Geometric structures on Lie algebras and the Hitchin flow

Dissertation
zur Erlangung des Doktorgrades
der Fakultät für Mathematik, Informatik
und Naturwissenschaften
der Universität Hamburg

vorgelegt
im Fachbereich Mathematik

von

Marco Freibert
aus Würzburg

Hamburg
2013

Als Dissertation angenommen vom Fachbereich
Mathematik der Universität Hamburg

Auf Grund der Gutachten von Prof. Dr. Vicente Cortés-Suárez
und Prof. Dr. Andrew Swann

Hamburg, den 10.04.2013

Prof. Dr. Ulf Kühn
Leiter des Fachbereichs Mathematik

Contents

Introduction

The subjects of this thesis are half-flat $\mathrm{SU}(3)$-structures, (co-)calibrated G_2-/G_2^*-structures and the Hitchin flow on Lie algebras. The major part of the thesis is devoted to the classification of the Lie algebras in certain classes which admit such structures. In the last chapter, we consider the Hitchin flow for cocalibrated G_2-structures on almost Abelian Lie algebras.

First, we explain what the mentioned structures are and why they are important in both mathematics and physics.

An $\mathrm{SU}(3)$-structure on a six-dimensional manifold M is a reduction P of the frame bundle $\mathcal{F}(M)$ of M to $\mathrm{SU}(3)$. P can equivalently be described by a pair (ω, ρ) of a two-form $\omega \in \Omega^2 M$ and a three-form $\rho \in \Omega^3 M$ whose common stabiliser in $\mathrm{GL}(T_x M)$ at each point $x \in M$ is conjugate to $\mathrm{SU}(3) \subseteq \mathrm{GL}(6, \mathbb{R})$. Here, ω and ρ are *stable* forms in the sense of Hitchin [Hi1], i.e. at each point $x \in M$ the orbits of ω_x and ρ_x are open under the natural action of $\mathrm{GL}(T_x M)$. Since $\mathrm{SU}(3) = \mathrm{SL}(3, \mathbb{C}) \cap \mathrm{SO}(6)$, (ω, ρ) induces an almost Hermitian structure (g, J) with fundamental two-form ω and a $(3,0)$-form Ψ of constant length with $\mathrm{Re}(\Psi) = \rho$. Similarly, a G_2-structure (resp. G_2^*-structure) on a seven-dimensional manifold M is a reduction P of $\mathcal{F}(M)$ of M to G_2 (resp. to G_2^*). In this case, we have an alternative description by a stable three-form $\varphi \in \Omega^3 M$ with pointwise stabiliser being conjugate to G_2 (resp. to G_2^*). Since $\mathrm{G}_2 \subseteq \mathrm{SO}(7)$ (resp. $\mathrm{G}_2^* \subseteq \mathrm{SO}_0(3,4)$), such a three-form induces a Riemannian metric (resp. pseudo-Riemannian metric of signature $(3,4)$), an orientation and thus a Hodge star operator $\star_\varphi$ on M.

The classes of half-flat $\mathrm{SU}(3)$-structures and of (co-)calibrated $\mathrm{G}_2/\mathrm{G}_2^*$-structures naturally appear when one distinguishes the corresponding G-structures $P \subseteq \mathcal{F}(M)$ via their intrinsic torsion. Therefore, recall that when G is a subgroup of $\mathrm{O}(p, n-p)$ and $\mathfrak{g} \subseteq \mathfrak{so}(p, n-p)$ is non-degenerate with respect to the Killing form of $\mathfrak{so}(p, n-p)$, the intrinsic torsion $\tau(P)$ of a G-structure P on an n-dimensional manifold M is a section of the vector bundle associated to the G-module $(\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ is the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{so}(p, n-p)$. Hence, one gets natural classes of G-structures by decomposing this G-module into indecomposable G-modules $V_1, \ldots, V_r$ and requiring that $\tau(P)$ lies pointwise in one or in a sum of the vector bundles associated to the V_i. If P is defined by tensor fields $T_1, \ldots, T_k$ on M as in the above cases, all information about the intrinsic torsion is contained in $\nabla^g T_1, \ldots, \nabla^g T_k$, where ∇^g is the Levi-Civita connection of the induced pseudo-Riemannian metric g of signature $(p, n-p)$. Thus, the most important class of G-structures with vanishing torsion consists exactly of those G-structures where all defining tensor fields are parallel and the holonomy principle shows that then the holonomy of g is a subgroup of G.

So SU(3)-structures (ω, ρ) with vanishing intrinsic torsion are those where $\nabla^g \omega = 0$ and $\nabla^g \rho = 0$. Then J is integrable, (M, g, J) is a Kähler manifold and Ψ is a nowhere vanishing holomorphic section of the canonical bundle. Thus, compact manifolds admitting an SU(3)-structure with vanishing intrinsic torsion are nothing but Calabi-Yau three-folds, a class of six-manifolds which plays a prominent role both in mathematics and in physics in the context of compactifications of 10-dimensional superstring theories. Coming back to arbitrary SU(3)-structures, a result of Chiossi and Salamon [ChiSa] shows that the intrinsic torsion of an SU(3)-structure is fully encoded in the exterior derivatives of ω, ρ and $J^*\rho$. The class of half-flat SU(3)-structures arises as the class of SU(3)-structures whose intrinsic torsion lies pointwise in a certain 21-dimensional G-submodule of the 42-dimensional G-module $\left(\mathbb{R}^6\right)^* \otimes \mathfrak{su}(3)^{\perp}$. Using the alternative description of Chiossi and Salamon, half-flat SU(3)-structures can be described as the SU(3)-structures fulfilling the equations $d\omega^2 = 0$ and $d\rho = 0$.

Similarly, G_2-/G_2^*-structures φ with vanishing intrinsic torsion are those with $\nabla^g \varphi = 0$ and they have holonomy contained in G_2 or G_2^*. By results of Fernández and Gray [FG] and Martín Cabrera [MC2], the intrinsic torsion is in this case fully determined by $d\varphi = 0$ and $d \star_\varphi \varphi = 0$. We are mainly interested in two classes of G_2-/G_2^*-structures which naturally appear via the distinction of the intrinsic torsion, namely the class of *calibrated* G_2-/G_2^*-structures, characterised by $d\varphi = 0$, and the class of *cocalibrated* G_2-/G_2^*-structures, which is characterised by $d \star_\varphi \varphi = 0$.

Besides their appearance as natural classes of G-structures, there are other stronger mathematical and physical motivations for studying half-flat SU(3)-structures and cocalibrated G_2-structures which we like to mention now.

Hitchin flow. The major mathematical motivation stems from Hitchin's flow equations [Hi1] for which half-flat SU(3)-structures and cocalibrated G_2-structures serve as initial values. Hitchin's flow equations are a kind of converse of the following facts. A seven-dimensional Riemannian manifold with holonomy contained in G_2 naturally induces a half-flat SU(3)-structure on each oriented hypersurface. Similarly, oriented hypersurfaces in eight-dimensional Riemannian manifolds whose holonomy is a subgroup of Spin(7) naturally carry cocalibrated G_2-structures. The Hitchin flow presented in [Hi1] embeds a compact six-dimensional manifold admitting a half-flat SU(3)-structure (resp. a compact seven-dimensional manifold with a cocalibrated G_2-structure) as an oriented hypersurface into a Riemannian manifold having holonomy contained in the group G_2 (resp. in the group Spin(7)). More precisely, Hitchin's flow equations are a system of partial differential equations for a one-parameter family $I \to \Omega^2 M \times \Omega^3 M$ of SU(3)-structures $t \mapsto (\omega(t), \rho(t))$ on a compact six-dimensional manifold M (resp. for a one-parameter family $I \to \Omega^3 M$ of G_2-structures $t \mapsto \varphi(t)$ on a compact seven-dimensional manifold M). If $(\omega(t_0), \rho(t_0))$ is half-flat for some $t_0 \in I$ (resp. $\varphi(t_0)$ is cocalibrated for some $t_0 \in I$), then a solution $(\omega(t), \rho(t))$ on I can be used to define a parallel G_2-structure on $M \times I$ (resp. a solution $\varphi(t)$ on I can be used to define a parallel Spin(7)-structure on $M \times I$). Recall that the groups G_2 and Spin(7) appear as exceptional cases in Berger's list [Ber1] of possible holonomy groups of irreducible non-symmetric simply-connected Riemannian manifolds and that it took over 30 years till Bryant [Br1] proved the existence of Riemannian manifolds with holonomy group G_2 and Spin(7). Today, we know explicit examples of complete Riemannian manifolds with exceptional holonomy [BrSa] and also know that there are compact manifolds with these holonomies [J1], [J2]. However, still not that many explicit examples of Riemannian manifolds with exceptional holonomy are known and the Hitchin flow is a useful tool for constructing such examples, cf. e.g. [AFISUV], [ApSa], [ChiFi], [CCGLPW], [CS], [Hi1] and [R3]. Hence, it is also of great interest to find examples of half-flat SU(3)-structures and cocalibrated G_2-structures on manifolds and to investigate which six- or seven-dimensional manifolds admit such structures at all.

In [CLSS], Hitchin's results have been reproved and it has been shown that the compactness assumption for the initial manifold can be dropped. Moreover, the same paper introduces a completely analogous Hitchin flow for one-parameter families of $\mathrm{SU}(1,2)$-structures and of $\mathrm{SL}(3,\mathbb{R})$-structures on a six-dimensional manifold leading in both cases to a pseudo-Riemannian manifold of signature $(3,4)$ with holonomy contained in G_2^* for half-flat initial value. Analogously to the SU(3)-case, half-flat $\mathrm{SU}(1,2)$- and $\mathrm{SL}(3,\mathbb{R})$-structures are defined as a pair $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$ of a stable two-form ω and a stable three-form ρ of certain kind with $d\omega^2 = 0$ and $d\rho = 0$. Furthermore, [CLSS] also introduces a Hitchin flow for one-parameter families of G_2^*-structures on seven-dimensional manifolds leading to pseudo-Riemannian manifolds of signature $(4,4)$ with holonomy contained in $\mathrm{Spin}_0(3,4)$ if one starts with a cocalibrated structure. Note that the groups G_2^* and $\mathrm{Spin}_0(3,4)$ appear as exceptional cases on Berger's list [Ber1] of possible holonomy groups of irreducible non-symmetric simply-connected pseudo-Riemannian manifolds and again Bryant [Br1] was the first who showed that pseudo-Riemannian manifolds with such holonomy groups exist many years after the publication of Berger's list. So the Hitchin flow is a useful tool for the construction of explicit examples of such metrics and a natural first step is to construct examples or to find obstructions to the existence of half-flat $\mathrm{SU}(1,2)$-/$\mathrm{SL}(3,\mathbb{R})$-structures and cocalibrated G_2^*-structures on six- or seven-dimensional manifolds.

In this thesis, we restrict ourselves to the left-invariant setting and consider left-invariant half-flat and cocalibrated structures on Lie groups. Since then everything can be considered as a problem on the associated Lie algebra $\mathfrak{g}$, we speak in the following of half-flat and cocalibrated structures on Lie algebras. Note that the defining differential equations for the initial G-structure reduce to algebraic equations on $\mathfrak{g}$ and Hitchin's flow equations reduce to a system of ordinary differential equations on $\mathfrak{g}$. Hence, the existence and uniqueness of solutions to the Hitchin flow is ensured. More generally, the existence and uniqueness is proved in the real-analytic setting in [CLSS]. Such a result is not valid in the smooth category [Br6].

Motivation from physics. (Compact) manifolds X possessing a G-structure with $\mathrm{G} \in \{\mathrm{SU}(3), \mathrm{G}_2, \mathrm{Spin}(7)\}$ appear in physics in the context of (Kaluza-Klein) compactifications of higher-dimensional supersymmetric theories like 10-dimensional superstring theories, 11-dimensional M-theory and their low energy limits given by 10- or 11-dimensional supergravities, respectively.

We give a rough idea why, see e.g. [De] for further details. The mentioned theories model our universe as a 10- or 11-dimensional Lorentzian manifold N. To meet our daily experience of three spatial and one time direction, one "compactifies" these theories and assumes in the simplest case that $N = \mathbb{R}^{3,1} \times X$ with a compact six- or seven-dimensional Riemannian manifold X whose size is so small that it is undetectable by our present instruments. Nevertheless, properties of X encode properties of the four-dimensional effective theory on $\mathbb{R}^{3,1}$. One important feature one wants to preserve in four dimensions is supersymmetry. This requires the existence of a nowhere vanishing spinor field on X. Hence, manifolds admitting a G-structure with G as above come into play since they can alternatively be described as oriented Riemannian manifolds with a nowhere vanishing spinor field.

In the above theories one usually assumes that X admits a parallel spinor field, cf. e.g. [CHSW] and [PT]. Then the holonomy is contained in G. For phenomenological reasons, a common further assumption is that the holonomy is even equal to G. Then the number of parallel spinor fields on X is minimal and one gets minimal supersymmetry in the effective four-dimensional theory. Physicists also deal with more general types of compactifications. These types include so-called background fluxes, D-branes, warped products or compactifications of the form $N = M \times X$ with a D-dimensional spacetime M, D not necessarily equal to four, and a $(10-D)$- or $(11-D)$-dimensional compact Riemannian manifold X. More generally, compactifications on non-compact asymptotically conical Riemannian manifolds with exceptional holonomy are considered, cf. e.g. [AW] and [GS].

The investigation of compactifications on six-dimensional manifolds admitting an SU(3) -structure with non-vanishing intrinsic torsion started in [Str]. Compactifications on six-dimensional manifolds with a half-flat SU(3)-structures first appeared in [GLMW] as mirror duals of compactifications on Calabi-Yau manifolds with NS three-form flux and are further studied in [GLM1] and [GLM2].

So far, compactifications on seven-dimensional manifolds admitting a cocalibrated G_2-structures seem to have received less attention. However, they might be of interest since there are examples, cf. e.g. [FI], [FIUV] and [Pu], which provide a (partial) solution to Strominger's equations [Str] in type II string theory.

Known Results. We summarise some known results on the subjects we are dealing with in this thesis.

The classification of the six-dimensional Lie algebras admitting a half-flat SU(3)-structure began with [ChiSw], [ChiFi] and [CT]. In these papers, the nilpotent Lie algebras admitting a half-flat SU(3)-structure with additional properties are classified. A few years later, Conti introduced in [C1] an obstruction to the existence of half-flat SU(3)-structures and used it to classify the nilpotent Lie algebras admitting half-flat SU(3)-structures without assuming any additional properties. In his PhD thesis [SHPhD], cf. also [SH], Schulte-Hengesbach refined Conti's obstruction and applied it to classify the direct sums of two three-dimensional Lie algebras admitting a half-flat SU(3)-structure. Note that the existence in both papers [C1], [SH] is proved by giving concrete examples of such structures. Schulte-Hengesbach also obtained partial classification results for such direct sums admitting other types of half-flat structures. Also, the problem of determining all such structures on a fixed Lie algebra up to isomorphism has been considered. In [SHPhD] and [CLSS] this problem has been solved for the Lie algebras $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ and $\mathfrak{h}_3 \oplus \mathfrak{h}_3$, for the first case see also [MaSa]. Moreover, the Hitchin flow has explicitly been solved on some Lie algebras. The most studied Lie algebra is $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$. Hitchin himself considered his flow on this Lie algebra in the same paper [Hi1] in which he introduced the flow. He found explicit examples of G_2-manifolds obtained before by [BGGG] and also the first example of a complete Riemannian metric with holonomy equal to G_2 obtained by Bryant and Salamon [BrSa]. Implicitly, as in [BGGG], Hitchin's flow equations on $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ have also been studied in [CGLP3], [CGLP4]. For a treatment of these examples which uses the Hitchin flow, we refer to [CCGLPW] and [MaSa]. Note that [CCGLPW] also studies the Hitchin flow on Lie algebras of the form $\mathfrak{su}(2) \oplus \mathfrak{h}$ for certain unimodular solvable Lie algebras $\mathfrak{h}$ via so-called group contractions. In [CLSS], the Hitchin flow has been studied on the Lie algebra $\mathfrak{h}_3 \oplus \mathfrak{h}_3$. There are solutions which define pseudo-Riemannian manifolds with holonomy equal to G_2 and G_2^*, respectively. The Hitchin flow on other two-step nilpotent Lie algebras has been considered in [ChiFi] and in [ApSa] and explicit examples with holonomy equal to G_2 are obtained.

Regarding classifications of Lie algebras admitting cocalibrated G_2-/G_2^*-structures, the results obtained in this thesis seem, to the best of the author's knowledge, to be the first ones. Note that in [R1], Reidegeld completely solved the existence problem of homogeneous cocalibrated G_2-structures on compact homogeneous seven-dimensional manifolds. The Hitchin flow for G_2-structures on the quaternionic Heisenberg algebra and two non-solvable Lie algebras has been considered in [AFISUV]. The initial cocalibrated G_2-structure there is constructed using a quaternionic contact structure on the corresponding Lie algebras and one obtains explicit metrics with holonomy equal to Spin(7). Note that one of these metrics already appeared in [GLPS].

The results of this thesis. Next, we give a summary of the main results of this thesis. We divide this summary according to the chapters of this thesis.

Results for almost Abelian Lie algebras admitting (co-)calibrated structures. An *almost Abelian* Lie algebra is a finite-dimensional Lie algebra $\mathfrak{g}$ admitting a codimension one Abelian ideal $\mathfrak{u}$. We classify the almost Abelian Lie algebras admitting calibrated or cocalibrated G_2-/G_2^*-structures, respectively, in Chapter 4. In the same chapter, we do the analogous classifications also for so-called $(G_2)_{\mathbb{C}}$-structures. Moreover, we classify the almost Abelian Lie algebras admitting a parallel G_2-/G_2^*-structure. In the case of parallel G_2^*-structures, we restrict ourselves to those for which $\mathfrak{u}$ is non-degenerate with respect to the induced pseudo-Riemannian metric. A parallel G_2-structure on a Lie algebra is flat according to [AK]. We show that for the particular case we are considering, the same is true also for parallel G_2^*-structures. The results on cocalibrated structures are already published in the author's paper [Fre1]. All other results have not been published yet.

An almost Abelian Lie algebra is fully determined by one endomorphism of the codimension one Abelian ideal $\mathfrak{u}$, namely $\mathrm{ad}(v)|_{\mathfrak{u}}$ for each $v \in \mathfrak{g}\backslash\mathfrak{u}$. We express the existence of the corresponding structure in most of the cases in terms of properties of the complex Jordan normal form of $\mathrm{ad}(v)|_{\mathfrak{u}}$. The results for G_2-structures are as follows:

Theorem 1. *Let $\mathfrak{g}$ be a seven-dimensional almost Abelian Lie algebra, $\mathfrak{u}$ be a codimension one Abelian ideal and $v \in \mathfrak{g}\backslash\mathfrak{u}$.*

(a) $\mathfrak{g}$ admits a calibrated G_2-structure if and only if the complex Jordan normal form of $\mathrm{ad}(v)|_{\mathfrak{u}}$ is given, up to a permutation of the Jordan blocks, by $\begin{pmatrix} J & 0 \\ 0 & \overline{J} \end{pmatrix}$ for a trace-free matrix $J \in \mathbb{C}^{3\times 3}$ in complex Jordan normal form.

(b) $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if the complex Jordan normal form of $\mathrm{ad}(v)|_{\mathfrak{u}}$ has the property that for all $m \in \mathbb{N}$ and all $\lambda \neq 0$ the number of Jordan blocks of size m with λ on the diagonal is the same as the number of Jordan blocks of size m with $-\lambda$ on the diagonal and the number of Jordan blocks of size $2m-1$ with 0 on the diagonal is even.

(c) $\mathfrak{g}$ admits a parallel G_2-structure if and only if $\mathrm{ad}(v)|_{\mathfrak{u}}$ is complex diagonalisable and the complex eigenvalues are given by ia, $-ia$, ib, $-ib$, $-i(a+b)$, $i(a+b)$ for some a, $b \in \mathbb{R}$.

The results for G_2^*- and $(G_2)_{\mathbb{C}}$-structures are more involved and can be found in Chapter 4. We emphasise that in the cocalibrated case, we do not consider the G_2-/G_2^*-structure φ itself but focus directly on the Hodge dual four-form $\star_\varphi\varphi$ without referring to φ. Therefore, note that a four-form Ψ is the Hodge dual of a G_2-/G_2^*-structure if and only if Ψ is a stable four-form of a certain kind, see Lemma 2.45.

For the proof we always proceed as follows. In all cases, we have to show the existence of closed three- or four-forms of a certain kind on $\mathfrak{g}$. We show that this is equivalent to the existence of three- or four-forms of specific type on $\mathfrak{u}$ such that $\mathrm{ad}(v)|_{\mathfrak{u}} \in \mathfrak{gl}(\mathfrak{u})$ is in the stabiliser Lie algebra of these forms for the natural action of $\mathrm{GL}(\mathfrak{u})$ on $\Lambda^*\mathfrak{u}^*$. The mentioned forms on $\mathfrak{u}$ are obtained from the corresponding forms on $\mathfrak{g}$ simply by restriction to $\mathfrak{u}$. The final step in the proof is to transfer the condition that there exist forms on $\mathfrak{u}$ of specific type for which $\mathrm{ad}(v)|_{\mathfrak{u}} \in \mathfrak{gl}(\mathfrak{u})$ lies in the stabiliser Lie algebra into properties of (the complex Jordan normal form of) $\mathrm{ad}(v)|_{\mathfrak{u}}$. Note that for the determination of the specific form of the induced four-form on $\mathfrak{u}$ in the cocalibrated case we do not use the algebraic invariants for orbits of k-forms of Westwick [W3], which is in contrast to our approach in [Fre1]. The proof we give in this thesis differs in this aspect from the one we gave in [Fre1].

Results for cocalibrated G_2-structures on direct sums. In Chapter 5, we classify the direct sums of four- and three-dimensional Lie algebras admitting a cocalibrated G_2-structure. These results are contained in the author's paper [Fre2]. For the direct sums of a four-dimensional non-unimodular Lie algebra $\mathfrak{g}_4$ and a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$, we are able to express the existence of cocalibrated G_2-structures solely in terms of the Lie algebra Betti numbers of $\mathfrak{g}_4$, $\mathfrak{g}_3$ and of the three-dimensional unimodular kernel $\mathfrak{u}$ of $\mathfrak{g}_4$.

Theorem 2. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional non-unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$. Denote by $\mathfrak{u}$ the unimodular kernel of $\mathfrak{g}_4$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) - h^2(\mathfrak{g}_4) + h^2(\mathfrak{g}_3) \leq 4$.*

The results on other types of direct sums of four- and three-dimensional Lie algebras are more complicated and can be found in Chapter 5. As for cocalibrated G_2-structures on almost Abelian case, we focus directly on the Hodge dual of a G_2-structure and do not consider the G_2-structure itself. The results are proved as follows.

Obstructions are found by methods analogous to the ones used in the almost Abelian case. In most of the cases, we consider again a splitting $\mathfrak{g} = \mathfrak{u}_0 \oplus \mathrm{span}(v)$ with $\mathfrak{u}_0$ being a six-dimensional unimodular ideal in $\mathfrak{g}$ and $v \in \mathfrak{g}\backslash\mathfrak{u}_0$, and the four-form $\Omega := \Psi|_{\mathfrak{u}} \in \Lambda^4\mathfrak{u}^*$ induced by the closed Hodge dual $\Psi \in \Lambda^4\mathfrak{g}^*$. In contrast to the almost Abelian Lie algebras, the induced three-form $\rho := (v \lrcorner \Psi)|_{\mathfrak{u}} \in \Lambda^3\mathfrak{u}^*$ gives us additional information and is used for finding obstructions. Note that in the process of finding such obstructions it is advantageous to use the above-mentioned algebraic invariants of Westwick [W3]. However, we do not use these algebraic invariants as prominently as in our paper [Fre2]. For example, the specific form of ρ is determined directly without using the concrete values of the algebraic invariants for the Hodge dual of a G_2-structure.

The existence is proved by different methods. Again, we do not work case-by-case but instead prove existence for several classes of direct sums at once. The essential ingredient in most of the cases is the openness of the orbit of all Hodge duals of a G_2-structure. We use this openness to prove a general proposition which ensures the existence of a cocalibrated G_2-structure on an arbitrary seven-dimensional manifold M if there is a Hodge dual $\Psi \in \Omega^4 M$ of a G_2-structure on M and a bounded four-form Φ lying in a certain subbundle of $\Lambda^4 T^*M$ such that $\Psi + \Phi$ is closed. The idea of the proof of this general proposition is to "rescale" Ψ and Φ in such a way that Ψ is still the Hodge dual of a G_2-structure, the sum stays closed and Φ gets small in comparison to Ψ. We then apply this general proposition to certain classes of direct sums of four- and three-dimensional Lie algebras. For the construction of Ψ with the necessary properties on the mentioned classes of direct sums we use the fact that one may build up the Hodge dual via certain two-forms on a four-dimensional subspace of $\mathfrak{g}$ and its orthogonal complement.

Results for half-flat structures. In Chapter 6, we present classification results for the six-dimensional Lie algebras possessing half-flat structures. We finish the classification of the decomposable six-dimensional Lie algebras admitting a half-flat $\mathrm{SU}(3)$-structure, which started with the classification of sums of three-dimensional Lie algebras admitting half-flat $\mathrm{SU}(3)$-structures in [SH]. Moreover, we classify the indecomposable solvable six-dimensional Lie algebras with five-dimensional nilradical admitting half-flat $\mathrm{SU}(3)$-structures and show that all indecomposable non-solvable six-dimensional Lie algebras possess a half-flat $\mathrm{SU}(3)$-structure. Altogether, these results almost completely solve the existence problem of half-flat $\mathrm{SU}(3)$-structures on six-dimensional Lie algebras. Only the classification of the indecomposable solvable six-dimensional Lie algebras with four-dimensional nilradical admitting a half-flat $\mathrm{SU}(3)$-structure remains open. Furthermore, we obtain some results on the (non-)existence of half-flat $\mathrm{SU}(1,2)$- and $\mathrm{SL}(3,\mathbb{R})$-structures on certain Lie algebras. Almost all the results presented in Chapter 6 are joint work with Schulte-Hengesbach and are published in the papers [FS1], [FS2]. Only one partial result on the existence of half-flat $\mathrm{SU}(1,2)$-/$\mathrm{SL}(3,\mathbb{R})$-structures on almost Abelian Lie algebras is not contained in these papers.

We changed parts of the proofs given in [FS1] and [FS2] since we are now able to use our classification results for Lie algebras admitting cocalibrated G_2-structures. The existence of a half-flat $\mathrm{SU}(3)$-structure on a given Lie algebra $\mathfrak{g}$ is proved in most cases by giving concrete examples. However, we give a direct proof that a six-dimensional almost

Abelian Lie algebra $\mathfrak{g}$ admits a half-flat SU(3)-structure if and only if $\mathfrak{g} \oplus \mathbb{R}$ admits a cocalibrated G_2-structure. Hence, we are able to identify directly all six-dimensional almost Abelian Lie algebras admitting half-flat SU(3)-structures. We still give the concrete examples of half-flat SU(3)-structures on almost Abelian Lie algebras obtained in [FS1] in the appendix. For disproving the existence, we further refine the obstructions used by Schulte-Hengesbach in [SH] further and make them more applicable for computer algebra systems like *Maple*. In fact, we use *Maple*, in particular the standard package "difforms" and the package "difforms2" developed by Schulte-Hengesbach, and apply the obstruction case-by-case. For the application, we refined all the involved lists [ABDO], [Mu5d], [Mu6d], [Tu1] of classes of Lie algebras by distinguishing the Lie algebras further by Lie algebra cohomology and by the dimension of the center. The necessary computations are again done with *Maple* using the package "LieAlgebraCohomology". These refinements may independently have interesting applications. We present a first application of these refinements to the classification of six-dimensional $(2,3)$-trivial Lie algebras.

Results for the Hitchin flow on almost Abelian Lie algebras. Chapter 7 contains the first results of an ongoing investigation of the Hitchin flow on seven-dimensional Lie algebras. We restrict ourselves to almost Abelian Lie algebras and the G_2 case. We prove the following theorem which states that in the mentioned situation, the maximal holonomy one may obtain via the Hitchin flow is SU(4).

Theorem 3. *Let $\mathfrak{g}$ be an almost Abelian seven-dimensional Lie algebra, φ_0 be a cocalibrated G_2-structure on $\mathfrak{g}$ and $0 \in (a,b) \ni t \mapsto \varphi(t)$ be a solution of Hitchin's flow equations with initial value $\varphi(0) = \varphi_0$. Then*

$$g := g_{\varphi(t)} + dt^2$$

defines a Riemannian metric on $\mathrm{G} \times I$ with holonomy contained in SU(4). *Here,* G *is any Lie group with Lie algebra $\mathfrak{g}$.*

For the proof of this theorem, we first show that Hitchin's flow equations are equivalent to certain algebraic and differential equations for the forms induced by φ_t and $\star_{\varphi_t}\varphi_t$ on $\mathfrak{u}$. In a second step we use the induced forms to write down a parallel SU(4)-structure on $G \times I$. To verify that the constructed SU(4)-structure is parallel, we apply a result of Martín Cabrera [MC4] which gives more manageable conditions when an SU(4)-structure is parallel. Moreover, we determine the moduli space of cocalibrated G_2-structures on the Lie algebras $\mathfrak{h}_3 \oplus \mathbb{R}^4$ and $\mathfrak{n}_{7,1}$, i.e. all such structures up to Lie algebra automorphism and scaling. We solve Hitchin's flow equations explicitly for the only element in the moduli space of $\mathfrak{h}_3 \oplus \mathbb{R}^4$ and for a two-parameter family in the moduli space of $\mathfrak{n}_{7,1}$. In the former case, we obtain an explicit Riemannian metric with holonomy equal to SU(2). This Riemannian metric is well-known. It is the Riemannian direct product of the Riemannian six-dimensional manifold obtained by the Hitchin flow for SU(3)-structures on $\mathfrak{h}_3 \oplus \mathbb{R}^3$ in

[ChiFi] and of $\mathbb{R}$ with the standard metric. For the two-parameter family on $\mathfrak{n}_{7,1}$ we get for "generic" parameter values the maximal possible holonomy group SU(4). Hence, we obtain an explicit two-parameter family of non-compact, non-complete Calabi-Yau four-folds of cohomogeneity one.

Structure of this thesis.

We give a short overview of the structure of this thesis. Note first that the notation and conventions we use throughout this thesis are summarised directly before Chapter 1. The reader may consult these pages if he is not sure about the meaning of some expression. The first three chapters are an introduction into all the concepts and notions we use in this thesis and the last four chapters contain the results of this thesis. In Chapter 1 and 2, we discuss various basic concepts on vector spaces and deal with all the examples of G-structures appearing in this thesis on a vector space level. The concepts introduced in these two chapters on vector spaces are transfered to manifolds and Lie groups in Chapter 3. Moreover, we also discuss some global concepts like the intrinsic torsion of a G-structure and the holonomy group of a pseudo-Riemannian manifold in that chapter. Most of the results given in the first three chapters are well-known and can be found in the literature. We would like to put some focus on Section 1.4, in which we give an introduction into the above-mentioned algebraic invariants of Westwick. Despite their importance for classifying seven-dimensional Lie algebras admitting cocalibrated G_2-structures in the author's papers [Fre1] and [Fre2], these invariants seem not to have gained much attention in the past. Moreover, the Sections 2.1 and 2.4 contain some results on two-forms and $(n-2)$-forms on n-dimensional vector spaces and on the Hodge dual of a G_2-structure, respectively, which are, to the best of the author's knowledge, not been written down explicitly in the literature. In Chapter 4, we classify the almost Abelian Lie algebras admitting (co-)calibrated G_2-/G_2^*-/$(G_2)_{\mathbb{C}}$-structures and parallel G_2-/G_2^*-structures. The problem to decide which of the direct sums of four- and three-dimensional Lie algebras admit a cocalibrated G_2-structure is solved in Chapter 5. In Chapter 6, we present and prove results on the (non-)existence of half-flat structures on six-dimensional Lie algebras. The results obtained on the Hitchin flow for cocalibrated G_2-structures on almost Abelian Lie algebras are stated and proved in Chapter 7. Directly after Chapter 7, we include an outlook which contains a summary of problems left open in this thesis and a discussion of possible future research directions. In the appendix, we give all the lists of Lie algebras up to dimension seven which play a role in this thesis. Note that these lists also contain our results on six-dimensional Lie algebras admitting a half-flat SU(3)-structures. Moreover, various other information can be read off these lists. The appendix also contains the concrete examples of half-flat SU(3)-structures and cocalibrated G_2-structures which are necessary to prove some of the classification results.

Acknowledgments

Above all, I am deeply indepted to my advisor Vicente Cortés for suggesting the topic of this thesis, and so drawing my attention to this interesting part of differential geometry, as well as for his continuous, encouraging, patient support in mathematical issues as also in all kinds of other issues during the last three years. I also would like to thank him for giving me the necessary freedom to develop my own ideas.

I would like to express my gratitude to Marisa Fernández, Andrew Swann and Anna Fino for inviting me to Bilbao, Odense and Turin, respectively, and for their kind hospitality during my stays.

I would like to thank my coauthor Fabian Schulte-Hengesbach for the pleasant and fruitful collaborations as well as for the nice time in our shared office. For the nice working atmosphere, I also want to thank all past and present members of our group and my other former office mates Philipp Levermann and Marc Lange as well as my present office mate Jennifer Maier.

Special thanks go to Malte Dyckmanns and Benedict Meinke for carefully reading parts of a draft version of this thesis and giving me useful comments on that version.

I acknowledge the University of Hamburg for financial support and hospitality during the last three years as well as the Research Training Group 1670 "Mathematics inspired by string theory and quantum field theory" for the financial support during the last months.

Besonders möchte ich auch meiner Verlobten Antonia für ihre Liebe, ihre Unterstützung und ihr Verständnis im letzten Jahr meiner Promotion danken.

Notation and conventions

We collect some notation and conventions we use throughout this thesis.

- Ground fields and Lie algebras:

  $\mathbb{F}$ always denotes the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. If we do not specify a ground field for a vector space or a Lie algebra at all, it should be clear from the context if it is $\mathbb{R}$ or $\mathbb{C}$. All appearing Lie algebras will be finite-dimensional. If G is a Lie group, $\mathfrak{g}$ or $L(G)$ denotes the associated Lie algebra.

- Structures on vector spaces:

  Let V be an n-dimensional $\mathbb{F}$-vector space. If $\mathbb{F}=\mathbb{R}$, we denote by $V_{\mathbb{C}}$ the complexification of V and for $\nu \in (\Lambda^n V^*)^{\otimes 2m}$, we write $\nu > 0$ if $\nu = \alpha^{2m}$ for some $\alpha \in \Lambda^n V^*$. If additionally V is oriented and $\tau \in (\Lambda^n V^*)^{\otimes(2m+1)}$, we write $\tau > 0$ if $\nu = \beta^{2m+1}$ for some positive oriented $\beta \in \Lambda^n V^*$. If $\mathbb{F}=\mathbb{C}$, we denote by $V_{\mathbb{R}}$ the realification of V. Let $\mathbb{F} \in \{\mathbb{R},\mathbb{C}\}$ be arbitrary and $(f_1,\ldots,f_n)$ be a basis of V. The dual basis of $(f_1,\ldots,f_n)$ is denoted by $\left(f^1,\ldots,f^n\right)$. The wedge product $\alpha_1 \wedge \alpha_2 \in \Lambda^{k_1+k_2}V^*$ of a k_1-form $\alpha_1 \in \Lambda^{k_1}V^*$ and a k_2-form $\alpha_2 \in \Lambda^{k_2}V^*$ is given by

  $$(\alpha_1 \wedge \alpha_2)(v_1,\ldots,v_{k_1+k_2}) = \frac{1}{k_1!k_2!}\sum_{\sigma\in S_{k_1+k_2}} \operatorname{sgn}(\sigma)\,\alpha_1(v_{\sigma(1)},\ldots,v_{\sigma(k_1)})\,\alpha_2(v_{\sigma(k_1+1)},\ldots,v_{\sigma(k_1+k_2)})$$

  for $v_1,\ldots,v_{k_1+k_2} \in V$. Moreover, we use the abbreviations

  $$f_{i_1\ldots i_k} := f_{i_1} \wedge \ldots \wedge f_{i_k} \in \Lambda^k V, \quad f^{j_1\ldots j_r} := f^{j_1} \wedge \ldots \wedge f^{j_r} \in \Lambda^r V^*.$$

  The contraction $X \lrcorner \rho \in \Lambda^{k-l}$ of an l-vector $X \in \Lambda^l V$ with a k-form $\rho \in \Lambda^k V^*$ is defined inductively by the usual contraction for $l=1$ and by $X \lrcorner \rho = v \lrcorner (Y \lrcorner \rho)$ for $X = Y \wedge v$ with $Y \in \Lambda^{l-1}V$, $v \in V$, and linear extension. By our convention, $T^{r,s}V := V^{\otimes r} \otimes (V^*)^{\otimes s}$ is the space of (r,s)-tensors on V. For an isomorphism $f : V \to W$ of $\mathbb{F}$-vector spaces and $(r,s) \in \mathbb{N}_0^2$ we define $\mathbb{F}$-linear maps $f_* : T^{r,s}V \to T^{r,s}W$ and $f^* : T^{r,s}W \to T^{r,s}V$ uniquely on decomposable (r,s)-tensors $A = v_1 \otimes \ldots v_r \otimes \alpha_1 \otimes \ldots \otimes \alpha_s \in T^{r,s}V$ and $B = w_1 \otimes \ldots w_r \otimes \beta_1 \otimes \ldots \otimes \beta_s \in T^{r,s}W$ by

  $$\begin{aligned} f_*A =& f(v_1) \otimes \ldots \otimes f(v_r) \otimes \alpha_1 \circ f^{-1} \otimes \ldots \otimes \alpha_s \circ f^{-1}, \\ f^*B =& f^{-1}(w_1) \otimes \ldots \otimes f^{-1}(w_r) \otimes \beta_1 \circ f \otimes \ldots \otimes \beta_s \circ f. \end{aligned}$$

If $s=0$ (resp. $r=0$), we define f_* (resp. f^*) in the same way for an arbitrary $\mathbb{F}$-linear map $f: V \to W$. The *natural action of* $\mathrm{GL}(V)$ *on* $T^{r,s}V$ is given by $f.A := f_* A = \left(f^{-1}\right)^* A$ for $f \in \mathrm{GL}(V)$ and $A \in T^{r,s}V$. We also write $f \cdot A$ instead of $f.A$. The *natural action of* $\mathfrak{gl}(V)$ *on* $T^{r,s}V$ is the one induced by the natural action of $\mathrm{GL}(V)$ on $T^{r,s}V$. Concretely, we have

$$\begin{aligned} g \cdot A = g.A = &\sum_{i=1}^{r} v_1 \otimes \ldots \otimes g(v_i) \otimes \ldots \otimes v_r \otimes \alpha_1 \otimes \ldots \otimes \alpha_s \\ &+ \sum_{j=1}^{s} v_1 \otimes \ldots \otimes v_r \otimes \alpha_1 \otimes \ldots \otimes -\alpha_j \circ g \otimes \ldots \otimes \alpha_s \end{aligned} \tag{1}$$

for $g \in \mathfrak{gl}(V)$ and $A = v_1 \otimes \ldots v_r \otimes \alpha_1 \otimes \ldots \otimes \alpha_s \in T^{r,s}V$ being a decomposable (r,s)-tensor on V. If an arbitrary group G acts on V, we denote by G_v the stabiliser subgroup of an element $v \in V$. If $\mathbb{F} = \mathbb{R}$, then a symmetric non-degenerate $(0,2)$-tensor g on V is called a *pseudo-Euclidean metric* and in the case that g is positive definite, we also say that g is an *Euclidean metric*. The signature of g is denoted by $(p, n-p)$ with p being the maximal dimension of a positive definite subspace of V. We write also $\mathrm{sign}(g) = (p, n-p)$ and set $\epsilon(g) := (-1)^{n-p}$. If $\mathbb{F} = \mathbb{C}$ and g is a symmetric non-degenerate bilinear form on V, we set $\epsilon(g) := 1$. A symmetric non-degenerate $\mathbb{F}$-bilinear form g on V induces a symmetric non-degenerate $\mathbb{F}$-bilinear form on $\Lambda^k V^*$, denoted by the same symbol g, by requiring that for an orthonormal basis $e_1, \ldots, e_n$ of (V, g) with $g(e_i, e_i) = \epsilon_i$, the set $\left\{ e^{i_1 \ldots i_k} \middle| 1 \leq i_1 < \ldots < i_k \leq n \right\}$ is an orthonormal basis of $\left(\Lambda^k V^*, g\right)$ with $g\left(e^{i_1 \ldots i_k}, e^{i_1 \ldots i_k}\right) = \epsilon_1 \cdot \ldots \cdot \epsilon_k$.

- Structures on $\mathbb{F}^n$:

  We denote by $(e_1, \ldots, e_n)$ the standard basis of $\mathbb{F}^n$. If there is no danger of confusion, we also use $(e_1, \ldots, e_n)$ to denote a chosen basis in an arbitrary n-dimensional $\mathbb{F}$-vector space. Moreover, $\langle \cdot, \cdot \rangle_{p,n-p} := \sum_{i=1}^{p} e^i \otimes e^i - \sum_{j=p+1}^{n} e^j \otimes e^j \in S^2 \left(\mathbb{R}^n\right)^*$ denotes the standard pseudo-Euclidean metric of signature $(p, n-p)$ on $\mathbb{R}^n$ and we also write $\langle \cdot, \cdot \rangle_n$ instead of $\langle \cdot, \cdot \rangle_{n,0}$. If $n = 2m$, we set $\langle \cdot, \cdot \rangle_{split} := \sum_{i=1}^{2m} (-1)^{i-1} e^i \otimes e^i \in S^2 \left(\mathbb{R}^{2m}\right)^*$. On $\mathbb{C}^n$, $\langle \cdot, \cdot \rangle_{n,\mathbb{C}} := \sum_{i=1}^{n} e^i \otimes e^i \in S^2 \left(\mathbb{C}^n\right)^*$ denotes the standard symmetric non-degenerate bilinear form. Moreover, we denote by $\omega_0 := \sum_{i=1}^{m} e^{2i-1} \wedge e^{2i} \in \Lambda^2 \left(\mathbb{F}\right)^{2m}$ the standard non-degenerate two-form on $\mathbb{F}^{2m}$ and, if $\mathbb{F} = \mathbb{R}$, we set $\omega_{p,m-p} := \sum_{i=1}^{p} e^{2i-1} \wedge e^{2i} - \sum_{j=p+1}^{m} e^{2j-1} \wedge e^{2j} \in \Lambda^2 \left(\mathbb{R}^{2m}\right)^*$ for all $p = 0, \ldots, m$.

- Annihilators and vector space decompositions:

  If V is a $\mathbb{F}$-vector space and A is a subset of V, we denote by

  $$A^0 := \{ \alpha \in V^* | \, \alpha(a) = 0 \, \forall a \in A \}$$

  the annihilator of A in V. If $V = W \oplus U$ as $\mathbb{F}$-vector spaces and $\pi_W : V \to W$ is the projection onto W along U, then $\pi_W^* : \Lambda^* W^* \to \Lambda^* V^*$ is injective. The image of π_W^* is $\Lambda^* U^0$. We use this to identify $\Lambda^* U^0$ with $\Lambda^* W^*$. If $\mathfrak{g} = \mathfrak{u} \oplus U$ is a real finite-dimensional Lie algebra which is the vector space direct sum of an ideal $\mathfrak{u}$ in $\mathfrak{g}$ and a vector subspace $U \subseteq \mathfrak{g}$, then the above injection also identifies the cochain complexes $(\Lambda^* U^0, \pi_{\Lambda^* U^0} \circ d_{\mathfrak{g}}|_{\Lambda^* U^0})$ and $(\Lambda^* \mathfrak{u}^*, d_{\mathfrak{u}})$, where $\pi_{\Lambda^* U^0} : \Lambda^* \mathfrak{g}^* \to \Lambda^* U^0$ is the projection onto $\Lambda^* U^0$ along $\mathfrak{u}^0 \wedge \Lambda^* \mathfrak{g}^*$. Using this identification, we write $d_{\mathfrak{u}}$ instead of $\pi_{\Lambda^* U^0} \circ d_{\mathfrak{g}}|_{\Lambda^* U^0}$. Note that if U is also an ideal in $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{u} \oplus U$ is a Lie algebra direct sum, then $\pi_{\Lambda^* U^0} \circ d_{\mathfrak{g}}|_{\Lambda^* U^0} = d_{\mathfrak{g}}|_{\Lambda^* \mathfrak{u}^*} = d_{\mathfrak{u}}$ in our identification. In this case we omit the index and simply write d.

- Matrices:

  We denote by $\mathrm{adj}(A) \in \mathbb{F}^{n \times n}$ the *adjugate matrix* of a matrix $A \in \mathbb{F}^{n \times n}$, which is defined by $\mathrm{adj}(A)_{ij} := (-1)^{i+j} \det(A(j,i))$ for all $i, j \in \{1, \ldots, n\}$, where $A(j,i)$ is the $(n-1) \times (n-1)$-matrix obtained from A by deleting the j-th row and i-th column. Note that if A is invertible, then $\mathrm{adj}(A) = \det(A) A^{-1}$. If $A_1 \in \mathbb{F}^{n_1 \times n_1}, \ldots, A_k \in \mathbb{F}^{n_k \times n_k}$, then $\mathrm{diag}(A_1, \ldots, A_k)$ denote the $(n_1 + \ldots + n_k) \times (n_1 + \ldots + n_k)$-matrix

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix}.$$

  For complex Jordan normal forms, we follow the standard convention which puts the 1s on the superdiagonal. We denote by $J_k(\lambda) \in \mathbb{C}^{k \times k}$ the $(k \times k)$-matrix consisting of one Jordan block of size k and we set

$$M_{a,b} := \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathbb{R}^{2 \times 2}$$

  for a, $b \in \mathbb{R}$. In each complex Jordan normal form we number consecutively the diagonal elements by $\lambda_1, \ldots, \lambda_n$ and the Jordan blocks by $1, \ldots, m$, both from the upper left to the lower right. Furthermore, we denote by $\mathrm{JB}(i)$ for all $i = 1, \ldots, n$, the number of the Jordan block in which the corresponding generalised eigenvector lies.

- Structures on manifolds:

  Unless stated otherwise, all manifolds M are smooth, finite-dimensional and connected. The only disconnected manifolds appearing in this thesis are certain Lie groups. Moreover, maps between manifolds are assumed to be smooth. We use the convention that an arbitrary symmetric non-degenerate $(0,2)$-tensor field on M is a *pseudo-Riemannian metric*. Moreover, we use the usual notation $\mathfrak{X}(M)$ or $\Omega^k(M)$ to denote the space of all vector fields or k-forms on the manifold M, respectively. Furthermore, the space $\Gamma\left(T^{r,s}M\right)$ of all (r,s)-tensor fields is denoted by $\mathcal{T}^{r,s}M$.

Chapter 1

Basic concepts and notions on vector spaces

1.1 G-structures on vector spaces

In this section, we introduce the notion of a G-*structure on a vector space* for a subgroup G of $\mathrm{GL}(n, \mathbb{R})$. We collect some basic facts and give an alternative description of a G-structure if G is the common stabiliser of an m-tuple of tensors on $\mathbb{R}^n$.

We begin with some preparatory definitions.

Definition 1.1. *Let V be a real n-dimensional vector space. A* frame on (the vector space) V *is an orderd basis $(v_1, \ldots, v_n)$ of V. The set of all frames on V is denoted by $\mathcal{F}(V)$.*

The set $\mathrm{Iso}(\mathbb{R}^n, V)$ of all linear isomorphisms from $\mathbb{R}^n$ to V is naturally isomorphic to $\mathcal{F}(V)$ via the isomorphism $\mathrm{Iso}(\mathbb{R}^n, V) \ni u \mapsto (u(e_1), \ldots, u(e_n)) \in \mathcal{F}(V)$. Thus, we also call an element of $\mathrm{Iso}(\mathbb{R}^n, V)$ a frame on (the vector space) V.

The natural right action of $\mathrm{GL}(n, \mathbb{R})$ on $\mathcal{F}(V)$ *is given by*

$$\mathrm{GL}(n, \mathbb{R}) \times \mathcal{F}(V) \ni (A, (v_1, \ldots, v_n)) \mapsto \left(\sum_{i=1}^{n} A_{i1} v_i, \ldots, \sum_{i=1}^{n} A_{in} v_i \right) \in \mathcal{F}(V). \tag{1.1}$$

The corresponding right action on $\mathrm{Iso}(\mathbb{R}^n, V)$ is given by

$$\mathrm{GL}(n, \mathbb{R}) \times \mathrm{Iso}(\mathbb{R}^n, V) \ni (A, u) \mapsto u \circ A \in \mathrm{Iso}(\mathbb{R}^n, V).$$

Note that the natural right action of $\mathrm{GL}(n, \mathbb{R})$ is simply transitive and so induces a free right action for every subgroup $\mathrm{G} \subseteq \mathrm{GL}(n, \mathbb{R})$ of $\mathrm{GL}(n, \mathbb{R})$. This right action is called the natural right action of G on $\mathcal{F}(V)$.

Now we are able to give the main definition of this section.

Definition 1.2. *Let $\mathrm{G} \subseteq \mathrm{GL}(n, \mathbb{R})$ be a subgroup of $\mathrm{GL}(n, \mathbb{R})$ and V be an n-dimensional real vector space. A* G-structure on (the vector space) V *is a* G*-orbit $P \subseteq \mathcal{F}(V)$ under* *the natural right action of* G *on* $\mathcal{F}(V)$. *We also call a* G*-orbit* P *in* $\mathrm{Iso}(\mathbb{R}^n, V)$ *under the natural right action of* G *a* G-structure on (the vector space) V.

Remark 1.3. *The definition of a* G*-structure does not depend solely on the abstract group* G *but also on how* G *is embedded into* $\mathrm{GL}(n, \mathbb{R})$. *For example, consider the two-dimensional real vector space* $V = \mathbb{R}^2$ *and the isomorphic subgroups* $\mathrm{G} = \{\mathrm{diag}(a, 1)\,|a \in \mathbb{R}^*\}$ *and* $\mathrm{H} = \{\mathrm{diag}(1, a)\,|a \in \mathbb{R}^*\}$ *of* $\mathrm{GL}(2, \mathbb{R})$. *Then the set* $P = \{(be_1, e_2)\,|b \in \mathbb{R}^*\}$ *is a* G*-structure but not an* H*-structure.*

Example 1.4. *(a) There is only one* $\mathrm{GL}(n, \mathbb{R})$*-structure on* V*, namely* $\mathcal{F}(V)$.

(b) There are exactly two $\mathrm{GL}^+(n, \mathbb{R})$*-structures on* V*, namely the two equivalence classes of ordered bases having the same orientation. Hence, a* $\mathrm{GL}^+(n, \mathbb{R})$*-structure is nothing but an orientation on* V.

(c) For $0 \le p \le n$, $\mathrm{O}(p, n-p)$*-structures are in one-to-one correspondence to pseudo-Euclidean metrics with signature* $(p, n-p)$*: Given an* $\mathrm{O}(p, n-p)$*-structure* P *on* V*, we get a pseudo-Euclidean metric of signature* $(p, n-p)$ *by declaring each frame in* P *to be an orthonormal basis. Conversely, suppose we have a pseudo-Euclidean metric* $\langle\cdot,\cdot\rangle$ *of signature* $(p, n-p)$. *Then the set* $P \subseteq \mathcal{F}(V)$ *of orthonormal frames with respect to* $\langle\cdot,\cdot\rangle$ *is an* $\mathrm{O}(p, n-p)$*-structure.*

(d) Let $n = 2m$ *be even. Then* $\mathrm{Sp}(n, \mathbb{R})$*-structures are in one-to-one correspondence to non-degenerate two-forms* ω *on* V*: Given an* $\mathrm{Sp}(n, \mathbb{R})$*-structure* P*, take a frame* $(v_1, \ldots, v_{2m}) \in P$ *and set* $\omega := \sum_{i=1}^m v^{2i-1} \wedge v^{2i} \in \Lambda^2 V^*$. *Here,* $v^1, \ldots, v^n \in V^*$ *is the dual basis of* $v_1, \ldots, v_n$. *The independence of* ω *on the particular choice of the frame* $(v_1, \ldots, v_{2m}) \in P$ *is a direct consequence of the fact that* P *is an* $\mathrm{Sp}(n, \mathbb{R})$*-orbit. Conversely, a non-degenerate two-form* $\omega \in \Lambda^2 V^*$ *on* V *induces an* $\mathrm{Sp}(n, \mathbb{R})$*-structure* P *on* V *by setting* $P := \left\{(v_1, \ldots, v_{2m}) \in \mathcal{F}(V)\,\middle|\,\omega = \sum_{i=1}^m v^{2i-1} \wedge v^{2i}\right\}$.

(e) Similarly, on an n*-dimensional real vector space* V, $\mathrm{SL}(n, \mathbb{R})$*-structures are in one-to-one correspondence with volume forms* $\mathrm{vol} \in \Lambda^n V^* \backslash \{0\}$ *on* V.

The last three examples gave a one-to-one correspondence between G-structures and tensors on V with stabiliser isomorphic to G. More generally, we have the following

Lemma 1.5. *Let* $S_i \in T^{r_i, s_i}\mathbb{R}^n$ *be an* (r_i, s_i)*-tensor for* $i = 1, \ldots, m$ *and* $\mathrm{G} \subseteq \mathrm{GL}(n, \mathbb{R})$ *be the common stabiliser subgroup in* $\mathrm{GL}(n, \mathbb{R})$ *of the tensors* $S_1, \ldots, S_m$ *under the natural action of* $\mathrm{GL}(n, \mathbb{R})$ *on* $T(\mathbb{R}^n)$. *Furthermore, let* V *be an* n*-dimensional real vector space. Then there exists a one-to-one correspondence between* G*-structures* $P \subseteq \mathcal{F}(V) \cong \mathrm{Iso}(\mathbb{R}^n, V)$ *on* V *and* m*-tuples* $(T_1, \ldots, T_m) \in T^{r_1, s_1}V \times \ldots \times T^{r_m, s_m}V$ *for which there exists* $u \in \mathrm{Iso}(\mathbb{R}^n, V)$ *such that* $u^* T_i = S_i$ *for* $i = 1, \ldots, m$. *The correspondence is as follows:*

- *If $P \subseteq \mathcal{F}(V)$ is a* G*-structure, then the associated m-tuple $(T_1, \ldots, T_m) \in T^{r_1,s_1}V \times \ldots \times T^{r_m,s_m}V$ is given by $T_i := \left(u^{-1}\right)^* S_i$ for $i = 1, \ldots, m$, where u is any element in P.*

- *If $(T_1, \ldots, T_m) \in T^{r_1,s_1}V \times \ldots \times T^{r_m,s_m}V$ is an m-tuple such that there exists $u \in \mathrm{Iso}(\mathbb{R}^n, V)$ with $u^*T_i = S_i$ for $i = 1, \ldots, m$, then the associated* G*-structure $P \subseteq \mathcal{F}(V)$ is given by the* G*-orbit $P := u \cdot \mathrm{G}$.*

Definition 1.6. *Let V be an n-dimensional real vector space and $(S_1, \ldots, S_m) \in T^{r_1,s_1}V \times \ldots \times T^{r_m,s_m}V$ be tensors on V. We say that $(T_1, \ldots, T_m) \in T^{r_1,s_1}\mathbb{R}^n \times \ldots \times T^{r_m,s_m}\mathbb{R}^n$ are* model tensors for *$(S_1, \ldots, S_m)$ if there exists $u \in \mathrm{Iso}\left(\mathbb{R}^n, V\right)$ such that $u^*S_i = T_i$ for $i = 1, \ldots, m$. In this case, we call $(u(e_1), \ldots, u(e_n)) \in \mathcal{F}(V)$ an* adapted basis for *$(S_1, \ldots, S_m)$. More generally, if $P \subseteq \mathcal{F}(V)$ is a* G*-structure, we call each element $(v_1, \ldots, v_n)$ in P an* adapted basis for *P.*

Remark 1.7.

- *In Example 1.4 (c), (d) or (e) we may choose $\langle \cdot, \cdot \rangle_{p,n-p} = \sum_{i=1}^{p} e^i \otimes e^i - \sum_{j=p+1}^{n} e^j \otimes e^j \in S^2\left(\mathbb{R}^n\right)^*$, $\omega_0 = \sum_{i=1}^{m} e^{2i-1} \wedge e^{2i} \in \Lambda^2\left(\mathbb{R}^{2m}\right)^*$ or $\mathrm{vol}_0 := e^{1\ldots n} \in \Lambda^n\left(\mathbb{R}^n\right)^*$ as model tensors, respectively.*

- *We include complex-valued (r, s)-tensors $S \in T^{r,s}V \otimes \mathbb{C}$ on real n-dimensional vector spaces V in our treatment by considering them as pair $(\mathrm{Re}(S), \mathrm{Im}(S))$ of (r, s)-tensors. E.g. S as above has model tensor $T \in T^{r,s}\mathbb{R}^n \otimes \mathbb{C} \cong T^{r,s}\mathbb{C}^n$ if $(\mathrm{Re}(S), \mathrm{Im}(S))$ has the model tensors $(\mathrm{Re}(T), \mathrm{Im}(T))$, which is equivalent to the existence of $u \in \mathrm{Iso}\left(\mathbb{R}^n, V\right)$ such that $u_{\mathbb{C}}^* S = T$. Similarly, we also include para-complex-valued (r, s)-tensors on real n-dimensional vector spaces.*

A G-structure naturally induces an H-structure for all subgroups H of $\mathrm{GL}(V)$ with $\mathrm{G} \subseteq \mathrm{H}$.

Definition 1.8. *Let $\mathrm{G} \subseteq \mathrm{GL}(V)$ be a subgroup, $P \subseteq \mathcal{F}(V)$ be a* G*-structure and $\mathrm{H} \subseteq \mathrm{GL}(V)$ be a subgroup such that $\mathrm{G} \subseteq \mathrm{H}$. The* H-enlargement of *P is the* H*-structure $u \cdot \mathrm{H}$ for some $u \in P$. Note that the definition does not depend on the chosen $u \in P$ since $\mathrm{G} \subseteq \mathrm{H}$.*

1.2 Cross products

This section delivers the model tensors and so also the subgroup G of $\mathrm{GL}(n, \mathbb{R})$ for most of the G-structures we are interested in. Therefore, we introduce the concept of an *r-fold $\mathbb{F}$-cross product X*. As one expects, the well-known cross product on $\mathbb{R}^3$ is a real two-fold cross product (with respect to the standard metric) and, more generally, the well-known $(n-1)$-fold cross product on $\mathbb{R}^n$ is a real $(n-1)$-fold cross product in our sense (with respect to the standard metric). Moreover, a complex structure on a $2m$-dimensional real vector space which is orthogonal to some pseudo-Euclidean metric is nothing but a real 1-fold cross product. Besides these examples and the generalisations to indefinite metrics and complex vectors spaces, there are essentially only two more cases, namely two-fold cross products in seven dimensions and three-fold cross products in eight dimensions. The definition of these two exceptional cases uses $\mathbb{F}$-*composition algebras* and we briefly recall some basics on these algebras. For the proofs and more details on $\mathbb{F}$-composition algebras, we refer the reader to [SV] and [CoSm] and for more background on cross products and also the proofs of some statements, we refer the reader to [BG1] and [Gr].

We begin with the definition of an r-fold cross product.

Definition 1.9. *Let V be an n-dimensional $\mathbb{F}$-vector space endowed with a non-degenerate symmetric bilinear form $g : V \times V \to \mathbb{F}$ and let $r \in \mathbb{N}$. An* r-fold cross product (on (V, g)) *is a multilinear map $X : V^r \to V$ such that*

$$g(X(v_1, \ldots, v_r), v_l) = 0 \tag{1.2}$$

and

$$g(X(v_1, \ldots, v_r), X(v_1, \ldots, v_r)) = \det((g(v_i, v_j))_{i,j}) \tag{1.3}$$

is true for all $v_1, \ldots, v_r \in V$ and all $l = 1, \ldots, r$. A morphism between an r-fold cross product X_1 on (V_1, g_1) and an r-fold cross product X_2 on (V_2, g_2) *is a linear map $f : V_1 \to V_2$ such that $f^*g_2 = g_1$ and $f(X_1(v_1, \ldots, v_r)) = X_2(f(v_1), \ldots, f(v_r))$ for all $v_1, \ldots, v_r \in V$.*

Remark 1.10.

- *If $r > 1$, then Equation (1.2) shows that the map*

$$\mathbb{R} \ni t \mapsto g(X(v + tw, v_1, \ldots, v_{i-1}, v + tw, v_i, \ldots, v_{r-2}), v + tw)$$

is the zero map for all $v, w, v_1, \ldots, v_{r-2} \in V$ and all $i = 1, \ldots, r-2$. The differential at $t = 0$ yields, using again Equation (1.2),

$$g(X(v, v_1, \ldots, v_{i-1}, v, v_i, \ldots, v_{r-2}), w) = 0$$

for all $v, w, v_1, \ldots, v_{r-2} \in V$ and all $i = 1, \ldots, r_2$. Hence $X : V^r \to V$ is skew-symmetric, i.e. a map $X : \Lambda^r V \to V$. Using again Equation (1.2), we get that $\varphi_X : V^{r+1} \to \mathbb{F}$, $\varphi_X(v_1, \ldots, v_{r+1}) := g(X(v_1, \ldots, v_r), v_{r+1})$ is an $(r+1)$-form, i.e. $\varphi_X \in \Lambda^{r+1}V^*$.

- *Let V be an n-dimensional $\mathbb{F}$-vector space endowed with a non-degenerate symmetric bilinear form g. There is no n-fold cross product on (V, g) and there is exactly one cross product $X : V^r \to V$ on (V, g) for $r > n$, namely $X \equiv 0$. A cross product X with $X \equiv 0$ is called* trivial. *Obviously, there are no trivial r-fold cross products on an n-dimensional $\mathbb{F}$-vector space for $r \geq n$.*

Definition 1.11. *Let (X, V, g) be an r-fold cross product. The $(r+1)$-form $\varphi_X \in \Lambda^{r+1}V^*$, $\varphi_X(v_1, \ldots, v_{r+1}) := g(X(v_1, \ldots, v_r), v_{r+1})$ is called* the $(r+1)$-form associated to X.

We provide the examples already mentioned in the introduction to this section.

Example 1.12. *(a) The* standard cross product

$$\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \quad \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} := \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}$$

is a 2-fold cross product on $\mathbb{R}^3$ with respect to the standard Euclidean metric on $\mathbb{R}^3$ with associated three-form equal to det.

(b) More generally, if g is a non-degenerate symmetric bilinear form on the n-dimensional $\mathbb{F}$-vector space V and there exists $\mathrm{vol} \in \Lambda^n V^*$ *with $g(\mathrm{vol}, \mathrm{vol}) = 1$, we may define an $(n-1)$-fold cross product $\star : \Lambda^{n-1}V \to V$ by the requirement that for fixed $v_1, \ldots, v_{n-1} \in V$ the element $\star(v_1 \wedge \ldots \wedge v_{n-1}) \in V$ fulfils $g(\star(v_1 \wedge \ldots \wedge v_{n-1}), w) = \mathrm{vol}(v_1, \ldots, v_{n-1}, w)$ for all $w \in V$. The n-form associated to $\star$ is* vol. *Note that for $\mathbb{F} = \mathbb{R}$, $\mathrm{vol} \in \Lambda^n V^*$ with $g(\mathrm{vol}, \mathrm{vol}) = 1$ exists if and only if the signature of g is $(n - 2q, 2q)$ for some $q \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$.*

(c) Let $J : V \to V$ be a 1-fold cross product on (V, g). Then the identities $g(Jv, v) = 0$, $g(Jv, Jv) = g(v, v)$ for all $v \in V$ and the non-degeneracy of g imply $J^2 = -\mathrm{id}_V$. One gets that the dimension of V has to be even. A 1-fold cross product on a real $2m$-dimensional vector space is nothing but a complex structure on (the vector space) V which is orthogonal with respect to the pseudo-Euclidean metric g, i.e. (g, J) is a pseudo-Hermitian structure *on V, cf. Section 2.3. The signature of g has to be $(2p, 2m - 2p)$ for some $p \in \{0, \ldots, m\}$.*

As already mentioned in the introduction, there are essentially two more cases of cross products which may be defined via eight-dimensional $\mathbb{F}$-composition algebras.

Definition 1.13. *A* composition algebra (over $\mathbb{F}$) (A, g) *consists of a (not necessarily associative) finite-dimensional unital $\mathbb{F}$-algebra A and a non-degenerate symmetric bilinear form $g : A \times A \to \mathbb{F}$ such that the* norm $N : A \to \mathbb{F}$, *defined by $N(a) := g(a, a)$, fulfils $N(a \cdot b) = N(a) \cdot N(b)$ for all $a, b \in A$. An eight-dimensional $\mathbb{F}$-composition algebra is called* $\mathbb{F}$-octonion algebra.

For a composition algebra (A, g), we set $\mathrm{Re}(A) := \mathbb{F} \cdot 1 \subseteq A$ *and call the elements in* $\mathrm{Re}(A)$ real. *Moreover, the elements in the subspace* $\mathrm{Im}(A) := (\mathbb{F} \cdot 1)^{\perp_g} = \mathrm{Re}(A)^{\perp_g}$ *are called* imaginary. *We have $A = \mathrm{Re}(A) \oplus \mathrm{Im}(A)$ as $\mathbb{F}$-vector spaces. Thus, for each $a \in A$ there exist unique $b \in \mathrm{Re}(A)$ and $c \in \mathrm{Im}(A)$ with $a = b + c$. We set* $\mathrm{Re}(a) := b$ *and*

$\mathrm{Im}(a) := c$ *and call* $\mathrm{Re}(a)$ *the* real part of a *and* $\mathrm{Im}(a)$ *the* imaginary part of a. *The* conjugation $\bar{\cdot} : A \to A$ *is defined by*

$$\overline{a} := \mathrm{Re}(a) - \mathrm{Im}(a)$$

for $a \in A$. *By [SV],* $\bar{\cdot}$ *is an* involution, *i.e.* $\overline{\overline{a}} = a$ *and* $\overline{ab} = \overline{b} \cdot \overline{a}$ *for* $a, b \in A$.

An isomorphism of $\mathbb{F}$-composition algebras (A, g), (B, h) *is an* $\mathbb{F}$*-algebra isomorphism* $f : A \to B$.

Remark 1.14.

- *By [SV, Corollary 1.2.4], the non-degenerate symmetric bilinear form* g *of a* $\mathbb{F}$*-composition algebra* (A, g) *is uniquely determined by the algebra* A. *Hence, each* $\mathbb{F}$*-algebra isomorphism* $f : A \to B$ *is automatically an isometry between* (A, g) *and* (B, h). *We sometimes suppress the metric* g *in the notation and only write* A *for the composition algebra* (A, g).
- *The automorphism group* $\mathrm{Aut}(A)$ *of a* $\mathbb{F}$*-composition algebra acts trivially on* $\mathrm{Re}(A)$ *and maps* $\mathrm{Im}(A)$ *again to* $\mathrm{Im}(A)$. *Hence, we may canonically consider* $\mathrm{Aut}(A)$ *as a subgroup of* $\mathrm{GL}(\mathrm{Im}(A))$.

The following examples of composition algebras are almost all well-known.

Example 1.15. *(a)* $(\mathbb{F}, g_{\mathbb{F}})$ *with* $g_{\mathbb{F}}(a, b) := a \cdot b$ *is, up to isomorphism, the only* 1*-dimensional* $\mathbb{F}$*-composition algebra. Moreover,* $\mathbb{C}$ *together with the real non-degenerate symmetric bilinear form* $g(z_1, z_2) := z_1\overline{z_2}$ *is a real two-dimensional composition algebra.*

(b) There is, up to isomorphism, one more real two-dimensional composition algebra. This composition algebra, called the para-complex numbers *plays a prominent role later in this thesis. It is defined as the real unital associative algebra generated by* 1 *and the symbol* e *subject to the relation* $e^2 = 1$ *and is denoted by* $\mathbb{C}_1$. *The corresponding pseudo-Euclidean metric of signature* $(1, 1)$ *is defined by* $g_{\mathbb{C}_1}(a_1 + b_1 e, a_2 + b_2 e) := a_1 a_2 - b_1 b_2$ *for* $a_1, a_2, b_1, b_2 \in \mathbb{R}$.

(c) The quaternions $\mathbb{H}$, *i.e. the unital real four-dimensional algebra generated by the symbols* i, j *subject to the relations* $ij = -ji$, $i^2 = j^2 = -1$, *together with* $g_{\mathbb{H}}(q_1, q_2) := q_1\overline{q_2}$ *provides an example of a real four-dimensional composition algebra. Here* $\overline{a + bi + cj + dij} := a - bi - cj - dij$ *for* $(a, b, c, d) \in \mathbb{R}^4$ *is the usual conjugation. Similarly, the* complex quaternions $\mathbb{H}_{\mathbb{C}}$, *defined as the unital complex four-dimensional algebra generated by the symbols* i, j *subject to the relations* $ij = -ji$, $i^2 = j^2 = -1$, *together with* $g_{\mathbb{H}_{\mathbb{C}}}(q_1, q_2) := q_1\overline{q_2}$ *is a complex four-dimensional composition algebra. Here, as in the real case,* $\overline{a + bi + cj + dij} := a - bi - cj - dij$ *for* $(a, b, c, d) \in \mathbb{C}^4$.

There is a procedure, called *Cayley-Dickson construction*, from which one can construct all $\mathbb{F}$-composition algebras starting with the one-dimensional one. We refer the reader to [SV] for the general construction and only use it implicitly to define the following $\mathbb{F}$-octonion algebras via the quaternions and the complex quaternions.

Definition 1.16. *Let $\epsilon \in \{-1, 1\}$ and define a real eight-dimensional unital algebra A_ϵ and a pseudo-Euclidean metric g_ϵ on A_ϵ by setting $A_\epsilon := \mathbb{H} \oplus \mathbb{H}$ as real vector spaces and by defining the multiplication on A_ϵ by*

$$(a, b)(c, d) := (ac + \epsilon \overline{d} b, da + b\overline{c})$$

and the pseudo-Euclidean metric g_ϵ by $g_{A_\epsilon}((a, b), (c, d)) := g_{\mathbb{H}}(a, c) - \epsilon g_{\mathbb{H}}(b, d)$ for all a, b, c, $d \in \mathbb{H}$. We denote (A_{-1}, g_{-1}) by $(\mathbb{O}, g_{\mathbb{O}})$ and call the elements of $\mathbb{O}$ octonions. *Moreover, we denote (A_1, g_1) by $(\mathbb{O}_s, g_{\mathbb{O}_s})$ and call the elements of $\mathbb{O}_s$* split-octonions.

In the complex case, we do a similar construction. We set $\mathbb{O}_{\mathbb{C}} := \mathbb{H}_{\mathbb{C}} \oplus \mathbb{H}_{\mathbb{C}}$ as complex vector spaces and define a multiplication on $\mathbb{O}_{\mathbb{C}}$ by

$$(a, b)(c, d) := (ac - \overline{d} b, da + b\overline{c})$$

and the non-degenerate symmetric complex bilinear form $g_{\mathbb{O}_{\mathbb{C}}}$ by $g_{\mathbb{O}_{\mathbb{C}}}((a, b), (c, d)) := g_{\mathbb{H}_{\mathbb{C}}}(a, c) + g_{\mathbb{H}_{\mathbb{C}}}(b, d)$ for all a, b, c, $d \in \mathbb{H}_{\mathbb{C}}$. Elements in $\mathbb{O}_{\mathbb{C}}$ are called complex octonions.

The $\mathbb{F}$-algebras just defined are all $\mathbb{F}$-octonion algebras up to isomorphism.

Theorem 1.17. *A $\mathbb{F}$-octonion algebra is neither commutative nor associative. The octonions and the split octonions are real octonion algebras and every real octonion algebra is isomorphic to exactly one of them. The complex octonions constitute the unique complex octonion algebra up to isomorphism.*

Proof. A proof may be found e.g. in [SV]. □

Remark 1.18. *Besides the $\mathbb{F}$-composition algebras given in Example 1.15 and Theorem 1.17, there are exactly, up to isomorphism, two more $\mathbb{F}$-composition algebras. Namely a four-dimensional real composition algebra with split signature and a two-dimensional complex composition algebra. Both play no role in this thesis and so we will not give a definition here and refer the reader to [SV].*

Before we come to the definition of a two-fold and a three-fold cross product via the multiplication on an $\mathbb{F}$-octonion algebra, we define some of the most important Lie groups for this thesis.

Definition 1.19. *Let $(A, \langle\cdot, \cdot\rangle)$ be an $\mathbb{F}$-octonion algebra. We set*

$$\mathrm{G}_2(A) := F \circ \mathrm{Aut}(A) \circ F^{-1}$$

where $F : \mathrm{Im}(A) \to \mathbb{F}^7$ *is the isomorphism which maps the ordered basis* $((i,0),(j,0),(k,0),(0,1),(0,i),(0,j),(0,-k))$ *of* $\mathrm{Im}(A) \cong \mathrm{Im}(\mathbb{H}_{\mathbb{F}}) \oplus \mathbb{H}_{\mathbb{F}}$ *to the standard ordered basis* $(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ *of* $\mathbb{F}^7$. *Moreover, we set* $\mathrm{G}_2 := \mathrm{G}_2(\mathbb{O})$, $\mathrm{G}_2^* := \mathrm{G}_2(\mathbb{O}_s)$ *and* $(\mathrm{G}_2)_{\mathbb{C}} := \mathrm{G}_2(\mathrm{Im}(\mathbb{O}_{\mathbb{C}}))$. *To unify the treatment of* G_2*- and* G_2^**-structures later in this thesis, we also set* $\mathrm{G}_2^1 := \mathrm{G}_2^*$ *and* $\mathrm{G}_2^{-1} := \mathrm{G}_2$.

Remark 1.20. *Bryant showed in [Br1] that* G_2 *is in* $\mathrm{SO}(7)$, G_2^* *is in* $\mathrm{SO}_0(3,4)$ *and* $(\mathrm{G}_2)_{\mathbb{C}}$ *is in* $\mathrm{SO}(n,\mathbb{C})$ *and that all these groups are connected. Moreover, he showed that* $(\mathrm{G}_2)_{\mathbb{C}}$ *is the simply-connected complex* 14*-dimensional Lie group whose Lie algebra is the exceptional simple Lie algebra* $(\mathfrak{g}_2)_{\mathbb{C}}$, G_2 *is the simply-connected real* 14*-dimensional Lie group whose Lie algebra* $\mathfrak{g}_2$ *is the compact real form of* $(\mathfrak{g}_2)_{\mathbb{C}}$ *and* G_2^* *is the connected real* 14*-dimensional Lie group with* $\pi_1(\mathrm{G}_2^*) = \mathbb{Z}_2$ *whose Lie algebra* $\mathfrak{g}_2^*$ *is the split real form of* $(\mathfrak{g}_2)_{\mathbb{C}}$.

Now we define the mentioned two- and three-fold $\mathbb{F}$-cross products.

Proposition 1.21. *Let* (A,g) *be an* $\mathbb{F}$*-octonion algebra.*

(a) The map $\times_A : \mathrm{Im}(A) \times \mathrm{Im}(A) \to \mathrm{Im}(A)$, *defined by*

$$a \times_A b = ab + g(a,b)e = ab - g(ab,e)e$$

for $a, b \in \mathrm{Im}(A)$, *is a two-fold* $\mathbb{F}$*-cross product on* $(\mathrm{Im}(A), g|_{\mathrm{Im}(A)})$.

(b) For any $\epsilon \in \{-1,1\}$ *the maps* $X_\epsilon^A : A^3 \to A$ *and* $Y_\epsilon^A : A^3 \to A$, *defined by*

$$\begin{aligned} X_\epsilon^A(a,b,c) :=& \epsilon(-(a\overline{b})c + g(a,b)c + g(b,c)a - g(c,a)b), \\ Y_\epsilon^A(a,b,c) :=& \epsilon(-a(\overline{b}c) + g(a,b)c + g(b,c)a - g(c,a)b) \end{aligned}$$

for $a, b, c \in A$, *are three-fold cross products on* $(A, \epsilon g)$.

Proof. For the proof, we refer to [BG1]. $\square$

Remark 1.22.

- *More generally, the proof given in [BG1] provides the existence of a functor from the category of* n*-dimensional* $\mathbb{F}$*-composition algebras (with the obvious morphisms) to the category of* $(n-1)$*-fold cross products which extends the assignment for* $\mathbb{F}$*-octonion algebras given in Proposition 1.21. This functor is fully faithful and essentially surjective and so yields an equivalence between the category of* n*-dimensional* $\mathbb{F}$*-composition algebras and the category of* $(n-1)$*-fold cross products. There is no such strong relation between three-fold* $\mathbb{F}$*-cross products and* $\mathbb{F}$*-composition algebras.*

- *As it is stated in Theorem 1.23, there is no isomorphism of cross products between* $(X_\epsilon^A, \epsilon g)$ *and* $(Y_\epsilon^A, \epsilon g)$. *But by [SV],* $g(\overline{a},\overline{b}) = g(a,b)$ *and so one can compute that* $X_\epsilon^A(\overline{a},\overline{b},\overline{c}) = \overline{Y_\epsilon^A(c,b,a)} = -\overline{Y_\epsilon^A(a,b,c)}$. *Hence, the stabilisers of* $(X_\epsilon^A, \epsilon g)$ *and*

$(Y_\epsilon^A, \epsilon g)$ in $\mathrm{GL}(A)$ are conjugate via the composition algebra conjugation. Thus, the same is true for the associated four-forms $\varphi_{X_\epsilon^A}$ and $\varphi_{X_\epsilon^A}$ in $\mathrm{GL}(A)$. Note that $\varphi_{X_1^A} = \varphi_{X_{-1}^A}$ and $\varphi_{Y_1^A} = \varphi_{Y_{-1}^A}$. To simplify the notation we set $\varphi_{X_A} := \varphi_{X_A^1}$ and $\varphi_{Y_A} := \varphi_{Y_A^1}$.

We have defined, up to isomorphism, all the possible $\mathbb{F}$-cross products. For the formulation of the classification theorem, recall that by our convention $\langle\cdot,\cdot,\rangle_{p,n-p} = \sum_{i=1}^p e^i \otimes e^i - \sum_{j=p+1}^n e^j \otimes e^j \in S^2(\mathbb{R}^n)^*$ and $\langle\cdot,\cdot\rangle_{n,\mathbb{C}} = \sum_{i=1}^n e^i \otimes e^i \in S^2(\mathbb{C}^n)^*$.

Theorem 1.23. *Let (X, V, g) be a non-trivial n-dimensional r-fold $\mathbb{F}$-cross product. Then (X, V, g) is isomorphic to exactly one of the following n-dimensional r-fold $\mathbb{F}$-cross products:*

(i) *$r = 1$, $n = 2m$ and $(J_{-1}, \mathbb{F}^n, \langle\cdot,\cdot\rangle)$, where J_{-1} is defined by $J_{-1}(e_{2i-1}) := -e_{2i}$, $J_{-1}(e_{2i}) := e_{2i-1}$ for $i = 1, \ldots, m$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{2p,2m-2p}$ for some $p \in \{0, \ldots, m\}$ if $\mathbb{F} = \mathbb{R}$ and $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{n,\mathbb{C}}$ if $\mathbb{F} = \mathbb{C}$.*

(ii) *$r = 2$, $n = 7$ and $(\mathrm{Im}(A), g|_{\mathrm{Im}(A)}, \times_A)$ for $A \in \{\mathbb{O}, \mathbb{O}_s, \mathbb{O}_\mathbb{C}\}$ and $\times_A$ defined as in Proposition 1.21 (a).*

(iii) *$r = 3$, $n = 8$ and $(X_\epsilon^A, A, \epsilon g)$ for $(A, \epsilon) \in \{(\mathbb{O}, 1), (\mathbb{O}, -1), (\mathbb{O}_s, 1), (\mathbb{O}_\mathbb{C}, 1)\}$ and X_ϵ^A defined as in Proposition 1.21 (b).*

(iv) *$r = 3$, $n = 8$ and $(Y_\epsilon^A, A, \epsilon g)$ for $(A, \epsilon) \in \{(\mathbb{O}, 1), (\mathbb{O}, -1), (\mathbb{O}_s, 1), (\mathbb{O}_\mathbb{C}, 1)\}$ and Y_ϵ^A defined as in Proposition 1.21 (b).*

(v) *$r = n - 1 \geq 2$ and $(\star, \mathbb{F}^n, \langle\cdot,\cdot\rangle)$, where $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{n-2q,2q}$ for some $q \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$ if $\mathbb{F} = \mathbb{R}$, $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{n,\mathbb{C}}$ if $\mathbb{F} = \mathbb{C}$ and $\star$ is in all cases constructed as in Example* 1.12 *(b) via the non-degenerate symmetric $\mathbb{F}$-bilinear form $\langle\cdot,\cdot\rangle$ and* $\det \in \Lambda^n(\mathbb{F}^n)^*$.

Proof. For the proof we refer to [BG1]. □

Finally, In this section, we consider the k-forms associated to the exceptional $(k-1)$-fold $\mathbb{F}$-cross products. We start with the two-fold cross products in seven dimensions.

Proposition 1.24. *For all $A \in \{\mathbb{O}, \mathbb{O}_s, \mathbb{O}_\mathbb{C}\}$ let $F : \mathbb{F}^7 \to \mathrm{Im}(A)$ be the isomorphism given in Definition 1.19 and for all $\epsilon \in \{-1, 1\}$ set*

$$\begin{aligned} \varphi_\epsilon &:= e^{123} - \epsilon\left(e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}\right) \in \Lambda^3\left(\mathbb{R}^7\right)^*, \\ \varphi_C &:= e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in \Lambda^3\left(\mathbb{C}^7\right)^*. \end{aligned} \tag{1.4}$$

Then $\varphi_{-1} = F^*\varphi_{\times_\mathbb{O}}$, $\varphi_1 = F^*\varphi_{\times_{\mathbb{O}_s}}$ and $\varphi_C = F^*\varphi_{\times_{\mathbb{O}_\mathbb{C}}}$, where the two-fold $\mathbb{F}$-cross product $\times_A$ is the one defined in Proposition 1.21 (a). The stabiliser of φ_ϵ in $\mathrm{GL}(7, \mathbb{R})$ is given by G_2^ϵ and the stabiliser of φ_C in $\mathrm{GL}(7, \mathbb{C})$ is given by $(\mathrm{G}_2)_\mathbb{C} \times \left\{\xi I_7 \,\middle|\, \xi \in \mathbb{C},\, \xi^3 = 1\right\}$.

Proof. The first part follows by direct calculation. The second part is proved in [Br1]. □

Finally, we come to the four-forms associated to the three-fold cross products in eight dimensions.

Proposition 1.25. *For all $A \in \{\mathbb{O}, \mathbb{O}_s, \mathbb{O}_\mathbb{C}\}$ let $G : \mathbb{F}^8 \to A$ be the isomorphism such that $G|_{\mathbb{F}^7} = F$ with F given in Definition 1.19 and $G(e_8) = 1$. Moreover, for all $\epsilon \in \{-1, 1\}$, let $\varphi_\epsilon \in \Lambda^3 \left(\mathbb{R}^7\right)^*$ and $\varphi_C \in \Lambda^3 \left(\mathbb{C}^7\right)^*$ be the three-forms defined in Equation (1.4) and set*

$$\begin{aligned} \Phi_\epsilon :=& e^8 \wedge \varphi_\epsilon + \epsilon \left(e^{1247} + e^{1256} + e^{1346} - e^{1357} - e^{2345} - e^{2367}\right) + e^{4567} \in \Lambda^4 \left(\mathbb{R}^8\right)^*, \\ \Phi_\mathbb{C} :=& e^8 \wedge \varphi_C - e^{1247} - e^{1256} - e^{1346} + e^{1357} + e^{2345} + e^{2367} + e^{4567} \in \Lambda^4 \left(\mathbb{C}^8\right)^*. \end{aligned} \tag{1.5}$$

Then $\Phi_{-1} = G^* \varphi_{X_\mathbb{O}}$, $\Phi_1 = G^* \varphi_{X_{\mathbb{O}_s}}$ and $\Phi_\mathbb{C} = G^* \varphi_{X_{\mathbb{O}_\mathbb{C}}}$, where $\varphi_{X_A}$is the four-form associated to the three-fold $\mathbb{F}$-cross product $X^1_\mathbb{O}$, cf. Remark 1.22. Moreover, the stabiliser of Φ_{-1} in $\mathrm{GL}(8, \mathbb{R})$ is given by $\mathrm{Spin}(7) \subseteq \mathrm{SO}(8)$, the stabiliser of Φ_1 in $\mathrm{GL}(8, \mathbb{R})$ is given by $\mathrm{Spin}_0(3, 4) \subseteq \mathrm{SO}_0(4, 4)$ and the one of $\Phi_\mathbb{C}$ is given by $\mathrm{Spin}(7, \mathbb{C}) \times \langle iI_8 \rangle \subseteq \mathrm{SO}(8, \mathbb{C}) \times \langle iI_8 \rangle$.

Proof. Again the first part is a direct calculation and the second is given in [Br1]. □

Remark 1.26. *The* Spin*-groups appearing in Proposition 1.25 are usually not defined as concrete subgroups of* $\mathrm{GL}(8, \mathbb{F})$*. So the statement of Proposition 1.25 is more exactly that the mentioned subgroups of* $\mathrm{GL}(8, \mathbb{F})$ *are isomorphic to the corresponding* Spin*-groups. The isomorphisms are obtained by observing that the real spin representations of* $\mathrm{Spin}(7)$ *and* $\mathrm{Spin}_0(3, 4)$ *are faithful and eight-dimensional and the same is true for the complex spin representation of* $\mathrm{Spin}(7, \mathbb{C})$*, cf. [LM].*

1.3 Stable forms

In the previous section, we have seen that certain types of k-forms arise from $(k-1)$-fold cross products, namely the associated ones. In particular, non-degenerate two-forms arise from 1-fold cross products in this way. In this sense one may consider the k-forms associated to $(k-1)$-fold cross products as a natural generalisation of non-degenerate two-forms to higher degrees. Another way of generalizing the concept of non-degenerate two-forms to higher degrees is discussed in this section. This concept was first introduced by Hitchin in [Hi1] and relies on the fact that the orbit of a non-degenerate two-form under the natural action of the general linear group is open. Forms with open orbit are called *stable*. In this section, we give a full classification of stable forms on real vector spaces and observe that the three-forms associated to a two-fold $\mathbb{F}$-cross product in seven dimensions are also stable. One important fact about stability of k-forms is that it is preserved under Hodge star operators (if $k \neq 0, n$). For that reason we start by recalling the definition of Hodge star operators:

Definition 1.27. *Let V be a real oriented n-dimensional vector space and g be a pseudo-Euclidean metric on V. The* Hodge star operator (associated to (V,g)) *is the linear map $\star : \Lambda^* V^* \to \Lambda^* V^*$ such that for a k-form $\psi \in \Lambda^k V^*$ the image $\star\psi$ under $\star$ is the unique $(n-k)$-form for which the identity*

$$g(\star\psi, \widetilde{\psi})\mathrm{vol} = \psi \wedge \widetilde{\psi} \tag{1.6}$$

is true for all $(n-k)$-forms $\widetilde{\psi} \in \Lambda^{n-k} V^*$. Here, vol *is the metric volume form on V, i.e.* vol *fulfils* $\mathrm{vol}(v_1, \ldots, v_n) = 1$ *for all oriented orthonormal bases $v_1, \ldots, v_n$ of (V,g). $\star\psi$ is called the* Hodge dual *of ψ.*

Let V be a complex n-dimensional vector space, g a complex symmetric non-degenerate bilinear form on V and choose a volume form $\mathrm{vol} \in \Lambda^n V^* \setminus \{0\}$ *such that there exists an orthonormal basis $v_1, \ldots, v_n$ of (V,g) with* $\mathrm{vol}(v_1, \ldots, v_n) = 1$. *Note that there are only two such choices, namely* vol *and* $-$vol. *Then we define the* Hodge star operator *$\star : \Lambda^* V^* \to \Lambda^* V^*$ associated to (V, g, vol) as in the real case by requiring that for a k-form $\psi \in \Lambda^{n-k} V^*$ the $(n-k)$-form $\star\psi$ is the unique $(n-k)$-form which fulfils Equation (1.6) for all $(n-k)$-forms $\widetilde{\psi} \in \Lambda^{n-k} V^*$. Again, $\star\psi$ is called the* Hodge dual *of ψ.*

Remark 1.28. *Let $V, g,$* vol *as in Definition 1.27. Recall that by our conventions $\epsilon(h) = (-1)^{n-p}$ for a pseudo-Euclidean metric h of signature $(p, n-p)$ and $\epsilon(k) = 1$ for a complex symmetric non-degenerate bilinear form k.*

- *The Hodge star operator associated to (V,g) is given by $\star = \epsilon(g)(\cdot \lrcorner \mathrm{vol}) \circ f^*$, where $f : V \to V^*$ is the linear map defined by $f(v) := g(v, \cdot)$.*
- *The restriction $\star|_{\Lambda^{n-1} V^*}$ is an $(n-1)$-fold $\mathbb{F}$-cross product on (V^*, g) if $\mathbb{F} = \mathbb{C}$ or if $\mathbb{F} = \mathbb{R}$ and the signature of g is $(n-2q, 2q)$ for some $q \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$, see Example 1.12 (b).*

Now we come to the main definition of this section.

Definition 1.29. *Let V be a finite-dimensional $\mathbb{F}$-vector space. A* stable form on V *is a k-form $\psi \in \Lambda^k V^*$ such that the orbit of ψ under the natural action of* $\mathrm{GL}(V)$ *on $\Lambda^k V^*$ is open. A k-form $\psi_1 \in \Lambda^k V_1^*$ on V_1 is* equivalent *to a k-form $\psi_2 \in \Lambda^k V_2^*$ on V_2 if there exists an isomorphism $f : V_1 \to V_2$ with $f^*\psi_2 = \psi_1$. In this case we also write $\psi_1 \sim \psi_2$ and observe that $f^*(\mathrm{GL}(V_2) \cdot \psi_2) = \mathrm{GL}(V_1) \cdot \psi_1$. So stability is preserved under equivalence.*

The stable k-forms for $k \in \{0, 1, 2, n-1, n\}$ are easily identified. Moreover, we already encountered in the previous section an example of a stable three-form in seven dimensions.

Example 1.30. *Let V be an n-dimensional $\mathbb{F}$-vector space*

(a) All non-zero one-forms, all non-zero $(n-1)$-forms and all volume forms $\mathrm{vol} \in \Lambda^n V^* \setminus \{0\}$ *on V are stable. No 0-form is stable.*

(b) *The two-forms on an n-dimensional vector space V of maximal rank form an open and dense subset of Λ^2V^*. Let $m \in \mathbb{N}_0$ be such that $n = 2m$ or $n = 2m+1$. Then the maximal rank of a two-form is $2m$ and if $\omega \in \Lambda^2V^*$ is of rank $2m$, a standard result in linear algebra tells us that there exists a basis $f^1, \ldots, f^n$ of V^* such that $\omega = \sum_{i=1}^m f^{2i-1} \wedge f^{2i}$. Hence, the set of two-forms of maximal rank is an orbit and so the stable two-forms are exactly those of maximal rank. Note that if $n = 2m$ is even, then the stable two-forms are exactly the non-degenerate ones.*

(c) *For arbitrary $\epsilon \in \{-1, 1\}$, the three-forms $\varphi_\epsilon \in \Lambda^3\left(\mathbb{R}^7\right)^*$ and $\varphi_C \in \Lambda^3\left(\mathbb{C}^7\right)^*$, both defined Equation (1.4), are stable. This follows from the fact that by Proposition 1.24 and Remark 1.20 the stabiliser of these three-forms is in each case 14-dimensional and $14 = 49 - 35 = \dim(\mathrm{GL}(7, \mathbb{F})) - \dim(\Lambda^3\left(\mathbb{F}^7\right)^*)$.*

In Proposition 1.33 below we show that stability is preserved under Hodge star operators. Hence, we might first restrict to the case $2k \leq n$. In this case, the dimension of $\mathrm{GL}(n, \mathbb{F})$ scales as n^2, whereas the dimension of $\Lambda^k\left(\mathbb{F}^n\right)^*$ scales as n^k. Hence, we expect that for $k > 2$, stability is a rare phenomenon. In fact, the next lemma tells us that for $k > 2$, there only can be stable k-forms for $k = 3$ and $n \in \{6, 7, 8\}$.

Lemma 1.31. *Let V be an n-dimensional $\mathbb{F}$-vector space and $\psi \in \Lambda^kV^*$ be a stable k-form with $2 < k \leq \frac{n}{2}$. Then $k = 3$ and $n \in \{6, 7, 8\}$.*

Proof. For $k \geq 4$, we have $n \geq 8$ and so

$$\binom{n}{k} \geq \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{24} \geq \frac{30\left(n^2 - n\right)}{24} \geq \frac{5}{4}\left(n^2 - \frac{n^2}{8}\right) = \frac{35}{32}n^2 > n^2.$$

Thus, there cannot exist a stable k-form with $k \geq 4$ and $2k \leq n$. For $k = 3$ and $n \geq 9$, we get

$$\frac{n(n-1)(n-2)}{6} \geq \frac{7}{6}\left(n^2 - n\right) \geq \frac{7}{6}\left(n^2 - \frac{n^2}{9}\right) = \frac{28}{27}n^2 > n^2$$

and so there is also no stable three-form in an n-dimensional vector space if $n \geq 9$. □

Remark 1.32. *Lemma 1.31 shows that the four-forms on eight-dimensional $\mathbb{F}$-vector spaces associated to three-fold $\mathbb{F}$-cross products are not stable. From the discussion above we see that these four-forms are the only $(r+1)$-forms associated to a non-trivial r-fold cross product which are not stable.*

In Definition 1.34, we present examples of stable three-forms in six and eight dimensions. But before we give these examples, we indicate how one gets a full description of all stable $(n-k)$-forms on an n-dimensional $\mathbb{F}$-vector space if one knows all stable k-forms on V. Besides, we also get the stabiliser group of the Hodge dual $\star\psi$ of an arbitrary k-form $\psi \in \Lambda^kV^*$ if we know the stabiliser of ψ.

Proposition 1.33. *Let V be an n-dimensional $\mathbb{F}$-vector space and g be a symmetric non-degenerate $\mathbb{F}$-bilinear form on V. If $\mathbb{F} = \mathbb{R}$, then assume that V is also oriented and if $\mathbb{F} = \mathbb{C}$, choose $\mathrm{vol} \in \Lambda^n V^*$ with $\mathrm{vol}(e_1, \ldots, e_n) = 1$ for some orthonormal basis $e_1, \ldots, e_n$ of (V, g). Let $\star : \Lambda^* V^* \to \Lambda^* V^*$ be the Hodge star operator associated to (V, g) or to (V, g, vol), respectively, and $k \in \{1, \ldots, n-1\}$. Furthermore, for a linear map $f \in \mathrm{End}(V)$, denote by f^t its transpose with respect to g. Then:*

(a) *For $\psi \in \Lambda^k V^*$ we have $\star(\mathrm{GL}(V) \cdot \psi \cup \mathrm{GL}(V) \cdot (-\psi)) = \mathrm{GL}(V) \cdot (\star\psi) \cup \mathrm{GL}(V) \cdot (-\star\psi)$.*

(b) *Let $\{\psi_i \in \Lambda^k V^* | i \in I\}$ be a system of representatives for all orbits of k-forms under the natural action of $\mathrm{GL}(V)$. Choose a subset $J \subseteq I$ such that ψ_{j_1} is not in the same $\mathrm{GL}(V)$-orbit as $-\psi_{j_2}$ for $j_1 \neq j_2$ and such that each k-form $\psi \in \Lambda^k V^*$ is in the $\mathrm{GL}(V)$-orbit of ψ_j or of $-\psi_j$ for some $j \in J$. Then the set*

$$\{\star\psi_j, -\star\psi_j | j \in J, -\star\psi_j \notin \mathrm{GL}(V) \cdot \star\psi_j\} \cup \{\star\psi_j | j \in J, -\star\psi_j \in \mathrm{GL}(V) \cdot \star\psi_j\}$$

is a system of representatives for all orbits of $(n-k)$-forms under the natural action of $\mathrm{GL}(V)$.

(c) *If V is a complex vector space or V is an oriented real vector space and k and $n-k$ are both odd, then $\star$ yields a bijection between the orbits in $\Lambda^k V^*$ and $\Lambda^{n-k} V^*$ under the natural $\mathrm{GL}(V)$-action*

(d) *If $\mathbb{F} = \mathbb{C}$, then $\mathrm{GL}(V)_{\star\psi} = \left\{\lambda h^{-t} \,\middle|\, \lambda^{n-k} = \det(h),\, h \in \mathrm{GL}(V)_\psi,\, \lambda \in \mathbb{C}^* \right\}$.*

(e) *If $\mathbb{F} = \mathbb{R}$, then $\mathrm{GL}^+(V)_{\star\psi} = \left\{ \det(h)^{\frac{1}{n-k}} h^{-t} \,\middle|\, h \in \mathrm{GL}^+(V)_\psi \right\}$ and $\mathrm{GL}(V)_{\star\psi} = \left\{ h^{-t} \,\middle|\, h.\psi = \frac{1}{\det(h)}\psi,\, h \in \mathrm{GL}(V) \right\}$.*

(f) *If $\psi \in \Lambda^k V^*$ is a stable k-form, then also $\star\psi \in \Lambda^{n-k} V^*$ is stable.*

Proof. (a) By Remark 1.28, we have $\star = \epsilon(g)(. \lrcorner \mathrm{vol}) \circ f^*$, where $f : V \to V^*$ is defined by $f(v) := g(v, .)$ and vol is the metric volume form if $\mathbb{F} = \mathbb{R}$. Hence, the identity

$$\star(h.\psi) = \frac{1}{\det(h)}(h^{-t}. \star \psi) \tag{1.7}$$

holds for all $h \in \mathrm{GL}(V)$.

Let $h \in \mathrm{GL}(V)$. Then there exists $\lambda \in \mathbb{F}^*$ and $\epsilon \in \{-1, 1\}$ such that $\lambda^{n-k} = \epsilon \det(h)$. Hence

$$\star(h.\psi) = \frac{1}{\det(h)}(h^{-t}. \star \psi) = (\lambda h^{-t}).(\epsilon \star \psi),$$

which shows $\star(\mathrm{GL}(V) \cdot \psi) \subseteq \mathrm{GL}(V) \cdot \star\psi \cup \mathrm{GL}(V) \cdot (-\star\psi)$. Since $\star^2|_{\Lambda^s V^*} = (-1)^{s(n-s)}\epsilon(g)\,\mathrm{id}|_{\Lambda^s V^*}$ is true for all $s \in \{0, \ldots, n\}$, we have $\mathrm{GL}(V) \cdot \star\psi \subseteq \star(\mathrm{GL}(V) \cdot \psi) \cup \star(\mathrm{GL}(V) \cdot (-\psi))$. This implies the statement.

(b) Suppose that there are $j_1, j_2 \in J$ with $j_1 \neq j_2$ and $\epsilon \in \{-1,1\}$ such that the forms $\star\psi_{j_1}$ and $\epsilon \star \psi_{j_2}$ are equivalent. Since $\star^2\big|_{\Lambda^s V^*} = (-1)^{s(n-s)}\epsilon(g)\,\mathrm{id}|_{\Lambda^s V^*}$ for all $s \in \{0,\ldots,n\}$, (a) shows that ψ_{j_1} is equivalent to ψ_{j_2} or to $-\psi_{j_2}$, which contradicts the choice of the set J. Thus, the elements of

$$\{\star\psi_j, -\star\psi_j | j \in J, \star\psi_j \notin \mathrm{GL}(V)(-\star\psi_j)\} \cup \{\star\psi_j | j \in J, \star\psi_j \in \mathrm{GL}(V)(-\star\psi_j)\}$$

represent pairwise different orbits.

Next, let $\Psi \in \Lambda^{n-k}V^*$ be given. By the choice of the set J, the k-form $\star\Psi$ is equivalent to $\epsilon\psi_{j_0}$ for some $j_0 \in J$ and some $\epsilon \in \{-1,1\}$. Again, since $\star^2\big|_{\Lambda^s V^*} = (-1)^{s(n-s)}\epsilon(g)\,\mathrm{id}|_{\Lambda^s V^*}$ for all $s \in \{0,\ldots,n\}$, (a) shows that Ψ is equivalent to $\delta\star\psi_{j_0}$ for some $\delta \in \{-1,1\}$ and hence to an element in

$$\{\star\psi_j, -\star\psi_j | j \in J, \star\psi_j \notin \mathrm{GL}(V)(-\star\psi_j)\} \cup \{\star\psi_j | j \in J, \star\psi_j \in \mathrm{GL}(V)(-\star\psi_j)\}.$$

(c) By (b), it suffices to show that in both cases each k-form is equivalent to its negative and also each $(n-k)$-form is equivalent to its negative. If V is a complex vector space and $p \in \{1,\ldots,n\}$ arbitrary, then there is $\lambda \in \mathbb{C}$ with $\lambda^{-p} = -1$ and so $(\lambda \mathrm{id}_V).\rho = -\rho$ for all $\rho \in \Lambda^p V^*$ Hence, ρ is equivalent to $-\rho$ for all p-forms $\rho \in \Lambda^p V^*$. If V is a real vector space and $p \in \{1,\ldots,n\}$ is odd, then $(-\mathrm{id}_V).\rho = -\rho$ and so ρ is equivalent to $-\rho$ for all $\rho \in \Lambda^p V^*$. This proves (c).

(d) Equation (1.7) shows that $\left\{\lambda h^{-t} \,\middle|\, \lambda^{n-k} = \det(h),\, h \in \mathrm{GL}(V)_\psi,\, \lambda \in \mathbb{C}^*\right\} \subseteq \mathrm{GL}(V)_{\star\psi}$. Using again $\star^2\big|_{\Lambda^s V^*} = (-1)^{s(n-s)}\epsilon(g)\,\mathrm{id}|_{\Lambda^s V^*}$ for all $s \in \{0,\ldots,n\}$, we get $\left\{\mu h^{-t} \,\middle|\, \mu^k = \det(h),\, h \in \mathrm{GL}(V)_{\star\psi},\, \mu \in \mathbb{C}^*\right\} \subseteq \mathrm{GL}(V)_\psi$. We use this inclusion now to prove $\mathrm{GL}(V)_{\star\psi} \subseteq \left\{\lambda h^{-t} \,\middle|\, h \in \mathrm{GL}(V)_\psi,\, \lambda^{n-k} = \det(h)\right\}$. Let $h \in \mathrm{GL}(V)_{\star\psi}$. Choose $\mu \in \mathbb{C}$ with $\mu^k = \det(h)$. Then $h_0 := \mu h^{-t} \in \mathrm{GL}(V)_\psi$ and so

$$\mu^k = \det(h) = \det\left(\mu h_0^{-t}\right) = \frac{\mu^n}{\det(h_0)},$$

i.e. $\mu^{n-k} = \det(h_0)$. Thus, $h \in \left\{\mu h_0^{-t} \,\middle|\, \mu^{n-k} = \det(h_0),\, h_0 \in \mathrm{GL}(V)_\psi,\, \mu \in \mathbb{C}^*\right\}$, which shows the statement.

(e) The first part follows exactly in the same way as part (d). For the second, note that (d) implies the identity

$$\mathrm{GL}(V)_{\star\psi} = \left\{\tilde{h} \in \mathrm{GL}(V) \,\middle|\, \tilde{h} = \lambda h^{-t},\, \lambda^{n-k} = \det(h),\, h \in \mathrm{GL}(V_{\mathbb{C}})_{\psi_{\mathbb{C}}},\, \lambda \in \mathbb{C}^*\right\},$$

where $\psi_{\mathbb{C}}$ is the complex k-linear extension of ψ. Let $\tilde{h} = \lambda h^{-t} \in \mathrm{GL}(V)_{\star\psi}$ be given. Set $h_0 := \frac{h}{\lambda}$ and $\mu := \lambda^k$. Note that $h_0 \in \mathrm{GL}(V)$ since $h_0^{-t} = \tilde{h} \in \mathrm{GL}(V)$. Moreover, $\det(h_0) = \frac{\det(h)}{\lambda^n} = \lambda^{-k} = \frac{1}{\mu}$ and so $\mu \in \mathbb{R}^*$. Furthermore, $h_0.\psi = \mu\psi = \frac{1}{\det(h_0)}\psi$ and so

$$\mathrm{GL}(V)_{\star\psi} \subseteq \left\{h_0^{-t} \,\middle|\, h_0.\psi = \frac{1}{\det(h_0)}\psi,\, h_0 \in \mathrm{GL}(V)\right\}.$$

The converse inclusion follows directly from Equation (1.7).

(f) It suffices to show that the stabiliser subgroups of ψ and $\star\psi$ have the same dimension since $\dim(\Lambda^k V^*) = \dim(\Lambda^{n-k} V^*)$. But this follows follows directly from (d) and (e). $\square$

By Lemma 1.31, a stable k-form with $3 \leq k \leq \frac{n}{2}$ fulfils $k = 3$ and $n \in \{6, 7, 8\}$. In seven dimensions, we encountered two real stable three-forms on $\mathbb{R}^7$ and one complex stable three-form on $\mathbb{C}^7$ in Example 1.30 (c). In Theorem 1.35 we will see that these are, up to equivalence, all. To classify the stable three-forms in six and eight dimensions up to equivalence, we first need to find stable three-forms in these dimensions.

Definition 1.34. *We define three-forms on $\mathbb{R}^6$ and $\mathbb{C}^6$ by*

$$\begin{aligned} \rho_\epsilon :=& e^{135} + \epsilon\left(e^{146} + e^{236} + e^{245}\right) \in \Lambda^3\left(\mathbb{R}^6\right)^*, \\ \rho_C :=& e^{135} - e^{146} - e^{236} - e^{245} \in \Lambda^3\left(\mathbb{C}^6\right)^*. \end{aligned} \tag{1.8}$$

Moreover, we define the complex three-form $\psi_{\mathfrak{sl}(3,\mathbb{C})} \in \Lambda^3\left(\mathfrak{sl}(3,\mathbb{C})\right)$ on the eight-dimensional complex simple Lie algebra $\mathfrak{sl}(3,\mathbb{C})$ by $\psi_{\mathfrak{sl}(3,\mathbb{C})}(u,v,w) := \kappa_{\mathfrak{sl}(3,\mathbb{C})}(u,[v,w])$ for $u,v,w \in \mathfrak{g}$, where $\kappa_{\mathfrak{sl}(3,\mathbb{C})}$ is the Killing form on $\mathfrak{sl}(3,\mathbb{C})$. Similarly, we define for each real form $\mathfrak{g} \in \{\mathfrak{sl}(3,\mathbb{R}), \mathfrak{su}(3), \mathfrak{su}(1,2)\}$ of $\mathfrak{sl}(3,\mathbb{C})$ a three-form $\psi_{\mathfrak{g}} \in \Lambda^3\mathfrak{g}^*$ by $\psi_{\mathfrak{g}}(u,v,w) := \kappa_{\mathfrak{g}}(u,[v,w])$ for $u,v,w \in \mathfrak{g}$.

The three-forms just defined turn out to be stable and allow us to classify all stable three-forms in six and eight dimensions.

Theorem 1.35. *(a) All the three-forms $\rho_\epsilon \in \Lambda^3\left(\mathbb{R}^6\right)^*$, $\rho_C \in \Lambda^3\left(\mathbb{C}^6\right)^*$ and $\psi_{\mathfrak{g}} \in \Lambda^3\mathfrak{g}^*$ defined in Definition 1.34 are stable and each stable three-form on a six- or eight-dimensional $\mathbb{F}$-vector space is equivalent to exactly one of these three-forms.*

(b) Each stable three-form on a seven-dimensional $\mathbb{F}$-vector space is equivalent to exactly one of the stable three-forms $\varphi_1, \varphi_{-1} \in \Lambda^3\left(\mathbb{R}^7\right)^*$, $\varphi_C \in \Lambda^3\left(\mathbb{C}^7\right)^*$ defined in Equation (1.4).

Proof. The classification in the complex case follows from the classification of prehomogeneous spaces given in [KiSa]. Since a real k-form is stable if and only if its complexification is stable, we get a full list of stable three-forms up to equivalence in six, seven and eight real dimensions by determining all real forms of the complex stable three-forms in these dimensions. This has been done in [Dj], where the real forms of all orbits of complex three-forms have been determined. Note that the classification of all orbits in six real dimensions has been well-known for a long time, cf. e.g. [Cap], and a classification of the orbits in seven real dimensions has also been given before in [W3]. For a summary of these results and other known results on real or complex stable three-forms in the mentioned dimensions, we refer the reader also to [LPV] and references therein. $\square$

Remark 1.36. *Lemma 1.31, Example 1.30 and Theorem 1.35 give us a complete classification of the equivalence classes of stable k-forms on an n-dimensional $\mathbb{F}$-vector space with $2k \leq n$. Using some results we prove later in this thesis, we may write down a complete list for arbitrary k. Namely, Lemma 2.4 and Proposition 2.5 will give us a full classification of stable $(n-2)$-forms up to equivalence, and Lemma 2.43 and Lemma 2.45 will yield a classification of stable four-forms in seven-dimensions up to equivalence. Moreover, by Proposition 1.33 a classification of the equivalence classes of stable five-forms in eight dimensions is given by the Hodge duals of the stable three-forms in eight-dimensions defined in Definition 1.34 for any Hodge star operator.*

We end this section by noting that for certain $(k, n) \in \mathbb{N}^2$ and all oriented real n-dimensional vector spaces V there is a non-zero differentiable $\mathrm{GL}^+(V)$-equivariant map $\phi : \Lambda^k V^* \backslash \{0\} \to \Lambda^n V^*$ such that $\phi(\rho) \neq 0$ if and only if $\rho \in \Lambda^k V^*$ is stable. This stems from the fact that for these particular values of (k, n), the stabiliser $\mathrm{GL}^+(V)_\rho$ in $\mathrm{GL}^+(V)$ of each stable k-form ρ is a subgroup of $\mathrm{SL}(V)$ and so the restriction of the mentioned map to the open orbits can be regarded as the $\mathrm{SL}(V)$-enlargement of the corresponding $\mathrm{GL}^+(V)_\rho$-structure (via certain model tensors). Note that the reference for the proof we give does not use enlargement theory but proves the result via the theory of prehomogeneous spaces [Ki].

Proposition 1.37. *Let V be a real oriented n-dimensional vector space and either $k \in \{2, n-2\}$ and $n = 2m$ even or $k \in \{3, n-3\}$ and $n = 6, 7, 8$. Then there exists a $\mathrm{GL}^+(V)$-equivariant map*

$$\phi : \Lambda^k V^* \to \Lambda^n V^*,$$

differentiable on $\Lambda^k V^* \backslash \{0\}$, such that $\phi^{-1}(0)$ is exactly the set of all non-stable k-forms.

If $\phi : \Lambda^k V^* \to \Lambda^n V^*$ is any such map, then for each stable k-form $\rho \in \Lambda^k V^*$ there exists a unique $(n-k)$-form $\hat{\rho} \in \Lambda^{n-k} V^*$ such that

$$d\phi_\rho(\alpha) = \hat{\rho} \wedge \alpha$$

for all $\alpha \in \Lambda^k V^*$. Moreover, the identity $\mathrm{GL}^+(V)_\rho = \mathrm{GL}^+(V)_{\hat{\rho}}$ is true and so also $\hat{\rho}$ is stable. Furthermore, the following relation between ρ and $\hat{\rho}$ holds:

$$\hat{\rho} \wedge \rho = \frac{n}{k} \phi(\rho) \tag{1.9}$$

Proof. This is proved in [CLSS]. Note that the proof there shows the properties only for one particular differentiable $\mathrm{GL}^+(V)$-equivariant map $\phi_0 : \Lambda^k V^* \to \Lambda^n V^*$ with $\phi_0(\rho) = 0$ exactly when $\rho \in \Lambda^k V^*$ is non-stable. On each of the open $\mathrm{GL}^+(V)$-orbits, any other ϕ as above is a non-zero multiple of ϕ_0 and so the properties also hold for ϕ. $\square$

In Chapter 2, a concrete description of the map $\phi : \Lambda^k V^* \to \Lambda^{n-k} V^*$ and the corresponding dual $(n-k)$-form $\hat{\rho}$ of a stable k-form $\rho \in \Lambda^k V^*$ are given for all the cases relevant in this thesis.

Remark 1.38. *A priori, the dual $(n-k)$-form $\hat{\rho}$ of a stable k-form ρ has nothing to do with the Hodge dual $\star\rho$ of ρ with respect to some non-degenerate bilinear form g. In particular, we do not need any pseudo-Euclidean metric to define $\hat{\rho}$. But it turns out that in all cases where the stabiliser of ρ is a subgroup of $\mathrm{O}(p, n-p)$ there is a tight connection to the Hodge dual defined by the induced bilinear form g.*

1.4 Algebraic invariants for orbits of k-forms

In this section, we deal with certain algebraic invariants of k-vectors on a finite-dimensional $\mathbb{F}$-vector space. These invariants give us information on the structure of particular k-forms and are used to obtain obstructions to the existence of cocalibrated structures in Chapter 5. They have partly been introduced by Westwick in [W3], where he used them to classify the orbits of three-vectors on a seven-dimensional real vector space under the natural action of $\mathrm{GL}(V)$. We recall this classification but formulate it for three-forms, as we also define the invariants directly for k-forms and not for k-vectors as Westwick did. If we deal with k-vectors, we consider them implicitly as k-forms on V^* via the natural isomorphism between V and V^{**}. For more background on some of the invariants and other related results, we also refer the reader to [Gu], [BG1], [Cap], [W1] and [W2].

Definition 1.39. *Let V be an n-dimensional $\mathbb{F}$-vector space and let $k \geq 1$. The* Grassman cone *$G_k(V^*)$ consists of all* decomposable *k-forms on V, i.e. of all those k-forms $\psi \in \Lambda^k V^*$ such that there are k one-forms $\alpha_1, \ldots, \alpha_k$ with $\psi = \alpha_1 \wedge \ldots \wedge \alpha_k$. The* length *$l(\psi)$ of an arbitrary k-form $\psi \in \Lambda^k V^*$ is defined as the minimal number m of decomposable k-forms $\psi_1, \ldots, \psi_m$ which is needed to write ψ as the sum of $\psi_1, \ldots, \psi_m$, i.e. $l(\psi) := \min\{m \in \mathbb{N}_0 \,|\, \exists\, \psi_1, \ldots, \psi_m \in G_k(V^*) : \psi = \sum_{i=1}^m \psi_i\}$. The* rank *$\mathrm{rk}(\psi)$ of ψ is the dimension of the subspace*

$$[\psi] := \bigcap \left\{ \psi \in \Lambda^k U | U \text{ is a subspace of } V^* \right\}$$

or, equivalently, the rank of the linear map $T : V \to \Lambda^{k-1} V^*$, $T(v) = v \lrcorner \psi$. $[\psi]$ is also called the support *(of ψ). Note that by definition $l(0) = 0$ and $\mathrm{rk}(0) = 0$.*

Next, let $\psi \in \Lambda^k V^*$, $\psi \neq 0$, be given. Choose $v \notin \ker(T)$ and a subspace W of V such that $W \oplus \mathrm{span}(v) \oplus \ker T = V$ is a direct vector space sum. We get a natural $(k-1)$-form $\rho(v, W) := (v \lrcorner \psi)|_W \in \Lambda^{k-1} W^*$ and a natural k-form $\Omega(W) := \psi|_W \in \Lambda^k W^*$ on W. From this construction, we obtain two more algebraic invariants $r(\psi)$ and $m(\psi)$ by looking at the *lengths of $\Omega(W)$ and $\rho(v,W)$ and minimizing over all possible v, W. More exactly, we set:*

$$r(\psi) := \min\left\{l(\Omega)|\Omega = \Omega(W) \in \Lambda^k W^*,\ \dim(W) = (\mathrm{rk}(\psi)-1),\ W \cap \ker T = \{0\}\right\},$$
$$m(\psi) := \min\left\{l(\rho)|\rho = \rho(v,W) \in \Lambda^{k-1} W^*,\ v \notin \ker T,\ W \oplus \mathrm{span}(v) \oplus \ker T = V\right\}.$$

For completeness, we set $r(0) := 0$ and $m(0) := 0$.

Remark 1.40.

- *If $\psi \in \Lambda^k V^*$ and $\alpha \in V^*$ such that $\alpha \notin [\psi]$, then $l(\psi) = l(\psi \wedge \alpha)$, cf. [BuGl, (2.2)].*
- *On a $2m$-dimensional $\mathbb{F}$-vector space, non-degenerate two-forms are exactly those with full rank $2m$. Hence, another way of generalizing the concept of non-degeneracy to forms of higher degree on an n-dimensional vector space is to call k-forms with full rank n* non-degenerate. *This generalisation has been done in [MaSw3], where also various other generalisations of non-degeneracy to higher forms are discussed.*
- *In [Cap], an algebraic invariant for k-forms $\psi \in \Lambda^k V^*$, called* B-longueur, *was considered. Therefore, let B be the set of all bases of V. For a fixed $b \in B$, set*

$$l_b(\psi) := \min\left\{m \in \mathbb{N}_0 \,\middle|\, \psi = \sum_{i=1}^{m} \psi_i \text{ s.t. } \forall j \in \{1,\ldots,m\} : \psi_j = \lambda_j \alpha_{j_1} \wedge \ldots \wedge \alpha_{j_k},\right.$$
$$\left.\lambda_j \in \mathbb{F},\ \alpha_{j_1},\ldots,\alpha_{j_m} \in b\right\}$$

The B-longueur of ψ *is defined as* $\min\{l_b(\psi)|b \in B\}$. *Of course, the B-longueur is greater or equal to the irreducible length of ψ and, in general, they do not coincide. E.g. the B-longueur of the three-form $\rho_{-1} \in \Lambda^3\left(\mathbb{R}^6\right)^*$ defined in Equation (1.8) is four and the length of it is three, cf. [Cap].*

- *An equivalent description of the numbers $r(\psi)$ and $m(\psi)$ is obtained as follows:*

Let $\alpha \in [\psi]$, $\alpha \neq 0$ and U be a complement of $\mathrm{span}(\alpha)$ *in $[\psi]$. Denote by $\rho(\alpha,U) \in \Lambda^{k-1}U$ and $\Omega(\alpha,U) \in \Lambda^k U$ the unique three- and four-form on V such that*

$$\psi = \rho(\alpha,U) \wedge \alpha + \Omega(\alpha,U).$$

Then

$$r(\psi) = \min\{l(\Omega)|\Omega = \Omega(\alpha,U) \in \Lambda^k U,\ \alpha \in [\psi]\backslash\{0\},\ U \oplus \mathrm{span}(\alpha) = [\psi]\},$$
$$m(\psi) = \min\{l(\rho)|\rho = \rho(\alpha,U) \in \Lambda^{k-1} U,\ \alpha \in [\psi]\backslash\{0\},\ U \oplus \mathrm{span}(\alpha) = [\psi]\}.$$

We will mostly work with this description.

For a k-form ψ, a given $v \in V\backslash\{0\}$ and a given subspace W of V with $\mathrm{span}(v) \oplus W = V$, the $(k-1)$-form $\rho(v,W) := (v \lrcorner \psi)|_W$ depends on both v and W. However, in the following sense it essentially only depends on v, and, in particular, the values of the algebraic invariants only depend on v:

Remark 1.41. *Let $\psi \in \Lambda^k V^*$ be a k-form and set $T : V \to \Lambda^{k-1} V^*$, $T(w) := w \lrcorner \psi$. Let $v \notin \ker T$ and let W_1, W_2 be two subspaces of V such that $V = \mathrm{span}(v) \oplus W_i \oplus \ker T$ for $i = 1, 2$. Set $\rho(v, W_i) := (v \lrcorner \psi)|_{W_i}$ for $i = 1, 2$ and denote by $\mathrm{pr}_{W_2} : V \to W_2$ the projection of V onto W_2 along $\mathrm{span}(v) \oplus \ker T$. Then $f : W_1 \to W_2$, $f := \mathrm{pr}_{W_2}|_{W_1}$ is an isomorphism with $f^* \rho(v, W_2) = \rho(v, W_1)$.*

If $f : W \to V$ is a linear isomorphism, then the induced map $f^* : \Lambda^k V^* \to \Lambda^k W^*$ is a linear isomorphism which obviously preserves the length of a k-form and also all the other algebraic invariants rk, r and m. In particular, these algebraic invariants are really invariants of $\mathrm{GL}(V)$-orbits in $\Lambda^k V^*$. Essentially there is only one more map which preserves the length [W1], namely a *dual isomorphism*. Note that we use a slightly different definition of a dual isomorphism as the one given e.g. in [KPRS].

Definition 1.42. *Let V be an n-dimensional vector space and $\mathrm{vol} \in \Lambda^n V^* \backslash \{0\}$ be a volume form. Then, for all $k \in \{1, \ldots, n-1\}$, the map $\delta : \Lambda^k V \to \Lambda^{n-k} V^*$ defined by*

$$\delta(X) := X \lrcorner \mathrm{vol}$$

for $X \in \Lambda^k V$ is called a dual isomorphism. *Note that any other dual isomorphism is a non-zero multiple of δ.*

Lemma 1.43. *Let V be an n-dimensional $\mathbb{F}$-vector space, $k \in \{1, \ldots, n-1\}$, $\delta : \Lambda^k V \to \Lambda^{n-k} V^*$ be a dual isomorphism. Then $l(X) = l(\delta(X))$ for all $X \in \Lambda^k V$. Hence, if $\star : \Lambda^* V^* \to \Lambda^* V^*$ is a Hodge star operator on V, then $l(\psi) = l(\star\psi)$ for all k-forms $\psi \in \Lambda^k V^*$.*

Proof. Let $\delta : \Lambda^k V \to \Lambda^{n-k} V^*$, $\delta(X) := X \lrcorner \mathrm{vol}$ be a dual isomorphism with $\mathrm{vol} \in \Lambda^n V^* \backslash \{0\}$. The image of a non-zero decomposable k-vector $Y = v_1 \wedge \ldots \wedge v_k$ on V is a non-zero $(n-k)$-form Ω which lies in $\Lambda^{n-k} [Y]^0$. Since the dimension of the annihilator $[Y]^0$ is $n - k$, Ω has to be decomposable. Hence, $l(X) \geq l(\delta(X))$ for all $X \in \Lambda^k V$. The inverse map of δ is also a dual isomorphism and we get the equality $l(X) = l(\delta(X))$ for all $X \in \Lambda^k V$. The statement for the Hodge dual follows since by Remark 1.28 the Hodge dual is the composition of a dual isomorphism with a linear isomorphism of the form f^*. $\square$

Remark 1.44. *In [Fre1], the author of this thesis showed that $r(\delta(X)) = m(X)$ and $m(\delta(X)) = r(X)$ if $r(X) > 0$ and $\mathrm{rk}(X) = n$ and the result is used to determine the values of the invariants for the orbits of Hodge duals of G_2^ϵ-structures and of $(\mathrm{G}_2)_\mathbb{C}$-structures. In this thesis, we use a different approach which also determines the model tensors of the induced three- and four-forms on a codimension one subspace of V, see Section 2.4.*

We end this section by recalling the classification of real three-forms in seven dimension by Westwick [W3] and also the classification of complex three-forms in seven dimensions

[Gu]. We add the values of the algebraic invariants rk, l, r, m in the real case determined by Westwick [W3] and the values of the algebraic invariants rk, l determined in [W2], [Cap], [Gu].

Proposition 1.45. *Let $\psi \in \Lambda^3 V^*$ be a three-form on a seven-dimensional real vector space V and $\Psi \in \Lambda^3 W^*$ be a three-form on a seven-dimensional complex vector space W.*

(a) ψ is equivalent to exactly one of the following three-forms on $\mathbb{R}^7$:

Table 1.1: Real three-forms in seven dimensions

ψ		$(\mathrm{rk}(\psi), l(\psi), m(\psi), r(\psi))$	$(\mathrm{rk}(\psi_{\mathbb{C}}), l(\psi_{\mathbb{C}}))$
Q_1	0	$(0,0,0,0)$	$(0,0)$
Q_2	e^{123}	$(3,1,1,0)$	$(3,1)$
Q_3	$e^{123}+e^{145}$	$(5,2,1,0)$	$(5,2)$
ρ_1	$e^{135}+e^{146}+e^{236}+e^{245}$	$(6,2,1,1)$	$(6,2)$
ρ_{-1}	$e^{135}-e^{146}-e^{236}-e^{245}$	$(6,3,2,2)$	$(6,2)$
ρ_0	$e^{126}-e^{135}+e^{234}$	$(6,3,1,1)$	$(6,3)$
P_1	$e^{123}+e^{145}+e^{267}$	$(7,3,1,1)$	$(7,3)$
R	$e^{123}+e^{145}+e^{167}+e^{246}-e^{257}$	$(7,4,1,2)$	$(7,3)$
P_2	$e^{123}+e^{237}+e^{267}-e^{357}+e^{456}+e^{567}$	$(7,3,1,2)$	$(7,3)$
S	$e^{145}+e^{167}+e^{246}-e^{257}+e^{347}+e^{356}$	$(7,4,2,3)$	$(7,3)$
P_3	$e^{123}+e^{145}+e^{167}$	$(7,3,1,0)$	$(7,3)$
P_4	$e^{123}+e^{145}+e^{167}+e^{246}$	$(7,4,1,1)$	$(7,4)$
φ_1	$-e^{123}+e^{145}-e^{167}+e^{246}+e^{257}+e^{347}-e^{356}$	$(7,4,2,2)$	$(7,4)$
φ_{-1}	$-e^{123}-e^{145}+e^{167}-e^{246}-e^{257}-e^{347}+e^{356}$	$(7,5,3,3)$	$(7,4)$

(b) Let $\psi_1,\, \psi_2 \in \Lambda^3\left(\mathbb{R}^7\right)^*$ be two different three-forms in Table 1.1. Then the complex-linear extensions $(\psi_1)_{\mathbb{C}} \in \Lambda^3\left(\mathbb{C}^7\right)^*$ and $(\psi_2)_{\mathbb{C}} \in \Lambda^3\left(\mathbb{C}^7\right)^*$ are equivalent if and only if $\{\psi_1, \psi_2\} \in \{\{\rho_1, \rho_{-1}\}, \{P_1, R\}, \{P_2, S\}, \{\varphi_1, \varphi_{-1}\}\}$. Moreover, Ψ is equivalent to the complex-linear extension of one of the three-forms in Table 1.1.

Note that Table 1.1 implies

Corollary 1.46. *Let V be a seven-dimensional real vector space. For $\psi_1, \psi_2 \in \Lambda^3 V^*$ we have $\psi_1 \in \mathrm{GL}(V)\cdot\psi_2$ if and only if*

$$(\rho(\psi_1), l(\psi_1), r(\psi_1), m(\psi_1)) = (\rho(\psi_2), l(\psi_2), r(\psi_2), m(\psi_2)).$$

Chapter 2

Interesting examples of G-structures

2.1 G-structures related to two-forms

In this section, we look at two-forms and $(n-2)$-forms on an n-dimensional $\mathbb{F}$-vector space V. We classify them up to equivalence and compute all the stabiliser subgroups. If $n=4$, we characterise subspace of $\Lambda^2 V^*$ in which each non-zero element has length two.

We start with two-forms. For those forms, the length is enough to distinguish them up to equivalence.

Lemma 2.1. *Let V be an n-dimensional $\mathbb{F}$-vector space and let $\omega \in \Lambda^2 V^*$. Then ω has length l if and only if $\omega^l \neq 0$ and $\omega^{l+1} = 0$ and this is equivalent to the existence of $2l$ linearly independent one-forms $\alpha_1, \ldots, \alpha_{2l} \in V^*$ such that $\omega = \sum_{i=1}^{l} \alpha_{2i-1} \wedge \alpha_{2i}$. In the case $l = \lfloor \frac{n}{2} \rfloor$ this is also equivalent to the stability of ω and if additionally n is even, also to the non-degeneracy of ω. Moreover, the map $\Lambda^2 V^* \to \mathbb{N}_0$, $\omega \mapsto l(\omega)$ induces a bijection between the $\mathrm{GL}(V)$-orbits of two-forms on V and $\left\{0, \ldots, \lfloor \frac{n}{2} \rfloor\right\}$.*

Proof. The last assertion in Lemma 2.1 follows from the previous ones. Hence, we only have to prove them. A proof of the first equivalence may be found in [BuGl, Theorem 2.11]. If $\omega^l \neq 0$ and $\omega^{l+1} = 0$, then, by the first equivalence, ω has length l. Hence, there exist $\omega_i \in G_2(V^*)$, $i = 1, \ldots, l$, with $\omega = \sum_{i=1}^{l} \omega_i$. We may choose one-forms $\alpha_j \in V^*$, $j = 1, \ldots, 2l$ such that $\omega_i = \alpha_{2i-1} \wedge \alpha_{2i}$. Then

$$\alpha_1 \wedge \ldots \wedge \alpha_{2l} = \omega_1 \wedge \ldots \wedge \omega_l = \frac{\omega^l}{l!} \neq 0.$$

Thus, $\alpha_1, \ldots, \alpha_{2l}$ are linearly independent and ω has the stated form. Conversely, if $\omega = \sum_{i=1}^{l} \alpha_{2i-1} \wedge \alpha_{2i}$ for linearly independent $\alpha_1, \ldots, \alpha_{2l} \in V^*$, then $\omega^l = l!\, \alpha_1 \wedge \ldots \wedge \alpha_{2l} \neq 0$ and $\omega^{l+1} = 0$. Moreover, such an ω has rank l and so it is stable if and only if $l = \lfloor \frac{n}{2} \rfloor$ by Example 1.30 and non-degenerate if additionally n is even. □

For the construction of cocalibrated G_2-structures in Chapter 5, we need k-dimensional subspaces of the two-forms on a real four-dimensional vector space, $k \in \{0,1,2,3\}$, in which each non-zero element is of length two. Such subspaces can be characterised as follows.

Lemma 2.2. *Let V be a real four-dimensional vector space, $k \in \{0,1,2,3\}$, $\omega_1,\ldots,\omega_k \in \Lambda^2 V^*$ be arbitrary two-forms on V, $\tau \in \Lambda^4 V^* \backslash \{0\}$ and π be an arbitrary permutation of $\{1,2,3\}$. Set $W := \mathrm{span}(\omega_1,\ldots,\omega_k)$, $\tilde{\omega}_1 := e^{12} + e^{34} \in \Lambda^2 \left(\mathbb{R}^4\right)^*$, $\tilde{\omega}_2 := e^{13} - e^{24} \in \Lambda^2 \left(\mathbb{R}^4\right)^*$, $\tilde{\omega}_3 := e^{14} + e^{23} \in \Lambda^2 \left(\mathbb{R}^4\right)^*$. Moreover, define the symmetric matrix $H = (h_{ij})_{ij} \in \mathbb{R}^{k\times k}$ by $\omega_i \wedge \omega_j = h_{ij}\tau$ for $i,j = 1,\ldots,k$. Then the following are equivalent:*

(i) W is k-dimensional and each element in $W\backslash\{0\}$ has length two.

(ii) There is an isomorphism $u : V \to \mathbb{R}^4$ such that $\left\{u^*\tilde{\omega}_{\pi(i)} | i = 1,\ldots,k\right\}$ is a basis of W.

(iii) H is definite.

(iv) There exists a Euclidean metric and an orientation on V such that W is a subspace of the space of all self-dual two-forms on V.

Proof. Condition (i) implies condition (ii) by [W3, Theorem 3.1] and [W3, Theorem 3.2]. The converse direction follows since $\tilde{\omega}_i \wedge \tilde{\omega}_j = 0$ for $i \neq j$ and so $\omega^2 \neq 0$ for all $\omega \in W\backslash\{0\}$ if $\left\{u^*\tilde{\omega}_{\pi(i)} | i = 1,\ldots,k\right\}$ is a basis of W. Since $\tilde{\omega}_1$, $\tilde{\omega}_2$, $\tilde{\omega}_3$ form a basis of the self-dual two-forms on $\mathbb{R}^4$ with respect to the standard Euclidean metric and orientation, we get the equivalence of (ii) and (iv). To prove the equivalence of (i) and (iii), let $\omega = \sum_{i=1}^k a_i\omega_i \in W$ with $a := (a_1,\ldots,a_k)^t \neq 0$. By Lemma 2.1, ω has length two if and only if

$$0 \neq \omega^2 = \sum_{i,j=1}^{k} a_i h_{ij} a_j \tau = a^t H a\, \tau,$$

i.e. if and only if $a^t H a \neq 0$. Hence, all elements in $W\backslash\{0\}$ have length two if and only if H is definite. □

Next, we compute the stabiliser subgroup of a two-form of length l under the natural action of the general linear group.

Proposition 2.3. *Let V be an n-dimensional $\mathbb{F}$-vector space and $\omega \in \Lambda^2 V^*$ be a two-form of length l. Set $\ker(\omega) := \{v \in V | \omega(v,\cdot) = 0\}$ and choose some complement W of $\ker(\omega)$ in V. Then the stabiliser subgroup $\mathrm{GL}(V)_\omega$ of ω under $\mathrm{GL}(V)$ is given by*

$$\begin{aligned} GL(V)_\omega = \big\{ f \in \mathrm{GL}(V) \,\big|\, & f|_W = f_1 + h,\ f|_{\ker(\omega)} = f_2,\ f_1 \in \mathrm{Sp}(\omega|_W, W), \\ & h \in \hom(W, \ker(\omega)),\ f_2 \in \mathrm{GL}(\ker(\omega)) \big\}. \\ \cong (\mathrm{Sp}(2l,\mathbb{F}) & \times \mathrm{GL}(n-2l,\mathbb{F})) \ltimes \mathbb{F}^{2l\times(n-2l)} \end{aligned}$$

Proof. Assume that $f \in \mathrm{GL}(V)$ stabilises ω. Then it also stabilises $\ker(\omega)$, i.e. $f|_{\ker(\omega)} \in \mathrm{GL}(\ker(\omega))$. The two-form $\omega|_W$ on W is non-degenerate and so we must have $f|_W = f_1 + h$ with $f_1 \in \mathrm{Sp}(\omega|_W, W)$, $h \in \hom(W, \ker(\omega))$. Hence, f has the stated form. Conversely, it is obvious that elements in $\mathrm{GL}(V)$ of the form as in the assertion stabilise ω. □

The results for two-forms imply the following results on the equivalence classes of $(n-2)$-forms:

Lemma 2.4. *Let V be an n-dimensional $\mathbb{F}$-vector space, $\Omega \in \Lambda^{n-2}V$ be an $(n-2)$-form on V and $l \in \left\{0, \ldots, \lfloor \frac{n}{2} \rfloor\right\}$. In a wedge product, denote by $\widehat{\alpha}$ a one-form which is omitted in this product. Then the following statements are true.*

(a) Ω is stable if and only if $l = \lfloor \frac{n}{2} \rfloor$.

(b) Let $\mathbb{F} = \mathbb{C}$ or $(l, n) \neq (m, 2m)$ for all odd $m \in \mathbb{N}$. Then Ω has length l if and only if there exists a basis $\alpha_1, \ldots, \alpha_n$ of V^* such that

$$\Omega = \sum_{i=1}^{l} \alpha_1 \wedge \ldots \widehat{\alpha_{2i-1}} \wedge \widehat{\alpha_{2i}} \wedge \ldots \wedge \alpha_n.$$

(c) Let $l = 2m$ for some $m \in \mathbb{N}$. Assume that $\mathbb{F} = \mathbb{C}$ or m is even. Then Ω has length m if and only if there exists a non-degenerate two-form $\omega \in \Lambda^2 V^*$ such that $\Omega = \frac{\omega^{m-1}}{(m-1)!}$.

(d) Let $\mathbb{F} = \mathbb{R}$ and $n = 2m$ for some odd $m \in \mathbb{N}$. Then Ω has length m if and only if there exists a basis $\beta_1, \ldots, \beta_n$ of V^* such that

$$\Omega = \pm \sum_{i=1}^{l} \beta_1 \wedge \ldots \widehat{\beta_{2i-1}} \wedge \widehat{\beta_{2i}} \wedge \ldots \wedge \beta_n.$$

This is the case if and only if there exists a non-degenerate two-form $\omega \in \Lambda^2 V^*$ such that $\Omega = \pm\frac{\omega^{m-1}}{(m-1)!}$. Moreover, for each non-degenerate two-form ω on V, $\frac{\omega^{m-1}}{(m-1)!}$ is not equivalent to $-\frac{\omega^{m-1}}{(m-1)!}$.

(e) If $\mathbb{F} = \mathbb{R}$ and $n = 2m$ for some odd m, the map $\Lambda^{n-2}V^* \to \mathbb{N}_0$, $\Omega \mapsto l(\Omega)$ induces a surjection between the orbits of $(n-2)$-forms on V and the set $\{0, \ldots, m\}$ such that each element in $\{0, \ldots, m-1\}$ has exactly one preimage and m has two preimages. In all other cases, the map $\Lambda^{n-2}V^* \to \mathbb{N}_0$, $\Omega \mapsto l(\Omega)$ induces a bijection between the orbits of $(n-2)$-forms on V and the set $\left\{0, \ldots, \lfloor \frac{n}{2} \rfloor\right\}$.

Proof. Let V be an n-dimensional $\mathbb{F}$-vector space. (a) follows directly from Proposition 1.33, Lemma 1.43 and Lemma 2.1. Moreover, note (e) follows directly from (b),(c) and (d). So it remains to prove (b), (c) and (d). If $n = 2m$, Lemma 2.1 gives the identity

$$\left\{ \sum_{i=1}^{m} \alpha_1 \wedge \ldots \widehat{\alpha_{2i-1}} \wedge \widehat{\alpha_{2i}} \wedge \ldots \wedge \alpha_n \,\middle|\, \alpha_1, \ldots, \alpha_{2m} \text{ basis of } V \right\}$$
$$= \left\{ \frac{\omega^{m-1}}{(m-1)!} \,\middle|\, \omega \in \Lambda^2 V^* \text{ non-degenerate} \right\}.$$

Choose a Hodge star operator $\star : \Lambda^2 V^* \to \Lambda^{n-2} V^*$, where in the case $\mathbb{F} = \mathbb{R}$ we choose the defining non-degenerate symmetric bilinear form to be positive definite. Furthermore, choose an ordered basis $(f_1, \ldots, f_n)$ of V which is oriented and orthonormal with respect to the structures which define $\star$. For $\mathbb{F} = \mathbb{C}$, oriented means $\mathrm{vol}(f_1, \ldots, f_n) = 1$. Then

$$\Omega_l := \star\left(\sum_{i=1}^{l} f^{2i-1} \wedge f^{2i}\right) = \sum_{i=1}^{l} f^1 \wedge \ldots \widehat{f^{2i-1}} \wedge \widehat{f^{2i}} \wedge \ldots \wedge f^n \in \Lambda^{n-2} V^*$$

for $l = 0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor$. By Lemma 1.43, Ω_l has length l. Moreover, if $\mathbb{F} = \mathbb{C}$, Lemma 2.2 and Proposition 1.33 show that $\left\{\Omega_l \,\middle|\, l \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}\right\}$ is a system of representatives of the orbits of $(n-2)$-forms on V. Hence, the statements for $\mathbb{F} = \mathbb{C}$ follow. Assume for the rest of the proof that $\mathbb{F} = \mathbb{R}$. In this case, Lemma 2.2 and Proposition 1.33 show that the set

$$\left\{\Omega_l \,\middle|\, -\Omega_l \in \mathrm{GL}(V)\Omega_l,\, l \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}\right\} \cup \left\{\Omega_l, -\Omega_l \,\middle|\, -\Omega_l \notin \mathrm{GL}(V)\Omega_l,\, l \in \left\{0, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}\right\}$$

is a system of representatives for the orbits of $(n-2)$-forms on V. Hence, the statement follows if we can show that Ω_l is not equivalent to $-\Omega_l$ if and only if $(l, n) = (m, 2m)$ for some odd m.

Consider first the case $2l < n$. Then $F.\Omega_l = -\Omega_l$ for $F \in \mathrm{GL}(V)$ defined by $F(e_i) := e_i$ for $i = 1, \ldots, n-1$ and $F(e_n) := -e_n$ and so Ω_l and $-\Omega_l$ are equivalent. If $n = 2m$, m even and $l = m$, then $G.\Omega_m = -\Omega_m$ for $G \in \mathrm{GL}(V)$ defined by $G(e_{2i-1}) := e_{2i}$ and $G(e_{2i}) := -e_{2i-1}$ for $i = 1, \ldots, m$ and so Ω_m is equivalent to $-\Omega_m$. To finish the proof, we show that Ω_m is not equivalent to $-\Omega_m$ if $n = 2m$ and m is odd. Assume the contrary, i.e. that there is some $h \in \mathrm{GL}(V)$ with $h.\Omega_m = -\Omega_m$. Consider the $\mathrm{GL}(V)$-module isomorphism $\kappa : \Lambda^{2m-2} V^* \to \Lambda^2 V \otimes \Lambda^{2m} V^*$ defined for $\Omega \in \Lambda^{2m-2} V^*$ by $\kappa(\Omega) := X \otimes \nu$ with $X \in \Lambda^2 V$, $\nu \in \Lambda^{2m} V^*$ such that $X \lrcorner \nu = \Omega$. Then $\Lambda^{2m} V \otimes \left(\Lambda^{2m} V^*\right)^{\otimes m} \cong \left(\Lambda^{2m} V^*\right)^{\otimes(m-1)}$ as $\mathrm{GL}(V)$-modules and so $\kappa(\Omega_m)^m \in \left(\Lambda^{2m} V^*\right)^{\otimes(m-1)}$. Thus,

$$\frac{1}{\det(h)^{m-1}} \kappa(\Omega_m)^m = h.\left(\kappa(\Omega_m)^m\right) = \kappa(h.\Omega_m)^m = (-1)^m \kappa(\Omega_m)^m = -\kappa(\Omega_m)^m$$

since m is odd. A short computation shows that $\kappa(\Omega_m)^m \neq 0$. Hence, $\det(h)^{m-1} = -1$, which is impossible since $m-1$ is even. Thus, Ω_m is not equivalent to $-\Omega_m$ in this case. □

Next, we compute the stabiliser groups of an $(n-2)$-form of length l. For the statement, note that by definition, $\det(\mathrm{id}_0) = 1$ for the only linear endomorphism $\mathrm{id}_0 : \{0\} \to \{0\}$ on the 0-dimensional vector space $\{0\}$ and so $\mathrm{sgn}(\det(\mathrm{id}_0)) = 1$.

Proposition 2.5. *Let V be an n-dimensional $\mathbb{F}$-vector space and $\Omega \in \Lambda^{n-2} V^*$ be of length $l \in \left\{1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\right\}$. Consider the map $F : V^* \to \Lambda^{n-1} V^*$, $F(\alpha) = \Omega \wedge \alpha$ and set $V_1 := \ker(F)^0 \subseteq V^{**} \cong V$. Choose some complement V_2 of V_1 in V. Then $\Omega = \frac{\omega^{l-1}}{(l-1)!} \wedge \nu$ for a non-degenerate two-form $\omega \in \Lambda^2 V_1^*$ and $\nu \in \Lambda^{n-2l} V_2^*$, where we use the decomposition $V_1 \oplus V_2$ to identify $(V_1 \oplus V_2)^*$ with $V_1^* \oplus V_2^*$. Moreover:*

(a) If $\mathbb{F} = \mathbb{C}$*, then*

$$\mathrm{GL}(V)_\Omega = \Big\{ f \in \mathrm{GL}(V) \,\Big|\, f|_{V_1} = \lambda f_1,\ f|_{V_2} = f_2 + h,\ \lambda^{2-2l} = \det(f_2), \\ f_1 \in \mathrm{Sp}(V_1, \omega),\ \lambda \in \mathbb{C}^*,\ f_2 \in \mathrm{GL}(V_2),\ h \in \hom(V_2, V_1) \Big\}.$$

(b) If $\mathbb{F} = \mathbb{R}$ *and* l *is even, then*

$$\mathrm{GL}(V)_\Omega = \Big\{ f \in \mathrm{GL}(V) \,\Big|\, f|_{V_1} = |\det(f_2)|^{\frac{1}{2-2l}} f_1,\ f|_{V_2} = f_2 + h,\ f_1 \in \mathrm{GL}(V_1), \\ f_1.\omega = \mathrm{sgn}(\det(f_2))\omega,\ f_2 \in \mathrm{GL}(V_2),\ h \in \hom(V_2, V_1) \Big\}.$$

(c) If $\mathbb{F} = \mathbb{R}$ *and* $l \neq 1$ *is odd, then*

$$\mathrm{GL}(V)_\Omega = \Big\{ f \in \mathrm{GL}(V) \,\Big|\, f|_{V_1} = \det(f_2)^{\frac{1}{2-2l}} f_1,\ f|_{V_2} = f_2 + h,\ f_1.\omega = \epsilon\omega, \\ \epsilon \in \{-1, 1\},\ f_2 \in \mathrm{GL}^+(V_2),\ h \in \hom(V_2, V_1) \Big\}.$$

and if $l = 1$*, then*

$$\mathrm{GL}(V)_\Omega = \{ f \in \mathrm{GL}(V) \,|\, f|_{V_1} = f_1,\ f|_{V_2} = f_2 + h,\ f_1 \in \mathrm{GL}(V_1), \\ f_2 \in \mathrm{SL}(V_2),\ h \in \hom(V_2, V_1) \}.$$

Proof. By Lemma 2.4, we may assume for the computation of the stabiliser of Ω that there exists a basis $F_1, \ldots, F_n$ of V such that in the dual basis $F^1, \ldots, F^n$ we have

$$\Omega = \sum_{i=1}^{l} F^1 \wedge \ldots \widehat{F^{2i-1}} \wedge \widehat{F^{2i}} \wedge \ldots \wedge F^n. \tag{2.1}$$

Then $V_1 = \mathrm{span}(F_1, \ldots, F_{2l})$. After possibly redefining $F_{2l+1}, \ldots, F_n$, we may assume that $V_2 = \mathrm{span}(F_{2l+1}, \ldots, F_n)$. Then $\Omega = \frac{\omega^{l-1}}{(l-1)!} \wedge \nu$ for $\omega := \sum_{i=1}^{l} F^{2i-1} \wedge F^{2i}$, $\nu := F^{2l+1} \wedge \ldots \wedge F^n$ and $\omega \in \Lambda^2 V_1^*$ is non-degenerate. Choose a Hodge star operator $\star$ such that $(F_1, \ldots, F_n)$ is an oriented orthonormal basis and such that in the real case the corresponding non-degenerate symmetric bilinear form is positive definite. Note that then $\Omega = \star\omega$. If $\mathbb{F} = \mathbb{C}$, then Proposition 1.33 and Proposition 2.3 imply

$$\mathrm{GL}(V)_\Omega = \{ \lambda h^{-t} \,|\, \lambda^{n-2} = \det(h),\ h|_{V_1} = h_1 + g,\ h|_{V_2} = h_2,\ \lambda \in \mathbb{C}^*,\ h_1 \in \mathrm{Sp}(\omega, V_1), \\ g \in \hom(V_1, V_2),\ h_2 \in \mathrm{GL}(V_2) \}.$$

Set $H := \lambda h^{-t}$. Now our choice of an orthonormal basis shows that $V_1 \perp V_2$ and $\mathrm{Sp}(\omega, V_1)$, considered as a subgroup of $\mathrm{GL}(V)$, is closed under transposition. Hence, $H|_{V_1} = \lambda H_1$ and $H|_{V_2} = H_2 + G$ with $H_1 \in \mathrm{Sp}(\omega, V_1)$, $G \in \hom(V_2, V_1)$ and $H_2 \in \mathrm{GL}(V_2)$ such that $\det(H_2) = \det\left(\lambda h_2^{-1}\right) = \frac{\lambda^{n-2l}}{\det(h_2)} = \frac{\lambda^{n-2l}}{\det(h)} = \lambda^{2-2l}$. This proves that the stabiliser is contained in the corresponding group in the statement. Conversely, a short computation shows that actually all elements in the group given in the statement stabilise Ω.

Now we come to the real case. By Proposition 1.33, we have

$$\mathrm{GL}(V)_\Omega = \left\{ h^{-t} \,\middle|\, h \in \mathrm{GL}(V),\ h.\omega = \frac{\omega}{\det(h)} \right\}.$$

Let $h \in \mathrm{GL}(V)$ with $h.\omega = \frac{\omega}{\det(h)}$. Obviously, we have $h|_{V_1} = h_1 + g$, $h|_{V_2} = h_2$ for certain $h_i \in \mathrm{GL}(V_i)$ for $i = 1, 2$ and $g \in \hom(V_1, V_2)$. For $H_1 := \frac{h_1}{\sqrt{|\det(h)|}}$ we have $\det(H_1) = \frac{\det(h_1)}{|\det(h)|^l}$, $H_1.\omega = \mathrm{sgn}(\det(h))\,\omega$ and so $\frac{1}{\det(H_1)}\omega^l = H_1.\omega^l = \mathrm{sgn}(\det(h))^l\omega^l$. Thus

$$\mathrm{sgn}(\det(h_1))|\det(h_1)|^{1-l}|\det(h_2)|^{-l} = \frac{\det(h_1)}{|\det(h)|^l} = \det(H_1) = \mathrm{sgn}(\det(h))^{-l}. \tag{2.2}$$

For l even, $\mathrm{sgn}(\det(h))^{-l} = 1$ and Equation (2.2) yields $\mathrm{sgn}(\det(h_1)) = 1$ and $|\det(h_1)| = |\det(h_2)|^{\frac{l}{1-l}}$. Hence, $\mathrm{sgn}(\det(h)) = \mathrm{sgn}(\det(h_2))$, $|\det(h)| = |\det(h_2)|^{\frac{1}{1-l}}$ and so $h_1 = |\det(h_2)|^{\frac{1}{2-2l}} H_1$. Since the transposition is the usual one if we identify $\mathrm{GL}(V)$ with $\mathrm{GL}(n, \mathbb{R})$ via the ordered basis $(F_1, \ldots, F_n)$, we obtain

$$\begin{aligned}\mathrm{GL}(V)_\Omega \subseteq \Big\{ f \in \mathrm{GL}(V) \,\Big|\, & f|_{V_1} = |\det(f_2)|^{\frac{1}{2-2l}} f_1,\ f|_{V_2} = f_2 + h,\ f_1 \in \mathrm{GL}(V_1), \\ & f_1.\omega = \mathrm{sgn}(\det(f_2))\omega,\ f_2 \in \mathrm{GL}(V_2),\ h \in \hom(V_2, V_1) \Big\}.\end{aligned}$$

The converse inclusion follows by direct calculation.

For l odd, $l \neq 1$, Equation (2.2) gives us $\mathrm{sgn}(\det(h_1)) = \mathrm{sgn}(\det(h))$ and $|\det(h_1)| = |\det(h_2)|^{\frac{l}{1-l}}$. Thus, $\mathrm{sgn}(\det(h_2)) = 1$, $|\det(h)| = |\det(h_2)|^{\frac{1}{1-l}} = \det(h_2)^{\frac{1}{1-l}}$ and so $h_1 = \det(h_2)^{\frac{1}{2-2l}} H_1$ with $H_1.\omega = \epsilon\,\omega$ for some $\epsilon \in \{-1, 1\}$. This shows

$$\begin{aligned}\mathrm{GL}(V)_\Omega \subseteq \Big\{ f \in \mathrm{GL}(V) \,\Big|\, & f|_{V_1} = \det(f_2)^{\frac{1}{2-2l}} f_1,\ f|_{V_2} = f_2 + h,\ f_1.\omega = \epsilon\omega, \\ & \epsilon \in \{-1, 1\},\ f_2 \in \mathrm{GL}^+(V_2),\ h \in \hom(V_2, V_1) \Big\}.\end{aligned}$$

Again the converse inclusion follows by direct calculation. The stabiliser in the case $l = 1$ is obviously as in the statement. □

Next, we define a $\mathrm{GL}^+(V)$-equivariant map $\phi : \Lambda^2 V^* \to \Lambda^{2m} V^*$ as in Proposition 1.37.

Definition 2.6. *Let V be a $2m$-dimensional real vector space. Let $\omega \in \Lambda^2 V^*$ be a two-form on V. We define $\phi : \Lambda^2 V^* \to \Lambda^{2m} V^*$ by*

$$\phi : \Lambda^2 V^* \to \Lambda^{2m} V^*,\ \phi(\omega) := \frac{\omega^m}{m!}.$$

ϕ is even $\mathrm{GL}(V)$-equivariant. Since the stable two-forms are exactly the non-degenerate ones, the set $\phi^{-1}(0)$ is, in fact, the set of all non-stable two-forms. Moreover, the dual stable $(2m-2)$-form $\hat{\omega}$ is given by $\hat{\omega} = \frac{\omega^{m-1}}{(m-1)!}$.

We end this section by defining a $\mathrm{GL}^+(V)$-equivariant map $\phi : \Lambda^{2m-2} V^* \to \Lambda^{2m} V^*$ as in Proposition 1.37. We omit the calculations necessary to check the claimed properties and instead refer to [CLSS] for some more details.

Remark 2.7. • *Let $\Omega \in \Lambda^{2m-2}V^*$ be a stable $(2m-2)$-form on a $2m$-dimensional real vector space and m be even. Using the $\mathrm{GL}(V)$-module isomorphism $\kappa : \Lambda^{2m-2}V^* \cong \Lambda^2 V \otimes \Lambda^{2m}V^*$, we may consider $\kappa(\Omega)^m$ as an element in $(\Lambda^{2m}V^*)^{\otimes(m-1)}$. We set*

$$\phi : \Lambda^{2m-2}V^* \to \Lambda^{2m}V^*,\ \phi(\Omega) := \left(\frac{\kappa(\Omega)^m}{m!}\right)^{\frac{1}{m-1}}.$$

ϕ is $\mathrm{GL}(V)$-equivariant. Moreover, $\phi(\Omega) \neq 0$ holds if and only if Ω is stable. By Lemma 2.4 there exists a stable two-form $\omega \in \Lambda^2 V^*$ such that $\Omega = \frac{\omega^{m-1}}{(m-1)!}$. One can compute that $\phi(\Omega) = \phi(\omega)$ and that the dual two-form $\hat{\Omega}$ is equal to $\frac{\omega}{m-1}$. In particular, $\omega \in \Lambda^2 V^*$ with $\Omega = \frac{\omega^{m-1}}{(m-1)!}$ is unique.

• *Let $\Omega \in \Lambda^{2m-2}V^*$ be a stable $(2m-2)$-form on a $2m$-dimensional oriented real vector space and now let m be odd. In this case, we set*

$$\phi : \Lambda^{2m-2}V^* \to \Lambda^{2m}V^*,\ \phi(\Omega) := \left|\frac{\kappa(\Omega)^m}{m!}\right|^{\frac{1}{m-1}}.$$

The map ϕ is $\mathrm{GL}^+(V)$-equivariant, and again $\phi(\Omega) \neq 0$ holds if and only if Ω is stable. For odd m, Lemma 2.4 yields the existence a stable two-form $\omega \in \Lambda^2 V^*$ which induces the given orientation and $\epsilon \in \{-1, 1\}$ such that $\Omega = \epsilon\frac{\omega^{m-1}}{(m-1)!}$. One can compute that $\phi(\Omega) = \phi(\omega)$ and that the dual two-form $\hat{\Omega}$ is given by $\epsilon\frac{\omega}{m-1}$. Hence, $\omega \in \Lambda^2 V^*$ with the property that it induces the given orientation and that there exists $\epsilon \in \{-1, 1\}$ with $\Omega = \epsilon\frac{\omega^{m-1}}{(m-1)!}$ is unique.

2.2 ϵ-complex structures

In this section, we deal with complex and para-complex structures on $2m$-dimensional real vector spaces. We unify the language as in [SHPhD] and speak of ϵ-complex structures, where $\epsilon = -1$ refers to complex and $\epsilon = 1$ to para-complex structures. After the basic definitions, we recall the well-known decompositions of the ϵ-complex k-forms induced by an ϵ-complex structure J. Next, we discuss ϵ-complex volume forms. We show how one can reconstruct J from such a volume form and that in the case of odd m we only need the real part of the volume form for the reconstruction of J. Lastly, we consider the particular case $m = 3$ and relate our results to the formalism of stable forms introduced in Section 1.3. Throughout this section, we follow closely [SHPhD]. More background on complex structures and related subjects may be found in any textbook on complex geometry like [Huy] or [Wells]. For para-complex structures, we refer the reader to [Kr].

We start with the main definitions of this section.

Definition 2.8. *Let $\epsilon \in \{-1, 1\}$.*

- *Let V be a $2m$-dimensional real vector space. An* ϵ-complex structure on V *is an endomorphism $J \in \mathrm{End}(V)$ such that $J^2 = \epsilon \mathrm{id}_V$ and such that if $\epsilon = 1$ we have* $\dim(V_+) = \dim(V_-) = m$ *for $V_+ := \mathcal{E}ig(J,1)$ and $V_- := \mathcal{E}ig(J,-1)$. A (-1)-complex structure is a* complex structure *in the usual sense and a 1-complex structure is also called a* para-complex structure.

- *The* ϵ-complex numbers *are defined as the real unital associative algebra generated by 1 and the symbol i_ϵ subject to the relation $i_\epsilon^2 = \epsilon \cdot 1$ and are denoted by $\mathbb{C}_\epsilon$. $\mathbb{C}_{-1} = \mathbb{C}$ are the usual complex numbers and the real unital associative algebra $\mathbb{C}_1$ are the para-complex numbers already mentioned Example 1.15 (d). From time to time, we write i instead of i_{-1} and e instead of i_1. We have $\mathbb{C}_\epsilon \cong \mathbb{R} \oplus \mathbb{R} i_\epsilon$ as real vector spaces. Thus, we may write an element $z \in \mathbb{C}_\epsilon$ as $z = a + b i_\epsilon$ with $a, b \in \mathbb{R}$.* $\mathrm{Re}(z) := a$ *is called the* real part of z *and* $\mathrm{Im}(z) := b$ *is called the* imaginary part of z. *Moreover, the map $z \mapsto \overline{z} := a - b i_\epsilon$ is called the* ϵ-complex conjugation *and $\overline{z}$ the* ϵ-complex conjugate of z. *For $\epsilon = -1$, $\bar{\cdot}$ is the usual* complex conjugation *and $\overline{z}$ the usual* complex conjugate of z *and for $\epsilon = 1$ we call $\bar{\cdot}$ also the* para-complex conjugation *and $\overline{z}$ the* para-complex conjugate of z. *Note that the notation is in accordance with the one of Section 1.2 if we consider $\mathbb{C}_\epsilon$ as a composition algebra with the pseudo-Euclidean metric $g_\epsilon(z, w) := z\overline{w}$.*

- *If V is a real n-dimensional vector space, then the free $\mathbb{C}_\epsilon$-module $V_{\mathbb{C}_\epsilon} := V \otimes_{\mathbb{R}} \mathbb{C}_\epsilon$ is called the* ϵ-complexification of V. *The (-1)-complexification is simply the usual* complexification *and the 1-complexification is also called* para-complexification. *To simplify the notation, we say that a free $\mathbb{C}_1$-module V is a* $\mathbb{C}_1$-vector space.

Remark 2.9.

- *$\mathbb{C}_1$ contains zero divisors, namely exactly the $z \in \mathbb{C}_1 \backslash \{0\}$ with $z\overline{z} = 0$.*

- *If J is an ϵ-complex structure on an n-dimensional real vector space V, then $n = 2m$ for some $m \in \mathbb{N}$ and V is a $\mathbb{C}_\epsilon$-vector space via $(a + b i_\epsilon) \cdot v := av + bJv$ for $a, b \in \mathbb{R}$ and $v \in V$. Note that for $\epsilon = 1$, V is, in fact, a $\mathbb{C}_1$-vector space since any real basis $v_1, \ldots, v_m$ of V_+ and any real basis $w_1, \ldots, w_m$ of V_- give the $\mathbb{C}_1$-basis $v_1 + w_1, \ldots, v_m + w_m$ of V. Conversely, if W is an m-dimensional $\mathbb{C}_\epsilon$-vector space, then W is a $2m$-dimensional real vector space and the multiplication with i_ϵ is an ϵ-complex structure on W. In this sense ϵ-complex structures on even-dimensional real vector spaces are the same as finite-dimensional $\mathbb{C}_\epsilon$-vector spaces. Moreover, all $\mathbb{C}_1$-modules which are finite-dimensional real vector spaces are free.*

The following example is the main example of an ϵ-complex structure. We will use it as a model tensor in the following.

Example 2.10. *An ϵ-complex structure on $\mathbb{R}^{2m}$ is given by*

$$J_\epsilon := \sum_{i=1}^{m} \left(e^{2i} \otimes e_{2i-1} + \epsilon e^{2i-1} \otimes e_{2i}\right) \tag{2.3}$$

We use these tensors as model tensors to identify ϵ-complex structures with $\mathrm{GL}(2m,\mathbb{R})_{J_\epsilon}$-structures.

For the use as a model tensor, we have to determine the stabiliser group of J_ϵ.

Definition 2.11. *The* ϵ-complex general linear group $\mathrm{GL}(m,\mathbb{C}_\epsilon) \subseteq \mathrm{GL}(2m,\mathbb{R})$ *is defined as the stabiliser of $J_\epsilon \in \mathrm{End}\left(\mathbb{R}^{2m}\right)$. For $\epsilon = -1$ it is given by the usual* complex general linear group $\mathrm{GL}(m,\mathbb{C})$ *embedded as a subgroup of* $\mathrm{GL}(2m,\mathbb{R})$. *With respect to the ordered basis $(e_1, e_3, \ldots, e_{2m-1}, e_2, \ldots, e_{2m})$ of $\mathbb{R}^{2m}$, $\mathrm{GL}(m,\mathbb{C})$ is given by the subgroup* $\left\{ \left. \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \right| A + iB \in \mathrm{GL}(m,\mathbb{C}) \right\}$ *of $\mathrm{GL}(2m,\mathbb{R})$. For $\epsilon = 1$, $\mathrm{GL}(m,\mathbb{C}_1)$ is also called the* para-complex general linear group *and is given by* $\mathrm{GL}(m,\mathbb{C}_1) = \mathrm{GL}\left(\left(\mathbb{R}^{2m}\right)_+\right) \times \mathrm{GL}\left(\left(\mathbb{R}^{2m}\right)_-\right) \cong \mathrm{GL}(m,\mathbb{R}) \times \mathrm{GL}(m,\mathbb{R})$ *with* $\left(\mathbb{R}^{2m}\right)_+ := \mathcal{E}ig(J_1, 1)$ *and* $\left(\mathbb{R}^{2m}\right)_- := \mathcal{E}ig(J_1, -1)$.

The splitting of the ϵ-complexification $V_{\mathbb{C}_\epsilon}$ into the eigenspaces of the $\mathbb{C}_\epsilon$-linear extension of an ϵ-complex structure $J \in \mathrm{End}(V)$ gives us corresponding splittings of the ϵ-complex k-forms.

Definition 2.12. *Let (V, J) be a $2m$-dimensional vector space V with an ϵ-complex structure J. We consider the ϵ-complexification $V_{\mathbb{C}_\epsilon}$ and the $\mathbb{C}_\epsilon$-*linear extension $J_{\mathbb{C}_\epsilon} \in \mathrm{End}(V_{\mathbb{C}_\epsilon})$ *of J, which is defined for $v_1 + i_\epsilon v_2 \in V_{\mathbb{C}_\epsilon}$, $v_1, v_2 \in V$, by*

$$J_{\mathbb{C}_\epsilon}(v_1 + i_\epsilon v_2) := J(v_1) + i_\epsilon J(v_2).$$

We set $V^{1,0} := \mathcal{E}ig(J_{\mathbb{C}_\epsilon}, i_\epsilon) \subseteq V_{\mathbb{C}_\epsilon}$ and $V^{0,1} := \mathcal{E}ig(J_{\mathbb{C}_\epsilon}, -i_\epsilon) \subseteq V_{\mathbb{C}_\epsilon}$ and observe that $V_{\mathbb{C}_\epsilon} = V^{1,0} \oplus V^{0,1}$ as $\mathbb{C}_\epsilon$-vector spaces, $V^{1,0} = \{w = v + \epsilon i_\epsilon J v \in V_{\mathbb{C}_\epsilon} | v \in V\}$ and $V^{0,1} = \overline{V^{1,0}}$.

We have a natural $\mathbb{C}_\epsilon$-isomorphism $(V_{\mathbb{C}_\epsilon})^* = (V^*)_{\mathbb{C}_\epsilon}$. Using this isomorphism, we simply write $V^*_{\mathbb{C}_\epsilon}$ and get the decomposition $V^*_{\mathbb{C}_\epsilon} = (V^*)^{1,0} \oplus (V^*)^{0,1}$ with $(V^*)^{1,0} := \mathcal{E}ig(J^*_{\mathbb{C}_\epsilon}, i_\epsilon) = \left(V^{0,1}\right)^0$ and $(V^*)^{0,1} := \mathcal{E}ig(J^*_{\mathbb{C}_\epsilon}, -i_\epsilon) = \left(V^{1,0}\right)^0$. More explicitly, the two spaces are given by

$$(V^*)^{1,0} = \{\alpha + \epsilon i_\epsilon J^* \alpha | \alpha \in V^*\}, \quad (V^*)^{0,1} = \{\alpha - \epsilon i_\epsilon J^* \alpha | \alpha \in V^*\} = \overline{(V^*)^{1,0}}.$$

For $p, q \in \mathbb{N}_0$ we set

$$\Lambda^{p,q} V^* := \Lambda^p (V^*)^{1,0} \wedge \Lambda^q (V^*)^{0,1} \subseteq \Lambda^{p+q} V^*_{\mathbb{C}_\epsilon} \tag{2.4}$$

and call the elements in $\Lambda^{p,q}V^*$ (p,q)-forms *or* forms of type (p,q). *We have* $\overline{\Lambda^{p,q}V^*} = \Lambda^{q,p}V^*$ *and* $\Lambda^k V^*_{\mathbb{C}} = \sum_{p=0}^k \Lambda^{p,k-p}V^*$. *For* $p \neq q$, *we set*

$$[[\Lambda^{p,q}V^*]] := \Lambda^k V^* \cap (\Lambda^{p,q}V^* \oplus \Lambda^{q,p}V^*) \tag{2.5}$$

and call the elements in $[[\Lambda^{p,q}V^*]]$ real forms of type (p,q) and (q,p). *Moreover, we set*

$$[\Lambda^{p,p}V^*] := \Lambda^k V^* \cap \Lambda^{p,p}V^* \tag{2.6}$$

and call the elements in $[\Lambda^{p,p}V^*]$ real forms of type (p,p). *Note that we have* $[[\Lambda^{p,q}V^*]] \otimes \mathbb{C}_\epsilon = \Lambda^{p,q}V^* \oplus \Lambda^{q,p}V^*$, $[\Lambda^{p,p}V^*] \otimes \mathbb{C}_\epsilon = \Lambda^{p,p}V^*$,

$$[[\Lambda^{p,q}V^*]] = \{\alpha + \overline{\alpha} | \alpha \in \Lambda^{p,q}V^*\}, \quad [\Lambda^{p,p}V^*] = \{\alpha + \overline{\alpha} | \alpha \in \Lambda^{p,p}V^*\} \tag{2.7}$$

and

$$\Lambda^{2l}V^* = \bigoplus_{p=0}^{l-1}[[\Lambda^{p,2l-p}V^*]] \oplus [\Lambda^{l,l}V^*], \quad \Lambda^{2l+1}V^* = \bigoplus_{p=0}^{l}[[\Lambda^{p,2l+1-p}V^*]]. \tag{2.8}$$

Remark 2.13.

- *In the para-complex case there is the natural decomposition* $V = V_+ \oplus V_-$ *and the corresponding decomposition* $V^* := (V^*)_+ \oplus (V^*)_-$. *Then* $[[\Lambda^{p,q}V^*]] = \Lambda^p (V^*)_+ \wedge \Lambda^q (V^*)_- \oplus \Lambda^q (V^*)_+ \wedge \Lambda^q (V^*)_-$ *and* $[\Lambda^{p,p}V^*] = \Lambda^p (V^*)_+ \wedge \Lambda^p (V^*)_-$.
- *Although* $\mathcal{E}ig(J_{\mathbb{C}_1}, \lambda)$ *is a well-defined* $\mathbb{C}_1$*-submodule of* $V_{\mathbb{C}_1}$ *for all* $\lambda \in \mathbb{C}_1$, *an element in* $V_{\mathbb{C}_1}$ *may have more than one eigenvalue with respect to* $J_{\mathbb{C}_1}$. *E.g. if* $v \in V_+$, *then* $v + ev \in V_{\mathbb{C}_1}$ *has both eigenvalue* $e = i_1$ *and* 1 *with respect to* $J_{\mathbb{C}_1}$, *which stems from the fact that* $v + ev$ *is linearly dependent in the* $\mathbb{C}_1$*-vector space* $V_{\mathbb{C}_1}$ *and that* $e - 1 \in \mathbb{C}_1$ *is a null-vector.*

As remarked in Section 1.1, volume forms are nothing but $\mathrm{SL}(n,\mathbb{R})$-structures. A natural question to ask is what kind of tensors are related to $\mathrm{SL}(m,\mathbb{C})$-structures, $\mathrm{SL}(m,\mathbb{C}) \subseteq \mathrm{GL}(2m,\mathbb{R})$, and what are the corresponding objects in the para-complex case.

Definition 2.14. *Let* V *be a* $2m$*-dimensional real vector space and* J *be an* ϵ*-complex structure on* V. *An* ϵ*-complex* m*-form* $\Psi \in \Lambda^m V_{\mathbb{C}_\epsilon}$ *is called* non-degenerate *if* $\Psi \wedge \overline{\Psi} \neq 0$. *An* ϵ*-complex* $(m,0)$*-form* Ψ *is called* ϵ-complex volume form *if* Ψ *is non-degenerate. Note that each non-zero complex* m*-form is non-degenerate.*

The basic example of an ϵ-complex volume form which we will use as a model tensor is the following.

Example 2.15. *An* ϵ*-complex volume form on* $(\mathbb{R}^{2m}, J_\epsilon)$ *is given by*

$$\Psi_\epsilon := \left(e^1 + \epsilon i_\epsilon e^2\right) \wedge \ldots \wedge \left(e^{2m-1} + \epsilon i_\epsilon e^{2m}\right) \tag{2.9}$$

Remark 2.16. *Note that in the para-complex case there are degenerate para-complex $(m,0)$-forms. An example is given by $(1+e)\Psi_1$ on $\mathbb{R}^{2m}$, Ψ_1 as in Equation (2.9). Note further that by induction on m one may show that for all $m \in \mathbb{N}$ the identity*

$$\Psi_1 = \left(f^{1\dots m} + f^{m+1\dots 2m}\right) + e\left(f^{1\dots m} - f^{m+1\dots 2m}\right)$$

is true, where $f^i := \frac{e^{2i-1}+e^{2i}}{\sqrt[m]{2}}$ and $f^{m+i} := \frac{e^{2i-1}-e^{2i}}{\sqrt[m]{2}}$ for $i = 1, \dots m$. Moreover, $(\mathbb{R}^m)^*_+ =$ $\mathrm{span}(f^1, \dots, f^m)$ and $(\mathbb{R}^m)^*_- = \mathrm{span}(f^{m+1}, \dots, f^{2m})$.

Definition 2.17. *The* ϵ-complex special linear group $\mathrm{SL}(m, \mathbb{C}_\epsilon)$ *is defined as* $\mathrm{GL}(2m, \mathbb{R})_{\Psi_\epsilon}$ *with the ϵ-complex volume form Ψ_ϵ on $\mathbb{R}^{2m}$ defined in Equation (2.9). Here,* $\mathrm{GL}(2m, \mathbb{R})$ *acts on $(\mathbb{R}^{2m})_{\mathbb{C}_\epsilon} \cong \mathbb{R}^{2m} \oplus i_\epsilon \mathbb{R}^{2m}$ in the natural way on each of the summands $\mathbb{R}^{2m}$ and $i_\epsilon \mathbb{R}^{2m}$. So $\mathrm{GL}(2m, \mathbb{R})_{\Psi_\epsilon} = \mathrm{GL}(2m, \mathbb{R})_{\mathrm{Re}(\Psi_\epsilon)} \cap \mathrm{GL}(2m, \mathbb{R})_{\mathrm{Im}(\Psi_\epsilon)}$ by definition. Obviously, $\mathrm{SL}(m, \mathbb{C}_{-1}) = \mathrm{SL}(m, \mathbb{C}) \subseteq \mathrm{GL}(2m, \mathbb{R})$ and from Remark 2.16 we get $\mathrm{SL}(m, \mathbb{C}_1) =$ $\mathrm{SL}\left(\left(\mathbb{R}^{2m}\right)_+\right) \times \mathrm{SL}\left(\left(\mathbb{R}^{2m}\right)_-\right) \cong \mathrm{SL}(m, \mathbb{R}) \times \mathrm{SL}(m, \mathbb{R})$.*

If an ϵ-complex m-form $\Psi \in \Lambda^m V^*_{\mathbb{C}_\epsilon}$ is an ϵ-complex volume form with respect to an ϵ-complex structure J on V, then Ψ has model tensor Ψ_ϵ. Conversely, suppose that we do not have an ϵ-complex structure J but we have an ϵ-complex m-form $\Psi \in \Lambda^m V^*_{\mathbb{C}_\epsilon}$ such that $u^*_{\mathbb{C}_\epsilon} \Psi = \Psi_\epsilon$ for some real isomorphism $u : \mathbb{R}^{2m} \to V$. The inclusion $\mathrm{SL}(m, \mathbb{C}_\epsilon) \subseteq \mathrm{GL}(m, \mathbb{C}_\epsilon)$ implies that Ψ induces an ϵ-complex structure J such that Ψ is an ϵ-complex volume form with respect to this structure. We also call $\Psi \in \Lambda^m V^*_{\mathbb{C}_\epsilon}$ with model tensor Ψ_ϵ an *ϵ-complex volume form*. Since $\mathrm{SL}(m, \mathbb{C}_\epsilon) \subseteq \mathrm{SL}(2m, \mathbb{R})$, Ψ also induces a real volume form. The following lemma gives a concrete description of these constructions.

Proposition 2.18. *Let V be a $2m$-dimensional real vector space and $m \geq 2$. Then*

$$\phi : \Lambda^m V^*_{\mathbb{C}_\epsilon} \to \Lambda^{2m} V^*, \quad \phi(\Psi) := \begin{cases} \frac{1}{4} \Psi \wedge \overline{\Psi}, & \text{if } m \text{ is even,} \\ \frac{1}{4 i_\epsilon} \Psi \wedge \overline{\Psi}, & \text{if } m \text{ is odd.} \end{cases}$$

maps the ϵ-complex volume forms $\Psi \in \Lambda^m V^*_{\mathbb{C}_\epsilon}$ to real volume forms on V. Moreover, any ϵ-complex volume form $\Psi \in \Lambda^m V^*_{\mathbb{C}_\epsilon}$ induces a unique ϵ-complex structure J such that Ψ is an ϵ-complex volume form in the sense of Definition 2.14 with respect to J. If we denote by $\kappa : \Lambda^{2m-1} V^* \to V \otimes \Lambda^{2m} V^*$ the natural $\mathrm{GL}(V)$-module isomorphism given by $\kappa(\psi) := w \otimes \nu$ for $\psi \in \Lambda^{2m-1} V^*$ and $w \in V$ and $\nu \in \Lambda^{2m} V^*$ with $w \lrcorner \nu = \psi$, J is defined by

$$J(v)\Phi(\Psi) = \begin{cases} \kappa\big(v \lrcorner \mathrm{Re}(\Psi) \wedge \mathrm{Im}(\Psi)\big) & \text{if } m \text{ is even,} \\ \kappa\big(v \lrcorner \mathrm{Re}(\Psi) \wedge \mathrm{Re}(\Psi)\big) & \text{if } m \text{ is odd,} \end{cases} \tag{2.10}$$

for $v \in V$.

Proof. For the proof one may consult, e.g., [SHPhD, Proposition 1.4]. $\square$

If $m \geq 3$ is odd, we show below in Proposition 2.21 that on an oriented vector space we may recover the complex structure J from the real part of an ϵ-complex volume form. To understand the construction abstractly on the level of enlargements of G-structures, we have to compute the stabiliser in $\mathrm{GL}^+(2m,\mathbb{R})$ of $\mathrm{Re}(\Psi_\epsilon)$. We do this for arbitrary $m \geq 3$.

Lemma 2.19. *Let $m \geq 3$, $\Psi_\epsilon \in \Lambda^m \mathbb{C}^{2m}$ be the ϵ-complex m-form defined in Equation (2.9) and $A := \mathrm{diag}(1,-1,1,-1,\ldots,1,-1)$, $B := \mathrm{diag}(-I_2, I_{2m-2})A \in \mathrm{GL}(2m,\mathbb{R})$. Then:*

$$\mathrm{GL}(2m,\mathbb{R})_{\mathrm{Re}(\Psi_\epsilon)} = \mathrm{SL}(m,\mathbb{C}_\epsilon) \rtimes \{A, I_{2m}\}, \quad \mathrm{GL}(2m,\mathbb{R})_{\mathrm{Im}(\Psi_\epsilon)} = \mathrm{SL}(m,\mathbb{C}_\epsilon) \rtimes \{B, I_{2m}\}.$$

Proof. For the proof note that $\mathrm{SL}(m,\mathbb{C}_\epsilon) = \mathrm{GL}(2m,\mathbb{R})_{\mathrm{Re}(\Psi_\epsilon)} \cap \mathrm{GL}(2m,\mathbb{R})_{\mathrm{Im}(\Psi_\epsilon)}$.

We only show $\mathrm{GL}(2m,\mathbb{R})_{\mathrm{Re}(\Psi_\epsilon)} = \mathrm{SL}(m,\mathbb{C}_\epsilon) \rtimes \{A, I_{2m}\}$. The computation of the stabiliser of $\mathrm{Im}(\Psi_\epsilon)$ is completely analogous. First, let $\epsilon = -1$. Let $g \in \mathrm{GL}(2m,\mathbb{R})_{\mathrm{Re}(\Psi_{-1})}$. The identity $\mathrm{Re}(\Psi_{-1}) = \frac{\Psi_{-1}}{2} + \frac{\overline{\Psi_{-1}}}{2} = \frac{g^*\Psi_{-1}}{2} + \frac{g^*\overline{\Psi_{-1}}}{2}$ is true. Now $\frac{\Psi_{-1}}{2}$, $\frac{g^*\Psi_{-1}}{2}$, $\frac{\overline{\Psi_{-1}}}{2}$ and $\frac{g^*\overline{\Psi_{-1}}}{2}$ are all decomposable as complex m-forms, $\frac{\Psi_{-1}}{2} \wedge \frac{\overline{\Psi_{-1}}}{2} \neq 0$ and $\frac{g^*\Psi_{-1}}{2} \wedge \frac{g^*\overline{\Psi_{-1}}}{2} \neq 0$. Since $m \geq 3$, [BuGl, Theorem 4.4] states that $g^*\Psi_{-1} = \Psi_{-1}$, $g^*\overline{\Psi_{-1}} = g^*\overline{\Psi_{-1}}$ or $g^*\Psi_{-1} = \overline{\Psi_{-1}}$, $g^*\overline{\Psi_{-1}} = \Psi_{-1}$. Since $A^*\Psi_{-1} = \overline{\Psi_{-1}}$, $A^*\overline{\Psi_{-1}} = \Psi_{-1}$, the assertion follows. Next, let $\epsilon = 1$ and $g \in \mathrm{GL}(2m,\mathbb{R})_{\mathrm{Re}(\Psi_1)}$. By Remark 2.16, there is a basis $f^1,\ldots,f^{2m}$ of V^* such that $\mathrm{Re}(\Psi_1) = f^{1\ldots m} + f^{m+1\ldots 2m}$. Again [BuGl, Theorem 4.4] shows that due to $m \geq 3$, g^* either stabilises both $f^1 \wedge \ldots \wedge f^m$ and $f^{m+1} \wedge \ldots \wedge f^{2m}$ or it interchanges the two decomposable forms. Moreover, if it stabilises both, it also stabilises Ψ_1. Hence, the statement follows from the fact that A^* interchanges $f^1 \wedge \ldots \wedge f^m$ and $f^{m+1} \wedge \ldots \wedge f^{2m}$. □

Let $m = 2l-1 \geq 3$ be odd. Then A, $B \in \mathrm{GL}(2m,\mathbb{R})$ as in Lemma 2.19 has determinant -1 and so $\mathrm{GL}^+(2m,\mathbb{R})_{\mathrm{Re}(\Psi_\epsilon)} = \mathrm{GL}^+(2m,\mathbb{R})_{\mathrm{Im}(\Psi_\epsilon)} = \mathrm{SL}(m,\mathbb{C}_\epsilon)$. This motivates us to call an m-form $\rho \in \Lambda^m V^*$ on a $2m$-dimensional oriented real vector space V with model tensor $\mathrm{Re}(\Psi_\epsilon)$ or $\mathrm{Im}(\Psi_\epsilon)$ an *ϵ-complex volume form* if $m = 2l-1 \geq 3$. By enlarging the corresponding $\mathrm{SL}(m,\mathbb{C}_\epsilon)$-structure, such a ρ induces an ϵ-complex structure J_ρ, a real volume form $\phi(\rho)$ on V and an ϵ-complex volume form Ψ with respect to J_ρ such that $\mathrm{Re}(\Psi) = \rho$ or $\mathrm{Im}(\Psi) = \rho$, respectively. The construction of J_ρ is as follows.

Definition 2.20. *Let $m = 2l-1$ be odd and $\rho \in \Lambda^m V^*$ be an m-form on a $2m$-dimensional oriented real vector space V. We define $K_\rho : V \to V \otimes \Lambda^{2m} V^*$ by*

$$K_\rho(v) := \kappa((v \lrcorner \rho) \wedge \rho),$$

where $\kappa : \Lambda^{2m-1} V^* \to V \otimes \Lambda^{2m} V^*$ is the natural isomorphism whose inverse is given by $\kappa^{-1}(v \otimes \nu) = v \lrcorner \nu$ for $v \in V$ and $\nu \in \Lambda^{2m} V^*$. Note that $(K_\rho \otimes \mathrm{id}_{\Lambda^{2m}V^*}) \circ K_\rho : V \to V \otimes (\Lambda^{2m} V^*)^{\otimes 2}$. Thus, we have

$$\lambda(\rho) := \frac{1}{2m} \mathrm{tr}\left((K_\rho \otimes \mathrm{id}_{\Lambda^{2m}V^*}) \circ K_\rho\right) \in (\Lambda^{2m} V^*)^{\otimes 2}. \tag{2.11}$$

We define the map $\phi : \Lambda^m V^* \to \Lambda^{2m} V^*$ *by*

$$\phi(\rho) := \sqrt{|\lambda(\rho)|} \in \Lambda^{2m} V^*. \tag{2.12}$$

If $\phi(\rho) \neq 0$*, we define* $J_\rho : V \to V$ *via*

$$J_\rho \phi(\rho) = K_\rho. \tag{2.13}$$

For $m = 2l - 1$ odd, one sees that $J_\epsilon^* \Psi_\epsilon = i_\epsilon \epsilon^{l-1} \Psi_\epsilon$ implies $J_\epsilon^* \mathrm{Re}(\Psi_\epsilon) = \epsilon^l \mathrm{Im}(\Psi_\epsilon)$ and $J_\epsilon^* \mathrm{Im}(\Psi_\epsilon) = \epsilon^{l-1} \mathrm{Re}(\Psi_\epsilon)$. Thus, $\left(\epsilon^{l-1} J_\epsilon\right)^* \mathrm{Im}(\Psi_\epsilon) = \epsilon^{-2(l-1)^2} \mathrm{Re}(\Psi_\epsilon) = \mathrm{Re}(\Psi_\epsilon)$. Hence, if an m-form $\rho \in \Lambda^m V^*$ on a $2m$-dimensional real vector space V has model tensor $\mathrm{Im}(\Psi_\epsilon)$ it also has model tensor $\mathrm{Re}(\Psi_\epsilon)$. This is why we restrict ourselves to the model tensor $\mathrm{Re}(\Psi_\epsilon)$ in the following proposition.

Proposition 2.21. *Let* V *be a* $2m$*-dimensional oriented real vector space and* $m = 2l-1 \geq 3$ *be odd. The map* $\phi : \Lambda^m V^* \to \Lambda^{2m} V^*$ *defined in Equation (2.12) is* $\mathrm{GL}^+(V)$*-equivariant. Assume that* $\rho \in \Lambda^m V^*$ *has model tensor* $\mathrm{Re}(\Psi_\epsilon)$*. Then* $\phi(\rho) \neq 0$*,* J_ρ *as defined in Equation (2.13) is an* ϵ*-complex structure on* V *and*

$$\Psi := \rho + i_\epsilon \epsilon^l J_\rho^* \rho \in \Lambda^m V_{\mathbb{C}_\epsilon}^* \tag{2.14}$$

is an ϵ*-complex volume form with respect to* J_ρ *with* $\mathrm{Re}(\Psi) = \rho$ *and* $\phi(\Psi) = \phi(\rho)$*. Furthermore,* $\epsilon\lambda(\rho) > 0$ *for* $\lambda : \Lambda^m V^* \to \left(\Lambda^{2m} V^*\right)^{\otimes 2}$ *defined in Equation (2.11). Moreover,* Ψ *is the unique* ϵ*-complex structure* $\tilde{\Psi} \in \Lambda^m V_{\mathbb{C}_\epsilon}^*$ *such that* $\mathrm{Re}\left(\tilde{\Psi}\right) = \rho$ *and such that the orientation induced by* $\tilde{\Psi}$ *is the given one. Furthermore, we have*

$$\phi(\rho) = \frac{\epsilon^l}{2} J_\rho^* \rho \wedge \rho \tag{2.15}$$

and

$$J_\rho^* \alpha(v) \phi(\rho) = \alpha \wedge (v \lrcorner \rho) \wedge \rho \tag{2.16}$$

for all $\alpha \in V^*$*.*

Proof. The $\mathrm{GL}^+(V)$-equivariance of ϕ is obvious. Let $\rho \in \Lambda^m V^*$ have model tensor $\mathrm{Re}(\Psi_\epsilon)$. If $u : \mathbb{R}^n \to V$ is such that $u^* \rho = \mathrm{Re}(\Psi_\epsilon)$, then $\Psi := \left(u_{\mathbb{C}_\epsilon}^{-1}\right)^* \Psi_\epsilon \in \Lambda^m V_{\mathbb{C}_\epsilon}$ is an ϵ-complex volume form with $\mathrm{Re}(\Psi) = \rho$. By choosing $\mathrm{Re}(\Psi_\epsilon) - i_\epsilon \mathrm{Im}(\Psi_\epsilon)$ instead of $\Psi_\epsilon = \mathrm{Re}(\Psi_\epsilon) + i_\epsilon \mathrm{Im}(\Psi_\epsilon)$, we may assume that Ψ induces the same orientation as ρ. Note therefore that $\phi(\Psi) = \frac{1}{2} \mathrm{Im}(\Psi) \wedge \mathrm{Re}(\Psi)$. By Proposition 2.18 we have $J_\Psi \phi(\Psi) = K_\rho$, where J_Ψ is the ϵ-complex structure induced by Ψ. Since $J_\Psi^2 = \epsilon \mathrm{id}_V$, we get $\lambda(\rho) = \epsilon \phi(\Psi)^2$. Thus, $\epsilon\lambda(\rho) = \phi(\Psi)^2 > 0$ and $\phi(\rho) = \phi(\Psi) \neq 0$. Hence, $J_\rho = J_\Psi$ and J_ρ is an ϵ-complex structure. Since Ψ is an $(m,0)$-form with respect to $J_\Psi = J_\rho$, the calculations directly above Proposition 2.21 show $\mathrm{Im}(\Psi) = \epsilon^l J_\rho^* \rho$ and so $\phi(\rho) = \phi(\Psi) = \frac{\epsilon^l}{2} J_\rho^* \rho \wedge \rho$. Since $\mathrm{Im}(\Psi)$ is determined by ρ, Ψ is unique. Moreover, using Equation (2.13), we may calculate

$$\left(J_\rho^* \alpha\right)(v) \phi(\rho) = J_\rho(v) \lrcorner \alpha \wedge \phi(\rho) = \alpha \wedge J_\rho(v) \lrcorner \phi(\rho) = \alpha \wedge \kappa^{-1}(K_\rho(v)) = \alpha \wedge (v \lrcorner \rho) \wedge \rho$$

for all $\alpha \in V^*$. □

We take a closer look at the case $m = 3$.

Proposition 2.22. *(a) Let V be a six-dimensional oriented real vector space. Then $\phi(\rho) \neq 0$ if and only if $\rho \in \Lambda^3 V^*$ is stable. This is the case if and only if ρ is an ϵ-complex volume form for some $\epsilon \in \{-1, 1\}$. Moreover, if ρ is stable, then the dual three-form $\hat{\rho}$ is given by $J_\rho^* \rho$.*

(b) The concrete values of K_ρ, $\lambda(\rho)$ and, if it is well-defined, of J_ρ for the three-forms ρ on $\mathbb{R}^6$ in Table 1.1 for the standard orientation of $\mathbb{R}^6$ are given in the following table:

Table 2.1: Invariants for three-forms in six dimensions

ρ	K_ρ	$\lambda(\rho)$	J_ρ
Q_1	0	0	-
Q_2	0	0	-
Q_3	$-2e^1 \otimes e_6 \otimes e^{123456}$	0	-
ρ_0	$2\left(e^1 \otimes e_4 + e^2 \otimes e_5 + e^3 \otimes e_6\right) \otimes e^{123456}$	0	-
ρ_1	$2J_1 \otimes e^{123456}$	$4\left(e^{123456}\right)^{\otimes 2}$	J_1
ρ_{-1}	$2J_{-1} \otimes e^{123456}$	$-4\left(e^{123456}\right)^{\otimes 2}$	J_{-1}

Proof. Let V be a 6-dimensional oriented real vector space and $\rho \in \Lambda^3 V^*$. Due to Proposition 1.45, ρ is equivalent to exactly one of the forms in Table 2.1 and ρ is an ϵ-complex structure if and only if it is equivalent to ρ_ϵ in Table 2.1 and so by Theorem 1.35 if and only if it is stable. Hence, "$\phi(\rho) \neq 0$ if and only if ρ is stable" follows directly from (b) and (b) is a straightforward computation. For a proof that the dual three-form $\hat{\rho}$ of a stable three-form $\rho \in \Lambda^3 V^*$ is equal to $J_\rho^* \rho$ we refer to [CLSS]. $\square$

2.3 (Special) ϵ-Hermitian structures

In this section, we consider (special) ϵ-Hermitian structures on $2m$-dimensional real vector spaces. Special ϵ-Hermitian structures are ϵ-Hermitian structures together with an ϵ-complex volume form of certain length. We recall the exact definitions and some basic facts of these structures. Using the results of Section 2.2, we show how one can reconstruct a special ϵ-Hermitian structure via a pair of a non-degenerate two-form ω and an ϵ-complex volume form Ψ fulfilling certain compatibility conditions. For odd m we prove that for the reconstruction we only need ω and $\mathrm{Re}(\Psi)$. Finally, we look at the case $m = 3$ and show that then a special ϵ-Hermitian structure can be recovered from a pair $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ of stable forms on V with $\omega \wedge \rho = 0$. Again, we closely follow [SHPhD].

We start with ϵ-Hermitian structures.

Definition 2.23. *Let V be a real $2m$-dimensional vector space. An* ϵ-Hermitian structure

(g, J) consists of a pseudo-Euclidean metric g on V and an ϵ-complex structure J such that $J^*g = -\epsilon g$. The fundamental two-form $\omega \in \Lambda^2 V^*$ (associated to (g, J)) *is defined by $\omega(v, w) := g(v, Jw)$. A 1-Hermitian structure is also called* para-Hermitian structure, *a (-1)-Hermitian structure is called* pseudo-Hermitian structure *or* Hermitian structure *in case g is positive definite.*

Remark 2.24.
- *$J^*g = -\epsilon g$ implies that the fundamental two-form $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$ is, in fact, a two-form, which is non-degenerate since g is non-degenerate.*
- *If V is a real $2m$-dimensional vector space and g is a pseudo-Euclidean metric, then a complex structure J such that (g, J) is a pseudo-Hermitian structure is the same as a one-fold cross product on (V, g). In particular, the signature of g is then $(2p, 2m-2p)$ for some $p \in \{0, \ldots, m\}$ by Example 1.12 (c).*
- *If (g, J) is a para-Hermitian structure on the real $2m$-dimensional vector space V, then $g(Jv, Jv) = -g(v, v)$ for all $v \in V$. Hence, g has necessarily signature (m, m).*

The following examples of ϵ-Hermitian structures on $\mathbb{R}^{2m}$ will be used as model tensors.

Example 2.25. *For $p \in \{0, \ldots, m\}$, $(\langle\cdot,\cdot\rangle_{2p,2m-2p}, J_{-1})$ is a pseudo-Hermitian structure on $\mathbb{R}^{2m}$, whereas $(\langle\cdot,\cdot\rangle_{split}, J_1)$ is a para-Hermitian structure on $\mathbb{R}^{2m}$. Here, $J_\epsilon \in \mathrm{End}\left(\mathbb{R}^{2m}\right)$, $\epsilon \in \{-1, 1\}$, is defined by Equation (2.3) and $\langle\cdot,\cdot\rangle_{2p,2m-2p} = \sum_{i=1}^{2p} e^i \otimes e^i - \sum_{j=2p+1}^{2m} e^j \otimes e^j \in S^2\left(\mathbb{R}^{2m}\right)^*$, $\langle\cdot,\cdot\rangle_{split} = \sum_{i=1}^{2m} (-1)^i e^i \otimes e^i \in S^2\left(\mathbb{R}^{2m}\right)^*$ by our conventions. The fundamental two-form is given by $\omega_{p,m-p} = \sum_{i=1}^{p} e^{2i-1} \wedge e^{2i} - \sum_{j=p+1}^{m} e^{2j-1} \wedge e^{2j}$ or $\omega_0 = \omega_{m,0} = \sum_{i=1}^{m} e^{2i-1} \wedge e^{2i}$, respectively.*

Remark 2.26.
- *The common stabiliser of $(\langle\cdot,\cdot\rangle_{2p,2m-2p}, J_{-1})$ is $U(p, m-p) \subseteq \mathrm{GL}(m, \mathbb{C}) \subseteq \mathrm{GL}(2m, \mathbb{R})$. To compute the common stabiliser of $(\langle\cdot,\cdot\rangle_{split}, J_1)$, note that both $\left(\mathbb{R}^{2m}\right)_+$ and $\left(\mathbb{R}^{2m}\right)_-$ are isotropic with respect to $\langle\cdot,\cdot\rangle_{split}$. Thus, $\langle\cdot,\cdot\rangle_{split}$ induces a non-degenerate bilinear pairing of $\left(\mathbb{R}^{2m}\right)_+$ and $\left(\mathbb{R}^{2m}\right)_-$. For $g \in \mathrm{End}\left(\left(\mathbb{R}^{2m}\right)_+\right)$ we denote by $g^t \in \mathrm{End}\left(\left(\mathbb{R}^{2m}\right)_-\right)$ the transpose with respect to the mentioned pairing. Then the common stabiliser of $(\langle\cdot,\cdot\rangle_{split}, J_1)$ is given by*

$$\left\{ f \in \mathrm{GL}(2m, \mathbb{R}) \,\middle|\, f|_{\left(\mathbb{R}^{2m}\right)_+} = f_1,\ f|_{\left(\mathbb{R}^{2m}\right)_-} = f_1^{-t},\ f_1 \in \mathrm{GL}\left(\left(\mathbb{R}^{2m}\right)_+\right) \right\} \cong \mathrm{GL}(m, \mathbb{R}).$$

  We call this group the para-unitary group. *To unify the treatment, we set $U^{-1}(p, m-p) := U(p, m-p)$ and denote, for arbitrary $p \in \{0, \ldots, m\}$, the para-unitary group by $U^1(p, m-p)$.*
- *Every pseudo-Hermitian structure (g, J) on a $2m$-dimensional real vector space has the pair $(\langle\cdot,\cdot\rangle_{2p,2m-2p}, J_{-1})$ for some $p \in \{0, \ldots, m\}$ as model tensors and every para-Hermitian structure (g, J) has the pair $(\langle\cdot,\cdot\rangle_{split}, J_1)$ as model tensors. Hence,*

$U^\epsilon(p, m-p)$-structures are nothing but ϵ-Hermitian structures such that for $\epsilon = -1$ the pseudo-Euclidean metric g has signature $(2p, 2m-2p)$.

- *A pair $(\omega, J) \in \Lambda^2 V^* \times \mathrm{End}(V)$ of a non-degenerate two-form ω and an ϵ-complex structure J with $J^*\omega = -\epsilon\omega$ defines a pseudo-Euclidean metric on V by $g(v, w) := -\epsilon\, \omega(Jv, w)$ for $v, w \in V$. The pair (g, J) is then an ϵ-Hermitian structure on V. The construction reflects the fact that $U(p, m-p)$ is the common stabiliser of $(\omega_{p,m-p}, J_{-1})$ and $U^1(p, m-p)$ is the common stabiliser of (ω_0, J_1).*
- *We may also construct an ϵ-Hermitian structure (g, J) via a pair $(g, \omega) \in \mathrm{S}^2V^* \times \Lambda^2 V^*$ consisting of a pseudo-Euclidean metric g and a non-degenerate two-form ω such that the endomorphism $J \in \mathrm{End}(V)$, uniquely defined by $g(\cdot, J\cdot) = \omega(\cdot, \cdot)$, is an ϵ-complex structure on V. This reflects the fact that $U(p, m-p)$ is also the common stabiliser of $(\langle\cdot,\cdot\rangle_{2p,2m-2p}, \omega_{p,m-p})$ and $U^1(p, m-p)$ is the one of $(\langle\cdot,\cdot\rangle_{split}, \omega_0)$.*

The following formula plays a crucial role to get obstructions to the existence of a half-flat SU(3)-structure, as will be discussed in Chapter 6.

Lemma 2.27. *Let V be a $2m$-dimensional real vector space and (g, J) be an ϵ-Hermitian structure on V with fundamental two-form ω. Then the identity*

$$\alpha \wedge J^*\beta \wedge \omega^{m-1} = \frac{1}{m} g(\alpha, \beta)\omega^m \tag{2.17}$$

is true for all $\alpha, \beta \in V^*$.

Proof. We only have to do the calculation for the corresponding model tensors on $\mathbb{R}^{2m}$ and for $\alpha, \beta \in \{e^1, \ldots, e^{2m}\}$, which is a straightforward task. □

Now, we come to special ϵ-Hermitian structures.

Definition 2.28. *A* special ϵ-Hermitian structure (g, J, Ψ) *on a $2m$-dimensional real vector space V consists of an ϵ-Hermitian structure (g, J) and an ϵ-complex volume form Ψ for J such that $g_{\mathbb{C}}(\Psi, \Psi) = (-1)^{m-p}2^m$ for $\epsilon = -1$ and g having signature $(2p, 2m-2p)$ and such that $g_{\mathbb{C}_1}(\Psi, \Psi) = 2^m$ for $\epsilon = 1$. Here, $g_{\mathbb{C}_\epsilon}$ denotes the ϵ-*complex sesquilinear extension *of g, i.e. $g_{\mathbb{C}_\epsilon}(\psi_1 \otimes z_1, \psi_2 \otimes z_2) := z_1\overline{z_2}g(\psi_1, \psi_2)$ for $\psi_1, \psi_2 \in \Lambda^m V^*$, $z_1, z_2 \in \mathbb{C}_\epsilon$. For $\epsilon = -1$, we also call (g, J, Ψ) a* special pseudo-Hermitian structure *and for $\epsilon = 1$ we say that (g, J, Ψ) is a* special para-Hermitian structure.

Remark 2.29. *Note that in the literature there is often a slightly different definition of a special ϵ-Hermitian structure in the sense that the condition on the norm of Ψ is dropped. Then a special ϵ-Hermitian structure in our sense is called* normalised.

We use the following standard examples of special ϵ-Hermitian structures on $\mathbb{R}^{2m}$ as model tensors for the corresponding G-structures.

Example 2.30. *A special pseudo-Hermitian structure on* $\mathbb{R}^{2m}$ *of signature* $(2p, 2m-2p)$ *is given by* $(\langle\cdot,\cdot\rangle_{2p,2m-2p}, J_{-1}, \Psi_{-1})$ *and a special para-Hermitian structure on* $\mathbb{R}^{2m}$ *is given by* $(\langle\cdot,\cdot\rangle_{split}, J_1, \Psi_1)$. *Here,* Ψ_ϵ, $\epsilon \in \{-1,1\}$, *is the model* ϵ*-complex volume form on* $\mathbb{R}^{2m}$ *given in Equation (2.9) and the other two tensors in each triple form the model pseudo- and para-Hermitian structures given in Example 2.25.*

Lemma 2.31. *Let* V *be a real* $2m$*-dimensional vector space. A triple* $(g, J, \Psi) \in S^2V^* \times \mathrm{End}(V) \times \Lambda^m V^*_{\mathbb{C}_\epsilon}$ *is a special* ϵ*-Hermitian structure if and only if* (g, J, Ψ) *has one of the triples in Example 2.30 as model tensors. Moreover, if* (g, J) *is an* ϵ*-Hermitian structure on* V *and* $\Psi \in \Lambda^{(m,0)}V^*$ *an* ϵ*-complex volume form with respect to* J*, then* (g, J, Ψ) *is a special* ϵ*-Hermitian structure if and only if*

$$\phi(\Psi) = \begin{cases} (-1)^{m-p}2^{m-2}\phi(\omega) & \text{if } \epsilon = -1 \text{ and } \mathrm{sign}(g) = (2p, 2m-2p), \\ (-1)^{l}2^{m-2}\phi(\omega) & \text{if } \epsilon = 1 \text{ and } m = 2l-1,\ 2l. \end{cases} \tag{2.18}$$

Here, ω *is the fundamental two-form associated to* (g, J)*.*

Proof. Remark 2.26 gave us model tensors for ϵ-Hermitian structures. Since $\Lambda^{(m,0)}V^*$ is one-dimensional, one may check that the condition on the ϵ-complex norm of Ψ in the definition of a special ϵ-Hermitian structure exactly means that special ϵ-Hermitian structures are those which have model tensors as in Example 2.30. Hence, we only have to check Equation (2.18) for the model tensors, which is a straightforward task. $\square$

Definition 2.32. *The* special ϵ-unitary group $\mathrm{SU}^\epsilon(p, m-p)$ *is for* $\epsilon = -1$ *the stabiliser of* $(\langle\cdot,\cdot\rangle_{2p,2m-2p}, J_{-1}, \Psi_{-1})$ *and for* $\epsilon = 1$ *and all* $p \in \{0,\ldots,m\}$ *it is the stabiliser of* $(\langle\cdot,\cdot\rangle_{split}, J_1, \Psi_1)$. $\mathrm{SU}^{-1}(p, m-p)$ *is equal to the usual special unitary group* $\mathrm{SU}(p, m-p)$ *and*

$$\begin{aligned} \mathrm{SU}^1(p, m-p) &= \left\{ f \in \mathrm{GL}(2m,\mathbb{R}) \,\middle|\, f|_{(\mathbb{R}^{2m})_+} = f_1,\ f|_{(\mathbb{R}^{2m})_-} = f_1^{-t},\ f_1 \in \mathrm{SL}\left(\left(\mathbb{R}^{2m}\right)_+\right) \right\} \\ &\cong \mathrm{SL}(m,\mathbb{R}), \end{aligned}$$

where the transpose is with respect to the non-degenerate bilinear pairing between $\left(\mathbb{R}^{2m}\right)_+$ *and* $\left(\mathbb{R}^{2m}\right)_-$ *induced by* $\langle\cdot,\cdot\rangle_{split}$. *By Lemma 2.31, special* ϵ*-Hermitian structures are nothing but* $\mathrm{SU}^\epsilon(p, m-p)$-structures *if we use the just mentioned tensors on* $\mathbb{R}^{2m}$ *as model tensors. Hence, we will use the terms* $\mathrm{SU}^\epsilon(p, m-p)$*-structure and* ϵ*-Hermitian structure interchangeably in the following. Moreover, we will also speak of an* $\mathrm{SL}(m,\mathbb{R})$-structure *instead of a* $\mathrm{SU}^1(p, m-p)$*-structure.*

If (g, J, Ψ) is a special ϵ-Hermitian structure, then Proposition 2.18 tells us that we may reconstruct J from Ψ. Hence, Remark 2.26 implies that (g, J, Ψ) can be reconstructed from (ω, Ψ), ω being the fundamental two-form. If m is odd, Proposition 2.21 shows that we can do better and only need to know $(\omega, \mathrm{Re}(\Psi))$. If we start abstractly with a pair

$(\omega, \Psi) \in \Lambda^2 V^* \times \Lambda^m V^*_{\mathbb{C}_\epsilon}$, ω non-degenerate and Ψ an ϵ-complex volume form, then the induced J has to fulfil $J^*\omega = -\epsilon\omega$. In Proposition 2.33, we show that this condition is equivalent to $\omega \wedge \Psi = 0$ and the analogue statement is true for the reconstruction in the case of odd m. For odd m, we restrict ourselves to pseudo-Hermitian structures (g, J, Ψ) of signature $(2m-4k, 4k)$, $k \in \mathbb{N}$. Note that then $\left(-g, J, \overline{\Psi}\right)$ is a pseudo-Hermitian structure of signature $(4k, 2m-4k)$ and we get all missing cases via this assignment.

Proposition 2.33. *Let V be a $2m$-dimensional real vector space.*

(a) Let $(\omega, \Psi) \in \Lambda^2 V^* \times \Lambda^m V^*_{\mathbb{C}_\epsilon}$ be a pair of a non-degenerate two-form ω and an ϵ-complex volume form Ψ such that $\omega \wedge \Psi = 0$ and such that Equation (2.18) is true. Then there is a unique special ϵ-Hermitian structure (g, J, Ψ) such that ω is the associated fundamental two-form, where J is defined by Equation (2.10) and $g := -\epsilon\omega(J\cdot, \cdot)$. Moreover, all special ϵ-Hermitian structures arise that way. Furthermore, if $z \in \mathbb{C}_\epsilon$ with $z\overline{z} = 1$, then $(\omega, z\Psi)$ induces the same ϵ-Hermitian structure as (ω, Ψ).

(b) Let $m \geq 3$ be odd and $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^m V^*$ be a pair of a non-degenerate two-form ω and an ϵ-complex volume form $\rho \in \Lambda^m V^*$ such that

$$\omega \wedge \rho = 0, \quad \phi(\rho) = 2^{m-2}\phi(\omega), \tag{2.19}$$

where we orient V via $\phi(\omega)$ if $\epsilon = -1$ and via $(-1)^l\phi(\omega)$ if $\epsilon = 1$ and $m = 2l-1$. Then there exists a unique special ϵ-Hermitian structure (g, J_ρ, Ψ) such that $\mathrm{Re}(\Psi) = \rho$, such that Ψ induces the given orientation and such that ω is the associated fundamental two-form. Moreover, if $\epsilon = -1$, then the signature of g is equal to $(2m-4k, 4k)$ for some $k \in \mathbb{N}$. The ϵ-complex structure J_ρ is defined by Equation (2.16), $g := -\epsilon\omega(J\cdot, \cdot)$ and Ψ is defined by Equation (2.14). Moreover, all special ϵ-Hermitian structures with the additional signature assumption for $\epsilon = -1$ arise this way.

(c) Let $m \geq 3$ be odd, $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^m V^*$ as in (b) and $\alpha \in \mathbb{R}$. If $\epsilon = -1$, then $(\omega, \cos(\alpha)\rho + \sin(\alpha)J^*_\rho\rho)$ induces the same pseudo-Hermitian structure as (ω, ρ). If $\epsilon = 1$, then $(\omega, \cosh(\alpha)\rho + \sinh(\alpha)J^*_\rho\rho)$ induces the same para-Hermitian structure as (ω, ρ).

Proof. (a) For the first part of the statement, the argument directly above Proposition 2.33 shows that we only have to check that $J^*\omega = -\epsilon\omega$ for the ϵ-complex structure J induced by Ψ is equivalent to $\omega \wedge \Psi = 0$. However, the decomposition $\Lambda^2 V^* = [[\Lambda^{2,0}V^*]] \oplus [\Lambda^{1,1}V^*]$ given in (2.8) shows that $J^*\omega = -\epsilon\omega$ exactly when $\omega \in [\Lambda^{1,1}V^*]$. Moreover, Ψ is a $(m, 0)$-form with respect to J and so $\omega \wedge \Psi = 0$ if and only if the $(0, 2)$-part of ω vanishes which, by Equation (2.8), implies that the $(0, 2)$-part of ω also vanishes. Hence, $\omega \wedge \Psi = 0$ if and only if $\omega \in [\Lambda^{1,1}V^*]$ and the first part follows. For the second part note that for $z \in \mathbb{C}_\epsilon$ with $z\overline{z} = 1$ we get $\phi(z\Psi) = \phi(\Psi)$ and so Equation (2.18) is fulfilled for $(\omega, z\Psi)$. The identity $v \lrcorner \Psi \wedge \Psi = 0$ implies $v \lrcorner \operatorname{Im}(\Psi) \wedge \operatorname{Re}(\Psi) = -v \lrcorner \operatorname{Re}(\Psi) \wedge \operatorname{Im}(\Psi)$ and $v \lrcorner \operatorname{Im}(\Psi) \wedge \operatorname{Im}(\Psi) = -\epsilon\, v \lrcorner \operatorname{Re}(\Psi) \wedge \operatorname{Re}(\Psi)$ for all $v \in V$. Using these formulas, Equation (2.10) yields that $z\Psi$ induces the same ϵ-complex structure as Ψ. Thus, $(\omega, z\Psi)$ induces the same ϵ-Hermitian structure as (ω, Ψ).

(b) Let $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^m V^*$ as in the statement. By Equation (2.14), the ϵ-complex structure Ψ induced by ρ is given by $\Psi = \rho + i_\epsilon \epsilon^l J_\rho^* \rho$ and it fulfils $\phi(\Psi) = \phi(\rho)$ by Proposition 2.21. Using the validity of the assertion in (a), we only have to check $\omega \wedge J_\rho^* \rho = 0$. Therefore, note that we have the identity $(\alpha + i_\epsilon \epsilon J_\rho^* \alpha) \wedge (\rho + i_\epsilon \epsilon^l J_\rho^* \rho) = 0$ for all $\alpha \in V^*$ since $(\alpha + i_\epsilon \epsilon J_\rho^* \alpha)$ is a $(1,0)$-form. Taking the imaginary part of this identity, we have $J_\rho^* \alpha \wedge \rho = -\epsilon^{l+1} \alpha \wedge J_\rho^* \rho$ and so

$$0 = J_\rho^* \alpha \wedge \rho \wedge \omega = -\epsilon^{l+1} \alpha \wedge J_\rho^* \rho \wedge \omega$$

for all $\alpha \in V^*$. Thus, $\omega \wedge J_\rho^* \rho = 0$.

(c) Follows directly from (a) and (b).

□

We also call a pair $(\omega, \Psi) \in \Lambda^2 V^* \times \Lambda^m V^*_{\mathbb{C}_\epsilon}$ as in Proposition 2.33 (a) a *special ϵ-Hermitian structure* or an *$\mathrm{SU}^\epsilon(p, m-p)$-structure* and if $\epsilon = 1$ we also speak of an *$\mathrm{SL}(m, \mathbb{R})$-structure*. Similarly, a pair $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^m V^*$ as in Proposition 2.33 (b) is called a *special ϵ-Hermitian structure* or an *$\mathrm{SU}^\epsilon(p, m-p)$-structure* and if $\epsilon = 1$ we also call (ω, ρ) an *$\mathrm{SL}(m, \mathbb{R})$-structure*.

We end the section by looking at the case $m = 3$. Then ϵ-complex volume forms $\rho \in \Lambda^3 V^*$ are exactly the stable forms by Proposition 2.22 and are characterised by the condition $\epsilon\lambda(\rho) > 0$.

Corollary 2.34. *Let V be a six-dimensional real vector space and let $\epsilon \in \{-1, 1\}$ be given. Then a pair (ω, ρ) is an $\mathrm{SU}^\epsilon(p, 3-p)$-structure for some $p \in \{1, 3\}$ if and only if both ω and ρ are stable, $\epsilon\lambda(\rho) > 0$, $\omega \wedge \rho = 0$ and $\phi(\rho) = 2\phi(\omega)$, where we use the orientation induced by ω to compute $\phi(\rho)$.*

2.4 G_2^ϵ-structures

In this section, we consider G_2^ϵ-structures and $(\mathrm{G}_2)_{\mathbb{C}}$-structures on vector spaces. Recall that $\mathrm{G}_2^1 = \mathrm{G}_2^*$ and $\mathrm{G}_2^{-1} = \mathrm{G}_2$, cf. Definition 1.19. We discuss well-known basic properties of these structures and refer to, e.g., [Br1], [J3] and [CLSS] for more background. Moreover, we also prove some results which are, to the best of the author's knowledge, not written down explicitly in the literature. All of these results are easy to obtain but turn out to be very useful for both getting obstructions to and proving the existence of cocalibrated G_2^ϵ-structures and cocalibrated $(G_2)_{\mathbb{C}}$-structures on Lie algebras in the Chapters 4 and 5.

By Lemma 1.5, G_2^ϵ-structures on a real seven-dimensional vector space V may equivalently be described as tensors $T \in T^{r,s}V$ having model tensor $S \in T^{r,s}\mathbb{R}^7$, with a tensor S whose stabiliser in $\mathrm{GL}(7,\mathbb{R})$ is equal to G_2^ϵ. In Proposition 1.24 we identified such a tensor S, namely the three-form $\varphi_\epsilon \in \Lambda^3\left(\mathbb{R}^7\right)^*$ defined in Equation (1.4). Moreover, Proposition 1.24 tells us that the stabiliser in $\mathrm{SL}(7,\mathbb{C})$ of the complex three-form $\varphi_C \in \Lambda^3\left(\mathbb{C}^7\right)^*$ is equal to $(G_2)_{\mathbb{C}}$. These remarks lead to the following definitions.

Definition 2.35.

- *Let V be a seven-dimensional real vector space. A three-form $\varphi \in \Lambda^3 V^*$ is called* G_2^ϵ-structure *for some $\epsilon \in \{-1,1\}$ if φ has the model tensor $\varphi_\epsilon \in \Lambda^3\left(\mathbb{R}^7\right)^*$ defined in Equation (1.4), i.e. if there exists an ordered basis $(f_1,\ldots,f_7)$ of V such that in the dual ordered basis $\left(f^1,\ldots,f^7\right)$ we have*

$$\varphi = f^{123} - \epsilon\left(f^{145} + f^{167} + f^{246} - f^{257} - f^{347} - f^{356}\right). \tag{2.20}$$

- *Let W be a seven-dimensional complex vector space. A pair $(\varphi,\mathrm{vol}) \in \Lambda^3W^* \times \Lambda^7W^*$ is called* $(G_2)_{\mathbb{C}}$-structure *if (φ,vol) is commonly equivalent to $(\varphi_C,\mathrm{vol}_0) \in \Lambda^3\left(\mathbb{C}^7\right)^* \times \Lambda^7\left(\mathbb{C}^7\right)^*$ with φ_C defined by Equation (1.4) and $\mathrm{vol}_0 := e^{1\ldots7} \in \Lambda^7\left(\mathbb{C}^7\right)^*$, i.e. if there exists an ordered basis $(f_1,\ldots,f_7)$ of W such that in the dual ordered basis $\left(f^1,\ldots,f^7\right)$ we have*

$$\varphi = f^{123} + f^{145} + f^{167} + f^{246} - f^{257} - f^{347} - f^{356},\ \mathrm{vol} = f^{1\ldots7}. \tag{2.21}$$

Remark 2.36.

- *Note that the common stabiliser of $(\varphi_C,\mathrm{vol}_0) \in \Lambda^3\left(\mathbb{C}^7\right)^* \times \Lambda^7\left(\mathbb{C}^7\right)^*$ is, in fact, $(G_2)_{\mathbb{C}} \subseteq \mathrm{GL}(7,\mathbb{C})$ by Proposition 1.24. We may embed $(G_2)_{\mathbb{C}}$ into $\mathrm{GL}(14,\mathbb{R})$ via the canonical identification of $\mathbb{C}^7$ with $\left(\mathbb{R}^{14},J_{-1}\right)$. In this way, if $(\varphi,\mathrm{vol}) \in \Lambda^3W^* \times \Lambda^7W^*$ is a $(G_2)_{\mathbb{C}}$-structure on the complex seven-dimensional vector space W, then $(\varphi,\mathrm{vol},m_i) \in \Lambda^3W_{\mathbb{R}}^* \otimes \mathbb{C} \times \Lambda^7W_{\mathbb{R}}^* \otimes \mathbb{C} \times \mathrm{End}(W_{\mathbb{R}})$ is a $(G_2)_{\mathbb{C}}$-structure in the sense of Section 1.1 on the real 14-dimensional vector space $W_{\mathbb{R}}$ via the model tensors $(\varphi_C,\mathrm{vol}_0,J_{-1}) \in \Lambda^3\left(\mathbb{R}^{14}\right)^* \otimes \mathbb{C} \times \Lambda^7\left(\mathbb{R}^{14}\right)^* \otimes \mathbb{C} \times \mathrm{End}(\mathbb{R}^{14})$. Here, $m_i : W_{\mathbb{R}} \to W_{\mathbb{R}}$ is the multiplication with i.*
- *If $\varphi \in \Lambda^3W^*$ is equivalent to $\varphi_C \in \Lambda^3\left(\mathbb{C}^7\right)^*$, then, by choosing an* adapted basis *$(f_1,\ldots,f_7)$ for φ, i.e. an ordered basis of W as in Equation (2.21), we may define a* seven-form *$\mathrm{vol} \in \Lambda^7W^*$ such that (φ,vol) is a $(G_2)_{\mathbb{C}}$-structure, e.g. by setting $\mathrm{vol} := f^{1\ldots7}$. This construction depends on the chosen adapted basis $(f_1,\ldots,f_7)$. Namely, if we choose a different adapted basis $(g_1,\ldots,g_7)$, then we get $g^{1\ldots7} = \xi^7 f^{1\ldots7} = \xi\mathrm{vol}$ for some third root of unity $\xi \in \mathbb{C}$.*

In Remark 1.20 we noted that $G_2 \subseteq SO(7)$ and $G_2^* \subseteq SO(3,4)$. Hence, a G_2-structure (resp. G_2^*-structure) $\varphi \in \Lambda^3 V^*$ induces a Euclidean metric (resp. pseudo-Euclidean metric of signature $(3,4)$) g_φ and a metric volume form $\phi(\varphi)$ on V. We give the concrete constructions.

Definition 2.37. *Let V be a seven-dimensional real vector space and $\varphi \in \Lambda^3 V^*$ be an arbitrary three-form. We define a symmetric bilinear map $b_\varphi : V \otimes V \to \Lambda^7 V^*$ by*

$$b_\varphi(v,w) := \frac{1}{6}(v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi \tag{2.22}$$

for $v, w \in V$. b_φ may be considered as a linear map $V \to V^* \otimes \Lambda^7 V^*$ and so $\det(b_\varphi) \in \left(\Lambda^7 V^*\right)^{\otimes 9}$. We set

$$\phi : \Lambda^3 V^* \to \Lambda^7 V^*, \ \phi(\varphi) := \det(b_\varphi)^{\frac{1}{9}}. \tag{2.23}$$

If $\phi(\varphi) \neq 0$, we define a symmetric bilinear form g_φ by

$$g_\varphi(v,w)\phi(\varphi) = b_\varphi(v,w) \tag{2.24}$$

for $v, w \in V$.

The map ϕ defined in Equation (2.23) has the same properties as the map with the same name in Proposition 1.37. Therefore, note that by Theorem 1.35 the set of all stable three-forms on V is exactly the set of all G_2- and G_2^*-structures on V.

Lemma 2.38. *Let V be a seven-dimensional real vector space.*

(a) The map $\phi : \Lambda^3 V^* \to \Lambda^7 V^*$ defined in Equation (2.23) is $GL(V)$-equivariant and $\phi^{-1}(0)$ is the set of all non-stable elements in $\Lambda^3 V^*$. That means $\phi(\varphi) \neq 0$ for a three-form $\varphi \in \Lambda^3 V^*$ if and only if φ is a G_2^ϵ-structure for some $\epsilon \in \{-1, 1\}$.

(b) If $\varphi \in \Lambda^3 V^*$ is a G_2-structure, then g_φ is a Euclidean metric on V, $\phi(\varphi)$ is a metric volume form for g_φ and each adapted basis $(f_1, \ldots, f_7)$ of φ is an oriented orthonormal basis for $(g_\varphi, \phi(\varphi))$.

(c) If $\varphi \in \Lambda^3 V^*$ is a G_2^*-structure, then g_φ is a pseudo-Euclidean metric of signature $(3,4)$ on V, $\phi(\varphi)$ is a metric volume form for g_φ and each adapted basis $(f_1, \ldots, f_7)$ of φ is an oriented orthonormal basis for $(g_\varphi, \phi(\varphi))$ such that $g(f_i, f_i) = 1$ for $i = 1, 2, 3$ and $g(f_j, f_j) = -1$ for $j = 4, 5, 6, 7$.

Proof. A proof of part (a) may be found in [CLSS]. Alternatively, one may prove (a) by computing $\phi(\varphi)$ for each three-form φ in Table 1.1. Part (b) and part (c) follow by straightforward calculations in an adapted basis. ∎

Similarly, since $(G_2)_{\mathbb{C}} \subseteq SO(7,\mathbb{C})$, a $(G_2)_{\mathbb{C}}$-structure $(\varphi, \mathrm{vol}) \in \Lambda^3 W^* \times \Lambda^7 W^*$ induces a non-degenerate symmetric complex bilinear form g_φ. Moreover, $SO(m,\mathbb{C}) \subseteq SO(m,m)$ and so a $(G_2)_{\mathbb{C}}$-structure also induces a pseudo-Euclidean metric g_{split} of split signature on $W_{\mathbb{R}}$. The concrete constructions are given in the following lemma.

Lemma 2.39. *Let V be a seven-dimensional real vector space and $W := V_{\mathbb{C}}$ its complexification.*

(a) A $(G_2)_{\mathbb{C}}$-structure $(\varphi, vol) \in \Lambda^3 W^* \times \Lambda^7 W^*$ induces a non-degenerate symmetric complex bilinear form g_φ on W by

$$g_\varphi(v,w)\mathrm{vol} := \frac{1}{6}(v \lrcorner \varphi) \wedge (w \lrcorner \varphi) \wedge \varphi \tag{2.25}$$

and a pseudo-Euclidean metric g_{split} of split signature $(7,7)$ on $W_{\mathbb{R}}$ by $g_{split} := \mathrm{Re}(g_\varphi)$. If $(f_1, \ldots, f_7)$ is an adapted basis for $(\varphi, \mathrm{vol}_{\mathbb{C}})$, then $g_\varphi(f_j, f_k) = \delta_{jk}$ for $j,k \in \{1,\ldots,7\}$ and $f_1, if_1, \ldots, f_7, if_7$ is an orthonormal basis for g_{split} such that $g_{split}(f_j,f_j) = 1$ and $g_{split}(if_j, if_j) = -1$ for $j = 1,\ldots,7$.

(b) If φ is a G_2^ϵ-structure on V, then $(\varphi_{\mathbb{C}}, \phi(\varphi)_{\mathbb{C}}) \in \Lambda^3 W^* \times \Lambda^7 W^*$ is a $(G_2)_{\mathbb{C}}$-structure on $W = V_{\mathbb{C}}$, where $(\varphi_{\mathbb{C}}, \phi(\varphi)_{\mathbb{C}})$ are the complex-linear extensions of $(\varphi, \phi(\varphi))$. Moreover, $g_{\varphi_{\mathbb{C}}}$ is the complex-linear extension of g_φ.

Proof. All parts follow by Lemma 2.38 or by simple calculations. $\square$

Remark 2.40. *Note that for a $(G_2)_{\mathbb{C}}$-structure (φ, vol) the non-degenerate symmetric complex bilinear form g_φ also depends on* vol *although we suppressed this dependence in the notation.*

We like to note the following properties of the stabiliser group of a G_2^ϵ-structure and of a $(G_2)_{\mathbb{C}}$-structure.

Lemma 2.41. *(a) The stabiliser of a G_2-structure $\varphi \in \Lambda^3 V^*$ on a seven-dimensional real vector space V acts transitively on the set of all lines in V and also on the set of all six-dimensional subspaces of V, respectively.*

(b) The stabiliser of a G_2^*-structure $\varphi \in \Lambda^3 V^*$ on a seven-dimensional real vector space V acts transitively on the set of positive lines, null lines and negative lines in V, respectively. It also acts transitively on the set of all six-dimensional subspaces of V of signature $(3,3)$ and $(2,4)$, respectively. Moreover, it acts transitively on the set of all degenerate six-dimensional subspaces of V.

(c) The stabiliser of a $(G_2)_{\mathbb{C}}$-structure $(\tilde{\varphi}, \mathrm{vol}) \in \Lambda^3 W^* \times \Lambda^7 W^*$ on a seven-dimensional complex vector space W acts transitively on the set of all non-null lines and on

the set of all null-lines. It also acts transitively on the set of all six-dimensional non-degenerate subspaces of V and of all six-dimensional degenerate subspaces, respectively.

Proof. The statements about transitive actions on lines with fixed signature can all be found in [Br1]. Since G_2^ϵ is contained in the corresponding orthogonal group, we get the transitivity of the action on all mentioned classes of non-degenerate six-dimensional subspaces by considering orthogonal complements of the corresponding non-null vectors. The transitivity on the class of all degenerate six-dimensional subspaces U in (b) and (c) follows from the fact that $U = u^\perp$ for a degenerate $u \in U$, i.e. an element in U such that $g(z,u) = 0$ for all $z \in U$. □

Remark 2.42. *One can do better than Lemma 2.41 and show that G_2 acts transitively on the unit sphere in $\mathbb{R}^7$ [Bo]. For each $\delta \in \{-1,1\}$, the transitivity of the action of G_2^* on the pseudo-sphere $S^\delta := \left\{v \in \mathbb{R}^7 | \langle v,v\rangle_{3,4} = \delta\right\} \subseteq \mathbb{R}^7$ follows using Lemma 2.41 since the linear automorphism of $\mathbb{R}^7$, defined by $g(e_{2i-1}) := -e_{2i-1}$ for $i = 1,2,3,4$ and $g(e_{2j}) := e_{2j}$ for $j = 1,2,3$, is in G_2^* and maps the vector e_1 of length -1 to $-e_1$ and the vector e_5 of length 1 to $-e_5$.*

By Lemma 2.38, a G_2^ϵ-structure $\varphi \in \Lambda^3 V^*$ on a seven-dimensional real vector space V induces a Hodge star operator $\star_\varphi$. Similarly, Lemma 2.39 shows that a $(\mathrm{G}_2)_\mathbb{C}$-structure $(\varphi, \mathrm{vol}) \in \Lambda^3 W^* \times \Lambda^7 W^*$ on a seven-dimensional complex vector space W induces a Hodge star operator $\star_\varphi$, where we again suppressed the dependence of $\star_\varphi$ on vol in the notation. The next lemma gives us a concrete description of the Hodge duals $\star_\varphi \varphi$ in all cases.

Lemma 2.43. *(a) If $\varphi \in \Lambda^3 V^*$ is a G_2^ϵ-structure on a seven-dimensional real vector space V, then the Hodge dual $\star_\varphi \varphi \in \Lambda^4 V^*$ is given in an adapted basis $f_1, \ldots, f_7$ for φ by*

$$\star_\varphi \varphi = \epsilon \left(f^{1247} + f^{1256} + f^{1346} - f^{1357} - f^{2345} - f^{2367}\right) + f^{4567} \tag{2.26}$$

and so has model tensor $\star_{\varphi_\epsilon} \varphi_\epsilon \in \Lambda^4 \left(\mathbb{R}^7\right)^*$. The dual three-form $\widehat{\varphi}$ of φ is $\widehat{\varphi} = \frac{1}{3} \star_\varphi \varphi$. The stabiliser of $\star_{\varphi_\epsilon}\varphi_\epsilon$ is $\mathrm{G}_2^\epsilon \times \{I_7, -I_7\} \cong \mathrm{G}_2^\epsilon \times \mathbb{Z}_2$.

(b) If $(\varphi, \mathrm{vol}) \in \Lambda^3 W^* \times \Lambda^7 W^*$ is a $(\mathrm{G}_2)_\mathbb{C}$-structure on a seven-dimensional complex vector space W, then the Hodge dual $\star_\varphi \varphi \in \Lambda^4 W^*$ is given in an adapted basis $f_1, \ldots, f_7$ for φ by

$$\star_\varphi \varphi = -f^{1247} - f^{1256} - f^{1346} + f^{1357} + f^{2345} + f^{2367} + f^{4567} \tag{2.27}$$

and so has model tensor $\star_{\varphi_C} \varphi_C \in \Lambda^4 \left(\mathbb{C}^7\right)^*$. The stabiliser of $\star_{\varphi_C}\varphi_C$ is $(\mathrm{G}_2)_\mathbb{C} \times \left\{\xi I_7 | \xi \in \mathbb{C}, \xi^4 = 1\right\} \cong (\mathrm{G}_2)_\mathbb{C} \times \mathbb{Z}_4$. Moreover, each stable four-form on W is equivalent to the four-form $\star_{\varphi_C}\varphi_C$ on $\mathbb{C}^7$.

Proof. (a) The determination of the Hodge dual is a straightforward computation. Proposition 1.33 shows that $\mathrm{GL}^+(7,\mathbb{R})_{\star_{\varphi_\epsilon}\varphi_\epsilon} = \mathrm{G}_2^\epsilon$. Since $(-I_7)^* \star_{\varphi_\epsilon} \varphi_\epsilon = \star_{\varphi_\epsilon}\varphi_\epsilon$, we have $\mathrm{GL}(7,\mathbb{R})_{\star_{\varphi_\epsilon}\varphi_\epsilon} = \mathrm{G}_2^\epsilon \times \{I_7, -I_7\}$. The identity $\widehat{\varphi} = \frac{1}{3} \star_\varphi \varphi$ only has to be shown for $\varphi = \varphi_\epsilon$. Proposition 1.37 tells us that $\mathrm{GL}^+(V)_{\hat{\varphi}_\epsilon} = \mathrm{G}_2^\epsilon$ and by [Br1] the only G_2^ϵ-invariant four-forms are the multiples of $\star_{\varphi_\epsilon}\varphi_\epsilon$. Hence, $\hat{\varphi}_\epsilon = \lambda \star_{\varphi_\epsilon} \varphi_\epsilon$ for some $\lambda \in \mathbb{R}^*$ and from Equation (1.9) we get $\lambda = \frac{1}{3}$.

(b) The form of the Hodge dual is a straightforward calculation. The rest follows directly from Proposition 1.33 and Theorem 1.35.

□

The stabilisers given in Lemma 2.43 show that a four-form $\Psi \in \Lambda^4 V^*$ on a real oriented seven-dimensional vector space V with model tensor $\star_{\varphi_\epsilon}\varphi_\epsilon$ induces a Euclidean metric g_Ψ if $\epsilon = -1$, a pseudo-Euclidean metric g_Ψ of signature $(3,4)$ if $\epsilon = 1$, a metric volume form $\phi(\Psi)$ and then also a G_2^ϵ-structure φ with $\star_\varphi\varphi = \Psi$. We now define the corresponding objects.

Definition 2.44. *Let V be a real oriented seven-dimensional vector space and $\Psi \in \Lambda^4 V^*$ be an arbitrary four-form. Let $\kappa : \Lambda^4 V^* \to \Lambda^3 V \otimes \Lambda^7 V^*$ be the natural isomorphism between these spaces. Define a symmetric bilinear map $b_\Psi : V^* \otimes V^* \to \Lambda^7 V \otimes \left(\Lambda^7 V^*\right)^{\otimes 3} \cong \left(\Lambda^7 V^*\right)^{\otimes 2}$ by*

$$b_\Psi(\alpha,\beta) := \frac{1}{6}(\alpha \lrcorner \kappa(\Psi)) \wedge (\beta \lrcorner \kappa(\Psi)) \wedge \kappa(\Psi) \tag{2.28}$$

for $\alpha, \beta \in V^*$. b_Ψ may be considered as a linear map $V^* \to V \otimes \left(\Lambda^7 V^*\right)^{\otimes 2}$. Then $\det(b_\Psi) \in \left(\Lambda^7 V^*\right)^{\otimes 12}$. We set

$$\phi : \Lambda^4 V^* \to \Lambda^7 V^*,\ \phi(\Psi) := |\det(b_\Psi)|^{\frac{1}{12}}. \tag{2.29}$$

If $\phi(\Psi) \neq 0$, we define a symmetric bilinear form g_Ψ on V^* by

$$g_\Psi(\alpha,\beta)\phi(\Psi)^{\otimes 2} = \mathrm{sgn}(\det(b_\Psi))b_\Psi(\alpha,\beta) \tag{2.30}$$

for $\alpha, \beta \in V^*$. Note that g_Ψ does not depend on the chosen orientation.

The map $\phi : \Lambda^4 V^* \to \Lambda^7 V^*$ defined in Equation (2.29) is a map as in Proposition 1.37.

Lemma 2.45. *Let V be a seven-dimensional oriented real vector space. Then*

(a) The map $\phi : \Lambda^4 V^* \to \Lambda^7 V^*$ defined by Equation (2.29) is $\mathrm{GL}^+(V)$-equivariant and $\phi^{-1}(0)$ is the set of all non-stable four-forms on V.

(b) Each stable four-form $\Psi \in \Lambda^4 V^*$ on V is equivalent to one and only one of the four-forms $\star_{\varphi_1}\varphi_1, -\star_{\varphi_1}\varphi_1, \star_{\varphi_{-1}}\varphi_{-1}, -\star_{\varphi_{-1}}\varphi_{-1}$ on $\mathbb{R}^7$. More exactly, if Ψ is stable and $\delta \in \{-1,1\}$, then:

(i) $\Psi \in \Lambda^4 V^*$ *has model tensor* $\delta \star_{\varphi_1} \varphi_1$ *if and only if* g_Ψ *is of signature* $(3,4)$ *and* $\delta \cdot \operatorname{sgn}(\det(b_\Psi)) = 1$. *If this is the case, then each adapted basis* $(f_1, \ldots, f_7)$ *for* Ψ *is an orthonormal basis for* g_Ψ *with* $g_\Psi(f_j, f_j) = 1$ *for* $j = 1,2,3$ *and* $g_\Psi(f_j, f_j) = -1$ *for* $j = 4,5,6,7$.

(ii) $\Psi \in \Lambda^4 V^*$ *has model tensor* $\delta \star_{\varphi_{-1}} \varphi_{-1}$ *if and only if* g_Ψ *is positive definite and* $\delta \cdot \operatorname{sgn}(\det(b_\Psi)) = 1$. *If this is the case, then each adapted basis* $(f_1, \ldots, f_7)$ *for* Ψ *is an orthonormal basis for* g_Ψ.

(c) Let $\Psi \in \Lambda^4 V^*$ *be stable, denote by* $\star_\Psi$ *the induced Hodge star operator and set* $\varphi := \star_\Psi \Psi \in \Lambda^3 V^*$. *Then* φ *is a* G_2*-structure if* g_Ψ *is positive definite and a* G_2^**-structure if* g_Ψ *is of signature* $(3,4)$. *Moreover,* $g_\varphi = g_\Psi$, $\phi(\Psi)$ *is a metric volume form,* $\phi(\varphi) = \operatorname{sgn}(\det(b_\Psi))\phi(\Psi)$ *and so* $\Psi = \operatorname{sgn}(\det(b_\Psi)) \star_\varphi \varphi$. *Furthermore, the dual three-form* $\widehat{\Psi} \in \Lambda^3 V^*$ *is given by* $\widehat{\Psi} = \frac{1}{4}\varphi$.

Proof. (a) If $\mathrm{vol} \in \Lambda^7 V^* \backslash \{0\}$ is fixed, then Proposition 1.33 and Remark 1.28 show that Ψ is stable if and only if $X \in \Lambda^3 V$ with $X \otimes \mathrm{vol} = \kappa(\Psi)$ is stable. Hence, the assertion is an immediate consequence of Lemma 2.38.

(b) By Proposition 1.33 and Theorem 1.35 each stable four-form Φ on V is equivalent to one of the four-forms $\star_{\varphi_1}\varphi_1, - \star_{\varphi_1} \varphi_1, \star_{\varphi_{-1}}\varphi_{-1}$ or $- \star_{\varphi_{-1}} \varphi_{-1}$. $\epsilon_1 \star_{\varphi_1} \varphi_1$ cannot be equivalent to $\epsilon_2 \star_{\varphi_{-1}} \varphi_{-1}$ for any $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ since the stabilisers are not isomorphic. Moreover, $\star_{\varphi_\epsilon}\varphi_\epsilon$ is not equivalent to $- \star_{\varphi_\epsilon} \varphi_\epsilon$ for $\epsilon \in \{-1,1\}$. Assume the contrary, i.e. there exists $g \in \mathrm{GL}(7, \mathbb{R})$ such that $g. \star_{\varphi_\epsilon} \varphi_\epsilon = - \star_{\varphi_\epsilon} \varphi_\epsilon$. Then $b_{g.\star_{\varphi_\epsilon}\varphi_\epsilon} = -b_{\star_{\varphi_\epsilon}\varphi_\epsilon}$ and so

$$\det(g)^{-12} \det(b_{\star_{\varphi_\epsilon}\varphi_\epsilon}) = \det(g.b_{\star_{\varphi_\epsilon}\varphi_\epsilon}) = \det(b_{g.\star_{\varphi_\epsilon}\varphi_\epsilon}) = -\det(b_{\star_{\varphi_\epsilon}\varphi_\epsilon}) \in \left(\Lambda^7 V^*\right)^{\otimes 12},$$

which is a contradiction since $\det(g)^{12} > 0$ and $\det(b_{\star_{\varphi_\epsilon}\varphi_\epsilon}) \neq 0$ by (a). Noting that both g_Ψ and $\operatorname{sgn}(\det(b_\Psi))$ do not depend on the chosen orientation on V, the rest of the assertion follows by calculating g_Ψ and $\operatorname{sgn}(\det(b_\Psi))$ for $\Psi = \delta \star_{\varphi_\epsilon} \varphi_\epsilon$.

(c) $-I_7$ is an orientation-reversing map on $\mathbb{R}^7$ which fixes $\delta \star_{\varphi_\epsilon} \varphi_\epsilon$ for all $\delta, \epsilon \in \{-1, 1\}$. Hence, part (c) implies that we only have to prove part (d) for $\Psi = \delta \star_{\varphi_\epsilon} \varphi_\epsilon$ with $\delta, \epsilon \in \{-1,1\}$ and the standard orientation on $\mathbb{R}^7$. Apart from the computation of the dual three-form $\hat{\Psi}$, this is a straightforward task. For the computation of the dual three-form $\hat{\Psi}$ we use the fact that $\hat{\Psi}$ is a G_2^ϵ-invariant three-form on $\mathbb{R}^7$ and so a multiple $\lambda\varphi_\epsilon$ of φ_ϵ by [Br1]. $\lambda = \frac{1}{4}$ is obtained from Equation (1.9).

□

Lemma 2.43 tells us that a G_2^ϵ-structure can alternatively be defined as a pair consisting of a four-form with model tensor $\star_{\varphi_\epsilon}\varphi_\epsilon$ and an orientation. Moreover, Lemma 2.45 shows that the Hodge dual of the four-form is a G_2^ϵ-structure in the sense of Definition 2.35. For the construction of G_2^ϵ-structures with certain properties in later chapters, we prefer the approach via four-forms. However, we will only call the corresponding three-form a G_2^ϵ-structure in the following. Similarly, a $(\mathrm{G}_2)_{\mathbb{C}}$-structure can be given by a pair of a four-form of certain kind and a volume form. This is the next lemma which is proved by direct calculation or using Lemma 2.45 and complex-linear extension.

Lemma 2.46. *Let W be a seven-dimensional complex vector space and $(\Psi, \mathrm{vol}) \in \Lambda^3 W^* \times \Lambda^4 W^*$ be such that there exists an ordered basis $(f_1, \ldots, f_7)$ of W with*

$$\Psi = -f^{1247} - f^{1256} - f^{1346} + f^{1357} + f^{2345} + f^{2367} + f^{4567}, \ \mathrm{vol} = f^{1\ldots7}.$$

Then $(\star_\Psi \Psi, \mathrm{vol})$ is a $(\mathrm{G}_2)_{\mathbb{C}}$-structure with adapted basis $(f_1, \ldots, f_7)$, where $\star_\Psi$ is the Hodge star operator induced by (Ψ, vol).

Remark 2.47. *If a four-form $\Psi \in \Lambda^4 W^*$ on a seven-dimensional complex vector space W is stable or, equivalently due to Lemma 2.43 (b), if there exists an ordered basis $(f_1, \ldots, f_7)$ with $\Psi = -f^{1247} - f^{1256} - f^{1346} + f^{1357} + f^{2345} + f^{2367} + f^{4567}$, then we may set $\mathrm{vol} := f^{1\ldots7}$ and Lemma 2.46 implies that $(\star_\Psi \Psi, \mathrm{vol}) \in \Lambda^3 W^* \times \Lambda^7 W^*$ is a $(\mathrm{G}_2)_{\mathbb{C}}$-structure. This construction depends on the chosen ordered basis $(f_1, \ldots, f_7)$. By choosing a different one, we get a multiple with some fourth root of unity.*

Next, let V be a seven-dimensional real vector space, $V = W \oplus \mathrm{span}(v)$ with $v \in V \backslash \{0\}$ and φ be a G_2^ϵ-structure on V. From the values of the algebraic invariants for φ given in Table 1.1 we get lower bounds for the lengths of $\omega := (v \lrcorner \varphi)|_W \in \Lambda^2 W^*$ and $\rho := \varphi|_W \in \Lambda^3 W^*$. In the case of a G_2-structure, the invariants tell us that the length of ω is at least three. Using Lemma 2.1, we get that the length of ω is three and that ω has model tensor $\omega_0 = \sum_{i=1}^3 e^{2i-1} \wedge e^{2i} \in \Lambda^2 \left(\mathbb{R}^6\right)^*$. We also determine the model tensor of ω for a G_2^*-structure depending on $\mathrm{sgn}(g_\varphi(v,v))$. Similarly, we determine the model tensor of ρ for arbitrary $\epsilon \in \{-1, 1\}$ depending on properties of W. Moreover, we also consider the Hodge dual $\star_\varphi \varphi$ and determine the model tensors of the naturally appearing three- and four-form on W depending on properties of v and W. In contrast to [Fre1], where only some of the mentioned model tensors have been determined, we do not use the algebraic invariants for the calculation but instead apply Lemma 2.41.

Proposition 2.48. *Let V be a seven-dimensional real vector space, φ be a G_2^ϵ-structure and $V = W \oplus \mathrm{span}(v)$ be a vector space decomposition with $\dim(W) = 6$, $v \in V^* \backslash \{0\}$. Set $\omega := (v \lrcorner \varphi)|_W \in \Lambda^2 W^*$, $\rho := \varphi|_W \in \Lambda^3 W^*$, $\tilde{\rho} := (v \lrcorner \star_\varphi \varphi)|_W \in \Lambda^3 W^*$ and $\Omega := \star_\varphi \varphi|_W \in \Lambda^4 W^*$. Moreover, let, for $\delta \in \{-1, 0, 1\}$, $\rho_\delta \in \Lambda^3 \left(\mathbb{R}^6\right)^*$ be the three-form given in Table 1.1, $\omega_0 = e^{12} + e^{34} + e^{56} \in \Lambda^2 \left(\mathbb{R}^6\right)^*$ as throughout this thesis and set $\omega_1 := e^{12} + e^{34} \in \Lambda^2 \left(\mathbb{R}^6\right)^*$ and $\Omega_1 := e^{1234} + e^{1256} \in \Lambda^2 \left(\mathbb{R}^6\right)^*$.*

(a) *The values of the algebraic invariants* rk, l, m *and* r *for the above-mentioned forms* ψ *on* $\mathbb{R}^6$ *can be found in the following table.*

Table 2.2: Algebraic invariants for certain forms on $\mathbb{R}^6$

ψ	$(\mathrm{rk}(\psi), l(\psi), m(\psi), r(\psi))$	ψ	$(\mathrm{rk}(\psi), l(\psi), m(\psi), r(\psi))$
ω_0	$(6,3,1,2)$	ω_1	$(4,2,1,1)$
ρ_{-1}	$(6,3,2,2)$	ρ_1	$(6,2,1,1)$
ρ_0	$(6,3,1,1)$	$\frac{1}{2}\omega_0^2$	$(6,3,2,1)$
Ω_1	$(6,2,1,0)$		

(b) ω *has model tensor* ω_0 *if* $\epsilon = -1$ *or if* $\epsilon = 1$ *and* $g_\varphi(v,v) \neq 0$. *Otherwise, i.e. if* $\epsilon = 1$ *and* v *is a null-vector,* ρ *has model tensor* ω_1.

(c) ρ *has model tensor* ρ_{-1} *if* $\epsilon = -1$ *or if* $\epsilon = 1$ *and* U *has signature* $(2,4)$. ρ *has model tensor* ρ_1 *if* $\epsilon = 1$ *and* U *has signature* $(3,3)$. *Otherwise, i.e. if* $\epsilon = 1$ *and* U *is degenerate,* ρ *has model tensor* ρ_0.

(d) $\tilde{\rho}$ *has model tensor* ρ_{-1} *if* $\epsilon = -1$. *If* $\epsilon = 1$, *then* ρ *has model tensor* $\rho_{-\mathrm{sgn}(g_\varphi(v,v))}$.

(e) Ω *has model tensor* $\frac{1}{2}\omega_0^2$ *if* $\epsilon = -1$ *or if* $\epsilon = 1$ *and* U *has signature* $(2,4)$. *If* $\epsilon = 1$ *and* U *has signature* $(3,3)$, Ω *has model tensor* $-\frac{1}{2}\omega_0^2$. *Finally,* Ω *has model tensor* Ω_1 *if* $\epsilon = 1$ *and* U *is degenerate.*

Proof. The values of the algebraic invariants for the three appearing three-forms in (a) are given in Table 1.1. The values of the algebraic invariants for the remaining forms are obvious if we take into account the results we obtained in Section 2.1.

For the rest of the proof, let $(f_1, \ldots, f_7)$ be an adapted basis for φ. By Lemma 2.41, to determine the model tensors of ρ and Ω for the different classes of six-dimensional subspaces W of V with fixed signature, we may choose an arbitrary six-dimensional subspace W in the corresponding class. For the determination of the model tensors of $\tilde{\rho}$ and ω, note that a three-form is obviously equivalent to all its non-zero multiples and by Lemma 2.1 a two-form is also equivalent to all of its non-zero multiples. Hence, Remark 1.41 and Lemma 2.41 imply that for the determination of the model tensors of ω and $\tilde{\rho}$ for the different classes of non-zero vectors v in V having the same sign $g_\varphi(v,v)$, we may choose an arbitrary vector v in the corresponding class and an arbitrary complement W of $\mathrm{span}(v)$ in V.

In the G_2-case we choose $v := f_1$ and $W := \mathrm{span}(f_2, \ldots, f_7)$. Then we get

$$\omega = f^{23} + f^{45} + f^{67}, \quad \rho = f^{246} - f^{257} - f^{347} - f^{356},$$
$$\tilde{\rho} = f^{357} - f^{346} - f^{256} - f^{247}, \quad \Omega = f^{2345} + f^{2367} + f^{4567}.$$

and see that all forms have the claimed model tensors.

In the G_2^*-case, $v_1 := f_1$ fulfils $g_\varphi(v_1, v_1) = 1$ and the six-dimensional subspace $W_1 := \operatorname{span}(f_2, \ldots, f_7)$ has signature $(2, 4)$. Computing all induced k-forms on W_1 concretely, it is obvious, as in the G_2-case, that they have the claimed model tensors. Moreover, $v_2 := f_7$ fulfils $g_\varphi(v_2, v_2) = -1$ and the six-dimensional subspace $W_2 := \operatorname{span}(f_1, \ldots, f_6)$ has signature $(3, 3)$. Again one directly sees that the induced k-forms on W_2 have the claimed model tensors. Finally, we have to look at the degenerate case, where we give some more details since the model tensors may not be as obvious as before. First, consider the degenerate subspace $W_3 := \operatorname{span}(f_1 + f_7, f_2, \ldots, f_6)$ and let $\left(F^1, F^2, \ldots, F^6\right)$ be a dual basis of $(f_1 + f_7, f_2, \ldots, f_6)$. The induced three- and four-form are given by

$$\begin{aligned}
\rho &= F^{123} - F^{145} - F^{246} + F^{125} + F^{134} + F^{356} \\
&= -F^6 \wedge \left(F^2 - F^4\right) \wedge F^4 - \left(-F^6\right) \wedge \left(F^3 + F^5\right) \wedge F^5 + \left(F^2 - F^4\right) \wedge \left(F^3 + F^5\right) \wedge F^1, \\
\Omega &= F^{1256} + F^{1346} - F^{2345} + F^{1236} - F^{1456} \\
&= \left(F^2 - F^4\right) \wedge \left(F^3 + F^5\right) \wedge F^{16} + \left(F^2 - F^4\right) \wedge \left(F^3 + F^5\right) \wedge F^{54}
\end{aligned}$$

and from our rewriting one sees that they have the claimed model tensors. Moreover, $v_3 := f_1 + f_7$ is a null-vector and on $W_4 := \operatorname{span}(f_1, \ldots, f_6)$ the induced two- and three-form are given by

$$\begin{aligned}
\omega &= f^{23} - f^{45} - f^{16} + f^{25} + f^{34} = f^{61} + \left(f^2 - f^4\right) \wedge \left(f^3 + f^5\right), \\
\tilde{\rho} &= -f^{124} + f^{256} + f^{346} + f^{135} + f^{236} - f^{456} \\
&= \left(-f^1\right) \wedge \left(f^2 - f^4\right) \wedge f^4 - \left(-f^1\right) \wedge \left(f^3 + f^5\right) \wedge f^5 + \left(f^2 - f^4\right) \wedge \left(f^3 + f^5\right) \wedge f^6.
\end{aligned}$$

Again we have rewritten the forms in such a ways that the model tensors are obviously the claimed ones. □

For $(\mathrm{G}_2)_\mathbb{C}$ we get similar results.

Proposition 2.49. *Let V be a seven-dimensional complex vector space, φ be a $(\mathrm{G}_2)_\mathbb{C}$-structure and $V = W \oplus \operatorname{span}(v)$ be a vector space decomposition with $\dim(W) = 6$ and $v \in V^* \backslash \{0\}$. Set $\omega := (v \lrcorner \varphi)|_W \in \Lambda^2 W^*$, $\rho := \varphi|_W \in \Lambda^3 W^*$, $\tilde{\rho} := (v \lrcorner \star_\varphi \varphi)|_W \in \Lambda^3 W^*$ and $\Omega := \star_\varphi \varphi|_W \in \Lambda^4 W^*$. Moreover, let $\omega_C := e^{12} + e^{34} + e^{56} \in \Lambda^2 \left(\mathbb{C}^6\right)^*$ and ρ_C be as in Equation (1.4). Furthermore, denote by $\rho_{0,C} \in \Lambda^3 \left(\mathbb{C}^6\right)^*$ the complex-linear extension of $\rho_0 \in \Lambda^3 \left(\mathbb{R}^6\right)^*$ as in Table 1.1.*

(a) ω has model tensor ω_C if $g_\varphi(v, v) \neq 0$ and model tensor $e^{12} + e^{34} \in \Lambda^2 \left(\mathbb{C}^6\right)^*$ if $g_\varphi(v, v) = 0$.

(b) ρ has model tensor $\rho_C \in \Lambda^3 \left(\mathbb{C}^6\right)^*$ if U is non-degenerate and model tensor $\rho_{0,C} \in \Lambda^3 \left(\mathbb{C}^6\right)^*$ if U is degenerate.

(c) $\tilde{\rho}$ has model tensor $\rho_C \in \Lambda^3 \left(\mathbb{C}^6\right)^*$ if $g_\varphi(v, v) \neq 0$ and model tensor $\rho_{0,C} \in \Lambda^3 \left(\mathbb{C}^6\right)^*$ if $g_\varphi(v, v) = 0$.

(d) Ω *has model tensor* $\frac{1}{2}\omega_C^2 \in \Lambda^4\left(\mathbb{C}^6\right)^*$ *if* U *is non-degenerate and model tensor* $e^{1234} + e^{1256} \in \Lambda^4\left(\mathbb{C}^6\right)^*$ *if* U *is degenerate.*

Proof. Using Remark 1.41, Lemma 2.41 and the fact that each complex k-form is equivalent to its λ-multiple for all $\lambda \in \mathbb{C}^*$, the proof follows by the same computations as the proof of Proposition 2.48. □

The length of the complex k-forms on $\mathbb{C}^6$ appearing as model tensors in Proposition 2.49 are given in Proposition 1.45 and so we know the values of the algebraic invariants r and m for the three-form part of a $(\mathrm{G}_2)_{\mathbb{C}}$-structure and also its Hodge dual. By Lemma 1.43, lengths are preserved under Hodge duals. Hence, Proposition 2.48 and Proposition 2.49 yield

Corollary 2.50. *Let* V *be a seven-dimensional real vector space,* $\varphi \in \Lambda^3 V^*$ *be a* G_2*-structure on* V*,* $\tilde{\varphi} \in \Lambda^3 V^*$ *be a* G_2^**-structure on* V *and* $(\overline{\varphi}, \mathrm{vol}) \in \Lambda^3 V_{\mathbb{C}}^* \times \Lambda^7 V_{\mathbb{C}}^*$ *be a* $(\mathrm{G}_2)_{\mathbb{C}}$*-structure. Then*

$$\begin{aligned}
&(\mathrm{rk}(\star_\varphi \varphi), l\left(\star_\varphi \varphi\right), m\left(\star_\varphi \varphi\right), r\left(\star_\varphi \varphi\right)) = (7,5,3,3),\\
&(\mathrm{rk}(\star_{\tilde{\varphi}} \tilde{\varphi}), l\left(\star_{\tilde{\varphi}} \tilde{\varphi}\right), m\left(\star_{\tilde{\varphi}} \tilde{\varphi}\right), r\left(\star_{\tilde{\varphi}} \tilde{\varphi}\right)) = (7,4,2,2),\\
&(\mathrm{rk}(\overline{\varphi}), l\left(\overline{\varphi}\right), m\left(\overline{\varphi}\right), r\left(\overline{\varphi}\right)) = (\mathrm{rk}(\star_{\overline{\varphi}} \overline{\varphi}), l\left(\star_{\overline{\varphi}} \overline{\varphi}\right), m\left(\star_{\overline{\varphi}} \overline{\varphi}\right), r\left(\star_{\overline{\varphi}} \overline{\varphi}\right)) = (7,4,2,2).
\end{aligned}$$

Proposition 2.48 and Proposition 2.49 are heavily used in Chapter 4 for the construction of (co)-calibrated G_2^ϵ-structures and (co)-calibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structures as well as for getting obstructions to the existence of such structures on almost Abelian Lie algebras. They are also used in Chapter 5 to get obstructions to the existence of cocalibrated G_2-structures on direct sums of four- and three-dimensional Lie algebras. At the end of this section, we provide some methods which are applied in the construction of examples of cocalibrated G_2-structures on those direct sums. Before, we like to mention a tight connection between $\mathrm{SU}^\delta(p, 3-p)$-structures on real six-dimensional vector spaces V and G_2^ϵ-structures on $V \oplus \mathbb{R}$. This connection allows us in Chapter 6 to transfer the existence problem of a half-flat $\mathrm{SU}(3)$-structure on a given real six-dimensional Lie algebra $\mathfrak{g}$ to the existence problem of a cocalibrated G_2-structure on $\mathfrak{g} \oplus \mathbb{R}$ with orthogonal splitting, which for some Lie algebras turns out to be very useful.

Proposition 2.51. *Let* W *be a seven-dimensional real vector space and* $V \subseteq W$ *be a six-dimensional subspace. Fix* $v \in W \backslash V$ *and an orientation on* V*. For*

$$(p, \delta, \epsilon) \in \{(1,-1,1), (3,1,1), (3,-1,-1)\}$$

there is a one-to-one correspondence between $\mathrm{SU}^\delta(p, 3-p)$*-structures* $(\omega, \rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ *on* V *and* G_2^ϵ*-structures* $\varphi \in \Lambda^3 W^*$ *on* W *such that* V *is orthogonal to* v *with respect to* g_φ *and* $g_\varphi(v, v) = -\delta$*. Moreover,* $g_\varphi = g_{(\omega,\rho)} \oplus -\delta\, \alpha \otimes \alpha$ *for* $\alpha \in V^0$ *with* $\alpha(v) = 1$*, where*

$g_{(\omega,\rho)}$ is the metric on V induced by (ω,ρ). If we identify V^* with v^0 via $W = V \oplus \operatorname{span}(v)$, the correspondence is given by

$$\Lambda^2 V^* \times \Lambda^3 V^* \ni (\omega, \rho) \mapsto \varphi := \omega \wedge \alpha + \rho \in \Lambda^3 W^*. \tag{2.31}$$

Note that then

$$\star_\varphi \varphi = -\frac{\delta}{2}\omega^2 + \delta J_\rho^* \rho \wedge \alpha. \tag{2.32}$$

The inverse construction is given by

$$\Lambda^3 W^* \ni \varphi \mapsto (\omega := (v \lrcorner \varphi)|_V, \rho := \varphi|_V) \in \Lambda^2 V^* \times \Lambda^3 V^*. \tag{2.33}$$

Proof. If $(\omega,\rho) \in \Lambda^2 V^* \times \Lambda^3 V^*$ is an $\mathrm{SU}^\delta(p, 3-p)$-structure, then, by definition, there exists a basis $f_1, \ldots, f_6$ such that in the dual basis $f^1, \ldots, f^6$ we have

$$\omega = f^{12} + (p-2)\left(f^{34} + f^{56}\right), \quad \rho = f^{135} + \delta\left(f^{146} + f^{236} + f^{245}\right)$$

and so, by setting $f^7 := \alpha$, we get

$$\varphi = f^{127} + (p-2)\left(f^{347} + f^{567}\right) + f^{135} + \delta\left(f^{146} + f^{236} + f^{245}\right).$$

We see that for $\delta = -1$, φ is a G_2^ϵ-structure with adapted dual basis $\left(f^7, -\epsilon f^2, \epsilon f^1, \epsilon f^4, -\epsilon f^3, -\epsilon f^5, -\epsilon f^6\right)$ and $\epsilon = -1$ if $p = 3$ and $\epsilon = 1$ if $p = 1$. If $\delta = 1$, φ is a G_2^*-structure with adapted dual basis $\left(-f^5, f^1, -f^3, f^6, f^7, f^4, -f^2\right)$. Using the dual basis just obtained, we get

$$\begin{aligned}\star_\varphi \varphi &= -\frac{\delta}{2}\left((p-2)\left(f^{1234} + f^{1256}\right) + f^{3456}\right) + f^{1367} + f^{1457} + f^{2357} + \delta f^{2467} \\ &= -\frac{\delta}{2}\omega^2 + \delta J_\rho^* \rho \wedge f^7.\end{aligned}$$

The statement about g_φ follows from the fact that $f_1, \ldots, f_6$ is an orthonormal basis for (ω,ρ), $f_1, \ldots, f_7$ is an orthonormal basis for φ and $g_\varphi(f_7, f_7) = \delta$.

For the converse direction, note first that if (ω,ρ) is an $\mathrm{SU}^\delta(p,3-p)$-structure on V, then $(-\omega,\rho)$ is also an $\mathrm{SU}^\delta(p,3-p)$-structure on V by Corollary 2.34. By Lemma 2.41, for arbitrary w_1, $w_2 \in W$ with $g_\varphi(w_1,w_1) = g_\varphi(w_2,w_2) \neq 0$, there exists an element in G_2^ϵ which maps w_1 to w_2 or to $-w_2$. Hence, to show that $((v \lrcorner \varphi)|_V, \varphi|_V) \in \Lambda^2 V^* \times \Lambda^3 V^*$ is an $\mathrm{SU}^\delta(p,3-p)$-structure on V, we may choose an arbitrary $v \in W$ with $g_\varphi(v,v) = -\delta$ and $V := v^\perp$ for the computation. Thus, the statement follows simply by inverting the above calculations. □

Next, we elaborate how one may build up a G_2-structure on a real seven-dimensional vector space V from two-forms on four- and three-dimensional complementary subspaces of V. This turns out to be useful for the construction of examples of cocalibrated G_2-structures on direct sums of four- and three-dimensional Lie algebras in Chapter 5. Therefore, we need *adapted splittings*.

Definition 2.52. *Let $\varphi \in \Lambda^3 V^*$ be a G_2-structure on a seven-dimensional real vector space V. A splitting $V = V_4 \oplus V_3$ is called* adapted (for φ) *if there exists an adapted basis $(f_1, \ldots, f_7)$ for φ such that $f_1, \ldots, f_4$ is a basis of V_4 and f_5, f_6, f_7 is a basis of V_3.*

The following lemma follows directly from Equation (2.26) and the fact that adapted bases are orthonormal.

Lemma 2.53. *Let V be a seven-dimensional real vector space, $\varphi \in \Lambda^3 V^*$ be a G_2-structure on V and $V = V_4 \oplus V_3$ be an adapted splitting. Then the decomposition $V = V_4 \oplus V_3$ is orthogonal with respect to g_φ and there exist a non-zero $\Omega_1 \in \Lambda^4 V_4^*$ and a non-zero $\Omega_2 \in \Lambda^2 V_4^* \wedge \Lambda^2 V_3^*$ such that*

$$\star_\varphi \varphi = \Omega_1 + \Omega_2. \tag{2.34}$$

Moreover, if $\tilde{\varphi} \in \Lambda^3 V$ is a G_2-structure with adapted basis $(F_1, \ldots, F_7)$, $F_j = \frac{1}{\lambda} f_j$ for $j = 1, 2, 3, 4$, $F_l = f_l$ for $l = 5, 6, 7$, then the splitting $V = V_4 \oplus V_3$ is also adapted for $\tilde{\varphi}$, $g_{\tilde{\varphi}}|_{V_4} = \lambda^2 g_\varphi|_{V_4}, g_{\tilde{\varphi}}|_{V_3} = g_\varphi|_{V_3}$ and

$$\star_{\tilde{\varphi}} \tilde{\varphi} = \lambda^4 \Omega_1 + \lambda^2 \Omega_2. \tag{2.35}$$

Remark 2.54. *An adapted splitting is also called* coassociative/associative splitting, *see [AS]. This is due to the fact that V_3 is a calibrated subspace for φ and V_4 is a calibrated subspace for $\star_\varphi \varphi$. However, since we do not need calibrations at all in this thesis, we prefer the term "adapted splitting".*

Given a splitting $V = V_4 \oplus V_3$ with $\dim(V_i) = i$, we may construct a G_2-structure with adapted splitting $V = V_4 \oplus V_3$ via two-forms of certain kind on V_4 and V_3 as follows.

Proposition 2.55. *Let V be a seven-dimensional real vector space and $V = V_4 \oplus V_3$ be a vector space decomposition of V into a real four-dimensional vector space V_4 and into a real three-dimensional vector space V_3. Fix $\tau \in \Lambda^4 V_4^* \backslash \{0\}$. Let $k \in \{0, 1, 2, 3\}$ and $\omega_i \in \Lambda^2 V_4^*$ for $i = 1, \ldots, k$ be such that the symmetric matrix $H = (h_{ij})_{ij} \in \mathbb{R}^{k \times k}$ defined by*

$$h_{ij} \tau = \omega_i \wedge \omega_j$$

is definite, where $k = 0$ means that there is no condition. Then V admits two-forms $\omega_{k+1}, \ldots, \omega_3 \in \Lambda^2 V_4^*$ such that for all bases $\nu_1, \ldots, \nu_3 \in \Lambda^2 V_3^*$ of $\Lambda^2 V_3^*$ the four-form

$$\Psi := \frac{1}{2} \omega_1^2 + \sum_{i=1}^{3} \omega_i \wedge \nu_i \tag{2.36}$$

is the Hodge Dual of a G_2-structure on V and $V = V_4 \oplus V_3$ is an adapted splitting.

Proof. Let $\tilde{\omega}_1 := e^{12} + e^{34} \in \Lambda^2 \left(\mathbb{R}^4\right)^*$, $\tilde{\omega}_2 := e^{13} - e^{24} \in \Lambda^2 \left(\mathbb{R}^4\right)^*$, $\tilde{\omega}_3 := e^{14} + e^{23} \in \Lambda^2 \left(\mathbb{R}^4\right)^*$. By Lemma 2.2, there exists an isomorphism $u : V_4 \to \mathbb{R}^4$ such that $u^* \tilde{\omega}_1, \ldots,$

$u^*\tilde{\omega}_k$ is a basis of $\mathrm{span}(\omega_1, \ldots, \omega_k)$. Since there is an automorphism of V_4 mapping $u^*\tilde{\omega}_1$ onto ω_1, we may, without loss of generality, assume that $\omega_1 = u^*\tilde{\omega}_1$. Let $A \in \mathbb{R}^{k\times k}$, $A = (a_{ij})_{ij}$ be such that $\omega_j = \sum_{i=1}^k a_{ij}\,(u^*\tilde{\omega}_i)$ for $j = 1, \ldots, k$. Set $f_i := u^{-1}(e_i) \in V_4$ for $i = 1, \ldots, k$ and set $\omega_l := u^*\tilde{\omega}_l$ for $l = k+1, \ldots, 3$. Since $\nu_1, \ldots, \nu_3$ is a basis, also $\tilde{\nu}_1, \ldots, \tilde{\nu}_3$ with $\tilde{\nu}_j = \sum_{i=1}^k a_{ji}\,\nu_i$ for $j = 1, \ldots, k$, $\tilde{\nu}_j := \nu_j$ for $j = k+1, \ldots, 3$ is a basis of V_3^*. Thus, there exists a basis f_5, f_6, f_7 of V_3 such that $\tilde{\nu}_1 = f^{56}$, $\tilde{\nu}_2 = -f^{57}$ and $\tilde{\nu}_3 = f^{67}$ and we can compute

$$\begin{aligned}
\Psi &= \frac{1}{2}\omega_1^2 + \sum_{i=1}^{3} \omega_i \wedge \nu_i = f^{1234} + \sum_{i,j=1}^{k} a_{ji}\,(u^*\tilde{\omega}_j) \wedge \nu_i + \sum_{i=k+1}^{3} u^*\tilde{\omega}_i \wedge \tilde{\nu}_i \\
&= f^{1234} + \sum_{j=1}^{3} (u^*\tilde{\omega}_j) \wedge \tilde{\nu}_j \\
&= f^{1234} + f^{1256} + f^{3456} - f^{1357} + f^{2457} + f^{1467} + f^{2367}
\end{aligned}$$

and we see that Ψ is the Hodge Dual of a G_2-structure with adapted basis $(f_7, f_1, f_2, f_3, f_4, f_6, -f_5)$. □

Remark 2.56. *The assertion of Proposition 2.55 has been used implicitly in the literature several times before, cf. e.g. [Br4] and [Ma].*

Moreover, we directly use the openness of the orbit of all Hodge duals of G_2-structures to construct cocalibrated G_2-structures. Actually, the orbit of all Hodge duals of G_2-structures is "uniformly" open in the following sense, cf. also [J3]:

Lemma 2.57. *There exists a universal constant $\epsilon_0 > 0$ such that if $\varphi \in \Lambda^3 V^*$ is a G_2-structure on a seven-dimensional real vector space V and $\Psi \in \Lambda^4 V^*$ is a four-form on V which fulfils*

$$\|\Psi - \star_\varphi \varphi\|_\varphi < \epsilon_0$$

for the norm $\|\cdot\|_\varphi$ induced by the Euclidean metric g_φ on V, then Ψ is the Hodge dual of a G_2-structure on V.

Proof. By Proposition 1.33 and Theorem 1.35, the orbit of all Hodge duals of G_2-structures is open. Fix some G_2-structure $\varphi_0 \in \Lambda^3\mathfrak{g}^*$. Then there exists a ball of radius $\epsilon_0 > 0$ in $(\Lambda^4 V^*, g_{\varphi_0})$ around $\star_{\varphi_0}\varphi_0$ such that each four-form in this ball is again the Hodge dual of a G_2-structure. ϵ_0 is the desired universal constant: Let $\varphi \in \Lambda^3\mathfrak{g}^*$ be any G_2-structure on V. Choose an automorphism $F : V \to V$ with $F^* \star_\varphi \varphi = \star_{\varphi_0}\varphi_0$. Then $F^* : (\Lambda^4 V^*, g_\varphi) \to (\Lambda^4 V^*, g_{\varphi_0})$ is an isometric isomorphism by Lemma 2.45. Thus, if $\Psi \in \Lambda^4\mathfrak{g}^*$ fulfils $\|\Psi - \star_\varphi\varphi\|_\varphi < \epsilon_0$, then $\|F^*\Psi - \star_{\varphi_0}\varphi_0\|_{\varphi_0} < \epsilon_0$. Hence, $F^*\Psi$ and so also Ψ is in the orbit of all Hodge duals of G_2-structures on V. □

2.5 Spin$^\epsilon(7)$-structures

In this section, we deal with Spin(7)- and $\mathrm{Spin}_0(3,4)$-structures on real eight-dimensional vector spaces. We discuss some basic properties of these structures and their relation to G_2^ϵ-structures on real seven-dimensional vector spaces.

In Proposition 1.25, we gave an example of a four-form on $\mathbb{R}^8$ with stabiliser equal to Spin(7) and one with stabiliser equal to $\mathrm{Spin}_0(3,4)$, namely Φ_{-1} and Φ_1, both defined in Equation (1.5). This leads to the following definition.

Definition 2.58. *Let V be a real eight-dimensional vector space. A four-form $\Phi \in \Lambda^4 V^*$ is called a* Spin(7)-structure *if it has model tensor $\Phi_{-1} \in \Lambda^4\left(\mathbb{R}^8\right)$, Φ_{-1} as in Equation (1.5). Φ is called a* $\mathrm{Spin}_0(3,4)$-structure *if it has model tensor $\Phi_1 \in \Lambda^4\left(\mathbb{R}^8\right)$, where Φ_1 is also defined in Equation (1.5). We set* $\mathrm{Spin}(7)^1 := \mathrm{Spin}_0(3,4)$ *and* $\mathrm{Spin}(7)^{-1} := \mathrm{Spin}(7)$. *Then $\Phi \in \Lambda^4 V^*$ is a* $\mathrm{Spin}^\epsilon(7)$*-structure for $\epsilon \in \{-1,1\}$ if and only if there exists an ordered basis $(f_1,\ldots,f_8)$ of V such that in the dual ordered basis $\left(f^1,\ldots,f^8\right)$ we have*

$$\begin{aligned}\Psi = &- f^{1238} + \epsilon\left(f^{1458} + f^{1678} + f^{2468} - f^{2578} - f^{3478} - f^{3568}\right)\\ &+ \epsilon\left(f^{1247} + f^{1256} + f^{1346} - f^{1357} - f^{2345} - f^{2367}\right) + f^{4567}.\end{aligned} \tag{2.37}$$

Spin(7) is a subgroup of SO(8) and $\mathrm{Spin}_0(3,4)$ is a subgroup of SO(4,4) by Proposition 1.25. Hence, a $\mathrm{Spin}^\epsilon(7)$-structure on an eight-dimensional vector space V induces a pseudo-Euclidean metric and a metric volume form on V via the corresponding standard model tensors. The concrete construction only in terms of Φ is somehow involved and may be found for the Spin(7)-case in [Kar]. We only give some part of the information in the next lemma and refer the reader for the proof of this lemma to [Kar] and [Br1].

Lemma 2.59. *Let V be an eight-dimensional real vector space and $\Phi \in \Lambda^4 V^*$ be a* $\mathrm{Spin}^\epsilon(7)$-*structure on V. Then Φ induces a pseudo-Euclidean metric g_Φ which is positive definite if $\epsilon = -1$ and of signature $(4,4)$ if $\epsilon = 1$. An orthonormal basis is in both cases given by an adapted basis $(f_1,\ldots,f_8)$ where for $\epsilon = -1$ we have $g_\Phi(f_i,f_i) = 1$ for $i = 1,2,3,8$ and $g_\Phi(f_j,f_j) = -1$ for $j = 4,5,6,7$. Moreover, Φ induces an orientation via* $\mathrm{vol}_\Phi := \frac{1}{14}\Phi^2$. vol_Φ *is a metric volume form with respect to g_Φ. Moreover, Φ is self-dual with respect to the induced Hodge star operator $\star_\Phi$.*

Equation (1.5) shows that a G_2^ϵ-structure on a seven-dimensional real vector space V induces a Spin^ϵ-structure on $V \oplus \mathbb{R}$. More generally, the analogous statement to Proposition 2.51 is true.

Proposition 2.60. *Let V be a seven-dimensional real vector space and $W \supseteq V$ be an eight-dimensional real vector space. Fix $v \in W \backslash V$. For $\epsilon \in \{-1,1\}$, there is a one-to-one correspondence between* G_2^ϵ*-structures $\varphi \in \Lambda^3 V^*$ on V and* $\mathrm{Spin}^\epsilon(7)$*-structures $\Phi \in \Lambda^4 W^*$ on W such that V is orthogonal to v with respect to g_Φ and $g_\Phi(v,v) = 1$.*

The correspondence is given by

$$\Lambda^3 V^* \ni \varphi \mapsto \Phi := \alpha \wedge \varphi + \star_\varphi \varphi \in \Lambda^3 W^* \tag{2.38}$$

where $\alpha \in V^0$ *with* $\alpha(v) = 1$. *The inverse construction is given by*

$$\Lambda^3 W^* \ni \Phi \mapsto \varphi := (v \lrcorner \Phi)\,|_V \in \Lambda^3 V^*. \tag{2.39}$$

Proof. A G_2^ϵ-structure on V induces a $\mathrm{Spin}^\epsilon(7)$-structure on W with orthogonal splitting $W = V \oplus \mathrm{span}(v)$ in the way given in the statement due to Equation (1.5) and the fact that adapted bases are orthonormal. Conversely, $\mathrm{Spin}^\epsilon(7)$ acts transitively on the pseudo-sphere by [Bo] and [Kath1] and so we may always assume that $v = f_8$ and $V = v^\perp = \mathrm{span}(f_1, \ldots, f_7)$ and the statement follows again from Equation (1.5). $\square$

Chapter 3

G-structures on manifolds and Lie groups

3.1 Basic definitions and relations

In this section, we look at G-structures on manifolds and related concepts. We start in Subsection 3.1.1 by defining these structures properly. If G is the common stabiliser of several tensors on $\mathbb{R}^n$, we show how one can describe G-structures equivalently by a collection of tensor fields. In Subsection 3.1.2, we introduce G-connections and use them to define the intrinsic torsion of a G-structure. We give an alternative description of the intrinsic torsion via minimal connections if G is a subgroup of $\mathrm{O}(p, n-p)$ such that $\mathfrak{g} \subseteq \mathfrak{so}(p, n-p)$ is non-degenerate. In Subsection 3.1.3, we discuss the holonomy of pseudo-Riemannian manifolds. We remind the reader of certain aspects of the classification of holonomy groups, in particular Berger's list. Afterwards, we state the well-known holonomy principle for pseudo-Riemannian manifolds, which relates the vanishing of the intrinsic torsion of a G-structure to the existence of parallel tensors fields and to the reduction of the holonomy group to a subgroup of G. Moreover, we remind the reader of the well-known theorem of Ambrose-Singer which enables to compute the holonomy algebra via the curvature.

Throughout the section, we assume that the reader is familiar with G-principal bundles, G-principal connections, associated vector bundles and related concepts. Nevertheless, we shortly recall all these concepts and some basic facts and refer for proofs to [Baum]. Other standard references are [J3],[KN], [Sa2] and [Ste].

3.1.1 G-structures on manifolds

Recall that if G is a Lie group, then a G-*principal bundle* is a locally trivial fibration (P, π, M) with fibre G such that P carries a G-right action which preserves the fibres and acts simply transitively on each of them. Note that we may arrange the local trivialisations $\pi^{-1}(U) \to U \times G$ in such a way that they are G-equivariant. We often simplify the notation and only write P for the triple (P, π, M). Moreover, recall that if H is a Lie subgroup of G, then a *reduction of P (to H)* is an H-principal bundle of the form $(Q, \pi|_Q, M)$, where Q is a submanifold of P invariant under the induced action of H on P. In this situation, we call (P, π, M) a G-*enlargement of Q*.

Example 3.4 below shows that reductions do not always exist. However, one can always enlarge a given H-principal bundle to a bigger group $\mathrm{G} \supseteq \mathrm{H}$.

Lemma 3.1. *Let* G *be a Lie group,* H *be a Lie subgroup of* G *and* (Q, p, M) *be an* H-*principal bundle. Then* H *acts on* $Q \times G$ *from the right by* $(q, g) \cdot h := \left(q \cdot h, h^{-1}g\right)$ *for* $h \in \mathrm{H}$, $q \in Q$, $g \in G$. *The triple* (P, π, M) *with*

$$P := (Q \times G)\,/\mathrm{H}$$

and $\pi : P \to M$, $\pi([q, g]) := p(q)$ *is a* G-*enlargement of* Q *with the obvious right-action of* G *when we identify* Q *with* $Q \times \{e\} \cong (Q \times \{e\})/\mathrm{H} \subseteq P$.

Proof. For a proof, one may consult, e.g., [Baum]. □

The only concrete examples of G-structures appearing in this thesis are the *frame bundle (of a manifold)*, which is a $\mathrm{GL}(n, \mathbb{R})$-principal bundle, and its reductions.

Definition 3.2. *Let* M *be an* n-*dimensional manifold. Set* $\mathcal{F}(M) := \bigcup_{x \in M} \mathcal{F}(T_x M)$ *and let* $\pi : \mathcal{F}(M) \to M$ *be the natural projection. Then one can endow* $\mathcal{F}(M)$ *with the structure of a smooth manifold such that* $\pi : \mathcal{F}(M) \to M$ *is a submersion. The triple* $(\mathcal{F}(M), \pi, M)$ *is called* the frame bundle of M. *We have a natural fibre-preserving* $\mathrm{GL}(n, \mathbb{R})$-*right action on* $\mathcal{F}(M)$ *which is simply transitive on the fibres, namely the one induced by the natural right action of* $\mathrm{GL}(n, \mathbb{R})$ *on* $T_x M$ *defined in Equation (1.1). This action makes* $(\mathcal{F}(M), \pi, M)$ *into a* $\mathrm{GL}(n, \mathbb{R})$-*principal bundle. Each section of* π *is called a* (global) frame (on M) *and a local section is called a* local frame.

For a closed subgroup G *of* $\mathrm{GL}(n, \mathbb{R})$, *a* G-structure on M *is a reduction* P *of the frame bundle* $\mathcal{F}(M)$ *of* M *to* G.

Remark 3.3. *If* P *is a* G-*structure, then each fibre* $P_x \subseteq \mathcal{F}(T_x M)$ *is a* G-*structure on the vector space* $T_x M$. *Hence, a* G-*structure* $P \subseteq \mathcal{F}(M)$ *is nothing but a family* $\{P_x\}_{x \in M}$ *of* G-*structures* P_x *on the vector spaces* $T_x M$ *which "smoothly" depends on* x *in the sense made precise in Definition 3.2.*

Example 3.4.

- *Analogously to Example 1.4,* $\mathrm{GL}^+(n, \mathbb{R})$-*structures are nothing but orientations on* M. *If* M *is oriented, we denote by* $\mathrm{GL}^+(M)$ *the corresponding* $\mathrm{GL}^+(n, \mathbb{R})$-*structure. Since there are non-orientable manifolds, reductions do not always exist.*

- *An $\{e\}$-structures is nothing but a global frame on M.*

As in Section 1.1, we want to give an alternative description of G-structures via tuples of tensor fields of certain kind. Therefore, we generalise the concept of model tensors from tensors to tensor fields on manifolds.

Definition 3.5. *Let M be an n-dimensional manifold and $T_i \in \mathcal{T}^{r_i,s_i}$ be (r_i, s_i)-tensor fields on M for $i = 1, \ldots, k$. The k-tuple $(T_1, \ldots, T_k)$ is said to* have the model tensors *$(S_1, \ldots, S_k) \in T^{r_1,s_1}\mathbb{R}^n \times \ldots \times T^{r_k,s_k}\mathbb{R}^n$ if the k-tuple $((T_1)_x, \ldots, (T_k)_x) \in T^{r_1,s_1}T_xM \times \ldots \times T^{r_k,s_k}T_xM$ has the model tensors $(S_1, \ldots, S_k)$ for all $x \in M$, i.e. if for all $x \in M$ there exists $u \in \mathcal{F}(T_xM)$ such that $\big(u^*(T_1)_x, \ldots, u^*(T_k)_x\big) = (S_1, \ldots, S_k)$.*

We get the following analogue of Lemma 1.5 on the manifold level.

Proposition 3.6. *Let M be an n-dimensional manifold and $\mathrm{G} \subseteq \mathrm{GL}(n, \mathbb{R})$ be a Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$ which is the common stabiliser of the tensors $S_1 \in T^{r_1,s_1}\mathbb{R}^n, \ldots, S_k \in T^{r_k,s_k}\mathbb{R}^n$. Then there is a one-to-one correspondence between G-structures and k-tuples $(T_1, \ldots, T_k) \in \Gamma\left(T^{r_1,s_1}M\right) \times \ldots \times \Gamma\left(T^{r_k,s_k}M\right)$ which have model tensors $(S_1, \ldots, S_k) \in T^{r_1,s_1}\mathbb{R}^n \times \ldots \times T^{r_k,s_k}\mathbb{R}^n$. The correspondence is as follows:*

- *If $P \subseteq \mathcal{F}(M)$ is a G-structure, then the associated k-tuple $(T_1, \ldots, T_k) \in \Gamma\left(T^{r_1,s_1}M\right) \times \ldots \times \Gamma\left(T^{r_k,s_k}M\right)$ is given at the point $x \in M$ by $(T_i)_x := \left(u^{-1}\right)^* S_i$ for $u \in P_x$ arbitrary and for $i = 1, \ldots, k$.*

- *Let $(T_1, \ldots, T_k) \in \Gamma\left(T^{r_1,s_1}M\right) \times \ldots \times \Gamma\left(T^{r_k,s_k}M\right)$ be a k-tuple with model tensors $(S_1, \ldots, S_k) \in T^{r_1,s_1}\mathbb{R}^n \times \ldots \times T^{r_k,s_k}\mathbb{R}^n$. Then the associated G-structure $P \subseteq \mathcal{F}(M)$ is given by*
$$P_x := \{u \in \mathcal{F}(T_xM)\,|u^*(T_i)_x = S_i \text{ for } i = 1, \ldots, k\}$$
for all $x \in M$.

Proof. Regarding Lemma 1.5, only the smoothness of P induced by the tensor fields $(T_1, \ldots, T_k)$ on M with model tensors $(S_1, \ldots, S_k)$ on $\mathbb{R}^n$ is not obvious. Since this is a local statement, we may assume $M = \mathbb{R}^n$ and consider the smoothness in $0 \in \mathbb{R}^n$. It suffices to show that there is a neighbourhood U of $0 \in \mathbb{R}^n$ and a smooth map $A : U \to \mathrm{GL}(n, \mathbb{R})$ such that $(A(x)^*(T_1)_x, \ldots, A(x)^*(T_k)_x) = (S_1, \ldots, S_k)$. Therefore, denote by $\mathcal{O}$ the orbit of $(S_1, \ldots, S_k)$ in $T^{r_1,s_1}\mathbb{R}^n \times \ldots \times T^{r_k,s_k}\mathbb{R}^n$ under the natural action of $\mathrm{GL}(n, \mathbb{R})$. By assumption, there exists $A_0 \in \mathrm{GL}(n, \mathbb{R})$ such that $(A_0^*(T_1)_0, \ldots, A_0^*(T_k)_0) = (S_1, \ldots, S_k)$. Choose a neighbourhood V of $A_0\mathrm{G}$ in $\mathrm{GL}(n, \mathbb{R})/\mathrm{G}$ which admits a smooth local section $s : V \to \mathrm{GL}(n, \mathbb{R})$ of $\mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})/\mathrm{G}$ with $s(A_0\mathrm{G}) = A_0$. Consider the smooth map
$$F : \mathbb{R}^n \times V \to \mathcal{O}, \quad F(x, A\mathrm{G}) := (s(A\mathrm{G})^*(T_1)_x, \ldots, s(A\mathrm{G})^*(T_k)_x)$$

for $x \in \mathbb{R}^n$ and $A \in \mathrm{GL}(n,\mathbb{R})$ with $AG \in V$. Note that the image of F lands in $\mathcal{O}$ exactly because $(T_1,\ldots,T_k)$ has model tensors $(S_1,\ldots,S_k)$. Since V and $\mathcal{O}$ have the same dimension, we may apply the implicit function theorem to get a smooth function $f : U \to V$ defined on an open neighbourhood U of 0 with $F(x, f(x)) = (S_1,\ldots,S_k)$. Hence, $A := s \circ f : U \to \mathrm{GL}(n,\mathbb{R})$ is the desired smooth function. □

In the situation of Proposition 3.6, we also call the k-tuple $(T_1,\ldots,T_k) \in \Gamma\left(T^{r_1,s_1}M\right) \times \ldots \times \Gamma\left(T^{r_k,s_k}M\right)$ *a* G*-structure on* M. Proposition 3.6 applies, e.g., to the cases $\mathrm{G} = \mathrm{O}(p,n-p)$, $\mathrm{Sp}(2m,\mathbb{R})$, $\mathrm{U}^\epsilon(p,m-p)$, $\mathrm{SU}^\epsilon(p,m-p)$, $\mathrm{G}_2^\epsilon, (\mathrm{G}_2)_\mathbb{C}$ and $\mathrm{Spin}^\epsilon(7)$ that we discussed on the vector space level in Chapter 2. All the concepts and definitions introduced in Chapter 2 which are related to particular G-structures on vector spaces can be extended to the entire manifold by defining them pointwise on each tangent space.

As an example, take $\mathrm{G} = \mathrm{G}_2^\epsilon$. A G_2^ϵ*-structure on (a seven-dimensional manifold)* M is a three-form $\varphi \in \Omega^3 M$ with model tensor $\varphi_\epsilon \in \Lambda^3\left(\mathbb{R}^7\right)^*$ defined in Equation (1.4). φ induces a pseudo-Riemannian metric $g_\varphi \in \Gamma\left(S^2T^*M\right)$ on M and a metric volume form $\phi(\varphi) \in \Omega^7 M$ by defining them pointwise on T_pM by Equation (2.24) and Equation (2.23), respectively. Also we get an induced Hodge star operator $\star_\varphi$ on the entire manifold.

Definition 3.7. *The* $\mathrm{O}(p,n-p)$*-structure associated to a pseudo-Riemannian metric* g *of signature* $(p,n-p)$ *on an* n*-dimensional manifold via the model tensor* $\langle\cdot,\cdot\rangle_{p,n-p}$ *is denoted by* $\mathrm{O}(M)$.

3.1.2 G-connections and intrinsic torsion

We briefly recall some well-known concepts in the theory of G-principal bundles. Therefore, let (P,π,M) be a fixed G-principal bundle in the following.

If $\rho : G \to \mathrm{GL}(V)$ is a representation of G on a real n-dimensional vector space V, then the *vector bundle* E *associated to* P *and* ρ is given by

$$E := (P \times V)/G. \tag{3.1}$$

Here, G acts from the right freely on $P \times V$ by $(p,v)\cdot g = \left(p \cdot g, \rho\left(g^{-1}\right)(v)\right)$. One can check that E is, in fact, a vector bundle of rank n over M with the obvious fibre-wise addition. Each G-principal bundle has an associated vector bundle via the adjoint representation.

Definition 3.8. *Let* P *be a* G*-principal bundle. The* adjoint bundle of P *is the vector bundle associated to* P *and the adjoint representation of* G *on the associated Lie algebra* $\mathfrak{g}$ *and is denoted by* $\mathfrak{g}(P)$.

The next important concept to recall is that of a G-*principal connection*. Therefore, first note that we have a natural subbundle of TP given by $V := \ker(d\pi)$. This subbundle is called the *vertical bundle*. A G-*principal connection* is a right-invariant *horizontal* distribution H on P, where *horizontal* means that $T_pP = V_p \oplus H_p$ for all $p \in P$.

If E is an associated vector bundle, then there is a natural map $H \mapsto \nabla^H$ from the set of all G-principal connections on P to the set of all (affine) connections on E. To define this map, let $C^\infty(P,V)^G$ be the $C^\infty(M)$-module of smooth G-equivariant functions $f : P \to V$, where G-equivariant means $f(p \cdot g) = \rho\left(g^{-1}\right) f(p)$ for $g \in \mathrm{G}$ and $p \in P$. The map $s \mapsto f_s$, $\Gamma(E) \to C^\infty(P,V)^G$ with $f_s(p) \in V$ uniquely defined by $s(\pi(p)) = [(p, f_s(p))] \in (P \times V)/G = E$ can be shown to be a $C^\infty(M)$-module isomorphism. Using this map to identify $\Gamma(E)$ with $C^\infty(P,V)^G$, ∇^H is given by $\left(\nabla^H\right)_X f = df\,(X^*) \in C^\infty(P,V)^G$ for $X \in \mathfrak{X}(M)$ and $f \in C^\infty(P,V)^G$. Here, $X^* \in \mathfrak{X}(P)$ is the unique *horizontal lift* of $X \in \mathfrak{X}(M)$ to P, i.e. $X^*_p \in H_p$ and $d\pi_p(X^*_p) = X_{\pi(p)}$ for all $p \in P$.

In general, the map $H \mapsto \nabla^H$ is not a bijection between the set of all G-principal connections on P and the set of all connections on the associated vector bundle E, cf. [J3]. However, for $\mathcal{F}(M)$ and the standard representation of $\mathrm{GL}(n,\mathbb{R})$ on $\mathbb{R}^n$ it is a bijection.

Proposition 3.9. *Let $\mathcal{F}(M)$ be the frame bundle of an n-dimensional manifold and P be a* G*-structure. Then the vector bundle of M associated to the standard representation of* $\mathrm{G} \subseteq \mathrm{GL}(n,\mathbb{R})$ *on* $\mathbb{R}^n$ *is (isomorphic to) the tangent bundle TM. The adjoint bundle* $\mathfrak{gl}(n,\mathbb{R})(\mathcal{F}(M))$ *is (isomorphic) to* $\mathrm{End}(TM) = T^*M \otimes TM$ *and the adjoint bundle* $\mathfrak{g}(P)$ *is a subbundle of* $\mathrm{End}(TM) = T^*M \otimes TM$. *Moreover, the above defined map $H \mapsto \nabla^H$ between* $\mathrm{GL}(n,\mathbb{R})$*-principal connections on $\mathcal{F}(M)$ and connections on M is a bijection.*

Proof. All assertions are proved in [Baum]. □

Remark 3.10. *More generally than the first assertion in Proposition 3.9, it is true that if P is a* G*-principal bundle, Q a reduction of P to* H *and* $\rho : G \to \mathrm{GL}(V)$ *a representation, then the vector bundle associated to P and ρ is isomorphic to the vector bundle associated to Q and* $\rho|_{\mathrm{H}}$.

We are interested in connections on M which are compatible with a given G-structure P in the sense that the corresponding $\mathrm{GL}(n,\mathbb{R})$-principal connection on $\mathcal{F}(M)$ is also a G-principal connection on P.

Definition 3.11. *Let P be a* G*-structure on an n-dimensional manifold M. We say that a connection ∇ on TM is a* G-connection *if the corresponding* $\mathrm{GL}(n,\mathbb{R})$*-principal connection H^∇ on the frame bundle $\mathcal{F}(M)$* reduces to P, *i.e. if H^∇ is a subbundle of TP. Note that then H^∇ is a* G*-principal connection on P and that ∇ is an* H*-connection for all Lie subgroups* H *of* $\mathrm{GL}(n,\mathbb{R})$ *with* $\mathrm{G} \subseteq \mathrm{H} \subseteq \mathrm{GL}(n,\mathbb{R})$. *Note further that to decide if a given connection ∇ is a* G*-connection one needs the concrete* G*-structure P and not only the abstract group* G *as the name seems to indicate.*

G-connections always exist and may be described as follows.

Lemma 3.12. *Let P be a G-structure on M. Then M admits a G-connection and the set of all G-connections on P is an affine space modelled on the real vector space $\Gamma\left(T^*M \otimes \mathfrak{g}(P)\right)$. If P is defined in the sense of Proposition 3.6 by the tensor fields $(T_1, \ldots, T_k) \in \mathcal{T}^{r_1,s_1}M \times \ldots \times \mathcal{T}^{r_k,s_k}M$, then a connection ∇ on TM is a G-connection if and only if $\nabla T_i = 0$ for $i = 1, \ldots, k$.*

Proof. The first assertion is proved, e.g., in [Baum]. The second also follows easily from the results in [Baum] but it is not stated directly there. Note however that [Sa2, Lemma 1.3] states the second assertion directly for the case of one tensor field, from which the case with k tensor fields follows by induction. □

Remark 3.13. *Let (M, g) be a pseudo-Riemannian manifold of dimension n and signature $(p, n-p)$. Then Lemma 3.12 states that the metric connections on M are exactly the $\mathrm{O}(p, n-p)$-connections on M. More generally, if G is a subgroup of $\mathrm{O}(p, n-p)$ and P is a G-structure on M, then each G-connection is metric with respect to the pseudo-Riemannian metric on M induced by P.*

An important invariant of a G-structure P is its *intrinsic torsion.*

Definition 3.14. *Let P be a G-structure and let $\sigma : T^*M \otimes \mathfrak{g}(P) \to \Lambda^2 T^*M \otimes TM$ be the anti-symmetrisation in the first two arguments using that $T^*M \otimes \mathfrak{g}(P) \subseteq T^*M \otimes T^*M \otimes TM$ by Proposition 3.9. The* intrinsic torsion *$\tau(P)$ of P is defined by*

$$\tau(P) := \left[T^{\nabla}(P)\right] \in \Gamma\left(\Lambda^2 T^*M \otimes TM/\mathrm{im}(\sigma)\right),$$

where ∇ is any G-connection on M and T^{∇} is its torsion. We call P torsion-free *if $\tau(P) = 0$. Note that by Lemma 3.12, the set of torsion tensors of G-connections is an affine space modelled on the real vector space $\Gamma(\mathrm{im}(\sigma))$. Hence, $\tau(P)$ is well-defined and P admits a torsion-free G-connection if and only if P is torsion-free.*

In the case when G is a subgroup of $\mathrm{O}(p, n-p)$ and $\mathfrak{g} \subseteq \mathfrak{so}(p, n-p)$ is non-degenerate with respect to the Killing form of $\mathfrak{so}(p, n-p)$, the description can be simplified as follows.

Remark 3.15. *Let M be an n-dimensional manifold, $\mathrm{G} \subseteq \mathrm{O}(p, n-p)$ and $\mathfrak{g}$ be the Lie algebra of G. Assume that $\mathfrak{g}$ is non-degenerate with respect to the Killing form of $\mathfrak{so}(p, n-p)$ and denote by $\mathfrak{g}^{\perp}$ the orthogonal complement of $\mathfrak{g}$ in $\mathfrak{so}(p, n-p)$. Then the adjoint action of $\mathrm{O}(p, n-p)$ on $\mathfrak{so}(p, n-p)$ induces by restriction an action of G on $\mathfrak{so}(p, n-p)$ and $\mathfrak{g}^{\perp}$ is a G-submodule of $\mathfrak{so}(p, n-p)$. We denote by $\mathfrak{g}^{\perp}(P)$ the vector bundle associated to P and the mentioned action of G on $\mathfrak{g}^{\perp}$. Since the map $\sigma_0 : (\mathbb{R}^n)^* \otimes \mathfrak{so}(p, n-p) \to \Lambda^2 (\mathbb{R}^n)^* \otimes \mathbb{R}^n$, defined as the anti-symmetrisation in the first two arguments, is an isomorphism, we obtain the following vector bundle isomorphisms.*

$$\Lambda^2 T^*M \otimes TM/\mathrm{im}(\sigma) \cong T^*M \otimes \mathfrak{so}(p, n-p)(O(M))/\left(T^*M \otimes \mathfrak{g}(P)\right) \cong T^*M \otimes \mathfrak{g}^{\perp}(P).$$

Definition 3.16. *Let M be an n-dimensional manifold and* G *be a subgroup of* $\mathrm{O}(p, n-p)$ *for some* $p \in \{0, 1 \ldots, n\}$ *such that* $\mathfrak{g} \subseteq \mathfrak{so}(p, n-p)$ *is non-degenerate with respect to the Killing form of* $\mathfrak{so}(p, n-p)$*. Moreover, let P be a* G*-structure on M, g be the induced pseudo-Riemannian metric of signature $(p, n-p)$ on M and ∇^g be the Levi-Civita connection of g. Then*

$$\nabla - \nabla^g \in \Gamma\left(T^*M \otimes \mathfrak{so}(p, n-p)(O(M))\right) = \Gamma\left(T^*M \otimes \mathfrak{g}(P) \oplus T^*M \otimes \mathfrak{g}^{\perp}(P)\right)$$

for all G*-connections ∇. Since the set of all* G*-connections is an affine space modelled on the vector space $\Gamma(T^*M \otimes \mathfrak{g}(P))$, there is a unique* G*-connection $\overline{\nabla}$ such that $\overline{\nabla} - \nabla^g$ is in $\Gamma\left(T^*M \otimes \mathfrak{g}^{\perp}(P)\right)$. This connection is called the* minimal connection (of P) *and using the vector bundle isomorphism in Remark 3.15, we get*

$$\tau(P) = \overline{\nabla} - \nabla^g. \tag{3.2}$$

Recall that the vector bundle $T^*M \otimes \mathfrak{g}^{\perp}(P)$ is the vector bundle associated to G and the representation $(\mathbb{R}^n)^* \otimes \mathfrak{g}^{\perp}$ of G. Decomposing this G-module into indecomposable G-submodules, we get a corresponding decomposition of the vector bundle $T^*M \otimes \mathfrak{g}^{\perp}(P) = \mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_l$. This gives us natural classes of G-structures P whose intrinsic torsion $\tau(P)$ at each point is contained in one or a sum of the subbundles $\mathcal{V}_i$. An equivalent way of describing these natural classes of G-structures is obtained by using the following lemma.

Proposition 3.17. *Let M be an n-dimensional manifold and* $\mathrm{G} \subseteq \mathrm{O}(p, n-p)$ *be a subgroup of* $\mathrm{O}(p, n-p)$ *such that $\mathfrak{g}$ is non-degenerate with respect to the Killing form of $\mathfrak{so}(p, n-p)$ and such that* G *is the common stabiliser of tensors $(S_1, \ldots, S_k) \in T^{r_1, s_1}\mathbb{R}^n \times T^{r_k, s_k}\mathbb{R}^n$. There is an injective vector bundle homomorphism*

$$\eta : T^*M \otimes \mathfrak{g}^{\perp}(P) \to T^*M \otimes (\mathcal{T}^{r_1, s_1}M \oplus \ldots \oplus \mathcal{T}^{r_k, s_k}M)$$

such that if P is a G*-structure defined by the tensor fields $(T_1, \ldots, T_k) \in \mathcal{T}^{r_1, s_1}M \times \ldots \times \mathcal{T}^{r_k, s_k}M$ with model tensors $(S_1, \ldots, S_k)$, then $\eta(\tau(P)) = -(\nabla^g T_1, \ldots \nabla^g T_k)$. Here, ∇^g is the Levi-Civita connection of the induced pseudo-Riemannian metric g of signature $(p, n-p)$ on M.*

Proof. We define a G-module homomorphism

$$\eta_0 : (\mathbb{R}^n)^* \otimes \mathfrak{g}^{\perp} \to (\mathbb{R}^n)^* \otimes (T^{r_1, s_1}\mathbb{R}^n \oplus \ldots \oplus T^{r_k, s_k}\mathbb{R}^n)$$

by

$$\eta_0(\alpha \otimes X) := \alpha \otimes (X.S_1, \ldots, X.S_k) \tag{3.3}$$

for $\alpha \in (\mathbb{R}^n)$ and $X \in \mathfrak{g}^{\perp}$. Since G is the common stabiliser of $(S_1, \ldots, S_k)$, the kernel of η_0 is trivial, i.e. η_0 is injective. Hence, η_0 induces an injective vector bundle homomorphism

$$\eta : T^*M \otimes \mathfrak{g}^{\perp}(P) \to T^*M \otimes (\mathcal{T}^{r_1, s_1}M \oplus \ldots \oplus \mathcal{T}^{r_k, s_k}M)$$

and

$$\eta\left(\overline{\nabla}-\nabla^g\right)=\left(\overline{\nabla}T_1-\nabla^gT_1,\ldots,\overline{\nabla}T_k-\nabla^gT_k\right)=-\left(\nabla^gT_1,\ldots,\nabla^gT_k\right)$$

due to Lemma 3.12. □

Hence, instead of decomposing the G-module $(\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp$ into indecomposable G-modules, we may equivalently decompose the G-module $\eta_0\left((\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp\right)$ into indecomposable G-modules and get the same natural classes of G-structures as before. We have a closer look at some examples in Section 3.2.

3.1.3 Holonomy theory

Recall for an n-dimensional pseudo-Riemannian manifold (M, g) of signature $(p, n-p)$ the holonomy group $Hol_x(g)$ at the point $x \in M$ is defined as

$$Hol_x(g) := \{P_\gamma | \gamma : [0,1] \to M \text{ piecewise smooth}, \gamma(0) = \gamma(1) = x\} \subseteq \mathrm{O}(T_xM, g_x),$$

where $P_\gamma : T_xM \to T_xM$ is the parallel transport map along γ with respect to the Levi-Civita connection ∇^g of g. By our convention, M is connected and so the holonomy groups at different points are conjugate to each other. Hence, for different points the representations of the holonomy groups on the corresponding tangent spaces are isomorphic. We call this representation the *holonomy representation* and denote it by $Hol(g)$. By identifying $\mathrm{O}(T_xM, g_x)$ with $\mathrm{O}(p, n-p)$, $Hol_x(g)$ becomes a subgroup of $\mathrm{O}(p, n-p)$ and the holonomy representation is then the standard representation of this group on $\mathbb{R}^n$. We denote the mentioned subgroup also by $Hol(g)$ and call it the *holonomy group of g*. Note that $Hol(g) \subseteq \mathrm{O}(p, n-p)$ is only defined up to conjugation in $\mathrm{O}(p, n-p)$. The *restricted holonomy group* $Hol_0(g)$ is constructed analogously to the holonomy group of g but instead of considering all loops at a given point we restrict to all contractible loops at the point. It is a subgroup of $\mathrm{SO}_0(p, n-p)$, defined up to conjugacy in $\mathrm{O}(p, n-p)$, and one can show that it is exactly the identity component of $Hol(g)$. Moreover, if $\left(\tilde{M}, \tilde{g}\right)$ is the pseudo-Riemannian universal covering of (M, g), then $Hol(\tilde{g}) = Hol_0(g)$.

A natural question which arises at this point is which subgroups of $\mathrm{SO}_0(p, n-p)$ can occur as holonomy groups of simply-connected pseudo-Riemannian manifolds (M, g). If (M, g) is simply-connected and additionally complete, then the well-known decomposition theorem of de Rham and Wu, cf. [dR] and [Wu], states that (M, g) is a pseudo-Riemannian product of *indecomposable* pseudo-Riemannian manifolds (M_i, g_i), $i = 1\ldots, k$, and $Hol(g) = Hol(g_1) \times \ldots \times Hol(g_k)$. Here, an *indecomposable* pseudo-Riemannian manifold is a pseudo-Riemannian manifold with indecomposable holonomy representation. Hence, the classification of the holonomy groups of simply-connected complete pseudo-Riemannian manifolds reduces to the classification of the holonomy groups of simply-connected complete indecomposable pseudo-Riemannian manifolds. We refer the reader to [Besse, §10.107] for a discussion of non-completeness in the Riemannian case.

An important subclass of simply-connected indecomposable complete pseudo-Riemannian manifolds is given by the class of simply-connected indecomposable pseudo-Riemannian symmetric spaces. These spaces can be treated completely algebraically. If (M, g) is even *irreducible*, i.e. the holonomy representation is irreducible, then a complete list of theses spaces and their holonomy groups has been obtained by É. Cartan [Car] in the Riemannian case and by Berger [Ber2] in the pseudo-Riemannian case. We refer the reader also to [Besse] or [He] for a modern treatment of the Riemannian case. Note that in the Riemannian case indecomposability is the same as irreducibility, which is not true in the pseudo-Riemannian due to the existence of invariant degenerate subspaces. Hence, the results of Berger do not cover all cases. For a nicely written summary of known results on indecomposable pseudo-Riemannian symmetric spaces we refer the reader to [KO].

For simply-connected indecomposable pseudo-Riemannian manifolds which are not locally symmetric, a general classification of the holonomy groups seems to be out of reach, cf. [GL] for a summary of known results. However, in the irreducible case, Berger found in 1955 [Ber1] an astonishingly short list of candidates for holonomy groups by purely algebraic methods.

Theorem 3.18 (Berger). *(a) Let (M, g) be a simply-connected irreducible n-dimensional Riemannian manifold which is not locally symmetric. Then $Hol(g)$ is either equal to* $\mathrm{SO}(n)$ *or contained in the following list:*

Table 3.1: Berger's list in the Riemannian case

n	$Hol(g)$	n	$Hol(g)$
$2m$	$\mathrm{U}(m)$	$2m$	$\mathrm{SU}(m)$
$4m$	$\mathrm{Sp}(m)$	$4m$	$\mathrm{Sp}(m)\mathrm{Sp}(1)$
7	G_2	8	$\mathrm{Spin}(7)$

(b) Let (M, g) be a simply-connected irreducible n-dimensional pseudo-Riemannian manifold of signature $(p, n-p)$, $p \notin \{0, n\}$ which is not locally symmetric. Then $Hol(g)$ is either equal to $\mathrm{SO}_0(p, n-p)$ *or contained in the following list:*

Table 3.2: Berger's list in the pseudo-Riemannian case

$n, (p, n-p)$	$Hol(g)$	$n, (p, n-p)$	$Hol(g)$
$2m, (2r, 2m-2r)$	$\mathrm{U}(r, m-r)$	$2m, (2r, 2m-2r)$	$\mathrm{SU}(r, m-r)$
$2m, (m, m)$	$\mathrm{SO}(m, \mathbb{C})$	-	-
$4m, (4r, 4m-4r)$	$\mathrm{Sp}(r, m-r)$	$4m, (4r, 4m-4r)$	$\mathrm{Sp}(r, m-r)\mathrm{Sp}(1)$

Table 3.2: Berger's list in the pseudo-Riemannian case

$n, (p, n-p)$	$Hol(g)$	$n, (p, n-p)$	$Hol(g)$
$4m, (2m, 2m)$	$\mathrm{Sp}(2m, \mathbb{R})\mathrm{SL}(2, \mathbb{R})$	$8m, (4m, 4m)$	$\mathrm{Sp}(2m, \mathbb{C})\mathrm{SL}(2, \mathbb{C})$
$7, (3, 4)$	G_2^*	$8, (4, 4)$	$\mathrm{Spin}_0(3, 4)$
$14, (7, 7)$	$(\mathrm{G}_2)_{\mathbb{C}}$	$16, (8, 8)$	$\mathrm{Spin}(7, \mathbb{C})$

Remark 3.19.

- *To be more precise, the subgroup $Hol(g)$ of $\mathrm{SO}_0(p, n-p)$ given in Theorem 3.18 is determined only up to conjugation with elements in $\mathrm{O}(p, n-p)$. Stated differently, Theorem 3.18 gives us the possible holonomy representations, where in each case the representation of $Hol(g)$ is the standard one on $\mathbb{R}^n$. For exact definitions of some of the above groups, we refer the reader to [Br3].*

- *A reduction of the holonomy group of a Riemannian manifold (M, g) to a proper subgroup of $\mathrm{SO}(n)$ has influence on the geometry. Namely, if $Hol(g)$ is a subgroup of $\mathrm{Sp}(m)\mathrm{Sp}(1)$, then g is Einstein and if $Hol(g)$ is a subgroup of $\mathrm{SU}(m)$, $\mathrm{Sp}(m)$, G_2 or $\mathrm{Spin}(7)$, then g is even Ricci-flat, cf. [Sa2].*

- *A question which arises at this point is if we can say anything about the possible holonomy groups of arbitrary affine connections on manifolds. Astonishingly, Hano and Ozeki showed in [HO] that any connected Lie subgroup of $\mathrm{GL}(n, \mathbb{R})$, $n \geq 2$, is the holonomy group of some affine connection on an open ball in $\mathbb{R}^n$.*

- *A natural restriction of the last problem is to ask for the possible irreducible restricted holonomies of torsion-free affine connections which are not locally symmetric. Berger also considered this problem in [Ber1]. He wrote down a list of connected Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$ and claimed that this list contains, up to a finite number, all possible irreducible restricted holonomies of torsion-free affine connections. This claim turned out to be wrong, cf. [CMS], where an infinite family of such holonomies not contained in the list in [Ber1] has been found. The problem has finally been solved by Merkulov and Schwachhöfer [MS1], [MS2]. For a more detailed summary of the history of the problem, we refer the reader to [MS1] and [Schw]. Note that [Schw] gives a different proof of the mentioned classification problem using Berger's original approach to the problem.*

We should emphasise that the two lists are not Berger's original lists. In the Riemannian case, Berger's list also contained the case of holonomy equal to $\mathrm{Spin}(9)$ in 16 dimensions. But Alekseevsky [Al] and, independently, Brown and Gray [BG2] showed that Riemannian manifolds with holonomy equal to $\mathrm{Spin}(9)$ are automatically locally symmetric. A similar argument may be applied to other real forms of $\mathrm{Spin}(9, \mathbb{C})$, which excludes two more possible candidates in the pseudo-Riemannian case, cf. [Br1]. Moreover, Bryant excluded one more case and added two cases of holonomies of simply-connected pseudo-Riemannian manifolds in [Br3].

Berger's Theorem gives only possible candidates of holonomy groups. In fact, they all occur as holonomy groups but it had taken some time to give an example for each of the cases in Table 3.1 and in Table 3.2, cf. [Br2] for a summary in the Riemannian case. For the last missing cases, namely G_2, G_2^*, $(\mathrm{G}_2)_{\mathbb{C}}$, $\mathrm{Spin}(7)$, $\mathrm{Spin}_0(3,4)$ and $\mathrm{Spin}(7,\mathbb{C})$, examples have been constructed by Bryant in [Br1] over 30 years after Berger published his list. We say that a pseudo-Riemannian manifold (M,g) has *exceptional holonomy* if its holonomy is equal to one of these groups. Note that the exceptional holonomy groups are exactly those which do not occur in series. For the Riemannian exceptional holonomies G_2 and $\mathrm{Spin}(7)$, complete examples have been constructed by Bryant and Salamon in [BrSa]. Finally, the existence of compact examples with these holonomies has been shown by Joyce in [J1] and [J2]. However, still not many explicit examples with exceptional holonomy are known and it is still of interest to find new ones. One method, which is one major motivation for classifying Lie algebras admitting half-flat structure or cocalibrated structures in this thesis, is the *Hitchin flow* [Hi1]. This flow yields, starting with a half-flat or cocalibrated structure, a (non-compact, non-complete) pseudo-Riemannian manifold with exceptional holonomy equal to G_2^{ϵ} or $\mathrm{Spin}^{\epsilon}(7)$, respectively. We discuss the Hitchin flow in more detail in Section 7.1.

Bryant's construction of examples of pseudo-Riemannian manifolds with exceptional holonomy heavily relies on the following theorem, which connects torsion-free G-structures to reductions of the holonomy group.

Theorem 3.20 (Holonomy principle)**.** *Let M be an n-dimensional manifold, P a* G*-structure on M with* G *being a subgroup of* $\mathrm{O}(p,n-p)$ *such that* $\mathfrak{g} \subseteq \mathfrak{so}(p,n-p)$ *is non-degenerate with respect to the Killing form of* $\mathfrak{so}(p,n-p)$*. Assume that P is defined by the tensor fields* $(T_1,\ldots T_k) \in \mathcal{T}^{r_1,s_1}M \times \ldots \times \mathcal{T}^{r_k,s_k}M$ *and denote by g the induced pseudo-Riemannian metric of signature $(p,n-p)$ on M and by ∇^g the Levi-Civita connection of g. Then the following are equivalent:*

(i) $\nabla^g T_i = 0$ *for all* $i = 1,\ldots,k$.

(ii) $\tau(P) = 0$.

Moreover, both (i) and (ii) imply $Hol(g) \subseteq \mathrm{G}$.

Proof. Proposition 3.17 gives the equivalence of (i) and (ii). Condition (i) implies $Hol(g) \subseteq \mathrm{G}$ by [J3, Proposition 2.5.2]. □

Remark 3.21. *Usually, cf. e.g. [Baum], a slightly different and more general assertion is called* holonomy principle*. This assertion states that for all $x \in M$ there is a one-to-one* *correspondence between $Hol_x(g)$ invariant tensors $S \in T^{r,s}T_xM$ and ∇^g-parallel tensor fields $T \in \mathcal{T}^{r,s}M$ with $T_x = S$.*

Suppose that $\mathrm{G} \subseteq \mathrm{O}(p, n-p)$ has the properties as in Theorem 3.20. Then the construction of a pseudo-Riemannian metric of signature $(p, n-p)$ with holonomy contained in $\mathrm{O}(p, n-p)$ can be done by constructing a torsion-free G-connection P. Equivalently, one may construct tensor fields $T_1, \ldots, T_k$ defining a G-structure P with $\nabla^g T_i = 0$ for $i = 1, \ldots, k$. We will see in Section 3.2 that in many cases the equations $\nabla^g T_1 = 0, \ldots, \nabla^g T_k = 0$ can be simplified a lot by regarding the decomposition of the G-module $(\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp$ or, equivalently, of the G-module $\eta_0\left((\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp\right)$ into indecomposable G-modules. But before we discuss concrete examples in more detail, we remind the reader of the Theorem of Ambrose-Singer which allows to compute the *holonomy algebra* $\mathfrak{hol}_x(g)$ at the point $x \in M$, i.e. the Lie algebra of the holonomy group $Hol_x(g)$ at the point $x \in M$, via the curvature.

Theorem 3.22 (Ambrose-Singer)**.** *Let (M, g) be a pseudo-Riemannian manifold and denote by R^g its curvature tensor. For a curve $\gamma : [0,1] \to M$ with $\gamma(0) = x$ and $\gamma(1) = y$ set $(\gamma^* R^g)(v, w) := (P_\gamma)^{-1} \circ (R^g)_y \left(P_\gamma(v), P_\gamma(w)\right) \circ P_\gamma$ for $v, w \in T_xM$, where P_γ is the parallel transport map along γ. Then the holonomy algebra $\mathfrak{hol}_x(g)$ of g at the point $x \in M$ is given by*

$$\mathfrak{hol}_x(g) = \mathrm{span}(\,(\gamma^* R^g)(v, w)|\, v, w \in T_xM,\ \gamma : [0,1] \to M,\ \gamma(0) = x,\ \gamma(1) = y,\ y \in M).$$

Remark 3.23. *Theorem 3.22 shows that* $\mathrm{span}(\,(R^g)_x(v, w)|\, v, w \in T_xM)$ *is a subspace of $\mathfrak{hol}_x(g)$. Often it suffices to compute the dimension of this space at certain points to compute the holonomy algebra. For example if we know that $Hol(g)$ is a subgroup of a connected Lie group* G *and there is one point $x \in M$ with* $\dim(\mathrm{span}(\,(R^g)_x(v, w)|\, v, w \in T_xM)) = \dim(\mathrm{G})$, *we can deduce that $Hol(g) = \mathrm{G}$.*

3.2 Intrinsic torsion of particular G-structures

In this section, we look at the intrinsic torsion of $\mathrm{U}^\epsilon(p, m-p)$-, $\mathrm{SU}^\epsilon(p, m-p)$-, G_2^ϵ- and $\mathrm{Spin}^\epsilon(7)$-structures. Recall that all these structures were treated on the vector space level in some detail in the Sections 2.3 - 2.5. Recall further that the intrinsic torsion of a G-structure P with G being a subgroup of $\mathrm{O}(q, n-q)$ such that $\mathfrak{g}$ is a non-degenerate subspace of $\mathfrak{so}(q, n-q)$ is a section of the vector bundle associated to P and the G-module $(\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp$ and that this G-module is equivalent to the G-module $\eta_0\left((\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp\right)$ with η_0 as in Equation (3.3). We use the results obtained in [ChiSa], [Fe], [FG], [GH], [GM], [Kath1], [MC4] and [SHPhD] and decompose the G-module $(\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp \cong \eta_0\left((\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp\right)$ into a sum of irreducible G-modules for (most of) the groups G mentioned at the beginning of this section. We do not give an explicit description of the summands here. Instead, we present in the relevant cases a description of the intrinsic torsion solely in terms of the exterior derivatives of the defining differential forms and other related differential forms and indicate how the different components of the intrinsic torsion appear as certain components of these derivatives. Some of the results have, to the best of the author's knowledge, only been written down explicitly in the literature for the Riemannian case. We transfer them to the pseudo-Riemannian case without going into much detail concerning this transfer since in all cases one may literally write down the same proof as in the Riemannian case.

Notation 3.24. *For a* G*-structure* P *on an* n*-dimensional manifold* M *with* $\mathrm{G} \subseteq \mathrm{O}(p, n-p)$ *such that* $\mathfrak{g} \subseteq \mathfrak{so}(p, n-p)$ *is non-degenerate with respect to the Killing form of* $\mathfrak{so}(p, n-p)$*, we denote by capital Latin letters (e.g.* W*) components of the decomposition of the* G*-module* $(\mathbb{R}^n)^* \otimes \mathfrak{g}^\perp$*. The corresponding subbundles of* $T^*M \otimes \mathfrak{g}^\perp(P)$ *are denoted by calligraphic letters (e.g.* $\mathcal{W}$*) as well as the class of* G*-structures with intrinsic torsion everywhere in the subbundle* $\mathcal{W}$*. Finally, we denote by small Latin letters (e.g.* w*) the part of the intrinsic torsion lying in* $\mathcal{W}$*.*

Remark 3.25.

- *All real finite-dimensional representations of real semisimple Lie groups are completely reducible, cf. [K]. This applies to* $\mathrm{SU}^\epsilon(p, m-p)$*,* G_2^ϵ *and* $\mathrm{Spin}^\epsilon(7)$*.*
- *The condition that* $\mathfrak{g}$ *is a non-degenerate subspace of* $\mathfrak{so}(p, n-p)$ *with respect to the Killing form of* $\mathfrak{so}(p, n-p)$ *is fulfilled in the cases* $\mathfrak{g} = \mathfrak{u}^\epsilon(p, m-p)$*,* $\mathfrak{g} = \mathfrak{su}^\epsilon(p, m-p)$*,* $\mathfrak{g} = \mathfrak{g}_2^\epsilon$ *and* $\mathfrak{g} = \mathfrak{spin}^\epsilon(7)$*. Therefore, note that the assertion is obviously true for the Euclidean cases* $\mathfrak{g} = \mathfrak{u}(m)$*,* $\mathfrak{su}(m)$*,* $\mathfrak{g}_2$*,* $\mathfrak{spin}(7)$*. Moreover, for an arbitrary real Lie subalgebra* $\mathfrak{g}$ *of* $\mathfrak{so}(p, n-p)$*,* $\mathfrak{g}$ *is non-degenerate in* $\mathfrak{so}(p, n-p)$ *if and only if the complexification* $\mathfrak{g}_\mathbb{C}$ *is non-degenerate in* $\mathfrak{so}(p, n-p)_\mathbb{C} = \mathfrak{so}(n, \mathbb{C})$

3.2.1 Intrinsic torsion of $\mathrm{SU}^\epsilon(p, m-p)$-structures

Before we discuss the intrinsic torsion of $\mathrm{SU}^\epsilon(p, m-p)$-structures, we summarise what is known about the intrinsic torsion of $\mathrm{U}^\epsilon(p, m-p)$-structures. Recall that an $\mathrm{U}^\epsilon(p, m-p)$-structure may be described by a pair of a two-form ω and a pseudo-Riemannian metric g satisfying a certain compatibility relation. Hence, its intrinsic torsion can be described by $\nabla^g\omega$ according to Proposition 3.17 since $\nabla^g g = 0$. By Equation (3.3), the $\mathrm{U}^\epsilon(p, m-p)$-module we want to decompose is given by

$$\begin{aligned} W &:= \left\{ \alpha \in \left(\mathbb{R}^{2m}\right)^* \otimes \Lambda^2 \left(\mathbb{R}^{2m}\right)^* \middle| \, \alpha(u, J_\epsilon v, J_\epsilon w) = \epsilon \alpha(u, v, w) \right\} \\ &= \left(\mathbb{R}^{2m}\right)^* \otimes \left[\left[\Lambda^{2,0} \left(\mathbb{R}^{2m}\right)^*\right]\right] \end{aligned}$$

with J_ϵ being the standard ϵ-complex structure on $\mathbb{R}^{2m}$. The decomposition of W into irreducible summands in the case $\mathrm{U}(m)$ is due to Gray and Hervella [GH]. If $m \geq 3$, then $W = W_1 \oplus W_2 \oplus W_3 \oplus W_4$ with irreducible non-zero $\mathrm{U}(m)$-submodules W_i. The classes $W_1, \ldots, W_4$ can literally be defined, with the obvious sign changes, also in the cases $\mathrm{U}(p, m-p)$ for arbitrary p and $\mathrm{U}^1(p, m-p) \cong \mathrm{GL}(m, \mathbb{R})$, and again $W = W_1 \oplus \ldots \oplus W_4$ as $\mathrm{U}^\epsilon(p, m-p)$-submodules. For $\epsilon = -1$, the decomposition stays irreducible, cf. [SHPhD], and for $\epsilon = 1$, the spaces W_i decompose further into two irreducible $\mathrm{U}^1(p, m-p) \cong \mathrm{GL}(m, \mathbb{R})$-summands, cf. [GM].

We like to mention some of the classes. First of all, the holonomy principle shows that the class $\{0\}$ consists exactly of the pseudo-Kähler or para-Kähler manifolds, respectively. Moreover, the class $\mathcal{W}_1$ is the class of nearly pseudo-Kähler or nearly para-Kähler manifolds, respectively, the class $\mathcal{W}_2$ are exactly those with $d\omega = 0$ and these run under the name almost pseudo-Kähler manifolds or almost para-Kähler manifolds, respectively, and the class $\mathcal{W}_3 \oplus \mathcal{W}_4$ consists of those with integrable J, i.e. they are the pseudo-Hermitian or para-Hermitian manifolds, respectively.

Next, we consider the intrinsic torsion of $\mathrm{SU}^\epsilon(p, m-p)$-structures. Since $(\mathfrak{su}^\epsilon(p, n-p))^\perp = (\mathfrak{u}^\epsilon(p, n-p))^\perp \oplus \mathbb{R} J_\epsilon$ as $\mathrm{SU}^\epsilon(p, m-p)$-modules, we get

$$V := \left(\mathbb{R}^{2m}\right)^* \otimes (\mathfrak{su}^\epsilon(p, n-p))^\perp \cong W_1 \oplus W_2 \oplus W_3 \oplus W_4 \oplus \left(\mathbb{R}^{2m}\right)^* = \sum_{i=1}^{5} W_i \tag{3.4}$$

as $\mathrm{SU}^\epsilon(p, m-p)$-modules with $W_5 := \left(\mathbb{R}^{2m}\right)^*$.

Consider first the case $m = 3$. This case has been treated in [ChiSa] and it has been shown that $W_1 = W_1^+ \oplus W_1^-$ and $W_2 = W_2^+ \oplus W_2^-$ as irreducible $\mathrm{SU}(3)$-modules and that W_3, W_4 and W_5 are irreducible. The classes W_i^+ and W_i^-, $i = 1, 2$, can again literally be defined as in the $\mathrm{SU}(3)$-case also for the $\mathrm{SU}(p, 3-p)$-case and for $\mathrm{SU}^1(p, 3-p) \cong \mathrm{SL}(3, \mathbb{R})$. Moreover, the decomposition $V = W_1^+ \oplus W_1^- \oplus W_2^+ \oplus W_2^- \oplus W_3 \oplus W_4 \oplus W_5$ is irreducible for $\mathrm{SU}(p, 3-p)$ with arbitrary p, cf. [SHPhD]. W_1 is the real two-dimensional $\mathrm{SU}^\epsilon(p, 3-p)$-module $\left[\left[\Lambda^{3,0}\left(\mathbb{R}^6\right)^*\right]\right]$ and it decomposes into the two real one-dimensional trivial $\mathrm{SU}^\epsilon(p, 3-p)$-modules $\mathbb{R} \cdot \rho_\epsilon$ and $\mathbb{R} \cdot (J_\epsilon)^* \rho_\epsilon$. Hence, we may identify w_1^+ and w_1^- with functions on M, which we do in the following. The space W_2 is a 16-dimensional $\mathrm{SU}^\epsilon(p, 3-p)$-module isomorphic to $\left[\left[\Lambda^{3,0}\left(\mathbb{R}^6\right)^*\right]\right] \otimes \left[\Lambda_0^{1,1}\left(\mathbb{R}^6\right)^*\right]$, and so isomorphic to $2\left[\Lambda_0^{1,1}\left(\mathbb{R}^6\right)^*\right]$. Here, $\left[\Lambda_0^{1,1}\left(\mathbb{R}^6\right)^*\right]$ are the real forms of type $(1,1)$ whose wedge product with ω_0^2 is 0. Thus, w_2^+ and w_2^- are real two forms on M whose wedge product with ω^2 is 0. The 12-dimensional $\mathrm{SU}^\epsilon(p, 3-p)$-module W_3 is equivalent to the $\mathrm{SU}^\epsilon(p, 3-p)$-module $\left[\left[\Lambda_0^{2,1}\left(\mathbb{R}^6\right)^*\right]\right]$, which are the real forms of type $(2,1)$ and $(1,2)$ whose wedge product with ω_0 vanishes. Hence, w_3 is a three-form on M such that the wedge product with ω vanishes. W_4 and W_5 are both equivalent to the $\mathrm{SU}^\epsilon(p, 3-p)$-module $\left(\mathbb{R}^6\right)^*$. Thus, w_4 and w_5 are one-forms on M. By [ChiSa] and [SHPhD], we have the following decomposition of the $\mathrm{SU}^\epsilon(p, 3-p)$-modules of all three-forms and four-forms on $\mathbb{R}^6$ into $\mathrm{SU}^\epsilon(p, 3-p)$-submodules, which is irreducible for $\epsilon = -1$:

$$\Lambda^3 \left(\mathbb{R}^6\right)^* = \mathbb{R} \cdot \rho_\epsilon \oplus \mathbb{R} \cdot J^*_{\rho_\epsilon}\rho_\epsilon \oplus \left[\left[\Lambda^{2,1}_0 \left(\mathbb{R}^6\right)^*\right]\right] \oplus \left(\mathbb{R}^6\right)^* \wedge \omega_0$$
$$\Lambda^4 \left(\mathbb{R}^6\right)^* = \mathbb{R} \cdot \omega_0^2 \oplus \left[\Lambda^{1,1}_0 \left(\mathbb{R}^6\right)^*\right] \wedge \omega_0 \oplus \left(\mathbb{R}^6\right)^* \wedge \rho_\epsilon.$$

Using the above mentioned identifications of the different components of the intrinsic torsion with certain differential forms on M and the just mentioned decompositions of the three- and four-forms, one can show, cf. [ChiSa] for SU(3) and [SHPhD] for arbitrary $\mathrm{SU}^\epsilon(p, 3-p)$, that the components of the intrinsic torsion can be recovered from the the exterior derivatives of the defining forms $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$ and of the pullback $J^*_\rho \rho$ as follows:

$$\begin{aligned} d\omega &= \frac{3}{2} w_1^- \rho - \frac{3}{2} w_1^+ J^*_\rho \rho + w_3 + w_4 \wedge \omega, \\ d\rho &= w_1^+ \omega^2 + w_2^+ \wedge \omega + w_5 \wedge \rho, \\ d(J^*_\rho \rho) &= w_1^- \omega^2 + w_2^- \wedge \omega - \epsilon (J^*_\rho w_5) \wedge \rho. \end{aligned} \tag{3.5}$$

Equation (3.5) gives us the following characterisation of the torsion-free $\mathrm{SU}^\epsilon(p, 3-p)$-structures:

Corollary 3.26. *Let $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$ be an $\mathrm{SU}^\epsilon(p, 3-p)$-structure on a six-dimensional manifold M. Then (ω, ρ) is torsion-free if and only if $d\omega = 0$, $d\rho = 0$ and $d(J^*_\rho \rho) = 0$.*

Many interesting classes of $\mathrm{SU}^\epsilon(p, 3-p)$-structures naturally appear by distinguishing them via their intrinsic torsion. In this thesis, we are only interested in the following class.

Definition 3.27. *Let $(\omega, \rho) \in \Omega^2 M \times \Omega^3 M$ be an $\mathrm{SU}^\epsilon(p, 3-p)$-structure on a six-dimensional manifold M. (ω, ρ) is called* half-flat *if the intrinsic torsion lies entirely in $\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$, i.e. if $w_1^+ = w_2^+ = w_4 = w_5 = 0$. By Equation (3.5), this is equivalent to $d\omega^2 = 2 d\omega \wedge \omega = 0$ and $d\rho = 0$. Therefore, note that a direct computation in a basis as in Lemma 2.1 shows that the wedge-product of a one-form with ω^2 vanishes if and only if the one-form itself is 0.*

Remark 3.28. *$W_1^- \oplus W_2^- \oplus W_3$ is a 21-dimensional $\mathrm{SU}^\epsilon(p, 3-p)$-submodule of the 42-dimensional $\mathrm{SU}^\epsilon(p, 3-p)$-module $\left(\mathbb{R}^6\right)^* \oplus \mathfrak{su}^\epsilon(p, 3-p)$. In this sense half-flat $\mathrm{SU}^\epsilon(p, 3-p)$-structures are "half torsion-free".*

For $m \geq 4$, we restrict to the $\mathrm{SU}(m)$-case. This case has been considered by Martín Cabrera in [MC4] and he showed that the decomposition $V = \sum_{i=1}^5 W_i$ is a decomposition into irreducible $\mathrm{SU}(m)$-modules. Moreover, he proves a nice characterisation of torsion-free $\mathrm{SU}(m)$-structures, which will play an important role in Section 7.2 to prove a reduction result for the holonomy of the Riemannian manifold obtained via the Hitchin flow on almost Abelian Lie algebras.

Proposition 3.29. *Let M be a $2m$-dimensional manifold, $m \geq 4$ and $(\omega, \Psi) \in \Omega^2 M \times \Omega^m M \otimes \mathbb{C}$ be an $\mathrm{SU}(m)$-structure on M. Then (ω, Ψ) is torsion-free if and only if $d\omega = 0$ and $d\,\mathrm{Re}(\Psi) = 0$.*

3.2.2 Intrinsic torsion of G_2^ϵ-structures

By Proposition 3.17, the intrinsic torsion of a G_2^ϵ-structure $\varphi \in \Omega^3 M$ on a seven-dimensional manifold M is given by $\nabla^{g_\varphi}\varphi$, g_φ being the induced pseudo-Riemannian metric. Denote for $A \in \{\mathbb{O}, \mathbb{O}_s\}$ by $F : \mathrm{Im}(A) \to \mathbb{R}^7$ the linear isomorphism defined in Definition 1.19 and set $\times_{-1} := F^* \times_{\mathbb{O}}$ and $\times_1 := F^* \times_{\mathbb{O}_s}$, where $\times_A$ is the real two-fold cross product on $\left(\mathrm{Im}(A), g_A|_{\mathrm{Im}(A)}\right)$. Then $\times_{-1}$ is a two-fold cross product on $\left(\mathbb{R}^7, \langle\cdot,\cdot\rangle_7\right)$ and $\times_1$ is a two-fold cross product on $\left(\mathbb{R}^7, \langle\cdot,\cdot\rangle_{3,4}\right)$. The G_2^ϵ-module $X := \eta_0\left(\left(\mathbb{R}^7\right)^* \otimes (\mathfrak{g}_2^\epsilon)^\perp\right)$ defined in Equation (3.3) is given by

$$X := \left\{\alpha \in \left(\mathbb{R}^7\right)^* \otimes \Lambda^3\left(\mathbb{R}^7\right)^* \middle| \alpha(u, v, w, v \times_\epsilon w) = 0 \,\forall u, v, w \in \mathbb{R}^7\right\}.$$

The decomposition of the G_2-module X into irreducible submodules has been done by Fernández and Gray in [FG]. We have $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$ with irreducible G_2-modules X_i, $i = 1, \ldots, 4$. We can define G_2^*-submodules X_i of X literally as the ones in the G_2-case and get $X = X_1 \oplus X_2 \oplus X_3 \oplus X_4$ as G_2^*-modules. The dimensions of the modules are given by $\dim(X_1) = 1$, $\dim(X_2) = 14$, $\dim(X_3) = 27$ and $\dim(X_4) = 7$ and they are also irreducible in the G_2^*-case by [Kath1]. Note that there is, up to equivalence, exactly one irreducible G_2^ϵ-module in each of the dimensions 1, 7, 14 and 27. Hence, X_1 is the trivial representation, X_2 is the adjoint representation, X_3 is the representation $S_0^2\left(\mathbb{R}^7\right)$ of trace-free symmetric two-tensors and X_4 the standard representation on $\mathbb{R}^7$. Thus, x_1 is a function on M and x_4 is a one-form on M. To interpret x_2 and x_3 as differential forms, we recall that by [FG] and [Kath1] we have

$$\Lambda^2\left(\mathbb{R}^7\right)^* = \Lambda^2_7 \oplus \Lambda^2_{14}, \qquad \Lambda^3\left(\mathbb{R}^7\right)^* = \mathbb{R}\cdot\varphi_\epsilon \oplus \Lambda^3_7 \oplus \Lambda^3_{27}$$

with

$$\Lambda^2_7 := \left\{\omega \in \Lambda^2\left(\mathbb{R}^7\right)^* \middle| \omega \wedge \varphi_\epsilon = 2 \star_{\varphi_\epsilon} \omega\right\}, \quad \Lambda^2_{14} := \left\{\omega \in \Lambda^2\left(\mathbb{R}^7\right)^* \middle| \omega \wedge \varphi_\epsilon = - \star_{\varphi_\epsilon} \omega\right\},$$
$$\Lambda^3_7 := \left\{\star_{\varphi_\epsilon}(\alpha \wedge \varphi_\epsilon) \middle| \alpha \in \left(\mathbb{R}^7\right)^*\right\}, \quad \Lambda^3_{27} := \left\{\psi \in \Lambda^3\left(\mathbb{R}^7\right)^* \middle| \psi \wedge \varphi_\epsilon = 0,\, \psi \wedge \star_{\varphi_\epsilon}\varphi_\epsilon = 0\right\}$$

as decompositions of the two- and the three-forms on $\mathbb{R}^7$ into irreducible G_2^ϵ-modules. Hence, x_2 may be considered as a two-form on M with $x_2 \wedge \varphi = - \star_\varphi x_2$ and x_3 as a three-form on M with $x_3 \wedge \varphi = 0$ and $x_3 \wedge \star_\varphi \varphi = 0$. Moreover, by applying the Hodge star operator, we get corresponding decompositions of the five- and four-forms. Using these decompositions, one can show, cf. [Br5] and [MC2], that the intrinsic torsion of a G_2-structure is encoded in the exterior derivatives of φ and $\star_\varphi\varphi$ as follows:

$$d\varphi = x_1 \star_\varphi \varphi + 3\, x_4 \wedge \varphi + \star_\varphi x_3, \quad d \star_\varphi \varphi = 4\, x_4 \wedge \star_\varphi\varphi + x_2 \wedge \varphi. \tag{3.6}$$

The proof can be transferred one-to-one to the G_2^*-case. Thus, Equation (3.6) holds also for G_2^*-structures and we obtain the following result:

Proposition 3.30. *A G_2^ϵ-structure φ on a seven-dimensional manifold M is torsion-free if and only if $d\varphi = 0$ and $d \star_\varphi \varphi = 0$.*

We have several interesting classes of G_2-structures.

Definition 3.31. *Let $\varphi \in \Omega^3 M$ be a G_2^ϵ-structure on a seven-dimensional manifold. φ is called* calibrated *if the intrinsic torsion is contained in $\mathcal{X}_2$, i.e. if $x_1 = x_3 = x_4 = 0$. By Equation (3.6) this is equivalent to $d\varphi = 0$. φ is called* nearly parallel *if the intrinsic torsion is contained in $\mathcal{X}_1$, i.e. if $x_2 = x_3 = x_4 = 0$. Equation (3.6) yields that this is equivalent to $d\varphi = \lambda \star_\varphi \varphi$ and $d \star_\varphi \varphi = 0$ for some constant $\lambda \in \mathbb{R}$. Finally, φ is called* cocalibrated *if the intrinsic torsion is contained in $\mathcal{X}_1 \oplus \mathcal{X}_3$, i.e. if $x_2 = x_4 = 0$. Equation (3.6) shows that this is equivalent to $d \star_\varphi \varphi = 0$.*

3.2.3 Intrinsic torsion of $\mathrm{Spin}^\epsilon(7)$-structures

Here, we shortly review the different classes of $\mathrm{Spin}^\epsilon(7)$-structures on eight-dimensional manifolds which appear when one decomposes the $\mathrm{Spin}^\epsilon(7)$-module $\left(\mathbb{R}^8\right)^* \otimes (\mathfrak{spin}^\epsilon(7))^\perp$ into irreducible components. All the results for $\mathrm{Spin}(7)$ can be found in [Fe] and [MC1] and we can transfer these results to the $\mathrm{Spin}_0(3,4)$-case similarly to the previous subsections.

The $\mathrm{Spin}^\epsilon(7)$-module $Z := \left(\mathbb{R}^8\right)^* \otimes (\mathfrak{spin}^\epsilon(7))^\perp$ decomposes into a sum $Z = Z_1 \oplus Z_2$ of two irreducible $\mathrm{Spin}^\epsilon(7)$-modules with $\dim(Z_1) = 48$ and $\dim(Z_2) = 8$. The class $\mathcal{Z}_1$ of $\mathrm{Spin}^\epsilon(7)$-structures $\Phi \in \Omega^4 M$ is characterised by $\star d\Phi \wedge \Phi = 0$ and the class $\mathcal{Z}_2$ is characterised by $d\Phi = \theta \wedge \Phi$ for the one-form $\theta \in \Omega^1 M$ defined by $\theta := -\frac{1}{7} \star (\star d\Phi \wedge \Phi)$. Hence, we obtain

Proposition 3.32. *Let $\Phi \in \Omega^4 M$ be a $\mathrm{Spin}^\epsilon(7)$-structure on an eight-dimensional manifold M. Then Φ is torsion-free if and only if $d\Phi = 0$.*

Remark 3.33. *Note that the fact that, in contrast to the G_2^ϵ-case, there only appears one equation for torsion-freeness in the $\mathrm{Spin}^\epsilon(7)$-case relies on the self-duality of a $\mathrm{Spin}^\epsilon(7)$-structure, cf. Lemma 2.59.*

3.3 Geometric structures on Lie algebras

Let G be a Lie group. Then (r,s)-tensors on the Lie algebra $\mathfrak{g} \cong T_e\mathrm{G}$ are in one-to-one correspondence with left-invariant (r,s)-tensor fields on G simply by extending them left-invariantly to G. The same is of course true for k-forms and symmetric k-tensors. Since the exterior derivative commutes with pullbacks, this correspondence induces a differential on the k-forms on $\mathfrak{g}$, called the *Chevalley-Eilenberg differential*. More exactly, we have

Definition 3.34. *Let $\mathfrak{g}$ be a real n-dimensional Lie algebra. Then the* Chevalley-Eilenberg differential $d : \Lambda^k\mathfrak{g}^* \to \Lambda^{k+1}\mathfrak{g}^*$ *is the* anti-derivation *of $\Lambda^*\mathfrak{g}^*$, i.e. d is linear and fulfils $d(\alpha_1 \wedge \alpha_2) = d(\alpha_1) \wedge \alpha_2 + (-1)^{k_1}\alpha_1 \wedge d(\alpha_2)$ for all $\alpha_1 \in \Lambda^{k_1}\mathfrak{g}^*$, $\alpha_2 \in \Lambda^{k_2}\mathfrak{g}^*$, which is uniquely defined by $d(\Lambda^0\mathfrak{g}^*) = \{0\}$ and by $d\alpha(v,w) = -\alpha([v,w])$ for $\alpha \in \mathfrak{g}^*$ and $v,w \in \mathfrak{g}$. The complex $(\Lambda^*\mathfrak{g}^*, d)$ is called the* Chevalley-Eilenberg cochain complex. *Note that it is, in fact, a complex since $d^2 = 0$ is equivalent to the Jacobi identity on $\mathfrak{g}$. The corresponding cohomology classes are denoted by $H^*(\mathfrak{g})$ and the dimension of $H^k(\mathfrak{g})$ is denoted by $h^k(\mathfrak{g})$ and is called the* k-th (Lie algebra) Betti number. *Note that always $h^0(\mathfrak{g}) = 1$. Hence, we normally omit this number and set $h^*(\mathfrak{g}) := \big(h^1(\mathfrak{g}), \ldots, h^n(\mathfrak{g})\big)$.*

A left-invariant H-structure P on G is canonically isomorphic as an H-structure to the trivial H-structure $\mathrm{G} \times P_e$. Hence, we may identify P with the H-structure P_e on $\mathfrak{g} \cong T_eG$. Restricting to left-invariant H-structures, the different classes of H-structures on G obtained in Section 3.2 via the distinction of the intrinsic torsion give us different classes of H-structures on $\mathfrak{g}$. Note that left-invariant connections ∇ on G are one-to-one to bilinear maps $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ by the identifications made at the beginning of this section. In this context, we also call a bilinear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ a *connection on* $\mathfrak{g}$ and denote it usually also by ∇. Note that ∇_X is an endomorphism of $\mathfrak{g}$ and so we get an induced bilinear map $\nabla : \mathfrak{g} \times T^{r,s}\mathfrak{g} \to T^{r,s}\mathfrak{g}$ for all $(r,s) \in \mathbb{N}_0^2$, which is exactly the one induced by the left-invariant connection on $\mathcal{T}^{r,s}\mathrm{G}$ one gets from the left-invariant connection ∇ on G. Note further that the Levi-Civita connection of a left-invariant pseudo-Riemannian metric on G provides an example of a left-invariant connection on G and we may speak of the *curvature tensor R^g of a pseudo-Euclidean metric g on* $\mathfrak{g}$. Replacing the Levi-Civita connection ∇^g on G by the corresponding connection $\nabla^g : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on $\mathfrak{g}$ and the exterior differential by the Chevalley-Eilenberg differential, we can transfer all the alternative descriptions of the different classes of left-invariant H-structures on G to the Lie algebra $\mathfrak{g}$. For example, a *half-flat* $\mathrm{SU}^\epsilon(p, 3-p)$*-structure on (a real six-dimensional Lie algebra)* $\mathfrak{g}$ is an $\mathrm{SU}^\epsilon(p, 3-p)$-structure $(\omega, \rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ on $\mathfrak{g}$ with $d\omega^2 = 0$ and $d\rho = 0$. Similarly, a *cocalibrated* G_2^ϵ*-structure on (a real seven-dimensional Lie algebra)* $\mathfrak{g}$ is a G_2^ϵ-structure $\varphi \in \Lambda^3\mathfrak{g}^*$ on $\mathfrak{g}$ with $d \star_\varphi \varphi = 0$. Another example is provided by *parallel* G_2*-structures on* $\mathfrak{g}$, i.e. G_2-structures $\varphi \in \Lambda^3\mathfrak{g}^*$ on $\mathfrak{g}$ with $\nabla^g\varphi = 0$. By Proposition 3.30, these structures can alternatively be described by the equations $d\varphi = 0$ and $d \star_\varphi \varphi = 0$. Parallel G_2-structures on Lie algebras are not particularly interesting.

Proposition 3.35. *A parallel* G_2*-structure $\varphi \in \Lambda^3\mathfrak{g}^*$ on a seven-dimensional Lie algebra $\mathfrak{g}$ induces a flat Euclidean metric g_φ on $\mathfrak{g}$.*

Proof. Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let g_φ be the induced left-invariant Riemannian metric on G. The holonomy principle states that $Hol(g)$ is a subgroup of

G_2. We noted in Remark 3.19 that (G, g) is Ricci-flat. By [AK], Ricci-flat Riemannian homogeneous spaces are flat and the result follows. □

Remark 3.36. *By Remark 3.19, there is an analogous statement as Proposition 3.35 for Riemannian manifolds with parallel* $\mathrm{SU}(m)$*- or* $\mathrm{Spin}(7)$*-structure. For pseudo-Riemannian manifolds, the result obtained in [AK] is no longer true. E.g. in [Kath2], Kath gives examples of pseudo-Riemannian symmetric spaces with parallel* G_2^**-structure, and so having holonomy in* G_2^**, which are Ricci-flat but not flat.*

Proposition 2.51 and the fact that we may replace ρ by J_ρ^* in the construction given in Proposition 2.51 due to Proposition 2.33 (b) imply the following relation between half-flat $\mathrm{SU}^\delta(p, 3-p)$-structures on a real six-dimensional Lie algebra $\mathfrak{g}$ and cocalibrated G_2^ϵ-structures on the real seven-dimensional Lie algebra $\mathfrak{g} \oplus \mathbb{R}$.

Proposition 3.37. *Let* $\mathfrak{g}$ *be a six-dimensional Lie algebra. Then:*

(a) $\mathfrak{g}$ *admits a half-flat* $\mathrm{SU}(3)$*-structure if and only if* $\mathfrak{g} \oplus \mathbb{R}$ *admits a cocalibrated* G_2*-structure such that* $\mathfrak{g}$ *is orthogonal to* $\mathbb{R}$*.*

(b) $\mathfrak{g}$ *admits a half-flat* $\mathrm{SU}(1,2)$*-structure (resp. a half-flat* $\mathrm{SL}(3,\mathbb{R})$*-structure) if and only if* $\mathfrak{g} \oplus \mathbb{R}$ *admits a cocalibrated* G_2^**-structure such that* $\mathfrak{g}$ *is non-degenerate of signature* $(2,4)$ *(resp. of signature* $(3,3)$*) and* $\mathfrak{g}$ *is orthogonal to* $\mathbb{R}$*.*

If (ω, ρ) *is a half-flat* $\mathrm{SU}^\delta(p, 3-p)$*-structure on* $\mathfrak{g}$*, then a cocalibrated* G_2^ϵ*-structure* φ *and its Hodge dual* $\star_\varphi \varphi$ *on* $\mathfrak{g} \oplus \mathbb{R}$ *with the corresponding value of* ϵ *and properties as in (a) or (b) are given by*

$$\varphi := \omega \wedge \alpha + J_\rho^* \rho, \quad \star_\varphi \varphi = -\frac{\delta}{2}\omega^2 + \rho \wedge \alpha \tag{3.7}$$

for $\alpha \in \mathfrak{g}^0 \backslash \{0\}$ *arbitrary. Conversely, if* φ *is a cocalibrated* G_2^ϵ*-structure on* $\mathfrak{g} \oplus \mathbb{R}$ *with properties as in (a) or (b), then a half-flat* $\mathrm{SU}^\delta(p, 3-p)$*-structure* (ω, ρ) *on* $\mathfrak{g}$ *with the corresponding values of* δ *and* p *is given by*

$$\omega := (v \lrcorner \varphi), \quad \rho := -v \lrcorner \star_\varphi \varphi \tag{3.8}$$

for arbitrary $v \in \mathbb{R}\backslash\{0\}$*.*

We end this section by observing that Corollary 2.34 implies the following obstruction to the existence of half-flat $\mathrm{SU}^\epsilon(p, 3-p)$-structures on a six-dimensional Lie algebra.

Proposition 3.38. *Let* $\mathfrak{g}$ *be a six-dimensional Lie algebra and* $\epsilon \in \{-1, 1\}$ *be fixed. If* $\epsilon\lambda(\rho) \leq 0$ *for all* $\rho \in Z^3(\mathfrak{g})$*, then* $\mathfrak{g}$ *does not admit any half-flat* $\mathrm{SU}^\epsilon(p, 3-p)$*-structure. If* $\lambda(\rho) = 0$ *for all* $\rho \in Z^3(\mathfrak{g})$*, then* $\mathfrak{g}$ *does not admit any half-flat structure at all.*

Chapter 4

(Co-)calibrated structures on almost Abelian Lie algebras

In this chapter, we concentrate on almost Abelian Lie algebras, i.e. Lie algebras $\mathfrak{g}$ which possess a codimension one Abelian ideal $\mathfrak{u}$. We determine the seven-dimensional almost Abelian Lie algebras $\mathfrak{g}$ which admit (co-)calibrated G_2^{ϵ}-/$(\mathrm{G}_2)_{\mathbb{C}}$-structures. Moreover, we classify the seven-dimensional almost Abelian Lie algebras possessing a parallel G_2- or G_2^*-structure, respectively, where we restrict ourselves in the latter case to those for which $\mathfrak{u}$ is non-degenerate. We show that then the induced pseudo-Euclidean metric on $\mathfrak{g}$ is flat, a result which is a priori clear in the G_2-case due to Proposition 3.35. Since almost Abelian Lie algebra are fully determined by the endomorphism $f := \mathrm{ad}(e_7)|_{\mathfrak{u}} \in \mathrm{End}(\mathfrak{u})$ for any $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, we express the condition of admitting the corresponding type of G_2^{ϵ}- or $(\mathrm{G}_2)_{\mathbb{C}}$-structure in terms of properties of (the complex Jordan normal form of) f.

We start in Section 4.1 by giving a brief review of almost Abelian Lie algebras. In Section, 4.2 we present and prove the classification results for almost Abelian Lie algebras admitting calibrated structure. In Section 4.3, we classify the almost Abelian Lie algebras which possess cocalibrated structures. The classification results on almost Abelian Lie algebras admitting parallel structures are given in Section 4.4. The results on the existence of cocalibrated structures are contained in the author's paper [Fre1], the results for calibrated and parallel structures have not been published yet.

4.1 Almost Abelian Lie algebras

In this section, we consider *almost Abelian* Lie algebras, i.e. finite-dimensional Lie algebras with codimension one Abelian ideals. We show how the exterior differential on k-forms can be described in an easy way and give a description of all closed k-forms on these Lie algebras. Finally we show how one can classify all such Lie algebras.

Definition 4.1. *An n-dimensional $\mathbb{F}$-Lie algebra $\mathfrak{g}$ is called* almost Abelian *if there exists an Abelian ideal $\mathfrak{u}$ of $\mathfrak{g}$ of dimension $n-1$.*

Remark 4.2. *An n-dimensional $\mathbb{F}$-Lie algebra $\mathfrak{g}$ is almost Abelian if and only if $\mathfrak{g} \cong \mathbb{F}^{n-1} \rtimes_{\varphi} \mathbb{F}$ for some linear map $\varphi : \mathbb{F} \to \mathrm{End}\left(\mathbb{F}^{n-1}\right)$. Since $\mathbb{F}$ is one-dimensional, an almost Abelian Lie algebra is fully determined by one linear endomorphism of $\mathbb{F}^{n-1}$, e.g. $\varphi(1)$.*

For the next lemma, recall that for an arbitrary $\mathbb{F}$-vector space V we have a natural action of the Lie group $\mathrm{GL}(V)$ and of Lie algebra $\mathfrak{gl}(V)$ on $\Lambda^* V^*$, cf. the paragraph before Equation (1) and the equation itself on page xvii. We denote by $\mathrm{GL}(V)_\rho$ the stabiliser subgroup of a k-form $\rho \in \Lambda^k V^*$ under this action of $\mathrm{GL}(V)$ and by $L(\mathrm{GL}(\mathfrak{u})_\rho)$ the associated Lie algebra.

Lemma 4.3. *Let $\mathfrak{g}$ be an n-dimensional almost Abelian $\mathbb{F}$-Lie algebra. Choose $e_n \in \mathfrak{g} \backslash \mathfrak{u}$, set $f := \mathrm{ad}(e_n)|_{\mathfrak{u}} \in \mathfrak{gl}(\mathfrak{u})$ and let $e^n \in \mathfrak{u}^0$ be such that $e^n(e_n) = 1$. Identifying the annihilator ${e_n}^0$ of e_n in $\mathfrak{g}$ with $\mathfrak{u}^*$ using the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathrm{span}(e_n)$, the following assertions are true:*

(a) $d\rho = e^n \wedge (f.\rho)$ and $d(e^n \wedge \rho) = 0$ for all $\rho \in \Lambda^k \mathfrak{u}^*$.

(b) A k-form $\rho \in \Lambda^k \mathfrak{u}^*$ is closed if and only if $f \in L(\mathrm{GL}(\mathfrak{u})_\rho)$.

Proof. Let $X, Y \in \mathfrak{g}$. Then $de^n(X,Y) = -e^n([X,Y]) = 0$ since $[X,Y] \in \mathfrak{u}$ and $e^n \in \mathfrak{u}^0$. This shows $de^n = 0$. Next, let $\alpha \in \mathfrak{u}^* \cong {e_n}^0$. Let X, $Y \in \mathfrak{u}$. Then $[X,Y] = 0$ and so $(d\alpha)(X,Y) = 0 = (e^n \wedge f.\alpha)(X,Y)$. Since

$$(d\alpha)(e_n, Y) = -\alpha([e_n, Y]) = -(\alpha \circ f)(Y) = (e^n \wedge (-\alpha \circ f))(e_n, Y) = (e^n \wedge f.\alpha)(e_n, Y),$$

we get $d\alpha = e^n \wedge f.\alpha$ for all $\alpha \in \mathfrak{u}^*$. But then $d\rho = e^n \wedge f.\rho$ for all $\rho \in \Lambda^k \mathfrak{u}^*$ follows immediately. Moreover, we get

$$d(e^n \wedge \rho) = -e^n \wedge d\rho = -e^n \wedge e^n \wedge f.\rho = 0$$

for all $\rho \in \Lambda^k \mathfrak{u}^*$ and (a) follows.

Now (a) shows that $\rho \in \Lambda^k \mathfrak{u}^* \cong \Lambda^k {e_n}^0$ is closed (with respect to the differential on $\mathfrak{g}$) if and only if $f.\rho = 0$ and by standard Lie theory this is equivalent to $f \in L(\mathrm{GL}(\mathfrak{u})_\rho)$. ☐

We already remarked that an n-dimensional almost Abelian Lie algebra is fully determined by an element of $\mathrm{End}\left(\mathbb{F}^{n-1}\right)$. However, different endomorphisms of $\mathbb{F}^{n-1}$ can lead to isomorphic n-dimensional almost Abelian Lie algebras. The following proposition investigates this phenomenon in more detail and gives a classification of almost Abelian Lie algebras:

Proposition 4.4. *Let $\mathfrak{g} = \mathbb{F}^{n-1} \rtimes_{\varphi} \mathbb{F}e_n$ and $\mathfrak{g}' = \mathbb{F}^{n-1} \rtimes_{\varphi'} \mathbb{F}e'_n$ be two n-dimensional almost Abelian $\mathbb{F}$-Lie algebras. Then $\mathfrak{g} \cong \mathfrak{g}'$ if and only if there exists $\gamma \in \mathbb{F}\backslash\{0\}$ such that $\varphi(e_n)$ and $\gamma\varphi'(e'_n)$ are conjugate in $\mathrm{GL}_{n-1}(\mathbb{F})$. Hence, $\mathfrak{g}$ is isomorphic to $\mathfrak{g}'$ if and only if there exists $\gamma \in \mathbb{F}\backslash\{0\}$ such that for all $m \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$ the number of Jordan blocks of size m with λ on the diagonal in the complex Jordan normal form of $\varphi(e_n)$ equals the number of Jordan blocks of size m with $\gamma\lambda$ on the diagonal in the complex Jordan normal form of $\gamma\varphi'(e'_n)$.*

Proof. "$\Rightarrow$":
Let $\mathfrak{g}$ be isomorphic to $\mathfrak{g}'$. If both $\mathfrak{g}$ and $\mathfrak{g}'$ are Abelian, there is nothing to show. Hence, we may assume for the rest of the proof that $\mathfrak{g}$ and $\mathfrak{g}'$ are both not Abelian.

We consider first the case that $\mathfrak{g}$ admits a codimension one Abelian ideal $\mathfrak{u}$ different from $\mathbb{F}^{n-1}$. Then $V := \mathfrak{u} \cap \mathbb{F}^{n-1}$ is an $(n-2)$-dimensional subspace of $\mathbb{F}^{n-1}$. Since $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{u} \cap \mathbb{F}^{n-1} = V$, we have $\varphi(e_n)(\mathbb{F}^{n-1}) \subseteq V$. Moreover, $\mathfrak{u} \neq \mathbb{F}^{n-1}$ implies the existence of $\lambda \neq 0$ and $w \in \mathbb{F}^{n-1}$ such that $u := w + \lambda e_n \in \mathfrak{u}$. Then, for all $v \in V$, the identities

$$\varphi(e_n)(v) = [e_n, v]_{\mathfrak{g}} = \frac{1}{\lambda}[\lambda e_n, v]_{\mathfrak{g}} = \frac{1}{\lambda}[u - w, v]_{\mathfrak{g}} = \frac{1}{\lambda}[u, v]_{\mathfrak{g}} = 0,$$

are true, where the last two identities follow from the fact that $\mathbb{F}^{n-1}$ and $\mathfrak{u}$ are Abelian. Hence $\varphi(e_n)|_V = 0$ and a Jordan normal form of $\varphi(e_n)$ is given by $\mathrm{diag}(J_2(0), 0, \ldots, 0)$, where $J_2(0)$ is the Jordan block of size two with 0 on the diagonal. The same is of course true for $\mathfrak{g}'$ and the statement follows for this case (note that then $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbb{F}^{n-3}$ with the three-dimensional Heisenberg algebra $\mathfrak{h}_3$).

So we may assume that the unique Abelian ideal of codimension one in $\mathfrak{g}$ and $\mathfrak{g}'$ is $\mathbb{F}^{n-1}$. Then each Lie algebra isomorphism $\Psi : \mathfrak{g} \to \mathfrak{g}'$ maps $\mathbb{F}^{n-1}$ isomorphically onto $\mathbb{F}^{n-1}$ and there has to be $\mathbb{F} \ni \gamma \neq 0$ and $w \in \mathbb{F}^{n-1}$ such that $\Psi(e_n) = \gamma e'_n + w$. So, for $\psi := \Psi|_{\mathbb{F}^{n-1}} \in \mathrm{GL}_{n-1}(\mathbb{F})$ and all $v \in \mathbb{F}^{n-1}$ we get

$$\begin{aligned}(\psi \circ \varphi(e_n))(v) &= \Psi([e_n, v]_{\mathfrak{g}}) = [\Psi(e_n), \Psi(v)]_{\mathfrak{g}'} = [\gamma e'_n + w, \psi(v)]_{\mathfrak{g}'} \\ &= \gamma(\varphi'(e'_n) \circ \psi)(v),\end{aligned}$$

which implies the statement.

"$\Leftarrow$":
Assume that there exists $\psi \in \mathrm{GL}_{n-1}(\mathbb{F})$ and $\gamma \in \mathbb{F}\backslash\{0\}$ such that $\varphi(e_n) = \psi^{-1} \circ (\gamma\varphi'(e'_n)) \circ \psi$. Inverting the above computation, we get that the map $\Psi : \mathfrak{g} \to \mathfrak{g}'$, defined by $\Psi(v + \alpha e_n) := \psi(v) + \alpha\gamma e'_n$ for $v \in \mathbb{F}^{n-1}$, $\alpha \in \mathbb{F}$, is a Lie algebra isomorphism. ☐

Remark 4.5.
- *For $\mathbb{F} = \mathbb{R}$ not all complex Jordan normal forms are possible for $\varphi(e_n)$. It is well-known that exactly those complex Jordan normal forms are possible in which for all $m \in \mathbb{N}$ and all $\lambda \in \mathbb{C}$ the number of complex Jordan blocks of size m with λ on the diagonal is the same as the number of complex Jordan blocks of size*

m with $\overline{\lambda}$ on the diagonal. Hence, not all complex almost Abelian Lie algebras are complexifications of real almost Abelian Lie algebras.

- *Proposition 4.4 gives us a classification of the real and the complex almost Abelian Lie algebras. One may, in principal, write down a complete list in each dimension as follows. One considers, step-by-step, all possible sizes of the Jordan blocks in the complex Jordan form for $\varphi(e_n)$, chooses the diagonal elements in each Jordan blocks as parameters and restricts these parameters in such a way that they are non-isomorphic for different parameter values but still give all isomorphism classes using the conditions given in Proposition 4.4.*
- *Proposition 4.4 may be reformulated in the way that the isomorphism classes of n-dimensional almost Abelian $\mathbb{F}$-Lie algebras which are not Abelian are in one-to-one correspondence to the orbits of $\mathrm{PGL}(n-1,\mathbb{F})$ on the projective space $\mathrm{P}(\mathrm{End}(\mathbb{F}^{n-1}))$. This is a stratified space with the largest strata having codimension $(n-2)$.*

4.2 Classification results for calibrated structures

In this section, we give a classification of the seven-dimensional almost Abelian Lie algebras $\mathfrak{g}$ which admit calibrated G_2^{ϵ}- or $(\mathrm{G}_2)_{\mathbb{C}}$-structures, respectively. Proposition 4.4 gives a classification of almost Abelian Lie algebras $\mathfrak{g}$ via the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$, where $\mathfrak{u}$ is an Abelian ideal in $\mathfrak{g}$ of codimension one and $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$. Except for the case of degenerate $\mathfrak{u}$, we express the condition that $\mathfrak{g}$ possesses a calibrated structure in terms of properties of the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$. For the admittance of a calibrated G_2^*- or $(\mathrm{G}_2)_{\mathbb{C}}$-structure with degenerate $\mathfrak{u}$ we only give an alternative description in terms of certain properties of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$. One can, in principle, obtain from that description also a description in terms of properties of the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$, but it is rather involved and not of that much help. Hence, we leave it out and end this section by giving an explicit list of all nilpotent almost Abelian Lie algebras admitting calibrated G_2-, G_2^*- or $(\mathrm{G}_2)_{\mathbb{C}}$-structures, respectively.

To treat the case of degenerate $\mathfrak{u}$, we need the following lemma.

Lemma 4.6. *Let $\rho_0 = e^{126} - e^{135} + e^{234} \in \Lambda^3\left(\mathbb{F}^6\right)^*$. Then*

$$\mathrm{GL}(6,\mathbb{F})_{\rho_0} = \left\{ \left. \begin{pmatrix} B & 0 \\ CB & \frac{B}{\det(B)} \end{pmatrix} \right| B \in \mathrm{GL}(3,\mathbb{F}),\, C \in \mathfrak{sl}(3,\mathbb{F}) \right\}$$

and

$$L(\mathrm{GL}(6,\mathbb{F})_{\rho_0}) = \left\{ \left. \begin{pmatrix} B & 0 \\ C & B - \mathrm{tr}(B) I_3 \end{pmatrix} \right| B \in \mathfrak{gl}(3,\mathbb{F}),\, C \in \mathfrak{sl}(3,\mathbb{F}) \right\}.$$

Proof. The group $\mathrm{GL}(6,\mathbb{R})_{\rho_0}$ for $\mathbb{F}=\mathbb{R}$ has been determined in [V]. We repeat the arguments to determine also the complex stabiliser of ρ_0. We do the computation for the real and complex case in parallel. A short computation shows that $V_0 := \mathrm{span}(e_4,e_5,e_6) = \left\{v\in\mathbb{F}^6 \,\middle|\, (v\lrcorner\rho_0)^2=0\right\}$. Moreover, if $A\in\mathrm{GL}(6,\mathbb{F})_{\rho_0}$ and $v\in V_0$, we get

$$A(v)\lrcorner\rho_0 = \rho_0(A(v),\cdot,\cdot) = (A.\rho_0)(A(v),\cdot,\cdot) = A.\,(v\lrcorner\rho_0)$$

and so $A(v)\in V_0$. Hence, we may write the dual map $A^*:\left(\mathbb{F}^6\right)^*\to\left(\mathbb{F}^6\right)^*$ as $A^t = A^* = \begin{pmatrix} B & BC\\ 0 & D\end{pmatrix}$ for $B,D\in\mathrm{GL}(3,\mathbb{F})$ and $C\in\mathfrak{gl}(3,\mathbb{F})$ with respect to the ordered basis $(e_1,\ldots,e_6)$. Applying A^* to ρ we get

$$\begin{aligned} e^{126}-e^{135}+e^{234} &= A^*\left(e^{126}-e^{135}+e^{234}\right)\\ &= Be^{12}\wedge De^6 - Be^{13}\wedge De^5 + Be^{23}\wedge De^6 + \mathrm{tr}(C)Be^{123}\\ &= Be^{12}\wedge De^6 - Be^{13}\wedge De^5 + Be^{23}\wedge De^6 + \mathrm{tr}(C)\det(B)e^{123}\end{aligned}$$

and so $\mathrm{tr}(C)=0$. We set $V_1 := \mathrm{span}(e_1,e_2,e_3)$. We have an isomorphism $F:\Lambda^2V_1^*\wedge V_0^*\to \mathrm{End}\,(V_0,V_1)$ given by $F(\omega\wedge\alpha)(v)=\alpha(v)\cdot\omega\lrcorner\,(e_{123})$ for $\omega\in\Lambda^2V_1^*$, $\alpha\in V_0^*$ and $v\in V_0$. We identify $\mathrm{End}\,(V_0,V_1)$ with $\mathfrak{gl}(3,\mathbb{R})$ via the basis (e_4,e_5,e_6) of V_0 and (e_1,e_2,e_3) of V_1. Then the identities $F(\rho_0)=I_3$ and

$$F\Big(Be^{12}\wedge De^6 - Be^{13}\wedge De^5 + Be^{23}\wedge De^6\Big) = \det(B)B^{-t}D^t$$

are true. Thus, $D=\frac{B}{\det(B)}$ and so

$$\mathrm{GL}(6,\mathbb{F})_{\rho_0} \subseteq \left\{ \begin{pmatrix} A & 0\\ BA & \frac{A}{\det(A)}\end{pmatrix} \,\middle|\, A\in\mathrm{GL}(3,\mathbb{F}),\ B\in\mathfrak{sl}(3,\mathbb{F})\right\}$$

The converse inclusion follows by inverting the above calculations. The computation of the associated Lie algebra $L(\mathrm{GL}(6,\mathbb{F})_{\rho_0})$ is straightforward.

□

We are now able to prove

Theorem 4.7. *Let $\mathfrak{g}$ be a seven-dimensional real almost Abelian Lie algebra and $\mathfrak{u}$ be a six-dimensional Abelian ideal in $\mathfrak{g}$.*

(a) The following are equivalent:

(i) $\mathfrak{g}$ admits a calibrated G_2-structure.

(ii) $\mathfrak{g}$ admits a calibrated G_2^*-structure such that $\mathfrak{u}$ has signature $(2,4)$ with respect to the induced pseudo-Euclidean metric on $\mathfrak{g}$.

(iii) *For any* $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, *there exist* A, $B \in \mathfrak{sl}(3,\mathbb{R})$ *and an ordered basis* $(e_1,\ldots,e_6)$ *of* $\mathfrak{u}$ *such that the transformation matrix of* $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ *with respect to* $(e_1,\ldots,e_6)$ *is given by*

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$$

(iv) *For any* $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, *the complex Jordan normal form of* $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ *is given, up to a permutation of the complex Jordan blocks, by* $\mathrm{diag}\left(J,\overline{J}\right)$ *for some trace-free matrix* $J \in \mathbb{C}^{3\times 3}$ *in complex Jordan normal form.*

(b) *The following are equivalent:*

(i) $\mathfrak{g}$ *admits a calibrated* G_2^**-structure such that* $\mathfrak{u}$ *has signature* $(3,3)$ *with respect to the induced pseudo-Euclidean metric on* $\mathfrak{g}$.

(ii) *For any* $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, *there exists a vector space decomposition* $\mathfrak{g} = V \oplus W$ *of* $\mathfrak{g}$ *into three-dimensional subspaces* V, W *such that*

$$\mathrm{ad}(e_7)|_{\mathfrak{u}} \in \{f \in \mathfrak{gl}(\mathfrak{g})\,|\,f|_V = f_V,\ f|_W = f_W,\ f_V \in \mathfrak{sl}(V),\ f_W \in \mathfrak{sl}(W)\}$$

(iii) *For any* $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, *the complex Jordan normal form of* $\mathrm{ad}(e_7)_{\mathfrak{u}}$ *is given, up to a permutation of the complex Jordan blocks, by* $\mathrm{diag}\,(J_1,J_2)$ *for trace-free matrices* $J_1,J_2 \in \mathbb{C}^{3\times 3}$ *which are complex Jordan normals form of real three-by-three matrices. That means, for* $i=1,2$, J_i *contains no Jordan block with a non-real number on the diagonal or exactly two Jordan blocks of size* 1 *with a non-real number and its complex conjugate on the diagonal, respectively.*

(c) *The following are equivalent:*

(i) $\mathfrak{g}$ *admits a calibrated* G_2^**-structure such that* $\mathfrak{u}$ *is degenerate with respect to the induced pseudo-Euclidean metric on* $\mathfrak{g}$.

(ii) *For any* $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, *there exists an ordered basis* $(e_1,\ldots,e_6)$ *of* $\mathfrak{u}$, $A \in \mathfrak{gl}(3,\mathbb{R})$ *and* $B \in \mathfrak{sl}(3,\mathbb{R})$ *such that the transformation matrix of* $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ *with respect to* $(e_1,\ldots,e_6)$ *is given by*

$$\begin{pmatrix} A & 0 \\ B & A-\mathrm{tr}(A)I_3 \end{pmatrix}$$

Proof. Choose $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$. Let $e^7 \in \mathfrak{u}^0$ with $e^7(e_7)=1$ and identify $\Lambda^k {e_7}^0$ with $\Lambda^k\mathfrak{u}^*$ using the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathrm{span}(e_7)$. If we say in the following that an element of $\Lambda^k\mathfrak{u}^*$ is closed, we always mean that the corresponding form in $\Lambda^k {e_7}^0$ is closed with respect to the differential of $\mathfrak{g}$.

Let $\varphi \in \Lambda^3\mathfrak{g}^*$ be a calibrated G_2^ϵ-structure. There are unique $\omega \in \Lambda^2\mathfrak{u}^*$, $\rho \in \Lambda^3\mathfrak{u}^*$ with $\varphi = \omega \wedge e^7 + \rho$. Lemma 4.3 implies

$$0 = d\varphi = d(\omega \wedge e^7 + \rho) = d\rho.$$

Proposition 2.48 tells us that the model tensor of ρ is ρ_{-1} if $\epsilon = 1$ and $\mathfrak{u}$ has signature $(2,4)$ or if $\epsilon = -1$, that ρ has model tensor ρ_1 if $\epsilon = 1$ and $\mathfrak{u}$ has signature $(3,3)$ and that ρ has model tensor ρ_0 if $\mathfrak{u}$ is degenerate.

Conversely, let $\rho \in \Lambda^3 {e_7}^0 \cong \Lambda^3\mathfrak{u}^*$ be closed with model tensor ρ_{-1}. Choose an arbitrary G_2-structure $\tilde{\varphi} \in \Lambda^3\mathfrak{g}^*$ and an arbitrary G_2^*-structure $\check{\varphi} \in \Lambda^3\mathfrak{g}^*$ such that $\mathfrak{u}$ has signature $(2,4)$ with respect to the induced pseudo-Euclidean metric. We decompose $\tilde{\varphi} = \tilde{\omega} \wedge e^7 + \tilde{\rho}$, $\check{\varphi} = \check{\omega} \wedge e^7 + \check{\rho}$ with $\tilde{\omega}$, $\check{\omega} \in \Lambda^2\mathfrak{u}^*$ and $\tilde{\rho}$, $\check{\rho} \in \Lambda^3\mathfrak{u}^*$. By Proposition 2.48, both $\tilde{\rho}$ and $\check{\rho}$ have model tensor ρ_{-1}. Hence, there are isomorphisms $\tilde{F}$, $\check{F} : \mathfrak{u} \to \mathfrak{u}$ with $\tilde{F}^*\tilde{\rho} = \rho = \check{F}^*\check{\rho}$. We define isomorphisms $\tilde{G}$, $\check{G} : \mathfrak{g} \to \mathfrak{g}$ by $\tilde{G}|_{\mathfrak{u}} := \tilde{F}$, $\check{G}_{\mathfrak{u}} := \check{F}$ and $\tilde{G}(e_7) := e_7 =: \check{G}(e_7)$. Then $\tilde{G}^*\tilde{\varphi}$ is a G_2-structure with $\tilde{G}^*\tilde{\varphi}|_{\mathfrak{u}} = \rho$ and the closure of ρ and Lemma 4.3 show that $\tilde{G}^*\tilde{\varphi}$ is closed. Moreover, by the same arguments $\check{G}^*\check{\varphi}$ is a calibrated G_2^*-structure with $\check{G}^*\check{\varphi}|_{\mathfrak{u}} = \rho$. Since $\check{G}$ is an isometry between $(\mathfrak{g}, g_{\check{G}^*\check{\varphi}})$ and $(\mathfrak{g}, g_{\check{\varphi}})$, the signature of $\mathfrak{u}$ is $(2,4)$ with respect to $g_{\check{G}^*\check{\varphi}}$. Similarly, we see that for each closed $\rho \in \Lambda^3\mathfrak{u}^*$ with model tensor ρ_1 there exists a calibrated G_2^*-structure $\hat{\varphi} \in \Lambda^3\mathfrak{g}^*$ with $\hat{\varphi}|_{\mathfrak{u}} = \rho$ and $\mathfrak{u}$ having signature $(3,3)$ with respect to $g_{\hat{\varphi}}$ and the analogous statement for closed ρ with model tensor ρ_0 and calibrated G_2^*-structures with degenerate $\mathfrak{u}$ is true.

Summarizing, the existence of a calibrated G_2^ϵ-structure such that $\mathfrak{u}$ has the desired property is equivalent to the existence of a closed three-form $\rho \in \Lambda^3 {e_7}^0 \cong \Lambda^3\mathfrak{u}^*$ with the corresponding model tensor mentioned above. By Lemma 4.3, the closure of ρ is equivalent to $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in L(\mathrm{GL}(V)_\rho)$. The identity component of the stabiliser of ρ_ϵ, $\epsilon \in \{-1, 1\}$, is, according to Lemma 2.19, equal to $\mathrm{SL}(3, \mathbb{C}_\epsilon) \subseteq \mathrm{GL}(6, \mathbb{R})$. This gives us the equivalence of (i)-(iii) in (a) and of (i) and (ii) in (b). The stabiliser of ρ_0 is given in Lemma 4.6 and we get the equivalence of (i) and (ii) in (c). The equivalence of (iii) and (iv) in (a) follows from the fact that $L(\mathrm{GL}(6, \mathbb{R})_{\rho_{-1}}) = i(\mathfrak{sl}(3, \mathbb{C}))$ for some injective $\mathbb{R}$-algebra homomorphism $i : \mathfrak{gl}(3, \mathbb{C}) \to \mathfrak{gl}(6, \mathbb{C})$ and that if J is a complex Jordan normal form for $A \in \mathfrak{gl}(3, \mathbb{C})$, then $\mathrm{diag}\left(J, \overline{J}\right)$ is a complex Jordan normal form for $i(A)$. The equivalence of (ii) and (iii) in (b) is obvious. $\square$

Remark 4.8. *In Section 4.3 we show that a seven-dimensional almost Abelian Lie algebra $\mathfrak{g}$ with codimension one Abelian ideal $\mathfrak{u}$ admits a cocalibrated G_2^*-structures such that $\mathfrak{u}$ has signature $(2,4)$ if and only if $\mathfrak{g}$ admits a cocalibrated G_2^*-structure such that $\mathfrak{u}$ has signature $(3,3)$. Moreover, the existence of a cocalibrated G_2^*-structure with non-degenerate $\mathfrak{u}$ implies the existence of a cocalibrated G_2^*-structure with degenerate $\mathfrak{u}$. The corresponding relations do not hold for calibrated G_2^*-structures:*

- *If the complex Jordan normal form of* $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ *is given by* $\mathrm{diag}(1+i, 1-i, 2+2i, 2-2i, -3-3i, -3+3i)$*, then Theorem 4.7 shows that* $\mathfrak{g}$ *admits a calibrated* G_2^**-structure such that* $\mathfrak{u}$ *has signature* $(2,4)$ *but neither one such that* $\mathfrak{u}$ *has signature* $(3,3)$ *nor one such that* $\mathfrak{u}$ *is degenerate.*

- *If the complex Jordan normal form of* $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ *is given by* $\mathrm{diag}(1,2,-3,4,5,-9)$*, then Theorem 4.7 shows that* $\mathfrak{g}$ *admits a calibrated* G_2^**-structure such that* $\mathfrak{u}$ *has signature* $(3,3)$ *but neither one such that* $\mathfrak{u}$ *has signature* $(2,4)$ *nor one such that* $\mathfrak{u}$ *is degenerate.*

- *If the complex Jordan normal form of* $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ *is given by* $\mathrm{diag}(1,2,3,-5,-4,-3)$*, then Theorem 4.7 shows that* $\mathfrak{g}$ *admits a calibrated* G_2^**-structure with degenerate* $\mathfrak{u}$ *but neither one where* $\mathfrak{u}$ *has signature* $(2,4)$ *nor one where* $\mathfrak{u}$ *has signature* $(3,3)$*.*

Remark 4.9. *Recently, results on the existence of calibrated* G_2*-structures on Lie algebras have been obtained. Namely, [CF] gives a full classification of the seven-dimensional nilpotent Lie algebras admitting a calibrated* G_2*-structure. Moreover, [FMOU] determines all the six-dimensional solvable Lie algebras* $\mathfrak{h}$ *admitting a so-called* symplectic half-flat $\mathrm{SU}(3)$-structure*. There is an analogous relation between symplectic half-flat* $\mathrm{SU}(3)$*-structures on* $\mathfrak{h}$ *and calibrated* G_2*-structures on* $\mathfrak{h}\oplus\mathbb{R}$ *as between half-flat* $\mathrm{SU}(3)$*-structures on* $\mathfrak{h}$ *and cocalibrated* G_2*-structures on* $\mathfrak{h}\oplus\mathbb{R}$*, cf. [FMOU]. Thus, the results obtained in [FMOU] give us a full list of the seven-dimensional solvable Lie algebras of the form* $\mathfrak{h}\oplus\mathbb{R}$ *admitting a calibrated* G_2*-structure such that the splitting* $\mathfrak{h}\oplus\mathbb{R}$ *is orthogonal. The results in [FMOU] show that a six-dimensional almost Abelian Lie algebra* $\mathfrak{h}$ *admits a symplectic half-flat* $\mathrm{SU}(3)$*-structure if and only if* $\mathfrak{h}\oplus\mathbb{R}$ *admits a calibrated* G_2*-structure. Analogously to the proof of Theorem 6.7, one may give a direct proof of this assertion.*

For $(\mathrm{G}_2)_{\mathbb{C}}$-structures we obtain the following result:

Theorem 4.10. *Let* $\mathfrak{g}$ *be a complex seven-dimensional almost Abelian Lie algebra and* $\mathfrak{u}$ *be a six-dimensional Abelian ideal in* $\mathfrak{g}$*.*

(a) The following are equivalent:

(i) $\mathfrak{g}$ *admits a calibrated* $(\mathrm{G}_2)_{\mathbb{C}}$*-structure such that* $\mathfrak{u}$ *is non-degenerate with respect to the induced non-degenerate symmetric complex bilinear form on* $\mathfrak{g}$*.*

(ii) For any $e_7\in\mathfrak{g}\backslash\mathfrak{u}$*, there exists a vector space decomposition* $\mathfrak{g}=V\oplus W$ *into three-dimensional subspaces* V *and* W *such that*

$$\mathrm{ad}(e_7)|_{\mathfrak{u}}\in\{f\in\mathfrak{gl}(\mathfrak{u})\,|\,f|_V=f_V,\ f|_W=f_W,\ f_V\in\mathfrak{sl}(V),\ f_W\in\mathfrak{sl}(W)\}\,.$$

(iii) For any $e_7\in\mathfrak{g}\backslash\mathfrak{u}$*, the complex Jordan normal form of* $\mathrm{ad}(e_7)_{\mathfrak{u}}$ *is given, up to a permutation of the complex Jordan blocks, by* $\mathrm{diag}\,(J_1,J_2)$ *for trace-free matrices* $J_1,J_2\in\mathbb{C}^{3\times3}$ *in complex Jordan normal form.*

(b) The following are equivalent:

(i) $\mathfrak{g}$ *admits a calibrated* $(\mathrm{G}_2)_{\mathbb{C}}$*-structure such that* $\mathfrak{u}$ *is degenerate with respect to the induced non-degenerate symmetric complex bilinear form on* $\mathfrak{g}$.

(ii) For any $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, *there exists an ordered basis* $(e_1, \ldots, e_6)$ *of* $\mathfrak{u}$, $A \in \mathfrak{gl}(3, \mathbb{C})$ *and* $B \in \mathfrak{sl}(3, \mathbb{C})$ *such that the transformation matrix of* $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ *with respect to* $(e_1, \ldots, e_6)$ *is given by*

$$\begin{pmatrix} A & 0 \\ B & A - \mathrm{tr}(A)I_3 \end{pmatrix}$$

Proof. The proof is completely analogous to the proof of Theorem 4.7. Using Proposition 2.49 and Lemma 4.3 we see as in the proof of Theorem 4.7 that the existence of a calibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure with non-degenerate $\mathfrak{u}$ (resp. degenerate $\mathfrak{u}$) is equivalent to the existence of a closed three-form $\rho \in \Lambda^3 {e_7}^0 \cong \Lambda^3 \mathfrak{u}^*$ with model tensor ρ_1 (resp. ρ_0), $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, and that this is equivalent to $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in L(\mathrm{GL}(\mathfrak{u})_\rho)$. The stabiliser of ρ_1 is given in Proposition 2.19 and the one of ρ_0 in Lemma 4.6. This establishes the equivalence of (i) and (ii) both in (a) and (b). The equivalence of (ii) and (iii) in (a) is obvious. □

We finish this section and use our results to determine the seven-dimensional nilpotent almost Abelian Lie algebra admitting calibrated G_2^ϵ-/$(\mathrm{G}_2)_{\mathbb{C}}$-structures. Note that the classification for the G_2-case already has been done in [CF]. Note further that a seven-dimensional almost Abelian $\mathbb{F}$-Lie algebra with six-dimensional Abelian ideal $\mathfrak{u}$ is nilpotent if and only if $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ is nilpotent for $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$ and this is the case if and only if the diagonal elements in the complex Jordan normal form are all 0. Thus, for each partition $n_1 + \ldots + n_k = 6$ of 6 with $n_1, \ldots, n_k \in \{1, \ldots, 6\}$, $n_1 \geq \ldots \geq n_k$, there is exactly one nilpotent almost Abelian Lie algebra, namely that one whose complex Jordan normal form has Jordan blocks of sizes $n_1, \ldots, n_k$, and these are all nilpotent seven-dimensional almost Abelian $\mathbb{F}$-Lie algebras. Therefore, in total we have 11 such nilpotent Lie algebras for both $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$. All of them have rational structure constants so each of them admits a co-compact lattice. Hence, if $\mathfrak{g}$ admits a calibrated G_2^ϵ-structure, we get a compact nilmanifold with calibrated G_2^ϵ-structure.

We obtain the following result, where we refer to the appendix for the names of the appearing Lie algebras.

Corollary 4.11. *Let* $\mathfrak{g}$ *be a seven-dimensional nilpotent almost Abelian* $\mathbb{F}$*-Lie algebra. Then:*

(a) If $\mathbb{F} = \mathbb{R}$, *then* $\mathfrak{g}$ *admits a calibrated* G_2*-structure if and only if* $\mathfrak{g} \in \{\mathbb{R}^7, A_{5,1} \oplus \mathbb{R}^2, \mathfrak{n}_{7,2}\}$.

(b) If $\mathbb{F} = \mathbb{R}$, *then* $\mathfrak{g}$ *admits a calibrated* G_2^**-structure.*

(c) If $\mathbb{F} = \mathbb{C}$, then $\mathfrak{g}$ admits a calibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure.

Proof. Theorem 4.7 shows that $\mathfrak{g}$ admits a calibrated G_2-structure if and only if the Jordan blocks of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ have the sizes $(1,1,1,1,1,1)$, $(2,2,1,1)$ or $(3,3)$. Hence, (a) follows. The proof of (c) is analogous to the one of (b) and we only show (b). Theorem 4.7 (a) and (b) show that $\mathfrak{g}$ admits a calibrated G_2^*-structure with non-degenerate $\mathfrak{u}$ if and only if the sizes of the Jordan blocks in the complex Jordan normal form are $(1,1,1,1,1,1)$, $(2,1,1,1,1)$, $(2,2,1,1)$, $(3,1,1,1)$, $(3,2,1)$ or $(3,3)$. To prove the assertion, it suffices, according to Theorem 4.7 (c), to give examples of $A \in \mathfrak{gl}(3,\mathbb{R})$ and $B \in \mathfrak{sl}(3,\mathbb{R})$ such that $\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$ has complex Jordan normal form with only zeros on the diagonal and with Jordan blocks of sizes $(2,2,2)$, $(4,1,1)$, $(4,2)$, $(5,1)$ and (6). Recall that by our convention, $J_m(\lambda)$ denotes a complex Jordan block of size m with $\lambda \in \mathbb{C}$ on the diagonal and further that the 1s in $J_m(\lambda)$ are on the superdiagonal. A complex Jordan normal form with Jordan blocks of sizes $(2,2,2)$ and only zeros on the diagonal may be achieved with $A = 0$ and $B \in \mathfrak{sl}(3,\mathbb{R})$ of rank three. Those where the blocks have the sizes $(4,1,1)$ or $(4,2)$ may be achieved with $A = \mathrm{diag}(J_2(0), 0)$ and $B = \mathrm{diag}\left(\begin{pmatrix} 1 & 0 \\ 1 & b \end{pmatrix}, -1-b\right)$ with $b = -1$ or $b = 1$, respectively. The sizes $(5,1)$ or (6) may be achieved with $A = J_3(0)$ and $B \in \mathfrak{sl}(3,\mathbb{R})$ with $b_{ij} = 0$ except $b_{21} = 1$ or $b_{31} = 1$, respectively. $\square$

4.3 Classification results for cocalibrated structures

In this section, we identify those seven-dimensional almost Abelian Lie algebras $\mathfrak{g}$ which admit cocalibrated structures. Analogous to Section 4.2, we express the condition of admitting such a structure entirely in terms of properties of the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$, where $\mathfrak{u}$ is an Abelian ideal of dimension six in $\mathfrak{g}$ and $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$. Moreover, also the proof follows the same lines as the determination of the Lie algebras admitting calibrated structures. In particular, we first express the condition of admitting a cocalibrated structure in terms of properties of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ and later transfer this into properties of the complex Jordan normal form. Since we do the transfer also for the degenerate case, this transfer requires more work than in the calibrated case. Hence, we postpone it and first prove the following proposition.

Proposition 4.12. *Let $\mathfrak{g}$ be a seven-dimensional almost Abelian $\mathbb{F}$-Lie algebra and $\mathfrak{u}$ be a six-dimensional Abelian ideal. Then:*

(a) If $\mathbb{F} = \mathbb{R}$, then $\mathfrak{g}$ admits a cocalibrated G_2^*-structure such that $\mathfrak{u}$ has signature $(2,4)$ if and only if $\mathfrak{g}$ admits a cocalibrated G_2^*-structure such that $\mathfrak{u}$ has signature $(3,3)$.

(b) *If $\mathbb{F}=\mathbb{R}$, then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $\mathfrak{g}$ admits a cocalibrated G_2^*-structure such that $\mathfrak{u}$ is non-degenerate and this is the case if and only if there exists a non-degenerate $\omega \in \Lambda^2\mathfrak{u}^*$ such that $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in \mathfrak{sp}(\mathfrak{u},\omega)$.*

(c) *If $\mathbb{F}=\mathbb{C}$, then $\mathfrak{g}$ admits a cocalibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure such that $\mathfrak{u}$ is non-degenerate if and only if there exists a non-degenerate $\omega \in \Lambda^2\mathfrak{u}^*$ such that $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in \mathfrak{sp}(\mathfrak{u},\omega)$.*

(d) *If $\mathbb{F}=\mathbb{R}$ (resp. $\mathbb{F}=\mathbb{C}$), then $\mathfrak{g}$ admits a cocalibrated G_2^*-structure (resp. $(\mathrm{G}_2)_{\mathbb{C}}$-structure) such that $\mathfrak{u}$ is degenerate if and only if there exists a two-dimensional subspace V_2, a complementary four-dimensional subspace V_4 and a non-degenerate two-form $\omega \in \Lambda^2 V_4^*$ on V_4 such that*

$$\mathrm{ad}(e_7)|_{\mathfrak{u}} \in \left\{ f \in \mathfrak{gl}(\mathfrak{u}) \,\middle|\, f|_{V_2} = f_2 + h,\ f|_{V_4} = -\frac{\mathrm{tr}(f_2)}{2}\mathrm{id}_{V_4} + f_4, \right.$$
$$\left. f_2 \in \mathfrak{gl}(V_2),\ h \in \hom(V_2,V_4),\ f_4 \in \mathfrak{sp}(V_4,\omega) \right\}.$$

Proof. Fix $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, let $e^7 \in \mathfrak{u}^0$ be that element with $e^7(e_7)=1$ and identify as usual $\Lambda^k {e_7}^0$ with $\Lambda^k\mathfrak{u}^*$ using the decomposition $\mathfrak{g}=\mathfrak{u}\oplus\mathrm{span}(e_7)$. Let $\varphi \in \Lambda^3\mathfrak{g}^*$ be a cocalibrated G_2^ϵ-structure (resp. cocalibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure). There exists $\rho \in \Lambda^3\mathfrak{u}^*$ and $\Omega \in \Lambda^4\mathfrak{u}^*$ with $\star_\varphi\varphi = \rho\wedge e^7 + \Omega$. Using Lemma 4.3 we get, as in the proof of Theorem 4.7, that $\Omega \in \Lambda^4 {e_7}^0 \cong \Lambda^4\mathfrak{u}^*$ is closed. Proposition 2.48 and Proposition 2.49 tell us that the model tensor of Ω is $\frac{1}{2}\omega_0^2 \in \Lambda^4\left(\mathbb{F}^6\right)^*$ or $-\frac{1}{2}\omega_0^2 \in \Lambda^4\left(\mathbb{F}^6\right)^*$ if $\mathfrak{u}$ is not degenerate and $e^{1234}+e^{1256} \in \Lambda^4\left(\mathbb{F}^6\right)^*$ if $\mathfrak{u}$ is degenerate. So the existence of a cocalibrated G_2^ϵ-structure (resp. cocalibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure) with non-degenerate $\mathfrak{u}$ implies the existence of a closed four-form $\Omega \in \Lambda^4 {e_7}^0 \cong \Lambda^4\mathfrak{u}^*$ with model tensor $\frac{1}{2}\omega_0^2 \in \Lambda^4\left(\mathbb{F}^6\right)^*$. With the use of Proposition 2.48, we can argue, similarly as in the proof of Theorem 4.7, that the existence of a cocalibrated G_2^ϵ-structure (resp. cocalibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure) with non-degenerate $\mathfrak{u}$ is even equivalent to the existence of a closed four-form $\Omega \in \Lambda^4 {e_7}^0 \cong \Lambda^4\mathfrak{u}^*$ with model tensor $\frac{1}{2}\omega_0^2 \in \Lambda^4\left(\mathbb{F}^6\right)^*$. In particular, (a) follows. Analogously, we get that the existence of a cocalibrated G_2^*-structure (resp. cocalibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure) with degenerate $\mathfrak{u}$ is equivalent to the existence of a closed four-form $\tilde{\Omega} \in \Lambda^4 {e_7}^0 \cong \Lambda^4\mathfrak{u}^*$ with model tensor $e^{1234}+e^{1256} \in \Lambda^4\left(\mathbb{F}^6\right)^*$. Using Lemma 4.3 we see that, both in the non-degenerate as in the degenerate case, the four-form Ω is closed if and only if $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in L(\mathrm{GL}(\mathfrak{u})_\Omega)$. By Lemma 2.4, $\Omega \in \Lambda^4\mathfrak{u}^*$ has model tensor $\frac{1}{2}\omega_0^2 \in \Lambda^4\left(\mathbb{F}^6\right)^*$ if and only if there exits a non-degenerate $\omega \in \Lambda^2\mathfrak{u}^*$ with $\Omega = \frac{1}{2}\omega^2$. By Proposition 2.5, the stabiliser group $\mathrm{GL}(\mathfrak{u})_\Omega$ of Ω is then equal to $\mathrm{Sp}(\mathfrak{u},\omega)$ and (b) and (c) follow. (d) follows form the concrete form of the stabiliser of $e^{1234}+e^{1256} \in \Lambda^4\left(\mathbb{F}^6\right)^*$, which is given in Proposition 2.5. □

Remark 4.13. *Regarding Proposition 4.12, it might be of interest to know whether or not the existence of a cocalibrated G_2-structure always implies the existence of a cocalibrated G_2^*-structure. We suppose not, but cannot provide a concrete counterexample.*

To transfer the conditions on $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ in Proposition 4.12, which are equivalent to the existence of a cocalibrated structure, in terms of the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$, we need to recall some well-known results, see e.g. [DPWZ], on the complex Jordan normal forms of elements in $\mathfrak{sp}(2n, \mathbb{F}) \subseteq \mathfrak{gl}(2n, \mathbb{F})$:

Proposition 4.14. *Let (V, ω) be a $\mathbb{F}$-symplectic vector space. Then a linear transformation $f \in \mathrm{GL}(V)$ is conjugate under the action of $\mathrm{GL}(V)$ to an element in $\mathfrak{sp}(V, \omega)$ if and only if the complex Jordan normal form of f has the property that for all $m \in \mathbb{N}$ and all $0 \neq \lambda$ the number of Jordan blocks of size m with λ on the diagonal equals the number of Jordan blocks of size m with $-\lambda$ on the diagonal and the number of Jordan blocks of size $2m-1$ with 0 on the diagonal is even.*

Proposition 4.12 and Proposition 4.14 allow us to prove

Theorem 4.15. *Let $\mathfrak{g}$ be a seven-dimensional almost Abelian $\mathbb{F}$-Lie algebra and $\mathfrak{u}$ be a codimension one Abelian ideal.*

(a) If $\mathbb{F} = \mathbb{R}$, then the following are equivalent:

- *(i) $\mathfrak{g}$ admits a cocalibrated G_2-structure.*
- *(ii) $\mathfrak{g}$ admits a cocalibrated G_2^*-structure such that the subspace $\mathfrak{u}$ is non-degenerate with respect to the induced pseudo-Euclidean metric on $\mathfrak{g}$.*
- *(iii) For any $e_7 \in \mathfrak{g} \backslash \mathfrak{u}$, $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in \mathfrak{gl}(\mathfrak{u})$ is in $\mathfrak{sp}(\mathfrak{u}, \omega)$, $\omega \in \Lambda^2 \mathfrak{u}^*$ being a non-degenerate two-form on $\mathfrak{u}$.*
- *(iv) For any $e_7 \in \mathfrak{g} \backslash \mathfrak{u}$, the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ has the property that for all $m \in \mathbb{N}$ and all $\lambda \neq 0$ the number of Jordan blocks of size m with λ on the diagonal is the same as the number of Jordan blocks of size m with $-\lambda$ on the diagonal and the number of Jordan blocks of size $2m-1$ with 0 on the diagonal is even.*

(b) If $\mathbb{F} = \mathbb{C}$, then the following are equivalent

- *(i) $\mathfrak{g}$ admits a cocalibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure such that the subspace $\mathfrak{u}$ is non-degenerate with respect to the induced non-degenerate complex symmetric bilinear form on $\mathfrak{g}$.*
- *(ii) For any $e_7 \in \mathfrak{g} \backslash \mathfrak{u}$, $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in \mathfrak{gl}(\mathfrak{u})$ is in $\mathfrak{sp}(\mathfrak{u}, \omega)$, $\omega \in \Lambda^2 \mathfrak{u}^*$ being a non-degenerate two-form on $\mathfrak{u}$.*
- *(iii) For any $e_7 \in \mathfrak{g} \backslash \mathfrak{u}$, the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ has the property that for all $m \in \mathbb{N}$ and all $\lambda \neq 0$ the number of Jordan blocks of size m with λ on the diagonal is the same as the number of Jordan blocks of size m with $-\lambda$ on the diagonal and the number of Jordan blocks of size $2m-1$ with 0 on the diagonal is even.*

(c) If $\mathbb{F}=\mathbb{R}$ (resp. $\mathbb{F}=\mathbb{C}$), then the following are equivalent:

(i) $\mathfrak{g}$ admits a cocalibrated G_2^*-structure (resp. cocalibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure).

(ii) For any $e_7\in\mathfrak{g}\backslash\mathfrak{u}$, there exists a two-dimensional subspace V_2, a complementary four-dimensional subspace V_4 and a non-degenerate two-form $\omega\in\Lambda^2V_4^*$ on V_4 such that $\mathrm{ad}(e_7)|_{\mathfrak{u}}\in\mathfrak{gl}(\mathfrak{u})$ is in

$$\{f\in\mathfrak{gl}(\mathfrak{u})\,|f|_{V_2}=f_2+h,\ f_2\in\mathfrak{gl}(V_2),\ h\in\hom(V_2,V_4),$$
$$\left.f|_{V_4}=-\frac{\mathrm{tr}(f_2)}{2}\mathrm{id}_{V_4}+f_4,\ f_4\in\mathfrak{sp}(V_4,\omega)\right\}.$$

(iii) For any $e_7\in\mathfrak{g}\backslash\mathfrak{u}$ the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}\in\mathfrak{gl}(\mathfrak{u})$ has the property that there exists a partition of $\{1,\ldots,6\}$ into three subsets I_1, I_2, I_3, each of cardinality two, such that the following is true:

(1) $\sum_{i\in I_1}\lambda_i=\sum_{i\in I_2}\lambda_i=-\sum_{i\in I_3}\lambda_i$.

(2) If there are $i_1\in I_1$, $i_2\in I_2$ such that $\mathrm{JB}(i_1)=\mathrm{JB}(i_2)$ then $\lambda_{i_1}=\lambda_{i_2}=-\frac{\sum_{i\in I_3}\lambda_i}{2}$ or $\mathrm{JB}(j_1)=\mathrm{JB}(j_2)$ for the uniquely determined $j_k\in I_k$ such that $\{i_k,j_k\}=I_k$, $k=1,2$.

(3) If there exists $i_2\in I_2$ such that $\mathrm{JB}(j)=\mathrm{JB}(i_2)$ for all $j\in I_1$ or if there exists $i_1\in I_1$ such that $\mathrm{JB}(j)=\mathrm{JB}(i_1)$ for all $j\in I_2$, then $\lambda_j=-\frac{\sum_{i\in I_3}\lambda_i}{2}$ for all $j\in I_1\cup I_2$ and $\mathrm{JB}(j)=\mathrm{JB}(k)$ for all $j,k\in I_1\cup I_2$.

Proof. (a) and (b) follow directly from Proposition 4.12 and Proposition 4.14. For the proof of (c), note that Proposition 4.12 shows that (ii) implies (i) Moreover, by Proposition 4.12, the implication "(i)$\Rightarrow$ (ii)" follows if we are able to show that the existence of a cocalibrated G_2^*-structure (resp. cocalibrated $(\mathrm{G}_2)_{\mathbb{C}}$-structure) with non-degenerate $\mathfrak{u}$ on $\mathfrak{g}$,implies condition (ii). Since (a) (resp. (b)) is already proved, we may also proceed as follows to finish the entire proof:

- First step: Show that condition (iv) in (a) (resp. (iii) in (b)) implies condition (iii) in (c).
- Second step: Show that the conditions (ii) and (iii) in (c) are equivalent.

First step:

Let $A\in\mathbb{C}^{6\times 6}$ be a matrix in complex Jordan normal form such that for all $m\in\mathbb{N}$ and all $0\neq\lambda\in\mathbb{C}$ the number of Jordan blocks of size m with λ on the diagonal is the same as the number of Jordan blocks of size m with $-\lambda$ on the diagonal and the number of Jordan blocks of size $2m-1$ with 0 on the diagonal is even. Number consecutively the diagonal elements of the complex Jordan normal form by $\lambda_1,\ldots,\lambda_6$. The assumptions on A imply that we can portion $\{1,\ldots,6\}$ as follows into three subsets I_1, I_2, I_3 of cardinality two:

- We can group the Jordan blocks with non-zero diagonal elements into pairs of Jordan blocks of the same size with λ and $-\lambda$, $\lambda \neq 0$ on the diagonal. Construct now subsets $I_1, \ldots, I_r$ of cardinality two by going successively through all these pairs of Jordan blocks and putting successively the two indices corresponding to the first,..., l-th, ... diagonal element in the two Jordan blocks in one I_k. By the index i corresponding to the l-th diagonal element in a certain Jordan block we mean that $i \in \{1, \ldots, 6\}$ such the i-th diagonal element of the big matrix A is the l-th diagonal element in the Jordan block.

- Similarly, we can group the Jordan blocks with zero on the diagonal and of odd size into pairs of the same size and construct subsets $I_{r+1}, \ldots, I_s$ taking successively all these pairs of Jordan blocks and putting again the two indices corresponding to the first,..., l-th, ... diagonal element in the two Jordan block in one I_k.

- Finally, we construct subsets $I_{s+1}, \ldots, I_3$ by taking successively the Jordan blocks with 0 in the diagonal of even size and putting together the two indices corresponding to the $(2l-1)$-th and $2l$-th diagonal element.

By construction, $\sum_{i \in I_k} \lambda_i = 0$ for all $k = 1, 2, 3$ and so condition (1) in Theorem 4.15 (c) (iii) is fulfilled. Moreover, if $i_1 \in I_1$, $i_2 \in I_2$ are such that $\mathrm{JB}(i_1) = \mathrm{JB}(i_2)$, then by construction also $\mathrm{JB}(j_1) = \mathrm{JB}(j_2)$ for the unique $j_k \in I_k$ such that $I_k = \{i_k, j_k\}$ for $k = 1, 2$. This show that condition (2) in Theorem 4.15 (c) (iii) is fulfilled. Finally, we argue that also condition (3) in Theorem 4.15 (c) (iii) is satisfied. Therefore, assume, without loss of generality, that there is $i_2 \in I_2$ such that $\mathrm{JB}(i_1) = \mathrm{JB}(j_1) = \mathrm{JB}(i_2)$, $\{i_1, j_1\} = I_1$. Then $\lambda_{i_1} + \lambda_{j_1} = 0$ and $\lambda_{i_1} = \lambda_{j_1} = \lambda_{i_2}$ imply $0 = \lambda_{i_1} = \lambda_{j_1} = \lambda_{i_2}$. By construction, the identity $\mathrm{JB}(i_1) = \mathrm{JB}(j_1) = \mathrm{JB}(i_2)$ implies that $\mathrm{JB}(j_2) = \mathrm{JB}(i_2)$ for $j_2 \in I_2$, $j_2 \neq i_2$. But then also $\lambda_{j_2} = 0$ and the first part is proved.

Second step:

For this part of the proof, we remind the reader that we follow that standard convention on the form of Jordan blocks which puts the 1s on the superdiagonal. We first show that condition (ii) implies condition (iii) in Theorem 4.15 (c). Let $f := \mathrm{ad}(e_7)|_{\mathfrak{u}}$, $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$. By assumption, we have a four-dimensional invariant subspace $V_4 \subseteq \mathfrak{u}$ and a two-dimensional complementary subspace $V_2 \subseteq \mathfrak{u}$ such that $f|_{V_2} = f_2 + h$, $f_2 \in \mathfrak{gl}(V_2)$, $h \in \mathrm{hom}(V_2, V_4)$ and $f|_{V_4} = f_4 - \frac{\mathrm{tr}(f_2)}{2}\mathrm{id}_{V_4}$ with $f_4 \in \mathfrak{sp}(V_4, \omega)$ for some non-degenerate two-form ω on V_4. To simplify the way of speaking, we say in the following that certain vectors $u_1, \ldots, u_s$ are a Jordan basis of a linear map if there is a permutation making them into a Jordan basis. Choose a Jordan basis $v_1, \ldots, v_4$ of f_4 and denote by $\mu_1, \ldots, \mu_4$ the corresponding diagonal elements. Then Proposition 4.14 tells us that, without loss of generality, $\mu_1 = -\mu_2$ and $\mu_3 = -\mu_4$. Set $\lambda_i := \mu_i - \frac{\mathrm{tr}(f_2)}{2}$. The vectors $v_1, \ldots, v_4$ are also a Jordan basis of $f|_{V_4}$. Moreover, v_i and v_j are in one Jordan block for f_4 with μ_i on the diagonal if and only if

v_i and v_j are in one Jordan block for $f_4 - \frac{\mathrm{tr}(f_2)}{2}\mathrm{id}_{V_4}$ with λ_i on the diagonal. By [GLR, Theorem 4.1.4], there is a Jordan basis $w_1, \ldots, w_6$ of f such that for all $i, j \in \{1, \ldots, 4\}$ the vectors v_i and v_j are in the same Jordan block for $f_4 - \frac{\mathrm{tr}(f_2)}{2}\mathrm{id}_{V_4}$ with λ_i on the diagonal if and only if w_i and w_j are in the same Jordan block for f with λ_i on the diagonal. Since the characteristic polynomial of f is the product of the characteristic polynomials of $f_4 - \frac{\mathrm{tr}(f_2)}{2}\mathrm{id}_{V_4}$ and f_2, the Jordan basis vectors w_5 or w_6 are in Jordan blocks with λ_5 or λ_6 on the diagonal, respectively, where λ_5, λ_6 are the roots of the characteristic polynomial of f_2. In particular, $\mathrm{tr}(f_2) = \lambda_5 + \lambda_6$. This allows us now to prove that the conditions (1) - (3) in Theorem 4.15 (c) are fulfilled for the sets $I_k := \{2k-1, 2k\}$, $k = 1, 2, 3$:

- We get

  $$\lambda_1 + \lambda_2 = \mu_1 + \mu_2 - \mathrm{tr}(f_2) = -\lambda_5 - \lambda_6, \quad \lambda_3 + \lambda_4 = \mu_3 + \mu_4 - \mathrm{tr}(f_2) = -\lambda_5 - \lambda_6,$$

  which is exactly condition (1).

- If w_{i_1} and w_{i_2} are in one Jordan block for f with $\lambda_{i_1} = \lambda_{i_2}$ on the diagonal for $i_1 \in \{1, 2\}$, $i_2 \in \{3, 4\}$, then v_{i_1} and v_{i_2} are in one Jordan block for f_4 with $\mu_{i_1} = \lambda_{i_1} + \frac{\lambda_5 + \lambda_6}{2}$ on the diagonal. We may have $\mu_{i_1} = \mu_{i_2} = 0$ and so $\lambda_{i_1} = \lambda_{i_2} = -\frac{\lambda_5+\lambda_6}{2}$. If this is not the case, Proposition 4.14 implies that f_4 has to contain two Jordan blocks of size two, one with μ_{i_1} and the other with $-\mu_{i_1}$ on the diagonal and so v_{j_1}, v_{j_2} are in one Jordan block, j_1, j_2 such that $\{i_1, j_1\} = \{1, 2\}$, $\{i_2, j_2\} = \{3, 4\}$. Hence, w_{j_1}, w_{j_2} are in one Jordan block. Thus, condition (2) is satisfied.

- If w_1, w_2 and w_{i_2} for some $i_2 \in \{3, 4\}$ or w_{i_1}, w_3 and w_4 for some $i_1 \in \{1, 2\}$ are in one Jordan block for f with λ on the diagonal, then v_1, v_2 and v_{i_2} or v_{i_1}, v_3 and v_4 are in one Jordan block for f_4 with $\lambda + \frac{\lambda_5+\lambda_6}{2}$ on the diagonal. But then Proposition 4.14 tells us that v_1, v_2, v_3 and v_4 are in one Jordan block for f_4 with 0 on the diagonal. Hence, w_1, w_2, w_3, w_4 are in one Jordan block for f with $-\frac{\lambda_5+\lambda_6}{2}$ on the diagonal. This is condition (3).

Finally, we show that condition (iii) implies condition (ii) in Theorem 4.15 (c). Let $A \in \mathbb{C}^{6\times 6}$ be in complex Jordan normal form and assume that it fulfils all the conditions in Theorem 4.15 (c) (iii). Let I_1, I_2 and I_3 be a partition of $\{1, \ldots, 6\}$ as in condition (iii). We may assume that $\mathrm{JB}(i_k) = \mathrm{JB}(i_3)$ for $i_k \in I_k$, $k = 1, 2$, $i_3 \in I_3$ implies $i_k < i_3$ simply by redefining I_k and I_3 if this is not the case (note therefore that $\lambda_{i_k} = \lambda_{i_3}$). Set $V_2 := \mathrm{span}(e_i | i \in I_3)$ and $V_4 := \mathrm{span}(e_j | j \in I_1 \cup I_2)$. Due to our assumption, V_4 is an invariant subspace for A. That means there are $A_2 \in \mathfrak{gl}(V_2)$, $H \in \mathrm{hom}(V_2, V_4)$ and $A_4 \in \mathfrak{gl}(V_4)$ such that $A|_{V_2} = A_2 + H$ and $A|_{V_4} = A_4$. Moreover, A_4 is in complex Jordan normal form and so $B := A_4 + \frac{\mathrm{tr}(A_2)}{2} I_4$ is also in complex Jordan normal form. We claim that B is conjugate to an element in $\mathfrak{sp}(4, \mathbb{F})$. Therefore, we have to check that B fulfils all the conditions in Proposition 4.14. We use the conditions (1) - (3) in Theorem 4.15 (c) (iii) to get information on the structure of B. First, the identity $\operatorname{tr}(A_2) = \sum_{i\in I_3} \lambda_i$ shows that the diagonal elements of B are given by $\mu_j = \lambda_j + \frac{\sum_{i\in I_3}\lambda_i}{2}$, $j \in I_1 \cup I_2$. Hence, we get the following properties of B:

(A) Condition (1) states that $\sum_{j\in I_k} \mu_j = \sum_{j\in I_k} \lambda_j + \sum_{i\in I_3} \lambda_i = 0$ for $k = 1, 2$.

(B) Condition (2) implies that if B contains a Jordan block of size 2 with $\mu = \lambda + \frac{\sum_{i\in I_3}\lambda_i}{2}$ on the diagonal, then $\lambda = -\frac{\sum_{i\in I_3}\lambda_i}{2}$, i.e. $\mu = 0$, or $\mu \neq 0$ and there is a different Jordan block of size 2. Property (A) implies that the value on the diagonal in this other Jordan block of size 2 has then to be equal to $-\mu$.

(C) Condition (3) states that there cannot be any Jordan block of size 3 in B and there can only be a Jordan block of size 4 in B if the diagonal elements are equal to 0.

Regarding (A) - (C), the Jordan blocks of size greater than one in B obviously fulfil all conditions in Proposition 4.14. To discuss those of size one, note that if there is at least one Jordan block of size one with $\mu \neq 0$ on the diagonal, then (B) and (C) directly imply that all the Jordan blocks in B with non-zero value on the diagonal must be of size one. Hence, (A) implies that the number of Jordan blocks of size one with $\mu \neq 0$ on the diagonal equals the number of Jordan blocks of size one with $-\mu$ on the diagonal.
Thus, we are left with the Jordan blocks of size one with 0 on the diagonal and have to show that their number is even. Suppose that their number is odd, i.e. it is one or three. If it was one, then (A) and (C) show that there is a Jordan block of size two with 0 on the diagonal. But then there has to be exactly one Jordan block of size one with a non-zero value on the diagonal, which we just excluded. If the number of Jordan blocks of size one with 0 on the diagonal was three, we again get that there is exactly one Jordan block of size one with a non-zero value on the diagonal. Thus, the number of Jordan blocks of size one with 0 on the diagonal has to be even and the statement is proved. □

Remark 4.16.
- *Theorem 4.15 (a) implies that seven-dimensional almost Abelian Lie algebras admitting a cocalibrated* G_2*-structure are necessarily unimodular. This is not true for arbitrary seven-dimensional Lie algebras, cf. Theorem 5.18.*
- *The admittance of a cocalibrated* G_2*-structure on an almost Abelian Lie algebra puts restrictions on the Lie algebra Betti numbers* $h^i(\mathfrak{g})$*. Since* $\mathfrak{g}$ *is unimodular, we have* $h^7(\mathfrak{g}) = 1$*. Moreover, Theorem 4.15 (a) implies the existence of a closed two-form* $\omega \in \Lambda^2 {e_7}^0$ *of length three. Hence,* $h^{2i}(\mathfrak{g}) > 0$ *for* $i = 1, 2, 3$*. A more thorough discussion of condition (iv) in Theorem 4.15 (a) implies the existence of three linearly independent closed two-forms in* $\Lambda^2 {e_7}^0$*. Hence,* $h^2(\mathfrak{g}) \geq 3$ *and also* $h^3(\mathfrak{g}) \geq 3$ *since there have to be three linearly independent non-exact closed two-forms in* $\Lambda^2 {e_7}^0$*.*

- *However, whether a seven-dimensional almost Abelian Lie algebra admits a cocalibrated* G_2*-structure or not cannot be decided solely by the Lie algebra cohomology. Therefore, note that the seven-dimensional almost Abelian Lie algebra* $\mathfrak{g} = \mathbb{R}^6 \rtimes \mathbb{R}e_7$ *with* $\mathrm{ad}(e_7)|_{\mathfrak{u}} = \mathrm{diag}(1,-1,2,-2,4,-4)$ *has the same Lie algebra cohomology as the one with* $\mathrm{ad}(e_7)|_{\mathfrak{u}} = \mathrm{diag}(1,-1,-1,-1,4,-2)$*, namely*

$$\big(h^1(\mathfrak{g}), h^2(\mathfrak{g}), h^3(\mathfrak{g}), h^4(\mathfrak{g}), h^5(\mathfrak{g}), h^6(\mathfrak{g}), h^7(\mathfrak{g})\big) = (1,3,3,3,3,1,1)\,.$$

  Theorem 4.15 (a) implies that the first Lie algebra admits a cocalibrated G_2*-structure while the second does not.*

We like to note the following consequences of Theorem 4.15 and Proposition 4.12.

Corollary 4.17. *Let* $\mathfrak{g}$ *be a real seven-dimensional almost Abelian Lie algebra and* $\mathfrak{u}$ *be a codimension one Abelian ideal.*

(a) If $\mathfrak{g}$ *admits a cocalibrated* G_2^**-structure with non-degenerate* $\mathfrak{u}$*, then it also admits a cocalibrated* G_2^**-structure with degenerate* $\mathfrak{u}$*.*

(b) $\mathfrak{g}$ *admits a cocalibrated* G_2^**-structure if and only if* $\mathfrak{g}_{\mathbb{C}}$ *admits a cocalibrated* $(\mathrm{G}_2)_{\mathbb{C}}$*-structure.*

Remark 4.18. *An interesting open question one may ask is if the analogue of Corollary 4.17 (b) holds for all real seven-dimensional Lie algebras. We do not think so but cannot give a concrete counterexample.*

We end this section by noting what Theorem 4.15 implies for the nilpotent almost Abelian Lie algebras. As in the case of a calibrated G_2^ϵ-structure, the interest stems from the fact that we get compact nilmanifolds with cocalibrated G_2^ϵ-structures.

Corollary 4.19. *Let* $\mathfrak{g}$ *be a nilpotent* $\mathbb{F}$*-Lie algebra of dimension seven with six-dimensional Abelian ideal* $\mathfrak{u}$*. Then:*

(a) If $\mathbb{F} = \mathbb{R}$*, then* $\mathfrak{g}$ *admits a cocalibrated* G_2*-structure if and only if* $\mathfrak{g} \notin \{\mathrm{A}_{4,1} \oplus \mathbb{R}^3, \mathfrak{n}_{6,1} \oplus \mathbb{R}, \mathfrak{n}_{6,2} \oplus \mathbb{R}\}$.

(b) If $\mathbb{F} = \mathbb{R}$*, then* $\mathfrak{g}$ *admits a cocalibrated* G_2^**-structure.*

(c) If $\mathbb{F} = \mathbb{C}$*, then* $\mathfrak{g}$ *admits a cocalibrated* $(\mathrm{G}_2)_{\mathbb{C}}$*-structure.*

4.4 Classification results for parallel structures

In the final section of this chapter, we restrict ourselves to the real case. We determine the seven-dimensional almost Abelian Lie algebras $\mathfrak{g}$ admitting parallel G_2^ϵ-structures. For simplicity, we consider only G_2^*-structures with non-degenerate six-dimensional Abelian ideal. We use the fact that the holonomy principle and Proposition 3.30 imply that a G_2^ϵ-structure $\varphi \in \Lambda^3\mathfrak{g}^*$ is parallel with respect to the induced pseudo-Euclidean metric if and only if φ is calibrated and cocalibrated. For the formulation of the statement, recall that we use the notation $M_{a,b}$ for the real two-by-two matrix $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

Theorem 4.20. *Let $\mathfrak{g}$ be a seven-dimensional real almost Abelian Lie algebra with six-dimensional Abelian ideal $\mathfrak{u}$.*

(a) *$\mathfrak{g}$ admits a parallel G_2-structure if and only if $\mathfrak{g}$ admits a basis $(e_1, \ldots, e_7)$ such that $(e_1, \ldots, e_6)$ is a basis of $\mathfrak{u}$ and there exist $a, b \in \mathbb{R}$ such that the transformation matrix of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ with respect to $(e_1, \ldots, e_6)$ is given by $\mathrm{diag}(M_{0,a}, M_{0,b}, M_{0,-a-b})$.*

(b) *$\mathfrak{g}$ admits a parallel G_2^*-structure such that $\mathfrak{u}$ has signature $(2,4)$ if and only if $\mathfrak{g}$ admits a basis $(e_1, \ldots, e_7)$ such that $(e_1, \ldots, e_6)$ is a basis of $\mathfrak{u}$ and such that the transformation matrix of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ with respect to $(e_1, \ldots, e_6)$ is given by one of the following matrices for certain $a \in \mathbb{R}^*$, $b, c, d, e \in \mathbb{R}$:*

$$\mathrm{diag}(M_{a,b}, M_{-a,b}, M_{0,-2b}),\ \mathrm{diag}\left(M_{0,c}, M_{0,d}, M_{0,-(c+d)}\right),\ \begin{pmatrix} M_{0,e} & I_2 & \\ & M_{0,e} & \\ & & M_{0,-2e} \end{pmatrix},$$

$$\begin{pmatrix} 0 & I_2 & \\ & 0 & I_2 \\ & & 0 \end{pmatrix}.$$

(c) *$\mathfrak{g}$ admits a parallel G_2^*-structure such that $\mathfrak{u}$ has signature $(3,3)$ if and only if for any $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$, the complex Jordan normal form of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ has the property that there is a partition $\{1, \ldots, 6\} = I \cup J$ with subsets I, J of cardinality three and a bijection $G : I \to J$ with*

(i) $\sum_{i \in I} \lambda_i = 0$,

(ii) $\lambda_{G(i)} = -\lambda_i$ *for all* $i \in I$,

(iii) $\mathrm{JB}(i_1) = \mathrm{JB}(i_2)$ *if and only if* $\mathrm{JB}(G(i_1)) = \mathrm{JB}(G(i_2))$ *for all* $i_1, i_2 \in I$,

(iv) $\mathrm{JB}(i) \neq \mathrm{JB}(j)$ *for all* $i \in I$, $j \in J$.

(d) *Parallel G_2-structures and parallel G_2^*-structure with non-degenerate $\mathfrak{u}$ are flat.*

Proof. The proof follows the same lines as the determination of the Lie algebras admitting calibrated or cocalibrated structures in the last two sections.

Let $\varphi \in \Lambda^3\mathfrak{g}^*$ be a parallel G_2^ϵ-structure with $\mathfrak{u}$ being non-degenerate. We may choose $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$ with $e_7 \perp_{g_\varphi} \mathfrak{u}$ and $g_\varphi(e_7, e_7) = -\delta \in \{-1, 1\}$. Here, $\delta = 1$ if $\epsilon = 1$ and $\mathfrak{u}$ has signature $(3,3)$. In all other case, $\delta = -1$. Let $e^7 \in \mathfrak{u}^0$ with $e^7(e_7) = 1$ and identify as usual $\Lambda^k {e_7}^0$ with $\Lambda^k \mathfrak{u}^*$. Proposition 2.51 tells us that we have a $\mathrm{SU}^\delta(p, 3-p)$-structure $(\omega, \rho) \in \Lambda^2\mathfrak{u}^* \times \Lambda^3\mathfrak{u}^*$ such that

$$\varphi = \omega \wedge e^7 + \rho, \quad \star_\varphi \varphi = -\frac{\delta}{2}\omega^2 + \delta J_\rho^* \rho \wedge e^7.$$

Here, $(\delta, \epsilon) = (-1,-1)$ implies $p = 3$ and $(\delta, \epsilon) = (-1, 1)$ implies $p = 1$. Moreover, Lemma 4.3 shows that $\frac{1}{2}\omega^2$ and ρ are closed and so $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in L\left(\mathrm{GL}(\mathfrak{u})_{\frac{1}{2}\omega^2} \cap \mathrm{GL}(\mathfrak{u})_\rho\right)$. By Proposition 2.5, $L(\mathrm{GL}(\mathfrak{u})_{\frac{1}{2}\omega^2}) = L(\mathrm{GL}(\mathfrak{u})_\omega)$ and so $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in L\left(\mathrm{GL}(\mathfrak{u})_{\frac{1}{2}\omega^2} \cap \mathrm{GL}(\mathfrak{u})_\rho\right)$ $\cong \mathfrak{su}^\delta(p, 3-p)$.

In particular, we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathrm{span}(e_7)$ into an Abelian ideal $\mathfrak{u}$ of $\mathfrak{g}$ and an Abelian subalgebra $\mathrm{span}(e_7)$ which acts skew-symmetric on the Abelian ideal $\mathfrak{u}$. Hence, in the G_2-case, the Euclidean metrics on $\mathfrak{g}$ are in Milnor's [Mi] class of flat Euclidean metrics on Lie algebras. However, one can show, doing the same calculations as in the Euclidean case, that the analogous class in the pseudo-Euclidean setting also consists solely of flat metrics. This shows (d). Note that the result in the Euclidean case also follows from Proposition 3.35, which has been proved using the result that Ricci-flat homogeneous spaces are flat [AK].

Conversely, assume that $f := \mathrm{ad}(e_7)|_{\mathfrak{u}}$ is contained for some, and hence for all, $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$ in a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(\mathfrak{u})$ which is conjugate via an isomorphism $\mathfrak{u} \cong \mathbb{R}^6$ to $\mathfrak{su}^\delta(p, 3-p)$ for some $(\delta, p) \in \{(-1,3), (-1,1), (1,3)\}$. Using the mentioned isomorphism, we may construct an $\mathrm{SU}^\delta(p, 3-p)$-structure $(\omega, \rho) \in \Lambda^2\mathfrak{u}^* \times \Lambda^3\mathfrak{u}^*$ on $\mathfrak{u}$ with $f.\omega = 0$ and $f.\rho = 0$. By Proposition 2.51, $\varphi := \omega \wedge e^7 + \rho$ is a G_2^ϵ-structure with Hodge dual $\star_\varphi \varphi = -\frac{\delta}{2}\omega^2 + \delta J_\rho^* \rho$. Here, $\epsilon = -1$ if $(\delta, p) = (-1, 3)$. Otherwise, $\epsilon = 1$. Lemma 4.3 tells us that φ and $\star_\varphi\varphi$ are both closed and hence φ is parallel.

Thus, $\mathfrak{g}$ admits a parallel G_2^ϵ-structure with $\mathfrak{u}$ being non-degenerate if and only if for any $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$ the linear endomorphism $\mathrm{ad}(e_7)|_{\mathfrak{u}}$ is contained in a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{gl}(\mathfrak{u})$ which is conjugate via an isomorphism $\mathfrak{u} \cong \mathbb{R}^6$ to $\mathfrak{su}^\delta(p, 3-p)$, where $\epsilon = -1$ if and only if $(\delta, p) = (-1, 3)$, $\epsilon = 1$ and $\mathfrak{u}$ has signature $(2,4)$ if and only if $(\delta, p) = (-1, 1)$ and $\epsilon = 1$ and $\mathfrak{u}$ has signature $(3,3)$ if and only if $\delta = 1$. Hence, (a) follows since the given matrices in the assertion are exactly the real Jordan normal forms of elements in $\mathfrak{su}(3) \subseteq \mathfrak{gl}(6, \mathbb{R})$. For (b), note that in [DPWZ], all the complex Jordan normal forms of elements in $\mathfrak{u}(1,2) \subseteq \mathfrak{gl}(3, \mathbb{C})$ are determined. To get all the complex Jordan normal forms of elements in $\mathfrak{su}(1,2) \subseteq \mathfrak{gl}(3,\mathbb{C})$, we only have to require additionally that they are trace-free. Hence, the possible complex Jordan normal forms of elements in $\mathfrak{su}(1,2) \subseteq \mathfrak{gl}(3, \mathbb{C})$ are

$$\mathrm{diag}(a + ib, -a + ib, -2ib), \ \mathrm{diag}(ic, id, -i(c+d)), \ \mathrm{diag}(J_2(ie), -2ie), \ J_3(0)$$

for $a \in \mathbb{R}^*$ and b, c, d, $e \in \mathbb{R}$ and we get the claimed real Jordan normal forms for elements in $\mathfrak{su}(1,2)$ considered as a subset of $\mathfrak{gl}(6,\mathbb{R})$. (c) follows from the fact that $\mathfrak{su}^1(p,3-p) = \left\{ \operatorname{diag}\left(A, -A^t\right) \in \mathfrak{gl}(6,\mathbb{R}) \middle| A \in \mathfrak{sl}(3,\mathbb{R}) \right\}$, cf. Definition 2.32. $\square$

Remark 4.21. *There are seven-dimensional almost Abelian Lie algebras which admit both a calibrated and a cocalibrated* G_2*-structures but no parallel* G_2*-structure. An example is provided by the nilpotent Lie algebra* $\mathfrak{n}_{7,2}$.

We look again at the nilpotent case. By [Mi, Theorem 2.4], a nilpotent Lie algebra $\mathfrak{g}$ admits a flat Riemannian metric if and only if $\mathfrak{g}$ is Abelian and so Proposition 3.35 shows that a nilpotent Lie algebra $\mathfrak{g}$ admits a parallel G_2-structure if and only if $\mathfrak{g}$ is Abelian. This is in accordance with Theorem 4.20. For the G_2^*-case with non-degenerate $\mathfrak{u}$ we get from Theorem 4.20:

Corollary 4.22. *Let* $\mathfrak{g}$ *be a seven-dimensional real nilpotent almost Abelian Lie algebra and let* $\mathfrak{u}$ *be a six-dimensional Abelian ideal in* $\mathfrak{g}$*. Then* $\mathfrak{g}$ *admits a parallel* G_2^**-structure with non-degenerate* $\mathfrak{u}$ *if and only if* $\mathfrak{g} \in \left\{\mathbb{R}^7, A_{5,1} \oplus \mathbb{R}^2, \mathfrak{n}_{7,2}\right\}$.

Chapter 5

Cocalibrated structures on direct sums

In this chapter, we give the classification of the direct sums of four- and three-dimensional Lie algebras which admit cocalibrated G_2-structures. The results are all contained in the author's paper [Fre2]. We start by recalling basic facts about three- and four-dimensional Lie algebras in Sections 5.1 and 5.2, respectively. In Section 5.3 we use the results obtained at the end of Section 2.4 to prove a general existence result for cocalibrated G_2-structure on manifolds. We apply this general result to our particular case of cocalibrated G_2-structures on direct sums $\mathfrak{g}_4 \oplus \mathfrak{g}_3$ of four- and three-dimensional Lie algebras using the structure theory of these Lie algebras obtained in the previous sections. In Section 5.4, we use Proposition 2.48 and again the structure theory to obtain obstructions to the existence of cocalibrated G_2-structures on the mentioned class of Lie algebras. Section 5.5 starts by presenting the main theorem of this chapter, which tells us exactly which sums of four- and three-dimensional Lie algebras admit cocalibrated G_2-structures. In the preceding, we give the proof of the main theorem using all previous results. We deal separately with the four cases which naturally appear by distinguishing whether $\mathfrak{g}_4$ or $\mathfrak{g}_3$ is unimodular or not.

5.1 Three-dimensional Lie algebras

The classification of three-dimensional Lie algebras is well-known for a long time [Bi] and given in the appendix in Table 7.1. We highlight some aspects of the classification.

Lemma 5.1. *Let $\mathfrak{g}$ be a three-dimensional unimodular Lie algebra.*

(a) There exists a basis e_1, e_2, e_3 of $\mathfrak{g}$ and $\tau_1, \tau_2, \tau_3 \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$ such that $de^i = \tau_i \sum_{j,k=1}^{3} \epsilon_{ijk} e^{jk}$ for $i = 1, 2, 3$.

(b) $d(\mathfrak{g}^*) \wedge \ker d|_{\mathfrak{g}^*} = \{0\}$.

(c) *There exists a linear map* $g : \Lambda^2\mathfrak{g}^* \to \ker d|_{\mathfrak{g}^*}$ *such that for the map* $G : \Lambda^2\mathfrak{g}^* \to \Lambda^3\mathfrak{g}^*$, $G(\omega) := \omega \wedge g(\omega)$ *for* $\omega \in \Lambda^2\mathfrak{g}^*$, *the identity* $G^{-1}(0) = d(\mathfrak{g}^*)$ *is true.*

(d) *If* $\tau_i\tau_j \geq 0$ *for all* $i, j \in \{1, 2, 3\}$, *i.e.* $\mathfrak{g} \notin \{\mathfrak{e}(1, 1), \mathfrak{so}(2, 1)\}$, *then* $F^{-1}(0) = \ker d|_{\mathfrak{g}^*}$, *where* $F : \mathfrak{g}^* \to \Lambda^3\mathfrak{g}^*$ *is defined by* $F(\alpha) := d(\alpha) \wedge \alpha$ *for* $\alpha \in \mathfrak{g}^*$.

Proof. We use the well-known part (a) [Bi] to show (b)-(d).

(b) Let $\omega = d\alpha$, $\alpha = \sum_{i=1}^3 a_ie^i \in \mathfrak{g}^*$ and $\beta = \sum_{i=1}^3 b_ie^i \in \mathfrak{g}^*$. Then

$$\omega = \sum_{i,j,k=1}^3 \tau_i a_i \epsilon_{ijk} e^{jk} \tag{5.1}$$

and so

$$\begin{aligned}\omega \wedge \beta &= \sum_{i,j,k,l=1}^3 \tau_i a_i b_l \epsilon_{ijk} e^{jkl} = \sum_{i,j,k,l=1}^3 \tau_i a_i b_l \epsilon_{ijk}\epsilon_{jkl} e^{123} \\ &= \left(\sum_{i=1}^3 2\tau_i a_i b_i\right) e^{123}.\end{aligned} \tag{5.2}$$

If $d\beta = \sum_{i,j,k=1}^3 \tau_i b_i \epsilon_{ijk} e^{jk} = 0$, then $\tau_i b_i = 0$ for all $i = 1, 2, 3$ and so $\omega \wedge \beta = 0$. This shows (b).

(c) Let $\omega \in \Lambda^2\mathfrak{g}^*$. Then $\omega = \sum_{i,j,k=1}^3 a_i \epsilon_{ijk} e^{jk}$ for unique $a_1, a_2, a_3 \in \mathbb{R}$. Set $g(\omega) := \sum_{i=1,\tau_i=0}^3 a_ie^i$. Then Equation (5.1) shows that $g(\omega) \in \ker d|_{\mathfrak{g}^*}$. Moreover,

$$\begin{aligned}\omega \wedge g(\omega) &= \sum_{i,j,k,l=1,\tau_l=0}^3 a_i a_l \epsilon_{ijk} e^{jkl} = \left(\sum_{i,j,k,l=1,\tau_l=0}^3 a_i a_l \epsilon_{jki}\epsilon_{jkl}\right) e^{123} \\ &= \left(\sum_{i,l=1,\tau_l=0}^3 2a_i a_l \delta_{il}\right) e^{123} = \left(\sum_{l=1,\tau_l=0}^3 2a_l^2\right) e^{123} = 0\end{aligned}$$

if and only if $\tau_l = 0$ implies $a_l = 0$ for $l = 1, 2, 3$. But Equation (5.1) shows that this is equivalent to $\omega \in d(\mathfrak{g}^*)$.

(d) The signs of the non-zero τ_i are all the same due to the assertion. Let $\alpha = \sum_{i=1}^3 a_ie^i \in \mathfrak{g}^*$, $a_1, a_2, a_3 \in \mathbb{R}$. Then Equation (5.2) implies that $d\alpha \wedge \alpha = 0$ if and only if $\sum_{i=1}^3 \tau_i a_i^2 = 0$ and this is the case if and only if $\tau_i a_i = 0$ for all $i = 1, 2, 3$. But Equation (5.1) states that this is equivalent to $\alpha \in \ker d|_{\mathfrak{g}^*}$.

□

The only two non-solvable three-dimensional Lie algebras are the simple ones, namely $\mathfrak{so}(3)$ and $\mathfrak{so}(2, 1)$. All other three-dimensional Lie algebras are almost Abelian: If $\mathfrak{g}$ is solvable and unimodular, then, by elementary Lie theory, there exists a codimension one ideal, which then has to be unimodular and so Abelian. If $\mathfrak{g}$ is not unimodular, then the unimodular kernel gives a codimension one Abelian ideal. Hence, Lemma 4.3 shows

Lemma 5.2. *Let* $\mathfrak{g}$ *be a three-dimensional solvable Lie algebra. Then* $\mathfrak{g}^*$ *admits a vector space decomposition* $\mathfrak{g}^* = W_2 \oplus \operatorname{span}(e^3)$ *with* W_2 *two-dimensional and* $de^3 = 0$ *such that there exists a linear map* $f : W_2 \to W_2$ *with* $d\alpha = f(\alpha) \wedge e^3$ *for all* $\alpha \in W_2$*. If* $\operatorname{tr}(f) \neq 0$, $\frac{\det(f)}{\operatorname{tr}(f)^2}$ *only depends on the Lie algebra* $\mathfrak{g}$*. Moreover,* $\operatorname{tr}(f) = 0$ *exactly when* $\mathfrak{g}$ *is unimodular.*

As we will see in Section 5.3, the existence of a *contact form* on the three-dimensional part $\mathfrak{g}_3$, i.e. of a one-form $\alpha \in \mathfrak{g}_3^*$ with $d(\alpha) \wedge \alpha \neq 0$, in a direct sum $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ ensures the existence of a cocalibrated G_2-structure on $\mathfrak{g}$ for certain four-dimensional Lie algebras $\mathfrak{g}_4$. Therefore, we recall the well-known classification of three-dimensional Lie algebras admitting contact forms, see e.g. [Di]

Lemma 5.3. *A three-dimensional Lie algebra does not admit a contact form if and only if* $\mathfrak{g}$ *is solvable and* f *as in Lemma 5.2 is a multiple of the identity. So* $\mathfrak{g}$ *admits a contact-form if and only if* $\mathfrak{g} \notin \{\mathbb{R}^3, \mathfrak{r}_{3,1}\}$.

5.2 Four-dimensional Lie algebras

A classification of all four-dimensional Lie algebra has first been obtained in [Mu4d] by Mubarakzyanov. We give a complete list in the Tables 7.2 and 7.3.

In [ABDO] it is proved that each four-dimensional solvable Lie algebra admits a codimension one unimodular ideal. Since the only simple Lie algebras up to dimension four are $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$, it is an immediate consequence of Levi's decomposition theorem that the non-solvable four-dimensional Lie algebras are exactly $\mathfrak{so}(3) \oplus \mathbb{R}$ and $\mathfrak{so}(2,1) \oplus \mathbb{R}$. This shows the first part of

Lemma 5.4. *Let* $\mathfrak{g}$ *be a four-dimensional Lie algebra. Then* $\mathfrak{g}$ *admits a codimension one unimodular ideal* $\mathfrak{u}$*.* $\mathfrak{u}$ *is unique if and only if* $\dim([\mathfrak{g}, \mathfrak{g}]) = 3$ *or* $\mathfrak{g}$ *is not unimodular. In these cases,* $\mathfrak{u}$ *the commutator ideal* $[\mathfrak{g}, \mathfrak{g}]$ *or the unimodular kernel of* $\mathfrak{g}$*, respectively.*

Proof. If $\mathfrak{g}$ is not unimodular, then the unimodular kernel has codimension one and each unimodular ideal of $\mathfrak{g}$ is an ideal of the unimodular kernel. Thus, a codimension one unimodular ideal has to coincide with the unimodular kernel. The commutator ideal $[\mathfrak{g}, \mathfrak{g}]$ is a unimodular ideal and contained in each codimension one ideal. Thus, the uniqueness statement follows also if $\dim([\mathfrak{g}, \mathfrak{g}]) = 3$.

If $\mathfrak{g}$ is unimodular and $\dim([\mathfrak{g}, \mathfrak{g}]) < 3$, then each three-dimensional subspace of $\mathfrak{g}$ containing $[\mathfrak{g}, \mathfrak{g}]$ is a unimodular ideal of $\mathfrak{g}$. In particular, there is more than one such ideal. $\square$

In Lemma 4.3 we saw that the exterior derivative has a particular nice form for Lie algebras admitting a codimension one Abelian ideal. More generally, if a Lie algebra admits a codimension one unimodular ideal we have

Lemma 5.5. *Let $\mathfrak{g}$ be an n-dimensional Lie algebra which admits a codimension one unimodular ideal $\mathfrak{u} \subseteq \mathfrak{g}$. Let $e_n \in \mathfrak{g}\backslash\mathfrak{u}$ and $e^n \in \mathfrak{u}^0$, $e^n(e_n) = 1$. Identifying as usual $\Lambda^* {e_n}^0 \cong \Lambda^*\mathfrak{u}^*$ via the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \operatorname{span}(e_n)$, the following statements are true:*

(a) $d_{\mathfrak{g}} e^n = 0$ and there exists $f \in \mathfrak{gl}(\mathfrak{u}^*)$ such that $d_{\mathfrak{g}}\alpha = d_{\mathfrak{u}}\alpha + f(\alpha) \wedge e^n$ for all $\alpha \in \mathfrak{u}^*$.

(b) $d_{\mathfrak{g}}(\omega \wedge e^n) = d_{\mathfrak{u}}(\omega) \wedge e^n$ for all $\omega \in \Lambda^*\mathfrak{u}^*$.

(c) $d_{\mathfrak{g}}(\Lambda^{n-2}\mathfrak{u}^*) \subseteq \Lambda^{n-2}\mathfrak{u}^* \wedge e^n$.

(d) $d_{\mathfrak{g}}(\Lambda^{n-2}\mathfrak{u}^* \wedge e^n) = \{0\}$. Moreover, $d_{\mathfrak{g}}(\Lambda^{n-1}\mathfrak{u}^*) = \{0\}$ exactly when $\mathfrak{g}$ is unimodular.

Proof. (a) For arbitrary $X, Y \in \mathfrak{g}$, the commutator $[X, Y]$ is in $\mathfrak{u}$. Hence $d_{\mathfrak{g}} e^n(X, Y) = -e^n([X, Y]) = 0$ and so $d_{\mathfrak{g}} e^n = 0$. It is clear that there are linear maps $f : \mathfrak{u}^* \to \mathfrak{u}^*$ and $g : \mathfrak{u}^* \to \Lambda^2\mathfrak{u}^*$ such that $d_{\mathfrak{g}}(\alpha) = g(\alpha) + f(\alpha) \wedge e^n$ for all $\alpha \in \mathfrak{u}^*$. For $Z, W \in \mathfrak{u}$ we have $[Z, W] \in \mathfrak{u}$ and

$$g(\alpha)(Z, W) = (d_{\mathfrak{g}}\alpha)(Z, W) = -\alpha([Z, W]) = (d_{\mathfrak{u}}\alpha)(Z, W).$$

From our identifications, we get $g(\alpha) = d_{\mathfrak{u}}(\alpha)$ and so (a).

(b) Part (a) implies that $d_{\mathfrak{g}}\omega = d_{\mathfrak{u}}\omega + f.\omega \wedge e^n$ for all $\omega \in \Lambda^k\mathfrak{u}^*$, where $(f, \omega) \mapsto f.\omega$ is the natural action of $f \in \mathfrak{gl}(\mathfrak{u}^*)$ on $\omega \in \Lambda^k\mathfrak{u}^*$. Then (a) implies $d_{\mathfrak{g}}(\omega \wedge e^n) = d_{\mathfrak{g}}(\omega) \wedge e^n = d_{\mathfrak{u}}(\omega) \wedge e^n$ as claimed.

(c) We have $d_{\mathfrak{g}}\omega = d_{\mathfrak{u}}\omega + f.\omega \wedge e^n$ for all $\omega \in \Lambda^{n-2}\mathfrak{u}^*$. But $\mathfrak{u}$ is unimodular, which is equivalent to the fact that all $(n-2)$-forms on $\mathfrak{u}$ are $d_{\mathfrak{u}}$-closed. Hence, $d_{\mathfrak{g}}\omega = f.\omega \wedge e^n \in \Lambda^{n-2}\mathfrak{u}^* \wedge e^n$ as claimed.

(d) Part (a) and (c) directly imply $d_{\mathfrak{g}}(\Lambda^{n-2}\mathfrak{u}^* \wedge e^n) = \{0\}$. Since $\mathfrak{g}$ is unimodular exactly when all $(n-1)$-forms are $d_{\mathfrak{g}}$-closed, the first part implies that $d_{\mathfrak{g}}(\Lambda^{n-1}\mathfrak{u}^*) = \{0\}$ exactly when $\mathfrak{g}$ is unimodular.

□

In Section 5.3, we relate the existence of cocalibrated G_2-structures on direct sums $\mathfrak{g}_4 \oplus \mathfrak{g}_3$ to the existence of subspaces of $\Lambda^2\mathfrak{g}_4^*$ in which each non-zero element is symplectic. The next lemma shows that almost all four-dimensional Lie algebras $\mathfrak{g}_4$ admit a, possibly trivial, subspace of $\Lambda^2\mathfrak{g}_4^*$ of the mentioned kind whose dimension solely depends on the Lie algebra Betti numbers of $\mathfrak{g}_4$ and of a codimension one unimodular ideal $\mathfrak{u}$ in $\mathfrak{g}_4$. Note that Ovando classified in [Ov] the symplectic four-dimensional Lie algebras and also all symplectic two-forms on them. The assertion of the next lemma is not stated in [Ov] but may be obtained from the explicit lists there. We do not use at all the results of [Ov] and instead give a direct proof.

Lemma 5.6. *Let $\mathfrak{g}$ be a four-dimensional Lie algebra and assume that*

(i) $\mathfrak{g}$ is almost Abelian with codimension one Abelian ideal $\mathfrak{u}$ or

(ii) $\mathfrak{g}$ is not unimodular and the unimodular kernel $\mathfrak{u}$ is not isomorphic to $e(1,1)$.

Then $\mathfrak{g}$ admits a subspace of $\Lambda^2\mathfrak{g}^*$ of dimension

$$D := h^2(\mathfrak{g}) - h^1(\mathfrak{g}) - h^1(\mathfrak{u}) + 4$$

in which each non-zero element is symplectic.

Proof. Fix a norm $\|\cdot\|$ on $\mathfrak{g}^* \oplus \Lambda^2\mathfrak{g}^*$ and identify $\Lambda^4\mathfrak{g}^* \cong \mathbb{R}$ for the rest of the proof. Choose an element $e_4 \in \mathfrak{g}\backslash\mathfrak{u}$ and let $e^4 \in \mathfrak{u}^0$ be such that $e^4(e_4) = 1$. As usual, we identify $e_4{}^0 \cong \mathfrak{u}^*$ via the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \operatorname{span}(e_7)$. By Lemma 5.5, there exists $f \in \mathfrak{gl}(\mathfrak{u}^*)$ such that $d_{\mathfrak{g}}\beta = d_{\mathfrak{u}}\beta + f(\beta) \wedge e^4$ for all $\beta \in \mathfrak{u}^*$. We fix a complement V of $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$ and set

$$W_\lambda := \left\{\omega + \lambda g(\omega) \wedge e^4 \middle| \omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}\right\} \subseteq \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge e^4$$

for $\lambda \neq 0$ with $g : \Lambda^2\mathfrak{u}^* \to \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ as in Lemma 5.1 (c), i.e. $g(\omega)\wedge\omega = 0$ if and only if $\omega \in d_{\mathfrak{u}}(\mathfrak{u}^*)$. We claim that there is $\lambda \neq 0$ such that each non-zero element in $U := d_{\mathfrak{g}}(V) + W_\lambda$ is symplectic and that the dimension of U is equal to $D = h^2(\mathfrak{g}) - h^1(\mathfrak{g}) - h^1(\mathfrak{u}) + 4$. Note that the closure of all elements in U is clear. We divide the proof into six steps.

Step I: All non-zero elements in $d_{\mathfrak{g}}(V)$ are symplectic and $d_{\mathfrak{g}}|_V : V \to d_{\mathfrak{g}}(V)$ is an isomorphism:

If $V = \{0\}$, then there is nothing to show. Otherwise our assumptions imply that $\mathfrak{g}$ is not unimodular and so $d_{\mathfrak{g}}(\Lambda^3\mathfrak{u}^*) \neq \{0\}$ by Lemma 5.5. Let $\alpha \in V\backslash\{0\}$. By definition of V, $d_{\mathfrak{u}}\alpha \neq 0$ and so Lemma 5.1 (d) tells us that $\Lambda^3\mathfrak{u}^* \ni d_{\mathfrak{u}}\alpha \wedge \alpha \neq 0$. Hence $d_{\mathfrak{g}}(d_{\mathfrak{u}}\alpha \wedge \alpha) \neq 0$ and so

$$d_{\mathfrak{g}}\alpha \wedge d_{\mathfrak{g}}\alpha = d_{\mathfrak{g}}(\alpha \wedge d_{\mathfrak{g}}\alpha) = d_{\mathfrak{g}}(\alpha \wedge d_{\mathfrak{u}}\alpha + \alpha \wedge f(\alpha) \wedge e^4) = d_{\mathfrak{g}}(\alpha \wedge d_{\mathfrak{u}}\alpha) \neq 0.$$

So $d_{\mathfrak{g}}\alpha$ is non-degenerate and, in particular, $d_{\mathfrak{g}}\alpha \neq 0$. This proves Step I.

Step II: $f(V)$ is a complement of $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$ and $d_{\mathfrak{g}}(V) \cap W_\lambda =$ $d_{\mathfrak{g}}(V) \cap \left(\ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge e^4\right) = \{0\}$ for all $\lambda \neq 0$:

The inequality $0 \neq d_{\mathfrak{g}}\alpha \wedge d_{\mathfrak{g}}\alpha = 2d_{\mathfrak{u}}\alpha \wedge f(\alpha) \wedge e^4$ for $\alpha \in V\backslash\{0\}$ implies that $f|_V$ is injective and so $\dim(V) = \dim(f(V))$. By Lemma 5.1 (b), $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge d_{\mathfrak{u}}(\mathfrak{u}^*) = \{0\}$. Thus, $f(V)$ is a complement of $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$. Let $\omega \in d_{\mathfrak{g}}(V) \cap \left(\ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge e^4\right)$. Then there are $\alpha \in V$, $\omega_1 \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$ and $\beta \in \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ such that

$$\omega = d_{\mathfrak{u}}\alpha + f(\alpha) \wedge e^4 = \omega_1 + \beta \wedge e^4.$$

This implies $f(\alpha) = \beta \in \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ and so, since $f(V)$ is a complement of $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$, $\beta = 0$. Now $f|_V$ is injective and so we must have $\alpha = 0$, which ultimately implies $\omega = 0$. This finishes the proof of Step II.

Step III: $\dim(d_{\mathfrak{g}}(V) \oplus W_\lambda) = h^2(\mathfrak{g}) - h^1(\mathfrak{g}) - h^1(\mathfrak{u}) + 4$:

Note that the dimension of W_λ is equal to the dimension of $\ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$ and that the dimension of $\ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{g}^*}$ is $h^2(\mathfrak{g}) + 4 - h^1(\mathfrak{g})$. Therefore it suffices to show

$$\ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{g}^*} = \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge e^4 \oplus d_{\mathfrak{g}}(V)$$

to get the statement about the dimension of $d_{\mathfrak{g}}(V) \oplus W_\lambda$. The inclusion "$\supseteq$" is obvious. For the other inclusion, let $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{g}^*}$. Then there exists $\omega_1 \in \Lambda^2\mathfrak{u}^*$ and $\beta \in \mathfrak{u}^*$ such that $\omega = \omega_1 + \beta \wedge e^4$. Since $f(V)$ is a complement of $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$, there exists $\alpha \in V$ with $\beta - f(\alpha) \in \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$. Then

$$\omega - (\beta - f(\alpha)) \wedge e^4 - d_{\mathfrak{g}}\alpha = \omega_1 + \beta \wedge e^4 - (\beta - f(\alpha)) \wedge e^4 - d_{\mathfrak{u}}\alpha - f(\alpha) \wedge e^4 = \omega_1 - d_{\mathfrak{u}}\alpha \in \Lambda^2\mathfrak{u}^*$$

and $\omega - (\beta - f(\alpha)) \wedge e^4 - d_{\mathfrak{g}}\alpha$ is $d_{\mathfrak{g}}$-closed. Hence, $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \oplus \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge e^4 \oplus d_{\mathfrak{g}}(V)$.

Step IV: $\ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \cap d_{\mathfrak{u}}(\mathfrak{u}^*) = \{0\}$:

Let $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*} \cap d_{\mathfrak{u}}(\mathfrak{u}^*)$. Then $\omega = d_{\mathfrak{u}}\beta$ for some $\beta \in \mathfrak{u}^*$ and $d_{\mathfrak{g}}\omega = 0$. We may assume that $\beta \in V$. But then

$$0 = d_{\mathfrak{g}}\omega = d_{\mathfrak{g}}(d_{\mathfrak{g}}\beta - f(\beta) \wedge e^4) = -d_{\mathfrak{u}}(f(\beta)) \wedge e^4.$$

Since $f(V)$ is a complement of $\ker d_{\mathfrak{u}}|_{\mathfrak{u}^*}$ in $\mathfrak{u}^*$ and $f|_V$ is injective we get $\beta = 0$ and so $\omega = 0$ as claimed.

Step V: Norm estimates:

Note first that the identity

$$(d_{\mathfrak{g}}\alpha)^2 = 2d_{\mathfrak{u}}\alpha \wedge f(\alpha) \wedge e^4$$

and the fact that $f|_V$ and $d_{\mathfrak{u}}|_V$ are injective imply the existence of a constant $A > 0$ such that

$$|(d_{\mathfrak{g}}\alpha)^2| \geq A \, \|\alpha\|^2 \,. \tag{5.3}$$

Note further the sign of $(d_{\mathfrak{g}}\alpha)^2 \in \Lambda^4\mathfrak{g}^* \cong \mathbb{R}$ for $\alpha \neq 0$ does not depend on α. Namely, let $F : V \to \mathbb{R}$, $F(\alpha) := (d_{\mathfrak{g}}\alpha)^2$. For $\dim(V) > 1$ the set $V \backslash \{0\}$ is connected, while $F(V \backslash \{0\})$ is disconnected if the sign depends on $\alpha \neq 0$, contradicting the continuity of F. If $\dim(V) = 1$ then the statement follows from the fact that F is homogeneous of degree two in α.

Next we consider the space W_λ for arbitrary $\lambda \neq 0$. Lemma 5.1 (c) tells us that

$$(\omega + \lambda g(\omega) \wedge e^4)^2 = 2\lambda \, \omega \wedge g(\omega) \wedge e^4 = 0$$

for $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$ implies $\omega \in d_{\mathfrak{u}}(\mathfrak{u}^*)$. But Step IV tells us that then $\omega = 0$. Thus, there exists $C > 0$, independent of λ, such that

$$|(\omega + \lambda g(\omega) \wedge e^4)^2| \geq C|\lambda| \, \|\omega\|^2 \tag{5.4}$$

for all $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$. Note that for fixed $\lambda \neq 0$, arguing as above, we see that the sign of $(\omega + \lambda g(\omega) \wedge e^4)^2 \in \mathbb{R}$ does not depend on ω. But it gets reversed if we reverse the sign of λ. Hence, we may assume that it is chosen such that $\omega_1^2 \cdot \omega_2^2 > 0$ for all $\omega_1 \in d_{\mathfrak{g}}(V)\backslash\{0\}$, $\omega_2 \in W_\lambda\backslash\{0\}$. By Lemma 5.1 (b), the identity $d_{\mathfrak{u}}\alpha \wedge g(\omega) = 0$ is true for all $\alpha \in V$ and $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$. Thus,

$$2d_{\mathfrak{g}}\alpha \wedge (\omega + \lambda g(\omega) \wedge e^4) = 2(d_{\mathfrak{u}}\alpha + f(\alpha) \wedge e^4) \wedge (\omega + \lambda g(\omega) \wedge e^4) = 2f(\alpha) \wedge e^4 \wedge \omega$$

and there exists a constant $B > 0$ such that

$$|2d_{\mathfrak{g}}\alpha \wedge (\omega + \lambda g(\omega) \wedge e^4)| \leq B \,\|\alpha\|\, \|\omega\| \,. \tag{5.5}$$

Step VI: All non-zero elements in $d_{\mathfrak{g}}(V) \oplus W_\lambda$ are symplectic for appropriate $\lambda \neq 0$:

Let $0 \neq \omega_0 = \omega_1 + \omega_2 \in d_{\mathfrak{g}}(V) \oplus W_\lambda$ with $\omega_1 = d_{\mathfrak{g}}\alpha \in d_{\mathfrak{g}}(V)$ for some $\alpha \in V$ and $\omega_2 = \omega + \lambda g(\omega) \wedge e^4 \in W_\lambda$ for some $\omega \in \ker d_{\mathfrak{g}}|_{\Lambda^2\mathfrak{u}^*}$. By the previous steps, we only have to consider the case when $\omega_1 \neq 0$ and $\omega_2 \neq 0$. Then both α and ω are not zero by the Equations (5.3) and (5.4). The discriminant of the polynomial $\omega_0^2 = (\omega_1 + X\omega_2)^2 = \omega_2^2 + 2X\,\omega_1 \wedge \omega_2 + X^2\,\omega_1^2$ is given by

$$(2\omega_1 \wedge \omega_2)^2 - 4\,\omega_1^2 \cdot \omega_2^2 \leq B^2\,\|\alpha\|^2\,\|\omega\|^2 - 4|\lambda|A\,C\,\|\alpha\|^2\,\|\omega\|^2 = (B^2 - 4|\lambda|A\,C)\,\|\alpha\|^2\,\|\omega\|^2\,,$$

where we used Equations (5.3), (5.4) and (5.5) and the fact that the sign of $\omega_1^2 \cdot \omega_2^2$ may be assumed to be positive. But for sufficiently large $|\lambda|$, independent of α and ω, this is negative and the quadratic polynomial in X does not have a real root. In particular, $X = 1$ is not a real root and so $\omega_0 = \omega_1 + \omega_2$ is non-degenerate. This finishes the proof. □

Remark 5.7.

- *Let $\mathfrak{g}$, $\mathfrak{u}$ and D be as in Lemma 5.6. Then D is, in fact, the maximal dimension of a subspace of $\Lambda^2\mathfrak{g}^*$ in which each non-zero element is symplectic. In the almost Abelian case, the maximality can easily be deduced using that d has a particular nice form by Lemma 4.3. If $\mathfrak{g}$ is not unimodular, the maximality can be deduced from Proposition 5.12 (a) and Theorem 5.18 (a) below.*
- *Lemma 5.6 applies to all but five Lie algebras:*
  - *The only non-unimodular four-dimensional Lie algebra with unimodular kernel $\mathfrak{u}$ isomorphic to $\mathfrak{e}(1,1)$ is $\mathfrak{r}_2 \oplus \mathfrak{r}_2$. In the basis given in Table 7.3, the two-form $e^{14} + e^{23}$ is symplectic. One is the maximal dimension of a subspace of $\Lambda^2\,(\mathfrak{r}_2 \oplus \mathfrak{r}_2)^*$ in which each non-zero element is symplectic, cf. [Ov].*
  - *The unimodular four-dimensional Lie algebras which do not admit a codimension one Abelian ideal are the two non-solvable ones $\mathfrak{so}(3)\oplus\mathbb{R}$ and $\mathfrak{so}(2,1)\oplus\mathbb{R}$ and two other Lie algebras, namely $A_{4,8}$ and $A_{4,10}$. All four do not admit any symplectic two-form.*

In Lemma 5.5, we gave a description of the exterior derivative of n-dimensional Lie algebras having a codimension one unimodular ideal $\mathfrak{u}$. If $n = 4$ and $\mathfrak{u} = \mathfrak{h}_3$, the next lemma shows that we can do better. For a proof, we refer the reader to [ABDO].

Lemma 5.8. *If $\mathfrak{g}$ is a four-dimensional Lie algebra $\mathfrak{g}$ which possesses an ideal $\mathfrak{u}$ isomorphic to $\mathfrak{h}_3$, then there exist an element $e_4 \in \mathfrak{g}\backslash\mathfrak{u}$, an element $e^1 \in \mathfrak{u}^* \cong {e_4}^0$, a two-dimensional subspace $V_2 \subseteq \mathfrak{u}^*$ with $\mathrm{span}(e^1) \oplus V_2 = \mathfrak{u}^*$, a linear map $F : V_2 \to V_2$ and a non-zero two-form $\nu \in \Lambda^2 V_2\backslash\{0\}$ such that $de^1 = \mathrm{tr}(F)e^{14} + \nu$, $d\alpha = F(\alpha) \wedge e^4$ for all $\alpha \in V_2$ and $de^4 = 0$. Here, e^4 is the element in $\mathfrak{u}^0$ with $e^4(e_4) = 1$. In this case, $\mathrm{tr}(F) = 0$ if and only if $\mathfrak{g}$ is unimodular.*

5.3 Existence

In this section, we state different existence results which are used in Section 5.5 to prove our main theorem. We first prove a general proposition which is true on any seven-manifold. This proposition is used afterwards to derive different more specific existence results for cocalibrated G_2-structures on Lie algebras.

For this purpose, we generalise the concept of adapted splittings to manifolds.

Definition 5.9. *Let M be a seven-dimensional manifold and $\varphi \in \Omega^3 M$ be a G_2-structure on M. We say that a vector bundle decomposition $TM = E_4 \oplus E_3$ is an* adapted splitting (for φ) *if for all $p \in M$ the vector space decomposition $T_pM = (E_4)_p \oplus (E_3)_p$ is an adapted splitting for $\varphi_p \in \Lambda^3 T_pM^*$ in the sense of Definition 2.52.*

Proposition 5.10. *Let M be a seven-dimensional manifold. Assume that there exists a G_2-structure φ on M which admits an adapted splitting $TM = E_4 \oplus E_3$ such that the following is true:*

(i) $\Omega_1 := (\star_\varphi \varphi)|_{E_4} \in \Gamma\left(\Lambda^4 E_4^*\right) \cong \Gamma\left(\Lambda^4 {E_3}^0\right) \subseteq \Gamma(\Lambda^4 T^*M)$ is closed.

(ii) There exists a bounded four-form $\Phi \in \Gamma\left(\Lambda^3 {E_3}^0 \wedge {E_4}^0\right)$ (i.e. $\|\Phi\|_{C_0} < \infty$) with $d\Phi = d\Omega_2$ for the four-form $\Omega_2 := \star_\varphi\varphi - \Omega_1 \in \Gamma\left(\Lambda^2 {E_3}^0 \wedge \Lambda^2 {E_4}^0\right)$.

Then M admits a cocalibrated G_2-structure, e.g. each G_2-structure $\varphi_\lambda \in \Omega^3(M)$ whose Hodge dual is given by

$$\Psi_\lambda := \lambda^4\Omega_1 + \lambda^2\Omega_2 - \lambda^2\Phi$$

for $\lambda \in \mathbb{R}$ with $|\lambda| > \frac{\|\Phi\|_{C_0}}{\epsilon_0}$. Here, ϵ_0 is the constant in Lemma 2.57

Proof. Let $p \in M$. By Lemma 2.53, $(\Omega_2)_p \in \Lambda^2{(E_3)_p}^0 \wedge \Lambda^2{(E_4)_p}^0$, $\sigma_\lambda := \lambda^4(\Omega_1)_p + \lambda^2(\Omega_2)_p$ is the Hodge-Dual of a G_2-structure on T_pM for all $\lambda \neq 0$ and $\left\|\lambda^3\Phi_p\right\|_\lambda = \|\Phi_p\|_1 = \|\Phi_p\|_{\varphi_p}$ for all $\lambda \neq 0$, where $\|\cdot\|_\lambda$ is the norm on T_pM induced by σ_λ. Thus,

$$\|(\Psi_\lambda)_p - \sigma_\lambda\|_\lambda = \left\|\lambda^2\Phi_p\right\|_\lambda = \frac{\|\Phi_p\|_{\varphi_p}}{|\lambda|} \leq \frac{\|\Phi\|_{C^0}}{|\lambda|} < \epsilon_0$$

for all $|\lambda| > \frac{\|\Phi\|_{C^0}}{\epsilon_0}$. Hence, Lemma 2.57 shows that Ψ_λ is then the Hodge dual of a G_2-structure on M. The assertion follows since Ψ_λ is closed by construction. $\square$

Remark 5.11.

- *The condition on the boundedness of Φ is trivially fulfilled if Φ is left-invariant or M is compact. Moreover, if the initial G_2-structure φ, the splitting $E_4 \oplus E_3$ and Φ are left-invariant, so is the induced cocalibrated G_2-structure.*

- *To prove an analogue of Proposition 5.10 in the left-invariant case for G_2- and also for G_2^*-structures we do not need at all a metric. We only need that the orbit of all Hodge duals is open in both cases. For the proof, let $\mathfrak{g}$ be a seven-dimensional Lie algebra $\mathfrak{g}$. The openness of the orbit implies that for any sequence $(A_n)_n$, $A_n \in \mathrm{GL}(\mathfrak{g})$, any Hodge dual $\Psi \in \Lambda^4\mathfrak{g}^*$ and any sequence $(\Phi_n)_n$, $\Phi_n \in \Lambda^4\mathfrak{g}^*$ with $\lim_{n\to\infty} \Phi_n = 0$ there is $N \in \mathbb{N}$ such that for all $n \geq N$ the four-form $\Psi + \Phi_n$ and so also the four-form $A_n^*(\Psi + \Phi_n)$ is a Hodge dual of the same type. Let now $\varphi \in \Lambda^3\mathfrak{g}^*$ be a G_2^ϵ-structure and $\mathfrak{g} = E_4 \oplus E_3$ be a splitting into a four-dimensional subspace E_4 and a three-dimensional subspace E_3 such that $\Psi := \star_\varphi \varphi = \Omega_1 + \Omega_2$ with $\Omega_1 \in \Lambda^4 E_4^*$, $\Omega_2 \in \Lambda^2 E_4^* \wedge \Lambda^2 E_3^*$, $d\Omega_1 = 0$ and such that there exists $\Phi \in \Lambda^3 E_4^* \wedge E_3^*$ with $d\Omega_2 = d\Phi$. Here, we identify, as usual, $E_4^* \cong {E_3}^0$ and $E_3^* \cong {E_4}^0$ via the decomposition $\mathfrak{g} = E_4 \oplus E_3$. Define $A_n \in \mathrm{GL}(\mathfrak{g})$ such that it acts by multiplication with n on E_4 and by the identity map on E_3 and set $\Phi_n := -\frac{\Phi}{n} \in \Lambda^3 E_4^* \wedge E_3^*$. Then our previous considerations show that*

$$\Psi_n := A_n^*(\star_\varphi \varphi + \Phi_n) = A_n^*\left(\Omega_1 + \Omega_2 - \frac{\Phi}{n}\right) = n^4\Omega_1 + n^2\Omega_2 - n^2\Phi$$

is, for n large enough, a Hodge dual of the same type as Ψ. Moreover, our assumptions imply that it is closed and so defines a cocalibrated G_2^ϵ-structure on $\mathfrak{g}$.

We apply Proposition 5.10 to the left-invariant case:

Proposition 5.12. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional Lie algebra $\mathfrak{g}_3$.*

(a) If $\mathfrak{g}_3$ is unimodular and there exists a $D := h^2(\mathfrak{g}_3)$-dimensional subspace W of $\Lambda^2\mathfrak{g}_4^*$ such that each non-zero element in W is a symplectic two-form, then $\mathfrak{g}$ admits a cocalibrated G_2-structure.

(b) Let $\mathfrak{g}_4 \in \{A_{4,12}, \mathfrak{r}_2 \oplus \mathfrak{r}_2\}$. If $\mathfrak{g}_3$ admits a contact-form α, then $\mathfrak{g}$ admits a cocalibrated G_2-structure.

(c) If $\mathfrak{g}_4$ is unimodular, admits a codimension one ideal $\mathfrak{u}$ isomorphic to $\mathfrak{h}_3$, $\mathfrak{g}_3$ is not unimodular and $h^1(\mathfrak{g}_4) + h^1(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) \geq 2$, then $\mathfrak{g}$ admits a cocalibrated G_2-structure.

Proof. (a) Choose a basis ν_1, ν_2, ν_3 of $\Lambda^2\mathfrak{g}_3^*$ such that $\nu_{D+1} = d\alpha_{D+1}, \ldots, \nu_3 = d\alpha_3$ is a basis of $d(\mathfrak{g}_3^*)$, $\alpha_{D+1}, \ldots, \alpha_3 \in \mathfrak{g}_3^*$. Note that there are $3 - D$ exact two-forms on $\mathfrak{g}_3$ since the unimodularity of $\mathfrak{g}_3$ is equivalent to the closure of all two-forms on $\mathfrak{g}_3$. Furthermore, choose a basis $\omega_1, \ldots, \omega_D$ of W. Then Lemma 2.2 and Proposition 2.55 imply that there exist two-forms $\omega_{D+1}, \ldots, \omega_3 \in \Lambda^2\mathfrak{g}_4^*$ such that

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2-structure with adapted splitting $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$. Since $d(\Lambda^2\mathfrak{g}_3^*) = 0$, the identity $d(\sum_{i=1}^3 \omega_i \wedge \nu_i) = d(-\sum_{i=D+1}^3 d\omega_i \wedge \alpha_i)$ is true and $\sum_{i=D+1}^3 d\omega_i \wedge \alpha_i \in \Lambda^3\mathfrak{g}_4^* \wedge \mathfrak{g}_3^*$. Hence, Proposition 5.10 implies the result.

(b) Let e^1, e^2, e^3, e^4 be a basis of $\mathfrak{g}_4^* \in \{A_{4,12}^*, (\mathfrak{r}_2 \oplus \mathfrak{r}_2)^*\}$ as in Table 7.3, i.e. $de^1 = e^{14} + e^{23}$, $de^2 = e^{24} - \epsilon e^{13}$, $de^3 = 0 = de^4$, where $\epsilon = 1$ if $\mathfrak{g}_4 = A_{4,12}$ and $\epsilon = -1$ if $\mathfrak{g}_4 = \mathfrak{r}_2 \oplus \mathfrak{r}_2$. Set $V_4^* := \mathrm{span}(e^4) \oplus \mathfrak{g}_3^*$, $V_3^* := \mathrm{span}(e^1, e^2, e^3)$. Let $\alpha_1 \in \mathfrak{g}_3^*$ be a contact form and set $\omega_1 := 2e^4 \wedge \alpha_1 - d\alpha_1 \in \Lambda^2 V_4^*$. Then $\omega_1^2 \neq 0$ and $d\left(\frac{1}{2}\omega_1^2\right) = 0$. Hence, if we set $\nu_1 := e^{12}$, $\nu_2 := e^{13}$, $\nu_3 := e^{23}$, Proposition 2.55 implies the existence of two-forms $\omega_2, \omega_3 \in \Lambda^2 V_4^*$ such that

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2-structure with adapted splitting $\mathfrak{g} = V_4 \oplus V_3$. Decompose $\omega_i = e^4 \wedge \alpha_i + \theta_i$ with $\alpha_i \in \mathfrak{g}_3^*$, $\theta_i \in \Lambda^2\mathfrak{g}_3^*$ for $i = 2, 3$. Then $d(\omega_1 \wedge \nu_1) = d(2e^4 \wedge \alpha_1 \wedge e^{12} - d\alpha_1 \wedge e^{12}) = 0$ and so the differential of the four-form $\sum_{i=1}^3 \omega_i \wedge \nu_i$ is given by

$$\begin{aligned} d\left(\sum_{i=1}^{3} \omega_i \wedge \nu_i\right) &= 0 + d(e^{134} \wedge \alpha_2 + e^{234} \wedge \alpha_3) + d(e^{13} \wedge \theta_2 + e^{23} \wedge \theta_3) \\ &= d(\epsilon e^{24} \wedge d\alpha_2 - e^{14} \wedge d\alpha_3) \\ &\quad + d(\epsilon(e^{24} \wedge \theta_2 - e^2 \wedge d\theta_2) - e^{14} \wedge \theta_3 + e^1 \wedge d\theta_3) \\ &= d(e^1 \wedge \rho_1 - \epsilon e^2 \wedge \rho_2). \end{aligned}$$

with $\rho_1 := -e^4 \wedge (d\alpha_3 + \theta_3) + d\theta_3 \in \Lambda^3 V_4^*$, $\rho_2 := -e^4 \wedge (d\alpha_2 + \theta_2) + d\theta_2 \in \Lambda^3 V_4^*$. Since $e^1 \wedge \rho_1 - \epsilon e^2 \wedge \rho_2$ is in $V_3^* \wedge \Lambda^3 V_4^*$, Proposition 5.10 implies the result.

(c) By Lemma 5.8 we may decompose $\mathfrak{g}_4^*$ into $\mathrm{span}(e^1) \oplus V_2 \oplus \mathrm{span}(e^4)$ for $e^1, e^4 \in \mathfrak{g}_4^*$ and a two-dimensional subspace V_2 such that $0 \neq de^1 \in \Lambda^2 V_2$, $d\alpha = F(\alpha) \wedge e^4$ for all $\alpha \in V_2$, $F : V_2 \to V_2$ a trace-free linear map, and $de^4 = 0$. Moreover, by Lemma 5.2 we may decompose $\mathfrak{g}_3^* = W_2 \oplus \mathrm{span}(e^7)$ with $e^7 \in \mathfrak{g}_3^*$ and a two-dimensional subspace W_2 such that $d\beta = G(\beta) \wedge e^7$ for all $\beta \in W_2$, $G : W_2 \to W_2$ a linear map which is not trace-free, and $de^7 = 0$. By rescaling e^7 we may assume that $\mathrm{tr}(G) = 1$.

We have $\ker d|_{\Lambda^2\mathfrak{g}_4^*} = \Lambda^2 V_2 \oplus V_2 \wedge e^4 \oplus \ker(F) \wedge e^1$. Thus, the identity

$$2 - \mathrm{rk}(F) + 3 = \dim(\ker(F)) + 3 = \dim(\ker d|_{\Lambda^2\mathfrak{g}_4^*}) = h^2(\mathfrak{g}_4) + 4 - h^1(\mathfrak{g}_4)$$

is true. Moreover, $\dim(\ker G) = h^1(\mathfrak{g}_3) - 1$ and so the condition in the statement is equivalent to $\dim(\ker G) \geq 2 - \mathrm{rk}(F)$. Hence, we may choose a basis α_1, α_2 of V_2, elements $\gamma_i \in V_2$, $1 \leq i \leq \mathrm{rk}(F)$, and a basis β_1, β_2 of W_2 such that $de^1 = \alpha_1 \wedge \alpha_2$, such that $\alpha_i = F(\gamma_i)$, $1 \leq i \leq \mathrm{rk}(F)$, is a basis of $F(V_2)$ and such that $\mathrm{span}(\beta_j | \mathrm{rk}(F)+1 \leq j \leq 2)$ is a subspace of $\ker G$. Set $V_4^* := \mathrm{span}(e^1) \oplus V_2 \oplus \mathrm{span}(e^7)$, $V_3^* := W_2 \oplus \mathrm{span}(e^4)$ and

$$\begin{aligned}
&\nu_1 := \beta_1 \wedge \beta_2, \quad \nu_2 := \beta_1 \wedge e^4, \quad \nu_3 := -\beta_2 \wedge e^4,\\
&\omega_1 := e^{71} - de^1 = e^{71} - \alpha_1 \wedge \alpha_2, \quad \omega_2 := e^7 \wedge \alpha_2 - e^1 \wedge \alpha_1,\\
&\omega_3 := e^7 \wedge \alpha_1 + e^1 \wedge \alpha_2.
\end{aligned}$$

Since ν_1, ν_2, ν_3 is a basis of $\Lambda^2 V_3^*$, Proposition 2.55 implies that

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2-structure with adapted splitting $V_4 \oplus V_3$. Moreover,

$$\begin{aligned}
d(\omega_1 \wedge \nu_1) &= d(e^{71} \wedge \beta_1 \wedge \beta_2 - de^1 \wedge \beta_1 \wedge \beta_2)\\
&= -e^7 \wedge de^1 \wedge \beta_1 \wedge \beta_2 + \mathrm{tr}(G) de^1 \wedge \beta_1 \wedge \beta_2 \wedge e^7 = 0
\end{aligned}$$

and so

$$\begin{aligned}
d\left(\sum_{i=1}^{3} \omega_i \wedge \nu_i\right) &= d\left(-\sum_{i=1}^{2} e^1 \wedge \alpha_i \wedge \beta_i \wedge e^4\right) = -\sum_{i=1}^{\mathrm{rk}(F)} F(\gamma_i) \wedge e^4 \wedge e^1 \wedge G(\beta_i) \wedge e^7\\
&= d\left(-\sum_{i=1}^{\mathrm{rk}(F)} \gamma_i \wedge e^1 \wedge G(\beta_i) \wedge e^7\right).
\end{aligned}$$

But $-\sum_{i=1}^{\mathrm{rk}(F)} \gamma_i \wedge e^1 \wedge G(\beta_i) \wedge e^7$ is in $V_3^* \wedge \Lambda^3 V_4^*$. Since F is trace-free, $d(\Lambda^4 V_4^*) = \{0\}$ and again Proposition 5.10 implies the result.

□

Remark 5.13. *The following generalisation of Proposition 5.12 (a) follows from Proposition 5.10 using Lemma 2.2:*
Let $M = N \times G$ be a seven-dimensional manifold such that N is a four-dimensional compact Riemannian manifold with trivial bundle of self-dual two-forms and such that G is a unimodular three-dimensional Lie group. If N admits $D := h^2(\mathfrak{g})$ ($\mathfrak{g}$ being the Lie algebra of G) self-dual, closed two-forms $\omega_i \in \Omega^2 N$ such that $\omega_i \wedge \omega_j = 0$ and $\omega_i^2 = \omega_j^2$ for $i \neq j$, then M admits a cocalibrated G_2-structure which is invariant under the left-action of G on $M = N \times G$ given by left-translation on the second factor.

$D = 0$ is allowed in Proposition 5.12 (a). Since each non-solvable four-dimensional Lie algebra $\mathfrak{g}$ is a Lie algebra direct sum $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{R}$ with $\mathfrak{h} \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$, $h^2(\mathfrak{so}(3)) = h^2(\mathfrak{so}(2,1)) = 0$ and $\mathfrak{so}(3), \mathfrak{so}(2,1)$ are the only three-dimensional non-solvable Lie algebras, we get

Corollary 5.14. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional Lie algebra $\mathfrak{g}_3$. If $\mathfrak{g}$ is not solvable, then $\mathfrak{g}$ admits a cocalibrated* G_2*-structure.*

5.4 Obstructions

In this section, we derive obstructions to the existence of cocalibrated G_2-structures on Lie algebras, which we use in subsections 5.5.1 - 5.5.4 to prove Theorem 5.18.

We first need the following technical lemma.

Lemma 5.15. *Let V be a six-dimensional vector space.*

(a) Let $V = V_3 \oplus W_3$ be a decomposition into two vector spaces of dimension three and let $\Omega = \Omega_1 + \Omega_2 \in \Lambda^4 V^*$ with $\Omega_1 \in \Lambda^2 V_3^* \wedge \Lambda^2 W_3^*$ and $\Omega_2 \in V_3^* \wedge \Lambda^3 W_3^*$ be a four-form of length three. Then the length of Ω_1 is also three.

(b) Let $V = V_4 \oplus V_2$ be a decomposition into a vector space V_4 of dimension four and a vector space V_2 of dimension two. Let ρ be a three-form with model tensor $\rho_{-1} \in \Lambda^3 \left(\mathbb{R}^6\right)^*$ such that $\rho \in \Lambda^2 V_4^* \wedge V_2^* \oplus V_4^* \wedge \Lambda^2 V_2^*$. Then, for any basis α_1, α_2 of V_2^*, the unique two-forms $\omega_1, \omega_2 \in \Lambda^2 V_4^*$ such that $\rho - \sum_{i=1}^2 \omega_i \wedge \alpha_i \in V_4^* \wedge \Lambda^2 V_2^*$ span a two-dimensional subspace in $\Lambda^2 V_4^*$ in which each non-zero element is of length two.

Proof. (a) Choose an arbitrary dual isomorphism $\delta : \Lambda^4 V^* \to \Lambda^2 V$. Then $\delta(\Omega_1) \in V_3 \wedge W_3$ and $\delta(\Omega_2) \in \Lambda^2 V_3$. By Lemma 1.43, the length of $\delta(\Omega)$ is three and Lemma 2.1 implies $0 \neq \delta(\Omega)^3 = (\delta(\Omega_1) + \delta(\Omega_2))^3 = \delta(\Omega_1)^3$. Thus, $\delta(\Omega_1)$ and so Ω_1 has length three.

(b) There is $\beta \in V_4^*$ such that $\rho = \omega_1 \wedge \alpha_1 + \omega_2 \wedge \alpha_2 + \beta \wedge \alpha_1 \wedge \alpha_2$. We have to show that $l(a\omega_1 + b\omega_2) = 2$ for all $(a, b) \neq (0, 0)$. Without loss of generality, we may assume $a \neq 0$ and then even $a = 1$. We rewrite ρ as

$$\rho = (\omega_2 + \beta \wedge \alpha_1) \wedge (\alpha_2 - b\alpha_1) + (\omega_1 + b\omega_2) \wedge \alpha_1.$$

Note that $(\omega_1 + b\omega_2) \wedge \alpha_1 \in \Lambda^3(V_4^* \oplus \mathrm{span}(\alpha_1))$ and $(\omega_2 + \beta \wedge \alpha_1) \in \Lambda^2(V_4^* \oplus \mathrm{span}(\alpha_1))$. By Proposition 1.45, $r(\rho) = 2$. Hence, $l((\omega_1 + b\omega_2) \wedge \alpha_1) \geq 2$ (consider $V^* = (V_4^* \oplus \mathrm{span}(\alpha_1)) \oplus \mathrm{span}(\alpha_2 - b\alpha_1))$ and so $l(\omega_1 + b\omega_2) \geq 2$. Since the maximal length of a two-form in four dimensions is two, we actually have $l(\omega_1 + b\omega_2) = 2$.

□

We are now ready to prove

Proposition 5.16. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$ such that $\mathfrak{g}_4$ admits a unique unimodular ideal $\mathfrak{u}$ of codimension one. If $\mathfrak{g}$ admits a cocalibrated G_2-structure, then*

$$h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) - h^2(\mathfrak{g}_4) + h^2(\mathfrak{g}_3) \leq 4.$$

Proof. Let Ψ be the Hodge dual of a cocalibrated G_2-structure. Fix an element $e_4 \in \mathfrak{g}\backslash\mathfrak{u}$ and let $e^4 \in \mathfrak{u}^0$ be such that $e^4(e_4) = 1$. We set

$$\Lambda^{i,j,k} := \Lambda^i\mathfrak{u}^* \wedge \Lambda^j\mathfrak{g}_3^* \wedge \Lambda^k\mathrm{span}(e^4)$$

and denote by $\theta^{i,j,k}$ the projection of θ into $\Lambda^{i,j,k}$ for all $i, j, k \in \mathbb{N}_0$ and all $(i+j+k)$-forms $\theta \in \Lambda^{i+j+k}\mathfrak{g}^*$. For the proof, we denote by d the exterior differential on $\mathfrak{g}$ and by $d_{\mathfrak{u}}$ the one on $\mathfrak{u}$. Lemma 5.5 implies the inclusions

$$d(\Lambda^{i,j,0}) \subseteq \Lambda^{i+1,j,0} \oplus \Lambda^{i,j,1} \oplus \Lambda^{i,j+1,0}, \quad d(\Lambda^{i,j,1}) \subseteq \Lambda^{i+1,j,1} \oplus \Lambda^{i,j+1,1}$$

for all $i, j \in \mathbb{N}_0$ and the unimodularity of $\mathfrak{u}$ and $\mathfrak{g}_3$ imply that for all $i \in \mathbb{N}_0$:

$$\begin{aligned} d(\Lambda^{2,i,0}) &\subseteq \Lambda^{2,i,1} \oplus \Lambda^{2,i+1,0}, \ d(\Lambda^{2,i,1}) \subseteq \Lambda^{2,i+1,1}, \\ d(\Lambda^{i,2,0}) &\subseteq \Lambda^{i+1,2,0} \oplus \Lambda^{i,2,1}, \ d(\Lambda^{i,2,1}) \subseteq \Lambda^{i+1,2,1}. \end{aligned}$$

We show that there are $D := h^2(\mathfrak{g}_3)$ linearly independent closed two-forms $\omega_1, \ldots, \omega_D \in \Lambda^2\mathfrak{g}_4^*$ such that $\mathrm{span}(\omega_1, \ldots, \omega_D) \cap \Lambda^{1,0,1} = \{0\}$. Note that $\dim(\ker d|_{\Lambda^{1,0,1}}) = h^1(\mathfrak{u})$ since $\ker d|_{\Lambda^{1,0,1}} = \ker d_{\mathfrak{u}}|_{\mathfrak{u}^*} \wedge e^4$ by Lemma 5.5. Hence, the existence of such $\omega_1, \ldots, \omega_D \in \Lambda^2\mathfrak{g}_4^*$ implies

$$\begin{aligned} &h^2(\mathfrak{g}_4) + 4 - h^1(\mathfrak{g}_4) = \dim(\ker d|_{\Lambda^2\mathfrak{g}^*}) \geq D + h^1(\mathfrak{u}) = h^2(\mathfrak{g}_3) + h^1(\mathfrak{u}) \\ &\Leftrightarrow\ h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) \leq 4. \end{aligned}$$

The two-forms $\omega_1, \ldots, \omega_D \in \Lambda^2\mathfrak{g}_4^*$ will be certain parts of $\Psi^{2,2,0} + \Psi^{1,2,1}$. Therefore, we decompose Ψ as

$$\Psi = \Omega + \rho \wedge e^4$$

with $\Omega \in \Lambda^4(\mathfrak{u}^* \oplus \mathfrak{g}_3^*)$, $\rho \in \Lambda^3(\mathfrak{u}^* \oplus \mathfrak{g}_3^*)$.

The first step of the proof is to show that the length of $\Omega^{2,2,0}$ is three. For that purpose, note that the identities

$$0 = (d\Psi)^{3,1,1} + (d\Psi)^{3,2,0} = d(\Omega^{3,1,0}), \quad 0 = (d\Psi)^{1,3,1} + (d\Psi)^{2,3,0} = d(\Omega^{1,3,0})$$

are true. If $\mathfrak{g}_4$ is not unimodular, then $d(\Lambda^{3,0,0}) = \Lambda^{3,0,1}$. Hence, $\Omega^{3,1,0} = 0$ in this case. If $\dim([\mathfrak{g}_4, \mathfrak{g}_4]) = 3$, then $d|_{\Lambda^{1,0,0}}$ and so $d|_{\Lambda^{1,3,0}}$ is injective and $\Omega^{1,3,0} = 0$ follows. We know from Lemma 5.4 that the uniqueness of the unimodular ideal $\mathfrak{u}$ implies that $\mathfrak{g}_4$ is not unimodular or $\dim([\mathfrak{g}_4,\mathfrak{g}_4]) = 3$. In both cases, Lemma 5.15 and the just obtained results show that then $l(\Omega^{2,2,0}) = 3$.

Next, we look at the $(2,2,1)$-component of $d\Psi$. This component is given by

$$0 = (d\Psi)^{2,2,1} = d(\Omega^{2,2,0}) + d(\rho^{2,1,0} \wedge e^4) + d(\rho^{1,2,0} \wedge e^4)$$

Hence, $d(\Omega^{2,2,0} + \rho^{1,2,0} \wedge e^4) = -d(\rho^{2,1,0} \wedge e^4) \in \Lambda^3\mathfrak{g}_4^* \wedge d(\mathfrak{g}_3^*)$ and so $d(\Omega^{2,2,0} + \rho^{1,2,0} \wedge e^4) \in d(\Lambda^2\mathfrak{g}_4^*) \wedge d(\mathfrak{g}_3^*)$. Let

$$\pi_k : \Lambda^k\mathfrak{g}_4^* \wedge \Lambda^2\mathfrak{g}_3^* \to (\Lambda^k\mathfrak{g}_4^* \wedge \Lambda^2\mathfrak{g}_3^*)/(\Lambda^k\mathfrak{g}_4^* \wedge d(\mathfrak{g}_3^*)) \cong \Lambda^k\mathfrak{g}_4^* \otimes H^2(\mathfrak{g}_3)$$

be the natural projection for $k \in \mathbb{N}$, where the last canonical isomorphism holds since $\mathfrak{g}_3$ is unimodular and so all two-forms on $\mathfrak{g}_3$ are closed. Moreover, the identity $\pi_3 \circ d = (d \otimes \mathrm{id}) \circ \pi_2$ is true. If we set $\Phi := \pi_2(\Omega^{2,2,0} + \rho^{1,2,0} \wedge e^4)$, we get the identity

$$(d \otimes \mathrm{id})(\Phi) = \pi_3(d(\Omega^{2,2,0} + \rho^{1,2,0} \wedge e^4)) = 0.$$

Write

$$\Phi = \sum_{i=1}^{D} \omega_i \otimes \nu_i$$

for $\omega_1, \ldots, \omega_D \in \Lambda^2\mathfrak{g}_4^*$ and some basis $\nu_1, \ldots, \nu_D$ of $H^2(\mathfrak{g}_3)$. Then $\omega_1, \ldots, \omega_D$ are all closed. By choosing a complement V of $d(\mathfrak{g}_3^*)$ in $\Lambda^2\mathfrak{g}_3^*$, we may identify $\nu_1, \ldots, \nu_D$ with elements in V and get

$$\Omega^{2,2,0} = \psi + \sum_{i=1}^{D} \omega_i^{2,0,0} \wedge \nu_i$$

with $\psi \in \Lambda^2\mathfrak{u}^* \wedge d(\mathfrak{g}_3^*)$. Since the length of $\Omega^{2,2,0}$ is three and the length of ψ is at most $\dim(d(\mathfrak{g}_3^*))$, the length of $\sum_{i=1}^{D} \omega_i^{2,0,0} \wedge \nu_i$ has to be $3 - \dim(d(\mathfrak{g}_3^*)) = D$ and so $\omega_1^{2,0,0}, \ldots, \omega_D^{2,0,0}$ have to be linearly independent. Thus, $\omega_1, \ldots, \omega_D$ are linearly independent and $\mathrm{span}(\omega_1, \ldots, \omega_D) \cap \Lambda^{1,0,1} = \{0\}$. This finishes the proof. □

Proposition 5.16 gives us an obstruction if the three-dimensional part is unimodular, whereas the next proposition gives us an obstruction if the three-dimensional part is not unimodular.

Proposition 5.17. *(a) Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of an almost Abelian four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional non-unimodular Lie algebra $\mathfrak{g}_3$. If $\mathfrak{g}$ admits a cocalibrated G_2-structure, then $\mathfrak{g}_4$ is unimodular and $\mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathbb{R}$.*

(b) Let $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathfrak{r}_2$ be a Lie algebra direct sum of a five-dimensional real almost Abelian Lie algebra $\mathfrak{g}_5$ which admits a codimension one Abelian ideal $\mathfrak{u}$ and of the two-dimensional real Lie algebra $\mathfrak{r}_2$. If $\mathfrak{g}$ admits a cocalibrated G_2-structure, then $\mathfrak{g}_5$ is unimodular.

Proof. (a) Let $\mathfrak{u}_3$ be an Abelian ideal in $\mathfrak{g}_4$. Choose an element $e_4 \in \mathfrak{g}_4 \backslash \mathfrak{u}_3$ and an element $e_7 \in \mathfrak{g}_3 \backslash \mathfrak{u}_2$, where $\mathfrak{u}_2$ is a codimension one Abelian ideal in $\mathfrak{g}_3$. Let $e^4 \in {\mathfrak{u}_3}^0 \subseteq \mathfrak{g}_4^*$, $e^4(e_4) = 1$ and $e^7 \in {\mathfrak{u}_2}^0 \subseteq \mathfrak{g}_3^*$, $e^7(e_7) = 1$. Let $\Psi \in \Lambda^4\mathfrak{g}^*$ be the Hodge dual of a cocalibrated G_2-structure, set $\Lambda^{i,j,k,l} := \Lambda^i\mathfrak{u}_3^* \wedge \Lambda^j\mathfrak{u}_2^* \wedge \Lambda^k\mathrm{span}(e^4) \wedge \Lambda^l\mathrm{span}(e^7)$ and denote by $\theta^{i,j,k,l}$ for each $s := (i+j+k+l)$-form $\theta \in \Lambda^s\mathfrak{g}^*$ the projection of θ onto $\Lambda^{i,j,k,l}$. By Lemma 5.5,

$$d(\Lambda^{i,j,k,l}) \subseteq \Lambda^{i,j,k+1,l} + \Lambda^{i,j,k,l+1}$$

for all $i, j, k, l \in \mathbb{N}_0$. Let Ω be the part of Ψ in $\Lambda^4\left(\mathfrak{u}_3^* \oplus \mathfrak{u}_2^* \oplus \mathrm{span}(e^4)\right)$, i.e.

$$\Omega = \Psi^{2,2,0,0} + \Psi^{3,1,0,0} + \Psi^{3,0,1,0} + \Psi^{2,1,1,0} + \Psi^{1,2,1,0}.$$

By Proposition 2.48, $r(\Omega) = 1$. Hence, $l(\Psi^{2,2,0,0} + \Psi^{3,1,0,0}) \geq 1$ and so $\Psi^{2,2,0,0} + \Psi^{3,1,0,0} \neq 0$. Moreover, the closure of Ψ implies

$$0 = (d\Psi)^{2,2,0,1} = d(\Psi^{2,2,0,0})^{2,2,0,1}, \quad 0 = (d\Psi)^{3,1,1,0} = d(\Psi^{3,1,0,0})^{3,1,1,0},$$
$$0 = (d\Psi)^{3,1,0,1} = d(\Psi^{3,1,0,0})^{3,1,0,1}.$$

Since $\mathfrak{g}_3$ is not unimodular, $d(\Lambda^2\mathfrak{u}_2^*) = \Lambda^3\mathfrak{g}_3^*$ and so $d(\Psi^{2,2,0,0})^{2,2,0,1} = 0$ implies $\Psi^{2,2,0,0} = 0$. Thus, $\Psi^{3,1,0,0} \neq 0$. If $\mathfrak{g}_4$ is non-unimodular, then $d(\Lambda^3\mathfrak{u}_3^*) = \Lambda^4\mathfrak{g}_4^*$ and so $d(\Psi^{3,1,0,0})^{3,1,1,0} \neq 0$, a contradiction. Hence, $\mathfrak{g}_4$ is unimodular. Similarly, if $d|_{\mathfrak{u}_2^*}$ is injective, then $d(\Psi^{3,1,0,0})^{3,1,0,1} \neq 0$, a contradiction. Thus, $d|_{\mathfrak{u}_2^*}$ is not injective and so $\mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathbb{R}$.

(b) The proof of part (b) is completely analogous to (a). Therefore, let $\Psi \in \Lambda^4\mathfrak{g}^*$ be the Hodge dual of a cocalibrated G_2-structure, let $\mathfrak{u}$ be an Abelian ideal of dimension four in $\mathfrak{g}_5$, $e_5 \in \mathfrak{g}_5 \backslash \mathfrak{u}$, $e^5 \in \mathfrak{u}^0 \subseteq \mathfrak{g}_5^*$ with $e^5(e_5) = 1$ and e^6, e^7 a basis of $\mathfrak{r}_2^*$ such that $de^6 = e^{67}$ and $de^7 = 0$. Similarly to (a), we set

$$\Lambda^{i,j,k,l} := \Lambda^i\mathfrak{u}^* \wedge \Lambda^j\mathrm{span}(e^6) \wedge \Lambda^k\mathrm{span}(e^5) \wedge \Lambda^l\mathrm{span}(e^7)$$

and denote for all $s := (i+j+k+l)$-forms $\theta \in \Lambda^s\mathfrak{g}^*$ the projection of θ onto $\Lambda^{i,j,k,l}$ by $\theta^{i,j,k,l}$. Then $d(\Lambda^{i,j,k,l}) \subseteq \Lambda^{i,j,k+1,l} + \Lambda^{i,j,k,l+1}$ as in (a). Moreover, arguing as in (a), we get $\Psi^{4,0,0,0} + \Psi^{3,1,0,0} \neq 0$. Since $de^6 \neq 0$, the identity

$$0 = (d\Psi)^{3,1,0,1} = d(\Psi^{3,1,0,0})^{3,1,0,1}$$

is true only if $\Psi^{3,1,0,0} = 0$. Thus, $\Psi^{4,0,0,0} \neq 0$. But then

$$0 = (d\Psi)^{4,0,1,0} = d(\Psi^{4,0,0,0})$$

implies that $\mathfrak{g}_5$ is unimodular by Lemma 5.5.

□

5.5 Main Results

We start this section by presenting the main result of this chapter, namely the classification of those direct sums $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ of a four-dimensional real Lie algebra $\mathfrak{g}_4$ and of a three-dimensional real Lie algebra $\mathfrak{g}_3$ which admit cocalibrated G_2-structures. The following four subsections are devoted to the proof of the main theorem using the results obtained in Section 5.3 and Section 5.4. Each of the four Subsections 5.5.1 - 5.5.4 treats exactly one of the four cases obtained by distinguishing whether $\mathfrak{g}_4$ or $\mathfrak{g}_3$ is unimodular or not.

The mentioned classification reads as follows.

Theorem 5.18. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional Lie algebra $\mathfrak{g}_4$ and of a three-dimensional Lie algebra $\mathfrak{g}_3$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if one of the following four conditions is fulfilled:*

(a) $\mathfrak{g}_4$ *is not unimodular,* $\mathfrak{g}_3$ *is unimodular and* $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) - h^2(\mathfrak{g}_4) + h^2(\mathfrak{g}_3) \leq 4$, *where* $\mathfrak{u}$ *is the unimodular kernel of* $\mathfrak{g}_4$.

(b) $\mathfrak{g}_4$ *is unimodular,* $\mathfrak{g}_3$ *is unimodular and at least one of the following conditions is true:*

- *(i)* $\mathfrak{g}_3 \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$
- *(ii)* $\mathfrak{g}_4 = \mathfrak{h} \oplus \mathbb{R}$ *for a three-dimensional unimodular Lie algebra* $\mathfrak{h}$.
- *(iii)* $\mathfrak{g} \in \{A_{4,1} \oplus e(2), A_{4,1} \oplus e(1,1), A_{4,8} \oplus e(1,1)\}$.

(c) $\mathfrak{g}_4$ *is unimodular,* $\mathfrak{g}_3$ *is not unimodular and at least one of the following conditions is true:*

- *(i)* $\mathfrak{g}_4 \in \{\mathfrak{so}(3) \oplus \mathbb{R}, \mathfrak{so}(2,1) \oplus \mathbb{R}\}$.
- *(ii)* $\mathfrak{g}_4$ *is almost Abelian,* $\mathfrak{g}_4 \notin \{\mathbb{R}^4, \mathfrak{h}_3 \oplus \mathbb{R}\}$ *and* $\mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathbb{R}$.
- *(iii) The commutator ideal* $[\mathfrak{g}_4, \mathfrak{g}_4]$ *of* $\mathfrak{g}_4$ *is isomorphic to* $\mathfrak{h}_3$.

(d) $\mathfrak{g}_4$ *is not unimodular,* $\mathfrak{g}_3$ *is not unimodular and at least one of the following conditions is true:*

- *(i)* $\mathfrak{g}_4 \in \{A_{4,12}, \mathfrak{r}_2 \oplus \mathfrak{r}_2\}$
- *(ii)* $\mathfrak{g} = A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$.
- *(iii) The unimodular kernel* $\mathfrak{u}$ *of* $\mathfrak{g}_4$ *is isomorphic to* $\mathfrak{h}_3$, $\mathfrak{g}_3 \neq \mathfrak{r}_2 \oplus \mathbb{R}$ *and* $\mathfrak{g} \notin \left\{ A_{4,9}^1 \oplus \mathfrak{r}_{3,\mu}, A_{4,9}^{\alpha} \oplus \mathfrak{r}_{3,1} \,\middle|\, \mu \in \left[-\frac{1}{3}, 0\right), \alpha \in \left(-1, -\frac{1}{3}\right] \right\}$.

5.5.1 $\mathfrak{g}_4$ not unimodular, $\mathfrak{g}_3$ unimodular

In this subsection we prove Theorem 5.18 (a). In the following, $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ always denotes a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional non-unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$. Furthermore, $\mathfrak{u}$ denotes the unimodular ideal of $\mathfrak{g}_4$.

Proposition 5.16 shows that if $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) > 4$, then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure, giving us one direction of Theorem 5.18 (a).

For the other direction, Lemma 5.6 and Proposition 5.12 (a) tell us that if $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) \leq 4$ and $\mathfrak{u} \neq e(1,1)$, then $\mathfrak{g}$ does admit a cocalibrated G_2-structure. By the Tables 7.2 and 7.3 or by Remark 5.7, the only four-dimensional non-unimodular Lie algebra $\mathfrak{g}_4$ with unimodular ideal $\mathfrak{u} = e(1,1)$ is $\mathfrak{g}_4 = \mathfrak{r}_2 \oplus \mathfrak{r}_2$. For $\mathfrak{g}_4 = \mathfrak{r}_2 \oplus \mathfrak{r}_2$, Lemma 5.3 and Proposition 5.12 (b) imply that $\mathfrak{g}_4 \oplus \mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{g}_3$ does admit a cocalibrated G_2-structure if $\mathfrak{g}_3 \neq \mathbb{R}^3$, i.e. if $h^2(\mathfrak{g}_3) \leq 2$. But $h^1(\mathfrak{r}_2 \oplus \mathfrak{r}_2) + h^1(e(1,1)) - h^2(\mathfrak{r}_2 \oplus \mathfrak{r}_2) = 2$. Hence, also in this case, $\mathfrak{g}_4 \oplus \mathfrak{g}_3$ admits a cocalibrated G_2-structure if and only if $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{g}_4) \leq 4$. This proves Theorem 5.18 (a).

5.5.2 $\mathfrak{g}_4$ unimodular, $\mathfrak{g}_3$ unimodular

Here, we prove Theorem 5.18 (b) and denote by $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ always a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$.

We begin with the case that $\mathfrak{g}_4$ is indecomposable. If $[\mathfrak{g}_4, \mathfrak{g}_4] = \mathbb{R}^3$, then Lemma 5.6, Proposition 5.12 (a) and Proposition 5.16 tell us that $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if

$$h^1(\mathfrak{g}_4) + 3 - h^2(\mathfrak{g}_4) + h^2(\mathfrak{g}_3) = h^1(\mathfrak{g}_4) + h^1(\mathbb{R}^3) - h^2(\mathfrak{g}_4) + h^2(\mathfrak{g}_3) \leq 4.$$

Table 7.3 tells us that always $h^1(\mathfrak{g}_4) - h^2(\mathfrak{g}_4) = 1$ in the considered cases. Hence, $\mathfrak{g}$ admits for these cases a cocalibrated G_2-structure exactly when $h^2(\mathfrak{g}_3) = 0$, i.e. when $\mathfrak{g}_3 \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$.

Next, we assume that $\mathfrak{g}_4$ is indecomposable but $[\mathfrak{g}_4, \mathfrak{g}_4] \neq \mathbb{R}^3$. By inspection of Table 7.3, $\mathfrak{g}_4 \in \{A_{4,1}, A_{4,8}, A_{4,10}\}$.

Let us begin with $\mathfrak{g}_4 \in \{A_{4,8}, A_{4,10}\}$. Then, in both cases, $h^1(\mathfrak{g}_4) + h^1(\mathfrak{u}) - h^2(\mathfrak{g}_4) = 3$, where $\mathfrak{u}$ is the unique unimodular ideal in $\mathfrak{g}_4$ which is isomorphic to $\mathfrak{h}_3$. Thus, Proposition 5.16 yields that $\mathfrak{g}$ does not admit a cocalibrated G_2-structure if $h^2(\mathfrak{g}_3) \geq 2$. Conversely, Corollary 5.14 tells us that if $h^2(\mathfrak{g}_3) = 0$, i.e. $\mathfrak{g}_3$ is not solvable, then $\mathfrak{g}$ does admit a cocalibrated G_2-structure. So we are left with the case that $h^2(\mathfrak{g}_3) = 1$, i.e. $\mathfrak{g}_3 \in \{e(2), e(1,1)\}$. For $\mathfrak{g} = A_{4,8} \oplus e(1,1)$, a cocalibrated G_2-structure is given in Table 7.13. All other cases do not admit a cocalibrated G_2-structure:

Lemma 5.19. *Let* $\mathfrak{g} \in \{A_{4,8} \oplus e(2), A_{4,10} \oplus e(2), A_{4,10} \oplus e(1,1)\}$*. Then* $\mathfrak{g}$ *does not admit a cocalibrated* G_2*-structure.*

Proof. Let e^1, e^2, e^3, e^4 be the basis of $\mathfrak{g}_4^*$, $\mathfrak{g}_4 \in \{A_{4,8}, A_{4,10}\}$ as in Table 7.3. Then there exists a linear, trace-free, invertible map $F : \mathrm{span}(e^2, e^3) \to \mathrm{span}(e^2, e^3)$ such that $de^1 = e^{23}$, $d\alpha = F(\alpha) \wedge e^4$, $de^4 = 0$ for all $\alpha \in \mathrm{span}(e^2, e^3)$. For $\mathfrak{g}_4 = A_{4,8}$ we have $F(e^2) = e^2$, $F(e^3) = -e^3$ whereas for $\mathfrak{g}_4 = A_{4,10}$ we have $F(e^2) = e^3$ and $F(e^3) = -e^2$. In particular, $\det(F) = -1$ if $\mathfrak{g}_4 = A_{4,8}$ and $\det(F) = 1$ if $\mathfrak{g}_4 = A_{4,10}$.

Let e^5, e^6, e^7 be a basis of $\mathfrak{g}_3^*$, $\mathfrak{g}_3 \in \{e(2), e(1,1)\}$ as in Table 7.1. Then there exists a linear, trace-free, invertible map $G : \mathrm{span}(e^5, e^6) \to \mathrm{span}(e^5, e^6)$ such that $d\beta = G(\beta) \wedge e^7$, $de^7 = 0$ for all $\beta \in \mathrm{span}(e^5, e^6)$. In both cases we have $G(e^5) = e^6$, whereas $G(e^6) = e^5$ if $\mathfrak{g}_3 = e(1,1)$ and $G(e^6) = -e^5$ if $\mathfrak{g}_3 = e(2)$. In particular, $\det(G) = -1$ if $\mathfrak{g}_3 = e(1,1)$ and $\det(G) = 1$ if $\mathfrak{g}_3 = e(2)$.

Let us now assume that $\Psi \in \Lambda^4\mathfrak{g}^*$ is a (closed) Hodge dual of a cocalibrated G_2-structure $\varphi \in \Lambda^3\mathfrak{g}^*$. We decompose Ψ uniquely into

$$\Psi = \rho \wedge e^1 + \Omega$$

with $\rho \in \Lambda^3(\mathrm{span}(e^2, e^3, e^4) \oplus \mathfrak{g}_3^*)$, $\Omega \in \Lambda^4(\mathrm{span}(e^2, e^3, e^4) \oplus \mathfrak{g}_3^*)$. Then

$$0 = d\Psi = d\rho \wedge e^1 - \rho \wedge e^{23} + d\Omega,$$

$d\Omega \in \Lambda^3\mathrm{span}(e^2, e^3, e^5, e^6) \wedge e^{47}$ (note that $de^{2356} = 0$) and $d\rho \in \Lambda^4(\mathrm{span}(e^2, e^3, e^4) \oplus \mathfrak{g}_3^*)$ imply $d\rho = 0$ and $\mathrm{pr}_{\mathrm{span}(e^{456}, e^{567})}(\rho) = 0$. Moreover, $\ker F = \{0\} = \ker G$ and $d\rho = 0$ imply $\mathrm{pr}_{\Lambda^3\mathrm{span}(e^2, e^3, e^5, e^6)}(\rho) = 0$.

Thus, $\rho = (\omega_1 + ae^{23}) \wedge e^4 + (\omega_2 + be^{23}) \wedge e^7 + \beta \wedge e^{47}$ for certain $\omega_1, \omega_2 \in \mathrm{span}(e^2, e^3) \wedge \mathrm{span}(e^5, e^6)$, $a, b \in \mathbb{R}$ and $\beta \in \mathrm{span}(e^2, e^3, e^5, e^6)$. Now Proposition 2.48 tells us that ρ has model tensor $\rho_{-1} \in \Lambda^3 \left(\mathbb{R}^6\right)^*$ and so Lemma 5.15 (b) yields that $\omega_1 + ae^{23}$ and $\omega_2 + be^{23}$ span a two-dimensional subspace in $\Lambda^2\mathrm{span}(e^2, e^3, e^5, e^6)$ in which each non-zero element has length two. This is equivalent to the requirement that ω_1 and ω_2 span such a two-dimensional subspace of $\Lambda^2\mathrm{span}(e^2, e^3, e^5, e^6)$ and Lemma 2.2 shows that this is equivalent to $\omega_1^2 \neq 0$ and $C - B^2 > 0$ for the numbers $B, C \in \mathbb{R}$ defined by $\omega_1 \wedge \omega_2 = B\omega_2^2$, $\omega_2^2 = C\omega_1^2$. By Lemma 2.1, there exists a basis α_1, α_2 of $\mathrm{span}(e^2, e^3)$ and α_3, α_4 of $\mathrm{span}(e^5, e^6)$ such that $\omega_1 = \alpha_1 \wedge \alpha_4 + \alpha_2 \wedge \alpha_3$. Since $d(\omega_1 \wedge e^4 + \omega_2 \wedge e^7) = d\rho = 0$, we must have $\omega_2 = F^{-1}(\alpha_1) \wedge G(\alpha_4) + F^{-1}(\alpha_2) \wedge G(\alpha_3)$. Thus, $C = \frac{\det(G)}{\det(F)}$. If $\mathfrak{g} \in \{A_{4,8} \oplus e(2), A_{4,10} \oplus e(1,1)\}$, then $C < 0$ leading to $C - B^2 < 0$. Thus, for these cases, there cannot exist a cocalibrated G_2-structure.

For the missing case $\mathfrak{g} = A_{4,10} \oplus e(2)$, let $\omega_1 := c_1e^{25} + c_2e^{26} + c_3e^{35} + c_4e^{36}$ be a general two-form in $\mathrm{span}(e^2, e^3) \wedge \mathrm{span}(e^5, e^6)$ of length two, i.e. with $c_1c_4 - c_2c_3 \neq 0$.

Then $\omega_2 = -c_4 e^{25} + c_3 e^{26} + c_2 e^{35} - c_1 e^{36}$, $B = -\frac{c_1^2+c_2^2+c_3^2+c_4^2}{2(c_1c_4-c_2c_3)}$, $C = 1$ and so

$$\begin{aligned} C - B^2 &= \frac{4(c_1c_4 - c_2c_3)^2 - (c_1^2 + c_2^2 + c_3^2 + c_4^2)^2}{4(c_1c_4 - c_2c_3)^2} \\ &= -\frac{((c_1 - c_4)^2 + (c_2 + c_3)^2)((c_1 + c_4)^2 + (c_2 - c_3)^2)}{4(c_1c_4 - c_2c_3)^2} < 0. \end{aligned}$$

Thus, $A_{4,10} \oplus e(2)$ does not admit a cocalibrated G_2-structure. □

Next we consider direct sums with $A_{4,1}$. The Lie algebra $A_{4,1}$ is almost Abelian and admits a symplectic two-form, e.g. $\omega = e^{14} + e^{23}$ in the basis e^1, e^2, e^3, e^4 given in Table 7.3. Hence, Proposition 5.12 (a) shows that $A_{4,1} \oplus \mathfrak{g}_3$ admits a cocalibrated G_2-structure if $h^2(\mathfrak{g}_3) \leq 1$, i.e. if $\mathfrak{g}_3 \notin \{\mathbb{R}^3, \mathfrak{h}_3\}$. The Lie algebra $\mathfrak{g} = A_{4,1} \oplus \mathbb{R}^3$ is almost Abelian and by Theorem 4.15 there does not exist a cocalibrated G_2-structure on $\mathfrak{g}$. Also $\mathfrak{g} = A_{4,1} \oplus \mathfrak{h}_3$ does not admit a cocalibrated G_2-structure.

Lemma 5.20. *Let $\mathfrak{g} = A_{4,1} \oplus \mathfrak{h}_3$. Then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure.*

Proof. Choose a basis $e^1, e^2, e^3, e^4, e^5, e^6, e^7$ of $A_{4,1} \oplus \mathfrak{h}_3$ as in Table 7.3 and Table 7.1, i.e.

$$de^1 = e^{24},\ de^2 = e^{34},\ de^3 = 0,\ de^4 = 0,\ de^5 = e^{67}\ de^6 = 0,\ de^7 = 0,$$

Assume that there exists a cocalibrated G_2-structure and let

$$\Psi = \sum_{1 \leq i < j < k < l \leq 7} a_{ijkl} e^{ijkl}$$

be its (closed) Hodge dual. Then a short computation shows that $a_{1567} = a_{2567} = a_{1256} = a_{1356} = a_{1257} = a_{1357} = a_{1235} = 0$. If we decompose Ψ uniquely into

$$\Psi = \Omega + e^1 \wedge \nu + e^{14} \wedge \omega,$$

with $\Omega \in \Lambda^4 \mathrm{span}(e^2, e^3, e^4, e^5, e^6, e^7)$, $\nu \in \Lambda^3 \mathrm{span}(e^2, e^3, e^5, e^6, e^7)$ and $\omega \in \Lambda^2 \mathrm{span}(e^2, e^3, e^5, e^6, e^7)$, then ν actually is in $\Lambda^3 \mathrm{span}(e^2, e^3, e^6, e^7)$ and so of length at most one. If we consider the decomposition $\big(\mathrm{span}(e^2, e^3, e^5, e^6, e^7) \oplus \mathrm{span}(e^4)\big) \oplus \mathrm{span}(e^1) = \mathfrak{g}^*$, Proposition 2.48 implies that the length of ν has to be at least two, a contradiction. □

So we are left with the case that $\mathfrak{g}_4$ is decomposable. Then $\mathfrak{g}_4$ is the Lie algebra direct sum of a three-dimensional unimodular Lie algebra $\mathfrak{h}$ and $\mathbb{R}$ and $\mathfrak{g}$ always admits a cocalibrated G_2-structure.

Proposition 5.21. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ be a Lie algebra direct sum of a four-dimensional unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional unimodular Lie algebra $\mathfrak{g}_3$. Moreover, let $\mathfrak{g}_4 = \mathfrak{h} \oplus \mathbb{R}$ be a Lie algebra direct sum of a three-dimensional unimodular Lie algebra $\mathfrak{h}$ and $\mathbb{R}$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure.*

Proof. We may assume that $h^2(\mathfrak{h}) \geq h^2(\mathfrak{g}_3)$. Moreover, we may assume that $\mathfrak{g}_4 = \mathfrak{h} \oplus \mathbb{R}$ does admit an Abelian ideal $\mathfrak{u}$ of codimension 1 since otherwise $\mathfrak{h} \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$ and Corollary 5.14 gives us the affirmative answer. By Künneth's formula, $h^1(\mathfrak{h}\oplus\mathbb{R}) = h^1(\mathfrak{h})+1$ and $h^2(\mathfrak{h} \oplus \mathbb{R}) = h^2(\mathfrak{h}) + h^1(\mathfrak{h})$. Thus

$$\begin{aligned} h^1(\mathfrak{h} \oplus \mathbb{R}) + h^1(\mathfrak{u}) + h^2(\mathfrak{g}_3) - h^2(\mathfrak{h} \oplus \mathbb{R}) &= h^1(\mathfrak{h}) + 1 + 3 + h^2(\mathfrak{g}_3) - h^2(\mathfrak{h}) - h^1(\mathfrak{h}) \\ &= h^2(\mathfrak{g}_3) - h^2(\mathfrak{h}) + 4 \leq 4, \end{aligned}$$

and Proposition 5.12 (a) implies the statement. □

5.5.3 $\mathfrak{g}_4$ unimodular, $\mathfrak{g}_3$ not unimodular

In this subsection, we prove Theorem 5.18 (c). In the following, $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ always denotes a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional non-unimodular Lie algebra $\mathfrak{g}_3$.

We start with the case that $\mathfrak{g}_4$ is almost Abelian. Then Proposition 5.17 (a) implies that if $\mathfrak{g}_3 \neq \mathfrak{r}_2 \oplus \mathbb{R}$, then $\mathfrak{g}$ does not admit a cocalibrated G_2-structure. So, in this case, it remains to consider sums of the form $\mathfrak{g}_4 \oplus \mathfrak{r}_2 \oplus \mathbb{R}$. This is done in Theorem 5.23 which tells us more generally when a direct sum of the form $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}_2$ where $\mathfrak{h}$ is a five-dimensional almost Abelian Lie algebra possesses a cocalibrated G_2-structure. For the proof of this theorem, we need the following

Lemma 5.22. *Let $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathfrak{r}_2$ be a Lie algebra direct sum of a five-dimensional unimodular almost Abelian Lie algebra $\mathfrak{g}_5$ and $\mathfrak{r}_2$. Let $\mathfrak{a}$ be an Abelian ideal of dimension four in $\mathfrak{g}_5$. Choose $e_5 \in \mathfrak{g}_5 \backslash \mathfrak{a}$ and let $e^5 \in \mathfrak{a}^0 \subseteq \mathfrak{g}_5^*$ be such that $e^5(e_5) = 1$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there exist two linearly independent two-forms $\omega_1, \omega_2 \in \Lambda^2\mathfrak{a}^* \cong \Lambda^2 {e_5}^0$ such that each non-zero linear combination is of length two and such that $d\omega_1 = \omega_2 \wedge e^5$.*

Proof. Let e^6, e^7 be a basis of $\mathfrak{r}_2^*$ such that $de^6 = e^{67}$, $de^7 = 0$. Assume first that $\mathfrak{g}$ admits a cocalibrated G_2-structure $\varphi \in \Lambda^3\mathfrak{g}^*$ with (closed) Hodge dual $\Psi \in \Lambda^4\mathfrak{g}^*$. Decompose Ψ uniquely into

$$\Psi = \Omega + \rho \wedge e^6$$

with $\Omega \in \Lambda^4(\mathfrak{g}_5^* \oplus \mathrm{span}(e^7))$, $\rho \in \Lambda^3(\mathfrak{g}_5^* \oplus \mathrm{span}(e^7))$. Since $d\Omega \in \Lambda^5(\mathfrak{g}_5^* \oplus \mathrm{span}(e^7))$ and $d(\rho \wedge e^6) \in \Lambda^4(\mathfrak{g}_5^* \oplus \mathrm{span}(e^7)) \wedge e^6$, the identities $d\Omega = 0 = d(\rho \wedge e^6)$ are true.

Set $\Lambda^{i,j,k} := \Lambda^i\mathfrak{a}^* \wedge \Lambda^j\mathrm{span}(e^5) \wedge \Lambda^k\mathrm{span}(e^7)$. For an $s := (i+j+k)$-form $\theta \in \Lambda^s(\mathfrak{g}_5^* \oplus \mathrm{span}(e^7))$ let $\theta^{i,j,k}$ be the projection of θ onto $\Lambda^{i,j,k}$. Lemma 5.5 implies $d(\Lambda^{i,0,k}) \subseteq \Lambda^{i,1,k}$ and $d(\Lambda^{i,1,k}) = 0$ for all $i, k \in \mathbb{N}_0$.

The closure of $\rho \wedge e^6$ implies $0 = d(\rho \wedge e^6) = d\rho \wedge e^6 - \rho \wedge e^{67}$ and so $0 = d\rho + \rho \wedge e^7$. Then the identities

$$0 = (d\rho + \rho \wedge e^7)^{3,0,1} = \rho^{3,0,0} \wedge e^7,\ 0 = (d\rho + \rho \wedge e^7)^{2,1,1} = d(\rho^{2,0,1}) + \rho^{2,1,0} \wedge e^7$$

are true. Thus, $\rho^{3,0,0} = 0$ and $d(\rho^{2,0,1}) = -\rho^{2,1,0} \wedge e^7$. This shows that

$$\rho = \omega_1 \wedge e^7 - \omega_2 \wedge e^5 + \alpha \wedge e^{57}$$

for ω_1, $\omega_2 \in \Lambda^{2,0,0}$, $\alpha \in \Lambda^{1,0,0}$ and that

$$\omega_2 \wedge e^{57} = -\rho^{2,1,0} \wedge e^7 = d(\rho^{2,0,1}) = d(\omega_1 \wedge e^7) = d\omega_1 \wedge e^7 \;\Leftrightarrow\; d\omega_1 = \omega_2 \wedge e^5.$$

By Proposition 2.48, ρ has model tensor $\rho_{-1} \in \Lambda^3 \left(\mathbb{R}^6\right)^*$ and Lemma 5.15 (b) yields that $V := \mathrm{span}(\omega_1, \omega_2)$ is two-dimensional and each non-zero element in V has length two.

Conversely, let ω_1, $\omega_2 \in \Lambda^2\mathfrak{a}^*$ be such that $d\omega_1 = \omega_2 \wedge e^5$ and such that ω_1, ω_2 are linearly independent and each non-zero linear combination of them is of length two. Set $V_4 := \mathfrak{a}^*$, $V_3 := \mathrm{span}(e^5) \oplus \mathfrak{r}_2^*$, $\nu_1 := e^{67} \in \Lambda^2 V_3$, $\nu_2 := e^{56} \in \Lambda^2 V_3$, $\nu_3 := e^{57} \in \Lambda^2 V_3$. By Lemma 2.2 and Proposition 2.55, there exists a two-form $\omega_3 \in \Lambda^2\mathfrak{a}^*$ such that

$$\Psi := \sum_{i=1}^{3} \omega_i \wedge \nu_i + \frac{1}{2}\omega_1^2$$

is the Hodge dual of a G_2-structure. By Lemma 5.5, $d(\Lambda^4\mathfrak{a}^*) = 0$ and $d(\Lambda^k\mathfrak{a}^* \wedge e^5) = 0$ for all $k \in \mathbb{N}_0$. Using these properties of d, a short computation shows that Ψ is closed. □

Lemma 5.22 allows us to prove

Theorem 5.23. *Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{r}_2$ be a Lie algebra direct sum of a five-dimensional almost Abelian Lie algebra $\mathfrak{h}$ and of $\mathfrak{r}_2$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $\mathfrak{h}$ is unimodular and $\mathfrak{h} \notin \left\{\mathbb{R}^5, \mathfrak{h}_3 \oplus \mathbb{R}^2, A_{5,7}^{-\frac{1}{3},-\frac{1}{3},-\frac{1}{3}}\right\}$.*

Proof. By Proposition 5.17 (b), $\mathfrak{h}$ has to be unimodular if $\mathfrak{g}$ admits a cocalibrated G_2-structure. So, for the rest we assume that $\mathfrak{h}$ is unimodular and let $e_5 \in \mathfrak{h}\backslash\mathfrak{a}$, $e^5 \in \mathfrak{a}^0 \subseteq \mathfrak{h}^*$, $e^5(e_5) = 1$. By Lemma 5.5, there exists a linear trace-free map $H : \mathfrak{a}^* \to \mathfrak{a}^*$ such that $d\alpha = H(\alpha) \wedge e^5$, $de^5 = 0$ for all $\alpha \in \mathfrak{a}^*$. Let e^6, e^7 be a basis of $\mathfrak{r}_2^*$ with $de^6 = e^{67}$, $de^7 = 0$. Then Lemma 5.22 tells us that $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there are two linearly independent two-forms ω_1, $\omega_2 \in \Lambda^2\mathfrak{a}^*$ such that $d\omega_1 = \omega_2 \wedge e^5$ and such that each non-zero linear combination is of length two.

We first prove that such a pair of two-forms always exists if there is a vector decomposition $\mathfrak{a}^* = V_2 \oplus W_2$ into two two-dimensional H-invariant subspaces such that the restrictions of H to V_2 and to W_2 are both not a multiple of the identity. In this case, we may choose for each $\lambda \neq 0$ a basis e^1, e^2 of V_2 and a basis e^3, e^4 of W_2 such that the restrictions of H to V_2 and W_2 with respect to the corresponding bases are given by

$$\begin{pmatrix} 0 & -\frac{\det(H|_{V_2})}{\lambda} \\ \lambda & \mathrm{tr}(H|_{V_2}) \end{pmatrix} \quad\text{and}\quad \begin{pmatrix} \mathrm{tr}(H|_{W_2}) & -\lambda \\ \frac{\det(H|_{W_2})}{\lambda} & 0 \end{pmatrix},$$

respectively. Set $\omega_1 := e^{14}+e^{23}$. Then ω_1 is of length two and $d\omega_1 = \big(\lambda(e^{13}-e^{24})+\omega_3\big)\wedge e^5$ with $\omega_3 := de^{23} \in \Lambda^2\mathfrak{a}^*$. Set $\omega_2 := \lambda(e^{13}-e^{24})+\omega_3$ and observe that $d\omega_1 = \omega_2 \wedge e^5$ and

$$\omega_1 \wedge \omega_2 = e_5 \lrcorner (\omega_1 \wedge d\omega_1) = e_5 \lrcorner \left(d\left(\frac{1}{2}\omega_1^2\right)\right) = 0$$

since $\mathfrak{g}_5$ is unimodular. Furthermore, observe that $C(\lambda)$, defined by

$$\omega_2^2 = \lambda^2 e^{1234} + 2\lambda(e^{13}-e^{24}) \wedge \omega_3 + \omega_3^2 = C(\lambda)\omega_1^2,$$

fulfils $C(\lambda) = \lambda^2 + \mathcal{O}(\lambda)$ as $\lambda \to \infty$. Thus, for $|\lambda|$ sufficiently large, $C(\lambda) > 0$ and Lemma 2.2 tells us that ω_1, ω_2 span a two-dimensional subspace in which each non-zero element is of length two. So all considered Lie algebras which admit such a splitting do admit a cocalibrated G_2-structure.

Next, we assume that $\mathfrak{a}^*$ does not admit a splitting as above and look at the possible real Jordan normal forms of H. Therefore, we remind the reader that by our convention, $J_m(a)$ denotes a Jordan block of size m with $a \in \mathbb{R}$ on the diagonal, where the 1s are on the superdiagonal, and $M_{b,c}$ denotes the real two-by-two matrix $\begin{pmatrix} b & c \\ -c & b \end{pmatrix}$. We get, after rescaling e^5, that there is a basis e^1, e^2, e^3, e^4 of $\mathfrak{a}^*$ such that H acts with respect to this basis as one of the following matrices:

$$\begin{pmatrix} J_3(a) & A \\ 0 & -3a \end{pmatrix}, \quad \begin{pmatrix} M_{0,1} & I_2 \\ 0 & M_{0,1} \end{pmatrix}, \quad \operatorname{diag}(M_{b,1}, -b, -b), \quad \operatorname{diag}(J_2(c), -c, -c),$$
$$\operatorname{diag}\left(f, -\frac{f}{3}, -\frac{f}{3}, -\frac{f}{3}\right), \qquad a,\ c,\ f,\ A \in \{0,1\},\ b \in \mathbb{R}^+.$$

In the first case, $\omega_1 := e^{12}+e^{34}-5e^{23}$ and $\omega_2 := -e^{24}+2a(-e^{12}+e^{34})+10ae^{23}+5e^{13}$ fulfil all desired conditions. In the second case, we may choose $\omega_1 := e^{12}-e^{34}$ and $\omega_2 := e^{14}-e^{23}$ and in the third case, $\omega_1 := e^{13}-e^{24}$ and $\omega_2 := e^{14}+e^{23}$ do the job. In the fourth case, we start with $c = 1$. Then $\omega_1 := e^{13}-e^{24}-\frac{1}{2}\left(e^{12}-e^{34}\right)$, $\omega_2 := e^{12}+e^{34}+e^{14}$ fulfil all desired conditions. If $c = 0$, then $\mathfrak{h} = \mathfrak{h}_3 \oplus \mathbb{R}^2$ and we already know by Proposition 5.16 that $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathbb{R}^2 \oplus \mathfrak{h}_3$ does not admit a cocalibrated G_2-structure. However, this also follows easily from the fact that in this case $d(\Lambda^2\mathfrak{a}^*) = \operatorname{span}(e^{135}, e^{145})$. In the last case, let $\omega_1 \in \Lambda^2\mathfrak{a}^*$ be of length two. Then there exist $\alpha \in \operatorname{span}(e^2, e^3, e^4)$ and $\omega \in \Lambda^2\operatorname{span}(e^2, e^3, e^4)$ such that $\omega_1 = \omega + \alpha \wedge e^1$. But then $d\omega_1 = \frac{2}{3}f\left(\omega - \alpha \wedge e^1\right) \wedge e^5$, i.e. $\omega_2 = \frac{2}{3}f\left(\omega - \alpha \wedge e^1\right)$ and so $\frac{2}{3}f\omega_1 + \omega_2 = \frac{4}{3}f\omega$ is of length one. Thus, $\mathfrak{g}$ does not admit a cocalibrated G_2-structure in this case, i.e. if $\mathfrak{h} \in \left\{\mathbb{R}^5, A_{5,7}^{-\frac{1}{3},-\frac{1}{3},-\frac{1}{3}}\right\}$. □

The only unimodular four-dimensional Lie algebras which are not almost Abelian are the two non-solvable ones $\mathfrak{so}(3) \oplus \mathbb{R}$, $\mathfrak{so}(2,1) \oplus \mathbb{R}$ and the two whose commutator ideal $\mathfrak{u}$ is isomorphic to $\mathfrak{h}_3$, namely $A_{4,8}$, $A_{4,10}$. Direct sums with the non-solvable four-dimensional Lie algebras admit cocalibrated G_2-structures by Corollary 5.14. Direct sums with $A_{4,8}$, $A_{4,10}$ admit cocalibrated G_2-structures by Proposition 5.12 (c) if $h^1(\mathfrak{g}_3) \geq 1$ (note that $h^1(\mathfrak{g}_4) - h^2(\mathfrak{g}_4) = 1$ for $\mathfrak{g}_4 \in \{A_{4,8}, A_{4,10}\}$ by Table 7.3) and by Corollary 5.14 if $h^1(\mathfrak{g}_3) = 0$, i.e. $\mathfrak{g}_3 \in \{\mathfrak{so}(3), \mathfrak{so}(2,1)\}$. This finishes the proof of Theorem 5.18 (c).

5.5.4 $\mathfrak{g}_4$ not unimodular, $\mathfrak{g}_3$ not unimodular

In this subsection, we prove Theorem 5.18 (d). In the following, $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ always denotes a seven-dimensional Lie algebra which is the Lie algebra direct sum of a four-dimensional non-unimodular Lie algebra $\mathfrak{g}_4$ and of a three-dimensional non-unimodular Lie algebra $\mathfrak{g}_3$. Furthermore, $\mathfrak{u}$ should always denote the unimodular kernel of $\mathfrak{g}_4$

By Proposition 5.17 (a), $\mathfrak{g}$ does not admit a cocalibrated G_2-structure if $\mathfrak{g}_4$ is almost Abelian, i.e. if $\mathfrak{u}$ is Abelian. If $\mathfrak{u} \in \{e(2), e(1,1)\}$, then $\mathfrak{g}_4 \in \{A_{4,12}, \mathfrak{r}_2 \oplus \mathfrak{r}_2\}$ and Proposition 5.12 (b) and Lemma 5.3 imply that $\mathfrak{g}$ admits a cocalibrated G_2-structure unless $\mathfrak{g}_3 = \mathfrak{r}_{3,1}$. But for $\mathfrak{g} = A_{4,12} \oplus \mathfrak{r}_{3,1}$ and $\mathfrak{g} = \mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_{3,1}$ cocalibrated G_2-structures can be found in Table 7.13.

Therefore, it remains to consider the case when the unimodular ideal $\mathfrak{u}$ is isomorphic to $\mathfrak{h}_3$. Then Lemma 5.8 tells us that we may decompose $\mathfrak{g}_4^* = \mathrm{span}(e^1) \oplus V_2 \oplus \mathrm{span}(e^4)$ with $e^1, e^4 \neq 0$ and $\dim(V_2) = 2$ such that $de^1 = \mathrm{tr}(F)e^{14} + \nu$ for $0 \neq \nu \in \Lambda^2 V_2$, such that for all $\alpha \in V_2$ the identity $d\alpha = F(\alpha) \wedge e^4$ for some linear map $F : V_2 \to V_2$ with $\mathrm{tr}(F) \neq 0$ is true and such that $de^4 = 0$. Moreover, by Lemma 5.2, we may decompose $\mathfrak{g}_3^* = W_2 \oplus \mathrm{span}(e^7)$ with $0 \neq e^7$ and W_2 two-dimensional such that for all $\beta \in W_2$ the identity $d\beta = G(\beta) \wedge e^7$ for some linear map $G : W_2 \to W_2$ with $\mathrm{tr}(G) \neq 0$ is true and such that $de^7 = 0$.

Proposition 5.24. *Let $\mathfrak{g}$, $\mathfrak{g}_4$, $\mathfrak{g}_3$, $\mathfrak{u}$, e^1, $e^4 \in \mathfrak{g}_4^* \backslash \{0\}$, $e^7 \in \mathfrak{g}_3^* \backslash \{0\}$, $V_2 \subseteq \mathfrak{g}_4^*$, $W_2 \subseteq \mathfrak{g}_3^*$ and $\nu \in \Lambda^2 V_2$ as above. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there are two linearly independent two-forms $\omega_1, \omega_2 \in V_2 \wedge W_2$, a non-zero two-form $\hat{\nu} \in \Lambda^2 W_2$ and some $\lambda \in \mathbb{R}$ such that the following conditions are fulfilled:*

(i) $d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) = 0$.

(ii) The two-forms $\tilde{\omega}_1 := \hat{\nu} + \omega_1$, $\tilde{\omega}_2 := \frac{\mathrm{tr}(F)}{\mathrm{tr}(G)}\hat{\nu} + \lambda\nu + \omega_2$ are linearly independent and each non-zero linear combination is of length two.

Proof. "$\Rightarrow$":
We set

$$\Lambda^{i,j,k,l} := \Lambda^i V_2 \wedge \Lambda^j W_2 \wedge \Lambda^k \mathrm{span}(e^4) \wedge \Lambda^l \mathrm{span}(e^7)$$

and denote, for an $s := (i + j + k + l)$-form $\Phi \in \Lambda^s(V_2 \oplus \mathrm{span}(e^4) \oplus \mathfrak{g}_3^*)$, by $\Phi^{i,j,k,l}$ the projection of Φ into $\Lambda^{i,j,k,l}$. Then we have

$$d(\Lambda^{i,j,0,0}) \subseteq \Lambda^{i,j,1,0} + \Lambda^{i,j,0,1},\ d(\Lambda^{i,j,1,0}) \subseteq \Lambda^{i,j,1,1},\ d(\Lambda^{i,j,0,1}) \subseteq \Lambda^{i,j,1,1},\ d(\Lambda^{i,j,1,1}) = \{0\}$$

for all $i, j \in \mathbb{N}_0$. Moreover, $d(\hat{\mu}) = -\mathrm{tr}(F)\hat{\mu}\wedge e^4$ for all $\hat{\mu} \in \Lambda^{2,0,0,0}$ and $d(\tilde{\mu}) = -\mathrm{tr}(G)\tilde{\mu}\wedge e^7$ for all $\tilde{\mu} \in \Lambda^{0,2,0,0}$.

Let $\Psi \in \Lambda^4(\mathfrak{g}_4 \oplus \mathfrak{g}_3)^*$ be the Hodge dual of a cocalibrated G_2-structure. Decompose Ψ into

$$\Psi = \Omega + e^1 \wedge \rho$$

with $\Omega \in \Lambda^4(V_2 \oplus \mathrm{span}(e^4) \oplus \mathfrak{g}_3^*)$, $\rho \in \Lambda^3(V_2 \oplus \mathrm{span}(e^4) \oplus \mathfrak{g}_3^*)$. Then

$$0 = d\Psi = d\Omega + (\mathrm{tr}(F)e^{14} + \nu) \wedge \rho - e^1 \wedge d\rho = e^1 \wedge (\mathrm{tr}(F)e^4 \wedge \rho - d\rho) + d\Omega + \nu \wedge \rho \quad (5.6)$$

implies $\Phi := \mathrm{tr}(F)e^4 \wedge \rho - d\rho = 0$. We look at different components of Φ. We have the identities

$$\begin{aligned}
0 = \Phi^{2,1,1,0} &= \mathrm{tr}(F)e^4 \wedge \rho^{2,1,0,0} - d(\rho^{2,1,0,0})^{2,1,1,0} = \mathrm{tr}(F)e^4 \wedge \rho^{2,1,0,0} - \mathrm{tr}(F)\rho^{2,1,0,0} \wedge e^4, \\
&= 2\mathrm{tr}(F)e^4 \wedge \rho^{2,1,0,0}, \\
0 = \Phi^{1,2,0,1} &= -d(\rho^{1,2,0,0})^{1,2,0,1} = -\mathrm{tr}(G)\rho^{1,2,0,0} \wedge e^7, \\
0 = \Phi^{2,0,1,1} &= \mathrm{tr}(F)e^4 \wedge \rho^{2,0,0,1} - d(\rho^{2,0,0,1}) = 2\mathrm{tr}(F)e^4 \wedge \rho^{2,0,0,1},
\end{aligned}$$

which imply $\rho^{2,1,0,0} = \rho^{1,2,0,0} = \rho^{2,0,0,1} = 0$. Moreover,

$$0 = \Phi^{0,2,1,1} = \mathrm{tr}(F)e^4 \wedge \rho^{0,2,0,1} - d(\rho^{0,2,1,0}) = \mathrm{tr}(F)e^4 \wedge \rho^{0,2,0,1} + \mathrm{tr}(G)e^7 \wedge \rho^{0,2,1,0},$$

i.e. $\frac{\mathrm{tr}(F)}{\mathrm{tr}(G)}e^4 \wedge \rho^{0,2,0,1} = -e^7 \wedge \rho^{0,2,1,0}$. Thus, ρ decomposes as

$$\rho = e^7 \wedge (\omega_1 + \hat{\nu}) + e^4 \wedge \left(\omega_2 + \frac{\mathrm{tr}(F)}{\mathrm{tr}(G)}\hat{\nu} + \lambda\nu\right) + e^{47} \wedge \alpha$$

with $\omega_1, \omega_2 \in \Lambda^{1,1,0,0}$, $\hat{\nu} \in \Lambda^{0,2,0,0}$, $\lambda \in \mathbb{R}$, $\alpha \in \Lambda^{1,0,0,0} \oplus \Lambda^{0,1,0,0}$. Proposition 2.49 and Lemma 5.15 (b) imply that $\tilde{\omega}_1 := \omega_1 + \hat{\nu}$ and $\tilde{\omega}_2 := \omega_2 + \frac{\mathrm{tr}(F)}{\mathrm{tr}(G)}\hat{\nu} + \lambda\nu$ span a two-dimensional subspace in which each non-zero element is of length two. Moreover,

$$0 = \Phi^{1,1,1,1} = \mathrm{tr}(F)e^4 \wedge \rho^{1,1,0,1} - d(\rho^{1,1,1,0}) - d(\rho^{1,1,0,1})$$

which shows that

$$\begin{aligned}
d(e^1 \wedge (\rho^{1,1,1,0} + \rho^{1,1,0,1})) =& (\nu + \mathrm{tr}(F)e^{14}) \wedge (\rho^{1,1,1,0} + \rho^{1,1,0,1}) - e^1 \wedge d(\rho^{1,1,1,0} + \rho^{1,1,0,1}) \\
=& \mathrm{tr}(F)e^{14} \wedge \rho^{1,1,0,1} - e^1 \wedge d(\rho^{1,1,1,0}) - e^1 \wedge d(\rho^{1,1,0,1}) \\
=& e^1 \wedge \Phi^{1,1,1,1} = 0.
\end{aligned}$$

Since $\rho^{1,1,1,0} = e^4 \wedge \omega_2$ and $\rho^{1,1,0,1} = e^7 \wedge \omega_1$, we get $d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) = 0$.

What is left to show is that $\hat{\nu} \neq 0$. Therefore, let $\tilde{\Omega}$ be the projection of Ψ onto the subspace $\Lambda^4(\mathrm{span}(e^1) \oplus V_2 \oplus W_2)$ (along $\sum_{i=1}^2 \Lambda^i(\mathrm{span}(e^1) \oplus V_2 \oplus W_2) \wedge \Lambda^{2-i}\mathrm{span}(e^4, e^7)$). By Proposition 2.48, $l(\tilde{\Omega}) \geq 1$, i.e. $\tilde{\Omega} \neq 0$. We may write $\tilde{\Omega}$ in terms of the components of ρ and Ω as

$$\tilde{\Omega} = e^1 \wedge \rho^{2,1,0,0} + e^1 \wedge \rho^{1,2,0,0} + \Omega^{2,2,0,0} = \Omega^{2,2,0,0}$$

and get $\Omega^{2,2,0,0} \neq 0$. Equation (5.6) gives us

$$0 = (d\Omega + \nu \wedge \rho)^{2,2,0,1} = d\left(\Omega^{2,2,0,0}\right)^{2,2,0,1} + \nu \wedge \rho^{0,2,0,1} = -\mathrm{tr}(G)\Omega^{2,2,0,0} \wedge e^7 + \nu \wedge \rho^{0,2,0,1}$$

and so $e^7 \wedge \hat{\nu} = \rho^{0,2,0,1} \neq 0$, i.e. $\hat{\nu} \neq 0$.

"$\Leftarrow$":

Assume that there exist two-forms $\omega_1, \omega_2 \in V_2 \wedge W_2$, $0 \neq \hat{\nu} \in \Lambda^2 W_2$ and $\lambda \in \mathbb{R}$ fulfilling all the conditions. Then $\tilde{\omega}_1$ as in the statement fulfils $0 \neq \tilde{\omega}_1^2 \in \Lambda^2 V_2 \wedge \Lambda^2 W_2$. Hence, there exists $0 \neq \tilde{\lambda} \in \mathbb{R}$ such that $\frac{\tilde{\lambda}}{2}\tilde{\omega}_1^2 = -\frac{1}{\mathrm{tr}(G)}\nu \wedge \hat{\nu}$. Set now $\theta_1 := \frac{1}{\tilde{\lambda}}e^{71}$, $\theta_2 := \frac{1}{\tilde{\lambda}}e^{41}$, $\theta_3 := e^{74} \in \Lambda^2\mathrm{span}(e^1, e^4, e^7)$. By assumption, $\tilde{\omega}_1$, $\tilde{\omega}_2$ as in the statement span a two-dimensional space in which each non-zero element has length two. Thus, we may apply Lemma 2.2 and Proposition 2.55 to $V_4^* := V_2 \oplus W_2$, $V_3^* := \mathrm{span}(e^1, e^4, e^7)$ and get the existence of a two-form $\tilde{\omega}_3 \in \Lambda^2 V_4^*$ such that

$$\Psi := \sum_{i=1}^{3} \tilde{\omega}_i \wedge \theta_i + \frac{1}{2}\tilde{\omega}_1^2$$

is the Hodge dual of a G_2-structure. Using $d\nu = -\mathrm{tr}(F)\nu \wedge e^4$, $d\hat{\nu} = -\mathrm{tr}(G)\hat{\nu} \wedge e^7$, we compute

$$\begin{aligned}
d\Psi &= \frac{1}{\tilde{\lambda}}d\left(\tilde{\omega}_1 \wedge e^{71} + \tilde{\omega}_2 \wedge e^{41}\right) + d\left(\tilde{\omega}_3 \wedge e^{74}\right) - \frac{1}{\tilde{\lambda} \cdot \mathrm{tr}(G)}d(\nu \wedge \hat{\nu}) \\
&= \frac{1}{\tilde{\lambda}}d\left(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}\right) + \frac{1}{\tilde{\lambda}}d\left(\hat{\nu} \wedge e^{71}\right) + \frac{1}{\tilde{\lambda}}d\left(\frac{\mathrm{tr}(F)}{\mathrm{tr}(G)}\hat{\nu} \wedge e^{41} + \lambda\nu \wedge e^{41}\right) \\
&\quad + \frac{\mathrm{tr}(F)}{\tilde{\lambda} \cdot \mathrm{tr}(G)}\nu \wedge \hat{\nu} \wedge e^4 + \frac{1}{\tilde{\lambda}}\nu \wedge \hat{\nu} \wedge e^7 \\
&= 0 - \frac{\mathrm{tr}(F)}{\tilde{\lambda}}\hat{\nu} \wedge e^{714} - \frac{1}{\tilde{\lambda}}\hat{\nu} \wedge e^7 \wedge \nu - \frac{\mathrm{tr}(F)}{\tilde{\lambda}}\hat{\nu} \wedge e^{741} - \frac{\mathrm{tr}(F)}{\tilde{\lambda} \cdot \mathrm{tr}(G)}\hat{\nu} \wedge e^4 \wedge \nu \\
&\quad + \frac{\mathrm{tr}(F)}{\tilde{\lambda} \cdot \mathrm{tr}(G)}\nu \wedge \hat{\nu} \wedge e^4 + \frac{1}{\tilde{\lambda}}\nu \wedge \hat{\nu} \wedge e^7 \\
&= 0.
\end{aligned}$$

□

Remark 5.25. *The two-form $\omega_1 \in V_2 \wedge W_2$ in Proposition 5.24 has to be of length two since $\tilde{\omega}_1 = \omega_1 + \hat{\nu}$ is of length two. By Lemma 2.2, there exists a basis e^2, e^3 of V_2 and a basis e^5, e^6 of W_2 such that $\omega_1 = e^{26} + e^{35}$. If $\det(G) \neq 0$, then the condition $d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) = 0$ implies that $\omega_2 = (F + \mathrm{tr}(F)\mathrm{id})(e^2) \wedge G^{-1}(e^6) + (F + \mathrm{tr}(F)\mathrm{id})(e^3) \wedge G^{-1}(e^5)$.*

Let us, nevertheless, start with $\det(G) = 0$.

Lemma 5.26. *Let $\mathfrak{g}$, $\mathfrak{g}_4$, $\mathfrak{g}_3$, $e^1, e^4 \in \mathfrak{g}_4^*$, $e^7 \in \mathfrak{g}_3^*$, V_2, $F: V_2 \to V_2$, W_2 and $G: W_2 \to W_2$ as in Proposition 5.24. Assume further that $\det(G) = 0$, i.e. $\mathfrak{g}_3 = \mathfrak{r}_2 \oplus \mathbb{R}$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $\det(F + \mathrm{tr}(F)\mathrm{id}) = 0$, i.e. $\mathfrak{g}_4 = A_{4,9}^{-\frac{1}{2}}$.*

Proof. "$\Rightarrow$:"
Assume that $\mathfrak{g}$ admits a cocalibrated G_2-structure. By Proposition 5.24 and Remark 5.25, there exists a basis e^2, e^3 of V_2 and a basis e^5, e^6 of W_2 such that $\omega_1 := e^{26} + e^{35}$ fulfils $d(\omega_1 \wedge e^{71}) \in d(V_2 \wedge W_2 \wedge e^{41}) = V_2 \wedge G(W_2) \wedge e^{741}$. Each element in $V_2 \wedge G(W_2) \wedge e^{741}$ is of length at most one due to $\det(G) = 0$. But

$$d(\omega_1 \wedge e^{71}) = ((F + \mathrm{tr}(F)\mathrm{id})(e^2) \wedge e^6 + (F + \mathrm{tr}(F)\mathrm{id})(e^3) \wedge e^5) \wedge e^{741}$$

is of length less than two if and only if $\det(F + \mathrm{tr}(F)\mathrm{id}) = 0$. Thus, $\det(F + \mathrm{tr}(F)\mathrm{id}) = 0$.
"$\Leftarrow$:"
We have $\det(F+\mathrm{tr}(F)\mathrm{id}) = 0 = \det(G)$ and $\mathrm{tr}(F+\mathrm{tr}(F)\mathrm{id}) = 3\mathrm{tr}(F) \neq 0, \mathrm{tr}(G) \neq 0$. Since both $F+\mathrm{tr}(F)\mathrm{id}$ and G are linear endomorphisms in two dimensions, this implies that they diagonalisable over the reals with one zero eigenvalue and one non-zero eigenvalue. We may, after rescaling e^4 and e^7, assume that the non-zero eigenvalue is equal to one in both cases and so $\mathrm{tr}(F) = \frac{1}{3}$ and $\mathrm{tr}(G) = 1$. Since $d(e^1 \wedge \alpha) = -e^1 \wedge (F+\mathrm{tr}(F)\mathrm{id})(\alpha) \wedge e^4$ for all $\alpha \in V_2$, there exists a basis e^2, e^3 of V_2 such that $de^{12} = 0$ and $de^{13} = -e^{134}$. Moreover, we may choose a basis e^5, e^6 of W_2 with $de^5 = 0$ and $de^6 = e^{67}$. Then the following two-forms fulfil all the conditions in Proposition 5.24:

$$\omega_1 := e^{25} - e^{36} + e^{26}, \quad \omega_2 := e^{25} - e^{36} - 2e^{35}, \quad \tilde{\omega}_1 := e^{56} + \omega_1, \quad \tilde{\omega}_2 := \frac{1}{3}e^{56} + \omega_2.$$

$\square$

If $\det(G) \neq 0$ and F and G are both not multiples of the identity, we get:

Lemma 5.27. *Let $\mathfrak{g}$, $\mathfrak{g}_4$, $\mathfrak{g}_3$, $e^1, e^4 \in \mathfrak{g}_4^*$, $e^7 \in \mathfrak{g}_3^*$, V_2, $F : V_2 \to V_2$, W_2 and $G : W_2 \to W_2$ as in Proposition 5.24. Assume further that F and G are both not multiples of the identity, i.e. $\mathfrak{g}_4 \neq A_{4,9}^1$ and $\mathfrak{g}_3 \neq \mathfrak{r}_{3,1}$. Then $\mathfrak{g}$ admits a cocalibrated G_2-structure.*

Proof. Set $H := -(F + \mathrm{tr}(F)\mathrm{id})$. Then also $H : V_2 \to V_2$ is not a multiple of the identity, not trace-free and $d(e^1 \wedge \alpha) = e^1 \wedge H(\alpha) \wedge e^4$ for all $\alpha \in V_2$. By rescaling e^4 appropriately, we may assume that $\mathrm{tr}(H) = -3$, i.e. $\mathrm{tr}(F) = 1$. Hence, we may choose a basis e^2, e^3 of V_2 such that the transformation matrix of H with respect to this basis is given by

$$\begin{pmatrix} 0 & \frac{\det(H)}{\det(G)} \\ -\det(G) & -3 \end{pmatrix}.$$

Moreover, by rescaling e^7 appropriately, we may assume that $\mathrm{tr}(G) = 1$. Hence, for all $a \in \mathbb{R}\backslash\{0\}$, we may choose a basis e^5, e^6 of W_2 such that the transformation matrix of G with respect to this basis is given by

$$\begin{pmatrix} 0 & -\frac{\det(G)}{a} \\ a & 1 \end{pmatrix}.$$

Set

$$\omega_1 := e^{25}+e^{36},\ \omega_2 := -\frac{\det(H)}{\det(G)a}e^{25}+\frac{3+a}{a}e^{35}-a\,e^{36},\ \tilde{\omega}_1 := e^{56}+\omega_1,\ \tilde{\omega}_2 := e^{56}-a\,e^{23}+\omega_2.$$

A short computation shows $d(\omega_1 \wedge e^{71} + \omega_2 \wedge e^{41}) = 0$. Moreover, $\tilde{\omega}_1^2 = 2e^{2536} \neq 0$ and $\tilde{\omega}_1 \wedge \tilde{\omega}_2 = B\tilde{\omega}_1^2$, $\tilde{\omega}_2^2 = C\tilde{\omega}_1^2$ with $B = -\frac{\det(H)}{2a\det(G)}$ and $C = a + \frac{\det(H)}{\det(G)}$. Hence,

$$C - B^2 = a + \frac{\det(H)}{\det(G)} - \frac{\det(H)^2}{4a^2\det(G)^2} > 0$$

for $a > 0$ large enough and so $\tilde{\omega}_1$, $\tilde{\omega}_2$ span a two-dimensional space in which each non-zero element has length two by Lemma 2.2. Thus, $\mathfrak{g}$ admits a cocalibrated G_2-structure by Proposition 5.24 □

Therefore, it remains to consider the cases when at least one of the maps F and G is (a multiple of) the identity:

Lemma 5.28. *Let $\mathfrak{g}$, $\mathfrak{g}_4$, $\mathfrak{g}_3$, $e^1, e^4 \in \mathfrak{g}_4^*$, $e^7 \in \mathfrak{g}_3^*$, V_2, $F : V_2 \to V_2$, W_2 and $G : W_2 \to W_2$ as in Proposition 5.24.*

(a) If F is a multiple of the identity, i.e. $\mathfrak{g}_4 = A_{4,9}^1$, then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $-\frac{3}{4}\mathrm{tr}(G)^2 > \det(G)$ or $\det(G) > 0$.

(b) If G is a multiple of the identity, i.e. $\mathfrak{g}_3 = \mathfrak{r}_{3,1}$, then $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if $\det(F) > -\frac{3}{4}\mathrm{tr}(F)^2$.

Remark 5.29. *Note that a real two-by-two matrix with negative determinant is always diagonalisable over the reals. The determinant of G is negative if the condition in Lemma 5.28 (a) is not fulfilled and the determinant of F is negative if the condition in Lemma 5.28 (b) is not fulfilled. Hence, it is easily checked that the condition on $\mathfrak{g}_3$ in Lemma 5.28 (a) is not fulfilled exactly when $\mathfrak{g}_3 \in \left\{\mathfrak{r}_{3,\mu} \,\middle|\, \mu \in \left[-\frac{1}{3}, 0\right)\right\}$ and that the condition on $\mathfrak{g}_4$ in Lemma 5.28 (b) is not fulfilled exactly when $\mathfrak{g}_4 \in \left\{A_{4,9}^{\alpha} \,\middle|\, \alpha \in \left(-1, -\frac{1}{3}\right]\right\}$. Hence, proving Lemma 5.28 finishes the proof of Theorem 5.18.*

Proof. (a) By rescaling e^4 we may assume that $\mathrm{tr}(F) = 2$, i.e. $F = \mathrm{id}$. Hence, Proposition 5.24 and Remark 5.25 tell us that $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there exists a basis e^2, e^3 of V_2, a basis e^5, e^6 of W_2, $\lambda, \alpha \in \mathbb{R}$, $\alpha \neq 0$ such that each non-zero linear combination of

$$\tilde{\omega}_{1,\alpha,\lambda} := \alpha e^{56} + e^{26} + e^{35},\ \tilde{\omega}_{2,\alpha,\lambda} := \frac{2}{\mathrm{tr}(G)}\alpha e^{56} + \lambda e^{23} + 3e^2 \wedge G^{-1}(e^6) + 3e^3 \wedge G^{-1}(e^5)$$

is of length two. A short computation shows

$$\tilde{\omega}_{1,\alpha,\lambda}^2 = 2e^{2356},\quad \tilde{\omega}_{1,\alpha,\lambda} \wedge \tilde{\omega}_{2,\alpha,\lambda} = \left(\alpha\lambda + \frac{3\mathrm{tr}(G)}{\det(G)}\right)e^{2356},$$
$$\tilde{\omega}_{2,\alpha,\lambda}^2 = \left(4\frac{\alpha\lambda}{\mathrm{tr}(G)} + 18\frac{1}{\det(G)}\right)e^{2356}$$

since for an invertible two-by-two matrix $\operatorname{tr}\left(G^{-1}\right) = \frac{\operatorname{tr}(G)}{\det(G)}$. Set $X := \alpha\lambda$. Then Lemma 2.2 tells us that each non-zero linear combination of $\tilde{\omega}_{1,\alpha,\lambda}$ and $\tilde{\omega}_{2,\alpha,\lambda}$ is of length two if and only if the quadratic polynomial

$$\begin{aligned}
&8\frac{X}{\operatorname{tr}(G)} + 36\frac{1}{\det(G)} - \left(X + \frac{3\operatorname{tr}(G)}{\det(G)}\right)^2 \\
&= -X^2 + \left(\frac{8}{\operatorname{tr}(G)} - 6\frac{\operatorname{tr}(G)}{\det(G)}\right)X + 36\frac{1}{\det(G)} - 9\frac{\operatorname{tr}(G)^2}{\det(G)^2}
\end{aligned}$$

in X with leading negative coefficient is positive for some $X \in \mathbb{R}$. Note that this expression does not depend on the basis we have chosen. Hence, $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if this quadratic polynomial is positive for some $X \in \mathbb{R}$ and this is true if and only if its discriminant is positive. The discriminant is given by

$$\left(6\frac{\operatorname{tr}(G)}{\det(G)} - 8\frac{1}{\operatorname{tr}(G)}\right)^2 - 4\cdot\left(9\frac{\operatorname{tr}(G)^2}{\det(G)^2} - 36\frac{1}{\det(G)}\right) = \frac{16(3\operatorname{tr}(G)^2 + 4\det(G))}{\det(G)\operatorname{tr}(G)^2},$$

and it is positive if and only if

$$-\frac{3}{4}\operatorname{tr}(G)^2 > \det(G) \quad \text{or} \quad \det(G) > 0.$$

(b) By rescaling e^7 we may assume $\operatorname{tr}(G) = 2$, i.e. $G = \mathrm{id}$. Then we see similarly as in the proof of part (a) that $\mathfrak{g}$ admits a cocalibrated G_2-structure if and only if there exists a basis e^2, e^3 of V_2, a basis e^5, e^6 of W_2, $\lambda, \alpha \in \mathbb{R}$, $\alpha \neq 0$ such that each non-zero linear combination of

$$\begin{aligned}
\tilde{\omega}_{1,\alpha,\lambda} &:= \alpha e^{56} + e^{26} + e^{35}, \\
\tilde{\omega}_{2,\alpha,\lambda} &:= \frac{\operatorname{tr}(F)}{2}\alpha e^{56} + \lambda e^{23} + (F + \operatorname{tr}(F)\mathrm{id})(e^2) \wedge e^6 + (F + \operatorname{tr}(F)\mathrm{id})(e^3) \wedge e^5
\end{aligned}$$

is of length two. If we set $X := \alpha\lambda$ as before, we find, analogously to the proof of (a), that the existence of a cocalibrated G_2-structure on $\mathfrak{g}$ is equivalent to the existence of $X \in \mathbb{R}$ such that $-X^2 - 4\operatorname{tr}(F)X - \operatorname{tr}(F)^2 + 4\det(F)$ is positive. Note therefore that for a two-by-two matrix $A \in \mathbb{R}^{2\times 2}$ we generally have $\det(A + \operatorname{tr}(A)I_2) = \det(A) + 2\operatorname{tr}(A)^2$. Now $-X^2 - 4\operatorname{tr}(F)X - \operatorname{tr}(F)^2 + 4\det(F)$ is positive for some $X \in \mathbb{R}$ exactly when the discriminant of this quadratic polynomial in X, which is given by $12\operatorname{tr}(F)^2 + 16\det(F)$, is positive. And this is the case if and only if

$$\det(F) > -\frac{3}{4}\operatorname{tr}(F)^2.$$

$\square$

Chapter 6

Half-flat structures on Lie algebras

In this chapter, we present the classification results for half-flat SU(3)-structures on certain classes of Lie algebras the author obtained together with Fabian Schulte-Hengesbach in the two papers [FS1] and [FS2]. Moreover, we present also some partial results on the classification of six-dimensional Lie algebras admitting other types of half-flat structures. Apart from one result on certain six-dimensional almost Abelian Lie algebras admitting half-flat structures of other types, also these results are joint work with Schulte-Hengesbach and already published in [FS1].

More exactly, we finish the classification of the decomposable six-dimensional Lie algebras which admit a half-flat SU(3)-structure. Therefore, we determine the direct sums of a four-dimensional Lie algebra and of a two-dimensional Lie algebra and the direct sums of a five-dimensional Lie algebra and $\mathbb{R}$ possessing a half-flat SU(3)-structure. These results are all contained in [FS1] and we also present the non-existence results on stable three-forms of certain type and on half-flat $SU(1,2)$- and half-flat $SL(3,\mathbb{R})$-structures on some of the considered decomposable Lie algebras given in [FS1]. Note that the direct sums of two three-dimensional Lie algebras which admit a half-flat SU(3)-structure have been determined before by Schulte-Hengesbach in [SH]. The analogous classification has been done by Conti [C1] for the class of six-dimensional nilpotent Lie algebras. We basically use a refinement of Conti's and Schulte-Hengesbach's obstructions to prove the non-existence of half-flat SU(3)-structures on the mentioned decomposable Lie algebras. Our obstruction has the advantage that it is easy to check using a computer algebra system. In fact, we use *Maple*, in particular the packages "difforms" and "difforms2", to check the obstruction. Existence is proved in most cases by giving an explicit example of a half-flat SU(3)-structure. We changed parts of the proofs given in [FS1] and use also the relation between half-flat SU(3)-structures on six-dimensional Lie algebras $\mathfrak{g}$ and cocalibrated G_2-structures on $\mathfrak{g} \oplus \mathbb{R}$. We give a direct proof that a six-dimensional almost Abelian Lie algebra $\mathfrak{g}$ admits a half-flat SU(3)-structure if and only if $\mathfrak{g} \oplus \mathbb{R}$ admits a cocalibrated G_2-structure and so get a full classification of the six-dimensional almost Abelian Lie algebras admitting a half-flat SU(3)-structure by Theorem 5.18. Moreover, we prove that in this case the six-dimensional almost Abelian Lie algebra $\mathfrak{g}$ admits half-flat structures of any type, a result not contained in [FS1] or [FS2]. We also apply the classification of the direct sums of four- and three-dimensional Lie algebras admitting a cocalibrated G_2-structure given in Theorem 5.18 to show that on certain decomposable six-dimensional Lie algebras there cannot exist a half-flat SU(3)-structure.

Moreover, we classify the indecomposable solvable six-dimensional Lie algebras with five-dimensional nilradical which admit a half-flat SU(3)-structure and show that all non-solvable six-dimensional Lie algebras possess such a structure. These results are all contained in [FS2]. The proofs are completely analogous to the paper [FS2]. We use again our refinement of Conti's obstruction but also apply some obstruction obtained by the relation between half-flat SU(3)-structures on a six-dimensional Lie algebra $\mathfrak{g}$ and cocalibrated G_2-structures on $\mathfrak{g} \oplus \mathbb{R}$. Existence is again proved by giving concrete examples. Note that by a result of Mubarakzyanov [Mu6d], a six-dimensional solvable indecomposable Lie algebra is nilpotent or the nilradical has dimension five or four. Hence, only the question which indecomposable solvable six-dimensional Lie algebras with four-dimensional nilradical admit a half-flat SU(3)-structure remains open. We emphasise that we refined the classification of indecomposable five-dimensional Lie algebras given in [Mu5d] and also the classification of six-dimensional Lie algebras with five-dimensional non-Abelian nilradical in [Mu6d] in order to obtain the mentioned classification results. This refinement is interesting in its own. We give an application of this refinement to the classification of six-dimensional $(2,3)$-trivial Lie algebras which is also contained in [FS2].

We start in Section 6.1 by giving a brief history of the results known before and also of the obstructions used by Conti in [C1] and by Schulte-Hengesbach in [SH] to obtain their results. Section 6.2 presents our refinement of these obstructions and also the above-mentioned obstruction obtained by the relation between half-flat SU(3)-structures on a six-dimensional Lie algebra $\mathfrak{g}$ and cocalibrated G_2-structures on $\mathfrak{g} \oplus \mathbb{R}$. In Section 6.3, we prove the classification results on six-dimensional Lie algebras admitting half-flat SU(3)-structures. Finally, Section 6.4 gives the result on the classification of six-dimensional $(2,3)$-trivial Lie algebras. Moreover, also the non-existence results on stable forms of certain kind and on half-flat SU(1,2)- or $\mathrm{SL}(3,\mathbb{R})$-structures in the decomposable case are presented in this section.

6.1 Known results and obstructions

The first steps towards a classification of the Lie algebras which admit half-flat SU(3)-structures have been done in [ChiSw], [ChiFi], [CT]. In these papers, a classification of the nilpotent six-dimensional Lie algebras admitting special kinds of half-flat SU(3)-structures has been given. The next step has been the following classification of the nilpotent Lie algebras admitting an arbitrary half-flat SU(3)-structures [C1] by Conti. For the names of the appearing Lie algebras, we refer the reader to the appendix.

Theorem 6.1 (Conti). *Let* $\mathfrak{g}$ *be a six-dimensional nilpotent Lie algebra. Then* $\mathfrak{g}$ *admits a half-flat* SU(3)*-structure if and only if*

(i) $\mathfrak{g}$ *is decomposable and* $\mathfrak{g} \in \left\{\mathbb{R}^6,\, \mathfrak{h}_3 \oplus \mathbb{R}^3, \mathfrak{h}_3 \oplus \mathfrak{h}_3, A_{5,i} \oplus \mathbb{R} \middle|\, i = 1,2,4,5,6\right\}$ *or*

(ii) $\mathfrak{g}$ *is indecomposable and* $\mathfrak{g} = \mathfrak{n}_{6,j}$ *for* $j \notin \{1,2,8,9,10,19,21,22\}$.

To prove Theorem 6.1, he introduces the concept of a *coherent splitting* on an arbitrary six-dimensional Lie algebra $\mathfrak{g}$, which is a splitting $\mathfrak{g}^* = V_2 \oplus V_4$ into a two-dimensional subspace V_2 and a four-dimensional subspace V_4 with $d\,(V_2) = \Lambda^2 V_2$ and $d\,(V_4) = \Lambda^2 V_2 \oplus V_2 \wedge V_4$. This splitting can be used to define a double complex $(\Lambda^{p,q}\mathfrak{g}^*, \delta_1, \delta_2)$. Conti shows that the triviality of the cohomology classes $H^{0,3}$ and $H^{0,4}$ implies that on $\mathfrak{g}$ there cannot exist a half-flat SU(3)-structure. He applies this obstruction then to eight nilpotent Lie algebras to exclude half-flat SU(3)-structures on them. The existence is proved by giving a concrete example of a half-flat SU(3)-structure in each case. There are two nilpotent Lie algebras, namely $\mathfrak{n}_{6,21}$ and $\mathfrak{n}_{6,22}$, which do not admit a coherent splitting and existence of half-flat SU(3)-structures on them is excluded by refining the methods. Basically, Conti uses Equation (2.16) and Equation (2.17). More exactly, he shows that the existence of a half-flat SU(3)-structure $(\omega,\rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ implies in both cases that the basis vector e^1 in the basis given in Table 7.6 fulfils $J_\rho^* e^1 \in \mathrm{span}(e^1, e^2, e^3)$. For that purpose he uses Equation (2.16), which states that

$$e^1 \wedge (v \lrcorner \rho) \wedge \rho = J_\rho^* e^1(v)\phi(\rho)$$

for all $v \in \mathfrak{g}$, and shows that $e^1 \wedge (w \lrcorner \psi) \wedge \psi = 0$ for all closed three-forms ψ and all $w \in \mathrm{span}(e_4, e_5, e_6)$. Afterwards, he computes that then the identity $e^1 \wedge J_\rho^* e^1 \wedge \sigma = 0$ for all closed four-forms $\sigma \in \Lambda^4\mathfrak{g}^*$, and so also for ω^2, is true. But this is a contradiction to Equation (2.17), which states that

$$\beta \wedge J_\rho^*\beta \wedge \omega^2 = \frac{1}{3} g(\beta,\beta)\omega^3 \neq 0$$

for all $\beta \in \mathfrak{g}^* \backslash \{0\}$.

Our obstruction given in Proposition 6.5 resembles this argumentation. It gives a direct obstruction for which one has to compute all closed three-forms and all closed four-forms on $\mathfrak{g}$ and can then easily check the obstruction. The obtained obstruction is built up in such a way that all computations can be done using a computer algebra system like *Maple*.

The next class of Lie algebras considered was the direct sums of two three-dimensional Lie algebras by Schulte-Hengesbach in [SH]. He gets an obstruction without introducing the double complex above by looking at the decisive steps in Conti's proof. He applies this obstruction to all but two cases of direct sums $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two three-dimensional Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ which do not admit a half-flat $\mathrm{SU}(3)$-structure. Existence is again proved by giving concrete examples of half-flat $\mathrm{SU}(3)$-structures. The missing two cases $\mathfrak{r}_2 \oplus \mathbb{R}^4$ and $\mathfrak{h}_3 \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ are treated separately. The first case is excluded by showing that all closed three-forms $\rho \in \Lambda^3 \left(\mathfrak{r}_2 \oplus \mathbb{R}^4\right)^*$ fulfil $\lambda(\rho) \geq 0$ and so $\mathfrak{r}_2 \oplus \mathbb{R}^4$ cannot admit a half-flat $\mathrm{SU}(3)$-structure by Proposition 3.38. The second case uses directly our main obstruction given below in Proposition 6.5 without stating it concretely.

The result obtained by Schulte-Hengesbach is given in Theorem 6.2. We rephrase it to make connection to the existence of cocalibrated G_2-structures on direct sums $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathbb{R}$ with $\dim(\mathfrak{g}_i) = 3$ for $i = 1, 2$.

Theorem 6.2 (Schulte-Hengesbach)**.** *A direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two three-dimensional Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ admits a half-flat $\mathrm{SU}(3)$-structure if and only if both $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are unimodular or exactly one of the Lie algebras is unimodular, say $\mathfrak{g}_{i_1}$ for some $i_1 \in \{1, 2\}$, and $h^2(\mathfrak{g}_{i_1}) \leq h^2(\mathfrak{g}_{i_2})$, where $i_2 \in \{1, 2\}$ is such that $\{i_1, i_2\} = \{1, 2\}$.*

Remark 6.3. *Theorem 6.2 and Theorem 5.18 imply that a direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two three-dimensional Lie algebras $\mathfrak{g}_1$, $\mathfrak{g}_2$ admits a half-flat $\mathrm{SU}(3)$-structure if and only if $\mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathbb{R}$ admits a cocalibrated G_2-structure. By Proposition 3.37, the existence of a half-flat $\mathrm{SU}(3)$-structure on a six-dimensional Lie algebra $\mathfrak{g}$ is equivalent to the existence of a cocalibrated G_2-structure on $\mathfrak{g} \oplus \mathbb{R}$ such that $\mathfrak{g}$ is orthogonal to $\mathbb{R}$. Hence, the non-existence of half-flat $\mathrm{SU}(3)$-structure in all cases in Theorem 6.2 follows independently also by our Theorem 5.18. However, the existence of half-flat $\mathrm{SU}(3)$-structures on the Lie algebras in Theorem 6.2 does not follow by Theorem 5.18 since we cannot ensure the existence of a cocalibrated G_2-structure such that $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is orthogonal to $\mathbb{R}$. In fact, in Remark 6.10 we present an example of a direct sum $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_2$ of a four-dimensional Lie algebra $\mathfrak{g}_4$ and a two-dimensional Lie algebra $\mathfrak{g}_2$ which does not admit a half-flat $\mathrm{SU}(3)$-structure but for which $\mathfrak{g}_4 \oplus \mathfrak{g}_2 \oplus \mathbb{R}$ admits a cocalibrated G_2-structure.*

Schulte-Hengesbach also classified the direct sums $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two three-dimensional Lie algebras admitting a half-flat $\mathrm{SU}(3)$-structure such that the decomposition is orthogonal. Moreover, he also classified the direct sums $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ admitting a half-flat $\mathrm{SL}(3, \mathbb{R})$-structure for which the decomposition is orthogonal and each summand is definite and did the same under the condition that the summands are the ± 1-eigenspaces of the para-complex structure. We do not give these classifications here. Instead, we mention his results that certain direct sums do not admit any half-flat $\mathrm{SU}(1, 2)$-structure. He achieved this result by showing $\lambda(\rho) \geq 0$ for all closed three-forms using *Maple* for the calculations.

The non-existence then follows by Proposition 3.38. Note that there is always a closed three-form with $\lambda(\rho) > 0$, i.e. with model tensor $\mathrm{Re}(\Psi_1)$, on direct sum $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ of two three-dimensional Lie algebras, e.g. $\rho = \nu_1 + \nu_2$ for arbitrary $\nu_i \in \Lambda^3\mathfrak{g}_i^*\backslash\{0\}$. Hence, one cannot get non-existence results for half-flat $\mathrm{SL}(3,\mathbb{R})$-structure via Proposition 3.38 on the considered direct sums.

Proposition 6.4. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_1 \in \{\mathbb{R}^3, \mathfrak{h}_3, \mathfrak{r}_2 \oplus \mathbb{R}\}$ and $\mathfrak{g}_2$ be three-dimensional non-unimodular and $\mathfrak{g}_2 \neq \mathfrak{r}_2 \oplus \mathbb{R}$. Then all closed three-forms $\rho \in \Lambda^3\mathfrak{g}^*$ on $\mathfrak{g}$ fulfil $\lambda(\rho) \geq 0$ and $\mathfrak{g}$ does not admit a half-flat $\mathrm{SU}(1,2)$-structure.*

6.2 New obstructions

We begin the new section by stating our main obstruction.

Proposition 6.5. *Let $\mathfrak{g}$ be a six-dimensional Lie algebra with a volume form $\nu \in \Lambda^6\mathfrak{g}^*\backslash\{0\}$. If there is a non-zero one-form $\alpha \in \mathfrak{g}^*$ satisfying*

$$\alpha \wedge \tilde{J}_\rho^*\alpha \wedge \sigma = 0 \tag{6.1}$$

for all closed three-forms $\rho \in \Lambda^3\mathfrak{g}^*$ and all closed four-forms $\sigma \in \Lambda^4\mathfrak{g}^*$, where $\tilde{J}_\rho^*\alpha$ is defined for $X \in \mathfrak{g}$ by

$$\tilde{J}_\rho^*\alpha(X)\,\nu = \alpha \wedge (X \lrcorner\, \rho) \wedge \rho, \tag{6.2}$$

then $\mathfrak{g}$ does not admit a half-flat $\mathrm{SU}(3)$-structure.

Proof. Suppose that $\alpha \in \mathfrak{g}^*$ is a non-zero one-form as in the statement and that, nevertheless, $(\omega, \rho) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ is a half-flat $\mathrm{SU}(3)$-structure on $\mathfrak{g}$. Then $\rho \in \Lambda^3\mathfrak{g}^*$ and $\frac{1}{2}\omega^2 \in \Lambda^4\mathfrak{g}^*$ are closed. Moreover, by Equation (2.16) there exists $\mu \in \mathbb{R}\backslash\{0\}$ such that $\tilde{J}_\rho^*\alpha = \mu J_\rho^*\alpha$. Hence, Equation (2.17) implies

$$\alpha \wedge \tilde{J}_\rho^*\alpha \wedge \frac{1}{2}\omega^2 = \frac{\mu}{6} g(\alpha,\alpha)\omega^3 \neq 0,$$

a contradiction. This shows the statement. $\square$

By Proposition 3.37 the non-existence of a cocalibrated G_2-structure on a direct sum $\mathfrak{g}_6 \oplus \mathbb{R}$ of a six-dimensional Lie algebra $\mathfrak{g}_6$ and of $\mathbb{R}$ implies the non-existence of a half-flat $\mathrm{SU}(3)$-structure on $\mathfrak{g}_6$. Hence, we get obstructions to the existence of half-flat $\mathrm{SU}(3)$-structures on certain classes of six-dimensional Lie algebras by the classification results for cocalibrated G_2-structures obtained in Theorem 4.15 and in Theorem 5.18. Moreover, the mentioned relation gives us also the following obstruction.

Proposition 6.6. *Let $\mathfrak{g}_6$ be a six-dimensional Lie algebra and set $\mathfrak{g}_7 := \mathfrak{g}_6 \oplus \mathbb{R}$. Choose a non-zero one-form $\alpha \in {\mathfrak{g}_6}^0$ in the annihilator ${\mathfrak{g}_6}^0$ of $\mathfrak{g}_6$ in $\mathfrak{g}_7$. For each pair $(\rho, \sigma) \in$*

$Z^3(\mathfrak{g}_6) \times Z^4(\mathfrak{g}_6)$ of a closed three-form and a closed four-form on $\mathfrak{g}_6$, we define a four-form $\Omega(\rho,\sigma) \in \Lambda^4\mathfrak{g}_7^*$ on $\mathfrak{g}_7$ as follows:

$$\Omega(\rho,\sigma) := \rho \wedge \alpha + \sigma.$$

If there exists a non-zero element $X \in \mathfrak{g}_7$ and a complement W of $\operatorname{span}(X)$ in $\mathfrak{g}_7$ such that for all pairs $(\rho,\sigma) \in Z^3(\mathfrak{g}_6) \times Z^4(\mathfrak{g}_6)$ the three-form $\tilde{\rho}(\rho,\sigma) := (X \lrcorner \Omega(\rho,\sigma))|_W \in \Lambda^3 W^*$ on W fulfils $\lambda(\tilde{\rho}) \geq 0$, then $\mathfrak{g}_6$ does not admit any half-flat $\mathrm{SU}(3)$-structure.

Proof. Let $\mathfrak{g}_6$, $\mathfrak{g}_7$, $\alpha \in {\mathfrak{g}_6}^0$ be as in the statement. Assume that $X \in \mathfrak{g}_7$ and $W \subseteq \mathfrak{g}_7$ as in the statement exist and that, nevertheless, $\mathfrak{g}_6$ admits a half-flat SU(3)-structure $(\omega,\rho) \in \Lambda^2\mathfrak{g}_6^* \times \Lambda^3\mathfrak{g}_6^*$. Set $\sigma := \frac{1}{2}\omega^2$. Then $(\rho,\sigma) \in Z^3(\mathfrak{g}_6) \times Z^4(\mathfrak{g}_6)$. By Proposition 3.37, the half-flat SU(3)-structure (ω,ρ) induces a cocalibrated G_2-structure φ on $\mathfrak{g}_7$ whose Hodge dual is given by

$$\star_\varphi \varphi = \rho \wedge \alpha + \sigma = \Omega(\rho,\sigma).$$

By Proposition 2.48, the three-form $\tilde{\rho}(\rho,\sigma) = (X \lrcorner \star_\varphi \varphi)|_W = (X \lrcorner \Omega(\rho,\sigma))|_W \in \Lambda^3 W^*$ on W has model tensor $\rho_{-1} \in \Lambda^3 \left(\mathbb{R}^6\right)^*$ and so fulfils $\lambda(\tilde{\rho}) < 0$ by Proposition 2.21, a contradiction. Hence, $\mathfrak{g}_6$ does not admit a half-flat SU(3)-structure. □

6.3 Results for half-flat structures

We first discuss the existence problem of half-flat SU(3)-structures on six-dimensional almost Abelian Lie algebras. This problem has completely been solved case-by-case for decomposable Lie algebras in [C1], [SH] and [FS1]. In [FS2], we excluded the existence of half-flat SU(3)-structures on indecomposable six-dimensional almost Abelian Lie algebras $\mathfrak{g}$ as follows. Theorem 4.15 shows that $\mathfrak{g} \oplus \mathbb{R}$ does not admit a cocalibrated G_2-structure if $\mathfrak{g}$ is an indecomposable six-dimensional almost Abelian Lie algebra. Thus, Proposition 3.37 implies the non-existence result. One observes that a six-dimensional almost Abelian Lie algebra $\mathfrak{g}$ admits a half-flat SU(3)-structure if and only if $\mathfrak{g} \oplus \mathbb{R}$ admits a cocalibrated G_2-structure. We give a direct proof of this statement below and also show that if one of these two equivalent conditions is fulfilled, then $\mathfrak{g}$ admits any kind of half-flat structure. Note that both the direct proof and the existence of other kinds of half-flat structures are not contained in [FS1], [FS2].

Theorem 6.7. *Let $\mathfrak{g}$ be a six-dimensional almost Abelian Lie algebra. Then the following are equivalent:*

(i) $\mathfrak{g} \oplus \mathbb{R}$ *admits a cocalibrated* G_2*-structure.*

(ii) $\mathfrak{g}$ *admits a half-flat* $\mathrm{SU}(3)$*-structure.*

If the equivalent conditions (i) and (ii) are fulfilled, then $\mathfrak{g}$ *also admits a half-flat* $\mathrm{SU}(1,2)$*-structure and a half-flat* $\mathrm{SL}(3,\mathbb{R})$*-structure. The six-dimensional almost Abelian Lie algebra* $\mathfrak{g}$ *fulfilling (i) and (ii) are:*

- $\mathfrak{g} = \mathfrak{g}_3 \oplus \mathbb{R}^3$ *with an arbitrary three-dimensional almost Abelian unimodular Lie algebra* $\mathfrak{g}_3$ *and*
- $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}$ *with* $\mathfrak{g}_5 = A_{5,1}$, $A_{5,2}$, $A_{5,7}^{-1,\alpha,-\alpha}$ *with* $\alpha \in \{-1\} \cup (0,1)$, $A_{5,8}^{-1}$, $A_{5,13}^{-1,0,\beta}$ *with* $\beta > 0$, $A_{5,14}^{0}$, $A_{5,15}^{-1}$, $A_{5,17}^{\gamma,-\gamma,1}$ *with* $\gamma > 0$, $A_{5,17}^{0,0,\delta}$ *with* $0 < \delta \leq 1$ *or* $A_{5,18}^{0}$.

Proof. (ii) implies (i) by Proposition 3.37. Next, we assume that (i) holds, i.e. that $\mathfrak{g} \oplus \mathbb{R}$ admits a cocalibrated G_2-structure $\varphi \in \Lambda^3 (\mathfrak{g} \oplus \mathbb{R})^*$. Let $\mathfrak{u}$ be an Abelian ideal of dimension five in $\mathfrak{g}$. Choose $f_6 \in \mathfrak{g} \backslash \mathfrak{u}$ and let $f^6 \in \mathfrak{g}^* \subseteq \mathfrak{g}^* \oplus \mathbb{R}^* = (\mathfrak{g} \oplus \mathbb{R})^*$ be the element in the annihilator of $\mathfrak{u}$ in $\mathfrak{g} \oplus \mathbb{R}$ with $f^6(f_6) = 1$. By Proposition 4.12 and Lemma 4.3 (b), there exists a closed two-form $\omega \in \Lambda^2 (\mathfrak{u}^* \oplus \mathbb{R}^*)$ of length three, where closed means here and for the rest of the proof that it is $d_{\mathfrak{g}}$-closed. Let $\alpha \in \mathbb{R}^* \backslash \{0\}$. We may decompose ω as

$$\omega = \alpha \wedge f^5 + \omega_0$$

with $f^5 \in \mathfrak{u}^* \backslash \{0\}$ and $\omega_0 \in \Lambda^2 \mathfrak{u}^*$. The length of ω_0 has to be two. Hence, Lemma 2.1 implies the existence of linearly independent elements $f^1, \ldots, f^4 \in \mathfrak{u}^*$ such that $\omega_0 = f^{12} + f^{34}$. Since $0 \neq \omega^3$, the one-forms $f^1, \ldots, f^5$ form a basis of $\mathfrak{u}^*$. Since ω and α are closed, also f^5 and ω_0 are closed. By Lemma 4.3, $d\left(\nu \wedge f^6\right)$ for all k-forms $\nu \in \Lambda^k \mathfrak{g}^*$. Thus, the pair $(\omega_1, \rho_1) \in \Lambda^2 \mathfrak{g}^* \times \Lambda^3 \mathfrak{g}^*$, defined by

$$\omega_1 := f^{56} + f^{14} + f^{23}, \quad \rho_1 := f^{512} - f^{543} - f^{613} - f^{642} = f^5 \wedge \omega_0 - f^{136} + f^{246},$$

is a half-flat SU(3)-structure on $\mathfrak{g}$, the pair $(\omega_2, \rho_2) \in \Lambda^2 \mathfrak{g}^* \times \Lambda^3 \mathfrak{g}^*$, defined by

$$\omega_2 := f^{56} - f^{14} - f^{23}, \quad \rho_2 := \rho_1,$$

is a half-flat $\mathrm{SU}(1,2)$-structure on $\mathfrak{g}$ and the pair $(\omega_3, \rho_3) \in \Lambda^2 \mathfrak{g}^* \times \Lambda^3 \mathfrak{g}^*$, defined by

$$\omega_3 := f^{56} + f^{13} + f^{24}, \quad \rho_3 := f^{512} + f^{534} + f^{614} + f^{632} = f^5 \wedge \omega_0 + f^{146} - f^{236},$$

is a half-flat $\mathrm{SL}(3,\mathbb{R})$-structure on $\mathfrak{g}$. The list of the six-dimensional Lie algebras $\mathfrak{g}$ fulfilling the equivalent conditions (i) and (ii) is obtained from Theorem 4.15. ☐

Remark 6.8. *Schulte-Hengesbach showed in [SH] that direct sums of the form* $\mathfrak{g}_3 \oplus \mathbb{R}^3$ *with unimodular* $\mathfrak{g}_3$ *admit half-flat* $\mathrm{SL}(3,\mathbb{R})$*-structures.*

Theorem 6.9. *Let* $\mathfrak{g}_4$ *be a four-dimensional Lie algebra.*

(a) *The direct sum* $\mathfrak{g}_4 \oplus \mathbb{R}^2$ *admits a half-flat* SU(3)*-structure if and only if* $\mathfrak{g}_4 = \mathfrak{g}_3 \oplus \mathbb{R}$ *for a unimodular three-dimensional Lie algebra* $\mathfrak{g}_3$.

(b) The direct sum $\mathfrak{g}_4 \oplus \mathfrak{r}_2$ *admits a half-flat* $\mathrm{SU}(3)$*-structure if and only if*

(i) $\mathfrak{g}_4$ *is unimodular and not in* $\left\{A_{4,5}^{-\frac{1}{2},-\frac{1}{2}}, \mathfrak{h}_3 \oplus \mathbb{R}, \mathbb{R}^4\right\}$ *or*

(ii) $\mathfrak{g}_4$ *is in* $\left\{A_{4,9}^{-\frac{1}{2}}, A_{4,12}, \mathfrak{r}_2 \oplus \mathfrak{r}_2\right\}$.

Proof. The following proof does not coincide with the one given in [FS1] where in most cases the obstruction given in Proposition 6.5 has been applied to show the non-existence of half-flat SU(3)-structures. We do not apply Proposition 6.5 directly at all in our proof.

(a) By Theorem 5.18, a direct sum of the form $\mathfrak{g}_4 \oplus \mathbb{R}^3$ with a four-dimensional Lie algebra $\mathfrak{g}_4$ admits a cocalibrated G_2-structure exactly when $\mathfrak{g}_4 = \mathfrak{g}_3 \oplus \mathbb{R}$ with $\mathfrak{g}_3$ being unimodular and three-dimensional. Hence, Proposition 3.37 shows that only the direct sums $\mathfrak{g}_4 \oplus \mathbb{R}^2 = \mathfrak{g}_3 \oplus \mathbb{R}^3$, $\mathfrak{g}_4 = \mathfrak{g}_3 \oplus \mathbb{R}$, with $\mathfrak{g}_3$ being three-dimensional and unimodular may admit half-flat SU(3)-structures. The existence of half-flat SU(3)-structures on these Lie algebras is proved in [SH], c.f. also the cited Theorem 6.2.

(b) By Theorem 5.18 and Proposition 3.37, only the sums $\mathfrak{g}_4 \oplus \mathfrak{r}_2$ with $\mathfrak{g}_4 = A_{4,9}^{-\frac{1}{2}}, A_{4,12}$, $\mathfrak{r}_2 \oplus \mathfrak{r}_2$ or $\mathfrak{g}_4$ being unimodular and $\mathfrak{g}_4 \notin \left\{\mathfrak{h}_3 \oplus \mathbb{R}, \mathbb{R}^4\right\}$ may admit a half-flat SU(3)-structure. For all these Lie algebras, except $\mathfrak{g}_4 = A_{4,5}^{-\frac{1}{2},-\frac{1}{2}}$, the existence of a half-flat SU(3)-structure either follows from [SH], cf. Theorem 6.2, or a concrete example of a half-flat SU(3)-structure is given in Table 7.9. For $\mathfrak{g} := A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$, the obstruction given in Proposition 6.5 cannot be applied directly. However, a different obstruction can be established as follows. Let $\left(e^1, \ldots, e^6\right)$ be a basis of $\left(A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2\right)^* \cong \left(A_{4,5}^{-\frac{1}{2},-\frac{1}{2}}\right)^* \oplus \mathfrak{r}_2^*$ such that $\left(e^1, \ldots, e^4\right)$ is the standard basis of $\left(A_{4,5}^{-\frac{1}{2},-\frac{1}{2}}\right)^*$ given in Table 7.3 and $\left(e^5, e^6\right)$ is a basis of $\mathfrak{r}_2^*$ which fulfils $de^5 = 0$, $de^6 = e^{56}$. We set $\nu := e^{123456} \in \Lambda^6\mathfrak{g}^*$ and define $\tilde{J}_\rho \in \mathrm{End}(\mathfrak{g})$ for a three-form $\rho \in \Lambda^3\mathfrak{g}^*$ by Equation (6.2). Then a straightforward calculation yields the identity

$$e^5 \wedge \tilde{J}_\rho^* e^4 \wedge \sigma = -e^4 \wedge \tilde{J}_\rho^* e^5 \wedge \sigma = (e^4 + \sqrt{2}e^5) \wedge \tilde{J}_\rho^*(e^4 + \sqrt{2}e^5) \wedge \sigma$$

for all closed three-forms ρ on $\mathfrak{g}$ and all closed four-forms σ on $\mathfrak{g}$. Suppose that $\mathfrak{g}$ admits a half-flat SU(3)-structure (ρ_0, ω_0). In particular, the forms ρ_0 and $\sigma_0 := \frac{1}{2}\omega_0^2$ are closed and fulfil the previous identity. Hence, if g_0 denotes the induced Euclidean metric, Equation (2.17) and the fact that $\tilde{J}_{\rho_0}$ is a non-zero multiple of J_{ρ_0} show

$$g_0\left(e^5, e^4\right) = -g_0\left(e^4, e^5\right) = g_0\left(e^4 + \sqrt{2}e^5, e^4 + \sqrt{2}e^5\right).$$

Since g_0 is symmetric, this implies that $e^4 + \sqrt{2}e^5$ is a null-vector, a contradiction. Hence, there cannot exist a half-flat SU(3)-structure on $\mathfrak{g} = A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$.

□

Remark 6.10. *We like to note an interesting consequence of Theorem 6.9. It provides, to the best of the author's knowledge, the first example in the literature of a six-dimensional Lie algebra* $\mathfrak{g}$ *which does not admit a half-flat* $\mathrm{SU}(3)$*-structure but for which* $\mathfrak{g} \oplus \mathbb{R}$ *admits a cocalibrated* G_2*-structure. Namely,* $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ *admits a cocalibrated* G_2*-structure due to Theorem 5.18 but Theorem 6.9 shows that* $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ *does not admit a half-flat* $\mathrm{SU}(3)$*-structure. Note that this shows that* $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2 \oplus \mathbb{R}$ *cannot admit a cocalibrated* G_2*-structure such that* $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ *and* $\mathbb{R}$ *are orthogonal by Proposition 3.37. Note further that Theorem 6.9 and Theorem 5.18 show that* $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ *is the only such example in the class of direct sums* $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_2$ *of a four-dimensional Lie algebra* $\mathfrak{g}_4$ *and of a two-dimensional Lie algebra* $\mathfrak{g}_2$*.*

The missing decomposable cases are those which are direct sums $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}$ of an indecomposable five-dimensional Lie algebra $\mathfrak{g}_5$, which is neither almost Abelian nor nilpotent, and of $\mathbb{R}$. For this class of Lie algebras, we obtain

Theorem 6.11. *Let* $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}$ *be a direct sum of a five-dimensional indecomposable Lie algebra, which is neither almost Abelian nor nilpotent, and of* $\mathbb{R}$*. Then* $\mathfrak{g}$ *admits a half-flat* $\mathrm{SU}(3)$*-structure if and only if*

$$\mathfrak{g}_5 \in \left\{ A_{5,19}^{-1,2},\ A_{5,19}^{-1,3},\ A_{5,19}^{2,-3},\ A_{5,30}^{0},\ A_{5,33}^{-1,-1},\ A_{5,35}^{0,-2},\ A_{5,36},\ A_{5,37},\ A_{5,40} \right\}$$

Proof. For all direct sums admitting a half-flat $\mathrm{SU}(3)$-structure, an explicit example can be found in Table 7.10. For the remaining direct sums $\mathfrak{g}_5 \oplus \mathbb{R}$, we apply Proposition 6.5 for all cases separately according to Table 7.4. Therefore, let $(e^1, \ldots, e^6)$ be a basis of $(\mathfrak{g}_5 \oplus \mathbb{R})^* = \mathfrak{g}_5^* \oplus \mathbb{R}^*$ such that $(e^1, \ldots, e^5)$ is the standard basis of $\mathfrak{g}^*$ given in Table 7.4 and e^6 spans $\mathbb{R}^*$. We claim that $\alpha = e^5$ is for all cases a one-form satisfying the obstruction condition (6.1). In fact, the equation can be efficiently verified by the computer algebra system *Maple* as follows. Let ρ be a three-form and σ a four-form involving altogether 35 coefficients when expressed with respect to the induced basis on forms. Due to our distinction of the Lie algebra classes in Table 7.4, the coefficient equations of $d\rho = d\sigma = 0$ can be solved in a closed form, independently of the parameters in the Lie bracket. Thus, the computer can almost instantaneously provide us with explicit expressions for the general closed three-form $\rho \in Z^3(\mathfrak{g})$ and also for the general closed four-form $\sigma \in Z^4(\mathfrak{g})$ by eliminating a number of parameters. Now, it is straightforward to compute $\tilde{J}_\rho$ via (6.2) with respect to the basis. The result allows us to verify Equation (6.1) for $\alpha = e^5$ and all $\rho \in Z^3(\mathfrak{g})$ and all $\sigma \in Z^4(g)$ for each of the remaining Lie algebras. □

Remark 6.12. *In [CS] and [CFS] the five-dimensional solvable Lie algebras* $\mathfrak{g}$ *admitting a* hypo $\mathrm{SU}(2)$-structure *are classified. There is a similar relation between hypo* $\mathrm{SU}(2)$*-structures on* $\mathfrak{g}$ *and half-flat* $\mathrm{SU}(3)$*-structures on* $\mathfrak{g} \oplus \mathbb{R}$ *as the one in Proposition 3.37* *between half-flat* $\mathrm{SU}(3)$*-structures and cocalibrated* G_2*-structures. Namely,* $\mathfrak{g}$ *admits a hypo* $\mathrm{SU}(2)$*-structure if and only if* $\mathfrak{g} \oplus \mathbb{R}$ *admits a half-flat* $\mathrm{SU}(3)$*-structure with orthogonal splitting* $\mathfrak{g} \oplus \mathbb{R}$*. For all five-dimensional solvable non-nilpotent Lie algebras* $\mathfrak{g}$ *admitting a hypo* $\mathrm{SU}(2)$*-structure we (or Schulte-Hengesbach in [SH]) independently found a half-flat* $\mathrm{SU}(3)$*-structure on* $\mathfrak{g} \oplus \mathbb{R}$*. However, for two indecomposable five-dimensional Lie algebras which do not admit a hypo* $\mathrm{SU}(2)$*-structure, namely* $A_{5,19}^{-1,3}$ *and* $A_{5,37}$*, we were able to find a half-flat* $\mathrm{SU}(3)$*-structure on the corresponding six-dimensional Lie algebras* $A_{5,19}^{-1,3} \oplus \mathbb{R}$ *and* $A_{5,37} \oplus \mathbb{R}$ *such that the summands are not orthogonal, cf. Table 7.10.*

Next we consider the non-solvable case. We show that all non-solvable indecomposable six-dimensional Lie algebras admit a half-flat SU(3)-structure. By our previous results, we even get the following result.

Theorem 6.13. *Let* $\mathfrak{g}$ *be a six-dimensional non-solvable Lie algebra. Then* $\mathfrak{g}$ *admits a half-flat* $\mathrm{SU}(3)$*-structure.*

Proof. By [Tu1], the indecomposable non-solvable Lie algebras of dimension less than six are $\mathfrak{so}(3)$, $\mathfrak{so}(2,1)$ and $A_{5,40}$. Direct sums with $\mathfrak{so}(3)$ and $\mathfrak{so}(2,1)$ admit a half-flat SU(3)-structure by Theorem 6.2, whereas the direct sum $\mathfrak{g}_{5,40} \oplus \mathbb{R}$ admits a half-flat SU(3)-structure by Theorem 6.11. The indecomposable six-dimensional non-solvable Lie algebras have also been determined in [Tu1] and are given in Table 7.5. They all admit a half-flat SU(3)-structure, where a concrete example in each case is given in Table 7.11. □

Finally, we attack the class of indecomposable solvable non-nilpotent six-dimensional Lie algebras. By a result of Mubarakzyanov [Mu6d], the nilradical of such a Lie algebra has either dimension five or four. Due to the complexity of the problem, see the classification of the mentioned Lie algebras with five-dimensional nilradical given in [Mu6d] (resp. with four-dimensional nilradical given in [Tu2]), we restrict ourselves to the case with five-dimensional nilradical and leave the other case open. Since the indecomposable solvable Lie algebras with five-dimensional nilradical admitting a half-flat SU(3)-structure have hardly anything in common, a simple characterisation seems not possible and we have to state our classification result in the following form.

Theorem 6.14. *An indecomposable solvable six-dimensional Lie algebra with five-dimensional nilradical admits a half-flat* $\mathrm{SU}(3)$*-structure if and only if it is contained in Table 7.12.*

Proof. First of all, Theorem 6.7 yields that all almost Abelian Lie algebras in our class, i.e. those with Abelian nilradical, do not admit a half-flat SU(3)-structure. As Table 7.7 contains all indecomposable Lie algebras with non-Abelian five-dimensional nilradical according to the classification of Mubarakzyanov [Mu6d] and Shabanskaya [Sha], it suffices to prove existence or non-existence in each case contained in the list. The existence problem is completely solved by the explicit examples given in Table 7.12. In the following, we prove the non-existence for the remaining Lie algebras.

For all these remaining Lie algebras, except $A_{6,39}^{-1,-1}$, $A_{6,41}^{-1}$ $A_{6,76}^{-1}$, $A_{6,78}$, $B_{6,3}^{0}$, $B_{6,4}^{1}$ and $B_{6,4}^{-1}$, we apply Proposition 6.5. In each case, we work in the basis $(e^1, \ldots, e^6)$ of $\mathfrak{g}^*$ given in Table 7.7. Analogously to the proof of Theorem 6.11, we show that $\alpha = e^6$ is a one-form fulfilling Equation (6.1) for all $\rho \in Z^3(\mathfrak{g})$ and all $\sigma \in Z^4(\mathfrak{g})$. That means, we start with a pair $(\rho, \sigma) \in \Lambda^3\mathfrak{g}^* \times \Lambda^4\mathfrak{g}^*$ of a three-form ρ and a four-form σ expressed with respect to the induced basis on forms using 35 coefficients in total. The classes in Table 7.7 are separated such that the space of closed forms has a fixed form. Thus, the general solution of the equations $d\rho = 0$ and $d\sigma = 0$ can be obtained by eliminating a certain amount of coefficients for each class. The computation of $\tilde{J}_\rho$ with respect to the given basis by Equation (6.2) allows us to verify equation Equation (6.1) for $\alpha = e^6$ and all $(\rho, \sigma) \in Z^3(\mathfrak{g}) \times Z^4(\mathfrak{g})$. All calculations can be executed conveniently in the computer algebra system *Maple*.

Unfortunately, Proposition 6.5 cannot be applied to the Lie algebras $A_{6,39}^{-1,-1}$, $A_{6,41}^{-1}$ $A_{6,76}^{-1}$, $A_{6,78}$, $B_{6,3}^{0}$, $B_{6,4}^{1}$ and $B_{6,4}^{-1}$. The following proof uses Proposition 6.6. Again, we compute the general closed three-form $\rho \in Z^3(\mathfrak{g})$ and the general closed four-form $\sigma \in Z^4(\mathfrak{g})$ with respect to the basis $(e^1, \ldots, e^6)$ given in Table 7.7. We choose $e^7 \in (\mathfrak{g} \oplus \mathbb{R})^*$ with $de^7 = 0$ such that $(e^1, \ldots, e^7)$ is a basis of $(\mathfrak{g} \oplus \mathbb{R})^* \cong \mathfrak{g}^* \oplus \mathbb{R}^*$ and compute $\Omega(\rho, \sigma) := \rho \wedge e^7 + \sigma \in \Lambda^4(\mathfrak{g} \oplus \mathbb{R})^*$. Afterwards, we compute for each of the seven Lie algebras the three-form $\tilde{\rho}(\rho, \sigma) = e_3 \lrcorner \Omega(\rho, \sigma) \in \Lambda^3 {e_3}^0$. When we compute $\lambda(\tilde{\rho}(\rho, \sigma))$, it turns out that it is in each case the square of a polynomial in the coefficients of the general closed three-form $\rho \in Z^3(\mathfrak{g})$ and of the general closed four-form $\sigma \in Z^4(\mathfrak{g})$ and so always non-negative. Thus, none of the seven Lie algebras admits a half-flat SU(3)-structure according to Proposition 6.6. □

Remark 6.15. *We like to remark that almost all the examples of half-flat* SU(3)*-structures on six-dimensional Lie algebras* $\mathfrak{g}$ *have been constructed case-by-case using Proposition 3.37. That means we first constructed a cocalibrated* G_2*-structure with orthogonal splitting on* $\mathfrak{g} \oplus \mathbb{R}$ *and then got an induced half-flat* SU(3)*-structure on* $\mathfrak{g}$*. The construction used the fact that the Chevalley-Eilenberg differential is, in most of the cases, particularly simple since* $\mathfrak{g} \oplus \mathbb{R}$ *is* almost nilpotent*, i.e.* $\mathfrak{g} \oplus \mathbb{R}$ *admits a nilpotent codimension one ideal.*

6.4 Other Results

The first aim of this section is to give some results on the non-existence of closed stable three-forms of certain kind on decomposable Lie algebras. In Proposition 6.4, we cited Schulte-Hengesbach's result [SH] on the non-existence of closed stable forms ρ with $\lambda(\rho) < 0$

on some direct sums of two three-dimensional Lie algebras. Moreover, we argued directly above Proposition 6.4 that there always is a closed stable form with $\lambda(\rho) > 0$ on the direct sum of two three-dimensional Lie algebras. We consider the same problem for the missing decomposable Lie algebras. In contrast to Schulte-Hengesbach's results, Proposition 6.16 shows that there are decomposable Lie algebras with $\lambda(\rho) = 0$ for all closed three-forms, i.e. there are decomposable Lie algebras which do not admit at all a closed stable three-form.

Proposition 6.16. *Let $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_2$ be a six-dimensional Lie algebra which is the direct sum of an indecomposable four-dimensional Lie algebra $\mathfrak{g}_4$ and of a two-dimensional Lie algebra $\mathfrak{g}_2$.*

(i) If $\mathfrak{g}_2 = \mathbb{R}^2$ and $\mathfrak{g}_4$ not in $\left\{A_{4,1}, A_{4,5}^{-1,1}, A_{4,9}^{-\frac{1}{2}}, A_{4,12}\right\}$, then $\lambda(\rho) \geq 0$ for all $\rho \in Z^3(\mathfrak{g})$ and $\mathfrak{g}$ does not admit a half-flat $\mathrm{SU}(1,2)$*-structure.*

(ii) If $\mathfrak{g}_2 = \mathfrak{r}_2$, the nilradical of $\mathfrak{g}_4$ is isomorphic to $\mathbb{R}^3$ and $h^*(\mathfrak{g}_4) = (1,0,0,0)$, then $\lambda(\rho) \geq 0$ for all $\rho \in Z^3(\mathfrak{g})$ and $\mathfrak{g}$ does not admit a half-flat $\mathrm{SU}(1,2)$*-structure.*

(iii) If $\mathfrak{g}_2 = \mathbb{R}^2$ and $h^*(\mathfrak{g}_4) = (1,0,0,0)$, then $\lambda(\rho) = 0$ for all $\rho \in Z^3(\mathfrak{g})$. Then $\mathfrak{g}$ does not admit at all a closed stable three-form and a half-flat structure of any kind.

Proof. The Lie algebras $\mathfrak{g}_4$ appearing in the statement may be identified when looking at Table 7.3. In the proof of Theorem 6.11, we explained that the general closed three-form ρ on each of the direct sums $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_2$ appearing in Proposition 6.16 is determined straightforwardly with computer support. Using *Maple* to calculate the quartic invariant $\lambda(\rho)$ of the general closed three-form ρ on each of the considered Lie algebras with the help of *Maple*, those with $\lambda = 0$ are easily identified. The cases with $\lambda \geq 0$ have been determined by applying the useful *Maple* function *factor* to $\lambda(\rho)$. The non-existence statements about closed stable forms of any kind and about half-flat structures of certain kind follow from Proposition 2.22 and Proposition 3.38, respectively. □

Analogously, we can prove the following proposition.

Proposition 6.17. *Let $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}$ be a six-dimensional Lie algebra which is a direct sum of an indecomposable five-dimensional Lie algebra $\mathfrak{g}_5$ and $\mathbb{R}$.*

(i) If the column $\lambda \geq 0$ in Table 7.4 is checked for $\mathfrak{g}_5$, then $\lambda(\rho) \geq 0$ for all $\rho \in Z^3(\mathfrak{g})$ and $\mathfrak{g}$ does not admit a half-flat $\mathrm{SU}(1,2)$*-structure.*

(ii) If the nilradical of $\mathfrak{g}_5$ is isomorphic to $\mathbb{R}^4$ and $h^3(\mathfrak{g}_5) = 0$, then $\lambda(\rho) = 0$ for all $\rho \in Z^3(\mathfrak{g})$. Then $\mathfrak{g}$ does not admit at all a closed stable three-form and a half-flat structure of any kind.

Remark 6.18. *Unfortunately, there seems to be no consistent pattern for the direct sums* $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}$ *of an indecomposable five-dimensional Lie algebra* $\mathfrak{g}_5$ *and* $\mathbb{R}$ *such that* $\lambda(\rho) \geq 0$ *for all* $\rho \in \Lambda^3\mathfrak{g}^*$ *except that the nilradical has to be either* $\mathbb{R}^4$ *or* $\mathfrak{h}_3 \oplus \mathbb{R}$*. Note that also the patterns we observed in the other cases may be just a simple coincidence.*

We finish this chapter by pointing out an interesting application of Table 7.7 to the classification of six-dimensional Lie algebras which are $(k-1, k)$-*trivial*, i.e. whose $(k-1)$-th and k-th Lie algebra cohomology vanishes. These Lie algebras play an analogous role for the study of multi-moment maps associated to closed geometries of degree k as semisimple Lie algebras do for the study of moment maps in symplectic geometry, see [MaSw1], [MaSw2] and [MaSw3]. For general k, one can read off the Tables 7.1 - 7.7 given in the appendix all $(k-1, k)$-trivial Lie algebras in the corresponding class. The most interesting case is the one of $(2, 3)$-trivial Lie algebras. A classification of $(2, 3)$-trivial Lie algebras up to dimension five has been established by Madsen and Swann in [MaSw1]. Using Table 7.7, one can get a full classification of $(2, 3)$-trivial Lie algebras in dimension six using the following theorem proved in [MaSw2].

Theorem 6.19 (Madsen, Swann)**.** *A Lie algebra* $\mathfrak{g}$ *is* $(2, 3)$*-trivial if and only if* $\mathfrak{g}$ *is solvable, the derived Lie algebra* $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ *is nilpotent of codimension one in* $\mathfrak{g}$ *and* $H^i(\mathfrak{n})^{\mathfrak{g}} = \{0\}$ *for* $i = 1, 2, 3$.

In particular, $(2, 3)$-trivial Lie algebras are indecomposable and they are either almost Abelian or can be found in Table 7.7. Thus, we obtain

Corollary 6.20. *A six-dimensional Lie algebra* $\mathfrak{g}$ *is* $(2, 3)$*-trivial if and only if it is one of the Lie algebras in Table 7.7 with* $h^2(\mathfrak{g}) = h^3(\mathfrak{g}) = 0$ *or if the nilradical* $\mathfrak{n}$ *of* $\mathfrak{g}$ *is isomorphic to* $\mathbb{R}^5$ *and the induced endomorphism* $\mathrm{ad}(v)\,|_{\Lambda^i\mathfrak{n}^*}$ *for an arbitrary* $v \in \mathfrak{g}\backslash\mathfrak{n}$ *has trivial kernel for* $i = 1, 2, 3$.

Chapter 7

Hitchin flow on almost Abelian Lie algebras

In this chapter, we look at Hitchin's flow equations whose solutions define pseudo-Riemannian metrics with holonomy contained in the exceptional holonomy groups G_2^ϵ and $\mathrm{Spin}^\epsilon(7)$. The starting value of this flow is a half-flat structure $\mathrm{SU}^\delta(p, 3-p)$-structure or a cocalibrated G_2^ϵ-structure, respectively. We present some results on the Hitchin flow for G_2-structures on almost Abelian Lie algebras. Most importantly, we show that in this case the holonomy of the Riemannian manifold obtained by the Hitchin flow always reduces further to a subgroup of SU(4). Moreover, we compute the Hitchin flow explicitly for certain initial cocalibrated G_2-structures on $\mathfrak{h}_3 \oplus \mathbb{R}^4$ and $\mathfrak{n}_{7,1}$. In the latter case, we obtain an explicit two-parameter family of non-compact, non-complete Calabi-Yau four-folds of cohomogeneity one. Note that these results are the first results of an ongoing investigation of the Hitchin flow on seven-dimensional Lie algebras and so many interesting questions remain unanswered in this chapter.

We start in Section 7.1 by giving a short review of the Hitchin flow on arbitrary six- or seven-dimensional manifolds. In Section 7.2 we look at the Hitchin flow for G_2-structures on real seven-dimensional almost Abelian Lie algebras and prove the mentioned reduction result of the holonomy of the induced eight-dimensional Riemannian manifold to a subgroup of SU(4). The *moduli space* of cocalibrated G_2-structures on the nilpotent almost Abelian Lie algebras $\mathfrak{h}_3 \oplus \mathbb{R}^4$ and $\mathfrak{n}_{7,1}$, i.e. all cocalibrated G_2-structures on these Lie algebras up to Lie algebra automorphisms and scalings, are determined in Section 7.3. Finally, the Hitchin flow and the holonomy of the induced Riemannian metrics for the entire moduli space on $\mathfrak{h}_3 \oplus \mathbb{R}^4$ and for a two-parameter family in the moduli space on $\mathfrak{n}_{7,1}$ is computed in Section 7.4.

7.1 Hitchin's flow equations

Manifolds admitting a parallel G_2^ϵ-structure (resp. a parallel $\mathrm{Spin}^\epsilon(7)$-structure) and so having holonomy contained in the exceptional holonomy group G_2^ϵ (resp. in the exceptional holonomy group $\mathrm{Spin}^\epsilon(7)$) naturally induce on hypersurfaces with the right signature half-flat $\mathrm{SU}^\delta(p,3-p)$-structures (resp. cocalibrated G_2^ϵ-structures) using pointwise the constructions given in Proposition 2.51 (resp. Proposition 2.60). The proof is easy and given below. Note that in [MC5] the intrinsic torsion of the induced SU(3)-structure is computed in terms of the intrinsic torsion of an arbitrary G_2-structure. Similar computations are done for Spin(7)-structures in [MC3].

Proposition 7.1. *(a) If $(p,\delta,\epsilon)\in\{(1,-1,1),(3,1,1),(3,-1,-1)\}$, $\varphi\in\Omega^3M$ is a parallel G_2^ϵ-structure on a seven-dimensional manifold M and N is an oriented hypersurface in M such that there exists a unit normal vector field $n\in\mathfrak{X}(M)$ with $g_\varphi(n,n)=-\delta$, then the pair $(\omega,\rho)\in\Omega^2N\times\Omega^3N$, defined by $\omega:=i^*(n\lrcorner\varphi)$ and $\rho:=i^*\varphi$, is a half-flat $\mathrm{SU}^\delta(p,3-p)$-structure on N. Here, $i:N\to M$ is the inclusion map.*

(b) If $\Phi\in\Omega^4M$ is a parallel $\mathrm{Spin}^\epsilon(7)$-structure on an eight-dimensional manifold M and N is an oriented hypersurface in M with a space-like unit normal vector field $n\in\mathfrak{X}(M)$, then $\varphi:=i^*(n\lrcorner\Phi)\in\Omega^3N$ is a cocalibrated G_2^ϵ-structure on N. Here, again $i:N\to M$ is the inclusion map.

Proof. (a) By Proposition 2.51, $(\omega,\rho)\in\Omega^2N\times\Omega^3N$ defined as in the statement is, in fact, an $\mathrm{SU}^\delta(p,3-p)$-structure. Moreover, Proposition 2.51 shows that $i^*(\star_\varphi\varphi)=-\frac{\delta}{2}\omega^2$. Hence, the closure of φ and $\star_\varphi\varphi$ imply the closure of $\rho=i^*\varphi$ and $\frac{1}{2}\omega^2$. Thus, (ω,ρ) is half-flat.

(b) Proposition 2.60 shows that $\varphi\in\Omega^3N$ defined as in the assertion is, in fact, a G_2^ϵ-structure and we get $i^*\Phi=\star_\varphi\varphi$. Hence, the closure of Φ implies the closure of $\star_\varphi\varphi=0$ and so φ is cocalibrated.

□

Conversely to Proposition 7.1, the Hitchin flow embeds a six-dimensional manifold with a half-flat $\mathrm{SU}^\delta(p,3-p)$-structure (resp. a seven-dimensional manifold with a cocalibrated G_2^ϵ-structure) into a seven-dimensional manifold with a parallel G_2^ϵ-structure (resp. into an eight-dimensional manifold with a parallel $\mathrm{Spin}^\epsilon(7)$-structure).

The Hitchin flow has been introduced by Hitchin [Hi1] on compact six-dimensional manifolds M and compact seven-dimensional manifolds N. In the six-dimensional case, it is a Hamilton flow on the product of the cohomology classes $\left[\frac{1}{2}\omega_0^2\right]\times[\rho_0]$, where $(\omega_0,\rho_0)\in\Omega^2M\times\Omega^3M$ is a half-flat SU(3)-structure on M. Here, one uses a natural symplectic two-form on the affine space $\left[\frac{1}{2}\omega_0^2\right] \times [\rho_0]$. The Hamilton function is constructed via a $\mathbb{Z}$-combination of the functionals on stable four-forms $\sigma \in \Omega^4 M$ and stable three-forms $\rho \in \Omega^3 M$ one gets by integrating the associated volume forms $\phi(\sigma)$, $\phi(\rho)$, cf. Proposition 1.37, over the entire manifold M. The solution with initial value (ω_0, ρ_0) on an interval I defines then a parallel G_2-structure on $M \times I$ and so a Riemannian metric with holonomy contained in G_2 on $M \times I$. In the seven-dimensional case, it is a gradient flow on the cohomology class $[\star_{\varphi_0}\varphi_0]$ of the Hodge dual of a cocalibrated G_2-structure $\varphi_0 \in \Omega^3 N$ on N. Here, one uses a natural non-degenerate symmetric bilinear form on $[\star_{\varphi_0}\varphi_0]$ and the functional on the stable four-forms $\Psi \in \Omega^4 N$ obtained by integrating the associated volume form $\phi(\Psi) \in \Omega^7 M$ over the entire manifold N. Analogously to the six-dimensional case, the solution with initial value φ_0 on an interval I defines then a parallel Spin(7)-structure on $N \times I$ and so a Riemannian metric with holonomy contained in Spin(7) on $N \times I$.

In [CLSS], the results of Hitchin are reproved by direct calculations without using any Hamilton or gradient flows. In particular, the compactness assumption can be dropped and the results generalise also to non-compact manifolds in a suitable way. Moreover, [CLSS] also generalises the Hitchin flow to half-flat $\mathrm{SU}(1,2)$- and $\mathrm{SL}(3,\mathbb{R})$-structures and to cocalibrated G_2^*-structures leading to pseudo-Riemannian manifolds with holonomy contained in the exceptional holonomy groups G_2^* or $\mathrm{Spin}_0(3,4)$-structures, respectively. Furthermore, [CLSS] gives an existence and uniqueness result in the real analytic category. Note that [Br6] shows that such a result is not valid in the smooth category. In the remainder of this section, we review all these results briefly and refer for proofs to the two mentioned papers [Hi1] and [CLSS]. We begin with the six-dimensional case.

Theorem 7.2. *Let $(p,\delta,\epsilon) \in \{(1,-1,1),(3,1,1),(3,-1,-1)\}$ and $(\omega_0,\rho_0) \in \Omega^2 M \times \Omega^3 M$ be a half-flat $\mathrm{SU}^\delta(p,3-p)$-structure on a six-dimensional manifold M. Assume that there exists a smooth 1-parameter family $I \to \Omega^2 M \times \Omega^3 M$, $t \mapsto (\omega(t),\rho(t))$ of stable forms, I being an open interval around 0, such that the identity $(\omega(0),\rho(0)) = (\omega_0,\rho_0)$ is true and such that (ω,ρ) fulfils the following partial differential equations on I, called* Hitchin's flow equations*:*

$$\dot{\rho} = d\omega, \qquad \frac{d}{dt}\left(\frac{1}{2}\omega^2\right) = dJ_\rho^*\rho. \tag{7.1}$$

Then $(\omega(t),\rho(t))$ is for all $t \in I$ a half-flat $\mathrm{SU}^\delta(p,3-p)$-structure and the three-form

$$\varphi := \omega \wedge dt + \rho \tag{7.2}$$

is a parallel G_2^ϵ-structure on $M \times I$. The induced pseudo-Riemannian metric g_φ on $M \times I$ with holonomy contained in G_2^ϵ is given by

$$g_\varphi = g_{(\omega(t),\rho(t))} - \delta\, dt^2. \tag{7.3}$$

Remark 7.3.

- *If $(\omega(t), \rho(t))$ is a solution of Hitchin's flow equations, then the orientation induced by $\omega(t)$ is the same for all $t \in I$. Hence, if we fix an orientation on M, we may recover $\omega(t)$ uniquely from $\frac{1}{2}\omega(t)^2$ by Remark 2.7. Thus, we may consider Hitchin's flow equations also as equations for a one-parameter family $(\sigma(t), \rho(t)) \in \Omega^4 M \times \Omega^3 M$ of stable four-forms $\sigma(t)$ and stable three-forms $\rho(t)$ on an oriented six-dimensional manifold M such that $(\sigma(0), \rho(0)) = \left(\frac{1}{2}\omega_0^2, \rho_0\right)$ for a half-flat $\mathrm{SU}^\delta(p, 3-p)$-structure (ω_0, ρ_0) on M inducing the given orientation.*

- *Since for all $p \in M$ the curve $I \ni t \mapsto (p, t)$ is a geodesic, $(M \times I, g)$ can only be complete if $I = \mathbb{R}$. In the Riemannian case, the Cheeger-Gromoll splitting theorem, cf. [ChGr], shows then that $(M \times I, g) \cong (M, g_{(\omega_0, \rho_0)}) \times (\mathbb{R}, dt^2)$ as Riemannian manifolds and we cannot have full holonomy G_2.*

- *The proof of Theorem 7.2 given in [CLSS] shows even more. Namely, if M is a seven-dimensional manifold with parallel G_2^ϵ-structure φ and $(M, g_\varphi) \cong (N \times I, h(t) - \delta\, dt^2)$ for a smooth one-parameter-family $I \ni t \mapsto h(t)$ of pseudo-Riemannian metrics on N, then the one-parameter family of $\mathrm{SU}^\delta(p, 3-p)$-structures induced on N fulfils Hitchin's flow equation.*

As already mentioned at the beginning of this section, on real analytic manifolds a solution of Hitchin's flow equations exists and is unique.

Theorem 7.4. *Let M be a real-analytic six-dimensional manifold, $(p, \delta, \epsilon) \in \{(1, -1, 1), (3, 1, 1), (3, -1, -1)\}$ and (ω_0, ρ_0) be a real-analytic half-flat $\mathrm{SU}^\delta(p, 3-p)$-structure on M.*

(a) There exists a unique maximal solution of Hitchin's flow equations with initial value (ω_0, ρ_0) which is defined on an open neighbourhood U of $M \times \{0\}$ in $M \times \mathbb{R}$. Hence, there is a parallel G_2^ϵ-structure φ on U and so induced the pseudo-Riemannian metric g_φ on U has holonomy contained in G_2^ϵ.

(b) Let f be a diffeomorphism of M and $\mu \in \mathbb{R}^*$. If (ω, ρ) is a solution of Hitchin's flow equations with initial value (ω_0, ρ_0), then $(f^*\omega, f^*\rho)$ is a solution of Hitchin's flow equations with initial value $(f^*\omega_0, f^*\rho_0)$ and $\left(\frac{1}{\mu^2}\omega(\mu t), \frac{1}{\mu^3}\rho(\mu t)\right)$ is a solution of Hitchin's flow equations with initial value $\left(\frac{1}{\mu^2}\omega_0, \frac{1}{\mu^3}\rho_0\right)$.

(c) If M is compact or a homogeneous space, then U as in (a) is of the form $U = M \times I$ for some open interval I around 0.

In seven dimensions the result is as follows.

Theorem 7.5. *Let $\epsilon \in \{-1, 1\}$ and $\varphi \in \Omega^3 M$ be a cocalibrated G_2^ϵ-structure on a seven-dimensional manifold M. Assume that there exists an open interval I around 0 and a* *smooth* 1*-parameter family* $I \to \Omega^3 M$, $t \mapsto \varphi(t)$, *of stable three-forms with* $\varphi(0) = \varphi_0$ *and such that* φ *fulfils the following partial differential equations on* I, *called* Hitchin's flow equations*:*

$$\frac{d}{dt} \star_\varphi \varphi = d\varphi. \tag{7.4}$$

Then $\varphi(t)$ *is a cocalibrated* G_2^ϵ*-structure for all* $t \in I$ *and the four-form*

$$\Phi := \star_\varphi \varphi + dt \wedge \varphi \tag{7.5}$$

is a parallel $\mathrm{Spin}^\epsilon(7)$*-structure on* $M \times I$*. The induced pseudo-Riemannian metric* g_Φ *on* $M \times I$ *with holonomy contained in* $\mathrm{Spin}^\epsilon(7)$ *is given by*

$$g_\Phi = g_{\varphi(t)} + dt^2. \tag{7.6}$$

Remark 7.6. *The analogous statements of Remark 7.3 are also true for the Hitchin flow for* G_2^ϵ*-structures. That means, the Hitchin flow preserves again the orientation on* M *and we may alternatively see Hitchin's flow equations on an oriented manifold* M *as equations for a one-parameter family of stable four-forms* $\Psi(t)$ *with* $\Psi(0) = \star_{\varphi_0}\varphi_0$*, where* $\varphi_0 \in \Omega^3 M$ *is a cocalibrated* G_2^ϵ*-structure on* M *which induces the given orientation. Furthermore, if* $(M \times I, g)$ *is complete, then we must have* $I = \mathbb{R}$ *and in the Riemannian case the Cheeger-Gromoll Splitting Theorem [ChGr] again implies then* $(M \times \mathbb{R}, g) \cong (M, g_{\varphi_0}) \times (\mathbb{R}, dt^2)$ *as Riemannian manifolds and so we cannot get full holonomy* $\mathrm{Spin}(7)$*. Note that in [Sto, Theorem 3.3], Stock gives an argument that* $I = \mathbb{R}$ *and* M *compact implies the triviality of the Hitchin flow and so that* φ_0 *is parallel. Finally, if* M *is an eight-dimensional manifold* M *with parallel* $\mathrm{Spin}^\epsilon(7)$*-structure* Φ *such that* (M, g_Φ) *is isometric to* $(N \times I, h(t) + dt^2)$ *for a seven-dimensional manifold* N *and a one-parameter-family* $I \ni t \mapsto h(t)$ *of pseudo-Riemannian metrics on* N*, then the induced one-parameter family of* G_2^ϵ*-structures on* N *fulfils Hitchin's flow equations.*

Again we have the following results in the real analytic category.

Theorem 7.7. *Let* M *be a real-analytic seven-dimensional manifold,* $\epsilon \in \{-1, 1\}$ *and* φ_0 *be a real-analytic cocalibrated* G_2^ϵ*-structure on* M*.*

(a) There exists a unique maximal solution of Hitchin's flow equations with initial value φ_0 *which is defined on an open neighbourhood* U *of* $M \times \{0\}$ *in* $M \times \mathbb{R}$*. Hence, there is a parallel* $\mathrm{Spin}^\epsilon(7)$*-structure* Φ *on* U *and so the pseudo-Riemannian metric* g_Φ *on* U *has holonomy contained in* $\mathrm{Spin}^\epsilon(7)$*.*

(b) Let f *be a diffeomorphism of* M *and* $\mu \in \mathbb{R}^*$*. If* φ *is a solution of Hitchin's flow equations with initial value* φ_0*, then* $f^*\varphi$ *is a solution of Hitchin's flow equations with initial value* $f^*\varphi_0$ *and* $\frac{1}{\mu^3}\varphi(\mu t)$ *is a solution of Hitchin's flow equations with initial value* $\frac{1}{\mu^3}\varphi_0$*.*

(c) *If M is compact or a homogeneous space, then U as in (a) is of the form $U = M \times I$ for some open interval I around* 0.

Remark 7.8. *Both in dimension six and seven, the Hitchin flow on a Lie group* G *preserves left-invariance if we start with a left-invariant initial value. Hence, Hitchin's flow equations are a system of ordinary differential equations on the associated Lie algebra* $\mathfrak{g}$. *Analogously to Theorem 7.4 (b) (resp. to Theorem 7.7 (b)), one gets the following property. If f is any Lie algebra automorphism of $\mathfrak{g}$, $\mu \in \mathbb{R}^*$ and $(\omega(t), \rho(t)) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ a solution of Hitchin's flow equations in six dimensions with initial value $(\omega_0, \rho_0) \in \Lambda^2\mathfrak{g}^* \times \Lambda^3\mathfrak{g}^*$ (resp. $\varphi(t) \in \Lambda^3\mathfrak{g}^*$ a solution of Hitchin's flow equations in seven dimensions with initial value $\varphi_0 \in \Lambda^3\mathfrak{g}^*$), then $\left(\frac{1}{\mu^2} f^*\omega(\mu t), \frac{1}{\mu^3} f^*\rho(\mu t)\right)$ is a solution of Hitchin's flow equations in six dimensions with initial value $\left(\frac{1}{\mu^2} f^*\omega_0, \frac{1}{\mu^3} f^*\rho_0\right)$ (resp. $\frac{1}{\mu^3} f^*\varphi(\mu t)$ is a solution of Hitchin's flow equations in seven dimensions with initial value $\frac{1}{\mu^3} f^*\varphi_0$). Assume that* G *is connected and simply-connected. Then each Lie algebra automorphism of $\mathfrak{g}$ lifts to a unique Lie group automorphism of* G *and the pseudo-Riemannian manifolds obtained from the Hitchin flow with initial values (ω_0, ρ_0) and $\left(\frac{1}{\mu^2} f^*\omega_0, \frac{1}{\mu^3} f^*\rho_0\right)$ (resp. φ_0 and $\frac{1}{\mu^3} f^*\varphi_0$) are homothetic. In particular, their holonomy groups are the same.*

7.2 Reduction of the holonomy

In this section, we look at the Hitchin flow on real seven-dimensional almost Abelian Lie algebras $\mathfrak{g}$ restricting to the G_2-case. From Theorem 7.5 we know that the solution of the Hitchin flow yields a Riemannian metric with holonomy contained in Spin(7). We prove that in the particular case of an almost Abelian Lie algebra the holonomy reduces further to a subgroup of SU(4). We do this by first showing that the Hitchin flow can alternatively be described by a certain system of algebraic and ordinary differential equations on a codimension one Abelian ideal $\mathfrak{u}$ in $\mathfrak{g}$. To prove this alternative description, we need to show the invariance of a particular subspace of the three-forms on $\mathfrak{u}$ under the action of $\mathrm{ad}(e_7)|_{\mathfrak{u}}$, $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$. Therefore, we introduce the following notation.

Notation 7.9. *Let V be a six-dimensional vector space. For a two-form $\omega \in \Lambda^2 V^*$ we define the subspace V_ω of $\Lambda^3 V^*$ by*

$$V_\omega := \{\rho \in \Lambda^3 V^* | \omega \wedge \rho = 0\}.$$

We are interested in the following situation.

Lemma 7.10. *Let $\mathfrak{g}$ be a seven-dimensional Lie algebra with six-dimensional Abelian ideal $\mathfrak{u}$. Let $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$ and denote by $F \in \mathrm{End}\left(\Lambda^3\mathfrak{u}^*\right)$ the endomorphism of $\Lambda^3\mathfrak{u}^*$ induced by $\mathrm{ad}(e_7)|_{\mathfrak{u}}$. If $\omega \in \Lambda^2\mathfrak{u}^* \cong \Lambda^2\mathrm{span}(e_7)^0$ is a $d_{\mathfrak{g}}$-closed two-form, then V_ω is an F-invariant subspace of $\Lambda^3\mathfrak{u}^*$.*

Proof. Let $\rho \in V_\omega$. By definition, $\omega \wedge \rho = 0$ and so Lemma 4.3 implies

$$0 = d(\omega \wedge \rho) = \omega \wedge d\rho = -\omega \wedge F(\rho) \wedge e^7.$$

Thus, $\omega \wedge F(\rho) = 0$ and so $F(\rho) \in V_\omega$. □

This allows us now to prove the mentioned alternative description of Hitchin's flow equations on $\mathfrak{g}$ by algebraic and ordinary differential equations on $\mathfrak{u}$.

Proposition 7.11. *Let $\mathfrak{g}$ be an almost Abelian seven-dimensional Lie algebra and $(a,b) \to \Lambda^3\mathfrak{g}^*$, $t \mapsto \varphi(t) \in \Lambda^3\mathfrak{g}^*$ be a smooth family of G_2-structures on $\mathfrak{g}$ with $0 \in (a,b)$ such that $\varphi(0)$ is cocalibrated. Let $\mathfrak{u}$ be a codimension one Abelian ideal in $\mathfrak{g}$ and $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$ be such that $e_7 \perp_{g_{\varphi(0)}} \mathfrak{u}$ and $g_{\varphi(0)}(e_7, e_7) = 1$. Moreover, let $e^7 \in \mathfrak{u}^0$ with $e^7(e_7) = 1$ and identify $\mathfrak{u}^*$ with ${e_7}^0$ via the decomposition $\mathfrak{g} = \mathfrak{u} \oplus \mathrm{span}(e_7)$. Let $\Omega(t) \in \Lambda^4\mathfrak{u}^*$, $\rho(t), \tilde{\rho}(t) \in \Lambda^3\mathfrak{u}^*$ and $\omega(t) \in \Lambda^2\mathfrak{u}^*$ be the unique elements with*

$$\star_{\varphi(t)}\varphi(t) = \Omega(t) + \rho(t) \wedge e^7, \quad \varphi(t) = \omega(t) \wedge e^7 + \tilde{\rho}(t).$$

Finally, denote by $F \in \mathrm{End}\left(\Lambda^3\mathfrak{u}^*\right)$ the linear map induced by $\mathrm{ad}(e_7)|_{\mathfrak{u}} \in \mathrm{End}(\mathfrak{u})$ on $\Lambda^3\mathfrak{u}^*$. Then $\varphi(t)$ solves Hitchin's flow equation

$$\frac{d}{dt} \star_{\varphi(t)} \varphi(t) = d\varphi(t)$$

if and only if for all $t \in I$ the three-form $\rho(t)$ is in $\mathcal{D} := \left\{\rho \in V_{\omega(0)} \,|\phi(\rho) \neq 0\right\}$ and the following is true:

(i) $\frac{d}{dt}\rho(t) = -F\left(\mu(\rho(t))J^*_{\rho(t)}\rho(t)\right)$.

(ii) $\tilde{\rho}(t) = \mu(\rho(t))J^*_{\rho(t)}\rho(t)$.

(iii) $\Omega(t) = \Omega(0) = \frac{1}{2}\omega(0)^2$.

(iv) $\omega(t) = \frac{\omega(0)}{\mu(\rho(t))}$.

Here, $\mu(\rho) := \sqrt{2\frac{\phi(\omega(0))}{\phi(\rho)}} \in \mathbb{R}$ for $\rho \in \Lambda^3\mathfrak{u}^*$ with $\phi(\rho) \neq 0$ and the orientation on $\mathfrak{u}$ we use to compute $\phi(\rho)$ is the one induced by $\phi(\omega(0))$.

Proof. We first assume that $\varphi(t)$ is a solution of Hitchin's flow equation. Note that $\mathcal{D}$ is open in $V_{\omega(0)}$ and each element in $\mathcal{D}$ is stable. By Lemma 7.10, F maps $V_{\omega(0)}$ into $V_{\omega(0)}$ and so

$$\dot{\nu}(t) = -F\left(\mu(\nu(t))J^*_{\nu(t)}\nu(t)\right), \quad \nu(0) = \rho(0)$$

is an initial value problem on $\mathcal{D}$. Let $\nu(t)$ be a solution on a maximal interval (a', b') in (a,b). The pair $(\omega(0), \nu(0)) = (\omega(0), \rho(0))$ is an SU(3)-structure on $\mathfrak{u}$ by Proposition 2.51 and Proposition 2.33. Hence, the pair $(\omega(0), \mu(\nu(t))\nu(t))$ is an SU(3)-structure on $\mathfrak{u}$ for all $t \in$

(a', b') by Corollary 2.34. Proposition 2.33 implies that also $\left(\omega(0), J^*_{\mu(\nu(t))\nu(t)}\left(\mu(\nu(t))\nu(t)\right)\right)$ $= \left(\omega(0), \mu(\nu(t))J^*_{\nu(t)}\nu(t)\right)$ is an SU(3)-structure on $\mathfrak{u}$ and that $J_{J^*_{\nu(t)}\nu(t)} = J_{\nu(t)}$ for all $t \in (a', b')$. Hence, Proposition 2.51 shows that

$$\begin{aligned}\tilde{\varphi}(t) :=& \frac{1}{\mu(\nu(t))}\omega(0) \wedge e^7 + \mu(\nu(t))J^*_{\nu(t)}\nu(t)\\ =& \,\omega(0) \wedge \left(\frac{1}{\mu(\nu(t))}e^7\right) + J^*_{\mu(\nu(t))\nu(t)}\left(\mu(\nu(t))\nu(t)\right)\end{aligned}$$

is a G_2-structure for all $t \in (a', b')$ with Hodge dual given by

$$\star_{\tilde{\varphi}(t)}\tilde{\varphi}(t) = \frac{1}{2}\omega(0)^2 + \mu(\nu(t))\nu(t) \wedge \left(\frac{1}{\mu(\nu(t))}e^7\right) = \frac{1}{2}\omega(0)^2 + \nu(t) \wedge e^7-$$

$(\omega(0), \tilde{\rho}(0))$ is an SU(3)-structure by Proposition 2.51 with $\rho(0) = -J^*_{\tilde{\rho}(0)}\tilde{\rho}(0)$. Thus, $\tilde{\rho}(0) = J^*_{\tilde{\rho}(0)}\rho(0) = J^*_{\rho(0)}\rho(0)$ by Proposition 2.33. Hence, $\tilde{\varphi}(0) = \omega(0) \wedge e^7 + J^*_{\rho(0)}\rho(0) = \omega(0) \wedge e^7 + \tilde{\rho}(0) = \varphi(0)$. Moreover,

$$\begin{aligned}\frac{d}{dt}\star_{\tilde{\varphi}(t)}\tilde{\varphi}(t) =& \,\dot{\nu}(t) \wedge e^7 = -F\left(\mu(\nu(t))J^*_{\nu(t)}\nu(t)\right) \wedge e^7 = d\left(\mu(\nu(t))J^*_{\nu(t)}\nu(t)\right)\\ =& \,d\left(\frac{1}{\mu(\nu(t))}\omega(0) \wedge e^7 + \mu(\nu(t))J^*_{\nu(t)}\nu(t)\right) = d\tilde{\varphi}(t)\end{aligned}$$

and so also $\tilde{\varphi}(t)$ solves Hitchin's flow equations with initial value $\varphi(0)$. Hence, the uniqueness result in Theorem 7.7 gives us $\tilde{\varphi}(t) = \varphi(t)$ and so also $\nu(t) = \rho(t)$ for all $t \in (a', b')$. Hence, the conditions (i)-(iv) as in the statement hold on (a', b'). What is left to show is that (a, b) is equal to the maximal interval of existence (a', b') of $\nu(t)$. Therefore, it suffices to show that $\lim_{t\to a'}\nu(t) = \lim_{t\to a'}\rho(t) = \rho(a')$ is in $\mathcal{D}$ and that also $\lim_{t\to b'}\nu(t) = \lim_{t\to b'}\rho(t) = \rho(b')$ is in $\mathcal{D}$. But this is clear since obviously $\rho(a')$ and $\rho(b')$ are in $V_{\omega(0)}$ and $\rho(a')$ and $\rho(b')$ are stable by Proposition 2.48.

Conversely, if $\rho(t)$ is in $\mathcal{D}$ for all $t \in (a, b)$ and $\rho(t)$, $\tilde{\rho}(t)$, $\omega(t)$ and $\Omega(t)$ fulfil (i)-(iv), then the above calculations show that $\frac{d}{dt}\star_{\varphi(t)}\varphi(t) = d\varphi(t)$. □

Proposition 7.11 allows us to prove the main theorem of this section.

Theorem 7.12. *Let $\mathfrak{g}$ be an almost Abelian seven-dimensional Lie algebra, $\varphi(0)$ be a cocalibrated G_2-structure on $\mathfrak{g}$ and $0 \in (a, b) \ni t \mapsto \varphi(t)$ be the solution of Hitchin's flow equations with initial value $\varphi(0)$. Then*

$$g := g_{\varphi(t)} + dt^2$$

defines a Riemannian metric on $\mathrm{G} \times I$ with holonomy contained in SU(4). *Here,* G *is any Lie group with Lie algebra $\mathfrak{g}$.*

Proof. Let $\mathfrak{u}$ be an Abelian ideal of codimension one in $\mathfrak{g}$ and $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$ be the unique element with $g_{\varphi(0)}(e_7, e_7) = 1$ and $e_7 \perp_{g_{\varphi(0)}} \mathfrak{u}$. We decompose $\varphi(t)$, $\star_{\varphi(t)}\varphi(t)$ as in Proposition 7.11 into

$$\varphi(t) = \omega(t) \wedge e^7 + \tilde{\rho}(t), \quad \psi(t) = \Omega(t) + \epsilon\rho(t) \wedge e^7$$

with $\omega(t) \in \Lambda^2\mathfrak{u}^*$, $\tilde{\rho}(t), \rho(t) \in \Lambda^3\mathfrak{u}^*$ and $\Omega(t) \in \Lambda^4\mathfrak{u}^*$. We define a two-form ω and a complex-valued four-form Ψ on $G \times I$ by

$$\omega := \omega(0) + dt \wedge \frac{e^7}{\mu(\rho(t))}, \quad \Psi := (\tilde{\rho}(t) - i\mu(\rho(t))\rho(t)) \wedge \left(dt - i\frac{e^7}{\mu(\rho(t))}\right), \tag{7.7}$$

where, as in Proposition 7.11, $\mu(\rho) := \sqrt{2\frac{\phi(\omega(0))}{\phi(\rho)}}$ for $\rho \in \Lambda^3\mathfrak{u}^*$ with $\phi(\rho) \neq 0$ and we choose the orientation on $\mathfrak{u}$ induced by $\phi(\omega(0))$. By Proposition 7.11, $\tilde{\rho}(t) = \mu(\rho(t))J^*_{\rho(t)}\rho(t)$ and $\tilde{\rho}(t) \in V_{\omega(0)}$ for all $t \in (a,b)$. Moreover,

$$\phi(\tilde{\rho}(t)) = \phi\left(\mu(\rho(t))J^*_{\rho(t)}\rho(t)\right) = \mu(\rho(t))^2\phi\left(J^*_{\rho(t)}\rho(t)\right) = 2\frac{\phi(\omega(0))}{\phi(\rho)}\phi(\rho) = 2\phi(\omega(0)).$$

Hence, the pair $(\omega(0), \tilde{\rho}(t))$ is an SU(3)-structure on $\mathfrak{u}$ for all $t \in (a,b)$. By Proposition 2.21,

$$\begin{aligned}\psi(t) :=& \tilde{\rho}(t) + iJ^*_{\tilde{\rho}(t)}\tilde{\rho}(t) = \mu(\rho(t))J^*_{\rho(t)}\rho(t) + iJ^*_{\mu(\rho(t))J^*_{\rho(t)}\rho(t)}\left(\mu(\rho(t))J^*_{\rho(t)}\rho(t)\right)\\ =&\mu(\rho(t))J^*_{\rho(t)}\rho(t) - i\mu(\rho(t))\rho(t) = \tilde{\rho}(t) - i\mu(\rho(t))\rho(t)\end{aligned} \tag{7.8}$$

is a complex volume form with respect to $J_{\tilde{\rho}(t)} = J_{\rho(t)}$ and $(\omega(0), \psi(t))$ is an SU(3)-structure on $\mathfrak{u}$ for all $t \in (a,b)$. Thus, (ω, Ψ) is an SU(4)-structure on $\mathrm{G} \times (a,b)$, invariant under the natural left-action of G on $\mathrm{G} \times (a,b)$. The induced metric $g_{(\omega,\Psi)}$ is given by

$$g_{(\omega,\Psi)} = g_{(\omega(0),\tilde{\rho}(t))} + \frac{e^7}{\mu(\rho(t))} \otimes \frac{e^7}{\mu(\rho(t))} + dt^2,$$

where $g_{(\omega(0),\tilde{\rho}(t))}$ is the metric on $\mathfrak{u}$ induced by the SU(3)-structure $(\omega(0), \tilde{\rho}(t))$. Since $\varphi(t) = \omega(0) \wedge \frac{e^7}{\mu(\rho(t))} + \tilde{\rho}(t)$, we get $g_{\varphi(t)} = g_{(\omega(0),\tilde{\rho}(t))} + \frac{e^7}{\mu(\rho(t))} \otimes \frac{e^7}{\mu(\rho(t))}$ by Proposition 2.51. Hence, $g_{(\omega,\Psi)} = g_{\varphi(t)} + dt^2 = g$.

To show that the holonomy of g is contained in SU(4), it suffices by Theorem 3.20 to show that (ω, Ψ) is torsion-free. By Proposition 3.29, the torsion vanishes if and only if $d\omega = 0$ and $d\,\mathrm{Re}(\Psi) = 0$. For the computations, we denote by d_7 the exterior derivative on G. By Theorem 4.15, we know that $f := \mathrm{ad}(e_7)|_{\mathfrak{u}}$ has to be in $\mathfrak{sp}(\mathfrak{u}, \omega(0))$ if $\varphi(0)$ is cocalibrated. Thus, $d_7(\omega(0)) = 0$ by Lemma 4.3 and so $d(\omega(0)) = 0$. Moreover, $d_7e^7 = 0$ by Lemma 4.3 and so $de^7 = 0$. Hence

$$d\omega = d(\omega(0)) - dt \wedge d\left(-\frac{e^7}{\mu(\rho(t))}\right) = 0.$$

By Equation (7.7) and Equation (7.8),

$$\mathrm{Re}(\Psi) = \mu(\rho(t))J^*_{\rho(t)}\rho(t) \wedge dt - \rho(t) \wedge e^7.$$

Denote by $F \in \mathrm{End}\left(\Lambda^3\mathfrak{u}^*\right)$ the endomorphism of $\Lambda^3\mathfrak{u}^*$ induced by $f \in \mathrm{End}(\mathfrak{u})$. Proposition 7.11 states that $\frac{d}{dt}\rho(t) = -F\left(\mu(\rho(t))J^*_{\rho(t)}\rho(t)\right)$. But then Lemma 4.3 implies

$$d_7\left(\mu(\rho(t))J^*_{\rho(t)}\rho(t)\right) = -F\left(\mu(\rho(t))J^*_{\rho(t)}\rho(t)\right) \wedge e^7 = \frac{d}{dt}\rho(t) \wedge e^7$$

and so

$$\begin{aligned} d\,\mathrm{Re}(\Psi) &= d\left(\mu(\rho(t))J^*_{\rho(t)}\rho(t)\right) \wedge dt - d(\rho(t)) \wedge e^7 \\ &= d_7\left(\mu(\rho(t))J^*_{\rho(t)}\rho(t)\right) \wedge dt - dt \wedge \frac{d}{dt}\rho(t) \wedge e^7 = 0. \end{aligned}$$

Thus, the holonomy of g is contained in $\mathrm{SU}(4)$. □

Remark 7.13. *We conjecture that an analogous holonomy reduction result is true in the* G_2^**-case if the subspace* $\mathfrak{u}$ *is non-degenerate with respect to* $g_{\varphi(0)}$*,* $\varphi(0)$ *being the initial cocalibrated* G_2^**-structure. One easily sees that, analogously to Proposition 7.11, the Hitchin flow reduces to a set of algebraic and ordinary differential equations for the induced forms on* $\mathfrak{u}$*. One can use these equations as in the proof of Theorem 7.12 to define an* $\mathrm{SU}(2,2)$*-structure* (ω,Ψ) *on* $\mathrm{G}\times I$ *with* $d\omega = 0$ *and* $d\,\mathrm{Re}(\Psi) = 0$ *if the signature of* $g_\varphi|_{\mathfrak{u}}$ *is* $(2,4)$ *or an* $\mathrm{SL}(4,\mathbb{R})$*-structure* (ω,Ψ) *on* $\mathrm{G}\times I$ *with* $d\omega = 0$ *and* $d\,\mathrm{Re}(\Psi) = 0$ *if the signature of* $g_\varphi|_{\mathfrak{u}}$ *is* $(3,3)$*, respectively. We suppose that this implies a holonomy reduction to a subgroup of* $\mathrm{SU}(2,2)$ *or to a subgroup of* $\mathrm{SL}(4,\mathbb{R})$*, respectively. To complete the proof, only an appropriate analogue of Proposition 3.29 is missing. Note that Cabrera's proof of Proposition 3.29 in [MC4] uses mainly representation theoretic arguments. So it should be possible to transfer the proof to the pseudo-Riemannian cases.*

7.3 Moduli spaces

In this section, we consider the *moduli spaces of cocalibrated* G_2*-structure* on seven-dimensional almost Abelian Lie algebras $\mathfrak{g}$. This space is by definition the set of all cocalibrated G_2-structures up to automorphisms of the Lie algebra and up to a scaling factor. Remark 7.8 shows that we may easily compute the Hitchin flow for an arbitrary initial value if we solve it for all initial values in the moduli space. We prove a result which simplifies the determination of the moduli space if the codimension one Abelian ideal $\mathfrak{u}$ in $\mathfrak{g}$ is unique and apply it to compute the moduli space on $\mathfrak{n}_{7,1}$. If $\mathfrak{u}$ is not unique, the proof of Proposition 4.4 shows that $\mathfrak{g} \in \left\{\mathbb{R}^7, \mathfrak{h}_3 \oplus \mathbb{R}^4\right\}$. Obviously, the moduli space on $\mathbb{R}^7$ consists of only one point. We prove that the same is true on $\mathfrak{h}_3 \oplus \mathbb{R}^4$.

We start with a proper definition of the mentioned moduli space.

Definition 7.14. *Let* $\mathfrak{g}$ *be a seven-dimensional Lie algebra. We set*

$$M^3_{\mathrm{G}_2}(\mathfrak{g}) := \left\{ \varphi \in \Lambda^3\mathfrak{g}^* \,\middle|\, \varphi \text{ is a cocalibrated } \mathrm{G}_2\text{-structure}\right\}$$

and

$$M^4_{\mathrm{G}_2}(\mathfrak{g}) := \left\{ \Psi \in \Lambda^4\mathfrak{g}^* \,\middle|\, d\Psi = 0,\ \Psi \text{ is the Hodge dual of a } \mathrm{G}_2\text{-structure}\right\}$$

Let $\mathbb{R}^* \subseteq \mathrm{GL}(\mathfrak{g})$ *be the subgroup given by* $\{a\,\mathrm{id}_{\mathfrak{g}}\,|a \in \mathbb{R}^*\}$*. Using the natural left action of elements in* $\mathrm{GL}(\mathfrak{g})$ *on* $\Lambda^3\mathfrak{g}^*$*, we set*

$$\mathcal{M}_{\mathrm{G}_2}(\mathfrak{g}) := M^3_{\mathrm{G}_2}(\mathfrak{g})/\left(\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*\right)$$

with the induced left-action of $\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*$ *on* $M^3_{\mathrm{G}_2}(\mathfrak{g})$ *and call* $\mathcal{M}_{\mathrm{G}_2}$ *the* moduli space of cocalibrated G_2-structures. *Let* $\left(\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*\right)_+$ *be those elements in* $\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*$ *with positive determinant and note that* $\mathcal{M}_{\mathrm{G}_2}(\mathfrak{g})$ *is naturally bijective to*

$$M^4_{\mathrm{G}_2}(\mathfrak{g})/\left(\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*\right)_+$$

via the map induced by the $\mathrm{GL}(\mathfrak{g})$*-equivariant map* $M^3_{\mathrm{G}_2}(\mathfrak{g}) \ni \varphi \mapsto \star_\varphi\varphi \in M^4(\mathrm{G}_2)(\mathfrak{g})$ *since* $\mathrm{GL}(\mathfrak{g})_{\star_\varphi\varphi} = \mathrm{GL}(\mathfrak{g})_\varphi \times \{-I_7, I_7\}$ *for all* G_2*-structures* $\varphi \in \Lambda^3\mathfrak{g}^*$*, cf. Lemma 2.43. We do not distinguish in the following between these descriptions.*

Remark 7.15. *In general, we do not endow* $\mathcal{M}_{\mathrm{G}_2}(\mathfrak{g})$ *with the quotient topology in this thesis since we only need the moduli space as a minimal set of initial values for the Hitchin flow on* $\mathfrak{g}$ *and do not consider topological issues like compactifications of the moduli space. Note however that if* $M^4_{\mathrm{G}_2}(\mathfrak{g})$ *is non-empty, then it is a non-empty open subset of the subspace* $\ker d|_{\Lambda^4\mathfrak{g}^*}$ *of* $\Lambda^4\mathfrak{g}^*$*. Hence,* $M^4_{\mathrm{G}_2}(\mathfrak{g})$ *is an embedded smooth submanifold of* $\Lambda^4\mathfrak{g}^*$ *and so also* $M^3_{\mathrm{G}_2}(\mathfrak{g})$ *is an embedded smooth submanifold of* $\Lambda^3\mathfrak{g}^*$*. Both have dimension* $\dim(\ker d|_{\Lambda^4\mathfrak{g}^*})$*. Moreover,* $\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*$ *is an embedded Lie subgroup of* $\mathrm{GL}(\mathfrak{g})$ *which acts smoothly on* $M^3_{\mathrm{G}_2}(\mathfrak{g})$*.*

The next proposition simplifies the computation of the moduli space.

Proposition 7.16. *Let* $\mathfrak{g}$ *be an oriented seven-dimensional almost Abelian Lie algebra and* $\mathfrak{u}$ *be a six-dimensional Abelian ideal of* $\mathfrak{g}$*. Choose* $e_7 \in \mathfrak{g}\backslash\mathfrak{u}$ *and set* $f := \mathrm{ad}(e_7)|_{\mathfrak{u}}$*. Set*

$$M^{ON}_{\mathrm{G}_2}(\mathfrak{g}) := \left\{ \Psi \in M^4_{\mathrm{G}_2}(\mathfrak{g}) \middle|\, g_\Psi(\mathfrak{u}, e_7) = 0,\ g_\Psi(e_7, e_7) = 1 \right\}.$$

Then each $\left(\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*\right)_+$*-orbit of an element in* $M^4_{\mathrm{G}_2}(\mathfrak{g})$ *intersects* $M^{ON}_{\mathrm{G}_2}(\mathfrak{g})$*. If* $\mathfrak{u}$ *is the unique codimension one Abelian ideal in* $\mathfrak{g}$*, i.e. if* $\mathfrak{g} \neq \mathfrak{h}_3 \oplus \mathbb{R}^4, \mathbb{R}^7$*, then* $\mathcal{M}_{\mathrm{G}_2}(\mathfrak{g})$ *is bijective to*

$$M^{ON}_{\mathrm{G}_2}(\mathfrak{g})/\mathrm{H}(\mathfrak{g}),$$

where $\mathrm{H}(\mathfrak{g})$ *is the subgroup of* $\left(\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*\right)_+$ *given, with respect to the decomposition* $\mathfrak{g} = \mathfrak{u} \oplus \mathrm{span}(e_7)$*, by*

$$\mathrm{H}(\mathfrak{g}) := \left\{ \begin{pmatrix} g & 0 \\ 0 & \mathrm{sgn}(\det(g)) \end{pmatrix} \middle|\, g \circ f = \frac{\mathrm{sgn}(\det(g))}{\lambda} f \circ g,\ g \in \mathrm{GL}(\mathfrak{u}),\ \lambda \in \mathbb{R}^* \right\} \tag{7.9}$$

Proof. Let $\Psi \in M^4_{\mathrm{G}_2}(\mathfrak{g})$. By Lemma 2.45 and Lemma 2.41, the stabiliser of Ψ in $\mathrm{GL}(\mathfrak{g})$ acts transitively on the set of six-dimensional subspaces of $\mathfrak{g}$. Hence, we may assume that there is a basis $e_1, \ldots, e_6 \in \mathfrak{u}$ of $\mathfrak{u}$, $\lambda \in \mathbb{R}\backslash\{0\}$ and $v \in \mathfrak{u}$ such that $e_1, \ldots, e_6, \lambda e_7 + v$ is an adapted basis for Ψ. Define a linear isomorphism $F \in \mathrm{GL}(\mathfrak{g})$ of $\mathfrak{g}$ by $F(e_i) := e_i$ for $i = 1, \ldots, 6$ and $F(e_7) := e_7 - \frac{1}{\lambda} v$. Then F is an automorphism of $\mathfrak{g}$, $\frac{1}{|\lambda|} F \in (\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*)_+$ and $\left(\frac{1}{|\lambda|} F\right).\Psi$ has the adapted basis $\frac{1}{|\lambda|} e_1, \ldots, \frac{1}{|\lambda|} e_6, \mathrm{sgn}(\lambda) e_7$. Since adapted bases are orthonormal, $\left(\frac{1}{|\lambda|} F\right).\Psi$ is in $M^{ON}_{\mathrm{G}_2}(\mathfrak{g})$.

Assume for the rest of the proof that $\mathfrak{u}$ is the unique codimension one Abelian ideal in $\mathfrak{g}$. By the proof of Proposition 4.4, we have $\mathfrak{g} \neq \mathbb{R}^7, \mathfrak{h}_3 \oplus \mathbb{R}^4$. Let $G = \lambda H$ with $H \in \mathrm{Aut}(\mathfrak{g})$ and $\lambda \in \mathbb{R}^*$ be an element in $(\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*)_+$ which fixes $M^{ON}_{\mathrm{G}_2}(\mathfrak{g})$. Since $G \in (\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*)_+$, we get $G(\mathfrak{u}) = \mathfrak{u}$ and $G(e_7) = \mu e_7 + v$ for certain $\mu \in \mathbb{R}^*$ and $v \in \mathfrak{u}$. Using that G fixes $M^{ON}_{\mathrm{G}_2}(\mathfrak{g})$ and that it is a linear isometry between $(\mathfrak{g}, g_\Psi)$ and $(\mathfrak{g}, g_{G.\Psi})$, we get

$$\begin{aligned} g_{G.\Psi}(v,v) &= g_{G.\Psi}(v,v) - g_{G.\Psi}(v, G(e_7)) + g_{G.\Psi}(v, G(e_7)) \\ &= g_{G.\Psi}(v,v) - g_{G.\Psi}(v,v) - \mu\, g_{G.\Psi}(v, e_7) + g_\Psi\left(G^{-1}(v), e_7\right) = 0. \end{aligned}$$

Thus, $v = 0$. Moreover,

$$1 = g_{G.\Psi}(e_7, e_7) = \frac{1}{\mu^2} g_{G.\Psi}(\mu e_7, \mu e_7) = \frac{1}{\mu^2} g_{G.\Psi}(G(e_7), G(e_7)) = \frac{1}{\mu^2} g_\Psi(e_7, e_7) = \frac{1}{\mu^2}$$

and so $\mu = \pm 1$. Set $g := G|_{\mathfrak{u}}$. Then μ has to be equal to $\mathrm{sgn}(\det(g))$ and so $H(e_7) = \frac{G(e_7)}{\lambda} = \frac{\mathrm{sgn}(\det(g))}{\lambda} e_7$. Thus

$$\begin{aligned} (g \circ f)(w) &= \lambda H([e_7, w]) = \lambda [H(e_7), H(w)] = \mathrm{sgn}(\det(g))[e_7, H(w)] \\ &= \frac{\mathrm{sgn}(\det(g))}{\lambda} (f \circ g)(w). \end{aligned}$$

Hence, each element in $(\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*)_+$ which stabilises $M^{ON}_{\mathrm{G}_2}(\mathfrak{g})$ is contained in $\mathrm{H}(\mathfrak{g})$ and conversely a short computation shows that $\mathrm{H}(\mathfrak{g})$ is a subgroup of $(\mathrm{Aut}(\mathfrak{g}) \times \mathbb{R}^*)_+$ and each element in it stabilises $M^{ON}_{\mathrm{G}_2}(\mathfrak{g})$. This proves the statement. $\square$

Obviously, $\mathcal{M}_{\mathrm{G}_2}\left(\mathbb{R}^7\right)$ consists of only one point. The same is true for $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbb{R}^4$:

Proposition 7.17. *Let $e_1, \ldots, e_7$ be a basis of $\mathfrak{h}_3 \oplus \mathbb{R}^4$ such that e_1, e_2, e_3 is a basis of $\mathfrak{h}_3$ with $[e_1, e_2] = e_3$, $[e_1, e_3] = [e_2, e_3] = 0$ and $e_4, \ldots, e_7$ be a basis of $\mathbb{R}^4$. Then $\{\varphi_0\}$ is bijective to $\mathcal{M}_{\mathrm{G}_2}\left(\mathfrak{h}_3 \oplus \mathbb{R}^4\right)$ via $\pi : M^3_{\mathrm{G}_2}\left(\mathfrak{h}_3 \oplus \mathbb{R}^4\right) \to \mathcal{M}_{\mathrm{G}_2}\left(\mathfrak{h}_3 \oplus \mathbb{R}^4\right)$ for*

$$\varphi_0 := e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356} \in M^3_{\mathrm{G}_2}\left(\mathfrak{h}_3 \oplus \mathbb{R}^4\right)$$

Proof. Set $\mathfrak{g} := \mathfrak{h}_3 \oplus \mathbb{R}^4$ and let $\varphi \in \Lambda^3 \mathfrak{g}^*$ be a cocalibrated G_2-structure. By [Br1], the stabiliser of φ acts transitively on the two-planes of $\mathfrak{g}$ and so also on the five-dimensional subspaces of $\mathfrak{g}$. Hence, we obtain an adapted basis $(f_1, \ldots, f_7)$ for φ such that f_3, f_4, f_5, f_6, f_7

is a basis of $V := \mathrm{span}(e_3) \oplus \mathbb{R}^4$. Then $f_1 = a_{11}e_1 + a_{12}e_2 + v_1$, $f_2 = a_{21}e_1 + a_{22}e_2 + v_2$ for $A = (a_{ij})_{ij} \in \mathrm{GL}(2,\mathbb{R})$ and $v_1, v_2 \in V$. Moreover, $f_j = \lambda_j e_3 + w_j$ with $\lambda_j \in \mathbb{R}$ and $w_j \in \mathbb{R}^4$ for $j = 3, \ldots, 7$. There exist $3 \leq j_1 < j_2 < j_3 < j_4 \leq 7$ such that w_{j_1}, w_{j_2}, w_{j_3}, w_{j_4} are linearly independent. The linear automorphism $F_1 \in \mathrm{GL}(\mathfrak{g})$, defined by $F_1(e_i) := f_i$ for $i = 1, 2$, $F_1(e_3) := \det(A)e_3$ and $F_1(e_k) := f_{j_{k-3}}$ for $k = 4, \ldots, 7$ is an automorphism of $\mathfrak{g}$. Let $j_5 \in \{3, 4, 5, 6, 7\}$ be the element different from $j_1, \ldots, j_4$. Then $\tilde{\varphi} := \left(F_1^{-1}\right).\varphi$ has an adapted basis $(E_1, \ldots, E_7)$ with $E_i = e_i$ for $i = 1, 2$, $E_{j_k} = e_{k+3}$ for $k = 1, 2, 3, 4$ and $E_{j_5} \in V$. Hence, the dual basis $(E^1, \ldots, E^7)$ fulfils $E^i = e^i$ for $i = 1, 2$, $E^{j_k} = e^{k+3} + a_k e^3$ for certain $a_k \in \mathbb{R}$, $k = 1, 2, 3, 4$, and $E^{j_5} = \lambda e^3$ for some $\lambda \in \mathbb{R}^*$. We show that $j_5 = 3$. If this is not the case, then the concrete form of the Hodge dual in terms of the dual adapted basis $\left(E^1, \ldots, E^7\right)$ given in Equation (2.26) shows that

$$\star_\varphi \varphi = \Omega_0 - \lambda e^3 \wedge \rho.$$

for certain $\Omega_0 \in \mathrm{span}(e^1, e^2) \wedge \Lambda^3 \mathfrak{g}^*$ and $\rho \in \Lambda^3 \left(\mathbb{R}^4\right)^*$ with $\rho \neq 0$. But $\star_\varphi \varphi$ cannot be closed since Ω_0 is closed and $d\left(e^3 \wedge \rho\right) = -e^{12} \wedge \rho \neq 0$. Hence, $j_5 = 3$. Using again Equation (2.26) to write down $\star_\varphi \varphi$ concretely, we see that $d\left(\star_\varphi \varphi\right) = 0$ forces $a_1 = a_2 = a_3 = a_4 = 0$. By applying an appropriate automorphism of $\mathfrak{g} = \mathfrak{h}_3 \oplus \mathbb{R}^4$ which respects the decomposition and acts trivially on $\mathfrak{h}_3$, we see that

$$\left(e_1, e_2, \frac{1}{\lambda} e_3, e_4, e_5, e_6, e_7\right)$$

is an adapted basis for $\tilde{\varphi}$. The linear isomorphism $F_2 \in \mathrm{GL}(\mathfrak{g})$ of $\mathfrak{g}$ defined by $F_2(e_i) := \lambda e_i$ for $i \neq 3$ and $F_2(e_3) := \lambda^2 e_3$ is an automorphism of $\mathfrak{g}$ and $\left(F_2, \frac{1}{\lambda}\right).\tilde{\varphi}$ has the adapted basis

$$(e_1, e_2, e_3, e_4, e_5, e_6, e_7)$$

Hence, $\left(F_2, \frac{1}{\lambda}\right).\tilde{\varphi} = \varphi_0$ and the statement follows. □

Finally, we determine $\mathcal{M}_{\mathrm{G}_2}(\mathfrak{n}_{7,1})$ with the use of Proposition 7.16. This case is particularly interesting since $\mathfrak{n}_{7,1}$ is nilpotent and admits a co-compact lattice. Moreover, as we will see in Proposition 7.22, the Hitchin flow yields full holonomy $\mathrm{SU}(4)$ for certain elements in $\mathcal{M}_{\mathrm{G}_2}(\mathfrak{n}_{7,1})$.

Lemma 7.18. *Let $e_1, \ldots, e_7$ be the basis of $\mathfrak{n}_{7,1}$ given in Table 7.8. Denote by $\mathfrak{u} = \mathrm{span}(e_1, \ldots, e_6)$ the unique codimension one Abelian ideal in $\mathfrak{n}_{7,1}$. Let $\mathrm{H}(\mathfrak{n}_{7,1})$ be the subgroup of $(\mathrm{Aut}(\mathfrak{n}_{7,1}) \times \mathbb{R}^*)_+$ defined by Equation (7.9) and $\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u}) := \mathrm{H}(\mathfrak{n}_{7,1}) \cap \mathrm{GL}(\mathfrak{u}) \subseteq \mathrm{GL}(\mathfrak{u})$. The action of $\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u})$ on*

$$\left\{\Omega \in \Lambda^4 \mathfrak{u}^* \,\middle|\, d\Omega = 0,\ \Omega = \frac{1}{2}\omega^2,\ \omega \in \Lambda^2 \mathfrak{u}^* \text{ non-degenerate}\right\}$$

has exactly two orbits represented by $\Omega_\pm := -e^{2356} \mp \left(e^{1346} + e^{1245}\right)$.

Proof. Set $V_1 := \operatorname{span}(e_1, e_2, e_3)$ and $V_2 := \operatorname{span}(e_4, e_5, e_6)$. A short computation shows

$$\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u}) = \left\{ \begin{pmatrix} A & \lambda BA \\ 0 & \lambda A \end{pmatrix} \middle| A \in \mathrm{GL}(3,\mathbb{R}),\ B \in \mathbb{R}^{3\times 3},\ \lambda \in \mathbb{R}^* \right\} = (\mathrm{GL}(3,\mathbb{R}) \times \mathbb{R}^*) \ltimes \mathbb{R}^{3\times 3},$$

where $\mathrm{GL}(3,\mathbb{R}) \times \mathbb{R}^*$ is the subgroup $\left\{ \begin{pmatrix} A & 0 \\ 0 & \lambda A \end{pmatrix} \middle| A \in \mathrm{GL}(3,\mathbb{R}),\ \lambda \in \mathbb{R}^* \right\}$ of $\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u})$ and $\mathbb{R}^{3\times 3}$ is the normal subgroup $\left\{ \begin{pmatrix} I_3 & B \\ 0 & I_3 \end{pmatrix} \middle| B \in \mathbb{R}^{3\times 3} \right\}$ of $\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u})$. The most general closed four-form can be computed to be $\Omega = \Omega_1 + \Omega_2$ with

$$\begin{aligned} \Omega_1 =& c_{11} e^{2356} + c_{22} e^{3164} + c_{33} e^{1245} + c_{12} \left(e^{2364} + e^{3156}\right) + c_{23} \left(e^{3145} + e^{1264}\right) \\ &+ c_{31} \left(e^{2345} + e^{1256}\right) \in \Lambda^2 V_1^* \wedge \Lambda^2 V_2^* = \Lambda^2 V_1^* \otimes \Lambda^2 V_2^* \\ \Omega_2 =& d_{23} e^{1456} - d_{13} e^{2456} + d_{12} e^{3456} \in V_1^* \wedge \Lambda^3 V_2^* = V_1^* \otimes \Lambda^3 V_2^* \end{aligned}$$

where all coefficients are arbitrary real numbers. We arrange them in a symmetric matrix $C = (c_{ij})_{ij} \in \mathbb{R}^{3\times 3}$ and an anti-symmetric matrix $D = (d_{ij})_{ij} \in \mathbb{R}^{3\times 3}$ by setting $c_{21} := c_{12}$, $c_{31} := c_{13}$, $c_{32} := c_{23}$ and $d_{21} := -d_{12}$, $d_{31} := -d_{13}\, d_{32} := -d_{23}$. Hence, we may describe the most general closed four-form by a pair $(C, D) \in \mathrm{Sym}(3) \times \mathfrak{so}(3)$ of a symmetric matrix C and an anti-symmetric matrix D in three dimensions. The subgroup $\mathrm{GL}(3,\mathbb{R}) \times \mathbb{R}^*$ acts on (C, D) by

$$\begin{aligned} &(\mathrm{GL}(3,\mathbb{R}) \times \mathbb{R}^*) \times (\mathrm{Sym}(3) \times \mathfrak{so}(3)) \ni ((A,\lambda),(C,D)) \mapsto \\ &\left(\frac{1}{\lambda^2 \det(A)^2} ACA^t, \frac{1}{\lambda^3 \det(A)^2} ADA^t \right) = \left(\frac{\operatorname{adj}(A^{-1})}{\lambda} C \left(\frac{\operatorname{adj}(A^{-1})}{\lambda} \right)^t, \frac{1}{\lambda^3 \det(A)^2} ADA^t \right) \end{aligned}$$

This can be seen by looking at the isomorphisms $\Lambda^2 V_1^* \otimes \Lambda^2 V_2^* \to V_1 \otimes V_1$ and $V_1^* \otimes \Lambda^3 V_2^* \to \Lambda^2 V_1$ given by

$$\begin{aligned} \Lambda^2 V_1^* \otimes \Lambda_2 V_2^* \ni (\omega_1 \otimes \omega_2) \to & \omega_1 \lrcorner\, e_{123} \otimes F_*(\omega_2 \lrcorner\, e_{456}), \\ V_1^* \otimes \Lambda_3 V_2^* \ni (\alpha \otimes \nu) \to & (\nu \lrcorner\, e_{456}) \cdot (\alpha \lrcorner\, e_{123}) \end{aligned}$$

where $F : V_2 \to V_1$ is the isomorphism given by $F(e_i) := e_{i-3}$ for $i = 4, 5, 6$. By Sylvester's law of inertia, there exists $\tilde{A} \in \mathrm{GL}(3,\mathbb{R})$ such that $\tilde{A} C \tilde{A}^t = \operatorname{diag}(\delta_1, \delta_2, \delta_3)$ with $\delta_1, \delta_2, \delta_3 \in \{1, 0, -1\}$ and $\delta_1 \geq \delta_2 \geq \delta_3$. By multiplying $\tilde{A}$ with $-I_3$, we may assume that $\tilde{A} \in \mathrm{GL}^+(3,\mathbb{R})$. Setting $A := \frac{\tilde{A}}{\sqrt{\det(\tilde{A})}}$ and $\lambda := 1$, we get $\frac{\operatorname{adj}(A^{-1})}{\lambda} = \frac{A}{\det(A)} = \tilde{A}$. Thus, each $\mathrm{H}^+(\mathfrak{n}_{7,1})$-orbit of an element $(C, D) \in \mathrm{Sym}(3) \times \mathfrak{so}(3)$ contains an element of the form $(\operatorname{diag}(\delta_1, \delta_2, \delta_3), \tilde{D})$ with $\delta_1, \delta_2, \delta_3 \in \{1, 0, -1\}$ and $\delta_1 \leq \delta_2 \leq \delta_3$.

Let $\Omega \in \Lambda^4 \mathfrak{u}^*$, $\Omega = \Omega_1 + \Omega_2$ with $\Omega_1 \in \Lambda^2 V_1^* \wedge \Lambda^2 V_2^*$ and $\Omega_2 \in V_1^* \wedge \Lambda^3 V_2^*$, be a four-form as in the statement and assume that the corresponding element in $\mathrm{Sym}(3) \times \mathfrak{so}(3)$ is given by $(\operatorname{diag}(\delta_1, \delta_2, \delta_3), \tilde{D})$ with $\delta_1, \delta_2, \delta_3 \in \{1, 0, -1\}$ and $\delta_1 \leq \delta_2 \leq \delta_3$. Moreover, let $\omega \in \Lambda^2 \mathfrak{u}^*$ be non-degenerate with $\Omega = \frac{1}{2}\omega^2$. By Lemma 2.4 and Lemma 5.15, the length of $\Omega_1 = \delta_1 e^{2356} + \delta_2 e^{1346} + \delta_3 e^{1245}$ is three and so $\delta_i \neq 0$ for $i = 1, 2, 3$. Moreover, Lemma 2.4 asserts that $\delta_1\delta_2\delta_3 = -1$ and so $(\delta_1, \delta_2, \delta_3) = (-1, -1, -1)$ or $(\delta_1, \delta_2, \delta_3) = (-1, 1, 1)$. Generally, an element $B \in \mathbb{R}^{3\times 3}$ acts on $(C, D) \in \mathrm{Sym}(3) \times \mathfrak{so}(3)$ such that $B.(C, D) = (C, D + G(B, C))$ for some bilinear map $G : \mathbb{R}^{3\times 3} \times \mathrm{Sym}(3) \to \mathfrak{so}(3)$. In our case, i.e. for $C = \mathrm{diag}(\delta_1, \delta_2, \delta_3)$, we obtain $G(B, C) = (g_{ij})_{ij} \in \mathfrak{so}(3)$, $g_{12} = \delta_1 b_{21} - \delta_2 b_{12}$, $g_{13} = \delta_1 b_{31} - \delta_3 b_{13}$ and $g_{23} = \delta_2 b_{32} - \delta_3 b_{23}$. Thus, we are able to find $\tilde{B} \in \mathbb{R}^{3\times 3}$ such that $\tilde{B}.(\mathrm{diag}(\delta_1, \delta_2, \delta_3), \tilde{D}) = (\mathrm{diag}(\delta_1, \delta_2, \delta_3), 0)$. Hence, each $\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u})$-orbit in

$$\left\{ \Omega \in \Lambda^4 \mathfrak{u}^* \,\middle|\, d\Omega = 0,\ \Omega = \frac{1}{2}\omega^2,\ \omega \in \Lambda^2 \mathfrak{u}^* \text{ non-degenerate} \right\}$$

contains $\Omega_+ = -e^{2356} - e^{1346} - e^{1245}$ or $\Omega_- = -e^{2356} + e^{1346} + e^{1245}$. That one $\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u})$-orbit cannot contain both follows by the uniqueness of $(\delta_1, \delta_2, \delta_3)$ in Sylvester's law of inertia. □

Lemma 7.18 allows us to compute the moduli space of $\mathfrak{n}_{7,1}$.

Proposition 7.19. *Let $e_1, \ldots, e_7 \in \mathfrak{n}_{7,1}$ be the basis of $\mathfrak{n}_{7,1}$ given in Table 7.8. Then the subset*

$$\left\{ -e^{2356} + \mathrm{sgn}(b)\left(e^{1245} + e^{1346}\right) + a e^{1237} - \frac{a^2b^2+4}{4b\mu} e^{2347} - \mu e^{1357} + e^{1267} + b e^{4567} \,\middle|\, \mu \in (0,1],\ a \geq 0,\ b \in \mathbb{R}^*,\ a \leq \frac{2}{|b|},\ \mu^2 \geq -\frac{a^2b^2+4}{4b} \right\}$$

of $\Lambda^4 \mathfrak{n}_{7,1}^*$ is bijective to $\mathcal{M}_{\mathrm{G}_2}(\mathfrak{n}_{7,1})$ via $\pi : M^4_{\mathrm{G}_2}(\mathfrak{n}_{7,1}) \to \mathcal{M}_{\mathrm{G}_2}(\mathfrak{n}_{7,1})$. The cocalibrated G_2-structure $\varphi_{a,b,\mu}$ having the above four-form with the same parameters as Hodge dual and inducing the orientation in which $(e_1, e_4, e_2, e_5, e_3, e_6, e_7)$ is oriented is given by

$$\begin{aligned}\varphi_{a,b,\mu} = {} & e^{147} - \mathrm{sgn}(b)\left(e^{257} + e^{367}\right) + \frac{a^2b^2-4}{4b} e^{123} - \frac{a(a^2b^2+4)}{8\mu} e^{234} - \frac{ab\mu}{2} e^{135} \\ & + \frac{ab}{2} e^{126} + b\mu e^{156} + \frac{a^2b^2+4}{4\mu} e^{246} - \frac{a^2b^2+4}{4} e^{345} - \frac{ab^2}{2} e^{456}.\end{aligned}$$

So $\mathcal{M}_{\mathrm{G}_2}(\mathfrak{n}_{7,1})$ is also bijective to $\left\{ \varphi_{a,b,\mu} \,\middle|\, \mu \in (0,1],\ b \in \mathbb{R}^*,\ 0 \leq a \leq \frac{2}{|b|},\ \mu^2 \geq -\frac{a^2b^2+4}{4b} \right\}$ via $\pi : M^3_{\mathrm{G}_2}(\mathfrak{n}_{7,1}) \to \mathcal{M}_{\mathrm{G}_2}(\mathfrak{n}_{7,1})$.

Proof. Let $\mathfrak{u} = \mathrm{span}(e_1, \ldots, e_6)$ be the unique codimension one Abelian ideal in $\mathfrak{n}_{7,1}$ and fix the orientation on $\mathfrak{u}$ in which the ordered basis $(e_1, e_4, e_2, e_5, e_3, e_6)$ is oriented. We remind the reader that $\frac{1}{2}\omega_1^2 = \frac{1}{2}\omega_2^2$ for non-degenerate two-forms $\omega_1, \omega_2 \in \Lambda^2 V^*$ on a real six-dimensional vector space V exactly when $\omega_1 = \pm\omega_2$, cf. Remark 2.7. Hence, Proposition 2.51 and Proposition 2.33 (c) show that

$$(\omega, \rho) \mapsto \Psi := \frac{1}{2}\omega^2 + \rho \wedge e^7.$$

defines a one-to-one correspondence between SU(3)-structures $(\omega, \rho) \in \Lambda^2\mathfrak{u}^* \times \Lambda^3\mathfrak{u}^*$ on $\mathfrak{u}$ such that the orientation induced by ω on $\mathfrak{u}$ coincides with the fixed one and Hodge duals $\Psi \in \Lambda^4\mathfrak{g}^*$ of G_2-structures on $\mathfrak{g}$ such that $g_\Psi(\mathfrak{u}, e_7) = 0$ and $g_\Psi(e_7, e_7) = 1$. The induced action of an element $\mathrm{diag}(g, \mathrm{sgn}(\det(g))) \in \mathrm{H}(\mathfrak{n}_{7,1})$ on the pair (ω, ρ) is given by $\mathrm{sgn}(\det(g))(g.\omega, g.\rho)$. Since $g \in \mathrm{H}(n_{7,1}, \mathfrak{u})$, Lemma 7.18 shows that each $\mathrm{H}(\mathfrak{n}_{7,1})$-orbit of an element in $M^{ON}_{\mathrm{G}_2}(\mathfrak{n}_{7,1})$ contains an element $\Psi \in \Lambda^4\mathfrak{n}^*_{7,1}$ such that the corresponding $\omega \in \Lambda^2\mathfrak{u}^*$ is equal to $\omega_+ := e^{14} + e^{25} + e^{36}$ or to $\omega_- := e^{14} - e^{25} - e^{36}$ and that there is no $\mathrm{H}(\mathfrak{n}_{7,1})$-orbit containing elements $\Psi_1 \in \Lambda^4\mathfrak{n}^*_{7,1}$ and $\Psi_2 \in \Lambda^4\mathfrak{n}^*_{7,1}$ such that the corresponding $\omega_1 \in \Lambda^2\mathfrak{u}^*$ and $\omega_2 \in \Lambda^2\mathfrak{u}^*$ are equal to ω_+ and ω_-, respectively. Thus, to determine $\mathcal{M}^{ON}_{\mathrm{G}_2}(\mathfrak{n}_{7,1})$, we have to determine, for $i = +, -$, the set of all $\rho \in \Lambda^3\mathfrak{u}^*$ such that $(\omega_i, \rho) \in \Lambda^2\mathfrak{u}^* \times \Lambda^3\mathfrak{u}^*$ is an $\mathrm{SU}(3)$-structure modulo the subgroup H_i of $\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u})$ which consists of the elements with positive determinant in $\mathrm{H}(\mathfrak{n}_{7,1}, \mathfrak{u})$ which stabilise ω_i and those of negative determinant which map ω_i onto $-\omega_i$. One obtains

$$\begin{aligned}
\mathrm{H}_+ &= \left\{ \begin{pmatrix} \frac{1}{\mu}C & \mu DC \\ 0 & \delta\mu C \end{pmatrix} \middle| \, C \in \mathrm{O}(3),\, D \in \mathrm{Sym}(3),\, \mu \in \mathbb{R}_+,\, \delta \in \{-1, 1\} \right\} \\
&= (\mathrm{O}(3) \times \mathbb{R}_+ \times \mathbb{Z}_2) \ltimes \mathrm{Sym}(3), \\
\mathrm{H}_- &= \left\{ \begin{pmatrix} \frac{1}{\mu}C & \mu DC \\ 0 & \delta\mu C \end{pmatrix} \middle| \, C \in \mathrm{O}(1,2),\, D \in \mathrm{Sym}(1,2),\, \mu \in \mathbb{R}_+,\, \delta \in \{-1, 1\} \right\} \\
&= (\mathrm{O}(1,2) \times \mathbb{R}_+ \times \mathbb{Z}_2) \ltimes \mathrm{Sym}(1,2),
\end{aligned}$$

where $\mathrm{Sym}(1,2) := \left\{ A \in \mathbb{R}^{3\times 3} \middle| \, (I_{1,2}A)^t = I_{1,2}A \right\}$.

Set $f_i := e_{i+3}$ for $i = 1, 2, 3$, $V_1 := \mathrm{span}(e_1, e_2, e_3)$ and $V_2 := \mathrm{span}(f_1, f_2, f_3)$. The most general three-form $\rho \in \Lambda^3\mathfrak{n}^*_{7,1}$ is given by

$$\rho = a\, e^{123} + \sum_{i,j=1}^{3} a_{ij} e^{i+1\, i+2} \wedge f^j + \sum_{i,j=1}^{3} b_{ij} e^i \wedge f^{j+1\, j+2} + b\, f^{123}$$

with $(a, A = (a_{ij})_{ij}, B = (b_{ij})_{ij}, b) \in \mathbb{R} \times \mathbb{R}^{3\times 3} \times \mathbb{R}^{3\times 3} \times \mathbb{R}$, where we compute the super- and subscripts modulo three. It is easy to check that $\rho \in V_{\omega_+}$, i.e. $\rho \wedge \omega_+ = 0$, exactly when A, $B \in \mathrm{Sym}(3)$ and that $\rho \in V_{\omega_-}$ if and only if A, $B \in \mathrm{Sym}(1,2)$. Note that this is a necessary condition for the pair $(\omega, \rho) \in \Lambda^2\mathfrak{u}^* \times \Lambda^3\mathfrak{u}^*$ being an $\mathrm{SU}(3)$-structure.

Let now $(\omega, \rho) \in \Lambda^2\mathfrak{u}^* \times \Lambda^3\mathfrak{u}^*$ be an $\mathrm{SU}(3)$-structure on $\mathfrak{u}$ with $\omega = \omega_+$ or $\omega = \omega_-$ and describe it equivalently by $(a, A, B, b) \in \mathbb{R}\times\mathrm{Sym}(3)^2\times\mathbb{R}$ or $(a, A, B, b) \in \mathbb{R}\times\mathrm{Sym}(1,2)^2\times\mathbb{R}$, respectively. First, we show that then $a \neq 0$ or $\det(A) \neq 0$. Assume the contrary, i.e. that $a = 0$ and $\det(A) = 0$. Then there exists $v \in V_2$ such that $v \lrcorner \sum_{i,j=1}^{3} a_{ij} e^{i+1\, i+2} \wedge f^j = 0$. Denote by $v^0 \subseteq V_2^*$ the annihilator of v in V_2 and consider it then as a subset of V^*. Let $\alpha \in v^0 \backslash \{0\}$ and $w \in V_1$. Then $\alpha \wedge \rho \in \Lambda^2 V_1^* \wedge \Lambda^2 v^0 \oplus V_1^* \wedge \Lambda^3 V_2^*$ and $w \lrcorner \rho \in V_1^* \wedge v^0 \oplus \Lambda^2 V_2^*$. Thus, $\alpha \wedge (w \lrcorner \rho) \wedge \rho = 0$ and Equation (2.16) shows that $J^*_\rho \alpha \in V_1^0 \cong V_2^*$. Since $\omega^2_\pm \in \Lambda^2 V_1^* \wedge \Lambda^2 V_2^*$, we obtain

$$\alpha \wedge J^*_\rho \alpha \wedge \omega^2_\pm = 0,$$

and Equation (2.17) gives us $g_{(\omega,\rho)}(\alpha,\alpha)=0$, a contradiction. Hence, $a\neq 0$ or $\det(A)\neq 0$.

The action of an element $(C,\mu,\delta)\in \mathrm{O}(p,3-p)\times\mathbb{R}_+\times\mathbb{Z}_2$ on a quadruple $(a,A,B,b)\in\mathbb{R}\times\mathrm{Sym}(p,3-p)^2\times\mathbb{R}$ is easily computed to be

$$
\begin{aligned}
(C,\mu,\delta).(a,A,B,b) &= \left(\frac{\mu^3 a}{\det(C)},\delta\mu\,\mathrm{adj}(C^{-1})A\left(C^{-t}\right)^t,\frac{1}{\mu}C^{-t}B\,\mathrm{adj}(C^{-1})^t,\frac{\delta b}{\mu^3\det(C)}\right)\\
&= \left(\frac{\mu^3 a}{\det(C)},\frac{\delta\mu CAC^{-1}}{\det(C)},\frac{C^{-t}BC^t}{\mu\det(C)},\frac{\delta b}{\mu^3\det(C)}\right).
\end{aligned}
$$

The action of the subgroup $\mathrm{Sym}(p,3-p)$ is more involved and not nicely described in terms of (a,A,B,b). However, for $(a,A,B,b)\in\mathbb{R}\times\mathrm{Sym}(p,3-p)^2\times\mathbb{R}$ and $D\in\mathrm{Sym}(p,3-p)$ we get

$$
D.(a,0,0,0)=(a,-aD,a\,\mathrm{adj}(D),-a\,\det(D))
$$

and $D.(0,A,B,b)=(0,A,B',b')$ for certain $B'\in\mathrm{Sym}(p,3-p)$ and $b'\in\mathbb{R}$. In particular, we see that $\mathrm{H}_\pm$ acts in such a way that it maps quadruples (a,A,B,b) where the first entry does not vanish (resp. vanishes) again to quadruples where the first entry does not vanish (resp. vanishes). We distinguish the cases $a=0$ and $a\neq 0$.

First case $a\neq 0$:
Let $(a,A,B,b)\in\mathbb{R}\times\mathrm{Sym}(p,3-p)^2\times\mathbb{R}$ be given with $a\neq 0$. By the properties of the action of the group $\mathrm{Sym}(p,3-p)$ on $\mathbb{R}\times\mathrm{Sym}(p,3-p)^2\times\mathbb{R}$ given above, we see that there exists $D\in\mathrm{Sym}(p,3-p)$ such that $D.(a,A,B,b)=(a,0,B',b')$ for certain $B'\in\mathrm{Sym}(p,3-p)$ and $b'\in\mathbb{R}$.

Assume first that B' is diagonalisable over the reals by conjugating with an element in $\mathrm{O}(p,3-p)$. Note that this is always possible for $p=3$. Then the H_+-/H_--orbit of (a,A,B,b) contains an element of the form $(a,0,\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3),b')$ for certain $\lambda_1,\lambda_2,\lambda_3\in\mathbb{R}$. A short computation shows that

$$
\lambda(\rho)=\left(a^2d^2+4a\lambda_1\lambda_2\lambda_3\right)\left(e^{142536}\right)^{\otimes 2}
$$

and so we must have $a\lambda_1\lambda_2\lambda_3=a\det(\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3))<0$. Hence, for appropriate $\mu_1,\mu_2,\mu_3\in\mathbb{R}^*$, we get

$$
\mathrm{diag}\left(-\frac{\mu_1}{a},-\frac{\mu_2}{a},-\frac{\mu_3}{a}\right).(a,0,\mathrm{diag}(\lambda_1,\lambda_2,\lambda_3),b')=(a,\mathrm{diag}(\mu_1,\mu_2,\mu_3),0,b'')
$$

for a certain $b''\in\mathbb{R}^*$. Obviously, $\mu_i\neq 0$ for some $i\in\{1,2,3\}$. Thus, the explicit description of the action of $\mathrm{O}(p,3-p)\times\mathbb{R}_+\times\mathbb{Z}_2$ on $(a,\mathrm{diag}(\mu_1,\mu_2,\mu_3),0,b'')$ given above shows that there is exactly one element in the $(\mathrm{O}(p,3-p)\times\mathbb{R}_+\times\mathbb{Z}_2)$-orbit of $(a,\mathrm{diag}(\mu_1,\mu_2,\mu_3),0,b'')$ which is of the form $(g,\mathrm{diag}(\tau,\mu,1),0,h)$ with $g>0$, $\mu\leq 1$ and additionally $\mu\geq 0$ and $\tau\leq\mu$ if $p=3$. In fact, to get uniqueness, we need in the case $p=3$ that $\mu\tau\neq 0$ and for $p=1$ we need $\mu>0$. But these conditions will follow from the computations given below. So suppose now that we have $\rho\in\Lambda^3\mathfrak{g}^*$ given by

$(g, \mathrm{diag}(\tau, \mu, 1), 0, h)$ with the above properties. Since (ω, ρ) is an $\mathrm{SU}(p, 3-p)$-structure, we must have $\phi(\rho) = 2\phi(\omega)$ by Corollary 2.34 and this is equivalent to

$$-4 = g^2h^2 + 4\mu h\tau \Leftrightarrow \tau = -\frac{g^2h^2+4}{4\mu h}.$$

Imposing this condition, the principal minors of the induced metric are given by

$$\mu, \frac{\epsilon(g^2h^2+4)}{4h}, \frac{(g^2h^2+4)^2}{16h^2}, \frac{(g^2h^2+4)^2}{16\mu h^2}, \frac{\epsilon(g^2h^2+4)}{4h}, 1$$

with $\epsilon = -1$ if $p = 3$ and $\epsilon = 1$ if $p = 1$. All these minors are positive if and only if $\mu > 0$ and $\epsilon h > 0$. Hence, $(g, \mathrm{diag}(\tau, \mu, 1), 0, h) = \left(g, \mathrm{diag}\left(-\frac{g^2h^2+4}{4\mu h}, \mu, 1\right), 0, h\right)$ with $g > 0$, $\mu > 0$ and $\epsilon h > 0$. Note that

$$(-1).\left(D_1.\left(g, \mathrm{diag}\left(-\frac{g^2h^2+4}{4\mu h}, \mu, 1\right), 0, h\right)\right) = \left(g, \mathrm{diag}\left(-\frac{g^2h^2+4}{4\mu h}, \mu, 1\right), 0, \frac{4}{g^2h}\right)$$

with $-1 \in \mathbb{Z}_2 \subseteq \mathrm{H}_+/\mathrm{H}_-$ and $D_1 := \mathrm{diag}\left(-\frac{2(g^2h^2+4)}{4\mu hg}, \frac{2\mu}{g}, \frac{2}{g}\right) \in \mathrm{Sym}(p, 3-p)$. Hence, we may additionally assume that $g \leq \frac{2}{|h|}$. Moreover, if $p = 3$, we must have $-\frac{g^2h^2+4}{4\mu h} \leq \mu$ and this is equivalent to $\mu^2 \geq -\frac{g^2h^2+4}{4h}$. This inequality is also true if $p = 1$ since then $h > 0$.

Hence, there is an element $\left(g, \mathrm{diag}\left(-\frac{g^2h^2+4}{4\mu h}, \mu, 1\right), 0, h\right)$ with $0 < \mu \leq 1$, $\epsilon h > 0$, $0 < g \leq \frac{2}{|h|}$ and $\mu^2 \geq -\frac{g^2h^2+4}{4h}$ in each H_+-/H_--orbit of an element (a, A, B, b) with $a \neq 0$. We claim that there is exactly one. Therefore, let $E \in \mathrm{H}_+/\mathrm{H}_-$ be such that it maps $\left(g, \mathrm{diag}\left(-\frac{g^2h^2+4}{4\mu h}, \mu, 1\right), 0, h\right)$ again to an element of the same form and with the same relations for the parameters. Write $E = CD$ with unique $C \in \mathrm{O}(p, 3-p) \times \mathbb{R}_+ \times \mathbb{Z}_2$ and $D \in \mathrm{Sym}(p, 3-p)$. The exact form of the action of C on elements in $\mathbb{R} \times \mathrm{Sym}(p, 3-p)^2 \times \mathbb{R}$ requires that D acts in such a way that the third entry of $D.(g, \mathrm{diag}(\tau, \mu, 1), 0, h)$, which lies in $\mathrm{Sym}(p, 3-p)$, is 0. This is a quadratic equation in the components of D and can be solved by the *Maple* function *solve*. The two solutions are $D = 0$ and $D = D_1$, D_1 as above. The uniqueness now follows by looking at the explicit form of $D_1.\left(g, \mathrm{diag}\left(-\frac{g^2h^2+4}{4\mu h}, \mu, 1\right), 0, h\right)$ computed above and taking into account our remark above that in each $(\mathrm{O}(p, 3-p) \times \mathbb{R}_+ \times \mathbb{Z}_2)$-orbit of an element of the form $(g', \mathrm{diag}(\tau', \mu', 1), 0, h')$ with $g' > 0$, $0 < \mu' \leq 1$, $\tau' \neq 0$ and additionally $\tau' \leq \mu'$ if $p = 3$, there is only the element itself which is of the same form and which fulfils the same relations.

Next, we have to consider $(a, 0, B', b')$ such that B' is not diagonalisable over the reals by conjugating with an element in $\mathrm{O}(p, 3-p)$. Then $p = 1$. Moreover, by [DPWZ], cf. also [MX], B' is then not diagonalisable over the reals at all and either B' can be brought by an element of $\mathrm{O}(1, 2)$ into a block diagonal matrix with a two-by-two block and a one-by-one block or the Jordan normal form of B' consists of one block of size three. Hence, B' is conjugate to

$$\begin{pmatrix} c_1 & c_3 & \\ -c_3 & c_2 & \\ & & c_4 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sigma & 0 & \frac{1}{\sqrt{2}} \\ 0 & \sigma & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sigma \end{pmatrix}$$

for certain c_1, c_2, c_3, c_4, $\sigma \in \mathbb{R}$ under the action of $\mathrm{O}(1,2)$. In the first case, the sub-block $\begin{pmatrix} c_1 & c_3 \\ -c_3 & c_2 \end{pmatrix}$ is not diagonalisable over the reals and so the determinant of it, given by $c_1c_2 + c_3^2$, has to be non-negative. Corollary 2.34 yields the identity $c_4 = -\frac{a^2b'^2+4}{4a(c_1c_2+c_3^2)}$. Imposing this relation, the second principal minor is given $-a^2(c_1c_2 + c_3^2)$ and so not positive. Hence, this case cannot occur. The second case can be excluded analogously. Corollary 2.34 yields $-4 = a^2b'^2 + 4a\sigma^3$ and imposing this relation, the second principal minor is given by $-a^2\sigma^2$ and is negative, a contradiction.

Second case $a = 0$:

Then $A \in \mathrm{GL}(3,\mathbb{R})$. We first show that there is exactly one element in the $\mathrm{Sym}(p, 3-p)$-orbit of $(0, A, B, b)$ which is of the form $(0, A, 0, c)$. It suffices to show that the linear map $\mathrm{Sym}(p, 3-p) \ni D \mapsto G(D) \in \mathrm{Sym}(p, 3-p)$, defined by $D.(0, A, 0, 0) = (0, A, G(D), g)$ with $g \in \mathbb{R}$, is a linear isomorphism for which we only have to show that its kernel is $\{0\}$. The three-form $\rho \in \Lambda^2 V_1^* \wedge V_2^*$ corresponding to $(0, A, 0, 0)$ is given by

$$\rho = \sum_{i,j=1}^{3} a_{ij} e^{i+1\, i+2} \wedge f^j \in \Lambda^2 V_1^* \wedge V_2^*.$$

Denote by $\tilde{\rho}$ the part of $D.\rho$ which is in $V_1^* \wedge \Lambda^2 V_2^*$, i.e. the one corresponding to $G(D)$. Then we have

$$\begin{aligned}
\tilde{\rho} &= \sum_{i,j,k=1}^{3} a_{ij}\, d_{i+1\,k}\; e^{i+2} \wedge f^{kj} - \sum_{i,j,k=1}^{3} a_{ij}\, d_{i+2\,k}\; e^{i+1} \wedge f^{kj} \\
&= \sum_{i,m,r,j,k=1}^{3} \epsilon_{imr}\, a_{ij}\, d_{mk}\; e^r \wedge f^{kj} = - \sum_{i,m,r,j,k,s=1}^{3} \epsilon_{imr}\, \epsilon_{jks}\, a_{ij}\, d_{mk}\; e^r \wedge f^{s+1\, s+2} \\
&= - \sum_{i,m,r,s=1}^{3} \epsilon_{imr}\, (a_i \times d_m)_s\; e^r \wedge f^{s+1\, s+2},
\end{aligned}$$

where $a_i := (a_{i1}, a_{i2}, a_{i3})^t \in \mathbb{R}^3$ for $i = 1, 2, 3$, $d_m := (d_{m1}, d_{m2}, d_{m3})^t \in \mathbb{R}^3$ for $m = 1, 2, 3$, $\times$ is the standard cross product on $\mathbb{R}^3$ given in Example 1.12 (a) and the subscript s denotes the s-th entry. This is 0 exactly when

$$a_1 \times d_2 = a_2 \times d_1,\; a_1 \times d_3 = a_3 \times d_1,\; a_2 \times d_3 = a_3 \times d_2.$$

Hence, $(a_1 \times a_2) \times (a_1 \times d_2) = (a_1 \times a_2) \times (a_2 \times d_1)$. Using the Grassmann identity $v_1 \times (v_2 \times v_3) = \langle v_1, v_3\rangle v_2 - \langle v_1, v_2\rangle v_3$ and the fact that a_1 and a_2 are linearly independent since $A \in \mathrm{GL}(3,\mathbb{R})$, we get that d_1 and d_2 are orthogonal to $\mathrm{span}(a_1 \times a_2)$. Doing the same for the other two equations, we get that for all $i, j \in \{1, 2, 3\}$, d_i is orthogonal to $\mathrm{span}(a_i \times a_j)$ and so to $a_i^{\perp}$. Hence, $d_i = \alpha_i a_i$ for certain $\alpha_i \in \mathbb{R}$, $i = 1, 2, 3$. Using again the above equations, we see that $\alpha_2 = -\alpha_1$, $\alpha_3 = -\alpha_1$ and $\alpha_3 = -\alpha_2$. Thus, $\alpha_1 = \alpha_2 = \alpha_3 = 0$

and the map G is a linear isomorphism. Thus, we may find $D \in \mathrm{Sym}(p, 3-p)$ such that $D.(0, A, B, b) = (0, A, 0, b')$ for some $b' \in \mathbb{R}$.

Analogously to the first case, we can exclude the case that A is not diagonalisable over the reals by conjugating with an element in $\mathrm{O}(p, 3-p)$. Then we may apply the same computations as in the first case and see that $(0, A, 0, b')$ contains a unique element of the form $\left(0, \mathrm{diag}\left(-\frac{1}{\mu h}, \mu, 1\right), 0, h\right)$ with $1 \geq \mu > 0$, $\epsilon h > 0$ and $\mu^2 \geq -\frac{1}{h}$, where again $\epsilon = -1$ if $p = 3$ and $\epsilon = 1$ if $p = 1$. This finishes the proof since the corresponding G_2-structures are easily computed. □

7.4 Hitchin flow on some examples

In this section, we consider the Hitchin flow on the Lie algebras $\mathfrak{h}_3 \oplus \mathbb{R}^4$ and $\mathfrak{n}_{7,1}$. We show that the Hitchin flow on $\mathfrak{h}_3 \oplus \mathbb{R}^4$ yields for all initial values a Riemannian metric with holonomy equal to $\mathrm{SU}(2)$. Moreover, we show that the Hitchin flow on $\mathfrak{n}_{7,1}$ with initial value $\varphi_{0,b,\mu} \in \Lambda^3\mathfrak{n}_{7,1}^*$ as in Proposition 7.19 yields for "generic" parameters b, μ a Riemannian metric with holonomy equal to the maximal possible one, namely $\mathrm{SU}(4)$.

We begin with the Hitchin flow on $\mathfrak{h}_3 \oplus \mathbb{R}^4$.

Proposition 7.20. *Set $f : \left(-\infty, \frac{2}{3}\right) \to \mathbb{R}$, $f(t) := \left(-\frac{3}{2}t + 1\right)^{\frac{2}{3}}$ and let H_3 be the simply-connected Lie group with Lie algebra $\mathfrak{h}_3$. The maximal solution $\varphi(t) : \left(-\infty, \frac{2}{3}\right) \to \Lambda^3\mathfrak{g}^*$ of the Hitchin flow on $\mathfrak{g} := \mathfrak{h}_3 \oplus \mathbb{R}^4$ with starting value $\varphi(0) = \varphi_0$, φ_0 as in Proposition 7.17, is given by*

$$\varphi(t) = \sqrt{f(t)}\left(e^{123} + e^{145} + e^{167} + e^{246} - e^{257}\right) - \frac{1}{\sqrt{f(t)}}\left(e^{347} + e^{356}\right)$$

and the induced Riemannian metric on $\mathrm{H}_3 \times \mathbb{R}^4 \times \left(-\infty, \frac{2}{3}\right)$ is given by

$$g = f(t)\left(e^1 \otimes e^1 + e^2 \otimes e^2\right) + \frac{1}{f(t)} e^3 \otimes e^3 + \sum_{i=4}^{7} e^i \otimes e^i + dt^2,$$

where $e^1, \ldots, e^7$ is a basis of the left-invariant one-forms on H_3 with $de^3 = -e^{12}$ and $de^i = 0$ for $i \neq 3$. Moreover, if $\tilde{\varphi}_0 \in \Lambda^3\left(\mathfrak{h}_3 \oplus \mathbb{R}^4\right)^*$ is an arbitrary cocalibrated G_2-structure on $\mathfrak{h}_3 \oplus \mathbb{R}^4$ and $\tilde{\varphi}(t) : I \to \Lambda^3\left(\mathfrak{h}_3 \oplus \mathbb{R}^4\right)^*$ is the solution of Hitchin's flow equations with initial value $\tilde{\varphi}(0) = \tilde{\varphi}_0$, then the induced Riemannian metric $g := g_{\tilde{\varphi}}(t) + dt^2$ on $\mathrm{H}_3 \times \mathbb{R}^4 \times I$ has holonomy equal to $\mathrm{SU}(2)$.

Proof. Obviously, $\varphi(0) = \varphi_0$. Moreover, one can compute that

$$\star_{\varphi(t)}\varphi(t) = -f(t)\left(e^{1247} + e^{1256}\right) - e^{1346} + e^{1357} + e^{2345} + e^{2367} + e^{4567}.$$

Note that the explicit form of $\star_{\varphi(t)}\varphi(t)$ also follows from the fact that

$$\left(\frac{1}{\sqrt{f(t)}} e_1, \frac{1}{\sqrt{f(t)}} e_2, \sqrt{f(t)} e_3, e_4, e_5, e_6, e_7\right)$$

is an adapted basis for $\varphi(t)$. Since $de^3 = -e^{12}$, $de^i = 0$ for $i \neq 3$ and $\dot{f}(t) = -\frac{1}{\sqrt{f(t)}}$, $\varphi(t)$ fulfils Hitchin's flow equation, i.e. $\frac{d}{dt} \star_{\varphi(t)} \varphi(t) = d\varphi(t)$.

The adapted basis given above shows that the Riemannian manifold $(\mathrm{H}_3 \times \left(-\infty, \frac{2}{3}\right),\ g := g_{\varphi(t)} + dt^2)$ is the direct product of the Riemannian manifold $(\mathrm{H}_3 \times \left(-\infty, \frac{2}{3}\right), h(t) + dt^2)$ and of the Riemannian manifold $\mathbb{R}^4$ endowed with the standard metric. Here, the metric $h(t)$ is given in the above left-invariant frame e_1, e_2, e_3 of H_3 by $\operatorname{diag}\left(f(t), f(t), \frac{1}{f(t)}\right) \in \mathrm{GL}(3,\mathbb{R})$. By Theorem 7.12, the holonomy is a subgroup of $\mathrm{SU}(4)$ which acts trivially on a four-dimensional subspace. Thus, $\mathrm{Hol}(g) = \{e\}$ or $\mathrm{Hol}(g) = \mathrm{SU}(2)$, cf. also [J3, Theorem 10.5.7]. A short computation shows that the Riemann curvature tensor does not vanish. Therefore, $\mathrm{Hol}(g) = \mathrm{SU}(2)$ for the initial value $\varphi_0 \in \Lambda^3 \left(\mathfrak{h}_3 \oplus \mathbb{R}^4\right)^*$. By Proposition 7.17, all cocalibrated G_2-structure $\tilde{\varphi}_0$ on $\mathfrak{h}_3 \oplus \mathbb{R}^4$ lie in the $\left(\mathrm{Aut}\left(\mathfrak{h}_3 \oplus \mathbb{R}^4\right) \times \mathbb{R}\right)$-orbit of φ_0. Since H_3 is simply-connected, all Lie algebra automorphisms lift to Lie group automorphisms and, by Remark 7.8, the Riemannian manifold obtained by the Hitchin flow with initial value $\tilde{\varphi}_0$ has also holonomy equal to $\mathrm{SU}(2)$. This finishes the proof. □

Remark 7.21. *The explicit Riemannian metric of holonomy* $\mathrm{SU}(2)$ *obtained in Proposition 7.20 is the Riemannian product of the Riemannian metric obtained in [ChiFi] by the Hitchin flow for some* $\mathrm{SU}(3)$*-structure on* $\mathfrak{h}_3 \oplus \mathbb{R}^3$ *and of* $\mathbb{R}$ *with the standard metric. This is no surprise since, more generally, if* $\mathfrak{g}$ *is any six-dimensional Lie algebra and* $(\omega(t), \rho(t))$ *is a solution of the Hitchin flow on* $\mathfrak{g}$ *with initial half-flat* $\mathrm{SU}(3)$*-structure* (ω_0, ρ_0)*, then, for any non-zero* $\alpha \in (\mathfrak{g} \oplus \mathbb{R})^*$ *lying in the annihilator of* $\mathfrak{g}$ *in* $\mathfrak{g} \oplus \mathbb{R}$*,* $\varphi(t) := \omega(t) \wedge \alpha + J^*_{\rho(t)}\rho(t)$ *is a solution of the Hitchin flow on* $\mathfrak{g} \oplus \mathbb{R}$ *with initial cocalibrated* G_2*-structure* $\varphi_0 := \omega_0 \wedge \alpha + J^*_{\rho_0}\rho_0$*, cf. Proposition 3.37 and [Sto].*

Next we consider the Hitchin flow on $\mathfrak{n}_{7,1}$. In Proposition 7.19, we described the moduli space of all cocalibrated G_2-structures by a set of three-forms $\varphi_{a,b,\mu} \in \Lambda^3\mathfrak{n}^*_{7,1}$ depending on three parameters a, b and μ. The solution of the Hitchin flow with arbitrary initial value $\varphi_{a,b,\mu}$ seems to be very hard to obtain. One reason for the difficulties is that the Euclidean metric $g_{\varphi_{a,b,\mu}}$ is, in general, not diagonal in the basis $e_1, \ldots, e_7$ of $\mathfrak{n}_{7,1}$ given in Table 7.8. However, if $a = 0$, it is diagonal and it possible to explicitly solve the Hitchin flow. One gets that the Euclidean metric $g_{\varphi(t)}$ stays diagonal and the Hitchin flow yields, for "generic" b and μ, a Riemannian metric with holonomy equal to $\mathrm{SU}(4)$, which is the maximal possible one by Theorem 7.12.

Proposition 7.22. *Let*

$$P := \left\{ (a, b, \mu) \in \mathbb{R}^3 \,\middle|\, b \neq 0,\ 0 \leq a \leq \frac{2}{|b|},\ 0 < \mu \leq 1,\ \mu^2 \geq -\frac{a^2b^2+4}{4b} \right\} \subseteq \mathbb{R}^3, \qquad (7.10)$$

be the space of parameter values for the moduli space of cocalibrated G_2*-structures on* $\mathfrak{n}_{7,1}$ *and denote for* $(a,b,\mu) \in P$ *by* $\varphi_{a,b,\mu} \in \Lambda^3\mathfrak{n}^*_{7,1}$ *the cocalibrated* G_2*-structure on* $\mathfrak{n}_{7,1}$ *given in Proposition 7.19. Moreover, let* $N_{7,1}$ *be a Lie group with Lie algebra* $\mathfrak{n}_{7,1}$*.*

(a) *Let* $(0,b,\mu)\in P$ *be given. Define functions* $f_i:\mathbb{R}\to\mathbb{R}$ *for* $i=1,2,3$ *by* $f_1(x):=-x+\frac{1}{b\mu}$*,* $f_2(x):=x+\mu$ *and* $f_3(x):=x+1$*. Set* $I_1:=\left(-\mu,\frac{1}{b\mu}\right)$ *and* $I_{-1}:=\left(\frac{1}{b\mu},\infty\right)$*. Let* $x:I_{max}\to\mathbb{R}$ *be the maximal solution of the initial value problem*

$$\frac{dx}{dt}=-\frac{1}{b\sqrt{bf_1(x)f_2(x)f_3(x)}},\quad x(0)=0. \tag{7.11}$$

Then the maximal solution $\varphi(t)$ *of the Hitchin flow on* $\mathfrak{n}_{7,1}$ *with initial value* $\varphi(0)=\varphi_{0,b,\mu}$ *is defined on the interval* I_{max}*,* $x(I_{max})=I_{\mathrm{sgn}(b)}$ *and* $\tilde{\varphi}(x):I_{\mathrm{sgn}(b)}\to\Lambda^3\mathfrak{n}^*_{7,1}$*, defined by* $\tilde{\varphi}(x(t)):=\varphi(t)$ *for all* $t\in I_{max}$*, is given by*

$$\begin{aligned}\tilde{\varphi}(x)=&\sqrt{bf_1(x)f_2(x)f_3(x)}\left(e^{147}-\mathrm{sgn}(b)\left(e^{257}+e^{367}\right)\right)-\frac{1}{b\sqrt{bf_1(x)f_2(x)f_3(x)}}e^{123}\\&+\mathrm{sgn}(b)\sqrt{\frac{bf_2(x)f_3(x)}{f_1(x)}}e^{156}+\sqrt{\frac{bf_1(x)f_3(x)}{f_2(x)}}e^{246}-\sqrt{\frac{bf_1(x)f_2(x)}{f_3(x)}}e^{345}.\end{aligned} \tag{7.12}$$

The induced Riemannian metric g *on* $N_{7,1}\times I_{\mathrm{sgn}(b)}$ *is given in the variable* x *by*

$$\begin{aligned}g=&\sum_{i=1}^{3}\frac{1}{|bf_i(x)|}\,e^i\otimes e^i+\sum_{j=4}^{6}|bf_{j-3}(x)|\,e^j\otimes e^j+\ b\,f_1(x)f_2(x)f_3(x)\,e^7\otimes e^7\\&+b^3f_1(x)f_2(x)f_3(x)\,dx\otimes dx.\end{aligned} \tag{7.13}$$

For an open and dense subset of $P_0:=P\cap\{0\}\times\mathbb{R}^2$*, the holonomy of* g *is equal to* $\mathrm{SU}(4)$*.*

(b) *There exists an open neighbourhood* $U\subseteq P$ *of* $\left(0,1,\frac{1}{2}\right)$ *in* P *such that for all* $(a,b,\mu)\in U$ *any solution* $\varphi:I_{a,b,\mu}\to\Lambda^3\mathfrak{n}^*_{7,1}$ *of Hitchin's flow equations with initial value* $\varphi(0)=\varphi_{a,b,\mu}$ *induces a Riemannian metric* $g_{a,b,\mu}$ *on* $N_{7,1}\times I_{(a,b,\mu)}$ *with holonomy equal to* $\mathrm{SU}(4)$*.*

Proof. Let $(0,b,\mu)\in P$. Note that $I_{\mathrm{sgn}(b)}$ is the maximal interval around $x=0$ for which $b\sqrt{bf_1(x)f_2(x)f_3(x)}\neq 0$. Hence, by separation of variables, the unique maximal solution $x:I_{max}\to\mathbb{R}$ of the initial value problem

$$\frac{dx}{dt}=-\frac{1}{b\sqrt{bf_1(x)f_2(x)f_3(x)}},\quad x(0)=0$$

fulfils $x(I_{max})=I_{\mathrm{sgn}(b)}$ and x is a strictly monotone diffeomorphism from I_{max} to $I_{\mathrm{sgn}(b)}$. We define $\varphi:I_{max}\to\Lambda^3\mathfrak{n}^*_{7,1}$ by $\varphi(t):=\tilde{\varphi}(x(t))$, where $\tilde{\varphi}:I_{\mathrm{sgn}(b)}\to\Lambda^3\mathfrak{n}^*_{7,1}$ is defined by Equation (7.12), and check that it is a solution of the Hitchin flow with initial value $\varphi_{0,b,\mu}$. Note that obviously the three-form $\varphi:I_{max}\to\Lambda^3\mathfrak{n}^*_{7,1}$ cannot be extended to the boundary of I_{max} since $\tilde{\varphi}$ cannot be extended to the boundary and so I_{max} is the maximal existence interval if φ is a solution of the mentioned initial value problem.

Obviously, we have $\varphi(0) = \tilde{\varphi}(x(0)) = \tilde{\varphi}(0) = \varphi_{0,b,\mu}$ since $x(0) = 0$. A dual adapted basis of $\tilde{\varphi}(x)$ is given by

$$\left(-\sqrt{bf_1(x)f_2(x)f_3(x)}e^7, \mathrm{sgn}(b)\sqrt{|bf_1(x)|}e^4, \frac{\mathrm{sgn}(b)}{\sqrt{|bf_1(x)|}}e^1, \sqrt{|bf_2(x)|}e^5, \frac{-\mathrm{sgn}(b)}{\sqrt{|bf_2(x)|}}e^2, \frac{-\mathrm{sgn}(b)}{\sqrt{|bf_3(x)|}}e^3, -\sqrt{|bf_3(x)|}e^6\right)$$

and so we obtain the identity

$$\begin{aligned}\star_{\tilde{\varphi}(x)}\tilde{\varphi}(x) = &- e^{2356} + \mathrm{sgn}(b)\left(e^{1245} + e^{1346}\right) - f_1(x)e^{2347} - f_2(x)e^{1357}\\ &+ f_3(x)e^{1267} + b^2 f_1(x)f_2(x)f_3(x)e^{4567}\end{aligned}$$

for the Hodge dual. Using $\frac{dx}{dt} = -\frac{1}{b\sqrt{bf_1(x)f_2(x)f_3(x)}}$, $f_1'(x) = -1$, $f_2'(x) = f_3'(x) = 1$ and $f_2(x)$, $f_3(x) > 0$, $bf_1(x) > 0$ for all $x \in I_{\mathrm{sgn}(b)}$, one obtains the identity

$$\begin{aligned}\frac{d}{dt}\left(\star_{\varphi(t)}\varphi(t)\right) &= \frac{d}{dt}\left(\star_{\tilde{\varphi}(x(t))}\tilde{\varphi}(x(t))\right)\\ &= -\frac{1}{b\sqrt{bf_1f_2f_3}}(x(t))\, e^{2347} + \frac{1}{b\sqrt{bf_1f_2f_3}}(x(t))\, e^{1357}\\ &\quad - \frac{1}{b\sqrt{bf_1f_2f_3}}(x(t))\, e^{1267} + b\left(\frac{f_2f_3 - f_1f_3 - f_1f_2}{\sqrt{bf_1f_2f_3}}\right)(x(t))\, e^{4567}\\ &= d(\tilde{\varphi}(x(t))) = d(\varphi(t)).\end{aligned}$$

Hence, $\varphi(t)$ is a solution of Hitchin's flow equations with initial value $\varphi_{0,b,\mu}$. Since adapted bases are orthonormal and $\frac{dx}{dt} = -\frac{1}{b\sqrt{bf_1(x)f_2(x)f_3(x)}}$, the induced Riemannian metric $g = g_{\tilde{\varphi}(x)} + \frac{1}{\left(\frac{dt}{dx}\right)^2}\, dx^2$ has the claimed form on $M := N_{7,1} \times I_{\mathrm{sgn}(b)}$. To determine the holonomy of g, we use *Maple* to compute the components of the Riemann curvature tensor $R^g \in \Omega^2 M \otimes \mathrm{End}(TM)$ with respect to the global frame $\left(e_1, \ldots, e_7, \frac{\partial}{\partial x}\right)$. Note that the components depend only on $x \in I_{\mathrm{sgn}(b)}$. By the theorem of Ambrose-Singer, cf. Theorem 3.22, $V_p := \mathrm{span}(R^g_p(v,w)|\, v, w \in T_pM)$ is a subspace of the holonomy algebra $\mathfrak{hol}_p(g)$ for all $p \in M$. For arbitrary $(0, b, \mu) \in P$, we compute $\dim(V_{(e,0)}) = 15$ with the use of *Maple* by determining the rank of $R^g_{(e,0)} \in \mathrm{End}\left(\Lambda^2 T_{(e,0)}M\right)$. However, *Maple* assumes "generic" parameter values b, μ, i.e. it ignores that certain polynomial combinations of b and μ can get zero. So we can only ensure $\dim(V_{(e,0)}) = 15$ for an open and dense subset of P_0. For this subset of P_0, we get that $\dim(Hol(g)) \geq 15$ and so $Hol(g) = \mathrm{SU}(4)$ since by Theorem 7.12 $Hol(g)$ is a Lie subgroup of the connected 15-dimensional Lie group $\mathrm{SU}(4)$. For the concrete values $(0, b, \mu) = \left(0, 1, \frac{1}{2}\right)$, we calculate $\dim(V_{(e,0)}) = 15$ and so $\dim(V_{(e,0)}) = 15$ is also true for any solution of the Hitchin flow with initial value $\varphi_{a,b,\mu} \in \Lambda^3\mathfrak{n}^*_{7,1}$ and (a, b, μ) in a small open neighbourhood U of $\left(0, 1, \frac{1}{2}\right)$ in P. Hence, (b) follows. □

Remark 7.23. *By Theorem 7.12, the Hitchin flow on an almost Abelian Lie algebra yields Riemannian metrics with holonomy contained in* SU(4)*. We saw that* SU(4) *can be achieved and that the holonomy can also be a proper non-trival subgroup of* SU(4)*, cf. Proposition 7.22 and Proposition 7.20, respectively. The holonomy group can also be trivial, which is the case when the initial value is one of the flat* G_2*-structures given in Theorem 4.20.*

Outlook

We encountered in this thesis several problems which remain unanswered and may be considered in future research. We remarked at the beginning of Chapter 7 that the investigation of the Hitchin flow for cocalibrated G_2^ϵ-structures on Lie algebras is an ongoing project. Hence, many of the open problems are related to the Hitchin flow. Nevertheless, there are still some interesting open questions related to the classification of certain structures on six- and seven-dimensional Lie algebras which remained unsolved in this thesis.

- In Remark 6.15 we already noted that for the construction of half-flat SU(3)-structures on certain six-dimensional almost nilpotent Lie algebras $\mathfrak{h}$, we constructed case-by-case cocalibrated G_2-structures on $\mathfrak{h} \oplus \mathbb{R}$ using that the Chevalley-Eilenberg differential on these Lie algebras is not too complicated. Hence, there is some hope to generalise our methods from the almost Abelian case to other types of almost nilpotent Lie algebras and get analogous classification results for these types.

- Another missing classification is the one of almost Abelian Lie algebras $\mathfrak{g}$ with codimension one Abelian ideal $\mathfrak{u}$ admitting a parallel G_2^*-structure with degenerate $\mathfrak{u}$. In Section 4.4, we saw that a parallel G_2^*-structure on such a Lie algebra with non-degenerate $\mathfrak{u}$ is flat. After the submission of this thesis, the author found examples of parallel G_2^*-structures on almost Abelian Lie algebras with degenerate $\mathfrak{u}$ which are not flat, similarly to the pseudo-Riemannian symmetric spaces found by Kath in [Kath2]. A further investigation of this phenomenon seems to be worthwhile in future work.

- In Chapter 5, we obtained a classification of the direct sums $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ of four-dimensional Lie algebras $\mathfrak{g}_4$ and three-dimensional Lie algebras $\mathfrak{g}_3$ admitting a cocalibrated G_2-structure. One possible direction for future work is to classify also the direct sums of four- and three-dimensional Lie algebras admitting cocalibrated G_2^*-structures. In Remark 5.11, we already showed that an analogue of Proposition 5.10 is true for left-invariant G_2^*-structures on Lie groups. Using this analogue, one obtains, analogously to the proof of Proposition 5.12, that direct sums $\mathfrak{g}_4 \oplus \mathfrak{g}_3$ with

$\mathfrak{g}_3 \in \{\mathfrak{so}(3), \mathfrak{so}(2,1), e(2), e(1,1)\}$ always admit cocalibrated G_2^*-structures and that direct sums $\mathfrak{g}_4 \oplus \mathfrak{h}_3$ admit cocalibrated G_2^*-structures if $\mathfrak{g}_4$ admits a symplectic two-form. Similarly, one can generalise the proof of Proposition 5.16 to get an analogue obstruction to the existence of cocalibrated G_2^*-structures as the one in Proposition 5.16 for the G_2-case. Besides, one may also try to adapt our methods to calibrated G_2^ϵ-structures to get a classification of the direct sums $\mathfrak{g} = \mathfrak{g}_4 \oplus \mathfrak{g}_3$ which admit such structures.

- Cocalibrated G_2-structures on arbitrary seven-dimensional manifolds admit a unique G_2-connection ∇^c such that the corresponding torsion tensor T^c is skew-symmetric, cf. [FI]. An investigation of this characteristic connection for cocalibrated G_2-structures on Lie algebras may turn out fruitful. For example, one may find non-flat cocalibrated G_2-structure with harmonic torsion tensor T^c on Lie algebras. Cocalibrated G_2-structures of this kind on arbitrary manifolds are partial solutions of Strominger's equations [Str] in type II superstring theory with constant dilaton, cf. [FI]. Examples of such structures have been found in [Fri1] and, very recently, have been further investigated in [Fri2].

- As already stated in Chapter 6, the existence problem of half-flat $\mathrm{SU}(3)$-structures on Lie algebras remains unsolved only for the class of six-dimensional indecomposable solvable Lie algebras with four-dimensional nilradical. A classification of all such Lie algebras has been obtained by Turkowski in [Tu2] and so one may try to finish the classification using this list. One major obstacle in this case is that these Lie algebras are not almost nilpotent and so the Chevalley-Eilenberg differential is more involved. In particular, the construction of examples is much harder, cf. Remark 6.15. Note that also the application of our obstruction may be more difficult since the exceptional case $A_{4,5}^{-\frac{1}{2},-\frac{1}{2}} \oplus \mathfrak{r}_2$ in Theorem 6.9 has four-dimensional nilradical.

For the Hitchin flow there are several interesting future research directions:

- First, one may try to prove the conjecture given in Remark 7.13. Namely, that the Hitchin flow on an almost Abelian Lie algebra $\mathfrak{g}$ with initial value a cocalibrated G_2^*-structure such that a codimension one Abelian ideal $\mathfrak{u}$ has signature $(2,4)$ or $(3,3)$ yields a pseudo-Riemannian manifold with holonomy contained in $\mathrm{SU}(2,2)$ or $\mathrm{SL}(4,\mathbb{R})$, respectively. Also the the Hitchin flow on $\mathfrak{g}$ with degenerate $\mathfrak{u}$ may be of interest.

- One may consider the moduli space of and then the Hitchin flow for cocalibrated G_2^*-structures on $\mathfrak{n}_{7,1}$. A subspace of this moduli space is given by three-forms of the form $\varphi_{0,b,\mu}$ with different values of b and μ as in the G_2-case. For these initial values, we conjecture that the outcome for "generic" parameter values is a pseudo-Riemannian manifold with holonomy equal to $\mathrm{SU}(2,2)$ or $\mathrm{SL}(4,\mathbb{R})$, respectively, depending on the signature of a codimension one Abelian ideal $\mathfrak{u}$.

- Similarly, one may look at the moduli spaces of and the Hitchin flow for cocalibrated G_2-structures on other almost Abelian Lie algebras. An interesting question is if one can characterise the cocalibrated G_2-structures on almost Abelian Lie algebras for which the Hitchin flow yields the maximal possible holonomy $\mathrm{SU}(4)$.

- There are examples of Lie algebras where the Hitchin flow for cocalibrated G_2-structures yields full holonomy $\mathrm{Spin}(7)$, cf. [AFISUV]. One may try to find more such examples and therefore consider also the moduli space of cocalibrated G_2-structures on Lie algebras which are not almost Abelian. Similarly, it is of interest to find also examples of cocalibrated G_2^*-structures where the Hitchin flow yields full holonomy $\mathrm{Spin}_0(3,4)$. In contrast, one may try to prove analogous holonomy reduction results for particular classes of Lie algebras as the one for seven-dimensional almost Abelian Lie algebras.

The most interesting future project is the investigation which of the incomplete pseudo-Riemannian manifolds $(\mathrm{G}\times I, g)$ with parallel G_2^ϵ-/$\mathrm{Spin}^\epsilon(7)$-structure Φ obtained by the Hitchin flow with left-invariant initial value on a Lie group G can be extended to a complete pseudo-Riemannian manifold. One natural assumption is that the extension is given by a complete manifold N with parallel G_2^ϵ-/$\mathrm{Spin}^\epsilon(7)$-structure Φ_N which admits a cohomogeneity one action of G preserving the G_2^ϵ-/$\mathrm{Spin}^\epsilon(7)$-structure on N and containing $\mathrm{G}\times I$ as an open dense subset such that Φ_N is a smooth extension of Φ to N. Note that the first complete Riemannian examples with exceptional holonomies given by Bryant and Salamon [BrSa] are of this form as well as many other explicit complete examples with parallel G_2- or $\mathrm{Spin}(7)$-structure, cf. e.g. [BGGG], [Cal], [CCGLPW], [ClSw], [CGLP1]-[CGLP4], [GS], [R2] and [R3]. Some of these explicit complete examples of G_2-holonomy manifolds arise as above from the Lie group $S^3\times S^3$, cf. [MaSa] for a unified treatment of these examples.

The related problem for the flow of so-called hypo $\mathrm{SU}(2)$-structures on nilpotent five-dimensional Lie groups N leading to six-dimensional Riemanian manifolds $(N\times I, g)$ with $Hol(g)\subseteq \mathrm{SU}(3)$ is considered in [C2]. It has been shown that Riemannian metrics obtained by the hypo flow cannot be extended to a complete Riemannian manifold in the above way unless they are a Riemannian product of N and $\mathbb{R}$. There is an ongoing project together with Florin Belgun and Oliver Goertsches which investigates the analogue question for the Hitchin flow for half-flat $\mathrm{SU}(3)$-structures and cocalibrated G_2-structures on nilpotent and split-solvable Lie algebras. Note that the explicit solutions of the Hitchin flow given in Section 7.4 cannot be extended in the above way to a complete Riemannian manifold since in all cases there is a fundamental vector field on $\mathrm{G}\times I$ whose length tends to infinity at the boundary of I.

Appendix

In the appendix, we solely consider *real* Lie algebras without mentioning this in the following explicitly. Some lists of Lie algebras appearing in the appendix are further subdivided into unimodular and non-unimodular Lie algebras. If there is no such subdivision, one may, nevertheless, easily identify the unimodular ones since an obvious characterisation is that the top-dimensional cohomology group does not vanish. To emphasise the unimodular Lie algebras in this case, the non-zero $h^{\dim(\mathfrak{g})}(\mathfrak{g})$ are written bold and underlined.

Table 7.1 contains all Lie algebras up to dimension three. The three-dimensional Lie algebras are further subdivided into the unimodular and the non-unimodular ones. The names for the non-unimodular Lie algebras in the first column have been adopted from [GOV]. In the second column, the Lie bracket is encoded dually. Here, $\left(e^1, \ldots, e^{\dim(\mathfrak{g})}\right)$ is a basis of $\mathfrak{g}^*$ and we write down the vector $\left(de^1, \ldots, de^{\dim(\mathfrak{g})}\right)$. The column labelled $\mathfrak{z}$ contains the dimension of the centre of $\mathfrak{g}$. In the last column, the vector $h^*(\mathfrak{g})$ of the dimensions of the corresponding Lie algebra cohomology groups is given. Note that $h^*(\mathfrak{g}) = \left(h^1(\mathfrak{g}), \ldots, h^{\dim(\mathfrak{g})}(\mathfrak{g})\right)$ by Definition 3.34.

Table 7.2 contains all four-dimensional Lie algebras which are the direct sum of a three-dimensional Lie algebra and $\mathbb{R}$. Again, we have further subdived the list into the unimodular Lie algebras and the non-unimodular Lie algebras. In the second column, the Lie bracket is encoded dually for a basis e^1, e^2, e^3, e^4 of $\mathfrak{g}^*$ in the same way as in Table 7.1. The next column contains the vector $h^*(\mathfrak{g})$ of the dimensions of the corresponding Lie algebra cohomology groups. The column labeled $\mathfrak{u}$ contains all isomorphism classes of unimodular codimension one ideals in $\mathfrak{g}$. If there are different isomorphic codimension one unimodular ideals, we remark it in a footnote. The next column, labeled $[\mathfrak{g}, \mathfrak{g}]$, contains the commutator ideal of $\mathfrak{g}$. Finally, in the last column, the integer $h^1(\mathfrak{g}) + h^1(\mathfrak{u}) - h^2(\mathfrak{g})$ is computed. If there is more than one isomorphism class of codimension one unimodular ideals $\mathfrak{u}$, then the different numbers are written next to each other, ordered according to the order in the column "$\mathfrak{u}$".

Table 7.3 contains all indecomposable four-dimensional Lie algebras and the Lie algebra $\mathfrak{r}_2 \oplus \mathfrak{r}_2$ ordered by nilradical. The first six columns are build up completely analogous to the ones in Table 7.2 and the names for the appearing Lie algebras are taken from [PSWZ].

However, in contrast to Table 7.2, there are four more columns which contain our results on the (non-)existence of half-flat SU(3)-structures and closed stable three-forms. Namely, the seventh column labeled hf $\oplus\,\mathfrak{r}_2$ is checked if and only if $\mathfrak{g}\oplus\mathfrak{r}_2$ admits a half-flat SU(3)-structure. Recall that $\mathfrak{g}\oplus\mathbb{R}^2$ never admits a half-flat SU(3)-structure. The column labeled $\lambda\geq 0\oplus\mathfrak{r}_2/\lambda\geq 0\oplus\mathbb{R}^2$ is checked if $\lambda(\rho)\geq 0$ for all closed three-forms ρ on $\mathfrak{g}\oplus\mathfrak{r}_2/\mathfrak{g}\oplus\mathbb{R}^2$. Similarly, the column "$\lambda=0\oplus\mathbb{R}^2$" is checked if $\lambda(\rho)=0$ for all closed three-forms ρ on $\mathfrak{g}\oplus\mathbb{R}^2$. None of the Lie algebras $\mathfrak{g}\oplus\mathfrak{r}_2$ satisfies $\lambda(\rho)=0$ for all closed three-forms ρ.

In Table 7.4, we listed all indecomposable five-dimensional Lie algebras ordered according to their nilradical. The names for the Lie algebras in the first column are taken from [PSWZ] and the second column again encodes the Lie bracket dually for a basis $e^1,\ldots,e^5$ of $\mathfrak{g}^*$. The column labeled $\mathfrak{z}$ contains the dimension of the center of $\mathfrak{g}$ and the next one the vector $h^*(\mathfrak{g})$. The column "hf" is checked if and only if $\mathfrak{g}\oplus\mathbb{R}$ admits a half-flat SU(3)-structure. Analogously, the columns "$\lambda\geq 0$" and "$\lambda=0$" are checked if $\lambda(\rho)\geq 0$ or $\lambda(\rho)=0$, respectively, for all closed three-forms ρ on $\mathfrak{g}\oplus\mathbb{R}$.

Table 7.5 contains all non-solvable indecomposable six-dimensional Lie algebras, Table 7.6 contains all nilpotent indecomposable six-dimensional Lie algebras and Table 7.7 contains all indecomposable six-dimensional Lie algebras with five-dimensional, non-Abelian nilradical. Table 7.6 is further subdivided into almost Abelian Lie algebras and those which are not almost Abelian and Table 7.7 is further subdivided by the different non-Abelian nilradicals which appear.

The notation and the Lie brackets in Table 7.5 are taken literally from [Tu1]. The Lie brackets in Table 7.6 are taken from [Mag]. In [Mag], the Lie algebras are only labeled by numbers from 1 to 22. We use the class symbol $\mathfrak{n}$ and the numbers given in [Mag] as index. Table 7.7 is based on the original list by Mubarakzyanov [Mu6d] and, apart from the obvious subdivision according to the number of free parameters and the Lie algebra cohomology, the list is modified as follows. On the one hand, some of Mubarakzyanov's classes $g_{6,n}$ are redundant since there is an isomorphism to one of the other classes for certain parameter values. On the other hand, Shabanskaya [Sha] found 6 new classes which are fitted in Table 7.7 according to their nilradical and denoted by $B_{6,i}$, $i=1,\ldots,6$. Moreover, a large number of isomorphisms for certain parameter values have been discovered by Shabanskaya [Sha] and by Schulte-Hengesbach and the author [FS2] resulting in a range restriction or vanishing of certain parameters. It turns out to be hard to assure that no further isomorphisms are possible due to the complexity and large amount of data. Lastly, a few parameter values are excluded because the corresponding Lie algebra is decomposable or nilpotent. Note that the reason for excluding parameter values is usually obvious when considering the matrix representing ad_{e_6} whereas non-obvious modifications are explained in footnotes. The names of the classes are modified such that the remaining parameters are written as exponents of the class symbol A and are denoted by a, b, c if continuous and by ε if discrete.

The Lie brackets in the Tables 7.5 - 7.7 are written as before in the well-known dual notation. In the column labeled $\mathfrak{z}$ the dimension of the center of the corresponding Lie algebra is given. The column labeled $\mathrm{h}^*(\mathfrak{g})$ contains the dimensions of the Lie algebra cohomology groups. The last column, labeled half-flat, is checked if and only if the Lie algebra under consideration admits a half-flat SU(3)-structure. Note that, in contrast to Table 7.5 and Table 7.7, the results on the existence of half-flat SU(3)-structures given in Table 7.6 have been obtained by Conti in [C1] and so are not results the author obtained together with Schulte-Hengesbach. Note further that all Lie algebras in Table 7.5 admit half-flat SU(3)-structures.

In Table 7.8, we give a list of all indecomposable nilpotent almost Abelian seven-dimensional Lie algebras. We introduce our own notation and give in the second column the names used in [Gong] for the corresponding Lie algebras. The Lie brackets, which are as usual encoded dually, are given, with the exception of $\mathfrak{n}_{7,1}$, in such a way that $\mathrm{ad}(e_7)|_{\mathrm{span}(e_1,\ldots,e_6)}$ is in Jordan normal form. Again the column "$\mathfrak{z}$" contains the dimension of the center and the column "$h^*(\mathfrak{g})$" the vector $h^*(\mathfrak{g})$. The column "cocalibrated" is checked exactly when the Lie algebra admits a cocalibrated G_2-structure. Similarly, the column "calibrated" is checked if and only if $\mathfrak{g}$ admits a calibrated G_2-structure.

Table 7.9, Table 7.10, Table 7.11 and Table 7.12 contains one example $(\omega,\rho)\in\Lambda^2\mathfrak{g}^*\times\Lambda^3\mathfrak{g}^*$ of a half-flat SU(3)-structure for each Lie algebra which admits such a structure in the class of direct sums of a four-dimensional and a two-dimensional Lie algebra not contained in [SH], in the class of direct sums of indecomposable five-dimensional Lie algebras with $\mathbb{R}$, in the class of non-solvable indecomposable six-dimensional Lie algebra and in the class of indecomposable six-dimensional Lie algebra with five-dimensional nilradical, respectively.

The examples of half-flat SU(3)-structures in the Tables 7.9 - 7.12 are given with respect to a basis $(e^1,\ldots,e^6)$ of $\mathfrak{g}^*$, where the choice of the basis for the Tables 7.9 and 7.10 is explained in a footnote and for Table 7.11 and Table 7.12, the basis is the one given in Table 7.5 and Table 7.7, respectively.

Moreover, in all the Tables 7.9 - 7.12 the Euclidean metric induced by the half-flat SU(3)-structure (ω,ρ) on $\mathfrak{g}$ is added. The label ONB indicates that the considered basis is orthonormal. Similarly, OB indicates that the considered basis is orthogonal. In this case, the norms of the non-unit basis vectors are given explicitly.

Finally, Table 7.13 contains (the dual bases of) adapted bases for cocalibrated G_2-structures on three different seven-dimensional Lie algebras $\mathfrak{g}$ which are Lie algebra direct sums of a four and a three-dimensional Lie algebra. These three cases are exceptional in the sense that they do not fulfil any of the different conditions we obtained in Chapter 5 which ensure the existence of a cocalibrated G_2-structure.

Table 7.1: Lie algebras up to dimension three

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$\mathrm{h}^*(\mathfrak{g})$
	one-dimensional		
$\mathbb{R}$	(0)	1	(1)
	two-dimensional		
$\mathfrak{r}_2$	$(0, e^{12})$	0	$(1,0)$
$\mathbb{R}^2$	$(0,0)$	2	$(2,1)$
	three-dimensional unimodular		
$\mathfrak{so}(3)$	$(e^{67}, -e^{57}, e^{56})$	0	$(0,0,1)$
$\mathfrak{so}(2,1)$	(e^{67}, e^{57}, e^{56})	0	$(0,0,1)$
$e(2)$	$(e^{67}, -e^{57}, 0)$	0	$(1,1,1)$
$e(1,1)$	$(e^{67}, e^{57}, 0)$	0	$(1,1,1)$
$\mathfrak{h}_3$	$(e^{67}, 0, 0)$	1	$(2,2,1)$
$\mathbb{R}^3$	$(0,0,0)$	3	$(3,3,1)$
	three-dimensional non-unimodular		
$\mathfrak{r}_2 \oplus \mathbb{R}$	$(e^{57}, 0, 0)$	1	$(2,1,0)$
$\mathfrak{r}_3$	$(e^{57} + e^{67}, e^{67}, 0)$	0	$(1,0,0)$
$\mathfrak{r}_{3,\mu}$	$(e^{57}, \mu e^{67}, 0)$, $-1 < \mu \leq 1$, $\mu \neq 0$	0	$(1,0,0)$
$\mathfrak{r}'_{3,\mu}$	$(\mu e^{57} + e^{67}, \mu e^{67} - e^{57}, 0)$, $\mu > 0$	0	$(1,0,0)$

Table 7.2: Four-dimensional Lie algebras which are a sum of a three-dimensional Lie algebra with $\mathbb{R}$

$\mathfrak{g}$	Lie bracket	$\mathrm{h}^*(\mathfrak{g})$	$\mathfrak{u}$	$[\mathfrak{g},\mathfrak{g}]$	$h^1(\mathfrak{g}) + h^1(\mathfrak{u}) - h^2(\mathfrak{g})$
	unimodular				
$\mathfrak{so}(3) \oplus \mathbb{R}$	$(e^{23}, -e^{13}, e^{12}, 0)$	$(1,0,1,1)$	$\mathfrak{so}(3)$	$\mathfrak{so}(3)$	1
$\mathfrak{so}(2,1) \oplus \mathbb{R}$	$(e^{23}, e^{13}, e^{12}, 0)$	$(1,0,1,1)$	$\mathfrak{so}(2,1)$	$\mathfrak{so}(2,1)$	1
$e(2) \oplus \mathbb{R}$	$(e^{23}, -e^{13}, 0, 0)$	$(2,2,2,1)$	$\mathbb{R}^3$, $e(2)$	$\mathbb{R}^2$	3, 1
$e(1,1) \oplus \mathbb{R}$	$(e^{23}, e^{13}, 0, 0)$	$(2,2,2,1)$	$\mathbb{R}^3$, $e(1,1)$	$\mathbb{R}^2$	3, 1
$\mathfrak{h}_3 \oplus \mathbb{R}$	$(e^{23}, 0, 0, 0)$	$(3,4,3,1)$	$\mathbb{R}^{3\dagger}$, $\mathfrak{h}_3$	$\mathbb{R}$	2, 1
$\mathbb{R}^4$	$(0,0,0,0)$	$(4,6,4,1)$	$\mathbb{R}^{3\ \ddagger}$	$\{0\}$	1
	non-unimodular				
$\mathfrak{r}_2 \oplus \mathbb{R}^2$	$(e^{14}, 0, 0, 0)$	$(3,3,1,0)$	$\mathbb{R}^3$	$\mathbb{R}$	3
$\mathfrak{r}_3 \oplus \mathbb{R}$	$(e^{14} + e^{24}, e^{24}, 0, 0)$	$(2,1,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^2$	4
$\mathfrak{r}_{3,\mu} \oplus \mathbb{R}$	$(e^{14}, \mu e^{24}, 0, 0)$, $-1 < \mu \leq 1$, $\mu \neq 0$	$(2,1,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^2$	4
$\mathfrak{r}'_{3,\mu} \oplus \mathbb{R}$	$(\mu e^{14} + e^{24}, -e^{14} + \mu e^{24}, 0, 0)$, $\mu > 0$	$(2,1,0,0)$	$\mathbb{R}^3$	$\mathbb{R}^2$	4

† There are several Abelian codimension one ideals, namely for all $(a,b) \neq 0$, $\mathrm{span}(e_1, ae_2 + be_3, e_4)$ is one.

‡ Although all codimension one unimodular ideals are isomorphic, there are of course different ones. Namely, all three-dimensional subspaces.

Table 7.3: Indecomposable four-dimensional Lie algebras and the Lie algebra $\mathfrak{r}_2 \oplus \mathfrak{r}_2$

$\mathfrak{g}$	Lie bracket	$\mathrm{h}^*(\mathfrak{g})$	$\mathfrak{u}$	$[\mathfrak{g},\mathfrak{g}]$	$h^1(\mathfrak{g})$ $+h^1(\mathfrak{u})$ $-h^2(\mathfrak{g})$	hf $\oplus\mathfrak{r}_2$	$\lambda\geq 0$ $\oplus\mathfrak{r}_2$	$\lambda\geq 0$ $\oplus\mathbb{R}^2$	$\lambda=0$ $\oplus\mathbb{R}^2$
	nilpotent, almost Abelian								
$A_{4,1}$	$(\mathrm{e}^{24},\mathrm{e}^{34},0,0)$	(2,2,2,$\underline{\mathbf{1}}$)	$\mathbb{R}^3,\mathfrak{h}_3$	$\mathbb{R}^2$	3,2	✓	–	–	–
	not nilpotent, almost Abelian, i.e. Nilradical $\mathbb{R}^3$								
$A^{\alpha}_{4,2}$	$(\alpha\mathrm{e}^{14},\mathrm{e}^{24}+\mathrm{e}^{34},\mathrm{e}^{34},0)$								
	$\alpha\notin\{-2,-1,0\}$	(1,0,0,0)	$\mathbb{R}^3$	$\mathbb{R}^3$	4	–	✓	✓	✓
	$\alpha=-2$	(1,0,1,$\underline{\mathbf{1}}$)	$\mathbb{R}^3$	$\mathbb{R}^3$	4	✓	–	✓	–
	$\alpha=-1$	(1,1,1,0)	$\mathbb{R}^3$	$\mathbb{R}^3$	3	–	–	✓	–
$A_{4,3}$	$(\mathrm{e}^{14},\mathrm{e}^{34},0,0)$	(2,2,1,0)	$\mathbb{R}^3$	$\mathbb{R}^2$	3	–	–	✓	–
$A_{4,4}$	$(\mathrm{e}^{14}+\mathrm{e}^{24},\mathrm{e}^{24}+\mathrm{e}^{34},\mathrm{e}^{34},0)$	(1,0,0,0)	$\mathbb{R}^3$	$\mathbb{R}^3$	4	–	✓	✓	✓
$A^{\alpha,\beta}_{4,5}$	$(\mathrm{e}^{14},\alpha\mathrm{e}^{24},\beta\mathrm{e}^{34},0)$								
[1]	$-1<\alpha\leq\beta\leq 1$, $\alpha\beta\neq 0$, $\beta\notin\{-\alpha,-(\alpha+1)\}$	(1,0,0,0)	$\mathbb{R}^3$	$\mathbb{R}^3$	4	–	✓	✓	✓
	$\beta=-(\alpha+1)$, $-1<\alpha<-\frac{1}{2}$	(1,0,1,$\underline{\mathbf{1}}$)	$\mathbb{R}^3$	$\mathbb{R}^3$	4	✓	–	✓	–
	$(\alpha,\beta)=(-\frac{1}{2},-\frac{1}{2})$	(1,0,1,$\underline{\mathbf{1}}$)	$\mathbb{R}^3$	$\mathbb{R}^3$	4	–	–	✓	–
	$\alpha=-1$, $\beta>0$, $\beta\neq 1$	(1,1,1,0)	$\mathbb{R}^3$	$\mathbb{R}^3$	3	–	–	✓	–
	$(\alpha,\beta)=(-1,1)$	(1,2,2,0)	$\mathbb{R}^3$	$\mathbb{R}^3$	2	–	–	–	–
$A^{\alpha,\beta}_{4,6}$	$(\alpha\mathrm{e}^{14},\beta\mathrm{e}^{24}+\mathrm{e}^{34},\mathrm{e}^{42}+\beta\mathrm{e}^{34},0)$								
	$\alpha>0$, $\beta\notin\{0,-\frac{1}{2}\alpha\}$	(1,0,0,0)	$\mathbb{R}^3$	$\mathbb{R}^3$	4	–	✓	✓	✓
	$\beta=-\frac{1}{2}\alpha$, $\alpha>0$	(1,0,1,$\underline{\mathbf{1}}$)	$\mathbb{R}^3$	$\mathbb{R}^3$	4	✓	–	✓	–
	$\beta=0$, $\alpha>0$	(1,1,1,0)	$\mathbb{R}^3$	$\mathbb{R}^3$	3	–	–	✓	–
	Nilradical $\mathfrak{h}_3$								
$A_{4,7}$	$(2\mathrm{e}^{14}+\mathrm{e}^{23},\mathrm{e}^{24}+\mathrm{e}^{34},\mathrm{e}^{34},0)$	(1,0,0,0)	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3	–	–	✓	✓
$A_{4,8}$	$(\mathrm{e}^{23},\mathrm{e}^{24},\mathrm{e}^{43},0)$	(1,0,1,$\underline{\mathbf{1}}$)	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3	✓	–	✓	–
$A^{\alpha}_{4,9}$	$((\alpha+1)\mathrm{e}^{14}+\mathrm{e}^{23},\mathrm{e}^{24},\alpha\mathrm{e}^{34},0)$								
	$-1<\alpha\leq 1$, $\alpha\notin\{-\frac{1}{2},0\}$	(1,0,0,0)	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3	–	–	✓	✓
	$\alpha=-\frac{1}{2}$	(1,1,1,0)	$\mathfrak{h}_3$	$\mathfrak{h}_3$	2	✓	–	–	–
	$\alpha=0$	(2,1,0,0)	$\mathfrak{h}_3$	$\mathbb{R}^2$	3	–	–	✓	–
$A_{4,10}$	$(\mathrm{e}^{23},\mathrm{e}^{34},\mathrm{e}^{42},0)$	(1,0,1,$\underline{\mathbf{1}}$)	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3	✓	–	✓	–
$A^{\alpha}_{4,11}$	$(2\alpha\mathrm{e}^{14}+\mathrm{e}^{23},\alpha\mathrm{e}^{24}+\mathrm{e}^{34},\mathrm{e}^{42}+\alpha\mathrm{e}^{34},0)$, $\alpha>0$	(1,0,0,0)	$\mathfrak{h}_3$	$\mathfrak{h}_3$	3	–	–	✓	✓
	Nilradical $\mathbb{R}^2$								
$A_{4,12}$	$\left(e^{14}+e^{23},e^{24}-e^{13},0,0\right)$	(2,1,0,0)	$e(2)$	$\mathbb{R}^2$	2	✓	–	–	–

[1] $A^{\alpha,-\alpha}_{4,5}\cong A^{-1,1/\alpha}_{4,5}$ for $\alpha\neq 0$ and $A^{-1,\beta}_{4,5}\cong A^{-1,-\beta}_{4,5}$.

Table 7.3: Indecomposable four-dimensional Lie algebras and the Lie algebra $\mathfrak{r}_2 \oplus \mathfrak{r}_2$

$\mathfrak{g}$	Lie bracket	$\mathrm{h}^*(\mathfrak{g})$	$\mathfrak{u}$	$[\mathfrak{g},\mathfrak{g}]$	$h^1(\mathfrak{g})$ $+h^1(\mathfrak{u})$ $-h^2(\mathfrak{g})$	hf $\oplus\mathfrak{r}_2$	$\lambda\geq 0$ $\oplus\mathfrak{r}_2$	$\oplus\mathbb{R}^2$	$\lambda=0$ $\oplus\mathbb{R}^2$
$\mathfrak{r}_2\oplus\mathfrak{r}_2$	$\left(e^{14}+e^{23}, e^{24}+e^{13}, 0, 0\right)$[2]	(2,1,0,0)	$e(1,1)$	$\mathbb{R}^2$	2	✓	–	–	–

Table 7.4: Indecomposable five-dimensional Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$\mathrm{h}^*(\mathfrak{g})$	hf	$\lambda\geq 0$	$\lambda=0$
	nilpotent, almost Abelian					
$A_{5,1}$	$(e^{35}, e^{45}, 0, 0, 0)$	2	(3,6,6,3,$\underline{1}$)	✓	–	–
$A_{5,2}$	$(e^{25}, e^{35}, e^{45}, 0, 0)$	1	(2,3,3,2,$\underline{1}$)	✓	–	–
	nilpotent, not almost Abelian					
$A_{5,3}$	$(e^{35}, e^{34}, e^{45}, 0, 0)$	2	(2,3,3,2,$\underline{1}$)	–	–	–
$A_{5,4}$	$(e^{24}+e^{35}, 0, 0, 0, 0)$	1	(4,5,5,4,$\underline{1}$)	✓	–	–
$A_{5,5}$	$(e^{25}+e^{34}, e^{35}, 0, 0, 0)$	1	(3,4,4,3,$\underline{1}$)	✓	–	–
$A_{5,6}$	$(e^{25}+e^{34}, e^{35}, e^{45}, 0, 0)$	1	(2,3,3,2,$\underline{1}$)	✓	–	–
	not nilpotent, almost Abelian, i.e. nilradical $\mathbb{R}^4$					
$A_{5,7}^{\alpha,\beta,\gamma}$	$(e^{15}, \alpha e^{25}, \beta e^{35}, \gamma e^{45}, 0)$					
[3]	$-1<\alpha\leq\beta\leq\gamma\leq 1$, $\alpha\beta\gamma\neq 0$, $\beta\notin\{-\alpha,-(\alpha+1)\}$, $\gamma\notin\{-\alpha,-(\alpha+1)$, $-\beta,-(\beta+1)$, $-(\alpha+\beta),-(\alpha+\beta+1)\}$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha=-1$, $-1<\beta\leq\gamma$, $\beta\gamma\neq 0$, $\gamma\notin\{-\beta,-\beta+1,-(\beta+1)\}$	0	(1,1,1,0,0)	–	✓	–
	$(\alpha,\beta)=(-1,-1)$, $\gamma\notin\{-1,0,1,2\}$	0	(1,2,2,0,0)	–	✓	–
	$(\alpha,\beta,\gamma)=(-1,-1,-1)$	0	(1,3,3,0,0)	–	✓	–
	$(\alpha,\beta,\gamma)=(-1,-1,1)$	0	(1,4,4,1,$\underline{1}$)	✓	–	–
	$(\alpha,\beta,\gamma)=(-1,-1,2)$	0	(1,2,3,1,0)	–	–	–
	$(\alpha,\gamma)=(-1,-\beta)$, $0<\beta<1$	0	(1,2,2,1,$\underline{1}$)	✓	–	–
	$(\alpha,\gamma)=(-1,-\beta-1)$, $\beta\notin\{0,1\}$	0	(1,1,2,1,0)	–	–	–
	$(\alpha,\beta,\gamma)=(1,1,-2)$	0	(1,0,3,3,0)	–	✓	–
	$\gamma=-(\alpha+\beta+1)$, $-1<\alpha\leq\beta\leq\gamma\leq 1$, $\alpha\beta\gamma\neq 0$, $\beta\neq-\alpha$	0	(1,0,0,1,$\underline{1}$)	–	✓	✓
	$\gamma=-(\beta+1)$, $\alpha\notin\{-1,0,1,\pm\beta,\pm\gamma\}$, $-1<\beta\leq-\frac{1}{2}$	0	(1,0,1,1,0)	–	✓	–
	$(\alpha,\gamma)=(1,-\beta-1)$, $\beta\leq-\frac{1}{2}$, $\beta\notin\{-2,-1\}$	0	(1,0,2,2,0)	–	✓	–
$A_{5,8}^{\alpha}$	$(e^{25}, 0, e^{35}, \alpha e^{45}, 0)$					
	$-1<\alpha\leq 1$, $\alpha\neq 0$	1	(2,2,1,0,0)	–	✓	–

[2] A relation of the standard basis f^1, f^2, f^3, f^4 of $\mathfrak{r}_2^*\oplus\mathfrak{r}_2^*$ with $(df^1, df^2, df^3, df^4)=(0, f^{12}, 0, f^{34})$ to our basis e^1, e^2, e^3, e^4 is given by $e^1=f^2+f^4$, $e^2=f^2-f^4$, $e^3=-\frac{1}{2}\left(f^1-f^3\right)$, $e^4=-\frac{1}{2}\left(f^1+f^3\right)$.

[3] $A_{5,7}^{\alpha,-\alpha,\gamma}\cong A_{5,7}^{-1,1/\alpha,\gamma/\alpha}$, $A_{5,7}^{\alpha,\beta,-(\alpha+\beta)}\cong A_{5,7}^{1/\alpha,\beta/\alpha,-(\beta/\alpha+1)}$, $A_{5,7}^{\alpha,\beta,-(\beta+1)}\cong A_{5,7}^{\alpha/\beta,1/\beta,-(1/\beta+1)}$

Table 7.4: Indecomposable five-dimensional Lie algebras – continued

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$\mathrm{h}^*(\mathfrak{g})$	hf	$\lambda\geq 0$	$\lambda=0$
	$\alpha=-1$	1	(2,3,3,2,$\underline{\mathbf{1}}$)	✓	–	–
$A_{5,9}^{\alpha,\beta}$	$(\mathrm{e}^{15}+\mathrm{e}^{25},\mathrm{e}^{25},\alpha\mathrm{e}^{35},\beta\mathrm{e}^{45},0)$					
[4]	$\alpha\leq\beta,\ \alpha\notin\{-2,-1,0\},$ $\beta\notin\{-2,-1,0,-\alpha,-(\alpha+1),-(\alpha+2)\}$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha=-2,\ \beta\notin\{-2,-1,0,1,2\}$	0	(1,0,1,1,0)	–	✓	–
	$(\alpha,\beta)\in\{(-2,-2),(-2,1)\}$	0	(1,0,2,2,0)	–	✓	–
	$(\alpha,\beta)\in\{(-2,-1),(-2,2)\}$	0	(1,1,2,1,0)	–	–	–
	$\alpha=-1,\ \beta\notin\{-2,-1,0,1\}$	0	(1,1,1,0,0)	–	✓	–
	$(\alpha,\beta)=(-1,-1)$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	–	✓	–
	$(\alpha,\beta)=(-1,1)$	0	(1,2,2,0,0)	–	✓	–
	$\beta=-\alpha,\ \alpha<0,\ \alpha\notin\{-2,-1\}$	0	(1,1,1,0,0)	–	✓	–
	$\beta=-(\alpha+1),\ \alpha\leq-\frac{1}{2},\ \alpha\notin\{-2,-1\}$	0	(1,0,1,1,0)	–	✓	–
	$\beta=-(\alpha+2),\ \alpha<-1,\ \alpha\neq-2$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
$A_{5,10}$	$(\mathrm{e}^{25},\mathrm{e}^{35},0,\mathrm{e}^{45},0)$	1	(2,2,2,1,0)	–	✓	–
$A_{5,11}^{\alpha}$	$(\mathrm{e}^{15}+\mathrm{e}^{25},\mathrm{e}^{25}+\mathrm{e}^{35},\mathrm{e}^{35},\alpha\mathrm{e}^{45},0)$					
	$\alpha\notin\{-3,-2,-1,0\}$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha=-3$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
	$\alpha=-2$	0	(1,0,1,1,0)	–	✓	–
	$\alpha=-1$	0	(1,1,1,0,0)	–	✓	–
$A_{5,12}$	$(\mathrm{e}^{15}+\mathrm{e}^{25},\mathrm{e}^{25}+\mathrm{e}^{35},\mathrm{e}^{35}+\mathrm{e}^{45},\mathrm{e}^{45},0)$	0	(1,0,0,0,0)	–	✓	✓
$A_{5,13}^{\alpha,\beta,\gamma}$	$(\mathrm{e}^{15},\alpha\mathrm{e}^{25},\beta\mathrm{e}^{35}+\gamma\mathrm{e}^{45},-\gamma\mathrm{e}^{35}+\beta\mathrm{e}^{45},0)$					
[5]	$-1<\alpha\leq 1,\ \alpha\neq 0,$ $\beta\notin\{-\frac{1}{2},0,-\frac{1}{2}\alpha,-\frac{1}{2}(\alpha+1)\},\ \gamma>0$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha=-1,\ \beta>0,\ \beta\notin\{0,\frac{1}{2}\},\ \gamma>0$	0	(1,1,1,0,0)	–	✓	–
	$(\alpha,\beta)=(-1,0),\ \gamma>0$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$(\alpha,\beta)=(-1,\frac{1}{2}),\ \gamma>0$	0	(1,1,2,1,0)	–	–	–
	$\beta=0,\ -1<\alpha\leq 1,\ \alpha\neq 0,\ \gamma>0$	0	(1,1,1,0,0)	–	✓	–
	$\beta=-\frac{1}{2},\ \alpha\notin\{-1,0,1\},\ \gamma>0$	0	(1,0,1,1,0)	–	✓	–
	$(\alpha,\beta)=(1,-\frac{1}{2}),\ \gamma>0$	0	(1,0,2,2,0)	–	✓	–
	$\beta=-\frac{1}{2}(\alpha+1),\ -1<\alpha\leq 1,\ \alpha\neq 0,\ \gamma>0$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
$A_{5,14}^{\alpha}$	$(\mathrm{e}^{25},0,\alpha\mathrm{e}^{35}+\mathrm{e}^{45},-\mathrm{e}^{35}+\alpha\mathrm{e}^{45},0)$					
	$\alpha\neq 0$	1	(2,2,1,0,0)	–	✓	–
	$\alpha=0$	1	(2,3,3,2,$\underline{\mathbf{1}}$)	✓	–	–
$A_{5,15}^{\alpha}$	$(\mathrm{e}^{15}+\mathrm{e}^{25},\mathrm{e}^{25},\alpha\mathrm{e}^{35}+\mathrm{e}^{45},\alpha\mathrm{e}^{45},0)$					
	$0<\lvert\alpha\rvert\leq 1,\ \alpha\notin\{-1,-\frac{1}{2}\}$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha=-1$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$\alpha=-\frac{1}{2}$	0	(1,0,1,1,0)	–	✓	–
	$\alpha=0$	1	(2,2,1,0,0)	–	✓	–
$A_{5,16}^{\alpha,\beta}$	$(\mathrm{e}^{15}+\mathrm{e}^{25},\mathrm{e}^{25},\alpha\mathrm{e}^{35}+\beta\mathrm{e}^{45},-\beta\mathrm{e}^{35}+\alpha\mathrm{e}^{45},0)$					

[4] $A_{5,9}^{\alpha,\beta}\cong A_{5,9}^{\beta,\alpha}$, $A_{5,9}^{\alpha,0}$ is decomposable.

[5] $A_{5,13}^{\alpha,\beta,0}=A_{5,7}^{\alpha,\beta,\beta}$, $A_{5,13}^{\alpha,\beta,\gamma}\cong A_{5,13}^{\alpha,\beta,-\gamma}$, $A_{5,13}^{-1,\beta,\gamma}\cong A_{5,13}^{-1,-\beta,-\gamma}$, $A_{5,13}^{0,\alpha,\beta}$ is decomposable.

Table 7.4: Indecomposable five-dimensional Lie algebras – continued

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	h*($\mathfrak{g}$)	hf	$\lambda \geq 0$	$\lambda = 0$
[6]	$\alpha \notin \{-1, -\frac{1}{2}, 0\}$, $\beta > 0$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha = -1$, $\beta > 0$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
	$\alpha = -\frac{1}{2}$, $\beta > 0$	0	(1,0,1,1,0)	–	✓	–
	$\alpha = 0$, $\beta > 0$	0	(1,1,1,0,0)	–	✓	–
$A_{5,17}^{\alpha,\beta,\gamma}$	$(\alpha e^{15} + e^{25}, -e^{15} + \alpha e^{25}, \beta e^{35} + \gamma e^{45}, -\gamma e^{35} + \beta e^{45}, 0)$					
[7]	$\alpha > 0$, $\beta \notin \{0, -\alpha\}$, $0 < \gamma \leq 1$	0	(1,0,0,0,0)	–	✓	✓
	$\beta = -\alpha$, $\alpha > 0$, $0 < \gamma < 1$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	✓
	$(\beta, \gamma) = (-\alpha, 1)$, $\alpha > 0$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$\alpha = 0$, $\beta > 0$, $\gamma > 0$	0	(1,1,1,0,0)	–	✓	–
	$(\alpha, \beta) = (0, 0)$, $0 < \gamma < 1$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$(\alpha, \beta, \gamma) = (0, 0, 1)$	0	(1,4,4,1,$\underline{\mathbf{1}}$)	✓	–	–
$A_{5,18}^{\alpha}$	$(\alpha e^{15} + e^{25} + e^{35}, -e^{15} + \alpha e^{25} + e^{45}, \alpha e^{35} + e^{45}, -e^{35} + \alpha e^{45}, 0)$					
	$\alpha > 0$	0	(1,0,0,0,0)	–	✓	✓
	$\alpha = 0$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	Nilradical $\mathfrak{h}_3 \oplus \mathbb{R}$					
$A_{5,19}^{\alpha,\beta}$	$(\alpha e^{15} + e^{23}, e^{25}, (\alpha - 1)e^{35}, \beta e^{45}, 0)$					
[8]	$0 < \alpha \leq 2$, $\alpha \notin \{\frac{1}{2}, 1\}$, $\beta \notin \{-1, 0, -2\alpha, -2\alpha + 1, -(\alpha + 1), -\alpha + 1\}$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = -1$, $\beta \notin \{0, -1, 2, 3\}$	0	(1,1,1,0,0)	–	–	–
	$(\alpha, \beta) = (-1, -1)$	0	(1,2,2,0,0)	–	–	–
	$(\alpha, \beta) = (-1, 2)$	0	(1,2,2,1,$\underline{\mathbf{1}}$)	✓	–	–
	$(\alpha, \beta) = (-1, 3)$	0	(1,1,2,1,0)	✓	–	–
	$\alpha = 0$, $\beta > 0$	1	(1,0,1,1,0)	–	✓	–
	$(\alpha, \beta) = (0, 1)$	1	(1,1,3,2,0)	–	–	–
	$\alpha = 1$, $\beta \notin \{-2, -1, 0\}$	0	(2,1,0,0,0)	–	✓	–
	$(\alpha, \beta) = (1, -2)$	0	(2,1,1,2,$\underline{\mathbf{1}}$)	–	–	–
	$(\alpha, \beta) = (1, -1)$	0	(2,2,2,1,0)	–	–	–
	$\beta = -1$, $\alpha \notin \{-1, 0, 1, \frac{1}{2}, 2\}$	0	(1,1,1,0,0)	–	✓	–
	$(\alpha, \beta) = (2, -1)$	0	(1,2,2,0,0)	–	✓	–
	$\beta = -(\alpha + 1)$, $\alpha \notin \{-1, 0, 1, \frac{1}{2}, 2\}$	0	(1,0,1,1,0)	–	–	–
	$(\alpha, \beta) = (2, -3)$	0	(1,0,2,2,0)	✓	–	–
	$\beta = -2\alpha$, $0 < \alpha \leq 2$, $\alpha \notin \{\frac{1}{2}, 1\}$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	–
$A_{5,20}^{\alpha}$	$(\alpha e^{15} + e^{23} + e^{45}, e^{25}, (\alpha - 1)e^{35}, \alpha e^{45}, 0)$					
	$\alpha \notin \{-1, -\frac{1}{2}, 0, \frac{1}{3}, \frac{1}{2}, 1\}$	0	(1,0,0,0,0)	–	✓	–
	$\alpha \in \{-1, \frac{1}{2}\}$	0	(1,1,1,0,0)	–	✓	–

[6] $A_{5,16}^{\alpha,\beta} \cong A_{5,16}^{\alpha,-\beta}$, $A_{5,16}^{\alpha,0} = A_{5,9}^{\alpha,\alpha}$

[7] $A_{5,17}^{\alpha,\beta,0} \cong A_{5,13}^{1,\alpha/\beta,1/\beta}$ for $\beta \neq 0$, $A_{5,17}^{\alpha,\beta,\gamma} \cong A_{5,17}^{\alpha,\beta,-\gamma} \cong A_{5,17}^{-\alpha,-\beta,\gamma} \cong A_{5,17}^{\beta/\gamma,\alpha/\gamma,1/\gamma}$ for $\gamma \neq 0$, $A_{5,17}^{\alpha,0,0}$ is decomposable.

[8] $A_{5,19}^{\alpha,\beta} \cong A_{5,19}^{\alpha/(\alpha-1),\beta/(\alpha-1)}$ for $\alpha \neq 1$, $A_{5,19}^{0,\beta} \cong A_{5,19}^{0,-\beta}$, $A_{5,19}^{\alpha,0}$ is decomposable.

Table 7.4: Indecomposable five-dimensional Lie algebras – continued

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$\mathrm{h}^*(\mathfrak{g})$	hf	$\lambda \geq 0$	$\lambda = 0$
	$\alpha \in \{-\frac{1}{2}, \frac{1}{3}\}$	0	(1,0,1,1,0)	–	–	–
	$\alpha = 0$	1	(2,1,1,2,$\underline{\mathbf{1}}$)	–	–	–
	$\alpha = 1$	0	(2,1,0,0,0)	–	✓	–
$A_{5,21}$	$(2e^{15} + e^{23}, e^{25}, e^{25} + e^{35}, e^{35} + e^{45}, 0)$	0	(1,0,0,0,0)	–	✓	–
$A_{5,22}$	$(e^{23}, 0, e^{25}, e^{45}, 0)$	1	(2,2,2,1,0)	–	–	–
$A^{\alpha}_{5,23}$	$(2e^{15} + e^{23}, e^{25}, e^{25} + e^{35}, \alpha e^{45}, 0)$					
	$\alpha \notin \{-4, -3, -1, 0\}$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = -4$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	–
	$\alpha = -3$	0	(1,0,1,1,0)	–	–	–
	$\alpha = -1$	0	(1,1,1,0,0)	–	✓	–
$A_{5,24}$ [9]	$(2e^{15} + e^{23} + e^{45}, e^{25}, e^{25} + e^{35}, 2e^{45}, 0)$	0	(1,0,0,0,0)	–	✓	–
$A^{\alpha,\beta}_{5,25}$	$(2\beta e^{15} + e^{23}, \beta e^{25} - e^{35}, e^{25} + \beta e^{35}, \alpha e^{45}, 0)$					
	$\alpha \neq 0,\ \beta \notin \{0, -\frac{1}{4}\alpha\}$	0	(1,0,0,0,0)	–	✓	–
	$\beta = 0,\ \alpha \neq 0$	1	(1,0,1,1,0)	–	✓	–
	$\beta = -\frac{1}{4}\alpha,\ \alpha \neq 0$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	–
$A^{\alpha,\varepsilon}_{5,26}$	$(2\alpha e^{15} + e^{23} + \varepsilon e^{45}, \alpha e^{25} - e^{35}, e^{25} + \alpha e^{35}, 2\alpha e^{45}, 0)$					
	$\alpha \neq 0,\ \varepsilon = \pm 1$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = 0,\ \varepsilon = \pm 1$	1	(2,1,1,2,$\underline{\mathbf{1}}$)	–	✓	–
$A_{5,27}$	$(e^{15} + e^{23} + e^{45}, 0, e^{35}, e^{35} + e^{45}, 0)$	0	(2,1,0,0,0)	–	✓	–
$A^{\alpha}_{5,28}$	$(\alpha e^{15} + e^{23}, (\alpha - 1)e^{25}, e^{35}, e^{35} + e^{45}, 0)$					
	$\alpha \notin \{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$	0	(1,0,0,0,0)	–	✓	–
	$\alpha = -2$	0	(1,0,1,1,0)	–	–	–
	$\alpha \in \{-1, \frac{1}{2}\}$	0	(1,1,1,0,0)	–	–	–
	$\alpha = -\frac{1}{2}$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	✓	–
	$\alpha = 0$	1	(1,1,2,1,0)	–	–	–
	$\alpha = 1$	0	(2,1,0,0,0)	–	✓	–
$A_{5,29}$	$(e^{15} + e^{24}, e^{25}, e^{45}, 0, 0)$	1	(2,2,1,0,0)	–	✓	–
	Nilradical $A_{4,1}$					
$A^{\alpha}_{5,30}$	$((\alpha+1)e^{15} + e^{24}, \alpha e^{25} + e^{34}, (\alpha - 1)e^{35}, e^{45}, 0)$					
	$\alpha \notin \{-2, -1, -\frac{1}{3}, 0, \frac{1}{2}, 1\}$	0	(1,0,0,0,0)	–	–	–
	$\alpha \in \{-2, \frac{1}{2}\}$	0	(1,1,1,0,0)	–	–	–
	$\alpha = -1$	1	(1,0,1,1,0)	–	–	–
	$\alpha = -\frac{1}{3}$	0	(1,0,0,1,$\underline{\mathbf{1}}$)	–	–	–
	$\alpha = 0$	0	(1,0,1,1,0)	✓	–	–
	$\alpha = 1$	0	(2,1,0,0,0)	–	–	–
$A_{5,31}$	$(3e^{15} + e^{24}, 2e^{25} + e^{34}, e^{35} + e^{45}, e^{45}, 0)$	0	(1,0,0,0,0)	–	–	–
$A^{\varepsilon}_{5,32}$	$(e^{15} + e^{24} + \varepsilon e^{35}, e^{25} + e^{34}, e^{35}, 0, 0),\ \varepsilon = \pm 1$	0	(2,1,0,0,0)	–	–	–
	Nilradical $\mathbb{R}^3$					

[9]The parameter in [PSWZ] is redundant.

Table 7.4: Indecomposable five-dimensional Lie algebras – continued

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$\mathrm{h}^*(\mathfrak{g})$	hf	$\lambda\geq 0$	$\lambda=0$
$A_{5,33}^{\alpha,\beta}$ [10]	$(\mathrm{e}^{14}, \mathrm{e}^{25}, \beta\mathrm{e}^{34} + \alpha\mathrm{e}^{35}, 0, 0)$					
	$\alpha, \beta \in \mathbb{R}^*$, $(\alpha,\beta) \neq (-1,-1)$	0	(2,1,0,0,0)	–	–	–
	$(\alpha,\beta) = (-1,-1)$	0	(2,1,1,2,$\underline{\mathbf{1}}$)	✓	–	–
$A_{5,34}^{\alpha}$	$(\alpha\mathrm{e}^{14} + \mathrm{e}^{15}, \mathrm{e}^{24} + \mathrm{e}^{35}, \mathrm{e}^{34}, 0, 0)$, $\alpha \in \mathbb{R}$	0	(2,1,0,0,0)	–	–	–
$A_{5,35}^{\alpha,\beta}$	$(\beta\mathrm{e}^{14} + \alpha\mathrm{e}^{15}, \mathrm{e}^{24} + \mathrm{e}^{35}, -\mathrm{e}^{25} + \mathrm{e}^{34}, 0, 0)$					
	$(\alpha,\beta) \notin \{(0,-2),(0,0)\}$	0	(2,1,0,0,0)	–	–	–
	$(\alpha,\beta) = (0,-2)$	0	(2,1,1,2,$\underline{\mathbf{1}}$)	✓	–	–
$A_{5,38}$	$(\mathrm{e}^{14}, \mathrm{e}^{25}, \mathrm{e}^{45}, 0, 0)$	1	(2,2,1,0,0)	–	–	–
$A_{5,39}$	$(\mathrm{e}^{14} + \mathrm{e}^{25}, -\mathrm{e}^{15} + \mathrm{e}^{24}, \mathrm{e}^{45}, 0, 0)$	1	(2,2,1,0,0)	–	–	–
	Nilradical $\mathfrak{h}_3$					
$A_{5,36}$	$(\mathrm{e}^{14} + \mathrm{e}^{23}, \mathrm{e}^{24} - \mathrm{e}^{25}, \mathrm{e}^{35}, 0, 0)$	0	(2,1,0,0,0)	✓	–	–
$A_{5,37}$	$(2\mathrm{e}^{14} + \mathrm{e}^{23}, \mathrm{e}^{24} + \mathrm{e}^{35}, -\mathrm{e}^{25} + \mathrm{e}^{34}, 0, 0)$	0	(2,1,0,0,0)	✓	–	–
	non-solvable, Nilradical $\mathbb{R}^2$					
$A_{5,40}$	$(2\mathrm{e}^{12}, -\mathrm{e}^{13}, 2\mathrm{e}^{23}, \mathrm{e}^{24} + \mathrm{e}^{35}, \mathrm{e}^{14} - \mathrm{e}^{25})$	0	(0,1,1,0,$\underline{\mathbf{1}}$)	✓	–	–

Table 7.5: Non-solvable indecomposable 6-dim. Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$\mathrm{h}^*(\mathfrak{g})$	hf
$L_{6,1}$	$(\mathrm{e}^{23}, -\mathrm{e}^{13}, \mathrm{e}^{12}, \mathrm{e}^{26} - \mathrm{e}^{35}, -\mathrm{e}^{16} + \mathrm{e}^{34}, \mathrm{e}^{15} - \mathrm{e}^{24})$	0	(0,0,2,0,0,$\underline{\mathbf{1}}$)	✓
$L_{6,2}$	$(\mathrm{e}^{23}, 2\mathrm{e}^{12}, -2\mathrm{e}^{13}, \mathrm{e}^{14} + \mathrm{e}^{25}, -\mathrm{e}^{15} + \mathrm{e}^{34}, \mathrm{e}^{45})$	1	(0,0,2,0,0,$\underline{\mathbf{1}}$)	✓
$L_{6,3}$	$(\mathrm{e}^{23}, 2\mathrm{e}^{12}, -2\mathrm{e}^{13}, \mathrm{e}^{14} + \mathrm{e}^{25} + \mathrm{e}^{46}, -\mathrm{e}^{15} + \mathrm{e}^{34} + \mathrm{e}^{56}, 0)$	0	(1,0,1,1,0,0)	✓
$L_{6,4}$	$(\mathrm{e}^{23}, 2\mathrm{e}^{12}, -2\mathrm{e}^{13}, 2\mathrm{e}^{14} + 2\mathrm{e}^{25}, \mathrm{e}^{26} + \mathrm{e}^{34}, -2\mathrm{e}^{16} + 2\mathrm{e}^{35})$	0	(0,0,2,0,0,$\underline{\mathbf{1}}$)	✓
$\mathfrak{so}(3,1)$	$(\mathrm{e}^{23} - \mathrm{e}^{56}, -\mathrm{e}^{13} + \mathrm{e}^{46}, \mathrm{e}^{12} - \mathrm{e}^{45}, \mathrm{e}^{26} - \mathrm{e}^{35}, -\mathrm{e}^{16} + \mathrm{e}^{34}, \mathrm{e}^{15} - \mathrm{e}^{24})$	0	(0,0,2,0,0,$\underline{\mathbf{1}}$)	✓

Table 7.6: Indecomposable nilpotent 6-dim. Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$\mathrm{h}^*(\mathfrak{g})$	hf
	almost Abelian			

[10] $A_{5,33}^{\alpha,0}$ and $A_{5,33}^{0,\beta}$ are decomposable.

Table 7.6: Indecomposable nilpotent 6-dim. Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
$\mathfrak{n}_{6,1}$	$(0,0,e^{12},e^{13},0,e^{15})$	2	$(3,6,8,6,3,1)$	–
$\mathfrak{n}_{6,2}$	$(0,0,e^{12},e^{13},e^{14},e^{15})$	1	$(2,3,4,3,2,1)$	–
	not almost Abelian			
$\mathfrak{n}_{6,3}$	$(0,0,0,e^{13},e^{23},e^{12})$	3	$(3,8,12,8,3,1)$	✓
$\mathfrak{n}_{6,4}$	$(0,0,0,0,e^{12},e^{13}+e^{24})$	2	$(4,8,10,8,4,1)$	✓
$\mathfrak{n}_{6,5}$	$(0,0,0,0,e^{13}+e^{24},e^{14}-e^{23})$	2	$(4,8,10,8,4,1)$	✓
$\mathfrak{n}_{6,6}$	$(0,0,0,e^{13},e^{14}+e^{23},e^{12})$	2	$(3,6,8,6,3,1)$	✓
$\mathfrak{n}_{6,7}$	$(0,0,0,e^{13},e^{14},e^{23})$	2	$(3,6,8,6,3,1)$	✓
$\mathfrak{n}_{6,8}$	$(0,0,e^{12},e^{13},e^{12},e^{25})$	2	$(3,5,6,5,3,1)$	–
$\mathfrak{n}_{6,9}$	$(0,0,e^{12},e^{13},0,e^{15}+e^{23})$	2	$(3,5,6,5,3,1)$	–
$\mathfrak{n}_{6,10}$	$(0,0,e^{12},0,e^{13}+e^{24},e^{14}-e^{23})$	2	$(3,5,6,5,3,1)$	–
$\mathfrak{n}_{6,11}$	$(0,0,e^{12},e^{13},e^{14},e^{23})$	2	$(2,4,6,4,2,1)$	✓
$\mathfrak{n}_{6,12}$	$(0,0,0,e^{13},0,e^{14}+e^{25})$	1	$(4,6,6,6,4,1)$	✓
$\mathfrak{n}_{6,13}$	$(0,0,0,e^{13},e^{12},e^{14}+e^{25})$	1	$(3,5,6,5,3,1)$	✓
$\mathfrak{n}_{6,14}^{\varepsilon}$	$(0,0,0,e^{13},e^{23},e^{14}+\varepsilon e^{25}),\ \varepsilon=-1,1$	1	$(3,5,6,5,3,1)$	✓
$\mathfrak{n}_{6,15}$	$(0,0,e^{12},e^{13},e^{12},e^{14}+e^{25})$	1	$(3,5,6,5,3,1)$	✓
$\mathfrak{n}_{6,16}$	$(0,0,0,e^{13},e^{14}+e^{23},e^{15}+e^{24})$	1	$(3,4,4,4,3,1)$	✓
$\mathfrak{n}_{6,17}$	$(0,0,e^{12},e^{13},0,e^{14}+e^{25})$	1	$(3,5,6,5,3,1)$	✓
$\mathfrak{n}_{6,18}^{\varepsilon}$	$(0,0,e^{12},e^{13},e^{23},e^{14}+\varepsilon e^{25})$, $\varepsilon=-1,1$	1	$(2,4,6,4,2,1)$	✓

Table 7.6: Indecomposable nilpotent 6-dim. Lie algebras

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
$\mathfrak{n}_{6,19}$	$(0,0,e^{12},e^{13},e^{14},e^{15}+e^{23})$	1	$(2,3,4,3,2,1)$	–
$\mathfrak{n}_{6,20}$	$(0,0,e^{12},e^{13},e^{14}+e^{23},e^{15}+e^{24})$	1	$(2,3,4,3,2,1)$	✓
$\mathfrak{n}_{6,21}$	$(0,0,e^{12},e^{23},e^{24},e^{15}+e^{34})$	1	$(2,2,2,2,2,1)$	–
$\mathfrak{n}_{6,22}$	$(0,0,e^{12},e^{23},e^{13}+e^{24},e^{15}+e^{34})$	1	$(2,2,2,2,2,1)$	–

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	Nilradical $\mathfrak{h}_3 \oplus \mathbb{R}^2$			
$A_{6,13}^{a,b,c}$ [11]	$((a+b)e^{16}+e^{23}, ae^{26}, be^{36}, e^{46}, ce^{56}, 0)$			
	$0<\lvert a\rvert\le\lvert b\rvert,\ -1<c\le 1,\ a\neq -1,$ $b\notin\{-1,-a,-2a,-(2a+1),-\frac{1}{2}(a+1),-(a+\frac{1}{2})\},$ $c\notin\{0,-(a+1),-(b+1),-(2a+b+1),-(2a+2b+1)\}$	0	(1,0,0,0,0,0)	–
	$a=0,\ b\notin\{-1,-\frac{1}{2},0\},\ -1<c\le 1,$ $c\notin\{0,-b,-2b,-b-1,-2b-1\}$	0	(2,1,0,0,0,0)	–
	$a=-1,\ b\notin\{-1,0,\frac{1}{2},1,2\},\ c\notin\{-1,0,1,-b,-2b,$ $-b-1,-b+1,-b+2,-2b+1,-2b+2\}$ *or* $c=-1,\ 0<\lvert a\rvert\le b,\ a\neq\pm 1,$ $b\notin\{1,-a,-2a,-2a\pm 1,-\frac{1}{2}a\pm\frac{1}{2},-a\pm\frac{1}{2}\}$ *or* $b=-2a,\ a\notin\{-1,0,\frac{1}{3},\frac{1}{2}\},\ -1<c\le 1,$ $c\notin\{0,-a,2a,3a,-1-a,-1+2a,-1+3a\}$	0	(1,1,1,0,0,0)	–
	$b=-(2a+1),\ a\notin\{-1,-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},0\},$ $c\notin\{-1,0,1,-a-1,2a,2a+2,3a+1,3a+2\}$ *or* $c=-(a+1),\ a\notin\{-1,0\},\ b\notin\{-1,0,\frac{1}{2},\pm a,$ $-\frac{a}{2},-2a,-(2a+1),-\frac{a}{2}\pm\frac{1}{2},\pm a+1,-(a+\frac{1}{2})\}$	0	(1,0,1,1,0,0)	–
	$b=-a,\ a>0,\ a\neq 1,\ -1<c\le 1,\ c\notin\{0,\pm a,-1\pm a\}$	1	(1,0,1,1,0,0)	–
	$b=-(a+\frac{1}{2}),\ a>-\frac{1}{4},\ a\notin\{0,\frac{1}{2}\},$ $c\notin\{0,\pm 1,\pm a,\pm(a+\frac{1}{2}),\pm(a-\frac{1}{2}),\pm(1+a)\}$ *or* $c=-(2a+b+1),\ a\notin\{-1,-\frac{1}{2},0\},\ b\notin\{-1,0,1,-\frac{1}{2}a,$ $\pm a,-2a,-(2a+1),-\frac{1}{2}(a+1),-(a+\frac{1}{2}),\pm(a+1)\}$	0	(1,0,0,1,1,0)	–

[11] $A_{6,13}^{a,b,c}\cong A_{6,13}^{b,a,c}\cong A_{6,13}^{a/c,b/c,1/c}$, $A_{6,13}^{0,0,c}$ and $A_{6,13}^{a,b,0}$ are decomposable.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$c=-(2a+2b+1)$, $a\notin\{-1,0\}$, $b\notin\{-1,0,-\frac{1}{2}a,-a,-2a,-(2a+1),-\frac{1}{2}(a+1),-(a+\frac{1}{2})\}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a,b)=(0,-1)$, $c\notin\{-1,1,2\}$ *or* $(a,c)=(0,-1)$, $b>0$, $b\notin\{\frac{1}{2},1\}$	0	(2,2,2,1,0,0)	–
	$(a,b)=(0,-\frac{1}{2})$, $c\notin\{-1,-\frac{1}{2},0,\frac{1}{2},1\}$ *or* $(a,c)=(0,-b-1)$, $-2\le b<0$, $b\notin\{-1,-\frac{1}{2}\}$	0	(2,1,1,2,1,0)	–
	$(a,c)=(0,-2b-1)$, $-1<b<0$, $b\neq-\frac{1}{2}$	0	(2,1,0,1,2,$\underline{\mathbf{1}}$)	–
	$(a,b)=(-1,\frac{1}{2})$, $c\notin\{-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,\frac{3}{2}\}$ *or* $(b,c)=(-2a,-1)$, $a>0$, $a\notin\{\frac{1}{3},\frac{1}{2},1\}$	0	(1,2,2,1,1,0)	–
	$(a,b)=(-1,-1)$, $c\notin\{-1,0,1,2,3,4\}$ *or* $(a,b)=(-1,2)$, $c\notin\{-4,-3,-2,-1,0,1\}$ *or* $(a,c)=(-1,-1)$, $b\notin\{-1,0,\frac{1}{2},1,\frac{3}{2},2,3\}$ *or* $(a,c)=(-1,1)$, $b\notin\{-2,-1,-\frac{1}{2},0,\frac{1}{2},1,2\}$ *or* $(a,c)=(-1,-b)$, $-1<b<1$, $b\notin\{0,\frac{1}{2}\}$	0	(1,2,2,0,0,0)	–
	$(a,b)=(-1,1)$, $c\notin\{-2,-1,0,1\}$	1	(1,1,3,2,0,0)	–
	$(a,c)=(-1,-2b+1)$, $b\notin\{-1,0,\frac{1}{2},1,2\}$ *or* $(b,c)=(-2a,2a-1)$, $a\notin\{-1,0,\frac{1}{3},\frac{1}{2}\}$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(b,c)=(-a,-1)$, $a>0$, $a\neq1$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(a,c)=(-1,-b-1)$, $b\notin\{-2,-1,0,\frac{1}{2},1,2,3\}$ *or* $(a,c)=(-1,-b+2)$, $b\notin\{-2,-1,0,\frac{1}{2},1,3\}$ *or* $(a,b)=(-\frac{2}{3},\frac{1}{3})$, $c\notin\{-\frac{4}{3},-1,-\frac{1}{3},0,\frac{2}{3},1\}$ *or* $(b,c)=(-2a-1,-1)$, $a\notin\{-\frac{3}{2},-1,-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},0\}$ *or* $(b,c)=(-2a,-1-a)$, $a\notin\{-1,-\frac{1}{3},-\frac{1}{4},0,\frac{1}{3},\frac{1}{2}\}$	0	(1,1,2,1,0,0)	–
	$(a,c)=(-1,-2b)$, $b\notin\{-2,-1,-\frac{1}{2},0,\frac{1}{2},1,2\}$ *or* $(a,c)=(-1,-b+1)$, $b\notin\{-1,0,\frac{1}{2},1,2\}$ *or* $(a,c)=(-1,-2b+2)$, $b\notin\{-1,0,\frac{1}{2},1,\frac{3}{2},2,3\}$ *or* $(b,c)=(-2a,3a-1)$, $a\notin\{-1,0,\frac{1}{4},\frac{1}{3},\frac{1}{2},1\}$ *or* $(b,c)=(-a-\frac{1}{2},-1)$, $a>-\frac{1}{4}$ $a\notin\{0,\frac{1}{2},1,\frac{3}{2}\}$	0	(1,1,1,1,1,0)	–
	$(b,c)=(-a,-1-a)$, $a\notin\{-1,-\frac{1}{2},0,1\}$	1	(1,0,2,3,1,0)	–
	$(b,c)=-(2a+1,a+1)$, $a\notin\{-2,-1,-\frac{3}{4},-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},0\}$ *or* $(b,c)=(-2a-1,1)$, $a\notin\{-2,-1,-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},0,\frac{1}{2}\}$ *or* $(b,c)=(-2a-1,2a)$, $a\notin\{-2,-1,-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},0,\frac{1}{2}\}$ *or* $(b,c)=(a,-a-1)$, $a\notin\{-1,-\frac{1}{3},-\frac{1}{4},0,\frac{1}{3},\frac{1}{2}\}$ *or* $(a,b)=(-\frac{1}{3},-\frac{1}{3})$, $c\notin\{-1,-\frac{2}{3},0,\frac{1}{3},1,\frac{4}{3}\}$	0	(1,0,2,2,0,0)	–
	$(b,c)=(-2a-1,3a+2)$, $-1<a<-\frac{1}{3}$, $a\notin\{-\frac{3}{4},-\frac{2}{3},-\frac{1}{2}\}$	0	(1,0,2,2,0,0)	✓

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$(b,c)=(-2a-1,3a+1),\ a\notin\{-1,-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},0,1\}$	0	(1,0,1,2,1,0)	✓
	$(b,c)=(-2a-1,2a+2),\ a\notin\{-\frac{3}{2},-1,-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},0,1\}$ *or* $(b,c)=-(a+\frac{1}{2},a+1),\ a\notin\{-2,-1,-\frac{3}{4},-\frac{1}{2},-\frac{1}{4},0,\frac{1}{2}\}$	0	(1,0,1,2,1,0)	–
	$(b,c)=(-a-\frac{1}{2},1),\ a>-\frac{1}{4},\ a\notin\{0,\frac{1}{2},1,\frac{3}{2}\}$ *or* $(b,c)=(-a-\frac{1}{2},a),\ a\notin\{-1,-\frac{1}{2},-\frac{1}{4},0,\frac{1}{4},\frac{1}{2},1\}$ *or* $(b,c)=(a,-3a-1),\ a\notin\{-1,-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},0,1\}$	0	(1,0,0,2,2,0)	–
	$(a,b,c)=(0,-1,1)$	0	(2,3,4,3,2,$\underline{\mathbf{1}}$)	✓
	$(a,b,c)=(0,-1,-1)$	0	(2,3,4,2,0,0)	–
	$(a,b,c)=(0,-\frac{1}{2},-1)$	0	(2,2,3,3,1,0)	–
	$(a,b,c)=(-\frac{1}{2},0,\frac{1}{2})$	0	(2,2,3,3,1,0)	✓
	$(a,b,c)=(0,-\frac{1}{2},1)$	0	(2,1,2,4,2,0)	–
	$(a,b,c)=(-\frac{1}{2},0,-\frac{1}{2})$	0	(2,1,2,4,2,0)	✓
	$(a,b,c)=(-1,-1,1)$	0	(1,4,4,0,0,0)	–
	$(a,b,c)=(-1,1,-1)$	1	(1,3,6,3,1,$\underline{\mathbf{1}}$)	✓
	$(a,b,c)\in\{(-1,\frac{1}{2},-1),\ (-1,\frac{1}{2},1)\}$	0	(1,3,3,2,2,0)	–
	$(a,b,c)\in\{(-1,\frac{1}{2},-\frac{1}{2}),\ (-1,2,-1)\}$	0	(1,3,3,1,1,0)	–
	$(a,b,c)\in\{(-1,-1,-1),\ (-1,2,1)\}$	0	(1,3,3,0,0,0)	–
	$(a,b,c)=(-1,1,1)$	1	(1,2,5,3,0,0)	–
	$(a,b,c)\in\{(-1,-1,3),\ (-1,2,-3)\}$	0	(1,2,4,2,1,$\underline{\mathbf{1}}$)	–
	$(a,b,c)\in\{(-1,\frac{1}{2},-\frac{3}{2}),\ (-1,\frac{1}{2},\frac{3}{2})\}$	0	(1,2,3,2,1,0)	–
	$(a,b,c)=(-\frac{2}{3},\frac{1}{3},-1)$	0	(1,2,3,2,1,0)	✓
	$(a,b,c)\in\{(-1,3,-1),\ (-1,-2,1)\}$	0	(1,2,3,1,0,0)	–
	$(a,b,c)\in\{(-1,\frac{1}{2},\frac{1}{2}),\ (-1,-1,2)\}$	0	(1,2,2,2,2,0)	–
	$(a,b,c)\in\{(-1,-1,4),(-1,2,-4),(-1,\frac{3}{2},-1),(-1,-\frac{1}{2},1)\}$	0	(1,2,2,1,1,0)	–
	$(a,b,c)=(-1,1,-2)$	1	(1,1,4,4,1,0)	✓
	$(a,b,c)\in\{(-\frac{2}{3},\frac{1}{3},1),\ (-\frac{2}{3},\frac{1}{3},-\frac{4}{3}),\ (-\frac{1}{3},-\frac{1}{3},-1)\}$	0	(1,1,3,2,0,0)	–
	$(a,b,c)\in\{(-1,3,-4),\ (-\frac{3}{2},2,-1)\}$	0	(1,1,2,2,1,0)	–
	$(a,b,c)=(-\frac{1}{4},-\frac{1}{2},\frac{1}{4})$	0	(1,1,2,2,1,0)	✓
	$(a,b,c)=(-\frac{1}{3},-\frac{1}{3},-\frac{2}{3})$	0	(1,0,4,4,0,0)	–

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$(a,b,c)=(-\frac{1}{3},-\frac{1}{3},1)$	0	(1,0,4,4,0,0)	✓
	$(a,b,c)\in\{(-2,3,1),\ (\frac{1}{2},-2,1)\}$	0	(1,0,3,3,0,0)	–
	$(a,b,c)=(-\frac{3}{4},\frac{1}{2},-\frac{1}{4})$	0	(1,0,3,3,0,0)	✓
	$(a,b,c)\in\{(-\frac{1}{4},-\frac{1}{4},\frac{3}{4}),\ (-\frac{1}{4},-\frac{1}{4},-\frac{3}{4})\}$	0	(1,0,2,3,1,0)	–
	$(a,b,c)=(\frac{3}{2},-2,1)$	0	(1,0,1,3,2,0)	–
	$(a,b,c)=(1,-3,4)$	0	(1,0,1,3,2,0)	✓
	$(a,b,c)\in\{(1,-\frac{3}{2},1),\ (1,1,-4)\}$	0	(1,0,0,3,3,0)	–
$A_{6,14}^{a,b}$	$((a+b)e^{16}+e^{23}+e^{56}, ae^{26}, be^{36}, e^{46}, (a+b)e^{56}, 0)$			
[12]	$\lvert a\rvert\le\lvert b\rvert,\ \ a\notin\{-1,0\},\ \ b\notin\{-1,0,-a,-\frac{3}{2}a,-2a,$ $-(a+1),\ -(a+\frac{1}{2}),\ -(a+\frac{1}{3}),\ -(2a+1),\ -\frac{1}{2}(a+1),$ $-\frac{1}{2}(3a+1),\ -\frac{1}{3}(2a+1)\}$	0	(1,0,0,0,0,0)	–
	$b=-a,\ a>0,\ a\neq 1$	1	(2,1,1,2,1,0)	–
	$b=0,\ a\notin\{0,-1,-\frac{1}{2},-\frac{1}{3}\}$	0	(2,1,0,0,0,0)	–
	$b=-1,\ a\notin\{-1,0,\frac{1}{3},\frac{1}{2},\frac{2}{3},1,\frac{3}{2},2\}$ *or* $b=-2a,\ a\notin\{-1,0,\frac{1}{4},\frac{1}{3},\frac{1}{2},1\}$ *or* $b=-(a+1),\ a\ge-\frac{1}{2},\ a\notin\{0,1,2\}$	0	(1,1,1,0,0,0)	–
	$b=-\frac{3}{2}a,\ a\notin\{-2,-1,0,\frac{1}{2},\frac{2}{3},\frac{2}{5},1,2\}$ *or* $b=-(2a+1),\ a\notin\{-2,-1,-\frac{3}{4},-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},0\}$	0	(1,0,1,1,0,0)	–
	$b=-\frac{1}{2}(3a+1),\ a\notin\{-1,-\frac{3}{5},-\frac{1}{2},-\frac{1}{3},-\frac{1}{5},0,\frac{1}{3},1\}$ *or* $b=-(a+\frac{1}{2}),\ a\ge-\frac{1}{4},\ a\notin\{0,\frac{1}{2},1\}$	0	(1,0,0,1,1,0)	–
	$b=-(a+\frac{1}{3}),\ a\ge-\frac{1}{6},\ a\notin\{0,\frac{1}{3},\frac{2}{3}\},$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a,b)=(0,0)$	1	(4,5,5,4,1,0)	–
	$(a,b)=(0,-1)$	0	(2,3,3,1,0,0)	–
	$(a,b)=(-1,1)$	1	(2,2,3,4,2,0)	–
	$(a,b)=(0,-\frac{1}{2})$	0	(2,1,1,3,2,0)	–
	$(a,b)=(0,-\frac{1}{3})$	0	(2,1,0,1,2,$\underline{\mathbf{1}}$)	–
	$(a,b)\in\{(-1,\frac{1}{2}),\ (1,-2)\}$	0	(1,2,2,1,1,0)	–
	$(a,b)\in\{(-1,-1),\ (-1,2)\}$	0	(1,2,2,0,0,0)	–
	$(a,b)\in\{(-1,\frac{2}{3}),\ (\frac{1}{3},-\frac{2}{3})\}$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–

[12] $A_{6,14}^{a,b}\cong A_{6,14}^{b,a}$

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$(a,b) \in \{(-1, \frac{3}{2}), (2, -3)\}$	0	(1,1,2,1,0,0)	–
	$(a,b) \in \{(\frac{1}{4}, -\frac{1}{2}), (-1, \frac{1}{3})\}$	0	(1,1,1,1,1,0)	–
	$(a,b) \in \{(-2, 3), (\frac{1}{2}, -\frac{3}{4})\}$	0	(1,0,2,2,0,0)	–
	$(a,b) \in \{(\frac{2}{5}, -\frac{3}{5})\}$	0	(1,0,1,2,1,0)	✓
	$(a,b) = (1, -\frac{3}{2})$	0	(1,0,1,2,1,0)	–
	$(a,b) \in \{(-\frac{1}{3}, -\frac{1}{3}), (-\frac{1}{5}, -\frac{1}{5})\}$	0	(1,0,0,2,2,0)	–
$A^{a}_{6,15}$	$((a+1)e^{16} + e^{23}, e^{26}, ae^{36}, e^{26} + e^{46}, e^{36} + ae^{56}, 0)$			
[13]	$-1 < a \leq 1,\ a \notin \{0, -\frac{1}{3}, -\frac{1}{2}, -\frac{2}{3}\}$	0	(1,0,0,0,0,0)	–
	$a = 0$	1	(2,2,1,0,0,0)	–
	$a = -1$	1	(1,2,4,2,1,$\underline{\mathbf{1}}$)	✓
	$a = -2$	0	(1,1,2,1,0,0)	–
	$a = -3$	0	(1,0,1,1,0,0)	–
	$a = -\frac{3}{2}$	0	(1,0,0,1,1,0)	–
$A_{6,16}$	$(e^{16} + e^{23} + e^{46}, e^{26}, 0, e^{26} + e^{46}, e^{36}, 0)$	1	(2,2,1,0,0,0)	–
$A^{\varepsilon,a}_{6,17}$	$(ae^{16} + e^{23} + \varepsilon e^{46}, ae^{26}, 0, e^{36}, e^{56}, 0)$			
	$\varepsilon = 0,\ a \notin \{-1, -\frac{1}{2}, 0\}$	1	(2,2,1,0,0,0)	–
	$\varepsilon = 0,\ a = 0$	2	(3,6,6,3,1,0)	–
	$\varepsilon = 1,\ a = 0$	1	(3,4,4,3,1,0)	–
	$\varepsilon = 0,\ a = -1$	1	(2,3,4,3,1,0)	–
	$\varepsilon = 0,\ a = -\frac{1}{2}$	1	(2,2,2,2,2,$\underline{\mathbf{1}}$)	–
$A^{a,b}_{6,18}$	$((a+1)e^{16} + e^{23}, ae^{26}, e^{36}, e^{36} + e^{46}, be^{56}, 0)$			
[14]	$a \notin \{-3, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0\},\ b \notin \{-2, -1, 0, -(a+1), -a, -(a+2), -(a+3), -(2a+1), -(2a+2), -(2a+3)\}$	0	(1,0,0,0,0,0)	–
	$a = 0,\ b \notin \{-3, -2, -1, 0\}$	0	(2,1,0,0,0,0)	–
	$a = -1,\ b \notin \{-2, -1, 0, 1\}$	1	(1,1,2,1,0,0)	–

[13] $A^{a}_{6,15} \cong A^{1/a}_{6,15}$

[14] $A^{a,0}_{6,18}$ is decomposable.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$a=-2$, $b\notin\{-2,-1,0,1,2,3\}$ *or* $a=-\frac{1}{2}$, $b\notin\{-\frac{5}{2},-2,-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2}\}$ *or* $b=-1$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,1\}$ *or* $b=-a$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,1,2\}$	0	(1,1,1,0,0,0)	–
	$a=-3$, $b\notin\{-2,-1,0,1,2,3,4,5\}$ *or* $b=-2$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,2\}$ *or* $b=-(a+1)$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,1\}$ *or* $b=-(a+2)$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,1\}$ *or* $b=-(2a+1)$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,\frac{1}{2},1,2\}$	0	(1,0,1,1,0,0)	–
	$a=-\frac{3}{2}$, $b\notin\{-2,-\frac{3}{2},-1,0,\frac{1}{2},1,\frac{3}{2},2\}$ *or* $b=-(a+3)$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,1,2\}$ *or* $b=-(2a+2)$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,1\}$	0	(1,0,0,1,1,0)	–
	$b=-(2a+3)$, $a\notin\{-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0\}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a,b)=(0,-1)$	0	(2,2,3,2,0,0)	–
	$(a,b)=(0,-2)$	0	(2,1,2,3,1,0)	–
	$(a,b)=(0,-3)$	0	(2,1,0,1,2,$\underline{\mathbf{1}}$)	–
	$(a,b)=(-1,-1)$	1	(1,2,4,2,1,$\underline{\mathbf{1}}$)	–
	$(a,b)=(-1,1)$	1	(1,2,4,2,0,0)	–
	$(a,b)\in\{(-\frac{1}{2},-1),\ (-2,-1),\ (-2,-1)\}$	0	(1,2,2,1,1,0)	–
	$(a,b)\in\{(-\frac{1}{2},\frac{1}{2}),\ (1,-1)\}$	0	(1,2,2,0,0,0)	–
	$(a,b)=(-1,-2)$	1	(1,1,3,3,1,0)	–
	$(a,b)\in\{(-3,3),\ (-\frac{1}{2},-2),\ (-2,1)\}$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(a,b)\in$ $\{(-\frac{1}{2},-\frac{3}{2}),(-\frac{1}{2},-\frac{1}{2}),(2,-2),(-3,-1),(-2,-2),(-2,3)\}$	0	(1,1,2,1,0,0)	–
	$(a,b)\in\{(-\frac{1}{2},-\frac{5}{2}),\ (-\frac{3}{2},-1),\ (-\frac{3}{2},\frac{3}{2})\}$	0	(1,1,1,1,1,0)	–
	$(a,b)\in\{(\frac{1}{2},-2),(1,-2),(1,-3),(-3,-2),(-3,1),(-3,2)\}$	0	(1,0,2,2,0,0)	–
	$(a,b)=(-3,5)$	0	(1,0,2,2,0,0)	✓
	$(a,b)\in\{(-\frac{3}{2},-2),(-\frac{3}{2},2),(-\frac{3}{2},\frac{1}{2}),(-\frac{3}{2},-\frac{1}{2}),(-3,4)\}$	0	(1,0,1,2,1,0)	–
	$(a,b)=(2,-5)$	0	(1,0,1,2,1,0)	✓
	$(a,b)\in\{(-\frac{3}{2},-\frac{3}{2}),\ (-\frac{3}{2},1),\ (1,-4)\}$	0	(1,0,0,2,2,0)	–
$A^{a}_{6,19}$	$((a+1)e^{16}+e^{23}+e^{56}, ae^{26}, e^{36}, e^{36}+e^{46}, (a+1)e^{56}, 0)$			
	$a\notin\{-3,-2,-\frac{3}{2},-\frac{4}{3},-1,-\frac{2}{3},-\frac{1}{2},0\}$	0	(1,0,0,0,0,0)	–

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$a = -1$	1	(2,2,3,3,1,0)	–
	$a = 0$	0	(2,1,0,0,0,0)	–
	$a = -2$	0	(1,2,2,1,1,0)	–
	$a = -\frac{1}{2}$	0	(1,1,1,0,0,0)	–
	$a = -\frac{3}{2}$	0	(1,0,1,2,1,0)	–
	$a \in \{-3, -\frac{2}{3}\}$	0	(1,0,1,1,0,0)	–
	$a = -\frac{4}{3}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A_{6,20}^{a}$	$(e^{16} + e^{23} + e^{46}, 0, e^{36}, e^{36} + e^{46}, ae^{56}, 0)$			
	$a \notin \{0, -1, -2, -3\}$	0	(2,1,0,0,0,0)	–
	$a = -1$	0	(2,2,2,1,0,0)	–
	$a = -2$	0	(2,1,1,2,1,0)	–
	$a = -3$	0	(2,1,0,1,2,$\underline{\mathbf{1}}$)	–
$A_{6,21}^{a,b}$	$(2ae^{16} + e^{23}, ae^{26}, e^{26} + ae^{36}, e^{46}, be^{56}, 0)$			
[15]	$-1 < b \leq 1$, $b \neq 0$, $a \notin \{-1, -\frac{1}{3}, -\frac{1}{4}, 0, -b,$ $-\frac{1}{3}b, -\frac{1}{4}b, -(b+1), -\frac{1}{3}(b+1), -\frac{1}{4}(b+1)\}$	0	(1,0,0,0,0,0)	–
	$a = 0$, $-1 < b \leq 1$, $b \neq 0$	1	(2,2,2,1,0,0)	–
	$a = -1$, $b \notin \{-1, 0, 1, 2\}$ *or* $b = -1$, $a > 0$, $a \notin \{\frac{1}{4}, \frac{1}{3}, 1\}$	0	(1,1,1,0,0,0)	–
	$b = -(a+1)$, $-2 \leq a < 0$, $a \notin \{-1, -\frac{1}{3}, -\frac{1}{4}\}$ *or* $a = -\frac{1}{3}$, $b \notin \{-1, -\frac{2}{3}, 0, \frac{1}{3}\}$	0	(1,0,1,1,0,0)	–
	$a = -\frac{1}{4}$, $b \notin \{-1, -\frac{3}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{3}{4}, 1\}$ *or* $b = -(3a+1)$, $-\frac{2}{3} \leq a < 0$, $a \notin \{-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}\}$	0	(1,0,0,1,1,0)	–
	$b = -(4a+1)$, $-\frac{1}{2} \leq a < 0$, $a \notin \{-\frac{1}{3}, -\frac{1}{4}\}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a, b) = (0, -1)$	1	(2,3,4,3,2,$\underline{\mathbf{1}}$)	–
	$(a, b) \in \{(-1, -1), (-1, 1)\}$	0	(1,2,2,0,0,0)	–
	$(a, b) \in \{(-1, 3), (-\frac{1}{3}, \frac{1}{3})\}$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(a, b) = (-\frac{1}{3}, -1)$	0	(1,1,2,1,0,0)	–
	$(a, b) \in \{(-1, 2), (-1, 4), (-\frac{1}{4}, -1)\}$	0	(1,1,1,1,1,0)	–
	$(a, b) \in \{(-\frac{1}{3}, -\frac{2}{3}), (-\frac{1}{3}, 1)\}$	0	(1,0,2,2,0,0)	–

[15] $A_{6,21}^{a,b} \cong A_{6,21}^{a/b,1/b}$, $A_{6,21}^{a,0}$ is decomposable.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$(\mathrm{a},\mathrm{b}) \in \{(-\frac{1}{4},-\frac{3}{4}),\ (-\frac{1}{3},\frac{4}{3})\}$	0	(1,0,1,2,1,0)	–
	$(\mathrm{a},\mathrm{b}) \in \{(-\frac{1}{4},-\frac{1}{4}),\ (-\frac{1}{4},1)\}$	0	(1,0,0,2,2,0)	–
$A^{\mathrm{a}}_{6,22}$	$(2\mathrm{a}e^{16}+e^{23}+e^{56}, \mathrm{a}e^{26}, e^{26}+\mathrm{a}e^{36}, e^{46}, 2\mathrm{a}e^{56}, 0)$			
	$\mathrm{a} \notin \{-1,-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},-\frac{1}{5},-\frac{1}{6},0\}$	0	(1,0,0,0,0,0)	–
	$\mathrm{a}=0$	1	(3,4,4,3,1,0)	–
	$\mathrm{a}=-\frac{1}{2},\ -1$	0	(1,1,1,0,0,0)	–
	$\mathrm{a}=-\frac{1}{3}$	0	(1,0,2,2,0,0)	–
	$\mathrm{a}=-\frac{1}{5},-\frac{1}{4}$	0	(1,0,0,1,1,0)	–
	$\mathrm{a}=-\frac{1}{6}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A^{\varepsilon,\mathrm{a}}_{6,23}$	$(2e^{16}+e^{23}+\varepsilon e^{56}, e^{26}, e^{26}+e^{36}, e^{36}+e^{46}, (2+\mathrm{a})e^{56}, 0)$			
[16]	$\varepsilon=0,\ \mathrm{a} \notin \{-7,-6,-5,-4,-3,-2\}$	0	(1,0,0,0,0,0)	–
	$\varepsilon=0,\ \mathrm{a}=-3$	0	(1,1,1,0,0,0)	–
	$\varepsilon=0,\ \mathrm{a}\in\{-4,-5\}$	0	(1,0,1,1,0,0)	–
	$\varepsilon=0,\ \mathrm{a}=-6$	0	(1,0,0,1,1,0)	–
	$\varepsilon=0,\ \mathrm{a}=-7$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$\varepsilon=1,\ \mathrm{a}=0$	0	(1,0,0,0,0,0)	–
$A^{\varepsilon}_{6,24}$	$(e^{23}+\varepsilon e^{46}, 0, e^{26}, e^{36}, e^{56}, 0)$			
	$\varepsilon=0$	2	(2,3,3,2,1,0)	–
	$\varepsilon=1$	1	(2,3,3,2,1,0)	–
$A^{\mathrm{a},\mathrm{b}}_{6,25}$	$((\mathrm{b}+1)e^{16}+e^{23}, e^{26}, \mathrm{b}e^{36}, \mathrm{a}e^{46}, e^{46}+\mathrm{a}e^{56}, 0)$			
[17]	$-1<\mathrm{b}\le 1,\ \mathrm{b}\notin\{-\frac{1}{2},0\},\ \mathrm{a}\notin\{-1,-\frac{1}{2},0,-\mathrm{b},-2(\mathrm{b}+1),$ $-\frac{1}{2}\mathrm{b},-(\mathrm{b}+1),-(2\mathrm{b}+1),-\frac{1}{2}(2\mathrm{b}+1),-(\mathrm{b}+2),-\frac{1}{2}(\mathrm{b}+2)\}$	0	(1,0,0,0,0,0)	–
	$\mathrm{a}=0,\ -1<\mathrm{b}\le 1,\ \mathrm{b}\notin\{0,-\frac{1}{2}\}$	1	(2,2,1,0,0,0)	–
	$\mathrm{b}=0,\ \mathrm{a}\notin\{-2,-1,-\frac{1}{2},0\}$	0	(2,1,0,0,0,0)	–
	$\mathrm{a}=-1,\ \mathrm{b}\notin\{-2,-1,-\frac{1}{2},0,\frac{1}{2},1,2\}$ *or* $\mathrm{b}=-2,\ \mathrm{a}\notin\{-1,-\frac{1}{2},0,1,\frac{3}{2},2,3\}$	0	(1,1,1,0,0,0)	–

[16]The parameter α in Mubarakzyanov's class $g^{\alpha,\varepsilon,h}_{6,23}$ can be normalised to 1 since $g^{0,\varepsilon,0}_{6,23}$ is nilpotent and $g^{0,0,h}_{6,23} \cong A^{0}_{6,24}$. $A^{0,-2}_{6,23}$ is decomposable.

[17]$A^{\mathrm{a},\mathrm{b}}_{6,25} \cong A^{\mathrm{a}/\mathrm{b},1/\mathrm{b}}_{6,25}$

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$a=-b-2,\ b\notin\{-4,-2,-\frac{3}{2},-1,-\frac{1}{2},0,1\}$ *or* $a=-\frac{1}{2},\ b\notin\{-2,-\frac{3}{2},-1,-\frac{3}{4},-\frac{1}{2},-\frac{1}{4},0,\frac{1}{2},1\}$	0	(1,0,1,1,0,0)	–
	$b=-1,\ a>0,\ a\notin\{\frac{1}{2},1\}$	1	(1,0,1,1,0,0)	–
	$a=-\frac{1}{2}b-1,\ b\notin\{-2,-1,-\frac{2}{3},-\frac{1}{2},0,1,2\}$ *or* $a=-2b-2,\ -1<b\leq 1,\ b\notin\{-\frac{3}{4},-\frac{2}{3},-\frac{1}{2},0\}$	0	(1,0,0,1,1,0)	–
	$a=-b-1,\ -1<b\leq 1,\ b\notin\{-\frac{1}{2},0\}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a,b)=(0,0)$	1	(3,4,3,1,0,0)	–
	$(a,b)=(0,-\frac{1}{2})$	1	(2,3,3,2,1,0)	–
	$(a,b)=(0,-1)$	2	(2,2,2,2,2,$\underline{\mathbf{1}}$)	–
	$(a,b)=(-1,0)$	0	(2,2,2,2,2,$\underline{\mathbf{1}}$)	✓
	$(a,b)\in\{(-\frac{1}{2},0),\ (-2,0)\}$	0	(2,1,1,2,1,0)	–
	$(a,b)=(-1,-\frac{1}{2})$	0	(1,2,2,1,1,0)	–
	$(a,b)=(-\frac{1}{2},-\frac{1}{2})$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(a,b)\in\{(-1,2),\ (-\frac{1}{2},-2),\ (3,-2)\}$	0	(1,1,2,1,0,0)	–
	$(a,b)\in\{(-1,-2),\ (-1,1)\}$	0	(1,2,2,0,0,0)	–
	$(a,b)=(-1,-1)$	1	(1,1,3,2,0,0)	–
	$(a,b)\in\{(-1,\frac{1}{2}),\ (\frac{3}{2},-2)\}$	0	(1,1,1,1,1,0)	–
	$(a,b)=(-\frac{1}{2},-1)$	1	(1,0,2,3,1,0)	–
	$(a,b)\in\{(-\frac{1}{2},-\frac{3}{2}),(-\frac{1}{2},-\frac{1}{4}),(-\frac{1}{2},1),(-3,1)\}$	0	(1,0,2,2,0,0)	–
	$(a,b)=(-\frac{1}{2},-\frac{3}{4})$	0	(1,0,1,2,1,0)	–
	$(a,b)\in\{(-\frac{3}{2},1),\ (-\frac{2}{3},-\frac{2}{3})\}$	0	(1,0,0,2,2,0)	–
$A^{a}_{6,26}$ [18]	$((a+1)e^{16}+e^{23}+e^{56},e^{26},ae^{36},(a+1)e^{46},e^{46}+(a+1)e^{56},0)$			
	$-1<a\leq 1,\ a\notin\{0,-\frac{1}{2},-\frac{2}{3},-\frac{3}{4}\}$	0	(1,0,0,0,0,0)	–
	$a=-1$	1	(2,2,2,2,2,$\underline{\mathbf{1}}$)	–
	$a=0$	0	(2,1,0,0,0,0)	–
	$a=-\frac{1}{2}$	0	(1,1,1,0,0,0)	–
	$a=-\frac{2}{3}$	0	(1,0,1,1,0,0)	–
	$a=-\frac{3}{4}$	0	(1,0,0,1,1,0)	–

[18] $A^{a}_{6,26}\cong A^{1/a}_{6,26}$

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
$A_{6,27}^{\varepsilon_1,\varepsilon_2,\mathrm{a}}$	$((\varepsilon_2+\mathrm{a})e^{16}+e^{23}+\varepsilon_1 e^{56}, \varepsilon_2 e^{26}, \mathrm{a}e^{36}, e^{36}+\mathrm{a}e^{46}, e^{46}+\mathrm{a}e^{56}, 0)$			
[19]	$\varepsilon_1=0,\ \ \varepsilon_2=1,\ \mathrm{a}>0$	0	(1,0,0,0,0,0)	–
	$\varepsilon_1=0,\ \ \varepsilon_2=1,\ \mathrm{a}=0$	1	(2,2,2,1,0,0)	–
	$\varepsilon_1\in\{0,1\},\ \varepsilon_2=0,\ \mathrm{a}=1$	0	(2,1,0,0,0,0)	–
$A_{6,28}^{\mathrm{a}}$	$(2e^{16}+e^{23}, e^{26}, e^{26}+e^{36}, \mathrm{a}e^{46}, e^{46}+\mathrm{a}e^{56}, 0)$			
	$\mathrm{a}\notin\{-4,-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0\}$	0	(1,0,0,0,0,0)	–
	$\mathrm{a}=0$	1	(2,2,1,0,0,0)	–
	$\mathrm{a}=-1$	0	(1,2,2,0,0,0)	–
	$\mathrm{a}=-3$	0	(1,0,2,2,0,0)	✓
	$\mathrm{a}=-\frac{1}{2}$	0	(1,0,1,1,0,0)	–
	$\mathrm{a}\in\{-4,-\frac{3}{2}\}$	0	(1,0,0,1,1,0)	–
	$\mathrm{a}=-2$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A_{6,29}$	$(2e^{16}+e^{23}+e^{56}, e^{26}, e^{26}+e^{36}, 2e^{46}, e^{46}+2e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A_{6,30}$	$(e^{23}, 0, e^{26}, e^{46}, e^{46}+e^{56}, 0)$	1	(2,2,2,1,0,0)	–
$A_{6,31}$	$(2e^{16}+e^{23}, e^{26}, e^{26}+e^{36}, e^{36}+e^{46}, e^{46}+e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A_{6,32}^{\mathrm{a,b,c}}$	$(2\mathrm{a}e^{16}+e^{23}, \mathrm{a}e^{26}-e^{36}, e^{26}+\mathrm{a}e^{36}, \mathrm{b}e^{46}, \mathrm{c}e^{56}, 0)$			
[20]	$\mathrm{a}>0,\ \lvert\mathrm{b}\rvert\le\lvert\mathrm{c}\rvert,\ \mathrm{b}\notin\{0,-4\mathrm{a}\},\ \mathrm{c}\notin\{0,-4\mathrm{a},-\mathrm{b},-(4\mathrm{a}+\mathrm{b})\}$	0	(1,0,0,0,0,0)	–
	$\mathrm{c}=-\mathrm{b},\ \mathrm{a}>0,\ \mathrm{b}>0,\ \mathrm{b}\neq 4\mathrm{a}$	0	(1,1,1,0,0,0)	–
	$\mathrm{a}=0,\ 0<\mathrm{b}\le\lvert\mathrm{c}\rvert,\ \mathrm{c}\neq-\mathrm{b}$	1	(1,0,1,1,0,0)	–
	$\mathrm{b}=-4\mathrm{a},\ \mathrm{a}>0,\ \mathrm{c}\notin\{0,\pm 4\mathrm{a}\}$	0	(1,0,0,1,1,0)	–
	$\mathrm{c}=-(4\mathrm{a}+\mathrm{b}),\ \mathrm{a}>0,\ \mathrm{b}\ge-2\mathrm{a},\ \mathrm{b}\neq 0$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(\mathrm{a},\mathrm{c})=(0,-\mathrm{b}),\ \mathrm{b}>0$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(\mathrm{b},\mathrm{c})=(-4\mathrm{a},4\mathrm{a}),\ \mathrm{a}>0$	0	(1,1,1,1,1,0)	–
	$(\mathrm{b},\mathrm{c})=(-4\mathrm{a},-4\mathrm{a}),\ \mathrm{a}>0$	0	(1,0,0,2,2,0)	–
$A_{6,33}^{\mathrm{a,b}}$	$(2\mathrm{a}e^{16}+e^{23}+e^{56}, \mathrm{a}e^{26}-e^{36}, e^{26}+\mathrm{a}e^{36}, \mathrm{b}e^{46}, 2\mathrm{a}e^{56}, 0)$			

[19] $A_{6,27}^{\varepsilon_1,\varepsilon_2,\mathrm{a}}\cong A_{6,27}^{-\varepsilon_1,-\varepsilon_2,-\mathrm{a}}$, $A_{6,27}^{\varepsilon_1,0,0}$ is nilpotent.

[20] $A_{6,32}^{\mathrm{a,b,c}}\cong A_{6,32}^{\mathrm{a,c,b}}\cong A_{6,32}^{-\mathrm{a},-\mathrm{b},-\mathrm{c}}$, $A_{6,32}^{\mathrm{a},0,\mathrm{c}}\cong A_{6,32}^{\mathrm{a,b},0}$ is decomposable, the parameter ε in Mubarakzyanov's class $g_{6,32}$ is redundant since $g_{6,32}\cong A_{6,33}$ for $\varepsilon\neq 0$.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
[21]	$a > 0,\ b \notin \{0, -2a, -4a, -6a\}$	0	(1,0,0,0,0,0)	–
	$a = 0,\ b > 0$	1	(2,1,1,2,1,0)	–
	$b = -2a,\ a > 0$	0	(1,1,1,0,0,0)	–
	$b = -4a,\ a > 0$	0	(1,0,0,1,1,0)	–
	$b = -6a,\ a > 0$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A_{6,34}^{\varepsilon,a,b}$	$(2ae^{16} + e^{23} + \varepsilon e^{56}, ae^{26} - e^{36}, e^{26} + ae^{36}, (2a+b)e^{46},$ $e^{46} + (2a+b)e^{56}, 0)$			
[22]	$\varepsilon = 0,\ a > 0,\ b \notin \{-2a, -4a, -6a\}$	0	(1,0,0,0,0,0)	–
	$\varepsilon = 0,\ b = -2a,\ a > 0$	1	(2,2,1,0,0,0)	–
	$\varepsilon = 0,\ a = 0,\ b > 0$	1	(1,0,1,1,0,0)	–
	$\varepsilon = 0,\ b = -6a,\ a > 0$	0	(1,0,0,1,1,0)	–
	$\varepsilon = 0,\ b = -4a,\ a > 0$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$\varepsilon = 1,\ b = 0,\ a > 0$	0	(1,0,0,0,0,0)	–
	$\varepsilon \in \{0,1\},\ (a,b) = (0,0)$	2	(2,2,2,2,2,$\underline{\mathbf{1}}$)	–
$A_{6,35}^{a,b,c}$	$((a+b)e^{16} + e^{23}, ae^{26}, be^{36}, ce^{46} - e^{56}, e^{46} + ce^{56}, 0)$			
[23]	$0 < a \le \lvert b\rvert,\ b \notin \{0, -a, -2a, \}$ $c \notin \{0, -\frac{1}{2}a, -\frac{1}{2}b, -(\frac{1}{2}a+b), -(\frac{1}{2}b+a), -(a+b)\}$	0	(1,0,0,0,0,0)	–
	$a = 0,\ b > 0,\ c \notin \{0, -b, -\frac{1}{2}b\}$	0	(2,1,0,0,0,0)	–
	$b = -2a,\ a > 0,\ c \notin \{0, -\frac{1}{2}a, a, \frac{3}{2}a\}$ *or* $c = 0,\ 0 < a \le \lvert b\rvert,\ b \notin \{-a, -2a\}$	0	(1,1,1,0,0,0)	–
	$b = -a,\ a > 0,\ c > 0,\ c \notin \{\frac{1}{2}a\}$	1	(1,0,1,1,0,0)	–
	$c = -\frac{1}{2}a,\ a > 0,\ b \notin \{0, -2a, -a, -\frac{1}{2}a, a\}$	0	(1,0,1,1,0,0)	–
	$c = -(\frac{1}{2}a + b),\ a > 0,\ b \notin \{0, -2a, -a, -\frac{1}{2}a, a\}$	0	(1,0,0,1,1,0)	–
	$c = -(a+b),\ 0 < a \le \lvert b\rvert,\ b \notin \{0, -2a, -a\}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a,c) = (0,0),\ b > 0$	0	(2,2,2,1,0,0)	–
	$(a,c) = (0, -\frac{1}{2}b),\ b > 0$	0	(2,1,1,2,1,0)	–
	$(a,c) = (0, -b),\ b > 0$	0	(2,1,0,1,2,$\underline{\mathbf{1}}$)	–

[21] $A_{6,33}^{a,b} \cong A_{6,33}^{-a,-b}$, $A_{6,33}^{a,0}$ is decomposable.
[22] $A_{6,34}^{\varepsilon,a,b} \cong A_{6,34}^{-\varepsilon,-a,-b}$
[23] $A_{6,35}^{a,b,c} \cong A_{6,35}^{b,a,c} \cong A_{6,35}^{-a,-b,-c}$, $A_{6,35}^{0,0,c}$ is decomposable.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$(b,c)=(-2a,0)$, $a>0$	0	(1,2,2,1,1,0)	–
	$(b,c)=(-2a,a)$, $a>0$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(b,c)=(-a,0)$, $a>0$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(b,c)=(-2a,-\frac{1}{2}a)$, $a>0$	0	(1,1,2,1,0,0)	–
	$(b,c)=(-2a,\frac{3}{2}a)$, $a>0$	0	(1,1,1,1,1,0)	–
	$(b,c)=(-a,\frac{1}{2}a)$, $a>0$	1	(1,0,2,3,1,0)	–
	$(b,c)=(a,-\frac{1}{2}a)$, $a>0$	0	(1,0,2,2,0,0)	–
	$(b,c)=(a,-\frac{3}{2}a)$, $a>0$	0	(1,0,0,2,2,0)	–
$A_{6,36}^{a,b}$	$(2ae^{16}+e^{23},ae^{26},e^{26}+ae^{36},be^{46}-e^{56},e^{46}+be^{56},0)$			
	$a>0$, $b\notin\{0,-2a,-\frac{3}{2}a,-\frac{1}{2}a\}$	0	(1,0,0,0,0,0)	–
	$a=0$, $b>0$	1	(2,2,2,1,0,0)	–
	$b=0$, $a>0$	0	(1,1,1,0,0,0)	–
	$b=-\frac{1}{2}a$, $a>0$	0	(1,0,1,1,0,0)	–
	$b=-\frac{3}{2}a$, $a>0$	0	(1,0,0,1,1,0)	–
	$b=-2a$, $a>0$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a,b)=(0,0)$	1	(2,3,4,3,2,$\underline{\mathbf{1}}$)	–
$A_{6,37}^{a,b,c}$	$(2ae^{16}+e^{23},ae^{26}-e^{36},e^{26}+ae^{36},be^{46}-ce^{56},ce^{46}+be^{56},0)$			
[24]	$a>0$, $b\notin\{0,-2a\}$, $c>0$, $(b,c)\notin\{(-a,1),(-3a,1)\}$	0	(1,0,0,0,0,0)	–
	$b=0$, $a>0$, $c>0$	0	(1,1,1,0,0,0)	–
	$a=0$, $b>0$, $c>0$	1	(1,0,1,1,0,0)	–
	$b=-2a$, $a>0$, $c>0$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(b,c)=(-a,1)$, $a>0$	0	(1,2,2,0,0,0)	–
	$(a,b)=(0,0)$, $c>0$, $c\neq 1$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(b,c)=(-3a,1)$, $a>0$	0	(1,0,2,2,0,0)	✓
	$(a,b,c)=(0,0,1)$	1	(1,3,6,3,1,$\underline{\mathbf{1}}$)	✓
$A_{6,38}^{a}$	$(2ae^{16}+e^{23},ae^{26}-e^{36},e^{26}+ae^{36},e^{26}+ae^{46}-e^{56},$ $e^{36}+e^{46}+ae^{56},0)$			

[24] $A_{6,37}^{a,b,0}\cong A_{6,32}^{a,b,b}$, $A_{6,37}^{a,b,c}\cong A_{6,37}^{a,b,-c}\cong A_{6,37}^{-a,-b,c}$, $A_{6,37}^{a,0,0}$ is decomposable.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$a > 0$	0	(1,0,0,0,0,0)	–
	$a = 0$	1	(1,2,4,2,1,$\underline{\mathbf{1}}$)	✓
$B_{6,1}$ [25]	$(e^{16} + e^{23} + e^{56}, e^{26}, 0, e^{36}, e^{56}, 0)$	1	(2,2,1,0,0,0)	–
	Nilradical $A_{4,1} \oplus \mathbb{R}$			
$A_{6,39}^{a,b}$	$((b+1)e^{16} + e^{45}, e^{15} + (b+2)e^{26}, ae^{36}, be^{46}, e^{56}, 0)$			
[26]	$a \notin \{-1, 0\}$, $b \notin \{-3, -2, -\frac{4}{3}, -1, -\frac{1}{2}, 0, -a, -(a+3),$ $-\frac{1}{3}(a+3), -\frac{1}{3}(a+4), -\frac{1}{2}(a+1), -\frac{1}{2}(a+4)\}$	0	(1,0,0,0,0,0)	–
	$b = 0$, $a \notin \{-4, -3, -1, 0\}$	0	(2,1,0,0,0,0)	–
	$a = -1$, $b \notin \{-3, -2, -\frac{3}{2}, -\frac{4}{3}, -1, -\frac{2}{3}, -\frac{1}{2}, 0, 1\}$ *or* $b = -3$, $a \notin \{-1, 0, 2, 3, 5, 6\}$ *or* $b = -\frac{1}{2}$, $a \notin \{-3, -\frac{5}{2}, -\frac{3}{2}, -1, 0, \frac{1}{2}\}$ *or* $b = -a$, $a \notin \{-1, 0, \frac{1}{2}, 1, \frac{4}{3}, \frac{3}{2}, 2, 3, 4\}$	0	(1,1,1,0,0,0)	–
	$b = -2$, $a \notin \{-1, 0, 2, 3\}$	1	(1,0,1,1,0,0)	–
	$b = -(a+3)$, $a \notin \{-5-3, -\frac{5}{2}, -2, -\frac{5}{3}, -1, 0\}$ *or* $b = -\frac{1}{2}(a+1)$, $a \notin \{-5, -1, 0, 1, \frac{5}{3}, 3, 5\}$	0	(1,0,1,1,0,0)	–
	$b = -1$, $a \notin \{-2, -1, 0, 1\}$	0	(1,0,1,1,0,0)	✓
	$b = -\frac{1}{2}(a+4)$, $a \notin \{-6, -4, -3, -2, -\frac{4}{3}, -1, 0, 2, 4\}$ *or* $b = -\frac{4}{3}$, $a \notin \{-\frac{5}{3}, -\frac{4}{3}, -1, 0, 1, \frac{4}{3}, \frac{5}{3}\}$	0	(1,0,0,1,1,0)	–
	$b = -\frac{1}{3}(a+1)$, $a \notin \{-6, -3, -\frac{3}{2}, -1, 0, 1, \frac{3}{2}, 3, 6\}$	0	(1,0,0,1,1,0)	✓
	$b = -\frac{1}{3}(a+4)$, $a \notin \{-4, -\frac{5}{2}, -1, 0, 2, 5\}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a, b) = (-1, 0)$	0	(2,2,2,1,0,0)	–
	$(a, b) = (-3, 0)$	0	(2,1,1,2,1,0)	✓
	$(a, b) = (-4, 0)$	0	(2,1,0,1,2,$\underline{\mathbf{1}}$)	–
	$(a, b) \in \{(-1, -3), (-1, -\frac{1}{2}), (-1, 1), (\frac{1}{2}, -\frac{1}{2}), (3, -3)\}$	0	(1,2,2,0,0,0)	–
	$(a, b) = (-1, -2)$	1	(1,1,3,2,0,0)	–
	$(a, b) = (1, -1)$	0	(1,1,3,2,0,0)	✓
	$(a, b) \in \{(-1, -1), (-\frac{5}{2}, -\frac{1}{2}), (5, -3)\}$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(a, b) = (2, -2)$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$(a, b) \in \{-(1, \frac{4}{3}), -(1, \frac{3}{2}), -(3, \frac{1}{2}), (2, -3), (\frac{4}{3}, -\frac{4}{3}), (4, -4)\}$	0	(1,1,1,1,1,0)	–

[25] $B_{6,1}$ is denoted by $n_{6,8}$ in [Sha].
[26] $A_{6,39}^{0,b}$ is decomposable.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$(a,b) \in \{(-\frac{3}{2},-\frac{1}{2}),(-1,-\frac{2}{3}),(\frac{3}{2},-\frac{3}{2}),(6,-3)\}$	0	(1,1,1,1,1,0)	✓
	$(a,b)=(3,-2)$	1	(1,0,2,3,1,0)	✓
	$(a,b)=(-2,-1)$	0	(1,0,2,3,1,0)	✓
	$(a,b)=(-5,2)$	0	(1,0,2,2,0,0)	–
	$(a,b)=\{(\frac{5}{3},-\frac{4}{3}),(-\frac{5}{3},-\frac{4}{3})\}$	0	(1,0,1,2,1,0)	–
	$(a,b)=(-\frac{4}{3},-\frac{4}{3})$	0	(1,0,0,2,2,0)	–
	$(a,b)=(-6,1),(1,-\frac{4}{3})$	0	(1,0,0,2,2,0)	✓
$A^{a}_{6,40}$	$((a+1)e^{16}+e^{45},e^{15}+(a+2)e^{26}+e^{36},(a+2)e^{36},ae^{46},e^{56},0)$			
	$a \notin \{-3,-\frac{5}{2},-2,-\frac{3}{2},-\frac{4}{3},-\frac{5}{4},-1,-\frac{1}{2},0\}$	0	(1,0,0,0,0,0)	–
	$a=-2$	1	(2,1,1,2,1,0)	–
	$a=0$	0	(2,1,0,0,0,0)	–
	$a=-1$	0	(1,1,2,1,0,0)	–
	$a \in \{-3,\ -\frac{1}{2}\}$	0	(1,1,1,0,0,0)	–
	$a=-\frac{5}{2}$	0	(1,0,1,1,0,0)	–
	$a=-\frac{4}{3}$	0	(1,0,0,1,1,0)	–
	$a=-\frac{5}{4}$	0	(1,0,0,1,1,0)	✓
	$a=-\frac{3}{2}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A^{a}_{6,41}$	$((a+1)e^{16}+e^{45},e^{15}+(a+2)e^{26},ae^{36}+e^{46},ae^{46},e^{56},0)$			
	$a \notin \{-3,-2,-\frac{3}{2},-\frac{4}{3},-1,-\frac{3}{4},-\frac{1}{2},-\frac{1}{3},0\}$	0	(1,0,0,0,0,0)	–
	$a=0$	1	(2,2,1,0,0,0)	–
	$a=-1$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$a \in \{-3,-\frac{1}{2}\}$	0	(1,1,1,0,0,0)	–
	$a=-2$	1	(1,0,1,1,0,0)	–
	$a \in \{-\frac{3}{2},-\frac{1}{3}\}$	0	(1,0,1,1,0,0)	–
	$a=-\frac{4}{3}$	0	(1,0,0,1,1,0)	–
	$a=-\frac{3}{4}$	0	(1,0,0,1,1,0)	✓
$A^{a}_{6,42}$	$((a+1)e^{16}+e^{45},e^{15}+(a+2)e^{26},e^{36}+e^{56},ae^{46},e^{56},0)$			
	$a \notin \{-4,-3,-\frac{5}{2},-2,-\frac{5}{3},-\frac{4}{3},-1,-\frac{1}{2},0\}$	0	(1,0,0,0,0,0)	–

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$a = 0$	0	(2,1,0,0,0,0)	–
	$a = -1$	0	(1,1,3,2,0,0)	✓
	$a \in \{-3, -\frac{1}{2}\}$	0	(1,1,1,0,0,0)	–
	$a = -4$	0	(1,0,1,1,0,0)	–
	$a = -2$	1	(1,0,1,1,0,0)	–
	$a \in \{-\frac{5}{2}, -\frac{4}{3}\}$	0	(1,0,0,1,1,0)	–
	$a = -\frac{5}{3}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A_{6,43}$	$(e^{45}, e^{15} + e^{26} + e^{36}, e^{36} + e^{56}, -e^{46}, e^{56}, 0)$	0	(1,1,2,1,0,0)	–
$A_{6,44}^{a}$	$(2e^{16} + e^{45}, e^{15} + 3e^{26}, ae^{36}, e^{46} + e^{56}, e^{56}, 0)$			
	$a \notin \{0, -1, -3, -4, -6, -7\}$	0	(1,0,0,0,0,0)	–
	$a = -1$	0	(1,1,1,0,0,0)	–
	$a \in \{-4, -3\}$	0	(1,0,1,1,0,0)	–
	$a = -6$	0	(1,0,0,1,1,0)	–
	$a = -7$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A_{6,45}$	$(2e^{16} + e^{45}, e^{15} + 3e^{26} + e^{36}, 3e^{36}, e^{46} + e^{56}, e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A_{6,46}$	$(2e^{16} + e^{45}, e^{15} + 3e^{26}, e^{36} + e^{46}, e^{46} + e^{56}, e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A_{6,47}^{\varepsilon,a}$	$(e^{16} + e^{45}, e^{15} + e^{26} + \varepsilon e^{46}, ae^{36}, e^{46}, 0, 0)$			
	$\varepsilon \in \{0, \pm 1\}$, $a \notin \{0, -1, -2, -3\}$	0	(2,1,0,0,0,0)	–
	$\varepsilon \in \{0, \pm 1\}$, $a = -1$	0	(2,2,2,1,0,0)	–
	$\varepsilon \in \{0, \pm 1\}$, $a = -2$	0	(2,1,1,2,1,0)	✓
	$\varepsilon \in \{0, \pm 1\}$, $a = -3$	0	(2,1,0,1,2,$\underline{\mathbf{1}}$)	✓
$A_{6,48}$	$(e^{16} + e^{45}, e^{15} + e^{26} + e^{36}, e^{36}, e^{46}, 0, 0)$			
$A_{6,49}^{\varepsilon}$	$(e^{16} + e^{45}, e^{15} + e^{26} + \varepsilon e^{46}, e^{56}, e^{46}, 0, 0)$, $\varepsilon \in \{0, \pm 1\}$	1	(2,2,1,0,0,0)	–
$A_{6,50}^{\varepsilon}$	$(e^{16} + e^{45}, e^{15} + e^{26} + \varepsilon e^{36}, e^{36} + e^{46}, e^{46}, 0, 0)$, $\varepsilon \in \{0, \pm 1\}$	0	(2,1,0,0,0,0)	–
$A_{6,51}^{\varepsilon}$	$(e^{45}, e^{15} + \varepsilon e^{46}, e^{36}, 0, 0, 0)$, $\varepsilon = \pm 1$ [27]	1	(3,4,4,3,1,0)	✓
$A_{6,52}^{\varepsilon}$	$(e^{45}, e^{15} + \varepsilon e^{46}, e^{36}, e^{56}, 0, 0)$, $\varepsilon \in \{0, \pm 1\}$	1	(2,3,3,2,1,0)	–
	Nilradical $A_{5,1}$			

[27] $A_{6,51}^{0}$ is decomposable.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
$A_{6,53}$	$(e^{35}, e^{45}, e^{36}, e^{46}, -e^{56}, 0)$	2	(1,0,3,5,2,0)	–
$A_{6,54}^{a,b}$	$(e^{16} + e^{35}, be^{26} + e^{45}, (1-a)e^{36}, (b-a)e^{46}, ae^{56}, 0)$			
[28]	$-1 < b \leq 1,\ b \notin \{-\frac{1}{2}, 0\},\ a \notin \{-1, 0, 1, 2, \pm b, 2b, \frac{1}{2}(b+1), \pm(b+1), b+\frac{1}{2}, \frac{1}{2}b+1, 2(b+1)\}$	0	(1,0,0,0,0,0)	–
	$a = 0,\ -1 < b \leq 1,\ b \notin \{-\frac{1}{2}, -1, 0\}$ *or* $a = 1,\ b \notin \{-2, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1\}$	0	(2,1,0,0,0,0)	–
	$a = b + 1,\ -1 < b < 1,\ b \notin \{-\frac{1}{2}, 0\}$	0	(1,1,1,1,1,0)	✓
	$a = -1,\ b \notin \{-4, -3, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, 1\}$ *or* $a = 2,\ b \notin \{-3, -2, -1, -\frac{1}{2}, 0, 1, \frac{3}{2}, 2, 3\}$ *or* $a = \frac{1}{2}(b+1),\ -1 < b < 1,\ b \notin \{-\frac{1}{2}, -\frac{1}{3}, 0, \frac{1}{3}\}$	0	(1,1,1,0,0,0)	–
	$b = 0,\ a \notin \{-1, 0, \frac{1}{2}, 1, 2\}$	1	(1,0,1,1,0,0)	–
	$a = -(b+1),\ -1 < b \leq 1,\ b \notin \{-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{3}, 0\}$ *or* $b = -1,\ a > 0,\ a \notin \{\frac{1}{2}, 1, 2\}$ *or* $a = b + \frac{1}{2},\ b \notin \{-2, -\frac{3}{2}, -1, -\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, 0, \frac{1}{2}, 1, \frac{3}{2}\}$	0	(1,0,1,1,0,0)	–
	$b = -2,\ a \notin \{-4, -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0, 1, 2\}$	0	(1,0,0,1,1,0)	–
	$a = 2(b+1),\ -1 < b \leq 1,\ b \notin \{-\frac{2}{3}, -\frac{1}{2}, 0\}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
	$(a,b) = (0,0)$	1	(3,4,3,1,0,0)	–
	$(a,b) = (1,1)$	0	(3,3,1,0,0,0)	–
	$(a,b) = (0,-1)$	0	(2,3,4,3,2,$\underline{\mathbf{1}}$)	✓
	$(a,b) = (1,0)$	1	(2,2,3,3,1,0)	✓
	$(a,b) \in \{(1,-1),\ (2,2)\}$	0	(2,2,2,1,0,0)	–
	$(a,b) \in \{(1,-2),\ (0,-2)\}$	0	(2,1,1,2,1,0)	–
	$(a,b) = (1, -\frac{1}{2})$	0	(2,1,0,1,2,$\underline{\mathbf{1}}$)	–
	$(a,b) = (2,1)$	0	(1,3,3,1,1,0)	✓
	$(a,b) = (2,-2)$	0	(1,2,2,1,1,0)	–
	$(a,b) = (-1,-2)$	0	(1,2,2,2,2,0)	✓
	$(a,b) \in \{(2,3),\ (-1,-3),\ (-1,1)\}$	0	(1,2,2,0,0,0)	–
	$(a,b) \in \{(-1,0),\ (\frac{1}{2},0)\}$	1	(1,1,3,2,0,0)	–
	$(a,b) \in \{(-1,-\frac{3}{2}),\ (2,0)\}$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–

[28] $A_{6,54}^{a,b} \cong A_{6,54}^{a/b,1/b}$

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$(a,b) \in \{(-1,-4),(2,-3),(2,-1),(2,\frac{3}{2})\}$	0	(1,1,2,1,0,0)	–
	$(a,b) \in \{(2,-\frac{1}{2}),\ (\frac{1}{4},-\frac{1}{2})\}$	0	(1,1,1,1,1,0)	–
	$(a,b) \in \{(\frac{3}{2},1),\ (\frac{1}{2},-1)\}$	0	(1,0,2,2,0,0)	–
	$(a,b) = (\frac{1}{3},-\frac{4}{3})$	0	(1,0,2,2,0,0)	✓
	$(a,b) = (-\frac{3}{2},-2)$	0	(1,0,1,2,1,0)	✓
$A^a_{6,55}$	$(e^{16}+e^{35}+e^{46},(a+1)e^{26}+e^{45},(1-a)e^{36},e^{46},ae^{56},0)$			
	$a \notin \{-4,-3,-2,-\frac{3}{2},-1,-\frac{1}{2},0,1,2,3\}$	0	(1,0,0,0,0,0)	–
	$a \in \{0,1\}$	0	(2,1,0,0,0,0)	–
	$a=-2$	0	(1,1,2,1,0,0)	–
	$a=-1$	1	(1,1,2,1,0,0)	–
	$a \in \{-\frac{1}{2},2\}$	0	(1,1,1,0,0,0)	–
	$a=3$	0	(1,0,1,1,0,0)	–
	$a \in \{-3,-\frac{3}{2}\}$	0	(1,0,0,1,1,0)	–
	$a=-4$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A^a_{6,56}$	$(e^{16}+e^{35},(1-b)e^{26}+e^{36}+e^{45},(1-a)e^{36},(1-2a)e^{46},ae^{56},0)$			
	$a \notin \{-1,0,\frac{1}{2},\frac{2}{3},\frac{3}{4},1,\frac{4}{3},\frac{3}{2},2,3\}$	0	(1,0,0,0,0,0)	–
	$a=1$	1	(2,2,3,3,1,0)	✓
	$a \in \{0,\frac{1}{2}\}$	0	(2,1,0,0,0,0)	–
	$a=2$	0	(1,1,2,1,0,0)	–
	$a \in \{-1,\frac{2}{3}\}$	0	(1,1,1,0,0,0)	–
	$a=\frac{3}{4}$	0	(1,0,1,1,0,0)	–
	$a \in \{\frac{3}{2},3\}$	0	(1,0,0,1,1,0)	–
	$a=\frac{4}{3}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A^a_{6,57}$	$(e^{16}+e^{35},2ae^{26}+e^{45},(1-a)e^{36},ae^{46}+e^{56},ae^{56},0)$			
	$a \notin \{-1,-\frac{2}{3},-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},0,1,2\}$	0	(1,0,0,0,0,0)	–
	$a=0$	1	(2,2,2,1,0,0)	–
	$a=1$	0	(2,1,0,0,0,0)	–
	$a=-1$	0	(1,1,1,1,1,0)	–

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$\mathrm{a}=2$	0	(1,1,1,0,0,0)	–
	$\mathrm{a}\in\{-\frac{1}{2},-\frac{1}{3}\}$	0	(1,0,1,1,0,0)	–
	$\mathrm{a}=-\frac{1}{4}$	0	(1,0,0,1,1,0)	–
	$\mathrm{a}=-\frac{2}{3}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A_{6,58}^{\varepsilon}$	$(3\mathrm{e}^{16}+\mathrm{e}^{35},2\mathrm{e}^{26}+\mathrm{e}^{36}+\mathrm{e}^{45},2\mathrm{e}^{36},\mathrm{e}^{46}+\varepsilon\mathrm{e}^{56},\mathrm{e}^{56},0)$, $\varepsilon\in\{0,1\}$	0	(1,0,0,0,0,0)	–
$A_{6,59}$ [29]	$(\mathrm{e}^{16}+\mathrm{e}^{35},\mathrm{e}^{45}+\mathrm{e}^{46},\mathrm{e}^{36},\mathrm{e}^{56},0,0)$	1	(2,2,2,1,0,0)	–
$A_{6,60}$ [30]	$(\mathrm{e}^{16}+\mathrm{e}^{35}+\mathrm{e}^{46},2\mathrm{e}^{26}+\mathrm{e}^{45},0,\mathrm{e}^{46}+\mathrm{e}^{56},\mathrm{e}^{56},0)$	0	(2,1,0,0,0,0)	–
$A_{6,61}^{\mathrm{a}}$	$(2\mathrm{e}^{16}+\mathrm{e}^{35},2\mathrm{a}\mathrm{e}^{26}+\mathrm{e}^{45},\mathrm{e}^{36}+\mathrm{e}^{56},(2\mathrm{a}-1)\mathrm{e}^{46},\mathrm{e}^{56},0)$			
	$\mathrm{a}\notin\{-2,-\frac{3}{2},-1,-\frac{3}{4},-\frac{1}{2},0,\frac{1}{4},\frac{1}{2}\}$	0	(1,0,0,0,0,0)	–
	$\mathrm{a}=\frac{1}{2}$	0	(2,1,0,0,0,0)	–
	$\mathrm{a}=0$	1	(1,1,2,1,0,0)	–
	$\mathrm{a}=-\frac{1}{2}$	0	(1,1,1,1,1,0)	–
	$\mathrm{a}=\frac{1}{4}$	0	(1,1,1,0,0,0)	–
	$\mathrm{a}\in\{-1,-\frac{3}{2}\}$	0	(1,0,1,1,0,0)	–
	$\mathrm{a}=-2$	0	(1,0,0,1,1,0)	–
	$\mathrm{a}=-\frac{3}{4}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A_{6,62}^{\varepsilon}$	$(2\mathrm{e}^{16}+\mathrm{e}^{35},\mathrm{e}^{26}+\mathrm{e}^{36}+\mathrm{e}^{45},\mathrm{e}^{36}+\varepsilon\mathrm{e}^{56},0,\mathrm{e}^{56},0)$, $\varepsilon\in\{0,1\}$	0	(2,1,0,0,0,0)	–
$A_{6,63}^{\mathrm{a}}$	$(\mathrm{e}^{16}+\mathrm{e}^{35},\mathrm{a}\mathrm{e}^{26}+\mathrm{e}^{45}+\mathrm{e}^{46},\mathrm{e}^{36},\mathrm{a}\mathrm{e}^{46},0,0)$			
	$\mathrm{a}\notin\{-2,-1,-\frac{1}{2},0\}$	0	(2,1,0,0,0,0)	–
	$\mathrm{a}=0$	1	(3,4,3,1,0,0)	–
	$\mathrm{a}=-1$	0	(2,2,2,2,2,$\underline{\mathbf{1}}$)	✓
	$\mathrm{a}\in\{-2,-\frac{1}{2}\}$	0	(2,1,1,2,1,0)	–
$A_{6,64}^{\varepsilon}$ [31]	$(\mathrm{e}^{16}+\mathrm{e}^{35}+\mathrm{e}^{46},\mathrm{e}^{26}+\varepsilon\mathrm{e}^{36}+\mathrm{e}^{45},\mathrm{e}^{36},\mathrm{e}^{46},0,0)$, $\varepsilon=\pm1$	0	(2,1,0,0,0,0)	–
$A_{6,65}^{\varepsilon,\mathrm{a}}$	$(\varepsilon\mathrm{e}^{16}+\mathrm{e}^{35},\mathrm{e}^{16}+\varepsilon\mathrm{e}^{26}+\mathrm{e}^{45},(\varepsilon-\mathrm{a})\mathrm{e}^{36},\mathrm{e}^{36}+(\varepsilon-\mathrm{a})\mathrm{e}^{46},\mathrm{a}\mathrm{e}^{56},0)$			
	$\varepsilon=1$, $\mathrm{a}\notin\{-2,-1,0,1,\frac{3}{2},2\}$	0	(1,0,0,0,0,0)	–

[29] The parameter h in Mubarakzyanov's class $g_{6,59}$ is redundant since it can be normalised for $h\neq0$ and $g_{6,59}=A_{6,63}^{0}$ for $h=0$.
[30] The parameter ω in Mubarakzyanov's class $g_{6,60}$ is redundant since $g_{6,60}=A_{6,55}^{1}$ for $\omega=0$.
[31] $A_{6,64}^{0}=A_{6,55}^{0}$

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$\varepsilon = 1,\ a = 1$	0	(2,2,1,0,0,0)	–
	$\varepsilon = 1,\ a = 0$	0	(2,1,0,0,0,0)	–
	$\varepsilon = 1,\ a = 2$	0	(1,1,1,1,1,0)	✓
	$\varepsilon = 1,\ a = -1$	0	(1,1,1,0,0,0)	–
	$\varepsilon = 0,\ a = 1$	1	(1,0,1,2,1,0)	–
	$\varepsilon = 1,\ a \in \{-2,\ \frac{3}{2}\}$	0	(1,0,1,1,0,0)	–
$A_{6,66}$	$(2e^{16} + e^{35}, e^{16} + 2e^{26} + e^{45}, e^{36} + e^{56}, e^{36} + e^{46}, e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A^{\varepsilon}_{6,68}$ [32]	$(e^{16} + e^{35} + ae^{46}, e^{16} + e^{26} + e^{45}, e^{36}, e^{36} + e^{46}, 0, 0)$, $\varepsilon \in \{0,1\}$	0	(2,1,0,0,0,0)	–
$A_{6,69}$	$(e^{16} + e^{35}, e^{16} + e^{26} + e^{45} + e^{46}, e^{36}, e^{36} + e^{46}, 0, 0)$	0	(2,1,0,0,0,0)	–
$A^{a,b}_{6,70}$	$(be^{16} - e^{26} + e^{35}, e^{16} + be^{26} + e^{45}, (b-a)e^{36} - e^{46},$ $e^{36} + (b-a)e^{46}, ae^{56}, 0)$			
	$a > 0,\ b \notin \{-\frac{1}{2}a, 0, \frac{1}{4}a, \frac{1}{2}a, a\}$	0	(1,0,0,0,0,0)	–
	$a = 0,\ b > 0$	0	(2,1,0,0,0,0)	–
	$b = \frac{1}{2}a,\ a > 0,$	0	(1,1,1,1,1,0)	✓
	$b = a,\ a > 0$	0	(1,1,1,0,0,0)	–
	$b = -\frac{1}{2}a,\ a > 0$ *or* $b = 0,\ a > 0$	0	(1,0,1,1,0,0)	–
	$b = \frac{1}{4}a,\ a > 0$	0	$(1,0,0,0,1,\underline{\mathbf{1}})$	–
	$(a, b) = (0, 0)$	0	$(2,3,4,3,2,\underline{\mathbf{1}})$	✓
	Nilradical $A_{5,2}$			
$A^{a}_{6,71}$	$((a+3)e^{16} + e^{25}, (a+2)e^{26} + e^{35}, (a+1)e^{36} + e^{45}, ae^{46}, e^{56}, 0)$			
	$a \notin \{-4, -3, -\frac{7}{3}, -2, -\frac{7}{4}, -\frac{3}{2}, -1, -\frac{1}{2}, 0\}$	0	(1,0,0,0,0,0)	–
	$a = 0$	0	(2,1,0,0,0,0)	–
	$a = -\frac{3}{2}$	0	(1,1,1,1,1,0)	✓
	$a \in \{-4, -\frac{1}{2}\}$	0	(1,1,1,0,0,0)	–
	$a = -3$	1	(1,0,1,1,0,0)	–
	$a = -2$	0	(1,0,1,1,0,0)	–
	$a = -1$	0	(1,0,1,1,0,0)	✓

[32] Mubarakzyanov's class $g_{6,67}$ is redundant since $g^{h}_{6,67} \cong A^{1,1/2}_{6,65}$ for all $h \in \mathbb{R}$.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$a = -\frac{7}{3}$	0	(1,0,0,1,1,0)	–
	$a = -\frac{7}{4}$	0	(1,0,0,0,1,$\underline{\mathbf{1}}$)	–
$A_{6,72}$	$(4e^{16} + e^{25}, 3e^{26} + e^{35}, 2e^{36} + e^{45}, e^{46} + e^{56}, e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A^{\varepsilon}_{6,73}$	$(e^{16} + e^{25} + \varepsilon e^{36}, e^{26} + e^{35} + \varepsilon e^{46}, e^{36} + e^{45}, e^{46}, 0, 0)$, $\varepsilon = \pm 1$	0	(2,1,0,0,0,0)	–
$A_{6,74}$	$(e^{16} + e^{25}, e^{26} + e^{35}, e^{36} + e^{45}, e^{46}, 0, 0)$	0	(2,1,0,0,0,0)	–
$A_{6,75}$	$(e^{16} + e^{25} + e^{46}, e^{26} + e^{35}, e^{36} + e^{45}, e^{46}, 0, 0)$	0	(2,1,0,0,0,0)	–
$B^{\varepsilon,a}_{6,2}$ [33]	$(e^{16} + e^{25} + \varepsilon e^{36} + a e^{46}, e^{26} + e^{35} + \varepsilon e^{46}, e^{36} + e^{45}, e^{46}, 0, 0)$, $\varepsilon = \pm 1$, $a \neq 0$	0	(2,1,0,0,0,0)	–
	Nilradical $A_{5,3}$			
$A^{a}_{6,76}$	$((2a+1)e^{16} + e^{25}, (a+1)e^{26} + e^{45}, e^{24} + (a+2)e^{36}, e^{46}, a e^{56}, 0)$			
[34]	$-1 < a \leq 1$, $a \notin \{0, -\frac{1}{3}, -\frac{1}{2}, -\frac{4}{5}\}$	0	(1,0,0,0,0,0)	✓
	$a = 0$	0	(2,1,0,0,0,0)	✓
	$a = -1$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	$a = -3$	0	(1,1,1,0,0,0)	✓
	$a = -2$	1	(1,0,1,1,0,0)	✓
	$a = -\frac{4}{5}$	0	(1,0,0,1,1,0)	✓
$A^{\varepsilon}_{6,77}$	$(e^{16} + e^{25} + \varepsilon e^{46}, e^{26} + e^{45}, e^{24} + 2e^{36}, e^{46}, 0, 0)$, $\varepsilon = \pm 1$	0	(2,1,0,0,0,0)	✓
$A_{6,78}$	$(-e^{16} + e^{25}, e^{45}, e^{24} + e^{36} + e^{46}, e^{46}, -e^{56}, 0)$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
$A_{6,79}$	$(3e^{16} + e^{25} + e^{36}, 2e^{26} + e^{45}, e^{24} + 3e^{36}, e^{46}, e^{46} + e^{56}, 0)$	0	(1,0,0,0,0,0)	✓
$B^{a}_{6,3}$	$(2a e^{16} + e^{45}, e^{15} + 3a e^{26} + e^{36}, e^{14} - e^{26} + 3a e^{36}, a e^{46} - e^{56}, e^{46} + a e^{56}, 0)$			
[35]	$a \neq 0$	0	(1,0,0,0,0,0)	✓
	$a = 0$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
$B^{\varepsilon}_{6,4}$	$(e^{45}, e^{15} + e^{36}, e^{14} - e^{26} + \varepsilon e^{56}, -e^{56}, e^{46}, 0)$, $\varepsilon = \pm 1$	0	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
	Nilradical $A_{5,4}$			

[33] $B_{6,2}$ is denoted by $n_{6,76}$ in [Sha].

[34] $A^{a}_{6,76} \cong A^{1/a}_{6,76}$, $A^{0}_{6,76} = A^{0}_{6,77}$

[35] $B_{6,3}$ and $B_{6,4}$ are denoted by $n_{6,83}$ and $n_{6,84}$ in [Sha], Mubarakzyanov's classes $g_{6,80}$ and $g_{6,81}$ are redundant since $g_{6,80} \cong A^{0}_{6,76}$ and $g^{\varepsilon}_{6,81} \cong A^{\varepsilon}_{6,77}$.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
$A_{6,82}^{\varepsilon,\mathrm{a},\mathrm{b}}$	$(2\varepsilon e^{16}+e^{24}+e^{35},(\varepsilon+a)e^{26},(\varepsilon+b)e^{36},(\varepsilon-a)e^{46},$ $(\varepsilon-b)e^{56},0)$			
[36]	$\varepsilon=1,\ 0\le a\le b,\ a\notin\{1,5\},\ b\notin\{1,5,2\pm a,4\pm a\}$	0	(1,0,0,0,0,0)	–
	$\varepsilon=1,\ a=1,\ b\ge 0,\ b\notin\{1,3,5\}$	0	(2,1,0,0,0,0)	–
	$\varepsilon=1,\ b=a+2,\ a>-1,\ a\notin\{0,1,3,5\}$	0	(1,1,1,0,0,0)	–
	$\varepsilon=1,\ b=a+4,\ a\ge -2,\ a\notin\{-1,0,1,5\}$	0	(1,0,1,1,0,0)	–
	$\varepsilon=1,\ a=5,\ b\ge 0,\ b\notin\{1,3,5,7,9\}$	0	(1,0,0,1,1,0)	–
	$\varepsilon=1,\ (a,b)=(1,1)$	0	(3,3,1,0,0,0)	–
	$\varepsilon=1,\ (a,b)=(1,3)$	0	(2,2,2,1,0,0)	–
	$\varepsilon=1,\ (a,b)=(1,5)$	0	(2,1,1,2,1,0)	–
	$\varepsilon=1,\ (a,b)=(0,2)$	0	(1,2,2,0,0,0)	–
	$\varepsilon=1,\ (a,b)\in\{(5,3),\ (5,7)\}$	0	(1,1,1,1,1,0)	–
	$\varepsilon=1,\ (a,b)=(0,4)$	0	(1,0,2,2,0,0)	–
	$\varepsilon=1,\ (a,b)=(5,9)$	0	(1,0,1,2,1,0)	✓
	$\varepsilon=1,\ (a,b)=(5,5)$	0	(1,0,0,2,2,0)	–
	$\varepsilon=0,\ a=1,\ 0<b<1$	1	$(1,1,2,1,1,\underline{\mathbf{1}})$	✓
	$\varepsilon=0,\ (a,b)=(1,0)$	1	$(3,3,2,3,3,\underline{\mathbf{1}})$	✓
	$\varepsilon=0,\ (a,b)=(1,1)$	1	$(1,3,6,3,1,\underline{\mathbf{1}})$	✓
$A_{6,83}^{\varepsilon,\mathrm{a}}$	$(2\varepsilon e^{16}+e^{24}+e^{35},(\varepsilon+a)e^{26},e^{26}+(\varepsilon+a)e^{36},$ $(\varepsilon-a)e^{46}-e^{56},(\varepsilon-a)e^{56},0)$			
[37]	$\varepsilon=1,\ a\ge 0,\ a\notin\{1,2,5\}$	0	(1,0,0,0,0,0)	–
	$\varepsilon=1,\ a=1$	0	(2,2,1,0,0,0)	–
	$\varepsilon=1,\ a=2$	0	(1,0,1,1,0,0)	–
	$\varepsilon=1,\ a=5$	0	(1,0,0,1,1,0)	–
	$\varepsilon=0,\ a=1$	1	$(1,1,2,1,1,\underline{\mathbf{1}})$	–
$A_{6,84}$	$(e^{24}+e^{35},e^{26},e^{56},-e^{46},0,0)$	1	$(2,2,2,2,2,\underline{\mathbf{1}})$	✓
$A_{6,85}^{\mathrm{a}}$	$(2e^{16}+e^{24}+e^{35},(a+1)e^{26},e^{36}+e^{56},(1-a)e^{46},e^{56},0)$			

[36] $A_{6,82}^{\varepsilon,a,b}\cong A_{6,82}^{\varepsilon,b,a}\cong A_{6,82}^{\varepsilon,-a,b}\cong A_{6,82}^{\varepsilon,a,-b}$

[37] $A_{6,83}^{0,0}$ is nilpotent, $A_{6,83}^{\varepsilon,a}\cong A_{6,83}^{\varepsilon,-a}$.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$a \geq 0,\ a \notin \{1,2,4,5\}$	0	(1,0,0,0,0,0)	–
	$a = 1$	0	(2,1,0,0,0,0)	–
	$a = 2$	0	(1,1,1,0,0,0)	–
	$a = 4$	0	(1,0,1,1,0,0)	–
	$a = 5$	0	(1,0,0,1,1,0)	–
$A_{6,87}$ [38]	$(2e^{16}+e^{24}+e^{35}, e^{26}, e^{36}+e^{56}, e^{36}+e^{46}, e^{26}+e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A_{6,88}^{\varepsilon,a,b}$	$(2\varepsilon e^{16}+e^{24}+e^{35}, (\varepsilon+a)e^{26}-be^{36}, be^{26}+(\varepsilon+a)e^{36}, (\varepsilon-a)e^{46}-be^{56}, be^{46}+(\varepsilon-a)e^{56}, 0)$			
[39]	$\varepsilon = 1,\ a \geq 0,\ a \notin \{1,2\},\ b > 0$	0	(1,0,0,0,0,0)	–
	$\varepsilon = 1,\ a = 1,\ b > 0$	0	(1,1,1,0,0,0)	–
	$\varepsilon = 1,\ a = 2,\ b > 0$	0	(1,0,1,1,0,0)	–
	$\varepsilon = 0,\ a = 1,\ b > 0$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	✓
	$\varepsilon = 0,\ (a,b) = (0,1)$	1	(1,3,6,3,1,$\underline{\mathbf{1}}$)	✓
$A_{6,89}^{\varepsilon,a,b}$	$(2\varepsilon e^{16}+e^{24}+e^{35}, (\varepsilon+b)e^{26}, \varepsilon e^{36}-ae^{56}, (\varepsilon-b)e^{46}, ae^{36}+\varepsilon e^{56}, 0)$			
[40]	$\varepsilon = 1,\ a > 0,\ b \geq 0,\ b \notin \{1,5\}$	0	(1,0,0,0,0,0)	–
	$\varepsilon = 1,\ b = 1,\ a > 0$	0	(2,1,0,0,0,0)	–
	$\varepsilon = 1,\ b = 5,\ a > 0$	0	(1,0,0,1,1,0)	–
	$\varepsilon = 0,\ a = 1,\ b \neq 0$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	✓
	$\varepsilon = 0,\ (a,b) = (1,0)$	1	(3,3,2,3,3,$\underline{\mathbf{1}}$)	✓
$A_{6,90}^{\varepsilon,a}$	$(2\varepsilon e^{16}+e^{24}+e^{35}, \varepsilon e^{26}+e^{46}, \varepsilon e^{36}+ae^{56}, \varepsilon e^{46}, -ae^{36}+\varepsilon e^{56}, 0)$			
[41]	$\varepsilon = 1,\ a \in \mathbb{R}$	0	(1,0,0,0,0,0)	–
	$\varepsilon = 0,\ a = \pm 1$	1	(2,2,2,2,2,$\underline{\mathbf{1}}$)	✓
$A_{6,93}^{\varepsilon,a}$	$(2\varepsilon e^{16}+e^{24}+e^{35}, \varepsilon e^{26}-ae^{56}, \varepsilon e^{36}-ae^{46}-e^{56}, e^{26}+ae^{36}+\varepsilon e^{46}, ae^{26}+\varepsilon e^{56}, 0)$			

[38]Mubarakzyanov's class $g_{6,86}$ is redundant since $g_{6,86} = A_{6,83}^{1,0}$.
[39]$A_{6,88}^{\varepsilon,a,0} \cong A_{6,82}^{\varepsilon,a,a}$ and $A_{6,88}^{\varepsilon,a,b} \cong A_{6,88}^{\varepsilon,-a,b} \cong A_{6,88}^{\varepsilon,a,-b}$
[40]$A_{6,89}^{\varepsilon,0,b} \cong A_{6,82}^{\varepsilon,b,0}$ and $A_{6,89}^{\varepsilon,a,b} \cong A_{6,89}^{\varepsilon,-a,b} \cong A_{6,89}^{\varepsilon,a,-b}$
[41]The Lie brackets of Mubarakzyanov's classes $g_{6,90}$, $g_{6,91} = A_{6,90}^{0,1}$ and $g_{6,93}$ are corrected in [Sha, Appendix G].

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
[42]	$\varepsilon = 1$, $a \geq 0$	0	(1,0,0,0,0,0)	–
	$\varepsilon = 0$, $a = 1$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	–
$B_{6,5}^{a,b}$ [43]	$(2e^{16} + e^{24} + e^{35}, e^{26} + ae^{46}, e^{36} + be^{56}, -ae^{26} + e^{46}, -be^{36} + e^{56}, 0)$, $a > 0$, $\lvert b\rvert < \lvert a\rvert$	0	(1,0,0,0,0,0)	–
$B_{6,6}^{a}$	$(e^{24} + e^{35}, e^{46}, ae^{56}, -e^{26}, -ae^{36}, 0)$			
[44]	$-1 < a < 1$, $a \neq 0$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	✓
	$a = \pm 1$	1	(1,3,6,3,1,$\underline{\mathbf{1}}$)	✓
	Nilradical $A_{5,5}$			
$A_{6,94}^{a}$	$((a+2)e^{16} + e^{25} + e^{34}, (a+1)e^{26} + e^{35}, ae^{36}, 2e^{46}, e^{56}, 0)$			
	$a \notin \{-5, -3, -2, -\frac{5}{3}, -\frac{3}{2}, -\frac{4}{3}, -1, -\frac{1}{2}, 0\}$	0	(1,0,0,0,0,0)	–
	$a = 0$	0	(2,1,0,0,0,0)	–
	$a = -2$	1	(1,1,2,1,1,$\underline{\mathbf{1}}$)	✓
	$a = -3$	0	(1,1,1,1,1,0)	✓
	$a = -\frac{1}{2}$	0	(1,1,1,0,0,0)	–
	$a = -1$	0	(1,0,1,1,0,0)	✓
	$a \in \{-5, -\frac{3}{2}\}$	0	(1,0,1,1,0,0)	–
	$a = -\frac{5}{3}$	0	(1,0,0,1,1,0)	✓
	$a = -\frac{4}{3}$	0	(1,0,0,1,1,0)	–
$A_{6,95}$	$(2e^{16} + e^{25} + e^{34} + e^{46}, e^{26} + e^{35}, 0, 2e^{46}, e^{56}, 0)$	0	(2,1,0,0,0,0)	–
$A_{6,96}$	$(3e^{16} + e^{25} + e^{34}, 2e^{26} + e^{35} + e^{46}, e^{36} + e^{56}, 2e^{46}, e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A_{6,97}$	$(4e^{16} + e^{25} + e^{34}, 3e^{26} + e^{35}, 2e^{36}, e^{36} + 2e^{46}, e^{56}, 0)$	0	(1,0,0,0,0,0)	–
$A_{6,98}^{\varepsilon}$	$(e^{16} + e^{25} + \varepsilon e^{26} + e^{34}, e^{26} + e^{35}, e^{36}, \varepsilon e^{56}, 0, 0)$			
	$\varepsilon = 0$	0	(3,3,1,0,0,0)	–

[42] Mubarakzyanov's class $g_{6,92}$ is redundant since

$$g_{6,92}^{\alpha,\mu_0,\nu_0} \cong g_{6,82}^{\alpha,\sqrt{-\mu_0\nu_0},\sqrt{-\mu_0\nu_0}} \quad \text{for } \mu_0\nu_0 < 0 \text{ and } \mu_0 = 0,\ \nu_0 = 0,$$
$$g_{6,92}^{\alpha,\mu_0,\nu_0} \cong g_{6,88}^{\alpha,0,\sqrt{\mu_0\nu_0}} \quad \text{for } \mu_0\nu_0 > 0,$$
$$g_{6,92}^{\alpha,\mu_0,\nu_0} \cong g_{6,83}^{\alpha,0} \quad \text{for } \mu_0 = 0,\ \nu_0 \neq 0 \text{ and } \mu_0 \neq 0,\ \nu_0 = 0.$$

[43] $B_{6,5}$ is denoted by $n_{6,95}$ in [Sha], $B_{6,5}^{a,b} \cong B_{6,5}^{b,a} \cong B_{6,5}^{-a,-b}$, $B_{6,5}^{a,a} \cong A_{6,92^*}^{1/a}$ where $A_{6,92^*}$ is the class mentioned in [CS].

[44] $B_{6,6}$ is denoted by $n_{6,96}$ in [Sha], $B_{6,6}^{a} \cong B_{6,6}^{1/a}$, $B_{6,6}^{0} \cong A_{6,89}^{0,1,0}$, $B_{6,6}^{1} \cong A_{6,92^*}^{0}$.

Table 7.7: Indecomposable 6-dim. Lie algebras with 5-dim. non-Abelian nilradical

$\mathfrak{g}$	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	hf
	$\varepsilon = 1$	0	(2,2,1,0,0,0)	–
	Nilradical $A_{5,6}$			
$A_{6,99}$	$(5e^{16}+e^{25}+e^{34}, 4e^{26}+e^{35}, 3e^{36}+e^{45}, 2e^{46}, e^{56}, 0)$	0	(1,0,0,0,0,0)	✓

Table 7.8: Indecomposable nilpotent almost Abelian 7-dim. Lie algebras

$\mathfrak{g}$	[Gong]	Lie bracket	$\mathfrak{z}$	$h^*(\mathfrak{g})$	cocalibrated	calibrated
$\mathfrak{n}_{7,1}$	(37A)	$\left(e^{47}, e^{57}, e^{67}, 0, 0, 0, 0\right)$	3	$(4, 12, 18, 18, 12, 4, 1)$	✓	–
$\mathfrak{n}_{7,2}$	(247A)	$\left(e^{27}, e^{37}, 0, e^{57}, e^{67}, 0, 0\right)$	2	$(3, 7, 13, 13, 7, 3, 1)$	✓	✓
$\mathfrak{n}_{7,3}$	(2457A)	$\left(e^{27}, e^{37}, e^{47}, 0, e^{67}, 0, 0\right)$	2	$(3, 7, 10, 10, 7, 3, 1)$	✓	–
$\mathfrak{n}_{7,4}$	(123457A)	$\left(e^{27}, e^{37}, e^{47}, e^{57}, e^{67}, 0, 0\right)$	1	$(2, 4, 6, 6, 4, 2, 1)$	✓	–

Table 7.9: Direct sums of a four-dimensional and a two-dimensional Lie algebra admitting a half-flat SU(3)-structure which are not a sum of two three-dimensional Lie algebras and so are not contained in [SH]

Lie algebra	Half-flat SU(3)-structure [45]
$A_{4,1} \oplus \mathfrak{r}_2$	$\omega = -e^{16}+e^{25}-e^{34}$, $\rho = e^{123}-e^{145}+e^{156}-e^{246}+e^{345}-2e^{356}$, $g = (e^1)^2+(e^2)^2+2(e^3)^2+(e^4)^2+(e^5)^2+2(e^6)^2-2e^1{\cdot}e^3+2e^4{\cdot}e^6$
$B^\beta \oplus \mathfrak{r}_2$, $\beta > 0$ [46]	$\omega = e^{15}+e^{24}+e^{36}$, $\rho = e^{123}-e^{146}+e^{256}+e^{345}$, ONB

[45]In each case except B^β, the exterior derivatives of the one-forms $e^1, \ldots, e^4$ are those given in Table 7.3 and $de^5 = 0$, $de^6 = e^{56}$.

[46]The family B^β, $\beta > 0$, with the Lie bracket $\left(\beta e^{14}-e^{24}, e^{14}, -\beta e^{34}, 0\right)$ unifies the cases $A_{4,2}^{-2}$, $A_{4,5}^{\alpha,-(\alpha+1)}$ for $-1 < \alpha < -\frac{1}{2}$ and $A_{4,6}^{\alpha,-\alpha/2}$ for $\alpha > 0$ since

$$B^\beta \cong A_{4,6}^{\alpha,-\alpha/2} \qquad \text{for } 0 < \beta < 2 \text{ and } \alpha = \tfrac{2\beta}{\sqrt{4-\beta^2}},$$
$$B^2 \cong A_{4,2}^{-2},$$
$$B^\beta \cong A_{4,5}^{\alpha,-(\alpha+1)} \qquad \text{for } \beta > 2 \text{ and } \alpha = -\tfrac{1}{2} - \tfrac{\sqrt{\beta^2-4}}{2\beta}.$$

Table 7.9: Direct sums of a four-dimensional and a two-dimensional Lie algebra admitting a half-flat SU(3)-structure which are not a sum of two three-dimensional Lie algebras and so are not contained in [SH]

Lie algebra	Half-flat SU(3)-structure [45]
$A_{4,8} \oplus \mathfrak{r}_2$	$\omega = -e^{14} + e^{16} - e^{24} + e^{25} + e^{34} + e^{35}$, $\rho = 2e^{123} + 4e^{124} + 4e^{134} - 2e^{156} - 2e^{234} + 2e^{236} - e^{245} + 3e^{246} - 3e^{256} + e^{345} + 3e^{346} + 3e^{356} + 12e^{456}$, $g = 2(e^1)^2 + 4(e^2)^2 + 4(e^3)^2 + 57(e^4)^2 + 2(e^5)^2 + 3(e^6)^2 + 4e^1{\cdot}e^2 - 4e^1{\cdot}e^3 - 18e^1{\cdot}e^4 + 2e^1{\cdot}e^6 - 4e^2{\cdot}e^3 - 26e^2{\cdot}e^4 - 2e^2{\cdot}e^5 + 4e^2{\cdot}e^6 + 26e^3{\cdot}e^4 - 2e^3{\cdot}e^5 - 4e^3{\cdot}e^6 - 18e^4{\cdot}e^6$
$A_{4,9}^{-\frac{1}{2}} \oplus \mathfrak{r}_2$	$\omega = e^{16} - 3e^{24} + 2e^{25} + e^{35}$, $\rho = \sqrt{3}\left(e^{124} + 2e^{134} - e^{135} + e^{146} - 2e^{156} + 2e^{236} + 4e^{245} - e^{345} + \frac{29}{2}e^{456}\right)$, $g = (e^1)^2 + 4(e^2)^2 + 4(e^3)^2 + 84(e^4)^2 + 17(e^5)^2 + 29(e^6)^2 - 18e^1e^4 + 8e^1e^5 + 4e^2e^3 + 16e^2{\cdot}e^6 - 4e^3{\cdot}e^6 - 75e^4{\cdot}e^5$
$A_{4,10} \oplus \mathfrak{r}_2$	$\omega = -e^{14} - e^{16} - e^{25} - e^{36}$, $\rho = e^{123} - e^{156} + e^{234} + e^{236} + e^{246} - e^{345} + e^{356} - e^{456}$, $g = (e^1)^2 + (e^2)^2 + (e^3)^2 + 2(e^4)^2 + (e^5)^2 + 3(e^6)^2 + 2e^1{\cdot}e^4 + 2e^1{\cdot}e^6 + 4e^4{\cdot}e^6$
$A_{4,12} \oplus \mathfrak{r}_2$	$\omega = e^{16} - 2e^{24} + e^{25} - e^{34} - e^{46}$, $\rho = e^{124} - 2e^{134} - e^{146} + e^{135} + e^{156} - e^{245} - e^{236} + 2e^{456}$, $g = (e^1)^2 + (e^2)^2 + (e^3)^2 + 9(e^4)^2 + 2(e^5)^2 + 3(e^6)^2 + 4e^1{\cdot}e^4 - 2e^1{\cdot}e^5 - 2e^2{\cdot}e^6 - 8e^4{\cdot}e^5 - 2e^3{\cdot}e^6$
$\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_2$ [47]	$\omega = e^{12} - e^{23} - e^{25} - e^{35} + e^{46}$, $\rho = e^{124} - e^{126} + 2e^{134} + 3e^{156} - e^{234} + e^{256} + e^{345} + 2e^{356}$, $g = 6(e^1)^2 + (e^2)^2 + 4(e^3)^2 + (e^4)^2 + 3(e^5)^2 + 2(e^6)^2 + 8e^1{\cdot}e^3 + 6e^1{\cdot}e^5 + 2e^2{\cdot}e^3 - 2e^2{\cdot}e^5 + 2e^3{\cdot}e^5 + 2e^4{\cdot}e^6$

[47] The exterior derivatives of the basis one-forms are $(0, e^{12}, 0, e^{34}, 0, e^{56})$.

Table 7.10: Direct sums of indecomposable non-nilpotent five-dimensional Lie algebras and $\mathbb{R}$ admitting a half-flat SU(3)-structure

Lie algebra	Half-flat SU(3)-structure[48]
$A_{5,7}^{-1,\beta,-\beta} \oplus \mathbb{R}$, $0 < \beta \leq 1$, $A_{5,13}^{-1,0,\gamma} \oplus \mathbb{R}$, $\gamma > 0$, $A_{5,17}^{0,0,\gamma} \oplus \mathbb{R}$, $0 < \gamma \leq 1$, $A_{5,8}^{-1} \oplus \mathbb{R}$, $A_{5,14}^{0} \oplus \mathbb{R}$	$\omega = -\mathrm{e}^{13} + \mathrm{e}^{24} + \mathrm{e}^{56}$, $\rho = \mathrm{e}^{126} + \mathrm{e}^{145} + \mathrm{e}^{235} + \mathrm{e}^{346}$, ONB
$A_{5,17}^{\alpha,-\alpha,1} \oplus \mathbb{R}$, $\alpha > 0$, $A_{5,15}^{-1} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{13} + \mathrm{e}^{24} - \mathrm{e}^{56}$, $\rho = \mathrm{e}^{125} + \mathrm{e}^{146} - \mathrm{e}^{236} - \mathrm{e}^{345}$, ONB
$A_{5,18}^{0} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{12} - \mathrm{e}^{34} - \mathrm{e}^{56}$, $\rho = \mathrm{e}^{136} + \mathrm{e}^{145} - \mathrm{e}^{235} + \mathrm{e}^{246}$, ONB
$A_{5,19}^{-1,2} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{13} + \mathrm{e}^{24} - 2\mathrm{e}^{25} - \mathrm{e}^{56}$, $\rho = -\mathrm{e}^{126} + \mathrm{e}^{145} - \mathrm{e}^{234} + \mathrm{e}^{346} - \mathrm{e}^{356}$, $g = (\mathrm{e}^1)^2 + 2(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + 2(\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 - 2\mathrm{e}^2 \cdot \mathrm{e}^6 - 2\mathrm{e}^4 \cdot \mathrm{e}^5$
$A_{5,19}^{-1,3} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{13} - 2\mathrm{e}^{25} - \mathrm{e}^{46}$, $\rho = \mathrm{e}^{126} - 2\mathrm{e}^{145} + \mathrm{e}^{234} + 2\mathrm{e}^{356}$, OB, $\|\|e_5\|\|^2 = 2$
$A_{5,19}^{2,-3} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{12} + 2\mathrm{e}^{35} - \mathrm{e}^{46}$, $\rho = \mathrm{e}^{134} + 2\mathrm{e}^{156} + \mathrm{e}^{236} + 2\mathrm{e}^{245}$, OB, $\|\|e_5\|\|^2 = 2$
$A_{5,30}^{0} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{16} + \mathrm{e}^{25} + \mathrm{e}^{34}$, $\rho = \mathrm{e}^{123} + 2\mathrm{e}^{145} - \mathrm{e}^{156} - \mathrm{e}^{246} - \mathrm{e}^{345} + \mathrm{e}^{356}$, $g = 2(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2(\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 - 2\mathrm{e}^1 \cdot \mathrm{e}^3 + 2\mathrm{e}^4 \cdot \mathrm{e}^6$
$A_{5,33}^{-1,-1} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{12} - \mathrm{e}^{36} - \mathrm{e}^{45}$, $\rho = -\mathrm{e}^{135} + \mathrm{e}^{146} + \mathrm{e}^{234} + \mathrm{e}^{256}$, ONB
$A_{5,35}^{0,-2} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{16} + \mathrm{e}^{25} + 3\mathrm{e}^{26} + \mathrm{e}^{34}$, $\rho = \mathrm{e}^{123} + \mathrm{e}^{145} + 2\mathrm{e}^{146} + \mathrm{e}^{245} + \mathrm{e}^{246} + \mathrm{e}^{356}$, $g = (\mathrm{e}^1)^2 + 2(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 5(\mathrm{e}^6)^2 + 2\mathrm{e}^1 \cdot \mathrm{e}^2 + 4\mathrm{e}^5 \cdot \mathrm{e}^6$

[48] In each case, the exterior derivatives of the one-forms $\mathrm{e}^1, \ldots, \mathrm{e}^5$ are those given in Table 7.4 and $d\mathrm{e}^6 = 0$.

Lie algebra	Half-flat SU(3)-structure
$A_{5,36} \oplus \mathbb{R}$	$\omega = \frac{1}{12}\mathrm{e}^{12} + \mathrm{e}^{13} + \mathrm{e}^{16} - \frac{1}{4}\mathrm{e}^{24} + \mathrm{e}^{46} + \mathrm{e}^{56}$, $\rho = -\frac{1}{6}\mathrm{e}^{124} + \frac{1}{12}\mathrm{e}^{125} - \mathrm{e}^{134} - \mathrm{e}^{135} + 4\mathrm{e}^{146} + 4\mathrm{e}^{236} + 3\mathrm{e}^{345} + 3\mathrm{e}^{456}$, $g = \frac{5}{12}(\mathrm{e}^1)^2 + \frac{1}{12}(\mathrm{e}^2)^2 + 12(\mathrm{e}^3)^2 + \frac{7}{4}(\mathrm{e}^4)^2 + \frac{1}{4}(\mathrm{e}^5)^2 + 28(\mathrm{e}^6)^2 + \frac{3}{2}\mathrm{e}^1\cdot\mathrm{e}^4 - \frac{1}{2}\mathrm{e}^1\cdot\mathrm{e}^5 + 2\mathrm{e}^2\cdot\mathrm{e}^6 + 24\mathrm{e}^3\cdot\mathrm{e}^6 - \mathrm{e}^4\cdot\mathrm{e}^5$
$A_{5,37} \oplus \mathbb{R}$	$\omega = -\frac{1}{3}\mathrm{e}^{16} + 3\mathrm{e}^{24} + \mathrm{e}^{35}$, $\rho = -\mathrm{e}^{125} + 3\mathrm{e}^{134} + 2\mathrm{e}^{146} + \mathrm{e}^{236} + 6\mathrm{e}^{345} - \frac{13}{3}\mathrm{e}^{456}$, $g = (\mathrm{e}^1)^2 + 3(\mathrm{e}^2)^2 + 3(\mathrm{e}^3)^2 + 3(\mathrm{e}^4)^2 + \frac{13}{3}(\mathrm{e}^5)^2 + \frac{13}{9}(\mathrm{e}^6)^2 + 4\mathrm{e}^1\cdot\mathrm{e}^5 - 4\mathrm{e}^3\cdot\mathrm{e}^6$
$A_{5,40} \oplus \mathbb{R}$	$\omega = \mathrm{e}^{14} + \mathrm{e}^{25} + \mathrm{e}^{34} - \mathrm{e}^{36}$, $\rho = \mathrm{e}^{124} - \mathrm{e}^{126} - \mathrm{e}^{135} + \mathrm{e}^{234} + \mathrm{e}^{456}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2(\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 - 2\mathrm{e}^4\cdot\mathrm{e}^6$

Table 7.11: Half-flat SU(3)-structures on non-solvable indecomposable 6-dim. Lie algebras

Lie algebra	Half-flat SU(3)-structure
$L_{6,1}$, $\mathfrak{so}(3,1)$	$\omega = \mathrm{e}^{14} + \mathrm{e}^{25} - \mathrm{e}^{36}$, $\rho = -\mathrm{e}^{126} - \mathrm{e}^{135} + \mathrm{e}^{234} + \mathrm{e}^{456}$, ONB
$L_{6,2}$	$\omega = -\mathrm{e}^{14} + \mathrm{e}^{25} + \mathrm{e}^{36}$, $\rho = -2\mathrm{e}^{125} - 2\mathrm{e}^{126} + 3\mathrm{e}^{135} + 2\mathrm{e}^{136} + \mathrm{e}^{156} + 2\mathrm{e}^{234} + \mathrm{e}^{245} + \mathrm{e}^{345} - \mathrm{e}^{346} - \frac{19}{4}\mathrm{e}^{456}$, $g = 2(\mathrm{e}^1)^2 + 4(\mathrm{e}^2)^2 + 6(\mathrm{e}^3)^2 + \frac{17}{2}(\mathrm{e}^4)^2 + \frac{53}{4}(\mathrm{e}^5)^2 + \frac{19}{2}(\mathrm{e}^6)^2 - 8\mathrm{e}^1\cdot\mathrm{e}^4 - 8\mathrm{e}^2\cdot\mathrm{e}^3 - 4\mathrm{e}^2\cdot\mathrm{e}^6 - 10\mathrm{e}^3\cdot\mathrm{e}^5 - 4\mathrm{e}^3\cdot\mathrm{e}^6 + 21\mathrm{e}^5\cdot\mathrm{e}^6$
$L_{6,3}$	$\omega = \mathrm{e}^{15} + \mathrm{e}^{24} - \mathrm{e}^{26} + \mathrm{e}^{36} + \mathrm{e}^{56}$, $\rho = 3\mathrm{e}^{124} + 11\mathrm{e}^{126} - \mathrm{e}^{134} - 2\mathrm{e}^{136} - 2\mathrm{e}^{156} + \mathrm{e}^{235} - \mathrm{e}^{246} + \mathrm{e}^{346} + \mathrm{e}^{456}$, $g = 5(\mathrm{e}^1)^2 + 14(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 18(\mathrm{e}^6)^2 - 4\mathrm{e}^1\cdot\mathrm{e}^4 - 18\mathrm{e}^1\cdot\mathrm{e}^6 - 6\mathrm{e}^2\cdot\mathrm{e}^3 - 4\mathrm{e}^2\cdot\mathrm{e}^5 + 8\mathrm{e}^4\cdot\mathrm{e}^6$
$L_{6,4}$	$\omega = -\frac{1}{2}\sqrt{3}\,(\mathrm{e}^{14} + \mathrm{e}^{15} - 2\mathrm{e}^{24} - \mathrm{e}^{25} + \mathrm{e}^{36})$, $\rho = \frac{1}{2}\sqrt{3}\,(\mathrm{e}^{123} + \mathrm{e}^{126} + \mathrm{e}^{134} + \mathrm{e}^{235} - \mathrm{e}^{456})$, $g = (\mathrm{e}^1)^2 + 2(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2(\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 - 2\mathrm{e}^1\cdot\mathrm{e}^2 + \mathrm{e}^1\cdot\mathrm{e}^4 + \mathrm{e}^1\cdot\mathrm{e}^5 - 2\mathrm{e}^2\cdot\mathrm{e}^4 - \mathrm{e}^2\cdot\mathrm{e}^5 + \mathrm{e}^3\cdot\mathrm{e}^6 + 2\mathrm{e}^4\cdot\mathrm{e}^5$

Table 7.12: Indecomposable 6-dim. Lie algebras with 5-dim. nilradical admitting a half-flat SU(3)-structure

Lie algebra	Half-flat SU(3)-structure
$A_{6,13}^{a,-(2a+1),3a+1}$, $a \neq 0$	$\omega = -e^{13} + 2ae^{26} + e^{45}$, $\rho = e^{124} + 2ae^{156} - e^{235} - 2ae^{346}$, OB, $\lVert e_6 \rVert = 2\lvert a \rvert$
$A_{6,13}^{0,-1,1}$	$\omega = e^{15} - e^{26} + e^{34}$, $\rho = -e^{124} - e^{136} - e^{235} - e^{456}$, ONB
$A_{6,13}^{a,-(2a+1),3a+2}$, $a \neq -1$	$\omega = -(2a+2)e^{16} - e^{23} + e^{45}$, $\rho = e^{124} + e^{135} + (2a+2)e^{256}$ $-(2a+2)e^{346}$, OB, $\lVert e_6 \rVert = 2\lvert a+1 \rvert$
$A_{6,13}^{-1,1,-1}$	$\omega = -e^{16} - e^{25} + e^{34}$, $\rho = -e^{124} - e^{135} + e^{236} - e^{456}$, ONB
$A_{6,14}^{\frac{2}{5},-\frac{3}{5}}$	$\omega = -e^{13} + \frac{4}{5}e^{26} + e^{45}$, $\rho = e^{125} - \frac{4}{5}e^{146} + e^{234} - \frac{4}{5}e^{356}$, OB, $\lVert e_6 \rVert = \frac{4}{5}$
$A_{6,15}^{-1}$	$\omega = e^{16} - e^{24} + e^{35}$, $\rho = -e^{125} - e^{134} - e^{236} - e^{456}$, ONB
$A_{6,18}^{-3,5}$	$\omega = 4e^{16} + e^{23} + e^{45}$, $\rho = e^{125} + e^{134} + 4e^{246} - 4e^{356}$, OB, $\lVert e_6 \rVert = 4$
$A_{6,18}^{2,-5}$	$\omega = -e^{13} + 4e^{26} + e^{45}$, $\rho = -e^{125} + 4e^{146} - e^{234} + 4e^{356}$, OB, $\lVert e_6 \rVert = 4$
$A_{6,25}^{-1,0}$	$\omega = -e^{15} + e^{24} - e^{36}$, $\rho = e^{126} + e^{134} - e^{146} + e^{235} + e^{256} - 2e^{456}$, $g = (e^1)^2+(e^2)^2+(e^3)^2+2(e^4)^2+2(e^5)^2+(e^6)^2-2e^1 \cdot e^5-2e^2 \cdot e^4$
$A_{6,28}^{-3}$	$\omega = -e^{16} + e^{24} + e^{35} - 4e^{45}$, $\rho = -e^{125} + e^{134} + e^{236} - 4e^{246} - e^{456}$, $g = (e^1)^2 + (e^2)^2 + (e^3)^2 + 17(e^4)^2 + (e^5)^2 + (e^6)^2 - 8e^3 \cdot e^4$
$A_{6,37}^{a,-3a,1}$, $a \neq 0$	$\omega = -4ae^{16} + e^{23} - e^{45}$, $\rho = e^{125} - e^{134} + 4ae^{246} + 4ae^{356}$, OB, $\lVert e_6 \rVert = 4\lvert a \rvert$
$A_{6,37}^{0,0,1}$	$\omega = -e^{16} + e^{24} + e^{35}$, $\rho = -e^{125} + e^{134} + e^{236} - e^{456}$, ONB
$A_{6,38}^{0}$	$\omega = -e^{16} - e^{25} + e^{34}$, $\rho = -e^{124} - e^{135} + e^{236} - e^{456}$, ONB
$A_{6,39}^{a,-\frac{1}{3}(a+3)}$, $a \notin \{\frac{3}{2}, 0\}$	$\omega = -e^{14} + e^{23} + \frac{2a+3}{2a-3}e^{34} + (\frac{2}{3}a - 1)e^{56}$, $\rho = (-\frac{2}{3}a+1)e^{126} - e^{135} + (-\frac{2}{3}a-1)e^{236} - e^{245} + (\frac{2}{3}a-1)e^{346}$, $g = (e^1)^2 + (e^2)^2 + \frac{8a^2+18}{(2a-3)^2}(e^3)^2 + (e^4)^2 + (e^5)^2$ $+\frac{1}{9}(2a-3)^2(e^6)^2 + \frac{-4a-6}{2a-3}e^1 \cdot e^3$

Table 7.12: Indecomposable 6-dim. Lie algebras with 5-dim. nilradical admitting a half-flat SU(3)-structure

Lie algebra	Half-flat SU(3)-structure
$A_{6,39}^{\frac{3}{2},-\frac{3}{2}}$	$\omega = -\mathrm{e}^{12} - \mathrm{e}^{34} + 2\mathrm{e}^{56}$, $\rho = \mathrm{e}^{135} + 2\mathrm{e}^{146} + 2\mathrm{e}^{236} - \mathrm{e}^{245}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 4(\mathrm{e}^6)^2$
$A_{6,39}^{\mathrm{a},-1}$, $\mathrm{a} \notin \{-1, 0\}$	$\omega = (1+\mathrm{a})\mathrm{e}^{16} + \mathrm{e}^{23} - \mathrm{e}^{34} + \mathrm{e}^{45}$, $\rho = -\mathrm{e}^{124} + \mathrm{e}^{135} + (1+\mathrm{a})\mathrm{e}^{236} + (1+\mathrm{a})\mathrm{e}^{256} + (1+\mathrm{a})\mathrm{e}^{346}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + 2(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + (1+\mathrm{a})^2\,(\mathrm{e}^6)^2 + 2\mathrm{e}^3 \cdot \mathrm{e}^5$
$A_{6,40}^{-\frac{5}{4}}$	$\omega = -\mathrm{e}^{13} - \mathrm{e}^{14} + \mathrm{e}^{23} - 3\mathrm{e}^{34} - \frac{1}{2}\mathrm{e}^{56}$, $\rho = \frac{1}{2}\mathrm{e}^{126} - \mathrm{e}^{135} + \mathrm{e}^{136}$ $+\mathrm{e}^{145} + \frac{1}{2}\mathrm{e}^{146} - \frac{3}{2}\mathrm{e}^{236} - \mathrm{e}^{245} - \frac{1}{2}\mathrm{e}^{246} - \frac{1}{2}\mathrm{e}^{346}$, $g = 2(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + 10(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + \frac{1}{2}(\mathrm{e}^6)^2 - 2\mathrm{e}^1 \cdot \mathrm{e}^2$ $+6\mathrm{e}^1 \cdot \mathrm{e}^3 + \mathrm{e}^5 \cdot \mathrm{e}^6$
$A_{6,41}^{-\frac{3}{4}}$	$\omega = -\mathrm{e}^{14} + \mathrm{e}^{23} - 2\mathrm{e}^{24} - \frac{2}{3}\mathrm{e}^{34} - \frac{3}{2}\mathrm{e}^{56}$, $\rho = \frac{3}{2}\mathrm{e}^{126} - \mathrm{e}^{135} + \frac{1}{4}\mathrm{e}^{136} + 2\mathrm{e}^{145} - \frac{1}{2}\mathrm{e}^{236} - \mathrm{e}^{245} + \frac{1}{2}\mathrm{e}^{345} - \frac{3}{2}\mathrm{e}^{346}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{41}{36}(\mathrm{e}^3)^2 + 5(\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + \frac{9}{4}(\mathrm{e}^6)^2$ $+\frac{2}{3}\mathrm{e}^1 \cdot \mathrm{e}^3 + \frac{1}{3}\mathrm{e}^2 \cdot \mathrm{e}^3 - 4\mathrm{e}^3 \cdot \mathrm{e}^4$
$A_{6,42}^{-1}$	$\omega = 2\mathrm{e}^{16} + \mathrm{e}^{23} - \mathrm{e}^{34} + \mathrm{e}^{45}$, $\rho = -\mathrm{e}^{124} + \mathrm{e}^{135} + 2\mathrm{e}^{236} + 2\mathrm{e}^{256} + 2\mathrm{e}^{346}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + 2(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 4(\mathrm{e}^6)^2 + 2\mathrm{e}^3 \cdot \mathrm{e}^5$
$A_{6,47}^{\varepsilon,-3}$, $\varepsilon \in \{0, \pm 1\}$	$\omega = \mathrm{e}^{12} + \mathrm{e}^{23} + \mathrm{e}^{34} + 2\mathrm{e}^{56}$, $\rho = -2\mathrm{e}^{126} - \mathrm{e}^{135} + 2\mathrm{e}^{146} + 2\mathrm{e}^{236} + \mathrm{e}^{245}$, $g = (\mathrm{e}^1)^2 + 2(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 4(\mathrm{e}^6)^2 - 2\mathrm{e}^2 \cdot \mathrm{e}^4$
$A_{6,47}^{-1,-2}$	$\omega = \mathrm{e}^{12} - \mathrm{e}^{23} + \mathrm{e}^{35} - \mathrm{e}^{46} + 7\mathrm{e}^{56}$, $\rho = 2\mathrm{e}^{126} - \mathrm{e}^{134} + 7\mathrm{e}^{135} - \mathrm{e}^{156} - 7\mathrm{e}^{236} - \mathrm{e}^{245} - 3\mathrm{e}^{356}$, $g = (\mathrm{e}^1)^2 + 5(\mathrm{e}^2)^2 + 10(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + 50(\mathrm{e}^5)^2 + (\mathrm{e}^6)^2$ $+6\mathrm{e}^1 \cdot \mathrm{e}^3 - 4\mathrm{e}^2 \cdot \mathrm{e}^5 - 14\mathrm{e}^4 \cdot \mathrm{e}^5$
$A_{6,47}^{0,-2}$	$\omega = \mathrm{e}^{12} - 2\mathrm{e}^{15} - \mathrm{e}^{35} + \mathrm{e}^{46} + \mathrm{e}^{56}$, $\rho = 2\mathrm{e}^{126} + \mathrm{e}^{134} + \mathrm{e}^{135} + \mathrm{e}^{156} - \mathrm{e}^{236} - \mathrm{e}^{245}$, $g = 5(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + 2(\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 + 4\mathrm{e}^1 \cdot \mathrm{e}^3 + 2\mathrm{e}^4 \cdot \mathrm{e}^5$
$A_{6,47}^{1,-2}$	$\omega = -\mathrm{e}^{15} + \mathrm{e}^{23} + \mathrm{e}^{46}$, $\rho = -\sqrt{2}\mathrm{e}^{126} - \frac{1}{2}\sqrt{2}\mathrm{e}^{134} + \frac{1}{2}\sqrt{2}\mathrm{e}^{245} + \sqrt{2}\mathrm{e}^{356}$, OB, $\|\|e_4\|\| = \frac{1}{\sqrt{2}}$, $\|\|e_6\|\| = \sqrt{2}$
$A_{6,51}^{\varepsilon}$, $\varepsilon = \pm 1$	$\omega = \mathrm{e}^{16} + \mathrm{e}^{23} - \mathrm{e}^{34} + \mathrm{e}^{45}$, $\rho = -\mathrm{e}^{124} + \mathrm{e}^{135} + \mathrm{e}^{236} + \mathrm{e}^{256} + \mathrm{e}^{346}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + 2(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 + 2\mathrm{e}^3 \cdot \mathrm{e}^5$
$A_{6,54}^{0,-1}$	$\omega = \mathrm{e}^{12} + \mathrm{e}^{34} + \mathrm{e}^{56}$, $\rho = \mathrm{e}^{136} + \mathrm{e}^{145} + \mathrm{e}^{235} - \mathrm{e}^{246}$, ONB

Table 7.12: Indecomposable 6-dim. Lie algebras with 5-dim. nilradical admitting a half-flat SU(3)-structure

Lie algebra	Half-flat SU(3)-structure
$A_{6,54}^{\frac{1}{3},-\frac{4}{3}}$	$\omega = 2\mathrm{e}^{16} + \mathrm{e}^{24} - \mathrm{e}^{35}$, $\rho = -\mathrm{e}^{125} - \frac{25}{16}\mathrm{e}^{134} + \frac{3}{2}\mathrm{e}^{146} + 2\mathrm{e}^{236} + \frac{3}{4}\mathrm{e}^{345} + 2\mathrm{e}^{456}$, $g = \frac{25}{16}(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{25}{16}(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 4(\mathrm{e}^6)^2 - \frac{3}{2}\mathrm{e}^1 \cdot \mathrm{e}^5 + 3\mathrm{e}^3 \cdot \mathrm{e}^6$
$A_{6,54}^{-\frac{3}{2},-2}$	$\omega = \mathrm{e}^{13} + \mathrm{e}^{25} + \mathrm{e}^{46} - \mathrm{e}^{56}$, $\rho = \mathrm{e}^{126} - \mathrm{e}^{145} - \mathrm{e}^{234} + \mathrm{e}^{235} - \mathrm{e}^{356}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + 2(\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 - 2\mathrm{e}^4 \cdot \mathrm{e}^5$
$A_{6,54}^{\mathrm{a},\mathrm{a}-1}$, $\mathrm{a} \neq 0$	$\omega = -\mathrm{e}^{13} - \mathrm{e}^{24} - \mathrm{a}\mathrm{e}^{56}$, $\rho = -\mathrm{a}\mathrm{e}^{126} - \mathrm{e}^{145} + \mathrm{e}^{235} + \mathrm{a}\mathrm{e}^{346}$, OB, $\lvert\lvert e_6 \rvert\rvert = \lvert \mathrm{a} \rvert$
$A_{6,56}^{1}$	$\omega = -\mathrm{e}^{13} - \mathrm{e}^{24} - \mathrm{e}^{56}$, $\rho = -\mathrm{e}^{126} - \mathrm{e}^{145} + \mathrm{e}^{235} + \mathrm{e}^{346}$, ONB
$A_{6,63}^{-1}$	$\omega = \mathrm{e}^{12} + \mathrm{e}^{34} + \mathrm{e}^{56}$, $\rho = \frac{5}{4}\mathrm{e}^{136} + \mathrm{e}^{145} - \frac{1}{2}\mathrm{e}^{146} + \mathrm{e}^{235} + \frac{1}{2}\mathrm{e}^{236} - \mathrm{e}^{246}$, $g = \frac{5}{4}(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{5}{4}(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 + \mathrm{e}^1 \cdot \mathrm{e}^2 - \mathrm{e}^3 \cdot \mathrm{e}^4$
$A_{6,65}^{1,2}$	$\omega = -\mathrm{e}^{13} - \mathrm{e}^{24} - 2\mathrm{e}^{56}$, $\rho = -2\mathrm{e}^{126} - \mathrm{e}^{145} + \mathrm{e}^{235} + 2\mathrm{e}^{346}$, OB, $\lvert\lvert e_6 \rvert\rvert = 2$
$A_{6,70}^{\mathrm{a},\frac{\mathrm{a}}{2}}$, $\mathrm{a} \neq 0$	$\omega = \mathrm{e}^{13} + \mathrm{e}^{24} + \mathrm{a}\mathrm{e}^{56}$, $\rho = -\mathrm{a}\mathrm{e}^{126} - \mathrm{e}^{145} + \mathrm{e}^{235} + \mathrm{a}\mathrm{e}^{346}$, OB, $\lvert\lvert e_6 \rvert\rvert = \lvert \mathrm{a} \rvert$
$A_{6,70}^{0,0}$	$\omega = -\mathrm{e}^{12} + \mathrm{e}^{34} - \mathrm{e}^{56}$, $\rho = -\mathrm{e}^{136} + \mathrm{e}^{145} - \mathrm{e}^{235} - \mathrm{e}^{246}$, ONB
$A_{6,71}^{-1}$	$\omega = \mathrm{e}^{12} + \mathrm{e}^{25} - 3\mathrm{e}^{36} - \mathrm{e}^{45} + 18\mathrm{e}^{56}$, $\rho = 3\mathrm{e}^{126} + \mathrm{e}^{135} - 6\mathrm{e}^{146} - \mathrm{e}^{234} - 6\mathrm{e}^{245} - 3\mathrm{e}^{456}$, $g = 2(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2(\mathrm{e}^4)^2 + 37(\mathrm{e}^5)^2 + 9(\mathrm{e}^6)^2$ $-2\mathrm{e}^1 \cdot \mathrm{e}^5 - 2\mathrm{e}^2 \cdot \mathrm{e}^4 - 12\mathrm{e}^3 \cdot \mathrm{e}^5$
$A_{6,71}^{-\frac{3}{2}}$	$\omega = \mathrm{e}^{12} - \mathrm{e}^{23} + \mathrm{e}^{34} + \mathrm{e}^{56}$, $\rho = -\mathrm{e}^{136} - \mathrm{e}^{145} - \mathrm{e}^{235} + \mathrm{e}^{246} - \mathrm{e}^{345}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + 2(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 + 2\mathrm{e}^1 \cdot \mathrm{e}^3$
$A_{6,76}^{\mathrm{a}}$, $\mathrm{a} \neq -1$	$\omega = \mathrm{e}^{13} + (4\mathrm{a}+4)\mathrm{e}^{26} - \frac{3}{4}\mathrm{e}^{34} + \mathrm{e}^{45}$, $\rho = \mathrm{e}^{124} + (3\mathrm{a}+3)\mathrm{e}^{136} + (4\mathrm{a}+4)\mathrm{e}^{156} + \mathrm{e}^{235} + (4\mathrm{a}+4)\mathrm{e}^{346}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{25}{16}(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 16\,(\mathrm{a}+1)^2\,(\mathrm{e}^6)^2$ $+\frac{3}{2}\mathrm{e}^3 \cdot \mathrm{e}^5$
$A_{6,77}^{\varepsilon}$, $\varepsilon = \pm 1$	$\omega = -\mathrm{e}^{13} - 4\mathrm{e}^{26} + \frac{3}{4}\mathrm{e}^{34} - \mathrm{e}^{45}$, $\rho = \mathrm{e}^{124} + 3\mathrm{e}^{136} + 4\mathrm{e}^{156} + \mathrm{e}^{235} + 4\mathrm{e}^{346}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{25}{16}(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 16(\mathrm{e}^6)^2 + \frac{3}{2}\mathrm{e}^3 \cdot \mathrm{e}^5$

Table 7.12: Indecomposable 6-dim. Lie algebras with 5-dim. nilradical admitting a half-flat SU(3)-structure

Lie algebra	Half-flat SU(3)-structure
$A_{6,79}$	$\omega = \mathrm{e}^{13} + 8\mathrm{e}^{26} - \frac{3}{4}\mathrm{e}^{34} + \mathrm{e}^{45}$, $\rho = \mathrm{e}^{124} + 6\mathrm{e}^{136} + 8\mathrm{e}^{156} + \mathrm{e}^{235} + 8\mathrm{e}^{346}$, $g = -(\mathrm{e}^1)^2 - (\mathrm{e}^2)^2 - \frac{25}{16}(\mathrm{e}^3)^2 - (\mathrm{e}^4)^2 - (\mathrm{e}^5)^2 - 64(\mathrm{e}^6)^2 - \frac{3}{2}\mathrm{e}^3 \cdot \mathrm{e}^5$
$B^{\mathrm{a}}_{6,3}$, $\mathrm{a} \neq 0$	$\omega = 8\mathrm{a}\mathrm{e}^{16} + \mathrm{e}^{23} - \frac{3}{4}\mathrm{e}^{34} + \mathrm{e}^{45}$, $\rho = -\mathrm{e}^{124} + \mathrm{e}^{135} + 6\mathrm{a}\mathrm{e}^{236} + 8\mathrm{a}\mathrm{e}^{256} + 8\mathrm{a}\mathrm{e}^{346}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{25}{16}(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 64\mathrm{a}^2(\mathrm{e}^6)^2 + \frac{3}{2}\mathrm{e}^3 \cdot \mathrm{e}^5$
$A^{0,1,\mathrm{b}}_{6,82}$, $0 \leq \mathrm{b} \leq 1$, $A_{6,84}$, $A^{0,1,\mathrm{b}}_{6,89}$, $\mathrm{b} \geq 0$, and $A^{0,\mathrm{a}}_{6,90}$, $\mathrm{a} = \pm 1$	$\omega = \mathrm{e}^{16} + \mathrm{e}^{23} + \mathrm{e}^{45}$, $\rho = -\mathrm{e}^{124} + \mathrm{e}^{135} + \mathrm{e}^{256} + \mathrm{e}^{346}$, ONB
$A^{1,5,9}_{6,82}$	$\omega = \mathrm{e}^{14} - 3\mathrm{e}^{24} - 12\mathrm{e}^{26} - \mathrm{e}^{35}$, $\rho = \mathrm{e}^{125} - 12\mathrm{e}^{136} - \mathrm{e}^{234} + 36\mathrm{e}^{236} - 12\mathrm{e}^{456}$, $g = (\mathrm{e}^1)^2 + 10(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 144(\mathrm{e}^6)^2 - 6\mathrm{e}^1 \cdot \mathrm{e}^2$
$A^{0,1,\mathrm{b}}_{6,88}$, $\mathrm{b} > 0$, $A^{0,0,1}_{6,88}$	$\omega = -\mathrm{e}^{16} - \mathrm{e}^{23} + \mathrm{e}^{45}$, $\rho = -\mathrm{e}^{125} + \mathrm{e}^{134} + \mathrm{e}^{246} + \mathrm{e}^{356}$, ONB
$B^{\mathrm{a}}_{6,6}$, $-1 \leq \mathrm{a} \leq 1$, $\mathrm{a} \neq 0$	$\omega = \mathrm{e}^{16} + \mathrm{e}^{23} + \mathrm{e}^{45}$, $\rho = -\mathrm{e}^{124} + \mathrm{e}^{135} + \mathrm{e}^{256} + \mathrm{e}^{346}$, ONB
$A^{-1}_{6,94}$	$\omega = \mathrm{e}^{14} + \mathrm{e}^{15} - 3\mathrm{e}^{16} - 3\mathrm{e}^{26} + \mathrm{e}^{34}$, $\rho = \mathrm{e}^{123} - 3\mathrm{e}^{146} + 3\mathrm{e}^{156} - \mathrm{e}^{245} + 3\mathrm{e}^{246} + 3\mathrm{e}^{356}$, $g = 2(\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 18(\mathrm{e}^6)^2 + 2\mathrm{e}^1 \cdot \mathrm{e}^3 - 6\mathrm{e}^5 \cdot \mathrm{e}^6$
$A^{-\frac{5}{3}}_{6,94}$	$\omega = \mathrm{e}^{12} - \frac{1}{7}\mathrm{e}^{23} + \frac{7}{2}\mathrm{e}^{25} + \mathrm{e}^{34} - \frac{7}{3}\mathrm{e}^{36} + \frac{7}{3}\mathrm{e}^{56}$, $\rho = \frac{1}{3}\mathrm{e}^{126} - \mathrm{e}^{135} + \frac{7}{3}\mathrm{e}^{146} + \mathrm{e}^{234} + \frac{7}{3}\mathrm{e}^{236} + \mathrm{e}^{245} + \frac{7}{6}\mathrm{e}^{256} + \frac{49}{6}\mathrm{e}^{456}$, $g = (\mathrm{e}^1)^2 + \frac{50}{49}(\mathrm{e}^2)^2 + 2(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + \frac{53}{4}(\mathrm{e}^5)^2 + \frac{49}{9}(\mathrm{e}^6)^2$ $-7\mathrm{e}^1 \cdot \mathrm{e}^5 + \frac{2}{7}\mathrm{e}^2 \cdot \mathrm{e}^4 - 2\mathrm{e}^3 \cdot \mathrm{e}^5$
$A^{-2}_{6,94}$	$\omega = \mathrm{e}^{14} - \mathrm{e}^{16} + 2\mathrm{e}^{24} - \mathrm{e}^{26} + \mathrm{e}^{35}$, $\rho = -\mathrm{e}^{125} + \mathrm{e}^{134} + \mathrm{e}^{236} - \mathrm{e}^{456}$, $g = (\mathrm{e}^1)^2 + 2(\mathrm{e}^2)^2 + (\mathrm{e}^3)^2 + 2(\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + (\mathrm{e}^6)^2 + 2\mathrm{e}^1 \cdot \mathrm{e}^2 - 2\mathrm{e}^4 \cdot \mathrm{e}^6$
$A^{-3}_{6,94}$	$\omega = -\mathrm{e}^{14} - \mathrm{e}^{25} - \frac{3}{2}\mathrm{e}^{34} - 3\mathrm{e}^{36}$, $\rho = 3\mathrm{e}^{126} - \mathrm{e}^{135} + \mathrm{e}^{234} - \frac{9}{2}\mathrm{e}^{236} - 3\mathrm{e}^{456}$, $g = (\mathrm{e}^1)^2 + (\mathrm{e}^2)^2 + \frac{13}{4}(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + (\mathrm{e}^5)^2 + 9(\mathrm{e}^6)^2 + 3\mathrm{e}^1 \cdot \mathrm{e}^3$
$A_{6,99}$	$\omega = \frac{4}{3}\mathrm{e}^{12} + \mathrm{e}^{14} + \frac{42}{19}\mathrm{e}^{23} + \mathrm{e}^{25} - \frac{63}{38}\mathrm{e}^{34} - 9\mathrm{e}^{36} - \frac{729}{38}\mathrm{e}^{56}$, $\rho = -9\mathrm{e}^{126} - \mathrm{e}^{135} + \mathrm{e}^{234} - \frac{567}{38}\mathrm{e}^{236} - \frac{81}{38}\mathrm{e}^{245} + 12\mathrm{e}^{256} + 9\mathrm{e}^{456}$, $g = (\mathrm{e}^1)^2 + \frac{25}{9}(\mathrm{e}^2)^2 + \frac{5413}{1444}(\mathrm{e}^3)^2 + (\mathrm{e}^4)^2 + \frac{8005}{1444}(\mathrm{e}^5)^2$ $+81(\mathrm{e}^6)^2 - \frac{63}{19}\mathrm{e}^1 \cdot \mathrm{e}^3 + \frac{8}{3}\mathrm{e}^2 \cdot \mathrm{e}^4 + \frac{81}{19}\mathrm{e}^3 \cdot \mathrm{e}^5$

Table 7.13: Dual adapted bases for cocalibrated G_2-structures in some exceptional cases

Lie algebra	dual adapted basis [49]
$A_{4,8} \oplus e(1,1)$	$\left(e^1, e^5, e^6, e^7, e^4, e^2, e^3\right)$
$A_{4,12} \oplus \mathfrak{r}_{3,1}$	$\left(e^7, -\frac{1}{3}\sqrt{5}\,e^1, \sqrt{5}\,e^4, e^2 - \frac{4}{5}\sqrt{5}\,e^5, e^3 + \frac{2}{5}\sqrt{5}\,e^6, e^5, e^6\right)$
$\mathfrak{r}_2 \oplus \mathfrak{r}_2 \oplus \mathfrak{r}_{3,1}$	$\left(\frac{9}{\sqrt{10}}e^1, e^2 + \frac{13}{9}e^5, e^5, e^3 + 3e^6, e^6, \frac{1}{2\sqrt{10}}e^7, \frac{1}{3\sqrt{10}}e^4\right)$

[49]In each case, $(e^1, \ldots, e^7)$ denotes a basis such that $e^1, \ldots, e^4$ satisfy the Lie algebra structure given in Table 7.3 and $e^5, \ldots, e^7$ satisfy the Lie algebra structure given in Table 7.1

Bibliography

[AS] S. Akbulut, S. Salur, *Deformations in* G_2 *manifolds*, Adv. Math. **217** (2008), no. 5, 2130-2140.

[Al] D. V. Alekseevsky, *Riemannian spaces with unusual holonomy groups*, Funct. Anal. Appl. **2** (1968), 97-105.

[AK] D. V. Alekseevsky , B. N. Kimel'fel'd, *The structure of homogeneous Riemannian spaces with zero Ricci curvature*, Funct. Anal. Appl. **9** (1975), no. 2, 5-11.

[ABDO] A. Andrada, M. L. Barberis, I. Dotti, G. Ovando, *Product structures on four dimensional solvable Lie algebras*, Homology Homotopy Appl. **7** (2005), no. 1, 9-37.

[AFISUV] L. C. de Andrés, M. Fernández, S. Ivanov, J. A. Santisteban, L. Ugarte, D. Vassilev, *Quaternionic Kähler and* Spin(7) *metrics arising from quaternionic contact Einstein structures*, Ann. Mat. Pura Appl. (4) (2012), Online First, DOI:10.1007/s10231-012-0276-8.

[ApSa] V. Apostolov, S. Salamon, *Kähler Reduction of Metrics with Holonomy* G_2, Commun. Math. Phys. **246** (2004), no. 1, 43-61.

[AW] M. Atiyah, E. Witten, *M-Theory Dynamics On A Manifold of* G_2 *Holonomy*, Adv. Theor. Math. Phys. **6** (2001), no. 1, 1-106.

[Baum] H. Baum, *Eichfeldtheorie: Eine Einführung in die Differentialgeometrie auf Faserbündeln*, Springer-Verlag, Berlin Heidelberg, 2009.

[Ber1] M. Berger, *Sur les groupes d'holonomie des variétés a connexion affine et des variétés riemanniennes*, Bull. Soc. Math. France **83** (1955), 279-330.

[Ber2] M. Berger, *Les espaces symétriques noncompacts*, Ann. Sci. Ec. Norm. Súer $\underline{7}$4 (1957), no. 2, 85-177.

[Besse] A. Besse, *Einstein manifolds*, Springer-Verlag, New York, 1987.

[Bi] L. Bianchi, *Sugli spazi a tre dimensioni che ammettono un gruppo continuo di movimenti*, Mem. Mat. Fis. Ital. Sci. Serie Terza **11** (1898), 267-352.

[Bo] A. Borel, *Some remarks about Lie groups transitive on spheres and tori*, Bull. Amer. Math. Soc. **55** (1949), no. 6, 580-587.

[BGGG] A. Brandhuber, J. Gomis, S. Gubser, S. Gukov, *Gauge theory at large N and new* G_2 *holonomy metrics*. Nuclear Phys. B **611** (2001), nos. 1-3, 179-204.

[BG1] R. B. Brown, A. Gray, *Vector cross products*, Comment. Math. Helv. **42** (1967), no. 1, 222-236.

[BG2] R. B. Brown, A. Gray, *Riemannian manifolds with holonomy group* Spin(9), Differential Geometry (in honour of Kentaro Yano), Kinokuniya, Tokyo, 1972, pp. 41-59.

[Br1] R. Bryant, *Metrics with Exceptional Holonomy*, Ann. of Math. **126** (1987), no. 3, 525-576.

[Br2] R. Bryant, *A survey of Riemannian metrics with special holonomy groups*, Proc. ICM Berkeley **1**, Amer. Math. Soc., 1987, pp. 505-514.

[Br3] R. Bryant, *Classical, exceptional and exotic holonomies: a status report.*, Actes de la Table Ronde de Géométrie Differentielle en l'Honneur de Marcel Berger, Collection SMF, Séminaires & Congrés **1**, Soc. Math. de France, 1996, pp. 93-166.

[Br4] R. Bryant, *Calibrated Embeddings in the Special Lagrangian and Coassociative Cases*, Ann. Global Anal. Geom. **18** (2000), nos. 3-4, 405-435.

[Br5] R. Bryant, *Some Remarks on* G_2*-structures*, Proceedings of 12^{th} Gökova Geometry-Topology Conference, International Press, 2006, pp. 75-109.

[Br6] R. Bryant, *Non-embedding and non-extension results in special holonomy*, The many facets of geometry, Oxford Univ. Press, Oxford, 2010, pp. 346-367.

[BrSa] R. Bryant, S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58** (1989), no. 3, 829-850.

[BuGl] H. Busemann, D. E. Glasco II, *Irreducible sums of simple multivectors*, Pac. J. of Math. **49** (1973), no. 1, 13-32.

[Cal] E. Calabi, *Metriques Kählériennes et fibrés holomorphes*, Ann. Scien. École Norm. Sup. **12** (1979), 269-294.

[CHSW] P. Candelas, G. T. Horowitz, A. Strominger, E. Witten, *Vacuum configurations for superstrings*, Nucl. Phys. B **258** (1985), 46-74.

[CaSt] R. Campoamor-Stursberg, *Some Remarks Concerning the Invariants of Rank One Solvable Real Lie Algebras*, Algebra Colloq. **12** (2005), no. 3, 497–518.

[Cap] B. Capdevielle, *Classification des formes trilinéaires alternées en dimension 6*, Enseignement Math. **18** (1972), 225–243.

[Car] É Cartan, *Sur une classe remarquable d'espaces de Riemann*, Bull. Soc. Math. France **4** (1926), 214-264.

[ChGr] J. Cheeger, D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geom. **6** (1971), no. 1, 119-128.

[CMS] Q.-S. Chi, S. Merkulov, L. Schwachhöfer, *On the existence of an infinite series of exotic holonomies*, Invent. Math. **126** (1996), no. 2, 391-411.

[ChiFi] S. Chiossi, A. Fino, *Conformally parallel* G_2*-structures on a class of solvmanifolds*, Math. Z. **252** (2006), no. 4, 825-848.

[ChiSa] S. Chiossi, S. Salamon, *The intrinsic torsion of* SU(3) *and* G_2 *structures*, Differential Geometry, Valencia, 2001, World Sci. Publ., River Edge, NJ, 2002, pp. 115-133.

[ChiSw] S. Chiossi, A. Swann, G_2*-structures with torsion from half-integrable nilmanifolds*, J. Geom. Phys. **54** (2006), no. 3, 262-285.

[CCGLPW] Z. W. Chong, M. Cvetič, G. W. Gibbons, H. Lü, C. N. Pope, P. Wagner, *General Metrics of* G_2 *Holonomy and Contraction Limits*, Nucl. Phys. B **638** (2002), no. 3, 459-482.

[ClSw] R. Cleyton, A. Swann, *Cohomogeneity-one* G_2*-structures*, J. Geom. Phys. **44** (2002), nos. 2-3, 202-220.

[C1] D. Conti, *Half-flat nilmanifolds*, Mathematische Annalen **350** (2011), no.1, 155-168.

[C2] D. Conti, SU(3)*-holonomy metrics from nilpotent Lie groups*, `arXiv:math/1108.2450`, (2011).

[CF] D. Conti, M. Fernández, *Nilmanifolds with a calibrated* G_2*-structure*, Differential Geom. Appl. **29** (2011), no. 4, 493-506.

[CFS] D. Conti, M. Fernández, J. A. Santisteban, *Solvable Lie algebras are not that hypo*, Transform. Groups **16** (2011), no. 1, 51-69.

[CS] D. Conti, S. Salamon, *generalised Killing spinors in dimension 5*, Trans. Amer. Math. Soc. **359** (2007), no. 11, 5319-5343.

[CT] D. Conti, A. Tomassini, *Special symplectic six-manifolds*, Q. J. Math. **58** (2007), no. 3, 297-311.

[CoSm] J. H. Conway, D. Smith *On Quaternions and Octonions: Their Geometry, Arithmetic and Symmetry*, A K Peters Ltd., Natick, 2003.

[CLSS] V. Cortés, T. Leistner, L. Schäfer, F. Schulte-Hengesbach, *Half-flat structures and special holonomy*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 1, 1-24.

[CGLP1] M. Cvetič, G. Gibbons, H. Lü, C. Pope, *Supersymmetric Non-singular Fractional D2-branes and NS-NS* 2*-branes*, Nucl. Phys. B **606** (2001), nos.1-2, 18-44.

[CGLP2] M. Cvetič, G. Gibbons, H. Lü, C. Pope, *Cohomogeneity one manifolds of* Spin(7) *and* G_2 *holonomy*, Ann. Phys. **300** (2002), no. 2, 139-184.

[CGLP3] M. Cvetič, G. Gibbons, H. Lü, C. Pope, *Supersymmetric* $M3$*-branes and* G_2 *Manifolds*, Nucl. Phys. B **620** (2002), nos. 1-2, 3-28.

[CGLP4] M. Cvetič, G. Gibbons, H. Lü, C. Pope, *A* G_2 *unification of the deformed and resolved conifolds*, Phys. Lett. B **534** (2002), nos. 1-4, 172-180.

[De] P. Deligne et al.(eds.), *Quantum Fields and Strings: A Course for Mathematicians*, Volume I & II, Amer. Math. Soc., Providence, 1999.

[dR] G. de Rham, *Sur la réductibilité d'un espace de Riemann*, Comm. Math. Helv. **26** (1952), 328-344.

[Di] A. Diatta, *Left invariant contact structures on Lie groups*, Differential Geom. Appl. **26** (2008), no. 5, 544-552.

[Dj] D. Ž. Djoković, *Classification of trivectors of an eight-dimensional real vector space*, Linear Multilinear Algebra **13** (1983), no. 1, 3-39.

[DPWZ] D. Ž. Djoković, J. Patera, P. Winternitz, H. Zassenhaus, *Normal forms of elements of classical real and complex Lie and Jordan algebras*, J. Math. Phys. **24** (1983), no. 6, 1363-1374.

[Fe] M. Fernández, *A Classification of Riemannian Manifolds with Structure Group* Spin(7), Ann. Mat. Pura Appl. (4) **143** (1986), no. 1 , 101-122.

[FG] M. Fernández, A. Gray, *Riemannian manifolds with structure group* G_2, Ann. Mat. Pura Appl. (4) **132** (1982), no. 1, 19-45.

[FMOU] M. Fernández, V. Manero, A. Otal, L. Ugarte, *Symplectic half-flat solvmanifolds*, Ann. Global Anal. Geom. **43** (2013), no. 4, 367-383.

[FIUV] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa, *Compact supersymmetric solutions of the heterotic equations of motion in dimensions* 7 *and* 8, Adv. Theor. Math. Phys. **15** (2011), no. 2, 245-284.

[Fre1] M. Freibert, *Cocalibrated structures on Lie algebras with a codimension one Abelian ideal*, Ann. Global Anal. Geom. **42** (2012), no. 4, 537-563.

[Fre2] M. Freibert, *Cocalibrated structures on products of four- and three-dimensional Lie groups*, Differential Geom. Appl. **31** (2013), no. 3, 349-373.

[FS1] M. Freibert, F. Schulte-Hengesbach, *Half-flat structures on decomposable Lie groups*, Transform. Groups **17** (2012), no. 1, 123-141.

[FS2] M. Freibert, F. Schulte-Hengesbach, *Half-flat structures on indecomposable Lie groups*, Transform. Groups **17** (2012), no. 3, 657-689.

[FI] T. Friedrich, S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. of Math. **6** (2002), no. 2, 303-336.

[Fri1] T. Friedrich, G_2*-manifolds with parallel characteristic torsion*, Differential Geom. Appl. **25** (2007), no. 6, 632-648.

[Fri2] T. Friedrich, *Cocalibrated* G_2*-manifolds with Ricci flat characteristic connection*, `arXiv::math/1303.7444`, (2013).

[FH] W. Fulton, J. Harris, *Representation Theory*, Springer-Verlag, New York, 1991.

[GM] P. M. Gadea, J. M. Masque, *Classification of Almost Parahermitian Manifolds*, Rend. Mat. Appl. (7) **11** (1991), no. 2, 377-396.

[GL] A. Galaev, T. Leistner, *Recent developments in pseudo-Riemannian holonomy theory*, Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lectures in Mathematics and Theoretical Physics **16**, de Gruyter, Berlin, 2010, pp. 581-629.

[GLPS] G. W. Gibbons, H. Lü, C. N. Pope, K. S. Stelle, *Supersymmetric domain walls from metrics of special holonomy*, Nucl. Phys. B **623** (2002), no. 1-2, 3-46.

[GLR] I. Gohberg, P. Lancaster, L. Rodman, *Invariant subspaces of matrices with applications*, John Wiley & Sons Inc., New York, 1986.

[Gong] M.-P. Gong, *Classification of nilpotent Lie algebras of dimension 7 (over algebraically closed fields and* $\mathbb{R}$*)*, PhD thesis, University of Waterloo, 1998.

[GOV] V. Gorbatsevich, A. Onishchik, and E. Vinberg, *Lie groups and Lie algebras III: Structure of Lie groups and Lie algebras*, Encyclopaedia of Mathematical Sciences **41**, Springer-Verlag, Berlin, 1994.

[Gr] A. Gray, *Vector cross products on manifolds*, Trans. Amer. Math. Soc. **141** (1969), 465-504.

[GH] A. Gray, L. M. Hervella, *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl. (4) **123** (1980), no. 1, 35-58.

[GS] S. Gukov, J. Sparks, *M-Theory on* Spin(7) *Manifolds*, Nucl. Phys. B **625** (2002), nos. 1-2, 3-69.

[Gu] G. B. Gurevich, *Foundations of the Theory of Algebraic Invariants*, P. Noordhoff, Groningen, 1964.

[GLM1] S. Gurrieri, A. Lukas, A. Micu, *Heterotic string compactified on half-flat manifolds*, Phys. Rev. D **70** (2004), 126009.

[GLM2] S. Gurrieri, A. Lukas, A. Micu, *Heterotic String Compactifications on Half-flat Manifolds II*, JHEP (2007), no. 12, 081, 35 pp.

[GLMW] S. Gurrieri, J. Louis, A. Micu, D. Waldram, *Mirror Symmetry in generalised Calabi-Yau Compactifications*, Nucl. Phys. B **654** (2003), 61-113.

[HO] J. Hano, H. Ozeki, *On the holonomy groups of linear connections*, Nagoya Math. J. **10** (1956), 97-100.

[He] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press, New York, 1978.

[Hi1] N. Hitchin, *Stable forms and special metrics*, Global differential geometry: the mathematical legacy of Alfred Gray, Contemporary Mathematics **288**, American Mathematical Society, Providence, 2001, pp. 70-89.

[Hi2] N. Hitchin, *The geometry of three-forms in six dimensions*, J. Differential Geom. **55** (2000), no. 3, 547-576.

[Huy] D. Huybrechts, *Complex Geometry: An Introduction*, Universitext, Springer-Verlag, Berlin, 2005.

[J1] D. Joyce, *Compact Riemannian* 7*-manifolds with holonomy* G_2 *I & II*, J. Differential Geom. **43** (1996), no. 2, 291-375.

[J2] D. Joyce, *Compact* 8*-manifolds with holonomy* Spin(7), Invent. Math. **123** (1996), no. 3, 507-552.

[J3] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press, Oxford, 2000.

[Kar] S. Karigiannis, *Deformations of* G_2 *and* Spin(7) *structures*, Can. J. Math. **57** (2005), no. 5, 1012-1055.

[Kath1] I. Kath, $\mathrm{G}^*_{2(2)}$*-structures on pseudo-Riemannian manifolds*, J. Geom. Phys. **27** (1998), nos. 3-4, 155-177.

[Kath2] I. Kath, *Indefinite symmetric spaces with* $G_{2(2)}$*-structure*, J. London Math. Soc. (2013), Online First, DOI: 10.1112/jlms/jds068.

[KO] I. Kath, M. Olbrich *The classification problem for pseudo-Riemannian symmetric spaces*, Transform. Groups **14** (2009), no. 4, 847-885.

[KPRS] A. Kasman, K. Pedings, A. Reiszl, T. Shiota, *Universality of Rank 6 Plücker Relations and Grassmann Cone Preserving Maps*, Proc. Amer. Math. Soc. **136** (2008), no. 1, 77-87.

[Ki] T. Kimura, *Introduction to prehomogeneous vector spaces*, Translations of Mathematical Monographs **215**, American Mathematical Society, Providence, 2003.

[KiSa] T. Kimura, M. Sato, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1-155.

[K] A. W. Knapp, *Lie Groups Beyond an Introduction. Second Edition*, Birkhäuser, Boston Basel Berlin, 2002.

[KN] S. Kobayashi, K. Nomizu, *Foundations of differential geometry I*, Interscience Publ., New York, 1963.

[Kr] M. Krahe, *Para-pluriharmonic maps and twistor spaces*, Handbook of pseudo-Riemannian geometry and supersymmetry, IRMA Lectures in Mathematics and Theoretical Physics **16**, de Gruyter, Berlin, 2010, pp. 497-559.

[LM] H. B. Lawson, M.L. Michelsohn, *Spin Geometry*, Princeton University Press, Princeton, 1989.

[LPV] H. V. Lê, M. Panák, J. Vanžura, *Manifolds admitting stable forms*, Comment. Math. Univ. Carolin. **49** (2008), no. 1, 101-117.

[Ma] T. B. Madsen, Spin(7)*-manifolds with three-torus symmetry*, J. Geom. Phys. **61** (2011), no. 11, 2285-2292.

[MaSa] T. B. Madsen, S. Salamon, *Half-flat structures on* $S^3 \times S^3$, Ann. Global Anal. Geom. (2013), Online First, DOI: 10.1007/s10455-013-9371-3.

[MaSw1] T. B. Madsen, A. Swann, *Homogeneous spaces, multi-moment maps and (2,3)-trivial algebras*, AIP Conf. Proc. **1360** (2011), 51–62.

[MaSw2] T. B. Madsen, A. Swann, *Multi-moment maps*, Adv. Math. **229** (2012), no. 4, 2287-2309.

[MaSw3] T. B. Madsen, A. Swann, *Closed forms and multi-moment maps*, Geom. Dedicata (2012), Online First, DOI: 10.1007/s10711-012-9783-4.

[Mag] L. Magnin, *Sur les algèbres de Lie nilpotentes de dimension* ≤ 7, J. Geom. Phys. **3** (1986), no. 1, 119-144.

[MC1] F. Martín Cabrera, *On Riemannian manifolds with* Spin(7)*-structure*, Publ. Math. Debrecen **46** (1995), nos. 3-4, 271-283.

[MC2] F. Martín Cabrera, *On Riemannian Manifolds with* G_2*-Structure*, Boll. Un. Mat. Ital. A (7) **10** (1996), no. 1, 99-112.

[MC3] F. Martín Cabrera, *Orientable hypersurfaces of Riemannian manifolds with* Spin(7)*-structure*, Acta Math. Hungar. **76** (1997), no. 3, 235-247.

[MC4] F. Martín Cabrera, *Special almost Hermitian geometry*, J. Geom. Phys. **55** (2005), no. 4, 450-470.

[MC5] F. Martín Cabrera, SU(3)*-structures on hypersurfaces of manifolds with* G_2*-structure*, Monatsh. Math. **148** (2006), no. 1, 29-50.

[Mi] J. Milnor, *Curvatures of left-invariant metrics on Lie groups*, Advances in Math. **21** (1976), no. 3, 293-329.

[MX] V. Mehrmann, H. Xu, *Structured Jordan canonical forms for structured matrices that are Hermitian, skew Hermitian or unitary with respect to indefinite inner products*, Electron. J. Linear Algebra **5** (1999), 67-103.

[MS1] S. Merkulov, L. Schwachhöfer, *Classification of irreducible holonomies of torsion-free affine connections*, Ann. of Math. (2) **150** (1999), no. 1, 77-149.

[MS2] S. Merkulov, L. Schwachhöfer, *Addendum to Classification of irreducible holonomies of torsion-free affine connections*, Ann. of Math. (2) **150** (1999), no. 3, 1177-1179.

[Mu4d] G. M. Mubarakzyanov, *On solvable Lie algebras (Russian)*, Izv. Vyssh. Uchebn. Zaved., Mat. **32** (1963), no. 1, 114-123.

[Mu5d] G. M. Mubarakzyanov, *Classification of real structures of Lie algebras of fifth order (Russian)*, Izv. Vyssh. Uchebn. Zaved., Mat. **34** (1963), no. 3, 99-106.

[Mu6d] G. M. Mubarakzyanov, *Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element (Russian)*, Izv. Vyssh. Uchebn. Zaved. Mat. **35** (1963), no. 4, 104–116.

[On] A. L. Onishchik, *Lectures on real semisimple Lie algebras and their representations*, ESI Lectures in Mathematics and Physics, European Mathematical Society (EMS), Zürich, 2004.

[Ov] G. Ovando, *Four dimensional symplectic Lie algebras*, Beiträge Algebra Geom. **47** (2006), no. 2, 419-434.

[PT] G. Papadopoulos, P. Townsend, *Compactifications of* $D = 11$ *supergravity on spaces of exceptional holonomy*, Phys. Lett. B **357** (1995), 300-306.

[PSWZ] J. Patera, R. T. Sharp, P. Winternitz, H. Zassenhaus, *Invariants of real low dimension Lie algebras*, J. Mathematical Phys. **17** (1976), no. 6, 986-994.

[Pu] C. Puhle, *The Killing spinor equation with higher order potentials*, J. Geom. Phys. **58** (2008), no. 10, 1355-1375.

[R1] F. Reidegeld, *Spaces admitting homogeneous* G_2*-structures*, Differential Geom. Appl. **28** (2010), no. 3, 301-312.

[R2] F. Reidegeld, *Special cohomogeneity-one metrics with* $Q^{1,1,1}$ *or* $M^{1,1,0}$ *as the principal orbit*, J. Geom. Phys. **60** (2010), no. 9, 1069-1088.

[R3] F. Reidegeld, *Exceptional holonomy and Einstein metrics constructed from Aloff-Wallach spaces*, Proc. Lond. Math. Soc. (3) **102** (2011), no. 6, 1127-1160.

[Sa1] S. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), no. 2-3, 311-333.

[Sa2] S. Salamon, *Riemannian geometry and holonomy groups*, Pitman Research Notes in Mathematics Series **201**, Longman Scientific & Technical, Harlow, 1989.

[SH] F. Schulte-Hengesbach, *Half-flat structures on products of three-dimensional Lie groups*, J. Geom. Phys. **60** (2010), no. 11, 1726-1740.

[SHPhD] F. Schulte-Hengesbach, *Half-flat structures on Lie groups*, PhD-thesis, University of Hamburg, 2010.

[Schw] L. Schwachhöfer, *Connections with irreducible holonomy representations*, Adv. Math. **160** (2001), no. 1, 1-80.

[Sha] A. Shabanskaya, *Classification of Six Dimensional Solvable Indecomposable Lie Algebras with a codimension one nilradical over* $\mathbb{R}$, PhD-thesis, University of Toledo, 2011.

[SV] T. A. Springer, F. D. Veldkamp, *Octonions, Jordan Algebras and Exceptional groups*, Springer-Verlag, Berlin Heidelberg, 2000.

[Ste] S. Sternberg, *Lectures on differential geometry*, Prentice-Hall, Englewood-Cliffs, 1963.

[Sto] S. Stock, *Gauge Deformations and Embedding Theorems for Special Geometries*, `arXiv:math/0909.5549`, (2009).

[Str] A. Strominger, *Superstrings with torsion*, Nucl. Phys. B **274** (1986), no. 2, 253-284.

[Tu1] P. Turkowski, *Low-dimensional real Lie algebras*, J. Math. Phys. **29** (1988), no. 10, 2139-2144.

[Tu2] P. Turkowski, *Solvable Lie algebras of dimension six*, J. Math. Phys. **31** (1990), no. 6, 1344-1350.

[Wells] R. O. Wells, *Differential Analysis on Complex Manifolds*, Springer-Verlag, New York, 1980.

[W1] R. Westwick, *Linear Transformations on Grassmann Spaces III*, Linear Multilinear Algebra **2** (1974), no. 3, 257-268.

[W2] R. Westwick, *Irreducible lengths of trivectors of rank seven and eight*, Pacific J. Math. **80** (1979), no. 2, 575-579.

[W3] R. Westwick, *Real trivectors of rank seven*, Linear Multilinear Algebra **10** (1981), no. 3, 183-204.

[Wu] H. Wu, *On the de Rham decomposition theorem*, Illinois J. Math. **8** (1964), no. 2, 291-311.

[V] J. Vanžura, *One kind of multisymplectic structures on* 6*-manifolds*, Steps in Differential Geometry, Proc. of the Colloquium in Differential Geometry, Debrecen, 2000, pp. 375-391.

Zusammenfasssung

In dieser Arbeit werden verschiedene geometrische Strukturen auf sechs- und siebendimensionalen Lie-Algebren untersucht. Besonderer Fokus wird dabei auf sogenannte halbflache SU(3)-Strukturen in sechs Dimensionen und kokalibrierte G_2-Strukturen in sieben Dimensionen gelegt. Diese beiden Strukturen treten natürlicherweise auf Hyperflächen in sieben- bzw. achtdimensionalen riemannschen Mannigfaltigkeiten mit Holonomie enthalten in der exzeptionellen Holonomiegruppe G_2 bzw. Spin(7) auf. Umgekehrt dienen diese beiden Strukturen als Startwerte von Evolutionsgleichungen, die 2001 von N. Hitchin eingeführt wurden. Die Lösungen dieser Evolutionsgleichungen erlauben es sieben- bzw. achtdimensionale riemannsche Mannigfaltigkeiten zu definieren, deren Holonomie eine Untergruppe von G_2 bzw. Spin(7) ist. Darüberhinaus werden Mannigfaltigkeiten mit halbflachen SU(3)-Strukturen und kokalibrierten G_2-Strukturen in der Physik im Kontext von Kompaktifizierungen zehndimensionaler Superstringtheorien betrachtet.

Das Hauptresultat dieser Dissertation ist die Klassifikation der direkten Summen von vier- und dreidimensionalen Lie-Algebren, die kokalibrierte G_2-Strukturen zulassen. Das analoge Klassifikationsproblem lösen wir auch für die Klasse der siebendimensionalen fast-abelschen Lie-Algebren. In dieser Klasse bestimmen wir auch diejenigen Lie-Algebren die kokalibrierte G_2^*-, kalibrierte G_2-/G_2^*- oder parallele G_2-/G_2^*-Strukturen besitzen.

Aufbauend auf Resultaten von D. Conti und F. Schulte-Hengesbach, vollenden wir eine Dimension niedriger die Klassifikation der zerlegbaren sechsdimensionalen Lie-Algebren, die eine halbflache SU(3)-Struktur besitzen. Anschließend betrachten wir das analoge Klassifikationsproblem für den unzerlegbaren Fall und lösen es vollständig bis auf die Klasse aller unzerlegbaren auflösbaren Lie-Algebren mit vierdimensionalem Nilradikal. Zusätzlich erzielen wir einige kleinere Resultate für die beiden pseudo-riemannschen Analoga von halbflachen SU(3)-Strukturen.

Im letzten Kapitel wenden wir uns dem Hitchin-Fluss auf siebendimensionalen fast-abelschen Lie-Algebren zu. Wir zeigen, dass in diesem Fall die Lösungen des Hitchin-Flusses keine Mannigfaltigkeiten mit Holonomie gleich Spin(7) liefern können, sondern dass die Holonomie maximal gleich SU(4) sein kann. Anschließend bestimmen wir alle kokalibrierten G_2-Strukturen auf einer bestimmten siebendimensionalen, nilpotenten, fast-abelschen Lie-Algebra modulo Lie-Algebren Isomorphismen und Skalierungen. Wir benutzen diese Klassifikation um den Hitchin-Fluss Hitchin-Fluss explizit für eine Zwei-Parameter-Familie von kokalibrierten G_2-Strukturen auf der eben genannten Lie-Algebra zu lösen. Als Ergebnis erhalten wir eine explizite Zwei-Parameter-Familie von riemannschen Metriken mit Holonomie gleich SU(4).

Lebenslauf

Persönliche Daten

Name:	Marco Freibert
Geburtsdatum:	16.07.1982
Geburtsort:	Würzburg

Schulausbildung

Juni 2002	Allgemeine Hochschulreife	Deutschhaus-Gymnasium Würzburg

Zivildienst

2002 - 2003	Zivildienst	Blindeninstitutsstiftung Würzburg

Hochschulausbildung

2003 - 2005	Grundstudium Physik (Diplom) und Mathematik (Diplom)	Julius-Maximilians-Universität Würzburg
August 2005	Vordiplom in Mathematik mit Nebenfach Physik	Julius-Maximilians-Universität Würzburg
Oktober 2005	Vordiplom in Physik mit Nebenfach Numerische Mathematik	Julius-Maximilians-Universität Würzburg
2005 - 2009	Hauptstudium Mathematik (Diplom)	Julius-Maximilians-Universität Würzburg
März 2009	Diplom in Mathematik mit Nebenfach Physik	Julius-Maximilians-Universität Würzburg
2009 - 2013	Promotion in Mathematik	Universität Hamburg