbb{N} \times \mathbb{N}_+$.

6.4.1 Technical lemmata and corollaries

The following lemmata and corollaries which are proved in the setting of Definition 6.12 are technical and will be needed to prove the subsequent results.

Lemma 6.13. [JR95, p.903] *Let* Y *be a discrete-time homogeneous Markov process with discrete state space* E*. Denote by* F *a non-empty subset of* E, $F \neq E$, *and by* F^c *its complement,* $F^c := E \setminus F \neq \emptyset$. *Assume that* Y *is irreducible and transient and that*

$$P(Y(m) = j, \tau > T) > 0$$

holds for all $m < T$ *and all* $j \in F^c$. *Then for all* $0 \leq m < n < T$ *and all* $j, k \in F^c$ *holds*

$$\begin{aligned} &P(Y(n) = k | Y(m) = j, \tau > T) = \\ &= \frac{P(\tau > T - n | Y(0) = k)}{P(\tau > T - m | Y(0) = j)} \cdot P(Y(n-m) = k, \tau > n - m | Y(0) = j). \end{aligned}$$

The proof presented in [JR95, p.903] is done for an absorbing process in continuous time, however the steps of the proof are similar for Lemma 6.13. Lemma 6.14 will be proved likewise.

Lemma 6.14. *Let* Y *be a discrete-time homogeneous Markov process with discrete state space* E*. Denote by* F *a non-empty subset of* E, $F \neq E$, *and by* F^c *its complement,* $F^c := E \setminus F \neq \emptyset$. *Assume that* Y *is irreducible and transient and that*

$$P(Y(m) = j, \sigma < T) > 0$$

holds for all $m > T$ *and all* $j \in F^c$. *Then for all* $0 \leq T < m < n$ *and all* $j, k \in F^c$ *holds*

$$\begin{aligned} &P(Y(n) = k | Y(m) = j, \sigma < T) = \\ &= \frac{P(Y(r) \in F^c, r > 0 | Y(0) = k)}{P(Y(r) \in F^c, r > 0 | Y(0) = j)} \cdot P(Y(n-m) = k, \tau > n - m | Y(0) = j). \end{aligned}$$

Proof. For all $0 \leq T < m < n$ and all $j, k \in F^c$ holds

$$
\begin{aligned}
&P(Y(n) = k | Y(m) = j, \sigma < T) = \\
&\overset{(\diamond 1)}{=} \frac{P(Y(n) = k, Y(m) = j, \sigma < T)}{P(Y(m) = j, \sigma < T)} \\
&\overset{(\diamond 2)}{=} \frac{P(Y(n) = k, Y(m) = j, Y(r) \in F^c, r \geq T)}{P(Y(m) = j, Y(r) \in F^c, r \geq T)} \\
&\overset{(\diamond 1)}{=} \frac{P(Y(r) \in F^c, r > n | Y(n) = k, Y(m) = j, Y(r) \in F^c, T \leq r \leq n)}{P(Y(r) \in F^c, r > m | Y(m) = j, Y(r) \in F^c, T \leq r \leq m)} \cdot \\
&\cdot \frac{P(Y(n) = k, Y(m) = j, Y(r) \in F^c, T \leq r \leq n)}{P(Y(m) = j, Y(r) \in F^c, T \leq r \leq m)} \\
&\overset{(\diamond 3)}{=} \frac{P(Y(r) \in F^c, r > n | Y(n) = k)}{P(Y(r) \in F^c, r > m | Y(m) = j)} \cdot \\
&\cdot \frac{P(Y(n) = k, Y(m) = j, Y(r) \in F^c, T \leq r \leq n)}{P(Y(m) = j, Y(r) \in F^c, T \leq r \leq m)} \\
&\overset{(\diamond 1)}{=} \frac{P(Y(r) \in F^c, r > n | Y(n) = k)}{P(Y(r) \in F^c, r > m | Y(m) = j)} \cdot \\
&\cdot \underbrace{P(Y(n) = k, Y(r) \in F^c, m \leq r \leq n | Y(m) = j, Y(r) \in F^c, T \leq r \leq m)}_{\overset{(\diamond 3)}{=} P(Y(n) = k, Y(r) \in F^c, m \leq r \leq n | Y(m) = j)} \\
&\overset{(\diamond 4)}{=} \frac{P(Y(r) \in F^c, r > 0 | Y(0) = k)}{P(Y(r) \in F^c, r > 0 | Y(0) = j)} \cdot \\
&\cdot P(Y(n-m) = k, Y(r) \in F^c, 0 \leq r \leq n-m | Y(0) = j) \\
&\overset{(\diamond 5)}{=} \frac{P(Y(r) \in F^c, r > 0 | Y(0) = k)}{P(Y(r) \in F^c, r > 0 | Y(0) = j)} \cdot P(Y(n-m) = k, \tau > n-m | Y(0) = j),
\end{aligned}
$$

where $\overset{(\diamond 1)}{=}$ is the definition of conditional probability, $\overset{(\diamond 2)}{=}$ is explained by

$$\{\omega : \sigma(\omega) < T\} = \{\omega : \sup\{n \geq 0 : Y(n)(\omega) \in F\} < T\} = \{\omega : Y(r)(\omega) \in F^c, r \geq T\},$$

$\overset{(\diamond 3)}{=}$ holds due to the Markov property and $\overset{(\diamond 4)}{=}$ is valid because Y is homogeneous, and finally $\overset{(\diamond 5)}{=}$ is explained by

$$\{\omega : \tau(\omega) > T\} = \{\omega : \inf\{n \geq 0 : Y(n)(\omega) \in F\} > T\} = \{\omega : Y(r)(\omega) \in F^c, 0 \leq r \leq T\}.$$

□

Lemma 6.15. *Let Y be a discrete-time homogeneous Markov process with discrete state space E. Denote by F a non-empty subset of E and by F^c its complement set, $F^c := E \setminus F \neq \emptyset$. For $0 \leq T < n < \infty$ and all $j \in F^c$ holds*

$$P(\sigma < T | Y(n) = j) = P(Y(r) \notin F, r > 0 | Y(0) = j) \cdot P(Y(r) \notin F, n \geq r \geq T | Y(n) = j).$$

Proof. For $0 \leq T < n < \infty$ and all $j \in F^c$ holds

$$
\begin{aligned}
&P(\sigma < T|Y(n) = j) = \\
&\overset{(\diamondsuit 1)}{=} \frac{P(\sigma < T, Y(n) = j)}{P(Y(n) = j)} \\
&\overset{(\diamondsuit 2)}{=} \frac{P(Y(r) \notin F, r \geq T, Y(n) = j)}{P(Y(n) = j)} \\
&\overset{(\diamondsuit 1)}{=} P(Y(r) \notin F, r > n|Y(n) = j, Y(r) \notin F, n \geq r \geq T) \cdot P(Y(r) \notin F, n \geq r \geq T|Y(n) = j) \\
&\overset{(\diamondsuit 3)}{=} P(Y(r) \notin F, r > n|Y(n) = j) \cdot P(Y(r) \notin F, n \geq r \geq T|Y(n) = j) \\
&\overset{(\diamondsuit 4)}{=} P(Y(r) \notin F, r > 0|Y(0) = j) \cdot P(Y(r) \notin F, n \geq r \geq T|Y(n) = j),
\end{aligned}
$$

where (again) $\overset{(\diamondsuit 1)}{=}$ is valid due to the definition of conditional probability, $\overset{(\diamondsuit 3)}{=}$ holds due to the Markov property, $\overset{(\diamondsuit 4)}{=}$ is valid because Y is homogeneous, and finally $\overset{(\diamondsuit 2)}{=}$ is explained by

$$\{\omega : \sigma(\omega) < T\} = \{\omega : \sup\{n \geq 0 : Y(n)(\omega) \in F\} < T\} = \{\omega : Y(r)(\omega) \notin F, r \geq T\}.$$

□

The following Corollary is a direct application of Lemma 6.14 and Lemma 6.15 for our uniformized process X:

Corollary 6.16. *Let X be the uniformized process from the two-node model (see Definition 6.1) on the state space $\mathbb{N}^2$. Denote by p^u the jump kernel of the uniformization. Let $F := \mathbb{N} \times \{0\}$ and $F^c := \mathbb{N} \times \mathbb{N}_+$. Denote by t_i jump times of the uniformized process X and let t_{i+1} denote the next jump time after t_i.*
Assume that for all $t_h > t_T$ and $j \in F^c$

$$P(X(t_h) = j, \sigma < t_T) > 0.$$

Then it holds for all $j, k \in F^c$ and $0 \leq t_T < t_h < t_{h+1}$

$$P(X(t_{h+1}) = k|X(t_h) = j, \sigma < t_T) = \frac{P(X(t_r) \in F^c, t_r > 0|X(0) = k)}{P(X(t_r) \in F^c, t_r > 0|X(0) = j)} \cdot p^u(j, k),$$

and for $0 \leq t_T < t_h < \infty$ and all $j \in F^c$ holds

$$
\begin{aligned}
&P(\sigma < t_T|X(t_h) = j) = \\
&= P(X(t_r) \notin F, t_r > 0|X(0) = j) \cdot P(X(t_r) \notin F, t_h \geq t_r \geq t_T|X(t_h) = j).
\end{aligned}
$$

Lemma 6.17. *Let X be the uniformized process from the two-node model (see Definition 6.1) on the state space $\mathbb{N}^2$. Denote by p^u the jump kernel of the uniformization. Let $F := \mathbb{N} \times \{0\}$ and $F^c := \mathbb{N} \times \mathbb{N}_+$. Denote by t_i jump times of the uniformized process X and let t_{i+1} denote the next jump time after t_i.*
Then for all $k \in F^c$ and $0 \leq t_T < t_h < t_{h+1}$ holds

$$
\begin{aligned}
&P(X(t_{h+1}) = k, X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) = \\
&= \sum_{j \in F^c} P(X(t_h) = j, X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot p^u(j, k).
\end{aligned}
$$

Proof. For all $k \in F^c$ and $0 \le t_T < t_h < t_{h+1}$ holds with X being homogeneous:

$$\begin{aligned} &P(X(t_{h+1}) = k, X(t_r) \in F^c, t_{h+1} \ge t_r \ge t_T) = \\ &= \sum_{j \in F^c} P(X(t_{h+1}) = k, X(t_h) = j, X(t_r) \in F^c, t_h \ge t_r \ge t_T) \\ &= \sum_{j \in F^c} P(X(t_h) = j, X(t_r) \in F^c, t_h \ge t_r \ge t_T) \cdot \\ &\quad \cdot \underbrace{P(X(t_{h+1}) = k | X(t_h) = j, X(t_r) \in F^c, t_h \ge t_r \ge t_T)}_{=P(X(t_{h+1})=k|X(t_h)=j)} \\ &= \sum_{j \in F^c} P(X(t_h) = j, X(t_r) \in F^c, t_h \ge t_r \ge t_T) \cdot p^u(j, k). \end{aligned}$$

□

6.4.2 Results

The first result is for the uniformized process X conditioned on not having been idle on the second component since some time - not necessarily from the start at time 0 - until some jump time t_h. Recall $F = \mathbb{N} \times \{0\}$, $F^c = \mathbb{N} \times \mathbb{N}_+$.

Proposition 6.18. *Consider the two-node model from Definition 6.1 with $r(2,0) > 0$ and $\mu_1 r(2,1) < \eta_1 r(2,0)$. Assume that the following conditions are fulfilled for all $k_1 \in \mathbb{N}$ and $0 \le t_T < t_h < t_{h+1}$ (where t_{h+1} is the next jump time after t_h):*

$$\sum_{k_2 \in \mathbb{N} \setminus \{0\}} P(X(t_h) = (k_1, k_2) | X(t_r) \in F^c, t_T \le t_r \le t_h) = \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{k_1}, \tag{6.34}$$

and

$$\begin{aligned} &P(X(t_h) = (k_1, 1) | X(t_r) \in F^c, t_T \le t_r \le t_h) \cdot \frac{\mu_2 r(2,0)}{\xi} = \\ &= \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{k_1} \cdot P(X(t_{h+1}) \in F | X(t_r) \in F^c, t_T \le t_r \le t_h) \cdot \sum_{i=0}^{k_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i \end{aligned} \tag{6.35}$$

which is a correlation property.
Then for all $k_1 \in \mathbb{N}$ holds

$$\sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1, k_2) | X(t_r) \in F^c, t_T \le t_r \le t_{h+1}) = \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{k_1}. \tag{6.36}$$

Proof. Denote by p^u the jump kernel of the uniformization with Poisson-ξ clock. For all $k_1 \in \mathbb{N}$ holds due to Lemma 6.17

$$\begin{aligned} &\sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1, k_2), X(t_r) \in F^c, t_{h+1} \ge t_r \ge t_T) = \\ &= \sum_{k_2=1}^{\infty} \sum_{j \in F^c} P(X(t_h) = j, X(t_r) \in F^c, t_h \ge t_r \ge t_T) \cdot p^u(j; k_1, k_2), \end{aligned}$$

with the definition of the conditional probability this is equivalent to

$$\sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1,k_2)|X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) \cdot P(X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) =$$
$$= \sum_{k_2=1}^{\infty} \sum_{j \in F^c} P(X(t_h) = j|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot$$
$$\cdot P(X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot p^u(j;k_1,k_2).$$

Deviding both sides of the equation by $P(X(t_r) \in F^c, t_h \geq t_r \geq t_T)$ yields

$$\frac{P(X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T)}{P(X(t_r) \in F^c, t_h \geq t_r \geq t_T)} \sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1,k_2)|X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) =$$
$$= \sum_{k_2=1}^{\infty} \sum_{(j_1,j_2) \in F^c} P(X(t_h) = (j_1,j_2)|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot p^u(j_1,j_2;k_1,k_2)$$
$$= \sum_{k_2=1}^{\infty} \Big(P(X(t_h) = (k_1,k_2)|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot p^u(k_1,k_2;k_1,k_2) +$$
$$+ P(X(t_h) = (k_1,k_2-1)|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot p^u(k_1,k_2-1;k_1,k_2) 1_{\mathbb{N}_+}(k_2-1)$$
$$+ P(X(t_h) = (k_1,k_2+1)|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot p^u(k_1,k_2+1;k_1,k_2)$$
$$+ P(X(t_h) = (k_1-1,k_2)|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot p^u(k_1-1,k_2;k_1,k_2) 1_{\mathbb{N}_+}(k_1)$$
$$+ P(X(t_h) = (k_1+1,k_2)|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot p^u(k_1+1,k_2;k_1,k_2)$$
$$+ P(X(t_h) = (k_1+1,k_2-1)|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot$$
$$\cdot p^u(k_1+1,k_2-1;k_1,k_2) 1_{\mathbb{N}_+}(k_2-1)$$
$$+ P(X(t_h) = (k_1-1,k_2+1)|X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot$$
$$\cdot p^u(k_1-1,k_2+1;k_1,k_2) 1_{\mathbb{N}_+}(k_1) \Big).$$

Plugging in the jump probabilities yields (recall $k_1 \geq 0$ is fixed)

$$\frac{P(X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T)}{P(X(t_r) \in F^c, t_h \geq t_r \geq t_T)} \sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1, k_2) | X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) =$$
$$= \frac{\xi - \lambda_1 - \lambda_2 - \mu_1(1 - r(1,1))1_{\mathbb{N}_+}(k_1) - \mu_2(1 - r(2,2))}{\xi} \cdot$$
$$\cdot \sum_{k_2=1}^{\infty} P(X(t_h) = (k_1, k_2) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)$$
$$+ \frac{\lambda_2}{\xi} \underbrace{\sum_{k_2=1}^{\infty} P(X(t_h) = (k_1, k_2 - 1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot 1_{\mathbb{N}_+}(k_2 - 1)}_{= \sum_{k_2=1}^{\infty} P(X(t_h) = (k_1,k_2) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)} +$$
$$+ \frac{\mu_2 r(2,0)}{\xi} \underbrace{\sum_{k_2=1}^{\infty} P(X(t_h) = (k_1, k_2 + 1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)}_{= \sum_{k_2=1}^{\infty} P(X(t_h) = (k_1,k_2) | X(t_r) \in F^c, t_h \geq t_r \geq t_T) - P(X(t_h) = (k_1,1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)} +$$
$$+ \frac{\lambda_1}{\xi} 1_{\mathbb{N}_+}(k_1) \sum_{k_2=1}^{\infty} P(X(t_h) = (k_1 - 1, k_2) | X(t_r) \in F^c, t_h \geq t_r \geq t_T) +$$
$$+ \frac{\mu_1 r(1,0)}{\xi} \sum_{k_2=1}^{\infty} P(X(t_h) = (k_1 + 1, k_2) | X(t_r) \in F^c, t_h \geq t_r \geq t_T) +$$
$$+ \frac{\mu_1 r(1,2)}{\xi} \underbrace{\sum_{k_2=1}^{\infty} P(X(t_h) = (k_1 + 1, k_2 - 1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T) 1_{\mathbb{N}_+}(k_2 - 1)}_{= \sum_{k_2=1}^{\infty} P(X(t_h) = (k_1+1,k_2) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)} +$$
$$+ \frac{\mu_2 r(2,1)}{\xi} 1_{\mathbb{N}_+}(k_1) \underbrace{\sum_{k_2=1}^{\infty} P(X(t_h) = (k_1 - 1, k_2 + 1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)}_{= \sum_{k_2=1}^{\infty} P(X(t_h) = (k_1-1,k_2) | X(t_r) \in F^c, t_h \geq t_r \geq t_T) - P(X(t_h) = (k_1-1,1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)}.$$

With (6.34) this is equivalent to

$$\frac{P(X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T)}{P(X(t_r) \in F^c, t_h \geq t_r \geq t_T)} \sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1, k_2) | X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) =$$
$$= \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{k_1} \cdot \frac{1}{\xi} \Big(\xi - (\underbrace{\lambda_1 + \mu_2 r(2,1)}_{\stackrel{(6.1)}{=} \eta_1 (1 - r(1,1))}) - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(k_1) +$$
$$+ \frac{\mu_1}{\eta_1} \underbrace{(\lambda_1 + \mu_2 r(2,1))}_{\stackrel{(6.1)}{=} \eta_1 (1 - r(1,1))} 1_{\mathbb{N}_+}(k_1) + \frac{\eta_1}{\mu_1} \mu_1 (1 - r(1,1)) \Big)$$
$$- \frac{\mu_2 r(2,0)}{\xi} P(X(t_h) = (k_1, 1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)$$
$$- \frac{\mu_2 r(2,1)}{\xi} 1_{\mathbb{N}_+}(k_1) P(X(t_h) = (k_1 - 1, 1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)$$

$$\Leftrightarrow$$

$$\frac{P(X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T)}{P(X(t_r) \in F^c, t_h \geq t_r \geq t_T)} \sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1, k_2) | X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) =$$
$$= \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1} - \frac{\mu_2 r(2,0)}{\xi} P(X(t_h) = (k_1, 1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T)$$
$$- \frac{\mu_2 r(2,1)}{\xi} 1_{\mathbb{N}_+}(k_1) P(X(t_h) = (k_1 - 1, 1) | X(t_r) \in F^c, t_h \geq t_r \geq t_T).$$

(6.35) yields

$$\frac{P(X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T)}{P(X(t_r) \in F^c, t_h \geq t_r \geq t_T)} \sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1, k_2) | X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) =$$
$$= \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1} \Big[1 - P(X(t_{h+1}) \in F | X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot \frac{1}{\mu_2 r(2,0)} \cdot$$
$$\Big(\mu_2 r(2,0) \sum_{i=0}^{k_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i + \frac{\mu_1}{\eta_1} \mu_2 r(2,1) 1_{\mathbb{N}_+}(k_1) \cdot \sum_{i=0}^{k_1-1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i\Big)\Big]$$
$$= \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1} \Big[1 - P(X(t_{h+1}) \in F | X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot$$
$$\Big(\sum_{i=0}^{k_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i + \frac{\mu_1 r(2,1)}{\eta_1 r(2,0)} 1_{\mathbb{N}_+}(k_1) \cdot \sum_{i=0}^{k_1-1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i\Big)\Big]$$
$$= \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1} \Big[1 - P(X(t_{h+1}) \in F | X(t_r) \in F^c, t_h \geq t_r \geq t_T) \cdot$$
$$\underbrace{\Big(\sum_{i=0}^{k_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i - 1_{\mathbb{N}_+}(k_1) \cdot \sum_{i=1}^{k_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i\Big)}_{=1}\Big]$$

$$\Leftrightarrow \sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1, k_2) | X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T) =$$
$$= \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1} \cdot$$
$$\cdot \underbrace{P(X(t_{h+1}) \in F^c | X(t_r) \in F^c, t_h \geq t_r \geq t_T) \frac{P(X(t_r) \in F^c, t_h \geq t_r \geq t_T)}{P(X(t_r) \in F^c, t_{h+1} \geq t_r \geq t_T)}}_{=1}.$$

□

Remark 6.19. *An analogical result for the conditional probabilities* $P(X(t_h) = k | \tau > t_h)$*,* $k \in F^c$*, can be deduced from Proposition 6.18 as a special case where* $t_T = 0$*:*

Consider the two-node model from Definition 6.1 with $\mu_1 r(2,1) < \eta_1 r(2,0)$ and $r(2,0) > 0$. Assume that the following conditions are fulfilled for all $k_1 \in \mathbb{N}$ and $0 \leq t_h < t_{h+1}$ where t_{h+1} denotes the next jump time of X after t_h:

$$\sum_{k_2 \in \mathbb{N}\setminus\{0\}} P(X(t_h) = (k_1, k_2) | \tau > t_h) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1}, \tag{6.37}$$

and

$$\begin{aligned} P(X(t_h) = (k_1, 1) | \tau > t_h) \cdot \frac{\mu_2 r(2,0)}{\xi} &= \\ = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1} \cdot P(X(t_{h+1}) \in F | \tau > t_h) \cdot \sum_{i=0}^{k_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i. \end{aligned} \tag{6.38}$$

Then for all $k_1 \in \mathbb{N}$ holds

$$\sum_{k_2=1}^{\infty} P(X(t_{h+1}) = (k_1, k_2) | \tau > t_{h+1}) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1}. \tag{6.39}$$

Remark 6.20. *Although it is not obvious in the general case, there are similarities of the formulas in Proposition 6.7 and in Proposition 6.18 which become visible when setting $t_T = t_h = t_0 = 0$ in Proposition 6.18:*
Consider the two-node model from Definition 6.1 with $\mu_1 r(2,1) < \eta_1 r(2,0)$ and $r(2,0) > 0$. We have from Proposition 6.18 (and Remark 6.19) that if for all $k_1 \in \mathbb{N}$ holds

$$\sum_{k_2 \in \mathbb{N}\setminus\{0\}} P(X(0) = (k_1, k_2) | X(0) \notin F) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1}$$

and

$$\begin{aligned} P(X(0) = (k_1, 1)) \cdot \frac{\mu_2 r(2,0)}{\xi} &= \\ = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1} \cdot P(X(t_1) \in F, X(0) \notin F) \cdot \sum_{i=0}^{k_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i \end{aligned}$$

then for the uniformized process X holds for all $k_1 \in \mathbb{N}$

$$\sum_{k_2=1}^{\infty} P(X(t_1) = (k_1, k_2) | X(t_1) \notin F, X(0) \notin F) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1}.$$

While Proposition 6.7 states that if for all $k_1 \in \mathbb{N}$ holds

$$\sum_{k_2 \in \mathbb{N}\setminus\{0\}} P(X(0) = (k_1, k_2)) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1}$$

and $P(X(0) \notin F) = 1$ and if furthermore

$$P(X(0) = (k_1, 1)) \cdot \frac{\mu_2 r(2,0)}{\xi} = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1} \cdot P(X(t_1) \in F) \cdot \sum_{i=0}^{k_1} \left(-\frac{\mu_1 r(2,1)}{\eta_1 r(2,0)}\right)^i$$

(note that $P(X(t_1) \in F) = P(X_2(0) = 1) \cdot \frac{\mu_2(1-r(2,2))}{\xi}$ holds due to $P(X_2(0) = 0) = 0$), then for the uniformized process X holds for all $k_1 \in \mathbb{N}$

$$\sum_{k_2=1}^{\infty} P(X(t_1) = (k_1, k_2)|X(t_1) \notin F) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{k_1}.$$

We here have analyzed the queue length probabilities of the stable node 1 conditioned on the unstable node 2 was not idle for some time.

It is also interesting to analyze the process after the last exit of the process out of the "bad" subset $F = \mathbb{N} \times \{0\}$ of the state space.
Such an analysis is motivated the following way: The queue length at node 2 will eventually diverge out of the state space to infinity - at least in the limit, so one may think of a time of a last exit of the queue length process out of $F = \mathbb{N} \times \{0\}$. If the event that node 2 runs idle is a critical event for a reproduction of queue length probabilities, then the reproduction of at least the marginal queue length probabilities of the stable node 1 for the jump times after the last exit out of this critical event could succeed.

Let us assume that for the uniformized queue length process X of the two-node model (see Definition 6.1) holds

$$P(\sigma < t_T) > 0$$

for $0 < t_T < \infty$, where t_T is a jump time. We then are interested whether the probability

$$P(X(t_h) = k|\sigma < t_T)$$

or

$$\sum_{k_2 \in \mathbb{N}_+} P(X(t_h) = (k_1, k_2)|\sigma < t_T),$$

where $k \in F^c = \mathbb{N} \times \mathbb{N}_+$, somehow remains invariant for all jump times $t_h \geq t_T$.
For $0 \leq t_T < t_h < t_{h+1}$, where t_{h+1} is the next jump time after t_h, holds for all $k \in F^c$:

$$\begin{aligned}
&P(X(t_{h+1}) = k|\sigma < t_T) = \\
&\overset{(\nabla 1)}{=} \sum_{j \in F^c} P(X(t_{h+1}) = k, X(t_h) = j|\sigma < t_T) \\
&\overset{(\nabla 2)}{=} \sum_{j \in F^c} P(X(t_{h+1}) = k|X(t_h) = j, \sigma < t_T) \cdot P(X(t_h) = j|\sigma < t_T) \\
&\overset{(\nabla 3)}{=} \sum_{j \in F^c} P(X(t_h) = j|\sigma < t_T) \cdot \frac{P(X(t_r) \in F^c, t_r > 0|X(0) = k)}{P(X(t_r) \in F^c, t_r > 0|X(0) = j)} \cdot p^u(j, k),
\end{aligned}$$

where $\overset{(\nabla 1)}{=}$ is the law of total probability, $\overset{(\nabla 2)}{=}$ is valid with the definition of conditional probability and $\overset{(\nabla 3)}{=}$ holds due to Corollary 6.16.

The equation

$$\begin{aligned}
&P(X(t_{h+1}) = k|\sigma < t_T) = \\
&= \sum_{j \in F^c} P(X(t_h) = j|\sigma < t_T) \cdot \frac{P(X(t_r) \in F^c, t_r > 0|X(0) = k)}{P(X(t_r) \in F^c, t_r > 0|X(0) = j)} \cdot p^u(j, k), \qquad (6.40)
\end{aligned}$$

which holds for $0 \leq t_T < t_h < t_{h+1}$ (where t_{h+1} is the next jump time after t_h) and for all $k \in F^c$, strongly reminds of the definition of *γ-invariant measures.*

Definition 6.21. [CSP99, p.84] *Let Y be a discrete-time homogeneous Markov process on the state space E with one-step transition probability matrix $p = (p(i,j) : i,j \in E)$. A collection of non-negative numbers $m = (m(i) : i \in E)$ with $m \neq 0$ satisfying*

$$\gamma \sum_{i \in E} m(i) \cdot p(i,j) = m(j), \quad j \in E,$$

is called a γ-invariant measure for p on E.

In [CSD06] a γ-invariant measure is defined for absorbing Markov chains:

Definition 6.22. [CSD06, p.451] *Let Z be an absorbing Markov chain on the state space E with absorbing state $\delta \in E$ and with transition probability matrix $p = (p(i,j) : i,j \in E)$. A collection $m := (m(j) : j \in E \setminus \{\delta\})$ of non-negative numbers such that m is non-trivial ($m \neq 0$) is called γ-invariant measure for p on $E \setminus \{\delta\}$ if*

$$\gamma \sum_{i \in E \setminus \{\delta\}} m(i) p(i,j) = m(j), \quad j \in E \setminus \{\delta\}.$$

Furthermore, a quasi-stationary distribution for such a chain Z is a proper probability distribution $m := (m(j) : j \in E \setminus \{\delta\})$ satisfying

$$P_m(Z(n) = j | \tau > n) = m(j), \quad j \in E \setminus \{\delta\},$$

for all $n \in \mathbb{N}$ where τ is the time to absorption in δ.

Remark 6.23. *In [CSD06] m of Definition 6.22 would be called $1/\gamma$-invariant. We reformulated this for readability and comparison reasons.*

Remark 6.24. *Definition 6.22 strongly resembles Definition 6.6 where $E = \{0, 1, 2, ...\}$ and $\delta = 0$.*

Remark 6.25. *The paper [CSD06] of P. Coolen-Schrijner and E. A. van Doorn as well as the articles [DS67] and [JR95] deal with discrete-time* absorbing *Markov chains. Most of the theory on quasi-stationary distributions is based on absorbing processes. However, there are surveys on other classes of processes, e.g., in the paper [CSP99] of P. Coolen-Schrijner and P. Pollett discrete-time Markov chains with linear, one-dimensional state space and with almost sure drift to infinity (i.e., they are transient) are analyzed with respect to quasi-stationarity. It is remarkable that the results in [CSP99] are proved by showing that the state probabilities of the original chain conditioned on not having left state 0 for the last time are equal to the state probabilities of its dual absorbing Markov chain conditioned on non-absorption.*

Our approach here is different from Definition 6.22: We are interested in the time after the last exit out of $\mathbb{N} \times \{0\}$ which is not a matter in the theory of absorbing processes. (Once absorbed, the process will stay in this state with probability one, thus there is no way out of it and a further analysis of the behavior of the process after absorption can be omitted.)

In (6.40) we do not consider all states in $E = \mathbb{N}^2$ but only the states in F^c since it holds from the definition of σ:

$$P(X(t_r) = i|\sigma < t_T) = 0 \quad \forall i \in F, t_r > t_T. \tag{6.41}$$

That is why we can use Definition 6.21:

If both of the following properties hold:

(i) the fraction
$$\frac{P(X(t_r) \in F^c, t_r > 0|X(0) = k)}{P(X(t_r) \in F^c, t_r > 0|X(0) = j)}$$
exists and is equal to some $\gamma \in (0, \infty)$ for all $j, k \in F^c$ for which $p^u(j, k) > 0$,

(ii) for all $k \in F^c, t_h > t_T, h \in \mathbb{N}$
$$P(X(t_{h+1}) = k|\sigma < t_T) = P(X(t_h) = k|\sigma < t_T) =: m(k), \tag{6.42}$$
and there is at least one state $j \in F^c$ for which $m(j) \neq 0$,

then (6.40) implies that $m = (m(k) : k \in \mathbb{N}^2)$ is a γ-invariant measure for p^u.
If additionally

$$\sum_{k\in\mathbb{N}^2} m(k) \overset{(6.42)}{=} \sum_{k\in\mathbb{N}^2} P(X(t_{h+1}) = k|\sigma < t_T) \overset{(6.41)}{=} \sum_{k\in F^c} P(X(t_{h+1}) = k|\sigma < t_T) = 1,$$

i.e., $(P(X(t_{h+1}) = k|\sigma < t_T) : k \in \mathbb{N}^2)$ is a proper probability distribution, then it is called a γ-invariant probability distribution for p^u.

First of all, it should be answered whether property (i) can be verified. The end of this section will consist of an analysis of the fraction to be considered:

$$\frac{P(X(t_r) \in F^c, t_r > 0|X(0) = k)}{P(X(t_r) \in F^c, t_r > 0|X(0) = j)}, j, k \in F^c. \tag{6.43}$$

Remark 6.26. *About the numerator and the denominator of the fraction* (6.43) *we can say the following:*
For all $k \in F^c$ holds

$$\begin{aligned} P(X(r) \in F^c, r \geq 0|X(0) = k) &\overset{(\triangle 1)}{=} \sum_{i\in F^c} P(X(r) \in F^c, r \geq 0, X(1) = i|X(0) = k) \\ &\overset{(\triangle 2)}{=} \sum_{i\in F^c} P(X(r) \in F^c, r \geq 0|X(1) = i, X(0) = k) \cdot p^u(k, i) \\ &\overset{(\triangle 3)}{=} \sum_{i\in F^c} P(X(r) \in F^c, r \geq 0|X(0) = i) \cdot p^u(k, i), \end{aligned}$$

where $\overset{(\triangle 1)}{=}$ holds with the law of total probability, $\overset{(\triangle 2)}{=}$ is valid with the definition of conditional probability and $\overset{(\triangle 3)}{=}$ holds because X is homogeneous and Markovian.
The equation

$$P(X(r) \in F^c, r \geq 0|X(0) = k) = \sum_{i\in F^c} P(X(r) \in F^c, r \geq 0|X(0) = i) \cdot p^u(k, i)$$

for all $k \in F^c$ *implies that the function*

$$P(X(r) \in F^c, r \geq 0 | X(0) = \bullet) : F^c \to [0,1]$$

is a harmonic function ([Bre99, Definition 2.1, p.179]) on F^c *for* $(p^u(j,k) : j,k \in F^c)$. *The substochastic matrix* $(p^u(j,k) : j,k \in F^c)$ *is a one-step (taboo) probability matrix as defined in [Chu67, Part I, §9, p.45]. The n-step taboo probability matrix with the taboo set* F *in the sense of [Chu67] is given by* $(P(X(t_n) = k, \tau > t_n | X(0) = j) : j,k \in F^c)$ *which is substochastic as well and has the semigroup property, [Chu67, p.54].*

Clearly, if $P(X(0) = k) > 0\ \forall k \in F^c$ (which we assume), it holds

$$P(X(r) \in F^c, r > 0 | X(0) = k) \in [0,1]$$

for all $k \in F^c$ due to the definition of a probability.
Moreover, for all $k \in F^c$ holds

$$P(X(r) \in F^c, r > 0 | X(0) = k) = \lim_{s \to \infty} P(\tau > s | X(0) = k),$$

because

$$\begin{aligned}\{\omega : X(r)(\omega) \notin F\ \forall r \geq 0\} &= \bigcap_{r=0}^{\infty} \{\omega : X(r)(\omega) \in F^c\} \\ &= \bigcap_{s=0}^{\infty} \{\omega : X(r)(\omega) \in F^c, 0 \leq r \leq s\} = \bigcap_{s=0}^{\infty} \{\omega : \tau(\omega) > s\}.\end{aligned}$$

For the existence of the fraction (6.43) it is important to know whether

$$P(X(r) \in F^c, r > 0 | X(0) = j) > 0 \quad \forall j \in F^c.$$

If there exists a state $j \in F^c$ for which $p^u(j,k) > 0$ for $k \in F^c$ and

$$P(X(r) \in F^c, r > 0 | X(0) = j) = 0$$

holds, then the process cannot be γ-invariant for p^u, see equation (6.40). That is why we assume (for a first approach) that for all $j \in F^c$ holds

$$P(X(r) \in F^c, r > 0 | X(0) = j) > 0 \tag{6.44}$$

if $p^u(j,k) > 0$ holds for $k \in F^c$. Then for all these $j \in F^c$ with (6.44) the fraction (6.43) exists and the equation

$$\frac{P(X(r) \in F^c, r > 0 | X(0) = k)}{P(X(r) \in F^c, r > 0 | X(0) = j)} = \lim_{s \to \infty} \frac{P(\tau > s | X(0) = k)}{P(\tau > s | X(0) = j)}$$

$(k \in F^c)$ is valid, see [Kön01, Regel I c), p.43]. The question of interest is whether this fraction lies in the interval $(0, \infty)$ and is invariant for all $j,k \in F^c$ or not.

Lemma 6.27. *Consider the two-node model from Definition 6.1 with uniformized queue length process* X *on the state space* $\mathbb{N}^2$ *and with jump kernel* p^u. *Let* $F := \mathbb{N} \times \{0\}$ *and* $F^c := \mathbb{N} \times \mathbb{N}_+$. *Denote by* t_i *jump times of the uniformized process* X. *Assume that* $P(\tau > t_s | X(0) = j) > 0$ *holds for all* $j \in F^c$ *and all jump times* t_s. *Then for all* $j,k \in F^c$ *holds*

$$p^u(k,j) \leq \frac{P(\tau > t_s | X(0) = k)}{P(\tau > t_s | X(0) = j)} \leq \frac{1}{p^u(j,j) \cdot p^u(j,k)} \quad \forall s \in \mathbb{N}.$$

Proof. We have

$$\frac{P(\tau > t_s|X(0)=k)}{P(\tau > t_s|X(0)=j)} = \frac{P(\tau > t_s|X(0)=k)}{P(\tau > t_{s-1}|X(0)=j)} \cdot \frac{P(\tau > t_{s-1}|X(0)=j)}{P(\tau > t_s|X(0)=j)}. \tag{6.45}$$

Similar to the proof of Lemma 2.3 in [JR95], it holds for all $j,k \in F^c$

$$\begin{aligned} P(\tau > t_{s-1}|X(0)=k) &\geq P(\tau > t_s|X(0)=k) \\ &= \sum_{i\in F^c} P(\tau > t_s, X(t_1)=i|X(0)=k) \\ &= \sum_{i\in F^c} P(\tau > t_s|X(t_1)=i, X(0)=k)\cdot p^u(k,i) \\ &= \sum_{i\in F^c} P(\tau > t_{s-1}|X(0)=i)\cdot p^u(k,i) \\ &\geq P(\tau > t_{s-1}|X(0)=j)\cdot p^u(k,j). \end{aligned} \tag{6.46}$$

(6.46) implies with $k=j$ and $p^u(j,j)>0$ for the second factor on the right-hand side of (6.45)

$$1 \leq \frac{P(\tau > t_{s-1}|X(0)=j)}{P(\tau > t_s|X(0)=j)} \leq \frac{1}{p^u(j,j)},$$

and for the first factor on the right-hand side of (6.45)

$$\begin{aligned} p^u(k,j) &\leq \frac{P(\tau > t_s|X(0)=k)}{P(\tau > t_{s-1}|X(0)=j)} \\ &\leq \frac{P(\tau > t_s|X(0)=k)}{P(\tau > t_{s-2}|X(0)=k)\cdot p^u(j,k)} \\ &\leq \frac{P(\tau > t_s|X(0)=k)}{P(\tau > t_s|X(0)=k)\cdot p^u(j,k)} = \frac{1}{p^u(j,k)}. \end{aligned}$$

□

Even if one could prove that the sequences

$$\left(\frac{P(\tau > t_s|X(0)=k)}{P(\tau > t_s|X(0)=j)} : s\in\mathbb{N}\right)$$

are unequal for different $j,k \in F^c$, this does not imply that the limit of the sequences will be different.

As shown in the above analysis there are certain difficulties to figure out whether there may be some kind of invariance of the probability $P(X(t_h)=k|\sigma < t_T)$, where $k\in F^c$, or $P(X_1(t_h)=k_1|\sigma<t_T)$, where $k_1\in\mathbb{N}$, for all jump times $t_h \geq t_T$ as well as for only one step. The problem is the existence of the fraction (6.43) and its equality for all states. This question remains unanswered in the present thesis and may be an interesting topic for further research.

6.5 Upper bound for the process of the stable subnetwork

In this section we will give an upper bound, which is geometrically distributed at all times, for the process of the stable part of the overall non-ergodic Jackson network.
One way to obtain this upper bound is by the pathwise coupling $X \leq_{st} X^+$ where X^+ is the process of the Jackson network with infinite supply at the unstable nodes, as done in the proof of Theorem 5.10 with $D = \emptyset$. Jackson networks with infinite supply where all nodes $i \in W$ without infinite supply are stable (i.e., $W \cap U = \emptyset$) have an ergodic homogeneous Markov process X_W^+ (see Corollary 4.27), so from Theorem 4.26 X_W^+ has a stationary and limiting distribution of the well known product form. Thus with $S = W$ and

$$X_S = (X_i : i \in S) \leq_{st} X_S^+$$

$X_S^+ = X_W^+$ already is the desired upper bound for X_S proved by pathwise coupling.

Another way which does not need a pathwise coupling to obtain the geometrically distributed upper bound for X_S where X is non-ergodic will be presented below. We will start with the two-node model, but the results are similarly obtained for Jackson networks with J nodes, $0 < J < \infty$.

The first result for the two-node model with uniformized queue length process is summarized in the following proposition:

Proposition 6.28. *Consider the two-node model, i.e., a Jackson network with two nodes where node 1 is stable and node 2 is unstable. Denote by X the uniformized queue length process, where the uniformization constant is $\xi \geq 2\cdot\sum_{i=1}^{2}(\lambda_i+\mu_i)$, on the partially ordered state space $(E, \prec) = (\mathbb{N}^2, \leq^2)$. Let p^u denote the kernel of the uniformization. Assume that the initial distribution $P^{X(0)}$ is such that for its marginal holds*

$$P(X_1(0) = n_1) = \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}, \quad n_1 \in \mathbb{N}. \tag{6.47}$$

Then for any (random) jump time t_n of the process $P^{X(t_n)} = P^{X(0)} \cdot (p^u)^n$ and for the stable marginal holds

$$P^{X_1(t_n)} \leq_{st} geo^0\left(1 - \frac{\eta_1}{\mu_1}\right).$$

Before proving Proposition 6.28, some preliminary information may be in order:

Definition 6.29. [KKO77, p.899] *Let $(E, \prec)$ be a partially ordered space.*
A set $A \subseteq E$ is increasing *($\equiv$ isotone $\equiv$ nondecreasing) if for $x, y \in E$ holds:*

$$(x \in A \text{ and } x \prec y) \quad \Rightarrow \quad y \in A.$$

A function $f : E \to \mathbb{R}$ is increasing *($\equiv$ isotone $\equiv$ nondecreasing) if for $x, y \in E$ holds:*

$$x \prec y \quad \Rightarrow \quad f(x) \prec f(y).$$

Lemma 6.30. *A Jackson network process X with non-decreasing service rates (which we do have here) is stochastically (strongly) monotone, [Mas87, p.366]. This means that the family of transition kernels is stochastically monotone with respect to $\leq^J$, i.e., for all states $x, y \in E = \mathbb{N}^J$ with $x \leq y$ coordinate-wise and all increasing sets $A \subseteq E = \mathbb{N}^J$ such that either $x \in A$ or $y \notin A$ holds:*

$$q(x, A) \leq q(y, A)$$

where $q(x, A) := \sum_{z \in A} q(x, z)$, [Mas87, Theorem 5.2, p.359].
Consider the uniformization of X with uniformization constant

$$\xi \geq 2 \cdot \sup_{z \in E}(-q(z, z)),$$

then the uniformized process is stochastically monotone as well, i.e., the family of the one-step transition kernels p^u of the uniformization is stochastically monotone with respect to $\leq^J$.

Proof. We will show that the monotonicity of the Jackson network process X in continuous time implies the monotonicity of the one-step transition kernel of the uniformization of this process:
We have to show that, if the states x, y are ordered $x \leq y$ coordinate-wise and A is an increasing set such that either $x \in A$ or $y \notin A$, then it follows for the one-step transition kernel p^u:

$$p^u(x, A) \leq p^u(y, A).$$

Let $x \leq y$ hold. According to [Mas87, p.360] we need to check only the following three cases:

- For $x, y \notin A$:

$$p^u(x, A) = \sum_{z \in A} \frac{q(x, z)}{\xi} = \frac{1}{\xi} q(x, A) \overset{(*)}{\leq} \frac{1}{\xi} q(y, A) = \sum_{z \in A} \frac{q(y, z)}{\xi} = p^u(y, A)$$

  where $(*)$ holds because of the monotonicity of the continuous-time X.

- For $x, y \in A$:

$$\begin{aligned} p^u(x, A) &= \sum_{z \in A \setminus \{x\}} \frac{q(x, z)}{\xi} + \Big(1 - \sum_{w \in E \setminus \{x\}} \frac{q(x, w)}{\xi}\Big) = 1 - \sum_{z \in A^c} \frac{q(x, z)}{\xi} \\ &= 1 - \frac{1}{\xi} q(x, A^c), \end{aligned}$$

  analogously it holds $p^u(y, A) = 1 - \frac{1}{\xi} q(y, A^c)$. With the monotonicity of the continuous-time X it holds for all $x, y \in A$:

$$q(x, A^c) \geq q(y, A^c)$$

  which implies $p^u(x, A) \leq p^u(y, A)$.

- For $x \notin A$, $y \in A$:

$$p^u(x, A) = \sum_{z \in A} \frac{q(x, z)}{\xi} = \frac{1}{\xi} q(x, A),$$

$$p^u(y, A) = \sum_{z \in A \setminus \{y\}} \frac{q(y, z)}{\xi} + \Big(1 - \sum_{w \in E \setminus \{y\}} \frac{q(y, w)}{\xi}\Big) = 1 - \sum_{z \in A^c} \frac{q(y, z)}{\xi}$$
$$= 1 - \frac{1}{\xi} q(y, A^c),$$

thus $p^u(x, A) \leq p^u(y, A)$ is equivalent to $q(x, A) + q(y, A^c) \leq \xi$ which holds if $\xi \geq 2 \cdot \sup_{z \in E}(-q(z, z))$, because:

$$q(x, A) + q(y, A^c) \leq q(x, E \setminus \{x\}) + q(y, E \setminus \{y\}) = -q(x, x) - q(y, y)$$
$$\leq 2 \cdot \sup_{z \in E}(-q(z, z)).$$

□

Lemma 6.31. *(i) For a real-valued random variable X it holds with the usual stochastic order ([MS02, Definition 1.2.1, p.2]) $X \leq_{st} X + 1$. [MS02, Property (T), pp.73-74]*

(ii) The projection $pr_1 : \mathbb{R}^J \to \mathbb{R}$, $pr_1(x_1, ..., x_J) = x_1$, is increasing with respect to $\leq^J$.

(iii) Denote by Tr the one-step transition operator for a uniformized process, which means for a uniformized process X where t_1 is the next jump time after 0 we have $Tr(X(0)) = X(t_1)$. If X and $\hat{X}$ have the same transition operator Tr and are stochastically monotone, then

$$X(0) \leq_{st} \hat{X}(0) \ \Rightarrow \ Tr(X(0)) = X(t_1) \leq_{st} \hat{X}(t_1) = Tr(\hat{X}(0)),$$

see [MS02, Corollary 5.2.12, p.186].

Proof of Proposition 6.28. Consider a non-ergodic Jackson network with two nodes as in the two-node model, so node 1 is stable and node 2 is unstable. Denote by X the uniformized queue length process, where the uniformization constant is

$$\xi \geq 2 \cdot \sum_{i=1}^{2} (\lambda_i + \mu_i),$$

on the state space $E = \mathbb{N}^2$. Thus X is stochastically monotone, see Lemma 6.30.
It follows from Lemma 6.31(i)

$$(X_1(0), X_2(0)) \leq_{st} (X_1(0), X_2(0) + 1). \tag{6.48}$$

Let $P^{(X_1(0), X_2(0))}$ be an initial distribution which fulfills (6.47) which is

$$P(X_1(0) = n_1) = \left(1 - \frac{\eta_1}{\mu_1}\right) \left(\frac{\eta_1}{\mu_1}\right)^{n_1}, \quad n_1 \in \mathbb{N}.$$

Then it holds for $P^{(X_1(0),X_2(0)+1)}$

$$\begin{aligned}\sum_{n_2=0}^{\infty} P(X_1(0)=n_1, X_2(0)+1=n_2) &= \sum_{n_2=1}^{\infty} P(X_1(0)=n_1, X_2(0)+1=n_2)\\ &= \sum_{n_2=1}^{\infty} P(X_1(0)=n_1, X_2(0)=n_2-1)\\ &= \sum_{n_2=0}^{\infty} P(X_1(0)=n_1, X_2(0)=n_2)\\ &= P(X_1(0)=n_1)\\ &= \left(1-\frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}. \end{aligned} \tag{6.49}$$

With (6.47) and (6.49) both vectors of random variables in (6.48) fulfill that the marginal distribution for the first coordinate (node 1) is the geometric distribution with rate $(1-\eta_1/\mu_1)$.
With the stochastic monotonicity of Jackson network processes and with Lemma 6.31(iii) (6.48) implies

$$\underbrace{Tr(X_1(0), X_2(0))}_{=(X_1(t_1),X_2(t_1))} \leq_{st} \underbrace{Tr(X_1(0), X_2(0)+1)}_{=:(X_1^{(1)}(t_1),X_2^{(1)}(t_1))}, \tag{6.50}$$

where $Tr(X_1(0), X_2(0)+1)$ fulfills (6.47) because Proposition 6.2 applies here, together with Lemma 6.31(ii) this implies

$$pr_1(X_1(t_1), X_2(t_1)) \leq_{st} pr_1(X_1^{(1)}(t_1), X_2^{(1)}(t_1)) \sim geo^0\left(1-\frac{\eta_1}{\mu_1}\right). \tag{6.51}$$

Again, with Lemma 6.31(i) it holds

$$(X_1(t_1), X_2(t_1)) \leq_{st} (X_1^{(1)}(t_1), X_2^{(1)}(t_1)) \leq_{st} (X_1^{(1)}(t_1), X_2^{(1)}(t_1)+1). \tag{6.52}$$

Since $(X_1^{(1)}(t_1), X_2^{(1)}(t_1))$ fulfills (6.47), the same computations as in (6.49) imply that $(X_1^{(1)}(t_1), X_2^{(1)}(t_1)+1)$ fulfills (6.47) as well, i.e.,

$$pr_1(X_1^{(1)}(t_1), X_2^{(1)}(t_1)+1) \sim geo^0\left(1-\frac{\eta_1}{\mu_1}\right). \tag{6.53}$$

We notate $(X_1^{(1)}(t_1), X_2^{(1)}(t_1)+1) =: (X_1^{(2)}(0), X_2^{(2)}(0))$. With monotonicity it holds

$$(X_1(t_2), X_2(t_2)) \leq_{st} Tr(X_1^{(1)}(t_1), X_2^{(1)}(t_1)) \leq_{st} \underbrace{Tr(X_1^{(2)}(0), X_2^{(2)}(0))}_{=(X_1^{(2)}(t_1),X_2^{(2)}(t_1))}, \tag{6.54}$$

where $(X_1^{(2)}(t_1), X_2^{(2)}(t_1))$ fulfills (6.47) because Proposition 6.2 applies here, together with Lemma 6.31(ii) this implies

$$pr_1(X_1(t_2), X_2(t_2)) \leq_{st} pr_1(X_1^{(2)}(t_1), X_2^{(2)}(t_1)) \sim geo^0\left(1-\frac{\eta_1}{\mu_1}\right). \tag{6.55}$$

This method goes on such that by induction we get for all $n \in \mathbb{N}$

$$P^{pr_1(X_1(t_n),X_2(t_n))} = P^{X_1(t_n)} \leq_{st} geo^0\left(1 - \frac{\eta_1}{\mu_1}\right). \tag{6.56}$$

□

The result (6.56) resembles the result given with the process of a Jackson network where node 2 has an infinite supply of work as upper bound which is ergodic:

$$X_1(t_n) \leq_{st} X_1^+(t_n) \sim geo^0\left(1 - \frac{\eta_1}{\mu_1}\right). \tag{6.57}$$

The difference between these two results is that (6.56) is obtained by analytical methods only, while (6.57) is proved by pathwise coupling of the processes.

Proposition 6.28 establishes the following result for the continuous-time queue length process X of the two-node model:

Theorem 6.32. *Consider the two-node model, i.e., a Jackson network with two nodes where node* 1 *is stable and node* 2 *is unstable. Denote by* X *the (continuous-time) queue length process on the state space* $E = \mathbb{N}^2$. *Assume that the initial distribution* $P^{X(0)}$ *is such that for its marginal holds*

$$P^{pr_1 \circ X(0)} = geo^0\left(1 - \frac{\eta_1}{\mu_1}\right)$$

Then for the stable marginal holds for all times $t \geq 0$

$$P^{pr_1 \circ X(t)} \leq_{st} geo^0\left(1 - \frac{\eta_1}{\mu_1}\right). \tag{6.58}$$

Proof. From Proposition 6.28 we have that

$$P^{pr_1 \circ X(t_n)} = \left(P^{X(t_n)}\right)^{\circ\ pr_1} = \left(P^{X(0)} \cdot (p^u)^n\right)^{\circ\ pr_1} \leq_{st} geo^0\left(1 - \frac{\eta_1}{\mu_1}\right).$$

Since with the uniformization kernel p^u and uniformization constant ξ it holds for the transition kernel $p(t)$ of the continuous-time process X:

$$p(t) = \sum_{n=0}^{\infty} e^{-\xi t} \frac{(\xi t)^n}{n!} (p^u)^n.$$

We have:

$$P^{X(t)} = P^{X(0)} \cdot p(t) = P^{X(0)} \cdot \sum_{n=0}^{\infty} e^{-\xi t} \frac{(\xi t)^n}{n!} (p^u)^n = \sum_{n=0}^{\infty} e^{-\xi t} \frac{(\xi t)^n}{n!} (P^{X(0)} \cdot (p^u)^n).$$

Applying the first projection implies:

$$\left(P^{X(t)}\right)^{\circ\ pr_1} = \sum_{n=0}^{\infty} e^{-\xi t} \frac{(\xi t)^n}{n!} \underbrace{\left(P^{X(0)} \cdot (p^u)^n\right)^{\circ\ pr_1}}_{\overset{(*)}{\leq}_{st} geo^0\left(1-\frac{\eta_1}{\mu_1}\right)} \overset{(**)}{\leq}_{st} geo^0\left(1 - \frac{\eta_1}{\mu_1}\right),$$

where $(*)$ holds with $\xi \geq 2 \cdot \sum_{i=1}^{2}(\lambda_i + \mu_i)$, see Proposition 6.28, and $(**)$ holds with the mixture property of $\leq_{st}$, see [MS02, Property (MI), p.74]. □

The distribution $geo^0\left(1-\frac{\eta_1}{\mu_1}\right)$ in Theorem 6.32 is an upper bound for the marginal distribution of the queue length process X, if the process is started with a distribution which has this geometrical distribution as marginal.
If the mean values of the marginal processes (from (6.58)) were equal or if there was equality in (6.58), we would call it a (stochastically) invariant marginal distribution. The mean values of the marginal processes at each time are not equal in general, as can be seen in the following proposition.

Proposition 6.33. *Consider the framework of Proposition 6.28. Assume that $P^{(X_1(0),X_2(0))}$ fulfills* (6.47)*. From the proof of Proposition 6.28 we know that*

$$(X_1(0), X_2(0)) \leq_{st} (X_1(0), X_2(0)+1) =: (X_1^{(1)}(0), X_2^{(1)}(0))$$

implies for the first jump time t_1

$$(X_1(t_1), X_2(t_1)) \leq_{st} (X_1^{(1)}(t_1), X_2^{(1)}(t_1)),$$

where $pr_1(X_1^{(1)}(t_1), X_2^{(1)}(t_1)) \sim geo^0\left(1-\frac{\eta_1}{\mu_1}\right)$.
If neither $r(2,1)=0$ (which would mean that node 1 is an ergodic $M/M/1$ system) nor $P(X_2(0)=0)=0$ (which would yield the framework of Proposition 6.2) holds, then

$$E[X_1(t_1)] < E[X_1^{(1)}(t_1)].$$

Proof. The mean value of $X_1^{(1)}(t_1)$ is

$$E[X_1^{(1)}(t_1)] = \sum_{n_1\in\mathbb{N}} n_1 \cdot P(X_1^{(1)}(t_1)=n_1) = \sum_{n_1\in\mathbb{N}} n_1 \cdot \left(1-\frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1}.$$

For the mean value of $X_1(t_1)$ we have to derive $P(X_1(t_1)=n_1)$ for all $n_1 \in \mathbb{N}$ first:

$$\begin{aligned}
&P(X_1(t_1)=n_1) = \sum_{n_2\in\mathbb{N}} P(X(t_1)=(n_1,n_2)) \\
&= \sum_{n_2\in\mathbb{N}} P(X(0)=(n_1,n_2)) \cdot \frac{1}{\xi}\Big(\xi-\lambda_1-\lambda_2-\mu_1(1-r(1,1))1_{\mathbb{N}_+}(n_1) \\
&\qquad -\mu_2(1-r(2,2))1_{\mathbb{N}_+}(n_2)\Big) \\
&+ \sum_{n_2\in\mathbb{N}} P(X(0)=(n_1-1,n_2)) \cdot \frac{\lambda_1}{\xi}1_{\mathbb{N}_+}(n_1) + \sum_{n_2\in\mathbb{N}} P(X(0)=(n_1,n_2-1)) \cdot \frac{\lambda_2}{\xi}1_{\mathbb{N}_+}(n_2) \\
&+ \sum_{n_2\in\mathbb{N}} P(X(0)=(n_1+1,n_2)) \cdot \frac{\mu_1 r(1,0)}{\xi} + \sum_{n_2\in\mathbb{N}} P(X(0)=(n_1,n_2+1)) \cdot \frac{\mu_2 r(2,0)}{\xi} \\
&+ \sum_{n_2\in\mathbb{N}} P(X(0)=(n_1+1,n_2-1)) \cdot \frac{\mu_1 r(1,2)}{\xi} \cdot 1_{\mathbb{N}_+}(n_2) \\
&+ \sum_{n_2\in\mathbb{N}} P(X(0)=(n_1-1,n_2+1)) \cdot \frac{\mu_2 r(2,1)}{\xi} \cdot 1_{\mathbb{N}_+}(n_1)
\end{aligned}$$

$\Leftrightarrow$

$$
\begin{aligned}
&P(X_1(t_1) = n_1) = \\
&= \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1, n_2)) \cdot \frac{1}{\xi}\Big(\xi - \lambda_1 - \mu_2 r(2,1) 1_{\mathbb{N}_+}(n_2) - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1)\Big) \\
&\quad + \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1 - 1, n_2)) \cdot \frac{\lambda_1 + \mu_2 r(2,1)}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) \\
&\quad + \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1 + 1, n_2)) \cdot \frac{\mu_1(1 - r(1,1))}{\xi} \\
&\quad - P(X(0) = (n_1 - 1, 0)) \cdot \frac{\mu_2 r(2,1)}{\xi} \cdot 1_{\mathbb{N}_+}(n_1) \\
&= \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1, n_2)) \cdot \frac{1}{\xi}\Big(\xi - \underbrace{(\lambda_1 + \mu_2 r(2,1))}_{=\eta_1(1 - r(1,1))} - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1)\Big) \\
&\quad + \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1 - 1, n_2)) \cdot \frac{1}{\xi}\underbrace{(\lambda_1 + \mu_2 r(2,1))}_{=\eta_1(1 - r(1,1))} \cdot 1_{\mathbb{N}_+}(n_1) \\
&\quad + \sum_{n_2 \in \mathbb{N}} P(X(0) = (n_1 + 1, n_2)) \cdot \frac{\mu_1(1 - r(1,1))}{\xi} \\
&\quad + [P(X(0) = (n_1, 0)) - P(X(0) = (n_1 - 1, 0) \cdot 1_{\mathbb{N}_+}(n_1)] \cdot \frac{\mu_2 r(2,1)}{\xi} \\
&\overset{(6.47)}{=} \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1} \cdot \frac{1}{\xi} \cdot \Big(\xi - \eta_1(1 - r(1,1)) - \mu_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) \\
&\qquad\qquad + \frac{\mu_1}{\eta_1}\eta_1(1 - r(1,1)) 1_{\mathbb{N}_+}(n_1) + \frac{\eta_1}{\mu_1}\mu_1(1 - r(1,1))\Big) \\
&\quad + [P(X(0) = (n_1, 0)) - P(X(0) = (n_1 - 1, 0) \cdot 1_{\mathbb{N}_+}(n_1)] \cdot \frac{\mu_2 r(2,1)}{\xi}
\end{aligned}
$$

$\Leftrightarrow$

$$
\begin{aligned}
&P(X_1(t_1) = n_1) = \\
&= \left(1 - \frac{\eta_1}{\mu_1}\right)\left(\frac{\eta_1}{\mu_1}\right)^{n_1} + [P(X(0) = (n_1, 0)) - P(X(0) = (n_1 - 1, 0) \cdot 1_{\mathbb{N}_+}(n_1)] \cdot \frac{\mu_2 r(2,1)}{\xi}.
\end{aligned} \tag{6.59}
$$

With (6.59) the mean value of $X_1(t_1)$ is

$$
\begin{aligned}
E[X_1(t_1)] &= \sum_{n_1 \in \mathbb{N}} n_1 \cdot P(X_1(t_1) = n_1) \\
&= E[X_1^{(1)}(t_1)] + \sum_{n_1 \in \mathbb{N}} n_1 \cdot P(X(0) = (n_1, 0)) \cdot \frac{\mu_2 r(2,1)}{\xi} \\
&\quad - \sum_{n_1 \in \mathbb{N}} n_1 \cdot P(X(0) = (n_1 - 1, 0)) \cdot 1_{\mathbb{N}_+}(n_1) \cdot \frac{\mu_2 r(2,1)}{\xi} \\
&= E[X_1^{(1)}(t_1)] - \underbrace{\frac{\mu_2 r(2,1)}{\xi}}_{>0 \text{ if } r(2,1)>0} \underbrace{\sum_{n_1 \in \mathbb{N}} P(X(0) = (n_1, 0))}_{=P(X_2(0)=0)}.
\end{aligned}
$$

□

Since $\leq_{st}$ holds in (6.58) and the mean values of the marginal processes are not equal at all times in general, we call the resulting upper bound distribution in Theorem 6.32 a *stochastically sub-invariant distribution*.

Proposition 6.28 (and therefore Theorem 6.32) only uses Proposition 6.2 which is a result for the two-node model, all other techniques hold for non-ergodic Jackson networks with an arbitrary finite number of nodes, too. Referencing Proposition 6.3 with $D = \emptyset$ yields with similar computations the same result for the marginals of the stable subnetwork in a non-ergodic Jackson network with J nodes, $0 < J < \infty$.
A similar result can be obtained for non-ergodic Jackson networks with unreliable nodes as well, because

$$(Y, X) \leq_{st} (Y, X + 1)$$

holds where Y can be considered as availability process and one may refer to Proposition 6.3 with $D \neq \emptyset$.

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Abstract

This thesis contributes to a better understanding of the behavior of queueing networks that cannot approach a classical equilibrium state.

We consider Jackson networks with unreliable nodes which randomly break down and are under repair for a random time. Our networks are described by Markov processes the states of which incorporate the availability status and the queue-lengths vector. For Jackson networks with unreliable nodes, it is known that, under ergodicity conditions, the Markovian availability–queue-lengths process has a stationary product-form distribution. Ergodicity conditions can be expressed as local rate conditions: The total arrival rate at each node has to be strictly less than its maximal service rate. If for some nodes the ergodicity condition is violated, the network process is not ergodic and there cannot exist a stationary distribution. Nevertheless, we are able to obtain the complete asymptotics for non-ergodic Jackson networks with unreliable nodes and show that the state distribution of the stable subnetworks, i.e., the set of nodes where the local rate condition is fulfilled, converges to a Jackson-type product-form distribution.

The characterization of the asymptotic behavior of non-ergodic Jackson networks with unreliable nodes strongly relies on a detailed investigation of another class of generalized Jackson networks, which is of interest for its own. In these networks some stations may have an additional buffer with an infinite supply of lower priority jobs (customers) served at that station whenever the station runs out of standard customers. Networks incorporating such buffers are called Jackson networks with infinite supply. We analyze the stationary and limiting behavior of these networks with reliable nodes as well as with unreliable nodes.

Our results offer a new way to measure and assess the performance of non-ergodic Jackson networks with unreliable nodes where steady-state methods cannot be applied because no steady state exists. Using the obtained limiting distributions we are able to compute long-time averages of the standard performance and availability metrics explicitly.

The limiting distribution of the stable subnetworks of a non-ergodic network process has a product form which resembles the product-form limiting and stationary distribution of an ergodic network process. The observed structural similarities of the asymptotic results for ergodic network processes and non-ergodic network processes on the stable subnetworks pose the following question: Assume that the non-ergodic network is started with an initial distribution which has the marginal limiting product-form distribution. Will this distribution be preserved over time? It turns out that the answer is in general negative.

However, there are subtleties which turn the question into the challenging problem to find conditions which transform the limiting product-form distribution into a quasi-stationary one, similar to those which occur when studying absorbing Markov processes. We prove several results which contribute to a better understanding of the principles behind such quasi-stationary behavior.

Zusammenfassung

Diese Arbeit liefert einen Beitrag zum besseren Verständnis des Verhaltens von Warteschlangennetzen, welche sich nicht in ein klassisches Gleichgewicht einschwingen können.

Es werden Jackson-Netzwerke mit unzuverlässigen Knoten, welche zufallsbeeinflusst ausfallen und daraufhin für eine zufällige Zeit repariert werden, betrachtet. Diese Netze werden durch Markov-Prozesse beschrieben, welche die zeitliche Entwicklung sowohl des Verfügbarkeitsstatus der Knoten im Netz als auch der vorliegenden Schlangenlängen je Knoten im System abbilden. Für Jackson-Netze mit unzuverlässigen Knoten ist bekannt, dass unter Ergodizitätsbedingungen der zugehörige Markovsche Verfügbarkeits- und Schlangenlängenprozess eine stationäre Verteilung in Produktform besitzt. Die Ergodizitätsbedingungen können als lokale Ratenbedingungen ausgedrückt werden: Die Gesamtankunftsrate an jedem Knoten muss strikt kleiner sein als dessen maximale Bedienrate (Kapazität). Ist für mindestens einen Knoten die Ergodizitätbedingung verletzt, so ist der Markov-Prozess nicht ergodisch und kann keine stationäre Verteilung besitzen. Dennoch kann die exakte Asymptotik für nicht-ergodische Jackson-Netze mit unzuverlässigen Knoten bewiesen werden und die Zustandsverteilung der stabilen Teilnetze (d.h. die Menge der Knoten, für welche die lokale Ratenbedingung erfüllt ist) konvergiert gegen eine Verteilung in Produktform.

Die Charakterisierung der Asymptotik nicht-ergodischer Jackson-Netze mit unzuverlässigen Knoten beruht auf einer detaillierten Untersuchung einer weiteren Klasse verallgemeinerter Jackson-Netze, die auch für sich genommen von Interesse ist. In diesen Netzen können einige Bedienstationen zusätzlich ausgestattet sein mit einem Puffer mit einem unbegrenzten Angebot an Aufträgen geringerer Priorität, welche immer dann abgearbeitet werden, wenn kein Standard-Auftrag aus dem Netz an dieser Station wartet. Netze mit solchen Puffern werden Jackson-Netze mit "infinite supply" genannt. Das stationäre und Grenz-Verhalten dieser Netzwerke mit zuverlässigen aber auch mit unzuverlässigen Knoten wird analysiert.

Die bewiesenen Ergebnisse ermöglichen es neue Methoden zu konstruieren zur Leistungsanalyse und -bewertung nicht-ergodischer Jackson-Netze mit unzuverlässigen Knoten, für welche stationäre Methoden nicht angewendet werden können, da keine stationäre Verteilung existiert. Die erhaltenen Grenzverteilungen nutzend, können Langzeitmittelwerte der Standardmaße für Leistungs- und Verfügbarkeitsanalysen explizit berechnet werden.

Die asymptotische Verteilung für stabile Teilnetze eines nicht-ergodischen Netzwerk-

prozesses hat eine Produktform, welche der Produktform der stationären und asymptotischen Verteilung in einem ergodischen Netzwerk stark ähnelt. Die beobachteten strukturellen Ähnlichkeiten der asymptotischen Ergebnisse für ergodische Netzwerkprozesse und für nicht-ergodische Netzwerkprozesse (auf den stabilen Teilen) werfen die folgende Frage auf: Angenommen das nicht-ergodische Netzwerk wird mit einer Verteilung gestartet, welche marginal (für das stabile Teilnetz) die Produktform besitzt. Wird diese marginale Produktform-Verteilung mit Verlauf der Zeit bestehen bleiben? Es stellt sich heraus, dass die Antwort im Allgemeinen negativ ist. Eine genauere Analyse führt dazu, die Fragestellung in das sich als schwer zu bewältigend herausstellende Problem zu transformieren, Bedingungen zu finden, wodurch die Grenzverteilung eine quasi-stationäre Verteilung wird, ähnlich wie jene quasi-stationären Verteilungen, die auftreten, wenn absorbierende Markov-Prozesse analysiert werden. Es werden Ergebnisse bewiesen, welche dazu beitragen, die hinter solch quasi-stationärem Verhalten der betrachteten Systeme stehenden Prinzipien zu verstehen.

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