A Study of Equivariant Hopf Algebras and Tensor Categories

through Topological Field Theories

Dissertation
zur Erlangung des Doktorgrades der Fakultät für Mathematik,
Informatik und Naturwissenschaften der Universität Hamburg

vorgelegt
im Fachbereich Mathematik von

Jennifer Maier

aus Hamburg

Hamburg
2013

Als Dissertation angenommen vom Fachbereich
Mathematik der Universität Hamburg

Auf Grund der Gutachten von Prof. Dr. Christoph Schweigert
und Prof. Dr. Peter Bantay

Hamburg, 16.01.2013

Prof. Dr. Ulf Kühn
Leiter des Fachbereichs Mathematik

Contents

1 Introduction

In this thesis we deal with modular categories as well as their equivariant versions. In particular, we discuss the correspondence between equivariant modular categories and extended 3-dimensional topological field theories and use this correspondence to construct certain equivariant modular categories. We also investigate, for G a finite group, G-equivariant structures on (weak) Hopf algebras and their strictification.

In the last few decades, the interaction of algebraic structures and low dimensional topology has been extensively investigated. Inspired by theoretical physics, the mathematical notion of topological field theories (TFTs) was introduced by Atiyah in [Ati88] and since then different variants have been established, as (fully) extended TFTs and equivariant TFTs. They have turned out to have various applications in pure mathematics, especially in representation theory, and furthermore they provide invariants of knots, links and of 3-manifolds [Tur10b].
Several classification results about TFTs of different types and the correspondent algebraic structure are already established. In dimension two the relevant algebraic structure are commutative Frobenius algebras. A Frobenius algebra is an algebra that has also the structure of a coalgebra, such that the coproduct is a morphism of bimodules (where the algebra is seen as a bimodule over itself with action by multiplication).
A $2d$ TFT is a symmetric monoidal functor from the category $\mathfrak{Cob}(1,2)$ of compact, closed, oriented smooth 1-manifolds and cobordisms to the category $\mathrm{Vect}_{\mathrm{K}}$ of vector spaces over a field K. Given a commutative Frobenius algebra, one defines a functor Z_A from the category $\mathfrak{Cob}(1,2)$ on objects by assigning to the circle the algebra A. The morphisms in $\mathfrak{Cob}(1,2)$ are cobordisms of compact, closed, oriented 1-manifolds, i.e. oriented 2-dimensional manifolds with ingoing and outgoing boundaries. These 2-dimensional cobordisms can be represented by generators and relations. That is to say, we have building blocks for the morphisms and the functor is determined by what it assigns to these blocks. By tensoriality of the TFT Z_A, it is clear that a 2-manifold Σ with n ingoing and m outgoing boundaries must be sent to a linear map $Z_A(\Sigma) : A^{\otimes n} \to A^{\otimes m}$ from the n-fold tensor product of A to the m-fold tensor product of A. The TFT Z_A assigns certain structure maps of the algebra A to certain building blocks, for example it assigns to the pair of pants , which in $\mathfrak{Cob}(1,2)$ is a morphism between two copies of the circle to the circle, the multiplication of the algebra. The Frobenius algebra axioms assure that Z_A is well defined, i.e. that the identities of the relations in $\mathfrak{Cob}(1,2)$ get sent to identities in $\mathrm{Vect}_{\mathrm{K}}$.

In fact, this assignment is one direction of a one-to-one correspondence, i.e. from any $2d$ TFT one gets a commutative Frobenius algebra (see also [Koc04]). This correspondence is even an equivalence of categories. So expressed in categorical terms, the category of $2d$ TFTs and the category of commutative Frobenius algebras are equivalent. Given a $2d$ TFT Z, one takes as A_Z the vector space that is assigned to the circle, the product is the image under Z of the pair of pants and the unit is the image of the cup, whereas the coproduct is the image of the co-pair of pants (ingoing and outgoing boundary are interchanged) and the counit is the image of the cap. From the relations that hold for the morphisms in $\mathfrak{Cob}(1,2)$ one can deduce the axioms of a commutative Frobenius algebra.

Now we will turn to extended $3d$ TFTs. Note that in this thesis we only consider an extension down to dimension one and not a full extension down to a point. So $3d$ TFT in this thesis will always mean 1-2-3 extended TFTs. We regard the $3d$ TFT as a symmetric monoidal 2-functor from a geometric category to an algebraic category. The source category is the bicategory $\mathfrak{Cob}(1,2,3)$ of 1-2-3-cobordisms, in which the objects are compact, closed, oriented smooth 1-manifolds, the 1-morphisms are surfaces with the objects as boundaries and 2-morphisms are 3-dimensional manifolds with boundaries and corners (see Definition 1.6). The target category is the 2-category $2\mathrm{Vect}_{\mathrm{K}}$ of 2-vector spaces over a field K (see Definition 1.2). In particular, objects are K-linear, abelian, finitely semi-simple categories.
Now for a given modular category $\mathcal{C}$, define a monoidal functor $Z_{\mathcal{C}} : \mathfrak{Cob}(1,2,3) \to 2\mathrm{Vect}_{\mathrm{K}}$ on the level of objects of $\mathfrak{Cob}(1,2,3)$ by assigning to the circle the category $\mathcal{C}$. Similar to the construction of a $2d$ TFT out of a Frobenius algebra, certain functors that are part of the definition of a modular category get assigned to certain 2-manifolds with boundary. For example $Z_{\mathcal{C}}$ assigns to the pair of pants, which in Cob(1,2,3) is a morphism between two copies of the circle to the circle, the tensor product, which is a functor $\mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}$. Here the symbol '$\boxtimes$' denotes the Deligne tensor product of two categories (see [Del90, Sec. 5] or [BK01, Def. 1.1.15] for the definition). This assignment is expected to be one direction of a one-to-one correspondence, i.e. $Z_{\mathcal{C}}$ can be extended to a monoidal functor $\mathfrak{Cob}(1,2,3) \to 2\mathrm{Vect}_{\mathrm{K}}$ and conversely, it is conjectured that one can extract a modular category from any extended $3d$ TFT.
However, there are two main obstacles to prove this correspondence: first of all, in contrast to the $2d$ TFT case, the 1- and 2-morphisms are not as easy to handle, since there is no presentation of them by generators and relations. And secondly, the axioms of a Frobenius algebra can all be expressed on the level of morphisms, whereas in a modular category the axiom for duality has

no expression in terms of functors and natural transformation, which makes it impossible to derive the categorical data purely from the $3d$ TFT axioms. A fairly basic example of a modular category is the representation category of the Drinfel'd double $\mathcal{D}(G)$ of a finite group (see [BK01]). A generalization of this modular category was proposed in [Ban10], where the representation category $\mathcal{M}(\mathcal{X})$ associated to a finite crossed module $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \partial, \mu)$ is considered. A crossed module consists of two finite groups $\mathcal{X}_1$ and $\mathcal{X}_2$, together with an action μ of the group $\mathcal{X}_1$ on the group $\mathcal{X}_2$ and a group morphism $\partial : \mathcal{X}_2 \to \mathcal{X}_1$ with compatibility conditions (see Definition 2.1). It was already a result of [Ban10] that the category $\mathcal{M}(\mathcal{X})$ is only modular in the case where it is equivalent to the representation category of the Drinfel'd double. Due to Bruguières ([Bru00]) there is a modularization procedure, that assigns to a premodular category with additional properties a unique modular category. This modularization was carried out in [Mai09] respectively in the resultant paper [MS11] for the representation category $\mathcal{M}(\mathcal{X})$ of a finite crossed module $\mathcal{X}$. The result was that the category gained from the modularization of the category $\mathcal{M}(\mathcal{X})$ is as well equivalent to the one associated to the representation category of the Drinfel'd double $\mathcal{D}(G)$ of the finite group $G = \operatorname{Im}\partial = \mathcal{X}_2/\ker\partial$. But in the modularization procedure another finite group J plays an important role, in the sense that modularization can be seen as the inverse of orbifolding with respect to the action of the group J. Thus we have an action of the group J on $\mathcal{D}(G)$-mod, and so the question arises, whether this category can be seen as the neutral component of a J-equivariant category. One of the main results of this thesis is the construction of such an embedding of the categories $\mathcal{M}(\mathcal{X})$ that arise from finite crossed modules $\mathcal{X}$ with injective boundary map into an equivariant category.

The embedding of the representation category $\mathcal{M}(\mathcal{X})$ into a J-equivariant category is realized by using equivariant Dijkgraaf-Witten theory, which is an equivariant extended $3d$ TFT. In order to define what a J-equivariant extended TFT is, we will, for a finite group J, consider the category $\mathfrak{Cob}^J(1,2,3)$ of 1-2-3 cobordisms, where all the manifolds are additionally equipped with a J-cover. On the other hand, a J-equivariant tensor category is an ordinary tensor category $\mathcal{C}$ equipped with a J-grading $\mathcal{C} = \bigoplus_{j\in J}\mathcal{C}_j$ and with a compatible categorical action of the group J, i.e. for each group element $j \in J$ we have a tensor functor $\phi_j \in \operatorname{Aut}(\mathcal{C})$ with $\phi_g(\mathcal{C}_h) \subset \mathcal{C}_{ghg^{-1}}$ as well as compositors $\alpha_{i,j} : \phi_i \circ \phi_j \to \phi_{ij}$ for every pair of elements $i, j \in J$. Now in contrast to ordinary modular categories, the braiding is modified by the group action, and the same applies to the twist. Ordinary modular categories appear as a special case of J-equivariant categories, when the group J is trivial.
As in the non-equivariant case, one expects a correspondence between ex-

tended J-equivariant $3d$ TFTs and J-modular categories. One part of this correspondence appeared in [Tur10a], where $(d+1)$-dimensional homotopy quantum field theories (HQFT's) are introduced. The source category of the HQFT has as objects d-dimensional manifolds M together with a connected CW-space X and a map $g : M \to X$ and as morphisms suitable cobordisms. It is further shown, how a J-equivariant category (there called crossed J-category) produces a 3-dimensional HQFT where $X = K(J,1)$ is an Eilenberg-MacLane space of J.
We are mainly involved with the other direction of the before mentioned correspondence, i.e. from a certain J-equivariant extended $3d$ TFT, which is an equivariant version of Dijkgraaf-Witten theory ([DW90]), we extract a J-equivariant category $\mathcal{C}^J(G)$, and as a main result, show that this category is J-modular. The category $\mathcal{C}^J(G)$ has as its neutral component the modular category gained from the modularization of the representation category $\mathcal{M}(\mathcal{X})$ of a finite crossed module $\mathcal{X}$, with $\mathcal{X}_2 = G$, injective boundary map and $J = \mathcal{X}_1/G$.

Another part of this thesis deals with the aspect of strictification of a categorical action of a group G on a G-equivariant tensor category $\mathcal{C}$. As already mentioned, such an action $(\phi_g)_{g\in G} \in \mathrm{Aut}(\mathcal{C})$ in particular contains the datum of compositors $\alpha_{g,h} : \phi_g \circ \phi_h \to \phi_{gh}$. It has been demonstrated by Müger [Tur10a, App. 5] that one can replace $\mathcal{C}$ by an equivalent category $\mathcal{C}^{str}$ with a *strict* action of G, i.e. the compositors are given by the identity: $\phi_g^{str} \circ \phi_h^{str} = \phi_{gh}^{str}$, and such that the equivalence $\mathcal{C} \cong \mathcal{C}^{str}$ is even compatible with the equivariant structure. Now, if one starts with the representation category of a G-Hopf algebra A, it is natural to ask whether it is always possible to 'strictify' the action on the Hopf algebra, i.e. to find another Hopf algebra A', which carries a strict group action such that A-mod is equivalent as a J-equivariant category to A'-mod. We show that one can not always find such a Hopf algebra A'. Yet we construct a *weak* Hopf algebra A^{str}, which is G-equivariant with strict G action and for which A^{str}-mod $\cong$ A-mod as G-equivariant categories.

Outline

This thesis is organized as follows:

Chapter I: We first give a short review of extended 1-2-3 extended TFTs and modular categories. In the second section we define, for a finite group G, a G-equivariant structure on tensor categories and a G-equivariant structure on Hopf algebras (G-Hopf algebras) and show how the category of modules of a G-Hopf algebra can be equipped with the structure of a G-equivariant tensor category with duality (Lemma 2.15). We further introduce the notion of G-ribbon categories and G-ribbon algebras and show that G-ribbon algebras yield G-ribbon categories as their representation categories (Proposition 2.18). In order to define G-modularity (see Definition 2.22) we give the definition of the orbifold category of a G-equivariant category (Definition 2.20). As a first original result we present the orbifold construction on the level of algebras: We define the *orbifold algebra* of a G-equivariant algebra and show that it corresponds to orbifolding of the representation category (see Proposition 2.27).

Chapter II: This chapter is dedicated to an easy example of a modular category and its generalization: We first recapitulate results about the Drinfel'd double $\mathcal{D}(G)$ of a finite group G (see [BK01] for a textbook reference) and its category of representations $\mathcal{D}(G)$-mod. Furthermore we give a short introduction to the categories of representations $\mathcal{M}(\mathcal{X})$ of a finite crossed module $\mathcal{X}$ (see Definition 2.1) of which $\mathcal{D}(G)$-mod is a special case. Summing up the results from [Ban10] and [Mai09], we will show in Proposition 2.4 that the modular category associated to a crossed module, the modularization of the category $\mathcal{M}(\mathcal{X})$, is again equivalent to the category of representations of a Drinfel'd double.

Chapter III: We first give an introduction to Dijkgraaf-Witten theory as an example for an extended $3d$ TFT. In section 2.4 we carry out calculations, that evaluate the given extended $3d$ TFT on certain 1-,2- and 3-dimensional manifolds, beyond the existing calculations in [Mor10, Section 4]. We show that, evaluated on the circle, this particular extended $3d$ TFT yields the representation category of the Drinfel'd double (see Proposition 2.16). We then define an equivariant version of the Dijkgraaf-Witten theory. These are essentially new results. The idea is to build a weak action on a group G by another group J out of a group extension (see Definition 3.1) and then take manifolds with J-twisted G-bundles as the source category. As mentioned above, an equivariant Dijkgraaf-Witten theory gives rise to an equivariant category $\mathcal{C}^J(G)$. In subsection 3.5 we implement some of the evaluations of the TFT on the circle and on certain 2- and 3-manifolds to gather more structure on the category $\mathcal{C}^J(G)$. In subsection 4.2 we introduce a Hopf algebra, which we call equivariant Drinfel'd double, in order to show some extra properties of $\mathcal{C}^J(G)$. Finally we use the orbifold construction on the algebra (orbifold algebra, see Definition 2.24 and Proposition 2.27) to show J-modularity of the category in Theorem 4.11.

Chapter IV: The fourth chapter contains two main results. As a first result, we show that it is not always possible to strictify the action of a group on a Hopf algebra within the category of Hopf algebras, i.e. to find a Hopf algebra

A' with A-mod $\cong$ A'-mod as equivariant categories (Theorem 2.2). Yet as a second result, in Theorem 3.1 we show that this kind of strictification is possible if we allow A' to be a weak Hopf algebra. That is to say, the Hopf algebra axioms of A' are weakened in contrast to the group action on A', which is strict. For this purpose we generalize the notion of an equivariant structure on a Hopf algebra to weak Hopf algebras (Definition 1.7). A summary of the different kinds of weakenings of a G-Hopf algebra can be found in subsection 4.1.

To sum up, this thesis contains the following main results:

- We construct an equivariant braided category $\mathcal{C}$ out of an equivariant extended $3d$ TFT from a normal group inclusion and show that this category $\mathcal{C}$ can be endowed with the structure of a ribbon category such that it is an equivariant modular category (Theorem 4.11 of Chapter III).
- We show that the group action on an equivariant Hopf algebra can not be strictified in the category of Hopf algebras (Theorem 2.2 of Chapter IV).
- We show that a strictification is possible if we allow weak Hopf algebras (see Theorem 3.1 of Chapter IV).

This thesis is based on the following publications:

[MS11] J. Maier and C. Schweigert. Modular categories from finite crossed modules. *J. Pure Appl. Algebra*, 215(9):2196–2208, 2011.

[MNS12] J. Maier, T. Nikolaus, and C. Schweigert. Equivariant Modular Categories via Dijkgraaf-Witten Theory. Adv. Theor. Math. Phys., 16(1):289–358, 2012

[MNS11] J. Maier, T. Nikolaus, and C. Schweigert. Strictification of weakly equivariant Hopf algebras. *accepted in Bull. Belg. Math. Soc., Arxiv preprint arXiv:1109.0236*, 2011.

Acknowledgment

First of all I would like to express my gratitude to my advisor Christoph Schweigert for his great support, his patience and his useful critiques on my work during the last few years. Secondly I thank David Evans for giving me the opportunity of an inspiring research stay of five months in Cardiff. I wish to thank Peter Bantay for his kind hospitality during my stay in Budapest. Special thanks to my co-author Thomas Nikolaus, who I very much enjoyed working with. I also would like to thank all other members of our group, in particular Alexander Barvels, for various interesting and helpful discussions. Finally I want to thank my officemates Ole Vollertsen, Stephanie Ziegenhagen, Marco Freibert and Grace Kennedy for the nice working atmosphere. This thesis is partially supported by the Research priority program SPP 1388 "Representation theory".

Chapter I

Extended TFTs and Equivariant Modular Categories

1 Extended $3d$ TFTs and Modular Categories

This section is intended to give a brief overview of extended $3d$ topological field theories and modular categories and how they are related. The definitions and the detailed construction of a $3d$ TFT from a modular category can be found in the books [Tur10b] and [BK01]. In contrast to them, we will formulate the extended $3d$ TFT in the language of 2-categories and 2-functors.

First we will recall the notion of an ordinary $3d$ TFT from [Ati88]. There Atiyah gave a definition that describes a topological field theory as a symmetric monoidal functor from a geometric category to an algebraic category. To make this definition explicit, let $\mathfrak{Cob}(2,3)$ be the category which has 2-dimensional compact oriented smooth manifolds as objects. Its morphisms $M : \Sigma \to \Sigma'$ are given by (orientation preserving) diffeomorphism classes of 3-dimensional, compact oriented cobordism from Σ to Σ' which we write as

$$\Sigma \hookrightarrow M \hookleftarrow \Sigma'.$$

Composition of morphisms is given by gluing cobordisms together along the boundary. The disjoint union of 2-dimensional manifolds and cobordisms equips this category with the structure of a symmetric monoidal category. For the algebraic category, we choose the symmetric tensor category Vect_K of finite dimensional vector spaces over an algebraically closed field K of characteristic zero.

Definition 1.1 (Atiyah)**.** *A* 3*d* *TFT is a symmetric monoidal functor*

$$Z : \mathfrak{C}\mathrm{ob}(2,3) \to \mathrm{Vect}_{\mathrm{K}}.$$

We now want to define an extended 3*d* TFT in a similar vein as a weak 2-functor from a certain geometric bicategory to an algebraic 2-category. For a survey on bicategories and 2-categories we refer to [Lei98]. Note that the source category will not be a 2-category but only a bicategory and therefore we will have to consider weak 2-functors.
The target category possesses additional properties which can be summarized by saying that the objects are 2-vector spaces and the morphisms are K-linear functors in the sense of [KV94]. There exist different definitions of 2-vector space and the ones we will work with are also called Kapranov Voevodski 2-vector spaces (or KV 2-vector spaces for short). But since there will be no confusion, in the following we will only use the term 2-vector space.

Definition 1.2.

1. *A 2-vector space (over a field* K*) is a* K*-linear, abelian, finitely semi-simple category. Here finitely semi-simple means that the category has finitely many isomorphism classes of simple objects and each object is a finite direct sum of simple objects.*

2. *Morphisms between 2-vector spaces are* K*-linear functors and 2-morphisms are natural transformations. We denote the 2-category of 2-vector spaces by* $2\mathrm{Vect}_{\mathrm{K}}$*.*

The category of vector spaces is a symmetric tensor category, and correspondingly, also in the category of 2-vector spaces, we have a tensor-product, the Deligne tensor product, which is subject of the next definition (see also [Del90, Sec. 5]).

Definition 1.3. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be abelian categories over a field* K*. Their* Deligne tensor product *is a* K*-linear category* $\mathcal{C} \boxtimes \mathcal{D}$ *together with a* K*-linear bifunctor* $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \boxtimes \mathcal{D}$ *that is right exact in each variable such that for every bifunctor* $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$ *that is right exact in each variable, there is a unique right exact functor* $\bar{F} : \mathcal{C} \boxtimes \mathcal{D} \to \mathcal{A}$ *such that* $\bar{F} \circ \boxtimes = F$*.*

In the case of finitely semisimple abelian categories over K we can give an equivalent description of the Deligne tensor product which was given in [BK01].

Lemma 1.4. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be finitely semisimple abelian categories over* k*. Define a category with*

- *objects: finite sums of the form* $\bigoplus X_i \boxtimes Y_i$ *with* $X_i \in \mathcal{C}, Y_i \in \mathcal{D}$*.*

- *morphisms:*
  $\overline{\mathrm{Hom}}(\bigoplus X_i \boxtimes Y_i, \bigoplus X'_i \boxtimes Y'_i) = \bigoplus_{i,j} \mathrm{Hom}_{\mathcal{C}}(X_i, X'_j) \otimes \mathrm{Hom}_{\mathcal{D}}(Y_i, Y'_j)$

This category fulfills the universal property of the Deligne tensor product. In particular the Deligne product of two finitely semisimple abelian categories is again finitely semisimple and abelian.

The proof of this lemma is by checking the universal property.

Remark 1.5. *The Deligne tensor product* $\boxtimes$ *endows* $2\mathrm{Vect}_{\mathrm{K}}$ *with the structure of a symmetric monoidal 2-category.*

The definition and the properties of symmetric monoidal bicategories (resp. 2-categories) can be found in [SP09, Ch. 3].
In the spirit of Definition 1.1, we formalize the properties of the extended theory Z by describing it as a monoidal 2-functor from a cobordism 2-category to the algebraic category $2\mathrm{Vect}_{\mathrm{K}}$. It remains to state the formal definition of the relevant geometric category. Here, we ought to be a little bit more careful, since we want to get a 2-category and hence cannot identify diffeomorphic 2-manifolds. For the definition of smooth manifolds with corners that are equipped with collars, as well as precise statements on how to address the difficulties in gluing these kinds of manifolds, we refer to [Mor06, 4.3]; here, we confine ourselves to the following short definition:

Definition 1.6. $\mathfrak{Cob}(1,2,3)$ *is the following symmetric monoidal bicategory:*

- *Objects are compact, closed, oriented 1-manifolds S.*
- *1-Morphisms are 2-dimensional, compact, oriented collared cobordisms $S \times I \hookrightarrow \Sigma \hookleftarrow S' \times I$.*
- *2-Morphisms are generated by diffeomorphisms of cobordisms fixing the collar and 3-dimensional collared, oriented cobordisms with corners M, up to diffeomorphisms preserving the orientation and boundary.*
- *Composition is by gluing along collars.*
- *The monoidal structure is given by disjoint union with the empty set $\emptyset$ as the monoidal unit.*

Remark 1.7. *The 1-morphisms are defined as collared surfaces, since in the case of extended cobordism categories, we consider surfaces rather than diffeomorphism classes of surfaces. A choice of collar is always possible, but not unique. The collars ensure that the glued surface has a well-defined smooth structure. Different choices for the collars yield equivalent 1-morphisms in* $\mathfrak{Cob}(1,2,3)$.

Obviously, extended cobordism categories can be defined in dimensions different from three as well. The definition of cobordism categories of arbitrary dimension can for example be found in [Lur09, Example 1.2.11]. We are now ready to give the definition of an extended TFT which goes essentially back to Lawrence [Law93]:

Definition 1.8. *An extended* 3*d TFT is a symmetric monoidal 2-functor*

$$Z: \quad \mathfrak{Cob}(1,2,3) \to 2\mathrm{Vect}_{\mathrm{K}} \ .$$

We pause to explain in which sense extended TFTs extend the TFTs defined in definition 1.1. To this end, we note that the monoidal 2-functor Z has to send the monoidal unit in $\mathfrak{Cob}(1,2,3)$ to the monoidal unit in $2\mathrm{Vect}_{\mathrm{K}}$. The monoidal unit in $\mathfrak{Cob}(1,2,3)$ is the empty set $\emptyset$, and the unit in $2\mathrm{Vect}_{\mathrm{K}}$ is the category $\mathrm{Vect}_{\mathrm{K}}$ of vector spaces over K. The functor Z restricts to a functor $Z|_{\emptyset}$ from the endomorphisms of $\emptyset$ in $\mathfrak{Cob}(1,2,3)$ to the endomorphisms of $\mathrm{Vect}_{\mathrm{K}}$ in $2\mathrm{Vect}_{\mathrm{K}}$. The monoidal equivalence $\mathrm{End}_{\mathfrak{Cob}(1,2,3)}(\emptyset) \cong \mathfrak{Cob}(2,3)$ follows directly from the definition of the two categories. Using the fact that the morphisms in $2\mathrm{Vect}_{\mathrm{K}}$ are additive (which follows from K-linearity of functors in the definition of 2-vector spaces), it is also easy to see that the equivalence of monoidal categories $\mathrm{End}_{2\mathrm{Vect}_{\mathrm{K}}}(\mathrm{Vect}_{\mathrm{K}}) \cong \mathrm{Vect}_{\mathrm{K}}$ holds: An additive functor $F : \mathrm{Vect}_{\mathrm{K}} \to \mathrm{Vect}_{\mathrm{K}}$ is determined by what it assigns to the one-dimensional vector space K. So let $F, G : \mathrm{Vect}_{\mathrm{K}} \to \mathrm{Vect}_{\mathrm{K}}$ be two K-linear functors and set $V_F = F(\mathrm{K})$, $V_G = G(\mathrm{K})$. A linear map from K to K is given by a scalar $\lambda \in \mathrm{K}$ and this gets sent to $\lambda \mathrm{id}_{V_F}$ resp. $\lambda \mathrm{id}_{V_G}$ by the K-linear functor F resp. G. Since all linear maps commute with multiplication by a scalar λ, a natural transformation $\eta : F \to G$ is a K-linear map from V_F to V_G. Hence we have deduced:

Lemma 1.9. *Let Z be an extended* 3*d TFT. Then $Z|_{\emptyset}$ is a* 3*d TFT in the sense of definition 1.1.*

At this point, the question arises whether a given (non-extended) $3d$ TFT can be extended. In general, there is no reason for this to be true. For Dijkgraaf-Witten theory, however, such an extension can be constructed based on ideas which we will describe in Chapter III. A very conceptual presentation of this construction based on important ideas of [Fre95] and [FQ93] can be found in [Mor10].

If we unwrap the Definition 1.8, we find that in particular, an extended $3d$ TFT Z assigns to the circle $\mathbb{S}^1$ a 2-vector space $Z(\mathbb{S}^1)$. From the geometric properties of the category $\mathfrak{Cob}(1,2,3)$ one can deduce much more structure on the 2-vector space $Z(\mathbb{S}^1)$ such as a tensor product and a braiding. In fact, for the example of a TFT Z we will consider in Chapter III, the category $Z(\mathbb{S}^1)$ can be shown to be a modular category, which is object of the next definitions.

Definition 1.10. *1. A* strict ribbon category *is a strict tensor category $\mathcal{C}$ with*

- duality*, i.e. for every object $X \in \mathcal{C}$ there exists an object $X^* \in \mathcal{C}$ together with morphisms*

$$b_X : \mathbf{1} \to X \otimes X^* \quad \textit{and} \quad d_X : X^* \otimes X \to \mathbf{1}$$

such that

$$(\mathrm{id}_X \otimes d_X)(b_X \otimes \mathrm{id}_X) = \mathrm{id}_X \quad \textit{and} \quad (d_X \otimes \mathrm{id}_{X^*})(\mathrm{id}_{X^*} \otimes b_X) = \mathrm{id}_{X^*}$$

- *a* braiding*: Let $\otimes^{op} : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ denote the opposite tensor product which switches the order of the factors $X \otimes^{op} Y = Y \otimes X$. Then a braiding is a natural isomorphism $c : \otimes \to \otimes^{op}$, that satisfies the Hexagon Axiom, that is, if we denote by α the associators in $\mathcal{C}$, for all objects $X, Y, Z \in \mathcal{C}$ the diagrams*

$$c_{X,Y\otimes Z} = (\mathrm{id}_X \otimes c_{Y,Z})(c_{X,Y} \otimes \mathrm{id}_Z)$$
$$c_{X\otimes Y,Z} = (c_{X,Y} \otimes \mathrm{id}_Z)(\mathrm{id}_X \otimes c_{Y,Z})$$

are required to commute.

- *a* twist*, i.e a natural isomorphism $\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ that satisfies for all objects $X, Y \in \mathcal{C}$ the relations*

$$\theta_{X\otimes Y} = (\theta_X \otimes \theta_Y)c_{Y,X}c_{X,Y}$$
$$\theta_{X^*} = (\theta_X)^*$$

2. Let $\mathcal{C}$ and $\mathcal{C}'$ be two ribbon categories with braidings c and c' and twists θ and θ'. A ribbon functor from $\mathcal{C}$ to $\mathcal{C}'$ is a tensor functor (F, F_2, F_0) that is braided, i.e the diagram

$$\begin{tikzcd}
F(X \otimes Y) \arrow[r, "F_2(X,Y)"] \arrow[d, "F(c_{X,Y})"] & F(X) \otimes F(Y) \arrow[d, "c'_{FX,FY}"] \\
F(Y \otimes X) \arrow[r, "F_2(Y,X)"] & F(Y) \otimes F(X)
\end{tikzcd}$$

commutes and that preserves the twist, i.e $F(\theta) = \theta'$.

Definition 1.11. *1. Let* K *be an algebraically closed field of characteristic zero. A* premodular tensor category *over* K *is a* K*-linear, abelian, finitely semisimple category $\mathcal{C}$ which has the structure of a ribbon category such that the tensor product is linear in each variable and the tensor unit is absolutely simple, i.e.* $\mathrm{End}(\mathbf{1}) = \mathrm{K}$.

2. Denote by $\Lambda_{\mathcal{C}}$ a set of representatives for the isomorphism classes of simple objects. The braiding on $\mathcal{C}$ allows to define the S-matrix *with entries in the field* K

$$s_{X,Y} := tr(R_{Y,X} \circ R_{X,Y}) \tag{I.1}$$

where $X, Y \in \Lambda_{\mathcal{C}}$. A premodular category is called modular*, if the S-matrix is invertible.*

2 Equivariant Modular Categories

Very much like ordinary modularity, G-modularity is a completeness requirement for the relevant tensor category that is suggested by principles of field theory. Indeed, it ensures that one can construct a G-equivariant topological field theory, see [Tur10a]. The definition of G-modularity is rather involved and we will not state it here but refer to [Kir04, Definition 10.1]. In [Kir04] it is also shown, that G-modularity of a category $\mathcal{C}$ is equivalent to ordinary modularity of its orbifold category. We will use this as the definition of G-modularity. In the second part of this section we will therefore introduce the notion of the orbifold category of an equivariant category. Thereafter we will give the definition of G-modularity (Definition 2.22). We are most interested in categories that appear as module categories over Hopf algebras and will hence give a corresponding G-equivariant structure on a Hopf algebra A, such that the representation category A-mod inherits a G-equivariant structure.

2.1 Equivariant Ribbon Categories and Equivariant Ribbon Algebras

We next introduce equivariant categories.

Definition 2.1. *Let G be a finite group and $\mathcal{C}$ a category.*

1. *A* categorical action *of the group G on the category $\mathcal{C}$ consists of the following data:*

   – *A functor $\phi_g : \mathcal{C} \to \mathcal{C}$ for every group element $g \in G$.*

– *A natural isomorphism* $\alpha_{g,h} : \phi_g \circ \phi_h \xrightarrow{\sim} \phi_{gh}$, *called* compositors, *for every pair of group elements* $g, h \in G$

such that the coherence conditions

$$\alpha_{gh,k} \circ \alpha_{g,h} = \alpha_{g,hk} \circ \phi_g(\alpha_{h,k}) \quad \text{and} \quad \phi_1 = \mathrm{id} \tag{I.2}$$

hold.

We use the notation ${}^gX := \phi_g(X)$ *for the image of an object* $X \in \mathcal{C}$ *under the functor* ϕ_g.

2. *If* $\mathcal{C}$ *is a monoidal category, we only consider actions by monoidal functors* ϕ_g *and require the natural transformations* $\alpha_{g,h}$ *to be monoidal natural transformations. In particular, for each group element* $g \in G$*, we have the additional datum of a natural isomorphism*

$$\gamma_g(X, Y) : {}^gX \otimes {}^gY \xrightarrow{\sim} {}^g(X \otimes Y)$$

for each pair of objects X, Y *of* $\mathcal{C}$ *such that the following diagrams commute:*

$$\begin{tikzcd}
{}^g({}^h(X)) \otimes {}^g({}^h(Y)) \arrow[rr, "{}^g\gamma_h(X,Y)\circ\gamma_g({}^hX,{}^kY)"] \arrow[d, "\alpha_{g,h}(X)\otimes\alpha_{g,h}(Y)"'] & & {}^g({}^h(X \otimes Y)) \arrow[d, "\alpha_{g,h}(X\otimes Y)"] \\
{}^{gh}X \otimes {}^{gh}Y \arrow[rr, "\gamma_{gh}(X,Y)"'] & & {}^{gh}(X \otimes Y)
\end{tikzcd}$$

(The data of a monoidal functor includes an isomorphism $\phi_g(1) \to 1$ *in principal, but in the sequel, the isomorphism will always be the identity and therefore we will suppress it in our discussion.)*

3. *A* G-equivariant *category* $\mathcal{C}$ *is a category with a grading* $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ *by the group* G *and a categorical action* $(\phi_G, \alpha_{g,h})$ *of* G*, subject to the compatibility requirement*

$$\phi_g \mathcal{C}_h \subset \mathcal{C}_{ghg^{-1}}.$$

4. *A* G-equivariant tensor category *is a* G*-equivariant monoidal category* $\mathcal{C}$*, subject to the compatibility requirement that the tensor product of two homogeneous elements* $X \in \mathcal{C}_g, Y \in \mathcal{C}_h$ *is again homogeneous,* $X \otimes Y \in \mathcal{C}_{gh}$.

Remark 2.2. *We remark that the condition* $\phi_1 = \mathrm{id}$ *in* (I.2) *should in general be replaced by an extra datum, an isomorphism* $\eta : \mathrm{id} \xrightarrow{\sim} \phi_1$ *and two coherence conditions which involve the compositors* $\alpha_{g,h}$. *The diagrams can be derived as follows: For any category* $\mathcal{C}$, *consider the category* $\mathrm{AUT}(\mathcal{C})$ *whose objects are automorphisms of* $\mathcal{C}$ *and whose morphisms are natural isomorphisms. The composition of functors and natural transformations endow* $\mathrm{AUT}(\mathcal{C})$ *with the natural structure of a strict tensor category. A categorical action of a finite group* G *on a category* $\mathcal{C}$ *then amounts to a tensor functor* $\phi : G \to \mathrm{AUT}(\mathcal{C})$, *where* G *is seen as a tensor category with only identity morphisms. The condition* $\phi_1 = \mathrm{id}$ *holds in the categories we are interested in, therefore we restrict the definition to the equality* $\phi_1 = \mathrm{id}$.

Similarly, for a monoidal category $\mathcal{C}$ *we consider the category* $\mathrm{AUT}mon(\mathcal{C})$ *whose objects are monoidal automorphisms of* $\mathcal{C}$ *and whose morphisms are monoidal natural automorphisms. The categorical actions we consider for monoidal categories are then tensor functors* $\phi : G \to \mathrm{AUT}mon(\mathcal{C})$. *For more details on this description of the* G*-action we refer to [Tur10a, Appendix 5].*

Definition 2.3. *A* braiding *on a* G*-equivariant tensor category* $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ *is a family of isomorphisms for every pair of objects* $X \in \mathcal{C}_g, Y \in \mathcal{C}$

$$c_{X,Y} : X \otimes Y \to {}^{g}Y \otimes X$$

which are natural in X *and* Y.

Moreover, a braiding is required to satisfy an analogue of the hexagon axioms, i.e. if we denote by α *the associator of the tensor category* $\mathcal{C}$, *we require for* $g, h \in G$, $X \in \mathcal{C}_g, Y \in \mathcal{C}_h$ *and* $Z \in \mathcal{C}$ *the following diagrams to commute:*

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "c_{X\otimes Y,Z}"] \arrow[dd, "\alpha"'] & {}^{gh}Z \otimes (X \otimes Y) \arrow[r, "\alpha^{-1}"] & ({}^{gh}Z \otimes X) \otimes Y \arrow[dr, "(\alpha_{g,h}\otimes \mathrm{id})\otimes \mathrm{id}"] & \\
 & & & ({}^{g}({}^{h}Z) \otimes X) \otimes Y \\
X \otimes (Y \otimes Z) \arrow[r, "\mathrm{id}\otimes c_{Y,Z}"] & X \otimes ({}^{h}Z \otimes Y) \arrow[r, "\alpha^{-1}"] & (X \otimes {}^{h}Z) \otimes Y \arrow[ur, "c_{X,Z}\otimes \mathrm{id}"'] &
\end{tikzcd} \tag{I.3}$$

$$\begin{tikzcd}
X \otimes (Y \otimes Z) \arrow[r, "c_{X,Y\otimes Z}"] \arrow[dd, "\alpha^{-1}"'] & {}^g(Y \otimes Z) \otimes X \arrow[r, "\gamma_g^{-1}\otimes \mathrm{id}"] & ({}^gY \otimes {}^gZ) \otimes X \arrow[dr, "\alpha"] & \\
 & & & {}^gY \otimes ({}^gZ \otimes X) \\
(X \otimes Y) \otimes Z \arrow[r, "c_{X,Y}\otimes \mathrm{id}"] & ({}^gY \otimes X) \otimes Z \arrow[r, "\alpha"] & {}^gY \otimes (X \otimes Z) \arrow[ur, "\mathrm{id}\otimes c_{X,Z}"'] &
\end{tikzcd} \tag{I.4}$$

and to be preserved under the action of G, i.e. the following diagram commutes for all objects X, Y with $X \in \mathcal{C}_h$ and $g, h \in G$

$$\begin{tikzcd}
{}^g(X \otimes Y) \arrow[r, "{}^g(c_{X,Y})"] \arrow[d, "\gamma_g"'] & {}^g({}^hY \otimes X) \arrow[r, "\gamma_g"] & {}^g({}^hY) \otimes {}^gX \arrow[d, "\alpha_{gh}(Y)\otimes \mathrm{id}"] \\
{}^gX \otimes {}^gY \arrow[r, "c_{{}^gX,{}^gY}"'] & {}^{ghg^{-1}}({}^gY) \otimes {}^gX \arrow[r, "\alpha_{ghg^{-1},g}(Y)\otimes \mathrm{id}"'] & {}^{gh}Y \otimes {}^gX
\end{tikzcd} \tag{I.5}$$

Remark 2.4. 1. *Note that a braided G-equivariant category is not, in general, a braided category. Its* neutral component $\mathcal{C}_1$ *with* $1 \in G$ *the neutral element, is a braided tensor category.*

2. *By replacing the underlying category by an equivalent category, one can replace a weak action by a strict action, compare [Tur10a, Appendix 5]. We still gave the definition of a weak action of a group on a category, since In our case, weak actions actually lead to simpler algebraic structures.*

Definition 2.5. 1. *A G-*equivariant ribbon category *is a G-braided category with dualities and a family of isomorphisms $\theta_X : X \to {}^gX$ for all $g \in G, X \in \mathcal{C}_g$, such that θ is compatible with duality, braiding and the action of G*

i.e the following diagrams are required to commute for $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$:

$$\begin{tikzcd}
{}^gX^\vee \arrow[r, "(\theta_X)^\vee"] \arrow[d, "\theta_{{}^gX^\vee}"'] & X^\vee \\
{}^{g^{-1}}({}^gX^\vee) \arrow[ur, "\alpha_{g^{-1},g}"'] &
\end{tikzcd} \tag{I.6}$$

$$
\begin{tikzcd}
X \otimes Y \arrow[rrr, "\theta_{X\otimes Y}"] \arrow[d, "\theta_X \otimes \theta_Y"'] & & & {}^{gh}(X \otimes Y) \\
{}^{g}X \otimes {}^{h}Y \arrow[dr, "c_{{}^{g}X,{}^{h}Y}"'] & & & {}^{ghg^{-1}}({}^{g}X) \otimes {}^{gh}Y \arrow[u, "\gamma_{gh}\circ(\alpha_{ghi^{-1},i}\otimes \mathrm{id})"'] \\
& {}^{g}({}^{h}Y) \otimes {}^{g}X \arrow[r, "\alpha_{g,h}\otimes \mathrm{id}"'] & {}^{gh}Y \otimes {}^{g}X \arrow[ur, "c_{{}^{gh}Y,{}^{g}X}"'] &
\end{tikzcd}
\tag{I.7}
$$

$$
\begin{tikzcd}
{}^{h}X \arrow[r, "{}^{h}\theta_X"] \arrow[d, "\theta_{{}^{h}X}"'] & {}^{h}({}^{g}X) \arrow[d, "\alpha_{h,g}"] \\
{}^{hgh^{-1}}({}^{h}X) \arrow[r, "\alpha_{hgh^{-1},h}"'] & {}^{hg}X
\end{tikzcd}
\tag{I.8}
$$

2. *A* G-premodular category *is a* K-*linear, abelian, finitely semi-simple* G-*equivariant ribbon category such that the tensor product is a* K-*bilinear functor and the tensor unit is absolutely simple.*

Remark 2.6. *The following facts directly follow from the definition of a* G*-equivariant ribbon category:*

1. *The neutral component* $\mathcal{C}_1$ *is itself a ribbon category. In particular, it contains the tensor unit of the* G*-equivariant tensor category.*

2. *The dual object of an object* $X \in \mathcal{C}_g$ *is in the category* $\mathcal{C}_{g^{-1}}$.

We now turn to equivariant structures on algebras.

Definition 2.7. *Let* A *be an (associative, unital) algebra over a field* K. A weak G-action *on* A *consists of algebra automorphisms* $\varphi_g \in \mathrm{Aut}(A)$ *for every element* $g \in G$*, and invertible elements* $c_{g,h} \in A$ *for every pair of elements* $g, h \in G$*, such that for all* $g, h, k \in G$ *the following conditions hold:*

$$\varphi_g \circ \varphi_h = \mathrm{Inn}_{c_{g,h}} \circ \varphi_{gh} \qquad \varphi_g(c_{h,k}) \cdot c_{g,hk} = c_{g,h} \cdot c_{gh,k} \quad \textit{and} \quad c_{1,1} = 1 \tag{I.9}$$

Here Inn_x *with* x *an invertible element of* A *denotes the algebra automorphism* $a \mapsto xax^{-1}$*. A weak action of a group* G *is called strict, if* $c_{g,h} = 1$ *for all pairs* $g, h \in G$.

Remark 2.8. *A weak action on a* K*-algebra A can be seen as a categorical action in the sense of 2.1 on the category which has one object and the elements of A as endomorphisms.*

Remark 2.9. *Note that our notion of a weak action $(\varphi_g, c_{g,h})$ of a group G on an algebra A corresponds to a weak action in the sense of [BCM86] together with the normal cocycle*

$$\begin{aligned}\sigma : \mathrm{K}[G] \times \mathrm{K}[G] &\to A^\times\\ (g,h) &\mapsto c_{g,h}\end{aligned}$$

that fulfills the cocycle and the twisted module condition of [BCM86].

We now want to relate a weak action $(\varphi_g, c_{g,h})$ of a group G on an algebra A to a categorical action on the representation category A-mod. To this end, we define for each element $g \in G$ a functor on objects by

$$^g(M,\rho) := (M, \rho \circ (\varphi_{g^{-1}} \otimes \mathrm{id}_M)) \tag{I.10}$$

and on morphisms by $^g f = f$. For the natural isomorphisms, we define

$$\alpha_{g,h}(M,\rho) := \rho\big((c_{h^{-1},g^{-1}})^{-1} \otimes \mathrm{id}_M\big). \tag{I.11}$$

The inversions in the above formulas make sure that the action on the level of categories really becomes a left action.

Lemma 2.10. *Given a weak action $(\varphi_g, c_{g,h})$ of G on a* K*-algebra A, these data define a categorical action on the category A-*mod *by the formulas I.10 and I.11.*

Proof. It is sufficient to show that the maps $\alpha_{g,h}$ defined in (I.11) are compositors in the sense of Definition 2.1. Let V be an A-module. We first show that $\alpha_{g,h}$ is a morphism $^g(^hV) \to {}^{gh}V$ in A-mod: Let $V \in A$-mod, then for every $v \in V, a \in A$, we have:

$$\begin{aligned}\alpha_{g,h}(V) \circ \rho_{^g(^hV)}(a \otimes v) &= (c_{h^{-1},g^{-1}})^{-1}\varphi_{h^{-1}} \circ \varphi_{g^{-1}}(a).v\\ &= \varphi_{(gh)^{-1}}(a)(c_{h^{-1},g^{-1}})^{-1}.v\\ &= \rho_{(^{gh}V)} \circ (\mathrm{id}_A \otimes \alpha_{g,h}(V))(a \otimes v),\end{aligned}$$

where we used the abbreviation $a.v := \rho(a \otimes v)$. The naturality of $\alpha_{g,h}$ follows from the fact that morphisms in A-mod commute with the action of A. The validity of the coherence condition (I.2) for the $\alpha_{g,h}$ follows from the second equation in (I.9), since the right side of (I.2) evaluated on an element $v \in V$ reads

$$\alpha_{g,hk} \circ \phi_g(\alpha_{h,k})(v) = (c_{(hk)^{-1},g^{-1}})^{-1}(c_{k^{-1},h^{-1}})^{-1}.v$$

and the left one is:

$$\alpha_{gh,k} \circ \alpha_{g,h}(v) = \left(c_{k^{-1},(gh)^{-1}}\right)^{-1} \varphi_{k^{-1}}\left(c_{h^{-1},g^{-1}}\right)^{-1}.v$$

□

It is known that the category of representations A-mod of a Hopf algebra A can be endowed with the structure of a tensor category with duality. In order to provide a weak action by tensor functors and tensor transformations on A-mod, the G-action on A needs to fulfill additional conditions, that are subject of the next definition.

Definition 2.11. *A weak G-action on a Hopf algebra A is a weak G-action $((\varphi_g)_{g\in G}, (c_{g,h})_{g,h\in G})$ on the underlying algebra which in addition satisfies the following properties:*

- *G acts by automorphisms of Hopf algebras.*
- *The elements $(c_{g,h})_{g,h\in G}$ are group-like, i.e $\Delta(c_{g,h}) = c_{g,h} \otimes c_{g,h}$.*

Lemma 2.12. *Given a weak action of G on a Hopf algebra A, the induced action on the tensor category* A-mod *of A-modules is by strict tensor functors and tensor transformations.*

Proof. It is straightforward to verify that Hopf algebra automorphisms induce strict tensor functors on the category of modules and that action by group-like elements induce tensor natural transformations. □

We next turn to an algebraic structure that yields G-equivariant tensor categories.

Definition 2.13. *A G-*Hopf algebra *over* K *is a Hopf algebra A with a weak G-action $((\varphi_g)_{g\in G}, (c_{g,h})_{g,h\in G})$ and a G-grading $A = \bigoplus_{g\in G} A_g$ such that:*

- *The algebra structure of A restricts to the structure of an associative algebra on each homogeneous component so that A is the direct sum of the components A_g as an algebra.*
- *The action of G is compatible with the grading, i.e. $\varphi_g(A_h) \subset A_{ghg^{-1}}$.*
- *The coproduct $\Delta : A \to A \otimes A$ respects the grading, i.e.*

$$\Delta(A_g) \subset \bigoplus_{p,q\in G, pq=g} A_p \otimes A_q \ .$$

Remark 2.14. *1. For the counit ϵ and the antipode S of a G-Hopf algebra, the compatibility relations with the grading $\epsilon(A_g) = 0$ for $g \neq 1$ and $S(A_g) \subset A_{g^{-1}}$ are immediate consequences of the definitions.*

2. For the unit 1 of a G-Hopf algebra, it follows directly from the definition that $1 = \sum_{h \in G} 1_h$, where each 1_h is a unit for the algebra A_h.

3. The restrictions of the structure maps endow the homogeneous component A_1 of A with the structure of a Hopf algebra with a weak G-action.

4. G-Hopf algebras with strict G-action have been considered under the name "G-crossed Hopf coalgebra" in [Tur10a, Chapter VII.1.2].

The category A-mod of finite-dimensional modules over a G-Hopf algebra inherits a natural duality from the duality of the underlying category of $\mathbb{K}$-vector spaces. From Lemma 2.12 we know that the weak action described in Lemma 2.10 is even a monoidal action, since G acts by Hopf algebra morphisms. Furthermore A-mod can be endowed with a grading by taking $(A\text{-mod})_g = A_g\text{-mod}$ as the g-homogeneous component. From the properties of a G-Hopf algebra one can finally deduce that the tensor product, duality and grading are compatible with the G-action. We have thus arrived at the following statement:

Lemma 2.15. *The category of representations of a G-Hopf algebra has a natural structure of a $\mathbb{K}$-linear, abelian G-equivariant tensor category with compatible duality.*

Proof. We show that the grading and the action on A-mod are compatible. Let $V \in A_h$-mod, then the h-component 1_h of the unit in A acts as the identity on V. We have to check, that ${}^gV \in A_{ghg^{-1}}$-mod, i.e that $1_{ghg^{-1}}$ acts as the identity on ${}^gV$. For $v \in {}^gV$, we have:

$$\rho_g(1_{ghg^{-1}} \otimes v) = \varphi_{g^{-1}}(1_{ghg^{-1}}).v = 1_h.v = v$$

This shows ${}^gV \in A_{ghg^{-1}}$-mod. □

The representation category of an ordinary quasi-triangular Hopf algebra is a braided tensor category. If the Hopf algebra has, moreover, a twist element, its representation category is even a ribbon category. We now present G-equivariant generalizations of these structures.

Definition 2.16. *Let A be a G-Hopf algebra.*

1. *A G-equivariant R-matrix is an invertible element $R = R_{(1)} \otimes R_{(2)} \in A \otimes A$ such that for $V \in (A\text{-mod})_g$, $W \in A\text{-mod}$, the map*

$$\begin{aligned} c_{V,W} : V \otimes W &\to {}^gW \otimes V \\ v \otimes w &\mapsto R_{(2)}.w \otimes R_{(1)}.v \end{aligned}$$

is a G-braiding on the category A-mod according to definition 2.3.

2. *A G-twist is an invertible element $\theta \in A$ such that for every object $V \in (A\text{-mod})_g$ the induced map*

$$\begin{aligned} \theta_V : &V \to^i V \\ &v \mapsto \theta^{-1}.v \end{aligned}$$

is a G-twist on A-mod as defined in 2.5.

*If A has an R-matrix and a twist, we call it a G-*ribbon-algebra.

Remark 2.17.
- *The structure of a G-ribbon algebra is not, in general, the structure of a ribbon Hopf algebra.*
- *The component A_1 with the obvious restrictions of R and θ is a ribbon algebra.*
- *The conditions that the category A-mod is braided resp. ribbon can be translated into algebraic conditions on the elements R and θ. Since we are mainly interested in the categorical structure we refrain from giving the definition in terms of algebraic conditions on the elements R and θ.*

Now we have introduced all the algebraic structure we need to state the following proposition.

Proposition 2.18. *The representation category of a G-ribbon algebra is a G-ribbon category.*

Remark 2.19. *In [Tur10a], Hopf algebras and ribbon Hopf algebras with strict G-action have been considered under the name ribbon Hopf G-coalgebras. In Chapter III we will give an illustrative example where the natural action is not strict.*

2.2 Orbifold Category and Orbifold Algebra

In this section we recall the notion of an orbifold category from [Kir04] and then give, in the manner of the last section, an orbifold construction also on the level of algebras, called the orbifold algebra. We show that for A-mod the category of representations of a G-equivariant algebra A, the orbifold category is equivalent to the category of representations of the orbifold algebra.

Definition 2.20. *Let $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$ be a G-equivariant category with G-action $(\phi_g, c_{g,h})$. The* orbifold category $\mathcal{C}^G$ *of $\mathcal{C}$ has:*

- *as objects pairs $(X, (\psi_g)_{g\in G})$ consisting of an object $X \in \mathcal{C}$ and a family of isomorphisms $\psi_g : {}^gX \to X$ with $g \in G$ such that*

$$\psi_g \circ {}^g\psi_h = \psi_{gh} \circ \alpha_{g,h}.$$

- *as morphisms $f : (X, \psi_g^X) \to (Y, \psi_h^Y)$ those morphisms $f : X \to Y$ in $\mathcal{C}$ for which*

$$\psi_g^Y \circ {}^g(f) = f \circ \psi_g^X$$

  holds for all $g \in G$.

In [Kir04], it has been shown that the orbifold category of a G-ribbon category is an ordinary, non-equivariant ribbon category:

Proposition 2.21. *1. Let $\mathcal{C}$ be a G-ribbon category. Then the orbifold category $\mathcal{C}^G$ is naturally endowed with the structure of a ribbon category by the following data:*

- *The tensor product of the objects $(X, (\psi_g^X))$ and $(Y, (\psi_g^Y))$ is defined as the object $(X \otimes Y, (\psi_g^X \otimes \psi_g^Y))$.*
- *The tensor unit for this tensor product is $\mathbf{1} = (\mathbf{1}, (\mathrm{id}))$*
- *The dual object of $(X, (\psi_g))$ is the object $(X^*, (\psi_g^*)^{-1})$, where X^* denotes the dual object in $\mathcal{C}$.*
- *The braiding of the two objects $(X, (\psi_g^X))$ and $(Y, (\psi_g^Y))$ with $X \in \mathcal{C}_g$ is given by the isomorphism $(\psi_g \otimes \mathrm{id}_X) \circ c_{X,Y}$, where $c_{X,Y} : X \otimes Y \to {}^gY \otimes X$ is the G-braiding in $\mathcal{C}$.*
- *The twist on an object $(X, (\psi_g))$ is $\psi_g \circ \theta$, where $\theta : X \to {}^gX$ is the twist in $\mathcal{C}$.*

2. If $\mathcal{C}$ is a G-premodular category, then the orbifold category $\mathcal{C}^G$ is even a premodular category.

We will now give a definition of G-modularity in terms of the orbifold category (see [Kir04] for more on G-modularity).

Definition 2.22. *A G-premodular category $\mathcal{C}$ is* G-modular *if its orbifold category $\mathcal{C}^G$ is modular.*

We have the following functors between a G-equivariant category $\mathcal{C}$ and its orbifold category $\mathcal{C}^G$:

$$\begin{aligned} F : \mathcal{C} &\to \mathcal{C}^G \\ X &\mapsto (\bigoplus_h {}^hX, \psi_g) \end{aligned}$$

where $\psi_g : \bigoplus_h {}^hX \to \bigoplus_h {}^g({}^hX) \to \bigoplus_h {}^{gh}X$ is the permutation of the summands.

$$\begin{aligned} F' : \mathcal{C}^G &\to \mathcal{C} \\ (X, (\psi_g)) &\mapsto X \end{aligned}$$

These functors are adjoint to each other:

$$\mathrm{Hom}_{\mathcal{C}}(F(X, (\psi_g)), Y) \cong \mathrm{Hom}_{\mathcal{C}^G}((X, (\psi_g)), F'(Y))$$

In [Kir04] the following proposition is proven:

Proposition 2.23. *Let $\mathcal{C}$ be a G-equivariant fusion category and let $A = F(\mathbf{1}) \in \mathcal{C}^G$. Then A has a natural structure of a commutative algebra in $\mathcal{C}^G$ with an action of G and the category of A-modules is naturally equivalent to the category $\mathcal{C}$.*

Moreover, the algebra $A = F(\mathbf{1})$ is isomorphic to $\mathrm{K}(G)$, the algebra of functions on G.
Thus, we have another natural functor from the category $\mathcal{C}^G$ to $\mathcal{C}$: induction on the algebra $\mathrm{K}(G)$:

$$\begin{aligned} \mathrm{Ind} : \mathcal{C}^G &\to \mathcal{C} \\ X &\mapsto (\mathrm{K}(G) \otimes X, m \otimes \mathrm{id}) \end{aligned}$$

In the case where the G-equivariant category turns up as the representation category of a G-equivariant Hopf algebra, it is even possible to describe the process of orbifolding on the level of (Hopf-)algebras. Given a G-equivariant algebra A, we introduce an orbifold algebra $\widehat{A}^G$ such that its representation category $\widehat{A}^G$-mod is isomorphic to the orbifold category of A-mod.

Definition 2.24. *Let A be an algebra with a weak G-action $(\varphi_g, c_{g,h})$. We endow the vector space $\widehat{A}^G := A \otimes \mathrm{K}[G]$ with a unital associative multiplication which is defined on two elements of the form $(a \otimes g)$, $(b \otimes h)$ with $a, b \in A$ and $g, h \in G$ by*

$$(a \otimes g)(b \otimes h) := (a\varphi_g(b)c_{g,h} \otimes gh) \ .$$

and unit

$$\mathbf{1} = (\mathbf{1}_A \otimes 1)$$

This algebra is called the orbifold algebra *$\widehat{A}^G$ of the G-equivariant algebra A with respect to the weak G-action.*

If A is even a G-Hopf algebra, it is possible to also endow the orbifold algebra with more structure. In order to define the coalgebra structure on the orbifold algebra, we use the standard coalgebra structure on the group algebra $\mathrm{K}[G]$ with coproduct $\Delta_G(g) = g \otimes g$ and counit $\epsilon_G(g) = 1$ on the canonical basis $(g)_{g \in G}$ and we define the coalgebra structure on $\widehat{A}^G$ as the tensor product of the two coalgebras. Explicitly, the coproduct and the counit take the following values on an element of the form $(a \otimes g)$, $(b \otimes h)$ with $a \in A$ and $g \in G$ The tensor product coalgebra on $A \otimes \mathrm{K}[G]$ has the coproduct and counit

$$\Delta(a \otimes g) = (\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_{\mathrm{K}[G]})(\Delta_A(a) \otimes g \otimes g), \quad \text{and} \quad \epsilon(a \otimes g) = \epsilon_A(a) \quad \text{(I.12)}$$

which is clearly coassociative and counital.
To show that this endows the orbifold algebra with the structure of a bialgebra, we first have to show that the coproduct Δ is a unital algebra morphism. This follows from the fact, that Δ_A is already an algebra morphism and that the action of G is by coalgebra morphisms. Next, we have to show that the counit ϵ is a unital algebra morphism as well. This follows from the fact that the action of G commutes with the counit and from the fact that the elements $c_{g,h}$ are group-like. The compatibility of the counit ϵ with the unit is obvious.
In a final step, one verifies that the endomorphism

$$S(a \otimes g) = (c_{g^{-1},g})^{-1}\varphi_{g^{-1}}(S_A(a)) \otimes g^{-1} \quad \text{(I.13)}$$

is an antipode. Altogether, one arrives at

Proposition 2.25. *If A is a G-Hopf-algebra, then the maps Δ, ϵ and S given in I.12 and I.13 equip the orbifold algebra $\widehat{A}^G$ with the structure of a Hopf algebra.*

Remark 2.26. *1. The algebra $\widehat{A}^G$ is not the fixed point subalgebra A^G of A; in general, the categories A^G-mod and $\widehat{A}^G$-mod are inequivalent.*

2. If the action of G on the algebra A is strict, then the algebra A is a module algebra over the Hopf algebra $\mathrm{K}[G]$ (i.e. an algebra in the tensor category $\mathrm{K}[G]$-mod). In that case the orbifold algebra is the smash product $A\#\mathrm{K}[G]$ (see [Mon93, Section 4] for the definitions).

The next proposition justifies the name "orbifold algebra" for $\widehat{A}^G$:

Proposition 2.27. *Let A be a G-Hopf algebra. Then there is an equivalence of tensor categories*

$$(A\text{-mod})^G \cong \widehat{A}^G\text{-mod} .$$

Proof. We first want to find an equivalence $F : (A\text{-mod})^G \to \widehat{A}^G\text{-mod}$

- An object of $(A\text{-mod})^G$ consists of a K-vector space M, an A-action $\rho : A \to \mathrm{End}(M)$ and a family of A-module morphisms $(\psi_g)_{g\in G}$. We define on the same K-vector space M the structure of an $\widehat{A}^G$ module by defining $\tilde{\rho}(a \otimes g) := \rho(a) \circ (\psi_{g^{-1}})^{-1}$.

  One next checks that, given two objects (M, ρ, ψ) and (M', ρ', ψ') in $(A\text{-mod})^G$, a K-linear map $f \in \mathrm{Hom}_{\mathrm{K}}(M, M')$ is in the subspace $\mathrm{Hom}_{(A\text{-mod})^J}(M, M')$ if and only if it is in the subspace $\mathrm{Hom}_{\widehat{A}^G\text{-mod}}(M, \tilde{\rho}), (M', \tilde{\rho}'))$.

  We can thus consider a K-linear functor

  $$F : (A\text{-mod})^G \to \widehat{A}^G\text{-mod} \tag{I.14}$$

  which maps on objects by $(M, \rho, \psi) \mapsto (M, \tilde{\rho})$ and on morphisms as the identity. This functor is clearly fully faithful.

  To show that the functor is also essentially surjective, we note that for any object $(M, \tilde{\rho})$ in $\widehat{A}^G$-mod, an object in $(A\text{-mod})^G$ can be obtained as follows: on the underlying vector space, we have the structure of an A-module by restriction, $\rho(a) := \tilde{\rho}(a \otimes 1_G)$. A family of equivariant morphisms is given by $\psi_g := (\tilde{\rho}(\mathbf{1} \otimes g^{-1}))^{-1}$. Clearly its image under F is isomorphic to $(M, \tilde{\rho})$. This shows that the functor F is an equivalence of categories, indeed even an isomorphism of categories.

- The functor F is also a strict tensor functor: consider two objects $F(M, \rho, \psi)$ and $F(M', \rho', \psi')$ in $(A\text{-mod})^G$. The functor F yields the following action of the orbifold Hopf algebra $\widehat{A}^G$ on the K-vector space $M \otimes_{\mathrm{K}} M'$:

  $$\tilde{\rho}_{M\otimes M'}(a \otimes g) = \rho \otimes \rho'(\Delta(a)) \circ ((\psi_{g^{-1}})^{-1} \otimes (\psi'_{g^{-1}})^{-1}) .$$

Since the coproduct on $\widehat{A}^G$ was just given by the tensor product of coproducts on A and $\mathrm{K}[G]$, this coincides with the tensor product of $F(M,\rho,\psi)$ and $F(M',\rho',\psi')$ in $\widehat{A}^G$-mod.

□

In a final step, we assume that the G-equivariant algebra A has the additional structure of a G-ribbon algebra. Then, by proposition 2.18, the category A-mod is a G-ribbon category and by proposition 2.21 the orbifold category $(A\text{-mod})^G$ is a ribbon category. The strict isomorphism (I.14) of tensor categories allows us to transport both the braiding and the ribbon structure to the representation category of the orbifold Hopf algebra $\widehat{A}^G$. General results [Kas95, Prop. 16.6.2] assert that this amounts to a natural structure of a ribbon algebra on $\widehat{A}^G$. In fact, we directly read off the R-matrix and the ribbon element. For example, the R-matrix $\hat{R}$ of $\hat{A}^G$ equals

$$\hat{R} = \hat{\tau}\; c_{\widehat{A}^G,\widehat{A}^G}(1_{\widehat{A}^G} \otimes 1_{\widehat{A}^G}) \in \widehat{A}^G \otimes \widehat{A}^G \;\;,$$

where the linear map $\hat{\tau}$ flips the two components of the tensor product $\widehat{A}^G \otimes \widehat{A}^G$. This expression can be explicitly evaluated, using the fact that $A\otimes \mathrm{K}[G]$ is an object in $(A\text{-mod})^G$ with A-module structure given by left action on the first component and that the morphisms ψ_g are given by left multiplication on the second component. We find for the R-matrix of $\widehat{A}^G$

$$\begin{aligned}\hat{R} &= \sum_{g,h\in G} (\mathrm{id}\otimes\psi_h)(\rho\otimes\rho)(R_{gh})((1_A\otimes 1_J)\otimes 1_A\otimes 1_G) \\ &= \sum_{g,h\in G} ((R_{g,h})_1\otimes 1_G)\otimes((R_{g,h})_2\otimes g)\end{aligned}$$

where R is the R-matrix of A. The twist element of $\widehat{A}^G$ can be computed similarly; one finds

$$\theta^{-1} = \sum_{g\in G}\psi_g\circ\rho(\theta_g)(1_{A_g}\otimes 1_G) = \sum_{g\in G}(1_A\otimes g^{-1})^{-1}(\theta_g\otimes 1_G)$$

We summarize our findings:

Corollary 2.28. *If A is a G-ribbon algebra, then the orbifold algebra $\widehat{A}^G$ inherits a natural structure of a ribbon algebra such that the equivalence of tensor categories in Proposition 2.27 is an equivalence of ribbon categories.*

Chapter II

Drinfel'd Double and Generalization

1 Drinfel'd Double of a Finite Group

In the following, let K be an algebraically closed field of characteristic zero and let G be a finite group. The *Drinfel'd double* $\mathcal{D}(G)$ of G is a finite dimensional Hopf algebra over K which is a special case of the quantum double construction [Dri87] with the Hopf algebra being the group algebra $\mathrm{K}[G]$ of G. The representation category of the Drinfel'd double $\mathcal{D}(G)$ provides a simple example of a modular tensor category.
The Hopf-algebra $\mathcal{D}(G)$ is defined as follows: As a vector space, $\mathcal{D}(G)$ is the tensor product $\mathrm{K}(G) \otimes \mathrm{K}[G]$ of the algebra of functions on G and the group algebra of G, i.e. $\mathcal{D}(G)$ has the canonical basis $(\delta_g \otimes h)_{g,h \in G}$. The algebra structure can be described as the smash product of the algebras $\mathrm{K}(G)$ and $\mathrm{K}[G]$ (see [Mon93]), an analogue of the semi-direct product for groups, whereas the coalgebra structure is just the tensor product of the two coalgebras. For two elements $g, h \in G$ let $\delta(g, h)$ take the value 1 if $g = h$ and 0 otherwise. In the canonical basis, we have:

- product:

$$(\delta_h \otimes g)(\delta_{h'} \otimes g') = \delta(h, gh'g^{-1})(\delta_g \otimes gg')$$

- unit:

$$\sum_{g \in G} \delta_g \otimes 1$$

- coproduct:

$$\Delta(\delta_h \otimes g) = \sum_{kl=h} (\delta_k \otimes g) \otimes (\delta_l \otimes g)$$

- counit:

$$\epsilon(\delta_h \otimes g) = \delta(h, 1)$$

It can easily be checked that this defines a bialgebra structure on $\mathrm{K}(G) \otimes \mathrm{K}[G]$ and that furthermore the linear map

$$S : (\delta_h \otimes g) \mapsto (\delta_{g^{-1}h^{-1}g} \otimes g^{-1})$$

is an antipode for this bialgebra so that $\mathcal{D}(G)$ is a Hopf algebra. Furthermore, the element

$$R := \sum_{g,h \in G} (\delta_g \otimes 1) \otimes (\delta_h \otimes g) \in \mathcal{D}(G) \otimes \mathcal{D}(G)$$

is a universal R-matrix, which fulfills the defining identities of a braided bialgebra. At last, the element

$$\theta := \sum_{g \in G} (\delta_g \otimes g^{-1}) \in \mathcal{D}(G)$$

is a ribbon-element in $\mathcal{D}(G)$, which gives $\mathcal{D}(G)$ the structure of a ribbon algebra (as defined in [Kas95, Def. XIV.6.1]).
The category $\mathcal{D}(G)$-mod is actually endowed with more structure than the one of a braided monoidal category. Since $\mathcal{D}(G)$ is a ribbon Hopf-algebra, the category of representations $\mathcal{D}(G)$-mod has also dualities and a compatible twist, i.e. has the structure of a ribbon category (see [Kas95, Proposition XIV.6.2] or [BK01, Def. 2.2.1] for the notion of a ribbon category). Moreover, the category $\mathcal{D}(G)$-mod is a 2-vector space over K in the sense of Definition 1.2 and thus, in particular, finitely semi-simple.
We will give a short description of the category $\mathcal{D}(G)$-mod:

- An object of $\mathcal{D}(G)$-mod is a vector space V with a G-grading

$$V = \bigoplus_{h \in G} V_h$$

and a group action by the group G, denoted by

$$\begin{aligned} \rho_g : V &\to V \\ v &\mapsto g.v \end{aligned}$$

subject to the compatibility condition

$$\rho_g V_h = V_{ghg^{-1}}$$

.

- The tensor product of two objects $(V = \bigoplus_{h\in G} V_h, \rho^V(g)_{g\in G})$ and $(W = \bigoplus_{h\in G} W_h, \rho^W(g)_{g\in G})$ is the vector space $V\otimes W$ with grading

  $$(V \otimes W)_h = \bigoplus_{kl=h} V_k \otimes V_l$$

  and the G-action is the tensor product $\rho^V(g) \otimes \rho^W(g)$ of the G-actions.

- The duality is inherited from the category of vector spaces, i.e. for an object $(V = \bigoplus_{h\in G} V_h, \rho^V(g)_{g\in G}$ the dual object is the dual vector space V^* with grading

  $$(V^*)_h = V^*_{h^{-1}}$$

  and the action of a group element $g \in G$ on a linear functional $\varphi \in V^*$ evaluated on a vector $v \in V$ is

  $$\rho_g\varphi\,(v) := \varphi(g^{-1}.v)$$

- The braiding is for $v \in V_h, w \in W$ given by

  $$v \otimes w \mapsto h.w \otimes v.$$

- The twist on a vector $v \in V_h$ is

  $$v \to h^{-1}.v$$

The ribbon structure of the category can directly be deduced from the ribbon structure of the algebra $\mathcal{D}(G)$ (see for example [Kas95] proof of Proposition XIV.6.2 for a reference).
Another way to look at the module category of $\mathcal{D}(G)$-mod is to consider it as the representation category of the action groupoid $G//G$, where the set of objects is G and G acts by conjugation on itself, i.e. the functor category $[G//G, \mathrm{Vect}]$. We can then exploit the character theory for action

groupoids from section 6.2 to calculate the S matrix. Denote by I the set of isomorphism classes of simple objects of $\mathcal{D}(G)$-mod $\cong [G//G, \mathrm{Vect}]$. For $i \in I$ let χ_i be the character of the representative of the simple object i (see Definition 6.4 of Chapter III). Then for $i, j \in I$ the (i, j)-th entry of the S-matrix is:

$$S_{i,j} = \frac{1}{|G|} \sum_{g \in G, m \in M} \chi_i(m, g) \chi_j(g, m).$$

Lemma 1.1. *The S-matrix of $\mathcal{D}(G)$*-mod *is invertible with $S^2 = \mathbf{1}$*

Proof. The proof is an application of the orthogonality relations (III.37) and (III.39) for the characters of the action groupoid $G//G$. □

Proposition 1.2. *The category $\mathcal{D}(G)$*-mod *is modular.*

2 Modular Categories from Finite Crossed Modules

We summarize the results from [Ban10] and [Mai09]. In [Ban10], Bantay introduced a category of representations for any crossed module and showed that this category is always premodular (cf Definition 1.11).
We start by stating the definition of a crossed module:

Definition 2.1. *A finite* crossed module $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mu, \partial)$ *consists of two finite groups $\mathcal{X}_1$ and $\mathcal{X}_2$, together with an action μ of $\mathcal{X}_1$ on $\mathcal{X}_2$ by group automorphisms, written as $\mu(m, g) = g.m$, and a group homomorphism, called the* boundary map, $\partial : \mathcal{X}_2 \to \mathcal{X}_1$ *that satisfies*

$$\partial(g.m) = g(\partial m)g^{-1} \quad \text{and} \quad (\partial n).m = nmn^{-1} \qquad \text{for all } m, n \in \mathcal{X}_2 \text{ and } g \in \mathcal{X}_1.$$

The representation category of a crossed module $\mathcal{X}$ is isomorphic to the module category of a Hopf algebra $\mathcal{B}(\mathcal{X})$, which is $\mathrm{K}(\mathcal{X}_1) \otimes \mathrm{K}(\mathcal{X}_2)$ as a vector space and has the following structure on the canonical basis $(\delta_m \otimes g)_{m \in \mathcal{X}_2, g \in \mathcal{X}_1}$

- product:

$$(\delta_m \otimes g)(\delta_{m'} \otimes g') = \delta(m, g.m)(\delta_m \otimes gg')$$

- unit:

$$\sum_{m \in \mathcal{X}_2} \delta_m \otimes 1$$

- coproduct:

$$\Delta(\delta_m \otimes g) = \sum_{kl=m} (\delta_k \otimes g) \otimes (\delta_l \otimes g)$$

- counit:

$$\epsilon(\delta_m \otimes g) = \delta(m, 1)$$

- antipode

$$S : (\delta_m \otimes g) \mapsto (\delta_{g^{-1}.m^{-1}} \otimes g^{-1})$$

Using the boundary map ∂ of $\mathcal{X}$, it is even possible to endow the Hopf algebra $\mathcal{B}(\mathcal{X})$ with a universal R-matrix and a ribbon-element, giving it the structure of a ribbon-algebra:
The R-matrix is

$$R := \sum_{m,n \in \mathcal{X}_2} (\delta_m \otimes 1) \otimes (\delta_n \otimes \partial m) \in \mathcal{B}(\mathcal{X}) \otimes \mathcal{B}(\mathcal{X})$$

and the twist is

$$\theta := \sum_{m \in \mathcal{X}_2} (\delta_m \otimes \partial m^{-1}) \in \mathcal{B}(\mathcal{X}).$$

Comparison with the structure maps of the Drinfel'd double shows that in the special case of a crossed module $\mathcal{X} = (G, G, \mathrm{ad}_G, \mathrm{id})$, where both of the groups are equal to a finite group G, action of the group G on itself is by conjugation $\mathrm{ad}_G(g, h) = ghg^{-1}$ and the boundary map is the identity, the algebra $\mathcal{B}(\mathcal{X})$ defined above is equal to the Drinfel'd double $\mathcal{D}(G)$ of the group G. The category $\mathcal{M}(\mathcal{X}) = \mathcal{B}(\mathcal{X})$-mod associated to a crossed module $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2, \mu, \delta)$ is known to be modular, if and only if the boundary map ∂ is an isomorphism [Nai10, Proposition 5.6]. In this case, the category $\mathcal{M}(\mathcal{X})$ is equivalent to the representation category of the Drinfel'd double of a finite group.
For a detailed discussion of the premodular tensor category $\mathcal{M}(\mathcal{X})$, including its character theory, we refer to [Ban10].

Bruguières [Bru00] (see also [Müg00]) has introduced the notion of modularization that associates to any premodular tensor category (obeying certain conditions) a modular tensor category. The modularization includes in particular a structure preserving functor that is dominant, which is a weaker form of essentially surjective:

Definition 2.2. *A functor $F : \mathcal{C} \to \mathcal{C}'$ is called dominant, if for every $Y \in \mathcal{C}'$ there exists an object $X \in \mathcal{C}$ and morphisms $\iota : Y \to F(X)$ and $\pi : F(X) \to Y$ such that $\pi \circ \iota = \mathrm{id}_Y$.*

The definition of a ribbon functor was given in 1.10 2.; now we are ready to recap the definition of a modularization:

Definition 2.3. *A modularization of a premodular category $\mathcal{C}$ is a dominant ribbon functor $F : \mathcal{C} \to \mathcal{C}'$ with $\mathcal{C}'$ a modular tensor category. A premodular category is called modularizable, if it admits a modularization.*

The modular category associated to the modularization is unique up to equivalence of braided tensor categories. The representation category $\mathcal{M}(\mathcal{X})$ of a finite crossed module $\mathcal{X}$ obeys the conditions [Ban10] for being modularizable. We state this in the following proposition:

Proposition 2.4. *The category $\mathcal{M}(\mathcal{X})$ of representations of a crossed module $\mathcal{X}$ is modularizable.*

Now the question arises whether modularization of the premodular category $\mathcal{M}(\mathcal{X})$ provides a source of new modular categories. A first main result of [Mai09] was a negative answer to this question, which is the following proposition:

Proposition 2.5. *The modularization yields a modular tensor category equivalent to the category for the Drinfel'd double of $\mathcal{X}_2/\ker\partial \cong \mathrm{Im}\,\partial$.*

Bruguières gave an explicit modularization procedure. We will give a short summary here. A certain subcategory of the premodular category, the symmetric center, is crucial in the construction of the modularization. It is subject of the next definition.

Definition 2.6. *Let $\mathcal{C}$ be a premodular category with braiding $c : \otimes \to \otimes^{op}$. The* symmetric center *of $\mathcal{C}$ is the full subcategory $\mathcal{Z}(\mathcal{C}) \subset \mathcal{C}$ of object $X \in \mathcal{C}$ for which $c_{X,Y} = c^{-1}_{Y,X}$ holds for all $Y \in \mathcal{C}$.*

The obstruction for a premodular category $\mathcal{C}$ to be modular is reflected by the fact that its symmetric center $\mathcal{Z}(\mathcal{C})$ is not trivial. However, if $\mathcal{C}$ is modularizable, the category $\mathcal{Z}(\mathcal{C})$ is a Tannakian category, and thus equivalent to the representation category of a finite group. In [Mai09] it was shown that the group corresponding to the Tannakian subcategory $\mathcal{Z}(\mathcal{C})$ is the semi-direct product

$$G(\mathcal{X}) := (\ker\partial)^* \rtimes_{\hat{\mu}} (\mathrm{coker}\,\partial) \tag{II.1}$$

Here $(\ker \partial)^*$ is the group of characters of the finite abelian group $\ker \partial$. The action μ of $\mathcal{X}_1$ on $\mathcal{X}_2$ restricts to an action of $\mathcal{X}_1$ on $\ker \partial \subset \mathcal{X}_2$ which factorizes to an action of $\operatorname{coker} \partial$ on $\ker \partial$. Furthermore we define on $(\ker \partial)^*$ an action $\hat{\mu}$ of $\operatorname{coker} \partial$ on the elements $g \in \operatorname{coker} \partial, \chi \in (\ker \partial)^*, x \in \ker \partial$ by $\hat{\mu}(g, \chi)(x) := \chi(g^{-1}x)$.
The algebra of functions on $G(\mathcal{X})$ then provides a commutative special symmetric Frobenius algebra in the premodular tensor category $\mathcal{M}(\mathcal{X})$. The category of left modules over this algebra can be shown to be a modular tensor category and the modularization is the induction-functor on the algebra of functions on $G(\mathcal{X})$.

Chapter III

Equivariant Modular Categories via Dijkgraaf-Witten Theory

In this chapter we will give an explicit construction of equivariant modular categories from equivariant Dijkgraaf-Witten Theory. This can also be seen as a continuation of the last chapter, since the (equivariant) modular categories we discover can be seen as an equivariant extension of the premodular category $\mathcal{M}(\mathcal{X})$ and its modularization (for crossed modules $\mathcal{X}$ with injective boundary map), as illustrated in diagram III.1.

1 Introduction

This chapter has two seemingly different motivations and, correspondingly, can be read from two different points of view, a more algebraic and a more geometric one. Both in the introduction and the main body of the chapter, we try to separate these two points of view as much as possible, in the hope to keep this chapter accessible for readers with specific interests.

1.1 Algebraic Motivation: Equivariant Modular Categories

Among tensor categories, modular tensor categories are of particular interest for representation theory and mathematical physics. The representation categories of several algebraic structures give examples of semisimple modular tensor categories:

1. Left modules over connected factorizable ribbon weak Hopf algebras with Haar integral over an algebraically closed field [NTV03].
2. Local sectors of a finite μ-index net of von Neumann algebras on $\mathbb{R}$, if the net is strongly additive and split [KLM01].
3. Representations of selfdual C_2-cofinite vertex algebras with an additional finiteness condition on the homogeneous components and which have semisimple representation categories [Hua05].

Despite this list and the rather different fields in which modular tensor categories arise, it is fair to say that modular tensor categories are rare mathematical objects. Arguably, the simplest incarnation of the first algebraic structure in the list is the Drinfel'd double $\mathcal{D}(G)$ of a finite group G introduced in Section 1 of Chapter II. Bantay [Ban10] has suggested finite crossed modules as a more general source for modular tensor categories (see also Section 2 of Chapter I): In this chapter we will only deal with finite crossed modules with an injective boundary map, i.e. we consider a pair, consisting of a finite group H and a normal subgroup $G \triangleleft H$. Bantay constructs a ribbon category which is, in a natural way, a representation category of a ribbon Hopf algebra $\mathcal{B}(G \triangleleft H)$. Unfortunately, it turns out that, for a proper subgroup inclusion, the category $\mathcal{B}(G \triangleleft H)$-mod is only premodular and not modular.

Still, the category $\mathcal{B}(G \triangleleft H)$-mod is modularizable in the sense of Bruguières [Bru00], and the next candidate for new modular tensor categories is the modularization of $\mathcal{B}(G \triangleleft H)$-mod. However, it has been shown [MS11] that this modularization is equivalent to the representation category of the Drinfel'd double $\mathcal{D}(G)$.

The modularization procedure of Bruguières is based on the observation that the violation of modularity of a modularizable tensor category $\mathcal{C}$ is captured in terms of a canonical Tannakian subcategory of $\mathcal{C}$. For the category $\mathcal{B}(G \triangleleft H)$-mod, this subcategory can be realized as the representation category of the the quotient group $J := H/G$ [MS11] (cf. line II.1). The modularization functor

$$\mathcal{B}(G \triangleleft H)\text{-mod} \to \mathcal{D}(G)\text{-mod}$$

is induction along the commutative Frobenius algebra given by the regular representation of J. This has the important consequence that the modularized category $\mathcal{D}(G)$ is endowed with a J-action.

Experience with orbifold constructions, see [Kir04, Tur10a] for a categorical formulation, raises the question of whether the category $\mathcal{D}(G)$-mod with this

J-action can be seen in a natural way as the neutral sector of a J-modular tensor category.
We thus want to complete the following square of tensor categories

$$\begin{tikzcd} \mathcal{D}(G)\text{-mod} \arrow[loop above, "J"] \arrow[r, hook] \arrow[d, shift left, "\text{orbifold}"] & ??? \arrow[loop above, "J"] \arrow[d, shift left, "\text{orbifold}"] \\ \mathcal{B}(G \triangleleft H)\text{-mod} \arrow[r, hook] \arrow[u, shift left, "\text{modularization}"] & ??? \arrow[u, shift left] \end{tikzcd} \tag{III.1}$$

Here vertical arrows pointing upwards stand for induction functors along the commutative algebra given by the regular representation of J, while downwards pointing arrows indicate orbifolding. In the upper right corner, we wish to place a J-modular category, and in the lower right corner its J-orbifold which, on general grounds [Kir04], has to be a modular tensor category. Horizontal arrows indicate the inclusion of neutral sectors.
In general, such a completion need not exist. Even if it exists, there might be inequivalent choices of J-modular tensor categories of which a given modular tensor category with J-action is the neutral sector [ENO10].

1.2 Geometric Motivation: Equivariant Extended TFT

Topological field theory is a mathematical structure that has been inspired by physical theories [Wit89] and which has developed into an important tool in low-dimensional topology. Recently, these theories have received increased attention due to the advent of *extended* topological field theories [Lur09, SP09]. The present chapter focuses on three-dimensional topological field theory.
Dijkgraaf-Witten theories provide a class of extended topological field theories. They can be seen as discrete variants of Chern-Simons theories, which provide invariants of three-manifolds and play an important role in knot theory [Wit89]. Dijkgraaf-Witten theories have the advantage of being particularly tractable and admitting a very conceptual geometric construction.
A Dijkgraaf-Witten theory is based on a finite group G; in this case the 'field configurations' on a manifold M are given by G-bundles over M, denoted by $\mathcal{A}_G(M)$. Furthermore, one has to choose a suitable action functional $S : \mathcal{A}_G(M) \to \mathbb{C}$ (which we choose here in fact to be trivial) on field configurations; this allows to make the structure suggested by formal path integration rigorous and to obtain a topological field theory. A conceptually very clear way to carry this construction out rigorously is described in [FQ93] and [Mor10], see Section 2 of this chapter for a review.

Let us now assume that as a further input datum we have another finite group J which acts on G. In this situation, we get an action of J on the Dijkgraaf-Witten theory based on G. But it turns out that this topological field theory together with the J-action does not fully reflect the equivariance of the situation: it has been an important insight that the right notion is the one of equivariant topological field theories, which have been another point of recent interest [Kir04, Tur10a]. Roughly speaking, equivariant topological field theories require that all geometric objects (i.e. manifolds of different dimensions) have to be decorated by a J-cover (see Definitions 3.11 and 3.13 for details). Equivariant field theories also provide a conceptual setting for the orbifold construction, one of the standard tools for model building in conformal field theory and string theory.
Given the action of a finite group J on a finite group G, these considerations lead to the question of whether Dijkgraaf-Witten theory based on G can be enlarged to a J-equivariant topological field theory. Let us pose this question more in detail:

- What exactly is the right notion of an action of J on G that leads to interesting theories? To keep equivariant Dijkgraaf-Witten theory as explicit as the non-equivariant theory, one needs notions to keep control of this action as explicitly as possible.

- Ordinary Dijkgraaf-Witten theory is mainly determined by the choice of field configurations $\mathcal{A}_G(M)$ to be G-bundles. As mentioned before, for J-equivariant theories, we should replace manifolds by manifolds with J-covers. We thus need a geometric notion of a G-bundle that is 'twisted' by this J-cover in order to develop the theory parallel to the non-equivariant one.

Based on an answer to these two points, we wish to construct equivariant Dijkgraaf-Witten theory as explicitly as possible.

1.3 Summary of the Results

In this chapter both the algebraic and the geometric problem we have just described are solved. In fact, the two problems turn out to be closely related. We first solve the problem of explicitly constructing equivariant Dijkgraaf-Witten and then use our solution to construct the relevant modular categories that complete the square (III.1).
Despite this strong mathematical interrelation, we have taken some effort to write this chapter in such a way that it is accessible to readers sharing only a geometric or algebraic interest. The geometrically minded reader might

wish to restrict his attention to Section 2 and 3, and only take notice of the result about J-modularity stated in theorem 4.11. An algebraically oriented reader, on the other hand, might simply accept the categories described in Proposition 3.22 together with the structure described in Propositions 3.23, 3.24 and 3.26 and then directly delve into Section 4.

For the benefit of all readers, we present an outline of all our findings. In Section 2, we review the pertinent aspects of Dijkgraaf-Witten theory and in particular the specific construction given in [Mor10]. Section 3 is devoted to the equivariant case: we observe that the correct notion of J-action on G is what we call a weak action of the group J on the group G; this notion is introduced in definition 3.1. Based on this notion, we can for every J-cover $P \to M$ very explicitly construct a category $\mathcal{A}_G(P \to M)$ of P-twisted G-bundles. For the definition and elementary properties of twisted bundles, we refer to section 3.2 and for a local description to appendix 6.1. We are then ready to construct equivariant Dijkgraaf Witten theory along the lines of the construction described in [Mor10]. This is carried out in section 3.3 and 3.4. We obtain a construction of equivariant Dijkgraaf-Witten theory that is so explicit that we can read off the category $\mathcal{C}^J(G)$ it assigns to the circle $\mathbb{S}^1$. The equivariant topological field theory induces additional structure on this category, which can also be computed by geometric methods due to the explicit control of the theory, and part of which we compute in section 3.5. This finishes the geometric part of our work. It remains to show that the category $\mathcal{C}^J(G)$ is indeed J-modular.

To establish the J-modularity of the category $\mathcal{C}^J(G)$, we have to resort to algebraic tools. Our discussion is based on Ch. VIII of [Tur10a] by A. Virélizier. At the same time, we explain the solution of the algebraic problems described in section 1.1. The Hopf algebraic notions we encounter in section 4, in particular Hopf algebras with a weak group action and their orbifold Hopf algebras might be of independent algebraic interest.
In section 4, we introduce the notion of a J-equivariant ribbon Hopf algebra. It turns out that it is natural to relax some strictness requirements on the J-action on such a Hopf algebra. Given a weak action of a finite group J on a finite group G, we describe in proposition 4.7 a specific ribbon Hopf algebra which we call the equivariant Drinfel'd double $\mathcal{D}^J(G)$. This ribbon Hopf algebra is designed in such a way that its representation category is equivalent to the geometric category $\mathcal{C}^J(G)$ constructed in section 3, compare proposition 4.8.
The J-modularity of $\mathcal{C}^J(G)$ is established via the modularity of its orbifold category. We introduced the notion of orbifold category and the corresponding notion of an orbifold category in Section 2.2 of Chapter I In the case

of the equivariant Drinfel'd double $\mathcal{D}^J(G)$, this orbifold algebra is shown to be isomorphic, as a ribbon Hopf algebra, to the Drinfel'd double of a finite group. This implies modularity of the orbifold theory and, by a result of [Kir04], J-modularity of the category $\mathcal{C}^J(G)$, cf. theorem 4.11.
In the course of our construction, we develop several notions of independent interest. In fact, this chapter might be seen as a study of the geometry of chiral backgrounds. It allows for various generalizations, some of which are briefly sketched in the conclusions. These generalizations include in particular twists by 3-cocycles in group cohomology and, possibly, even the case of non-semi simple chiral backgrounds.

2 Dijkgraaf-Witten Theory and Drinfel'd double

This section contains a short review of Dijkgraaf-Witten theory as an extended three-dimensional topological field theory, covering the contributions of many authors, including in particular the work of Dijkgraaf-Witten [DW90], of Freed-Quinn [FQ93] and of Morton [Mor10]. We explain how these extended $3d$ TFTs give rise to modular tensor categories. These specific modular tensor categories are the representation categories of a well-known class of quantum groups, the Drinfel'd doubles of finite groups, that where reviewed in Section 1 of Chapter II.
While this section does not contain original material, we present the ideas in such a way that equivariant generalizations of the theories can be conveniently discussed. In this section, we also introduce some categories and functors that we need for later sections.

2.1 Motivation for Dijkgraaf-Witten Theory

We start with a brief motivation for Dijkgraaf-Witten theory from physical principles. A reader already familiar with Dijkgraaf-Witten theory might wish to take at least notice of Definition 2.2 and of Proposition 2.3.
It is an old, yet successful idea to extract invariants of manifolds from quantum field theories, in particular from quantum field theories for which the fields are G-bundles with connection, where G is some group. In this chapter we mostly consider the case of a finite group and only occasionally make reference to the case of a compact Lie group.
Let M be a compact oriented manifold of dimension 1,2 or 3, possibly with boundary. As the *'space' of field configurations*, we choose G bundles with

connection,

$$\mathcal{A}_G(M) := \mathcal{B}un_G^{\nabla}(M).$$

This way, we in fact really assign to a manifold a groupoid, rather than an actual space. The morphisms of the category take gauge transformations into account. We will nevertheless keep on calling it 'space' since the correct framework to handle $\mathcal{A}_G(M)$ is as a stack on the category of smooth manifolds.
Moreover, another piece of data specifying the model is a function defined on manifolds of a specific dimension,

$$S : \mathcal{A}_G(M) \to \mathbb{C}$$

called the *action*. In the simplest case, when G is a finite group, a field configuration is given by a G-bundle, since all bundles are canonically flat and no connection data are involved. Then, the simplest action is given by $S[P] := 0$ for all G-bundles P. In the case of a compact, simple, simply connected Lie group G, consider a 3-manifold M. In this situation, each G-bundle P over M is globally of the form $P \cong G \times M$, because $\pi_1(G) = \pi_2(G) = 0$. Hence a field configuration is given by a connection on the trivial bundle which is a 1-form $A \in \Omega^1(M, \mathfrak{g})$ with values in the Lie algebra of G. An example of an action yielding a topological field theory that can be defined in this situation is the Chern-Simons action

$$S[A] := \int_M \langle A \wedge dA \rangle - \frac{1}{6} \langle A \wedge A \wedge A \rangle$$

where $\langle \cdot, \cdot \rangle$ is the basic invariant inner product on the Lie algebra $\mathfrak{g}$.
The heuristic idea is then to introduce an invariant $Z(M)$ for a 3-manifold M by integration over all field configurations:

$$Z(M) := \text{"} \int_{\mathcal{A}_G(M)} d\phi \; e^{iS[\phi]} \text{"}.$$

Warning 2.1. *In general, this path integral has only a heuristic meaning. In the case of a finite group, however, one can choose a counting measure $d\phi$ and thereby reduce the integral to a well-defined finite sum. The definition of Dijkgraaf-Witten theory [DW90] is based on this idea.*

Instead of giving a well-defined meaning to the invariant $Z(M)$ as a path-integral, we exhibit some formal properties these invariants are expected to satisfy. To this end, it is crucial to allow for manifolds that are not closed, as well. This allows to cut a three-manifold into several simpler three-manifolds

with boundaries so that the computation of the invariant can be reduced to the computation of the invariants of simpler pieces.
Hence, we consider a 3-manifold M with a 2-dimensional boundary ∂M. We fix boundary values $\phi_1 \in \mathcal{A}_G(\partial M)$ and consider the space $\mathcal{A}_G(M, \phi_1)$ of all fields ϕ on M that restrict to the given boundary values ϕ_1. We then introduce, again at a heuristic level, the quantity

$$Z(M)_{\phi_1} := \text{''} \int_{\mathcal{A}_G(M,\phi_1)} d\phi \; e^{iS[\phi]} \text{''}. \tag{III.2}$$

The assignment $\phi_1 \mapsto Z(M)_{\phi_1}$ could be called a 'wave function' on the space $\mathcal{A}_G(\partial M)$ of boundary values of fields. These 'wave functions' form a vector space $\mathcal{H}_{\partial M}$, the *state space*

$$\mathcal{H}_{\partial M} := \text{''} L^2\big(\mathcal{A}_G(\partial M), \mathbb{C}\big) \text{''}$$

that we assign to the boundary ∂M. The transition to wave functions amounts to a linearization. The notation L^2 should be taken with a grain of salt and should indicate the choice of an appropriate vector space for the category $\mathcal{A}_G(\partial M)$; it should not suggest the existence of any distinguished measure on the category.
In the case of Dijkgraaf-Witten theory based on a finite group G, the space of states has a basis consisting of δ-functions on the set of isomorphism classes of field configurations on the boundary ∂M:

$$\mathcal{H}_{\partial M} = \mathbb{C}\big\langle \delta_{\phi_1} \mid \phi_1 \in Iso\mathcal{A}_G(\partial M) \big\rangle.$$

In this way, we associate finite dimensional vector spaces $\mathcal{H}_\Sigma$ to compact oriented 2-manifolds Σ. The heuristic path integral in equation (III.2) suggests to associate to a 3-manifold M with boundary ∂M an element

$$Z(M) \in \mathcal{H}_{\partial M} \ ,$$

or, equivalently, a linear map $\mathbb{C} \to \mathcal{H}_{\partial M}$.
A natural generalization of this situation are cobordisms $M : \Sigma \to \Sigma'$, where Σ and Σ' are compact oriented 2-manifolds. A cobordism is a compact oriented 3-manifold M with boundary $\partial M \cong \bar{\Sigma} \sqcup \Sigma'$ where $\bar{\Sigma}$ denotes Σ, with the opposite orientation. To a cobordism, we wish to associate a linear map

$$Z(M) : \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$$

by giving its matrix elements in terms of the path integral

$$Z(M)_{\phi_0,\phi_1} := \text{''} \int_{\mathcal{A}_G(M,\phi_0,\phi_1)} d\phi \; e^{iS[\phi]} \text{''}$$

with fixed boundary values $\phi_0 \in \mathcal{A}_G(\Sigma)$ and $\phi_1 \in \mathcal{A}_G(\Sigma')$. Here $\mathcal{A}_G(M, \phi_0, \phi_1)$ is the space of field configurations on M that restrict to the field configuration ϕ_0 on the ingoing boundary Σ and to the field configuration ϕ_1 on the outgoing boundary Σ'. One can now show that the linear maps $Z(M)$ are compatible with gluing of cobordisms along boundaries. (If the group G is not finite, additional subtleties arise; e.g. $Z(M)_{\phi_0,\phi_1}$ has to be interpreted as an integral kernel.)
Atiyah [Ati88] gave a definition of a topological field theory that formalizes these properties: it describes a topological field theory as a symmetric monoidal functor from a geometric tensor category to an algebraic category. To make this definition explicit, let $\mathfrak{Cob}(2,3)$ be the category which has 2-dimensional compact oriented smooth manifolds as objects. Its morphisms $M : \Sigma \to \Sigma'$ are given by (orientation preserving) diffeomorphism classes of 3-dimensional, compact oriented cobordism from Σ to Σ' which we write as

$$\Sigma \hookrightarrow M \hookleftarrow \Sigma'.$$

Composition of morphisms is given by gluing cobordisms together along the boundary. The disjoint union of 2-dimensional manifolds and cobordisms equips this category with the structure of a symmetric monoidal category. For the algebraic category, we choose the symmetric tensor category $\mathrm{Vect}_{\mathrm{K}}$ of finite dimensional vector spaces over an algebraically closed field K of characteristic zero. For convenience, we recall Definition 1.1 from chapter I.

Definition 2.2 (Atiyah)**.** *A 3d TFT is a symmetric monoidal functor*

$$Z : \mathfrak{Cob}(2,3) \to \mathrm{Vect}_{\mathrm{K}}.$$

Let us set up such a functor for Dijkgraaf-Witten theory, i.e. fix a finite group G and choose the trivial action $S : A_G(M) \to \mathbb{C}$, i.e. $S[P] = 0$ for all G-bundles P on M. Then the path integrals reduce to finite sums over 1 hence simply count the number of elements in the category $\mathcal{A}_G$. Since we are counting objects in a category, the stabilizers have to be taken appropriately into account, for details see e.g. [Mor08, Section 4]. This is achieved by the *groupoid cardinality* (which is sometimes also called the Euler-characteristic of the groupoid Γ)

$$|\Gamma| := \sum_{[g] \in Iso(\Gamma)} \frac{1}{|Aut(g)|}.$$

A detailed discussion of groupoid cardinality can be found in [BD01] and [Lei08].
We summarize the discussion:

Proposition 2.3 ([DW90],[FQ93])**.** *Given a finite group G, the following assignment Z_G defines a 3d TFT: to a closed, oriented 2-manifold Σ, we assign the vector space freely generated by the isomorphism classes of G-bundles on Σ,*

$$\Sigma \quad \longmapsto \quad \mathcal{H}_\Sigma := \mathrm{K}\big\langle \delta_P \mid P \in Iso\mathcal{A}_G(\Sigma) \big\rangle \ .$$

To a 3 dimensional cobordism M, we associate the linear map

$$Z_G\Big(\Sigma \hookrightarrow M \hookleftarrow \Sigma'\Big): \quad \mathcal{H}_\Sigma \to \mathcal{H}_{\Sigma'}$$

with matrix elements given by the groupoid cardinality of the categories $\mathcal{A}_G(M, P_0, P_1)$:

$$Z_G(M)_{P_0,P_1} := \big|\mathcal{A}_G(M, P_0, P_1)\big| \ .$$

Remark 2.4. *1. In the original paper [DW90], a generalization of the trivial action $S[P] = 0$, induced by an element η in the group cohomology $H^3_{Gp}\big(G, \mathrm{U}(1)\big)$ with values in $\mathrm{U}(1)$, has been studied. We postpone the treatment of this generalization to future work. In the following, the term Dijkgraaf-Witten theory refers to the 3d TFT of Proposition 2.3 or its extended version.*

2. *In the case of a compact, simple, simply-connected Lie group G, a definition of a 3d TFT by a path integral is not available. Instead, the combinatorial definition of Reshetikin-Turaev [RT91] can be used to set up a 3d TFT which has the properties expected for Chern-Simons theory.*

3. *The vector spaces $\mathcal{H}_\Sigma$ can be described rather explicitly. Since every compact, closed, oriented 2-manifold is given by a disjoint union of surfaces Σ_g of genus g, it suffices to compute the dimension of $\mathcal{H}_{\Sigma_g}$. This can be done using the well-known description of moduli spaces of flat G-bundles in terms of homomorphisms from the fundamental group $\pi_1(\Sigma_g)$ to the group G, modulo conjugation,*

$$Iso\mathcal{A}_G(\Sigma_g) \cong \mathrm{Hom}(\pi_1(\Sigma_g), G)/G$$

which can be combined with the usual description of the fundamental group $\pi_1(\Sigma_g)$ in terms of generators and relations. In this way, one finds that the space is one-dimensional for surfaces of genus 0. In the case of surfaces of genus 1, it is generated by pairs of commuting group elements, modulo simultaneous conjugation.

4. Following the same line of argument, one can show that for a closed 3-manifold M, one has

$$\left|\mathcal{A}_G(M)\right| = \left|\mathrm{Hom}(\pi_1(M), G)\right| \ / \ |G| \ .$$

This expresses the 3-manifold invariants in terms of the fundamental group of M.

2.2 Dijkgraaf-Witten Theory as an Extended TFT

Up to this point, we have considered a version of Dijkgraaf-Witten theory which assigns invariants to closed 3-manifolds $Z(M)$ and vector spaces to 2-dimensional manifolds Σ. Iterating the argument that has lead us to consider three-manifolds with boundaries, we might wish to cut the two-manifolds into smaller pieces as well, and thereby introduce two-manifolds with boundaries into the picture.

Hence, we drop the requirement on the 2-manifold Σ to be closed and allow Σ to be a compact, oriented 2-manifold with 1-dimensional boundary $\partial\Sigma$. Given a field configuration $\phi_1 \in \mathcal{A}_G(\partial\Sigma)$ on the boundary of the surface Σ, we consider the space of all field configurations $\mathcal{A}_G(\Sigma, \phi_1)$ on Σ that restrict to the given field configuration ϕ_1 on the boundary $\partial\Sigma$. Again, we linearize the situation and consider for each field configuration ϕ_1 on the 1-dimensional boundary $\partial\Sigma$ the vector space freely generated by the isomorphism classes of field configurations on Σ,

$$\mathcal{H}_{\Sigma,\phi_1} := \text{"}L^2\big(\mathcal{A}_G(\Sigma, \phi_1)\big)\text{"} = \mathbb{C}\big\langle \delta_\phi \mid \phi \in Iso\mathcal{A}_G(\Sigma, \phi_1)\big\rangle.$$

The object we associate to the 1-dimensional boundary $\partial\Sigma$ of a 2-manifold Σ is thus a map $\phi_1 \mapsto \mathcal{H}_{\Sigma,\phi_1}$ of field configurations to vector spaces, i.e. a complex vector bundle over the space of all fields on the boundary. In the case of a finite group G, we prefer to see these vector bundles as objects of the functor category from the essentially small category $\mathcal{A}_G(\partial\Sigma)$ to the category $\mathrm{Vect}_{\mathbb{C}}$ of finite-dimensional complex vector spaces, i.e. as an element of

$$\mathrm{Vect}(\mathcal{A}_G(\partial\Sigma)) = \big[\mathcal{A}_G(\partial\Sigma), \mathrm{Vect}_{\mathbb{C}}\big].$$

Thus the extended version of the theory assigns the category $Z(S) = [\mathcal{A}_G(S), \mathrm{Vect}_{\mathbb{C}}]$ to a one dimensional, compact oriented manifold S. These categories possess certain additional properties which can be summarized by saying that they are 2-vector spaces as defined in 1.2

Proposition 2.5. *[Mor10] Given a finite group G, there exists an extended 3d TFT Z_G which assigns the categories*

$$\big[\mathcal{A}_G(S), \mathrm{Vect}_{\mathrm{K}}\big]$$

to 1-dimensional, closed oriented manifolds S and whose restriction $Z_G|_\emptyset$ is (isomorphic to) the Dijkgraaf-Witten TFT described in proposition 2.3.

Remark 2.6. *One can iterate the procedure of extension and introduce the notion of a fully extended TFT which also assigns quantities to points rather than just 1-manifolds. It can be shown that Dijkgraaf-Witten theory can be turned into a fully extended TFT, see [FHLT09]. The full extension will not be needed in the present chapter.*

2.3 Construction via 2-Linearization

In this subsection, we describe the construction of the extended $3d$ TFT of Proposition 2.5in more detail. An impatient reader may skip this subsection and should still be able to understand most of this chapter. He might, however, wish to take notice of the technique of 2-linearization in Proposition 2.9 which is also an essential ingredient in our construction of equivariant Dijkgraaf-Witten theory in the sequel of this chapter.
As emphasized in particular by Morton [Mor10], the construction of the extended TFT is naturally split into two steps, which have already been implicitly present in preceding sections. The first step is to assign to manifolds and cobordisms the configuration spaces $\mathcal{A}_G$ of G bundles. We now restrict ourselves to the case when G is a finite group. The following fact is standard:

- The assignment $M \mapsto \mathcal{A}_G(M) := \mathcal{B}un_G$ is a contravariant 2-functor from the category of manifolds to the 2-category of groupoids. Smooth maps between manifolds are mapped to the corresponding pullback functors on categories of bundles.

A few comments are in order: for a connected manifold M, the category $\mathcal{A}_G(M)$ can be replaced by the equivalent category given by the action groupoid $\mathrm{Hom}\big(\pi_1(M), G\big)//G$ where G acts by conjugation. In particular, the category $\mathcal{A}_G(M)$ is essentially finite, if M is compact. It should be appreciated that at this stage no restriction is imposed on the dimension of the manifold M.
The functor $\mathcal{A}_G(-)$ can be evaluated on a 2-dimensional cobordism $S \hookrightarrow \Sigma \hookleftarrow S'$ or a 3-dimensional cobordism $\Sigma \hookrightarrow M \hookleftarrow \Sigma'$. It then yields diagrams of the form

$$\mathcal{A}_G(S) \longleftarrow \mathcal{A}_G(\Sigma) \longrightarrow \mathcal{A}_G(S')$$
$$\mathcal{A}_G(\Sigma) \longleftarrow \mathcal{A}_G(M) \longrightarrow \mathcal{A}_G(\Sigma').$$

Such diagrams are called spans. They are the morphisms of a symmetric monoidal bicategory $\mathfrak{S}$pan of spans of groupoids as follows (see e.g. [DPP04] or [Mor06]):

- Objects are (essentially finite) groupoids.
- Morphisms are spans of essentially finite groupoids.
- 2-Morphisms are isomorphism classes of spans of span-maps.
- Composition is given by forming weak fiber products.
- The monoidal structure is given by the cartesian product $\times$ of groupoids.

Proposition 2.7 ([Mor10]). *$\mathcal{A}_G$ induces a symmetric monoidal 2-functor*

$$\widetilde{\mathcal{A}_G} : \mathfrak{Cob}(1,2,3) \to \mathfrak{S}\mathrm{pan}.$$

This functor assigns to a 1-dimensional manifold S the groupoid $\mathcal{A}_G(S)$, to a 2-dimensional cobordism $S \hookrightarrow \Sigma \hookleftarrow S'$ the span $\mathcal{A}_G(S) \longleftarrow \mathcal{A}_G(\Sigma) \longrightarrow \mathcal{A}_G(S')$ and to a 3-cobordism with corners a span of span-maps.

Proof. It only remains to be shown that composition of morphisms and the monoidal structure is respected. The first assertion is shown in [Mor10, theorem 2] and the second assertion follows immediately from the fact that bundles over disjoint unions are given by pairs of bundles over the components, i.e. $\mathcal{A}_G(M \sqcup M') = \mathcal{A}_G(M) \times \mathcal{A}_G(M')$. □

The second step in the construction of extended Dijkgraaf-Witten theory is the 2-linearization of [Mor08]. As we have explained in section 2.1, the idea is to associate to a groupoid Γ its category of vector bundles $\mathrm{Vect}_{\mathrm{K}}(\Gamma)$. If Γ is essentially finite, the category of vector bundles is conveniently defined as the functor category $\big[\Gamma, \mathrm{Vect}_{\mathrm{K}}\big]$. If K is algebraically closed of characteristic zero, this category is a 2-vector space, see [Mor08, Lemma 4.1.1].

- The assignment $\Gamma \mapsto \mathrm{Vect}_{\mathrm{K}}\big(\Gamma\big) := \big[\Gamma, \mathrm{Vect}_{\mathrm{K}}\big]$ is a contravariant 2-functor from the bicategory of (essentially finite) groupoids to the 2-category of 2-vector spaces. Functors between groupoids are sent to pullback functors.

We next need to explain what 2-linearization assigns to spans of groupoids. To this end, we use the following lemma due to [Mor08, 4.2.1]:

Lemma 2.8. *Let $f : \Gamma \to \Gamma'$ be a functor between essentially finite groupoids. Then the pullback functor $f^* : \mathrm{Vect}\big(\Gamma'\big) \to \mathrm{Vect}\big(\Gamma\big)$ admits a 2-sided adjoint $f_* : \mathrm{Vect}\big(\Gamma\big) \to \mathrm{Vect}\big(\Gamma'\big)$, called the pushforward.*

Two-sided adjoints are also called 'ambidextrous' adjoint, see [Bar09, Ch. 5] for a discussion. We use this pushforward to associate to a span

$$\begin{tikzcd} & \Lambda \arrow[dl, "s_1"'] \arrow[dr, "t_1"] & \\ \Gamma & & \Gamma' \end{tikzcd} \tag{III.3}$$

of (essentially finite) groupoids the *'pull-push'-functor*

$$(p_1)_* \circ (p_0)^* : \quad \mathrm{Vect}_{\mathbb{K}}\big(\Gamma\big) \longrightarrow \mathrm{Vect}_{\mathbb{K}}\big(\Gamma'\big).$$

A similar construction [Mor08] associates to spans of span-morphisms a natural transformation. We will give a sketch here:
Given a 2-morphism in $\mathfrak{S}$pan, i.e. a diagram of groupoids of the form:

$$\begin{tikzcd} & \Lambda \arrow[dl, "s_1"'] \arrow[dr, "t_1"] & \\ \Gamma \arrow[r, phantom, "\Downarrow \theta_s"] & K \arrow[u, "s"'] \arrow[d, "t"] \arrow[r, phantom, "\Downarrow \theta_t"] & \Gamma' \\ & \Lambda' \arrow[ul, "s_2"] \arrow[ur, "t_2"'] & \end{tikzcd} \tag{III.4}$$

which is commutative only up to natural isomorphisms

$$\theta_s : s_1 \circ s \to s_2 \circ t \quad \text{and} \quad \theta_t : t_1 \circ s \to t_2 \circ t \tag{III.5}$$

the pullback $s^* : [\Lambda, \mathrm{Vect}] \to [K, \mathrm{Vect}]$ of s has the pushforward $s_* : [K, \mathrm{Vect}] \to [\Lambda, \mathrm{Vect}]$ as a two-sided adjoint (see Lemma 2.8) that comes with natural transformations

$$\eta_{s,R} : \mathrm{id} \to s_* s^* \quad \epsilon_{s,R} : s^* s_* \to \mathrm{id} \tag{III.6}$$

$$\eta_{s,L} : \mathrm{id} \to s^* s_* \quad \epsilon_{s,L} : s_* s^* \to \mathrm{id} \tag{III.7}$$

and analogously the pullback $t^* : [\Lambda', \mathrm{Vect}] \to [K, \mathrm{Vect}]$ of t has the pushforward $t_* : [K, \mathrm{Vect}] \to [\Lambda', \mathrm{Vect}]$ as a two-sided adjoint and we have natural transformations $\eta_{t,R}$, $\epsilon_{t,R}$, $\eta_{t,L}$ and $\epsilon_{t,L}$. Now we assign to the diagram (III.4) the composition

$$(\theta_t)_* \bullet (\epsilon_{t,L} \circ \eta_{s,R}) \bullet \theta_s^* \tag{III.8}$$

where the symbol '$\bullet$' denotes the horizontal composition of the 2-morphisms, '$\circ$' denotes the vertical composition and $(\theta_t)_*$ resp. θ_s^* denote the pushforward resp. pullback of the natural isomorphism θ_t and θ_s that appear in diagram (III.4). In form of a diagram, the formula (III.8) is:

$$
\begin{tikzcd}
 & {[\Lambda, \mathrm{Vect}]} \arrow[rr, "\mathrm{id}"] \arrow[dr, "s^*"] & {} \arrow[d, phantom, "\Downarrow \eta_{s,R}"] & {[\Lambda, \mathrm{Vect}]} \arrow[dr, "(t_1)_*"] & \\
{[\Gamma, \mathrm{Vect}]} \arrow[ur, "s_1^*"] \arrow[dr, "s_2^*"'] \arrow[rr, phantom, "\Downarrow \theta_s^*"] & & {[K, \mathrm{Vect}]} \arrow[ur, "s_*"] \arrow[dr, "t_*"] \arrow[rr, phantom, "\Downarrow (\theta_t)_*"] \arrow[d, phantom, "\Downarrow \epsilon_{t,L}"] & & {[\Gamma', \mathrm{Vect}]} \\
 & {[\Lambda', \mathrm{Vect}]} \arrow[rr, "\mathrm{id}"'] \arrow[ur, "t^*"] & {} & {[\Lambda', \mathrm{Vect}]} \arrow[ur, "(t_2)_*"'] &
\end{tikzcd}
\tag{III.9}
$$

Taking all the assignments on the level of objects, 1- and 2-morphisms of $\mathfrak{Span}$ into account we arrive at the following:

Proposition 2.9 ([Mor08]). *The assignment $\Gamma \mapsto \mathrm{Vect}_{\mathrm{K}}(\Gamma)$ can be extended to a symmetric monoidal 2-functor on the category of spans of groupoids*

$$\widetilde{\mathcal{V}_{\mathrm{K}}} : \mathfrak{Span} \to 2\mathrm{Vect}_{\mathrm{K}}.$$

This 2-functor is called 2-linearization.

Proof. The proof that $\widetilde{\mathcal{V}_{\mathrm{K}}}$ is a 2-functor is in [Mor08, Theorem 7.1.2]. For a product $\Gamma \times \Gamma'$ of essentially finite groupoids Γ and Γ, there is a canonical equivalence $\eta_2 : (\Gamma, \Gamma')\mathrm{Vect}_{\mathrm{K}}\big(\Gamma\big) \boxtimes \mathrm{Vect}_{\mathrm{K}}\big(\Gamma'\big) \xrightarrow{\sim} \mathrm{Vect}_{\mathrm{K}}\big(\Gamma \times \Gamma'\big)$, which assigns to a pair of functors in $\mathrm{Vect}_{\mathrm{K}}\big(\Gamma\big) \boxtimes \mathrm{Vect}_{\mathrm{K}}\big(\Gamma'\big)$ their tensor product and likewise to a pair of natural transformations their tensor product (by the universal property of the Deligne tensor product it suffices to define a functor on pairs of objects and morphisms). The tensor unit in $\mathfrak{Span}$ is the one object groupoid $\mathbf{1}$ with only the identity as morphisms. There is an isomorphism $\eta_0 : [\mathbf{1}, \mathrm{Vect}_{\mathrm{K}}] \to \mathrm{Vect}_{\mathrm{K}}$ from the functor category $[\mathbf{1}, \mathrm{Vect}_{\mathrm{K}}]$ to the category $\mathrm{Vect}_{\mathrm{K}}$ of vector spaces over K, which is the tensor unit in 2Vect. One can check, that η_2 and η_0 provide the 2-functor $\widetilde{\mathcal{V}_{\mathrm{K}}}$ with a monoidal structure. □

Arguments similar to the ones in [DPP04, prop 1.10] which are based on the universal property of the span category can be used to show that such an extension is essentially unique.

We are now in a position to give the functor Z_G described in Proposition 2.5 which is Dijkgraaf-Witten theory as an extended $3d$ TFT as the composition of functors

$$Z_G := \widetilde{\mathcal{V}_{\mathrm{K}}} \circ \widetilde{\mathcal{A}_G} : \quad \mathfrak{C}\mathrm{ob}(1,2,3) \longrightarrow 2\mathrm{Vect}_{\mathrm{K}}.$$

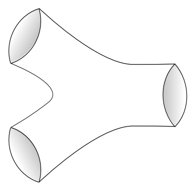
It follows from Propositions 2.7 and 2.9 that Z_G is an extended $3d$ TFT in the sense of Definition 1.8. For the proof of Proposition 2.5, it remains to be shown that $Z_G|_\emptyset$ is the Dijkgraaf-Witten $3d$ TFT from proposition 2.3; this follows from a calculation which can be found in [Mor10, Section 5.2].

2.4 Derivation of the Braided Category

The goal of this subsection is a more detailed discussion of extended Dijkgraaf-Witten theory Z_G as described in Proposition 2.5. Our focus is on the object assigned to the 1-manifold $\mathbb{S}^1$ given by the circle with its standard orientation. We start our discussion by evaluating an arbitrary extended $3d$ TFT Z as in definition 1.8 on certain manifolds of different dimensions:

1. To the circle $\mathbb{S}^1$, the extended TFT Z assigns a K-linear, abelian finitely semisimple category $\mathcal{C}_Z := Z(\mathbb{S}^1)$.

2. To the two-dimensional sphere with three boundary components, two incoming and one outgoing, also known as the *pair of pants*,

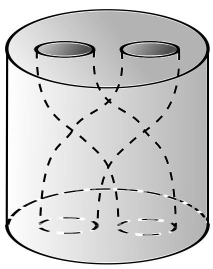


   the TFT associates a functor

$$\otimes : \ \mathcal{C}_Z \boxtimes \mathcal{C}_Z \to \mathcal{C}_Z \quad ,$$

   which turns out to provide a tensor product on the category $\mathcal{C}_Z$.

3. The figure



shows a 2-morphism between two three-punctured spheres, drawn as the upper and lower lid. The outgoing circle is drawn as the boundary of the big disk. To this cobordism, the TFT associates a natural transformation

$$\otimes \Rightarrow \otimes^{\mathrm{opp}}$$

which turns out to be a braiding.

Moreover, the TFT provides coherence cells, in particular associators and relations between the given structures. This endows the category $\mathcal{C}_Z$ with much additional structure. This structure can be summarized as follows:

Proposition 2.10. *For Z an extended $3d$ TFT, the category $\mathcal{C}_Z := Z(\mathbb{S}^1)$ is naturally endowed with the structure of a braided tensor category.*

For details, we refer to [Fre95] [Fre94] [Fre99] and [CY99]. This is not yet the complete structure that can be extracted: from the braiding-picture above it is intuitively clear that the braiding is not symmetric; in fact, the braiding is 'maximally non-symmetric' in a precise sense that is explained in Definition 1.11 of chapter I. We show this in the next section for the category obtained from the Dijkgraaf-Witten extended TFT.

We now specialize to the case of extended Dijkgraaf-Witten TFT Z_G. We first determine the category $\mathcal{C}(G) := \mathcal{C}_Z$; it is by definition the functor category

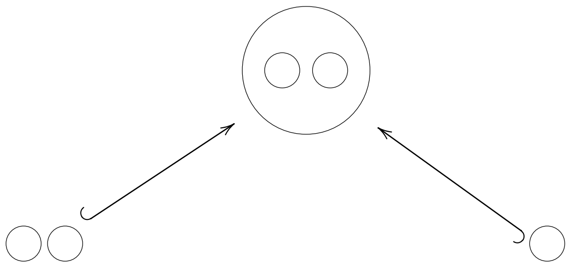
$$\mathcal{C}(G) = \left[\mathcal{A}_G(\mathbb{S}^1), \mathrm{Vect}_{\mathrm{K}}\right].$$

It is a standard result in the theory of coverings that G-covers on $\mathbb{S}^1$ are described by group homomorphisms $\pi_1(\mathbb{S}^1) \to G$ and their morphisms by group elements acting by conjugation. Thus the category $\mathcal{A}_G(\mathbb{S}^1)$ is equivalent to the action groupoid $G//G$ for the conjugation action. As a consequence, we obtain the abelian category $\mathcal{C}(G) \cong [G//G, \mathrm{Vect}_{\mathrm{K}}]$. We spell out this functor category explicitly:

Proposition 2.11. *For the extended Dijkgraaf-Witten $3d$ TFT, the category $\mathcal{C}(G)$ associated to the circle $\mathbb{S}^1$ is given by the category of G-graded vector spaces $V = \bigoplus_{h \in G} V_g$ together with a G-action on V such that for all $g, h \in G$*

$$g.V_h \subset V_{ghg^{-1}} \quad .$$

As a next step we determine the tensor product on $\mathcal{C}(G)$. It can be derived from the image of the pair of pants under the $3d$ TFT, i.e the following cobordism:



(III.10)

Since the fundamental group of the pair of pants (which is the manifold in the middle) is the free group on two generators, the relevant category of G-bundles is equivalent to the action groupoid $(G \times G)//G$ where G acts by simultaneous conjugation on the two copies of G. The 2-linearization $\widetilde{\mathcal{V}_{\mathrm{K}}}$ on the span

$$\begin{tikzcd}
& (G \times G)//G \arrow[dl, "s_1"'] \arrow[dr, "t_1"] & \\
(G//G) \times (G//G) & & G//G
\end{tikzcd}$$

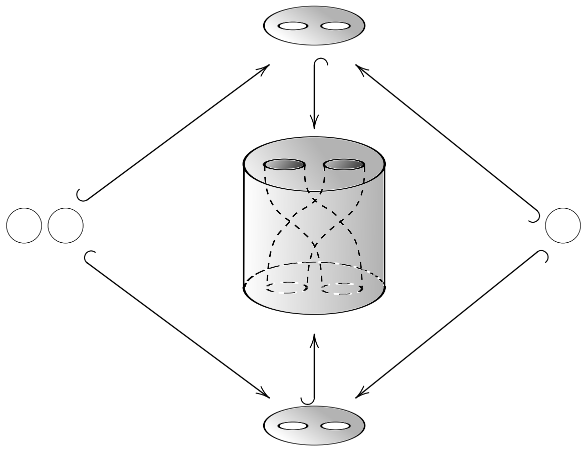
is treated in detail in [Mor10, Rem. 5]; the result of this calculation via the pull-push construction yields the following tensor product:

Proposition 2.12. *The tensor product of two objects V and W in $\mathcal{C}(G)$, which are vector-spaces over* K *with G-grading $V = \bigoplus_{h \in G} V_h$ and $W = \bigoplus_{h \in G} W_h$ and compatible G-action, is given by the G-graded vector space*

$$(V \otimes W)_g = \bigoplus_{st=g} V_s \otimes W_t$$

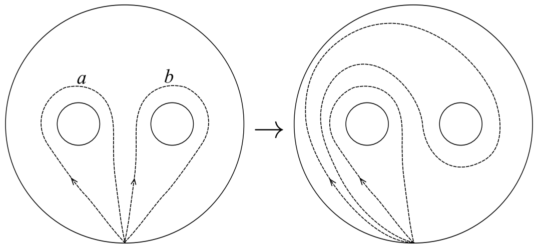
together with the G-action $g.(v, w) = (gv, gw)$. The associators are the obvious ones induced by the tensor product in $\mathrm{Vect}_{\mathrm{K}}$.

In the same vein, the braiding can be calculated. We present the details of this calculation to illustrate the method of the 2-linearization explained in Section 2.3. The braiding comes from the following diagram of manifolds and inclusions:



(III.11)

Note that the inclusion on the lower left exchanges the two boundary circles compared to the inclusion on the upper left. The fundamental group of the pair of pants and the fundamental group of the 3-manifold in the middle is in both cases the free group on two generators. We consider the fundamental groups as groupoids with one object. For the ingoing boundary circles on the left we therefore get a groupoid with two non-connected objects and $\mathbb{Z}$ as automorphism-groups. The groupoid associated to the outgoing boundary circle on the right hand side is the one object groupoid with $\mathbb{Z}$ as the group of automorphisms. The inclusions induce functors on the groupoids. The functors induced from the inclusions of the ingoing boundary clearly takes the two objects to the one object in the target category. On morphisms the functors are inclusions of the group $\mathbb{Z}$ into $\mathbb{Z}*\mathbb{Z}$, where for the lower inclusion, the entries get exchanged.
The inclusion of the pair of pants into the bottom boundary of the 3-manifold in the middle, induces the identity on objects, but as for the morphisms, we have to take into account the effect on the generators a and b of the fundamental group $\mathbb{Z}$ by the braiding. We display it in the following picture:



(III.12)

So on the generators $a, b \in \mathbb{Z} * \mathbb{Z}$ of the morphisms, the assignment of the functor is $a \mapsto aba^{-1}$ and $b \mapsto a$.

In order to get a span of spans of groupoids we take the categories of G-bundles $\mathcal{A}_G(M)$ for the respective manifolds M and pullbacks of the functors induced by the inclusions. As discussed above, the category of G-bundles on the manifolds in the middle is equivalent to the action groupoid $(G \times G)//G$ where G acts by simultaneous conjugation on the two copies of G. For the two boundary circles on the left hand side, the category of G-bundles is the product category $G//G \times G//G$ with adjoint action. This can easily be seen from the equivalance of the category of G-bundles with the category of functors from the fundamental groupoid to the group G considered as a one-object-groupoid, i.e. $\mathcal{A}_G(\mathbb{S}^1 \sqcup \mathbb{S}^1) \cong [\pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1), G]$.

$$\begin{tikzcd}
 & (G \times G)//G \arrow[dl, "p"'] \arrow[dr, "m"] & \\
G//G \times G//G & (G \times G)//G \arrow[u, "\mathrm{id}"'] \arrow[d, "b"] & G//G \\
 & (G \times G)//G \arrow[ul, "\bar{p}"] \arrow[ur, "m"'] &
\end{tikzcd} \tag{III.13}$$

The functors in diagram (III.13) are given on an object (h_1, h_2) and a morphism g in $(G \times G)//G$ by:

$$\begin{aligned}
p : (h_1, h_2) &\mapsto (h_1, h_2) \\
(g) &\mapsto (g, g) \\
\bar{p} : (h_1, h_2) &\mapsto (h_2, h_1) \\
(g) &\mapsto (g, g) \\
m : (h_1, h_2) &\mapsto h_2 h_1 \\
(g) &\mapsto g \\
b : (h_1, h_2) &\mapsto (h_1 h_2 h_1^{-1}, h_1) \\
(g) &\mapsto g
\end{aligned}$$

The endofunctor b^* on $(G \times G)//G \cong [\pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1), G]$ is derived from the effect of the braiding on the fundamental group $\pi_1(\mathbb{S}^1) \times \pi_1(\mathbb{S}^1)$ of $\mathbb{S}^1 \sqcup \mathbb{S}^1$ displayed in (III.12). We note that the left side of diagram (III.13) is commutative only up to a natural isomorphism $\theta : p \Rightarrow \bar{p} \circ b$, with

$$\theta(h_1, h_2) : (h_1, h_2) \overset{(1,h_1)}{\mapsto} (h_1, h_1 h_2 h_1^{-1}). \tag{III.14}$$

The right hand side of (III.13) is strictly commutative.

We calculate the pullback $b^* : [(G \times G)//G, \mathrm{Vect}] \to [(G \times G)//G, \mathrm{Vect}]$ of b. Let V be an object in the category $[(G \times G)//G, \mathrm{Vect}]$ i.e. a vector space with $G \times G$-grading and compatible G-action. Then for $(h_1, h_2) \in G \times G$ we have

$$b^*(V)_{(h_1,h_2)} = V_{b(h_1,h_2)} = V_{(h_1 h_2 h_1^{-1}, h_1)}$$

and the action of $g \in G$ on $v \in b^*(V)$ is the same action as on V. Since the functor b is an isomorphism of categories with inverse $b^{-1} : (h_1, h_2) \mapsto (h_2, h_2^{-1} h_1 h_2)$ on objects, its pullback b^* is also an isomorphism with inverse $(b^{-1})^*$ and the unit and counit of the adjunctions are identities. Thus we only need to calculate

$$c := (\mathrm{id}_m)_* \bullet \theta^* \tag{III.15}$$

where θ^* is the pullback of θ from (III.14): On an object in $[(G//G)^2, \mathrm{Vect}]$ which is a pair of vector spaces (V, W) with $G \times G$-grading $(V, W) = \bigoplus_{h_1,h_2} (V_{h_1}, W_{h_2})$ and $G \times G$-action $(\rho(g_1), \rho(g_2)) : (V, W) \to (V, W)$, the pullback θ looks as follows:

$$\theta^*(V, W)(h_1, h_2) : V_{h_1} \otimes W_{h_2} \xrightarrow{\tau \circ \rho(1,h_1)} W_{h_1 h_2 h_1^{-1}} \otimes V_{h_1}, \tag{III.16}$$

where the map τ switches the factors of the tensor product. Thus we get for the map c from III.15:

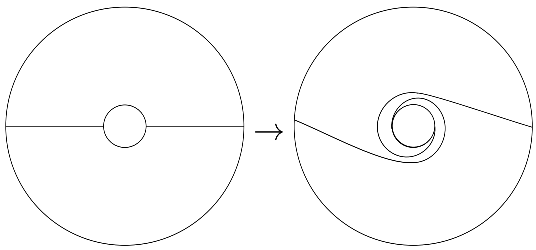
$$\begin{aligned} c(V \otimes W)(h_1, h_2) : V_{h_1} \otimes W_{h_2} &\to W_{h_1 h_2 h_1^{-1}} \otimes V_{h_1} \\ v \otimes w &\mapsto h_1.w \otimes v \end{aligned}$$

where $h_1.w := \rho(h_1)(w)$ denotes the action of the element $h_1 \in G$ on W. We thus have arrived at the following proposition:

Proposition 2.13. *The braiding $V \otimes W \to W \otimes V$ is for $v \in V_g$ and $w \in W$ given by*

$$c : v \otimes w \mapsto g.w \otimes v.$$

We now turn to another 2-morphism in $\mathfrak{Cob}(1, 2, 3)$, the *Dehn twist*. It is an automorphism of the identity 1-morphism of the circle, i.e. the cylinder. The Dehn twist is a rotation of the outgoing boundary of the cylinder by 2π and can be visualized by the following picture, where the inner circle is the ingoing boundary and the outer circle is the outgoing boundary:



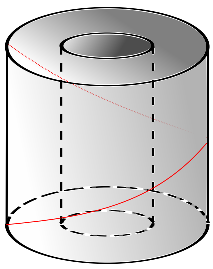
We can further parametrize the Dehn twist as follows: We embed the cylinder in the complex plane $\mathbb{C}$ as the set

$$C = \{z \in \mathbb{C} | \frac{1}{2} \leq |z| \leq 1\} = \{r \cdot e^{i\varphi} \in \mathbb{C} | r \in [\frac{1}{2}, 1], \varphi \in [0, 2\pi]\}$$

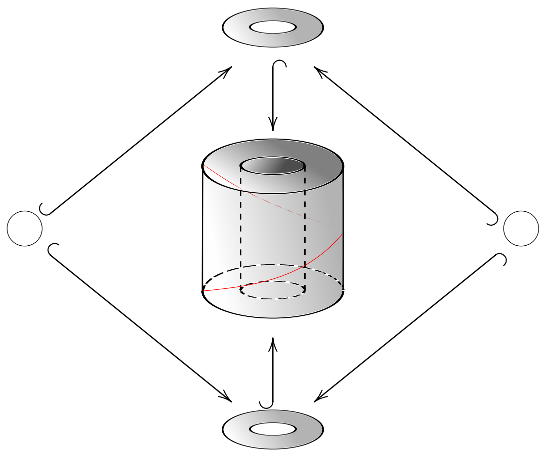
Then the Dehn twist is the following map:

$$\begin{aligned} D : C &\to C \\ re^{i\varphi} &\mapsto re^{4\pi(1-r)} \end{aligned}$$

The corresponding three-manifold with boundary and corners can be obtained by gluing the two cylinders over C together along the map D. We draw the corresponding three-manifold as:

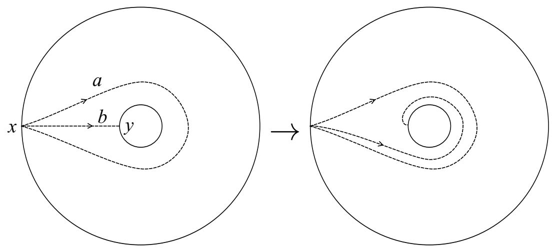


The upper lid is the source and the lower lid is the target cylinder. The lines on the outer surface indicate the rotation of the outer boundary circle by 2π. Then we have the following diagram of inclusions of manifolds:



The circle on the left hand side is mapped to the inner circle of the annulus on the top and at the bottom, whereas the circle on the right hand side is mapped to the outer circle of the respective annuli. The upper annulus is mapped to the top of the hollow cylinder and the bottom annulus is mapped to the bottom of the hollow cylinder.

Note that in this case it is not enough to consider the fundamental group of the cylinder and the hollow cylinder, as the Dehn twist does not change the homotopy class of a path with the same start and endpoint. Instead, for the cylinder we need to look at a slightly bigger but equivalent groupoid $\bar{\pi}_1$, which has as objects two points, one on each boundary, and as morphisms homotopy classes of paths with these points as start- resp. endpoints. This groupoid can be described as the groupoid with two objects x (for a point on the inner boundary) and y (for a point on the outer boundary) and with two morphisms $a \in \mathrm{Hom}(x, x)$ and $b \in \mathrm{Hom}(x, y)$ as generators. Thus in $\bar{\pi}_1$ the groups of automorphisms are $\mathrm{Aut}(x) = \mathrm{Aut}(y) = \mathbb{Z}$. The Dehn twist has the following effect on the generators a and b of the fundamental groupoid:



(III.17)

For the hollow cylinder we choose two points on the on the respective boundary circles of the upper lid. We get the same fundamental groupoid $\bar{\pi}_1$ as for the cylinder.

The inclusions of manifolds induce functors of the fundamental groupoids. The fundamental groupoid $\pi_1(\mathbb{S}^1)$ of the circle has only one object and $\mathbb{Z}$ as group of automorphisms. The inclusions on the left side induce a functor $\pi_1(\mathbb{S}^1) \to \bar{\pi}_1$, where the one object in $\pi_1(\mathbb{S}^1)$ gets mapped to x and the generator 1 of $\mathbb{Z}$ gets mapped to a. The inclusions on the right hand side induce a functor $\pi_1(\mathbb{S}^1) \to \bar{\pi}_1$, where the only object of $\pi_1(\mathbb{S}^1)$ gets mapped to the object y and the generator 1 of $\mathbb{Z}$ gets mapped to bab^{-1}. From the inclusions in the middle we get two different functors. The inclusions on the top induces the identity functor on the groupoid $\bar{\pi}_1$. For the inclusion on the bottom, we have to take the twist into account and so while the functor is the identity on objects, the generator a gets mapped to itself but the generator b gets mapped to ba^{-1}, since the Dehn twist transforms a path between two points on the different boundaries to one that is wrapped around the inner boundary ones more, as indicated in picture (III.17).

Now by taking the functor categories $\Gamma = [bar\pi_1, G]$ and $[\pi_1(\mathbb{S}^1), G] = G//G$ from the groupoids $\bar{\pi}_1$ and $\pi_1(\mathbb{S}^1)$ to the group G seen as a one-object groupoid, and pullbacks of the respective functors obtained from the inclusions and we get the following span of spans of groupoids:

$$\begin{tikzcd} & \Gamma \arrow[dl, "s_1"'] \arrow[dr, "t_1"] & \\ G//G & \Gamma \arrow[u, "\mathrm{id}"'] \arrow[d, "t"] & G//G \\ & \Gamma \arrow[ul, "s_2"] \arrow[ur, "t_2"'] & \end{tikzcd} \tag{III.18}$$

The groupoid Γ looks as follows:

- objects: $(h_1, h_2) \in G^2$
- morphisms: $(g_1, g_2) \in G^2, (g_1, g_2).(h_1, h_2) = (g_1 h_1 g_1^{-1}, g_2 h_2 g_1^{-1})$

Indeed a functor from $\bar{\pi}_1$ to the one-object-groupoid G is a pair of group elements, since it amounts to assigning to each of the two generators of the morphisms in $\bar{\pi}_1$ a morphism in G which is just a group element. Likewise, a natural transformation amounts to the choice of two group elements, one for each object in $\bar{\pi}_1$ and the effect of a natural transformation on an object can be derived from the naturality axiom. The morphisms of diagram (III.18) are given as follows:

$$\begin{aligned} s_1, s_2 : (h_1, h_2) &\mapsto h_1 \\ (g_1, g_2) &\mapsto g_1 \\ t_1, t_2 : (h_1, h_2) &\mapsto h_2 h_1 h_2^{-1} \\ (g_1, g_2) &\mapsto g_2 \\ t : (h_1, h_2) &\mapsto (h_1, h_2 h_1^{-1}) \\ (g_1, g_2, g_3, g_4) &\mapsto (g_1, g_2) \end{aligned}$$

It can easily be seen, that all the above functors are equivalences of categories, with the following quasi-inverses:

$$\begin{aligned} s_1', s_2' : h &\mapsto (h, 1) \\ g &\mapsto (g, g) \\ t_1', t_2' : h &\mapsto (h, 1) \\ g &\mapsto (g, g) \\ t' : (h_1, h_2) &\mapsto (h_1, h_2 h_1) \\ (g_1, g_2) &\mapsto (g_1, g_2) \end{aligned}$$

This is very useful for dealing with the pushforwards in the 2-linearization, since we have the following lemma:

Lemma 2.14. *Let $f : \mathcal{G} \to \mathcal{G}'$ be an equivalence of categories, with quasi-inverse f'. Then the pullback $f^* : [\mathcal{G}', \mathrm{Vect}] \to [\mathcal{G}, \mathrm{Vect}]$ is also an equivalence of categories with inverse $(f')^*$.*

Proof. Assume the functors $f : \mathcal{G} \to \mathcal{G}'$ and $f' : \mathcal{G}' \to \mathcal{G}$ together with the natural isomorphisms $\eta : \mathrm{id}_{\mathcal{G}'} \to f \circ f'$ and $\epsilon : f' \circ f \to \mathrm{id}_{\mathcal{G}}$ constitute an equivalence of the categories $\mathcal{G}$ and $\mathcal{G}'$.
Then the pullbacks, $\eta^* : \mathrm{id}_{[\mathcal{G}',\mathrm{Vect}]} \to (f')^* \circ f^*$ and $\epsilon^* : f^* \circ (f')^* \to \mathrm{id}_{[\mathcal{G}',\mathrm{Vect}]}$, given on functors $F \in \mathcal{G}', G \in \mathcal{G}$ and objects $X \in \mathcal{G}', Y \in \mathcal{G}$ by $\eta^*(F)(X) = F(\eta(X))$ and $\epsilon^*(G)(Y) = G(\epsilon(Y))$ are also natural isomorphisms. This gives an equivalence of categories between $[\mathcal{G}', \mathrm{Vect}]$ and $[\mathcal{G}, \mathrm{Vect}]$. □

By applying Lemma 2.14, we only need to use pullbacks in order to calculate the natural transformation from the Dehn twist. We replace the right hand side of diagram (III.18) by one with inverse functors:

$$\begin{tikzcd}
\Gamma \arrow[d, "\mathrm{id}"'] & & \\
\hat{\Gamma} & \Downarrow \theta'_r & G//G \arrow[llu, "t'_1"'] \arrow[lld, "t'_2"] \\
\Gamma \arrow[u, "t'"] & &
\end{tikzcd} \tag{III.19}$$

This diagram only commutes up to a natural isomorphism $\theta'_r : t'_1 \Rightarrow t' \circ t'_2$, since we have on objects $t'_1(h) = (h, 1)$, $t' \circ t'_2(h) = (h, h)$ and on morphisms $t'_1(g) = t' \circ t'_2(g) = (g, g)$. The natural isomorphism is given on an object $h \in G//G$ by $\theta'_r(h) = (h^{-1}, 1)$. The right hand side of diagram (III.18) is strictly commutative, since we have on objects: $s_1(h_1, h_2) = s_2 \circ t(h_1, h_2) = h_1$ and on morphisms $s_1(g_1, g_2) = s_2 \circ t(g_1, g_2) = g_1$.
The pullback $(\theta'_r)^*$ of the natural isomorphisms θ'_r can be calculated to be:

$$\begin{aligned}
&\theta'^*_r : (t'_1)^* \Rightarrow (t' \circ t'_2)^* \\
&\theta'^*_r(V)_{|V_{(h)}} = \rho(h^{-1}, 1) : V_{(h,1)} \to V_{(h,h)}
\end{aligned}$$

Now let $V = ((V_h)_{h \in G}, (\rho_g)_{g \in G}$ be an object in $[G//G, \mathrm{Vect}_{\mathrm{K}}]$, i.e. a G-graded vector space with compatible G-action. Equation (III.8) reduces to the following natural isomorphism:

$$\begin{aligned}\theta := \theta_r'^* \bullet \mathrm{id}^* : (s' \circ t_1')^* \circ (s_1)^* \Rightarrow (t' \circ t_2')^* \circ (s_2 \circ t)^* \\ \theta(V)_{|V_h} = \theta_r'^* \bullet \mathrm{id}^*(V)_{|V_h} &= \theta_r'^*((s_1)^*(V))_{|V_h} \circ (s' \circ t_1')^*(\mathrm{id}^*(V)_{|V_{(h,1)}}) \\ &= (s_1)^*(\rho)(h^{-1}, 1)_{|V_h} \\ &= \rho(h^{-1})_{|V_h}\end{aligned}$$

Now we can state the following:

Proposition 2.15. *The twist on an object V of $[G//G, \mathrm{Vect}_{\mathrm{K}}]$ and a vector $v \in V_h$ is given by*

$$\theta : v \mapsto h^{-1}.v \tag{III.20}$$

2.5 Drinfel'd Double and Modularity

The braided tensor category $\mathcal{C}(G)$ we just computed from the last section has a well-known description as a familiar representation category. In Section 1 of Chapter II we discussed the Drinfel'd double $\mathcal{D}(G)$ of a finite group G and its representation category. In fact, comparison with Propositions 2.12 and 2.13 with the category $\mathcal{D}(G)$-mod described in section 1 of chapter II shows that we have the following equivalence:

Proposition 2.16. *The category $\mathcal{C}(G)$ is isomorphic, as a braided tensor category, to the category $\mathcal{D}(G)$-mod.*

And as a direct consequence of this and Proposition 1.2 we get:

Proposition 2.17. *The category $\mathcal{C}(G) \cong \mathcal{D}(G)$-mod is modular.*

3 Equivariant Dijkgraaf-Witten Theory

We are now ready to turn to the construction of equivariant generalizations of the results of section 2. We denote again by G a finite group. The equivariance will be given with respect to another finite group J that acts on G in a way we will have to explain. As usual, 'twisted sectors' [VW95] have to be taken into account for a consistent equivariant theory. A description of these twisted sectors in terms of bundles twisted by J-covers is one important result of this section.

3.1 Weak Actions and Extensions

Our first task is to identify the appropriate definition of a J-action. The first idea that comes to mind – a genuine action of the group J acting on G by group automorphisms – turns out to need a modification. For reasons that will become apparent in a moment, we only require an action up to inner automorphism.

Definition 3.1. *1. A* weak action *of a group J on a group G consists of a collection of group automorphisms $\rho_j : G \to G$, one automorphism for each $j \in J$, and a collection of group elements $c_{i,j} \in G$, one group element for each pair of elements $i, j \in J$. These data are required to obey the relations:*

$$\rho_i \circ \rho_j = \mathrm{Inn}_{c_{i,j}} \circ \rho_{ij} \qquad \rho_i(c_{j,k}) \cdot c_{i,jk} = c_{i,j} \cdot c_{ij,k} \quad \text{and} \quad c_{1,1} = 1$$

for all $i, j, k \in J$. Here Inn_g denotes the inner automorphism $G \to G$ associated to an element $g \in G$. We will also use the short hand notation ${}^{j}g := \rho_j(g)$.

2. Two weak actions $(\rho_j, c_{i,j})$ and $(\rho'_j, c'_{i,j})$ of a group J on a group G are called isomorphic, if there is a collection of group elements $h_j \in G$, one group element for each $j \in J$, such that

$$\rho'_j = \mathrm{Inn}_{h_j} \circ \rho_j \quad \text{and} \quad c'_{i,j} \cdot h_{ij} = h_i \cdot \rho_i(h_j) \cdot c_{i,j}$$

Remark 3.2. *1. If all group elements $c_{i,j}$ equal the neutral element, $c_{i,j} = 1$, the weak action reduces to a strict action of J on G by group automorphisms.*

2. A weak action induces a strict action of J on the group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ of outer automorphisms.

3. In more abstract terms, a weak action amounts to a (weak) 2-group homomorphism $J \to \mathrm{AUT}(G)$. Here $\mathrm{AUT}(G)$ denotes the automorphism 2-group of G. This automorphism 2-group can be described as the monoidal category of endofunctors of the one-object-category with morphisms G. The group J is considered as a discrete 2-group with only identities as morphisms. Compare also Remark 2.2. For more details on 2-groups, we refer to [BL04].

Weak actions are also known under the name *Dedecker cocycles*, due to the work [Ded60]. The correspondence between weak actions and extensions of groups is also termed *Schreier theory*, with reference to [Sch26]. Let us briefly sketch this correspondence:

- Let $(\rho_j, c_{i,j})$ be a weak action of J on G. On the set $H := G \times J$, we define a multiplication by

$$(g,i) \cdot (g',j) := \big(g \cdot \,{}^{i}(g') \cdot c_{i,j}\ ,\ ij\big). \tag{III.21}$$

One can check that this turns H into a group in such a way that the sequence $G \to H \to J$ consisting of the inclusion $g \mapsto (g,1)$ and the projection $(g,j) \mapsto j$ is exact.

- Conversely, let $G \longrightarrow H \xrightarrow{\pi} J$ be an extension of groups. Choose a set theoretic section $s : J \to H$ of π with $s(1) = 1$. Conjugation with the group element $s(j) \in H$ leaves the normal subgroup G invariant. We thus obtain for $j \in J$ the automorphism $\rho_j(g) := s(j)\ g\ s(j)^{-1}$ of G. Furthermore, the element $c_{i,j} := s(i)s(j)s(ij)^{-1}$ is in the kernel of π and thus actually contained in the normal subgroup G. It is then straightforward to check that $(\rho_j, c_{i,j})$ defines a weak action of J on G.

- Two different set-theoretic sections s and s' of the extension $G \to H \to J$ differ by a map $J \to G$. This map defines an isomorphism of the induced weak actions in the sense of definition 3.1.2.

We have thus arrived at the

Proposition 3.3 (Dedecker, Schreier)**.** *There is a 1-1 correspondence between isomorphism classes of weak actions of J on G and isomorphism classes of group extensions $G \to H \to J$.*

Remark 3.4.

1. *The correspondence from Proposition 3.3 can be turned into an equivalence of categories. Since we do not need such a statement in the following, we refrain from giving a precise formulation of the equivalence.*

2. *Under this correspondence,* strict actions *of J on G correspond to* split extensions. *This can be easily seen as follows: given a* split *extension $G \to H \to J$, one can choose the section $J \to H$ as a group homomorphism and thus obtains a* strict *action of J on G. Conversely for a strict action of J on G it is easy to see that the group constructed in equation (III.21) is a semidirect product and thus the sequence of groups splits. In order to take all extensions into account, we thus really need to consider weak actions.*

3.2 Twisted Bundles

It is a common lesson from field theory that in an equivariant situation, one has to include "twisted sectors" to obtain a complete theory. Our next task is to construct the parameters labeling twisted sectors for a given weak action of a finite group J on G, with corresponding extension $G \to H \to J$ of groups and chosen set-theoretic section $J \to H$. We will adhere to a two-step procedure, similar to the procedure in the non-equivalent case outlined in Section 2.3. To this end, we will first construct a category of twisted bundles for any smooth manifold. Then, the linearization functor can be applied to spans of such categories.
We start our discussion of twisted G-bundles with the most familiar case of the circle $M = \mathbb{S}^1$.
The isomorphism classes of G-bundles on $\mathbb{S}^1$ are in bijection to connected components of the free loop space $\mathcal{L}BG$ of the classifying space BG:

$$Iso\big(\mathcal{A}_G(\mathbb{S}^1)\big) = \mathrm{Hom}_{\mathrm{Ho(Top)}}(\mathbb{S}^1, BG) = \pi_0(\mathcal{L}BG).$$

Given a (weak) action of J on G, one can introduce twisted loop spaces. For any element $j \in J$, we have a group automorphism $j : G \to G$ and thus a homeomorphism $j : BG \to BG$. The j-twisted loop space is then defined to be

$$\mathcal{L}^j BG := \big\{ f : [0,1] \to BG \mid f(0) = j \cdot f(1) \big\}.$$

Our goal is to introduce for every group element $j \in J$ a category $\mathcal{A}_G(\mathbb{S}^1, j)$ of j-twisted G-bundles on $\mathbb{S}^1$ such that

$$Iso\big(\mathcal{A}_G(\mathbb{S}^1, j)\big) = \pi_0(\mathcal{L}^j BG) \ .$$

In the case of the circle $\mathbb{S}^1$, the twist parameter was a group element $j \in J$. A more geometric description uses a family of J-covers P_j over $\mathbb{S}^1$, with $j \in J$. The cover P_j is uniquely determined by its monodromy j for the base point $1 \in \mathbb{S}^1$ and a fixed point in the fiber over 1. A concrete construction of the cover P_j is given by the quotient $P_j := [0,1] \times J/\sim$ where $(0,i) \sim (1, ji)$ for all $i \in J$. In terms of these J-covers, we can write

$$\mathcal{L}^j BG = \big\{ f : P_j \to BG \mid f \text{ is } J\text{-equivariant} \big\}.$$

This description generalizes to an arbitrary smooth manifold M. The natural twist parameter in the general case is a J-cover $P \xrightarrow{J} M$.
Suppose, we have a weak J-action on G and construct the corresponding extension $G \to H \xrightarrow{\pi} J$. The category of bundles we need are H-lifts of the given J-cover:

Definition 3.5. *Let J act weakly on G. Let $P \xrightarrow{J} M$ be a J-cover over M.*

- *A P-twisted G-bundle over M is a pair (Q, φ), consisting of an H-bundle Q over M and a smooth map $\varphi : Q \to P$ over M that is required to obey*
$$\varphi(q \cdot h) = \varphi(q) \cdot \pi(h)$$
for all $q \in Q$ and $h \in H$. Put differently, a $P \xrightarrow{J} M$-twisted G-bundle is a lift of the J-cover P reduction along the group homomorphism $\pi : H \to J$.

- *A morphism of P-twisted bundles (Q, φ) and (Q', φ') is a morphism $f : Q \to Q'$ of H-bundles such that $\varphi' \circ f = \varphi$.*

- *We denote the category of P-twisted G-bundles by $\mathcal{A}_G\big(P \to M\big)$. For $M = \mathbb{S}^1$, we introduce the abbreviation $\mathcal{A}_G\big(\mathbb{S}^1, j\big) := \mathcal{A}_G\big(P_j \to \mathbb{S}^1\big)$ for the standard covers of the circle.*

Remark 3.6. *There is an alternative point of view on a P-twisted bundle (Q, φ): the subgroup $G \subset H$ acts on the total space Q in such a way that the map $\varphi : Q \to P$ endows Q with the structure of a G-bundle on P. Both the structure group H of the bundle Q and the bundle P itself carry an action of G; for twisted bundles, an equivariance condition on this action has to be imposed. Unfortunately this equivariance property is relatively involved; therefore, we have opted for the definition in the form given above.*

A morphism $f : P \to P'$ of J-covers over the same manifold induces a functor $f_* : \mathcal{A}_G\big(P \to M\big) \to \mathcal{A}_G\big(P' \to M\big)$ by $f_*(Q, \varphi) := (Q, f \circ \varphi)$. Furthermore, for a smooth map $f : M \to N$, we can pull back the twist data $P \to M$ and get a *pullback functor* of twisted G-bundles:
$$f^* : \mathcal{A}_G\big(P \to N\big) \to \mathcal{A}_G\big(f^*P \to M\big)$$
by $f^*(Q, \varphi) = (f^*Q, f^*\varphi)$. Before we discuss more sophisticated properties of twisted bundles, we have to make sure that our definition is consistent with 'untwisted' bundles:

Lemma 3.7. *Let the group J act weakly on the group G. For G-bundles twisted by the trivial J-cover $M \times J \to M$, we have a canonical equivalence of categories*
$$\mathcal{A}_G\big(M \times J \to M\big) \cong \mathcal{A}_G(M).$$

Proof. We have to show that for an element $(Q,\varphi) \in \mathcal{A}_G\big(M \times J \to M\big)$ the H-bundle Q can be reduced to a G-bundle. Such a reduction is the same as a section of the associated fiber bundle $\pi_*(Q) \in \mathcal{B}un_J(M)$ see e.g. [Bau09, Satz 2.14]). Now $\varphi : Q \to M \times J$ induces an isomorphism of J-covers $Q \times_H J \cong (M \times J) \times_H J \cong M \times J$ so that the bundle $Q \times_H J$ is trivial as a J-cover and in particular admits global sections.
Since morphisms of twisted bundles have to commute with these sections, we obtain in that way a functor $\mathcal{A}_G\big(M \times J \to M\big) \to \mathcal{A}_G(M)$. Its inverse is given by extension of G-bundles on M to H-bundles on M. □

We also give a description of twisted bundles using standard covering theory; for an alternative description using Čech-cohomology, we refer to appendix 6.1. We start by recalling the following standard fact from covering theory, see e.g. [Hat02, 1.3] that has already been used to prove proposition 2.11: for a finite group J, the category of J-covers is equivalent to the action groupoid $\mathrm{Hom}(\pi_1(M), J)//J$. (Note that this equivalence involves choices and is not canonical.)
To give a similar description of twisted bundles, fix a J-cover P. Next, we choose a basepoint $m \in M$ and a point p in the fiber P_m over m. These data determine a unique group morphism $\omega : \pi_1(M,m) \to J$ representing P.

Proposition 3.8. *Let J act weakly on G. Let M be a connected manifold and P be a J-cover over M represented after the choices just indicated by the group homomorphism $\omega : \pi_1(M) \to J$. Then there is a (non-canonical) equivalence of categories*

$$\mathcal{A}_G\big(P \to M\big) \cong \mathrm{Hom}^\omega\big(\pi_1(M), H\big)//G$$

where we consider group homomorphisms

$$\mathrm{Hom}^\omega\big(\pi_1(M), H\big) := \big\{\mu : \pi_1(M) \to H \mid \pi \circ \mu = \omega\big\}$$

whose composition restricts to the group homomorphism ω describing the J-cover P. The group G acts on $\mathrm{Hom}^\omega\big(\pi_1(M), H\big)$ via pointwise conjugation using the inclusion $G \to H$.

Proof. Let $m \in M$ and $p \in P$ over m be the choices of base point in the J-cover $P \to M$ that lead to the homomorphism ω. Consider a $(P \to M)$ twisted bundle $Q \to M$. Since $\varphi :\ Q \to P$ is surjective, we can choose a base point q in the fiber of Q over m such that $\varphi(q) = p$. The group homomorphism $\pi_1(M) \to H$ describing the H-bundle Q is obtained by lifting closed paths in M starting in m to paths in Q starting in q. They are mapped under φ to lifts of the same path to P starting at p, and these lifts are just described by the group homomorphism $\omega : \pi_1(M) \to J$ describing the cover P. If the end point of the path in Q is qh for some $h \in H$, then by the defining property of φ, the lifted path in P has endpoint $\varphi(qh) = \varphi(q)\pi(h) = p\pi(h)$. Thus $\pi \circ \mu = \omega$.

□

Remark 3.9. *For non-connected manifolds, a description as in proposition 3.8 can be obtained for every component. Again the equivalence involves choices of base points on M and in the fibers over the base points. This could be fixed by working with pointed manifolds, but pointed manifolds cause problems when we consider cobordisms. Alternatively, we could use the fundamental groupoid instead of the fundamental group, see e.g. [May99].*

Example 3.10. *We now calculate the categories of twisted bundles over certain manifolds using Proposition 3.8.*

1. *For the circle $\mathbb{S}^1$, $\omega \in \mathrm{Hom}(\pi_1(\mathbb{S}^1), J) = \mathrm{Hom}(\mathbb{Z}, J)$ is determined by an element $j \in J$ and the condition $\pi \circ \mu = \omega$ requires $\mu(1) \in H$ to be in the preimage $H_j := \pi^{-1}(j)$ of j. Thus, we have $\mathcal{A}_G(\mathbb{S}^1, j) \cong H_j//G$.*
2. *For the 3-Sphere $\mathbb{S}^3$, all twists P and all G-bundles are trivial. Thus, we have $\mathcal{A}_G(P \to \mathbb{S}^3) \cong \mathcal{A}_G(\mathbb{S}^3) \cong pt//G$.*

3.3 Equivariant Dijkgraaf-Witten Theory

The key idea in the construction of equivariant Dijkgraaf-Witten theory is to take twisted bundles $\mathcal{A}_G(P \to M)$ as the field configurations, taking the place of G-bundles in Section 2. We then cannot expect to get invariants of closed 3-manifolds M, but rather invariants of 3-manifolds M together with a twist datum, i.e. a J-cover P over M. Analogous statements apply to manifolds with boundary and cobordisms. Therefore we need to generalize the Definition 1.6 of Chapter I of the category $\mathfrak{Cob}(1,2,3)$ to an equivariant version, i.e. we need to endow it with an extra datum of a J-cover over each manifold.

Definition 3.11. $\mathfrak{Cob}^J(1,2,3)$ *is the following symmetric monoidal bicategory:*

- *Objects are compact, closed, oriented 1-manifolds S, together with a J-cover $P_S \xrightarrow{J} S$.*
- *1-Morphisms are collared cobordisms*

$$S \times I \hookrightarrow \Sigma \hookleftarrow S' \times I$$

where Σ is a 2-dimensional, compact, oriented cobordism, together with a J-cover $P_\Sigma \to \Sigma$ and isomorphisms

$$P_\Sigma|_{(S\times I)} \xrightarrow{\sim} P_S \times I \qquad \text{and} \qquad P_\Sigma|_{(S'\times I)} \xrightarrow{\sim} P_{S'} \times I.$$

over the collars.

- *2-Morphisms are generated by*
  - *orientation preserving diffeomorphisms $\varphi : \Sigma \to \Sigma'$ of cobordisms fixing the collar together with an isomorphism $\widetilde{\varphi} : P_\Sigma \to P_{\Sigma'}$ covering φ.*
  - *3-dimensional collared, oriented cobordisms with corners M with cover $P_M \to M$ together with covering isomorphisms over the collars (as before) up to diffeomorphisms preserving the orientation and boundary.*
- *Composition is by gluing cobordisms and covers along collars.*
- *The monoidal structure is given by disjoint union.*

Remark 3.12. *In analogy to remark 1.7, we point out that the isomorphisms of covers are defined over the collars, rather than only over the the boundaries. This endows the glued cover with a well-defined smooth structure.*

Definition 3.13. *An extended* 3d *J-TFT is a symmetric monoidal 2-functor*

$$Z : \mathfrak{Cob}^J(1,2,3) \to 2\mathrm{Vect}_{\mathrm{K}}.$$

Just for the sake of completeness, we will also give a definition of non-extended J-TFT. Therefore define the symmetric monoidal category $\mathfrak{Cob}^J(2,3)$ to be the endomorphism category of the monoidal unit $\emptyset$ in $\mathfrak{Cob}(1,2,3)$. More concretely, this category has as objects closed, oriented 2-manifolds with J-cover and as morphisms J-cobordisms between them.

Definition 3.14. *A (non-extended)* 3d *J-TFT is a symmetric monoidal functor*

$$\mathfrak{Cob}^J(2,3) \to \mathrm{Vect}_{\mathrm{K}}.$$

Similarly as in the non-equivariant case (Lemma 1.9), we get

Lemma 3.15. *Let Z be an extended* 3d *J-TFT. Then $Z|_\emptyset$ is a (non-extended)* 3d *J-TFT.*

Now we can state the main result of this section:

Theorem 3.16. *For a finite group G and a weak J-action on G, there is an extended 3d J-TFT called Z_G^J which assigns the categories*

$$\mathrm{Vect}_{\mathrm{K}}\big(\mathcal{A}_G(P \to S)\big) = \big[\mathcal{A}_G(P \to S), \mathrm{Vect}_{\mathrm{K}}\big]$$

to 1-dimensional, closed oriented manifolds S with J-cover $P \to S$.

We will give a proof of this theorem in the next sections. Having twisted bundles at our disposal, the main ingredient will again be the 2-linearization described in Section 2.3.

3.4 Construction via Spans

As in the case of ordinary Dijkgraaf-Witten theory, cf. Section 2.3, equivariant Dijkgraaf-Witten Z_G^J theory is constructed as the composition of the symmetric monoidal 2-functors

$$\widetilde{\mathcal{A}_G} : \mathfrak{C}\mathrm{ob}^J(1,2,3) \to \mathfrak{S}\mathrm{pan} \quad \text{and} \quad \widetilde{\mathcal{V}_{\mathrm{K}}} : \mathfrak{S}\mathrm{pan} \to 2\mathrm{Vect}_{\mathrm{K}}.$$

The second functor will be exactly the 2-linearization functor of proposition 2.9. Hence we can limit our discussion to the construction of the first functor $\widetilde{\mathcal{A}_G}$. As it will turn out, our definition of twisted bundles is set up precisely in such a way that the construction of the corresponding functor in Proposition 2.7 can be generalized.

Our starting point is the following observation:

- The assignment $(P_M \overset{J}{\to} M) \longmapsto \mathcal{A}_G(P_M \overset{J}{\to} M)$ of twisted bundles to a twist datum $P_M \to M$ constitutes a contravariant 2-functor from the category of manifolds with J-cover to the 2-category of groupoids. Maps between manifolds with cover are mapped to the corresponding pullback functors of bundles.

From this functor which is defined on manifolds of any dimension, we construct a functor $\widetilde{\mathcal{A}_G}$ on J-cobordisms with values in the 2-category $\mathfrak{S}\mathrm{pan}$ of spans of groupoids, where the category $\mathfrak{S}\mathrm{pan}$ is defined in section 2.3. To an object in $\mathfrak{C}\mathrm{ob}^J(1,2,3)$, i.e. to a J-cover $P_S \to M$, we assign the category $\mathcal{A}_G(P_S \to S)$ of J-covers. To a 1-morphism $P_S \hookrightarrow P_\Sigma \hookleftarrow P'_S$ in $\mathfrak{C}\mathrm{ob}^J(1,2,3)$, we associate the span

$$\mathcal{A}_G(P_S \to S) \leftarrow \mathcal{A}_G(P_\Sigma \to \Sigma) \to \mathcal{A}_G(P_{S'} \to S') \tag{III.22}$$

and to a 2-morphism of the type $P_\Sigma \hookrightarrow P_M \hookleftarrow P_{\Sigma'}$ the span

$$\mathcal{A}_G(P_\Sigma \to \Sigma) \leftarrow \mathcal{A}_G(P_M \to M) \to \mathcal{A}_G(P_{\Sigma'} \to \Sigma'). \tag{III.23}$$

We have to show that this defines a symmetric monoidal functor $\widetilde{\mathcal{A}_G}$: $\mathfrak{Cob}^J(1,2,3) \to \mathfrak{Span}$.
In particular, we have to show that the composition of morphisms is respected.

Lemma 3.17. *Let $P_\Sigma \to \Sigma$ and $P_{\Sigma'} \to \Sigma'$ be two 1-morphisms in $\mathfrak{Cob}^J(1,2,3)$ which can be composed at the object $P_S \to S$ to get the 1-morphism*

$$P_\Sigma \circ P_{\Sigma'} := \left(P_\Sigma \sqcup_{P_S \times I} P_{\Sigma'} \to \Sigma \sqcup_{S\times I} \Sigma'\right) ,$$

where $I = [0,1]$ is the standard interval. (Recall that we are gluing over collars.) Then the category $\mathcal{A}_G\big(P_\Sigma \circ P_{\Sigma'}\big)$ is the weak pullback of $\mathcal{A}_G(P_\Sigma \to \Sigma)$ and $\mathcal{A}_G(P_{\Sigma'} \to \Sigma')$ over $\mathcal{A}_G(P_S \to S)$.

Proof. By definition the category

$$\mathcal{A}_G\big(P_\Sigma \circ P_{\Sigma'}\big)$$

has as objects twisted G-bundles over the 2-manifold $\Sigma \sqcup_{S\times I} \Sigma' =: N$. The manifold N admits an open covering $N = U_0 \cup U_1$ with $U_0 = \Sigma \setminus S$ and $U_1 = \Sigma' \setminus S$ where the intersection is the cylinder $U_0 \cap U_1 = S \times (0,1)$. By construction, the restrictions of the glued bundle $P_N \to N$ to U_0 and U_1 are given by $P_\Sigma \setminus P_S$ and $P_{\Sigma'} \setminus P_S$.
The natural inclusions $U_0 \to \Sigma$ and $U_1 \to \Sigma'$ induce equivalences

$$\mathcal{A}_G(P_\Sigma \to \Sigma) \xrightarrow{\sim} \mathcal{A}_G(P_N|_{U_0} \to U_0)$$
$$\mathcal{A}_G(P_{\Sigma'} \to \Sigma') \xrightarrow{\sim} \mathcal{A}_G(P_N|_{U_1} \to U_1)$$

Analogously, we have an equivalence

$$\mathcal{A}_G\big(P_N|_{U_0\cap U_1} \to U_0 \cap U_1\big) \xrightarrow{\sim} \mathcal{A}_G(P_S \to S) .$$

At this point, we have reduced the claim to an assertion about descent of twisted bundles which we will prove in corollary 3.20. This corollary implies that $\mathcal{A}_G(P_N \to N)$ is the weak pullback of $\mathcal{A}_G(P_N|_{U_0} \to U_0)$ and $\mathcal{A}_G(P_N|_{U_1} \to U_1)$ over $\mathcal{A}_G\big(P_N|_{U_0\cap U_1}\big)$. Since weak pullbacks are invariant under equivalence of groupoids, this shows the claim. □

We now turn to the promised results about descent of twisted bundles. Let $P \to M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M, where for the sake of generality we allow for arbitrary open coverings. We want to show that twisted bundles can be glued together like ordinary bundles; while the precise meaning of this statement is straightforward, we briefly summarize the relevant definitions for the sake of completeness:

Definition 3.18. *Let $P \to M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M. The descent category $\mathcal{D}\mathrm{esc}(U_\alpha, P)$ has*

- *Objects: families of $P|_{U_\alpha}$-twisted bundles Q_α over U_α, together with isomorphisms of twisted bundles $\varphi_{\alpha\beta} : Q_\alpha|_{U_\alpha \cap U_\beta} \xrightarrow{\sim} Q_\beta|_{U_\alpha \cap U_\beta}$ satisfying the cocycle condition $\varphi_{\alpha\beta} \circ \varphi_{\beta\gamma} = \varphi_{\alpha\gamma}$.*
- *Morphisms: families of morphisms $f_\alpha : Q_\alpha \to Q'_\alpha$ of twisted bundles such that over $U_{\alpha\beta}$ we have $\varphi'_{\alpha\beta} \circ (f_\alpha)|_{U_{\alpha\beta}} = (f_\beta)|_{U_{\alpha\beta}} \circ \varphi_{\alpha\beta}$.*

Proposition 3.19 (Descent for twisted bundles)**.** *Let $P \to M$ be a J-cover over a manifold M and $\{U_\alpha\}$ be an open covering of M. Then the groupoid $\mathcal{A}_G(P \to M)$ is equivalent to the descent category $\mathcal{D}\mathrm{esc}(U_\alpha, P)$.*

Proof. Note that the corresponding statements are true for H-bundles and for J-covers. Then the description in Definition 3.5 of a twisted bundle as an H-bundle together with a morphism of the associated J-cover immediately implies the claim. □

Corollary 3.20. *For an open covering of M by two open sets U_0 and U_1 the category $\mathcal{A}_G(P \to M)$ is the weak pullback of $\mathcal{A}_G(P|_{U_0} \to U_0)$ and $\mathcal{A}_G(P|_{U_1} \to U_1)$ over $\mathcal{A}_G(P|_{U_0 \cap U_1} \to U_0 \cap U_1)$.*

In order to prove that the assignment (III.22) and (III.23) really promotes $\mathcal{A}_G$ to a symmetric monoidal functor $\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \to \mathfrak{Span}$, it remains to show that $\mathcal{A}_G$ preserves the monoidal structure.
Now a bundle over a disjoint union is given by a pair of bundles over each component. Thus, for a disjoint union of J-manifolds $P \to M = (P_1 \sqcup P_2) \to (M_1 \sqcup M_2)$, we have $\mathcal{A}_G(P \to M) \cong \mathcal{A}_G(P_1 \to M_1) \times \mathcal{A}_G(P_2 \to M_2)$. Note that the manifolds M, M_1 and M_2 can also be cobordisms. The isomorphism of categories is clearly associative and preserves the symmetric structure. Together with lemma 3.17, this proves the next proposition.

Proposition 3.21. *$\mathcal{A}_G$ induces a symmetric monoidal functor*

$$\widetilde{\mathcal{A}_G} : \mathfrak{Cob}^J(1,2,3) \to \mathfrak{Span}$$

which assigns the spans (III.22) *and* (III.23) *to 2 and 3-dimensional cobordisms with J-cover.*

3.5 Twisted Sectors and Fusion

We next proceed to evaluate the J-equivariant TFT Z_G^J constructed in the last section on the circle, as we did in Section 2.4 for the non-equivariant

TFT. We recall from Section 3.2 the fact that over the circle $\mathbb{S}^1$ we have for each $j \in J$ a standard cover P_j. The associated category

$$\mathcal{C}(G)_j := Z_G^J\big(P_j \to \mathbb{S}^1\big)$$

is called the j-twisted sector of the theory; the sector $\mathcal{C}(G)_1$ is called the neutral sector. By Lemma 3.7, we have an equivalence $\mathcal{A}_G(P_1 \to \mathbb{S}^1) \cong \mathcal{A}_G(\mathbb{S}^1)$; hence we get an equivalence of categories $\mathcal{C}(G)_1 \cong \mathcal{C}(G)$, where $\mathcal{C}(G)$ is the category arising in the non-equivariant Dijkgraaf-Witten model, we discussed in Section 2.4. We have already computed the twisted sectors as abelian categories in Example 3.10 and note the result for future reference:

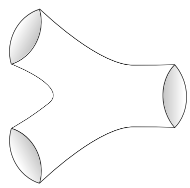
Proposition 3.22. *For the j-twisted sector of equivariant Dijkgraaf-Witten theory, we have an equivalence of abelian categories*

$$\mathcal{C}(G)_j \cong [H_j//G, \mathrm{Vect}_{\mathrm{K}}] \ ,$$

where $H_j//G$ is the action groupoid given by the conjugation action of G on $H_j := \pi^{-1}(j)$. More concretely, the category $\mathcal{C}(G)_j$ is equivalent to the category of H_j-graded vector spaces $V = \bigoplus_{h \in H_j} V_h$ together with a G-action on V such that

$$g.V_h \subset V_{ghg^{-1}} .$$

As a next step, we want to make explicit additional structure on the categories $\mathcal{C}(G)_j$ coming from certain cobordisms. Therefore, consider the pair of pants $\Sigma(2,1)$:



The fundamental group of $\Sigma(2,1)$ is the free group on two generators. Thus, given a pair of group elements $j, k \in J$, there is a J-cover $P_{j,k}^{\Sigma(2,1)} \to \Sigma(2,1)$ which restricts to the standard covers P_j and P_k on the two ingoing boundaries and to the standard cover P_{jk} on the outgoing boundary circle. (To find a concrete construction, one should fix a parametrization of the pair of pants $\Sigma(2,1)$.) The cobordism $P_{j,k}^{\Sigma(2,1)}$ is a morphism

$$P_{j,k}^{\Sigma(2,1)} : \big(P_j \to \mathbb{S}^1\big) \sqcup \big(P_k \to \mathbb{S}^1\big) \longrightarrow \big(P_{jk} \to \mathbb{S}^1\big) \tag{III.24}$$

in the category $\mathfrak{Cob}^J(1,2,3)$. Applying the equivariant TFT-functor Z_G^J yields a functor

$$\otimes_{jk} : \quad \mathcal{C}(G)_j \boxtimes \mathcal{C}(G)_k \longrightarrow \mathcal{C}(G)_{jk}.$$

We describe this functor in terms of the equivalent categories of graded vector spaces as a functor

$$H_j//G\text{-mod} \times H_k//G\text{-mod} \to H_{jk}//G\text{-mod} \ .$$

Proposition 3.23. *For* $V = \bigoplus_{h \in H_j} V_h \in H_j//G\text{-mod}$ *and* $W = \bigoplus W_h \in H_k//G\text{-mod}$ *the product* $V \otimes_{jk} W \in H_{jk}//G\text{-mod}$ *is given by*

$$(V \otimes_{jk} W)_h = \bigoplus_{st=h} V_s \otimes W_t$$

together with the action $g.(v \otimes w) = g.v \otimes g.w$.

Proof. As a first step we have to compute the span $\widetilde{\mathcal{A}_G}(P^{\Sigma(2,1)}_{j,k})$ associated to the cobordism $P^H_{j,k}$. From the description of twisted bundles in Proposition 3.8 and the fact that the fundamental group of $\Sigma(2,1)$ is the free group on two generators, we derive the following equivalence of categories:

$$\mathcal{A}_G\big(P^{\Sigma(2,1)}_{jk} \to \Sigma(2,1)\big) \cong (H_j \times H_k)//G \ .$$

Here we have $H_j \times H_k = \{(h, h') \in H \times H \mid \pi(h) = j,\ \pi(h') = k\}$, on which G acts by simultaneous conjugation. This leads to the span of action groupoids

$$H_j//G \times H_k//G \longleftarrow (H_j \times H_k)//G \longrightarrow H_{jk}//G$$

where the left map is given be projection to the factors and the right hand map by multiplication. Applying the 2-linearization functor $\widetilde{\mathcal{V}_{\mathbb{K}}}$ from Proposition 2.9 amounts to computing the corresponding pull-push functor. This yields the result. $\square$

Next, we consider the 2-manifold $\Sigma(1,1)$ given by the cylinder over $\mathbb{S}^1$, i.e. $\Sigma(1,1) = \mathbb{S}^1 \times I$:

There exists a cover $P^{\Sigma(1,1)}_{j,x} \to \Sigma(1,1)$ for $j, x \in J$ that restricts to P_j on the ingoing circle and to $P_{xjx^{-1}}$ on the outgoing circle. The simplest way to construct such a cover is to consider the cylinder $P_{xjx^{-1}} \times I \to \mathbb{S}^1 \times I$ and to use the identification of $P^{\Sigma(1,1)}_{j,x}$ over (a collaring neighborhood of) the outgoing circle by the identity and over the ingoing circle the identification by the morphism $P_{\Sigma(1,1)}|_{\mathbb{S}^1\times 1} = P_j \to P_{x^{-1}jx}$ given by conjugation with x. In this way, we obtain a cobordism that is a 1-morphism

$$P^{\Sigma(1,1)}_{j,x} : (P_j \to \mathbb{S}^1) \longrightarrow (P_{xjx^{-1}} \to \mathbb{S}^1) \tag{III.25}$$

in the category $\mathfrak{Cob}^J(1,2,3)$ and hence induces a functor

$$\phi_x : \mathcal{C}(G)_j \to \mathcal{C}(G)_{xjx^{-1}}.$$

We compute the functor on the equivalent action groupoids explicitly:

Proposition 3.24. *The image under ϕ_x of an object $V = \bigoplus V_h \in H_j//G$-mod is the graded vector space with homogeneous component*

$$\phi_x(V)_h = V_{s(x^{-1})hs(x^{-1})^{-1}}$$

for $h \in H_{xjx^{-1}}$ and with G-action on $v \in V_h$ given by $s(x^{-1})gs\,(x^{-1})^{-1} \cdot v$.

Proof. As before we compute the span $\widetilde{\mathcal{A}_G}(P^{\Sigma(1,1)}_{j,x})$. Using explicitly the equivalence given in the proof of proposition 3.8, we obtain the span of action groupoids

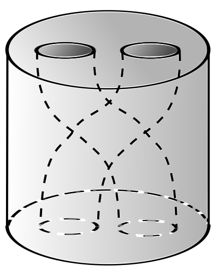
$$H_j//G \leftarrow H_{xjx^{-1}}//G \to H_{xjx^{-1}}//G$$

where the right-hand map is the identity and the left-hand map is given by

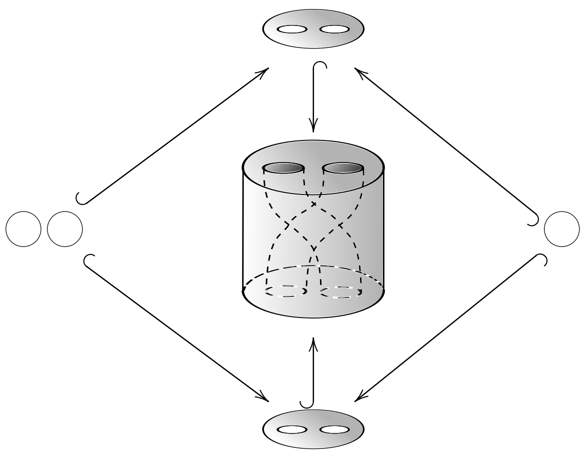
$$(h,g) \mapsto \left(s(x^{-1})hs(x^{-1})^{-1}, s(x^{-1})gs(x^{-1})^{-1}\right) .$$

Computing the corresponding pull-push functor, which here in fact only consists of a pullback, shows the claim. □

Finally we come to the structure corresponding to the braiding of section 2.4. Note that the cobordism that interchanges the two ingoing circles of the pair of pants $\Sigma(2,1)$, as in the following picture,



can also be realized as the diffeomorphism $F : \Sigma(2,1) \to \Sigma(2,1)$ of the pair of pants that rotates the ingoing circles counterclockwise around each other and leaves the outgoing circle fixed. In this picture, we think of the cobordism as the cylinder $\Sigma(2,1) \times I$ where the identification with $\Sigma(2,1)$ on the top is the identity and on the bottom is given by the diffeomorphism F. More explicitly, denote by $\tau : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^1 \times \mathbb{S}^1$ the map that interchanges the two copies. We then again consider the following diagram (the same as diagram III.11) in the 2-category $\mathfrak{Cob}(1,2,3)$:



Our next task is to lift this situation to manifolds with J-covers. On the ingoing pair of pants, we take the J cover $P^{\Sigma(2,1)}_{jk}$. We denote the symmetry isomorphism in $\mathfrak{Cob}^J(1,2,3)$ by τ as well. Applying the diffeomorphism of the pair of pants explicitly, one sees that the outgoing pair of pants will have monodromies jkj^{-1} and j on the ingoing circles. Hence we have to apply a J-cover $P^{\Sigma(1,1)}_{j,k}$ of the cylinder $\Sigma(1,1)$ first to one insertion. The next lemma asserts that then the 2-morphism in $\mathfrak{Cob}^J(1,2,3)$ is fixed:

Lemma 3.25. *In the 2-category* $\mathfrak{Cob}^J(1,2,3)$*, there is a unique 2-morphism*

$$\hat{F} : \quad P^{\Sigma(2,1)}_{j,k} \Longrightarrow \left(P^{\Sigma(2,1)}_{jkj^{-1},j}\right) \circ \tau \circ \left(\mathrm{id} \sqcup P^{\Sigma(1,1)}_{j,k}\right)$$

that covers the 2-morphism F in $\mathfrak{Cob}(1,2,3)$*.*

Proof. First we show that a morphism $\tilde{F} : P^{\Sigma(2,1)}_{jkj^{-1},j} \to P^{\Sigma(2,1)}_{j,k}$ can be found that covers the diffeomorphism $F : \Sigma(2,1) \to \Sigma(2,1)$. This morphism is most easily described using the action of F on the fundamental group $\pi_1(\Sigma(2,1))$ of the pair of pants. The latter is a free group with two generators which can be chosen as the paths a, b around the two ingoing circles, $\pi_1(\Sigma(2,1)) = \mathbb{Z} * \mathbb{Z} = \langle a, b \rangle$. Then the induced action of F on the generators is $\pi_1(F)(a) = aba^{-1}$

and $\pi_1(F)(b) = a$. Hence, we find on the covers $F^*P_{j,k} \cong P_{jkj^{-1},j}$. This implies that we have a diffeomorphism $\tilde{F} : P_{jkj^{-1},j} \to P_{j,k}$ covering F.
To extend $\tilde{F}$ to a 2-morphism in $\mathfrak{Cob}^J(1,2,3)$, we have to be a bit careful about how we consider the cover $P^{\Sigma(2,1)}_{jkj^{-1},j} \to \Sigma(2,1)$ of the pair of pants as a 1-morphism. In fact, it has to be considered as a morphism $(P_j \to \mathbb{S}^1) \sqcup (P_k \to \mathbb{S}^1) \longrightarrow (P_{jk} \to \mathbb{S}^1)$ where the ingoing components are first exchanged and then the identification of $P_k \to \mathbb{S}^1$ and $P_{jkj^{-1}} \to \mathbb{S}^1$ via the conjugation isomorphisms $P^{\Sigma(1,1)}_{j,k}$ induced by covers of the cylinders is used first, compare the lower arrows in the preceding commuting diagram. This yields the composition $\big(P^{\Sigma(2,1)}_{jkj^{-1},j}\big) \circ \tau \circ \big(\mathrm{id} \sqcup P^{\Sigma(1,1)}_{j,k}\big)$ on the right hand side of the diagram. □

The next step is to apply the TFT functor Z^J_G to the 2-morphism $\hat{F}$. The target 1-morphism of $\hat{F}$ can be computed using the fact that Z^J_G is a symmetric monoidal 2-functor; we find the following functor $\mathcal{C}(G)_j \otimes \mathcal{C}(G)_k \to \mathcal{C}(G)_{jk}$:

$$Z^J_G\Big(\big(P^{\Sigma(2,1)}_{jkj^{-1},j}\big) \circ \tau \circ \big(\mathrm{id} \sqcup P^{\Sigma(1,1)}_{j,k}\big)\Big) = (-)^j \otimes^{op}_{jkj^{-1},j} (-)$$

We thus have the functor which acts on objects as $(V,W) \mapsto \phi_j(W) \otimes V$ for $V \in \mathcal{C}(G)_j$ and $W \in \mathcal{C}(G)_k$.
Then $c := Z^J_G(\hat{F})$ is a natural transformation $(-) \otimes_{j,k} (-) \Longrightarrow (-)^j \otimes^{op}_{jkj^{-1},j} (-)$ i.e. a family of isomorphisms

$$c_{V,W} : V \otimes_{j,k} W \xrightarrow{\sim} \quad \phi_j(W) \otimes_{jkj^{-1},j} V \tag{III.26}$$

in $\mathcal{C}(G)_{jk}$ for $V \in \mathcal{C}(G)_j$ and $W \in \mathcal{C}(G)_k$.
We next show how this natural transformation is expressed when we use the equivalent description of the categories $\mathcal{C}(G)_j$ as vector bundles on action groupoids:

Proposition 3.26. *For $V = \bigoplus V_h \in H_j//G$-mod and $W = \bigoplus W_h \in H_k//G$-mod the natural isomorphism $c_{V,W} : V \otimes W \to \phi_j(W) \otimes V$ is given by*

$$v \otimes w \mapsto (s(j^{-1})h).w \otimes v$$

for $v \in V_h$ with $h \in H_j$ and $w \in W$.

Proof. We first compute the 1-morphism in the category $\mathfrak{Span}$ of spans of finite groupoids that corresponds to the target 1-morphism $\big(P^{\Sigma(2,1)}_{jkj^{-1},j}\big) \circ \tau \circ \big(id \sqcup P^{\Sigma(1,1)}_{j,k}\big)$. From the previous proposition, we obtain the following zig-zag diagram:

$$H_j//G \times H_k//G \leftarrow H_{jkj^{-1}}//G \times H_j//G \leftarrow (H_{jkj^{-1}} \times H_k)//G \to H_{jk}//G \ .$$

The first morphism is given by the morphisms implementing the J-action that has been computed in the proof of proposition 3.24, composed with the exchange of factors. The second 1-morphism is obtained from the two projections and the last 1-morphism is the product in the group H.
Thus, the 2-morphism $\hat{F}$ from Lemma 3.25 yields a 2-morphism $\hat{F}_G$ in the diagram

$$\begin{tikzcd}
 & & H_j \times H_k//G \arrow[dll] \arrow[dr] \arrow[dd, Rightarrow, "\hat{F}_G"] & \\
H_j//G \times H_k//G & & & H_{jk}//G \\
 & H_{jkj^{-1}}//G \times H_j//G \arrow[ul] & (H_{jkj^{-1}} \times H_j)//G \arrow[l] \arrow[ur] &
\end{tikzcd}$$

where $\hat{F}_G$ is induced by the equivariant map $(h, h') \mapsto (hh'h^{-1}, h)$. Once the situation is presented in this way, one can carry out explicitly the calculation along the lines described in [Mor10, Section 4.3] and obtain the result. □

A similar discussion can in principle be carried out to compute the associators. More generally, structural morphisms on $H//G$-mod can be derived from suitable 3-cobordisms. The relevant computations become rather involved. On the other hand, the category $H//G$-mod also inherits structural morphisms from the underlying category of vector spaces. We will use in the sequel the latter type of structural morphism.

4 Equivariant Drinfel'd Double

The goal of this section is to show that the category $\mathcal{C}^J(G) := \bigoplus_{j \in J} \mathcal{C}(G)_j$ comprising the categories we have constructed in proposition 3.22 has a natural structure of a J-modular category.

4.1 The Equivariant Braided Structure

Let

$$1 \to G \to H \xrightarrow{\pi} J \to 1 \qquad \text{(III.27)}$$

be an exact sequence of finite groups. The normal subgroup G acts on H by conjugation; denote by $H//G$ the corresponding action groupoid. We consider the functor category $H//G\text{-mod} := [H//G, \mathrm{Vect}_{\mathbb{K}}]$. The category

$H//G$-mod is the category of H-graded vector spaces, endowed with an action of the subgroup G such that $g.V_h \subset V_{ghg^{-1}}$ for all $g \in G, h \in H$.
An immediate corollary of proposition 3.22 is the following description of the category $\mathcal{C}^J(G) := \bigoplus_{j \in J} \mathcal{C}(G)_j$ as an abelian category:

Proposition 4.1. *The category $\mathcal{C}^J(G)$ is equivalent, as an abelian category, to the category $H//G$-mod. In particular, the category $\mathcal{C}^J(G)$ is a 2-vector space in the sense of definition 1.2.*

Proof. With $H_j := \pi^{-1}(j)$, Proposition 3.22 gives the equivalence $\mathcal{C}(G)_j \cong H_j//G$-mod of abelian categories. The equivalence of categories $\mathcal{C}^J(G) \cong H//G$-mod now follows from the decomposition $H = \bigsqcup_{j \in J} H_j$. By [Mor08, Lemma 4.1.1], the representation category of a finite groupoid is a 2-vector space. □

Representation categories of finite groupoids are very close in structure to representation categories of finite groups. In particular, there is a complete character theory that describes the simple objects, see Appendix 6.2.
The category $H//G$-mod has a natural structure of a monoidal category: the tensor product of two objects $V = \oplus_{h \in H} V_h$ and $W = \oplus_{h \in H} W_h$ is the vector space $V \otimes W$ with H grading given by $(V \otimes W)_h := \oplus_{h_1 h_2 = h} V_{h_1} \otimes W_{h_2}$ and G action given by $g.(v \otimes w) = g.v \otimes g.w$. The associators are inherited from the underlying category of vector spaces.

Proposition 4.2. *Consider the exact sequence* (III.27) *of finite groups. Any choice of a a set-theoretic section $s : J \to H$ allows us to endow the abelian category $H//G$-mod with the structure of a J-equivariant tensor category as follows: the functor ϕ_j is given by shifting the grading from h to $s(j)hs(j)^{-1}$ and replacing the action by g by the action of $s(j)gs(j)^{-1}$. The isomorphism $\alpha_{i,j} : \phi_i \circ \phi_k \to \phi_{ij}$ is given by left action of the element*

$$\alpha_{i,j} = s(i)s(j)s(ij)^{-1} \ .$$

The fact that the action is only a weak action thus accounts for the failure of s to be a section in the category of groups.

Proof. Only the coherence conditions $\alpha_{ij,k} \circ \alpha_{i,j} = \alpha_{i,jk} \circ \phi_i(\alpha_{j,k})$ remain to be checked. By the results of Dedecker and Schreier, cf. proposition 3.3, the group elements $s(i)s(j)s(ij)^{-1} \in G$ are the coherence cells of a weak group action of J on H. By definition 3.1, this implies the coherence identities, once one takes into account that that composition of functors is written in different order than group multiplication. □

In subsection 3.5 we derived more structure on the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ from the geometry of extended cobordism categories. In particular, from the pair of pants, we get functors $\otimes_{jk} : \mathcal{C}(G)_j \boxtimes \mathcal{C}(G)_k \to \mathcal{C}(G)_{jk}$, spelled out explicitly in proposition 3.23, which we collect into a functor

$$\otimes : \mathcal{C} \boxtimes \mathcal{C} \to \mathcal{C}. \tag{III.28}$$

Another structure are the isomorphisms $V\otimes W \to \phi_j(W)\otimes V$ for $V \in \mathcal{C}(G)_j$, described in proposition 3.26. Together with the associators, this suggests to endow the category $\mathcal{C}^J(G)$ with a structure of a braided J-equivariant tensor category:

Remark 4.3. *The J-equivariant monoidal category $H//G$-mod has a natural braiding isomorphism that has been described in proposition 3.26*

We use the equivalence of abelian categories between $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ and $H//G$-mod to endow the category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ with associators. The category has now enough structure so that we can state our next result:

Proposition 4.4. *The category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$, with the tensor product functor from (III.28), can be endowed with the structure of a braided J-equivariant tensor category such that the isomorphism $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j \cong H//G$-mod becomes an isomorphism of braided J-equivariant tensor categories.*

Proof. The compatibility with the grading is implemented by definition via the graded components $\otimes_{jk}$ of $\otimes$ and the graded components of $c_{V,W}$. It remains to check that the action is by tensor functors and that the braiding satisfies the hexagon axiom. The second boils down to a simple calculation and the first is seen by noting that the action is essentially an index shift which is preserved by tensoring together the respective components. □

4.2 Equivariant Drinfel'd Double

In order to identify the structure of a J-modular tensor category on the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$, we need in addition dualities and a twist.

We will not be able to directly endow the geometric category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ with the structure of a J-equivariant ribbon category. Rather, we will realize an equivalent category as the category of modules over a suitable algebra.

Explicitly, we want to construct a J-ribbon algebra given a finite group G with a weak J-action, such that our found category is the representation category of this algebra and then use Proposition 2.18 of Chapter I.

We start from the well-known fact reviewed in Section 1 of Chapter II that the Drinfel'd double $\mathcal{D}(H)$ of the finite group H is a ribbon Hopf algebra. The double $\mathcal{D}(H)$ has a canonical basis $(\delta_h \otimes g)$ indexed by pairs of elements $g, h \in H$. Let $G \subset H$ be a subgroup. We are interested in the vector subspace $\mathcal{D}^J(G)$ spanned by the basis vectors $(\delta_h \otimes g)$ with $h \in H$ and $g \in G$.

Lemma 4.5. *The structure maps of the Hopf algebra $\mathcal{D}(H)$ restrict to the vector subspace $\mathcal{D}^J(G)$ in such a way that the latter is endowed with the structure of a Hopf subalgebra.*

The Drinfel'd double $\mathcal{D}(H)$ of a group H has the structure of a ribbon algebra. However, neither the R-matrix nor the the ribbon element yield an R-matrix or a ribbon element of $\mathcal{D}^J(G) \subset \mathcal{D}(H)$. Rather, this Hopf subalgebra can be endowed with the structure of a J-ribbon Hopf algebra as in Definition 2.16 of Chapter I. To this end, consider the partition of the group H into the subsets $H_j := \pi^{-1}(j) \subset H$, where π is the projection $H \to J$ in the exact sequence (III.27) It gives a J-grading of the algebra $\mathcal{D}^J(G)$ as a direct sum of subalgebras:

$$\mathcal{D}^J(G)_j := \langle \delta_h \otimes g \rangle_{h \in H_j, g \in G} \quad .$$

The set-theoretical section s gives a weak action of J on $\mathcal{D}^J(G)$ that can be described by its action on the canonical basis of $\mathcal{D}^J(G)_j$:

$$\varphi_j(\delta_h \otimes g) := \quad (\delta_{s(j)hs(j)^{-1}} \otimes s(j)gs(j)^{-1}) \ ;$$

the coherence elements are

$$c_{ij} := \sum_{h \in H} \delta_h \otimes s(i)s(j)s(ij)^{-1} \quad .$$

Proposition 4.6. *The Hopf algebra $\mathcal{D}^J(G)$, together with the grading and weak J-action derived from the weak J-action on the group G, has the structure of a J-Hopf algebra.*

Proof. It only remains to check the compatibility relations of grading and weak J-action with the Hopf algebra structure that have been formulated in definition 2.13. The fact that $\varphi_i(\mathcal{D}^J(G)_j) \subset \mathcal{D}^J(G)_{iji^{-1}}$ is immediate, since $s(i)H_js(i^{-1}) \subset H_{iji^{-1}}$. The axioms are essentially equivalent to the property that conjugation commutes with the product and coproduct of the Drinfel'd double. $\square$

We now turn to the last piece of structure, an R-matrix and a twist element in $\mathcal{D}^J(G)$. Consider the element $R = \sum_{ij} R_{i,j} \in \mathcal{D}^J(G) \otimes \mathcal{D}^J(G)$ with homogeneous elements $R_{i,j}$ defined as

$$R_{i,j} := \sum_{h_1 \in H_i, h_2 \in H_j} (\delta_{h_1} \otimes 1) \otimes (\delta_{h_2} \otimes s(i^{-1})h_1). \tag{III.29}$$

(Note that $\pi(s(i^{-1})h_1) = 1$ for $h_1 \in H_i$ and thus $s(i^{-1})h_1 \in G$.)
The element $R_{i,j}$ is invertible with inverse

$$R_{i,j}^{-1} = \sum_{h_1 \in H_i, h_2 \in H_j} (\delta_{h_1} \otimes 1) \otimes (\delta_{h_2} \otimes h_1^{-1} s(i^{-1})^{-1}).$$

We also introduce a twist element $\theta = \sum_{j \in J} \theta_j \in \mathcal{D}^J(G)$ with

$$\theta_j^{-1} := \sum_{h \in H_j} \delta_h \otimes s\big(j^{-1}\big)h \in \mathcal{D}^J(G)_j \tag{III.30}$$

for every element $j \in J$.

Proposition 4.7. *The elements R and θ endow the J-Hopf algebra $\mathcal{D}^J(G)$ with the structure of a J-ribbon algebra that we call the J-Drinfel'd double of G.*

Proof. By Definition 2.16 we have to check that the induced transformations on the level of categories satisfy the axioms for a braiding and a twist. We first compute the induced braiding by using the R-matrix given in (III.29): For $V \in (\mathcal{D}^J(G)\text{-mod})_j = \mathcal{D}^J(G)_j\text{-mod}$ and $W \in \mathcal{D}^J(G)\text{-mod}$, we get the linear map

$$\begin{aligned} c_{V,W} : V \otimes W &\to {}^jW \otimes V \\ v \otimes w &\mapsto s(j^{-1})h.w \otimes v \quad \text{for} \quad v \in V_h \end{aligned}$$

First of all, we show that this is a morphism in $\mathcal{D}^J(G)$-mod. Let $v \in V_h$ and $w \in W_{h'}$. We have $v \otimes w \in (V \otimes W)_{hh'}$ and $c_{V,W}(v \otimes w) = s(j^{-1})h.w \otimes v \in ({}^jW \otimes V)_{hh'}$, since the element $s(j^{-1})h.w$ is in the component $(s(j^{-1})hh'h^{-1}s(j^{-1})^{-1})$ of W which is the component $hh'h^{-1}$ of ${}^jW$. So $c_{V,W}$ respects the grading. As for the action of G, we observe that for $g \in G$, the element $g.v$ lies in the component $V_{ghg^{-1}}$, and so

$$\begin{aligned} c_{V,W}(g.(v \otimes w)) &= \big(s(j^{-1})ghg^{-1}\big)\, g.w \otimes g.v = s(j^{-1})gh.w \otimes g.v \\ g.c_{V,W}(v \otimes w) &= \big(s(j^{-1})gs(j^{-1})^{-1}\big)\, s(j^{-1})h.w \otimes g.v = s(j^{-1})gh.w \otimes g.v \end{aligned}$$

which shows that $c_{V,W}$ commutes with the action of G.
Furthermore, it needs to be checked, that $c_{V,W}$ satisfies the hexagon axioms (I.3) and (I.4), which in our case reduce to the two diagrams

$$\begin{tikzcd}
U \otimes V \otimes W \arrow[r, "c_{U\otimes V,W}"] \arrow[d, "\mathrm{id}_U \otimes c_{V,W}"'] & {}^{ij}W \otimes U \otimes V \\
U \otimes {}^j W \otimes V \arrow[r, "c_{U,{}^jW}\otimes \mathrm{id}_V"'] & {}^i({}^jW) \otimes U \otimes V \arrow[u, "\alpha_{i,j}\otimes \mathrm{id}_U"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
U \otimes V \otimes W \arrow[r, "c_{U,V\otimes W}"] \arrow[d, "c_{U,V}\otimes \mathrm{id}_W"'] & {}^iV \otimes {}^i W \otimes U \\
{}^iV \otimes U \otimes W \arrow[ur, "\mathrm{id}_{{}^iV}\otimes c_{U,W}"'] &
\end{tikzcd}$$

where $U \in \mathcal{C}_i, V \in \mathcal{C}_j$ and we suppressed the canonical associators in $\mathcal{D}^J(G)$-mod. And at last we need to check the compatibility of the braiding and the J-action, i.e. diagram (I.5). All of that can be proven by straightforward calculations, which show, that the morphism induced by the element R is really a braiding in the module category.
We now compute the morphism given by the action with the inverse twist element (III.30). For $V \in (\mathcal{D}^J(G)\text{-mod})_j = \mathcal{D}^J(G)_j$-mod we get the linear map:

$$\begin{aligned}
\theta_V : V &\to {}^jV \\
v &\mapsto s(j^{-1})h.v
\end{aligned}$$

for $v \in V_h$. This map is compatible with the H-grading since for $v \in V_h$, we have $s(j^{-1})h.v \in V_{s(j^{-1})hs(j^{-1})^{-1}} = ({}^jV)_h$ and it is also compatible with the G-action, since

$$\begin{aligned}
\theta_V(g.v) &= s(j^{-1})(ghg^{-1})g.v = s(j^{-1})gh.v \\
g.\theta_V(v) &= \left(s(j^{-1})gs(j^{-1})^{-1}\right) s(j^{-1})h.v = s(j^{-1})gh.v
\end{aligned}$$

So the induced map is a morphism in the category $\mathcal{D}^J(G)$. In order to proof that it is a twist, we have to check the commutativity of diagrams (I.6)-(I.8) as well as for every $i, j \in J$ and object V in the component j, the commutativity of the diagrams This again can be checked by straightforward calculation. $\square$

We are now ready to come back to the J-equivariant tensor category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ described in Proposition 4.4. From this proposition, we know that the category $\mathcal{C}^J(G)$ is equivalent to $H//G$-mod $\cong \mathcal{D}^J(G)$-mod as a J-equivariant tensor category. Also J-action and tensor product coincide with the ones on $\mathcal{D}^J(G)$-mod. Moreover, the equivariant braiding of $\mathcal{C}^J(G)$ computed in proposition 3.26 coincides with the braiding on $\mathcal{D}^J(G)$ computed in the proof of the last proposition 4.7.
This allows us to transfer also the other structure on the representation category of the J-Drinfel'd double $\mathcal{D}^J(G)$ described in proposition 4.7 to the category $\mathcal{C}^J(G)$:

Proposition 4.8. *The J-equivariant tensor category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ described in Proposition 4.4 can be endowed with the structure of a J-premodular category such that it is equivalent, as a J-premodular category, to the category* $\mathcal{D}^J(G)$-mod.

Remark 4.9. *At this point, we have constructed in particular a J-equivariant braided fusion category $\mathcal{C}^J(G) = \bigoplus_{j\in J}\mathcal{C}(G)_j$ (see [ENO10]) with neutral component $\mathcal{C}(G)_1 \cong \mathcal{D}(G)$-mod from a weak action of the group J on the group G, or in different words, from a 2-group homomorphisms $J \to \mathrm{AUT}(G)$ with $\mathrm{AUT}(G)$ the automorphism 2-group of G.*
In this remark, we very briefly sketch the relation to the description of J-equivariant braided fusion categories with given neutral sector $\mathcal{B}$ in terms of 3-group homomorphisms $J \to \underline{Pic}(\mathcal{B})$ given in [ENO10]. Here $\underline{Pic}(\mathcal{B})$ denotes the so called Picard 3-group whose objects are invertible module-categories of the category $\mathcal{B}$. The group structure comes from the tensor product of module categories which can be defined since the braiding on $\mathcal{B}$ allows to turn module categories into bimodule categories.
Using this setting, we give a description of our J-equivariant braided fusion category $\mathcal{D}^J(G)$-mod in terms of a functor $\Xi : J \to \underline{Pic}(\mathcal{D}(G))$. To this end, we construct a 3-group homomorphism $\mathrm{AUT}(G) \to \underline{Pic}(\mathcal{D}(G))$ and write Ξ as the composition of this functor and the functor $J \to \mathrm{AUT}(G)$ defining the weak J-action.
The 3-group homomorphism $\mathrm{AUT}(G) \to \underline{Pic}(\mathcal{D}(G))$ is given as follows: to an object $\varphi \in \mathrm{AUT}(G)$ we associate the twisted conjugation groupoid $G//^{\varphi}G$, where G acts on itself by twisted conjugation, $g.x := gx\varphi(g)^{-1}$. This yields the category $G//^{\varphi}G$-mod $:= [G//^{\varphi}G, \mathrm{Vect}_{\mathbb{K}}]$ which is naturally a module category over $\mathcal{D}(G)$-mod. Morphisms $\varphi \to \psi$ in $\mathrm{AUT}(G)$ are given by group elements $g \in G$ with $g\varphi g^{-1} = \psi$; to such a morphism we associate the functor $L_g : G//^{\varphi}G \to G//^{\psi}G$ given by conjugating with $g \in G$ on objects and morphisms. This induces functors of module categories $G//^{\varphi}G$-mod $\to$

$G//^{\psi}G$-mod. Natural coherence data exists; one then shows that this really establishes the desired 3-group homomorphism.

4.3 Equivariant Modularity

It remains to show that the J-equivariant ribbon category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ described in 4.7 is J-modular. To this end, we will use the orbifold category of the J-equivariant category as defined in Definition 2.20 of Chapter 1.
Recall that we gave the definition of J-modularity of a category in terms of modularity of its orbifold category (Definition 2.22 of Chapter I). Our problem is thus reduced to showing modularity of the orbifold category of $\mathcal{D}^J(G)$-mod.
We will therefore show in this subsection that the orbifold category of the J-equivariant ribbon category $\mathcal{C}^J(G)$-mod is J-modular.
Since we have already seen in Corollary 2.28 of Chapter I that the orbifold category is equivalent, as a ribbon category, to the representation category of the orbifold Hopf algebra, it suffices to compute this Hopf algebra explicitly. Our final result asserts that this Hopf algebra is an ordinary Drinfel'd double:

Proposition 4.10. *The* K*-linear map*

$$\begin{array}{rccc} \Psi : & \widehat{\mathcal{D}^J(G)}^J & \to & \mathcal{D}(H) \\ & (\delta_h \otimes g \otimes j) & \mapsto & (\delta_h \otimes gs(j)) \end{array} \tag{III.31}$$

is an isomorphism of ribbon algebras, where the Drinfel'd double $\mathcal{D}(H)$ is taken with the standard ribbon structure introduced in Section 1 of Chapter II.

This result immediately implies the equivalence

$$(\widehat{\mathcal{D}^J(G)}\text{-mod})^J \cong \mathcal{D}(H)\text{-mod}$$

of ribbon categories and thus, by Proposition 2.17, the modularity of the orbifold category, so that we have finally proven:

Theorem 4.11. *The category $\mathcal{C}^J(G) = \bigoplus_{j\in J} \mathcal{C}(G)_j$ has a natural structure of a J-modular tensor category.*

Proof of proposition 4.10. We show by direct computations that the linear map Ψ preserves product, coproduct, R-matrix and twist element:

- Compatibility with the product:

$$\begin{aligned}
\Psi((\delta_h \otimes g \otimes j)(\delta'_h \otimes g' \otimes j')) &= \Psi((\delta_h \otimes g) \cdot {}^{j}(\delta'_h \otimes g')c_{j,j'} \otimes jj') \\
&= \Psi\left((\delta_h \otimes g) \cdot (\delta_{s(j)h's(j)^{-1}} \otimes s(j)g's(j)^{-1}) \cdot \sum_{h'' \in H} (\delta_{h''} \otimes s(j)s(j')s(jj')^{-1} \right. \\
&= \Psi\left(\delta(h, gs(j)hs(j)^{-1}g^{-1})(\delta_h \otimes gs(j)g's(j')s(jj')^{-1}) \otimes jj'\right) \\
&= \delta(h, gs(j)hs(j)^{-1}g^{-1})(\delta_h \otimes gs(j)g's(j')) \\
&= (\delta_h \otimes gs(j)) \cdot (\delta_{h'} \otimes g's(j')) \\
&= \Psi(\delta_h \otimes g \otimes j)\Psi(\delta_{h'} \otimes g' \otimes j')
\end{aligned}$$

- Compatibility with the coproduct:

$$\begin{aligned}
(\Psi \otimes \Psi)\Delta(\delta_h \otimes g \otimes j) &= \sum_{h'h''=h} \Psi(\delta_{h'} \otimes g \otimes j) \otimes \Psi(\delta_{h''} \otimes g \otimes j) \\
&= \sum_{h'h''=h} (\delta_{h'} \otimes gs(j)) \otimes (\delta_{h''} \otimes gs(j)) \\
&= \Delta\left(\Psi(\delta_h \otimes g \otimes j)\right)
\end{aligned}$$

- The R-matrix of the orbifold algebra $\widehat{\mathcal{D}^J(G)}^J$ can be determined using the lines preceding Corollary 2.28 and the definition of the R-Matrix of $\mathcal{D}^J(G)$ given in (III.29):

$$R = \sum_{j,j' \in J} \sum_{h \in H_j, h' \in H_{j'}} (\delta_h \otimes 1_G \otimes 1_J) \otimes (1 \otimes 1_G \otimes j^{-1})^{-1}(\delta_{h'} \otimes s(j^{-1})h \otimes 1_J)$$

This implies

$$\begin{aligned}
(\Psi \otimes \Psi)(R) &= \sum_{j,j' \in J} \sum_{h \in H_j, h' \in H_{j'}} \Psi(\delta_h \otimes 1_G \otimes 1_J) \otimes \Psi(1 \otimes 1_G \otimes j^{-1})^{-1} \\
&\qquad \cdot \Psi(\delta_{h'} \otimes s(j^{-1})h \otimes 1_J) \\
&= \sum_{j,j' \in J} \sum_{h \in H_j, h' \in H_{j'}} (\delta_h \otimes 1_H) \otimes (1 \otimes s(j^{-1})^{-1}) \cdot (\delta_{h'} \otimes s(j^{-1})h) \\
&= \sum_{h\ h' \in H} (\delta_h \otimes 1) \otimes (\delta_{h'} \otimes h),
\end{aligned}$$

which is the standard R-matrix of the Drinfel'd double $\mathcal{D}(H)$.

- The twist in $\widehat{\mathcal{D}^J(G)}^J$ is by corollary 2.28 equal to

$$\theta^{-1} = \sum_{j\in J} \sum_{h\in H_j} (\delta_h \otimes s(j^{-1})h \otimes 1_J)$$

  and thus it gets mapped to the element

$$\begin{aligned}\Psi(\theta^{-1}) &= \sum_{j\in J} \sum_{h\in H_j} \Psi(1 \otimes 1_G \otimes j^{-1})^{-1} \Psi(\delta_h \otimes s(j^{-1})h \otimes 1_J) \\ &= \sum_{j\in J} \sum_{h\in H_j} (1 \otimes s(j^{-1})^{-1}) \cdot (\delta_h \otimes s(j^{-1})h) \\ &= \sum_{h\in H} (\delta_h \otimes h)\end{aligned}$$

  which is the inverse of the twist element in $\mathcal{D}(H)$.

□

4.4 Summary of all Tensor Categories involved

We summarize our findings by discussing again the four tensor categories mentioned in the introduction, in the square of equation (III.1), thereby presenting the explicit solution of the algebraic problem described in section 1.1. Given a finite group G with a weak action of a finite group J, we get an extension $1 \to G \to H \to J \to 1$ of finite groups, together with a set-theoretic section $s : J \to H$.

Proposition 4.12.
We have the following natural realizations of the categories in question in terms of categories of finite-dimensional representations over finite-dimensional ribbon algebras:

1. *The premodular category introduced in [Ban10] is* $\mathcal{B}(G \triangleleft H)$-mod. *As an abelian category, it is equivalent to the representation category* $G//H$-mod *of the action groupoid* $G//H$*, i.e. to the category of* G*-graded* $\mathbb{K}$*-vector spaces with compatible action of* H*.*

2. *The modular category obtained by modularization is* $\mathcal{D}(G)$-mod. *As an abelian category, it is equivalent to* $G//G$-mod.

3. *The* J*-modular category constructed in this chapter is* $\mathcal{D}^J(G)$-mod. *As an abelian category, it is equivalent to* $H//G$-mod.

4. *The modular category obtained by orbifolding from the J-modular category* $\mathcal{D}(G)$-mod *is equivalent to* $\mathcal{D}(H)$-mod. *As an abelian category, it is equivalent to* $H//H$-mod.

Equivalently, the diagram in equation (III.1), has the explicit realization:

$$\begin{tikzcd} \mathcal{D}(G)\text{-mod} \arrow[loop above, "J"] \arrow[r, hook] \arrow[d, shift left, "orbifold"] & \mathcal{D}^J(G)\text{-mod} \arrow[loop above, "J"] \arrow[d, shift left, "orbifold"] \\ \mathcal{B}(G \triangleleft H)\text{-mod} \arrow[r, hook] \arrow[u, shift left, "modularization"] & \mathcal{D}(H)\text{-mod} \arrow[u, shift left] \end{tikzcd} \tag{III.32}$$

We could have chosen the inclusion in the lower line as an alternative starting point for the solution of the algebraic problem presented in introduction 1.1. Recall from the introduction that the category $\mathcal{B}(G \triangleleft H)$-mod contains a Tannakian subcategory that can be identified with the category of representations of the quotient group $J = H/G$. The Tannakian subcategory and thus the category $\mathcal{B}(G \triangleleft H)$-mod contain a commutative Frobenius algebra given by the algebra of functions on J; recall that the modularization function was just induction along this algebra. The image of this algebra under the inclusion in the lower line yields a commutative Frobenius algebra in the category $\mathcal{D}(H)$-mod. In a next step, one can consider induction along this algebra to obtain another tensor category which, by general results [Kir04, Theorem 4.2] is a J-modular category.
In this approach, it remains to show that this J-modular tensor category is equivalent, as a J-modular tensor category, to $\mathcal{D}^J(G)$-mod and, in a next step that the modularization $\mathcal{D}(G)$-mod can be naturally identified with the neutral sector of the J-modular category. This line of thought has been discussed in [Kir01, Lemma 2.2] including the square (III.32) of Hopf algebras. Our results directly lead to a natural Hopf algebra $\mathcal{D}^J(G)$ and additionally show how the various categories arise from extended topological field theories which are built on clear geometric principles and through which all additional structure of the algebraic categories become explicitly computable.

5 Outlook

Our results very explicitly provide an interesting class J-modular tensor categories. All data of these theories, including the representations of the modular group $SL(2,\mathbb{Z})$ on the vector spaces assigned to the torus, are directly accessible in terms of representations of finite groups. Also series of examples

exist in which closed formulae for all quantities can be derived, e.g. for the inclusion of the alternating group in the symmetric group.
Our results admit generalizations in various directions. In fact, in this chapter, we have only studied a subclass of Dijkgraaf-Witten theories. The general case requires, apart from the choice of a finite group G, the choice of an element of

$$H^3_{Gp}(G, U(1)) = H^4(\mathcal{A}_G, \mathbb{Z}) .$$

This element can be interpreted [Wil08] geometrically as a 2-gerbe on $\mathcal{A}_G$. It is known that in this case a quasi-triangular Hopf algebra can be extracted that is exactly the one discussed in [DPR90]. Indeed, our results can also be generalized by including the additional choice of a non-trivial element

$$\omega \in H^4_J(\mathcal{A}_G, \mathbb{Z}) \equiv H^4(\mathcal{A}_G//J, \mathbb{Z}) .$$

Only all these data together allow to investigate in a similar manner the categories constructed by Bantay [Ban10] for crossed modules with a boundary map that is not necessarily injective any longer. We plan to explain this general case in a subsequent publication.

6 Appendix

6.1 Cohomological Description of Twisted Bundles

In this appendix, we give a description of P-twisted bundles as introduced in definition 3.5 in terms of local data. This local description will also serve as a motivation for the term ‘twisted’ in twisted bundles. Recall the relevant situation: $1 \to G \to H \xrightarrow{\pi} J \to 1$ is an exact sequence of groups. Let $P \xrightarrow{J} M$ be a J-cover. A P-twisted bundle on a smooth manifold M is an H-bundle $Q \to M$, together with a smooth map $\varphi : Q \to P$ such that $\varphi(qh) = \varphi(q)\pi(h)$ for all $q \in Q$ and $h \in H$.

We start with the choice of a contractible open covering $\{U_\alpha\}$ of M, i.e. a covering for which all open sets U_α are contractible. Then the J-cover P admits local sections over U_α. By choosing local sections s_α, we obtain the cocycle

$$j_{\alpha\beta} := s_\alpha^{-1} \cdot s_\beta : U_\alpha \cap U_\beta \to J$$

describing P.
Let (Q, φ) be a P-twisted G-bundle over M. We claim that we can find local sections

$$t_\alpha : U_\alpha \to Q$$

of the H-bundle Q which are compatible with the local section of the J-cover P in the sense that $\varphi \circ t_\alpha = s_\alpha$ holds for all α.
To see this, consider the map $\varphi : Q \to P$; restricting the H-action on Q along the inclusion $G \to H$, we get a G-action on Q that covers the identity on P. Hence Q has the structure of a G-bundle over P. Note that the image of s_α is contractible, since U_α is contractible. Thus the G-bundle $Q \to P$ admits a section s'_α over the image of s_α. Then $t_\alpha := s'_\alpha \circ s_\alpha$ is a section of the H-bundle $Q \to M$ that does the job.
With these sections $t_\alpha : U_\alpha \to Q$, we obtain the cocycle description

$$h_{\alpha\beta} := t_\alpha^{-1} \cdot t_\beta : U_\alpha \cap U_\beta \to H$$

of Q.
The set underlying the group H is isomorphic to the set $G \times J$. The relevant multiplication on this set depends on the choice of a section $J \to H$; it has been described in equation (III.21):

$$(g, i) \cdot (g', j) := \left(g \cdot \ {}^{i}(g') \cdot c_{i,j} \ , \ ij\right) \ .$$

This allows us to express the H-valued cocycles $h_{\alpha\beta}$ in terms of J-valued and G-valued functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G \ .$$

By the condition $\varphi \circ t_\alpha = s_\alpha$, the J-valued functions are determined to be the J-valued cocycles $j_{\alpha\beta}$. Using the multiplication on the set $G \times J$, the cocycle condition $h_{\alpha\beta} \cdot h_{\beta\gamma} = h_{\alpha\gamma}$ can be translated into the following condition for $g_{\alpha\beta}$

$$g_{\alpha\beta} \cdot \ {}^{j_{\beta\gamma}}\left(g_{\beta\gamma}\right) \cdot c_{j_{\alpha\beta}, j_{\beta\gamma}} = g_{\alpha\gamma} \tag{III.33}$$

over $U_\alpha \cap U_\beta \cap U_\gamma$. This local expression can serve as a justification of the term P-twisted G-bundle.

We next turn to morphisms. A morphism f between P-twisted bundles (Q, φ) and (Q', ψ) which are represented by twisted cocycles $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ is represented by a coboundary

$$l_\alpha := (t'_\alpha)^{-1} \cdot f(t_\alpha) : U_\alpha \to H$$

between the H-valued cocycles $h_{\alpha\beta}$ and $h'_{\alpha\beta}$. Since f satisfies $\psi \circ f = \varphi$, the J-component $\pi \circ l_\alpha : U_\alpha \to H \to J$ is given by the constant function to $e \in J$. Hence the local data describing the morphism f reduce to a family of functions

$$k_\alpha : U_\alpha \to G.$$

Under the multiplication (III.21), the coboundary relation $l_\alpha \cdot h_{\alpha\beta} = h'_{\alpha\beta} \cdot l_\beta$ translates into

$$k_\alpha \cdot \,^{e}\big(g_{\alpha\beta}\big) \cdot c_{e,j_{\alpha\beta}} = g'_{\alpha\beta} \cdot \,^{j_{\alpha\beta}}\big(k_\beta\big) \cdot c_{j_{\alpha\beta},e}$$

One can easily conclude from the definition 3.1 of a weak action that $^{e}g = g$ and $c_{e,g} = c_{g,e} = e$ for all $g \in G$. Hence this condition reduces to the condition

$$k_\alpha \cdot g_{\alpha\beta} = g'_{\alpha\beta} \cdot \,^{j_{\alpha\beta}}\big(k_\beta\big) \quad . \tag{III.34}$$

We are now ready to present a classification of P-twisted bundles in terms of Čech-cohomology.
Therefore we define the relevant cohomology set:

Definition 6.1. *Let $\{U_\alpha\}$ be a contractible cover of M and $(j_{\alpha\beta})$ be a Čech-cocycle with values in J.*

- *A $(j_{\alpha\beta})$-twisted Čech-cocycle is given by a family*

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$$

  satisfying relation (III.33).

- *Two such cocycles $g_{\alpha\beta}$ and $g'_{\alpha\beta}$ are cobordant if there exists a coboundary, that is a family of functions $k_a : U_\alpha \to G$ satisfying relation (III.34).*

- *The twisted Čech-cohomology set $\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G)$ is defined as the quotient of twisted cocycles modulo coboundaries.*

Warning 6.2. *It might be natural to guess that twisted Čech-cohomology $\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G)$ agrees with the preimage of the class $[j_{\alpha\beta}]$ under the map $\pi_* : \check{\mathrm{H}}^1(M,H) \to \check{\mathrm{H}}^1(M,J)$. This turns out to be wrong: The natural map*

$$\begin{array}{rcl} \check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G) & \to & \check{\mathrm{H}}^1(M,H) \\ [g_{\alpha\beta}] & \mapsto & [(g_{\alpha,\beta}, j_{\alpha\beta})] \end{array} \;,$$

is, in general, not injective. The image of this map is always the fiber ${\pi_*}^{-1}[j_{\alpha\beta}]$.

We summarize our findings:

Proposition 6.3. *Let P be a J-cover of M, described by the cocycle $j_{\alpha\beta}$ over the contractible open cover $\{U_\alpha\}$. Then there is a canonical bijection*

$$\check{\mathrm{H}}^1_{j_{\alpha\beta}}(M,G) \cong \Big\{ \textit{Isomorphism classes of } P\textit{-twisted } G\textit{-bundles over } M \Big\}$$

6.2 Character Theory for Action Groupoids

In this subsection, we explicitly work out a character theory for finite action groupoids $M//G$; in the case of $M = pt$, this theory specializes to the character theory of a finite group (cf. [Isa94] and [Ser77]). In the special case of a finite action groupoid coming from a finite crossed module, a character theory including orthogonality relation has been presented in [Ban10]. In the sequel, let K be a field and denote by $[M//G, \mathrm{Vect}_{\mathrm{K}}]$ the category of K-linear representations of $M//G$.

Definition 6.4. *Let $((V_m)_{m\in M}, (\rho_g)_{g\in G})$ be a* K- *linear representation of the action groupoid $M//G$ and denote by $P(m)$ the projection of $V = \bigoplus_{n\in M} V_n$ to the homogeneous component V_m. We call the function*

$$\begin{aligned} \chi : M \times G \to \mathrm{K} \\ \chi(m,g) := \mathrm{Tr}_V(\rho_g \circ P(m)) \end{aligned}$$

the character *of the representation.*

Example 6.5. *On the* K*-vector space $H := \mathrm{K}(M)\otimes\mathrm{K}[G]$ with canonical basis $(\delta_m \otimes g)_{m\in M, g\in G}$, we define a grading by $H_m = \bigoplus_{g\in G} \mathrm{K}(\delta_{g.m} \otimes g)$ and a group action by $\rho_g(\delta_m \otimes h) = \delta_m \otimes gh$. This defines an object in $[M//G, \mathrm{Vect}_{\mathrm{K}}]$, called the regular representation. The character is easily calculated in the canonical basis and found to be*

$$\chi_H(m,g) = \sum_{(n,h)\in M\times G} \delta(g,1)\delta(h.m,n) = \delta(g,1)|G|$$

Definition 6.6. *We call a function*

$$f : M \times G \to \mathrm{K}$$

an action groupoid class function *on $M//G$, if it satisfies*

$$f(m,g) = 0 \text{ if } g.m \neq m \quad \text{and} \quad f(h.m, hgh^{-1}) = f(m,g) \ .$$

The character of any finite dimensional representation is a class function. From now on, we assume that the characteristic of K does not divide the order $|G|$ of the group G. This assumption allows us to consider the following normalized non-degenerate symmetric bilinear form

$$\langle f, f'\rangle := \frac{1}{|G|} \sum_{g\in G, m\in M} f(m,g^{-1})f'(m,g). \tag{III.35}$$

If $K = \mathbb{C}$, one can show, precisely as in the case of groups, that the equality $\chi(m, g^{-1}) = \overline{\chi(m, g)}$ holds, which allows to introduce the hermitian scalar product

$$(\chi, \chi') := \frac{1}{|G|} \sum_{g \in G, m \in M} \overline{\chi(m, g)} \chi'(m, g) \; . \tag{III.36}$$

Lemma 6.7. *Let* K *be algebraically closed. The characters of irreducible* $M//G$*-representations are orthogonal and of unit length with respect to the bilinear form (III.35):*

$$\langle \chi_i, \chi_j \rangle = \delta(i, j) \tag{III.37}$$

for $i, j \in I$

Proof. The proof proceeds as in the case of finite groups: for a linear map $f : V \to W$ on the vector spaces underlying two irreducible representations, one considers the intertwiner

$$f^0 = \frac{1}{|G|} \sum_{g \in G, m \in M} \rho_W(g^{-1}) P_W(m) f P_V(m) \rho_V(g). \tag{III.38}$$

and applies Schur's lemma. □

A second orthogonality relation

$$\sum_{i \in I} \chi_i(m, g) \chi_i(n, h^{-1}) = \sum_{z \in G} \delta(n, z.m) \delta(h, zgz^{-1}) \tag{III.39}$$

can be derived as in the case of finite groups, as well.
Combining the orthogonality relations with the explicit form for the character of the regular representation from Example 6.5, we derive in the case of an algebraically closed field whose characteristic does not divide the order $|G|$ the following statement:

Lemma 6.8. *Every irreducible representation* V_i *is contained in the regular representation with multiplicity* $d_i := \dim_{\mathrm{K}} V_i$*.*

As a consequence, the following generalization of Burnside's Theorem holds:

Proposition 6.9. *Denote by* $(V_i)_{i \in I}$ *a set of representatives for the isomorphism classes of simple representations of the action groupoid and by* $d_i := \dim_{\mathrm{K}} V_i$ *the dimension of the simple object. Then*

$$\sum_{i \in I} |d_i|^2 = |M||G|$$

Proof. One combines the relation $\dim H = \sum_{i\in I} d_i \dim V_i$ from Lemma 6.8 with the relation $\dim H = |M||G|$. $\square$

In complete analogy to the case of finite groups, one then shows:

Proposition 6.10. *The irreducible characters of $M//G$ form an orthogonal basis of the space of class functions with respect to the scalar product (III.35).*

The above proposition allows us to count the number of irreducible representations. On the set

$$A := \{(m,g)|g.m = m\} \subset M \times G$$

the group G naturally acts by $h.(m,g) := (h.m, hgh^{-1})$. A class function of $M//G$ is constant on G-orbits of A; it vanishes on the complement of A in $M \times G$. We conclude that the number of irreducible characters equals the number of G-orbits of A.
This can be rephrased as follows: the set A is equal to the set of objects of the inertia groupoid $\Lambda(M//G) := [\bullet//\mathbb{Z}, M//G]$. Thus the number of G-orbits of A equals the number of isomorphism classes of objects in $\Lambda(M//G)$, thus $|I| = |Iso(\Lambda(M//G))|$.

Chapter IV

Strictification

In the last chapter we constructed for a finite group G a 3-dimensional G-equivariant topological field theory which is a generalization of the well-known Dijkgraaf-Witten theory [DW90, FQ93] and extracted a G-braided category $\mathcal{C}$ from it (the group was called J in the last chapter). The category $\mathcal{C}$ could be equipped with the structure of a G-ribbon category, and furthermore it was shown to be G-modular.
In general the action of the group G on a G-modular category $\mathcal{C}$ is given by tensor functors $\phi_g : \mathcal{C} \to \mathcal{C}$ together with compositors $\phi_g \circ \phi_h \xrightarrow{\sim} \phi_{gh}$, subject to coherence laws for threefold products (see Definition 2.1). It has been demonstrated by Müger [Tur10a, App. 5] that one can replace $\mathcal{C}$ by an equivalent category $\mathcal{C}^{str}$ with a *strict* action of G, i.e. there the compositors are given by the identity: $\phi_g \circ \phi_h = \phi_{gh}$, and such that the equivalence $\mathcal{C} \cong \mathcal{C}^{str}$ is even compatible with the equivariant structure.
Now consider the G-modular category $\mathcal{C}$ which belongs to our equivariant Dijkgraaf-Witten theory introduced in Section 3 of Chapter III. Although the category $\mathcal{C}$ can relatively easily be described abstractly, the orbifold construction and proof of the G-modularity of $\mathcal{C}$ is rather difficult. Therefore we realized $\mathcal{C}$ as the representation category of an algebra A, which we called the equivariant Drinfel'd double. The fact that $\mathcal{C}$ is a tensor category is reflected by the fact that A has the additional structure of a Hopf algebra. Furthermore there is also an algebraic structure on A belonging to the G-action on the representation category. This structure is not just a G-action on A, as one might naively expect, but a *weak* G-action, which is an action by Hopf algebra automorphisms $\varphi_g : A \to A$ such that $\varphi_g \circ \varphi_h$ equals φ_{gh} only up to an inner automorphism of A. This weakening of the G-action reflects the fact that the action on the category is only weak in the sense that we have coherent isomorphisms $\phi_g \circ \phi_h \xrightarrow{\sim} \phi_{gh}$ of functors rather than equalities. In order to accommodate the example of the algebra A, we had to introduce

the notion of Hopf algebra with weak G-action ([MNS12, definition 4.13]), generalizing the notion of Hopf algebra with strict G-action considered in existing literature [Tur10a, Vir02].

In the light of Müger's observation that one can replace a G-equivariant tensor category $\mathcal{C}$ by an equivalent category $\mathcal{C}^{str}$ with strict G-action it is a natural question to ask whether one can replace a Hopf algebra A with weak G-action by a Hopf algebra A^{str} with strict G-action such that the representation categories are equivalent as tensor categories. A first result of this chapter asserts that this is not possible in general, see Theorem 2.2. The reason is that the Hopf algebra axioms are too restrictive: the tensor product of the representation category is, in the case of Hopf algebras, directly inherited from the underlying tensor product of vector spaces. Weak Hopf algebras [BNS99, BS00, NV02] have been introduced to provide a more flexible notion for the tensor product. Note that the qualifier weak here refers to a weakening of the bialgebra axioms (i.e. a weakening of the unitality of the coproduct or, equivalently, of the counitality of the product) and should not be confused with 'weak G-action'. We refer to the appendix for a table summarizing the situation.
Thus, a refined version of the question posed above would be whether one can replace a Hopf algebra A with weak G-action by a *weak* Hopf algebra A^{str} with *strict* G-action such that the representation categories are equivalent. The second main result of the present chapter is to show that this is indeed possible, see Theorem 3.1. The given concrete construction of A^{str} is inspired by Müger's strictification procedure [Tur10a, Appendix 5] on the level of categories. But our construction is presented in an independent manner which requires no knowledge about orbifold categories and other constructions that enter in the categorical strictification.

1 Equivariant Weak Hopf Algebras and their Representation Categories

In this section, we will first give an introduction to weak Hopf algebras and then generalize the notion of equivariant Hopf algebras to them. The notion of a weak Hopf algebra was first introduced in [BS96] as C^*-weak Hopf algebras. In the following let K be an algebraically closed field of characteristic zero. In [BNS99] an equivalent definition of weak Hopf algebra can be found and this is the one we will use. A weak Hopf algebra is still both an associative and coassociative algebra. The weakening of the Hopf algebra axioms appears in the compatibility of the coproduct with the unit and the compatibility of the product with the counit. These weakenings are dual concepts in the sense that for a finite dimensional weak Hopf algebra, as for a finite dimensional Hopf algebra, its dual is again a weak Hopf algebra. The category of representations of a weak Hopf algebra can still be endowed with the structure of a tensor category with duality. The reconstruction theorem [JS91] assures that one can recover a Hopf algebra from its representation category and a fiber functor into the category $\mathrm{Vect}_{\mathrm{K}}$ of finite dimensional vector spaces over K. However, a finitely semisimple tensor category $\mathcal{C}$ can be recovered as the category of modules over a Hopf algebra if and only if there is a fiber functor from $\mathcal{C}$ to $\mathrm{Vect}_{\mathrm{K}}$. In general there is no reason for a tensor category to have a fiber functor into $\mathrm{Vect}_{\mathrm{K}}$. So in general $\mathcal{C}$ can not be recovered as the category of modules over a Hopf algebra. But in [Hay99] it was shown that it is always possible to realize it as the module category over a *weak* Hopf algebra.

In the case of an equivariant category $\mathcal{C}$, a related problem appears. It is known ([Tur10a]) that weak actions of a group on a category can be 'strictified', i.e. for any G-equivariant category there is an equivalence (which is compatible with the equivariant structure) to a G-equivariant category $\mathcal{C}'$ on which the G-action is strict. Now the question arises whether this strictification of the action can also be done on the G-Hopf algebra, if $\mathcal{C} \cong H$-mod as G-equivariant categories. In this chapter we show that it is not always possible to find the strictified category again as the module category over a Hopf algebra, but that indeed it is always possible to recover it as the module category over a weak Hopf algebra.

We start by giving the definition of a weak bialgebra from [BNS99].

Definition 1.1. *Let A be a* K*-vector space that has both an algebra structure (A, m, η) and a coalgebra structure (A, Δ, ϵ). A is called a* weak bialgebra *if*

- *Δ is an algebra morphism, i.e.*

$$\Delta(xy) = \Delta(x)\Delta(y) \qquad \text{for all } x, y \in A$$

- *The coproduct has the following compatibility with the unit:*

$$(\Delta \otimes \mathrm{id})\Delta(\mathbf{1}) = (\Delta(\mathbf{1}) \otimes \mathbf{1})(\mathbf{1} \otimes \Delta(\mathbf{1})) = (\mathbf{1} \otimes \Delta(\mathbf{1}))(\Delta(\mathbf{1}) \otimes \mathbf{1})$$

- *The counit has the following compatibility with the product:*

$$\epsilon(xyz) = \epsilon(xy_{(1)})\epsilon(y_{(2)}z) = \epsilon(xy_{(2)})\epsilon(y_{(1)}z)$$

for all $x, y, z \in A$.

Definition 1.2. *A weak bialgebra A is called a* weak Hopf algebra *if there exists a* $\mathbb{K}$*-linear map $S : A \to A$, called the antipode, that satisfies the following conditions:*

$$x_{(1)}S(x_{(2)}) = \epsilon(\mathbf{1}_{(1)}x)\mathbf{1}_{(2)} \tag{IV.1}$$
$$S(x_{(1)})x_{(2)} = \mathbf{1}_{(1)}\epsilon(x\mathbf{1}_{(2)}) \tag{IV.2}$$
$$S(x_{(1)})x_{(2)}S(x_{(3)}) = S(x) \tag{IV.3}$$

The right sides of the equations (IV.1) and (IV.2) also define the *target* and *source counital maps.* For $x \in A$ they are:

$$\epsilon_t(x) = \epsilon(\mathbf{1}_{(1)}x)\mathbf{1}_{(2)} \tag{IV.4}$$
$$\epsilon_s(x) = \mathbf{1}_{(1)}\epsilon(x\mathbf{1}_{(2)}) \tag{IV.5}$$

Their images in A are called the *target* and *source subalgebras* $A_t := \epsilon_t(A)$ and $A_s := \epsilon_s(A)$. They play an important role in the category of left resp. right modules over A. It can be shown, that they have the following equivalent description:

Lemma 1.3.

$$A_t = \{x \in A | \Delta(x) = 1_{(1)}x \otimes 1_{(2)}\} \tag{IV.6}$$
$$A_s = \{x \in A | \Delta(x) = 1_{(1)} \otimes x1_{(2)}\} \tag{IV.7}$$

Recall that in the the definition (2.13) of a weak action of a group on a Hopf algebra, the compositors are required to be grouplike. There is a generalization of the notion of grouplike elements to the case of a weak Hopf algebras. One distinguishes left and right grouplike elements:

Definition 1.4. *Let A be a weak Hopf algebra. An element $x \in A$ is called* left*, resp.* right grouplike *if $\epsilon_{t/s}(x) \in (A_{t/s})^\times$ and*

$$(x\epsilon_s(x)x^{-1} \otimes x)\Delta(\mathbf{1}) = \Delta(x) = \Delta(\mathbf{1})(x \otimes x)$$

resp.

$$(x \otimes x)\Delta(\mathbf{1}) = \Delta(g) = \Delta(\mathbf{1})(x \otimes \epsilon_t(x)^{-1}x).$$

As in [Vec03] is shown, there is an equivalent definition of grouplike elements in a weak Hopf algebra:

Lemma 1.5. *An element $x \in A$ is left/right grouplike if and only if x is invertible and $\Delta(x) = (x \otimes x)\Delta(\mathbf{1})$ (resp. $\Delta(x) = \Delta(\mathbf{1})(x \otimes x)$).*

We now look at the categories of left resp. right modules over a weak Hopf algebra. The following result shows that they have the same properties as the ones of a Hopf algebra.

Proposition 1.6. *The category of left resp. right modules over a weak Hopf algebra A has the structure of a tensor category with duality, where the tensor product of two modules V, W is given by*

$$V \otimes W := \Delta(\mathbf{1}) V \otimes_{\mathrm{K}} W \tag{IV.8}$$

resp.

$$V \otimes W := V \otimes_{\mathrm{K}} W \Delta(\mathbf{1}) \tag{IV.9}$$

and the tensor unit is the target resp. source counital algebra. The dual object of an A-module V is the dual space V^* where the action of $x \in A$ on an element $\phi \in V^*$ evaluated on an element $v \in V$ is given by

$$(\phi.x)(v) := \phi(S(x)v) \text{ resp. } (x.\phi).(v) := \phi(vS(x))$$

We now turn to equivariant structure on a weak Hopf algebra.
In the following, let G be a finite group. In Section 2.1 of Chapter I we gave the definition of an equivariant Hopf algebra. With a slight change on the requirement for the compositors, the definition of a G-weak Hopf algebra is analogously. For convenience, we will state this generalization of Definition 2.13 of Chapter I here:

Definition 1.7. *A* G-weak Hopf algebra *over* K *is a weak Hopf algebra A with a weak G-action $((\varphi_g)_{g\in G}, (c_{g,h})_{g,h\in G})$ and a G-grading $A = \bigoplus_{g\in G} A_g$ such that:*

- *The algebra structure of A restricts to the structure of an associative algebra on each homogeneous component so that A is the direct sum of the components A_g as an algebra.*
- *The action of G is compatible with the grading, i.e. $\varphi_g(A_h) \subset A_{ghg^{-1}}$.*
- *The coproduct $\Delta: A \to A \otimes A$ respects the grading, i.e.*

$$\Delta(A_g) \subset \bigoplus_{p,q\in G, pq=g} A_p \otimes A_q \ .$$

- *The elements $(c_{g,h})_{g,h\in G}$ are right grouplike.*

Remark 1.8. *Weak Hopf algebras with weak G-action give a special case of G-weak Hopf algebra, where the grading is concentrated in degree* 1*. Thus all results of this chapter imply analogous results where the term G-weak Hopf algebra is replaced by Hopf algebra with weak G-action.*

For G-weak Hopf algebras we have a similar result to Lemma I2.15.

Lemma 1.9. *The category of representations of a G-Hopf algebra inherits the natural structure of a* $\mathbb{K}$*-linear, abelian G-equivariant tensor category with dualities.*

2 Strictification of the Group Action

As mentioned in the introduction of this chapter, the action of the group G on a G-equivariant tensor category $\mathcal{C}$ can always be strictified (see [Tur10a, App. 5]), i.e. there is an equivalent G-equivariant tensor category $\mathcal{C}^{str}$ with strict G-action (all compositors are identities). If one starts with the representation category of a G-Hopf algebra A, it is natural to ask whether this strictification leads to the representation category of another G-Hopf algebra with strict G-action. We will make this precise in the next definition. A G-equivariant functor between G-equivariant tensor categories is a tensor functor F together with natural isomorphisms

$$\psi_g : F({}^gM) \xrightarrow{\sim} {}^gF(M)$$

such that for every pair $g, h \in G$ the obvious coherence diagrams of morphisms from $F({}^{gh}M)$ to ${}^{gh}F(M)$ commute. See also [Tur10a, Appendix 5, Def. 2.5].

Definition 2.1.
1. *Let A be a Hopf algebra with weak G-action. A* strictification *of A is a weak Hopf algebra B with* strict *G-action and an equivalence*
$$A\text{-mod} \xrightarrow{\sim} B\text{-mod}$$
of tensor categories with G-action.

2. *Let A be a G-Hopf algebra. A* strictification *of A is a G-weak Hopf algebra B with* strict *G-action and an equivalence*
$$A\text{-mod} \xrightarrow{\sim} B\text{-mod}$$
of G-equivariant tensor categories.

We will now show that it is in general not possible to find a strictification that is a Hopf algebra, rather than a weak Hopf algebra. This shows that we really have to allow for weak Hopf algebras as strictifications. In the next chapter we then show that a strictification as a weak Hopf algebra always exists.

Consider the weak action of $\mathbb{Z}/2 \times \mathbb{Z}/2 = \{1, t_1, t_2, t_1t_2\}$ on the group algebra $\mathbb{C}[\mathbb{Z}/2]$ of $\mathbb{Z}/2 = \{1, t\}$ given by

$$\varphi_g = \mathrm{id} \qquad \text{for all } g \in \mathbb{Z}/2 \times \mathbb{Z}/2$$

and non-trivial compositors given by the grouplike elements $c_{g,h} \in \mathbb{C}[\mathbb{Z}/2]$ as in the following table:

$h\backslash g$	1	t_1	t_2	t_1t_2
1	1	1	1	1
t_1	1	t	t	1
t_2	1	1	1	1
t_1t_2	1	t	t	1

(IV.10)

In [MNS12, Section 3.1] we showed how weak actions correspond to extensions of groups together with the choice of a set theoretic section. In this case, the relevant extension is given by the exact sequence of groups

$$\mathbb{Z}/2 \to D_4 \to \mathbb{Z}/2 \times \mathbb{Z}/2\,,$$

where D_4 denotes the dihedral group of order 8. The inclusion of $\mathbb{Z}/2$ into D_4 is given by mapping the nontrivial t element of $\mathbb{Z}/2$ to the rotation by π. The projection to $\mathbb{Z}/2 \times \mathbb{Z}/2$ is given by mapping the rotation $a \in D_4$ by $\frac{\pi}{2}$ to the first generator t_1 and the reflection $b \in D_4$ to the second generator t_2. The set theoretic section is defined by $s : \mathbb{Z}/2 \times \mathbb{Z}/2 \to D_4$ with $s(1) = 1, s(t_1) = a, s(t_2) = b, s(t_1t_2) = ab$.

Theorem 2.2. *There is no strictification as a Hopf algebra of* $\mathbb{C}[\mathbb{Z}/2]$ *with the weak* $\mathbb{Z}/2 \times \mathbb{Z}/2$*-action with compositors as displayed in* (IV.10).

Remark 2.3. *Note that the algebra* $\mathbb{C}[\mathbb{Z}/2]$ *is not a priori endowed with a grading by* $\mathbb{Z}/2$*. We can consider it as being trivially graded.*

For the proof of proposition 2.2 we need the following elementary facts:

Lemma 2.4. *Let* $A = \mathbb{C}[G]$ *be the complex group algebra of a finite abelian group* G*.*

1. *Let A' be an arbitrary Hopf algebra. If* A-mod $\cong A'$-mod *as tensor categories, then $A \cong A'$ as algebras (not necessarily as Hopf algebras).*

2. *The natural endomorphisms of the identity functor* $\mathrm{Id} : A\text{-mod} \to A\text{-mod}$ *are given by the action of elements in A. More precisely there is an isomorphism of algebras*

$$A \xrightarrow{\sim} \mathrm{End}(\mathrm{Id})_{A\text{-mod}} \ .$$

3. *Let $\varphi : A \to A$ be an algebra automorphism such that the restriction functor res_φ* : A-mod $\to A$-mod *is naturally isomorphic to the identity functor. Then* $\varphi = \mathrm{id}$.

Proof. 1.) By the reconstruction theorem we know that we can recover the Hopf algebra A as endomorphisms of the fibre functor $F : A\text{-mod} \to \mathbb{C}\text{-mod}$ and A' as endomorphisms of the fibre functor $G : A\text{-mod} \xrightarrow{\sim} A'\text{-mod} \to \mathbb{C}\text{-mod}$. Now we claim that the underlying functors of F and G are naturally isomorphic. To this end note that for each simple representation V_i of A we have $V_i^n \cong 1$ where n is the order of the group. Thus we have $F(V_i) \cong \mathbb{C} \cong G(V_i)$ by the fact that F and G are tensor-functors. But it is easy to see that the $\mathbb{C}$-linearity and the fact that A-mod is semisimple then already show that F and G are isomorphic as functors between abelian categories. This implies that $A \cong \mathrm{End}(F) \cong \mathrm{End}(G) \cong A'$. Note that the functors F and G still might have different tensor functor structures, leading to different Hopf algebra structures on A and A'.

2.) This follows from the fact that A is abelian and from the fact that the center of an algebra is isomorphic to the endomorphisms of the identity functor on its representation category.

3) The functor res_φ is an equivalence of categories. Hence it sends simple objects to simple objects. That means it acts on simple characters $\chi : G \to \mathbb{C}^*$. By the fact that this functor is naturally isomorphic to the identity this action has to be trivial. Hence we know $\chi \circ \varphi = \chi$ for each character χ. Because G is abelian, the characters form a basis of the dual space A^*. Thus $\varphi^* = \mathrm{id}$ which implies $\varphi = \mathrm{id}$.

□

Proof of Theorem 2.2. Assume that there is a Hopf algebra H with a strict action of $\mathbb{Z}/2$ by Hopf algebra automorphisms φ_g together with an equivalence of categories A-mod $\to H$-mod. By Lemma 2.4(1) we know that the underlying algebra of H is isomorphic to $\mathbb{C}[\mathbb{Z}/2]$. We choose an isomorphism

and transport the action φ_g on H to an action φ'_g on $\mathbb{C}[\mathbb{Z}/2]$ (which is now only an action by algebra automorphisms and not necessarily by Hopf algebra automorphisms). By assumption there are now natural isomorphisms $res_{\varphi'_g} \xrightarrow{\sim} res_{\varphi_g} = \mathrm{Id}$ hence by lemma 2.4(3) we have $\varphi'_g = \mathrm{id}$.
Now we have both times the trivial action on the Hopf algebra $\mathbb{C}[\mathbb{Z}/2]$, once with the nontrivial compositors $c_{g,h}$ as displayed in table (IV.10) above and once with the trivial compositors. By Lemma 2.4(2), an isomorphism between the two induced actions on the representation categories is induced by invertible elements $(a_g \in \mathbb{C}[\mathbb{Z}/2])_{g \in \mathbb{Z}/2 \times \mathbb{Z}/2}$ such that

$$a_{gh} \cdot c_{g,h} = a_g \cdot a_h \qquad \text{for all } g, h \in \mathbb{Z}/2 \times \mathbb{Z}/2 \tag{IV.11}$$

We show that such elements can not exist: Assume, there are invertible elements $(a_g \in \mathbb{C}[\mathbb{Z}/2])_{g \in \mathbb{Z}/2 \times \mathbb{Z}/2}$ that fulfill (IV.11). In particular we have, by setting $g = h = 1$ in (IV.11), $a_1^2 = a_1$, and since the elements a_g are invertible, it follows that $a_1 = 1$. One concludes similarly, by setting $g = h = t_2$ resp. $g = h = t_1 t_2$, that $a_{t_2}^2 = 1$ and $a_{t_1 t_2}^2 = 1$. Now if we set $g = t_1$ and $h = t_2$ in (IV.11) and take the square of the resulting equation, we get $a_{t_1}^2 = 1$, but clearly those elements don't fulfill the equation $a_1 t = a_{t_1}^2$, which is (IV.11) with $g = h = t_1$. This contradicts the existence of the strictification Hopf algebra H of the $\mathbb{Z}/2 \times \mathbb{Z}/2$-equivariant Hopf algebra $\mathbb{C}(\mathbb{Z}/2)$. □

3 Existence of a Strictification

In this section, we will successively prove the following theorem which holds for Hopf algebras over an arbitrary field K.

Theorem 3.1. *1. For any Hopf algebra with weak G-action there exists a strictification in the sense of definition 2.1(1).*

2. For any G-Hopf algebra there exists a strictification in the sense of definition 2.1(2).

Note that the first part of Theorem 3.1 follows from the second part if we consider a Hopf algebra with weak G-action as a G-Hopf algebra with grading concentrated in degree 1, see also Remark 1.8(4). Therefore we will only prove the second part.
In the following let A be a G-Hopf algebra with unit 1_A, counit ϵ_A, coproduct Δ_A and a weak G-action $((\varphi_g)_{g \in G}, (c_{g,h})_{g,h \in G})$. The plan of this section is to construct step by step a strictification A^{str}.
In section 3.1 we construct A^{str} as an algebra, in section 3.2 we endow it with a weak Hopf algebra structure and finally in section 3.3 we turn it into

a G-weak Hopf algebra with strict G-action. Along the way, we also provide the necessary equivalences of the representation categories

$$F : A\text{-mod} \xrightarrow{\sim} A^{str}\text{-mod}$$

and show that they preserve all the structure involved. This implies that A^{str} is a strictification, which proves Theorem 3.1.

3.1 The Algebra

In the following we use the notation $\mathrm{K}(G)$ for the K-vector space of functions on the finite group G, with distinguished basis $(\delta_g)_{g\in G}$. By $\mathrm{K}[G]$ we denote the K-vector space underlying the group algebra with basis $(g)_{g\in G}$.

Definition 3.2. *Set $A^{str} = \mathrm{K}(G) \otimes_{\mathrm{K}} A \otimes_{\mathrm{K}} \mathrm{K}[G]$ as a vector space and define a multiplication on the generators of A^{str} by*

$$(\delta_g \otimes a \otimes h)(\delta_{g'} \otimes a' \otimes h') = \delta(gh', g')(\delta_{g'} \otimes a\varphi_h(a')c_{h,h'} \otimes hh') \qquad \text{(IV.12)}$$

where $\delta(gh', g')$ is the Kronecker delta, i.e. $\delta(gh', g') = 1$ if $gh' = g'$ and $\delta(gh', g') = 0$ otherwise. This multiplication has the unit

$$1 = \sum_{g\in G} \delta_g \otimes 1_A \otimes 1. \qquad \text{(IV.13)}$$

It can easily be checked that the product and the unit defined in (IV.12) and (IV.13) endow A^{str} with the structure of an associative unital algebra.

We next define a functor $F : A\text{-mod} \to A^{str}\text{-mod}$: Let M be an object in A-mod. Define an object in A^{str}-mod which is $M \otimes_{\mathrm{K}} \mathrm{K}[G]$ as a vector space and has the following right action of the algebra A^{str}: On an element of the form $(m \otimes k)$ with $m \in M, k \in G$, the action of $(\delta_g \otimes a \otimes h)$ reads:

$$(m \otimes k).(\delta_g \otimes a \otimes h) := \delta(kh, g)(m.\varphi_k(a)c_{k,h} \otimes g) \qquad \text{(IV.14)}$$

One checks that this really defines a right action of A^{str}. For a morphism $f \in \mathrm{Hom}_A(M, N)$ we consider the morphism $f \otimes \mathrm{id}_{\mathrm{K}[G]} \in \mathrm{Hom}_{A^{str}}(M \otimes \mathrm{K}[G], N \otimes \mathrm{K}[G])$. Together this defines a functor:

$$F : A\text{-mod} \to A^{str}\text{-mod} \qquad \text{(IV.15)}$$

Proposition 3.3. *The functor F is an equivalence of abelian categories.*

Proof. We show that F is essentially surjective and fully faithful.
For the essential surjectivity, note that an object N in A^{str}-mod has a G-grading $N = \bigoplus_{g \in G} N_g$ with $N_g := N.(\delta_g \otimes 1_A \otimes 1)$. Endow the subspace $N_1 := N.(\delta_1 \otimes 1_A \otimes 1) \subset N$ with the structure of an A-module by setting $n.a := n.(\delta_1 \otimes a \otimes 1)$ for an element $n \in N_1$ and $a \in A$.
Define the K-linear map

$$\Theta : F(N_1) = N_1 \otimes \mathrm{K}[G] \to N$$

by

$$(n \otimes g) \mapsto n.(\delta_g \otimes 1_A \otimes g).$$

It is easy to see that Θ is an isomorphism with inverse
$n \mapsto \sum_{g \in G} n.(\delta_1 \otimes (c_{g^{-1},g})^{-1} \otimes g^{-1}) \otimes g$. In order to see that Θ is a morphism in A^{str}-mod, note that the action of A^{str} on $F(N_1)$ is given by

$$(n \otimes k).(\delta_g \otimes a \otimes h) = \delta(kh, g) n.(\delta_1 \otimes \varphi_k(a) c_{k,h} \otimes 1) \otimes g$$

Hence we have

$$\begin{aligned}
\Theta((n \otimes k).(\delta_g \otimes a \otimes h)) &= \delta(kh, g) \Theta(n.(\delta_1 \otimes \varphi_k(a) c_{k,h} \otimes 1) \otimes g) \\
&= \delta(kh, g) n.(\delta_1 \otimes \varphi_k(a) c_{k,h} \otimes 1)(\delta_g \otimes 1_A \otimes g) \\
&= \delta(kh, g) n.(\delta_g \otimes \varphi_k(a) c_{k,h} \otimes g)
\end{aligned}$$

and

$$\begin{aligned}
\Theta(n \otimes k).(\delta_g \otimes a \otimes h) &= n.(\delta_k \otimes 1_A \otimes k)(\delta_g \otimes a \otimes h) \\
&= \delta(kh, g) n.(\delta_g \otimes \varphi_k(a) c_{k,h} \otimes g).
\end{aligned}$$

This shows that $F(N_1) \cong N$ as A^{str}-modules and thus essential surjectivity. It is clear that F is faithful. In order to see that F is also full, consider for two A-modules M, N a morphism $f \in \mathrm{Hom}_{A^{str}}(F(M), F(N))$. We have

$$f(m \otimes k).(\delta_g \otimes 1_A \otimes 1) = f((m \otimes k).(\delta_g \otimes 1_A \otimes 1)) = \delta(k, g) f(m \otimes k),$$

so $f(m \otimes k) \in N \otimes \mathrm{K}k$ and we have $f = \sum_{g \in G} f_g \otimes \mathrm{id}_{\mathrm{K}g}$ for some $f_g \in \mathrm{Hom}_{\mathrm{K}}(M, N)$. From the definition (IV.14) of the action of A^{str}, it is clear that $f_1 \in \mathrm{Hom}_A(M, N)$. Now, since f commutes with the action of A^{str}, we have

$$\begin{aligned}
\delta(kh, g) f_g(m.\varphi_k(a) c_{k,h}) \otimes g &= f((m \otimes k).(\delta_g \otimes a \otimes h)) \\
&= f(m \otimes k).(\delta_g \otimes a \otimes h) \\
&= \delta(kh, g) f_k(m).\varphi_k(a) c_{k,h} \otimes g
\end{aligned}$$

and (by setting $a = 1_A$, $k = 1$ and $h = g$) we get $f_g = f_1$ for all $g \in G$ and therefore f is of the form $f = f_1 \otimes \mathrm{id}_{\mathrm{K}[G]}$. $\square$

3.2 The Weak Hopf Algebra Structure

We need the strictification algebra A^{str} to have more structure in order for its representation category to be a tensor category. In fact, we want it to be a weak Hopf algebra (cf. Definition 1.2)

Proposition 3.4. *The linear maps $\Delta : A^{str} \to A^{str} \otimes A^{str}$ and $\epsilon : A^{str} \to \mathrm{K}$ defined on the generators of A^{str} by*

$$\begin{aligned}\Delta(\delta_g \otimes a \otimes h) &= \sum_{(a)} (\delta_g \otimes a_{(1)} \otimes h) \otimes (\delta_g \otimes a_{(2)} \otimes h) \ , \\ \epsilon(\delta_g \otimes a \otimes h) &= \epsilon_A(a)\end{aligned}$$

endow A^{str} with the structure of a weak bialgebra. Furthermore, the linear map $S : A^{str} \to A^{str}$ given by

$$S(\delta_g \otimes a \otimes h) = (\delta_{gh^{-1}} \otimes c^{-1}_{h^{-1},h} \cdot \varphi_{h^{-1}}\big(S_A(a)\big) \otimes h^{-1})$$

is an antipode for A^{str}, where S_A is the antipode of A.

Proof. The maps Δ and ϵ are a coassociative coproduct and a counit on A^{str}, as they are just the structural maps of the tensor product coalgebra of $\mathrm{K}(G)$, A and $\mathrm{K}[G|$ (where we consider the diagonal coproduct on both $\mathrm{K}(G)$ and $\mathrm{K}[G]$). We show that Δ is also a morphism of algebras, i.e. that

$$(m \otimes m) \circ (\mathrm{id} \otimes \tau \otimes \mathrm{id})(\Delta \otimes \Delta) = \Delta \circ m. \tag{IV.16}$$

If we plug in two elements $(\delta_g \otimes a \otimes h), (\delta_{g'} \otimes a' \otimes h')$, we get for the left hand side of (IV.16)

$$\begin{aligned}&\sum_{(a)} (\delta_g \otimes a_{(1)} \otimes h) \cdot (\delta_{g'} \otimes a'_{(1)} \otimes h') \otimes (\delta_g \otimes a_{(2)} \otimes h) \cdot (\delta_{g'} \otimes a'_{(2)} \otimes h') \\ &= \sum_{(a)} \delta(gh', g')(\delta_{g'} \otimes a_{(1)}\varphi_h(a'_{(1)})c_{h,h'} \otimes hh') \otimes (\delta_{g'} \otimes a_{(2)}\varphi_h(a'_{(2)})c_{h,h'} \otimes hh') \ ,\end{aligned}$$

and for the right hand side

$$\begin{aligned}&\delta(gh', g')\Delta(\delta_{g'} \otimes a\varphi_h(a')c_{h,h'} \otimes hh') \\ &= \delta(gh', g') \sum_{(a)} (\delta_{g'} \otimes (a\varphi_h(a')c_{h,h'})_{(1)} \otimes hh') \otimes (\delta_{g'} \otimes (a\varphi_h(a')c_{hh'})_{(2)} \otimes hh') \\ &= \sum_{(a)} \delta(gh', g')(\delta_{g'} \otimes a_{(1)}\varphi_h(a'_{(1)})c_{h,h'} \otimes hh') \otimes (\delta_{g'} \otimes a_{(2)}\varphi_h(a'_{(2)})c_{h,h'} \otimes hh') \ .\end{aligned}$$

All equations follow just by definition of the product and coproduct, except for the last one, where we used that the coproduct in A is a morphism of algebras, that the elements $c_{g,h}$ are group-like and that the action of G on A is a coalgebra-morphism.
Further equations concerning the compatibilities of the product with the counit, the coproduct with the unit and the antipode can be checked directly. □

In a weak Hopf algebra H the target and source counital maps are defined on an element $h \in H$ by

$$\begin{aligned} \epsilon_t(h) &:= (\epsilon \otimes \mathrm{id}_H)(\Delta(1)(h \otimes 1)) \\ \epsilon_s(h) &:= (\mathrm{id}_H \otimes \epsilon)((1 \otimes h)\Delta(1)) \end{aligned}$$

The maps ϵ_t and ϵ_s are idempotents. The image of H under them are called the target and source counital subalgebras

$$\begin{aligned} H_t &:= \epsilon_t(H) \\ H_s &:= \epsilon_s(H). \end{aligned}$$

The category of right modules over H can be endowed with the structure of a tensor category, where the tensor product of two modules M, N is defined via the coproduct on the following vector space:

$$M \bar{\otimes} N := (M \otimes_{\mathrm{K}} N)\Delta(1) \ .$$

The tensor unit is the source counital subalgebra H_s with H-action given by $z.h := \epsilon_s(zh)$ for $h \in H, z \in H_s$.

Lemma 3.5. *For the algebra A^{str}, the target and source counital maps are given by*

$$\epsilon_t(\delta_g \otimes a \otimes h) = \epsilon_A(a)(\delta_{gh^{-1}} \otimes 1_A \otimes 1) \tag{IV.17}$$
$$\epsilon_s(\delta_g \otimes a \otimes h) = \epsilon_A(a)(\delta_g \otimes 1_A \otimes 1) \tag{IV.18}$$

and the target and source counital subalgebras are

$$A_t^{str} \cong A_s^{str} \cong \mathrm{K}(G) \ .$$

Proof. We calculate ϵ_t on an element $(\delta_g \otimes a \otimes h) \in A^{str}$:

$$\begin{aligned} \epsilon_t(\delta_g \otimes a \otimes h) &= (\epsilon \otimes \mathrm{id})(\Delta(1)((\delta_g \otimes a \otimes h) \otimes 1)) \\ &= (\epsilon \otimes \mathrm{id})((\delta_g \otimes a \otimes h) \otimes (\delta_{gh^{-1}} \otimes 1_A \otimes 1)) \\ &= \epsilon_A(a)(\delta_{gh^{-1}} \otimes 1_A \otimes 1) \ . \end{aligned}$$

The calculation for the source counital map is completely parallel. Choose a basis $(a_i)_{i\in I}$ of the algebra A with $a_j = 1_A$ for a fixed $j \in I$, then a general element $b \in A^{str}$ is of the form

$$b = \sum_{g,h\in G, i\in I} \lambda(g,h,i)(\delta_g \otimes a_i \otimes h)$$

with $\lambda(g,h,i) \in \mathrm{K}$. We have:

$$\Delta(b) = \sum_{g,h\in G, i\in I} \sum_{(a_i)} \lambda(g,h,i)(\delta_g \otimes (a_i)_{(1)} \otimes h) \otimes (\delta_g \otimes (a_i)_{(2)} \otimes h)$$

$$\Delta(1)(b \otimes 1) = \sum_{g,h\in G, i\in I} \lambda(g,h,i)(\delta_g \otimes a_i \otimes h) \otimes (\delta_g \otimes 1_A \otimes h)$$

By equating coefficients, we get $\lambda(g,h,i) = 0$ for $h \neq 1, i \neq j$ and therefore:

$$A_t^{str} = \langle \delta_g \otimes 1_A \otimes 1, g \in G \rangle \cong \mathrm{K}(G) \ .$$

An analog calculation shows the same result for A_s^{str}. □

Proposition 3.6. *The equivalence $F : A$-mod $\xrightarrow{\sim} A^{str}$-mod can be promoted to an equivalence of tensor categories.*

Proof. The tensor unit in the representation category of the weak Hopf algebra A^{str} is given by the source counital subalgebra, which is by Lemma 3.5 isomorphic to $\mathrm{K}(G)$. The weak Hopf algebra A^{str} acts on the source counital subalgebra as follows: for an element $(\delta_k \otimes 1_A \otimes 1) \in A_s^{str}$ and $(\delta_g \otimes a \otimes h) \in A^{str}$, we have

$$(\delta_k \otimes 1_A \otimes 1).(\delta_g \otimes a \otimes h) = \epsilon_s((\delta_k \otimes 1_A \otimes 1)(\delta_g \otimes a \otimes h)) = \delta(kh,g)\epsilon_A(a)(\delta_g \otimes 1_A \otimes 1) \ .$$

The action of an element $a \in A$ on the tensor unit K in A-mod is by multiplication with $\epsilon_A(a)$. So we get for the image of the tensor unit under F the vector space $\mathrm{K} \otimes \mathrm{K}[G]$ with A^{str}-action

$$(\lambda \otimes k).(\delta_g \otimes a \otimes h) = \delta(kh,g)\epsilon_A(a)\lambda \otimes g \ .$$

We clearly have an isomorphism $F(\mathbf{1}) \to \mathbf{1}$ in A^{str}-mod given by

$$\eta_0 : (\lambda \otimes k) \mapsto \lambda\delta_k.$$

Let $M, N \in A$-mod. We have

$$F(M)\bar{\otimes}F(N) = \langle (m \otimes g \otimes n \otimes g), m \in M, n \in N, g \in G \rangle.$$

Thus the linear map

$$\eta_2(M,N) : (m \otimes g \otimes n \otimes g) \mapsto (m \otimes n \otimes g)$$

is an isomorphism $F(M)\bar{\otimes}F(N) \xrightarrow{\sim} F(M \otimes N)$. It can be checked to commute with the action of A^{str} and is natural in M, N. Moreover the isomorphisms η_2 and η_0 clearly satisfy the required coherence axioms. We have therefore established that (F, η_0, η_2) is a tensor functor. □

3.3 G-Action and G-Grading

We will now define a G-equivariant structure on A^{str} that induces a G-equivariant structure on the category A^{str}-mod. The last step of proving theorem 3.1 is then to show that the categories A-mod and A^{str}-mod are even equivalent as G-equivariant categories.

Definition 3.7. *On the weak Hopf algebra A^{str} we have a strict left action φ^{str} of the group G given by translation in the first factor. Explicitly, an element $g' \in G$ acts on an element $(\delta_g \otimes a \otimes h) \in A^{str}$ by*

$$\varphi^{str}_{g'}(\delta_g \otimes a \otimes h) = (\delta_{g'g} \otimes a \otimes h) \ .$$

The strict G-action on A^{str} gives us a strict left G-action on the category A^{str}-mod by setting $\phi_g(M, \rho) = (M, \rho \circ (\mathrm{id}_M \otimes \varphi^{str}_{g^{-1}})$. We will now establish, that the equivalence A-mod $\cong A^{str}$ is compatible with the G-actions.

Proposition 3.8. *The equivalence $F : A$-mod $\xrightarrow{\sim} A^{str}$-mod given in (IV.15) respects the G-action of the two categories, i.e. for every element $g \in G$ there are natural isomorphisms*

$$\psi_g : F({}^gM) \ \xrightarrow{\sim} \ {}^gF(M)$$

such that for every pair $g, h \in G$ the obvious coherence diagrams of morphisms from $F({}^{gh}M)$ to ${}^{gh}F(M)$ commute.

Proof. For $M \in A$-mod consider the linear map $\psi_g : M \otimes \mathrm{K}[G] \to M \otimes \mathrm{K}[G]$ defined by

$$\psi_g : (m \otimes k) \mapsto (m.c_{g^{-1},k} \otimes g^{-1}k).$$

We first show that ψ_g is a morphism of A^{str}-modules. To distinguish the actions on the different modules we use the notation "$\star$" for the A^{str}-action on $F({}^gM)$ and ${}^gF(M)$, "$\odot$" for the action on $F(M)$ and "." for the A-action on M.

$$\begin{aligned}
&\psi_g\big((m\otimes k)\star(\delta_x\otimes a\otimes h)\big)\\
&\quad=\psi_g\big(\delta(kh,x)(m.\varphi_{g^{-1}}(\varphi_k\big(a)c_{k,h}\big)\otimes x)\big)\\
&\quad=\delta(kh,x)\ m.\varphi_{g^{-1}}(\varphi_k\big(a)c_{k,h}\big)c_{g^{-1},kh}\otimes g^{-1}kh\\
&\quad=\delta(kh,x)\ m.c_{g^{-1},k}\varphi_{g^{-1}k}(a)(c_{g^{-1},k})^{-1}\varphi_{g^{-1}}(c_{k,h})c_{g^{-1},kh}\otimes g^{-1}kh\\
&\quad=\delta(g^{-1}kh,g^{-1}x)\ m.c_{g^{-1},k}\varphi_{g^{-1}k}(a)c_{g^{-1}k,h}\otimes g^{-1}kh\\
&\quad=(m.c_{g^{-1},k}\otimes g^{-1}k)\odot(\delta_{g^{-1}x}\otimes a\otimes h)\\
&\quad=\psi_g(m\otimes k)\star(\delta_x\otimes a\otimes h)\ .
\end{aligned}$$

Moreover we have to verify that the ψ_g satisfy a coherence condition for two indices g and h. This condition can be checked similarly to the above computation, using the cocycle condition for the elements $c_{g,h}$. □

Definition 3.9. *We define a G-grading in the sense of 2.13 on the algebra A^{str} by:*

$$(A^{str})_h=\bigoplus_{g\in G}(\mathrm{K}(\delta_g)\otimes A_{g^{-1}hg})\otimes \mathrm{K}[G]\ . \tag{IV.19}$$

The following Lemma can then be checked directly.

Lemma 3.10. *The algebra A^{str} is a G-weak Hopf algebra with strict G-action, i.e a weak Hopf algebra with strict G-action and compatible G-grading.*

Note that the grading on A resp. A^{str} gives a grading on the representation category by $(A\text{-mod})_h := A_h\text{-mod}$ resp. $(A^{str}\text{-mod})_h=(A^{str})_h\text{-mod}$.

Proposition 3.11. *The equivalence $F: A\text{-mod}\xrightarrow{\sim}A^{str}\text{-mod}$ given in (IV.15) respects the G-grading of the two categories, i.e. for every element $h\in G$, we have*

$$F((A\text{-mod})_h)\subset(A^{str}\text{-mod})_h.$$

Proof. Let $M\in(A\text{-mod})_h=A_h\text{-mod}$. We know that the action by the unit $(\mathbf{1}_A)_h$ of A_h is the identity on M. We need to show, that the h-component of the unit in A^{str}, which is $\mathbf{1}_h=\sum_{g\in G}(\delta_g\otimes(1_A)_{g^{-1}hg}\otimes 1)$, acts as an idempotent on $F(M)=M\otimes\mathrm{K}[G]$. In fact it even acts as the identity: for any element of the form $(m\otimes k)$, we have:

$$\begin{aligned}
(m\otimes k).\left(\sum_{g\in G}\delta_g\otimes(1_A)_{g^{-1}hg}\otimes 1\right)&=\sum_{g\in G}(\delta(k,g)m.\varphi_k((1_A)_{g^{-1}hg})c_{k,1}\otimes k)\\
&=(m.\varphi_k((1_A)_{k^{-1}hk})\otimes k)\\
&=(m.(1_A)_h\otimes k)\\
&=(m\otimes k)
\end{aligned}$$

where in the first equality we used the definition of the action of A^{str} on $M \otimes \mathrm{K}[G]$ given in (IV.14), in the third equality the fact that G acts by unital algebra morphisms and in the last equality that M is in the h-component of A-mod.
So we have $F(M).\mathbf{1}_h = F(M)$; therefore $F(M)$ lies in the component $(A^{str}\text{-mod})_h$.
□

4 Equivariant R-Matrix and Ribbon-Element

In [MNS12] we considered G-equivariant categories with a G-braiding and a G-twist as additional data (G-ribbon categories). For the definition see [Tur10a, Kir04]. Since those categories were our main motivation to study the strictification in terms of algebras, we want to say a few words about the G-ribbon structure.
The definition of a G-equivariant R-matrix is rather involved even in the strict Hopf algebra case. We will refrain here from stating the axioms for it explicitly, but we will instead make an equivalent definition:

Definition 4.1. *Let A be a G-(weak) Hopf algebra.*

1. *A G-*equivariant R-matrix *is an element* $R = R_1 \otimes R_2 \in \Delta^{op}(1)(A \otimes A)\Delta(1)$ *such that for* $V \in (A\text{-mod})_g$, $W \in A$-mod, *the map*

$$\begin{aligned} c_{VW} : V \otimes W &\to {}^gW \otimes V \\ v \otimes w &\mapsto w.R_2 \otimes v.R_1 \end{aligned}$$

   is a G-braiding, in particular a morphism of A-modules.

2. *A G-*twist *is an invertible element $\theta \in A$ such that for every object* $V \in (A\text{-mod})_g$ *the induced map*

$$\begin{aligned} \theta_V : V &\to {}^gV \\ v &\mapsto v.\theta^{-1} \end{aligned}$$

   is a G-twist in A-mod.

A G-(weak) ribbon-algebra *is a G-(weak) Hopf algebra A with a G-equivariant R-matrix and a G-twist.*

Lemma 4.2. *A G-weak Hopf algebra A can be endowed with the structure of a G-weak ribbon algebra if and only if the representation category* A-mod *has the structure of a G-ribbon category.*

Proof. If A is a G-ribbon algebra, it follows from the definition that A-mod is a G-ribbon category. If on the other hand, A-mod is a G-ribbon category with G-braiding c and G-twist θ, define an R-matrix and a twist of A by

$$R = \tau \circ c_{A,A}(1_A \otimes 1_A) \quad \text{and} \quad \theta = \theta_A(1)^{-1} \ . \tag{IV.20}$$

For $v \in V$, $w \in W$ let $\bar{v} : A \to V$, $\bar{w} : A \to W$ be the A-linear maps with $\bar{v}(1_A) = v, \bar{w}(1_A) = w$. We then have

$$\tau((v \otimes w).R) = \tau(\bar{v} \otimes \bar{w}(R)) = (\bar{w} \otimes \bar{v})c_{A,A}(1_A \otimes 1_A) = c_{V,W}(v \otimes w) \ ,$$

$$v.\theta^{-1} = v.(\theta_A(1_A)) = \bar{v}(\theta_A(1_A)) = \theta_V \bar{v}(1_A) = \theta_V(v).$$

Thus R and θ satisfy the conditions of definition 4.1 by construction. □

As an immediate consequence of lemma 4.2, we have:

Corollary 4.3. *If A is a G-ribbon algebra, the strictification algebra A^{str} inherits the structure of a G-weak ribbon algebra such that the equivalence* $F : A\text{-mod} \to A^{str}\text{-mod}$ *is an equivalence of G-ribbon categories.*

4.1 Table summarizing Terminology

The following table summarizes the terminology for Hopf algebras with an action of a finite group G and their weakenings. We consider two types of weakenings: a weakening of the G-action corresponding to the two rows of the table, and a weakening of the unitality of the coproduct, corresponding to the two columns of the table.

Each square contains three different entries, depending on additional structure on the Hopf algebra. The objects in 1. only have the G-action and no additional structure (see Definition 2.11). The objects in 2. are equipped with a G-grading with the compatibilities introduced in Definition 2.13. The objects in 3. have, in addition to the G-equivariant structure, a G-equivariant R-matrix and a G-twist as introduced in Definition 4.1.

	Hopf algebra	weak Hopf algebra
strict G-action	1. Hopf algebra with strict G-action 2. G-Hopf algebra with strict G-action 3. G-ribbon algebra with strict G-action	1. weak Hopf algebra with strict G-action 2. G-weak Hopf algebra with strict G-action 3. G-weak ribbon algebra with strict G-action
weak G-action	1. Hopf algebra with weak G-action 2. G-Hopf algebra 3. G-ribbon algebra	1. weak Hopf algebra with weak G-action 2. G-weak Hopf algebra 3. G-weak ribbon algebra

Bibliography

[Ati88] M. Atiyah. Topological quantum field theories. *Publications Mathématiques de l'IHÉS*, 68(1):175–186, 1988.

[Ban10] P. Bantay. Characters of crossed modules and premodular categories. In *Moonshine: the first quarter century and beyond*, volume 372 of *London Math. Soc. Lecture Note Ser.*, pages 1–11. Cambridge Univ. Press, Cambridge, 2010.

[Bar09] B. Bartlett. The geometry of unitary 2-representations of finite groups and their 2-characters. *Applied Categorical Structures*, 19:175–232, 2009.

[Bau09] H. Baum. *Eichfeldtheorie: Eine Einführung in die Differentialgeometrie auf Faserbündeln.* Springer-Lehrbuch Masterclass. Springer-Verlag Berlin Heidelberg, 2009.

[BCM86] R.J. Blattner, M. Cohen, and S. Montgomery. Crossed products and inner actions of Hopf algebras. *Trans. Amer. Math. Soc.*, 298(2):671–711, 1986.

[BD01] J.C. Baez and J. Dolan. From finite sets to Feynman diagrams. In Björn Engquist and Wilfried Schmid, editors, *Mathematics Unlimited - 2001 and Beyond*, pages 29–50. Springer, 2001.

[BK01] B. Bakalov and A.A. Kirillov. *Lectures on tensor categories and modular functors.* American Mathematical Society, 2001.

[BL04] J.C. Baez and A.D. Lauda. Higher-dimensional algebra. V. 2-groups. *Theory Appl. Categ.*, 12(14):423–491, 2004.

[BNS99] G. Böhm, F. Nill, and K. Szlachányi. Weak Hopf algebras. I. Integral theory and C^*-structure. *J. Algebra*, 221(2):385–438, 1999.

[Bru00] A. Bruguières. Catégories prémodulaires, modularisations et invariants des variétés de dimension 3. *Math. Annal.*, 316:215–236, 2000.

[BS96] G. Böhm and K. Szlachányi. A coassociative C^*-quantum group with nonintegral dimensions. *Lett. Math. Phys.*, 38(4):437–456, 1996.

[BS00] G. Böhm and K. Szlachányi. Weak Hopf algebras. II. Representation theory, dimensions, and the Markov trace. *J. Algebra*, 233(1):156–212, 2000.

[CY99] L. Crane and D. Yetter. On algebraic structures implicit in topological quantum field theories. *Journal of Knot Theory and its Ramifications*, 8:125–164, 1999.

[Ded60] P. Dedecker. Les foncteurs ExtΠ, H2 Π et H2 Π non abéliens. *CR Acad. Sc. Paris*, 258:4891–4895, 1960.

[Del90] P. Deligne. Catégories tannakiennes, The Grothendieck Festschrift, Vol. II, 111–195. *Progr. Math*, 87, 1990.

[DPP04] R.J.M.G. Dawson, R. Paré, and D.A. Pronk. Universal properties of span. *Theory and Appl. of Categories*, 13(4):61–85, 2004.

[DPR90] R. Dijkgraaf, V. Pasquier, and P. Roche. Quasi Hopf algebras, group cohomology and orbifold models. *Nucl. Phys. B 18 (1990) 60*, 1990.

[Dri87] V. G. Drinfel′d. Quantum groups. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Berkeley, Calif., 1986)*, pages 798–820, Providence, RI, 1987. Amer. Math. Soc.

[DW90] R. Dijkgraaf and E. Witten. Topological gauge theories and group cohomology. *Comm. Math. Phys.*, 129(2):393–429, 1990.

[ENO10] P. Etingof, D. Nikshych, and V. Ostrik. Fusion categories and homotopy theory. *Quantum Topol.*, 1(3):209273, 2010.

[FHLT09] D.S. Freed, M.J. Hopkins, J. Lurie, and C. Teleman. Topological quantum field theories from compact Lie groups. *arXiv*, 905:367, 2009.

[FQ93] D.S. Freed and F. Quinn. Chern-Simons theory with finite gauge group. *Comm. Math. Phys.*, 156(3):435–472, 1993.

[Fre94] D.S. Freed. Higher algebraic structures and quantization. *Commun. Math. Phys.*, 159(2):343–398, 1994.

[Fre95] D.S. Freed. Classical Chern-Simons theory, Part 1. *Adv. Math.*, 133:237–303, 1995.

[Fre99] D.S. Freed. Quantum groups from path integrals. In G. Semenoff and L. Vinet, editors, *Particles and fields*, CRM Series on mathematical physics, pages 63–108. Springer Verlag, 1999.

[Hat02] A. Hatcher. *Algebraic topology.* Cambridge Univ Press, 2002.

[Hay99] T. Hayashi. A canonical Tannaka duality for finite seimisimple tensor categories. *Arxiv preprint math/9904073*, 1999.

[Hua05] Y.-Z. Huang. Vertex operator algebras, the Verlinde conjecture and modular tensor categories. *Proc. Natl. Acad. Sci. USA*, 102:5352–5356, 2005.

[Isa94] I. M. Isaacs. *Character theory of finite groups.* Dover Publications Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York; MR0460423 (57 #417)].

[JS91] A. Joyal and R. Street. An introduction to Tannaka duality and quantum groups. In *Category theory (Como, 1990)*, volume 1488 of *Lecture Notes in Math.*, pages 413–492. Springer, Berlin, 1991.

[Kas95] C. Kassel. *Quantum groups*, volume 155 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[Kir01] A.A. Kirillov. Modular categories and orbifold models II. Preprint math.QA/0110221, 2001.

[Kir04] A.A. Kirillov. On G-equivariant modular categories. arXiv:math/0401119v1 [math.QA], 2004.

[KLM01] Y. Kawahigashi, R. Longo, and M. Müger. Multi-interval subfactors and modularity of representations in conformal field theory. *Commun. Math. Phys.*, 219:631–669, 2001.

[Koc04] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 2004.

[KV94] M.M. Kapranov and V.A. Voevodsky. 2-Categories and Zamolodchikov Tetrahedra Equations. *Summer Research Institute on Algebraic Groups and Their Generalizations, July 6-26, 1991, Pennsylvania State University*, 56(Part 1):177, 1994.

[Law93] R.J. Lawrence. Triangulations, categories and extended topological field theories. In L. Kauffman and R. Baadhio, editors, *Quantum Topology, Ser. Knots & Everything*, volume 3, pages 191–208. World Scientific, 1993.

[Lei98] T. Leinster. Basic bicategories. *arXiv preprint math/9810017*, 1998.

[Lei08] T. Leinster. The Euler characteristic of a category. *Documenta Mathematica*, 13:21–49, 2008.

[Lur09] J. Lurie. On the Classification of Topological Field Theories. Preprint arXiv:0905.0465v1 [math.CT], 2009.

[Mai09] J. Maier. Modularisierung der Darstellungskategorie verallgemeinerter Drinfel'd Doppel. Master's thesis, Universität Hamburg, 2009.

[May99] J.P. May. *A concise course in algebraic topology.* University of Chicago Press, 1999.

[MNS11] J. Maier, T. Nikolaus, and C. Schweigert. Strictification of weakly equivariant Hopf algebras. *accepted in Bull. Belg. Math. Soc., Arxiv preprint arXiv:1109.0236*, 2011.

[MNS12] J. Maier, T. Nikolaus, and C. Schweigert. Equivariant Modular Categories via Dijkgraaf-Witten Theory. *Adv. Theor. Math. Phys.*, 16(1):289–358, 2012.

[Mon93] S. Montgomery. *Hopf algebras and their actions on rings.* American Mathematical Society, 1993.

[Mor06] J.C. Morton. Double bicategories and double cospans. *Arxiv preprint math/0611930*, 2006.

[Mor08] J.C. Morton. 2-Vector Spaces and Groupoids. Preprint arXiv:0810.2361, 2008.

[Mor10] J.C. Morton. Extended TQFT, Gauge Theory, and 2-Linearization. Preprint arXiv:1003.5603v2 [math.QA], 2010.

[MS11] J. Maier and C. Schweigert. Modular categories from finite crossed modules. *J. Pure Appl. Algebra*, 215(9):2196–2208, 2011.

[Müg00] M. Müger. Galois theory for braided tensor categories and the modular closure. *Adv. Math.*, 150(2):151–201, 2000.

[Nai10] D. Naidu. Crossed pointed categories and their equivariantizations. *Pacific J. Math.*, 247(2):477–496, 2010.

[NTV03] D. Nikshych, V. Turaev, and L. Vainerman. Quantum groupoids and invariants of knots and 3-manifolds. *Topology and its Appl.*, 127:91–123, 2003.

[NV02] D. Nikshych and L. Vainerman. Finite quantum groupoids and their applications. In *New directions in Hopf algebras*, volume 43 of *Math. Sci. Res. Inst. Publ.*, pages 211–262. Cambridge Univ. Press, Cambridge, 2002.

[RT91] N.Yu. Reshetikhin and V.G. Turaev. Invariants of 3-manifolds via link polynomials and quantum groups. *Inv. Math.*, 103(547), 1991.

[Sch26] O. Schreier. Über die Erweiterung von Gruppen I. *Monatsh. Math. Phys*, 34(1):165–180, 1926.

[Ser77] J.P. Serre. *Linear representations of finite groups.* Lecture Notes in Mathematics 1500. Springer Verlag, 1977.

[SP09] C. Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Quantum Field Theories.* PhD thesis, Ph. D. thesis, UC Berkeley. Also available at http://sites. google. com/site/chrisschommerpriesmath, 2009.

[Tur10a] V. Turaev. *Homotopy Quantum Field Theory.* With appendices by M. Müger and A. Virélizier. European Mathematical Society, Zürich, 2010.

[Tur10b] V. Turaev. *Quantum invariants of knots and 3-manifolds*, volume 18 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, revised edition, 2010.

[Vec03] P. Vecsernyés. Larson-Sweedler theorem and the role of grouplike elements in weak Hopf algebras. *J. Algebra*, 270(2):471–520, 2003.

[Vir02] A. Virelizier. Hopf group-coalgebras. *J. Pure Appl. Algebra*, 171(1):75–122, 2002.

[VW95] C. Vafa and E. Witten. On orbifolds with discrete torsion. *Journal of Geometry and Physics*, 15(3):189–214, 1995.

[Wil08] S. Willerton. The twisted Drinfeld double of a finite group via gerbes. *Algebr. Geom. Topol. 8 (2008), no. 3, 1419-1457*, 2008.

[Wit89] E. Witten. Quantum field theory and the Jones polynomial. *Commun. Math. Phys.*, 121:351–399, 1989.

Zusammenfassung

In dieser Arbeit untersuchen wir modulare und äquivariant-modulare Tensorkategorien. Modulare Tensorkategorien sind eine Quelle für 3-dimensionale topologische Feldtheorien (3d TFTs) und somit für Invarianten von 3d Mannigfaltigkeiten und Links. Für jede endliche Gruppe G gibt es eine Verallgemeinerung der Definition von 3d TFT, eine sogenannte G-äquivariante 3d TFT. Die relevante algebraische Struktur ist in diesem Fall eine G-äquivariant-modulare Kategorie. Wir geben in dieser Arbeit eine geometrische Konstruktion einer Klasse von G-modularen Kategorien, indem wir die Korrespondenz von G-äquivarianten erweiterten 3d TFTs und G-äquivariant-modularen Kategorien ausnutzen.
In Kapitel I erinnern wir zunächst an die Zusammenhänge von G-Hopf Algebren, G-modularen Tensorkategorien und G-äquivarianten TFTs. Ein wesentliches neues Resultat dieses Kapitels ist die Orbifold-Konstruktion, die auf G-äquivarianten Kategorien existiert, auf dem Niveau von Hopf Algebren.
In Kapitel II analysieren wir die Modularisierung (im Sinne von Bruguières) einer prämodularen Kategorie anhand des Beispiels der Darstellungskategorie endlicher gekreuzter Moduln. Sie ergibt die Kategorie $\mathcal{D}(G)$-mod der Moduln des Drinfel'd Doppel einer Gruppe G, jedoch mit der zusätzlichen Struktur der Wirkung einer weiteren endliche Gruppe.
Dies ist der Ausgangspunkt der Konstruktion einer äquivarint-modularen Kategorie in Kapitel III, das den Kern dieser Arbeit darstellt. Unsere Konstruktion basiert auf einer normalen Untergruppe $G \triangleleft H$ einer endlichen Gruppe H. Wir betrachten zu diesem Zweck *erweiterte* TFTs, die nicht nur 3d und 2d, sondern auch noch 1d Mannigfaltigkeiten eine Größe zuordnen; genauer gesagt, betrachten wir eine Erweiterung von Dijkgraaf-Witten Theorien. Diese ordnet dem Kreis die Darstellungskategorie $\mathcal{D}(G)$-mod zu. Ausgehend von der erweiterten Dijkgraaf-Witten Theorie und der exakten Sequenz $G \hookrightarrow H \to G/H =: J$ entwickeln wir eine J-äquivariante erweiterte 3d TFT Z_G^J. Aus Z_G^J gewinnen wir eine J-äquivariante Tensorkategorie und zeigen dann durch algebraische Überlegungen, dass diese Kategorie J-modular ist.
In Kapitel IV untersuchen wir die Striktifizierung der Gruppenwirkung auf G-äquivarianten Kategorien. Die Gruppe G wirke auf einer Kategorie $\mathcal{C}$ durch Endofunktoren $\phi_g \in \mathrm{End}(\mathcal{C})$ für jedes Gruppenelement $g \in G$, so dass $\phi_g \circ \phi_h \xrightarrow{\sim} \phi_{gh}$. Es ist bekannt, dass es eine zu $\mathcal{C}$ äquivalente Kategorie gibt, auf der die Gruppenwirkung strikt ist, d.h. $\phi_g \circ \phi_h = \phi_{gh}$. Die in Kapitel 3 konstruierte Kategorie ist die Modulkategorie einer Hopf Algebra A, die nur eine *schwache* Wirkung der Gruppe G trägt. Dies führt zu der Frage, ob es eine Hopf Algebra A^{str} gibt, auf die G strikt wirkt, so dass A-mod $\cong A^{str}$-mod als G-äquivariante Tensorkategorien. Wir zeigen sowohl, dass es eine *schwache* Hopf Algebra (im Sinne von Böhm, Nill, Szlachányi) gibt, die dies erfüllt, als auch, dass im Allgemeinen keine gewöhnliche Hopf Algebra mit strikter Wirkung und mit dieser Eigenschaft existiert.

Lebenslauf

Persönliche Daten

Name:	Jennifer Maier
Geburtstag:	07.02.1984
Geburtsort:	Hamburg

Bildungsgang

2009-2013	Doktorandin an der Universität Hamburg
01/2011 – 05/2011	Marie Curie Research Fellowship, Cardiff University
2009	Diplom in Mathematik
2003	Abitur am Gymnasium Ohmoor, Hamburg