Gauge Theories with Nonunitary Parallel Transport

Dissertation
zur Erlangung des Doktorgrades
des Fachbereichs Physik
der Universität Hamburg

vorgelegt von
Thorsten Prüstel
aus Bremerhaven

Hamburg 2003

Gutachter der Doktorarbeit:	Prof. Dr. G. Mack Prof. Dr. K. Fredenhagen
Gutachter der Disputation:	Prof. Dr. G. Mack Prof. Dr. J. Louis
Datum der Disputation:	28.11.2003
Vorsitzender des Prüfungsausschusses:	Dr. H.D. Rüter
Vorsitzender des Promotionsausschusses:	Prof. Dr. R. Wiesendanger
Dekan des Fachbereichs Physik:	Prof. Dr. G. Huber

Abstract

In all gauge theories, including general relativity, parallel transporters are of fundamental importance. They are unitary maps between vector spaces at different space time points. It is proposed to abandon unitarity as a general requirement. The proposal is motivated by the fact that in discrete calculus and differential geometry unitarity is not a natural requirement, and by a desire to interpret Higgs fields geometrically.

The present thesis discusses gauge theories with nonunitary parallel transport both in the continuum and on a graph.

Vierbein fields can be identified as parts of nonunitary parallel transporters in the conventional four space time dimensions. Assuming invertibility of the parallel transporters it is shown how general relativity fits into this framework. Metricity is obtained automatically without need to assume it.

Going to the lattice, Higgs fields can be interpreted as associated with nonunitary parallel transport in extra dimensions. A gauge theoretic model based on joint work with C. Lehmann and G. Mack is presented which can explain how quarks of different flavor can acquire different masses by spontaneous symmetry breaking and what is the difference between colour and flavor.

Zusammenfassung

In allen Eichtheorien einschließlich der Allgemeinen Relativitätstheorie sind Paralleltransporter von grundlegender Bedeutung. Dabei handelt es sich um unitäre Abbildungen zwischen Vektorräumen an verschiedenen Punkten der Raum-Zeit. Es wird vorgeschlagen, die allgemeine Forderung nach Unitarität fallenzulassen. Der Vorschlag wird motiviert durch den Umstand, dass im Rahmen eines diskreten Kalküls und einer diskreten Differenzialgeometrie Unitarität keine natürliche Forderung ist, und durch den Wunsch Higgsfeldern eine geometrische Interpretation zu geben.

Die vorliegende Arbeit behandelt Eichtheorien mit nichtunitärem Paralleltransport sowohl im Kontinuun als auch auf Graphen.

Vierbeinfelder können als Teil eines nichtunitären Paralleltransporters in den herkömmlichen vier Raum-Zeit Dimensionen identifiziert werden. Unter der Annahme invertierbarer Paralleltransporter wird gezeigt, wie die Allgemeine Relativitätstheorie sich in diesen Rahmen einfügt. Metrizität ergibt sich automatisch.

Auf dem Gitter ist es möglich, Higgsfelder als verknüpft mit einem nichtunitären Paralleltransport in zusätzlichen Dimensionen aufzufassen.

Basierend auf gemeinsamer Arbeit mit C. Lehmann und G. Mack wird ein eichtheoretisches Modell vorgestellt, das erklärt, wie Quarks unterschiedlichen Flavors verschiedene Massen durch spontane Symmetriebrechung erlangen und das eine Erklärung für den Unterschied zwischen Colour and Flavor liefert.

Contents

Chapter 1

Introduction

The aim of fundamental physics is to describe and explain a variety of phenomena by using a minimum of basic principles and concepts. In this way nature becomes understandable and its complexity can be reduced. Every physical theory is built on a set of primary principles. This provides a criterion to distinguish between more and less fundamental theories. A theory will be the more fundamental the less structure is assumed a priori. The a priori structure consists of the basic principles and all the axioms of the mathematical theories that are used to formulate the theory.

One of the major developments of twentieth-century physics has been the recognition that all known fundamental interactions are governed by gauge theories. The power of gauge theories results from their poor a priori structure. General relativity is built on two principles only, the principle of relativity, or general covariance, and the equivalence principle. When appropriately interpreted these principles are also operative in the gauge theories of elementary particle physics. These principles strongly constrain equations of motion.

It is important to note that the principle of relativity is in fact a statement of absence of a priori structure. Before general relativity emerged it was assumed that space is equipped with an a priori structure which defines the notion of a straight line. This is equivalent to the assumption that it is a priori possible to compare directions at different points in space. This a priori structure is eleminated in general relativity and in gauge theory. To compare vectors it is necessary to parallel transport vectors from one point to the other. The result is given by a map $\mathcal{U}(C)$, called parallel transporter, which is dynamically determined.

However, the principle of relativity is not pushed to its logical conclusion. Since one assumes a differentiable manifold the a priori definition of a straight line persists in the infinitesimally small. Therefore we follow Mack [29, 30] and propose to push Einsteins principle to the extreme, i.e. we suggest the strategy of lessening what is assumed as a priori structure.

As a first step, we replace the differentiable manifold by a discrete one which has less a priori structure. It turns out not to be necessary to impose a differentiable structure (which determines what is a straight line in the infinitesimal small in the continuum) in addition to the topology. In the framework we will use, conventional notions of locality and the notion of a point retain their meaning. This is important, because without locality complexity of nature becomes unmanageable.

The corresponding gauge theory is similar to that in the continuum, nevertheless there is a profound difference.

In the classical case, the parallel transporters are unitary in an appropriate sense. Given a path C, let $-C$ be the path traversed in the opposite direction. One demands that

$$\mathcal{U}(-C)\mathcal{U}(C) = \mathbf{1}. \tag{1.1}$$

i.e. parallel transport $back \circ forth = identity$. In the algebraic approach to the discrete case, unitarity in the sense of (1.1) cannot be required in a natural way. Therefore we permit nonunitary parallel transporters. In this way one is led to a theory which involves additional degrees of freedom. We shall show how these additional degrees of freedom associated with a nonunitary parallel transport along an extra dimension can describe Higgs fields.

It turns out that also in the continuum the concept of nonunitary parallel transport is useful. General relativity appears as gauge theory of a special type, due to the appearance of tetrads and of the particular form of the action. In our framework the tetrads can be interpreted as associated with nonunitary parallel transporters along the conventional four space time dimensions.

There is another reason why one might want to generalize the notion of gauge theories. As discussed, the purpose of physics is complexity reduction. The concept of renormalization group enables one to manage complexity by construction of simplified models, also known as effective theories, which live on coarser scales. Higgs fields are associated with parallel transporters along extra directions. If one were to use unitary parallel transporters for this purpose both in the theory from which one starts and for all effective theories deduced from it by real space renormalization group transformations, then one would end up with a nonrenormalizable 4-dimensional effective theory, a gauged nonlinear σ-model. This suggests to admit renormalization group transformations which lead to nonunitary block-parallel transporters.

Because of the peculiar behaviour of parallel transporters under gauge transformations, the Higgs potential will possess a characteristic biinvariance property. It will be exploited to point out a mechanism by which quarks of different generations may aquire different masses, and to propose a model wherein also the Cabbibo Kobayashi Maskawa matrix can be computed in principle.

The outline of this thesis is as follows. In the next chapter we briefly review conventional gauge theories from a geometrical point of view. Chapter 3 introduces the basic concepts of gauge theories with nonunitary parallel transport. We employ the developed tools to deal with gravity.

In chapter 4 and chapter 5 we will leave the continuum and review the formalism of Dimakis and Müller-Hoissen. We extend it and introduce the concepts and tools of nonunitary parallel transport in the framework of semicommutative differential geometry on a graph.

In chapter 6 the tools and the language described in chapter 4 and 5 will be employed to reveal the geometry of Higgs fields.

Finally, chapter 7 presents a gauge theoretic model which can explain how quarks of different flavor can acquire different masses by spontaneous symmetry breaking. The masses depend exponentially on the positions of minima of a Higgs potential.

We conclude with a summary of our results and an outlook.

Chapter 2

Geometry of gauge theories

In this chapter we give a brief review of gauge theories from a geometrical point of view. We focus on the structural assumptions which are made in the standard formulation to motivate possible generalizations.

The power of Yang-Mills theories stems from the fact that they are built on two general principles only which appear as true principles of nature. They are the Naheinformationsprinzip and the equivalence principle. When appropriately interpreted, they are the same principles which are also operative in general relativity. Therefore gauge theories provide a unified geometric framework which allows to deal with all fundamental interactions.

Nevertheless we know that conventional gauge theories are not sufficient to describe nature. In gauge theories of elementary particles one is forced to add Higgs fields in an ad hoc manner. Their introduction increases the number of free parameters considerably. This comes as no surprise, since they lack a geometric meaning, reflecting the fact that Higgs fields are not required by basic principles of the standard theory. As a consequence many features of the standard model cannot be explained.

2.1 Naheinformationsprinzip

A crucial feature of gauge theories, including general relativity, is the validity of the Naheinformationsprinzip, which forbids direct exchange of information at a distance.

More precisely, in gauge theories one deals with bundles of vector spaces V_x attached to the points x of a space time manifold $\mathcal{M}$. The content of the Naheinformationsprinzip was clearly formulated in the pioneering paper of Yang and Mills. It asserts that there is no a priori way of comparing vectors in different vector spaces V_x and V_y, $x \neq y$. Instead one needs parallel transporters

$$\mathcal{U}(C) : V_x \to V_y \tag{2.1}$$

along paths C in $\mathcal{M}$ from x to y. They are linear maps. The following properties are assumed

i.) Parallel transport along the empty path : $x \to x$ is trivial,

$$\mathcal{U}(C) = \mathbf{1}. \tag{2.2}$$

ii.) Composition rule: If a path is composed from two paths, $C = C_2 \circ C_1$, then the parallel transporter is similarly composed,

$$\mathcal{U}(C_2 \circ C_1) = \mathcal{U}(C_2)\mathcal{U}(C_1). \tag{2.3}$$

The right hand side involves the composition of two maps.

Assuming that each nonempty path can be composed from basic small pieces, $C = b_n \circ \ldots \circ b_1$, it follows that

$$\mathcal{U}(C) = \mathcal{U}(b_n)\ldots\mathcal{U}(b_1), \tag{2.4}$$

and only the parallel transporters $\mathcal{U}(b)$ are needed. A connection is an assignment of a parallel transporter $\mathcal{U}(C)$ to every path C in such a way that (2.2) and (2.3) holds.

Denoting by $-C$ the path C traversed in the opposite direction, we may define an *algebraic* $*$-operation which maps parallel transporters into parallel transporters in the opposite direction,

$$\mathcal{U}(C)^* := \mathcal{U}(-C). \tag{2.5}$$

As usual for a $*$-operation, it interchanges factors,

$$(\mathcal{U}(C_2)\mathcal{U}(C_1))^* = \mathcal{U}(C_1)^*\mathcal{U}(C_2)^*. \tag{2.6}$$

Traditionally, one makes the assumption

$$\mathcal{U}(C)^* = \mathcal{U}(C)^{-1}. \tag{2.7}$$

In other words, $\mathcal{U}(-C)\mathcal{U}(C) = \mathbf{1}$, or $back \circ forth = identity$.

The existence of parallel transporters is a consequence of the Naheinformationsprinzip. However, in the standard formulation additional structure is assumed. The vector spaces V_x come equipped with a positive definite scalar product $\langle\, ,\, \rangle$, whereas in general relativity the fibers carry an indefinite scalar product. Furthermore, one requires that the parallel transporters preserve the length of vectors v in V_x, i.e.

$$\langle v, w\rangle_x = \langle \mathcal{U}(C)v, \mathcal{U}(C)w\rangle_y \tag{2.8}$$

for $v, w \in V_x$. Note that (2.8) follows from (2.7) if the $*$-operation is at the same time the adjoint map with respect to the scalar product $\langle\, ,\, \rangle_x$.

Additionally we see that $\mathcal{U}(C)$ is a unitary map from V_x to V_y. Therefore we call such parallel transporters unitary parallel transporters.

There are further requirements. The fundamental physical laws relate only physical quantities at infinitesimally close points of space time. The notion of infinitesimally close requires a topology. Moreover, since the dynamical equations of motion are differential equations, one needs a differential structure, i.e. a smooth manifold $\mathcal{M}$.

In the continuum one deals with vector potentials rather than with parallel transporters. Their definition depends on a choice of moving frame, besides the parallel transporters. A moving frame furnishes orthonormal bases $\boldsymbol{e}(x) = (e_1(x), \ldots, e_N(x))$ in V_x with

$$\langle e_\alpha(x), e_\beta(x)\rangle_x = \delta_{\alpha\beta}. \tag{2.9}$$

The Naheinformationsprinzip requires that there are no preferred moving frames.

The orthonormal bases $\boldsymbol{e}(x)$ form the fibers of a principal fiber bundle whose structure group is the gauge group G. Parallel transport of vectors induces parallel transport of bases and thereby a connection on a principal fiber bundle.

Gauge transformations are determined by matrices $\boldsymbol{S}(x) \in G$. A (passive) gauge transformation is a change of moving frame

$$e_\alpha(x) \to e_\alpha(x)' = e_\beta(x) S^\beta{}_\alpha(x). \tag{2.10}$$

This transformation preserves orthonormality.

The moving frame allows one to convert maps into matrices. In this way, parallel transporters get converted to parallel transport matrices $\boldsymbol{U}(C) := (U^\alpha{}_\beta(C))$

$$\mathcal{U}(C) e_\alpha(x) = e_\beta(y) U^\beta{}_\alpha(C). \tag{2.11}$$

The parallel transport matrix along infinitesimal paths C from a point x with coordinates x^μ to $x + \delta x$ with coordinates $x^\mu + \delta x^\mu$ defines the vector potential $\boldsymbol{A}_\mu(x) = (A^\alpha{}_{\beta\mu}(x))$. When differentiability assumptions are made, one can write

$$\boldsymbol{U}(C) = \mathbf{1} - \boldsymbol{A}_\mu(x)\delta x^\mu. \tag{2.12}$$

The parallel transport matrices take values in the gauge group G, whereas $\boldsymbol{A}_\mu(x) \in LieG$.

Under a change of moving frame the parallel transport matrices $\boldsymbol{U}(C)$ change according to

$$\boldsymbol{U}(C) \to \boldsymbol{U}(C)' = \boldsymbol{S}(y)^{-1} \boldsymbol{U}(C) \boldsymbol{S}(x). \tag{2.13}$$

Consequently, the vector potentials transforms as follows,

$$\boldsymbol{A}_\mu(x) \to \boldsymbol{A}_\mu(x)' = \boldsymbol{S}(x)^{-1} \boldsymbol{A}_\mu(x) \boldsymbol{S}(x) + \boldsymbol{S}(x)^{-1} \partial_\mu \boldsymbol{S}(x). \tag{2.14}$$

Given $x \in \mathcal{M}$, the holonomy group H_x consists of all parallel transporters $\mathcal{U}(C)$ around closed paths $C : x \to x$. If $\mathcal{M}$ is connected, then the holonomy groups are independent of x modulo isomorphism. Their equivalence class is called the holonomy group of the connection.

Parallel transporters can be used to define covariant derivatives in a standard way. Let b be a straight infinitesimal path from x to $x + \delta x$ and ψ a field taking values $\psi(x)$ in V_x. Then the covariant derivatives are defined by

$$D_\mu \psi(x) \delta x^\mu = \mathcal{U}(C)^{-1} \psi(x + \delta x) - \psi(x). \tag{2.15}$$

Combining (2.12) and (2.15), one gets

$$D_\mu e_\alpha(x) = e_\beta(x) A^\beta{}_{\alpha\mu}(x). \tag{2.16}$$

If we expand $\psi(x) = e_\alpha(x)\psi^\alpha(x)$ then it follows that

$$D_\mu \psi(x) = (D_\mu \psi(x))^\alpha e_\alpha(x) \tag{2.17}$$

with

$$(D_\mu \psi(x))^\alpha = \partial_\mu \psi^\alpha(x) + A^\alpha{}_{\beta\mu}(x) \psi^\beta(x). \tag{2.18}$$

In matrix notation the last equation may be rewritten as

$$D_\mu \boldsymbol{\psi}(x) = (\partial_\mu + \boldsymbol{A}_\mu(x)) \boldsymbol{\psi}(x). \tag{2.19}$$

The curvature tensor $\mathcal{F}_{\mu\nu}(x) : V_x \to V_y$ can be obtained by considering parallel transport around an infinitesimal quadrangle $\Box$ in the $\mu\nu$ plane

$$\mathcal{U}(\Box) =: \mathbf{1} - \mathcal{F}_{\mu\nu}(x) \delta x^\mu \delta x^\nu. \tag{2.20}$$

The curvature matrix $\boldsymbol{F}_{\mu\nu}(x) = (F^\alpha{}_{\beta\mu\nu}(x)$ is given by

$$\mathcal{F}_{\mu\nu}(x) e_\alpha(x) = e_\beta(x) F^\alpha{}_{\beta\mu\nu}(x), \tag{2.21}$$

where

$$\boldsymbol{F}_{\mu\nu}(x) = \partial_\mu \boldsymbol{A}_\nu(x) - \partial_\nu \boldsymbol{A}_\mu(x) + [\boldsymbol{A}_\mu(x), \boldsymbol{A}_\nu(x)]. \tag{2.22}$$

It transforms under gauge transformations according to

$$\boldsymbol{F}_{\mu\nu}(x) \to \boldsymbol{S}(x)^{-1} \boldsymbol{F}_{\mu\nu}(x) \boldsymbol{S}(x). \tag{2.23}$$

2.2 Equivalence principle

Let us now turn to the second fundamental principle, i.e. the principle of equivalence. It demands that the free field equation for the components ψ is valid at an arbitrary space time point x if a suitable moving frame is chosen in a neighbourhood of x. This requirement fixes the equations of motion uniquely. For a general moving frame one has to write down the free field equation and to replace ∂_μ by $\partial_\mu + \boldsymbol{A}_\mu$. This procedure is known as the "minimal coupling" recipe in textbooks. Under a change of moving frame, i.e. passive gauge transformations, $\boldsymbol{A}_\mu$ changes according to (2.14). By choosing it properly one can transform the vector potential to zero at the chosen point x and the principle of equivalence is satisfied.

It is important to note that the principle of equivalence is very restrictive. In fact it singles out minimal coupling to gauge fields as only form of interaction in nature. In particular, Higgs models involving fundamental scalar fields which have self-interactions through some potential are ruled out. Also Yukawa interactions are forbidden.

2.3 Critique of the standard formalism

The structural assumptions of gauge theories give rise to criticism. Firstly, we saw that space time is treated as a differentiable manifold. As discussed in the introduction, the notion of a differentiable structure requires an a priori definition of a straight line in the infinitesimally small.

However, it is widely believed that a differentiable manifold becomes an empty concept when it comes to physics at the Planck scale.

In general relativity, general covariance demands that there should be no preferred coordinate system. However, an a priori defined preferred class of coordinate systems is assumed. We propose to push the principle of general covariance to its logical conclusion and postulate that the fundamental physical equations should make sense without any reference to coordinates whatever.

The first step in this direction is to replace a differentiable manifold by an arbitrary directed graph [13, 14, 15, 33]. In this way the a priori definition of a straight line in the infinitesimally small is eliminated. The notion of infinitesimally close is replaced by nearest neighbour relations. More precisely, certain relations between vertices of the graph are singled out as direct relations, called links, and all others are obtained from them by composition. In this way, the links of a graph provide a substitute for topology.

In order to formulate gauge theories on graphs one needs a differential calculus. Surprisingly it turns out that the links already specify a differential structure. The corresponding differential geometry is compatible with conventional locality properties, particularly the notion of a point retains its meaning.

Müller-Hoissen and Dimakis showed that the infintesimal approach employing vector potentials can also be used for arbitrary directed graphs [14, 11, 12]. The familiar formulae of continuum gauge theory can be retained literally on the lattice.

In spite of all this, a profound difference between the continuum case and the discrete case exists. In either case a vector potential is only defined upon specifying a moving frame. We saw that in the continuum the vector potential takes its values in a Lie algebra of matrices, which is naturally a vector space. In the discrete case, the vector potential must also take values in a vector space, because it specifies elements of an algebra. When the parallel transport matrices take their values in a group, the vector potential will be in the group algebra.

In the algebraic approach to the discrete case, there is no natural way to impose the unitarity condition as a requirement on the vector potential. Although an antihermiticity property $\boldsymbol{A}(x) = -\boldsymbol{A}^*$ can be formulated, it does not imply unitarity of $\mathcal{U}(b)$ any more. Therefore it is natural in this context to admit nonunitary parallel transporters. There are two stages to the generalization:

i.) $\mathcal{T}(b)$ are invertible, but $\mathcal{T}(b)^* \neq \mathcal{T}(b)^{-1}$

ii.) $\mathcal{T}(b)$ is noninvertible, i.e. $\mathcal{T}(b)^{-1}$ does not exist at all.

A comprehensive examination of step ii.) is beyond the scope of this thesis.

Under appropriate conditions, one can write down a polar decomposition of the parallel transporters. The unitary factor represents a conventional gauge field. We will discuss in this work how the selfadjoint factor (or its generalization) can accomodate Higgs fields and the tetrads of general relativity.

Thus, by lessening the a priori structure we shall arrive at a generalization of gauge theories which can cover a much wider range of phenomena than conventional ones.

Chapter 3

Nonunitary parallel transport and gravity

In the literature the question has been repeatedly asked, whether Einstein's general relativity is a "true" gauge theory. From the point of view of the present work, the crucial and defining property of a gauge theory is the validity of the Naheinformationsprinzip. Undoubtedly Einstein's general relativity satisfies this principle. But although general relativity and the gauge theories governing the dynamics of elementary particles obey very much the same basic principles, they are different both in their variables and in their action. More precisely, apart from the vector potential a vierbein field appears, which has no analogue in Yang-Mills theories, and the Einstein-Hilbert action is linear in the curvature, while the Yang-Mills action is quadratic in the field strength.

Thus one might be motivated to generalize the notion of gauge theories. We proceed in the spirit of the Einsteinian principle of minimal a priori structure and first give up the requirement that the parallel transporters should be unitary. In addition we abandon the second standard requirement that there should exist a bilinear or sesquilinear form on fibers V_x which is invariant under parallel transport. In the context of general relativity, with V_x = the tangent space $T_x\mathcal{M}$ to the manifold $\mathcal{M}$ at x, this amounts to abandoning the postulate that the connection is metric.

When one or the other of these assumptions is violated, we speak of nonunitary parallel transporters.

In this chapter we show how to recover general relativity without these assumptions, provided the parallel transporters remain invertible. There is a canonical way of constructing a metric and a metric connection (which involves unitary parallel transporters) in the more general framework of nonunitary gauge theories. Following Palatini, the further condition of vanishing torsion is thought to arise from variation of the action.

Assuming invertibility of parallel transporters, there will be in general two holonomy groups, the holonomy group H generated by all parallel transporters along loops and their inverses, and the unitary gauge group G which is generated by the unitary parallel transporters. They will be considered in section 3.3. It is not obvious at this stage that $G \subset H$, but this will follow from the fact that they must be Lie groups. The field strength and vector potential take their values in the Lie algebra of H, and matter fields must transform according to representations of H. Nevertheless only G admits an interpretation as a local symmetry.

For general relativity, $G = SO(1,3)$ and the holonomy groups H are the de Sitter groups $SO(1,4)$ or $SO(2,3)$ (or rather their simply connected covering groups). The second possibility $SO(2,3)$ is distinguished by admitting chiral fermions.

Vierbein and spin connection become identifyable pieces of a single de Sitter vector potential. The resulting de Sitter field strength can be used to cast the Einstein Hilbert action in quasi-Maxwellian form.

We show that, somewhat surprisingly, also classical massive bodies can be treated within the de Sitter theory and that their equation of motion is de Sitter covariant.

Finally we study an extension of general relativity by considering a conformal holonomy group.

3.1 A priori assumptions of the standard formalism

Let us review the structural assumptions which are made in the standard formalism.

In classical general relativity one deals with a four-dimensional differentiable space time manifold $\mathcal{M}$ and a dynamically determined geometry on $\mathcal{M}$. The geometry provides a connection in the tangent bundle, which specifies the parallel transport of tangent vectors along a path C from x to y

$$\mathcal{U}(C) : T_x\mathcal{M} \to T_y\mathcal{M}. \tag{3.1}$$

Furthermore there is a Lorentzian metric $g(x) : T_x\mathcal{M} \times T_x\mathcal{M} \to \mathbb{R}$ on $\mathcal{M}$. The connection is compatible with $g(x)$ in the sense that the metric is invariant under parallel transport. Following Palatini, the dynamical equations of motion for both the metric and the connection are derived from a variational principle. The vanishing of the torsion is one of these field equations.

In the vierbein approach, general relativity becomes more similar to gauge theories. The connection in the tangent bundle can be thought to be constructed in two steps:

i.) The dynamics determine a connection in a vector bundle $\mathcal{V}$ over $\mathcal{M}$. The fibers V_x are isomorphic to the fourdimensional real representation space of the Lorentz group $SO(1,3)$, $V_x \approx V^{(\frac{1}{2},\frac{1}{2})} \approx \mathbb{R}^4$. In addition, there is a Lorentz invariant bilinear form $\langle\, ,\, \rangle_x$ of signature $(+---)$ on the fibers. The connection provides parallel transporters. They are linear maps which leave the bilinear form invariant

$$\mathcal{U}(C) \quad : \quad V_x \to V_y, \tag{3.2}$$
$$\langle \mathcal{U}(C)v, \mathcal{U}(C)w\rangle_y \quad = \quad \langle v, w\rangle_x \tag{3.3}$$

for all $v, w \in V_x$. In general relativity one wants to parallel transport not just vectors in an abstract vector space V_x, but also tangent vectors. In order that $\mathcal{U}(C)$ can do that, a second step has to be performed.

ii.) The fibers V_x need to be identified with the tangent spaces $T_x\mathcal{M}$. The identification is provided by a vierbein field, which is not a priori given, but also determined dynamically. It specifies an invertible map from the tangent space to the internal vector space

$$e(x) : T_x\mathcal{M} \to V_x \tag{3.4}$$

for every $x \in \mathcal{M}$. Since V_x is equipped with a bilinear form, a length can be assigned to tangent vectors through the vierbein field. In other words, the bilinear form on the abstract fibers becomes a Lorentz metric on $\mathcal{M}$ via

$$g(x)(X,Y) := \langle e(x)X, e(x)Y\rangle_x \tag{3.5}$$

for all $X, Y \in T_x\mathcal{M}$. Since the bilinear form is invariant under parallel transport, the metric tensor is also invariant under parallel transport

$$g(y)(\mathcal{U}(C)^{T\mathcal{M}}X, \mathcal{U}(C)^{T\mathcal{M}}Y) = g(x)(X,Y), \tag{3.6}$$

with

$$\mathcal{U}(C)^{T\mathcal{M}} \quad : \quad T_x\mathcal{M} \to T_y\mathcal{M}, \tag{3.7}$$
$$\mathcal{U}(C)^{T\mathcal{M}} \quad := \quad e(y)^{-1} \circ \mathcal{U}(C) \circ e(x). \tag{3.8}$$

In this way, general relativity may be considered as a special kind of a gauge theory with gauge group $SO(1,3)$, and with an additional vierbein field, which does not appear in gauge theories of elementary particles.

The Einstein-Palatini action is a functional of the vector potential associated with $\mathcal{U}(C)$ and of the vierbein.

Fundamental matter is described by wave functions for spin $\frac{1}{2}$ particles rather than by fourvectors. Thus one is forced to consider the parallel transport of spinors. Because of the structural assumptions of standard differential geometry, the parallel transport of vectors in an arbitrary representation space of the gauge group determines the parallel transport of vectors in any representation space. Actually, it suffices to consider the representation spaces of left- and right-handed Weyl spinors as internal spaces, since others can be constructed as sums or tensor products.

For Weyl spinors, the fibers are isomorphic to 2-dimensional complex representation spaces of $SL(2,\mathbb{C})$

$$V_x^+ \approx V^{(\frac{1}{2},0)} \approx \mathbb{C}^2 \quad \text{and} \quad V_x^- \approx V^{(0,\frac{1}{2})} \approx \mathbb{C}^2. \tag{3.9}$$

$V^{(\frac{1}{2},0)}$ is equipped with an antisymmetric bilinear (symplectic) form

$$\langle\, ,\, \rangle : V^{(\frac{1}{2},0)} \times V^{(\frac{1}{2},0)} \to \mathbb{C}. \tag{3.10}$$

It is required to be invariant under parallel transport, like the bilinear form in $V^{(\frac{1}{2},\frac{1}{2})}$ was. A basis in the vector space $V^{(\frac{1}{2},0)}$ is called "orthogonal", or admissible, if

$$< e_a(x), e_b(x) >= \epsilon_{ab}, \tag{3.11}$$

where $\epsilon = (\epsilon_{ab})$ is the 2-dimensional antisymmetric tensor with $\epsilon_{12} = +1$. A moving frame provides an admissible basis in V_x^+ for every x. The matrices $\boldsymbol{S}$ of the corresponding gauge group have to satisfy

$$\epsilon^{-1}\boldsymbol{S}^t\epsilon = \boldsymbol{S}^{-1}. \tag{3.12}$$

It follows that the gauge group is the quantum mechanical Lorentz group, i.e. the two fold cover of $SO(1,3)$, $G = Spin(1,3) = SL(2,\mathbb{C})$.

As mentioned above, Dirac spinors, fourvectors and arbitrary tensors can be formed as sums and tensor products.

Fourvectors are elements of $V^{(\frac{1}{2},0)} \otimes V^{(0,\frac{1}{2})}$. More precisely, $V^{(\frac{1}{2},\frac{1}{2})}$ is a real subspace of the complex representation space $V^{(\frac{1}{2},0)} \otimes V^{(0,\frac{1}{2})}$. This identification can be used to construct a moving frame in $V^{(\frac{1}{2},\frac{1}{2})}$. A basis is given by the linear combinations

$$e_\alpha(x) := (\boldsymbol{\sigma}_\alpha)^{a\dot{b}} e_a(x) \otimes e_{\dot{b}}(x), \tag{3.13}$$

where $\boldsymbol{\sigma}_i$ are the Pauli matrices for $i = 1,2,3$ and $\boldsymbol{\sigma}_0 = \mathbf{1}$. The real vector space V_x^+ contains vectors

$$v(x) = v^\alpha(x) e_\alpha(x) \tag{3.14}$$

with real coefficients $v^\alpha(x)$.

A choice of moving frame converts parallel transporters $\mathcal{U}(C)$ into parallel transport matrices $\boldsymbol{U}(C)$. The parallel transport matrices $\mathbf{U}(C)^W \in$

$SL(2,\mathbb{C})$ for vectors in $V_x^+ \approx V^{(\frac{1}{2},0)}$ and $\boldsymbol{U}(C) \in SO(1,3)$ for vectors in $V_x \approx V^{(\frac{1}{2},\frac{1}{2})}$ are related by the fundamental formula of spinor calculus

$$\boldsymbol{S}\boldsymbol{\sigma}_\alpha \boldsymbol{S}^\dagger = \sigma_\beta \Lambda^\beta{}_\alpha(\boldsymbol{S}) \qquad \text{for } \boldsymbol{S} \in SL(2,\mathbb{C}), \tag{3.15}$$

$$\boldsymbol{X}\boldsymbol{\sigma}_\alpha + \boldsymbol{\sigma}_\alpha \boldsymbol{X}^\dagger = \sigma_\beta \Lambda^\beta{}_\alpha(\boldsymbol{X}) \qquad \text{for } \boldsymbol{X} \in \mathfrak{sl}(2,\mathbb{C}), \tag{3.16}$$

where † is the Hermitian adjoint of a matrix. This formula yields the Lorentz transformation $\Lambda(\boldsymbol{S})$ which is associated with $\boldsymbol{S} \in SL(2,\mathbb{C})$, and similarly for elements of the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. As a consequence the parallel transport matrices $\boldsymbol{U}(C)^W$ and the the spinorial field strength matrix $\boldsymbol{F}_{\mu\nu}(x)$ obey the following relations

$$\boldsymbol{U}(C)^W \boldsymbol{\sigma}_\alpha \boldsymbol{U}(C)^{W\dagger} = \boldsymbol{\sigma}_\beta U(C)^\beta{}_\alpha \tag{3.17}$$

$$\boldsymbol{F}_{\mu\nu}\boldsymbol{\sigma}_\alpha + \boldsymbol{\sigma}_\alpha \boldsymbol{F}^\dagger_{\mu\nu} = \boldsymbol{\sigma}_\beta R^\beta{}_{\alpha\mu\nu}, \tag{3.18}$$

where $R^\beta{}_{\alpha\mu\nu}$ are the anholonomic components of the Lorentz field strength. Let $\boldsymbol{A}_\mu(x) = (\boldsymbol{A}^\alpha{}_{\beta\mu}(x))$ be the $\mathfrak{so}(1,3)$ valued vector potential which corresponds to an orthonormal moving frame $(e_\alpha(x))$ in $V_x \approx V^{(\frac{1}{2},\frac{1}{2})}$. By introducing matrices $\tilde{\boldsymbol{\sigma}}_i := -\boldsymbol{\sigma}_i$ and $\tilde{\boldsymbol{\sigma}}_0 := \boldsymbol{\sigma}_0$ obeying

$$\boldsymbol{\sigma}_\alpha \tilde{\boldsymbol{\sigma}}_\beta + \boldsymbol{\sigma}_\beta \tilde{\boldsymbol{\sigma}}_\alpha = 2\eta_{\alpha\beta}, \tag{3.19}$$

with $(\eta_{\alpha\beta}) = diag(+1,-1,-1,-1)$, one defines

$$\boldsymbol{A}^W_\mu(x) := \frac{1}{4} A^{\alpha\beta}_\mu(x)(\boldsymbol{\sigma}_\alpha \tilde{\boldsymbol{\sigma}}_\beta - \boldsymbol{\sigma}_\beta \tilde{\boldsymbol{\sigma}}_\alpha). \tag{3.20}$$

$\boldsymbol{A}^W_\mu(x)$ enables one to construct parallel transporters $\mathcal{U}^W(C)$ for Weyl spinors, which preserve the symplectic form (3.10). Conversely, if the parallel transport of Weyl spinors is given, fourvectors which can be made from Weyl spinors can also be parallel transported.

In the same manner, the parallel transport of fourvectors determines parallel transport of Dirac spinors. Now the fibers V_x^D are isomorphic to the direct sum $V_x^D \approx V^{(\frac{1}{2},0)} \oplus V^{(0,\frac{1}{2})} \approx \mathbb{C}^4$. The corresponding vector potential is defined by

$$\boldsymbol{A}^D_\mu(x) := \frac{1}{4} A^{\alpha\beta}{}_\mu(x)[\boldsymbol{\gamma}_\alpha, \boldsymbol{\gamma}_\beta] \tag{3.21}$$

where Dirac γ-matrices are employed, which satisfy standard anticommutation relations

$$\{\boldsymbol{\gamma}_\alpha, \boldsymbol{\gamma}_\beta\} = 2\eta_{\alpha\beta}. \tag{3.22}$$

The vector potential $\boldsymbol{A}^D_\mu(x)$ can be used to define parallel transporters $\mathcal{U}^D(C)$ for Dirac spinors. There is a scalar product (sesquilinear form) on V_x^D

$$\langle \Psi, \Phi \rangle^D_x := \bar{\Psi}(x)\Phi(x) := \Psi^\dagger(x)\boldsymbol{\beta}\Phi(x), \tag{3.23}$$

cp. appendix C, which is left invariant by $\mathcal{U}^D(C)$.

3.2 Parallel transporters and their polar decomposition

Above we saw that in all gauge theories, including general relativity, parallel transporters

$$\mathcal{U}(C) : V_x \to V_y \tag{3.24}$$

along paths C between points x, y of the space time manifold play a basic role. We further saw that traditionally one assumes that

$$\mathcal{U}(-C)\mathcal{U}(C) = \mathbf{1}, \tag{3.25}$$

i.e.

$$\mathcal{U}(-C) = \mathcal{U}(C)^{-1}. \tag{3.26}$$

We propose to abandon this requirement and to permit also parallel transporters which violate the unitarity condition (3.25). In the case of a discrete manifold, (3.25) appears actually as no natural requirement. However, in the continuum, the situation is quite different. Actually, in this case the parallel transporters must satisfy (3.25), as one can easily show by using vector potentials. In appendix A we will see how this can be overcome.

In the following we assume that there are nonunitary parallel transporters on a differentiable manifold.

Let us restrict our attention to a nonunitary gauge theory where all parallel transporters are invertible.

The vector potential and its split into two pieces $\boldsymbol{E}_\mu(x)$ and $\boldsymbol{A}_\mu(x)$ is defined by considering infinitesimal paths b from x to $x+\delta x$ and its opposite $-b$,

$$\boldsymbol{T}(\pm b) = \mathbf{1} - (\boldsymbol{E}_\mu(x) \pm \boldsymbol{A}_\mu(x))\delta x^\mu. \tag{3.27}$$

Especially, it follows that

$$\boldsymbol{T}(-b)\boldsymbol{T}(b) = \mathbf{1} - 2\boldsymbol{E}_\mu(x)\delta x^\mu \neq \mathbf{1}. \tag{3.28}$$

The unitary parallel transport matrices along infinitesimal paths are now defined by

$$\boldsymbol{U}(b) = \mathbf{1} - \boldsymbol{A}_\mu(x)\delta x^\mu. \tag{3.29}$$

The corresponding maps $\mathcal{U}(b)$ satisfy $\mathcal{U}(b)^* = \mathcal{U}(b)^{-1}$.

The formulae (3.27), (3.29) define a polar decomposition of the parallel transporters along *infinitesimal* paths b, cp.[10].

Theorem 3.1 (Polar decomposition of vector potentials) *The split of the vector potential* $\boldsymbol{B}_\mu(x)$

$$\boldsymbol{B}_\mu(x) = \boldsymbol{E}_\mu(x) + \boldsymbol{A}_\mu(x) \tag{3.30}$$

defines a polar decomposition of the parallel transporters along infinitesimal paths b,

$$\mathcal{T}(b) = \mathcal{U}(b)\mathcal{P}(b) \tag{3.31}$$
$$\boldsymbol{P}(b) = \mathbf{1} - \boldsymbol{E}_\mu(x)\delta x^\mu. \tag{3.32}$$

The first factor is self adjoint in the sense that $\mathcal{P}(b)^* \equiv \mathcal{P}(-b) = \mathcal{P}(b) : V_x \to V_x$ *for our* $*$*-operation, and the second factor is unitary, i.e. it satisfies* $\mathcal{U}(b)^* = \mathcal{U}(b)^{-1}$.

The proof is obvious.

There is also a generalization of the polar decomposition of parallel transporters along finite paths C

$$\mathcal{T}(C) = \mathcal{U}(C)\mathcal{P}(C), \tag{3.33}$$
$$\mathcal{U}(C) : V_x \to V_y, \tag{3.34}$$
$$\mathcal{P}(C) : V_x \to V_x, \tag{3.35}$$

but for finite paths $C : x \to y$ the factor $\mathcal{P}(C)$ satisfies neither $\mathcal{P}(C)^* = \mathcal{P}(C)$ nor the composition law $\mathcal{P}(C_2)\mathcal{P}(C_1) = \mathcal{P}(C_2 \circ C_1)$. Instead, we demand that $\mathcal{P}(C)$ is selfadjoint for *infinitesimal* paths $b : x \to x+\delta x$, and the unitary parallel transporters $\mathcal{U}(C)$ obey the composition law $\mathcal{U}(C_2)\mathcal{U}(C_1) = \mathcal{U}(C_2 \circ C_1)$. This fixes the decomposition. A general formula for $\mathcal{P}(C)$ will be presented below.

The unitary factor $\mathcal{U}(C)$ is defined by reference to the vector potential $\boldsymbol{A}_\mu(x)$

$$\boldsymbol{U}(C) = T\exp\left(-\int_C \boldsymbol{A}_\mu(x)dx^\mu\right), \tag{3.36}$$

where T is ordering with respect to the parameter τ. The following theorem states a formula for $\boldsymbol{P}(C)$ [32].

Theorem 3.2 *Given the path C parametrized by* $\tau \in [\tau_f, \tau_i]$, *write* $\boldsymbol{U}[\tau_2, \tau_1]$ *for the unitary parallel transporters along the piece of C from* $C(\tau_1)$ *to* $C(\tau_2)$. *Define the covariant line integral*

$$\int_C \boldsymbol{E}_\mu(x)Dx^\mu := \int_{\tau_i}^{\tau_f} \boldsymbol{U}[\tau_f, \tau_i]^{-1}\boldsymbol{E}_\mu(x(\tau))\boldsymbol{U}[\tau_f, \tau_i]d\tau. \tag{3.37}$$

Then

$$\boldsymbol{P}(C) = T\exp\left(-\int_C \boldsymbol{E}_\mu(x)Dx^\mu\right). \tag{3.38}$$

Proof 3.1 *The path C can be decomposed into infinitesimal pieces* $C = b_N \circ \ldots b_1, N \to \infty$. *Inserting the polar decomposition for the infinitesimal pieces, one obtains the formula*

$$\boldsymbol{T}(C) = \boldsymbol{U}(b_N)\boldsymbol{P}(b_N)\ldots\boldsymbol{P}(b_2)\boldsymbol{U}(b_1)\boldsymbol{P}(b_1), \tag{3.39}$$

in the limit $N \to \infty$. The $\boldsymbol{P}$-factors can be pushed to the right, using $\boldsymbol{PU} = \boldsymbol{UP}'$, where $\boldsymbol{P}' := \boldsymbol{U}^{-1}\boldsymbol{PU}$. As a result one arrives at formula (3.38).

Let us now turn our attention to gauge transformations. Local gauge transformations $\boldsymbol{S}(x)$ are linear transformations of moving frames,

$$e_\alpha(x) \to e'_\alpha(x) = e_\beta(x) S^\beta{}_\alpha(x). \tag{3.40}$$

Parallel transport matrices $\boldsymbol{T}(C)$ along paths C from x to y transform according to

$$\boldsymbol{T}(C) \to \boldsymbol{T}(C)' = \boldsymbol{S}(y)^{-1}\boldsymbol{T}(C)\boldsymbol{S}(x). \tag{3.41}$$

The transformation behaviour of the pieces of the vector potential is given by the next theorem.

Theorem 3.3 (Tranformation laws) *Under a local gauge transformation $\boldsymbol{S}(x) \in G = SL(2, \mathbb{C})$ the pieces of the vector potential $\boldsymbol{B}_\mu(x)$ transform according to*

$$\begin{aligned} \boldsymbol{E}_\mu(x) \to \boldsymbol{E}'_\mu(x) &= \boldsymbol{S}^{-1}(x)\boldsymbol{E}_\mu(x)\boldsymbol{S}(x), & (3.42)\\ \boldsymbol{A}_\mu(x) \to \boldsymbol{A}'_\mu(x) &= \boldsymbol{S}^{-1}(x)\boldsymbol{A}_\mu(x)\boldsymbol{S}(x) + \boldsymbol{S}(x)^{-1}\partial_\mu \boldsymbol{S}(x). & (3.43) \end{aligned}$$

Proof 3.2 *Combining formula (3.41) and (3.27), we arrive at (3.42).*

We note that $\boldsymbol{E}_\mu(x)$, which will later be identified with the spinorial form of the vierbein, transforms homogeneously.

3.3 Holonomy groups

Consider a gauge theory with possibly nonunitary parallel transporters $\mathcal{T}(C)$. In this case two gauge groups arise. Given $x \in \mathcal{M}$, the holonomy group H_x consists of all invertible parallel transporters $\mathcal{T}(C)$ around closed paths $C : x \to x$ and their inverses, which do not need to be parallel transporters. If $\mathcal{M}$ is connected and all parallel transporters $\mathcal{T}(C), C : x \to x$ are invertible, then the holonomy groups for different x are isomorphic. Select $\hat{x}$, write $H := H_{\hat{x}}$, and call this the holonomy group for short. In the continuum, i.e. for space time *manifolds*, H must be a Lie group.

Let us mention that a gauge theory involving noninvertible parallel transporters does not lead to holonomy groups at all, but to semigroups which may, moreover, depend on $\hat{x}$.

Definition 3.1 *Let H be a holonomy group. A $*$-operation in H is defined as a map $H \to H$ which takes $\mathcal{T}(C) \mapsto \mathcal{T}(C)^* := \mathcal{T}(-C)$.*

Note that the $*$-operation is an involutive antiautomorphism of H, i.e. $g^{**} = g$ and $(g_1 g_2)^* = g_2^* g_1^*$.

The unitary parallel transporters $\mathcal{U}(C)$ of section 3.2 around closed loops $C : \hat{x} \to \hat{x}$ form the unitary gauge group G. They obey $\mathcal{U}(C)^* = \mathcal{U}(C)^{-1}$. Again, G is independent, modulo isomorphisms, of the choice of $\hat{x}$.

It is not obvious that $G \subset H$, but under weak conditions it is true, as we shall show in theorem 3.5. To prepare for it, we introduce yet another group L which we call the loop group. Its elements are composed of parallel transporters, either $\mathcal{U}(C_i)$ or $\mathcal{T}(C_i)$ along pieces $C_1, \ldots, C_n$ of a closed path $C = C_n \circ \ldots \circ C_1 : \hat{x} \to \hat{x}$. Clearly one has $G \subseteq L$ and $H \subseteq L$. Theorem 3.5 asserts that under certain conditions $H = L$ and therefore $G \subseteq H$. Let us assume that this is the case. Then the Lie algebra of G can be characterized as a subalgebra of the Lie algebra of H as follows.

The $*$-operation $\mathcal{T}(C) \to \mathcal{T}(C)^* = \mathcal{T}(-C)$ induces an automorphism of the holonomy group

$$\begin{aligned} \Theta &: H \to H \\ g &\to \Theta(g) := g^{*-1}. \end{aligned} \tag{3.44}$$

Actually Θ is an involutive automorphism, i.e.

$$\Theta(g_1 g_2) = \Theta(g_1)\Theta(g_2) \quad \text{and} \tag{3.45}$$
$$\Theta^2 = \mathbf{1}. \tag{3.46}$$

This automorphism passes to an involutive automorphism, also denoted by Θ of the Lie algebra $\mathfrak{h}$ of H. Due to (3.46) Θ induces a split

$$\mathfrak{h} = \mathfrak{u} + \mathfrak{p} \tag{3.47}$$

where $\mathfrak{u}$ consists of elements X of $\mathfrak{h}$ with $\Theta(X) = +X$ and $\mathfrak{p}$ consists of those elements with $\Theta(X) = -X$. As a result we find that $\mathfrak{u}$ is a subalgebra of the Lie algebra of the holonomy group, i.e.

$$[\mathfrak{u}, \mathfrak{u}] \subset \mathfrak{u} \tag{3.48}$$

and that $ad(\mathfrak{u})$ leaves $\mathfrak{p}$ invariant, i.e.

$$[\mathfrak{u}, \mathfrak{p}] \subset \mathfrak{p}. \tag{3.49}$$

Since

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{u}, \tag{3.50}$$

we see that $\mathfrak{u}$ is actually a symmetric subalgebra of $\mathfrak{h}$ with respect to Θ.

The elements g of the subgroup with Lie algebra $\mathfrak{u}$ satisfy $\Theta(g) = g$, i.e. they are unitary in the sense that

$$g^* = g^{-1}. \tag{3.51}$$

Thus, $\mathfrak{u}$ is the Lie algebra of G.

Later, when we shall consider gauge theory on a graph, we need a decomposition on the group level. It is provided by the next theorem [34].

Theorem 3.4 (Polar decomposition) *Let $\mathcal{T}(C) \in \mathcal{N} \subset H$, where $\mathcal{N}$ is a sufficiently small neighbourhood of the identity in H. Suppose that there is an involutive antiautomorphism $g \to g^*$ of H, which passes to an involutive antiautomorphism of the Lie algebra of H.*

Then $\mathcal{T}(C)$ can be uniquely represented in the form

$$\mathcal{T}(C) = \mathcal{U}(C)\mathcal{P}(C), \tag{3.52}$$

where $\mathcal{U}(C)$ and $\mathcal{P}(C)$ satisfy

$$\mathcal{U}(C)^* = \mathcal{U}(C)^{-1} \tag{3.53}$$

$$\mathcal{P}(C)^* = \mathcal{P}(C), \tag{3.54}$$

i.e. the first factor $\mathcal{U}(C)$ is unitary, the second factor $\mathcal{P}(C)$ self adjoint, and both are close to the identity.

The restriction to a neighbourhood of the identity may be unwelcome. Conditions for the existence of polar decomposition in a general system theoretic context were examined in [10].

Proof 3.3 *Let $\mathcal{N}$ be a sufficiently small neighbourhood of the identity in H such that for $\mathcal{T}(C) \in \mathcal{N}$ we have a unique representation $\mathcal{T}(C)^*\mathcal{T}(C) = e^X, X \in \mathfrak{h}$ small. Then also $\mathcal{T}(C)^*\mathcal{T}(C))^{1/2} = e^{X/2}$ and within $\mathcal{N}$ there are uniquely determined elements*

$$\mathcal{P}(C) := (\mathcal{T}(C))^*\mathcal{T}(C))^{1/2} \tag{3.55}$$

$$\mathcal{U}(C) := \mathcal{T}(C)(\mathcal{T}(C))^*\mathcal{T}(C))^{-1/2} \tag{3.56}$$

with $\mathcal{T}(C) = \mathcal{U}(C)\mathcal{P}(C)$. Clearly, one finds $\mathcal{P}(C)^* = \mathcal{P}(C)$ and $\mathcal{U}(C)^* = \mathcal{U}(C)^{-1}$.

Let us consider an example.

Example 3.1 *Let $H = SL(n, \mathbb{C}), \Theta(g) := g^{\dagger -1}$, where $\dagger$ denotes the Hermitian adjoint of a matrix. Then, Θ is an automorphism, and $\Theta^2 = \mathbf{1}$.*

$g \in G$ if $g = g^{\dagger -1}$, i.e. $g^{-1} = g^\dagger$, i.e. g is unitary in the conventional sense. Hence, $G = SU(n)$.

Suppose g satisfies

$$\Theta(g) = g^{-1}, \textit{ i.e. } \quad g^\dagger = g. \tag{3.57}$$

As a consequence, the $\mathcal{P}$ factor is a positive-definite Hermitian matrix. Thus, our polar decomposition is then just the ordinary polar decomposition of a matrix.

We shall use this example in chapter 7, where we interpret the Higgs as a parallel transporter along an extra dimension.

Note that in the example the involution is just the well-known Cartan involution, i.e. an involution whose fixed point set is the Lie algebra of the maximal compact subgroup $SU(2)$ of H.

We emphasize that in general our involution is not equal to the Cartan involution. For instance, when we treat gravity, the unitary gauge group is the noncompact Lorentz group.

Definition 3.2 *Let H be a group which is equipped with an involutive automorphism Θ and let $g^* := \Theta(g)^{-1}$. A $*$-representation of H is a representation of H given by operators $\tau(g) : V \to V$ where V is a real or complex vector space which is equipped with a nondegenerate bilinear or sesquilinear form $\langle\, ,\, \rangle$ such that*

$$\langle v, \tau(g)w\rangle = \langle \tau(g^*)v, w\rangle. \tag{3.58}$$

Unitary representations are the special case associated with the trivial automorphism $\Theta(g) = g$.

Our $*$-operation is an algebraic operation $\mathcal{T}(C)^* = \mathcal{T}(-C)$. We are interested in situations where this is at the same time the adjoint map between vector spaces with a bilinear or sesquilinear form, respectively.

Theorem 3.5 *Suppose that $\mathcal{M}$ is connected. Given parallel transporters $\mathcal{T}(C)$ in a vector bundle $\mathcal{V}$ over $\mathcal{M}$ with fibers $V_x \approx V = V_{\hat{x}}$, suppose that the induced representation of the holonomy group H on V can be made into a $*$-representation by a choice of a bilinear or sesquilinear form on V. Then,*

i.) if $G \subset H$ then the fibers can be equipped with bilinear or sesquilinear forms $\langle , \rangle_x$, respectively, such that

$$\langle \mathcal{U}(C)v, \mathcal{U}(C)w\rangle_y = \langle v, w\rangle_x \tag{3.59}$$

for all $x, y \in \mathcal{M}, v, w \in V_x$ and all paths $C : x \to y$.

ii.) If H and G are simply connected Lie groups, then $L = H$ and $G \subset H$. Moreover,

$$\langle v, \mathcal{T}(C)w\rangle_y = \langle \mathcal{T}(C)^* v, w\rangle_x \tag{3.60}$$

for all paths $C : x \to y$ between abritary sites $x, y \in \mathcal{M}$ and all $v \in V_y, w \in V_x$.

Note that eq. (3.59) is a special case of (3.60).

In general relativity we are mainly interested in real vector spaces with an indefinite bilinear form. We shall use the theorem with the following

Lemma 3.1 *Suppose the real vector space V carries a representation of the holonomy group H with automorphism $g \to \Theta(g) = g^{*-1}$. Let V' be the dual space. Then $V \otimes V'$ can be equipped with a bilinear form to make it into a $*$-representation space.*

Proof 3.4 *Proof of theorem 3.5:*

Given x, choose a path $C : x \to \hat{x}$. Define the bilinear or sesquilinear form $\langle , \rangle_x$ in V_x by

$$\langle v, w\rangle_x := \langle \mathcal{U}(C)v, \mathcal{U}(C)w\rangle_{\hat{x}}, \tag{3.61}$$

where $\mathcal{U}(C)$ are unitary parallel transporters introduced in section 3.2. First we show that the scalar product does not depend on the choice of the path C. To see this let C' be another path from x to $\hat{x}$. Then $L = C \circ (-C') : \hat{x} \to \hat{x}$ is a closed path, therefore $\mathcal{U}(L)$ is an element of the unitary gauge group with $\mathcal{U}(L)^* = \mathcal{U}(L)^{-1}$. As $\mathcal{U}(-C)\mathcal{U}(C) = \mathbf{1}$ it follows that $\mathcal{U}(C) = \mathcal{U}(C)\mathcal{U}(-C')\mathcal{U}(C') = \mathcal{U}(L)\mathcal{U}(C')$. Consequently,

$$\begin{aligned}\langle \mathcal{U}(C)v, \mathcal{U}(C)w\rangle_{\hat{x}} &= \langle \mathcal{U}(L)\mathcal{U}(C')v, \mathcal{U}(L)\mathcal{U}(C')w\rangle_{\hat{x}} \\ &= \langle \mathcal{U}(C')v, \mathcal{U}(C')w\rangle_{\hat{x}}.\end{aligned} \tag{3.62}$$

due to the $*$-property of the representation of the holonomy group and the unitarity of the parallel transporters. This proves independence of the choice of C.

To show property (3.59) consider a path $C : x \to y$ and let C' be a path from x to $\hat{x}$. Then $C_y = C' \circ (-C)$ is a path from y to $\hat{x}$. By the definition of the scalar product it follows that

$$\begin{aligned}\langle \mathcal{U}(C)v, \mathcal{U}(C)w\rangle_y &:= \langle \mathcal{U}(C_y)\mathcal{U}(C)v, \mathcal{U}(C_y)\mathcal{U}(C)w\rangle_{\hat{x}} \\ &= \langle \mathcal{U}(C')v, \mathcal{U}(C')w\rangle_{\hat{x}} =: \langle v, w\rangle_x.\end{aligned} \tag{3.63}$$

This proves part i.
Proof of part ii.
Next we prove the first assertion $G \subset H$. Let $\mathcal{N}$ be a sufficiently small neighbourhood of the identity in H such that for $\mathcal{T}(C) \in \mathcal{N}$ we have a unique representation $\mathcal{T}(C)^*\mathcal{T}(C) = e^X, X \in \mathfrak{h}$ small. Then also $(\mathcal{T}(C)^*\mathcal{T}(C))^{-1/2} = e^{-X/2} \in H$ and therefore $\mathcal{U}(C) = \mathcal{T}(C)(\mathcal{T}(C)^*\mathcal{T}(C))^{-1/2} \in H$. This shows that $LieG \subset LieH$. Since simply connected Lie groups are uniquely determined by their Lie algebra, this implies $G \subset H$.

To prove the remaining assertions, we choose a radial gauge. It suffices to restrict attention to a neighbourhood of $\hat{x}$. For every x in it, choose a path C_x from x to $\hat{x}$, and identify V_x together with its bilinear (or sesquilinear) form with $V_{\hat{x}}$ and its bilinear or sesquilinear form. Thereby parallel transporters $\mathcal{T}(C)$ and $\mathcal{U}(C)$ get identified with elements of the holonomy group and unitary gauge group, respectively. Since $G \subset H$, arbitrary products, as appear in the loop group, also become elements of H. Therefore $L \subset H$, *hence $L = H$. Finally, by the identification of parallel transporters $\mathcal{T}(C)$ for arbitrary paths C with elements of the holonomy group H, the statement (3.60) follows from the assumption that we deal with a $*$-representation of H. This completes the proof of theorem 3.5.*

Proof 3.5 (Lemma 3.1) *We show that the bilinear form*

$$\langle v \otimes \xi, w \otimes \chi \rangle := \xi(w)\chi(v) \qquad v, w \in V,\ \xi, \chi \in V' \tag{3.64}$$

on $V \otimes V'$ possesses the $*$-property, i.e. we have to show that

$$\langle v \otimes \xi, (\tau^{\otimes}(g))(w \otimes \chi) \rangle = \langle (\tau^{\otimes}(g^*))(v \otimes \xi), w \otimes \chi \rangle, \tag{3.65}$$

where the representation $\tau^{\otimes}(g) : V \otimes V' \to V \otimes V'$ is given by

$$(\tau^{\otimes}(g))(v \otimes \xi) := \tau(g)v \otimes \tau'(g)\xi, \tag{3.66}$$

and τ' denotes the representation carried by the dual space V', which is defined as

$$(\tau'(g)\xi)(v) := \xi(\tau(g^*)v), \qquad v \in V, \xi \in V'. \tag{3.67}$$

A simple calculation yields

$$\begin{aligned} \langle v \otimes \xi, (\tau^{\otimes}(g))(w \otimes \chi) \rangle &= \langle v \otimes \xi, \tau(g)w \otimes \tau'(g)\chi \rangle \\ &= \xi(\tau(g)w)\chi(\tau(g^*)v) \\ &= \langle \tau(g^*)v \otimes \tau'(g^*)\xi, w \otimes \chi \rangle \\ &= \langle (\tau^{\otimes}(g^*)(v \otimes \xi), w \otimes \chi \rangle. \end{aligned} \tag{3.68}$$

q.e.d.

We saw above that parallel transporters $\mathcal{T}(C)$ and $\mathcal{U}(C)$ can be identified with elements of the holonomy group H and unitary gauge group G, respectively. Therefore the polar decomposition of vector potentials (3.30) may be restated by using the antiautomorphism of definition 3.1 which passes to the Lie algebra of H

$$\boldsymbol{B}_\mu(x) = \boldsymbol{E}_\mu(x) + \boldsymbol{A}_\mu(x) \tag{3.69}$$

with

$$\boldsymbol{E}_\mu(x)^* = +\boldsymbol{E}_\mu(x) \tag{3.70}$$

$$\boldsymbol{A}_\mu(x)^* = -\boldsymbol{A}_\mu(x). \tag{3.71}$$

3.4 Generalized metricity

Traditionally, one demands in general relativity that parallel transport of tangent vectors preserves a metric.

In the approach based on nonunitary parallel transporters $\mathcal{T}(C)$ there is no such demand.

Instead one defines the $*$-operation on parallel transporters by $\mathcal{T}(C)^* = \mathcal{T}(-C)$ and one uses this to define a metric connection.

We assume that the parallel transport $\mathcal{T}(C)$ on some space of spinors $V_x \ni \psi(x)$ is defined. We write V'_x for the space of real-linear maps

$$f : V_x \to \mathbb{C}, \quad v \to f(v). \tag{3.72}$$

Both complex linear and antilinear maps will be of interest later on. Parallel transport of fibers V_x passes to parallel transport of fibers V'_x in a canonical way,

$$(\mathcal{T}(C)f)(v) := f(\mathcal{T}(-C)v) \tag{3.73}$$

for $C : x \to y$, and therefore also to $V_x \otimes V'_x$.

Given a choice of moving frame, the piece $\boldsymbol{E}_\mu(x)$ of the vector potential determined by $\mathcal{T}(C)$ according to eq.(3.27) defines a linear map $V_x \to V_x$. The space $End(V_x)$ of such linear map is canonically isomorphic to $V_x \otimes V'_x$, because $u \otimes f$ defines a map $v \to uf(v)$. Therefore we may regard $\boldsymbol{E}_\mu(x)$ as an element

$$\boldsymbol{E}_\mu(x) \in V_x \otimes V'_x. \tag{3.74}$$

Now we proceed to construct a possibly degenerate metric on complexified tangent space $\mathbb{C}T_x\mathcal{M}$. In the general context in which we are working so far, there is no guarantee that it will have the right signature for general relativity, nor that it defines parallel transport of real tangent vectors. To ascertain this, the spinor space and the holonomy group will have to be chosen appropriately, as we shall see in detail.

Combining theorem 3.5 and lemma 3.1, $V_x \otimes V'_x$ gets equipped with a nondegenerate bilinear form such that the $*$-property (3.60) holds. The same holds therefore for the unitary factors $\mathcal{U}(C)$ in the generalized polar decomposition. Unitarity reads $\mathcal{U}(C)^* = \mathcal{U}(C)^{-1}$, consequently

$$\langle \mathcal{U}(C)w, \mathcal{U}(C)z\rangle_y = \langle w, z\rangle_x, \tag{3.75}$$

where $\langle\, , \rangle_x$ denotes the bilinear form on $V_x \otimes V'_x$.

Since $\boldsymbol{E}_\mu(x) \in End(V_x) \approx V_x \otimes V'_x$, the 1-form $\boldsymbol{E}(x) := \boldsymbol{E}_\mu(x)dx^\mu$ defines a map

$$\boldsymbol{E}(x) : T_x\mathcal{M} \to V_x \otimes V'_x \tag{3.76}$$

from the tangent space of $\mathcal{M}$ to the space of vectors $v \in V_x \otimes V'_x$. This furnishes a bilinear form on $T_x\mathcal{M}$, i.e. a metric, via

$$g_{\mu\nu}(x) := \langle \partial_\mu, \partial_\nu \rangle_x := \langle \boldsymbol{E}(\partial_\mu), \boldsymbol{E}(\partial_\nu)\rangle_x. \tag{3.77}$$

If the range of the map obeys

$$\boldsymbol{E}(x)\,[T_x\mathcal{M}] = V_x \otimes V'_x, \tag{3.78}$$

then $\boldsymbol{E}$ identifies $V_x \otimes V'_x$ with the tangent space $T_x\mathcal{M}$ and parallel transport in the space of vectors passes to a metric preserving parallel transport of tangent vectors. In general relativity, the condition (3.78) is satisfied.

It is of interest to study the case

$$E(x)\,[T_x\mathcal{M}] =: W_x \subset V_x \otimes V'_x. \tag{3.79}$$

This will be done in appendix B.

3.5 Nonunitary parallel transport of Dirac spinors

Let us now apply our general formalism developed so far to parallel transport of Dirac spinors. We saw in section 3.1 that parallel transport of four-vectors determines the parallel transport of Dirac spinors. Conversely, if the parallel transport of Dirac spinors is given, four-vectors which can be made from Dirac spinors can also be parallel transported. But to obtain the parallel transport of tangent vectors ∂_μ to $\mathcal{M}$, tangent vectors need to be identified with four-vectors $v = (v^\alpha)$. This requires the vierbein e^α_μ. Its square yields the metric $g_{\mu\nu}(x) = e^\alpha_\mu(x) e^\beta_\nu(x) \eta_{\alpha\beta}$.

We propose to incorporate the vierbein into the connection, defining a new vector potential

$$\begin{aligned} \boldsymbol{B}_\mu(x) &:= \frac{1}{2\mathfrak{l}} e^\alpha_\mu(x) \boldsymbol{\gamma}_\alpha + \frac{1}{8} A^{\alpha\beta}{}_\mu(x) [\boldsymbol{\gamma}_\alpha, \boldsymbol{\gamma}_\beta] & (3.80) \\ &=: \boldsymbol{E}^D_\mu(x) + \boldsymbol{A}^D_\mu(x), & (3.81) \end{aligned}$$

cp. (3.21). As a difference of two vector potentials, the e^α_μ-term transforms homogeneously under Lorentz gauge transformations, as it must be. The matrices $\frac{1}{2}\boldsymbol{\gamma}_\alpha$ and $\frac{1}{4}[\boldsymbol{\gamma}_\alpha, \boldsymbol{\gamma}_\beta]$ furnish a representation of the generators $M_{\alpha 4}$ and $M_{\alpha\beta}$ of the Lie algebra $\mathfrak{so}(1,4)$

$$[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac} + \eta_{ad} M_{bc} \tag{3.82}$$

with $a, b, c, d = 0, \ldots, 4$ and $\eta_{44} = -1$. Defining

$$P_\alpha := M_{\alpha 4}/\mathfrak{l}, \tag{3.83}$$

where $\mathfrak{l}$ has dimensions of length, the de Sitter algebra takes the form

$$[M_{\alpha\beta}, M_{\gamma\delta}] = \eta_{\beta\gamma} M_{\alpha\delta} - \eta_{\alpha\gamma} M_{\beta\delta} - \eta_{\beta\delta} M_{\alpha\gamma} + \eta_{\alpha\delta} M_{\beta\gamma} \tag{3.84}$$

$$[M_{\alpha\beta}, P_\gamma] = \eta_{\beta\gamma} P_\alpha - \eta_{\alpha\gamma} P_\beta \tag{3.85}$$

$$[P_\alpha, P_\beta] = \frac{1}{\mathfrak{l}^2} M_{\alpha\beta}. \tag{3.86}$$

If we introduce

$$\boldsymbol{P}_\alpha := \frac{1}{2l}\boldsymbol{\gamma}_\alpha \quad \text{and} \quad \boldsymbol{M}_{\alpha\beta} := \frac{1}{4}[\boldsymbol{\gamma}_\alpha, \boldsymbol{\gamma}_\beta] \tag{3.87}$$

the vector potential can be rewritten as

$$\boldsymbol{B}_\mu(x) = e^\alpha_\mu(x)\boldsymbol{P}_\alpha + \frac{1}{2}A^{\alpha\beta}{}_\mu(x)\boldsymbol{M}_{\alpha\beta}. \tag{3.88}$$

We note that the incorporation of the vierbein necessitates the introduction of a length scale.

The vector potential $\boldsymbol{B}_\mu(x)$ is associated with a connection whose holonomy group is the two fold cover $Spin(1,4)$ of a de Sitter group. To identify the two pieces of the vector potential we introduce an involutive antiautomorphism,

$$\begin{aligned} \Phi \quad &: \quad \mathfrak{so}(1,4) \to \mathfrak{so}(1,4), \\ \boldsymbol{X} \quad &\to \quad \Phi(\boldsymbol{X}) := \boldsymbol{\beta}\boldsymbol{X}^\dagger\boldsymbol{\beta}^{-1}, \end{aligned} \tag{3.89}$$

where $\boldsymbol{\beta}$ is defined by (3.23). Obviously we have

$$\Phi^2 = \mathbf{1} \quad \text{and} \quad \Phi([\boldsymbol{X}_1, \boldsymbol{X}_2]) = [\Phi(\boldsymbol{X}_2), \Phi(\boldsymbol{X}_1)]. \tag{3.90}$$

Since $\Phi(\boldsymbol{\gamma}_\alpha) = \boldsymbol{\gamma}_\alpha$, cp. appendix C, the generators of the Lorentz group have negative eigenvalues and the "momentum" generators[1] positive ones. This leads to a split of the vector potential

$$\Phi(\boldsymbol{E}^D_\mu(x)) \quad = \quad +\boldsymbol{E}^D_\mu(x) \tag{3.91}$$

$$\Phi(\boldsymbol{A}^D_\mu(x)) \quad = \quad -\boldsymbol{A}^D_\mu(x). \tag{3.92}$$

The antiautomorphism Φ passes to an involutive antiautomorphism, also denoted by Φ of the two fold cover of the de Sitter group $Spin(1,4)$ with $\Phi((g_1 g_2)) = \Phi(g_2)\Phi(g_1)$ for $g_1, g_2 \in Spin(1,4)$. Due to (3.91) and (3.92) one concludes

$$\Phi(g) = g^{-1} \Leftrightarrow g \in SL(2,\mathbb{C}) = Spin(1,3). \tag{3.93}$$

Using the vector potential (3.80) one can construct a de Sitter parallel transport of Dirac spinors along paths $C : x \to y$. Furthermore, the antiautomorphism Φ enables one to define a $*$-operation $\mathcal{T}^D(C) \to \mathcal{T}^D(C)^*$

$$\mathcal{T}^D(C)^* \quad : \quad V^D_y \to V^D_x \tag{3.94}$$

$$\mathcal{T}^D(C)^* e_a(y) \quad : \quad = e_b(x)\Phi\left(\boldsymbol{T}(C)^D\right)^b{}_a \tag{3.95}$$

where $\boldsymbol{T}^D(C)$ is the de Sitter parallel transport matrix along a path $C : x \to y$.

[1] Note that $\boldsymbol{P}_\alpha$ is not a generator of a space time symmetry, but acts on an internal vector space only.

Due to (3.93) the parallel transporters constructed by the Lorentz part of the vector potential satisfy the unitarity condition

$$\mathcal{U}^D(C)^* = \mathcal{U}^D(C)^{-1}. \tag{3.96}$$

However, in general the parallel transporters constructed from the de Sitter vector potential are not unitary

$$\mathcal{T}^D(C)^* \neq \mathcal{T}^D(C)^{-1}, \tag{3.97}$$

but permit a decomposition in two factors. More precisely, let $\mathcal{T}^D(b)$ be a de Sitter parallel transporter along an infinitesimal path $b : x \to x + \delta x$. Then it splits into a self adjoint factor

$$\begin{aligned}\mathcal{P}^D(b)e_a(x) &= e_b(x)[\delta^b{}_a - (E^D_\mu(x))^b{}_a \delta x^\mu] & (3.98)\\ &= e_b(x)\Phi(\boldsymbol{P}(b))^b{}_a = \mathcal{P}^D(b)^* e_a(x) & (3.99)\end{aligned}$$

involving the vierbein and a unitary part $\mathcal{U}^D(b)^* = \mathcal{U}^D(b)^{-1}$ involving the spin connection, respectively:

$$\mathcal{T}^D(b) = \mathcal{U}^D(b)\mathcal{P}^D(b). \tag{3.100}$$

Moreover we have

$$\langle \mathcal{T}(C)^*\psi(y), \phi(x)\rangle_x = \langle \psi(y), \mathcal{T}(C)\phi(x)\rangle_y, \tag{3.101}$$

where $\langle\, ,\, \rangle_x$ is the indefinite scalar product in V^D_x, cp. (3.23). Note that the scalar product is not invariant under de Sitter parallel transport. Due to (3.100) and (3.101) it follows that $\langle\, ,\, \rangle_x$ is invariant under Lorentz parallel transport, as it should be.

We see that in our approach there is no demand of a metric connection. Instead one defines a $*$-operation on parallel transporters. In this way a scalar product is singled out which obeys (3.101). As a consequence of (3.100) and $\mathcal{U}^D(b)^* = \mathcal{U}^D(b)^{-1}$, the scalar product is invariant under unitary parallel transport.

Let us mention that there is an alternative to the de Sitter group $SO(1,4)$. Actually, it is also possible to choose

$$\boldsymbol{E}^D_\mu(x) := \frac{1}{2l} e^\alpha_\mu(x)\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_5. \tag{3.102}$$

This choice leads to the (anti)-de Sitter group with Lie algebra $\mathfrak{so}(2,3)$, which offers some advantages. First, the Majorana condition on Dirac spinors is left invariant under anti- de Sitter parallel transport, while it is not under de Sitter parallel transport. Later it will turn out that Weyl fermions can only be accomodated with a holonomy group $H = Spin(2,3)$.

Let us remind ourselves that the Majorana condition on Dirac spinors is defined as

$$\psi^{\boldsymbol{C}} := \boldsymbol{C}\bar{\psi}^t = \boldsymbol{C}\beta^t\psi^{c.c.} \stackrel{!}{=} \psi, \tag{3.103}$$

where $\boldsymbol{C}$ denotes the charge conjugation matrix and $c.c$ means complex conjugation.

Invariance under parallel transport requires

$$(\boldsymbol{T}(C)\psi)^{\boldsymbol{C}} = \boldsymbol{T}(C)\psi^{\boldsymbol{C}}. \tag{3.104}$$

(3.104) can only be satisfied if

$$\boldsymbol{C}\boldsymbol{\beta}^t\boldsymbol{E}_\mu^{Dc.c.}(x) = \boldsymbol{E}_\mu^D(x)\boldsymbol{C}\boldsymbol{\beta}^t. \tag{3.105}$$

Using the relations of the appendix C, it turns out that (3.104) is only satisfied for the choice (3.102), since we have

$$\boldsymbol{C}\boldsymbol{\beta}^t(\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_5)^{c.c.} = \boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_5\boldsymbol{C}\boldsymbol{\beta}^t \tag{3.106}$$

but

$$\boldsymbol{C}\boldsymbol{\beta}^t(\boldsymbol{\gamma}_\alpha)^{c.c.} = -\boldsymbol{\gamma}_\alpha\boldsymbol{C}\boldsymbol{\beta}^t. \tag{3.107}$$

3.6 Einstein-Hilbert action

We shall use the language of forms and write the vector potential as $\boldsymbol{B}(x) := \boldsymbol{B}_\mu(x)dx^\mu$.

Next, let us compute the field strength associated with the nonunitary parallel transporters $\mathcal{T}(C)$. The general formula is

$$\boldsymbol{F}^{\mathcal{T}} = d\boldsymbol{B} + \boldsymbol{B}\wedge\boldsymbol{B} = \frac{1}{2}\boldsymbol{F}^{\mathcal{T}}_{\mu\nu}\, dx^\mu \wedge dx^\nu, \tag{3.108}$$

where $\boldsymbol{F}^{\mathcal{T}}_{\mu\nu}$ is given by

$$\begin{aligned}
\boldsymbol{F}^{\mathcal{T}}_{\mu\nu} &= \partial_\mu\boldsymbol{B}_\nu(x) - \partial_\nu\boldsymbol{B}(x) + [\boldsymbol{B}_\mu(x), \boldsymbol{B}_\nu(x)] && (3.109)\\
&= \frac{1}{2}\left(\partial_{[\mu}e^\alpha{}_{\nu]}(x) + A^\alpha{}_{\beta[\mu}(x)e^\beta{}_{\nu]}(x)\right)\boldsymbol{P}_\alpha + && (3.110)\\
&+ \frac{1}{2}\left(\partial_{[\mu}A^\alpha{}_{\beta\nu]}(x) + A^\alpha{}_{\gamma[\mu}(x)A^\gamma{}_{\beta\nu]}(x) + \frac{1}{l^2}e^\alpha{}_{[\mu}(x)e^\beta{}_{\nu]}(x)\right)\boldsymbol{M}_{\alpha\beta}.
\end{aligned}$$

We find that the field strength associated with the de Sitter parallel transport decomposes in two different parts. Apart from the unitary field strength, i.e. the Lorentz curvature $R^{\alpha\beta}{}_{\mu\nu}(x)$

$$F^{\mathcal{U}\alpha\beta}{}_{\mu\nu} := \partial_{[\mu}A^\alpha{}_{\beta\nu]}(x) + A^\alpha{}_{\gamma[\mu}(x)A^\gamma{}_{\beta\nu]}(x) =: R^{\alpha\beta}{}_{\mu\nu}(x), \tag{3.111}$$

it includes the torsion tensor

$$F^{\mathcal{T}\alpha 4}{}_{\mu\nu}(x) = \frac{1}{\mathfrak{l}} T^{\alpha}{}_{\mu\nu}(x) = \frac{1}{\mathfrak{l}}(\partial_{[\mu} e^{\alpha}{}_{\nu]}(x) + A^{\alpha}{}_{\beta[\mu}(x) e^{\beta}{}_{\nu]}(x)), \tag{3.112}$$

which appears as the coefficient of the momentum generator. We can rewrite $\boldsymbol{F}^{\mathcal{T}}(x)$ as

$$\boldsymbol{F}^{\mathcal{T}}_{\mu\nu}(x) = \boldsymbol{F}^{\mathcal{U}}_{\mu\nu}(x) + \boldsymbol{F}^{e}_{\mu\nu}(x) + \boldsymbol{T}_{\mu\nu}(x), \tag{3.113}$$

where we have defined $\boldsymbol{F}^{e}_{\mu\nu} := \frac{1}{\mathfrak{l}^2} e^{\alpha}{}_{[\mu}(x) e^{\beta}{}_{\nu]}(x) \boldsymbol{M}_{\alpha\beta}$.

The $*$-operation enables one to identify both parts of the nonunitary field strength. More precisely, we have

$$\left(\boldsymbol{F}^{\mathcal{U}}_{\mu\nu}(x) + \boldsymbol{F}^{e}_{\mu\nu}(x)\right)^* = -(\boldsymbol{F}^{\mathcal{U}}_{\mu\nu}(x) + \boldsymbol{F}^{e}_{\mu\nu}(x)) \tag{3.114}$$

$$\boldsymbol{T}_{\mu\nu}(x)^* = +\boldsymbol{T}_{\mu\nu}(x). \tag{3.115}$$

In this way, the case of vanishing torsion can be expressed as the vanishing of the even part of the nonunitary field strength

$$\boldsymbol{T}_{\mu\nu}(x) = 0 = \frac{1}{2}\left(\boldsymbol{F}^{\mathcal{T}}_{\mu\nu} + \boldsymbol{F}^{\mathcal{T}*}_{\mu\nu}\right). \tag{3.116}$$

The odd part of the field strength,

$$\boldsymbol{F}_{\mu\nu}(x) := \frac{1}{2}\left(\boldsymbol{F}^{\mathcal{T}}_{\mu\nu} - \boldsymbol{F}^{\mathcal{T}*}_{\mu\nu}\right), \tag{3.117}$$

contains the unitary field strength, i.e. the Lorentz field strength and an additional term $\boldsymbol{F}^{e}_{\mu\nu}$, steming from the nonunitary piece of the de Sitter vector potential.

It is important to note that unitary gauge transformations do not mix even and odd part of the field strength. In this way it is possible to write down gauge invariant actions by using the odd part of the field strength only.

Let $F^{\mathcal{U}}_{\mu\nu} := (F^{\mathcal{U}\alpha\beta}{}_{\mu\nu})$ and $F^{e}_{\mu\nu} := (F^{e\alpha\beta}{}_{\mu\nu})$. Then the Einstein-Hilbert action can be cast in a quasi Maxwellian form.

$$S_{E-H} = \frac{1}{2}\int d^4x\sqrt{-g} g^{\mu\rho} g^{\nu\sigma}\, tr(F^{\mathcal{U}}_{\mu\nu} F^{e}_{\rho\sigma}) \tag{3.118}$$

$$= -\frac{1}{16\pi k}\int d^4x\sqrt{-g}\,\mathcal{R}, \tag{3.119}$$

if we identify

$$\frac{1}{16\pi k} = \frac{1}{\mathfrak{l}^2}. \tag{3.120}$$

Thus the length scale $\mathfrak{l}$ appears to be associated with the gravitational coupling constant.

Let us prove (3.118) by a quick calculation

$$\begin{aligned} g^{\mu\rho} g^{\nu\sigma}\, tr(F^{\mathcal{U}}_{\mu\nu} F^{e}_{\rho\sigma}) &= \frac{1}{\mathfrak{l}^2} g^{\mu\rho} g^{\nu\sigma} R^{\alpha}{}_{\beta\mu\nu} \left(e^{\beta}_{\rho} e_{\sigma\alpha} - e^{\beta}_{\sigma} e_{\rho\alpha} \right) \\ &= -\frac{2}{\mathfrak{l}^2} R^{\rho\sigma}{}_{\rho\sigma} = -\frac{2}{\mathfrak{l}^2} \mathcal{R}. \end{aligned} \tag{3.121}$$

It is interesting to note that also a cosmological constant can be included by a different choice of action. Let $F := (F^{\mathcal{U}\alpha\beta}{}_{\mu\nu} + F^{e\alpha\beta}{}_{\mu\nu})$. Since

$$\begin{aligned} g^{\mu\rho} g^{\nu\sigma}\, tr(F^{e}_{\mu\nu} F^{e}_{\rho\sigma}) &= \frac{1}{\mathfrak{l}^4} g^{\mu\rho} g^{\nu\sigma} \left(-e^{\alpha}{}_{\mu} e_{\nu\beta} + e^{\alpha}{}_{\nu} e_{\mu\beta}\right) \left(-e^{\beta}_{\rho} e_{\sigma\alpha} + e^{\beta}_{\sigma} e_{\rho\alpha} \right) \\ &= \frac{2}{\mathfrak{l}^4} \left(\eta^{\alpha\beta} \eta_{\beta\alpha} - \delta^{\alpha}{}_{\alpha} \delta^{\beta}{}_{\beta} \right) = -24/\mathfrak{l}^4, \end{aligned} \tag{3.122}$$

we get

$$\frac{1}{2} \int d^4x \sqrt{-g} g^{\mu\rho} g^{\nu\sigma}\, tr(F_{\mu\nu} F^{e}_{\rho\sigma}) = - \int d^4x \sqrt{-g} \left(\frac{1}{\mathfrak{l}^2} \mathcal{R} - 2\Lambda \right), \tag{3.123}$$

where the cosmological constant Λ is

$$\Lambda := -\frac{6}{\mathfrak{l}^4}. \tag{3.124}$$

Of course, such a large cosmological constant is unacceptable. One might speculate that quantum fluctuations will cancel Λ almost completely such that a very small cosmological constant is left.

It is interesting to note that Einstein gravity including a cosmological constant can be obtained in two other ways. In fact, either one can start from a Yang-Mills type action

$$\int \mathrm{tr}(\boldsymbol{F} \wedge \star \boldsymbol{F}), \tag{3.125}$$

where $\star$ denotes the Hodge-operator and $\boldsymbol{F} = \frac{1}{2}(\boldsymbol{F}^{\mathcal{T}}_{\mu\nu} - \boldsymbol{F}^{\mathcal{T}*}_{\mu\nu})$ the odd part of the nonunitary field strength, or from an action which is similar to that of a topological field theory, i.e. an action involving no Hodge operator,

$$\int \mathrm{tr}(\boldsymbol{D} \wedge \boldsymbol{F}), \tag{3.126}$$

where $\boldsymbol{D}$ is defined as $\boldsymbol{D} := \boldsymbol{F}\boldsymbol{\gamma}_5$ and tr denotes a trace over Dirac indices.

Let us first focus on the Yang-Mills formulation. Using the well-known formulae

$$dx^{\mu} \wedge dx^{\nu} \wedge dx^{\rho} \wedge dx^{\sigma} = \epsilon^{\mu\nu\rho\sigma} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \epsilon^{\mu\nu\rho\sigma} d^4x, \tag{3.127}$$

$$\star\, dx^{\rho} \wedge dx^{\sigma} = \det(e) g^{\rho\rho'} g^{\sigma\sigma'} \epsilon_{\rho'\sigma'\tau\upsilon} dx^{\tau} \wedge dx^{\upsilon} \tag{3.128}$$

and

$$\mathrm{tr}(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = (\eta_{\alpha\beta}\eta_{\gamma\delta} - \eta_{\alpha\gamma}\eta_{\beta\delta} + \eta_{\alpha\delta}\eta_{\beta\gamma}) \tag{3.129}$$

we arrive at

$$\int \mathrm{tr}(\boldsymbol{F} \wedge \star\boldsymbol{F}) \propto \int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma}) \tag{3.130}$$

where on the right hand side tr means $\mathrm{tr}(F_{\mu\nu}F_{\rho\sigma}) = F^{\alpha}{}_{\beta\mu\nu}F^{\beta}{}_{\alpha\rho\sigma}$. The Yang-Mills action splits in three different parts

$$\begin{aligned}\int \mathrm{tr}(\boldsymbol{F} \wedge \star\boldsymbol{F}) &\propto \int d^4x\sqrt{-g}g^{\mu\rho}g^{\nu\sigma}\mathrm{tr}\Big(F^{\mathcal{U}}_{\mu\nu}F^{\mathcal{U}}_{\rho\sigma} + \\ &+ 2F^{\mathcal{U}}_{\mu\nu}F^{e}_{\rho\sigma} + F^{e}_{\mu\nu}F^{e}_{\rho\sigma}\Big).\end{aligned} \tag{3.131}$$

We have already shown that the mixed term $F^{\mathcal{U}}_{\mu\nu}F^{e}_{\rho\sigma}$ leads to the Einstein-Hilbert action and the last term to a cosmological constant, respectively.

The third term is quadratic in the Lorentz curvature. Such actions have been discussed in the literature as corrections to Einstein gravity. In the low energy regime its influence can be neglected.

Let us now return to the topological formulation. Using

$$\mathrm{tr}(\gamma_5\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = -4i\epsilon_{\alpha\beta\gamma\delta} \tag{3.132}$$

and again (3.127) we obtain

$$\begin{aligned}\int \mathrm{tr}(\boldsymbol{D} \wedge \boldsymbol{F}) \propto \int d^4x\, \epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}\Big(F^{\mathcal{U}\alpha\beta}{}_{\mu\nu}F^{\mathcal{U}\gamma\delta}{}_{\rho\sigma} + \\ +2F^{\mathcal{U}\alpha\beta}{}_{\mu\nu}F^{e\gamma\delta}{}_{\rho\sigma} + F^{e\alpha\beta}{}_{\mu\nu}F^{e\gamma\delta}{}_{\rho\sigma}\Big).\end{aligned} \tag{3.133}$$

Now the first term is a topological invariant, more precisely it is the integrand of the Gauss-Bonnet topological invariant and therefore it does not contribute to the dynamics, at least classically. The third term can be readily identified with a cosmological constant, wheras the second term again yields the Einstein-Hilbert action.

3.7 Matter action

For a massless Dirac field, the Lagrangian is

$$\mathcal{L}_{matter} = \frac{1}{2}(\bar{\psi}e^{\mu}_{\alpha}\gamma^{\alpha}D_\mu\psi - D_\mu\bar{\psi}e^{\mu}_{\alpha}\gamma^{\alpha}\psi), \tag{3.134}$$

where D_μ denotes the the ordinary covariant derivative defined as

$$D_\mu = \partial_\mu + \frac{1}{8}A^{\alpha\beta}{}_{\mu}[\gamma_\alpha, \gamma_\beta]. \tag{3.135}$$

In a gauge theory with nonunitary parallel transporters it comes as no surprise that two different covariant derivatives appear. First there is the covariant derivative associated with the nonunitary parallel transporters, which is defined as

$$\mathcal{T}(C)e_\alpha(x) =: e_\alpha(x+\delta x) - D^{\mathcal{T}}_\mu e_\alpha(x)\delta x^\mu. \tag{3.136}$$

Using (3.80), we obtain

$$D^{\mathcal{T}}_\mu = \partial_\mu + \frac{1}{2\ell} e^\alpha_\mu \gamma_\alpha + \frac{1}{8} A^{\alpha\beta}{}_\mu [\gamma_\alpha, \gamma_\beta]. \tag{3.137}$$

$D^{\mathcal{T}}_\mu$ is covariant under H-transformations.

In addition, we have the covariant derivative associated with unitary parallel transport

$$\mathcal{U}(C)e_\alpha(x) = e_\alpha(x+\delta x) - D^{\mathcal{U}}_\mu e_\alpha(x)\delta x^\mu, \tag{3.138}$$

with

$$D^{\mathcal{U}}_\mu = \partial_\mu + \frac{1}{8} A^{\alpha\beta}_\mu [\boldsymbol{\gamma}_\alpha, \boldsymbol{\gamma}_\beta]. \tag{3.139}$$

$D^{\mathcal{U}}_\mu$ is just the ordinary covariant derivative (3.135), as it should be. Obviously, it is covariant under G- but not H- transformations.

Invoking the involutive automorphism Θ defined in section 3.3, (3.44), the unitary covariant derivative appears as the even part of the H-covariant derivative

$$D^+_\mu = \frac{1}{2}(D^{\mathcal{T}}_\mu + \Theta(D^{\mathcal{T}}_\mu)) = D^{\mathcal{U}}_\mu. \tag{3.140}$$

This motivates to introduce another kind of covariant derivative, which is odd under the automorphism

$$D^-_\mu = \frac{1}{2}(D^{\mathcal{T}}_\mu - \Theta(D^{\mathcal{T}}_\mu)) = \frac{1}{2\ell} e^\alpha_\mu \boldsymbol{\gamma}_\alpha =: D^e_\mu. \tag{3.141}$$

The three different derivatives can be employed to express the different pieces of the de Sitter field strength, analogously to the standard formalism, i.e. $[D^{\mathcal{U}}_\mu, D^{\mathcal{U}}_\nu]$ and $[D^-_\mu, D^-_\nu]$ yield $\boldsymbol{F}^{\mathcal{U}}_{\mu\nu}$ and $\boldsymbol{F}^e_{\mu\nu}$, respectively, whereas $D^{\mathcal{U}}_{[\mu}, D^-_{\nu]}$ gives $\boldsymbol{T}_{\mu\nu}$.

We saw that the Einstein-Hilbert action could be written schematically as $F^{\mathcal{U}}F^e$. The matter action can be cast in an analogous form

$$\mathcal{L}_{matter} = \frac{\ell}{2}(\bar{\psi} D^{e\,\mu} D^{\mathcal{U}}_\mu \psi - D^{\mathcal{U}}_\mu \bar{\psi} D^{e\mu} \psi). \tag{3.142}$$

Especially, we see that the Dirac operator is the product of the odd and even part of the covariant derivative associated with nonunitary parallel transport.

Let us remind ourselves that an FF^e term yields the Einstein-Hilbert action and a cosmological constant. This might motivate to consider as a matter action term

$$\mathfrak{l}\bar{\psi}D^{e\mu}D^{\mathcal{T}}_{\mu}\psi + \ldots\ . \tag{3.143}$$

A short calculation shows that in this way apart from the ordinary Dirac action one obtains a "geometric" mass term, which appears in a purely geometric theory. Obviously, this mechanism cannot explain the mass generation of known elementary particles, since it is of order $\mathfrak{l}^{-1}$, i.e. the Planck mass. Nevertheless it is an amusing fact that the geometric mass term and the cosmological constant have the same origin.

Let us emphasize that there is another possibility to rewrite the Dirac matter action. Write $D := D_\mu dx^\mu$ and $\boldsymbol{E}(x) := \frac{1}{2\mathfrak{l}}e^{\alpha}{}_{\mu}(x)\boldsymbol{\gamma}_\alpha dx^\mu$, then we have

$$\int d^4x\, \det(e^{\alpha}_{\mu})\bar{\psi}e^{\mu}_{\alpha}\boldsymbol{\gamma}_\alpha D_\mu\psi \propto \mathfrak{l}^3 \int \bar{\psi}\boldsymbol{E}\wedge\boldsymbol{E}\wedge\boldsymbol{E}\boldsymbol{\gamma}_5\wedge D\psi. \tag{3.144}$$

(3.144) can be shown by using

$$\epsilon^{\mu\nu\rho\sigma}e^{\alpha}_{\mu}e^{\beta}_{\nu}e^{\gamma}_{\rho} \propto \epsilon^{\alpha\beta\gamma\delta}e^{\sigma}_{\delta}\det(e), \tag{3.145}$$

$$\boldsymbol{\gamma}_5\boldsymbol{\gamma}^\delta = -i/3!\epsilon^{\alpha\beta\gamma\delta}\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_\beta\boldsymbol{\gamma}_\gamma, \tag{3.146}$$

and (3.127).

It is a remarkable fact that both the Einstein-Hilbert action and the matter action are polynomial in e^{α}_{μ} and $A^{\alpha}{}_{\beta\mu}$. This might motivate to reconsider the issue of renormalizability of gravity, see also [25].

3.8 Classical equations of motion

Up to now our discussion was based on the parallel transport of Dirac spinors. It turns out, somewhat surprisingly, that also classical massive particles can be treated within the de Sitter framework.

Let us recall that in general relativity a classical point particle is described by its four-vector $u^\alpha(\tau) := \frac{dx^\alpha(\tau)}{d\tau}$, where (u^α) are the components with respect to an orthonormal basis. The equation of motion is

$$\left(\frac{D}{d\tau}u(\tau)\right)^\alpha := \frac{d}{d\tau}u^\alpha + A^{\alpha}{}_{\beta\mu}u^\beta u^\mu = 0. \tag{3.147}$$

Equivalently, the dynamic is determined by

$$(\mathcal{U}(C)u(\tau))^\alpha = u(\tau+\delta\tau)^\alpha. \tag{3.148}$$

It is interesting that within the de Sitter framework the same equation describes the motion, one has just to replace $\mathcal{U}(C)$ by $\mathcal{T}(C)$

$$(\mathcal{T}(C)u(\tau))^\alpha = u(\tau+\delta\tau)^\alpha. \tag{3.149}$$

To see this, one has to define the de Sitter parallel transport of four-vectors. We exploit the fact that the four-velocity is a future-directed, timelike four-vector, i. e. $u^0 > 0$ and $\eta(u,u) := \eta_{\alpha\beta}u^\alpha u^\beta > 0$. We identify u with an equivalence class of future-directed, lightlike five-vectors and define their de Sitter parallel transport.

Let (v^α) be the components of a future-directed, timelike four-vector, i.e.

$$(v^\alpha) \in \mathcal{C}_+ := \{(w^\alpha)|w^0 > 0 \text{ and } \eta(w,w) > 0\}. \tag{3.150}$$

Define

$$v_4 := ||v||_{1,3} = (v_\alpha \eta^{\alpha\beta} v_\beta)^{\frac{1}{2}} \quad \text{and} \quad \eta^{44} = -1. \tag{3.151}$$

Now we can define a future-directed, lightlike five-vector (v_α, v_4)

$$(v_\alpha, v_4) \in \mathcal{C}_+^{(1,4)} := \{(w_\alpha, w_4)|w_0 > 0, ||(v_\alpha, v_4)||_{1,4} := v_\alpha \eta^{\alpha\beta} v_\beta + v_4\, \eta^{44}\, v_4 = 0\}. \tag{3.152}$$

Evidently, multiplication with a positive real number yields again an element in $\mathcal{C}_+^{(1,4)}$. The resulting equivalence classes are elements (”rays”) of a real projective space

$$\mathbb{P}^{1,3} := \{[v]|v \in \mathcal{C}_+^{1,4}\}, \tag{3.153}$$

where

$$[v] := \{w|w = \lambda v, \lambda > 0\}. \tag{3.154}$$

u satisfies $||u||_{1,3} = 1$. Every four-velocity determines uniquely a ray. Since every ray contains a vector with $u^4 = 1$, the converse is also true.

Elements of $SO(1,4)$ act as pseudorotations in $\mathbb{R}^5$ and map lightlike vectors to lightlike ones. The parallel transport of rays can be defined as

$$\mathcal{T}(C)[v] := [\mathcal{T}(C)v]. \tag{3.155}$$

Let us now define the de Sitter parallel transport of five-vectors. The de Sitter vector potential can be compactly rewritten as

$$\boldsymbol{B}_\mu(x) = \frac{1}{2}\, {}^{dS}\Gamma^{ab}{}_\mu(x) \boldsymbol{M}_{ab}, \quad a,b = 0,\dots,4 \tag{3.156}$$

with

$$\boldsymbol{M}_{\alpha\beta} := \frac{1}{4}[\boldsymbol{\gamma}_\alpha, \boldsymbol{\gamma}_\beta] \quad \text{and} \quad \boldsymbol{M}_{\alpha 4} := \frac{1}{2}\boldsymbol{\gamma}_\alpha. \tag{3.157}$$

Comparison with (3.80) leads to

$${}^{dS}\Gamma^{\alpha\beta}{}_\mu(x) := A^{\alpha\beta}{}_\mu(x) \tag{3.158}$$

$${}^{dS}\Gamma^{\alpha 4}{}_\mu(x) := \mathfrak{l}^{-1} e^\alpha_\mu(x). \tag{3.159}$$

Now the parallel transport of five-vectors may be defined as

$$\begin{aligned} (\mathcal{T}(b)v)^a &:= v^a(x+\delta x) - [\partial_\mu v(x)^a + {}^{dS}\,\Gamma^a{}_{b\mu} v(x)^b]\delta x^\mu & (3.160)\\ &=: v(x+\delta x)^a - (D^{\mathcal{T}}_\mu v(x))^a \delta x^\mu. & (3.161) \end{aligned}$$

This may be rewritten

$$
\begin{aligned}
(\mathcal{T}(b)v(\tau))^\alpha &= v(\tau+\delta\tau)^\alpha - \left(\frac{D^{\mathcal{T}}}{d\tau}v\right)^\alpha \delta\tau \\
&= v(\tau+\delta\tau)^\alpha - \left(\frac{d}{d\tau}v^\alpha + A^\alpha{}_{\beta\mu}v^\beta u^\mu - \frac{e^\alpha_\mu}{\mathfrak{l}}v^4 u^\mu\right)\delta\tau \\
&= v(\tau+\delta\tau)^\alpha - \left(\frac{D^{\mathcal{U}}}{d\tau}v\right)^\alpha + \frac{1}{\mathfrak{l}}v^4 u(\tau)^\alpha \delta\tau. \qquad (3.162)
\end{aligned}
$$

Since u^α is a four-vector, we have $u_\alpha \eta^{\alpha\beta} u_\beta = 1$, and u can be identified with the five-vector $(u_\alpha, 1)$. Due to (3.147) and (3.162) it follows that

$$(\mathcal{T}(C)u(\tau))^\alpha = u(\tau+\delta\tau)^\alpha(1+\frac{1}{\mathfrak{l}}\delta\tau). \qquad (3.163)$$

Since $(1+\frac{1}{\mathfrak{l}}\delta\tau) > 0$ we get

$$\mathcal{T}(C)[u(\tau)] = [u(\tau+\delta\tau)]. \qquad (3.164)$$

As a result, the equation of motion takes the form

$$\frac{D^{\mathcal{T}}}{d\tau}[u(\tau)] = 0. \qquad (3.165)$$

For a representative of the equivalence class, the equation of motion is

$$\left(\frac{D^{\mathcal{T}}}{d\tau}u(\tau)\right)^a = u(\tau)^a. \qquad (3.166)$$

Obviously, (3.166) is de Sitter covariant.

Recall that the energy-momentum tensor of a classical point particle is determined by its four-velocity

$$T_{\mu\nu}(x) = m\int d\tau\,(-g(x))^{-1/2}\delta^4(x-C(\tau))u_\mu(\tau)u_\nu(\tau). \qquad (3.167)$$

Thus, also the de Sitter parallel transport of $T_{\mu\nu}$ is defined.

3.9 Parallel transport of Weyl spinors

Up to now we assumed that matter was described by Dirac spinors. However, it is possible to define nonunitary parallel transport also for Weyl spinors, assuming $H = Spin(2,3)$. These parallel transporters will be real linear but not complex linear. In the following we shall consider lefthanded spinors $\xi \in V^+$ for definiteness sake.

Let $\mathcal{C}$ be the operator of complex conjugation, and define the following real linear transformations of V^+

$$\boldsymbol{\rho}_\alpha \;:=\; \boldsymbol{\sigma}_\alpha \epsilon \mathcal{C}/\mathfrak{l} \tag{3.168}$$

$$\boldsymbol{\rho}_{\alpha\beta} \;:=\; \boldsymbol{\sigma}_\alpha \tilde{\boldsymbol{\sigma}}_\beta - \boldsymbol{\sigma}_\beta \tilde{\boldsymbol{\sigma}}_\alpha, \tag{3.169}$$

where ϵ is the antisymmetric tensor in two dimensions. They are Lorentz covariant in the sense that

$$\boldsymbol{S}\boldsymbol{\rho}_\alpha \boldsymbol{S}^{-1} = \boldsymbol{\rho}_\beta \Lambda^\beta{}_\alpha(\boldsymbol{S}) \tag{3.170}$$

for $\boldsymbol{S} \in SL(2,\mathbb{C})$, and satisfy the same commutation relations as $\boldsymbol{\gamma}_\alpha \boldsymbol{\gamma}_5$ and $[\boldsymbol{\gamma}_\alpha, \boldsymbol{\gamma}_\beta]$, in particular

$$[\boldsymbol{\rho}_\alpha, \boldsymbol{\rho}_\beta] = -\mathfrak{l}^{-2} \boldsymbol{\rho}_{\alpha\beta}. \tag{3.171}$$

Therefore they generate the Lie algebra $\mathfrak{so}(2,3)$. The vector potential associated with the nonunitary prallel transport of Weyl spinors is

$$\boldsymbol{B}_\mu(x) \;:=\; \frac{1}{2} e^\alpha_\mu \boldsymbol{\rho}_\alpha + \frac{1}{8} A^{\alpha\beta}{}_\mu(x) \boldsymbol{\rho}_{\alpha\beta} \tag{3.172}$$

$$=: \; \boldsymbol{E}^W_\mu(x) + \boldsymbol{A}^W_\mu(x). \tag{3.173}$$

Note that $\boldsymbol{\rho}_\alpha$ involves the operator of time-inversion $T = \epsilon\mathcal{C}$

$$\boldsymbol{\rho}_\alpha = \boldsymbol{\sigma}_\alpha T. \tag{3.174}$$

In place of eq.(3.168), (3.169) one could define

$$\boldsymbol{\rho}_\alpha \;:=\; i \boldsymbol{\sigma}_\alpha \epsilon \mathcal{C}/\mathfrak{l} \tag{3.175}$$

$$\boldsymbol{\rho}_{\alpha\beta} \;:=\; \boldsymbol{\sigma}_\alpha \tilde{\boldsymbol{\sigma}}_\beta - \boldsymbol{\sigma}_\beta \tilde{\boldsymbol{\sigma}}_\alpha. \tag{3.176}$$

They satisfy the same commutation relations. Apparently, it is not possible to accomodate the Lie algebra $\mathfrak{so}(1,4)$ here.

The $*$-operation is given by the involutive anti-automorphism

$$\Phi(\boldsymbol{X}) := \epsilon^{-1} \boldsymbol{X}^t \epsilon \tag{3.177}$$

for $\boldsymbol{X} \in \mathfrak{so}(2,3)$. It can be used to identify the two pieces of the vector potential

$$\Phi(\boldsymbol{E}^W_\mu(x)) = +\boldsymbol{E}^W_\mu(x) \tag{3.178}$$

$$\Phi(\boldsymbol{A}^W_\mu(x)) = -\boldsymbol{A}^W_\mu(x). \tag{3.179}$$

The anti-automorphism passes to an involutive anti-automorphism of the two fold cover of the anti-de Sitter group. The elements of the unitary gauge group $G = SL(2,\mathbb{C})$ are characterised by

$$\Phi(g) = g^{-1} \Leftrightarrow g \in SL(2,\mathbb{C}). \tag{3.180}$$

Due to (3.178), (3.179) the polar decomposition of parallel transporters along infinitesimal paths is

$$\begin{aligned} \boldsymbol{T}(C) &= \mathbf{1} - \boldsymbol{B}_\mu(x)\delta x^\mu, & (3.181) \\ \boldsymbol{P}(C) &= \mathbf{1} - \boldsymbol{E}_\mu(x)\delta x^\mu, & (3.182) \\ \boldsymbol{U}(C) &= \mathbf{1} - \boldsymbol{A}_\mu(x)\delta x^\mu. & (3.183) \end{aligned}$$

So far, everything looks very similiar to the Dirac case. But beware: Only the unitary parallel transporters are $\mathbb{C}$-linear. This fact requires some care in certain calculations. For details of the "Weyl-formalism", see the diploma thesis of F. Neugebohrn [34].

3.9.1 Metricity

We saw in section 3.4 that the metric tensor can be constructed in the following way

$$g(\partial_\mu, \partial_\nu) := tr(\boldsymbol{E}(\partial_\mu)\boldsymbol{E}(\partial_\nu)). \tag{3.184}$$

In the above formalism one has

$$\boldsymbol{E}(x) = e^\alpha_\mu(x)\boldsymbol{\sigma}_\alpha \epsilon \mathcal{C} dx^\mu. \tag{3.185}$$

Therefore we arrive at

$$\begin{aligned} tr(\boldsymbol{E}(\partial_\mu)\boldsymbol{E}(\partial_\nu)) &= tr(e^\alpha_\mu(x)e^\beta_\nu(x)\boldsymbol{\sigma}_\alpha\tilde{\boldsymbol{\sigma}}_\beta) & (3.186) \\ &= 2e^\alpha_\mu(x)e^\beta_\nu(x)\eta_{\alpha\beta} & (3.187) \\ &= 2g_{\mu\nu}(x). & (3.188) \end{aligned}$$

As expected, we get the metric tensor of general relativity.

3.9.2 Einstein-Hilbert action

Let us finally mention that also in the "Weyl-formalism" the Einstein-Hilbert action can be obtained quite analogously to the Dirac case. One gets

$$S_{E-H} = \Im\, tr \int_{\mathcal{M}} (\boldsymbol{F}^e \wedge \boldsymbol{F}^{\mathcal{U}}) = \Im\, tr \int_{\mathcal{M}} (\boldsymbol{E} \wedge \boldsymbol{E} \wedge \boldsymbol{F}^{\mathcal{U}}). \tag{3.189}$$

The complete action takes again a polynomial form and can serve as a starting point to study Einstein gravity on a simplicial reduced irregular graph, see [34].

3.10 Conformal gravity

Finally, let us sketch a possible generalization of general relativity.

4×4 Dirac gamma matrices furnish a representation of the Lie algebra $\mathfrak{so}(2,4)$ of the conformal group. This allows us to consider a conformal holonomy group.

The conformal connection incorporates an additional vector field $\kappa_\mu(x)$ and an axial vierbein ${}^5e^\alpha_\mu(x)$:

$$\boldsymbol{B}_\mu := \frac{1}{2}\kappa_\mu(x)\boldsymbol{\gamma}_5 + \frac{1}{2}e^\alpha_\mu(x)\boldsymbol{\gamma}_\alpha + \frac{1}{2}\,{}^5e^\alpha_\mu(x)\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_5 + \frac{1}{8}A^{\alpha\beta}_\mu(x)[\boldsymbol{\gamma}_\alpha,\boldsymbol{\gamma}_\beta]. \quad (3.190)$$

The resulting theory involves additional dynamical variables and is no longer equivalent to Einstein gravity.

The matrices $\boldsymbol{\gamma}_\alpha$ and $\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_5$, respectively, do not correspond to "physical" transformations. Actually it turns out to be convenient to introduce new fields corresponding to the generators of translations and special conformal translations:

$$\frac{1}{2}e^\alpha_\mu(x)\boldsymbol{\gamma}_\alpha + \frac{1}{2}\,{}^5e^\alpha_\mu(x)\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_5 =: {}^Re^\alpha_\mu\boldsymbol{\gamma}_\alpha\mathcal{P}_R +{}^L e^\alpha_\mu(x)\boldsymbol{\gamma}_\alpha\mathcal{P}_L. \quad (3.191)$$

where the left-and righthanded vierbein are defined by

$$ {}^Le^\alpha_\mu = \frac{1}{2}(e^\alpha_\mu - {}^5e^\alpha_\mu) \quad (3.192)$$

$$ {}^Re^\alpha_\mu = \frac{1}{2}(e^\alpha_\mu + {}^5e^\alpha_\mu), \quad (3.193)$$

and $\mathcal{P}_{L,R}$ denote the chiral projectors $\frac{1}{2}(1 \mp \boldsymbol{\gamma}_5)$. Note that the left-and righthanded vierbein are associated with the generators of translations and special conformal translations, respectively.

The antiautomorphism on the conformal Lie algebra is defined as in the de Sitter case. Since $\Phi(\boldsymbol{\gamma}_5) = -\boldsymbol{\gamma}_5$ and $\Phi(\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_5) = +\boldsymbol{\gamma}_\alpha\boldsymbol{\gamma}_5$ it follows that

$$\boldsymbol{E}_\mu(x) = {}^Re^\alpha_\mu\boldsymbol{\gamma}_\alpha\mathcal{P}_R +{}^L e^\alpha_\mu(x)\boldsymbol{\gamma}_\alpha\mathcal{P}_L, \quad (3.194)$$

$$\boldsymbol{A}_\mu(x) = \frac{1}{2}\kappa_\mu(x)\boldsymbol{\gamma}_5 + \frac{1}{8}A^{\alpha\beta}{}_\mu(x)[\boldsymbol{\gamma}_\alpha,\boldsymbol{\gamma}_\beta]. \quad (3.195)$$

Consequently, the unitary gauge group is $G = Spin(1,3) \times D(1)$, where $D(1)$ denotes the noncompact dilatation group which is isomorphic to the positive real numbers.

It is important to note that the appearance of two distinct vierbein fields implies a deep modification of general relativity. Now there are three different "metrics"

$$ {}^{LL}g_{\mu\nu} := {}^Le^\alpha_\mu\,{}^Le_{\alpha\nu} \quad (3.196)$$

$$ {}^{RR}g_{\mu\nu} := {}^Re^\alpha_\mu\,{}^Re_{\alpha\nu} \quad (3.197)$$

$$ {}^{LR}g_{\mu\nu} := {}^Le^\alpha_\mu\,{}^Re_{\alpha\nu} \quad (3.198)$$

with different behaviour under scale transformations. In particular, it follows that ${}^{LR}g$ is invariant under $\boldsymbol{\gamma}_5$-scale transformations. It is important to note that as a result of (3.196) and (3.197) one is faced with a theory involving two different lightcones.

The odd piece of the nonunitary field strength associated with the vector potential (3.190) is

$$\frac{1}{2}(\boldsymbol{F}^{\mathcal{T}}_{\mu\nu}(x) - \boldsymbol{F}^{\mathcal{T}*}_{\mu\nu}(x)) = \frac{1}{2}(F^{\mathcal{T}\alpha\beta}{}_{\mu\nu}(x)\boldsymbol{M}_{\alpha\beta} + F^{\mathcal{T}45}{}_{\mu\nu}(x)\boldsymbol{D}), \quad (3.199)$$

where $\boldsymbol{M}_{\alpha\beta}$ and $\boldsymbol{D}$ denote the generators of the Lorentz group and the Dilatation group, respectively. The corresponding components of the nonunitary field strength are

$$F^{\mathcal{T}\alpha\beta}{}_{\mu\nu} = R^{\alpha\beta}{}_{\mu\nu} + 2\,{}^{L}e^{\alpha}_{[\mu}\,{}^{R}e_{\beta\nu]} - 2\,{}^{R}e^{\alpha}_{[\mu}\,{}^{L}e_{\beta\nu]}, \quad (3.200)$$

and

$$F^{\mathcal{T}45}{}_{\mu\nu} = \partial_{[\mu}\kappa_{\nu]} + 2\,{}^{L}e_{\gamma[\mu}\,{}^{R}e^{\gamma}_{\nu]} =: f_{\mu\nu} + 2\,{}^{L}e_{\gamma[\mu}\,{}^{R}e^{\gamma}_{\nu]} = f_{\mu\nu} + {}^{LR}g_{[\mu\nu]}. \quad (3.201)$$

κ_μ can be interpreted as the Weyl vector field and $f_{\mu\nu}$ as the length curvature (Weyl's "Streckenkrümmung"), [17], see also [37, 38]. Note that these components are invariant under scale transformations.

The even part of $\boldsymbol{F}^{\mathcal{T}}_{\mu\nu}$ is

$$\frac{1}{2}(\boldsymbol{F}^{\mathcal{T}}_{\mu\nu}(x) + \boldsymbol{F}^{\mathcal{T}*}_{\mu\nu}(x)) = \frac{1}{2}({}^{L}T^{\alpha}{}_{\mu\nu}(x)\boldsymbol{K}_{\alpha} + {}^{R}T^{\alpha}{}_{\mu\nu}(x)\boldsymbol{P}_{\alpha}), \quad (3.202)$$

where $\boldsymbol{K}_\alpha$ and $\boldsymbol{P}_\alpha$ denote the generators of special conformal translations and translations, respectively. The components

$${}^{L}T^{\alpha}{}_{\mu\nu} = \partial_{[\mu}\,{}^{L}e^{\alpha}_{\nu]} + A^{\alpha}{}_{\gamma[\mu}\,{}^{L}e^{\gamma}_{\nu]} + {}^{L}e^{\alpha}_{[\mu}\kappa_{\nu]} \quad (3.203)$$

$${}^{R}T^{\alpha}{}_{\mu\nu} = \partial_{[\mu}\,{}^{R}e^{\alpha}_{\nu]} + A^{\alpha}{}_{\gamma[\mu}\,{}^{R}e^{\gamma}_{\nu]} - {}^{R}e^{\alpha}_{[\mu}\kappa_{\nu]} \quad (3.204)$$

can be considered as a "left-and righthanded" torsion, respectively. Actually we can write

$${}^{L}T^{\alpha}{}_{\mu\nu} = \mathcal{D}_{[\mu}\,{}^{L}e^{\alpha}_{\nu]} \quad (3.205)$$

and

$${}^{R}T^{\alpha}{}_{\mu\nu} = \mathcal{D}_{[\mu}\,{}^{R}e^{\alpha}_{\nu]}, \quad (3.206)$$

where $\mathcal{D}_\mu$ is the Weyl-covariant derivative

$$\mathcal{D}_{\mu}\,{}^{L,R}e^{\alpha}_{\nu} := \partial_{\mu}\,{}^{L,R}e^{\alpha}_{\nu} + A^{\alpha}{}_{\beta\mu}\,{}^{L,R}e^{\beta}_{\nu} \mp {}^{L,R}e^{\alpha}_{\mu}\kappa_{\nu}. \quad (3.207)$$

We find that the nonunitary field strength describes generalizations of pseudo-Riemannian spaces which are known as Weyl spaces [17]. Furthermore, there are two torsion tensors. Their physical interpretation is not clear. For $f_{\mu\nu} = 0$ a Weyl space is equivalent to a pseudo-Riemannian space.

A natural generalization of the Einstein-Hilbert action is

$$\int d^4x \sqrt{det(\,^L e^\alpha_\mu) det(\,^R e^\beta_\nu)}\,^L e^\mu{}_\alpha\,^R e^\nu_\beta [R^{\alpha\beta}{}_{\mu\nu} + \eta^{\alpha\beta} f_{\mu\nu}]. \tag{3.208}$$

For $^L e^\alpha_\mu \to e^\alpha_\mu,\ ^R e^\alpha_\mu \to e^\alpha_\mu$ and $f_{\mu\nu} = 0$ we get back general relativity.

The field equations are not determined yet. A complete understanding of this theory with its "exotic" features is still missing.

Chapter 4

Discrete differential calculus

On any associative algebra over $\mathbb{C}$ or $\mathbb{R}$ a differential calculus can be defined, which generalizes the calculus of differential forms on a differentiable manifold. This structure has been studied for noncommutative algebras in many applications [1]. Noncommutative differential calculus was originally invented to give up the notion of a point. However, we are interested in the case, where the algebra of functions remains commutative. In this "semi-commutative" framework conventional notions of locality and the notion of a point retain their meaning.

In this chapter we briefly review a differential calculus on arbitrary directed graphs which was developed by Dimakis and Müller-Hoissen [14, 33]. In addition we shall construct an integral calculus which is as close as possible to the classical simplicial calculus on triangulated manifolds and prove Stokes law. For compatibility and proper covariance under noninjective maps, the differential calculus has to be adjusted by simplicial reduction and extended to pseudographs.

4.1 Motivation

In classical differential geometry the exterior derivative d is characterized by three properties:

- nilpotence
$$d^2 = 0 \tag{4.1}$$

- alternating Leibniz rule
$$d(\omega \wedge \xi) = d\omega \wedge \xi + (-1)^n \omega \wedge d\xi \tag{4.2}$$
for all n-forms ω and arbitrary forms ξ

- covariance
$$\Phi_\star \circ d = d \circ \Phi_\star \tag{4.3}$$

where $\Phi : \mathcal{M} \to \mathcal{M}'$ is a homeomorphic map of manifolds and
$\Phi_\star : \Omega(\mathcal{M}') \to \Omega(\mathcal{M})$ is the pull-back.

One fundamental difficulty in formulating a differential calculus on arbitrary directed graphs Γ is the breakdown of Leibniz rule. Actually it is a well-known fact that the Leibniz rule is not obeyed by the ordinary finite difference derivative on the lattice. This can be rectified by changing the commutation relations of 1-forms with functions. The algebra of functions however remains commutative.

To be more explicit, consider the D-dimensional unit lattice $\mathcal{M} := \mathbb{Z}^D$. Its sites can be labelled by integers $\{x^\mu\}, \mu = 1 \dots D$. Let $\mathcal{A}$ be the commutative algebra of real or complex functions on $\mathcal{M}$, with pointwise multiplication. The coordinates x^μ are functions on $\mathcal{M}$. The finite difference derivative is defined by[1]

$$\partial_\mu f(x) = f(x + \hat{\mu}) - f(x), \tag{4.4}$$

where $x + \hat{\mu}$ denotes the nearest neighbour of x in $\hat{\mu}$ direction. Instead of the Leibniz rule, ∂_μ satisfies

$$\partial_\mu (fg)(x) = (\partial_\mu f(x))g(x) + f(x + \hat{\mu})\partial_\mu g(x). \tag{4.5}$$

However, it turns out that the exterior derivative d, which acts on functions according to

$$df(x) = \sum_\mu dx^\mu \partial_\mu f(x), \tag{4.6}$$

will indeed satisfy the Leibniz rule if the usual commutativity of functions with 1-forms will be replaced by the relation

$$f(x)dx^\mu = dx^\mu f(x + \hat{\mu}). \tag{4.7}$$

Using algebraic notation, eq. (4.7) can be rewritten as

$$f_\mu dx^\mu = dx^\mu f, \tag{4.8}$$

where f_μ is the translated function defined by $f_\mu(y) := f(y - \hat{\mu})$.

Now consider the value $dx^\mu(z) := e^{yz}$ of the 1-form dx^μ at z as associated with the edge of Γ from z to $y = z + \hat{\mu}$. Specializing to the function $f = e^z$, supported only at z, we have $f_\mu = e^y$ and consequently arrive at the relation

$$e^y e^{yz} = e^{yz} e^z. \tag{4.9}$$

We will see in the next section that (4.9) generalizes to arbitrary directed graphs without need for a preferred coordinate system.

[1]Note that here $f(x) := f \circ x$ is a function.

Furthermore we shall discuss the remarkable fact that directed graphs behave like discrete manifolds. It is obvious that their links specify a substitute for topology, but it is surprising that one does not need an additional specification of a differential structure. It turns out that the corresponding exterior derivative d satisfies (4.1)-(4.3) if Φ is assumed to be bijective. However, covariance under noninjective maps requires to extend the calculus to pseudographs.

4.2 Universal differential calculus on a directed graph

First we recall the definition of a differential calculus over an associative not necessarily commutative algebra.

Definition 4.1 *Let $\mathcal{A}$ be an associative unital Algebra. A differential calculus $(\Omega(\mathcal{A}), d)$ on $\mathcal{A}$ consists of a $\mathbb{Z}$-graded associative algebra over $\mathbb{R}$, respectively $\mathbb{C}$*

$$\Omega(\mathcal{A}) = \bigoplus_{n \geq 0} \Omega^n(\mathcal{A}) \tag{4.10}$$

and a ($\mathbb{R}-$ respectively $\mathbb{C}$-) linear map

$$d : \Omega^n(\mathcal{A}) \to \Omega^{n+1}(\mathcal{A}) \tag{4.11}$$

which is nilpotent

$$d^2 = 0 \tag{4.12}$$

and obeys the graded Leibniz rule

$$d(\omega_1 \omega_2) = (d\omega_1)\omega_2 + (-1)^n \omega_1 d\omega_2 \tag{4.13}$$

for all $\omega_1 \in \Omega^n(\mathcal{A})$, $\omega_2 \in \Omega(\mathcal{A})$.
$(\Omega^1(\mathcal{A}), d)$ will be called first order differential calculus.

Let $\mathcal{A} = \Omega^0(\mathcal{A})$. By definition, $\Omega(\mathcal{A})$ is a bimodule for $\mathcal{A}$, i.e. multiplication with elements $f \in \mathcal{A}$ from the right and left is defined.

Definition 4.2 *If d generates the spaces $\Omega^n(\mathcal{A})$ for $n > 0$ in the sense that*

$$\Omega^n(\mathcal{A}) = \mathcal{A} d\Omega^{n-1}(\mathcal{A})\mathcal{A} \tag{4.14}$$

the differential calculus will be called minimal.

Using the Leibniz rule, every element of $\Omega^n(\mathcal{A})$ can be written as a linear combination of monomials $a_0 da_1 da_2 \dots da_n, a_i \in \mathcal{A}$. The action of d is then given by

$$d(a_0 da_1 da_2 \dots da_n) = da_0 da_1 da_2 \dots da_n. \tag{4.15}$$

Associated with any $\mathcal{A}$ there is a "largest" differential calculus $(\tilde{\Omega}, \delta)$ on $\mathcal{A}$. It can be constructed explicitly in terms of tensor products. Consider the submodule of $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ given by

$$ker(m : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \rightarrow \mathcal{A}), \quad m(a \otimes_{\mathbb{C}} b) := ab. \tag{4.16}$$

This submodule is generated by elements of the form $\mathbf{1} \otimes_{\mathbb{C}} a - a \otimes_{\mathbb{C}} \mathbf{1}$ with $a \in \mathcal{A}$. Then the map

$$\delta : \mathcal{A} \rightarrow ker(m : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \rightarrow \mathcal{A}) \tag{4.17}$$

defined by

$$\delta a := \mathbf{1} \otimes_{\mathbb{C}} a - a \otimes_{\mathbb{C}} \mathbf{1} \tag{4.18}$$

obeys the graded Leibniz rule (4.13) for $\omega_1, \omega_2 \in \mathcal{A}$.
Now let $\tilde{\Omega}^1 := ker(m : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \rightarrow \mathcal{A})$. Define the space of n-forms as

$$\tilde{\Omega}^n := \underbrace{\tilde{\Omega}^1 \otimes_{\mathcal{A}} \tilde{\Omega}^1 \otimes_{\mathcal{A}} \ldots \otimes_{\mathcal{A}} \tilde{\Omega}^1}_{n-times} \subset \underbrace{\tilde{\Omega}^1 \otimes_{\mathbb{C}} \tilde{\Omega}^1 \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \tilde{\Omega}^1}_{(n+1)-times}, \tag{4.19}$$

cp. [24]. The linear operator δ can be extended to the whole differential algebra using the Leibniz rule with respect to the product $\otimes_{\mathcal{A}}$. The resulting differential algebra is universal in the sense that every other minimal differential calculus can be derived from it as a quotient $\tilde{\Omega}/\mathcal{J}$, where $\mathcal{J}$ is a differential ideal, i.e. a two-sided ideal in $\tilde{\Omega}$ with the property $\delta\mathcal{J} \subset \mathcal{J}$. More precisely, we have

Theorem 4.1 *Each differential algebra Ω is the image of a homomorphism π of differential algebras*

$$\pi : \tilde{\Omega} \rightarrow \Omega, \tag{4.20}$$

where $\tilde{\Omega}$ is the univesral differential algebra. Therefore Ω is the quotient $\Omega = \tilde{\Omega}/ker\pi$, where the kernel of π is a two-sided differential ideal in $\tilde{\Omega}$.

In the following we will be interested in the case where $\mathcal{A}$ is the commutative algebra of all $\mathbb{R}$- or $\mathbb{C}$-valued functions on a discrete set $\mathcal{M}$ of points $x, y, z \ldots$, which shall be identified with the vertices of a directed graph Γ. A natural basis of $\mathcal{A}$ consists of functions $e^x, x \in \mathcal{M}$, defined by

$$e^x(y) = \delta_{xy} \quad \text{for all } x, y \in \mathcal{M}. \tag{4.21}$$

They satisfy the multiplication law

$$e^x e^y = \delta_{xy} e^x. \tag{4.22}$$

The algebra $\mathcal{A}$ has a unit,

$$\mathbf{1} = \sum_x e^x. \tag{4.23}$$

Each $f \in \mathcal{A}$ can then be written as

$$f = \sum_{x \in \mathcal{M}} f(x) e^x, \quad f(x) \in \mathbb{C}. \tag{4.24}$$

Let us introduce the 1-forms

$$e^{xy} = \begin{cases} e^x de^y & \text{for } x \neq y \\ 0 & \text{otherwise.} \end{cases} \tag{4.25}$$

Their importance arises from the following

Lemma 4.1 *$(e^{xy})_{x,y \in \mathcal{M}}$ is a basis over $\mathbb{C}$ of the space $\tilde{\Omega}^1$ of universal 1-forms.*

The basis 1-forms (e^{xy}) can be used to obtain reduced differential calculi in a convenient way.

Lemma 4.2 *All first order differential calculi are obtained from the universal one by setting some of the e^{xy} to zero.*

Proof 4.1 *The proof uses theorem (4.1), for details see [33].*

Let us now regard the elements of $\mathcal{M}$ as vertices, henceforth called sites, of a directed graph Γ and associate with each nonvanishing e^{xy} an arrow from the point x to y. In this way a first order differential calculus on a discrete set can be represented by a directed graph. In particular, the universal differential algebra corresponds to the complete graph, i.e. all the vertices are connected pairwise by arrows in both directions. Deleting some of the arrows leads to a graph which represents a reduction of the universal differential algebra.

Conversely, given a digraph, it determines uniquely a differential calculus Ω_Γ generated by the algebra of functions defined on the set of vertices and 1-forms attached to the directed edges of Γ. Thus, in a differential calculus on the directed graph Γ, $e^{xy} \neq 0$ if and only if Γ contains a directed edge from y to x.

Because of the bijective correspondence between first order differential calculi and directed graphs, the latter behave like discrete manifolds in the sense that Γ contains already all the information about the differential calculus without the need of an additional specification.

Let us add that the described differential calculus on a directed graph is also appropriate to deal with complex systems, cp. [28].

We stress that the differential calculus on a directed graph may be still too large. Again we can reduce it by division by a differential ideal. More precisely, we have

Theorem 4.2 (Universal minimal differential algebra on a directed graph) *[Dimakis and Müller-Hoissen]*
There exists a minimal differential calculus (Ω_Γ, d) *on* Γ *which is universal in the sense that any other minimal differential calculus on* Γ *is obtained from it by dividing* Ω_Γ *by a differential ideal* $\mathcal{J}$*.*

Set

$$\rho = \sum_{x,y} e^{xy}. \tag{4.26}$$

Then

1. $e^z e^{xy} = \delta_{zx} e^{xy}, \qquad e^{xy} e^z = \delta_{yz} e^{xy}.$

2. The generators e^{xy} *satisfy the following relations. Whenever there is no edge from* y *to* x *in* Γ *then*

$$e^y \rho^2 e^x = \sum_z e^{yz} e^{zx} = 0. \tag{4.27}$$

3. For functions $f \in \mathcal{A}$*,*

$$df = \sum_{xy} [f(y) - f(x)] e^{xy} = \rho f - f\rho. \tag{4.28}$$

In particular, $d\mathbf{1} = 0$*.*

4.

$$de^{xy} = \rho e^{xy} - e^x \rho^2 e^y + e^{xy} \rho. \tag{4.29}$$

Proof 4.2 *Part 1:* $e^z e^{xy} = \delta_{zx} e^{xy}$ *follows from the definition of* e^{xy} *and the multiplication law (4.22).*

Let $x \neq y$ *and consider the quantity* $e^{xy} e^z = e^x (de^y) e^z = e^x (d(e^y e^z) - e^y de^z)$*. The second term vanishes because of the multiplication law (4.22), and the first is equal to* $e^x d(\delta_{yz} e^y) = \delta_{yz} e^{yz}$*. q.e.d.*

Part 3: First we show that $d\mathbf{1} = 0$*. This follows from the Leibniz rule since* $d\mathbf{1} = d(\mathbf{11}) = (d\mathbf{1})\mathbf{1} + \mathbf{1}d\mathbf{1} = 2d\mathbf{1}$*.*

From eq.(4.23) and the definition of e^{xy} *it follows then that* $e^z de^z = -\sum_x e^{zx}$*. Because of* $\mathbb{C}$*-linearity, this implies* $df = \sum_z f(z) de^z = \sum_z f(z) \sum_x e^x de^z = \sum_z f(z) [e^z de^z + \sum_{x \neq z} e^x de^z] = \sum_x \sum_z f(z) [-e^{zx} + e^{xz}]$*. Assertion 3 follows from this by a change of summation variable.*

Part 4: $de^{xy} = d(e^x de^y) = de^x de^y$ *because of the Leibniz rule and* $d^2 = 0$*. Inserting* $de^x = [\rho, e^x]$ *from part 3, one obtains the reult.*

Part 2: If there is no link fro y *to* x*,* $e^{xy} = 0$ *and therefore* $de^{xy} = 0$*. Inserting the result of part 4 yields the assertion.*

Concatenation of the 1-forms e^{xy} leads to the n-forms

$$e^{x_n \dots x_0} := e^{x_n x_{n-1}} e^{x_{n-1} x_{n-2}} \dots e^{x_1 x_0}, \tag{4.30}$$

which can be rewritten as follows

$$e^{x_n \dots x_0} = e^{x_n} \rho e^{x_{n-1}} \rho \dots \rho e^{x_0}. \tag{4.31}$$

They satisfy the relations

$$e^{y_n \dots y_0} e^{x_n \dots x_0} = \delta_{y_0 x_n} e^{y_n \dots y_1 x_n \dots x_0} \tag{4.32}$$

and especially

$$f e^{x_n \dots x_0} = f(x_n) e^{x_n \dots x_0}, \quad e^{x_n \dots x_0} f = e^{x_n \dots x_0} f(x_0) \tag{4.33}$$

for $f \in \mathcal{A}$. Starting with the universal first order differential calculus, the n-forms $e^{x_n \dots x_0}$ constitute a basis of $\tilde{\Omega}^n$ over $\mathbb{C}$. In the case of a differential calculus on an arbitrary directed graph Γ, where some of the e^{xy} are zero, the possibilities to build nonvanishing higher forms are obviously restricted. Moreover, constraints are imposed on them, cp. part 2. of theorem 4.2. Note that the 1-forms (e^{xy}) provide also in this case a basis of Ω^1_Γ.

Generally, if the algebra $\mathcal{A}$ has an involution, i.e. an $\mathbb{C}$-antilinear map

$$\star : \mathcal{A} \to \mathcal{A}, \quad f \to f^\star \tag{4.34}$$

such that

$$f^{\star\star} = f, \quad (fh)^\star = h^\star f^\star \qquad \text{for all}\, f, h \in \mathcal{A}, \tag{4.35}$$

one can also make Ω an involutive algebra by defining

$$(df)^\star = -df^\star \tag{4.36}$$

$$(\omega_1 \omega_2)^\star = \omega_2^\star \omega_1^\star \tag{4.37}$$

for all $f \in \mathcal{A}, \omega_1, \omega_2 \in \Omega$.

In the present case, the involution in $\mathcal{A}$ is just given by complex conjugation

$$f^\star(x) := \overline{f(x)}, \quad f \in \mathcal{A}. \tag{4.38}$$

When there is an edge from x to y whenever there is an edge from y to x we call the graph bidirectional. In the case of a bidirectional graph, there is a $\star$-operation on the graph (reversal of direction of edges), and a corresponding $\star$-operation in the algebra Ω_Γ,

$$(e^{xy})^\star = e^{yx}. \tag{4.39}$$

Using (4.36), (4.37) and

$$d(e^x e^y) = 0 \Rightarrow de^x e^y = -e^x de^y, \tag{4.40}$$

one finds that the $\star$-operation in Ω_Γ can be identified with the involution.

Note that for a bidirectional graph the 1-form ρ is self adjoint in the sense that

$$\rho^\star = \rho. \tag{4.41}$$

We retain a $\star$-operation for graphs which are not bidirectional by adjoining adjoints of edges where necessary, distinguishing them as virtual edges. The virtual edges are not counted among the edges proper.

Let us now repeat the explicit construction of the differential algebra for the case that the algebra is given by the complex-valued functions on $\mathcal{M}$. This will be needed for the definition of an integral calculus in the next section.

One can identify $\underbrace{\mathcal{A} \otimes_{\mathbb{C}} \ldots \otimes_{\mathbb{C}} \mathcal{A}}_{(n+1)-times}$ with the $\mathbb{C}$-vector space of maps from $\underbrace{\mathcal{M} \times \ldots \times \mathcal{M}}_{(n+1)-times} \to \mathbb{C}$. If $f \in \mathcal{A}$ the action of d is given by

$$df(x_0, x_1) := (\mathbf{1} \otimes_{\mathbb{C}} f - f \otimes_{\mathbb{C}} \mathbf{1})(x_0, x_1) = f(x_1) - f(x_0). \tag{4.42}$$

In this way, the space of 1-forms Ω^1 can be identified with the space of functions of two variables vanishing on the diagonal. Analogously, Ω^n is identified with the space of functions of $n+1$ variables vanishing on any subset of $\mathcal{M}^{n+1}$ which contains a two-dimensional diagonal

$$f(x_0, \ldots, x_{i-1}, x, x, x_{i+2}, \ldots, x_n) = 0. \tag{4.43}$$

For $f \in \Omega^n$ the differential $df \in \Omega^{n+1}$ is defined by

$$df(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i f(x_0, \ldots, \hat{x}_i, \ldots, x_{n+1}). \tag{4.44}$$

One introduces a product between elements of $\mathcal{A}$ and Ω^n

$$(gf)(x_0, \ldots, x_n) := g(x_0) f(x_0, \ldots, x_n) \tag{4.45}$$
$$(fg)(x_0, \ldots, x_n) := f(x_0, \ldots, x_n) g(x_n) \tag{4.46}$$

for all $g \in \mathcal{A}, f \in \Omega^n$ such that Ω^n becomes an $\mathcal{A}$-bimodule. The multiplication law (4.45) can be extended to a product of a n-form with a m-form

$$(fg)(x_0, \ldots, x_{n+m}) := f(x_0, \ldots, x_n) g(x_n, \ldots, x_{n+m}). \tag{4.47}$$

Finally, the involution is given by

$$(f^\star)(x_0, \ldots, x_n) := (-1)^n \overline{f(x_n, \ldots, x_0)}. \tag{4.48}$$

Consequently, one finds $(df)^\star = (-1)^n d(f^\star)$ for $f \in \Omega^n$.

4.3 Simplices and integrals

Obviously, the introduction of an integral calculus requires the definition of appropriate objects, which can be integrated over. This shall be done in a way which is as close as possible to the classical simplicial calculus on triangulated manifolds.

Given an edge b, we denote by $-b$ the edge in the opposite direction, if it exists. It may actually be absent form Γ. If it is present, we call b a bidirectional edge. The following simplicial constructions are of interest mainly for graphs whose edges are all bidirectional. A path C of length n shall be a sequence $b_{n-1}, \ldots, b_0$ of n edges $b_i : x_i \to x_{i+1}$, $x_i \in \mathcal{M}$, $i = 0, \ldots, n-1$. A possibly overcomplete basis $\{\omega_C\}$ of Ω^n_Γ may be labelled by paths C of length n,

$$\omega_C = e^{x_n x_{n-1}} \ldots e^{x_1 x_0}. \tag{4.49}$$

Note that $e^z \omega_C = \omega_C$ if $z = x_n$ and 0 otherwise. Using (4.21), (4.42) and (4.47), one can associate with any n-form ω_C a function of $n+1$ sites, also denoted by ω_C, via

$$\omega_C(y_n, \ldots, y_0) = \delta_{y_n x_n} \ldots \delta_{y_0 \ldots x_0}. \tag{4.50}$$

Given C, we denote by $-C$ the path in the opposite direction. It may involve virtual edges if the graph is not bidirectional.

Definition 4.3 (Simplices in a graph) *Given the directed graph Γ, an oriented n-simplex is an equivalence class $\Delta = [x_0, \ldots, x_n]$ of sequences of $n+1$ distinct sites $x_i \in \mathcal{M}$ modulo even permutations, with the property that for every $i, j, 0 \le i \ne j \le n, \langle x_i, x_j \rangle$ is an edge of Γ. If π is an odd permutation, then $[x_{\pi 0}, \ldots, x_{\pi n}]$ is the oriented simplex obtained from it by reversing the orientation. It is also denoted by $-[x_0, \ldots, x_n]$. The boundary $\partial\Delta$ is defined as a formal sum of oriented $(n-1)$ simplices with integer coefficients (i.e. as a (n-1)-chain) in the standard way,*

$$\partial\Delta = \sum_{i=0}^{n} (-1)^i [x_0, \ldots \hat{x}_i, \ldots x_n] \tag{4.51}$$

Subsets of $x_n, \ldots, x_0$ with $k+1$ elements define simplices which are called k-faces of Δ.

Now we are ready to integrate n-forms $\omega \in \Omega_\Gamma$ over domains which are formal sums of oriented simplices with integer coefficients (chains).

Definition 4.4 (Integration) *The integral of n-forms ω over oriented n-simplices is defined by*

$$\int_{\Delta=[x_0,\ldots,x_n]} \omega := \mathcal{S}\omega(x_0 \ldots x_n) \tag{4.52}$$

where $\mathcal{S}$ is antisymmetrization, viz.

$$S\omega(x_0 \dots x_n) = \frac{1}{(n+1)!} \sum_{\pi} sign(\pi)\, \omega(x_{\pi 0} \dots x_{\pi n}), \tag{4.53}$$

where sum over all permutations π of $0, \dots, n$ is understood.

Let us state some properties of integrals.

Theorem 4.3 (Stokes law)

$$\int_{\Delta} d\omega = \int_{\partial\Delta} \omega \quad for\, \omega \in \Omega_{\Gamma}. \tag{4.54}$$

At this stage we are still not as close to the classical calculus as possible. Typically there exist n-forms ω such that $\int_{\Delta} \omega = 0$ for all Δ; therefore the pairing of forms and chains is degenerate. This can be remedied in part as follows by imposing further relations on forms.

We say that $C \subset \Delta$ if the path C visits only 0-simplices of Δ.

Theorem 4.4 (Ideal) *1. The forms ω_C for C such that there exists no simplex Δ with $C \subset \Delta$ span a differential left and right ideal $\mathcal{J}$ of Ω_{Γ}.*
2. $\Omega' = \Omega_{\Gamma}/\mathcal{J}$ is a differential algebra spanned by ω_C with $C : x \to y$ such that $C \subset \Delta$ for some simplex Δ.

Let us say that a path C meanders if there is no oriented simplex Δ with $C \subset \Delta$. A path of length 0 never meanders.

The construction of Ω' enables one also to introduce the notion of a dimension of a graph. Actually, considering a bidirectional edge $\langle y, x\rangle$, one can build arbitrary high forms like $e^{yx}e^{xy}\dots$. Consequently, $\Omega^n_{\Gamma} \neq \emptyset$ for arbitrary high n. Ω' however is spanned by ω_C associated with nonmeandering paths. Therefore we find the following remark

Remark 4.1 *There is $m \in \mathbb{N}$ such that ${\Omega'}^{r} = \emptyset$ for all $r > m$.*

Proof 4.3 *Let $S^n_{\Gamma} := \{\Delta | \Delta is\ n\text{-}simplex\ of\ \Gamma\}$ denotes the set of all n-simplices of a graph Γ. Then we have by definition of simplices that for some $m \in \mathbb{N}$ $S^r_{\Gamma} = \emptyset$, $r > m$. As Ω' is spanned by ω_C with $C \subset \Delta$, the assertion follows.*

Note that if the graph Γ consists of n vertices one has

$$S^r_{\Gamma} = \emptyset \quad \text{for all}\, r > n. \tag{4.55}$$

Remark 4.1 suggests the following definition

Definition 4.5 (Dimension of a graph) *The dimension D of a finite graph Γ is defined by*

$$D := max\{n \,|\, {\Omega'}^{n} \neq \emptyset\}. \tag{4.56}$$

Given a path $C = (x_n, \ldots, x_0)$ from x_0 to x_n, such that $x_0, \ldots, x_n$ are the 0-faces of a n-simplex Δ, and a permutation π of $0 \ldots n$ with $\pi(0) = 0, \pi(n) = n$, then

$$\pi C := (x_{\pi(n)}, \ldots, x_{\pi(0)}) \tag{4.57}$$

is also a path from x_0 to x_n. Let $\mathcal{J}^n_\Delta$ be the set of n-forms

$$\mathcal{J}^n_\Delta := \{\omega_{\pi C} - sign\pi\omega_C \,|\, \omega_C \in \Omega', C \subset \Delta\} \tag{4.58}$$

with π as above. Furthermore let $\mathcal{J}^n$ be the subbimodule of Ω^n, which is generated by all $\mathcal{J}^n_\Delta$. Then we have

Theorem 4.5 *$\mathcal{J}' := \bigoplus_{n=3} \mathcal{J}^n$ is a differential left and right ideal of Ω.*

Note that $\mathcal{J}^0 = \mathcal{J}^1 = \mathcal{J}^2 = \emptyset$. Now we may impose the following relation on basis elements

$$\omega_{\pi C} - sign\pi\omega_C = 0. \tag{4.59}$$

The resulting differential algebra Ω'' is obtained by dividing Ω' by $\mathcal{J}'$

$$\Omega'' = \sum \Omega''^n := \Omega'/\mathcal{J}'. \tag{4.60}$$

More precisely, one has

Theorem 4.6 (Duality) *Given 0-faces x, y of the oriented n-simplex $\Delta = [x_0, \ldots, x_n]$, with $x \neq y$ if $n > 0$, define the n-form $\omega_{\Delta,x,y} \in \Omega''^n$ by*

$$\omega_{\Delta,x,y} = sign\pi\omega_C \tag{4.61}$$

where $\Delta C : x \to y$ is an arbitrary path of the form

$$C = (x_{\pi 0} = y, x_{\pi 1} \ldots, x_{\pi(n-1)}, x_{\pi n} = x). \tag{4.62}$$

Then
1. $\omega_{\Delta,x,y}$ span Ω''^n.
2. $\int_\Delta \omega_{\Delta',x,y} = \delta_{\Delta,\Delta'}$.

Remark 4.2 *Define $\omega_\Delta = \sum_{x,y} \omega_{\Delta,x,y}$. Then*
1. $\omega_{\Delta,x,y} = e^y \omega_\Delta e^x$.
2. Sums of ω_Δ with coefficients in $\mathcal{A}$ span Ω''^n.
3. $\int_\Delta \omega_{\Delta'} = c_n \delta_{\Delta,\Delta'}$
with $c_n = n(n+1)$ if $n > 0$, and $c_0 = 1$.

4.4 Calculus on pseudographs and singular simplices

In the classical theory, if $f : \mathcal{M} \to \mathcal{M}'$ is a homoeomorphic map of manifolds, and ω' is a n-form on $\mathcal{M}'$, the pullback $f_\star\omega'$ is defined as a n-form on $\mathcal{M}$ and

$$df_\star\omega' = f_\star d\omega'. \tag{4.63}$$

Let us emphasize, that it is not required that f is an immersion, i.e. two distinct points in a small neighbourhood may be mapped into the same point in $\mathcal{M}'$ by f. Moreover, simplices on $\mathcal{M}$ may be mapped into singular simplices in $\mathcal{M}'$, i.e. the vertices of the simplex need not be assigned to distinct points on $\mathcal{M}'$.

The differential calculus on graphs we have considered so far does not admit either possibility.

Maps of graphs map edges from x to y into edges from $f(x)$ to $f(y)$. Since a graph has no edges from x to x, two distinct points x, y are always mapped into distinct points $f(x)$ and $f(y)$. Moreover, it is essential in the proof of Stokes theorem that there are edges between any two vertices of a simplex. Therefore its vertices must be distinct sites.

These limitations are unwelcome when one wishes to do multiscale analysis. The idea is that space time points are only distinguished when they can be separated by a detector whose resolution determines a length scale. When the length scale is increased, two formerly distinct sites may become identified. An important special case of multiscale analysis is dimensional reduction. It occurs when the extension of space time in some extra dimension falls below the length scale.

This motivates the desire to extend the calculus to pseudographs Γ. They are more general than graphs by admitting edges from x to x. We are mainly interested in pseudographs with (up to) two directed edges from x to x, which are mapped one into the other by the $\star$-operation (reversal of direction), and corresponding generators e^{xx} and $e^{xx\star}$ of Ω^1_Γ. The resulting calculus is not minimal in the sense of definition 4.2 because $e^{xx} \neq e^x df e^x$ for any $f \in \mathcal{A}$.

Theorem 4.7 (Calculus on pseudographs) *Let Γ' be the pseudograph with $*$-operation obtained from a directed graph Γ by adding for some or all sites x pairs of adjoint edges from x to x. Retain the notation*

$$\rho = \sum_{y \neq x} e^{xy}, \tag{4.64}$$

and the definitions of df and $de^{xy}, y \neq x$ (part 3 and part 4 of theorem4.2).
1. The calculus (Ω_Γ, d) can be extended to a calculus $(\Omega_{\Gamma'}, d)$ on the pseudograph by adding generators e^{xx} and $e^{xx\star}$ subject to the relations

$$(e^{xx})^2 = 0 = (e^{xx\star})^2, \quad e^{xx}e^{xx*} \neq 0, \quad e^y e^{xx} = \delta_{yx} e^{xx}. \tag{4.65}$$

and setting

$$de^{xx} = \rho e^{xx} - e^x\rho^2 e^x + e^{xx}\rho + e^{xx}e^{xx\star} + e^{xx\star}e^{xx} \tag{4.66}$$
$$de^{xx\star} = \rho e^{xx\star} - e^x\rho^2 e^x + e^{xx\star}\rho + e^{xx}e^{xx\star} + e^{xx\star}e^{xx}. \tag{4.67}$$

2. For homeomorphisms $f : \Gamma \to \Gamma'$ of pseudographs define the pullback $f_\star$ as follows. For n-forms $\omega' = \sum_{C'} \alpha_{C'}\omega_{C'}$ on Γ',

$$f_\star\omega' = \sum_{C'} \alpha_{C'} \sum_{C:f(C)=C'} \omega_C. \tag{4.68}$$

Then

$$df_\star\omega' = f_\star d\omega'. \tag{4.69}$$

Singular simplices in a pseudograph Γ' are constructed as maps of the graph Γ_n associated with the unit n-simplex Δ_n to Γ', for all nonnegative integers n.

We recall that Δ_n consists of the points $(x_0, \dots, x_n)$ of $\mathbb{R}^{n+1}$ satisfying $\sum_{i=0}^n x_i = 1$, $x_i \geq 0$, $i = 0, \dots, n$. For each integer $i = 0, 1, \dots, n$, the point $v_i = (\delta_{i0}, \dots, \delta_{in})$ of $\mathbb{R}^{n+1}$ is a point of Δ_n and will be referred to as its i-th vertex. Δ_n is the convex hull of the set of its vertices. Its 1-faces consist of convex linear combinations $x = tv_i + (1-t)v_j, 0 \leq t \leq 1$, $i \neq j$ of two distinct vertices. They can be given an orientation in two ways. The orientation is reversed by the $\star$-operation. The vertices and the oriented 1-faces of Δ_n form the graph Γ_n associated with Δ_n.

Definition 4.6 (Singular Simplices) *A singular n-simplex in a pseudograph Γ' is a map ξ of the graph Γ_n to Γ' whose restriction to edges $b = <x, y>$ obeys*

$$-(\xi b) = \xi(-b). \tag{4.70}$$

Chains are formal sums of singular simplices with integer coefficients.
The integral of a n-form ω' on Γ' over a singular n-simplex ξ is defined by

$$\int_\xi \omega' = \sum_\pi \frac{1}{(n+1)!} sign(\pi)\xi_\star\omega(v_{\pi 0}, \dots, v_{\pi n}). \tag{4.71}$$

The sum is over the $(n+1)!$ permutations of integers $0, 1, \dots, n$.

Since $\xi_\star\omega'$ is a n-form on a graph Γ_n, the definition (4.50) applies.

The word singular refers to the fact that the map ξ need not be injective. When it is not, we call ξ a truely singular simplex. We emphasize that the integral of a n-form over a truely singular n-simplex does not have to vanish.

One verifies, that the definitions are equivalent to the old ones for graphs Γ'.

Theorem 4.8 (Covariance) *Let $f : \Gamma \to \Gamma'$ be a map of pseudographs, ω' a n-form on Γ', and ξ a singular simplex in Γ. Then*

$$\int_{\xi} f_{\star}\omega' = \int_{f\xi} \omega', \tag{4.72}$$

where $f\xi$ is the singular simplex in Γ' obtained by composing maps ξ and f.

4.5 Proofs

4.5.1 Proof of Stokes law

We will need a formula for the exterior derivative of elements of the basis ω_C of the space Ω^n_Γ of n-forms.

Proposition 4.1 (Exterior derivative of basic n-forms) *Let $\Theta_{xy} = 1$ if there is an edge Γ from y to x, and $\Theta_{xy} = 0$ otherwise, and let $C = (x_0, \ldots, x_n), n \geq 1$. Then*

$$\begin{aligned} d\omega_C(y_0, \ldots, y_{n+1}) = \Theta_{y_1 y_0}\omega_C(\hat{y}_0, y_1 \ldots, y_{n+1}) \\ + \sum_{j=1}^{n}(-1)^j \Theta_{y_{j-1}y_j}\Theta_{y_j y_{j+1}}\omega_C(y_0, \ldots, \hat{y}_j, \ldots, y_{n+1}) \\ + (-1)^{n+1}\omega_C(y_0, \ldots, y_n, \hat{y}_{n+1}). \end{aligned} \tag{4.73}$$

Hatted arguments $\hat{y}_j$ are to be omitted.

We will also need a basic combinatorial lemma.

Lemma 4.3 *Let S_{n+2} be the group of permutations of integers $0, \ldots, n+1$ and $0 \leq k \leq n+1$. Then*

$$\begin{aligned} \sum_{j=0}^{n+1} \sum_{\pi \in S_{n+2}:\pi j = k} (-1)^j sign(\pi)\omega(y_{\pi 0}, \ldots \hat{y}_{\pi j} \ldots y_{\pi(n+1)}) = \\ (n+2)!(-1)^k \mathcal{S}\omega(y_0, \ldots \hat{y}_k \ldots, y_{\pi(n+1)}). \end{aligned} \tag{4.74}$$

This lemma will be a corollary of another lemma,

Lemma 4.4 *Given k,*

$$\sum_{\pi \in S_{n+2}:\pi j = k} (-1)^j sign(\pi)\omega(y_{\pi 0}, \ldots, \hat{y}_{\pi j}, \ldots, y_{\pi(n+1)}) \tag{4.75}$$

is independent of $j = 0, \ldots, n+1$.

Proof 4.4 *lemma 2: It suffices to show that the expression remains unchanged when $j+1$ is substituted for j. Replace j by $j+1$ and make a variable transformation on π by multiplying it with the transposition of j and $j+1$. Then $(-1)^j$ and $sign(\pi)$ both change by a factor (-1), and the whole expression remains unchanged. q.e.d.*
lemma 3: Since the summand of the j-summation is independent of j by lemma 2, it may be evaluated for $j=k$. The assertion follows then from the fact that there are $n+2$ equal terms in the sum over j, and from the definition of $\mathcal{S}$. q.e.d.

Proof 4.5 (Proof of Proposition 1) *By definition,*

$$\omega_C = e^{x_0x_1}e^{x_1x_2}\dots e^{x_{n-1}x_n}. \tag{4.76}$$

Applying the Leibniz rule yields

$$d\omega_C = (de^{x_0x_1})e^{x_1x_2}\dots e^{x_{n-1}x_n} - e^{x_0x_1}(de^{x_1x_2})\dots e^{x_{n-1}x_n} \\ \pm \dots - (-1)^n e^{x_0x_1}e^{x_1x_2}\dots de^{x_{n-1}x_n} \tag{4.77}$$

Inserting assertion 4 of theorem 4.2, we obtain

$$d\omega_C = (\rho e^{x_0x_1} - e^{x_0}\rho^2 e^{x_1} + e^{x_0x_1}\rho)e^{x_1x_2}\dots e^{x_{n-1}x_n} \\ - e^{x_0x_1}(\rho e^{x_1x_2} - e^{x_1}\rho^2 e^{x_2} + e^{x_1x_2}\rho)e^{x_2x_3}\dots e^{x_{n-1}x_n} \pm \dots \\ - (-1)^n e^{x_0x_1}\dots e^{x_{n-2}x_{n-1}}(\rho e^{x_{n-1}x_n} - e^{x_{n-1}}\rho^2 e^{x_n} + e^{x_{n-1}x_n}\rho) \tag{4.78}$$

Because the sum $\rho = \sum e^{xy}$ runs only over $x \neq y$, and the multiplication law (1.) of theorem 4.2 holds, terms $e^{x_{i-1}x_i}\rho e^{x_ix_{i+1}}$ vanish. Hence

$$d\omega_C = \rho\omega_C + \sum_{i=0}^{n-1}(-1)^i e^{x_0}e^{x_0x_1}\dots e^{x_{i-1}x_i}\rho^2\hat{e}^{x_ix_{i+1}}\dots e^{x_{n-1}x_n}e^{x_n} - (-1)^n\omega_C\rho \\ = \sum_x{}' \omega_{(x,x_0,\dots,x_n)} + \sum_{i=0}^{n-1}(-1)^i \sum_x{}' \omega_{(x_0,\dots,x_i,x,x_{i+1},\dots,x_n)} - (-1)^n\omega_{(x_0,\dots,x_n,x)}, \tag{4.79}$$

where it is understood that the sums $\sum_x'$ run only over such x that the subscripts of the summands are actually paths.

This can be made explicit with the help of the Θ-symbols, viz.

$$d\omega_C = \sum_x \Big\{\Theta_{xx_0}\omega_{(x,x_0,\dots,x_n)} - \sum_{i=0}^{n-1}(-1)^i\Theta_{x_ix}\Theta_{xx_{i+1}}\omega_{(x_0,\dots x_i,x,x_{i+1},\dots x_n)} \\ - (-1)^n\Theta_{x_nx}\omega_{(x_0\dots x_n,x)}\Big\} \tag{4.80}$$

Inserting the definition (4.50) we evaluate

$$d\omega_C(y_0,\dots,y_{n+1}) = \sum_x \big\{\Theta_{xx_0}\delta_{xy_0}\omega_C(y_1,\dots,y_{n+1})$$
$$- \sum_{i=0}^{n-1}(-1)^i\Theta_{x_ix}\Theta_{xx_{i+1}}\delta_{xy_{i+1}}\omega_C(y_0,\dots,\hat{y}_{i+1},\dots,y_{n+1})$$
$$- (-1)^n\Theta_{x_nx}\delta_{xy_{n+1}}\omega_C(y_0,\dots,y_n)\big\}. \quad (4.81)$$

The sum over x can be executed. Remembering the Kronecker δs in the definition (4.50), the x subscripts of Θ-symbols may be replaced by y-subscripts. After a change of variable $i+1=j$ we finally obtain the assertion of proposition 4.1. q.e.d.

Proof 4.6 (Stokes Theorem) *We wish to show*

$$\int_\Delta d\omega = \int_{\partial\Delta}\omega \quad (4.82)$$

for $(n+1)$-simplices $\Delta = [y_0,\dots,y_{n+1}]$. Using the definition of the integral, and proposition 4.1, the left hand side evaluates to

$$\int_\Delta d\omega = \frac{1}{(n+2)!}\sum_{\pi\in S_{n+2}} sign(\pi)\, d\omega(y_{\pi0},\dots,y_{\pi(n+1)})$$
$$= \sum_{\pi\in S_{n+2}} sign(\pi)\big\{\Theta_{y_{\pi1}y_{\pi0}}\omega(\hat{y}_{\pi0},y_{\pi1}\dots,y_{\pi(n+1)})$$
$$+ \sum_{j=1}^{n}(-1)^j\Theta_{y_{\pi(j-1)}y_{\pi j}}\Theta_{y_{\pi j}y_{\pi(j+1)}}\omega(y_{\pi0},\dots,\hat{y}_{\pi j},\dots,y_{\pi(n+1)})$$
$$+ (-1)^{n+1}\Theta_{y_{\pi n}y_{\pi(n+1)}}\omega(y_{\pi1},\dots,y_{\pi n},\hat{y}_{\pi(n+1)})\big\} \quad (4.83)$$

By definition of simplices, there are links from y_i to y_k for any $j \neq k$. Therefore the Θ-factors are all equal to 1 and we arrive finally at

$$\int_\Delta d\omega = \sum_{\pi\in S_{n+2}} sign(\pi)\sum_{j=0}^{n+1}(-1)^j\omega(y_{\pi0},\dots,\hat{y}_{\pi j}\dots,y_{\pi(n+1)}) \quad (4.84)$$

Next we consider the right hand side. Inserting the definition of the boundary, we get

$$\int_{\partial\Delta}\omega = \sum_{i=0}^{n+1}(-1)^i\int_{[y_0,\dots\hat{y}_i\dots y_{n+1}]}\omega$$
$$= \sum_{i=0}^{n+1}(-1)^i\mathcal{S}\omega(y_{\pi0},\dots,\hat{y}_{\pi i},\dots,y_{\pi(n+1)}). \quad (4.85)$$

Lemma 1 asserts that the left hand side and the right hand side are equal. q.e.d.

Proof 4.7 (Theorem 4.4) *part 1: Let $\mathcal{J}$ be spanned by forms ω_C associated with paths $C : x \to z$ which meander. Let $C' : z \to w$ be arbitrary paths. We must show that $\omega_C\omega_{C'} \in \mathcal{J}$ and $\omega_{C'}\omega_C \in \mathcal{J}$ for all paths C'. If $w \neq x$ then $\omega_C\omega_{C'} = 0$ and otherwise*

$$\omega_C\omega_{C'} = \omega_{C\circ C'}. \tag{4.86}$$

Similarly, if $y \neq z$, then $\omega_{C'}\omega_C = 0$ and otherwise

$$\omega_{C'}\omega_C = \omega_{C'\circ C}. \tag{4.87}$$

We must therefore show that whenever C meanders, then $C_1 = C' \circ C$ meanders for arbitrary paths C' such that the decomposition is defined, and similarly for $C \circ C'$. If $C = (x_n, \dots, x_0)$ then $C' \circ C = (x_m, \dots, x_n, \dots, x_0)$ with $m \geq n$. The meandering condition on C means that for some i, j with $i \neq j, 0 \leq i, j \leq n$, $< x_j, x_i >$ is not an edge of Γ. But then it is a forteriori true that for some i, j with $i \neq j$, $0 \leq i, j \leq m$, $< x_j, x_i >$ is not an edge of Γ. Therefore $C' \circ C$ meanders. Similarly, $C \circ C'$ meanders. This completes the proof that $\mathcal{J}$ is a left and right ideal.

part 2: Since Ω_Γ is spanned by ω_C with C arbitrary paths, and the division by $\mathcal{J}$ sets $\omega_C = 0$ for meandering paths, $\Omega' = \Omega_\Gamma/\mathcal{J}$ is spanned by ω_C for nonmeandering paths C.

Proof 4.8 (Theorem 4.5) *Let $\mathcal{J}^n$ be spanned by n-forms $\xi_C := \omega_{\pi C} - sign(\pi)\omega_C$, where $\omega_C \in \Omega'^n$ and C is a path from x to y with $C \subset \Delta$ for some n-simplex Δ. Let $C' : z \to w$ be paths with $C' \subset \Delta$ for some m-simplex Δ'. We have to show that $\xi_C\omega_{C'} \in \mathcal{J}^{n+m}$ and $\omega_{C'}\xi_C \in \mathcal{J}^{n+m}$ for all nonmeandering paths C' of length m. If $w \neq x$ then $\xi_C\omega_{C'} = 0$ and otherwise*

$$\xi_C\omega_{C'} = \omega_{\pi C\circ C'} - sign(\pi)\omega_{C\circ C'}. \tag{4.88}$$

Similarly, if $y \neq z$, then $\omega_{C'}\omega_C = 0$ and otherwise

$$\omega_{C'}\xi_C = \omega_{C'\circ\pi C} - sign(\pi)\omega_{C'\circ C}. \tag{4.89}$$

We have therefore to show that there is a permutation π' such that

$$\omega_{C'\circ\pi C} - sign(\pi)\omega_{C'\circ C} = \omega_{\pi'(C'\circ C)} - sign(\pi')\omega_{C'\circ C}. \tag{4.90}$$

and $\pi'(0) = 0$ and $\pi'(m+n) = m+n$. If $C = (x_n, \dots, x_0)$ and $C' = (x_{m+n}, \dots, x_n)$, then $C' \circ C = (x_{m+n}, \dots, x_0)$. Define

$$\pi'(x_i) := \begin{cases} \pi(x_i) & for \quad 0 \leq i \leq n \\ x_i & for \quad n < i \leq m+n. \end{cases} \tag{4.91}$$

Obviously, $\omega_{C'}\xi_C$ can be written as the right hand side of eq.(4.87), thus $\xi_C \in \mathcal{J}^{n+m}$. Similarly, $\xi_C\omega_{C'} \in \mathcal{J}^{n+m}$. This completes the proof that $\mathcal{J}'$ is a left and right ideal of Ω'.

Chapter 5

Gauge theories on graphs

In this chapter a generalization of gauge theory and differential geometry on directed graphs will be considered. As the semicommutative calculus of chapter 4 forms the basis, it is possible to deal with non-abelian gauge theories without being forced to give up conventional locality properties, in sharp contrast to Connes' version of differential geometry [1, 18].

Classically, there are two different possible approaches to differential geometry on manifolds: the infinitesimal approach using vector potentials and the global approach employing parallel transporters. It was shown by Dimakis and Müller-Hoissen [11, 12] that the infinitesimal approach can also be used for discrete sets. We shall discuss both approaches and extend them to the case where the fibers do not have constant dimensions.

We argue that in the algebraic approach to the discrete case it is quite natural to admit nonunitary, possibly even noninvertible parallel transporters. Further we will consider a polar decomposition of parallel transporters. Finally, a discrete analogue of the principle of equivalence shall be discussed.

5.1 Infinitesimal approach

In the classical theory, the infinitesimal approach defines connections by reference to covariant derivatives (or Cartan Ehresmann connection forms) and proceeds to introducing vector potentials and computing field strength or curvature forms. One proves that a connection in this sense defines parallel transporters [23].

The covariant derivative acts on smooth sections $\psi \in \Gamma^\infty(\mathcal{M}, \mathcal{V})$ of vector bundles $\mathcal{V}$. Products ψf of sections and elements $f \in C^\infty(\mathcal{M})$ of the algebra of smooth functions are pointwise defined. Therefore the space of smooth sections is a right $C^\infty(\mathcal{M})$-module.

For convenience, let us recall some basic definitions.

Definition 5.1 (Module) *A right module $\mathfrak{V}$ over an algebra $\mathcal{A}$ is finitely*

generated if there is a finite number of elements $(E_\alpha)_{\alpha=1\ldots n}$ in $\mathfrak{V}$ such that every $\psi \in \mathfrak{V}$ can be written as $\sum_{\alpha=1}^n E_\alpha \psi^\alpha$ for some $\psi^1, \ldots, \psi^n$ in $\mathcal{A}$.
The elements $(E_\alpha)_{\alpha=1\ldots n}$ form a basis of $\mathfrak{V}$ if $0 = \sum_{\alpha=1}^n E_\alpha \psi^\alpha$ implies $\psi^\alpha = 0$, for all $\alpha = 1, \ldots, n$.
The right module $\mathfrak{V}$ is called free if it has a basis; it is called projective if it is a submodule of a free module $\mathfrak{W}$, i.e., there exists a free module $\mathfrak{W}$ and a submodule $\mathfrak{U}$ such that $\mathfrak{W} = \mathfrak{U} \oplus \mathfrak{V}$.

The prototype of a free module is $\mathcal{A}^n := \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{A}$. Any of its elements can be thought of as an n-dimensional vector with entries in $\mathcal{A}$ and can be written uniquely as a linear combination $\sum_{\alpha=1}^n E_\alpha \psi^\alpha$.

Actually, the fact that a vector bundle $\mathcal{V}$ is completely characterised by the space of its smooth sections, thought of as a right module, is the origin of the algebraic analogue of vector bundles. More precisely, the Serre-Swan theorem [18] asserts that locally trivial, finite-dimensional complex vector bundles over a compact Hausdorff space $\mathcal{M}$ correspond canonically to finitely generated, projective modules over the algebra $C^\infty(\mathcal{M})$. Conversely, if $\mathfrak{V}$ is a finitely generated, projective module over $C^\infty(\mathcal{M})$, the fiber $\mathcal{V}_x$ of the associated bundle $\mathcal{V}$ over the point $x \in \mathcal{M}$ is

$$\mathcal{V}_x := \mathfrak{V}/\mathfrak{V}\mathcal{I}_x, \tag{5.1}$$

where the ideal $\mathcal{I}_x$ corresponding to the point x is given by

$$\mathcal{I}_x = \{f \in C^\infty(\mathcal{M}) | f(x) = 0\}. \tag{5.2}$$

The above considerations suggest that in a noncommutative setting the notion of a vector bundle has to be replaced by a finitely generated, projective module over a noncommutative algebra. In this case, as the notion of a point ceases to be meaningful, also the interpretation as a "bundle of fibers over points" clearly makes no sense, reflecting the loss of locality.

In the case of the semicommutative calculus on graphs the situation is different. Also in this case the analogue of vector bundles is provided algebraically by modules over an algebra $\mathcal{A}$. However, as the algebra of functions on a graph is commutative, the notion of a fiber remains valid. More precisely, if $\mathfrak{V}$ is a right $\mathcal{A}$-module, then $\mathfrak{V}e^x =: V_x$ is a finite dimensional complex vector space for every x. It can be considered as the fiber over x. As a vector space, V_x is spanned by a basis $(E_\alpha(x))_{\alpha=1\ldots n}$ with $E_\alpha(x) := E_\alpha e^x$. Because of eq. (4.23) one obtains a right $\mathcal{A}$-module basis by setting

$$\mathfrak{V} \ni E_\alpha = \sum_{x \in \mathcal{M}} E_\alpha(x) e^x \tag{5.3}$$

We are interested in the case, where n may depend on $x, n = n(x)$. Thus we admit the possibility that the fibers may have different dimensions. This might be considered as a consequence of the Naheinformationsprinzip: the dimension of the fibers is no longer globally defined, but becomes a local concept. Then the notion of a moving frame has to be adjusted by the following

Definition 5.2 (Moving frame) *Let* $n := max_x dim V_x$. *A moving frame* $\mathbf{E} = (E_\alpha)_{\alpha=1\dots n}$ *is a basis for* $\mathfrak{V}$ *which is constructed as follows:*
For every x *one chooses a basis* $E_\alpha(x) \in V_x, \alpha = 1 \dots dim V_x$. *Setting* $E_\alpha(x) = 0$ *for* $\alpha > dim V_x$, *one obtains*

$$E_\alpha = \sum_x E_\alpha(x) e^x. \tag{5.4}$$

Let us now turn to the central notion of a connection [1, 18], which can be used as a starting point for both the infinitesimal and global approach to differential geometry on abritrary graphs.

Definition 5.3 *A connection on a right* $\mathcal{A}$*-modul* $\mathfrak{V}$ *is a* $\mathbb{C}$*-linear map*

$$\nabla : \mathfrak{V} \to \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1, \tag{5.5}$$

such that

$$\nabla(\psi f) = \nabla(\psi) f + \psi \otimes_A df \tag{5.6}$$

for all $f \in \mathcal{A}$ *and* $\psi \in \mathfrak{V}$.

∇ is also called the exterior covariant derivative. It extends uniquely as a $\mathbb{C}$-linear operator on the space of $\mathfrak{V}$-valued forms

$$\nabla : \mathfrak{V} \otimes_{\mathcal{A}} \Omega \to \mathfrak{V} \otimes_{\mathcal{A}} \Omega, \tag{5.7}$$

by requiring that

$$\nabla(\psi \otimes_{\mathcal{A}} \omega) = \nabla(\psi)\omega + \psi \otimes_{\mathcal{A}} d\omega \tag{5.8}$$

for all $\psi \in \mathfrak{V}$ and $\omega \in \Omega$. This can be rewritten as a graded Leibniz rule:

$$\nabla(\Psi\omega) = \nabla\Psi\omega + (-1)^r \Psi d\omega \tag{5.9}$$

for all $\Psi \in \mathfrak{V} \otimes_{\mathcal{A}} \Omega^r$ and all $\omega \in \Omega$.

Definition 5.4 *The curvature* $\mathcal{F}$ *of a connection* ∇ *is a right* $\mathcal{A}$*-module homomorphism*

$$\mathcal{F} : \mathfrak{V} \to \mathfrak{V} \otimes_{\mathcal{A}} \Omega^2 \tag{5.10}$$

defined by

$$\mathcal{F}(E) = \nabla^2 E. \tag{5.11}$$

It extends to a map

$$\mathcal{F} : \mathfrak{V} \otimes_{\mathcal{A}} \Omega \to \mathfrak{V} \otimes_{\mathcal{A}} \Omega \tag{5.12}$$

with the property

$$\mathcal{F}(\psi \otimes_{\mathcal{A}} \omega) = \mathcal{F}(\psi)\omega. \tag{5.13}$$

The definitions (5.3) and (5.4) are generally valid in differential geometry. However, we specialize to semicommutative differential geometry and turn to the consideration of vector potentials, which are only defined upon choosing a moving frame. Actually the situation is similar to that in the classical case.

Definition 5.5 (Vector potential) *Given a moving frame, the vector potential is defined as a matrix-valued 1-form* $\boldsymbol{A} = (A^\alpha{}_\beta) = \sum_{x,y} \boldsymbol{A}_{yx} e^{yx}, \alpha, \beta = 1 \dots n$ *by*

$$\nabla E_\alpha = E_\beta \otimes_{\mathcal{A}} A^\beta{}_\alpha. \tag{5.14}$$

Sums over repeated indices are understood. The field strength $\boldsymbol{F} = \sum_{x,y,z} \boldsymbol{F}_{zyx} e^{zyx}$ *is a matrix-valued 2-form defined in terms of the curvature by*

$$\mathcal{F}(E_\alpha) = E_\beta \otimes_{\mathcal{A}} F^\beta{}_\alpha. \tag{5.15}$$

Theorem 5.1 *1.) Let* $\psi = E_\alpha \psi^\alpha \in \mathfrak{V}$*. Then*

$$\nabla\psi = E_\alpha \otimes_{\mathcal{A}} (d\psi^\alpha + A^\alpha{}_\beta \psi^\beta). \tag{5.16}$$

2.) The field strength is given by

$$\boldsymbol{F} = d\boldsymbol{A} + \boldsymbol{A}\boldsymbol{A}. \tag{5.17}$$

Proof 5.1 *see reference [33].*

5.2 Endomorphism-valued forms

In conventional gauge theory endomorphism-valued forms play an important role [6]. We shall now introduce this concept in the context of semicommutative differential geometry. In chapter 6 we will use the exterior covariant derivative d_∇ of endomorphism-valued forms to formulate and study a generalized Yang-Mills action. Also the Bianchi identity takes an elegant form, if it is formulated with the help of d_∇. Moreover, in [34] the use of endomorphism-valued forms is necessary to get a discrete version of torsion. First we start with some preparatory considerations.

5.2.1 Exterior covariant derivative of $End_{\mathcal{A}}(\mathfrak{V})$-valued forms

Let $\mathfrak{V}$ be a right $\mathcal{A}$-module. Then the conjugate or dual space

$$\mathfrak{V}' := \{\alpha' : \mathfrak{V} \to \mathcal{A} | \alpha'(\psi f) = \alpha'(\psi) f,\ \psi \in \mathfrak{V}, f \in \mathcal{A}\} \tag{5.18}$$

is a left $\mathcal{A}$-module with

$$(f\alpha')(\psi) := f(\alpha'(\psi)) \tag{5.19}$$

for all $\alpha' \in \mathfrak{V}', \psi \in \mathfrak{V}, f \in \mathcal{A}$.
The basis of $\mathfrak{V}'$ is denoted by $(E^\alpha)_{\alpha=1\ldots n}$ and it satisfies

$$E^\alpha(E_\beta) = \delta^\alpha_\beta \mathbf{1}. \tag{5.20}$$

A basis E^α of the dual module provides a basis $E^\alpha(x) := e^x E^\alpha, \alpha = 1, \ldots, n$ of the dual vector space V'_x for every x. Again, because of eq. (4.23) one has

$$\mathfrak{V}' \ni E^\alpha = \sum_{x \in \mathcal{M}} e^x E^\alpha(x). \tag{5.21}$$

If $\mathfrak{V} = \mathcal{A}^n$, then the entries of $\mathfrak{V}'$ can be regarded as row vectors.

In order to identify $\mathfrak{V} \otimes_\mathcal{A} \mathfrak{V}'$ with $End_\mathcal{A}(\mathfrak{V}) = Hom_\mathcal{A}(\mathfrak{V}, \mathfrak{V})$, we use the following result from module theory [21].

Theorem 5.2 *Let $\mathfrak{E}, \mathfrak{F}$ be two right $\mathcal{A}$-modules. Define a map*

$$\Phi_{\mathfrak{E},\mathfrak{F}} : \mathfrak{F} \otimes_\mathcal{A} \mathfrak{E}' \to Hom_\mathcal{A}(\mathfrak{E}, \mathfrak{F}) \tag{5.22}$$

by

$$\Phi_{\mathfrak{E},\mathfrak{F}}(\psi \otimes_\mathcal{A} \alpha')(\phi) := \psi \alpha'(\phi) \tag{5.23}$$

for all $\alpha' \in \mathfrak{E}', \phi \in \mathfrak{E}, \psi \in \mathfrak{F}$. Then $\Phi_{\mathfrak{E},\mathfrak{F}}$ is an isomorphism from $\mathfrak{F} \otimes_\mathcal{A} \mathfrak{E}'$ to $Hom_\mathcal{A}(\mathfrak{E}, \mathfrak{F})$ iff $\mathfrak{E}$ is finitely generated and projective.

Proof 5.2 *see [21].*

If we set $\mathfrak{E} = \mathfrak{V}$ and $\mathfrak{F} = \mathfrak{V}$, we see that in the cases we are interested in one can identify $\mathfrak{V} \otimes_\mathcal{A} \mathfrak{V}'$ with $End_\mathcal{A}(\mathfrak{V}) = Hom_\mathcal{A}(\mathfrak{V}, \mathfrak{V})$. If we set $\mathfrak{F} = \mathfrak{V} \otimes_\mathcal{A} \Omega^n$ one can also write vector potential and field strength as an endomorphism-valued 1-form and 2-form, respectively

$$A = E_\alpha \otimes_\mathcal{A} A^\alpha{}_\beta \otimes_\mathcal{A} E^\beta \in \mathfrak{V} \otimes_\mathcal{A} \Omega^1 \otimes_\mathcal{A} \mathfrak{V}' \tag{5.24}$$

$$\mathcal{F} = E_\alpha \otimes_\mathcal{A} F^\alpha{}_\beta \otimes_\mathcal{A} E^\beta \in \mathfrak{V} \otimes_\mathcal{A} \Omega^2 \otimes_\mathcal{A} \mathfrak{V}'. \tag{5.25}$$

Note that the representation (5.24),(5.25) is not unique, nevertheless it is often useful. For example, it allows us to define a trace of an endomorphism-valued r-form.

Definition 5.6 (Trace) *The trace of an endomorphism-valued r-form is a map*

$$Tr : \mathfrak{V} \otimes_\mathcal{A} \Omega^r \otimes_\mathcal{A} \mathfrak{V}' \to \Omega^r \tag{5.26}$$

defined by

$$Tr(\Psi) = Tr(E_\alpha \otimes_\mathcal{A} \Psi^\alpha{}_\beta \otimes_\mathcal{A} E^\beta) := E^\beta(E_\alpha)\Psi^\alpha{}_\beta = \Psi^\alpha{}_\alpha \tag{5.27}$$

for all $\Psi \in \mathfrak{V} \otimes_\mathcal{A} \Omega^r \otimes_\mathcal{A} \mathfrak{V}'$.

The space of endomorphisms $End_{\mathcal{A}}(\mathfrak{V})$ becomes an algebra, if a multiplication is introduced by

$$(\psi \otimes_{\mathcal{A}} \alpha') \circ (\phi \otimes_{\mathcal{A}} \beta') := \psi \otimes_{\mathcal{A}} \alpha'(\phi)\beta' = \psi\alpha'(\phi) \otimes_{\mathcal{A}} \beta'. \tag{5.28}$$

Now we can regard $\mathfrak{V}$ also as an left $End_{\mathcal{A}}(\mathfrak{V})$-module, actually it is an $End_{\mathcal{A}}(\mathfrak{V})$-$\mathcal{A}$-bimodule.

Endomorphism-valued 0-forms induce endomorphisms of the fibers in the sense that

$$E_\alpha \otimes_{\mathcal{A}} E^\beta = E_\alpha \otimes_{\mathcal{A}} \sum_x e_x E^\beta = E_\alpha \sum_x e_x \otimes_{\mathcal{A}} E^\beta = \sum_x E_\alpha(x) \otimes_{\mathcal{A}} E^\beta(x). \tag{5.29}$$

Definition 5.7 *A connection ∇' on the dual left $\mathcal{A}$-modul $\mathfrak{V}'$ is a map*

$$\nabla' : \mathfrak{V}' \to \Omega^1 \otimes_{\mathcal{A}} \mathfrak{V}' \tag{5.30}$$

and defined by

$$d(\alpha'(\psi)) = (\nabla'\alpha')(\psi) + \alpha'(\nabla\psi). \tag{5.31}$$

We extend the duality contraction with an element in $\Omega \otimes_{\mathcal{A}} \mathfrak{V}'$ and $\mathfrak{V} \otimes_{\mathcal{A}} \Omega$ by defining

$$\omega \otimes_{\mathcal{A}} \alpha'(\psi) := \omega\alpha'(\psi) \qquad \alpha'(\psi \otimes_{\mathcal{A}} \omega) := \alpha'(\psi)\omega \tag{5.32}$$

for all $\omega \in \Omega, \alpha' \in \mathfrak{V}', \psi \in \mathfrak{V}$.

Theorem 5.3 *Given a moving frame the vector potenial $\boldsymbol{A}' = (A'^{\alpha}{}_{\beta})$ of the dual connection ∇' is*

$$A'^{\alpha}{}_{\beta} = -A^{\alpha}{}_{\beta}. \tag{5.33}$$

Proof 5.3 *If we set $\alpha' = E^\alpha$ and $\psi = E_\alpha$, we get from (5.31)*

$$\nabla' E^\alpha(E_\beta) = -E^\alpha(\nabla E_\beta). \tag{5.34}$$

Because of (5.32) we conclude that

$$\nabla' E^\alpha = A'^{\alpha}{}_{\beta} \otimes_{\mathcal{A}} E^\beta = -A^{\alpha}{}_{\beta} \otimes_{\mathcal{A}} E^\beta. \tag{5.35}$$

Now we are ready to introduce the exterior covariant derivative of endomorphisms.

Definition 5.8 *The exterior covariant derivative of endomorphisms d_∇ is a $\mathbb{C}$-linear map*

$$d_\nabla : \mathfrak{V} \otimes_{\mathcal{A}} \mathfrak{V}' \to \mathfrak{V} \otimes \Omega^1 \otimes \mathfrak{V}' \tag{5.36}$$

defined by

$$d_\nabla(\psi \otimes_{\mathcal{A}} \alpha') := \nabla\psi \otimes_{\mathcal{A}} \alpha' + \psi \otimes_{\mathcal{A}} \nabla'\alpha' \tag{5.37}$$

for all $\alpha' \in \mathfrak{V}', \psi \in \mathfrak{V}$.

It extends uniquely as a $\mathbb{C}$-linear operator on the space of $End_{\mathcal{A}}(\mathfrak{V})$-valued r-forms

$$d_\nabla : \mathfrak{V} \otimes_{\mathcal{A}} \Omega^r \otimes_{\mathcal{A}} \mathfrak{V}' \to \mathfrak{V} \otimes_{\mathcal{A}} \Omega^{r+1} \otimes_{\mathcal{A}} \mathfrak{V}' \tag{5.38}$$

by requiring that

$$d_\nabla(\psi \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \alpha') = \nabla\psi \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \alpha' + \psi \otimes_{\mathcal{A}} d\omega \otimes \alpha' + (-1)^r \psi \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \nabla'\alpha' \tag{5.39}$$

for all $\psi \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \alpha' \in \mathfrak{V} \otimes_{\mathcal{A}} \Omega^r \otimes_{\mathcal{A}} \mathfrak{V}'$.
The exterior covariant derivative can be rewritten as a graded commutator

$$d_\nabla \kappa = [\nabla, \kappa] := \nabla \circ \kappa - (-1)^r \kappa \circ \nabla. \tag{5.40}$$

for all $\kappa \in \mathfrak{V} \otimes \Omega^r \otimes \mathfrak{V}'$.
To see this, let $\kappa := \psi \otimes \omega \otimes \alpha'$ and $\phi \in \mathfrak{V}$. By definition we have

$$(d_\nabla \kappa)(\phi) = (\nabla\psi)\omega\alpha'(\phi) + \psi \otimes_{\mathcal{A}} d\omega\alpha'(\phi) + (-1)^r \psi \otimes_{\mathcal{A}} \omega \nabla'\alpha'(\phi). \tag{5.41}$$

Using (5.31) it follows that

$$\begin{aligned}
(d_\nabla \kappa)(\phi) &= \nabla\psi\omega\alpha'(\phi) + \psi \otimes_{\mathcal{A}} d\omega\alpha'(\phi) + \\
&+ (-1)^r \psi \otimes_{\mathcal{A}} \omega d(\alpha'(\phi)) - (-1)^r \psi \otimes_{\mathcal{A}} \omega\alpha'(\nabla\phi) \\
&= \nabla((\psi \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \alpha')(\phi)) - (\psi \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \alpha')(\nabla\phi) \\
&= (\nabla \circ \kappa)(\phi) - (\kappa \circ \nabla)(\phi).
\end{aligned} \tag{5.42}$$

With the help of d_∇ the Bianchi identity can be written in an elegant way, further the proof becomes almost trivial.

Theorem 5.4 (Bianchi identity) *The field strengh $\mathcal{F}$ fulfills the Bianchi identity*

$$d_\nabla \mathcal{F} = 0. \tag{5.43}$$

This can be rewritten in terms of $\boldsymbol{F} = \sum \boldsymbol{F}_{zyx} e^{zyx}$:

$$d_\nabla \boldsymbol{F} := d\boldsymbol{F} + [\boldsymbol{A}, \boldsymbol{F}] = 0. \tag{5.44}$$

Proof 5.4 *Proof of eq.(5.43) is trivial, $\nabla^3 - \nabla^3 = 0$. For proof of (5.44) we employ matrix notation: $(E_\alpha) := \boldsymbol{E}, (E^\alpha) := \boldsymbol{E}'$*

$$\begin{aligned}
d_\nabla \mathcal{F} &= \nabla\boldsymbol{E} \otimes_{\mathcal{A}} \boldsymbol{F} \otimes_{\mathcal{A}} \boldsymbol{E}' + \boldsymbol{E} \otimes_{\mathcal{A}} d\boldsymbol{F} \otimes_{\mathcal{A}} \boldsymbol{E}' + \boldsymbol{E} \otimes_{\mathcal{A}} \boldsymbol{F} \otimes_{\mathcal{A}} \nabla'\boldsymbol{E}' \\
&= \boldsymbol{E} \otimes_{\mathcal{A}} \boldsymbol{A}\boldsymbol{F} \otimes_{\mathcal{A}} \boldsymbol{E}' + \boldsymbol{E} \otimes_{\mathcal{A}} d\boldsymbol{F} \otimes_{\mathcal{A}} \boldsymbol{E}' - \boldsymbol{E} \otimes_{\mathcal{A}} \boldsymbol{F}\boldsymbol{A} \otimes_{\mathcal{A}} \boldsymbol{E}' \\
&= \boldsymbol{E} \otimes_{\mathcal{A}} (d\boldsymbol{F} + [\boldsymbol{A}, \boldsymbol{F}]) \otimes_{\mathcal{A}} \boldsymbol{E}' = 0.
\end{aligned} \tag{5.45}$$

q.e.d.

Similar to the classical case, a change of moving frame will be referred to as a (passive) gauge transformation.

Definition 5.9 *A gauge transformation* $\boldsymbol{G}$ *is a* $n \times n$ *invertible matrix valued function* $\boldsymbol{G} = \sum_x \boldsymbol{G}(x) e_x$ *which is obtained from* $\boldsymbol{g}(x) \in GL(dimV_x, \mathbb{C})$ *by setting*

$$G^{\alpha}{}_{\beta}(x) = \begin{cases} g^{\alpha}{}_{\beta}(x) & if \;\; \alpha, \beta = 1, \ldots, dimV_x \\ \delta^{\alpha}{}_{\beta} & otherwise. \end{cases} \tag{5.46}$$

Gauge transformations act on moving frames according to

$$E_{\alpha} \to E_{\alpha} = E_{\beta} G^{\beta}{}_{\alpha} = \sum_{x \in \mathcal{M}} E_{\beta}(x) G^{\beta}{}_{\alpha}(x). \tag{5.47}$$

Later we shall distinguish between unitary and nonunitary gauge transformations. Only the group of unitary gauge transformations will be a local symmetry.

Theorem 5.5 (Transformation Laws) *Under a gauge transformation*

$$\boldsymbol{A} \to \boldsymbol{G}^{-1} \boldsymbol{A} \boldsymbol{G} + \boldsymbol{G}^{-1} d\boldsymbol{G} \tag{5.48}$$

$$\boldsymbol{A}_{yx} \to \boldsymbol{G}^{-1}(y) \boldsymbol{A}_{yx} \boldsymbol{G}(x) + \boldsymbol{G}^{-1}(y)[\boldsymbol{G}(x) - \boldsymbol{G}(y)] \tag{5.49}$$

$$\boldsymbol{F} \to \boldsymbol{G}^{-1} \boldsymbol{F} \boldsymbol{G} \tag{5.50}$$

$$\boldsymbol{F}_{zyx} \to \boldsymbol{G}^{-1}(z) \boldsymbol{F}_{zyx} \boldsymbol{G}(x). \tag{5.51}$$

Proof 5.5 *see [34, 35].*

Note the locality properties of the field strength. $e^z \boldsymbol{F} e^x \neq 0$ does not imply $z = x$, cp. (2.23)

5.3 Parallel transport

Now we turn to the global approach to semicommutative differential geometry. Here the crucial objects are parallel transporters. Classically, the global approach starts from parallel transporters $\mathcal{U}(C) : V_x \to V_y$ along paths C from x to y which map the fiber V_x at x to the fiber V_y at y. Covariant derivatives are introduced by reference to parallel transporters. In a noncommutative setting the notion of a parallel transporter becomes invalid. In the semicommutative case the notion of a parallel transporter remains valid and appears naturally. Again the starting point is the connection.

In fact, any connection induces a map

$$\mathcal{U} : \mathfrak{V} \to \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1 \tag{5.52}$$

which is defined by

$$\mathcal{U}(E_{\alpha}) := E_{\alpha} \otimes_{\mathcal{A}} \rho + \nabla E_{\alpha}, \tag{5.53}$$

where $\rho = \sum_{x,y} e^{xy}$, cp. theorem 4.2, eq. (4.26). As a consequence of (5.6) and (4.28) one finds that $\mathcal{U}$ is a right $\mathcal{A}$-homomorphism

$$\mathcal{U}(\psi f) = \mathcal{U}(\psi) f \tag{5.54}$$

for all $\psi \in \mathfrak{V}, f \in \mathcal{A}$. $\mathcal{U}$ is called the parallel transport 1-form associated with the connection ∇. Using (5.54), we obtain

$$\mathcal{U}(E_\alpha(x)) = \mathcal{U}(E_\alpha) e^x. \tag{5.55}$$

Thus $\mathcal{U}(E_\alpha(x))$ can be expressed as follows

$$\begin{aligned} \mathcal{U}(E_\alpha) e^x &= \left(\sum_{y,z} E_\beta U^\beta{}_{yz\alpha} \otimes_{\mathcal{A}} e^{yz} \right) e^x \\ &= \sum_y E_\beta(y) U^\beta{}_{yx\alpha} \otimes_{\mathcal{A}} e^{yx}, \end{aligned} \tag{5.56}$$

where the matrix $\boldsymbol{U}_{yx} = (U^\alpha{}_{yx\beta})$ will be called the parallel transport matrix associated withe link $\langle y, x \rangle$ from x to y. Further we see that via

$$E_\alpha(x) \rightarrow \mathcal{U}_{yx} E_\alpha(x) := E_\beta(y) U^\beta{}_{yx\alpha} \tag{5.57}$$

the parallel transport 1-form defines a $\mathbb{C}$-linear map between the fibers of the right module

$$\mathcal{U}_{yx} : V_x \rightarrow V_y. \tag{5.58}$$

$\mathcal{U}_{yx}$ shall be called parallel transporter along the edge $\langle y, x \rangle$. It can be regarded as the analogue of the parallel transporters used in classical differential geometry.

As $\mathcal{U}_{yx} \in Hom_{\mathbb{C}}(V_x, V_y) = V_y \otimes_{\mathbb{C}} V'_x$ it can be written in the following form

$$\mathcal{U}_{yx} = E_\beta(y) U^\beta{}_{yx\alpha} \otimes_{\mathbb{C}} E^\alpha(x) =: E_\beta(y) U^\beta{}_{yx\alpha} E^\alpha(x), \tag{5.59}$$

We will often suppress the tensor product over $\mathbb{C}$. Note that the parallel transporter would vanish if one used in (5.59) the tensor product over $\mathcal{A}$. As $\mathcal{U} \in Hom_{\mathcal{A}}(\mathfrak{V}, \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1)$, the parallel transport 1-form $\mathcal{U}$ can be written as an element in $\mathfrak{V} \otimes_{\mathcal{A}} \Omega^1 \otimes_{\mathcal{A}} \mathfrak{V}'$

$$\mathcal{U} = \sum_{y,x} E_\beta(y) U^\beta{}_{yx\alpha} \otimes_{\mathcal{A}} e^{yx} \otimes_{\mathcal{A}} E^\alpha(x). \tag{5.60}$$

Our previous considerations are collected in the following definition

Definition 5.10 (Parallel transport) *Given a moving frame (E_α) and a connection the parallel transport 1-form*

$$\mathcal{U} : \mathfrak{V} :\rightarrow \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1 \tag{5.61}$$

is defined by

$$\mathcal{U}(E_\alpha) := E_\alpha \otimes_{\mathcal{A}} \rho + \nabla E_\alpha. \tag{5.62}$$

The parallel transporter along an edge $\langle y, x\rangle$

$$\mathcal{U}_{yx} : V_x \to V_y \tag{5.63}$$

is defined by

$$\mathcal{U}(E_\alpha) = \sum_{x,y} \mathcal{U}_{yx}(E_\alpha(x)) \otimes_{\mathcal{A}} e^{yx}, \tag{5.64}$$

where

$$\mathcal{U}_{yx} = E_\alpha(y) \otimes_{\mathbb{C}} U^\alpha{}_{yx\beta} E^\beta(x). \tag{5.65}$$

The matrix $\boldsymbol{U}_{yx}$ *with entries* $U^\alpha{}_{yx\beta}$ *defined by (5.56) will be called the parallel transport matrix.*

Let $\psi = \sum_x E_\alpha(x)\psi^\alpha(x) \in \mathfrak{V}$. Then we have

$$\mathcal{U}(\psi) = \sum_x \mathcal{U}(E_\alpha(x)\psi^\alpha(x)) = \sum_{y,x} E_\beta(y) U^\beta{}_{yx\alpha} \otimes_{\mathcal{A}} e^{yx}\psi^\alpha(x) \tag{5.66}$$

$$= \sum_{y,x} E_\beta(y) \otimes_{\mathcal{A}} (\boldsymbol{U}_{yx}\boldsymbol{\psi}(x))^\beta e^{yx}. \tag{5.67}$$

Treating the elements $\psi \in \mathfrak{V}$ as column vectors $\boldsymbol{\psi} := (\psi^\alpha)$, one can employ the following matrix notation for the parallel transport 1-form

$$\boldsymbol{\psi} \to \boldsymbol{U}\boldsymbol{\psi} = \sum_{y,x} \boldsymbol{U}_{yx}\boldsymbol{\psi}(x) e^{yx}, \qquad \boldsymbol{U} := \sum_{yx} \boldsymbol{U}_{yx} e^{yx}. \tag{5.68}$$

This notation is often used by Dimakis and Müller-Hoissen.

We have started with a connection and used it to introduce parallel transport 1-forms. Conversely, any right $\mathcal{A}$-homomorphism

$$\mathcal{U} : \mathfrak{V} :\to \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1 \tag{5.69}$$

allows one to define a connection

$$\nabla\psi := -\psi \otimes_{\mathcal{A}} \rho + \mathcal{U}(\psi) \tag{5.70}$$

in the sense of definition 5.3, as can easily be shown. More precisely, we have the following

Theorem 5.6 *Let* $\mathcal{U}$ *be a right* $\mathcal{A}$*-homomorpism*

$$\mathcal{U} : \mathfrak{V} :\to \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1. \tag{5.71}$$

Then the map

$$\nabla : \mathfrak{V} :\to \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1 \tag{5.72}$$

defined by

$$\nabla\psi := -\psi \otimes_{\mathcal{A}} \rho + \mathcal{U}(\psi) \tag{5.73}$$

for all $\psi \in \mathfrak{V}$ *satisfies (5.6).*

Proof 5.6

$$\begin{aligned}\nabla(\psi f) &= -\psi f \otimes_{\mathcal{A}} \rho + \mathcal{U}(\psi f) \\ &= \psi \otimes_{\mathcal{A}} df - \psi \otimes_{\mathcal{A}} \rho f + \mathcal{U}(\psi) f \\ &= \psi \otimes_{\mathcal{A}} df + \nabla \psi f. \end{aligned} \tag{5.74}$$

The relation between vector potential and parallel transport 1-form is given by

Theorem 5.7 *i.)Given a moving frame (E_α), the parallel transport matrix and the vector potential are related by* [1]

$$\boldsymbol{U} = \sum_{y,x} \boldsymbol{U}_{yx} e^{yx} = \rho + \boldsymbol{A} = \sum_{y,x} e^{yx}(\mathbf{1} + \boldsymbol{A}_{yx}) \tag{5.75}$$

ii.) Let $\psi = E_\alpha \psi^\alpha \in \mathfrak{V}$. Then the covariant derivative can be written as

$$\nabla \psi = E_\alpha \otimes_{\mathcal{A}} (U^\alpha{}_\beta \psi^\beta - \psi^\alpha \rho), \tag{5.76}$$

with $U^\alpha{}_\beta = \sum_{y,x} U^\alpha{}_{yx\beta} e^{yx}$.
iii.)The field strength is

$$\boldsymbol{F}_{zyx} = \sum_{z,y,x} (\boldsymbol{U}_{zy} \boldsymbol{U}_{yx} - \boldsymbol{U}_{zx}) \tag{5.77}$$

Proof 5.7 *see [34, 35, 33].*

Recall that gauge transformations are changes of moving frames. Vector potentials depend on a choice of moving frame, in contrast with parallel transport 1-forms.

Theorem 5.8 *The parallel transport form $\mathcal{U}$ is gauge invariant and therefore independent of a choice of moving frame.*

Corollary 5.1 *i.) The parallel transport matrix $(\boldsymbol{U}_{yx})$ along the edge from x to y transforms under gauge transformations according to*

$$\boldsymbol{U}_{yx} \to \boldsymbol{G}(y)^{-1} \boldsymbol{U}_{yx} \boldsymbol{G}(x) \tag{5.78}$$

and the corresponding matrix 1-form

$$\boldsymbol{U} = \sum_{x \neq y} \boldsymbol{U}_{yx} e^{yx} \tag{5.79}$$

transforms according to

$$\boldsymbol{U} \to \boldsymbol{G}^{-1} \boldsymbol{U} \boldsymbol{G}. \tag{5.80}$$

ii.) The difference $\boldsymbol{B} - \boldsymbol{A}$ of two vector potentials transforms under gauge transformations like a parallel transport matrix 1-form.

[1] Note the different signs in 5.75 and 2.12. By choosing appropriate basic 1-forms e^{yx} one can obtain a minus sign in the discrete case also. But then other formulae change, and for our purpose it is more convenient to work with (5.75), cp. [35, 2, 34].

Proof 5.8 *We employ matrix notation. The dual basis will be denoted by* $\boldsymbol{E}'$. *Further we will repeatedly use identities following from* $e^x e^{zy} = \delta_{xz} e^{zy}$ *and* $e^{zy} e^x = \delta_{yx} e^{zy}$, *in particular* $\boldsymbol{G}(y) e^{yx} = \boldsymbol{G} e^{yx}$, *and* $e^{yx} \boldsymbol{G}(x) = e^{yx} \boldsymbol{G}$, *and similarly for* $\boldsymbol{E}$ *and* $\boldsymbol{E}'$.

Given a parallel transport 1-form $\mathcal{U}$, *the gauge transformed parallel transport 1-form is*

$$\begin{aligned} \mathcal{U}' &= \sum_{xy} \boldsymbol{E}(y)\boldsymbol{G}(y) \otimes_{\mathcal{A}} [e^{yx} + \boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{G} - \boldsymbol{G}^{-1} d\boldsymbol{G}] \otimes_{\mathcal{A}} \boldsymbol{G}^{-1}\boldsymbol{E}'(x) \\ &= \sum_{xy} \boldsymbol{E}\boldsymbol{G} \otimes_{\mathcal{A}} [e^{yx} + \boldsymbol{G}^{-1}\boldsymbol{A}\boldsymbol{G} - \boldsymbol{G}^{-1}(\boldsymbol{G}(y) - \boldsymbol{G}(x))e^{yx}] \otimes_{\mathcal{A}} \boldsymbol{G}^{-1}\boldsymbol{E}' \\ &= \boldsymbol{E} \otimes_{\mathcal{A}} \boldsymbol{A} \otimes_{\mathcal{A}} \boldsymbol{E}' + \sum_{yx} \boldsymbol{E} \otimes_{\mathcal{A}} e^{yx} \otimes_{\mathcal{A}} \boldsymbol{E}' \\ &= \mathcal{U}. \end{aligned} \tag{5.81}$$

Proof 5.9 (Corollary) *Part i.) is an immediate consequence of the theorem.*
Part ii.): In the difference of two vector potentials the inhomogeneous term in the transformation law cancels out. Comparing with eq. (5.80) we see that the resulting transformation law is the same as for the parallel transport matrix.

5.4 Nonunitary parallel transport

5.4.1 Motivation

In the previous section we frequently encountered situations within the semi-commutative framework, which are very similiar to those in the classical case. Nevertheless, there is a profound difference between the continuum and the discrete case.

In either case, given a connection, a vector potential is only defined upon specifying a moving frame. Therefore the vector potential is a matrix valued 1-form $A^\alpha{}_{\beta\mu}e^\mu$, where e^μ, which substitutes for dx^μ, is attached to links $\mu = \langle y, x\rangle$ from x to y. But in the continuum, $\boldsymbol{A}_\mu$ takes its values in a Lie algebra of matrices, which is naturally a vector space. In the discrete case, the vector potential must also take values in a vector space, because it specifies elements of an algebra. When the parallel transport matrices take their values in a group, the discrete vector potential will be in the group algebra. To illustrate these points, let us consider the familiar case of lattice gauge theory.

In lattice gauge theory the gauge fields are provided by parallel transporters sitting on the links of the lattice. The parallel transporters are connected with the vector potentials $\boldsymbol{A}_\mu(x)$ in the continuum by

$$\boldsymbol{U}_\mu(x) = e^{-a\boldsymbol{A}_\mu(x)}, \tag{5.82}$$

where a is the lattice constant. But unlike the continuum case, the vector potential $\boldsymbol{A}_\mu(x)$ is not the vector potential used to define a covariant derivative. To see this let us define a covariant derivative of the form

$$\begin{aligned} D_\mu\psi(x) &= \partial_\mu\psi(x) + \boldsymbol{B}_\mu(x)\psi(x+a\mu) + \boldsymbol{C}_\mu(x)\psi(x) \\ &= \frac{1}{a}(\psi(x+a\mu) - \psi(x)) + \boldsymbol{B}_\mu(x)\psi(x+a\mu) + \boldsymbol{C}_\mu(x)\psi(x). \end{aligned}$$

$\boldsymbol{B}_\mu(x)$ and $\boldsymbol{C}_\mu(x)$ are supposed to compensate for the lack of gauge covariance of the lattice derivative. Under gauge transformations $\boldsymbol{G}(x)$, the covariant derivative has to satisfy

$$D'_\mu\psi(x)' = \boldsymbol{G}(x)D_\mu\psi(x) \tag{5.83}$$

Consequently we have

$$\begin{aligned} &\tfrac{1}{a} \quad (\boldsymbol{G}(x+a\mu)\psi(x+a\mu) - \boldsymbol{G}(x)\psi(x)) + \\ &+ \quad \boldsymbol{B}_\mu(x)'\boldsymbol{G}(x+a\mu)\psi(x+a\mu) + \boldsymbol{C}'_\mu(x)\boldsymbol{G}(x)\psi(x) \\ &= \quad \boldsymbol{G}(x)\big(\frac{1}{a}(\psi(x+a\mu) - \psi(x)) + \boldsymbol{B}_\mu(x)\psi(x+a\mu) + \boldsymbol{C}_\mu(x)\psi(x)\big). \end{aligned}$$

Comparing coefficients of $\psi(x+a\mu)$ and $\psi(x)$ gives

$$\boldsymbol{C}_\mu(x)' = \boldsymbol{G}(x)\boldsymbol{C}_\mu(x)\boldsymbol{G}(x)^{-1}, \tag{5.84}$$

$$\begin{aligned}\boldsymbol{B}_\mu(x)' &= \boldsymbol{G}(x)\boldsymbol{B}_\mu(x)\boldsymbol{G}(x+a\mu)^{-1} + \frac{1}{a}(\boldsymbol{G}(x)\boldsymbol{G}(x+a\mu)^{-1} - 1) \\ &= \boldsymbol{G}(x)\boldsymbol{B}_\mu(x)\boldsymbol{G}(x+a\mu)^{-1} + \boldsymbol{G}(x)\partial_\mu \boldsymbol{G}(x)^{-1}. \end{aligned}\tag{5.85}$$

It is consistent to set $\boldsymbol{C}_\mu(x) = 0$. For $\boldsymbol{B}_\mu(x)$ we then find a transformation rule quite similiar to the transformation behaviour of the continuum vector potentials.In order to be consistent with the transformation rule (5.85), the vector potential has to be related to the parallel transporter by

$$\boldsymbol{B}_\mu(x) = \frac{1}{a}(\boldsymbol{U}_\mu(x) - \mathbf{1}). \tag{5.86}$$

As the parallel transporters take values in the gauge group, eq. (5.86) shows that the vector potential $\boldsymbol{B}_\mu(x)$ takes values in the group algebra. The connection with the Lie algebra valued vector potentials $\boldsymbol{A}_\mu(x)$ in the continuum is given by

$$\boldsymbol{U}_\mu(x) = e^{-a\boldsymbol{A}_\mu(x)} = 1 + a\boldsymbol{B}_\mu(x). \tag{5.87}$$

Here $\boldsymbol{A}_\mu(x)$ denotes the smooth vector potentials in the continuum being evaluated at the lattice points $x^\mu = n^\mu a$. Then $a\boldsymbol{A}_\mu(x) \to 0$ as $a \to 0$ and consequently one finds

$$-\boldsymbol{B}_\mu(x) \underset{a\to 0}{\longrightarrow} \boldsymbol{A}_\mu(x). \tag{5.88}$$

Eq. (5.87) shows that the continuum vector potential $\boldsymbol{A}_\mu(x)$ and the vector potential $\boldsymbol{B}_\mu(x)$ are quite different as $\boldsymbol{B}_\mu(x)$ includes all higher order terms with respect to the lattice constant and is clearly not Lie algebra valued. In particular, unitarity of the parallel transporters does not appear as a natural requirement on the vector potential $\boldsymbol{B}_\mu(x)$. This observation can be generalized to the case of gauge theories on graphs. So one is faced with the following situation: Classically, unitarity of the parallel transporters follows from antihermiticity of the vector potential. However, in the algebraic approach to the discrete case there is no natural way to impose unitarity as a requirement on the vector potential. Actually we will see that an antihermiticity property does not imply unitarity of the parallel transporters any more. Therefore it is natural in this context to admit nonunitary, possibly even noninvertible parallel transporters.

5.4.2 Hermitian structures over modules

To proceed with our discussion concerning nonunitary parallel transport, we have to introduce Hermitian structures over modules. In particular, a Hermitian structure will allow us to formulate a discrete analogue of the "$*$-property" of the parallel transporters considered in section 3.3, cp. eq.(3.60).

Classically, any complex vector bundle can be endowed with a Hermitian structure. The conventional practice is to define a positive definite sesquilinear form $\langle\, ,\, \rangle$ on each fiber of the bundle, which has to vary smoothly with x. The noncommutative point of view is to eliminate the notion of a point. Consequently, what remains is a pairing $\mathfrak{V} \times \mathfrak{V} \to \mathcal{A}$ on a finite projective right $\mathcal{A}$-module with values in the algebra $\mathcal{A}$. However, we will see that in the semicommutative setting a Hermitean structure induces a Hermitian inner product on each fiber $V_x = \mathfrak{V} e_x$ of the module over the discrete manifold. So again we find a situation being quite similar to its classical counterpart.

The properties of a Hermitian structure are collected in the following definition.

Definition 5.11 (Hermitian Structure) *A Hermitean structure over a complex vector bundle $\mathfrak{V}$ is a sesqui-linear map*

$$\mathfrak{V} \times \mathfrak{V} \to \mathcal{A} \tag{5.89}$$

such that for all $f \in \mathcal{A}$ and all $\psi, \phi, \varphi \in \mathfrak{V}$
i.)$\langle \psi, \phi + \varphi \rangle = \langle \psi, \phi \rangle + \langle \psi, \varphi \rangle$
ii.) $\langle \psi, \phi f \rangle = \langle \psi, \phi \rangle f$
iii.) $\langle \psi, \psi \rangle > 0, \quad \langle \psi, \psi \rangle = 0 \Leftrightarrow \psi = 0$
iv.) $\langle \psi, \phi \rangle = \overline{\langle \phi, \psi \rangle}$.

As a consequence one has $\langle f\psi, \phi \rangle = \overline{f} \langle \psi, \phi \rangle$ The dual module $\mathfrak{V}'$ can be used to express a condition of non degeneracy of an Hermitian structure.

Definition 5.12 *A Hermitian structure over $\mathfrak{V}$ is called non degenerate if the map*

$$\mathfrak{V} \to \mathfrak{V}', \quad \alpha \to \langle \alpha, \, \rangle \tag{5.90}$$

is an isomorphism.

In this sense, the map can be regarded as metric on $\mathfrak{V}$. Thus a Hermitian structure enables one to identify the module in a basis independent way with its dual module.

For the case of a free module $\mathcal{A}^n$ there is a canonical Hermitean structure given by

$$\langle \psi, \phi \rangle := \boldsymbol{\psi}^\dagger \boldsymbol{\phi} = \sum_{i=1}^{n} \overline{\psi}_i \phi^i \tag{5.91}$$

where $\boldsymbol{\psi} = (\psi_1, \dots, \psi_n), \boldsymbol{\phi} = (\phi^1, \dots, \phi^n) \in \mathcal{A}^n$.

As already mentioned, a Hermitian structure also gives rise to an inner product on each fiber. Since $\langle\, ,\, \rangle$ is $\mathcal{A}$-valued it can be expressed as follows

$$\langle\, ,\, \rangle = \sum_x \langle\, ,\, \rangle_x e_x, \tag{5.92}$$

where $\langle\, ,\, \rangle_x$ denotes a $\mathbb{C}$-valued map on $\mathfrak{V} \times \mathfrak{V}$. Actually, as the notation indicates, it is a map defined defined on $V_x \times V_x$, as can easily be seen

$$\begin{aligned} \langle \psi, \phi \rangle &= \sum_{x,y} \langle \psi(x) e_x, \phi(y) e_y \rangle \\ &= \sum_{x,y} e_x \langle \psi(x), \phi(y) \rangle e_y \\ &= \sum_x \langle \psi(x), \phi(x) \rangle e_x = \sum_x \langle \psi(x), \phi(x) \rangle_x e_x. \end{aligned} \tag{5.93}$$

So we have the following theorem

Theorem 5.9 *A Hermitian structure over a semicommutative complex vector bundle $\mathfrak{V}$ induces on each fiber V_x a Hermitian inner product*

$$\langle\, ,\, \rangle_x : V_x \times V_x \to \mathbb{C} \tag{5.94}$$

such that for all $f, g \in \mathbb{C}$ and all $\psi(x), \phi(x), \varphi(x) \in V_x$
i.)$\langle \psi(x), \phi(x) + \varphi(x) \rangle_x = \langle \psi(x), \phi(x) \rangle_x + \langle \psi(x), \varphi(x) \rangle_x$
ii.) $\langle \psi(x), g\phi(x) \rangle_x = \langle \psi(x), \phi(x) \rangle_x g$
iii.) $\langle \psi(x), \psi(x) \rangle_x \geq 0$,
iv.) $\langle \psi(x), \phi(x) \rangle_x = \overline{\langle \phi(x), \psi(x) \rangle_x}$.
It is called non degenerate if the map

$$V_x \to V_x', \quad \psi(x) \to \langle \psi(x),\, \rangle_x \tag{5.95}$$

is an vector space isomorphism.

Let us finally emphasize that we shall also be interested in bilinear structures inducing bilinear inner products. This case can be treated in a similar way, see reference [34].

5.4.3 Nonunitary parallel transporters

We saw in section 5.3 that in the context of semicommutative differential geometry a parallel transporter [2] $\mathcal{T}_{yx}$ along $\langle y, x \rangle$ is a map

$$\mathcal{T}_{yx} : V_x \to V_y, \tag{5.96}$$

where $V_x := \mathfrak{V} e^x$. Further we argued that in this context it is natural to admit nonunitary parallel transporter

$$\mathbf{1} \neq \mathcal{T}_{xy} \mathcal{T}_{yx} : V_x \to V_x. \tag{5.97}$$

[2] Changing notation, we reserve the letter $\mathcal{U}$ for unitary parallel transporters, denoting more general ones by $\mathcal{T}$. Further we write $\boldsymbol{B}$ for the vector potential associated with $\mathcal{T}$ and $\boldsymbol{A}$ for the vector potential associated with the unitary parallel transporters $\mathcal{U}$.

We restrict attention to the situation where all fibers V_x have the same dimension and all parallel transporters are invertible. The case of non-invertible parallel transporter is much more complicated and will not be considered.

If all parallel transporters are invertible, the situation is similar to that in the continuum. More precisely, a unique polar decomposition exists and there are two different holonomy groups.

5.4.4 Holonomy groups

Consider a gauge theory on a graph with possibly nonunitary parallel transporters. We assume that all parallel transporters are invertible. For each site x consider loops $C : x \to x$ composed of links $\langle x_0, x_n\rangle, \langle x_n, x_{n-1}\rangle, \ldots \langle x_1, x_0\rangle$, with $x_i \neq x_{i-1}$. Define parallel transporters along these loops

$$\mathcal{T}(C) := \mathcal{T}_{x_0 x_n}\mathcal{T}_{x_n x_{n-1}} \ldots \mathcal{T}_{x_1 x_0} : V_x \to V_x. \tag{5.98}$$

Then we have

Theorem 5.10 *Let $\mathcal{T}_{xy}$ be the possibly nonunitary parallel transporters. Assume that the graph or pseudograph is connected. Then*

The groups H_x generated by the parallel transporters $\mathcal{T}(C) : x \to x$ and their inverses are independent of x modulo isomorphism. Their equivalence class is called the holonomy group of the connection and will be denoted by H.

The proof is trivial.

Next, we define a $*$-operation in the holonomy group H.

Definition 5.13 *A $*$-operation in the holonomy group H is defined as a map $H \to H$ which takes*

$$\mathcal{T}(C) \mapsto \mathcal{T}(C)^* := \mathcal{T}(-C). \tag{5.99}$$

It has the standard properties

$$(\mathcal{T}(C_2)\mathcal{T}(C_1))^* = \mathcal{T}(C_1)^*\mathcal{T}(C_2)^* \quad \textit{and} \quad \mathcal{T}(C)^{**} = \mathcal{T}(C). \tag{5.100}$$

Furthermore, we define.

$$\mathcal{T}^*_{yx} := \mathcal{T}_{xy} \tag{5.101}$$

5.4.5 Polar decomposition

If H is a Lie group, we can employ theorem 3.4 to make a polar decomposition of the discrete parallel transporter $\mathcal{T}_{xy}$

$$\mathcal{T}_{xy} = \mathcal{U}_{xy}\mathcal{P}_{xy} \tag{5.102}$$

with a unitary factor

$$\mathcal{U}_{yx} : V_x \to V_y, \qquad \mathcal{U}_{yx}^* = \mathcal{U}_{yx}^{-1}, \tag{5.103}$$

and a self-adjoint factor

$$\mathcal{P}_{yx} : V_x \to V_x, \qquad \mathcal{P}_{yx}^* = \mathcal{P}_{yx}. \tag{5.104}$$

But beware: We saw in theorem 3.4 that both factors are close to the identity of H. Therefore, in general one needs extra assumptions, cp. [34, 10]. Upon introducing a moving frame the maps $\mathcal{T}_{yx}$,$\mathcal{U}_{yx}$,and $\mathcal{P}_{yx}$ are converted to matrices and the polar decomposition (5.102) is translated into a matrix decomposition

$$\boldsymbol{T}_{xy} = \boldsymbol{U}_{xy}\boldsymbol{P}_{xy}. \tag{5.105}$$

Note that in general (5.105) is not the standard polar decomposition of matrices.

The parallel transport matrices are associated with vector potentials via

$$\boldsymbol{T}_{yx} = \mathbf{1} + \boldsymbol{B}_{yx} \tag{5.106}$$

$$\boldsymbol{U}_{yx} = \mathbf{1} + \boldsymbol{A}_{yx}. \tag{5.107}$$

The polar decomposition (5.105) yields a polar decomposition of the corresponding vector potentials

Theorem 5.11 (Polar decomposition of vector potentials) *The vector potentials introduced above are related by*

$$\boldsymbol{B}_{yx} = \boldsymbol{U}_{yx}(\boldsymbol{P}_{yx} - \mathbf{1}) + \boldsymbol{A}_{yx}. \tag{5.108}$$

Proof 5.10

$$\begin{aligned} \boldsymbol{B}_{yx} &= \boldsymbol{U}_{yx}\boldsymbol{P}_{yx} - \mathbf{1} \\ &= \boldsymbol{U}_{yx}\boldsymbol{P}_{yx} - \boldsymbol{U}_{yx} + \boldsymbol{U}_{yx} - \mathbf{1} \\ &= \boldsymbol{U}_{yx}(\boldsymbol{P}_{yx} - \mathbf{1}) + \boldsymbol{A}_{yx}. \end{aligned} \tag{5.109}$$

Let us now turn to the unitary gauge group.

Theorem 5.12 *The groups G_x associated with the vector potential $\boldsymbol{A}_{yx}$ and generated by the parallel transporters $\mathcal{U}(C) : x \to x$ are independent of x modulo isomorphism. Their equivalence class is called the Unitary Gauge Group and shall be denoted by G.*

The proof is trivial.

Note that in applications only the unitary gauge group is a local symmetry, because the action involves the scalar product, which is preserved only by the elements of the unitary gauge group. Nevertheless, matter fields must make up representation spaces of the holonomy group, since they cannot be parallel transported otherwise. By construction, these representation spaces are at the same time representation spaces for the unitary gauge group, but irreducible representations may become reducible. This is a mechanism whereby fields may be required to make up larger multiplets than required by local symmetry. The construction principle "local from global symmetry via vector potentials" is thereby transcended.

Let us assume that the vector spaces V_x come equipped with a nondegenerate bi- or sesquilinearform. We are again interested in situations where the algebraic $*$-operation $\mathcal{T}_{yx} \mapsto \mathcal{T}^*_{yx}$ yields at the same time the adjoint map between vector spaces with a bi- or sesquilinearform.

Here we focus on a Hermitian structure. For later convenience we introduce also for matrix-valued forms $\boldsymbol{M} := \sum_{x,y} \boldsymbol{M}_{yx} e^{yx}$ a $*$-operation defined by

$$\boldsymbol{M}^* := \sum_{x,y} \boldsymbol{M}^\dagger_{yx} e^{yx\star} = \sum_{x,y} \boldsymbol{M}^\dagger_{yx} e^{xy}, \tag{5.110}$$

where $\dagger$ denotes the Hermitian adjoint of a matrix.

In the literature about noncommutative geometry, one finds the following standard definition of a unitary connection [1, 18].

Definition 5.14 *A connection ∇ on $\mathfrak{V}$ is said to be unitary or to be compatible with the Hermitian structure if*

$$d\langle\psi, \phi\rangle = -\langle\nabla\psi, \phi\rangle + \langle\psi, \nabla\phi\rangle. \tag{5.111}$$

In (5.111) the Hermitian structure has to be extended to $\mathcal{A}$-linear maps

$$\langle\, , \,\rangle : \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1 \times \mathfrak{V} \to \mathcal{A}, \quad \langle\psi \otimes_{\mathcal{A}} \omega, \phi\rangle := \omega^\star \langle\psi, \phi\rangle \tag{5.112}$$

$$\langle\, , \,\rangle : \mathfrak{V} \times \mathfrak{V} \otimes_{\mathcal{A}} \Omega^1 \to \mathcal{A}, \quad \langle\psi, \phi \otimes_{\mathcal{A}} \omega\rangle := \langle\psi, \phi\rangle\omega \tag{5.113}$$

for all $\psi, \phi \in \mathfrak{V}$ and all $\omega \in \Omega^1$.

Definition 5.14 is in general use, both in the classical and in the noncommutative framework. Classically, a connection satisfying condition (5.111) leads indeed to unitary parallel transporters in our sense. In the noncommutative setting an interpretation is more difficult as the notion of a parallel transporter ceases to be meaningful. Nevertheless one can show that (5.111) results in an (anti)-hermiticity property of the vector potential.

It is a remarkable fact that in the semicommutative framework condition (5.111) does not force the parallel transporters to be unitary, although it implies also in the semicommutative case an (anti)-hermitian vector potential. Instead, the parallel transporters associated to a connection obeying (5.111) have only to satisfy

$$\langle \mathcal{T}_{yx}\psi(x), \phi(y)\rangle_y = \langle\psi(x), \mathcal{T}_{xy}\phi(y)\rangle_x, \tag{5.114}$$

cp. theorem 3.5, eq.(3.60)

Theorem 5.13 *Let ∇ be a connection on a bidirectional graph.*
Then,
1.) ∇ satisfies condition (5.111) $\Leftrightarrow$ the associated parallel transporters fulfill

$$\langle \mathcal{T}_{yx}\psi(x), \phi(y)\rangle_y = \langle\psi(x), \mathcal{T}_{xy}\phi(y)\rangle_x. \tag{5.115}$$

2.) ∇ satisfies condition (5.111) $\Leftrightarrow$ the associated vector potential is Hermitian

$$\boldsymbol{B}^* = \boldsymbol{B} \Leftrightarrow \boldsymbol{B}^\dagger_{yx} = \boldsymbol{B}_{xy} \tag{5.116}$$

Proof 5.11 *part 1.)*

$$\begin{aligned}\langle\nabla\psi, \phi\rangle &= -\langle\psi \otimes_{\mathcal{A}} \rho, \phi\rangle + \langle\mathcal{T}(\psi), \phi\rangle \\ &= -\rho^\star\langle\psi, \phi\rangle + \sum_{x\neq y}\langle\mathcal{T}_{yx}\psi(x) \otimes_{\mathcal{A}} e^{yx}, \phi\rangle \\ &= -\rho\langle\psi, \phi\rangle + \sum_{x\neq y} e^{yx*}\langle\mathcal{T}_{yx}\psi(x), \phi\rangle \\ &= -\rho\langle\psi, \phi\rangle + \sum_{x\neq y} e^{xy}\langle\mathcal{T}_{yx}\psi(x), \phi(y)\rangle_y \end{aligned} \tag{5.117}$$

A similar calculation leads to

$$\langle\psi, \nabla\phi\rangle = -\langle\psi, \phi\rangle\rho + \sum_{x\neq y}\langle\psi(y), \mathcal{T}_{yx}\phi(x)\rangle_y e^{yx}. \tag{5.118}$$

Using $d\langle\psi, \phi\rangle = [\rho, \langle\psi, \phi\rangle]$ and $d\langle\psi, \phi\rangle + \langle\nabla\psi, \phi\rangle = \langle\psi, \nabla\phi\rangle$, one arrives at

$$\sum_{x\neq y}\langle\mathcal{T}_{yx}\psi(x), \phi(y)\rangle_y e^{xy} = \sum_{x\neq y}\langle\psi(x), \mathcal{T}_{xy}\phi(y)\rangle_x e^{xy}. \tag{5.119}$$

Taking into account that the basis 1-forms are linearly independent for any reduced differential calculus, we can finally conclude that

$$\langle \mathcal{T}^*_{xy}\psi(x), \phi(y)\rangle_y = \langle\psi(x), \mathcal{T}_{xy}\phi(y)\rangle_x, \tag{5.120}$$

with $\mathcal{T}^*_{xy} = \mathcal{T}_{yx}$.
part 2.) Eq (5.120) leads to

$$\sum_{x\neq y}(\boldsymbol{B}_{yx}\boldsymbol{\psi}(x))^\dagger\boldsymbol{\phi}(y)e^{xy} = \sum_{x\neq y}\boldsymbol{\psi}(x)^\dagger\boldsymbol{B}_{xy}\boldsymbol{\phi}(y)e^{xy} \tag{5.121}$$

It follows that

$$\sum_{xy}\boldsymbol{B}^\dagger_{xy}e^{yx} = \sum_{xy}\boldsymbol{B}_{xy}e^{xy} \Rightarrow \boldsymbol{B}^* = \boldsymbol{B}. \tag{5.122}$$

Note that it is crucial that the graph is assumed to be bidirectional. Otherwise $e^{yx\star}$ may vanish and ρ may not be self-adjoint $\rho^\star \neq \rho$.

The polar decomposition of vector potentials and parallel transporters, respectively leads to a polar decompostion of the connection. Indeed we know from theorem 5.6 that parallel transporters determine a connection. Thus in the nonunitary case there are two connections.

Definition 5.15 *Given a moving frame and a polar decomposed vector potential, consider the possibly nonunitary connection*

$$\nabla^{\mathcal{T}} E_\alpha := E_\beta \otimes_{\mathcal{A}} B^\beta{}_\alpha. \tag{5.123}$$

The unitary connection is defined by

$$\nabla^{\mathcal{U}} E_\alpha := E_\beta \otimes_{\mathcal{A}} A^\beta{}_\alpha. \tag{5.124}$$

As a consequence, two kinds of field strength arise, associated with $\nabla^{\mathcal{T}}$ and $\nabla^{\mathcal{U}}$ respectively.

Definition 5.16 *The nonunitary field strength is defined by*

$$\mathcal{F}^{\mathcal{T}} := \nabla^{\mathcal{T}} \circ \nabla^{\mathcal{T}}. \tag{5.125}$$

The unitary field strength is defined by

$$\mathcal{F}^{\mathcal{U}} := \nabla^{\mathcal{U}} \circ \nabla^{\mathcal{U}}. \tag{5.126}$$

Now it is crucial that the field strength $\mathcal{F}^{\mathcal{T}}$ describes additional degrees of freedom. In reference [34] it is shown that these additional degrees of freedom can accomodate vierbein fields. In the next chapter we shall see that they also describe Higgs fields in theories with extra dimensions.

5.4.6 Principle of equivalence

One of the corner stones of conventional gauge theories is the principle of equivalence. Naturally the question arises, if a principle of equivalence can be formulated within the framework of nonunitary parallel transporters on discrete manifolds. To answer the question, let us recall the content of the classical principle of equivalence. It asserts that the free field equation for matter fields is valid at one point x, if a suitable moving frame is chosen in a neighbourhood of x. Consequently, it is always possible to make the vector potential $\boldsymbol{A}_\mu$ vanish at any one point by a suitable choice of a moving frame.

However, in general it is not possible to choose a moving frame to make $\boldsymbol{A}_\mu$ vanish in a finite region. Otherwise it would be in general possible to transform the field strength away.

To analyse the situation in the discrete case, we have to consider the transformation law of the vector potential or more conveniently, of the parallel transport matrices (5.78). At first we shall restrict our attention on unitary parallel transport.

It is obvious that

$$\boldsymbol{U}_{yx} = \mathbf{1} \Leftrightarrow \boldsymbol{A}_{yx} = 0. \tag{5.127}$$

So a gauge transformation $\boldsymbol{G}(x)$ making $\boldsymbol{U}_{yx} = \mathbf{1}$ also forces the vector potential to vanish. By inspection of (5.78) we see that, given a parallel transport matrix $\boldsymbol{U}_{yx}$, $\boldsymbol{G}(x)$ has to satisfy

$$\boldsymbol{U}_{yx} = \boldsymbol{G}(y)\boldsymbol{G}(x)^{-1} \tag{5.128}$$

in order to transform $\boldsymbol{A}_{yx}$ to zero. In this way we find that for any point x all the vector potentials $(\boldsymbol{A}_{yx})_{y\in\mathcal{M}}$ can be transformed to zero.

What is about the field strength? Consider as an example a bidirectional graph consisting of three points x, y, z and the field strength component $\boldsymbol{F}_{zyx}$. Theorem 5.7 tells us that

$$\boldsymbol{F}_{zyx} = \boldsymbol{U}_{zy}\boldsymbol{U}_{yx} - \boldsymbol{U}_{zx}. \tag{5.129}$$

If the field strength could be gauged away, the gauge transformation would have to satisfy

$$\boldsymbol{U}_{zy} = \boldsymbol{G}(z)\boldsymbol{G}(y)^{-1}, \tag{5.130}$$

$$\boldsymbol{U}_{yx} = \boldsymbol{G}(y)\boldsymbol{G}(x)^{-1}, \tag{5.131}$$

$$\boldsymbol{U}_{zx} = \boldsymbol{G}(z)\boldsymbol{G}(x)^{-1}. \tag{5.132}$$

Note that in general it is not possible to find gauge transformations satisfying (5.130)-(5.132) simultaneously. For example, if the gauge transformation satisfies (5.130) and (5.131), then $\boldsymbol{G}(z)\boldsymbol{G}(x)^{-1}$ is already fixed.

We conclude that also in the discrete case a principle of equivalence is valid in the sense that by a choice of a suitable moving frame the vector potentials $(\boldsymbol{A}_{zx})_{z\in\mathcal{M}}$ belonging to unitary parallel transport can be transformed to zero for any chosen point x. However, in general this is not possible for $(\boldsymbol{A}_{zx})_{z,x\in\mathcal{M}}$, preventing that the field strength can be gauged away.

There are important exceptions, as in the discrete case there are components of the field strength which posess no classical counterpart, for example

$$\boldsymbol{F}_{xyx} = \boldsymbol{T}_{xy}\boldsymbol{T}_{yx} - \mathbf{1}. \tag{5.133}$$

Clearly, $\boldsymbol{F}_{xyz}$ vanishes anyway if the parallel transporters are unitary, $\boldsymbol{T}_{yx} = \boldsymbol{T}_{xy}^{-1}$. Thus we have to reconsider our arguments for the nonunitary case. Actually, there is a profound difference.

Eq (5.128) shows that in general even locally, i.e. for fixed x, the vector potential $\boldsymbol{B}_{yx}$ associated with the nonunitary parallel transporters can not be transformed away, if one permits only $\boldsymbol{G} \in G$. Instead one has to employ gauge transformations which take values in the holonomy group H. But we have argued that only the unitary gauge group is a local symmetry. In this sense, a gauge theory involving nonunitary parallel transporters violates the principle of equivalence.

Let us finally mention that also gauge transformations taking their value in H do not enable one to gauge the field strength $\boldsymbol{F}_{xyx}$ away, as they would have to satisfy

$$\boldsymbol{T}_{yx} = \boldsymbol{G}(y)\boldsymbol{G}(x)^{-1} \tag{5.134}$$

$$\boldsymbol{T}_{xy} = \boldsymbol{G}(x)\boldsymbol{G}(y)^{-1}. \tag{5.135}$$

Obviously, this would only be possible if $\boldsymbol{T}_{yx} = \boldsymbol{T}_{xy}^{-1}$, in contradiction with our assumptions.

Chapter 6

Geometry of Higgs fields

In this chapter our aim is to reveal the geometry of Higgs fields. We will use the language and tools developed in chapter 4 and 5 to study gauge theories on a symmetric lattice in arbitrary dimensions. It was shown by Dimakis and Müller-Hoissen [11, 12, 14] that in the particular case of unitary parallel transporters one obtains in this way conventional lattice gauge theory.

However, we are interested in constructing a gauge theory, which automatically includes a Higgs potential and kinetic terms for Higgs fields. This can be achieved by specializing the n-dimensional symmetric lattice theory to a 5-dimensional theory, where the gauge fields along the extra dimension are assumed to be nonunitary. We will compute the generalized Yang-Mills action and show that it can be expressed entirely in terms which have a geometric meaning. Further we point out that our tools provide a suitable starting point for extensions of the standard model, which have been proposed by C. T. Hill and others [20, 8, 9]. In these models the transverse lattice slices appears as branes and every brane carries a 3+1 dimensional gauge theory.

Since we shall be interested in a reinterpretation of the standard model, it will be necessary to thin the degrees of freedom and to arrive in this way at a 2-brane system. This may be thought of as a result of a real space renormalization group flow.

In the 2-brane picture the matter fields live on the branes and the different chiralities communicate via a nonunitary Higgs parallel transporter connecting left and right handed matter.

6.1 Symmetric lattice

The involution introduced in chapter 4 is defined in a natural way on the universal differential algebra. Also it was found to be crucial to study nonunitary parallel transport, for example in the formulation of the $*$-property of the parallel transporters, cp. theorem 5.13.

However, the involution is in general not consistent with reductions of the universal algebra, as it requires that with $e^{xy} = 0$ one must also have $e^{yx} = 0$. Such a reduction possessing this property is called symmetric [14].

A particular example of a symmetric reduction is the symmetric lattice. It is obtained by choosing $\mathcal{M} := \mathbb{Z}^n = \{x = (x^\mu) | \mu = 0, \dots n-1, x^\mu \in \mathbb{Z}\}$ and by imposing the relations

$$e^{xy} \neq 0 \Leftrightarrow y = x + \hat{\mu} \quad \text{or} \quad y = x - \hat{\mu} \quad \text{for some}\, \mu, \tag{6.1}$$

where [1] $\hat{\mu} := (\mu^\nu) := (\delta^\nu_\mu)$. The resulting graph is a hypercubic lattice without distinguished directions, i.e. both arrows are present between connected vertices.

It turns out to be convenient to introduce a variable ϵ, which takes values in $\{\pm 1\}$. In addition, we define $e_x^{\epsilon\mu} := e^{x+\epsilon\mu, x}$ and, more generally,

$$e_x^{\epsilon_1\mu_1 \dots \epsilon_r\mu_r} := e_{x+\epsilon_1\mu_1}^{\epsilon_2\mu_2 \dots \epsilon_r\mu_r} e_x^{\epsilon_1\mu_1}. \tag{6.2}$$

In particular, $e_x^{\epsilon\mu\epsilon'\nu} = e^{x+\epsilon\mu+\epsilon'\nu, x+\epsilon\mu, x}$.

It is important to realize that as a consequence of (6.1) there are relations between higher forms, especially 2-forms. Acting with d on the identity

$$e^{x+\epsilon\mu+\epsilon'\nu, x} = 0, \tag{6.3}$$

yields

$$e^{x+\epsilon\mu+\epsilon'\nu, x+\epsilon\mu, x} = -e^{x+\epsilon\mu+\epsilon'\nu, x+\epsilon'\nu, x} \tag{6.4}$$

for $\epsilon\mu + \epsilon'\nu \neq 0$. These relations are supplemented with corresponding relations for the case $\epsilon\mu + \epsilon'\nu = 0$:

$$e^{x, x+\mu, x} = -e^{x, x-\mu, x}. \tag{6.5}$$

The relations (6.4) and (6.5) can be considered as a discrete analogue of the classical relations

$$dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu. \tag{6.6}$$

They are important for the calculation of the field strength of the symmetric lattice. Actually it was found that within the framework of the symmetric lattice calculus it is possible to recover ordinary lattice gauge theory [14].

Nevetheless there is a profound difference. In fact, classically, expressions like $dx^\mu \wedge dx^{-\mu}$ are meaningless. Therefore in those cases a component $\boldsymbol{F}_{\mu,-\mu}(x)$ of the field strength makes no sense at all. But in the case of the symmetric lattice calculus non-vanishing 2-forms $e_x^{\mu,-\mu}$ naturally appears. As a consequence the resulting field strength also contains components like

[1] In the following we shall often omit the hat over μ.

$\boldsymbol{F}_{\mu,-\mu}(x)$, which do not have to vanish, since also nonunitary parallel transporters are admitted.

In this way the resulting gauge theory is able to describe additional degrees of freedom, which can be interpreted as a Higgs field.

To calculate the field strength for the symmetric lattice, let us start from the general formula for the generalized field strength associated with possibly nonunitary parallel transporters, cp. theorem 5.7:

$$\boldsymbol{F}^{\mathcal{T}} = \sum_{i,j,k} e^{kji}(\boldsymbol{T}_{kj}\boldsymbol{T}_{ji} - \boldsymbol{T}_{ki}) \ .$$

Inserting $j = x + \epsilon\mu$ and $k = x + \epsilon\mu + \epsilon'\nu$ yields

$$\begin{aligned}
\boldsymbol{F}^{\mathcal{T}} &= \sum_{x,\epsilon,\mu,\epsilon',\nu} e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu,x}(\boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu}\boldsymbol{T}_{x+\epsilon\mu,x} - \boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x}) \\
&= \frac{1}{2}\sum_{x,\epsilon,\mu,\epsilon',\nu} e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu,x}(\boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu}\boldsymbol{T}_{x+\epsilon\mu,x} - \boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x}) \\
&+ \frac{1}{2}\sum_{x,\epsilon,\mu,\epsilon',\nu} \underbrace{e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon'\nu,x}}_{=-e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu,x}}(\boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon'\nu}\boldsymbol{T}_{x+\epsilon'\nu,x} - \boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x}) \\
&= \frac{1}{2}\sum_{x,\epsilon,\mu,\epsilon',\nu} e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu,x}(\boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu}\boldsymbol{T}_{x+\epsilon\mu,x} - \\
&+ \boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon'\nu}\boldsymbol{T}_{x+\epsilon'\nu,x}) \\
&=: \frac{1}{2}\sum_{x,\epsilon,\mu,\epsilon',\nu} e_x^{\epsilon\mu,\epsilon'\nu}\boldsymbol{F}^{\mathcal{T}}_{\epsilon\mu,\epsilon'\nu}(x).
\end{aligned} \tag{6.7}$$

The geometric information encoded in $\boldsymbol{F}^{\mathcal{T}}$ can be revealed by employing the exterior covariant derivative of endomorphism-valued forms. We find, cp. 5.40,

Theorem 6.1

$$\boldsymbol{F}^{\mathcal{T}} = \frac{1}{2}d_{\nabla}\boldsymbol{T} = \frac{1}{2}\sum_{x,\epsilon,\mu,\epsilon',\nu} e_x^{\epsilon\mu,\epsilon'\nu}\boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}_{\epsilon'\nu}(x), \tag{6.8}$$

where $\boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}_{\epsilon'\nu}(x)$ *is defined as*

$$\boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}_{\epsilon'\nu}(x) := \boldsymbol{T}_{\epsilon'\nu}(x+\epsilon\mu)\boldsymbol{T}_{\epsilon\mu}(x) - \boldsymbol{T}_{\epsilon\mu}(x+\epsilon'\nu)\boldsymbol{T}_{\epsilon'\nu}(x). \tag{6.9}$$

Note that we have introduced the notation $\boldsymbol{T}_{\epsilon\mu}(x) := \boldsymbol{T}_{x+\epsilon\mu,x}$.

Proof 6.1 *By definition,*

$$d_{\nabla}\boldsymbol{T} = d\boldsymbol{T} + \boldsymbol{B}\boldsymbol{T} + \boldsymbol{T}\boldsymbol{B} \tag{6.10}$$

Now we use 4.29

$$\begin{aligned} d\boldsymbol{T} &= d\sum_{x,\epsilon\mu} \boldsymbol{T}_{x+\epsilon\mu,x} e^{x+\epsilon\mu,x} = \rho\boldsymbol{T} + \boldsymbol{T}\rho - \sum_{x,\epsilon\mu} \boldsymbol{T}_{x+\epsilon\mu,x} e^{x+\epsilon\mu}\rho^2 e^x \\ &= \rho\boldsymbol{T} + \boldsymbol{T}\rho, \end{aligned} \tag{6.11}$$

because $e^{x+\epsilon\mu}\rho^2 e^x$ *has to vanish for a symmetric lattice. Furthermore, we have*

$$\boldsymbol{B} = \boldsymbol{T} - \rho. \tag{6.12}$$

Consequently, we arrive at

$$\begin{aligned} d_\nabla \boldsymbol{T} &= \rho\boldsymbol{T} + \boldsymbol{T}\rho + \boldsymbol{T}^2 - \rho\boldsymbol{T} + \boldsymbol{T}^2 - \boldsymbol{T}\rho \\ &= 2\boldsymbol{T}^2. \end{aligned} \tag{6.13}$$

$$\begin{aligned} \boldsymbol{T}^2 &= \sum_{x,\epsilon,\mu,\epsilon',\nu} e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu,x} \boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu} \boldsymbol{T}_{x+\epsilon\mu,x} \\ &= \frac{1}{2}\sum_{x,\epsilon,\mu,\epsilon',\nu} e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu,x} \boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu} \boldsymbol{T}_{x+\epsilon\mu,x} \\ &+ \frac{1}{2}\sum_{x,\epsilon,\mu,\epsilon',\nu} \underbrace{e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon'\nu,x}}_{=-e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu,x}} \boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon'\nu} \boldsymbol{T}_{x+\epsilon'\nu,x} \\ &= \frac{1}{2}\sum_{x,\epsilon,\mu,\epsilon',\nu} e^{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu,x} (\boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon\mu} \boldsymbol{T}_{x+\epsilon\mu,x} - \\ &+ \boldsymbol{T}_{x+\epsilon\mu+\epsilon'\nu,x+\epsilon'\nu} \boldsymbol{T}_{x+\epsilon'\nu,x}) \\ &= \boldsymbol{F}^{\mathcal{T}}. \end{aligned} \tag{6.14}$$

q.e.d.

Unfortunately, (6.8) is not right for general graphs.

Remark 6.1 *In general,*

$$\boldsymbol{F}^{\mathcal{T}} \neq \frac{1}{2} d_\nabla \boldsymbol{T}, \tag{6.15}$$

because $e^k\rho^2 e^i$ *does not have to nanish. Thus one gets*

$$\begin{aligned} d_\nabla \boldsymbol{T} &= 2\boldsymbol{T}^2 - \sum_{k,i} \boldsymbol{T}_{ki} e^k \rho^2 e^i \\ &= \sum_{k,j,i} \big(2\boldsymbol{T}_{kj}\boldsymbol{T}_{ji} - \boldsymbol{T}_{ki}\big) e^{kji} \\ &\neq \sum_{k,j,i} \big(\boldsymbol{T}_{kj}\boldsymbol{T}_{ji} - \boldsymbol{T}_{ki}\big) e^{kji} = 2\boldsymbol{F}^{\mathcal{T}}. \end{aligned} \tag{6.16}$$

Since

$$\boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}_{\epsilon'\nu}(x) = -\boldsymbol{D}^{\mathcal{T}}_{\epsilon'\nu}\boldsymbol{T}_{\epsilon\mu}(x), \tag{6.17}$$

the components $\boldsymbol{F}^{\mathcal{T}}_{\epsilon\mu,\epsilon'\nu}(x)$ of the field strength are antisymmetric. For later use we note that

$$(\boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}_{\epsilon'\nu}(x))^{\dagger} = \boldsymbol{D}^{\mathcal{T}}_{-\epsilon\mu}\boldsymbol{T}_{-\epsilon'\nu}(x+\epsilon\mu+\epsilon'\nu). \tag{6.18}$$

We saw that in the framework of nonunitary parallel transport there are two different connections. Therefore we can define a second kind of covariant derivative associated with unitary parallel transport

$$\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{T}_{\epsilon\mu}(x) := \boldsymbol{T}_{\epsilon'\nu}(x+\epsilon\mu)\boldsymbol{U}_{\epsilon\mu}(x) - \boldsymbol{U}_{\epsilon\mu}(x+\epsilon'\nu)\boldsymbol{T}_{\epsilon'\nu}(x). \tag{6.19}$$

Notice that $\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}$ is only covariant under unitary gauge transformations, in contrast to $\boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}$.

The field strength associated with the unitary connection can be written as

$$\boldsymbol{F}^{\mathcal{U}}_{\epsilon\mu\epsilon'\nu}(x) = \boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{U}_{\epsilon'\nu}(x). \tag{6.20}$$

According to the two possibilities

$$\epsilon\mu+\epsilon'\nu \neq 0 \quad \text{and} \quad \epsilon\mu+\epsilon'\nu = 0, \tag{6.21}$$

the field strength splits in two qualitatively different parts

$$\begin{aligned}\boldsymbol{F}^{\mathcal{T}} &= \frac{1}{2}\sum_{x,\epsilon\mu\neq-\epsilon'\nu} e_x^{\epsilon\mu,\epsilon'\nu}\left(\boldsymbol{T}_{\epsilon'\nu}(x+\epsilon\mu)\boldsymbol{T}_{\epsilon\mu}(x) - \boldsymbol{T}_{\epsilon\mu}(x+\epsilon'\nu)\boldsymbol{T}_{\epsilon'\nu}(x)\right)\\ &\quad+\frac{1}{2}\sum_{x,\epsilon,\mu} e_x^{\epsilon\mu,-\epsilon\mu}\left(\boldsymbol{T}_{-\epsilon\mu}(x+\epsilon\mu)\boldsymbol{T}_{\epsilon\mu}(x) - \boldsymbol{T}_{\epsilon\mu}(x-\epsilon\mu)\boldsymbol{T}_{-\epsilon\mu}(x)\right)\\ &=: \frac{1}{2}\sum_{x,\mu\neq\nu} e_x^{\epsilon\mu,\epsilon'\nu}\boldsymbol{F}^{\mathcal{T}}_{\epsilon\mu,\epsilon'\nu}(x) + \frac{1}{2}\sum_{x,\epsilon,\mu} e_x^{\epsilon\mu,-\epsilon\mu}\boldsymbol{F}^{\mathcal{T}}_{\epsilon\mu,-\epsilon\mu}(x).\end{aligned} \tag{6.22}$$

Formula (6.22) shows explicitly that the field strength $\boldsymbol{F}^{\mathcal{T}}$ involves additional degrees of freedom due to the existence of $\boldsymbol{F}^{\mathcal{T}}_{\epsilon\mu,-\epsilon\mu}(x)$. Therefore a nonunitary gauge theory is richer than a conventional gauge theory and opens the possibility to adress issues which are not known to ordinary gauge theories.

In fact, we shall see that these additional degrees of freedom lead to a suitable Higgs potential.

If all parallel transporters are unitary $\boldsymbol{T}_{\epsilon\mu}(x) = \boldsymbol{U}_{\epsilon\mu}(x)$, the second part of the field strength has to vanish, as is easily seen

$$\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{U}_{-\epsilon\mu}(x) = \boldsymbol{U}_{-\epsilon\mu}(x+\epsilon\mu)\boldsymbol{U}_{\epsilon\mu}(x) - \boldsymbol{U}_{\epsilon\mu}(x-\epsilon\mu)\boldsymbol{U}_{-\epsilon\mu}(x) \tag{6.23}$$

$$= \boldsymbol{U}_{-\epsilon\mu}(x+\epsilon\mu)\boldsymbol{U}^{-1}_{-\epsilon\mu}(x+\epsilon\mu) - \boldsymbol{U}_{\epsilon\mu}(x-\epsilon\mu)\boldsymbol{U}_{\epsilon\mu}(x-\epsilon\mu)^{-1} = 0. \tag{6.24}$$

6.2 Generalized Yang-Mills action

To write down a Yang-Mills action, one has to introduce an appropriate inner product on the space of differential forms. We follow [14] and define a map $\langle\,,\,\rangle_\Omega : \Omega \times \Omega \to \mathbb{C}$, which is defined by

$$\left\langle e_i^{\epsilon_1\mu_1,\ldots,\epsilon_r\mu_r}, e_j^{\epsilon'_1\nu_1,\ldots,\epsilon'_s\nu_s} \right\rangle_\Omega := (2l^2)^{-r}\delta_{r,s}\delta_{i,j}\delta^{\epsilon'_1\nu_1,\ldots,\epsilon'_r\nu_r}_{\epsilon_1\mu_1,\ldots,\epsilon_r\mu_r}\,, \tag{6.25}$$

where $l \in \mathbb{R}^+$ and

$$\delta^{\epsilon'_1\nu_1,\ldots,\epsilon'_r\nu_r}_{\epsilon_1\mu_1,\ldots,\epsilon_r\mu_r} := \delta^{\epsilon'_1\nu_1}_{[\epsilon_1\mu_1}\cdots\delta^{\epsilon'_r\nu_r}_{\epsilon_r\mu_r]} = \sum_{k=1}^{r}(-1)^{k+1}\delta^{\epsilon'_1\nu_1}_{\epsilon_k\mu_k}\delta^{\epsilon'_2\nu_2,\ldots,\epsilon'_r\nu_r}_{\epsilon_1\mu_1,\ldots,\hat{\epsilon_k}\hat{\mu_k},\ldots,\epsilon_r\mu_r}\,. \tag{6.26}$$

For 2-forms one gets

$$\left\langle e_i^{\epsilon_1\mu_1,\epsilon_2\mu_2}, e_j^{\epsilon'_1\nu_1,\epsilon'_2\nu_2} \right\rangle_\Omega = (4l^4)^{-1}\delta_{i,j}\delta^{\epsilon'_1\nu_1,\epsilon'_2\nu_2}_{\epsilon_1\mu_1,\epsilon_2\mu_2} \tag{6.27}$$

Finally the inner product on Ω has to be extended to matrix-valued forms by

$$\langle \boldsymbol{\psi}, \boldsymbol{\phi}\rangle_\Omega = \sum_{i_1\ldots i_r j_1\ldots j_s} \boldsymbol{\psi}^\dagger_{i_1\ldots i_r}\,\langle e_{i_1\ldots i_r}, e_{j_1\ldots j_s}\rangle_\Omega\,\boldsymbol{\phi}_{j_1\ldots j_s}\,. \tag{6.28}$$

Now we are ready to write down a generalized Yang-Mills action. It is defined by

$$S_{YM} := tr\langle \boldsymbol{F}^{\mathcal{T}}, \boldsymbol{F}^{\mathcal{T}}\rangle_\Omega. \tag{6.29}$$

Note that the action is only invariant under unitary gauge transformations, but not under H-transformations, according to our general theory in chapter 5.

Using (6.25) and (6.28), we arrive at

$$S_{YM} = \frac{1}{8l^2} tr \sum_{x,\epsilon\mu,\epsilon'\nu} \boldsymbol{F}^{\mathcal{T}\dagger}_{\epsilon\mu\epsilon'\nu}(x)\boldsymbol{F}^{\mathcal{T}}_{\epsilon\mu\epsilon'\nu}(x) \tag{6.30}$$

$$= \frac{1}{8l^2} tr \Bigg[\sum_{x,\epsilon\mu\neq\epsilon'\nu} \boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}^\dagger_{\epsilon'\nu}(x)\boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}_{\epsilon'\nu}(x) + $$

$$+ \; tr\sum_{x,\epsilon\mu} \boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}^\dagger_{-\epsilon\mu}(x)\boldsymbol{D}^{\mathcal{T}}_{\epsilon\mu}\boldsymbol{T}_{-\epsilon\mu}(x)\Bigg]. \tag{6.31}$$

We see that the Yang-Mills action can be written in a kinetic type form.

More explicitly, we have

$$
\begin{aligned}
S_{YM} &= \frac{1}{4l^2} tr \Big[\sum_{x,\epsilon\mu\neq\epsilon'\nu} \big\{ \boldsymbol{T}_{-\epsilon\mu}(x+\epsilon\mu)\boldsymbol{T}_{-\epsilon'\nu}(x+\epsilon\mu+\epsilon'\nu)\boldsymbol{T}_{\epsilon'\nu}(x+\epsilon\mu)\boldsymbol{T}_{\epsilon\mu}(x) + \\
&- \boldsymbol{T}_{-\epsilon'\nu}(x+\epsilon'\nu)\boldsymbol{T}_{-\epsilon\mu}(x+\epsilon\mu+\epsilon'\nu)\boldsymbol{T}_{\epsilon'\nu}(x+\epsilon\mu)\boldsymbol{T}_{\epsilon\mu}(x) \big\} + \\
&+ \sum_{x,\epsilon\mu} \big\{ \boldsymbol{T}_{-\epsilon\mu}(x+\epsilon\mu)\boldsymbol{T}_{\epsilon\mu}(x)\boldsymbol{T}_{-\epsilon\mu}(x+\epsilon\mu)\boldsymbol{T}_{\epsilon\mu}(x) + \\
&- \boldsymbol{T}_{\epsilon\mu}(x-\epsilon\mu)\boldsymbol{T}_{-\epsilon\mu}(x)\boldsymbol{T}_{-\epsilon\mu}(x+\epsilon\mu)\boldsymbol{T}_{\epsilon\mu}(x) \big\} \Big]. \qquad (6.32)
\end{aligned}
$$

We emphasize that the described formalism includes conventional lattice gauge theory. More precisely, in the special case that all parallel transporters are unitary we recover the ordinary Wilson action of conventional lattice gauge theory. In this case we have already seen that the diagonal components $\boldsymbol{F}^{\mathcal{T}}_{\epsilon\mu,-\epsilon\mu}(x)$ of the field strength has to vanish and so we are left with

$$
\begin{aligned}
S_{Wilson} &= tr\langle \boldsymbol{F}^{\mathcal{U}}, \boldsymbol{F}^{\mathcal{U}} \rangle_\Omega = \frac{1}{8l^2} tr \sum_{x,\epsilon\mu,\epsilon'\nu} (\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{U}_{\epsilon'\nu}(x))^\dagger \boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{U}_{\epsilon'\nu}(x) \\
&= \frac{1}{8l^2} tr \sum_{x,\epsilon\mu,\epsilon'\nu} (\boldsymbol{D}^{\mathcal{U}}_{-\epsilon\mu}\boldsymbol{U}_{-\epsilon'\nu}(x+\epsilon\mu+\epsilon'\nu)\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{U}_{\epsilon'\nu}(x) \\
&= \frac{1}{4l^2} tr \sum_{x,\epsilon\mu,\epsilon'\nu} \big[\mathbf{1} - \\
&+ \boldsymbol{U}_{-\epsilon'\nu}(x+\epsilon'\nu)\boldsymbol{U}_{-\epsilon\mu}(x+\epsilon\mu+\epsilon'\nu)\boldsymbol{U}_{\epsilon'\nu}(x+\epsilon\mu)\boldsymbol{U}_{\epsilon\mu}(x) \big].
\end{aligned}
$$

This is the Wilson action of conventional lattice gauge theory.

6.3 Geometry of Higgs fields

Up to now we have constructed general nonunitary gauge theories in arbitrary dimensions. To obtain theories being more close to the standard model, we specialize to a 5-dimensional theory with a finite lattice describing the extra dimension. Furthermore we assume that only the gauge fields propagating along the extra dimension are nonunitary.

The model lives on the discrete manifold $\mathbb{Z}^4 \times \boldsymbol{N}$, where $\boldsymbol{N} = \{0, \ldots, N-1\}$ shall be considered as a subset of $\mathbb{Z}$ equipped with the induced differential calculus. $\mathbb{Z}^4 \times \boldsymbol{N}$ may be viewed as a N-brane system, where the N transverse lattice slices are the branes. Recently such multi-brane systems have received considerably interest. It was shown that they can be used to study extra-dimensional extensions of the standard model. Note that the choice $\mathbb{Z}^4 \times \boldsymbol{N}$ corresponds to a model with free boundary conditions, known as aliphatic model. Another possibility is to consider

$\mathbb{Z}^4 \times \mathbb{Z}_N$. This choice leads to a periodic model in which the zeroth and Nth brane are linked together with an additional nonunitary parallel transporter.

To deal with a theory living on $\mathbb{Z}^4 \times \boldsymbol{N}$ it is convenient to change our notation slightly. Let $x = (x^\mu) \in \mathbb{Z}^4, \mu = 0, \ldots 3$ and define $x_i := x + i\hat{5} \in \mathbb{Z}^4 \times \boldsymbol{N}, i = 0, 1, \ldots N-1$.

$\mathbb{Z}^4 \times \boldsymbol{N}$ is a symmetric graph, but is no longer a symmetric lattice. In particular, the 1-forms $e^{x_0, x_{N-1}}, e^{x_{N-1}, x_{N-1}+5}$ and e^{x_0, x_0-5} vanish. In contrast, $\mathbb{Z}^4 \times \mathbb{Z}_N$ corresponds to a reduction $e^{xy} \neq 0 \Leftrightarrow y = x + \mu \; modN$ or $y = x + \mu \; modN$. Therefore in this case the 1-forms $e^{x_0, x_{N-1}}, e^{x_{N-1}, x_0}$ exist.

As a consequence of the vanishing "boundary 1-forms", the corresponding field strength of the N-brane system has to be modified sligthly. Using the additional constraint

$$e^{x_i+\epsilon\mu+5, x_i} = 0 \Rightarrow e^{x_i+\epsilon\mu+5, x_i+\epsilon\mu, x_i} = -e^{x_i+\epsilon\mu+5, x_i+5, x_i}, \tag{6.33}$$

the calculation of the field strength is very similar to the symmetric lattice case

Starting again from the general formula an easy but tedious calculation yields [2]

$$\begin{aligned}
\boldsymbol{F}^{\mathcal{T}} = & \frac{1}{2}\sum_{i=0}^{N} \sum_{x,\epsilon\mu,\epsilon'\nu,} e^{x_i+\epsilon\mu+\epsilon'\nu, x_i+\epsilon\mu, x_i} \boldsymbol{F}^{\mathcal{U}}_{\epsilon\mu,\epsilon'\nu}(x_i) + \\
& + \frac{1}{2}\sum_{i=1}^{N-1} \sum_{x,\epsilon\mu,\epsilon'\nu,} e^{x_i+\epsilon\mu+\epsilon'5, x_i+\epsilon\mu, x_i} \boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_{\epsilon'5}(x_i) + \\
& + \frac{1}{2}\sum_{x,\epsilon\mu} e^{x_0+\epsilon\mu+5, x_0+\epsilon\mu, x_0} \boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_5(x_0) + \\
& + \frac{1}{2}\sum_{x,\epsilon\mu} e^{x_N+\epsilon\mu-5, x_N+\epsilon\mu, x_N} \boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_{-5}(x_N) + \\
& + \frac{1}{2}\sum_{i=2}^{N-2} \sum_{x,\epsilon,\epsilon',} e^{x_i+\epsilon 5+\epsilon' 5, x_i+\epsilon 5, x_i} \boldsymbol{F}^{\mathcal{U}}_{\epsilon 5,\epsilon' 5}(x_i) + \\
& + \sum_{i=0,N-1} \sum_{x} e^{x_i, x_i+5, x_i} \left[\boldsymbol{T}_{-5}(x_i+5)\boldsymbol{T}_{+5}(x_i) - \boldsymbol{1}\right] \\
& + \sum_{i=1,N} \sum_{x,\epsilon\mu,\epsilon'\nu,} e^{x_i, x_i-5, x_i} \left[\boldsymbol{T}_{+5}(x_i+5)\boldsymbol{T}_{-5}(x_i) - \boldsymbol{1}\right]
\end{aligned} \tag{6.34}$$

Note that by assumption we have $\boldsymbol{F}^{\mathcal{T}}_{\epsilon\mu,\epsilon'\nu} = \boldsymbol{F}^{\mathcal{U}}_{\epsilon\mu,\epsilon'\nu}$. At this point it is worth digressing a little and reviewing the work of C.T. Hill and others.

Recently there has been considerable interest in theories of extra dimensions emerging not far from the weak scale. The motivation arises from

[2]For later convenience we consider a (N+1)-brane system.

string theories which admit an arbitrary hierarchy between the compactification scale of the extra dimensions and the fundamental string scale.

Extra dimensions can not be observed directly, but rather they appear in accelerator experiments as new particles, which are known as Kaluza-Klein (KK) modes. These are the excited modes of existing fields that propagate in the compact extra dimension.

As KK-modes begin to appear in accelerator experiments, one may ask how are they to be described in an effective 4-dimensional Lagrangian, without an a priori knowledge of the existence of the extra dimensions themselves. Asking this question has led to a new class of KK models known as deconstruction. Independently, the "Harvard group" of Arkani-Hamed, Cohen and Georgi [4, 5, 3] and the "Fermilab group" of Hill, Pokorski, Wang and Cheng [20, 8, 9] have employed a lattice to describe the extra dimensions. More precisely, the 3+1 dimensions of space-time are considered to be continuos, while the extra compact dimensions are latticized. This construction is known as a transverse lattice [7]. By using the lattice technique, the extra dimensions can be integrated out and so one obtains a gauge invariant effective 4-dimensional Lagrangian describing KK-modes.

To get a flavor for deconstruction, let us consider as an example an extension of QCD with $N+1$ gauge groups $SU(3)_i$. The 3+1 dimensional Lagrangian is

$$\mathcal{L} = -\frac{1}{4}\sum_{i=0}^{N} F^a_{i\mu\nu}F^{ia\mu\nu} + tr\sum_{n=1}^{N} D_\mu \Phi_i^\dagger D^\mu \Phi_i \tag{6.35}$$

There are N Higgs fields Φ_i and the kth field transforms as an $(\bar{\mathbf{3}}_i, \mathbf{3}_{i-1})$ representation. The covariant derivative is defined as $D_\mu = \partial_\mu + ig\sum_{i=0}^{N} A^a_{i\mu}L^a_i$. g is a dimensionless gauge coupling constant being common to all of the $SU(3)_i$ gauge groups. L^a_i are the generators of the ith $SU(3)_i$, where a denotes the color index. One has $[L_i, L_j] = 0$ for $i \neq j$ and when the covariant derivative acts upon Φ_i one has a commutator of the gauge part with Φ_i, i.e. $L_i^{a\dagger}$ acts on the left and L^a_{i-1} on the right.

By introducing a potential for each Higgs field by hand, they develop a vacuum expectation value v and may be parameterized as

$$\Phi_i \to v\exp(\phi^a_i L^a_i/v). \tag{6.36}$$

As a result, the kinetic terms of the Higgs fields lead to a mass matrix for the gauge fields

$$\frac{1}{2}v^2 g^2 \sum_{i=1}^{N}(A^a_{(i-1)\mu} - A^a_{i\mu})^2 \tag{6.37}$$

By diagonalization one obtains the eigenvalues

$$M_n = 2vg\sin\left(\frac{\pi n}{2(N+1)}\right), \quad n = 0, \dots, N \tag{6.38}$$

Consequently, for small n one is faced with a KK tower of masses given by

$$M_n \approx \frac{gv\pi n}{N+1}, \quad n \ll N, \tag{6.39}$$

where $n = 0$ corresponds to the zero-mode gluon. In order to match on to the spectrum of the KK modes, one has to require

$$\frac{gv\pi}{N+1} = \frac{\pi}{R}, \tag{6.40}$$

where R is the size of the compactified extra dimension. In this way, the 3+1 dimensional theory given by (6.35) with N+1 gauge groups and N Higgs fields Φ_i provides a gauge invariant description of the first n KK modes by generating the same spectrum [3].

Now the point is that the described 3+1 dimensional gauge theory correponds to a transverse lattice description of a full 4+1 dimensional gauge theory with lattice size R and lattice constant l. This construction describes a foliation of branes, each spaced by the lattice cut-off l. The number of branes is $N + 1 = R/l + 1$.

Now let us return to the nonunitary gauge theory on the discrete manifold $\mathbb{Z}^4 \times \boldsymbol{N}$. The short review presented above shows that our formalism developed so far provides suitable tools to deal also with deconstruction models. Actually, in the framework of nonunitary gauge theories it is quite natural to place different gauge groups on different branes. The nonunitary parallel transporters connecting the branes can be viewed as the described Higgs fields Φ_i. In particular, they possess the "right" transformation properties. In this way the action

$$S_{YM} = tr\langle \boldsymbol{F}^{\mathcal{T}}, \boldsymbol{F}^{\mathcal{T}} \rangle_\Omega, \tag{6.41}$$

where $\boldsymbol{F}^{\mathcal{T}}$ is given by (6.34), provides a full lattice description of the transverse lattice model.

Taking into account the orthogonality of the 1-forms, one gets

$$\begin{aligned} S_{YM} = &\frac{1}{8l^2} tr \sum_{i=0}^{N} \sum_{x,\epsilon\mu,\epsilon'\nu} \boldsymbol{F}^{\mathcal{U}\dagger}_{\epsilon\mu\epsilon'\nu}(x_i) \boldsymbol{F}^{\mathcal{U}}_{\epsilon\mu\epsilon'\nu}(x_i) + \\ &+ \frac{1}{8l^2} tr \sum_{i=1}^{N-1} \sum_{x,\epsilon\mu,\epsilon'} (\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_{\epsilon' 5}(x_i))^\dagger (\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_{\epsilon' 5}(x_i) + \\ &+ \frac{1}{8l^2} tr \sum_{x,\epsilon\mu} (\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_5(x_0))^\dagger (\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_5(x_0) + \\ &+ \frac{1}{8l^2} tr \sum_{x,\epsilon\mu} (\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_{-5}(x_N))^\dagger (\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu} \boldsymbol{T}_{-5}(x_N) + \dots , \end{aligned} \tag{6.42}$$

[3] This is the spectrum assuming free boundary conditions. With periodic boundary conditions one has $N + 1$ Higgs fields, because there is an additional field linking the first gauge group to the last. As a consequence, the spectrum is changed. One gets a zero-mode corresponding to A^a_5 and the KK modes are doubled.

where the dots indicate further terms involving only nonunitary parallel transporters.

We have repeatedly seen that $\sum_{x,\epsilon\mu,\epsilon'\nu} \boldsymbol{F}^{\mathcal{U}\dagger}_{\epsilon\mu\epsilon'\nu}(x_i)\boldsymbol{F}^{\mathcal{U}}_{\epsilon\mu\epsilon'\nu}(x_i)$ yields the Wilson action, so in the continuum limit the first sum in (6.42) gives actually N+1 copies of an ordinary 3+1 continuum gauge theory, cp. (6.35).

To see, whether the other terms in (6.42) yield the kinetic terms of the Higgs fields, we write $\boldsymbol{T}_5(x_i) =: \Phi_{i+1}(x)$ and expand

$$\begin{aligned} \boldsymbol{T}_5(x_i+\mu) &= \Phi_{i+1}(x+\mu) = \Phi_{i+1}(x) + l\partial_\mu \Phi_{i+1}(x) + \mathcal{O}(l^2) && (6.43)\\ \boldsymbol{U}_\mu(x_i) &= \mathbf{1} - lA^i_\mu(x) + \mathcal{O}(l^2). && (6.44)\end{aligned}$$

Note that in (6.44) $A^i_\mu(x)$ means the Lie algebra valued vector potential.

With (6.43) and (6.44) we arrive at

$$\begin{aligned} \boldsymbol{D}^{\mathcal{U}}_\mu \boldsymbol{T}_5(x_i) &= l\big(\partial_\mu \Phi_{i+1}(x) + A_{(i+1)\mu}(x)\Phi_{i+1}(x)\\ &- \Phi_{i+1}(x)A_{i\mu}(x)\big) + \mathcal{O}(l^2) && (6.45)\\ &= lD_\mu \Phi^{i+1}(x) + \mathcal{O}(l^2). && (6.46)\end{aligned}$$

Thus we recover also the kinetic terms for the Higgs fields in (6.35).

6.3.1 2-brane system

We have seen that a general N-brane system describes extensions of the standard model. The standard model itself may be obtained if we start from a N-brane system and thin the degrees of freedom, thus arriving at a 2-brane system which can be pictured by the discrete manifold $\mathbb{Z}^4 \times \mathbf{2}$ with $\mathbf{2} = \{0,1\}$. Again the gauge fields propagating along the fifth extra dimension are assumed to be nonunitary. The resulting field strength is easily obtained from our formula (6.34):

$$\begin{aligned} \boldsymbol{F}^{\mathcal{T}} = \frac{1}{2}\sum_{i=L,R}\sum_{x,\epsilon\mu,\epsilon'\nu,} & e^{x_i+\epsilon\mu+\epsilon'\nu, x_i+\epsilon\mu, x_i}\boldsymbol{F}^{\mathcal{U}}_{\epsilon\mu,\epsilon'\nu}(x_i) + \\ & +\frac{1}{2}\sum_{x,\epsilon\mu,\epsilon'\nu,} e^{x_L+\epsilon\mu-5, x_L+\epsilon\mu, x_L}\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{T}_{-5}(x_L) + \\ & +\frac{1}{2}\sum_{x,\epsilon\mu,\epsilon'\nu,} e^{x_R+\epsilon\mu+5, x_R+\epsilon\mu, x_R}\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{T}_5(x_R) + \\ + \sum_{x,\epsilon\mu,\epsilon'\nu,} & e^{x_R, x_R+5, x_R}\big[\boldsymbol{T}_{-5}(x_R+5)\boldsymbol{T}_{+5}(x_R) - \mathbf{1}\big] + \\ + \sum_{x,\epsilon\mu,\epsilon'\nu,} & e^{x_L, x_L-5, x_L}\big[\boldsymbol{T}_{+5}(x_L-5)\boldsymbol{T}_{-5}(x_L) - \mathbf{1}\big]. \end{aligned} \qquad (6.47)$$

One finds that the field strength splits into three different parts. Since the corresponding two forms are orthogonal to each other

$$\langle e^{\epsilon\mu,\epsilon'\nu}_x, e^{\epsilon\mu,5}_x\rangle = 0 = \langle e^{\epsilon\mu,\epsilon'\nu}_x, e^{5,-5}_x\rangle = \langle e^{\epsilon\mu,5}_x, e^{5,-5}_x\rangle, \qquad (6.48)$$

the split of the field strength leads to a corresponding split of the generalized Yang-Mills action into three qualitatively different parts.

$$
\begin{aligned}
S_{YM} &= tr\langle \boldsymbol{F}^{\mathcal{T}}, \boldsymbol{F}^{\mathcal{T}}\rangle_\Omega \\
&= \frac{1}{8l^2} tr\Big[\sum_{i=L,R}\sum_{x,\epsilon\mu\neq\epsilon'\nu} \boldsymbol{F}^{\mathcal{U}\dagger}_{\epsilon\mu\epsilon'\nu}(x_i)\boldsymbol{F}^{\mathcal{U}}_{\epsilon\mu\epsilon'\nu}(x_i) + \\
&+ \sum_{x,\epsilon\mu}(\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{T}_{-5}(x_L))^\dagger \boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{T}_{-5}(x_L) + \sum_{x,\epsilon\mu}(\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{T}_5(x_R))^\dagger \boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{T}_5(x_R)\Big] + \\
&+ \frac{1}{4l^2} tr\Big[\sum_x \big[\boldsymbol{T}_{-5}(x_R+5)\boldsymbol{T}_{+5}(x_R) - \mathbf{1}\big]^2 + \\
&+ \sum_x \big[\boldsymbol{T}_{-5}(x_L-5)\boldsymbol{T}_{-5}(x_L) - \mathbf{1}\big]^2\Big]. \qquad (6.49)
\end{aligned}
$$

We find that the first part of the generalized action yields the Wilson actions for two 3+1 dimensional ordinary gauge theories, one living on the "left-handed" brane, the other carried by the "right-handed". The second part is already written in a kinetic type form. Actually, we shall see in the next chapter that these terms provide the usual kinetic terms of the standard model Higgs. The third part gives a Higgs potential, which vanishes, if the parallel transport in the fifth direction is unitary. Let us write $\boldsymbol{T}_5(x_R) := \Phi(x)$ and $\boldsymbol{T}_{-5}(x_L) := \Phi^\dagger(x)$. Then we have

$$
\begin{aligned}
& tr(\sum_x [\boldsymbol{F}^{\mathcal{T}\dagger}_{5,-5}(x_L)\boldsymbol{F}^{\mathcal{T}}_{5,-5}(x_L) + \boldsymbol{F}^{\mathcal{T}\dagger}_{-5,5}(x_R)\boldsymbol{F}^{\mathcal{T}}_{-5,5}(x_R)] \\
&\qquad = 2tr\sum_x (\Phi^\dagger(x)\Phi(x) - \mathbf{1})^2 =: \sum_x V_{\text{Higgs}}(\Phi). \qquad (6.50)
\end{aligned}
$$

Note that by defining $\boldsymbol{T}_{x_R,x_R-5} = \mathbf{1} = \boldsymbol{T}_{x_L,x_L+5}$ also the Higgs potential can be expressed as a covariant derivative, cp. (6.9)

$$\boldsymbol{F}^{\mathcal{T}}_{5,-5}(x_L) = \boldsymbol{D}^{\mathcal{T}}_5 \boldsymbol{T}_{-5}(x_L) \qquad (6.51)$$

Thus again we find that the complete action can be recast in a kinetic type form.

We discussed in chapter 4 that the emergence of pseudographs may be viewed as a result of a multiscale analysis, which also includes dimensional reduction. Therefore it is natural to apply our formalism of gauge theories on pseudographs to our Higgs model considered in this subsection. Thus we shall arrive at a 4-dimensional theory, where the Higgs fields are treated as nonunitary gauge fields attached to the loops of the pseudograph.

The vector potential on a pseudograph can be decomposed as

$$\boldsymbol{B} = \boldsymbol{A}_{xy}e^{xy} + \boldsymbol{P}_{xx}e^{xx} + \boldsymbol{P}^\dagger_{xx}e^{xx\star} \qquad (6.52)$$

The field strength is defined by

$$\boldsymbol{F}^{\mathcal{T}} := d\boldsymbol{B} + \boldsymbol{B}\boldsymbol{B} \tag{6.53}$$

Using (4.66) one obtains

$$\begin{aligned}
\boldsymbol{F}^{\mathcal{T}} &= d\boldsymbol{A} + \boldsymbol{A}\boldsymbol{A} + \sum_x \big[(\boldsymbol{P}_{xx}\boldsymbol{P}^\dagger_{xx} - \mathbf{1})e^{xx}e^{xx\star} + (\boldsymbol{P}^\dagger_{xx}\boldsymbol{P}_{xx} - \mathbf{1})e^{xx\star}e^{xx}\big] + \\
&+ \sum_{x,y}[(\boldsymbol{P}_{xx} - \mathbf{1}) + (\boldsymbol{P}^\dagger_{xx} - \mathbf{1})]e^{xyx} + \\
&+ \sum_{x,y}\big[(\boldsymbol{U}_{xy}\boldsymbol{P}_{yy} - \boldsymbol{U}_{xy})e^{xy}e^{yy} + (\boldsymbol{U}_{xy}\boldsymbol{P}^\dagger_{yy} - \boldsymbol{U}_{xy})e^{xy}e^{yy\star} + \\
&+ (\boldsymbol{P}_{xx}\boldsymbol{U}_{xy} - \boldsymbol{U}_{xy})e^{xx}e^{xy} + (\boldsymbol{P}^\dagger_{xx}\boldsymbol{U}_{xy} - \boldsymbol{U}_{xy})e^{xx\star}e^{xy}\big].
\end{aligned} \tag{6.54}$$

Next, let us apply our general formula for the field strength to the symmetric lattice $\mathbb{Z}^4$ supplemented with loops attached to every point $x \in \mathbb{Z}^4$. We obtain

$$\begin{aligned}
\boldsymbol{F}^{\mathcal{T}} &= d\boldsymbol{A} + \boldsymbol{A}\boldsymbol{A} + \sum_x \big[(\boldsymbol{P}_{xx}\boldsymbol{P}^\dagger_{xx} - \mathbf{1})e^{xx}e^{xx\star} + (\boldsymbol{P}^\dagger_{xx}\boldsymbol{P}_{xx} - \mathbf{1})e^{xx\star}e^{xx}\big] + \\
&+ \sum_{x,\epsilon\mu}[(\boldsymbol{P}_{xx} - \mathbf{1}) + (\boldsymbol{P}^\dagger_{xx} - \mathbf{1})]e^{x,x+\epsilon\mu,x} + \\
&+ \sum_{x,\epsilon\mu}\big[(\boldsymbol{U}_{x+\epsilon\mu,x}\boldsymbol{P}_{xx} - \boldsymbol{U}_{x+\epsilon\mu,x})e^{x+\epsilon\mu,x}e^{xx} + \\
&+ (\boldsymbol{U}_{x+\epsilon\mu,x}\boldsymbol{P}^\dagger_{xx} - \boldsymbol{U}_{x+\epsilon\mu,x})e^{x+\epsilon\mu,x}e^{xx\star} + \\
&+ (\boldsymbol{P}_{x+\epsilon\mu,x+\epsilon\mu}\boldsymbol{U}_{x+\epsilon\mu,x} - \boldsymbol{U}_{x+\epsilon\mu,x})e^{x+\epsilon\mu,x+\epsilon\mu}e^{x+\epsilon\mu,x} + \\
&+ (\boldsymbol{P}^\dagger_{x+\epsilon\mu,x+\epsilon\mu}\boldsymbol{U}_{x+\epsilon\mu,x} - \boldsymbol{U}_{x+\epsilon\mu,x})e^{x+\epsilon\mu,x+\epsilon\mu\star}e^{x+\epsilon\mu,x}\big].
\end{aligned} \tag{6.55}$$

To compute the Yang-Mills action we have to extend the inner product (6.25) on forms involving e^{xx} and $e^{xx\star}$, respectively. We define

$$\langle e^{xx}e^{xx\star}, e^{xx}e^{xx\star}\rangle_\Omega = (2l)^{-2} = \langle e^{xx\star}e^{xx}, e^{xx\star}e^{xx}\rangle_\Omega \tag{6.56}$$

and

$$\langle e^{xx}e^{xx\star}, e^{xx\star}e^{xx}\rangle_\Omega = 0 = \langle e^{xx\star}e^{xx}, e^{xx}e^{xx\star}\rangle_\Omega. \tag{6.57}$$

Furthermore we write

$$e^{\epsilon\mu,0}_x := e^{x+\epsilon\mu,x}e^{xx}, \quad e^{\epsilon\mu,0\star}_x := e^{x+\epsilon\mu,x}e^{xx\star} \tag{6.58}$$

and analogoulsy we define $e^{0,\epsilon\mu}_x$ and $e^{0\star,\epsilon\mu}_x$. Now we can apply (6.25) by treating 0 and $0\star$ in the same way as $\epsilon\mu$ and $\epsilon'\nu$. For example, we get

$$\langle e^{\epsilon\mu,0}_x, e^{0,\epsilon\mu}_x\rangle_\Omega = -(2l)^{-2}. \tag{6.59}$$

With these definitions and assuming $\boldsymbol{P}_{xx}^{\dagger} = \boldsymbol{P}_{xx}$ we arrive at the following action

$$\begin{aligned} S_{YM} &= S_{Wilson} + \frac{1}{4l^2}\Big[2tr\sum_x [\boldsymbol{P}_{xx}^2 - \mathbf{1}]^2 + 4tr\sum_x [\boldsymbol{P}_{xx} - \mathbf{1}]^2 + \\ &+ \; tr\sum_{x,\epsilon\mu} (\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{P}(x))^{\dagger}\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{P}(x)\Big], \end{aligned} \tag{6.60}$$

where we have defined $\boldsymbol{P}(x) := \boldsymbol{P}_{xx}$. Note that we find now a true 4-dimensional theory in the sense that there is no sum of several Wilson actions, in contrast to the N-brane models considered before. Nevertheless the generalized Yang-Mills action contains a suitable Higgs potential, and a kinetic type term, which is defined as

$$\boldsymbol{D}^{\mathcal{U}}_{\epsilon\mu}\boldsymbol{P}(x) := \boldsymbol{P}(x+\epsilon\mu)\boldsymbol{U}_{\epsilon\mu}(x) - \boldsymbol{U}_{\mu}(x)\boldsymbol{P}(x). \tag{6.61}$$

6.4 Reinterpretation of the standard model

In the last section it was shown that a generalized Yang-Mills action involving nonunitary parallel transporter automatically includes a Higgs potential and kinetic Higgs terms. The present section is based on joint work with M. Olschewsky and B. Angermann [35, 2].

We propose a reinterpretation of the standard model. More precisely, we interpret the Higgs field as a nonunitary parallel transporter along an extra dimension through a space time defect connecting left-and right handed matter. In this picture, the unitary gauge group is conventional. The holonomy group H is generated by the unitary gauge group, the conventional Higgs field, Kobayashi-Maskawa and mass matrix.

Matter fields must make up representation spaces of the holonomy group, quarks of all three generations belong to a single irreducible representation of the holonomy group H. These representation spaces are at the same time representation spaces for the unitary gauge group, but irreducible representation spaces may become reducible. Thus the matter fields are forced to make up larger multiplets than required by local symmetry.

6.4.1 The bilayered membrane

We assume that left and right handed matter fermions live on the opposite boundaries of a domain wall. Equivalently, one can imagine that they sit on two different $3+1$ dimensional branes embedded in a 5-dimensional space, similar to the models studied above. The space time domain between the two boundaries will be called bulk later on. Here we look at an effective theory, in which the distance between the two branes is one lattice spacing.

Such a scenario shall also be called *bilayered membrane*.

We confine ourselfes to the quark sector of the standard model and neglect color. In addition we imagine that all three generations of left handed quarks belong to one irreducible representation space of the holonomy group H. In the standard model the representation space of H is a six-dimensional complex vector space $V_L \otimes \mathbb{C}^3$, where V_L is the two dimensional representation space of the left handed gauge group $G_L = SU(2) \times U(1)$ and $\mathbb{C}^3$ describes flavor. Similarly, the right handed quarks belong to an isomorphic 6-dimensional representation space $(V_R \oplus \tilde{V}_R) \otimes \mathbb{C}^3$ of H, where V_R and $\tilde{V}_R$ are 1-dimensional representation spaces of the right handed gauge group $G_R = U(1)_Y$ with hypercharge $Y = +2/3$ and $Y = -1/3$, respectively. The unitary gauge group is $G_R = U(1)_Y$.

The Higgs fields in our sense map between them

$$\Phi : (V_R \oplus \tilde{V}_R) \times \mathbb{C}^3 \to V_L \otimes \mathbb{C}^3 \tag{6.62}$$

$$\Phi^\dagger : V_L \otimes \mathbb{C}^3 \to (V_R \oplus \tilde{V}_R) \otimes \mathbb{C}^3. \tag{6.63}$$

It is well known that conventional gauge fields connect same chiralities, L to L and R to R, respectively. Note that our Higgs is thought to be a nonunitary parallel transport along an extra dimension connecting different chiralities.

To make contact with the formalism developed above the vector spaces $(V_R \oplus \tilde{V}_R) \times \mathbb{C}^3$ and $V_L \otimes \mathbb{C}^3$ are considered as fibers of a right module over the discrete manifold $\mathbb{Z}^4 \times \boldsymbol{N}$. Let $\mathfrak{V}$ denotes the module, then the fibers are given by

$$\mathfrak{V}e^R := \mathfrak{V}_R := (V_R \oplus \tilde{V}_R) \otimes \mathbb{C}^3, \quad \mathfrak{V}e^L := \mathfrak{V}_L := V_L \otimes \mathbb{C}^3. \tag{6.64}$$

The Higgs and its adjoint get identified with

$$\Phi(x) := \boldsymbol{T}_{+5}(x_R), \quad \Phi(x)^\dagger := \boldsymbol{T}_{-5}(x_L), \tag{6.65}$$

cp. section 6.3.1.

In chapter 7 we shall argue that nonunitary parallel transporters naturally emerge after a renormalization group step, even if one starts from a fundamental theory which involves unitary Higgs fields only. The bilayered membrane may be viewed as the endpoint of a renormalization group flow. There will be a Higgs potential which is to be determined as a result of the renormalization group flow. We will see in chapter 7 that the quark masses could be determined if the Higgs potential could be computed. The value of Φ at the minimum of the Higgs potential will contain the information that goes into the quark masses.

Writing φ for the conventional Higgs doublet,

$$\varphi = \begin{pmatrix} \varphi_0 \\ \varphi_+ \end{pmatrix} \qquad \tilde{\varphi} = \begin{pmatrix} -\bar{\varphi}_+ \\ \bar{\varphi}_0 \end{pmatrix}, \tag{6.66}$$

with

$$\varphi \quad : \quad V_R \to V_L \tag{6.67}$$
$$\tilde{\varphi} \quad : \quad \tilde{V}_R \to V_L, \tag{6.68}$$

the Higgs parallel transporter Φ through the defect will have the form

$$\Phi = (\varphi M_U, \tilde{\varphi} M_D) = \begin{pmatrix} \varphi_0 M_U & -\bar{\varphi}_+ M_D \\ \varphi_+ M_U & \bar{\varphi}_0 M_D. \end{pmatrix}. \tag{6.69}$$

Neglecting fluctuations of the Higgs field Φ around the minimum of the Higgs potential, M_U, M_D will be constant 3×3 matrices

$$M_U, M_D : \mathbb{C}^3 \to \mathbb{C}^3. \tag{6.70}$$

The mass matrices can be diagonalized through a biunitary transformation,

$$M_U = A_L^\dagger m_U A_R, \qquad m_U = \mathrm{diag}(m_u, m_c, m_t) \tag{6.71}$$
$$M_D = B_L^\dagger m_U B_R, \qquad m_D = \mathrm{diag}(m_d, m_s, m_b), \tag{6.72}$$

where m_u, ect. denotes the quark masses and A_L, A_R, B_L, B_R are unitary 3×3 matrices. Note that the Kobayashi Maskawa matrix is

$$C_{CKM} = A_L B_L^\dagger. \tag{6.73}$$

Going to unitary gauge $\varphi_0 = \rho, \varphi_+ = 0$, and transforming away A_R, B_R by a basis change, the Higgs parallel transporter Φ can be polar decomposed as follows

$$\Phi = \Phi_{\mathcal{U}} \begin{pmatrix} m_U & 0 \\ 0 & m_D \end{pmatrix} \rho = \begin{pmatrix} A_L^\dagger m_U & 0 \\ 0 & B_L^\dagger m_D \end{pmatrix} \rho. \tag{6.74}$$

The result of parallel transporting forth and back along the fifth dimension through the defect is

$$\Phi^\dagger \Phi = \begin{pmatrix} m_U^2 & 0 \\ 0 & m_D^2 \end{pmatrix} \rho. \tag{6.75}$$

We find that if the Higgs was a unitary parallel transporter, all quark masses would be equal.

Let us now consider the elements of the holonomy group H_{tot} of the whole theory, including the boundaries of the defect. Its elements are the parallel transporters along paths which may pass throught the defect, possibly several times forth and back, and their inverses. The parallel transporters along pieces of path below the defect will be of the form

$$U(C) = U(C)_L \otimes \mathbf{1}_{3\times 3}, \tag{6.76}$$

where

$$U(C)_L = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \in SU(2) \times U(1) \tag{6.77}$$

denotes standard parallel transporters. The parallel transporters along pieces of path above the defect will be unitary matrices $U_R \otimes \mathbf{1}_{3\times 3}$, where U_R are diagonal 2×2 matrices whose nonvanishing entries are representation operators of $U(1)$.

If we go to unitary gauge, the parallel transporter along a closed path which passes once across the defect, forth and back, will be of the form

$$U_R \Phi^\dagger U_L \Phi = U_R \begin{pmatrix} m_U & 0 \\ 0 & m_D \end{pmatrix} \begin{pmatrix} U_{11}\mathbf{1} & C_{CKM} U_{12} \\ C^\dagger_{CKM} U_{21} & U_{22}\mathbf{1} \end{pmatrix} \begin{pmatrix} m_U & 0 \\ 0 & m_D \end{pmatrix} \rho^2. \tag{6.78}$$

Note that the Kobayashi-Maskawa matrix is involved. Furthermore we notice that the difference of the quark masses is responsible for the fact that the holonomy group H is larger than the unitary gauge group. It is important to realize that since H involves nonunitary matrices in the conventional sense, it has to be a noncompact group. Noncompact groups can have expanding and compressing factors, respectively. This will be exploited in the next chapter to explain the huge differences of the fermion masses.

Let us recall the Yukawa term of the standard model [22]

$$\mathcal{L}_{Yukawa} = \overline{q}_L \varphi M_U q_R + \overline{q}_L \tilde{\varphi} M_U q_R + h.c.. \tag{6.79}$$

Its underlying geometry can be revealed by employing the covariant derivatives introduced in chapter 5, eq. (5.76). Let $\psi(x_L) := q_L, \psi(x_R) := q_R^{tot} := (q_R, \tilde{q}_R)^t$ and define

$$D_5 q_L(x) \;:=\; \boldsymbol{T}_{+5}(x_R)\psi(x_R) - \Psi(x_L) = \Phi(x) q_R^{tot}(x) - q_L(x) \tag{6.80}$$

$$D_{-5} q_R^{tot}(x) \;:=\; \boldsymbol{T}_{-5}(x_L)\psi(x_L) - \Psi(x_R) = \Phi(x)^\dagger q_L(x) - q_R^{tot}(x) \tag{6.81}$$

cp. (6.65). The Yukawa interaction now appears as a consequence of a generalized minimal coupling principle, which has a geometric meaning. In fact, due to $\bar{q}_L q_L = 0 = \bar{q}_R^{tot} q_R^{tot}$ we find [4]

$$\mathcal{L}_{Yukawa} = \bar{q}_L D_5 q_L + \bar{q}_R^{tot} D_{-5} q_R^{tot}, \tag{6.82}$$

it takes the form of a Dirac matter Lagrangian.

[4] We regard q_L as Dirac spinor with $(\mathbf{1} + \boldsymbol{\gamma}_5) q_L = 0$.

Chapter 7

Origin of quark masses and CKM-matrix

Up to now we have discussed the geometric interpretation of Higgs fields. More precisely, we have shown that Higgs fields are associated with nonunitary parallel transporters along extra directions.

If one were to use unitary parallel transporters for this purpose both in the fundamental theory from which one starts and for all effective theories deduced from it by real space renormalization group transformations, then one would end up with a nonrenormalizable 4-dimensional effective theory, a gauged nonlinear σ-model. This suggests to admit RG-transformations which lead to nonunitary block parallel transporters in the linear span of the gauge group.

When the parallel transporters become nonunitary there appears a Higgs potential, which possesses a characteristic biinvariance property due to the peculiar behaviour of parallel transporters under gauge transformations. Its importance was first noticed in the study of the RG-flow of a 1-dimensional model [27].

This chapter is based on joint work with G. Mack and C. Lehmann [26]. We propose a model which exploits these features to explain the origin of the splitting of the masses of quarks of different flavors. If the theory is an ordinary gauge theory to begin with and the nonunitary parallel transporter and their potential appear only as a result of a RG-flow, then the quark mass ratios and the Cabbibo-Kobayashi-Maskawa matrix are computable in principle.

7.1 Lessons from renormalization group

The crucial feature of our model is a characteristic biinvariance property of the Higgs potential. To understand its origin, one is led to the question how Higgs fields come into an effective local theory at intermediate scales.

Effective theories have an ultraviolett-cutoff M with the dimension of a mass. M can be lowered by a renormalization group transformation. We assume that the cutoff is introduced in the form of a lattice.

Our hypothesis is that Higgs fields make their appearance in higher dimensional field theories. At the level of effective theories, they are nonunitary parallel transporters Φ along the links of the lattice in the extra direction(s). The Higgs fields take their values in a group H, which is typically noncompact and larger than the unitary gauge group. They can arise from unitary parallel transporters, i.e. ordinary gauge theories without Higgs fields, through real space renormalization group transformations.

This mechanism can be illustrated in part by a 1-dimensional model [27]. Let us discuss the effect of the conventional 4 space time dimensions which have been ignored in the 1-dimensional model. We call these the perpendicular directions.

In the 1-dimensional model, the ordinary gauge theory is a fix point. So one needs a theory in which there are some, albeit arbitrarily small, fluctuations of the length of Φ to start a nontrivial flow.

When the conventional four dimensions are taken into account, two things are expected to happen.

Coarsening in the perpendicular directions, which is involved in the construction of a block spin Φ from unitary parallel transporters $\boldsymbol{u}$ on the links along the extra dimension will produce Φ in the linear span of the unitary gauge group G. Noninvertible Φ will have measure zero.

At some scale one will be prevented from reducing to a ordinary gauge theory by integrating out the length of Φ. More precisely, the selfadjoint factor p in the polar decomposition $\Phi = up$ can not be integrated out, because otherwise the effective theory will become nonlocal in the perpendicular directions. This will happen when the mass m determined by the curvature at the minima of the Higgs potential falls below the cutoff scale M.

Consequently, there will be a domain of scales where the effective theory needs a Higgs field for its locality.

When the parallel transporter Φ becomes nonunitary, a Higgs potential $V(\Phi)$ with a biinvariance property arises. More precisely, under unitary gauge transformations the parallel transporter Φ transforms as

$$\Phi = \phi(x, y) \rightarrow g(x)\phi(x, y)g(y)^{-1}. \tag{7.1}$$

Thus, gauge invariance requires that V is G-biinvariant in the sense that

$$V(g_1 \Phi g_2) = V(\Phi) \tag{7.2}$$

for all $g_1, g_2 \in G, \Phi \in H$.

Actually this biinvariance property is the essential feature, which will be exploited by our model to explain how quarks of different flavor can aquire different masses by spontaneous symmetry breaking. More precisely, their masses are determined by minima of the Higgs potential $V(\Phi)$. To give a taste of this mechanism, let us assume that an element $\Phi \in H$ can be factored as follows

$$\Phi = \rho g_1 d(\eta) g_2 \tag{7.3}$$

with $g_1, g_2 \in G, \rho \in \mathbb{R}_*$, and with

$$d(\eta) = \begin{pmatrix} e^{\eta_1} & & \\ & \dots & \\ & & e^{\eta_r} \end{pmatrix} \tag{7.4}$$

from a maximal noncompact abelian subgroup T of the holonomy group H. As a consequence of G-biinvariance the Higgs potential can be written as a function which depends on ρ and (η_i) only

$$V(\Phi) = V(\rho d(\eta)) = \mathcal{V}(\rho, (\eta_i)). \tag{7.5}$$

Let us assume that G is the maximal compact subgroup of H. It follows from standard group theoretical results [36] that there exists a group[1] $W_{\mathfrak{t}}$ - the Weyl group of the pair (H, T) - which consists of linear maps $\pi : \eta \mapsto \pi(\eta)$ with $\pi^2 = 1$ (reflections) such that there exists $w_\pi \in G$ with the property

$$w_\pi d(\eta) w_\pi^\dagger = d(\pi(\eta)). \tag{7.6}$$

This Weyl group $W_{\mathfrak{t}}$ is a discrete group of symmetries of the Higgs potential

$$\mathcal{V}(\pi(\eta)) = \mathcal{V}(\eta), \tag{7.7}$$

steeming from the G-biinvariance property.

Next, we consider a simple example to show that flavor mass splitting occurs if the discrete symmetry under $W_{\mathfrak{t}}$ is broken spontaneously, i.e. if the orbits of of minima of $\mathcal{V}$ do not consist of a single point.

For definiteness consider $H = SL(2, \mathbb{C})$ and $G = SU(2)$. Matrices $A \in SL(2, \mathbb{C})$ may be parameterized as $A = g_1 d(\eta) g_2$ with $g_1, g_2 \in G$, η real and $d(\eta) = diag(e^{-\eta/2}, e^{\eta/2})$. There is $w \in G$ such that $w d(\eta) w^\dagger = d(-\eta)$. Consequently, $\mathcal{V}(\eta) = V(d(\eta))$ obeys

$$\mathcal{V}(\eta) = \mathcal{V}(-\eta). \tag{7.8}$$

Now suppose that the orbit of the minima under the discrete group $W_{\mathfrak{t}} = \{1, w\}$ is nontrivial, then the minima of $\mathcal{V}(\eta)$ are at $\pm\hat{\eta} \neq 0$. Thus, the interaction term becomes

$$\bar{\psi}\Phi\psi = \sum_{\alpha=1}^{2} \bar{\psi}'_\alpha m_\alpha \psi_\alpha \tag{7.9}$$

[1] $\mathfrak{t} := LieT$

with $\bar{\psi}' = \psi g_1, \psi' = g_2\psi$, and masses

$$m_1 = \rho e^{-\eta}, \qquad m_2 = \rho e^{\eta}, \quad \eta \neq 0. \tag{7.10}$$

We have arrived at fermions of two flavors with different masses. Note the exponential dependence of the mass ratio on the position $\pm\eta$ of the minima. We see here a natural mechanism to produce large mass ratios.

7.2 Model description

We now consider a model living on a 5-dimensional space time, called bulk, with two four dimensional boundaries R and L.

Right handed quarks $q_R^U = (u_R, c_R, t_R)$ and $q_R^D = (d_R, s_R, b_R)$ and all right handed leptons live on R and lefthanded quarks $q_L = (d_L, u_L; s_L, c_L; b_L, t_L)$ and left handed leptons live on L. There may be Dirac fermions in the bulk, presumably equally many as there are chiral fermions on the boundaries.

Quarks of the same weak hypercharge but belonging to different generations are in the same irreducible representation space of the holonomy group H. In the present model the irreducible representations of H remain irreducible when restricted to G.

The unitary gauge group G_R of R is $U(1) \times SU(3)_c$ (weak hypercharge and colour). It is shared by the bulk and by L. Since the photon interacts equally with righthanded and lefthanded quarks and mediates interactions between them, we need an abelian gauge field which propagates in the bulk that separates them. In the following we will disregard colour except for a comment at the end on the possibility of admitting nonunitary trans-bulk colour parallel transporter. The elements $u_R = e^{i\vartheta/6}$ of G_R are then parameterized by an angle ϑ and act on right handed quarks according to

$$\begin{aligned} \begin{pmatrix} q_R^D \\ q_R^U \end{pmatrix} &\mapsto t_R(u_R) \begin{pmatrix} q_R^D \\ q_R^U \end{pmatrix}, \\ t_R(u_R) &= \begin{pmatrix} u_R^D & 0 \\ 0 & u_R^U \end{pmatrix} = \begin{pmatrix} e^{-i\vartheta/3} & 0 \\ 0 & e^{2i\vartheta/3} \end{pmatrix} \end{aligned} \tag{7.11}$$

and on left handed quarks as $q_L \mapsto t_L(u_R) q_L$, $t_L(u_R) = e^{i\vartheta/6}$, as in the standard model. There will be no massless modes other than photons as we shall see.

The unitary gauge group of L is $G_R \times G_L$, with $G_L = SU(2)$. G_L acts on left handed quarks as in the standard model. It consists of matrices

$$u_L = \begin{pmatrix} u_{11}\mathbf{1} & u_{12}\mathbf{1} \\ u_{21}\mathbf{1} & u_{22}\mathbf{1}, \end{pmatrix} \tag{7.12}$$

where $\mathbf{1}$ is the 3×3 unit matrix.

The unitary gauge group of the bulk is $G_R \times G$, $G = G^D \times G^U = SU(3) \times SU(3)$ or $SO(3) \times SU(3)$ or $SU(3) \times SO(3)$. We do not know which of the two different groups is associated with U or D. The elements of G are unitary 6×6 matrices of the form

$$\phi = \begin{pmatrix} \phi^D & 0 \\ 0 & \phi^U \end{pmatrix} . \tag{7.13}$$

At the level of effective theory, the parallel transporters Φ in the bulk are nonunitary, and are elements of the bulk-holonomy group $G_R \times H$, with $H = \mathbb{R}_* H^D \times \mathbb{R}_* H^U$, where $H^D = SL(3,\mathbb{R})$ or $SL(3,\mathbb{C})$, $H^U = SL(3,\mathbb{C})$, or the other way round. $\mathbb{R}_* H^D$ consists of positive-real multiples of matrices in H^D. Parallel transporters Φ have the form

$$\Phi = \begin{pmatrix} \rho^D \phi^D u_R^D & \mathbf{0} \\ \mathbf{0} & \rho^U \phi^U u_R^U \end{pmatrix} \tag{7.14}$$

with $\phi^D \in H^D$ etc., $u_R^{D,U}$ as in eq.(7.11), and $\rho^D, \rho^U \in \mathbb{R}_*$. If H has two factors, there will be two additional gauge coupling constants g_D, g_U.

At least one of the two groups G^D, G^U needs to be a group of complex matrices. Otherwise there will be no CP-violation. The choice of two different groups would make it understandable why the mass splittings in the D-family and in the U-family are different.

The salient feature of the model is that the elements of G_L and of G can both act on q_L but do not commute. As a result, the local G-invariance is broken by the L-boundary. This symmetry breaking is responsible for the appearance of a Cabbibo-Kobayashi-Maskawa (CKM) matrix. It would appear that also the G_L-invariance on L is broken by interaction with the bulk. But it turns out that this is just the customary Higgs mechanism. How the standard model's Higgs doublet emerges when one passes to a G_L-covariant description will be seen below.

Next we imagine that real space renormalization group transformations have been applied to the theory in the bulk until a lattice spacing in the fifth direction has been reached which equals the width d. Let us assume that d is so small that the gauge theories on the boundaries can be discretized to lattice gauge theories with lattice spacing d and with appropriate coupling constants without further ado, using appropriate- e.g. domain wall-lattice Dirac-Weyl operators. The resulting effective theory will describe the physics at distances $>> d$. Except for the G-symmetry breaking by the L-boundary, it is a defect model (bilayered membrane) of the type examined in chapter 6. There it was shown that the plaquette term involving parallel transport along the extra dimension is a kinetic term for the Higgs and possesses a natural differential geometric meaning. The effective theory lives on a lattice with links connecting sites (x, R) on R and (x, L) on L. We write x in place of (x, R) or (x, L) when it is clear from the context which site is meant. We denote by $\Phi(x)$ the parallel transporters across the bulk along the aforementioned link.

Since the renormalization group transformations must preserve loclity, the renormalization group calculations can be performed to a good approximation in a 5-dimensional theory wit ∞ extension in the fifth direction. Therefore the emerging Higgs potential $V(\Phi)$ for $\Phi = diag(\rho^D\phi^D, \rho^U\phi^U)t_R(u_R) \in H \times G_R$ will be G-invariant and independent of the factor $t_R(u_R) \in G_R$. Consequently it has the G-biinvariance property (7.2).

Apart from standard lattice gauge theory terms on the boundaries there will be the following terms in the effective action S_{eff}

$$\begin{aligned} S_\Phi &= \sum_x \Big\{ \overline{q}_L(x)\Phi(x)q_R(x). \qquad (7.15) \\ &\quad - \sum_\mu tr\ g^{-2}\big(P_\mu - \Phi(x)^\dagger\Phi(x)\big) + V(\Phi)\Big\} + h.c. \\ P_\mu &= \Phi(x+\mu)t_R(u_{R\mu}(x))^\dagger\Phi(x)^\dagger t_L(u_{R\mu}(x,L))u_{L\mu}(x) \end{aligned}$$

plus lepton mass terms, where $\hat{\mu}$ is the lattice vector in μ-direction on R or L, and $g^{-2} = diag(g_D^{-2}, g_U^{-2})$. The expression involves parallel transporters $u_{R\mu}(x)$, $u_{L\mu}(x)$ from $x+\mu$ to x on R and L besides parallel transporters Φ. The field $\Phi(x)$ parallel transports from (x,R) to (x,L). The P_μ-term will be called the plaquette term for short ($\mu = 0...3$).

It turns out that the plaquette term is G-invariant. The only terms in the action which are not G-invariant live on L, viz. $\sum \overline{q}_L(x+\hat{\mu})\gamma^\mu u_{L\mu}(x)^\dagger q_L(x)$.

Our groups are such that elements of H^D can be factored as

$$\phi^D = A_L^{D\dagger} d(\eta^D) A_R^D$$

with $A_R^D, A_L^D \in G^D$, and similarly for ϕ^U. Here $\eta = (\eta_1, \eta_2, \eta_3) \in \mathfrak{t}$ lives on the plane $\sum_i \eta_i = 0$, η_i real, and

$$d(\eta) = diag(e^{\eta_1}, e^{\eta_2}, e^{\eta_3}).$$

Because of G-biinvariance, V does not depend on the A-factors, whence

$$V(\Phi) = \mathcal{V}(\rho^D, \rho^U; \eta^D, \eta^U). \qquad (7.16)$$

Diagonal matrices $d(\eta)$ make up a maximal noncompact abelian subgroup T^D of H^D and $T^U$of H^U. For the groups considered here, to every permutation π of $\{1,2,3\}$ there exists $w_\pi^D \in G^D$ such that

$$w_\pi d(\eta) w_\pi^\dagger = d(\pi\eta), \qquad (7.17)$$

where $\pi(\eta_1, \eta_2, \eta_3) := (\eta_{\pi 1}, \eta_{\pi 2}, \eta_{\pi_3})$, and similarly for T^U. It follows from G-biinvariance that

$$\mathcal{V}(\rho^D, \rho^U; \pi\eta^D, \pi'\eta^U) = \mathcal{V}(\rho^D, \rho^U; \eta^D, \eta^U). \qquad (7.18)$$

Pairs $(w_\pi, w_{\pi'})$ form the Weyl group $W_\mathfrak{t}$ of the Lie algebra $\mathfrak{t}$ of $T = T^D \times T^U$; it is a discrete group of symmetries of $\mathcal{V}$. The symmetry comes from G-invariance. Minima of $\mathcal{V}$ form orbits under $W_\mathfrak{t}$. These minima will determine the quark masses. The symmetry is spontaneously broken if the orbit contains more than one point. If it is completely broken then the quark masses $\{m_d, m_s, m_b\}$ are all distinct, and so are $\{m_u, m_c, m_t\}$. This is the situation in the real world. If $(\rho_0^D, \rho_0^U; \eta^D, \eta^U)$ is the minimum with $\eta_1^D < \eta_2^D < \eta_3^D$ etc., then

$$\begin{aligned} m_d &= \rho_0^D e^{\eta_1^D}\,,\ m_s = \rho_0^D e^{\eta_2^D}\,,\ m_b = \rho_0^D e^{\eta_3^D}\,, \\ m_u &= \rho_0^U e^{\eta_1^U}\,,\ m_c = \rho_0^U e^{\eta_2^U}\,,\ m_t = \rho_0^U e^{\eta_3^U}\,. \end{aligned}$$

From the action S of eq.(7.15), we recover a discretized version of the standard model under the assumption that fluctuations of $\Phi(x)$ away from a x-independent value may be ignored except for fluctuations of ρ^U/ρ_0^U and ρ^D/ρ_0^D around 1. In particular, the first term in the action eq.(7.15) produces quark mass terms. The factors $A_R^{D,U}$ can be transformed away. Writing $q_L = (q_L^D, q_L^U)^T$,

$$\begin{aligned} \overline{q}_L(x)\Phi(x)q_R(x) &= \overline{q}_L(x)m_D q_R^{D\prime}(x)\rho^D(x)/\rho_0^D \\ &\quad +\overline{q}_L(x)m_U q_R^{U\prime}(x)\rho^U(x)/\rho_0^U \end{aligned}$$

with $q_L^{D\prime} = A_L^D q_L^D$, $q_L^{U\prime} = A_L^U q_L^U$, and mass matrices $m_D = diag(m_d, m_s, m_b)$ etc..

The masses are determined by the minima of $V(\Phi)$. We will see later how $A_L^{D,U}$ acquire definite values modulo action of the unbroken symmetry $G_{diag} = G^D \cap G^U$. The lattice gauge field $u_R(x) \in U(1)$ attached to the links across the bulk can be gauged away, and the minimization of the u_R-factor in the plaquette term constrains

$$u_R(x, R, \mu) = u_R(x, L, \mu). \tag{7.19}$$

There will be fluctuations away from this equality. They are described by a neutral massive vector boson Z' with lattice field

$$z'_\mu(x) = u_R(x, R, \mu)u_R(x, L, \mu)^\dagger\ . \tag{7.20}$$

It is massive because a real multiple of the trans-bulk PT $u_R(x)$ acts as a Higgs field for it.

After these fixations, the effective action reduces to a 4-dimensional effective action which is a lattice approximation of the standard model action in unitary gauge, *except* that the gauge group is $SU(2)_L \times U(1)_L \times U(1)_R$, which is broken by a Higgs doublet φ and a complex scalar Higgs singlet ξ. Accordingly there is a neutral massive vector meson Z' besides Z, and a second Higgs particle (if it is not too heavy). We write

$$\begin{aligned} \rho^D(x) &= \sigma(x)\rho(x)\ , \\ \rho^U(x) &= \sigma^2(x)\rho(x)\ . \end{aligned} \tag{7.21}$$

ρ and σ are what remains of φ and ξ in unitary gauge.

7.3 Kinetic term for the Higgs

We expect that the term

$$\sum_{x,\mu} trg^{-2}\left[\Phi^\dagger(x)\Phi(x) - \Phi(x+\mu)t_R(u_{R\mu}(x,R))^\dagger\Phi^\dagger(x)u_{L\mu}(x)t_L(u_{R\mu}(x,L))\right]$$

yields the standard kinetic term for the Higgs. To see this the gauge fixing has to be undone as follows. One introduces an interface $g(x) \in SU(2)$ between bulk and L-boundary, setting $g(x) = \mathbf{1}$ initially, and replaces $\Phi(x)$ by $g(x)\Phi(x)$. The action is now invariant under $SU(2)$-gauge transformations $v(x)$ which act on q_L and $u_{L\mu}$ as usual, and which take $g(x) \mapsto v(x)g(x)$. Because of $SU(2)$-gauge-invariance, we may integrate over $g(x)$. Given $g(x)$ and $\rho(x), \rho(x) > 0$, one can define a Higgs doublet field

$$\varphi(x) := \rho(x)\hat{\varphi}(x) \tag{7.22}$$

as follows. To every $g \in SU(2)$ there is a complex 2-vector $\hat{\varphi} := (\varphi_+, \varphi_0)^t$ of unit length such that $g = L[\hat{\varphi}]$ where $L[\hat{\varphi}] \in SU(2)$ is uniquely determined by the requirement that

$$\hat{\varphi} = L[\hat{\varphi}]\hat{\varphi}_0, \tag{7.23}$$

where $\hat{\varphi}_0 = (1,0)^t$. Uniqueness holds because the little group of $\hat{\varphi}_0$ in $SU(2)$ is trivial,

$$g\hat{\varphi}_0 = \hat{\varphi}_0 \Rightarrow g = \mathbf{1}, \tag{7.24}$$

for $g \in SU(2)$. As a result of (7.23) the interface $L[\hat{\varphi}]$ has to take the form

$$L[\hat{\varphi}] = \begin{pmatrix} \varphi_+ & \bar{\varphi}_0 \\ \varphi_0 & -\bar{\varphi}_+ \end{pmatrix}. \tag{7.25}$$

Furthermore it follows from (7.23) that under $SU(2)$ gauge transformations $v(x)$ which act on $\hat{\varphi}$ as $\hat{\varphi} \mapsto v\hat{\varphi}$ the interface $L[\hat{\varphi}]$ transforms according

$$L[\hat{\varphi}] \mapsto vL[\hat{\varphi}]. \tag{7.26}$$

For later use we note that

$$L[\hat{\varphi}]^\dagger uL[\hat{\varphi}'] = \begin{pmatrix} \hat{\varphi}^\dagger u\hat{\varphi}' & \hat{\varphi}^\dagger u\epsilon^{-1}\hat{\varphi}'^\dagger \\ \hat{\varphi}\epsilon^{-1}u\hat{\varphi}' & \hat{\varphi}\epsilon u\epsilon^{-1}\hat{\varphi}'^\dagger \end{pmatrix} \tag{7.27}$$

for $u \in SU(2)$. Since $\epsilon u\epsilon^{-1} = \bar{\epsilon}$ it follows

$$(L[\hat{\varphi}]^\dagger uL[\hat{\varphi}'])_{11} = \overline{(L[\hat{\varphi}]^\dagger uL[\hat{\varphi}'])_{22}}, \tag{7.28}$$

as it must be.

Integrating over $\varphi(x)$ is equivalent to integrating over $g(x)$ and $\rho(x)$. The extra complex Higgs field is

$$\xi(x) = \sigma(x) u_R^D(x), \tag{7.29}$$

whence $\rho^D \phi^D u_R^D = \xi \phi^D \rho$ and $\rho^U \phi^U u_R^U = \xi^{\dagger 2} \phi^U \rho$.

Let us consider the described procedure in more detail. After introducing the interface, the plaquette term takes the form

$$P_\mu = \Phi(x+\mu) t_R(u_{R\mu}(x,R))^\dagger \Phi(x)^\dagger L[\hat{\varphi}(x)]^\dagger t_L(u_{R\mu}(x,L)) u_{L\mu}(x) L[\varphi(x+\mu)].$$

Now we use

$$t_L(v) L[\hat{\varphi}] = L[t_\varphi(v)\varphi] t_R(v) \tag{7.30}$$

for $v \in U(1)$ where t_φ is defined as

$$t_\varphi(v) := e^{i\vartheta/2} \mathbf{1}_2. \tag{7.31}$$

Using (7.31), (7.11) and (7.25), eq.(7.30) can be proved by a simple calculation. Note that $t_R(u_{R\mu}(x,L))$ commutes with $\Phi(x+\mu)$. In this way one gets a plaquette term of $u_{R\mu}$ parallel transporters

$$tr P_\mu = tr\big(\Phi(x+\mu) t_R(u_R(\partial p)) \Phi(x)^\dagger L[\hat{\varphi}(x)]^\dagger u_{L\mu}(x) L[\varphi(x+\mu)]\big) \tag{7.32}$$

where $t_R(u_R(\partial p))$ is defined as

$$t_R(u_R(\partial p)) := t_R(u_{R\mu}(x,L) u_R(x+\mu) u_{R\mu}(x,R)^\dagger u_R(x)^\dagger. \tag{7.33}$$

(7.32) suggests to introduce a new field

$$w_\mu(x) \ = L[\varphi(x)]^\dagger u_{L\mu}(x) L[t_\varphi(u_{R\mu}(x))\varphi] \tag{7.34}$$

Since under $SU(2)$ gauge transformations

$$u_{L\mu}(x) \to v(x) u_{L\mu}(x) v(x)^\dagger \tag{7.35}$$

one finds that $w_\mu(x)$ is a gauge invariant object.

Now the trace of the plaquette term can be written as

$$tr(P_\mu(x)) = tr\big(t_R(u_{R\mu}(\partial p)) \Phi(x)^\dagger w_\mu(x) \Phi(x+\mu)\big). \tag{7.36}$$

Next, we expand

$$t_R(u_R(\partial p)) = \mathbf{1} + [t_R(u_R(\partial p)) - \mathbf{1}] \tag{7.37}$$

and

$$\Phi(x)^\dagger w_\mu(x) \Phi(x+\mu) = \Phi(x)^\dagger \Phi(x) + [\Phi(x)^\dagger w_\mu(x) \Phi(x+\mu) - \Phi(x)^\dagger \Phi(x)] \tag{7.38}$$

and consider the second term of (7.37) and (7.38) to be small. As a result, we arrive at

$$\begin{aligned}\sum_{x,\mu} tr g^{-2}[\Phi(x)^\dagger\Phi(x) - P_\mu(x)] &= \sum_x tr g^{-2}\Big(\Phi(x)^\dagger\Phi(x)[\mathbf{1} - u_R(\partial p)] + \\ &+ \Phi(x)^\dagger\Phi(x) - \Phi(x)^\dagger w_\mu(x)\Phi(x+\mu)\Big).\end{aligned}$$

First, let us focus on

$$K_\mu := tr\, g^{-2}(\Phi(x)^\dagger\Phi(x) - \Phi^\dagger(x)w_\mu(x)\Phi(x+\mu)). \tag{7.39}$$

The trace over 6×6-matrices can be decomposed as

$$\begin{aligned}K_\mu &= g_D^{-2} tr[\phi^D(x)^\dagger\phi^D(x) + \phi^D(x)^\dagger w_{DD\mu}(x)\phi^D(x+\mu)] + \\ &+ g_U^{-2} tr[\phi^U(x)^\dagger\phi^U(x) + \phi^U(x)^\dagger w_{UU\mu}(x)\phi^U(x+\mu)],\end{aligned} \tag{7.40}$$

where tr on the right hand side denotes a trace over 3×3 matrices. To recover the standard model, we make the assumption that only the scalar factors of $\Phi(x)$ depend on x

$$\phi^D(x) = \rho_D(x)A_L^{D\dagger}d(\eta^D)A_R^D \tag{7.41}$$

$$\phi^U(x) = \rho_U(x)A_L^{U\dagger}d(\eta^U)A_R^U. \tag{7.42}$$

Thus we are lead to

$$\begin{aligned}K_\mu &= g_D^{-2} tr\, d(\eta^D)[\rho_D^2(x) + \rho_D(x)w_{DD\mu}(x)\rho_D(x+\mu)] + \\ &+ g_U^{-2} tr\, d(\eta^U)[\rho_U^2(x) + \rho_U(x)w_{UU\mu}(x)\rho_U(x+\mu)].\end{aligned} \tag{7.43}$$

Furthermore we assume that

$$\begin{aligned}\rho_D(x) &= \rho(x)\langle\rho_D\rangle/\rho_0 \\ \rho_U(x) &= \rho(x)\langle\rho_U\rangle/\rho_0.\end{aligned} \tag{7.44}$$

Consequently we get

$$\begin{aligned}K_\mu &= g_D^{-2}\langle\rho_D\rangle^2/\rho_0^2\, tr\, d(\eta^D)[\rho^2(x) + \rho(x)w_{DD\mu}(x)\rho(x+\mu)] + \\ &+ g_U^{-2}\langle\rho_D\rangle^2/\rho_0^2\, tr\, d(\eta^U)[\rho^2(x) + \rho(x)w_{UU\mu}(x)\rho(x+\mu)].\end{aligned} \tag{7.45}$$

Due to (7.27) and (7.22) one finds that

$$\rho(x)w_{DD\mu}(x)\rho(x+\mu) = \varphi(x)^\dagger u_{L\mu}(x)t_\varphi^{1/2}(u_{R\mu}(x,L))\varphi(x+\mu). \tag{7.46}$$

Note that

$$w_{DD\mu}(x) = \overline{w_{UU\mu}(x)}, \tag{7.47}$$

cp. (7.28). Eq. (7.46) and (7.47) show that $\varphi(x)$ has the same hypercharge and isospin assignments as the standard model Higgs field. Recall that the kinetic term of a scalar field $\varphi(x)$ on a lattice is

$$\frac{1}{2a^2}\Big(\varphi(x)^\dagger\varphi(x) - \varphi(x)^\dagger u_\mu(x)\varphi(x+\mu)\Big). \tag{7.48}$$

Therefore we can conclude that (7.39) actually yields the kinetic terms of the standard model Higgs.

Let us return to the first term on the right hand side of eq (7.48). We have already mentioned that the lattice gauge field $u_R(x) \in U(1)$ attached to the links across the bulk may be considered as part of an additional complex scalar Higgs singlet ξ. Then we can write, cp.(7.20), (7.29)

$$\Phi(x)^\dagger\Phi(x)[1 - u_R(\partial p)] = \rho(x)^2 t_R(\xi(x))^\dagger[1 - z'_\mu(x)]t_R(\xi(x)), \tag{7.49}$$

where $t_R(\xi(x)) := diag(\xi(x), \xi(x)^{\dagger 2})$. We see that the additional neutral vector boson couples to the scalar Higgs field ξ. Therefore, when the gauge group $SU(2)\times U(1)_L\times U(1)_R$ is broken to the standard model one the extra vector boson becomes massive.

The standard models Higgs potential V_H is given by

$$V_H(\rho,\sigma) = min\ \mathcal{V}(\rho\sigma, \rho\sigma^2; \eta^D, \eta^U).$$

It depends on two arguments because we have two Higgs fields. By definition, the minimum of V_H is at $\rho = \rho_0$, $\sigma = \sigma_0$, where σ_0 is the vacuum expectation value of ξ. But beware: Conventional Higgs fields differ from our φ, ξ by normalization factors.

In the presence of leptons one needs extra nonunitary parallel transporter. To give mass to the charged leptons, one needs a parallel transporter $\phi^l(\mathbf{x}) \in SL(3,\mathbf{C})$ which enters into the charged leptons mass term $\propto \bar{e}_R\phi^l l_L$. The most economical choice is

$$\phi^l = \phi^D. \tag{7.50}$$

To give mass to neutrinos one needs still another PT ϕ^ν and possibly a Majorana mass term for right handed neutrinos, in order to invoke the seesaw mechanism. We refrain from speculating what ϕ^ν might be.

We add a comment on colour [29]. Because G_R gauge transformations on R and L are independent, there is a colour group $SU(3)_{cL}\times SU(3)_{cR}$ to begin with which is broken to the diagonal $SU(3)_c$ because the $SU(3)_c$ factor in the cross-bulk PT acts as a Higgs field for it. As a result there will be an axigluon [16].

Let us consider the possibility of admitting nonunitary cross-bulk colour PT $\chi(\mathbf{x})$ in a noncompact colour holonomy group H_c which substitutes for the factor $SU(3)_c$ multiplying H. χ enters as a factor in Φ. Therefore $V(\Phi)$

depends on it. It is a complex 3×3 matrix and admits a decomposition $\chi = ru_1 d(\eta_c)u_2$ with $u_1,\, u_2 \in SU(3)_c$; let us assume that the factor r is real. Under a gauge transformation $(v_1, v_2) \in SU(3)_c \times SU(3)_c$, $\chi \mapsto v_1 \chi v_2^\dagger$. If the symmetry is to be broken down to the diagonal subgroup, the minimum of V, considered as a function of χ, must be at $\eta_c = 0$, i.e. $d(\eta_c) = \mathbf{1}$ and $\chi \in \mathbb{R}_* SU(3)_c$. The expectation value of the factor r, which occurs in the hadronic PT ϕ^D but not in its leptonic brother ϕ^l, will determine the ratio between charged lepton masses and D-quark masses if the economical choice (7.50) of the leptonic PT is adopted.

In conclusion, consider V as a function of η in a maximal noncompact abelian subalgebra $\mathfrak{t}$ of $LieH$. The difference between colour and flavor is that the orbits of the minima of V under the Weyl group $W_\mathfrak{t}$ of $\mathfrak{t}$ are trivial for colour and nontrivial for flavor. In other words the $W_\mathfrak{t}$-symmetry is spontaneously broken for flavor, but not for colour.

Let us finally turn to the CKM matrix. Assuming there are really two independent factors G^D, G^U in G, the CKM matrix C could be transformed away if it were not for the breaking of G-invariance by the L-boundary. C is therefore not determined by the minima of V but could be obtained as follows.

$A_L^D(x)$ and $A_L^U(x)$ are dynamical fields of the effective theory, because $\Phi(x)$ depends on them. Let $W(A_L^D, A_L^U)$ be the effective action for these fields alone obtained by integrating out the quark-fields and gauge fields associated with G_R and G_L, as well as A_R^D, A_R^U and the fluctuations of η^D, η^U and of ρ^D, ρ^U away from the minimum of V. The quark masses determined by minimization of V enter as parameters into W. The CKM field

$$C(x) = A_L^D(x) A_L^{U\dagger}(x)$$

is invariant under global G_{diag}-transformations, and so is W. The CKM matrix C is determined by the ground state of the theory with action W. In tree approximation (which may be accurate enough or not), it is determined by the minimum of the restriction of W to constant fields C; this is a calculation essentially within the standard model. It would be interesting to deal with the full effective action by numerical means.

Finally, let us comment on a GUT extension of our model. The assignment of fermions to boundaries is compatible with the action of the Pati-Salam subgroup $SU(4)_c \times SU(2)_L \times SU(2)_R$ of the GUT group $SO(10)$, but not with the action of $SO(10)$ itself. One may speculate that on scales shorter than the GUT scale, $SO(10)$ is an unbroken local symmetry of the bulk and light fermions migrate to the boundary when $SU(2)_L \times SU(2)_R$ breaks in the bulk, surviving temporarily on the boundaries. Our model is supposed to be valid at still larger scales, where $SU(2)_R$ is also broken on R. Migration of modes from bulk to branes have been discussed in the literature [19].

Chapter 8

Summary and outlook

In this thesis we have discussed gauge theories with nonunitary parallel transport both on a graph and in the continuum.

Starting from the proposal to abandon unitarity of paralleltransporters, we introduced the basic concepts of nonunitary gauge theories in the continuum in chapter 3. We showed that the vierbein can be interpreted as part of a de Sitter parallel transport in the conventional four space time dimensions.

We learned how general relativity can be recovered within the developed framework. A quasi-Maxwellian form of the Einstein-Hilbert action was presented, and it turned out that the complete action, including the matter action, can be written in a form which is polynomial both in the vierbein and in the spin connection.

Using polar decomposition of the nonunitary parallel transporter it was shown that there is a canonical way of constructing a metric and a metric connection with unitary parallel transporters in the general framework.

Finally we sketched a generalization of Einstein's theory of gravity.

In chapter 4 and chapter 5 we introduced the basic concepts and language of nonunitary parallel transport in the context of semicommutative differential geometry on a graph. We followed Dimakis and Müller-Hoissen, adding an integral calculus and the concept of endomorphism-valued forms.

Employing the tools of chapter 4 and 5, the geometry of Higgs fields was investigated in chapter 6. The Higgs was interpreted as associated with a nonunitary parallel transport in extra dimension(s). It turned out that generalized Yang-Mills actions automatically include a Higgs potential and kinetic terms for the Higgs fields. We argued that the framework of nonunitary gauge theories on a graph also provides tools to deal with deconstruction models. Finally, a reinterpretation of the standard model was discussed.

In chapter 7 we proposed a gauge theoretic model which explains how quarks of different flavor can acquire different masses by spontaneous symmetry breaking and what is the difference between colour and flavor.

Let us discuss at the end of this thesis a loose list of open questions that arise from the obtained results.

- First one would like to extend the model of chapter 7. Color and the leptonic sector has to be included. Are there GUT-extensions of the proposed model? It would be interesting to deal with the full effective action by numerical means. The dream is to compute quark masses and the CKM-matrix.

- The polynomial form of the gravity gravity served as a starting point for discretization. One may want to develop a “real space” renormalization group for graphs to calculate the flow of the discretized action. The issue of renormalizability of gravity might be reconsidered. Can the flow explain a tiny cosmological constant?

- It would be interesting to investigate the generalization of gravity associated with a conformal holonomy group.

- The model in chapter 7 might be combined with our formulation of gravity. For instance, one might consider a de Sitter group both on the left- and righthanded boundary, and a conformal holonomy group in the bulk. Can the Higgs field be interpreted as fifth component of a vielbein?

- There is a correspondence between string and gauge theories. Does a stringy description of nonunitary gauge theories exist?

- ...

Appendix A

Nonunitary parallel transport on differentiable manifolds

We saw that vector potentials are defined after a choice of moving frame has been made. It furnishes bases $(e_\alpha(x))$ in V_x. In this way, maps $\mathcal{T}(C)$ are converted to matrices $\boldsymbol{T}(C)$ via

$$\mathcal{T}(C)e_\alpha(x) = e_\beta(y)\boldsymbol{T}(C)^\beta{}_\alpha \tag{A.1}$$

and similarly for $\mathcal{U}(C)$. We also saw that parallel transport along infinitesimal paths $b : x \to x + \delta x$ determines the vector potential

$$\mathcal{T}(C)e_\alpha(x) = 1 - \boldsymbol{B}_\mu(x)\delta x^\mu. \tag{A.2}$$

Consider an infinitesimal path from x to $x + \delta x$. The inverse path $-b$ goes from $x + \delta x$ to x. As a consequence, the parallel transport matrix is

$$\begin{aligned} \boldsymbol{T}(-C) &= \mathbf{1} + \boldsymbol{B}_\mu(x + \delta x)\delta x^\mu \\ &= \mathbf{1} + \boldsymbol{B}_\mu(x)\delta x^\mu + \mathcal{O}((\delta x)^2)) \\ &= \boldsymbol{T}(C)^{-1}. \end{aligned} \tag{A.3}$$

Therefore, as long as only differentiable paths C are permitted, all parallel transporters are forced to be unitary in the sense of (3.25) and (3.26), respectively.

But there is a third possibility, which lies between the case of differentiable paths and the case of a discrete manifold. We propose to consider parallel transport along continous paths which are no longer differentiable. More precisely, the paths possess one-sided derivatives, which need not be equal. In reference [31] it will be shown that a sufficiently large class of paths possessing this property actually exists.

Let C be a path which is parameterized by t. The left-handed derivative is defined as

$$\frac{d_+}{dt}C(t) := \lim_{s\to 0, s>0} \frac{C(t+s)-C(t)}{s}, \tag{A.4}$$

the right-handed is defined as

$$\frac{d_-}{dt}C(t) := \lim_{s\to 0, s>0} \frac{C(t)-C(t-s)}{s}, \tag{A.5}$$

respectively. Now one can introduce two kind of "velocities" which transform differently under $t \to -t$

$$v(t) \;:=\; \frac{1}{2}\left(\frac{d_+}{dt}C(t) + \frac{d_-}{dt}C(t)\right), \tag{A.6}$$

$$u(t) \;:=\; \frac{1}{2}\left(\frac{d_+}{dt}C(t) - \frac{d_-}{dt}C(t)\right). \tag{A.7}$$

Due to the definitions (A.4) and (A.5) we find that under $t \to -t$ the velocities tranform as follows

$$v(-t) = -v(t) \tag{A.8}$$

$$u(-t) = u(t). \tag{A.9}$$

Only $v(t)$ transforms like an ordinary velocity. In fact, in the special case that C is differentiable, i.e. left- and right-handed derivative are equal, $u(t)$ vanishes and $v(t)$ becomes the ordinary velocity.

Consider an infinitesimal piece $b : x \to x + \delta x$ of a continous path which is parameterised in the following way

$$b : [-\delta\tau/2, \delta\tau/2] \to \mathcal{M} \tag{A.10}$$

with

$$b(-\delta\tau) = x \quad \text{and} \quad b(\delta\tau) = x + \delta x. \tag{A.11}$$

The inverse path $-b : x + \delta x \to x$ is given by

$$-b : [-\delta\tau/2, \delta\tau/2] \to \mathcal{M} \tag{A.12}$$

with

$$-b(t) = b(-t). \tag{A.13}$$

Corresponding with the split of the forward derivative $\frac{d_+}{dt}C$ into two pieces u and v, the deviation from $\mathbf{1}$ of the parallel transport matrix along such a path splits into two pieces

$$\boldsymbol{T}(b) = \mathbf{1} - \boldsymbol{A}_\mu(-\delta\tau/2)v(-\delta\tau/2)^\mu\delta\tau - \boldsymbol{E}_\mu(-\delta\tau/2)u(-\delta\tau/2)^\mu\delta\tau \tag{A.14}$$

The parallel transport matrix along the inverse path $-b$ becomes

$$\begin{aligned} \boldsymbol{T}(-b) &= \mathbf{1} + \boldsymbol{A}_\mu(\delta\tau/2)v(-\delta\tau/2)^\mu\delta\tau - \boldsymbol{E}_\mu(\delta\tau/2)u(-\delta\tau/2)^\mu\delta\tau \\ &= \mathbf{1} + \boldsymbol{A}_\mu(-\delta\tau/2)v(-\delta\tau/2)^\mu\delta\tau - \boldsymbol{E}_\mu(-\delta\tau/2)u(-\delta\tau/2)^\mu\delta\tau + \mathcal{O}((\delta\tau)^2) \end{aligned}$$

As a result, $\boldsymbol{T}(-b) \circ \boldsymbol{T}(b) \neq \mathbf{1}$.

Let us discuss the appearance of nonunitary parallel transport in the continuum from a slightly different point of view. We saw in chapter 2 that parallel transporters are a consequence of the Naheinformationsprinzip. According to this principle, information has to be transferred by exchange of signals which propagate in space time along a path C. Assume that a signal has arrived at an observer at y which wishes to send the information back to x along exact by the inverse path $-C$. However, this is impossible. There are a lot of paths which cannot be distinguished physically, but which differ by a certain small amount.

The validity of (3.25) depends on the exact space time paths chosen. Clearly, if the actual path chosen deviates from $-C$, the parallel transporter will be nonunitary.

In this way, the appearance of nonunitary parallel transporters might be viewed as a residual effect of the small scale structure of space time. They are the effective description which encodes the unknown physics on the Planck scale.

Appendix B

Generalized metricity II

We saw in chapter that the vierbein defines a map

$$\boldsymbol{E}(x) := \boldsymbol{E}_\mu(x)dx^\mu : T_x\mathcal{M} \to V_x \otimes V_x'. \tag{B.1}$$

Furthermore we saw that if

$$\boldsymbol{E}(x)\,[T_x\mathcal{M}] = V_x \otimes V_x', \tag{B.2}$$

one arrives at general relativity.

Now we assume that $\boldsymbol{E}(x)$ is not surjective, i.e.

$$\boldsymbol{E}(x)\,[T_x\mathcal{M}] =: W_x \subset V_x \otimes V_x'. \tag{B.3}$$

The underlying idea is as follows. We discussed that the internal space of gravity gets identified with the tangent space of space time only dynamically. On the other hand, traditionally it is assumed that there is an a priori distinction between the internal spaces of gravity and gauge theory. It might appear more natural to assume instead that also this distinction is of dynamical origin. In this appendix we sketch first steps towards such a theory , but a lot of work remain to be done.

As a result of (B.3) the "inverse" vierbein is only pseudoinverse. Therefore we can define a projector

$$\pi(x) := \boldsymbol{E}(x) \circ \boldsymbol{E}^{-1}(x) : V_x \otimes V_x' \to W, \qquad \pi^2 = \pi. \tag{B.4}$$

Consequently, the space $V_x \otimes V_x'$ decomposes as follows

$$V_x \otimes V_x' = \pi[V_x \otimes V_x'] + (\mathbf{1} - \pi)[V_x \otimes V_x'] =: W_x + Z_x \tag{B.5}$$

Similarly, the unitary vectorpotential $\boldsymbol{A}_\mu(x) : V_x \otimes V_x' \to V_x \otimes V_x'$ decomposes

$$\begin{aligned}\boldsymbol{A}_\mu(x) \;&=\; \pi(x)\boldsymbol{A}_\mu(x)\pi(x) + \pi(x)\boldsymbol{A}_\mu(x)(\mathbf{1} - \pi) + \\ &\quad +(\mathbf{1} - \pi(x))\boldsymbol{A}_\mu(x)\pi(x) + (\mathbf{1} - \pi(x))\boldsymbol{A}_\mu(x)(\mathbf{1} - \pi(x)).\end{aligned} \tag{B.6}$$

The covariant derivative $D_\mu := \partial_\mu + \boldsymbol{A}_\mu$ acting on $w \in W_x$ splits into two parts

$$\nabla_\mu w = (\partial_\mu + \pi \boldsymbol{A}_\mu \pi + (1-\pi)\boldsymbol{A}_\mu \pi) w. \tag{B.7}$$

This suggests decomposing the covariant derivative into a tangent and normal coponent, analogously to the ADM formalism [6]

$$\begin{aligned} D_\mu w &= \Big((1-\pi)(\partial_\mu + \boldsymbol{A}_\mu) + \pi(\partial_\mu + \boldsymbol{A}_\mu) \Big) w \\ &=: (D_\mu w)_\perp + (D_\mu w)_\parallel . \end{aligned} \tag{B.8}$$

Notice that $(D_\mu w)_\parallel$ provides a connection in the sense that

$$(D_\mu(fw))_\parallel = (\partial_\mu f) w + (\nabla_\mu w)_\parallel, \tag{B.9}$$

for $f \in C^\infty(\mathcal{M})$.

Furthermore the connection is metric with respect to the bilinear form in W_x, which is induced by the bilinear form in $V_x \otimes V_x'$

$$\begin{aligned} \langle \ , \ \rangle_{V_x \otimes V_x'} &= \langle \ , \ \rangle_{W_x} + \langle \ , \ \rangle_{Z_x} \\ &:= \langle \pi \bullet, \pi \bullet \rangle_{V_x \otimes V_x'} + \langle (1-\pi) \bullet, (1-\pi) \bullet \rangle_{V_x \otimes V_x'} . \end{aligned} \tag{B.10}$$

To see this, let $v, w \in W_x$ and $D_\mu = \partial_\mu + \boldsymbol{A}_\mu$. Then,

$$\begin{aligned} \partial_\mu \langle v, w \rangle_{V_x \otimes V_x'} &= \langle \nabla_\mu v, w \rangle_{V_x \otimes V_x'} + \langle v, \nabla_\mu w \rangle_{V_x \otimes V_x'} \\ &= \langle (\nabla_\mu v)_\perp, w \rangle_{V_x \otimes V_x'} + \langle v, (\nabla_\mu w)_\perp \rangle_{V_x \otimes V_x'} + \\ &+ \langle (\nabla_\mu v)_\parallel, w \rangle_{V_x \otimes V_x'} + \langle v, (\nabla_\mu w)_\parallel \rangle_{V_x \otimes V_x'} \\ &= \langle (\nabla_\mu v)_\parallel, w \rangle_{W_x} + \langle v, (\nabla_\mu w)_\parallel \rangle_{W_x} = \partial_\mu \langle v, w \rangle_{W_x} \end{aligned} \tag{B.11}$$

$\pi \boldsymbol{A}_\mu \pi$ describes gravity, but Z_x appears as an internal space. Consequently, $(1-\pi)\boldsymbol{A}_\mu(1-\pi)$ may be viewed as a gauge field, whereas $(1-\pi)\boldsymbol{A}_\mu \pi$ is analogous to the extrinsic curvature in the ADM analysis of gravity.

Let $(e_\alpha(x))_\alpha, (e_i(x))_i$ be a basis in W_x and Z_x, respectively. The field strength is

$$\mathcal{F}(\partial_\mu, \partial_\nu) e_\alpha(x) = [D_\mu, D_\nu] e_\alpha(x) \tag{B.12}$$

$$= e_\beta(x) F^\beta{}_{\alpha\mu\nu} + e_j(x) F^j{}_{\alpha\mu\nu}. \tag{B.13}$$

In components we have

$$F^\beta{}_{\alpha\mu\nu} = \partial_{[\mu} A^\beta{}_{\alpha\nu]} + A^\beta{}_{\gamma[\mu} A^\gamma{}_{\alpha\nu]} + A^\beta{}_{i[\mu} A^i{}_{\alpha\nu]} \tag{B.14}$$

$$=: R^\beta{}_{\alpha\mu\nu} + \mathcal{T}^\beta{}_{\alpha\mu\nu}, \tag{B.15}$$

where $R^\beta{}_{\alpha\mu\nu} = \partial_{[\mu} A^\beta{}_{\alpha\nu]} + A^\beta{}_{\gamma[\mu} A^\gamma{}_{\alpha\nu]}$ denotes the Riemann curvature tensor, and

$$F^j{}_{\alpha\mu\nu} = \partial_{[\mu} A^j{}_{\alpha\nu]} + A^j{}_{\gamma[\mu} A^\gamma{}_{\alpha\nu]} + A^j{}_{i[\mu} A^i{}_{\alpha\nu]}, \tag{B.16}$$

with

$$(A^{\alpha}{}_{\beta\mu}) := \pi \boldsymbol{A}_{\mu}\pi \tag{B.17}$$
$$(A^{i}{}_{\beta\mu}) := (1-\pi)\boldsymbol{A}_{\mu}\pi \tag{B.18}$$
$$(A^{i}{}_{j\mu}) := (1-\pi)\boldsymbol{A}_{\mu}(1-\pi) \tag{B.19}$$

Note that the components $F^{\alpha}{}_{\beta\mu\nu}$ can be used to construct a generalized Ricci tensor and Ricci scalar, respectively.

Furthermore we have

$$\mathcal{F}(\partial_{\mu},\partial_{\nu})e_i(x) = [D_{\mu},D_{\nu}]e_i(x) \tag{B.20}$$
$$= e_j(x)F^{j}{}_{i\mu\nu} + e_{\alpha}(x)F^{\alpha}{}_{i\mu\nu} \tag{B.21}$$

with

$$F^{j}{}_{i\mu\nu} = \partial_{[\mu}A^{j}{}_{i\nu]} + A^{j}{}_{k[\mu}A^{k}{}_{i\nu]} + A^{j}{}_{\gamma[\mu}A^{\gamma}{}_{i\nu]} \tag{B.22}$$
$$=: {}^{YM}F^{j}{}_{i\mu\nu} + K^{j}{}_{i\mu\nu}, \tag{B.23}$$

where ${}^{YM}F^{j}{}_{i\mu\nu}$ denotes the Yang-Mills field strength,$K^{j}{}_{i\mu\nu}$ is the last term in (B.22), and

$$F^{\alpha}{}_{i\mu\nu} = \partial_{[\mu}A^{\alpha}{}_{i\nu]} + A^{\alpha}{}_{k[\mu}A^{k}{}_{i\nu]} + A^{\alpha}{}_{\gamma[\mu}A^{\gamma}{}_{i\nu]}. \tag{B.24}$$

Appendix C

Dirac algebra

Let us recall some basic facts about Dirac matrices. We restrict our considerations to spinors associated to 4-dimensional Minkowski space with $\eta_{\alpha\beta} = diag(+1, -1, -1, -1)$. The basic theorem asserts that a complex spinor space V together with a quadruple of linear operators γ_α acting irreducibly on V and satisfying $\{\gamma_\alpha \gamma_\beta\} = 2\eta_{\alpha\beta}$ is uniquely determined up to equivalence and that $dim V = 4$.

Since the quadruples $-\gamma_\alpha^t, \gamma_\alpha^\dagger, -\bar{\gamma}_\alpha$ act irreducibly on the dual V', complex-conjugate $\bar{V}'$ and complex vector space $\bar{V}$, and satisfy the same anticommutation relations, it follows that there exists equivalence maps $B : V \to V', \beta : V \to \bar{V}', C : V \to V'$, such that

$$-\gamma_\alpha^t = B\gamma_\alpha B^{-1}, \quad \gamma_\alpha^\dagger = \beta\gamma_\alpha\beta^{-1}, \quad -\bar{\gamma}_\alpha = C\gamma_\alpha C^{-1}. \tag{C.1}$$

$\boldsymbol{\beta}, \boldsymbol{C}$, ect. denote the corresponding matrix representations.

One can show that β, C and B define on V a Hermitian form, a real structure, and a symplectic form. The adjoint and charge-conjugate spinor are defined by

$$\bar{\psi} = \psi^\dagger \beta, \quad \psi^C := C\bar{\psi}^t = C\beta\psi^{c.c} \tag{C.2}$$

for $\psi \in V$. Furthemore one defines

$$\boldsymbol{\gamma}_5 := i\boldsymbol{\gamma_0\gamma_1\gamma_2\gamma}_3, \quad \{\boldsymbol{\gamma_5}, \boldsymbol{\gamma}_\alpha\} = 0, \quad \boldsymbol{\gamma}_5^2 = \mathbf{1}. \tag{C.3}$$

Let us add some properties of β and C (more precisely, of their matrix representations) which are needed in chapter 3

$$\begin{aligned} \boldsymbol{\beta}^\dagger &= \boldsymbol{\beta}, \quad \boldsymbol{C} = -\boldsymbol{C}^t, & \text{(C.4)} \\ (\boldsymbol{\beta\gamma}_\alpha)^\dagger &= \boldsymbol{\beta\gamma}_\alpha, \quad (\boldsymbol{\gamma}_\alpha \boldsymbol{C})^t = \boldsymbol{\gamma}_\alpha \boldsymbol{C}, & \text{(C.5)} \\ \boldsymbol{\beta\gamma}_5\boldsymbol{\beta}^{-1} &= \boldsymbol{\gamma}_5^\dagger, \quad \boldsymbol{C}^{-1}\boldsymbol{\gamma}_5\boldsymbol{C} = \boldsymbol{\gamma}_5^t. & \text{(C.6)} \end{aligned}$$

Bibliography

[1] A.Connes. *Non-commutative Geometry.* Academic Press, New York, 1994.

[2] Bastian Angermann. *Verallgemeinerte Eichtheorien auf diskreten Mannigfaltigkeiten.* 2002. Diploma thesis.

[3] Nima Arkani-Hamed, Andrew G. Cohen, and Howard Georgi. Accelerated unification. 2001.

[4] Nima Arkani-Hamed, Andrew G. Cohen, and Howard Georgi. (de)constructing dimensions. *Phys. Rev. Lett.*, 86:4757–4761, 2001.

[5] Nima Arkani-Hamed, Andrew G. Cohen, and Howard Georgi. Electroweak symmetry breaking from dimensional deconstruction. *Phys. Lett.*, B513:232–240, 2001.

[6] John Baez and Javier P. Muniain. *Gauge fields, knots and gravity.* World Scientific Publishing Co. Pte. Ltd., 1994.

[7] William A. Bardeen, Robert B. Pearson, and Eliezer Rabinovici. Hadron masses in quantum chromodynamics on the transverse lattice. *Phys. Rev.*, D21:1037, 1980.

[8] Hsin-Chia Cheng, Christopher T. Hill, Stefan Pokorski, and Jing Wang. The standard model in the latticized bulk. *Phys. Rev.*, D64:065007, 2001.

[9] Hsin-Chia Cheng, Christopher T. Hill, and Jing Wang. Dynamical electroweak breaking and latticized extra dimensions. *Phys. Rev.*, D64:095003, 2001.

[10] Mathias de Riese. *Polar Decomposition in General Complex Systems and Discrimination of Gauge, Higgs and Vierbein Fields.* 2001. Diploma thesis.

[11] A. Dimakis, F. Muller-Hoissen, and T. Striker. From continuum to lattice theory via deformation of the differential calculus. *Phys. Lett.*, B300:141–144, 1993.

[12] A. Dimakis, F. Muller-Hoissen, and T. Striker. Noncommutative differential calculus and lattice gauge theory. *J. Phys.*, A26:1927–1949, 1993.

[13] Aristophanes Dimakis and Folkert Muller-Hoissen. Differential calculus and gauge theory on finite sets. *J. Phys.*, A27:3159–3178, 1994.

[14] Aristophanes Dimakis and Folkert Muller-Hoissen. Discrete differential calculus graphs, topologies and gauge theory. *J. Math. Phys.*, 35:6703–6735, 1994.

[15] Aristophanes Dimakis, Folkert Muller-Hoissen, and Francois Vanderseypen. Discrete differential manifolds and dynamic on networks. *J. Math. Phys.*, 36:3771–3791, 1995.

[16] Paul H. Frampton and Sheldon L. Glashow. Unifiable chiral color with natural gim mechanism. *Phys. Rev. Lett.*, 58:2168, 1987.

[17] T. Fulton, F. Rohrlich, and L. Witten. Conformal invariance in physics. *Rev. Mod. Phys.*, 34:442–457, 1962.

[18] J. M. Gracia-Bondia, J. C. Varilly, and H. Figueroa. Elements of noncommutative geometry. Boston, USA: Birkhaeuser (2001) 685 p.

[19] S. Groot Nibbelink, H. P. Nilles, and M. Olechowski. Instabilities of bulk fields and anomalies on orbifolds. *Nucl. Phys.*, B640:171–201, 2002.

[20] Christopher T. Hill, Stefan Pokorski, and Jing Wang. Gauge invariant effective lagrangian for kaluza-klein modes. *Phys. Rev.*, D64:105005, 2001.

[21] P. Hilton and Y.-C. Wu. *A course in modern algebra.* John Wiley, New York, 1974.

[22] Kerson Huang. *Quarks, leptons and gauge theory.* World Scientific Publishing Co. Pte. Ltd., 1982.

[23] Shoshichi Kobayashi and Katsumi Nomizu. *Foundations of differential geometry Volume I.* John Wiley, 1996.

[24] Giovanni Landi. *An introduction to noncommutative spaces and their geometries.* Springer Verlag, 1997.

[25] O. Lauscher and M. Reuter. Is quantum einstein gravity nonperturbatively renormalizable? *Class. Quant. Grav.*, 19:483–492, 2002.

[26] C. Lehmann, G. Mack, and T. Prustel. Origin of quark masses and kobayashi maskawa matrix in a gauge theory with nonunitary parallel transporters. 2003.

[27] Claudia Lehmann and Gerhard Mack. Gauge theories with nonunitary parallel transporters: A soluble higgs model. 2003.

[28] G. Mack. Universal dynamics, a unified theory of complex systems: Emergence, life and death. *Commun. Math. Phys.*, 219:141–178, 2001.

[29] Gerhard Mack. To gauge theory from a minimum of a priori structure. *Proc. Steklov. Math. Institute.* http://voltaire.desy.de.

[30] Gerhard Mack. Pushing einstein's principles to the extreme. 1997.

[31] Gerhard Mack and Thorsten Prustel. Gauge theories with nonunitary parallel transporters. 2003. in preparation.

[32] Gerhard Mack, Thorsten Prustel, and Mathias de Riese. Gauge theories with nonunitary parallel transporters general relativity as example. 2004. in preparation.

[33] F. Muller-Hoissen. Introduction to noncommutative geometry of commutative algebras and applications in physics. Prepared for 2nd Mexican School on Gravitation and Mathematical Physics, Tlaxcala, Mexico, 1-7 Dec 1996.

[34] Falk Neugebohrn. *Allgemeine Relativitaetstheorie auf diskreten Mannigfaltigkeiten.* 2003. Diploma thesis.

[35] Michael Olschewsky. *Higgsfelder als verallgemeinerte Eichfelder.* 2002. Diploma thesis.

[36] Nolan R. Wallach. *Harmonic analysis on homogeneous spaces.* New York: Dekker, 1973.

[37] H. Weyl. Gravitation und elektrizitaet. *Sitzungsber. Preuss. Akad. Wiss.*, ::465–480, 1918.

[38] H. Weyl. Eine neue erweiterung der relativitaetstheorie. *Ann. Physik (Leipzig)*, 59:101–133, 1919.

Danksagung

An erster Stelle möchte ich mich für die interessante Aufgabenstellung bei Herrn Prof. Dr. Mack bedanken.
Durch seine lehrreiche Betreuung, unermüdliche Hilfsbereitschaft und Intuition hat er großen Anteil am Zustandekommen dieser Arbeit.

Es ist mir weiter eine Freude, Falk Neugebohrn für die exzellente Zusammenarbeit und Unterstützung zu danken. Großer Dank gebührt ebenso Michael Olschewsky und Michael Röhrs, die mir besonders in der Endphase meiner Arbeit aufopferungsvoll zur Seite standen.

Für zahlreiche interessante Gespräche über Physik und den "Rest" danke ich Dorothea Bahns, Olaf Hohm, Muharrem Küskü und Tobias Schlegelmich.

Mein Dank gilt auch Timo Behnke, Johannes Kückens und Krisztian Peters, ohne die vieles langweiliger gewesen wäre.

Maximilian Krüger danke ich für seine ganz besondere Freundschaft und Geduld.

Dank an Tina Kaschwich für ihre Hilfsbereitschaft, Du bist die einzig wahre Latex-Göttin.

Kaum in Worte kann ich den Dank an meine Freundin Bianca von Scheidt für ihre Liebe und Geduld fassen.

Ganz besonders danke ich meiner Mutter und meinem Vater.

Der Deutschen Forschungsgemeinschaft (DFG) danke ich für die finanzielle Unterstützung.