Control-constrained parabolic optimal control problems on evolving surfaces – theory and variational discretization

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Contents

Introduction

We investigate linear-quadratic parabolic optimal control problems on evolving material hypersurfaces in $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$. In addition, we consider elliptic state equations on stationary surfaces. The state equations can be seen as models for diffusion-driven processes taking place on surfaces, such as evolving bio-membranes. Primarily however, they are but academic examples, employed in order to investigate the changes an optimal control problem undergoes as we substitute its stationary euclidean domain by a curved and moving one.
Following [DE07], we consider parabolic state equations in their weak form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Gamma(t)} y\,\varphi\,\mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{b}\,y\,\varphi\,\mathrm{d}\Gamma(t) = \int\limits_{\Gamma(t)} y\,\dot{\varphi}\,\mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} u\,\varphi\,\mathrm{d}\Gamma(t)\,, \quad y(0) = y_0\,,$$

where $\Gamma = \{\Gamma(t)\}^{t\in[0,T]}$ is a family of C^2-smooth, compact n-dimensional surfaces in $\mathbb{R}^{n+1}$, evolving smoothly in time with velocity V, and $\dot{\varphi} = \partial_t\varphi + V\nabla\varphi$ denotes the material derivative of a smooth test function φ. We define unique weak solutions for the state equation under low regularity assumptions on the data u, y_0. In particular we allow for $y_0 \in L^2(\Gamma(0))$. The idea is to introduce distributional material derivatives in the sense of [LM68] and a $W(0,T)$-like solution space.
The stationary diffusion equation on a fixed surface Γ reads

$$\int\limits_{\Gamma} \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{c}\,y\,\varphi\,\mathrm{d}\Gamma = \int\limits_{\Gamma} u\,\varphi\,\mathrm{d}\Gamma\,, \forall \varphi \in H^1(\Gamma)\,.$$

Both in the stationary and the instationary case each surface is approximated by a triangulation Γ^h on which a finite element scheme for the state equation is formulated along the lines of [Dzi88] and [DE07], respectively. Here we assume $n = 1, 2, 3$ in order that the interpolation be well defined. The approximation error of this discretization of the state equation decomposes into a finite element error, arising from the projection onto a finite dimensional Ansatz space, and a geometrical part which is due to the approximation of Γ by Γ^h. We prove convergence results for the parabolic equations under weak regularity assumptions.
The state equations define linear control-to-state operators. Using these, we formulate control constrained optimal control problems along with their necessary optimality conditions where the adjoint state equations appear. The optimal control problems are subjected to variational discretization, see [Hin05], by replacing Γ and the state equation by their finite dimensional approximations. The variationally discretized problems are amenable to an implementable

semismooth Newton algorithm. In both cases we prove convergence of the discretized optimal controls.

In the elliptic case we also discuss in some detail the implementation of a globalized semismooth Newton algorithm for the control problem, involving a new merit function. In the parabolic setting a suitable scalar product is formulated in order to arrive at an easily computable discrete adjoint scheme.

Our analytical findings are complemented with numerical examples.

This work is structured as follows. We begin with a very short introduction into the setting in Chapter 1, introducing some basic concepts for hypersurfaces such as signed distance functions and Sobolev spaces. We then investigate elliptic optimal control problems on stationary hypersurfaces in Chapter 2 along with their numerical treatment rounding up the chapter with some numerical examples.

It follows a detailed study of the properties of the parabolic equation in Chapter 3. In order to formulate well posed optimal control problems, we proof the existence of an appropriate weak solution in Section 3.4, complementing the existence results from [DE07]. Afterwards, we examine the space- and time-discretization of the state equation in Sections 3.6 and 3.7 and prove optimal L^2-error bounds.

Finally, Chapter 4 is devoted to parabolic optimal control. Using the results from Section 3.4 we formulate control constrained optimal control problems. We then apply variational discretization in the sense of [Hin05] to achieve fully implementable optimization algorithms and end the work with some numerical examples.

Chapter 1

Some facts on compact embedded hypersurfaces

The purpose of this chapter is to collect a number of results that are used in later chapters. Among other things, the existence of sufficiently smooth distance functions will be proved and we investigate properties of piecewise smooth functions.

Consider a connected compact n-dimensional abstract manifold $\mathcal{M}$ of class C^k, $k \geq 2$. Suppose now that $\mathcal{M}$ is embedded into $\mathbb{R}^{n+1}$, i.e., it exists an injective C^k-map $f : \mathcal{M} \to \mathbb{R}^{n+1}$ such that its differential $D_{\mathcal{M}} f_m : T_m\mathcal{M} \to \mathbb{R}^{n+1}$ is injective for all $m \in \mathcal{M}$. Its image $\Gamma = f(\mathcal{M})$ is then a compact hypersurface in $\mathbb{R}^{n+1}$.

What is more, the hypersurface Γ is closed and connected and by the Jordan-Brouwer separation theorem it divides $\mathbb{R}^{n+1} \setminus \Gamma$ in exactly two open connected components. Those are the bounded interior Ω of Γ and its unbounded exterior $\mathbb{R}^{n+1} \setminus \overline{\Omega}$. Hence the exterior unit normal vector field $\nu : \Gamma \to \mathbb{R}^{n+1}$ induces a natural orientation on Γ.

A proof of the Jordan-Brouwer theorem that extends to the C^2-case can be found in [Lim88]; there it is assumed that $\mathcal{M}$ is orientable which is true for any such surface, compare [Bre97, Ch. VI, Cor. 8.9]. Also [Bre97, Ch. VI, Cor. 8.8] ,while being more difficult to read, includes a stronger, purely topological version of the Jordan-Brouwer theorem.

Throughout this work we regard the tangential spaces $T_\gamma\Gamma$ as subsets of $\mathbb{R}^{n+1}$, i.e.,

$$T_{f(m)}\Gamma = D_{\mathcal{M}} f_m(T_m\mathcal{M})\,.$$

The surrounding space $\mathbb{R}^{n+1}$ determines a Riemannian metric on $\mathcal{M}$ via the identity $g(\cdot,\cdot) = \langle D_{\mathcal{M}} f(\cdot), D_{\mathcal{M}} f(\cdot)\rangle_{\mathbb{R}^{n+1}}$. Instead of $(\Gamma, \langle\cdot,\cdot\rangle_{\mathbb{R}^{n+1}})$ we simply write Γ. For notational convenience we often write

$$v_1 v_2 \text{ instead of } \langle v_1, v_2\rangle_{\mathbb{R}^{n+1}} \quad \text{and} \quad \|v_1\| \text{ instead of } \|v_1\|_{\mathbb{R}^{n+1}}\,, \quad \forall v_1, v_2 \in T_\gamma\Gamma\,. \qquad (1.0.1)$$

In Section 1.1 we introduce the signed distance function $d : \mathbb{R}^{n+1} \to \mathbb{R}$ for Γ, as well as the projection $\Pi_\Gamma : \mathbb{R}^{n+1} \to \Gamma$ onto Γ and investigate their regularity.

In Section 1.2 we do the same for a family of surfaces $\Gamma(t) = \Phi^0_t(\Gamma_0)$ evolving from an initial surface Γ_0 under a flow Φ.

Section 1.3 is devoted to Sobolev spaces on manifolds. Finally, a loose collection of results that are more or less linked to Sobolev spaces, such as uniform elliptic $H^2(\Gamma)$-stability estimates, can be found in the Appendices 1.A-1.C. The notation in Section 1.3 and the Appendices 1.A-1.C differs slightly from that in the rest of the work, as it is only here that local representations and maps are explicitly used.

1.1 Signed distance function and projection

The distance of a point $x \in \mathbb{R}^{n+1}$ to the surface Γ

$$\mathrm{dist}_\Gamma : \mathbb{R}^{n+1} \to \mathbb{R}\,, \quad \mathrm{dist}_\Gamma(x) = \inf_{\gamma \in \Gamma} \|\gamma - x\|$$

is Lipschitz continuous with Lipschitz constant 1. This fact, which holds true for any subset $A \subset \mathbb{R}^{n+1}$, is easily shown using minimizing sequences.
Nevertheless, in order to analyze the quality of approximations of Γ, it will turn out to be more convenient to work with a signed version of dist_Γ, the so-called signed distance function

$$d(x) = \begin{cases} -\mathrm{dist}_\Gamma(x), & \text{if } x \in \Omega \\ \mathrm{dist}_\Gamma(x), & \text{if } x \notin \Omega \end{cases}$$

with Ω as above. The function d exhibits additional regularity in a tubular neighborhood of Γ with radius $\epsilon > 0$

$$\mathcal{N}_\epsilon = \left\{ x \in \mathbb{R}^{n+1} \mid |d(x)| < \epsilon \right\}\,.$$

Closely linked to d are the unique projection onto Γ

$$\Pi_\Gamma : \mathcal{N}_\epsilon \to \Gamma\,, \quad x \mapsto \arg\min_{\gamma \in \Gamma} \|\gamma - x\|\,,$$

and the exterior unit normal field.

Definition 1.1.1. An exterior unit normal vector or outwards-pointing unit normal vector of the C^k-surface $\Gamma \subset \mathbb{R}^{n+1}$, $k \geq 1$, at $\gamma \in \Gamma$ is the uniquely defined unit vector $\nu \in \mathbb{R}^{n+1}$ which is perpendicular to $T_\gamma\Gamma$ and for which there exists $\delta > 0$ such that for $t \in (0, \delta)$

$$\begin{aligned} \gamma + t\nu &\in \mathbb{R}^{n+1} \setminus \overline{\Omega} \\ \gamma - t\nu &\in \ \Omega\,. \end{aligned}$$

The exterior normal field of Γ is the map $\nu : \Gamma \to \mathbb{R}^{n+1}$ that associates each point $\gamma \in \Gamma$ with its respective exterior unit normal direction.

The regularity of d, ν and Π_Γ as well as the well-definedness of the latter are the subject of the current section.
Using the defining properties of ν, i.e., the orthogonality relation and $\|\nu\| = 1$, one can conclude by the implicit function theorem that $\nu \in C^{k-1}(\Gamma)$.
Regarding Π_Γ, observe that from standard optimization theory it follows that for

$$y \in \arg\min_{\gamma \in \Gamma} \|\gamma - x\|$$

the vector $x - y$ is perpendicular to the tangent space $T_y\Gamma$. Hence we have

$$x = y \pm \nu(y)\|x - y\| = y + \nu(y)d(x)\,. \tag{1.1.1}$$

Also, since Γ is compact, the set $\arg\min_{\gamma\in\Gamma} \|\gamma - x\|$ is not empty.

Theorem 1.1.2 ([DZ94, Thm. 5.6]). *Let Γ denote the C^k-smooth boundary of a bounded domain Ω in $\mathbb{R}^{n+1}$, $k \geq 2$. Then there exists $\epsilon > 0$ such that $d \in C^k(\mathcal{N}_\epsilon)$ and for $x \in \mathcal{N}_\epsilon$ the set $\arg\min_{\gamma\in\Gamma} \|\gamma - x\|$ is a singleton. Further one has $\Pi_\Gamma \in C^{k-1}(\mathcal{N}_\epsilon, \Gamma)$.*

Proof. By [DZ94, Thm. 5.6] there exists a ball $B_{\epsilon(\gamma)}(\gamma)$, $\epsilon(\gamma) > 0$ for each $\gamma \in \Gamma$ such that the assertions of the theorem hold inside $B_{\epsilon(\gamma)}(\gamma)$. One can assume w.l.o.g. that the radii $\epsilon(\gamma) \in (0, \infty]$ are maximized, i.e., one of the three properties stated in the theorem does not hold on $B_\delta(\gamma)$ if $\delta > \epsilon(\gamma)$. Now, Γ being closed and bounded, a common compactness argument proves that the $\epsilon(\gamma)$ are bounded away from zero, thus establishing the theorem: Assume $\epsilon(\gamma)$, $\gamma \in \Gamma$ are not bounded away from 0. Then there exists a convergent sequence $\gamma_k \to \bar{\gamma}$, such that $\epsilon(\gamma_k) \to 0$. But for some $k \in \mathbb{N}$ the ball $B_{\epsilon(\gamma_k)}(\gamma_k)$ lies entirely in $B_{\frac{1}{2}\epsilon(\bar{\gamma})}(\bar{\gamma})$, contradicting the maximality of $\epsilon(\gamma_k)$. ■

Inside $\mathcal{N}_\epsilon$ the distance d has the important property

$$\nabla d(x) = \nu(\Pi_\Gamma(x))\,. \tag{1.1.2}$$

This relation, apart from complying with intuition, is also easy to prove. For $x \in \mathcal{N}_\epsilon$ (1.1.1) reads

$$x = \Pi_\Gamma(x) + \nu(\Pi_\Gamma(x))d(x)\,,$$

which we can derive for x to obtain

$$\mathrm{id}_{\mathbb{R}^{n+1}} = D\Pi_\Gamma(x) + d(x)D_\Gamma\nu(\Pi_\Gamma(x))D\Pi_\Gamma(x) + \nu(\Pi_\Gamma(x))\nabla d(x)^T\,.$$

Multiply by $\nu(\Pi_\Gamma(x))^T$ from the left to get (1.1.2). Observe that $\nu(\Pi_\Gamma(x))$ is orthogonal on the columns of $D\Pi_\Gamma(x)$, and that $\nu(\gamma)^T D_\Gamma\nu(\gamma) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\gamma}\|\nu(\gamma)\|^2 = 0$.

1.2 Time dependent signed distance function and projection

In the context of evolving surfaces we require signed distance functions that also exhibit a certain time regularity as well as tubular neighborhoods $\mathcal{N}_\epsilon$ whose radius ϵ does not depend on the time variable $t \in [0, T]$, $T > 0$.

We will consider hypersurfaces $\Gamma(t)$ that evolve from an initial connected compact hypersurface without boundary $\Gamma(0) = \Gamma_0$ of class C^k, $k \geq 2$. To this end consider an open set $\mathcal{U} \subset \mathbb{R}^{n+1}$ and a velocity field $V \in C^k(\mathcal{U} \times \mathbb{R}, \mathbb{R}^{n+1})$. Let further $\mathcal{U}_0 \subset \mathcal{U}$ denote an open, bounded, and connected neighborhood of Γ_0. Then by compactness of $\overline{\mathcal{U}_0}$ there exists some $T, \delta > 0$ such that the initial value problems

$$y(0; 0, x_0) = x_0 \in \mathcal{U}_0\,, \quad \forall t \in (-\delta, T + \delta) : \frac{\mathrm{d}}{\mathrm{d}t}y(t; 0, x_0) = V(y(t; 0, x_0), t)$$

admit unique solutions $y(\,\cdot\,;0,x_0)$, compare [Wal98, §12, Thm. VI]. Hence for all initial times $s \in (-\delta, T+\delta)$ the problems

$$y(s;s,x_s) = x_s \in \mathcal{U}_s\,, \quad \forall t \in (-\delta, T+\delta) : \frac{\mathrm{d}}{\mathrm{d}t} y(t;s,x_s) = V(y(t;s,x_s),t)$$

with $U_s = y(s;0,\mathcal{U}_0)$ are uniquely solvable for $y(t;s,x_s)$. The map

$$\Phi : \mathcal{D} \to \mathbb{R}^n\,, \quad (t,s,x_s) \mapsto \Phi_t^s(x_s) = y(t;s,x_s)$$

which acts on the open domain

$$\mathcal{D} = (-\delta, T+\delta) \times \bigcup_{s\in(-\delta,T+\delta)} \{s\} \times \mathcal{U}_s \subset \mathbb{R}^{n+3}$$

satisfies $\Phi \in C^k(\mathcal{D}, \mathbb{R}^{n+1})$, compare [Wal98, §12, Cor. XI] and is called the flow of the vector field V. Moreover, by definition of the maps $\Phi_t^s : \mathcal{U}_s \to \mathcal{U}_t$ there holds

$$\Phi_t^s \circ \Phi_s^r = \Phi_t^r\,, \text{ and } \Phi_t^t = \mathrm{id}_{\mathcal{U}_t}\,,$$

in particular all Φ_t^s are C^k-diffeomorphisms. Now $\Gamma(t) = \Phi_t^0(\Gamma_0)$ is the translation of Γ_0 along the velocity field V, and by Φ_t^s we synonymously denote the restriction $\Phi_t^s : \Gamma(s) \mapsto \Gamma(t)$.
For all $t \in [0,T]$ the surface $\Gamma(t)$ is closed and divides $\mathbb{R}^{n+1} \setminus \Gamma(t)$ into two open connected sets, again by the Jordan-Brouwer separation theorem. Those are the bounded interior Ω_t of $\Gamma(t)$ and its unbounded exterior $\mathbb{R}^{n+1} \setminus \overline{\Omega_t}$. Because $\Phi_\cdot^s$ is a homotopy between the surfaces $\Gamma(s)$ and $\Gamma(t)$ it seems natural that Φ_t^s maps the interior Ω_s of Γ_s to the interior Ω_t of Γ_t.

Lemma 1.2.1. *For all $s,t \in [0,T]$ there holds $\Phi_t^s(\Omega_s \cap \mathcal{U}_s) = \Omega_t \cap \mathcal{U}_t$.*

Proof. We show that the property holds in a small environment around any fixed $s \in [0,T]$ and thus globally.
Because $\Gamma(s)$ is compact, there exists $x_s \in \mathcal{U}_s$ such that $\|x_s\| > \max_{\gamma\in\Gamma(s)} \|\gamma\|$. In particular the half-line $\mathscr{L} = \{tx_s \mid t \in [1,\infty)\}$ lies entirely in $\mathbb{R}^{n+1} \setminus \overline{\Omega_s}$, which is the unbounded one of the two connected components of $\mathbb{R}^{n+1}\setminus\Gamma(s)$. Now choose $\epsilon_s > 0$ such that for $t \in (s-\epsilon_s, s+\epsilon_s)$ both $x_s \in \mathcal{U}_t$ and $\|x_s\| > \max_{\gamma\in\Gamma(t)} \|\gamma\|$ holds. Hence $\mathscr{L}$ also lies entirely in the unbounded component $\mathbb{R}^n \setminus \overline{\Omega_t}$. Because there are only two connected components, one concludes

$$\Phi_t^s(\Omega_s \cap \mathcal{U}_s) = \Omega_t \cap \mathcal{U}_t\,.$$

Suppose the property does not hold globally. Then there exists $t \in [0,T]$ such that $t = \inf\left\{\tau \in [0,T] \;\middle|\; \Phi_\tau^0(\Omega_0 \cap \mathcal{U}_0) \neq \Omega_\tau \cap \mathcal{U}_\tau\right\}$. Apply the local result at t to get a contradiction. ■

As a consequence one gets the following Lemma.

Lemma 1.2.2. *Assume $s,t \in [0,T]$ and $x \in \Omega_s \cap \mathcal{U}_s$ such that $x \in \mathbb{R}^{n+1} \setminus \overline{\Omega_t \cap \mathcal{U}_t}$. Then for some $\theta \in (\min(s,t), \max(s,t))$ one has $x \in \Gamma(\theta)$.*

Proof. By Lemma 1.2.1 we can rewrite the relation $x \in \mathbb{R}^{n+1} \setminus \overline{\Omega_t \cap \mathcal{U}_t}$ as $\Phi_s^t(x) \in \mathbb{R}^{n+1} \setminus \overline{\Omega_s \cap \mathcal{U}_s}$. The curve $\Phi_\cdot^t(x)$ is a path connecting $x = \Phi_t^t(x)$ and $\Phi_s^t(x)$ which hits $\Gamma(s)$ because those points are lying in different connected components. ■

Next let us mention that we can express the outer unit normal field ν_t of $\Gamma(t)$ through that of $\Gamma(0)$.

Lemma 1.2.3. *Let ν_0 denote the outer unit normal field of Γ_0. Then for $t \in [0,T]$, $\gamma \in \Gamma_0$ the outer unit normal field of $\Gamma(t)$ at $\Phi_t^0(\gamma)$ is given by*

$$\nu_t(\Phi_t^0(\gamma)) = \frac{(D\Phi_t^0(\gamma))^{-T}\nu_0(\gamma)}{\|(D\Phi_t^0(\gamma))^{-T}\nu_0(\gamma)\|} \in C^{k-1}(\Gamma_0 \times [0,T], \mathbb{R}^{n+1})\,.$$

Note that at this point we do not prove that ν_t continues to point outwards throughout the interval $[0,T]$. This is done in the proof of Theorem 1.2.4.

Proof. The image of $D\Phi_t^0(\gamma))T_\gamma\Gamma(0)$ is the tangential space of $\Gamma(t)$ and since

$$\langle (D\Phi_t^0(\gamma))^{-T}\nu_0(\gamma), D\Phi_t^0(\gamma))v\rangle_{\mathbb{R}^{n+1}} = \nu_0(\gamma)^T(D\Phi_t^0(\gamma))^{-1}D\Phi_t^0(\gamma))v = 0$$

for all tangential vectors $v \in T_\gamma\Gamma(0)$ of Γ_0 we conclude the orthogonality of ν_t on $T_{\Phi_t^0(\gamma)}$. ■

We can now proceed to prove the main result of this section, the smoothness of the signed distance function of $\Gamma(t)$

$$d(x,t) = \begin{cases} -\mathrm{dist}_{\Gamma(t)}(x), & \text{if } x \in \Omega_t \\ \mathrm{dist}_{\Gamma(t)}(x), & \text{if } x \notin \Omega_t \end{cases}\,. \tag{1.2.1}$$

Theorem 1.2.4. *There exists $\epsilon > 0$ such that the signed distance function*

$$d : \mathbb{R}^{n+1} \times [0,T] \to \mathbb{R}$$

is C^k-smooth in the space-time domain

$$\mathcal{N}_\epsilon = \big\{(x,t) \in \mathbb{R}^{n+1} \times [0,T] \;\big|\; \mathrm{dist}_{\Gamma(t)}(x) < \epsilon\big\}$$

and $\Pi_{\Gamma(\cdot)}(\cdot) \in C^{k-1}(\mathcal{N}_\epsilon, \mathbb{R}^{n+1})$.

Parts of the proof have been inspired by that of [BG88, Thm. 2.7.12].

Proof. Consider the Riemannian C^k-manifold $\mathcal{N} = \Gamma_0 \times \mathbb{R} \times (-\delta, T+\delta)$ and the C^{k-1}-mapping

$$E : \mathcal{N} \to \mathbb{R}^{n+2}\,, \quad (\gamma, d, t) \mapsto (\Phi_t^0(\gamma) + \nu_t(\Phi_t^0(\gamma))d, t)^T\,,$$

which has the following two important properties:

1. $E(\cdot, 0, \cdot) : \Gamma_0 \times [0,T] \to \mathbb{R}^{n+2}$ is injective,
2. $D_{\mathcal{N}}E$ is invertible everywhere on $\Gamma_0 \times \{0\} \times [0,T] \subset \mathcal{N}$.

The first follows immediately from our assumptions on Φ^0_t; in order to perceive the second property we compute the differential of E

$$D_{\mathcal{N}}E(\gamma,0,t) = \begin{pmatrix} D_{\Gamma_0}\Phi^0_t(\gamma) & \nu_t(\Phi^0_t(\gamma)) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which is a surjective mapping due to the orthogonality of ν_t on the image of $D_{\Gamma_0}\Phi^0_t$. Hence, because of $\dim(\mathcal{N}) = \dim(\mathbb{R}^{n+2})$, it is also bijective.
Thus we have that E is a local diffeomorphism at each point of $\Gamma_0 \times \{0\} \times [0,T]$, and $D_{\mathcal{N}}E$ is invertible on some tube $\Gamma_0 \times (-\tilde{\epsilon},\tilde{\epsilon}) \times [0,T]$. By compactness we now can argue that for some $0 < \epsilon \leq \tilde{\epsilon}$ the map E constitutes a diffeomorphism between $\Gamma_0 \times (-\epsilon,\epsilon) \times [0,T]$ and its image $\mathcal{N}_\epsilon$:
Assume that there exist two sequences $(\gamma_k, d_k, t_k), (\tilde{\gamma}_k, \tilde{d}_k, t_k) \in \Gamma_0 \times \mathbb{R} \times [0,T]$ such that $d_k, \tilde{d}_k \to 0$ while $E(\gamma_k, d_k, t_k) = E(\tilde{\gamma}_k, \tilde{d}_k, t_k)$. Then we can extract subsequences converging towards $(\gamma, 0, t)$ and $(\tilde{\gamma}, 0, t)$ and by continuity of E we get $E(\gamma,0,t) = E(\tilde{\gamma},0,t)$. But by property 1. the map E is injective on $\Gamma_0 \times \{0\} \times [0,T]$. Thus $\gamma = \tilde{\gamma}$ contradicting E being a local diffeomorphism around $(\gamma, 0, t)$.
Now we know that for any $(x,t) \in \mathcal{N}_\epsilon$ the pre-image $(\gamma_0(x,t), d^1(x,t), t)$ is a singleton, and the component functions

$$\gamma_0(x,t) : \mathcal{N}_\epsilon \to \Gamma_0\,, \quad d^1(x,t) : \mathcal{N}_\epsilon \to (-\epsilon,\epsilon)\,, \quad \text{as well as} \quad \Pi^1_{\Gamma(t)}(x,t) = \Phi^0_t \circ \gamma_0 : \mathcal{N}_\epsilon \to \Gamma(t)$$

are C^{k-1}-smooth.
Next we show the identities $d^1 = d$ and $\Pi^1_{\Gamma(\cdot)} = \Pi_{\Gamma(\cdot)}$ on $\mathcal{N}_\epsilon$. Observe that as in the argument preceding (1.1.1) the necessary optimality conditions for $y \in \arg\min_{\gamma\in\Gamma(t)} \operatorname{dist}_{\Gamma(t)}(x)$ are

$$x - y = \lambda \nu_t(y) \quad \text{for some } \lambda \in \mathbb{R}\,.$$

But since E is bijective on $\Gamma_0 \times (-\epsilon,\epsilon) \times [0,T]$, the only $(y,\lambda) \in \Gamma(t) \times (-\epsilon,\epsilon)$ that satisfies this condition is $(\Pi^1_{\Gamma(t)}(x), d^1(x,t))$. Because of $|\lambda| = \|x-y\|$ all y that might satisfy this relation for larger $\lambda \notin (-\epsilon,\epsilon)$ clearly do not lie in $\arg\min_{\gamma\in\Gamma(t)} \operatorname{dist}_{\Gamma(t)}(x)$. Hence we proved $\Pi^1_{\Gamma(t)}(x) = \Pi_{\Gamma(t)}(x)$ for $(x,t) \in \mathcal{N}_\epsilon$.
Also we immediately get that

$$\text{either } d^1(x,t) = d(x,t) \text{ or } d^1(x,t) = -d(x,t)\,. \tag{1.2.2}$$

Now we show that $\nu_t(\Pi_{\Gamma(t)}(x))$ remains an outward-pointing vector for all $t \in [0,T]$. It suffices to prove that for any fixed $s \in [0,T]$ there exists some $\epsilon_s > 0$ such that $\nu_t(\Pi_{\Gamma(t)}(x))$ does not change its orientation on $(s-\epsilon_s, s+\epsilon_s)$. Pick any $(x,s) \in \mathcal{N}_\epsilon$ with $d(x,s) = \frac{\epsilon}{2}$. There exists ϵ_s such that the $\frac{\epsilon}{4} < d(x,\cdot) < \epsilon$ on $[s-\epsilon_s, s+\epsilon_s]$. Since by Lemma 1.2.2 the only way for x to pass from the inside to the outside or vice versa is by the surface Γ_t passing through it, i.e., $d(x,t) = 0$ for some t, the point x remains on the same side of Γ_t for all $t \in (s-\epsilon_s, s+\epsilon_s)$. As a consequence, choosing $0 < \delta \leq \frac{\epsilon}{4}$ in Definition 1.1.1 ,we see that the vector

$$\nu_t(\Pi_{\Gamma(t)}(x)) = \frac{x - \Pi_{\Gamma(t)}(x)}{d^1(x,t)}$$

retains its orientation throughout $(s - \epsilon_s, s + \epsilon_s)$.
Considering (1.2.2) it follows that d^1 is the restriction to $\mathcal{N}_\epsilon$ of the signed distance function d from (1.2.1).
The additional regularity of d follows from the identities $\nabla_x d(x,t) = \nu_t(\Pi_{\Gamma(t)}(x)) \in C^{k-1}(\mathcal{N}_\epsilon)$, compare (1.1.2), and

$$\partial_t d(x,t) = \langle \nu_t(\Pi_{\Gamma(t)}(x)), \partial_t \Phi_t^0(\gamma_0(x,t)) \rangle \in C^{k-1}(\mathcal{N}_\epsilon)\,. \tag{1.2.3}$$

The second relation is proved similarly to the first one. Starting with

$$x = \Pi_{\Gamma(t)}(x) + d(x,t)\nu_t(\Pi_{\Gamma(t)}(x))\,,$$

we differentiate for t and multiply by $\nu_t(\Pi_{\Gamma(t)}(x))^T$ to derive (1.2.3) using

$$\nu_t(\Pi_{\Gamma(t)}(x))^T D_{\Gamma_0}\Phi_t^0(\gamma_0(x,t))\partial_t\gamma_0(x,t) = 0\,, \quad d(x,t)\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nu_t(\Pi_{\Gamma(t)}(x))\|^2 = 0\,,$$

in the process, i.e., the orthogonality relation and the fact that ν_t is normalized. ∎

Remark 1.2.5. All results of this section also generalize to the case of a Hölder continuously differentiable flow $\Phi \in C^{k,\gamma}(\mathcal{D}, \mathbb{R}^{n+1})$, $k \geq 2$ and a compact, closed $C^{k,\gamma}$-hypersurface Γ_0. One then gets $d \in C^{k,\gamma}(\mathcal{N}_\epsilon, \mathbb{R})$ and $\Pi_{\Gamma(\cdot)}(\cdot) \in C^{k-1,\gamma}(\mathcal{N}_\epsilon, \mathbb{R}^{n+1})$.

1.3 Sobolev spaces on manifolds

The standard way to introduce Sobolev spaces on an abstract manifold is by completion of the C^k-functions with respect to the respective Sobolev norm. This may also be the reason for that the $W^{k,\infty}$-spaces are usually omitted since they cannot be constructed in this way. We follow this approach and in addition give a definition for $W^{k,\infty}$ on an n-dimensional Riemannian manifold $(\mathcal{M}, g)$ of class C^K, $k \leq K$.
Through the Riemannian metric tensor g we can define the length of curves and the distance $\mathrm{dist}(\cdot,\cdot)$ of two points of $\mathcal{M}$, turning $(\mathcal{M}, \mathrm{dist})$ into a metric space that is topologically equivalent to $\mathcal{M}$.
The notation in this section differs from that in the rest of this work. This is because, excepting the Appendices 1.A-1.C, this is the only section where we actually work with local charts. Apart from naming these charts we need to label the components of tensors such as the metric tensor. From the viewpoint of later chapters these local objects are not visible; the metric then arises from the embedding of $T_m\mathcal{M}$ into $\mathbb{R}^{n+1}$ and is just referred to as $\cdot$ or as $\langle\cdot,\cdot\rangle_{\mathbb{R}^{n+1}}$. Also, the definition of Sobolev spaces of course does not rely on $\mathcal{M}$ being a hypersurface in an euclidean space. Hence, a more general notation seems appropriate.
The Riemannian structure of $(\mathcal{M}, g)$ gives rise to the kth covariant Levi-Civita derivative $\nabla^k u$ of a function $u \in C^k(\mathcal{M})$. The derivative $\nabla^k u$ is then a tensor field of type $(k, 0)$, i.e., k-times covariant and 0-times contravariant. Following [Heb00] let us define the norm of a $(k,0)$-tensor T

$$|T| = \left(\sum_{i_1 \ldots i_k = 1}^{n} \sum_{j_1 \ldots j_k = 1}^{n} g^{i_1 j_1} \ldots g^{i_k j_k} T_{i_1 \ldots i_k} T_{j_1 \ldots j_k} \right)^{\frac{1}{2}} \tag{1.3.1}$$

where $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ and $T_{i_1 \dots i_k} = T(\frac{\partial}{\partial x_{i_1}}, \dots, \frac{\partial}{\partial x_{i_k}})$ are the components of the tensors g and T in local coordinates $\frac{\partial}{\partial x_i}$, and g^{ij} are chosen such that $\sum_{j=1}^n g_{ij} g^{jk} = \delta_{ik}$, where δ_{ij} denotes Kronecker's delta. In other words the matrix (g^{ij}) is the inverse of the matrix (g_{ij}). In particular for the gradient ∇u of a function u one has

$$\|\nabla u\|_g = (g(\nabla u, \nabla u))^{\frac{1}{2}} = |\nabla^1 u|,$$

because in local coordinates it's components read $(\nabla u)_i = \sum_{j=1}^n g^{ij} (\nabla^1 u)_j$. Remember that the gradient is defined through the relation

$$g(\nabla u, \frac{\partial}{\partial x_i}) = \nabla^1 u(\frac{\partial}{\partial x_i}), \quad i = 1, \dots, n.$$

Now for $1 \le p < \infty$ the $W^{k,p}$-norm of some smooth function $u \in C^k(\mathcal{M})$ is

$$\|u\|_{k,p} = \left(\sum_{i=0}^{k} \int_{\mathcal{M}} |\nabla^i u|^p \, \mathrm{d}v(g) \right)^{\frac{1}{p}},$$

where $v(g) = \sqrt{\det(g)} \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$ is the volume form associated with g. Observe that while $\det(g) = \det(g_{ij})$ does depend on the local basis the 1-form $v(g)$ does not. Let us mention that in later parts of this work, where we deal with hypersurfaces on which metric tensor and volume form are prescribed by the surrounding space, we will just denote integrals as $\int_\Gamma \mathrm{d}\Gamma$.

Now for $k \le K$ let

$$C^{k,p} = \left\{ u \in C^k(\mathcal{M}) \mid \|u\|_{k,p} \text{ is finite} \right\}.$$

Let us define Sobolev spaces and state some fundamental properties, compare [Heb00, Prop. 2.1-2.3].

Lemma and Definition 1.3.1. *Let $0 \le k \le K$ and $1 \le p < \infty$. By $W^{k,p}(\mathcal{M}) \subset L^p(\mathcal{M})$ we denote the completion of $C^{k,p}$ with respect to the norm $\|\cdot\|_{k,p}$. $W^{k,p}(\mathcal{M})$ is a Banach space. On compact manifolds the $W^{k,p}$-spaces do not depend on the metric g.*
Let further $p \in (1, \infty)$, then $W^{k,p}(\mathcal{M})$ is reflexive. The spaces $W^{k,2}(\mathcal{M})$ are Hilbert spaces and often are referred to as $H^k(\mathcal{M})$. By $H^{-k}(\mathcal{M})$ we denote the completion of $L^2(\mathcal{M})$ with respect to the dual norm of $H^k(\mathcal{M})$.

One has the usual compact embeddings, see [Heb00, Thm 2.9].

Lemma 1.3.2. *Let $\mathcal{M}$ be compact. For integers $k \ge 0$ and $l \ge 1$, such that $k + l \le K$, and real numbers $p, q \ge 1$ with $p < nq/(n - lq)$ the embedding $W^{k+l,q}(\mathcal{M}) \subset W^{k,p}(\mathcal{M})$ is compact. For $q > n$ the space $W^{1,q}(\mathcal{M})$ is a compact subspace of the space of Hölder continuous functions $C^{0,\gamma}(\mathcal{M})$ for any Hölder exponent $\gamma \in (0,1)$ such that $(1 - \gamma) q > n$.*

While it is possible to define a scalar product on $(k, 0)$-tensors, that induces the norm $|\cdot|$ from (1.3.1), we content ourselves with pointing out that for $k = 1$ the scalar product associated with the $W^{1,2}$-norm can be written as

$$\langle u, w \rangle_{1,2} = \int_{\mathcal{M}} uw \, \mathrm{d}v(g) + \int_{\mathcal{M}} g(\nabla u, \nabla w) \, \mathrm{d}v(g).$$

In order to facilitate some of the following proofs we make some technical assumptions on $(\mathcal{M}, g)$, which are met in many practical cases, among those all surfaces considered in this work.

Assumption 1.3.3. For $(\mathcal{M}, g)$ there exists an atlas $(\mathcal{U}_i, \phi_i)_{i\in I}$ with subordinate partition of unity $\{\eta_i\}_{i\in I}$, such that the pullbacks of g via ϕ_i^{-1} are uniformly positive definite and bounded, i.e., for any tangential vector $v = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i}$ there holds

$$2\sum_{i=1}^n v_i^2 \geq \sum_{i,j=1}^n v_i g_{ij} v_j \geq \frac{1}{2}\sum_{i=1}^n v_i^2 \,. \tag{1.3.2}$$

Observe that for the inverse matrix (g^{ij}) Equation (1.3.2) implies also

$$2\sum_{i=1}^n v_i^2 \geq \sum_{i,j=1}^n v_i g^{ij} v_j \geq \frac{1}{2}\sum_{i=1}^n v_i^2 \,. \tag{1.3.3}$$

In particular Assumption 1.3.3 is satisfied for all compact Riemannian manifolds $(\mathcal{M}, g)$, as can be seen by choosing normal coordinates at every point $m \in \mathcal{M}$ in a sufficiently small neighborhood $\mathcal{U}_m$ and then passing to finitely many $\mathcal{U}_m$ covering $\mathcal{M}$.
We are now going to make use of these assumptions in order to interpret the concept of weak derivatives on manifolds. Unlike in a vector space, here the partial derivatives of a function as well as the components of tensors are local objects.

Lemma and Definition 1.3.4. *Let Assumption 1.3.3 hold for $(\mathcal{M}, g)$. An element $u \in W^{k,p}(\mathcal{M})$ possesses* weak covariant derivatives $\nabla^l u$, $l = 1\dots k$. *In a local chart $(\mathcal{U}_i, \phi_i)$, $i \in I$, the components of $\nabla^l u$ are the $L^p(\mathcal{U}_i)$-limits of the components of $\nabla^l \varphi_r$, where $\{\varphi_r\}_{r\in\mathbb{N}} \subset C^{k,p}(\mathcal{M})$ tends to u in the $W^{k,p}$-norm. For a.e. $m \in \mathcal{M}$ the object $\nabla^l u(m)$ is an $(l, 0)$-tensor. Also there exist* weak partial derivatives $\partial_{i_1\dots i_l} u \in L^p(\mathcal{U}_i)$, *$l \leq k$, that are the L^p-limits of $\partial_{i_1\dots i_l}\varphi_r$.*

Proof. Consider a $W^{k,p}$ Cauchy sequence $\{\varphi_r\}_{r\in\mathbb{N}} \subset C^{k,p}(\mathcal{M})$ converging by definition towards its $W^{k,p}$-limit $u \in W^{k,p}(\mathcal{M})$. Let $(\mathcal{U}, \phi)$ denote a local chart as by Assumption 1.3.3. Using the Cholesky decomposition $g^{ij} = \sum_{r=1}^n c^{ir} c^{jr}$ we can rewrite the tensor norm from (1.3.1) as

$$|T| = \left(\sum_{i_1 j_1=1}^n \sum_{r_2\dots r_k=1}^n g^{i_1 j_1} \tilde{T}_{i_1 r_2 \dots r_k} \tilde{T}_{j_1 r_2\dots r_k}\right)^{\frac{1}{2}} \tag{1.3.4}$$

with $\tilde{T}_{i_1 r_2\dots r_k} = \sum_{i_2\dots i_k=1}^n c^{i_2 r_2}\dots c^{i_k r_k} T_{i_1 i_2\dots i_k}$.
Now one can apply (1.3.3) to the expression (1.3.4). Iterating this argument over the indices $i_1 \dots i_k$ and $j_1 \dots j_k$ one proves the estimate

$$|\nabla^l(\varphi_r - \varphi_s)|^2 \geq \frac{1}{2^l} \sum_{i_1\dots i_l=1}^n |\nabla^l(\varphi_r - \varphi_s)_{i_1\dots i_l}|^2 \,. \tag{1.3.5}$$

The components of $\nabla^l\varphi_r$, $l \leq k$, are thus $L^p(\mathcal{U}_i)$ Cauchy sequences converging to a limit denoted $(\nabla^l u)_{i_1\dots i_l} \in L^p(\mathcal{U}_i)$.

For a.e. $m \in \mathcal{M}$ the $(\nabla^l u)_{i_1 \dots i_l}$ are the components of a $(l, 0)$-tensor, because the transformation rule for coordinate changes $(\nabla^l \varphi_r)_{i_1 \dots i_l} \mapsto \widetilde{(\nabla^l \varphi_r)}_{j_1 \dots j_l}$ is a linear mapping and thus holds a.e. after passing to the L^p-limit.
By the definition of the covariant derivative one has

$$\frac{\partial}{\partial x_\alpha}(\nabla^l \varphi_r)_{i_1 \dots i_l} = (\nabla^{l+1} \varphi_r)_{\alpha i_1 \dots i_l} + \sum_{m=1}^{l} \sum_{\beta=1}^{n} \Gamma^\beta_{\alpha i_m} (\nabla^l \varphi_r)_{i_1 \dots i_{m-1} \, \beta \, i_{m+1} \dots i_l} \,, \tag{1.3.6}$$

the n^3 Christoffel symbols Γ^i_{jk} being C^K functions. In view of (1.3.6) the $L^p(\mathcal{U}_i)$-convergence of the covariant derivatives of φ_r implies $L^p(\mathcal{U}_i)$-convergence of the partial derivatives $\frac{\partial}{\partial x_\alpha}(\nabla^l \varphi_r)_{i_1 \dots i_l}$ in $L^p(\mathcal{U}_i)$, given that $l + 1 \le k$. Successively differentiating (1.3.6) with respect to $\alpha_2 \dots \alpha_{k-1}$, one shows by induction over the number of partial derivatives that $\partial_{\alpha_1 \dots \alpha_s}(\nabla^l)_{i_1 \dots i_l} u \in L^p(\mathcal{U}_i)$ for $l + s \le k$. Hence the weak partial derivatives exist. ∎

Next, we also want to properly define $W^{k,\infty}(\mathcal{M})$.

Lemma and Definition 1.3.5. *Let Assumption 1.3.3 hold for $(\mathcal{M}, g)$, and additionally assume $\int_{\mathcal{M}} \mathrm{d}v(g) < \infty$. Equipped with the norm*

$$\|u\|_{k,\infty} = \max_{l=0 \dots k} \| |\nabla^l u| \|_{L^\infty(\mathcal{M},g)}$$

the space

$$W^{k,\infty}(\mathcal{M}) = \left\{ u \in W^{k,1} \mid \|u\|_{k,\infty} \text{ is finite} \right\}$$

is a Banach space.

Proof. A $W^{k,\infty}$-Cauchy sequence $\{\varphi_i\}_{i \in \mathbb{N}}$ possesses a $W^{k,1}$-limit $\varphi \in W^{k,1}(\mathcal{M}, g)$.
Now let $0 \le l \le k$. Since countable unions of null sets are again of measure 0, there exists a set $\mathcal{N}$ of $v(g)$-measure 0 which does not depend on l, such that for all $m \in \mathcal{M} \setminus \mathcal{N}$ we have $|\nabla^l \varphi_i(m) - \nabla^l \varphi_j(m)| \le \epsilon$ for $i, j \ge c_\epsilon$ independent of m. Since the space of $(l, 0)$ tensors is complete we obtain the pointwise limit $\nabla^l \varphi_i(m) \to T^l(m)$ with

$$\| |T^l| \|_{L^\infty(\mathcal{M},g)} \le \| |\nabla^l \varphi_{c_\epsilon}| \|_{L^\infty(\mathcal{M},g)} + \epsilon \,.$$

Note that $|T^l|$ is measurable since it is the pointwise a.e. limit of measurable functions. Finally the uniform bound

$$|\nabla^l \varphi_i(m) - T^l(m)| = \lim_{j \to \infty} |\nabla^l \varphi_i(m) - \nabla^l \varphi_j(m)| \le \epsilon$$

implies uniform convergence on $\mathcal{M} \setminus \mathcal{N}$. In particular $\| |\nabla^l \varphi_i - T^l| \|_{L^\infty(\mathcal{M},g)} \to 0$ implies $\nabla^l \varphi = T^l$ and thus $\varphi \in W^{k,\infty}(\mathcal{M})$ and $\|\varphi_i - \varphi\|_{k,\infty} \to 0$. ∎

1.3.1 Manifolds with boundary and Stokes' formula

In order to work with partitions of smooth surfaces that arise from finite element methods it is useful to consider manifolds with boundary. Since the patches those partitions consist of usually exhibit a certain lack of regularity at the boundary let us also allow for corners.

Definition 1.3.6. An n-dimensional *manifold with boundary* is a Hausdorff space $\mathcal{M}$ that is locally homeomorph to the half space $\{x \in \mathbb{R}^n \mid x_1 \geq 0\}$. The set of points that are mapped onto $\{x \in \mathbb{R}^n \mid x_1 = 0\}$ are called the *boundary* $\partial\mathcal{M}$ of $\mathcal{M}$.
An n-dimensional C^2-smooth *manifold with corners* $\mathcal{M}$ is locally diffeomorph to sets

$$H_i = \{x \in \mathbb{R}^n \mid x_j \geq 0, \text{ for } 1 \leq j \leq n-i\}\,, \quad 0 \leq i \leq n\,.$$

More precisely $\mathcal{M}$ is a manifold with boundary such that

1. for every $m \in \mathcal{M}$ there exists a neighborhood $\mathcal{U}_m \subset \mathcal{M}$ and a homeomorphism

$$\varphi_m : \mathcal{U}_m \to H_{i_m} \cap V_m\,,$$

with $\varphi_m(m) = 0$ and V_m an $\mathbb{R}^n$-neighborhood of 0.

2. the changes of coordinates $\varphi_{m_1} \circ \varphi_{m_2}^{-1}$ are C^2-diffeomorphisms wherever $\mathcal{U}_{m_1} \cap \mathcal{U}_{m_2} \neq \emptyset$.

Each point $m \in \mathcal{M}$ is then called a *corner of dimension* i_m, and the boundary is characterized by $\partial\mathcal{M} = \{m \in \mathcal{M} \mid \text{dimension of } m \text{ is not } n\}$. Note that because of the second condition the dimension i_m does not depend on the choice of the chart φ_m.

Note that since the boundary of a Riemannian manifold has $v(g)$-measure 0 its presence does not change the preceding definitions of an lemmata on Sobolev spaces.
On manifolds with corners Stokes' formula holds, compare [Tay11, Ch. 1,Prop. 13.4].

Theorem 1.3.7 (Stokes' formula)**.** *On an oriented C^2-manifold $\mathcal{M}$ of dimension n with corners, whose boundary $\partial\mathcal{M}$ is equipped with its natural orientation, there holds*

$$\int_{\mathcal{M}} d\beta = \int_{\partial\mathcal{M}} \beta$$

for every compactly supported n-form of class C^1.

A consequence of this is the divergence theorem for vector fields. The divergence of a differentiable vector field F

$$\mathrm{div}_{\mathcal{M}} F = \frac{1}{\sqrt{\det(g)}} \sum_{i=1}^{n} \partial_i \left(\sqrt{\det(g)} F_i \right)$$

is a measure of the change of a volume under the flow of F.

Theorem 1.3.8 (Divergence theorem)**.** *If $(\mathcal{M}, g)$ is a compact oriented Riemannian C^2-manifold with corners and F a C^1-vector field, then*

$$\int\limits_{\mathcal{M}} \mathrm{div}_{\mathcal{M}} F \, \mathrm{d}v(g) = \int\limits_{\partial\mathcal{M}} g(F, \mu) \, \mathrm{d}v_\partial(g)\,,$$

where μ is the outward pointing normal to $\partial\mathcal{M}$, and $v_\partial(g)$ denotes the volume induced by g on $\partial\mathcal{M}$.

Proof. The proof preceding [Tay11, Ch. 2, Thm. 2.1] applies. ■

For $u \in C^2(\mathcal{M})$ the Laplace-Beltrami operator is defined as

$$\Delta_{\mathcal{M}} u = \mathrm{div}_{\mathcal{M}} \nabla_{\mathcal{M}} u\,.$$

as a consequence of the divergence theorem one has

$$\int_{\mathcal{M}} \varphi(-\Delta_{\mathcal{M}} u)\,\mathrm{d}v(g) = \int_{\mathcal{M}} g(\nabla_{\mathcal{M}}\varphi, \nabla_{\mathcal{M}} u)\,\mathrm{d}v(g) - \int_{\partial\mathcal{M}} g(\nabla_{\mathcal{M}} u, \mu)\,\mathrm{d}v_{\partial}(g)\,, \tag{1.3.7}$$

for a smooth test function $\varphi \in C^1(\mathcal{M})$, which gives rise to the weak formulation of elliptic problems.

The main result of this section is fairly easy to prove in euclidean space, whereas on a manifold $\mathcal{M}$ its proof is rather technical. It says that a piecewise smooth, globally continuous function on $\mathcal{M}$ is an element of $W^{1,\infty}(\mathcal{M})$ and Lipschitz continuous.

Lemma 1.3.9. *Consider a compact n-dimensional orientable C^2-manifold $\mathcal{M} = \bigcup_{j\in J} \mathcal{M}_j$ with or without boundary. $\partial\mathcal{M}$ is assumed to be C^2-smooth.*

The partition $\{\mathcal{M}_j\}_{j\in J}$, J finite, consists of n-dimensional orientable compact C^2-manifolds M_j with corners such that

$$\mathcal{M}_i \cap \mathcal{M}_j \subset \partial\mathcal{M}_i \cap \partial\mathcal{M}_j \Leftrightarrow i \neq j\,,$$

and the exterior normals μ_i of $\mathcal{M}_i$ and μ_j of $\mathcal{M}_j$, $i \neq j$ satisfy $\mu_j = -\mu_i$ a.e. on the submanifold $\partial\mathcal{M}_i \cap \partial\mathcal{M}_j$. In particular $\partial\mathcal{M}_i \cap \partial\mathcal{M}_j \cap \partial\mathcal{M}_k$ for pair-wise distinct indices i, j, k is a set of $\partial\mathcal{M}_i \cap \partial\mathcal{M}_j$-measure zero.

A function $u \in C(\mathcal{M}, \mathbb{R})$ that is piecewise smooth ($u \in C^1(\mathcal{M}_j, \mathbb{R})$, $j \in J$) lies in $W^{1,\infty}(\mathcal{M})$ and is Lipschitz continuous.

Proof. Because $\mathcal{M}$ is compact, Assumption 1.3.3 holds with a finite index set I, and we have a uniformly positive definite atlas $(\mathcal{U}_i, \phi_i)_{i\in I}$ with subordinate partition of unity $\{\eta_i\}_{i\in I}$. We show that $\eta_i u$ is in $W^{1,\infty}(\mathcal{M})$ and Lipschitz; this proves the assertion since I is finite.

We proceed by showing the existence of weak partial derivatives of $\tilde{u} = (\eta_i u\sqrt{\det g}) \circ \phi_i^{-1}$ in $L^\infty(\{x \in \mathbb{R}^n \mid x_1 \geq 0\})$. Let $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support in $\{x \in \mathbb{R}^n \mid x_1 > 0\}$ and denote $\tilde{\varphi} = \varphi \circ \phi_i^{-1}$. Then one applies Theorem 1.3.8 and makes use of the assumptions on the partition $\{\mathcal{M}_j\}_{j\in J}$ to get

$$\begin{aligned}
\sum_{j\in J} \int_{\phi_i(\mathcal{U}_i\cap\mathcal{M}_j)} \partial_\alpha\tilde{u}\varphi + \tilde{u}\partial_\alpha\varphi\,\mathrm{d}x &= \sum_{j\in J} \int_{\phi_i(\mathcal{U}_i\cap\mathcal{M}_j)} \mathrm{div}\,(\tilde{u}\varphi e_\alpha)\,\mathrm{d}x \\
&= \sum_{j\in J} \int_{\mathcal{M}_j} \mathrm{div}_{\mathcal{M}}\left(\eta_i u\tilde{\varphi}\frac{\partial}{\partial x_\alpha}\right)\mathrm{d}v(g) \\
= \sum_{j\in J} \int_{\partial\mathcal{M}_j} \eta_i u\tilde{\varphi}\, g(\frac{\partial}{\partial x_\alpha}, \mu_j)\,\mathrm{d}v(g) &= \int_{\partial\mathcal{M}} \eta_i u\tilde{\varphi}\, g(\frac{\partial}{\partial x_\alpha}, \mu)\,\mathrm{d}v(g) = 0\,.
\end{aligned}$$

Hence $\tilde{u} \in W^{1,\infty}(\{x \in \mathbb{R}^n \mid x_1 \geq 0\})$. But then $(\eta_i u) \circ \phi_i^{-1}$ is, too, because $\frac{1}{\sqrt{\det(g)}} \circ \phi_i^{-1} \in C^1(\{x \in \mathbb{R}^n \mid x_1 \geq 0\})$. Observe that since the boundary $\{x \in \mathbb{R}^n \mid x_1 = 0\}$ of the domain of $(\eta_i u) \circ \phi_i^{-1}$ is smooth this also implies Lipschitz continuity

$$(\eta_i u) \circ \phi_i^{-1} \in C^{0,1}(\{x \in \mathbb{R}^n \mid x_1 \geq 0\}) .$$

First let us argue the $W^{1,\infty}$-smoothness of $\eta_i u$. By convolution with smoothing functions one obtains a sequence $(\varphi_k)_{k \in \mathbb{N}}$ in $C^2(\phi_i(\mathcal{U}_i))$ approximating $(\eta_i u) \circ \phi_i^{-1}$ in the $W^{1,1}$-norm. By Assumption 1.3.3 this implies convergence of $\varphi_k \circ \phi_i$ towards $\eta_i u$ in the $W^{1,1}(\mathcal{M})$-norm. We already proved the boundedness of the weak gradient of $(\eta_i u) \circ \phi_i^{-1}$; boundedness of the weak gradient $\nabla(\eta_i u)$ hence follows again by Assumption 1.3.3.
As to the Lipschitz continuity of $\eta_i u$ note that for $m_1, m_2 \notin \operatorname{supp}(\eta_i u)$ the relation

$$|\eta_i u(m_1) - \eta_i u(m_2)| \leq \mathrm{d}(m_1, m_2)$$

holds trivially. Hence let us consider $m_1 \in \operatorname{supp}(\eta_i u)$ and $m_2 \in \mathcal{U}_i$. Observe that a shortest path connecting m_1 with m_2 or, more general, connecting paths may leave the set $\mathcal{U}_i$. We make use of the fact that the support of η_i is a compact set. Hence let $\delta > 0$ denote its distance from the boundary of $\mathcal{U}_i$. Then the length of each path between m_1 and m_2 that leaves $\mathcal{U}_i$ is bounded from below by δ.
On the other hand there holds (1.3.2), and thus the lengths of all paths not leaving $\mathcal{U}_i$ are bounded from below by $\frac{1}{2}\|\phi_i(m_1) - \phi_i(m_2)\|$. Using the Lipschitz constant L_i of $(\eta_i u) \circ \phi_i^{-1}$ one obtains

$$\min(\delta, \frac{1}{2L_i}|\eta_i u(m_1) - \eta_i u(m_2)|) \leq \mathrm{d}(m_1, m_2) .$$

Now $|\eta_i u| \leq C$ for some bound $C \geq 0$ which finally leads to

$$|\eta_i u(m_1) - \eta_i u(m_2)| \min(\frac{\delta}{2C}, \frac{1}{2L_i}) \leq \mathrm{d}(m_1, m_2) ,$$

yielding the possibly quite large Lipschitz constant $(\min(\frac{\delta}{2C}, \frac{1}{2L_i}))^{-1}$ for $\eta_i u$. ∎

The Lipschitz constants constructed in the proof of Lemma 1.3.9 are by no means optimal. In many situations one obtains better results by using local convexity results, that facilitate the previous proof. As a drawback their standard form requires a little more regularity.

Lemma 1.3.10 (Local convexity)**.** *Consider a manifold without boundary $\mathcal{M}$ of class C^3. Then for each $m \in \mathcal{M}$ there exists $\delta > 0$ such that the geodesic ball $B_\delta(m) = \{\tilde{m} \in \mathcal{M} \mid \operatorname{dist}(\tilde{m}, m) < \delta\}$ is convex, i.e., that every two points in $B_\delta(m)$ can be joined by a shortest path that lies entirely in $B_\delta(m)$.*

Proof. A proof that generalizes to our case can be found in [Aub82, Thm. 1.36]. ∎

Corollary 1.3.11. *Consider a compact manifold without boundary $\mathcal{M}$ of class C^3. Then Assumption 1.3.3 holds with the additional property that the index set I is finite and that all $\mathcal{U}_i$, $i \in I$ are convex.*

Another possibility to improve the estimates on the Lipschitz constants consists in the application of [MMV98], the assumptions of which are a bit harder to check, in particular in the setting with boundary.

1.A The parabolic tube $\Gamma \times [0, T]$

In this appendix we investigate the practically relevant case where $(\mathcal{M}, g)$ is a hypersurface of $\mathbb{R}^{n+2}$, namely the product $\mathcal{M} = \Gamma \times [0, T]$ of a compact hypersurface of $\mathbb{R}^{n+1}$ without boundary and the compact interval $[0, T]$, $T > 0$, equipped with the $\mathbb{R}^{n+2}$ scalar product $g(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{\mathbb{R}^{n+2}}$.
Let us verify that Assumption 1.3.3 holds with finite index set I and thus prove the following corollary to Lemma 1.3.9.

Corollary 1.A.1. *On the parabolic tube* $(\Gamma \times [0, T], \langle \cdot, \cdot \rangle_{\mathbb{R}^{n+2}})$ *any continuous function that is piecewise smooth on a partition that satisfies the prerequisites of Lemma 1.3.9 lies in* $W^{1,\infty}(\Gamma \times [0, T])$ *and is Lipschitz continuous.*

The Corollary is due to the fact that starting from an atlas $(\mathcal{U}_i, \phi_i)_{i \in I}$ for Γ one obtains a canonical one for $\Gamma \times [0, T]$, namely $(\mathcal{U}_i \times [0, T], (\phi_i, \mathrm{id}_{\mathbb{R}}))_{i \in I}$. If $(\mathcal{U}_i, \phi_i)_{i \in I}$ fulfills the requirements of Assumption 1.3.3 so does $(\mathcal{U}_i \times [0, T], (\phi_i, \mathrm{id}_{\mathbb{R}}))_{i \in I}$.
Let $g^{\Gamma}(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{\mathbb{R}^{n+1}}$ denote the Riemannian tensor on Γ. Condition (1.3.2) holds for the metric tensor g

$$g_{ij} = \begin{cases} \delta_{ij} & \text{if } i = n+1 \text{ or } j = n+1 \\ g^{\Gamma}_{ij} & 1 \leq i, j < n+1 \end{cases} \tag{1.A.1}$$

because it does for g^{Γ}_{ij}:

$$2 \sum_{i=1}^{n+1} v_i^2 \geq \sum_{i,j=1}^{n+1} v_i g_{ij} v_j = v_{n+1}^2 + \sum_{i,j=1}^{n} v_i g^{\Gamma}_{ij} v_j \geq \frac{1}{2} \sum_{i=1}^{n+1} v_i^2 \,.$$

Using the structure (1.A.1) of the components of g one can show that the geodesics on $\Gamma \times [0, T]$ decompose into $(\alpha c(t), \beta t)$, $\alpha, \beta \in \mathbb{R}$, where $c(t)$ is a geodesic in Γ. Hence if the $\mathcal{U}_i$ are convex then the sets $\mathcal{U}_i \times [0, T]$ are convex, too. We have thus the following variant of Corollary 1.3.11 allowing for better Lipschitz constants in Lemma 1.3.9.

Corollary 1.A.2. *Consider a compact hypersurface without boundary* Γ *of class* C^3*. Then Assumption 1.3.3 holds for the tube* $\Gamma \times [0, T]$ *with the additional property that the index set* I *is finite and that all* $\mathcal{U}_i$, $i \in I$ *are convex.*

1.B Uniform stability of elliptic problems on a moving surface

In the situation of Section 1.2 we will require interior regularity estimates for the following elliptic state equation. For $y \in L^2(\Gamma(t))$ find $z \in H^1(\Gamma(t))$ such that

$$\int_{\Gamma(t)} g(\nabla_{\Gamma} z, \nabla_{\Gamma} \varphi) + \mu z \varphi \, \mathrm{d}\Gamma(t) = \langle y, \varphi \rangle_{L^2(\Gamma(t))} \,, \forall \varphi \in H^1(\Gamma(t)) \,. \tag{1.B.1}$$

where $\mu \in C^1(\Gamma(t))$ and $\mu \geq 1$. Here, the metric g is given by the $\mathbb{R}^{n+1}$ scalar product.
Problem (1.B.1) admits a unique solution and it is easy to see that $\|z\|_{H^1(\Gamma(t))} \leq \|y\|_{H^{-1}(\Gamma(t))}$.
But, as shows the next Lemma, the solution z even lies in $H^2(\Gamma(t))$.

Lemma 1.B.1. *Let* $\Gamma(t) = \Phi_t^0(\Gamma_0)$ *as in Section 1.2. For the solution* z *of* (1.B.1) *there holds*

$$\|z\|_{H^2(\Gamma(t))} \leq C\|y\|_{L^2(\Gamma(t))}\,,$$

and C *does not depend on* $t \in [0,T]$.

Observe that the lemma easily adapts to the case $\mu \equiv 0$. The estimate then reads

$$\|z\|_{H^2(\Gamma(t))} \leq C\|y\|_{L^2(\Gamma(t))} + \|z\|_{H^1(\Gamma(t))}\,,$$

and the solution z of (1.B.1) is not unique.

Proof. For Γ_0 there exists a finite atlas $(\mathcal{U}_i, \phi_i)_{i\in I}$, I finite, with subordinate partition of unity $\{\eta_i\}_{i\in I}$ as in Assumption 1.3.3. Hence $(\Phi_t^0(\mathcal{U}_i), \phi_i \circ \Phi_0^t)_{i\in I}$ is an atlas for $\Gamma(t)$ with subordinate partition of unity $(\eta_i \circ \Phi_0^t)_{i\in I}$. Now choose $\Omega_i \subset \phi_i(\mathcal{U}_i)$ compact, such that the support $\Omega_i^0 = \operatorname{supp}(\tilde{\eta}_i)$ of the pulled-back function $\tilde{\eta}_i = \eta_i \circ \phi_i^{-1}$ lies in the interior of Ω_i.
Due to the smoothness of the flow Φ the components $g_{ij}(x,t)$ of the metric of $\Gamma(t)$ are C^2-smooth. Because $[0,T]$, the unit sphere in $\mathbb{R}^n$, and the Ω_i are compact sets there exist bounds $\underline{c}, \overline{c} \in \mathbb{R}$ such that

$$0 < \underline{c} \leq \sqrt{\det(g(x,t))}\,,\ \sum_{i,j=1}^n v_i g_{ij}(x,t) v_j\,,\ \sum_{i,j=1}^n v_i g^{ij}(x,t) v_j \leq \overline{c} \tag{1.B.2}$$

for all $x \in \Omega_i$, $t \in [0,T]$ and $v \in \mathbb{R}^n$ with $\|v\|_{\mathbb{R}^n} = 1$. For every $H^1(\Omega)$-function φ with support in Ω_i the variational problem (1.B.1) can be rewritten as

$$\int\limits_{\phi_i(\mathcal{U}_i)} \left(\nabla_{\mathbb{R}^n}\tilde{z}(x) G(x,t) \nabla_{\mathbb{R}^n}\varphi(x) + \tilde{\mu}(x)\tilde{z}(x)\varphi(x)\right)\sqrt{\det(g(x,t))}\,\mathrm{d}x = \langle \tilde{y}, \varphi\rangle_{L^2(\phi_i(\mathcal{U}_i))}\,.$$

Here G denotes the matrix with entries g_{ij} and

$$\tilde{z} = z \circ (\phi_i \circ \Phi_0^t)^{-1}\,,\quad \tilde{\mu} = \mu \circ (\phi_i \circ \Phi_0^t)^{-1}\,,\quad \tilde{y} = y \circ (\phi_i \circ \Phi_0^t)^{-1}\,.$$

Because of (1.B.2) we are in the position to apply an interior regularity estimate like [GT98, Thm 8.8] and exploit $\mu \geq 1$ to get

$$\|\tilde{z}\|_{H^2(\Omega_i^0)} \leq c\|\tilde{y}\|_{L^2(\Omega_i)}\,, \tag{1.B.3}$$

with C independent of $t \in [0,T]$. Now we use (1.B.2) again to bound the right hand side of (1.B.3) through

$$\sqrt{\underline{c}}\,\|\tilde{y}\|_{L^2(\Omega_i)} \leq \|y\|_{L^2(\Phi_t^0(\mathcal{U}_i))}\,.$$

A short computation yields $\|\tilde{\eta}_i\tilde{z}\|_{H^2(\Omega_i^0)} \leq C\|\tilde{z}\|_{H^2(\Omega_i^0)}$. It remains to show that $\|\eta_i z\|_{H^2(\Gamma(t))} \leq C\|\tilde{\eta}_i\tilde{z}\|_{H^2(\Omega_i^0)}$. Using convolution, $\tilde{\eta}_i\tilde{z}$ can be approximated in the $H^2(\Omega_i^0)$-norm by $C^2(\Omega_i)$-functions $(\varphi_k)_{k\in\mathbb{N}}$, that vanish outside Ω_i. Now let $\varphi \in C^2(\phi_i(\mathcal{U}_i))$ whose support lies inside Ω_i and $\tilde{\varphi} = \varphi \circ \phi_i \circ \Phi_0^t \in C^2(\Gamma(t))$. One has

$$(\nabla^1\tilde{\varphi})_{i_1} = \frac{\partial}{\partial x_{i_1}}\tilde{\varphi}\,,\quad (\nabla^2\tilde{\varphi})_{i_1 i_2} = \frac{\partial^2}{\partial x_{i_1}\partial x_{i_2}}\tilde{\varphi} - \sum_{m=1}^n \Gamma_{i_1 i_2}^m \frac{\partial}{\partial x_{i_m}}\tilde{\varphi}\,. \tag{1.B.4}$$

Because of (1.B.2) we can estimate $\|\frac{\partial^l}{\partial x_{i_1}\dots\partial x_{i_l}}\tilde{\varphi}\|_{L^2(\Gamma(t))} \le \sqrt{\bar{c}}\,\|\frac{\partial^l}{\partial x_{i_1}\dots\partial x_{i_l}}\varphi\|_{L^2(\Omega_i)}$ for $l = 0, 1, 2$. The Christoffel symbols $\Gamma^m_{i_1 i_2} \in C^1(\Gamma(t))$ are uniformly bounded with respect to $t \in [0, T]$. Hence, due to (1.B.4), the components of the covariant derivative can be estimated through the partial derivatives.
We used the Cholesky decomposition of (g^{ij}) to prove (1.3.5). A similar argument, involving (1.B.2) instead of (1.3.3), yields

$$|\nabla^l \varphi|^2 \le \bar{c}^l \sum_{i_1\dots i_l=1}^{n} |\nabla^l \varphi_{i_1\dots i_l}|^2 \,,$$

and ends up with

$$\|\tilde{\varphi}\|_{H^2(\Gamma(t))} = \left(\sum_{i=0}^{2} \int_{\Gamma(t)} |\nabla^i \tilde{\varphi}|^2 \,\mathrm{d}\Gamma(t) \right)^{\frac{1}{2}} \le c \left(\|\varphi\|_{H^2(\Omega_i)} + n^3 C \|\varphi\|_{H^1(\Omega_i)} \right) .$$

Hence, φ_k being a Cauchy sequence in $H^2(\Omega_i)$, the pulled-back functions $\tilde{\varphi}_k$ form a Cauchy sequence in $H^2(\Gamma(t))$ whose limit is $\eta_i z$ (because its $L^2(\Gamma(t))$-limit is $\eta_i z$), and we have

$$\|\eta_i z\|_{H^2(\Gamma(t))} \le C \|\tilde{\eta}_i \tilde{z}\|_{H^2(\Omega_i^0)} \le C\sqrt{\bar{c}}\,\|y\|_{L^2(\Gamma(t))} \,.$$

Summing up over $i \in I$ we obtain the desired estimate. ∎

1.C Semismoothness on manifolds

Let us make sure that the Newton methods applied in later chapters do converge locally superlinearly. In the situation of Section 1.1, assume that we want to compute the solution $\bar{u}$ of the equation

$$G(u) = u - P_{[a,b]}\big(Q(u)\big) = 0 \in L^2(\Gamma) \,, \tag{1.C.1}$$

where $P_{[a,b]}$ is the point-wise projection onto the interval $[a, b]$ and $Q : L^2(\Gamma) \to H^1(\Gamma)$ a continuous linear operator.
For the fast local convergence of Newton's method in $L^2(\Gamma)$ we require that

$$\sup_{M \in \partial G(\bar{u}+\delta u)} \|G(\bar{u} + \delta u) - G(\bar{u}) - M\delta u\|_{L^2(\Gamma)} = o(\|\delta u\|_{L^2(\Gamma)}) \quad \text{as } \|\delta u\|_{L^2(\Gamma)} \to 0 \,,$$

for some set-valued mapping $\partial G : \Gamma \rightrightarrows \mathcal{L}(L^2(\Gamma), L^2(\Gamma))$. An operator $G : L^2(\Gamma) \to L^2(\Gamma)$ with this property is called semismooth at $\bar{u}$ with generalized derivative ∂G if in addition it is continuous at $\bar{u}$ and ∂G is non-empty in a neighborhood of $\bar{u}$, compare [Ulb11, Def 3.1].
If G is semismooth, a Newton algorithm for (1.C.1) converges locally superlinearly if there exists $C > 0$ such that in a neighborhood of the solution $\bar{u}$ each set ∂G contains at least one element M with $\|M^{-1}\|_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))} \le C$.
One way to prove semismoothness of (1.C.1) is now to prove a generalization of [Ulb11, Thm. 3.49] on compact surfaces. This, while being more time consuming, will in general lead to

better constants than the following approach. Here we satisfy ourselves with a short argument using local charts.
One has $\|v\|_{L^p(\Gamma)} \le C\|v\|_{H^1(\Gamma)}$ with $p > 2$ depending on n, see Lemma 1.3.2. In order to apply results known for Euclidean space, one locally prolongates the operator G via pull-backs onto $L^2(\phi_i(\mathcal{U}_i))$, $i \in I$, using a finite atlas as in Assumption 1.3.3. As a consequence of (1.3.2) one has equivalence of the norms

$$\underline{c}_p\|v\|_{L^p(\phi_i(\mathcal{U}_i))} \le \|v \circ \phi_i\|_{L^p(\mathcal{U}_i)} \le \overline{c}_p\|v\|_{L^p(\phi_i(\mathcal{U}_i))}\,, \quad p \ge 2\,,$$

and thus the pull-back $\phi_i^* : L^p(\phi_i(\mathcal{U}_i)) \to L^p(\mathcal{U}_i)$, $v \mapsto v \circ \phi_i$ is a linear homeomorphism with inverse

$$\phi_i^{*-1} = \phi_i^{-1*} : L^p(\mathcal{U}_i) \to L^p(\phi_i(\mathcal{U}_i))\,, \quad v \mapsto v \circ \phi_i^{-1}\,.$$

Consider further the linear continuous restriction operator

$$\mathfrak{r} : L^p(\Gamma) \to L^p(\mathcal{U}_i)\,, \quad v \mapsto v\big|_{\mathcal{U}_i}$$

and its L^p-L^q-adjoint $\mathfrak{p} = \mathfrak{r}^* : L^q(\mathcal{U}_i) \to L^q(\Gamma)$ which is the prolongation by zero. Note that we can consider $\mathfrak{p}$ a continuous operator from $L^p(\mathcal{U}_i)$ into $L^p(\Gamma)$.
Now the operator

$$\tilde{G}_i = \phi_i^{-1*} \circ \mathfrak{r} \circ G : L^2(\Gamma) \to L^2(\phi_i(\mathcal{U}_i))\,, \quad u \mapsto \phi_i^{-1*}\mathfrak{r}(u) - P_{[a,b]}\big(\phi_i^{-1*}\mathfrak{r}(Q(u))\big)$$

is semismooth everywhere, because the operator $\phi_i^{-1*} \circ \mathfrak{r} \circ Q$ maps $L^2(\Gamma)$ continuously into $L^p(\phi_i(\mathcal{U}_i))$ for some $p > 2$, compare [Ulb11, Thm. 3.49], and one has

$$\partial\tilde{G}_i(u) = \phi_i^{-1*} \circ \mathfrak{r} - \partial P_{[a,b]}\big(Q(u)\big) \circ \phi_i^{-1*} \circ \mathfrak{r} \circ Q\,.$$

The pull-back ϕ_i^*, the prolongation $\mathfrak{p}$ and the point-wise multiplication with the smooth function η_i from the partition of unity are linear continuous operators and do not affect semismoothness. Hence $\eta_i G = \eta_i \mathfrak{p}\phi_i^*\phi_i^{-1*}\mathfrak{r}G = \eta_i\left(\mathfrak{p} \circ \phi_i^* \circ \tilde{G}_i\right) : L^2(\Gamma) \to L^2(\Gamma)$ is semismooth.
Finally, the sum $G = \sum_{i\in I} \eta_i G$ of semismooth operators is again semismooth.
The generalized differential $\partial P_{[a,b]}\big(Q(u)\big)$ consists of point-wise multiplications with L^∞-functions, among those the indicator function of the inactive set, compare Section 2.2. Since it is a point-wise operation one has

$$\partial G(u) = \sum_{i\in I} \eta_i \mathfrak{p}\phi_i^*\partial\tilde{G}_i(u) = \mathrm{id}_{L^2(\Gamma)} - \partial P_{[a,b]}\big(Q(u)\big) \circ Q\,.$$

Chapter 2

Elliptic optimal control on stationary surfaces

The following chapter is devoted to the numerical treatment of the following linear-quadratic optimal control problem on a n-dimensional, sufficiently smooth compact hypersurface without boundary $\Gamma \subset \mathbb{R}^{n+1}$, $n = 1, 2, 3$.

$$\begin{aligned} \min_{u \in L^2(\Gamma),\, y \in H^1(\Gamma)} \mathbf{O}(u, y) &= \frac{1}{2}\|y - y_d\|^2_{L^2(\Gamma)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{subject to} \quad & u \in U_{ad} \text{ and} \\ & \int_\Gamma \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{c} y \varphi \,\mathrm{d}\Gamma = \int_\Gamma u\varphi \,\mathrm{d}\Gamma\,, \forall \varphi \in H^1(\Gamma) \end{aligned} \tag{$\mathbb{P}$}$$

with $U_{ad} = \big\{v \in L^2(\Gamma) \mid a \le v \le b\big\}$, $a < b \in \mathbb{R}$. For simplicity we will assume Γ to be compact without boundary and $\mathbf{c} \equiv 1$. In Section 2.4 we briefly investigate the case $\mathbf{c} \equiv 0$, in Section 2.5 we give an example on a surface with boundary.
It follows by standard arguments that $(\mathbb{P})$ admits a unique solution $u \in U_{ad}$ with unique associated state $y = y(u) \in H^2(\Gamma)$.
Our numerical approach uses variational discretization applied to $(\mathbb{P})$, see [Hin05] and [HPUU09], on a discrete surface Γ^h approximating Γ. A globalized semismooth Newton strategy is proposed, inspired by results from [Grä08]; [GK09], compare also [HV12b].
The discretization of the state equation in $(\mathbb{P})$ is achieved by the finite element method developed in [Dzi88], where a priori error estimates for finite element approximations of the Poisson problem for the Laplace-Beltrami operator are provided. Let us mention that uniform estimates are presented in [Dem09], and steps towards a posteriori error control for elliptic PDEs on surfaces are taken by Demlow and Dziuk in [DD07].
For alternative approaches for the discretization of the state equation by finite elements see, e.g., the work of Burger [Bur08] and the references therein.

We assume that Γ is of class C^3. The surface $\Gamma \subset \mathbb{R}^{n+1}$, as a compact hypersurface without boundary, is orientable with an exterior unit normal field ν and hence the zero level set of a signed distance function d such that $|d(x)| = \mathrm{dist}(x,\Gamma)$ and $\nu(x) = \frac{\nabla d(x)}{\|\nabla d(x)\|}$ for $x \in \Gamma$, compare Chapter 1. Further, there exists an neighborhood $\mathcal{N} \subset \mathbb{R}^{n+1}$ of Γ, such that d is also of class C^3 on $\mathcal{N}$ and the projection

$$\Pi_\Gamma : \mathcal{N} \to \Gamma\,, \quad \Pi_\Gamma(x) = x - d(x)\nabla d(x) \tag{2.0.1}$$

is unique, see Theorem 1.1.2. Note also that $\nabla d(x) = \nu(\Pi_\Gamma(x))$.
As in (1.0.1) we denote the metric tensor simply as $v_1 v_2$, for $v_1, v_2 \in T_\gamma\Gamma$, considering $T_\gamma\Gamma$ a subset of $\mathbb{R}^{n+1}$.
We use the Laplace-Beltrami operator $-\Delta_\Gamma = -\mathrm{div}_\Gamma \nabla_\Gamma$, compare (1.3.7), in its weak form i.e. $-\Delta_\Gamma : H^1(\Gamma) \to H^1(\Gamma)^*$

$$y \mapsto \int_\Gamma \nabla_\Gamma y \nabla_\Gamma(\,\cdot\,)\,\mathrm{d}\Gamma \in H^1(\Gamma)^*\,.$$

Let S denote the prolongated, restricted solution operator of the state equation

$$S : L^2(\Gamma) \to L^2(\Gamma)\,, \quad u \mapsto y \qquad -\Delta_\Gamma y + \mathbf{c}y = u\,,$$

which is compact and for $\mathbf{c} \equiv 1$ constitutes a linear homeomorphism onto $H^2(\Gamma)$, compare Appendix 1.B.
By standard arguments we get the following necessary (and here also sufficient) conditions for optimality of $u \in U_{ad}$

$$\langle \nabla_u \mathbf{O}(u, y(u)), v - u\rangle_{L^2(\Gamma)} = \langle \alpha u + S^*(Su - y_d), v - u\rangle_{L^2(\Gamma)} \geq 0\,, \quad \forall v \in U_{ad}\,. \tag{2.0.2}$$

We rewrite (2.0.2) as

$$u = \mathrm{P}_{U_{ad}}\left(-\frac{1}{\alpha}S^*(Su - y_d)\right)\,, \tag{2.0.3}$$

denoting by $\mathrm{P}_{U_{ad}}$ the L^2-orthogonal projection onto U_{ad}.

2.1 Discretization

We now discretize ($\mathbb{P}$) by use of polyhedral approximations Γ^h for Γ. Following Dziuk, we consider surfaces $\Gamma^h = \bigcup_{i \in I_h} T_h^i$ consisting of triangles T_h^i with corners on Γ whose maximum diameter is denoted by h. With finite element error bounds in mind we assume the family of triangulations Γ^h to be regular in the usual sense that the angles of all triangles are bounded away from zero uniformly in h.
In the following we assume $h > 0$ sufficiently small such that $\Gamma^h \subset \mathcal{N}$. In addition we assume that Π_Γ from (2.0.1) constitutes a homeomorphism between Γ^h and Γ. In order to compare functions defined on Γ^h with functions on Γ we use Π_Γ to lift a function $y \in L^2(\Gamma^h)$ to Γ

$$y^l(\Pi_\Gamma(x)) = y(x) \quad \forall x \in \Gamma^h\,,$$

and for $y \in L^2(\Gamma)$ we define the inverse lift

$$y_l(x) = y(\Pi_\Gamma(x)) \quad \forall x \in \Gamma^h\,.$$

The lift operation $(\cdot)_l : L^2(\Gamma) \to L^2(\Gamma^h)$ then defines a linear homeomorphism with inverse $(\cdot)^l$. Moreover, there exists $c_{\text{int}} > 0$ such that

$$1 - c_{\text{int}} h^2 \leq \|(\cdot)_l\|^2_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma^h))}, \|(\cdot)^l\|^2_{\mathcal{L}(L^2(\Gamma^h),L^2(\Gamma))} \leq 1 + c_{\text{int}} h^2 \,, \tag{2.1.1}$$

as shows the following lemma.

Lemma and Definition 2.1.1. *Denote by δ_h the Jacobian of $\Pi_\Gamma|_{\Gamma^h} : \Gamma^h \to \Gamma$ which is defined in the relative interior of each triangle. One has $\delta_h = \frac{\mathrm{d}\Gamma}{\mathrm{d}\Gamma^h} = |\det(M)|$ where $M \in \mathbb{R}^{n\times n}$ represents the Derivative $\mathrm{D}\Pi_\Gamma(\gamma) : T_\gamma \Gamma^h \to T_{\Pi_\Gamma(\gamma)}\Gamma$ with respect to arbitrary orthonormal bases of the respective tangential space. For small $h > 0$ there holds*

$$\sup_\Gamma |1 - \delta_h| \leq c_{\text{int}} h^2 \,.$$

Now $\delta_h^{-1} = |\det(M^{-1})|$, and by the change of variable formula we have

$$\left| \int_{\Gamma^h} v_l \,\mathrm{d}\Gamma^h - \int_\Gamma v \,\mathrm{d}\Gamma \right| = \left| \int_\Gamma v\delta_h^{-1} - v \,\mathrm{d}\Gamma \right| \leq c_{\text{int}} h^2 \|v\|_{L^1(\Gamma)} \,.$$

Proof. see [DE07, Lemma 5.1] ∎

Problem ($\mathbb{P}$) is approximated by the following sequence of optimal control problems

$$\begin{aligned} \min_{u\in L^2(\Gamma^h),\, y\in Y_h} \quad & \mathbf{O}(u, y) = \frac{1}{2}\|y - (y_d)_l\|^2_{L^2(\Gamma^h)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma^h)} \\ \text{subject to} \quad & u \in U^h_{ad} \text{ and} \\ & y = S_h u \,, \end{aligned} \tag{$\mathbb{P}_h$}$$

with the admissible set $U^h_{ad} = \left\{ v \in L^2(\Gamma^h) \mid a \leq v \leq b \right\}$ and the discretized state space

$$Y_h = \left\{ \varphi \in C^0\left(\Gamma^h\right) \;\middle|\; \forall i \in I_h : \varphi|_{T^i_h} \in \mathcal{P}^1(T^i_h) \right\}$$

of piecewise linear, globally continuous functions on Γ^h. Let us avoid the issue of defining $H^1(\Gamma^h)$ on the non-smooth surface Γ^h and instead endow Y_h with the norm

$$\|\varphi\|^2_{Y_h} = \int_{\Gamma^h} \nabla_{\Gamma^h}\varphi \nabla_{\Gamma^h}\varphi + \varphi^2 \,\mathrm{d}\Gamma^h \,.$$

Problem ($\mathbb{P}_h$) may be regarded as the extension of variational discretization introduced in [Hin05] to optimal control problems on surfaces.
In [Dzi88] it is explained, how to implement a discrete solution operator $S_h : L^2(\Gamma^h) \to L^2(\Gamma^h)$, mapping $L^2(\Gamma^h)$ onto Y_h, such that

$$\|(\cdot)^l S_h (\cdot)_l - S\|_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))} \leq C_{\text{FE}} h^2 \,, \tag{2.1.2}$$

which we will use throughout this chapter. See in particular [Dzi88, Equation (6)] and [Dzi88, 7. Lemma]. The method works as follows. Let $u \in L^2(\Gamma)$, and solve

$$\int_{\Gamma^h} \nabla_{\Gamma^h} y_h \nabla_{\Gamma^h} \varphi + \mathbf{c} y_h \varphi \, \mathrm{d}\Gamma^h = \int_{\Gamma^h} u_l \varphi \, \mathrm{d}\Gamma^h \,, \quad \forall \varphi \in Y_h$$

for $y_h \in Y_h$ in order to compute $y_h^l = (\cdot)^l S_h (\cdot)_l u$.
We choose $L^2(\Gamma^h)$ as control space, because in general we cannot evaluate $\int_\Gamma v \, \mathrm{d}\Gamma$ exactly, whereas the expression $\int_{\Gamma^h} v_l \, \mathrm{d}\Gamma^h$ for piecewise polynomials v_l can be computed up to machine accuracy. Also, the operator S_h is self-adjoint, while $((\cdot)^l S_h (\cdot)_l)^* = (\cdot)_l{}^* S_h (\cdot)^{l*}$ is not. The adjoint operators of $(\cdot)_l$ and $(\cdot)^l$ have the shapes

$$\forall v \in L^2(\Gamma^h): \; ((\cdot)_l)^* v = \delta_h^{-1} v^l \,, \quad \forall v \in L^2(\Gamma): \; ((\cdot)^l)^* v = \delta_h v_l \,, \tag{2.1.3}$$

hence evaluating $(\cdot)_l{}^*$ and $(\cdot)^{l*}$ requires knowledge of the Jacobians δ_h^{-1} and δ_h which may not be known analytically.
Similar to $(\mathbb{P})$, problem $(\mathbb{P}_h)$ possesses a unique solution $u_h \in U_{ad}^h$ which satisfies

$$u_h = \mathrm{P}_{U_{ad}^h} \left(-\frac{1}{\alpha} p_h(u_h) \right) . \tag{2.1.4}$$

Here $P_{U_{ad}^h} : L^2(\Gamma^h) \to U_{ad}^h$ is the $L^2(\Gamma^h)$-orthogonal projection onto U_{ad}^h and for $v \in L^2(\Gamma^h)$ the adjoint state is $p_h(v) = S_h^*(S_h v - (y_d)_l) \in Y_h$.

Observe that the projections $\mathrm{P}_{U_{ad}}$ and $\mathrm{P}_{U_{ad}^h}$ coincide with the pointwise projection $\mathrm{P}_{[a,b]}$ on Γ and Γ^h, respectively, and hence

$$\left(\mathrm{P}_{U_{ad}^h} (v_l) \right)^l = \mathrm{P}_{U_{ad}} (v) \tag{2.1.5}$$

for any $v \in L^2(\Gamma)$.
Let us now investigate the relation between the optimal control problems $(\mathbb{P})$ and $(\mathbb{P}_h)$.

Theorem 2.1.2 (Order of Convergence)**.** *Let $u \in L^2(\Gamma)$, $u_h \in L^2(\Gamma^h)$ be the solutions of* $(\mathbb{P})$ *and* $(\mathbb{P}_h)$*, respectively. Then for sufficiently small $h > 0$ there holds*

$$\begin{aligned} \alpha \left\| u_h^l - u \right\|_{L^2(\Gamma)}^2 + \left\| y_h^l - y \right\|_{L^2(\Gamma)}^2 \leq \frac{1 + c_{\text{int}} h^2}{1 - c_{\text{int}} h^2} \Bigg(\dots \\ \frac{1}{\alpha} \left\| \left((\cdot)^l S_h^* (\cdot)_l - S^* \right) (y - y_d) \right\|_{L^2(\Gamma)}^2 + \left\| \left((\cdot)^l S_h (\cdot)_l - S \right) u \right\|_{L^2(\Gamma)}^2 \Bigg) , \end{aligned} \tag{2.1.6}$$

with $y = Su$ and $y_h = S_h u_h$.

The following proof along the lines of [HPUU09, Thm. 3.4] can be found in [HV12a].

Proof. From (2.1.5) it follows that the projection of $-\left(\frac{1}{\alpha} p(u)\right)_l$ onto U_{ad}^h is u_l

$$u_l = \mathrm{P}_{U_{ad}^h} \left(-\frac{1}{\alpha} p(u)_l \right) .$$

We insert u_h into the corresponding variational inequality to obtain

$$\langle -\frac{1}{\alpha}p(u)_l - u_l, u_h - u_l\rangle_{L^2(\Gamma^h)} \leq 0\,.$$

On the other hand one plugs u_l into the variational inequality corresponding to (2.1.4) and gets

$$\langle -\frac{1}{\alpha}p_h(u_h) - u_h, u_l - u_h\rangle_{L^2(\Gamma^h)} \leq 0\,.$$

Adding these inequalities yields

$$\begin{aligned}\alpha\|u_l - u_h\|^2_{L^2(\Gamma^h)} \leq & \langle (p_h(u_h) - p(u)_l)\,, u_l - u_h\rangle_{L^2(\Gamma^h)} \\ = & \langle p_h(u_h) - S_h^*(y - y_d)_l, u_l - u_h\rangle_{L^2(\Gamma^h)} + \langle S_h^*(y-y_d)_l - p(u)_l, u_l - u_h\rangle_{L^2(\Gamma^h)}\,.\end{aligned}$$

The first addend is estimated via

$$\begin{aligned}\langle p_h(u_h) - S_h^*(y-y_d)_l, u_l - u_h\rangle_{L^2(\Gamma^h)} &= \langle y_h - y_l, S_h u_l - y_h\rangle_{L^2(\Gamma^h)} \\ &= -\|y_h - y_l\|^2_{L^2(\Gamma^h)} + \langle y_h - y_l, S_h u_l - y_l\rangle_{L^2(\Gamma^h)} \\ &\leq -\frac{1}{2}\|y_h - y_l\|^2_{L^2(\Gamma^h)} + \frac{1}{2}\|S_h u_l - y_l\|^2_{L^2(\Gamma^h)}\,.\end{aligned}$$

The second addend satisfies

$$\langle S_h^*(y-y_d)_l - p(u)_l, u_l - u_h\rangle_{L^2(\Gamma^h)} \leq \frac{\alpha}{2}\|u_l - u_h\|^2_{L^2(\Gamma^h)} + \frac{1}{2\alpha}\|S_h^*(y-y_d)_l - p(u)_l\|^2_{L^2(\Gamma^h)}\,.$$

Together this yields

$$\alpha\|u_l - u_h\|^2_{L^2(\Gamma^h)} + \|y_h - y_l\|^2_{L^2(\Gamma^h)} \leq \frac{1}{\alpha}\|S_h^*(y-y_d)_l - p(u)_l\|^2_{L^2(\Gamma^h)} + \|S_h u_l - y_l\|^2_{L^2(\Gamma^h)}$$

The claim follows using (2.1.1) for sufficiently small $h > 0$. ■

Because both S and S_h are self-adjoint, quadratic convergence follows directly from (2.1.6). For operators that are not self-adjoint one can use

$$\|(\cdot)_l{}^* S_h^* (\cdot)^{l*} - S^*\|_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))} \leq C_{\mathrm{FE}} h^2\,. \tag{2.1.7}$$

which is a consequence of (2.1.2). Equation (2.1.3) and Lemma 2.1.1 imply

$$\|((\cdot)_l)^* - (\cdot)^l\|_{\mathcal{L}(L^2(\Gamma^h),L^2(\Gamma))} \leq c_{\mathrm{int}} h^2\,, \quad \|((\cdot)^l)^* - (\cdot)_l\|_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma^h))} \leq c_{\mathrm{int}} h^2\,. \tag{2.1.8}$$

Combine (2.1.6) with (2.1.7) and (2.1.8) to prove quadratic convergence for arbitrary linear elliptic state equations.

2.2 Implementation

In order to solve (2.1.4) numerically, we proceed as in [Hin05] using the finite element techniques for PDEs on surfaces developed in [Dzi88] combined with the semismooth Newton techniques from [HIK03] and [Ulb03] applied to the equation

$$G_h(u_h) = u_h - \mathrm{P}_{[a,b]}\left(-\frac{1}{\alpha}p_h(u_h)\right) = 0\,. \tag{2.2.1}$$

Since the operator p_h continuously maps $v \in L^2(\Gamma^h)$ into Y_h, Equation (2.2.1) is semismooth and thus is amenable to the Newton method that converges locally superlinearly.

Remark 2.2.1. Although the considerations of Appendix 1.C do not apply directly they show that the operator $(\cdot)^l \circ G_h \circ (\cdot)_l$ is semismooth; use (2.1.1) for the L^2-stability of $(\cdot)_l$ and Lemma 3.6.5(*1.*) for the H^1-stability of $(\cdot)^l$ on Y_h. The semismoothness of G_h is then a consequence of (2.1.1).

A representative of the generalized derivative ∂G_h of G_h at $v \in L^2(\Gamma^h)$ is given by

$$DG_h(v) = \left(I + \frac{\mathbb{1}_{\mathcal{I}(p_h^v)}}{\alpha} S_h^* S_h\right).$$

Here and in the following let $\mathbb{1}_{\mathcal{I}(p_h^v)} : \Gamma^h \to \{0,1\}$ denote the indicator function of the inactive set

$$\mathcal{I}\left(-\frac{1}{\alpha}p_h\right) = \left\{\gamma \in \Gamma^h \,\middle|\, a < -\frac{1}{\alpha}p_h(v)[\gamma] < b\right\},$$

which – in an abuse of notation – we will refer to as $\mathcal{I}(p_h^v)$ most of the time. Thus

$$\mathbb{1}_{\mathcal{I}(p_h^v)} = \begin{cases} 1 \text{ on } \mathcal{I}(-\frac{1}{\alpha}p_h(v)) \subset \Gamma^h \\ 0 \text{ elsewhere on } \Gamma^h \end{cases},$$

although the endomorphism of $L^2(\Gamma^h)$ defined through the pointwise multiplication with the function $\mathbb{1}_{\mathcal{I}(p_h^v)}$ will also be denoted as $\mathbb{1}_{\mathcal{I}(p_h^v)} : L^2(\Gamma^h) \to L^2(\Gamma^h)$.
A semismooth Newton step for (2.2.1) at $v \in L^2(\Gamma^h)$ reads

$$DG_h(v)\delta v = \left(I + \frac{\mathbb{1}_{\mathcal{I}(p_h^v)}}{\alpha} S_h^* S_h\right)\delta v = -v + \mathrm{P}_{[a,b]}\left(-\frac{1}{\alpha}p_h(v)\right) = -G_h(v)\,. \tag{2.2.2}$$

Reformulating the equation by means of the next iterate $v^+ = v + \delta v$

$$\left(I + \frac{\mathbb{1}_{\mathcal{I}(p_h^v)}}{\alpha} S_h^* S_h\right) v^+ = \mathrm{P}_{[a,b]}\left(-\frac{1}{\alpha}p_h(v)\right) + \frac{\mathbb{1}_{\mathcal{I}(p_h^v)}}{\alpha} S_h^* S_h v\,, \tag{2.2.3}$$

we see that v^+ lies in the following finite dimensional subspace of $L^2(\Gamma^h)$

$$Y_h^+ = \left\{\mathbb{1}_{\mathcal{I}(p_h^v)}\varphi_1 + \mathbb{1}_{\hat{\mathcal{A}}(p_h^v)}\varphi_2 + \mathbb{1}_{\check{\mathcal{A}}(p_h^v)}\varphi_3 \mid \varphi_1, \varphi_2, \varphi_3 \in Y_h\right\},$$

with the indicator functions of the active sets

$$\check{\mathcal{A}}\,(p_h^v) = \left\{\gamma \in \Gamma^h \mid \left(-\frac{1}{\alpha}p_h(y_h(v))\right)[\gamma] \le a)\right\} \text{ and } \hat{\mathcal{A}}\,(p_h^v) = \left\{\gamma \in \Gamma^h \mid \left(-\frac{1}{\alpha}p_h(y_h(v))\right)[\gamma] \ge b)\right\}.$$

One can represent the $\mathbb{1}$-functions in a computer program by resolving the borders of the inactive set $\mathcal{I}(p_h^v)$, which are level sets of piecewise linear functions. Hence, although the algorithm operates in $L^2(\Gamma^h)$ the iterates v^+ can be represented with about constant effort.
The Newton algorithm applied to Equation (2.2.1) now reads

1. Choose a starting point $v \in L^2(\Gamma^h)$ that lies sufficiently close to the solution u_h of (2.2.1).

2. Do until convergence: Solve (2.2.3) for v^+; set $v := v^+$.

Here Equation (2.2.3) is solved for the next iterate v^+ by performing three steps

1. Set $\mathbb{1}_{\check{\mathcal{A}}(p_h^v)} v^+ = a$ and $\mathbb{1}_{\hat{\mathcal{A}}(p_h^v)} v^+ = b$.

2. Solve

$$\left(I + \frac{\mathbb{1}_{\mathcal{I}(p_h^v)}}{\alpha} S_h^* S_h\right) \mathbb{1}_{\mathcal{I}(p_h^v)} v^+ = \frac{\mathbb{1}_{\mathcal{I}(p_h^v)}}{\alpha} \left(S_h^*(y_d)_l - S_h^* S_h \left(\mathbb{1}_{\check{\mathcal{A}}(p_h^v)} + \mathbb{1}_{\hat{\mathcal{A}}(p_h^v)}\right) v^+\right)$$

   for $\mathbb{1}_{\mathcal{I}(p_h^v)} v^+$ by CG iteration over $L^2(\mathcal{I}(p_h^v))$.

3. Set $v^+ = \mathbb{1}_{\mathcal{I}(p_h^v)} v^+ + \mathbb{1}_{\check{\mathcal{A}}(p_h^v)} v^+ + \mathbb{1}_{\hat{\mathcal{A}}(p_h^v)} v^+$.

Observe that step 2. is possible since $\left(I + \frac{1}{\alpha} \mathbb{1}_{\mathcal{I}(p_h^v)} S_h^* S_h\right) \mathbb{1}_{\mathcal{I}(p_h^v)}$ constitutes a positive definite endomorphism of $L^2(\mathcal{I}(p_h^v))$. Note also that all parts of the preceding algorithm can be implemented; in order to do so one has to keep track of the active and inactive sets. Details can be found in [HV11] and [HV12b] .

Remark 2.2.2. Considering how we solved (2.2.3), i.e., step 1. and 2., we conclude

$$\|DG_h^{-1}(v)\|_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))} \leq 1 + \frac{\|S_h\|^2_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))}}{\alpha}, \quad \forall v \in L^2(\Gamma)\,.$$

2.3 Globalization

In order to formulate a globally convergent algorithm, we want to apply inexact Armijo line-search along the lines of [HV12b]. For this purpose we construct in the present section a sufficiently smooth merit function which is inspired by [Grä08]; [GK09]. There, a merit function was proposed in a finite dimensional setting, relying on S^{-1} rather than S, which turned out to be an obstacle when trying to carry over the results to an infinite dimensional setting, compare [Grä08, Rem. 3].
In addition we want the algorithm to be mesh-independent in so far as the number of Newton steps does not increase if $h \to 0$. Mesh-independent behavior of the algorithm can only be expected if already the operator G_h is mesh-independently semismooth. Here, we make a slightly stronger but nevertheless reasonable assumption in order to ensure fast local convergence of the algorithm. The $W^{1,\infty}$-convergence estimates from [Dem09, Thm. 3.2] can be used to prove mesh independent differentiability as in Assumption 2.3.1 under additional reasonable assumptions on the adjoint state $p(u)$. Those are

- $p(u) \in C^2(\Gamma)$ and

- $\nabla_\Gamma p(u) \neq 0$ along the set $\{\gamma \in \Gamma \mid p(u)[\gamma] = a \vee p(u)[\gamma] = b\}$.

For a proof on open sets $\Omega \in \mathbb{R}^2$ see [HV12b].

Assumption 2.3.1. There exists $h_0, \eta > 0$, such that for $0 \le h \le h_0$ the operator $G_h : L^2(\Gamma^h) \to L^2(\Gamma^h)$ is uniformly strictly Fréchet differentiable with $\frac{1}{2}$-Hölder continuous derivative on the $L^2(\Gamma^h)$-ball $B_\eta(u_h)$, i.e., there exists $C > 0$, independent of h, such that for $u_1, u_2 \in B_\eta(u_h)$

$$\|G_h(u_1) - G_h(u_2) - DG_h(u_2)(u_1 - u_2)\|_{L^2(\Gamma^h)} \le C\|u_1 - u_2\|^{\frac{3}{2}}_{L^2(\Gamma^h)}, \tag{2.3.1}$$

and

$$\|DG_h(u_1) - DG_h(u_2)\|_{\mathcal{L}(L^2(\Gamma^h), L^2(\Gamma^h))} \le C\|S_h(u_1 - u_2)\|^{\frac{1}{2}}_{L^2(\Gamma^h)} \tag{2.3.2}$$

To begin with, following [Grä08]; [GK09], we introduce a multiplier $w \in L^2(\Gamma^h)$ associated with the equality constraints and formulate a damping strategy by means of the Lagrange dual function $\ell_h : L^2(\Gamma^h) \to \mathbb{R}$, which will serve as the merit function

$$\ell_h(w) = -\inf_{u,y \in L^2(\Gamma^h)} \Big(\underbrace{\frac{1}{2}\|y - y_d^h\|^2_{L^2(\Gamma^h)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma^h)} + \chi_{U^h_{ad}}(u) - \langle w, y - S_h u\rangle_{L^2(\Gamma^h)}}_{\mathcal{L}(u,y,w)} \Big),$$

where y_d^h is the $L^2(\Gamma^h)$-orthogonal projection of y_d onto Y_h. By

$$\chi_{U^h_{ad}} = \begin{cases} 0 & \text{on } U^h_{ad} \\ \infty & \text{on } L^2(\Gamma^h) \setminus U^h_{ad} \end{cases}$$

we denote the characteristic function of the set U^h_{ad} in the sense of convex analysis.
It turns out that ℓ_h is differentiable with Lipschitz continuous derivative and strongly convex. Note that all results from this section hold for $h \ge 0$ with

$$\Gamma^0 = \Gamma, \quad S_0 = S : L^2(\Gamma) \to L^2(\Gamma), \quad U^0_{ad} = U_{ad},$$

except that, in general, Algorithm 2.3.3 cannot be implemented for $h = 0$.

Lemma 2.3.2 (Lagrange dual function)**.** *The function $\ell_h : L^2(\Gamma^h) \to \mathbb{R}$ is strongly convex and Fréchet differentiable with Lipschitz continuous $L^2(\Gamma^h)$-gradient*

$$\nabla \ell_h(w) = y(w) - S_h u(w),$$

where $y(w) = w + y_d^h$ and $u(w) = P_{[a,b]}\left(-\frac{1}{\alpha} S_h^* w\right)$ are the unique minimizers of the Lagrange function $\mathcal{L}(u, y, w)$ for any given $w \in L^2(\Gamma^h)$.

Proof. The proof is inspired by the one given in [GK09, Thm. 2.1], compare also [HV12b, Lem. 4.1] . First we note that the minimization problem in the definition of $\ell_h(w)$ can be rewritten as

$$\begin{aligned}\ell_h(w) = &-\min_{y \in L^2(\Gamma^h)} \left(\frac{1}{2}\|y - y_d^h\|^2_{L^2(\Gamma^h)} - \langle w, y\rangle_{L^2(\Gamma^h)}\right)\ldots \\ &- \min_{u \in L^2(\Gamma^h)} \left(\frac{\alpha}{2}\|u\|^2_{L^2(\Gamma^h)} + \chi_{U^h_{ad}}(u) + \langle w, S_h u\rangle_{L^2(\Gamma^h)}\right),\end{aligned}$$

with unique minimizers $u(w)$ and $y(w)$, respectively. For $y(w)$ this is obvious. The necessary condition for the problem in u

$$\langle \alpha u(w) + S_h^* w, v - u(w)\rangle_{L^2(\Gamma^h)} \geq 0\,, \quad \forall v \in U_{ad}^h$$

looks very much like (2.0.2) and likewise guarantees a unique solution $u(w) = P_{[a,b]}\left(-\frac{1}{\alpha} S_h^* w\right)$. Note that from the viewpoint of convex analysis this condition reads

$$-\alpha u(w) - S_h^* w \in \partial \chi_{U_{ad}^h}(u(w))\,, \tag{2.3.3}$$

because the subdifferential $\partial\chi_{U_{ad}^h}$ is the Fréchet normal cone of the convex set U_{ad}^h.
As to the smoothness of ℓ_h observe

$$\ell_h(w) = -\frac{1}{2}\|w\|^2_{L^2(\Gamma^h)} + \langle w, w + y_d^h\rangle_{L^2(\Gamma^h)} + \psi^*(-S_h^* w)\,,$$

$\psi^* : L^2(\Gamma^h) \to L^2(\Gamma^h)$ being the polar function of the convex functional $\psi : L^2(\Gamma^h) \to L^2(\Gamma^h)$, $\psi(u) = \frac{\alpha}{2}\|u\|^2_{L^2(\Gamma^h)} + \chi_{U_{ad}^h}(u)$ defined as

$$\psi^*(v) = \sup_{u \in L^2(\Gamma^h)} \Big(\langle v, u\rangle_{L^2(\Gamma^h)} - \psi(u)\Big)\,.$$

Since ψ is convex and lower semicontinuous we have the property $v \in \partial\psi(u) \Leftrightarrow u \in \partial\psi^*(v)$, see [ET99, Cor. 5.2]. Just like (2.3.3) the equation

$$v \in \alpha u(w) + \partial\chi_{U_{ad}^h}(u(w))$$

admits a unique solution $u = P_{[a,b]}(\frac{1}{\alpha}v)$ for any $v \in L^2(\Gamma^h)$ and we conclude that $\partial\psi^*(v)$ is single valued, thus ψ^* is Gâteaux differentiable, compare [ET99, Prop. 5.3]. Since $\partial\psi^*(v) = P_{[a,b]}\left(\frac{1}{\alpha}v\right)$ the function ψ^* is even Lipschitz continuously Fréchet differentiable. Hence

$$D\ell_h(w)v = \langle w + y_d^h, v\rangle_{L^2(\Gamma^h)} + \left\langle P_{[a,b]}\left(-\frac{1}{\alpha}S_h^* w\right), (-S_h^*)v\right\rangle_{L^2(\Gamma^h)}\,,$$

and the formula for the gradient follows.
The strong convexity now is a consequence of the monotonicity of the derivative

$$\begin{aligned}\langle \nabla\ell_h(w_1) - \nabla\ell_h(w_2), w_1 - w_2\rangle_{L^2(\Gamma^h)} &= \|w_1 - w_2\|^2_{L^2(\Gamma^h)} \cdots \\ &+ \left\langle P_{[a,b]}\left(-\frac{1}{\alpha}S_h^* w_1\right) - P_{[a,b]}\left(-\frac{1}{\alpha}S_h^* w_2\right), -S_h^*(w_1 - w_2)\right\rangle_{L^2(\Gamma^h)} \\ &\geq \|w_1 - w_2\|^2_{L^2(\Gamma^h)}\end{aligned} \tag{2.3.4}$$

which again follows by a short computation from the definition of the orthogonal projection $P_{[a,b]} = P_{U_{ad}^h}$. ∎

The gradient $\nabla\ell_h$ is semismooth, compare Remark 2.2.1, and we are now in the position to apply a semismooth Newton strategy to the dual problem

$$\min_{w \in L^2(\Gamma^h)} \ell_h(w)\,. \tag{$\mathbb{P}_h'$}$$

Due to strong convexity problem ($\mathbb{P}'_h$) admits a unique solution w^* satisfying $\nabla \ell_h(w^*) = 0$. Hence $u(w^*)$ solves (2.1.4) and thus also ($\mathbb{P}_h$), compare Lemma 2.3.2; there is no duality gap. A semismooth Newton step for ($\mathbb{P}'_h$) reads

$$\Big(I + \frac{1}{\alpha} S_h \mathbb{1}_{S_h^* w} S_h^*\Big)\delta w = -(w + y_d^h) + S_h P_{[a,b]}\left(-\frac{1}{\alpha} S_h^* w\right) = -\nabla \ell_h(w)\,. \tag{2.3.5}$$

Equation (2.3.5) can be obtained by applying S_h to both sides of Equation (2.2.2), if we think of w as a function of the iterate v, i.e.

$$w = w(v) = S_h v - y_d^h\,, \tag{v-w}$$

and take into account that $S_h^* y_d = S_h^* y_d^h$. Thus both Newton iterations are equivalent because the next iterate v^+ in (2.2.3) only depends on the adjoint state $p_h(y_h(v)) = S_h^* w(v)$.
Step (2.3.5) is implementable since beginning with the first iteration there holds $w^+ \in Y_h$. Observe also that the generalized Hessian on the left hand side of (2.3.5) is a self-adjoint and positive definite endomorphism of $L^2(\Gamma^h)$. Finally, ℓ_h is easy to evaluate since by Lemma 2.3.2 we have

$$\ell_h(w) = \frac{1}{2}\|w\|^2_{L^2(\Gamma^h)} - \frac{\alpha}{2}\|u(w)\|^2_{L^2(\Gamma^h)} + \langle w, y_d^h - S_h u(w)\rangle_{L^2(\Gamma^h)}\,.$$

Thus ℓ_h is a suitable as merit function for a globalized Newton algorithm.

Algorithm 2.3.3 (Damped Newton-Algorithm)**.** $w := w_0 \in L^2(\Gamma^h)$, $\beta \in (0,1)$ given.
Do until convergence

i) Solve Equation (2.3.5) for δw. Set $\lambda := 1$.

ii) If $\ell_h(w + \lambda\delta w) > \ell_h(w) + \frac{1}{3}\lambda \underbrace{\langle \nabla \ell_h(w), \delta w\rangle_{L^2(\Gamma^h)}}_{\leq 0 \text{ by } (2.3.5)}$, set $\lambda := \beta\,\lambda$ and return to ii).

iii) Set $w := w + \lambda\delta w$. Return to i).

As mentioned above, in the sense of (v-w) we can interpret Algorithm 2.3.3 as a Newton algorithm with respect to v, but working exclusively on $w(v)$.
Since ℓ_h is sufficiently smooth the number of damping steps in the line search algorithm is bounded, and Algorithm 2.3.3 converges for any initial value w_0.

Lemma 2.3.4 (Global convergence)**.** *Let $L = 1 + \frac{\|S_h\|^2}{\alpha}$ denote the Lipschitz constant of $\nabla \ell_h$ and $\beta \in (0,1)$ as in Algorithm 2.3.3. Let u_h and w^* denote the solutions of ($\mathbb{P}_h$) and ($\mathbb{P}'_h$), respectively. Then we have*

i) A step with damping parameter $\lambda \leq \beta^{K(L,\beta)}$ is always accepted, where

$$K(L,\beta) = \frac{\log(2) - \log(3L)}{\log \beta}\,.$$

ii) Hence Algorithm 2.3.3 converges for arbitrary initial data $w_0 \in L^2(\Gamma^h)$, in the sense that $w \to w^*$, $u(w) \to u_h$ and $v \to u_h$ in $L^2(\Gamma^h)$.

iii) The stopping criterion $\|\nabla \ell_h(w)\|_{L^2(\Gamma^h)} \le \mathrm{Tol}$ *is reasonable since*

$$\|\nabla \ell_h(w)\|_{L^2(\Gamma^h)} \ge \|w - w^*\|_{L^2(\Gamma^h)} .$$

Proof. i) By the mean value theorem one has $\Theta \in (0,1)$ such that

$$\begin{aligned} \ell_h(w + \lambda\delta w) &= \ell_h(w) + \lambda\langle \nabla \ell_h(w + \Theta\lambda\, \delta w), \delta w\rangle_{L^2(\Gamma^h)} \\ &\le \ell_h(w) + \lambda \left(\langle \nabla \ell_h(w), \delta w\rangle_{L^2(\Gamma^h)} + \lambda L \|\delta w\|^2_{L^2(\Gamma^h)} \right) . \end{aligned}$$

Using (2.3.5), standard arguments deliver that sufficient descent is achieved for $\beta^k L \le \frac{2}{3}$.
ii)+iii) Denoting by λ_k the damping parameter generated in step ii) of Algorithm 2.3.3 with associated iterates w_k the descend condition reads

$$-\frac{1}{3}\lambda_k \langle \nabla \ell_h(w_k), \delta w_k\rangle_{L^2(\Gamma^h)} \le \ell_h(w_k) - \ell_h(w_{k+1})$$

Because ℓ_h is bounded from below by $\ell_h(w^*)$, summation over k yields

$$\begin{aligned} \ell_h(w_0) - \ell_h(w^*) \ge \ell_h(w_0) - \liminf_{k\to\infty} \ell_h(w_k) &\ge -\sum_{k=1}^{\infty} \frac{\lambda_k}{3} \langle \nabla \ell_h(w_k), \delta w_k\rangle_{L^2(\Gamma^h)} \\ &\ge \sum_{k=1}^{\infty} \frac{2\beta}{9L} \|\delta w_k\|^2_{L^2(\Gamma^h)} , \end{aligned} \tag{2.3.6}$$

where we again use (2.3.5) in the last estimate. Hence $\delta w_k \to 0$ and thus $\nabla \ell_h(w_k) \to 0$. Inserting $\nabla \ell_h(w^*) = 0$ into the strong convexity relation (2.3.4) we arrive at the estimate $\|\nabla \ell_h(w)\|_{L^2(\Gamma^h)} \ge \|w - w^*\|_{L^2(\Gamma^h)}$ and thus finally at $w \to w^*$. $u(w) \to u_h$ follows by continuity and the fact that $u_h = u(w^*)$.
Finally, because $p_h(v) = S_h^* w \to p_h(u_h)$ and $S_h \delta v = \delta w \to 0$, we conclude from (2.2.2) that $v^+ \to u_h$. Since in addition $1 \ge \lambda > \frac{2\beta}{3L}$, the damped sequence v also converges towards u_h.
■

Remark 2.3.5. One can continue the estimate (2.3.6) by

$$\ell_h(w_0) - \ell_h(w^*) \ge \sum_{k=1}^{\infty} \frac{2\beta}{9L^3} \|\nabla \ell_h(w_k)\|^2_{L^2(\Gamma^h)} \ge \sum_{k=1}^{\infty} \frac{2\beta}{9L^3} \|w_k - w^*\|^2_{L^2(\Gamma^h)} ,$$

the left-hand side of which is bounded independently of h. As an immediate consequence we can bound the number of steps ending outside of any ball $B_\epsilon(w^*)$ by $\frac{5L^3}{\beta\epsilon^2}(\ell(w_0) - \ell(w^*))$.

Until now we made no use of Assumption 2.3.1. Its purpose is to ensure that the damping steps generated by Algorithm 2.3.3 do not affect the fast local convergence of the Newton iteration.

Lemma 2.3.6 (Transition to fast local convergence)**.** *Let* $(\mathbb{P})$ *satisfy Assumption 2.3.1 and denote the solution of* $(\mathbb{P}_h)$ *by* u_h*. Given* $\beta \in (0,1)$ *there exists* $\tilde\eta > 0$*, independent of* $h > 0$*, such that the Algorithm 2.3.3, upon reaching the ball* $B_{\tilde\eta}(u_h) \subset L^2(\Gamma^h)$*, proceeds with full Newton steps, i.e. with* $\lambda = 1$*.*

Proof. As a consequence of (2.3.1) and the boundedness of DG_h^{-1}, see Remark 2.2.2, there holds

$$\|v^+ - u_h\|_{L^2(\Gamma^h)} \leq C_{\text{Newt}} \|v - u_h\|_{L^2(\Gamma^h)}^{\frac{3}{2}}$$

for $v \in B_\eta(u_h)$.
Upon reaching a ball $B_{\tilde{\eta}}(u_h)$, $\tilde{\eta} \leq \min(\eta, \frac{1}{C_{\text{Newt}}^2})$, there holds $\|v^+ - u_h\|_{L^2(\Gamma^h)} \leq \|v - u_h\|_{L^2(\Gamma^h)}$ and thus

$$\|\delta v\|_{L^2(\Gamma^h)} \leq 2\|v - u_h\|_{L^2(\Gamma^h)}\,, \tag{2.3.7}$$

and the undamped Newton iteration hence converges $\frac{1}{2}$-superlinearly.
Lemma 2.3.4 ensures that this happens after finitely many steps, compare Remark 2.3.5.
The function $v \mapsto \ell_h(w(v))$ is $C^{2,\frac{1}{2}}$-smooth in the ball $B_\eta(u_h)$ from Assumption 2.3.1. Hence, using Taylor expansion and (2.3.5) together with the improved Hölder continuity from (2.3.2) we get

$$\begin{aligned}\ell_h(S_h(v + \delta v)) &= \ell_h(S_h v) + \langle \nabla \ell_h(S_h v), S_h \delta v\rangle_{L^2(\Gamma^h)} \dots \\ &\quad + \frac{1}{2}\langle S_h \delta v, \left(I + \frac{1}{\alpha} S_h \mathbb{1}_{p_h(v+\Theta\delta v)} S_h^*\right) S_h \delta v\rangle_{L^2(\Gamma^h)} \\ &\leq \ell_h(S_h v) - \frac{1}{2}\langle S_h \delta v, \left(I + \frac{1}{\alpha} S_h \mathbb{1}_{\mathcal{I}(p_h^v)} S_h^*\right) S_h \delta v\rangle_{L^2(\Gamma^h)} + C\|S_h \delta v\|_{L^2(\Gamma^h)}^{\frac{5}{2}}\,.\end{aligned}$$

decrease $\tilde{\eta} > 0$ if necessary such that by (2.3.7) δv becomes sufficiently small to conclude

$$\ell_h(S_h(v + \delta v)) \leq \ell_h(S_h(v)) - \frac{1}{3}\langle S_h \delta v, \left(I + \frac{1}{\alpha} S_h \mathbb{1}_{\mathcal{I}(p_h^v)} S_h^*\right) S_h \delta v\rangle_{L^2(\Gamma^h)}\,.$$

■

If, after sufficiently many iterations, the choice $\lambda = 1$ is always accepted, locally super-linear convergence occurs with respect to both v and $w = w(v)$.
Note that one could compute the next full-step iterate $v^+ = v + \delta v$ following a full step from w by solving (2.2.3) at any time during the iteration, whereas $v + \lambda\delta v$, $\lambda \neq 1$ is not so readily accessible.
Another possible globalization for our semismooth Newton method was described in [Ulb11, Alg. 7.27]. The trust-region algorithm proposed there uses a merit function based either on a Fisher-Burmeister function or on the objective $\mathbf{O}$ itself. For a number of other trust-region approaches that are not directly linked to our method we refer to the references in [Ulb11].

It is not difficult to see, that the fixed-point equation for problem $(\mathbb{P}_h)$

$$u_h = P_{[a,b]}\left(-\frac{1}{\alpha} S_h^*(S_h u_h - y_d)\right)$$

can be solved by fixed-point iteration that converges globally for $\alpha > \|S_h\|_{\mathcal{L}(L^2(\Gamma^h), L^2(\Gamma^h))}^2$, see [Hin03]; [Hin05]. A similar global convergence result holds for the undamped Newton algorithm.

Lemma 2.3.7. *Consider Algorithm 2.3.3 without damping, i.e., $\lambda := 1$ in every step. The undamped algorithm converges globally if $h > 0$ is sufficiently small and*

$$\alpha > \frac{4}{3}\|S\|^2_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))}\,.$$

Proof. See [Vie07]. ■

Similar results have been published in [BIK99].

2.4 The case $\mathbf{c} \equiv 0$

In this section we investigate the case $\mathbf{c} \equiv 0$ which corresponds to a stationary, purely diffusion-driven process. Since Γ has no boundary, in this case total mass must be conserved, i.e., the state equation admits a solution only for controls with mean value zero. For such a control the state is uniquely determined up to a constant. Thus the admissible set U_{ad} has to be changed to

$$U_{ad} = \big\{v \in L^2(\Gamma) \mid a \le v \le b\big\} \cap L^2_0(\Gamma)\,, \text{ where } L^2_0(\Gamma) := \left\{ v \in L^2(\Gamma) \;\middle|\; \int\limits_\Gamma v \,\mathrm{d}\Gamma = 0 \right\}\,,$$

and $a < 0 < b$. Problem $(\mathbb{P})$ then admits a unique solution (u, y) and there holds $\int\limits_\Gamma y \,\mathrm{d}\Gamma = \int\limits_\Gamma y_d \,\mathrm{d}\Gamma$. W.l.o.g we assume $\int\limits_\Gamma y_d \,\mathrm{d}\Gamma = 0$ and therefore only need to consider states with mean value zero. The state equation now reads $y = \tilde{S}u$ with the solution operator $\tilde{S} : L^2_0(\Gamma) \to L^2_0(\Gamma)$ of the equation $-\Delta_\Gamma y = u$, $\int\limits_\Gamma y \,\mathrm{d}\Gamma = 0$.

Using the injection $L^2_0(\Gamma) \stackrel{\imath}{\to} L^2(\Gamma)$, $\tilde{S}$ is prolongated as an operator $S : L^2(\Gamma) \to L^2(\Gamma)$ by $S = \imath\tilde{S}\imath^*$. The adjoint $\imath^* : L^2(\Gamma) \to L^2_0(\Gamma)$ of $\imath$ is the L^2-orthogonal projection onto $L^2_0(\Gamma)$. The unique solution of $(\mathbb{P})$ is again characterized by (2.0.3), where the orthogonal projection now takes the form

$$\mathrm{P}_{U_{ad}}(v) = \mathrm{P}_{[a,b]}(v + m)$$

with $m \in \mathbb{R}$ such that

$$\int\limits_\Gamma \mathrm{P}_{[a,b]}(v + m)\,\mathrm{d}\Gamma = 0\,.$$

If for $v \in L^2(\Gamma)$ the inactive set $\mathcal{I}(v + m) = \{\gamma \in \Gamma \mid a < v[\gamma] + m < b\}$ is non-empty, the constant $\mathrm{m} = \mathrm{m(v)}$ is uniquely determined by $v \in L^2(\Gamma)$. Hence, the solution $u \in U_{ad}$ satisfies

$$u = \mathrm{P}_{[a,b]}\left(-\frac{1}{\alpha}p(u) + m\left(-\frac{1}{\alpha}p(u)\right)\right)\,,$$

with $p(u) = S^*(Su - \imath^* y_d) \in H^2(\Gamma)$ denoting the adjoint state and $m(-\frac{1}{\alpha}p(u)) \in \mathbb{R}$ is implicitly given by $\int\limits_\Gamma u \,\mathrm{d}\Gamma = 0$. Note that $\imath^*\imath$ is the identity on $L^2_0(\Gamma)$.

In $(\mathbb{P}_h)$ we now replace U_{ad}^h by $U_{ad}^h = \left\{v \in L^2(\Gamma^h) \mid a \le v \le b\right\} \cap L_0^2(\Gamma^h)$. Similar as in (2.1.4), the unique solution u_h then satisfies

$$u_h = \mathrm{P}_{U_{ad}^h}\left(-\frac{1}{\alpha}p_h(u_h)\right) = \mathrm{P}_{[a,b]}\left(-\frac{1}{\alpha}p_h(u_h) + m_h\left(-\frac{1}{\alpha}p_h(u_h)\right)\right)\,, \tag{2.4.1}$$

with $p_h(v_h) = S_h^*(S_h v_h - \imath_h^*(y_d)_l) \in Y_h$ and $m_h(-\frac{1}{\alpha}p_h(u_h)) \in \mathbb{R}$ the unique – excepting the cases $\check{\mathcal{A}}(p_h^v) = \Gamma$ and $\hat{\mathcal{A}}(p_h^v) = \Gamma$ – constant such that $\int_{\Gamma^h} u_h \,\mathrm{d}\Gamma^h = 0$. Note that $m_h\left(-\frac{1}{\alpha}p_h(u_h)\right)$ is semismooth with respect to u_h and thus Equation (2.4.1) is amenable to a semismooth Newton method.

The discretization error between the problems $(\mathbb{P}_h)$ and $(\mathbb{P})$ now decomposes into two components, one introduced by the discretization of U_{ad} through the discretization of the surface, the other by discretization of S.

For the first error component we need to investigate the relation between $\mathrm{P}_{U_{ad}^h}(u_l)$ and $\mathrm{P}_{U_{ad}}(u)$, which is now slightly more involved than Equation (2.1.5).

Lemma 2.4.1. *There exists a constant $C_m > 0$ depending only on Γ, $|a|$ and $|b|$ such that for all $v \in L^2(\Gamma)$ with $\int_{\mathcal{I}(v+m(v))} \mathrm{d}\Gamma > 0$, and for $0 < h < h_v$ sufficiently small, the value of $m_h(v_l)$ is unique and*

$$|m_h(v_l) - m(v)| \le \frac{C_m}{\int_{\mathcal{I}(v+m(v))} \mathrm{d}\Gamma} h^2\,,$$

where $h_v > 0$ depends on v.

Proof. For $v \in L^2(\Gamma)$, $\epsilon > 0$ choose $\delta > 0$ and $h > 0$ so small that the set

$$\mathcal{I}_v^\delta = \left\{\gamma \in \Gamma^h \mid a + \delta \le v_l(\gamma) + m(v) \le b - \delta\right\}$$

satisfies $\int_{\mathcal{I}_v^\delta} \mathrm{d}\Gamma^h(1+\epsilon) \ge \int_{\mathcal{I}(v+m(v))} \mathrm{d}\Gamma$. Decreasing h further if necessary ensures

$$\frac{Ch^2}{\int_{\mathcal{I}_v^\delta} \mathrm{d}\Gamma^h} \le (1+\epsilon)\frac{Ch^2}{\int_{\mathcal{I}(v+m(v))} \mathrm{d}\Gamma} \le \delta\,,$$

with $C = c_{\text{int}} \max(|a|,|b|) \int_\Gamma \mathrm{d}\Gamma$. Because of $\int_{\mathcal{I}_v^\delta} \mathrm{d}\Gamma^h > 0$, the monotonous function $M_v^h : \mathbb{R} \to \mathbb{R}$

$$M_v^h(x) = \int_{\Gamma^h} \mathrm{P}_{[a,b]}\left(v_l + x\right)\,\mathrm{d}\Gamma^h\,,$$

is strictly monotonous near $m(v)$. Since $\int_\Gamma \mathrm{P}_{[a,b]}\left(v + m(v)\right)\,\mathrm{d}\Gamma = 0$, Lemma 2.1.1 yields

$$|M_v^h(m(v))| \le c_{\text{int}} \|\mathrm{P}_{[a,b]}\left(v + m(v)\right)\|_{L^1(\Gamma)} h^2 \le Ch^2\,.$$

Let us assume w.l.o.g. $-Ch^2 \le M_v^h(m(v)) \le 0$. Due to (strict) monotonicity of $M_v^h(\cdot)$ this implies $m(v) \le m_h(v_l)$ where, for the time being, we consider $m_h(v_l)$ a set-valued function. Then again, since $\frac{Ch^2}{\int_{\mathcal{I}_v^\delta} \mathrm{d}\Gamma^h} \le \delta$, we conclude

$$M_v^h\left(m(v) + \frac{Ch^2}{\int\limits_{\mathcal{I}_v^\delta} \mathrm{d}\Gamma^h}\right) \ge M_v^h\left(m(v)\right) + \int\limits_{\mathcal{I}_v^\delta} \frac{Ch^2}{\int\limits_{\mathcal{I}_v^\delta} \mathrm{d}\Gamma^h} \mathrm{d}\Gamma^h = M_v^h\left(m(v)\right) + Ch^2 \ge 0\,,$$

and again by strict monotonicity of $M_v^h(\cdot)$ it follows $m_h(v_l) \le m(v) + \frac{Ch^2}{\int_{\mathcal{I}_v^\delta} \mathrm{d}\Gamma^h}$. Altogether we get $0 \le m_h(v_l) - m(v) \le \frac{Ch^2}{\int_{\mathcal{I}_v^\delta} \mathrm{d}\Gamma^h} \le \frac{(1+\epsilon)C}{\int_{\mathcal{I}(v+m(v))} \mathrm{d}\Gamma} h^2$. This proves the claim.
Finally, $m_h(v_l)$ is single-valued since the inactive set $\mathcal{I}(v_l + m_h(v_l))$ contains $\mathcal{I}_v^\delta$ and is thus of positive measure. ∎

Because

$$\left(\mathrm{P}_{U_{ad}^h}\left(v_l\right)\right)^l - \mathrm{P}_{U_{ad}}\left(v\right) = \mathrm{P}_{[a,b]}\left(v + m_h(v_l)\right) - \mathrm{P}_{[a,b]}\left(v + m(v)\right)\,,$$

we get the following corollary.

Corollary 2.4.2. *Let $v \in L^2(\Gamma)$ with $\int_{\mathcal{I}(v+m(v))} \mathrm{d}\Gamma > 0$. With C_m and $h_v > 0$ as in Lemma 2.4.1 there holds for $0 < h < h_v$*

$$\left\|\left(\mathrm{P}_{U_{ad}^h}\left(v_l\right)\right)^l - \mathrm{P}_{U_{ad}}\left(v\right)\right\|_{L^2(\Gamma)} \le C_m \frac{\sqrt{\int_\Gamma \mathrm{d}\Gamma}}{\int_{\mathcal{I}(v+m(v))} \mathrm{d}\Gamma} h^2\,.$$

Note that since for $u \in L^2(\Gamma)$ the adjoint $p(u)$ is a continuous function on Γ, the corollary is applicable for $v = -\frac{1}{\alpha}p(u)$.
The following theorem can be proved along the lines of Theorem 2.1.2.

Theorem 2.4.3. *Let $u \in L^2(\Gamma)$, $u_h \in L^2(\Gamma^h)$ be the solutions of $(\mathbb{P})$ and $(\mathbb{P}_h)$, respectively, in the case $\mathbf{c} \equiv 0$. Let $\tilde{u}_h = \left(\mathrm{P}_{U_{ad}^h}\left(-\frac{1}{\alpha}p(u)_l\right)\right)^l$. Then there holds for $\epsilon > 0$ and $0 \le h < h_\epsilon$*

$$\alpha\|u_h^l - \tilde{u}_h\|_{L^2(\Gamma)}^2 + \left\|y_h^l - y\right\|_{L^2(\Gamma)}^2 \le (1+\epsilon)\left(\frac{1}{\alpha}\left\|\left((\cdot)^l S_h^*(\cdot)_l - S^*\right)(y - y_d)\right\|_{L^2(\Gamma)}^2 \cdots \right.$$
$$\left. + \left\|(\cdot)^l S_h(\cdot)_l \tilde{u}_h - y\right\|_{L^2(\Gamma)}^2\right)\,.$$

Using Corollary 2.4.2 we conclude from the theorem

$$\|u_h^l - u\|_{L^2(\Gamma)} \le C\left(\frac{1}{\alpha}\left\|\left((\cdot)^l S_h^*(\cdot)_l - S^*\right)(y - y_d)\right\|_{L^2(\Gamma)} + \frac{1}{\sqrt{\alpha}}\left\|\left((\cdot)^l S_h(\cdot)_l - S\right)u\right\|_{L^2(\Gamma)} \cdots \right.$$
$$\left. + \left(1 + \frac{\|S\|_{\mathcal{L}(L^2(\Gamma),L^2(\Gamma))}}{\sqrt{\alpha}}\right)\frac{C_m\sqrt{\int_\Gamma \mathrm{d}\Gamma}h^2}{\int_{\mathcal{I}(-\frac{1}{\alpha}p(u)+m(-\frac{1}{\alpha}p(u)))} \mathrm{d}\Gamma}\right),$$

the latter part of which is the error introduced by the discretization of U_{ad}. Hence one has h^2-convergence of the optimal controls.
Equation (2.4.1) is amenable to a semismooth Newton method as described in Section 2.2. The algorithm however needs to take the scalar quantity $m_h\left(-\frac{1}{\alpha}p_h(v)\right)$ into account for each iterate $v \in L^2(\Gamma^h)$. The functional $m_h\left(-\frac{1}{\alpha}p_h(\cdot)\right)$ can be shown to be semismooth with generalized derivative $\frac{1}{\int_{\Gamma^h} \mathbb{1}_{\mathcal{I}(p_h^v)}\,\mathrm{d}\Gamma^h} \int_{\Gamma^h} \frac{\mathbb{1}_{\mathcal{I}(p_h^v)}}{\alpha} S_h^* S_h \,\mathrm{d}\Gamma^h$ and is evaluated by performing a Newton algorithm on

$$\int_{\Gamma^h} \mathrm{P}_{[a,b]}\left(-\frac{1}{\alpha}p_h(v) + m_h\right) \mathrm{d}\Gamma^h = 0\,.$$

2.5 Numerical examples

The figures show some selected Newton steps v^+. Note that jumps of the color-coded function values are well observable along the border between active and inactive set. For all examples Newton's method is initialized with $u_0 \equiv 0$.
The meshes are generated from a macro triangulation through congruent refinement, new nodes are projected onto the surface Γ. The maximal edge length h in the triangulation is not exactly halved in each refinement, but up to an error of order $O(h^2)$. Therefore we just compute our estimated order of convergence (EOC) according to

$$EOC_i = \frac{\ln \|u_{h_{i-1}} - u_l\|_{L^2(\Gamma^{h_{i-1}})} - \ln \|u_{h_i} - u_l\|_{L^2(\Gamma^{h_i})}}{\ln(2)}.$$

For different refinement levels, the tables show relative errors, measured in L^2- and L^∞-norms, and the corresponding EOCs along with the number of Newton iterations before the desired accuracy of 10^{-9} with respect to w is reached.
The stopping criterion is that of Lemma 2.3.4(iii).
The theoretical findings from Section 2.3 are confirmed by our examples that exhibit mesh-independent behavior in terms of Newton steps.
We had showed that there is a mesh-independent upper bound to the number of Newton iterations before the desired accuracy is reached, compare Remark 2.3.5. Under certain conditions on the adjoint state $p(u)$ the behavior of the damped Newton algorithm is also mesh-independent in the sense that it transitions into the undamped iteration and converges superlinearly towards the solution u_h of $(\mathbb{P}_h)$, see Lemma 2.3.6. These assumptions are met by all our examples, since the surface gradient of the (here) smooth function $-\frac{1}{\alpha}p(u)$ is bounded away from zero along the border of the inactive set.
Accordingly for all examples the damping is not active during the last few Newton steps.

Example 2.5.1 (Sphere I)**.** We consider the problem

$$\min_{u\in L^2(\Gamma),\, y\in H^1(\Gamma)} \mathbf{O}(u,y) \text{ subject to } -\Delta_\Gamma y + y = u - r, \quad -1 \le u \le 1 \tag{2.5.1}$$

with Γ the unit sphere in $\mathbb{R}^3$ and $\alpha = 10^{-7}$. We choose $y_d = 52\alpha x_3(x_1^2 - x_2^2)$, to obtain the solution

$$u = r = \min\big(1, \max\big(-1, 4x_3(x_1^2 - x_2^2)\big)\big)$$

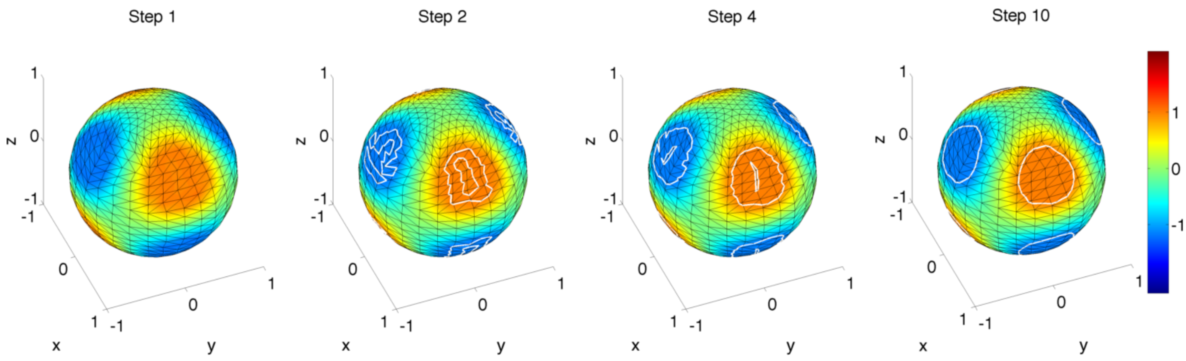

Figure 2.1: Selected full Steps v^+ computed for Example 2.5.1 on the twice refined sphere.

reg. refs.	0	1	2	3	4	5
L^2-error	2.3789e-01	5.1804e-02	1.2326e-02	2.9491e-03	7.0647e-04	1.7358e-04
L^2-EOC	-	2.38	2.11	2.07	2.06	2.03
L^∞-error	3.4918e-01	9.2848e-02	2.6760e-02	1.1637e-02	2.3391e-03	5.0757e-04
L^∞-EOC	-	2.06	1.83	1.21	2.32	2.21
# Steps	5	18	16	13	14	14

Table 2.1: L^2-error, EOC and number of iterations for Example 2.5.1.

of (2.5.1).

The presented theory can be extended to include surfaces with boundary, [Dzi88, §6], given that the boundary is piecewise C^1-smooth and $\Pi_\Gamma|_{\Gamma^h} : \Gamma^h \to \Gamma$ is surjective. Both conditions are satisfied in the following example; in particular there holds $\partial\Gamma^h = \partial\Gamma$.

Example 2.5.2. Let $\Gamma = \big\{(x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_3 = x_1 x_2 \text{ and } x_1, x_2 \in (0,1)\big\}$ and $\alpha = 10^{-6}$. For

$$\min_{u\in L^2(\Gamma),\, y\in H^1(\Gamma)} \mathbf{O}(u,y) \text{ subject to } -\Delta_\Gamma y = u - r, \quad y = 0 \text{ on } \partial\Gamma \quad -0.5 \le u \le 0.5\,,$$

by proper choice of y_d (via symbolic differentiation), we get the unique solution

$$u = r = \max\big(-0.5, \min\big(0.5, \sin(\pi x_1)\sin(\pi x_2)\big)\big)\,.$$

reg. refs.	0	1	2	3	4	5
$L2$-error	4.2746e-01	7.9834e-02	1.9234e-02	4.3798e-03	1.0638e-03	2.6082e-04
EOC	-	2.42	2.05	2.13	2.04	2.03
L^∞-error	5.0171e-01	1.8055e-01	4.1925e-02	1.1035e-02	2.8747e-03	7.1757e-04
L^∞-EOC	-	1.47	2.11	1.93	1.94	2.00
# Steps	7	7	7	8	8	8

Table 2.2: L^2-error, EOC and number of iterations for Example 2.5.2.

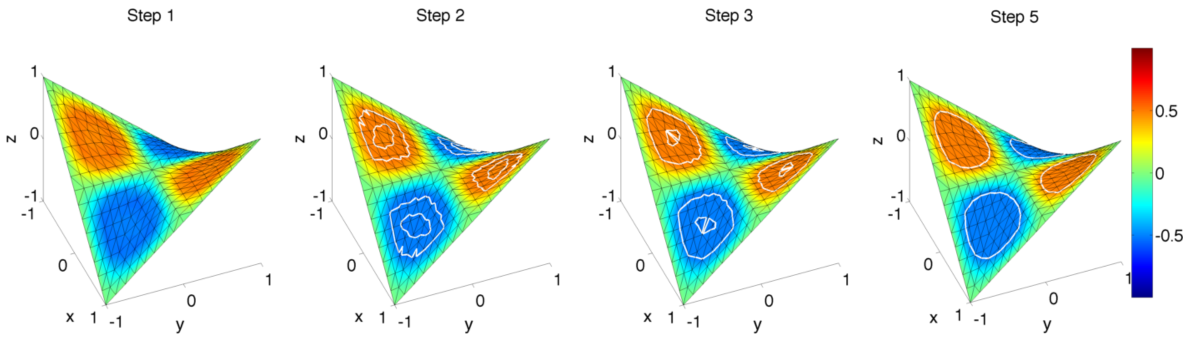

Figure 2.2: Selected full Steps v^+ computed for Example 2.5.2 on the thrice refined grid.

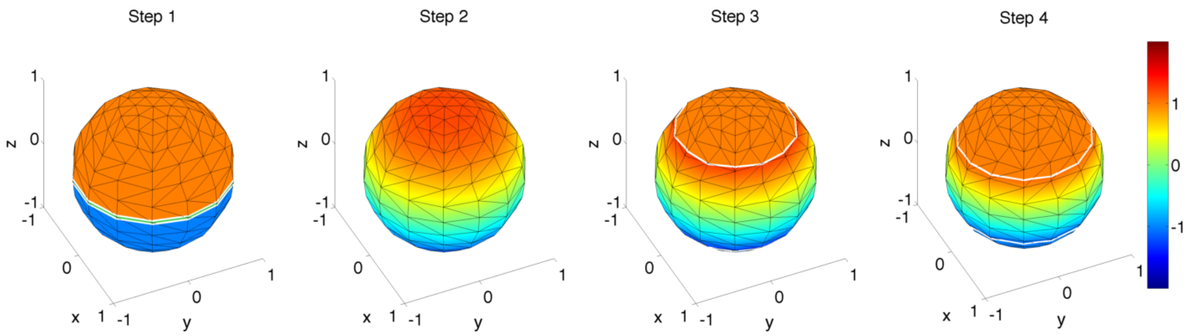

Figure 2.3: First four full Steps v^+ computed for Example 2.5.3 with $\alpha = 10^{-2}$ on once refined sphere.

Example 2.5.2, although $\mathbf{c} \equiv 0$, is also covered by the theory in Sections 2.1-2.3, as by the Dirichlet boundary conditions the state equation remains uniquely solvable for $u \in L^2(\Gamma)$. In the last two examples we apply the variational discretization to optimization problems, that involve zero-mean-value constraints as in Section 2.4.

Example 2.5.3 (Sphere II)**.** We consider

$$\min_{u \in L^2(\Gamma),\, y \in H^1(\Gamma)} \mathbf{O}(u, y) \text{ subject to } -\Delta_\Gamma y = u\,, \quad -1 \leq u \leq 1\,, \quad \int_\Gamma y \, \mathrm{d}\Gamma = \int_\Gamma u \, \mathrm{d}\Gamma = 0\,,$$

reg. refs.	1	2	3	4	5	6
L^2-error	1.2035e-02	3.0743e-03	7.7422e-04	1.9401e-04	4.8531e-05	1.1441e-05
EOC	-	2.01	2.00	2.00	2.00	2.08
# Steps	2	2	2	2	2	2
L^2-error	2.0121e-01	5.7649e-02	1.5928e-02	4.2907e-03	1.1039e-03	2.6252e-04
EOC	-	1.84	1.86	1.89	1.96	2.07
# Steps	16	16	15	16	15	16

Table 2.3: L^2-error, EOC and number of iterations for Example 2.5.3, first for $\alpha = 10$, second for $\alpha = 10^{-5}$.

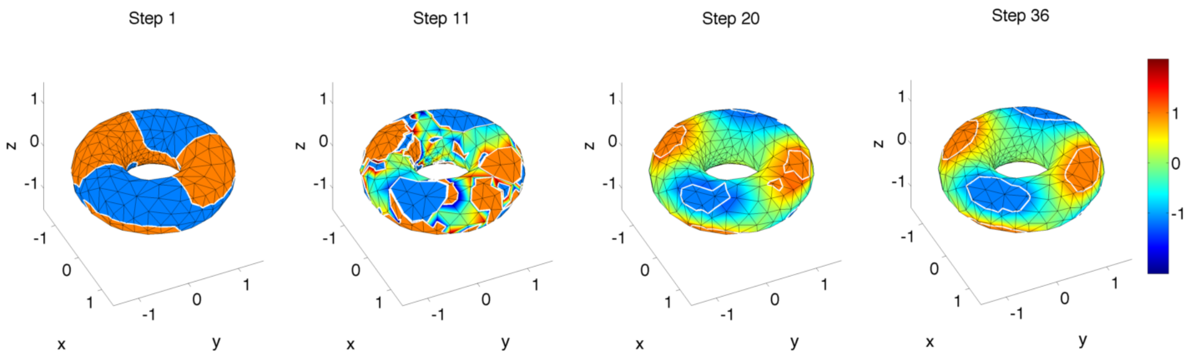

Figure 2.4: Selected full Steps v^+ computed for Example 2.5.4 on the once refined torus.

reg. refs.	0	1	2	3	4	5
$L2$-error	1.7903e-01	5.2558e-02	1.4498e-02	3.7472e-03	9.4760e-04	2.3754e-04
EOC	-	1.90	1.96	1.98	1.99	2.00
# Steps	6	6	5	2	2	2
$L2$-error	1.2100e-01	3.2415e-02	7.6855e-03	1.8145e-03	4.4676e-04	1.0955e-04
EOC	-	2.04	2.20	2.11	2.03	2.03
# Steps	66	62	45	27	22	18

Table 2.4: L^2-error, EOC and number of iterations for Example 2.5.4, first for $\alpha = 1$, second for $\alpha = 10^{-6}$.

with Γ the unit sphere in $\mathbb{R}^3$. Set $\alpha = 10, 10^{-5}$ and

$$y_d(x_1, x_2, x_3) = 4\alpha x_3 + \begin{cases} \ln(x_3 + 1) + C\,, & \text{if} \quad 0.5 \le x_3 \\ x_3 - \frac{1}{4}\text{arctanh}(x_3)\,, & \text{if} \quad -0.5 \le x_3 \le 0.5 \\ -C - \ln(1 - x_3)\,, & \text{if} \quad x_3 \le -0.5 \end{cases},$$

where C is chosen for y_d to be continuous. The solution according to these parameters is

$$u = \min\big(1, \max\big(-1, 2x_3\big)\big)\,.$$

Example 2.5.4 (Torus)**.** Let $\alpha = 1, 10^{-6}$ and

$$\Gamma = \left\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 \;\middle|\; \sqrt{x_3^2 + \left(\sqrt{x_1^2 + x_2^2} - 1\right)^2} = \frac{1}{2} \right\}$$

the 2-torus embedded in $\mathbb{R}^3$. By symbolic differentiation we compute y_d, such that

$$\min_{u \in L^2(\Gamma),\, y \in H^1(\Gamma)} \mathbf{O}(u, y) \text{ subject to } -\Delta_\Gamma y = u - r, \quad -1 \le u \le 1\,, \quad \int_\Gamma y \,\mathrm{d}\Gamma = \int_\Gamma u \,\mathrm{d}\Gamma = 0$$

is solved by

$$u = r = \max\big(-1, \min\big(1, 5x_1x_2x_3\big)\big)\,.$$

As the presented tables clearly demonstrate, the examples show the expected convergence behavior.

Chapter 3

Parabolic equations on moving surfaces

In this chapter parabolic state equations on evolving hypersurfaces in $\mathbb{R}^{n+1}$ are investigated. In order to formulate well-posed optimal control problems in the next chapter, we prove existence and uniqueness of weak solutions for the state equation, in the sense of vector-valued distributions. In the process, dealing with time-varying domains and function spaces, we introduce spaces that are simple cases of measurable Hilbertian sums.
The state equation is then discretized in space using the surface finite element method proposed in [DE07]. The time discretization is carried out through a discontinuous Galerkin scheme. In order to obtain a scheme the adjoint of which is computable, we provide the control space with an equivalent norm that can be evaluated using only a finite number of surfaces $(\Gamma(t_m))_{m=0}^{M}$.
We prove convergence of both the spatial and the time-discretization under weak regularity assumptions. In particular we allow for initial data $y_0 \in L^2(\Gamma(T))$.

Following [DE07], we consider the state equation in its weak form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma(t)} y \, \varphi \, \mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{b} \, y \, \varphi \, \mathrm{d}\Gamma(t) = \int\limits_{\Gamma(t)} y \, \dot{\varphi} \, \mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} u \, \varphi \, \mathrm{d}\Gamma(t) \,, \tag{3.0.1}$$

with $y(0) = y_0$. Here, $\Gamma = \left\{\Gamma(t)\right\}^{t\in[0,T]}$ is a family of C^2-smooth, compact n-dimensional surfaces in $\mathbb{R}^{n+1}$, evolving smoothly in time with velocity V, as described in Section 1.2. Further assume u sufficiently smooth and let $\dot{\varphi} = \partial_t \varphi + V \nabla \varphi$ denote the material derivative of a smooth test function φ.
We start by defining unique weak solutions for the state equation. The idea is to to introduce distributional material derivatives in the sense of [LM72] and a $W(0,T)$-like solution space. As a consequence, a large part of the theory developed around $W(0,T)$ for fixed domains applies, compare for example [LM68], [LM72], and [Lio71]. In order to define integrability and (weak) differentiability we use the flow Φ of the velocity field V to pull back functions from the varying domain $\Gamma(t)$, $t \in [0,T]$ onto a fixed surface $\Gamma(s)$.

The reason for our interest in $W(0,T)$-like spaces is that $W(0,T)$ and more regular variants, such as $H^1((0,T), H^2(\Omega))$, have long since been recognized as the suitable framework for optimal control, compare for example [Trö05, Thm. 3.10] and preceding commentary.
An alternative approach to prove existence of weak solutions along the lines of [LSU68] is taken in [Sch10], that entirely avoids the notion of vector-valued distributions. As a result this approach does not yield weak time-derivatives nor a bijective state-to-control mapping. It is also harder to prove uniqueness of the solutions and useful embedding theorems such as [LM72, Thm. 3.1] are not accessible.
In the second part of this chapter we assume $n = 1, 2, 3$ and prove optimal L^2-error bounds for a discretization of (3.0.1). For the spacial discretization we apply the surface finite element approach from [DE07] . The time-discretization is achieved through a discontinuous Galerkin finite element approach.
Recent works also deal with the discretization of (3.0.1), both in space, compare [DE10], and time, see [DLM11] and [DE11]. A finite volume method for evolving surfaces was introduced in [LNR11].
In [DE10] order-optimal error bounds in terms of the norm $\sup_{t\in[0,T]} \|\cdot\|_{L^2(\Gamma(t))}$ are derived for the discretization of the state equation, assuming a slightly higher regularity of the state than we require in Section 3.6 and 3.7, where we derive $\left(\int_0^T \|\cdot\|^2_{L^2(\Gamma(t))}\mathrm{d}t\right)^{\frac{1}{2}}$-like bounds.
A class of Runge-Kutta methods to tackle the space-discretized problem is investigated in [DLM11], assuming among other things that one can evaluate f in a point-wise fashion, i.e. that $f(t) \in L^2(\Gamma(t))$ is well defined. For a fully discrete approach and the according error bounds see [DE11]. There a backwards Euler method is considered for time discretization whose implementation resembles our discontinuous Galerkin approach in Section 3.7. Yet while the approach in [DE11] ultimately leads to $\sup_{t\in[0,T]} \|\cdot\|_{L^2(\Gamma(t))}$-convergence, we allow for non-smooth right-hand sides and thus cannot expect to obtain such strong convergence estimates.
This chapter is structured as follows. After giving a brief overview of the setting we start with a very short introduction into vector-valued distributions. We then prepare in Section 3.2 and 3.3 the setting in which to look for a solution of (3.0.1).
The proof of existence of an appropriate weak solution is conducted in Section 3.4, complementing the existence results from [DE07] and [Sch10].
In Sections 3.6 and 3.7 we examine the space- and time-discretization of the state equation.

Before we can properly formulate (3.0.1), let us summarize the results of Section 1.2 and our assumptions regarding the family $\left\{\Gamma(t)\right\}_{t\in[0,T]}$.

Assumption 3.0.5. The hypersurface $\Gamma_0 = \Gamma(0) \subset \mathbb{R}^{n+1}$ is C^2-smooth and compact without boundary. Γ evolves along a C^2-smooth velocity field $V : \mathbb{R}^{n+1} \times [0,T] \to \mathbb{R}^{n+1}$ with flow $\Phi : \mathbb{R}^{n+1} \times [0,T]^2 \to \mathbb{R}^{n+1}$, such that its restriction $\Phi_t^s(\,\cdot\,) : \Gamma(s) \to \Gamma(t)$ is a diffeomorphism for every $s, t \in [0,T]$.

The assumption gives rise to a second representation of $\Gamma(t)$ as the level set

$$\Gamma(t) = \left\{x \in \mathbb{R}^{n+1} \mid d(x,t) = 0\right\} ,$$

of a continuous signed distance function $d : \mathbb{R}^{n+1} \to \mathbb{R}$ with $|d(x,t)| = \operatorname{dist}(x, \Gamma(t))$.
Also $\Gamma(t)$ is orientable with a smooth unit normal field $\nu(\cdot, t)$.
Further, we have $d(\cdot) \in C^2(\mathcal{N}_\epsilon)$ for some neighborhood $\mathcal{N}_\epsilon$ of $\bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}$ in $\mathbb{R}^{n+2}$ as defined in Theorem 1.2.4; there holds $\nabla d(\gamma, t) = \nu(\gamma, t)$ for $\gamma \in \Gamma(t)$.
Using d we can define the projection

$$\Pi_{\Gamma(t)} : \left\{x \in \mathbb{R}^{n+1} \mid \operatorname{dist}(x, \Gamma(t)) < \epsilon\right\} \to \Gamma(t)\,, \quad \Pi_{\Gamma(t)}(x) = x - d(x,t)\nabla d(x,t)\,. \tag{3.0.2}$$

We consider the surface gradient an element of $\mathbb{R}^{n+1}$ and denote the metric tensor like in (1.0.1). In the following we will write ∇_Γ instead of $\nabla_{\Gamma(t)}$ wherever it is clear which surface $\Gamma(t)$ the gradient relates to.

3.1 Vector-valued distributions

We are going to exploit results on vector-valued distributions, which we recall here for completeness. In order to define weak derivatives consider $\mathcal{D}((0,T))$, the space of real valued C^∞-smooth functions with compact support in $(0,T)$. Fix $s \in [0,T]$. Each $y \in L^2((0,T), H^1(\Gamma(s)))$ defines a vector-valued distribution $\mathcal{T}_y : \mathcal{D}((0,T)) \to H^1(\Gamma(s))$ through the $H^1(\Gamma(s))$-valued integral $\int\limits_{[0,T]} y(t)\varphi(t)\,\mathrm{d}t$.
Its distributional derivative is said to lie in $L^2((0,T), H^{-1}(\Gamma(s)))$ iff it can be represented by $w \in L^2((0,T), H^{-1}(\Gamma(s)))$ in the following sense

$$\mathcal{T}_y'(\varphi) = \int\limits_{[0,T]} y(t)\varphi'(t)\,\mathrm{d}t = -\int\limits_{[0,T]} w(t)\varphi(t)\,\mathrm{d}t \in H^1(\Gamma(s))\,, \quad \forall \varphi \in \mathcal{D}((0,T))\,, \tag{3.1.1}$$

and we write $y' = w$. Note that by H^{-1} we denote the representation of the dual $(H^1)^*$ which arises from $L^2 \supset H^1$ by completion.

Lemma and Definition 3.1.1. *For $s \in [0,T]$, the space*

$$W_s(0,T) = \left\{v \in L^2((0,T), H^1(\Gamma(s))) \;\middle|\; v' \in L^2((0,T), H^{-1}(\Gamma(s)))\right\}$$

with scalar product $\int_0^T \langle\cdot,\cdot\rangle_{H^1(\Gamma(s))} + \langle(\cdot)', (\cdot)'\rangle_{H^{-1}(\Gamma(s))}\mathrm{d}t$ is a Hilbert space.

1. *$W_s(0,T)$ is compactly embedded into $C([0,T], L^2(\Gamma(s)))$, the space of continuous L^2-valued functions.*

2. *Denote by $\mathcal{D}([0,T], H^1(\Gamma(s)))$ the space of C^∞-smooth $H^1(\Gamma(s))$-valued test functions on $[0,T]$. The inclusion $\mathcal{D}([0,T], H^1(\Gamma(s))) \subset W_s(0,T)$ is dense.*

3. *For two functions $v, w \in W_s(0,T)$ the product $\langle v(t), w(t)\rangle_{L^2(\Gamma(s))}$ is absolutely continuous with respect to $t \in [0,T]$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma(s)} v(t)w(t)\,\mathrm{d}\Gamma(s) = \langle v', w\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))} + \langle v, w'\rangle_{H^1(\Gamma(s)),H^{-1}(\Gamma(s))}\,,$$

a.e. in $(0,T)$, and as a consequence there holds the integration by parts formula

$$\int_{[r,t]} \langle v', w\rangle_{H^{-1},H^1}\, \mathrm{d}\tau = \langle v(t), w(t)\rangle_{L^2(\Gamma(s))} - \langle v(r), w(r)\rangle_{L^2(\Gamma(s))} - \int_{[r,t]} \langle v, w'\rangle_{H^1,H^{-1}}\, \mathrm{d}\tau\,.$$

For a proof of the lemma, see [LM68, Ch. I,Thms. 3.1 and 2.1]. Assertion *3.* follows from *2.* by approximation with smooth functions. One can also use the integration by parts formular to prove the embedding into $C((0,T), L^2(\Gamma(s)))$, see [Eva98, Ch. 5,Thm. 3]. For further references see [Trö05, Thm. 3.10].
Our approach to weak material derivatives relies on the following equivalent formulation of condition (3.1.1)

$$\forall \varphi \in \mathcal{D}((0,T), H^1(\Gamma(s))) : \quad \int_{[0,T]} \langle y(t), \varphi'(t)\rangle_{L^2(\Gamma(s))} + \langle w(t), \varphi(t)\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))}\, \mathrm{d}t = 0\,, \tag{3.1.2}$$

which defines the weak derivative $y' = w$ of a function $y \in L^2((0,T), H^1(\Gamma(s)))$ via its $L^2((0,T), L^2(\Gamma(s)))$-scalar product with elements of $\mathcal{D}((0,T), H^1(\Gamma(s)))$.
The identity (3.1.2) follows from (3.1.1) by Lemma 3.1.1[*3.*]. On the other hand (3.1.1) is a consequence of (3.1.2). To see this, test (3.1.2) with $\psi v \in \mathcal{D}((0,T), H^1(\Gamma(s)))$, where $\psi \in \mathcal{D}((0,T))$ and $v \in H^1(\Gamma(s))$.

3.2 Preliminaries

The scope of this section is to provide the technical background for the definitions in the following section.
We start by defining the strong material derivative for smooth functions $f \in C^1(\mathbb{R}^{n+1} \times [0,T])$, namely the derivative

$$\dot{f}(x,t) = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} f(\Phi_s^t(x), s) = \nabla f(x,t) V(x,t) + \partial_t f(x,t)\,, \tag{3.2.1}$$

along trajectories of the velocity field V. The material derivative has the following properties.

Lemma 3.2.1. *Let f be sufficiently smooth. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} f\, \mathrm{d}\Gamma(t) = \int_{\Gamma(t)} \dot{f} + f \operatorname{div}_{\Gamma} V\, \mathrm{d}\Gamma(t),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} \|\nabla_\Gamma f\|^2\, \mathrm{d}\Gamma(t) = \int_{\Gamma(t)} 2\nabla_\Gamma f \nabla_\Gamma \dot{f} - 2\nabla_\Gamma f (D_\Gamma V) \nabla_\Gamma f + \|\nabla_\Gamma f\|^2 \operatorname{div}_{\Gamma} V\, \mathrm{d}\Gamma(t)\,,$$

with $\operatorname{div}_{\Gamma(t)} V = \sum_{i=1}^{n+1} \nabla^i_{\Gamma(t)} V^i$ *and* $(D_{\Gamma(t)} V)_{ij} = \nabla^j_{\Gamma(t)} V^i$. *The components* $\nabla^i_{\Gamma(t)}$ *denote the components of the gradient as a vector in* $\mathbb{R}^{n+1}$ *conforming to our notation.*

A proof and details can be found in the Appendix of [DE07]. Note that the surface divergence $\operatorname{div}_{\Gamma(t)}$ from the Lemma generalizes the one introduced in Section 1.3.1 which only applied to tangential vector fields.

Lemma and Definition 3.2.2. *Denote by $J_t^s(\,\cdot\,) = \det \mathrm{D}_{\Gamma(s)}\Phi_t^s(\,\cdot\,)$ the Jacobian determinant of the restricted flow $\Phi_t^s : \Gamma(s) \to \Gamma(t)$, meaning that $J_t^s(\,\cdot\,)$ is the determinant of the matrix representation of $\mathrm{D}_{\Gamma(s)}\Phi_t^s(\,\cdot\,)$ with respect to orthonormal bases of the respective tangent space. By Assumption 3.0.5 $J_t^s \in C^1([0,T] \times \Gamma(s))$ and there exists $C_J > 0$, such that for all $s,t \in [0,T]$*

$$\frac{1}{C_J} \leq \min_{\gamma\in\Gamma(s)} J_t^s(\gamma) \leq \max_{\gamma\in\Gamma(s)} J_t^s(\gamma) \leq C_J\,.$$

Given Assumption 3.0.5, consider the family $\left\{L^2(\Gamma(t))\right\}_{t\in[0,T]}$. Then for $v \in L^2(\Gamma(t))$ we introduce the pull-back

$$\phi_t^s v = v\big(\Phi_t^s(\,\cdot\,)\big) \in L^2(\Gamma(s))\,,$$

which is a linear homeomorphism from $L^2(\Gamma(t))$ into $L^2(\Gamma(s))$ for any $s,t \in [0,T]$. Moreover ϕ_t^s is a linear homeomorphism from $H^1(\Gamma(t))$ into $H^1(\Gamma(s))$. Thus finally the adjoint operator, $\phi_t^{s*} : H^{-1}(\Gamma(s)) \to H^{-1}(\Gamma(t))$ is also a linear homeomorphism. There exist constants $C_{L^2(\Gamma)}, C_{H^1(\Gamma)}$ independent of s,t, such that for all $v \in L^2(\Gamma(t))$, or $v \in H^1(\Gamma(t))$ respectively, and for all $s,t \in [0,T]$

$$\|\phi_t^s v\|_{H^1(\Gamma(s))} \leq C_{H^1(\Gamma)}\|v\|_{H^1(\Gamma(t))}\,, \quad \|\phi_t^s v\|_{L^2(\Gamma(s))} \leq C_{L^2(\Gamma)}\|v\|_{L^2(\Gamma(t))}\,,$$

and thus finally $\|\phi_t^{s*}\|_{\mathcal{L}(H^{-1}(\Gamma(s)),H^{-1}(\Gamma(t)))} \leq C_{H^1(\Gamma)}$.
Furthermore there holds $\partial_t J_t^s = \phi_t^s(\operatorname{div}_{\Gamma(t)} V) J_t^s$.

Proof. As to the smoothness of J_t^s it is a consequence of

$$J_t^s = \frac{\sqrt{\det(g_{ij}^t)}}{\sqrt{\det(g_{ij}^s)}}\,,$$

where $g_{ij}^s = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\rangle_{\mathbb{R}^{n+1}}$ and $g_{ij}^t = \langle D\Phi_t^s \frac{\partial}{\partial x_i}, D\Phi_t^s \frac{\partial}{\partial x_j}\rangle_{\mathbb{R}^{n+1}}$denote the metric tensors in corresponding local charts.
For $s,t \in [0,T]$ we have

$$\int\limits_{\Gamma(t)} v^2 \,\mathrm{d}\Gamma(t) = \int\limits_{\Gamma(s)} (\phi_t^s v)^2 J_t^s \,\mathrm{d}\Gamma(s)$$

and thus $\|\phi_t^s v\|_{L^2(\Gamma(s))} \leq C_{L^2(\Gamma)}\|v\|_{L^2(\Gamma(t))}$, with $C_{L^2(\Gamma)} = C_J^{\frac{1}{2}}$.
For H^1 equivalence consider $\varphi \in H^1(\Gamma(t))$. Now

$$\int\limits_{\Gamma(t)} \nabla_\Gamma\varphi\nabla_\Gamma\varphi \,\mathrm{d}\Gamma(t) = \int\limits_{\Gamma(s)} (\nabla_\Gamma\phi_t^s\varphi)^T D\Phi_t^s (D\Phi_t^s)^T \nabla_\Gamma\phi_t^s\varphi J_t^s \,\mathrm{d}\Gamma(s)\,.$$

Because $D\Phi_t^s$ is regular one has $v^T D\Phi_t^s (D\Phi_t^s)^T v > 0$ for all $s,t \in [0,T]$ and $v \in \mathbb{R}^{n+1} \setminus \{0\}$, and a compactness argument yields $v^T D\Phi_t^s (D\Phi_t^s)^T v > C_{D\Phi}\|v\|^2_{\mathbb{R}^{n+1}} > 0$. Thus there holds

$$\|\phi_t^s\varphi\|_{H^1(\Gamma(s))} \leq C_{H^1(\Gamma)}\|\varphi\|_{H^1(\Gamma(t))}\,. \tag{3.2.2}$$

with $C_{H^1(\Gamma)} = \sqrt{\max\{C_J, \frac{C_J}{C_{D\Phi}}\}}$. By a density argument one proves that (3.2.2) holds for all $\varphi \in H^1(\Gamma(t))$: an $H^1(\Gamma(t))$ Cauchy sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ translates into an $H^1(\Gamma(s))$ Cauchy sequence $\phi_t^s\varphi_k$ and one easily shows (3.2.2) for their limits.

Now $\|\cdot\|_{H^1(\Gamma(t))}$ and $\|\phi_t^s(\,\cdot\,)\|_{H^1(\Gamma(s))}$ are two equivalent norms on $H^1(\Gamma(t))$. Hence also their dual norms are equivalent. The norm of $f \in H^{-1}(\Gamma(s))$ can now be expressed by means of $H^1(\Gamma(t))$ as

$$\sup_{w\in H^1(\Gamma(s))} \frac{\langle f, w\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))}}{\|w\|_{H^1(\Gamma(s))}} = \sup_{v\in H^1(\Gamma(t))} \frac{\langle \phi_t^{s*} f, v\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))}}{\|\phi_t^s v\|_{H^1(\Gamma(s))}}\,, \tag{3.2.3}$$

and the bound on the norm of ϕ_t^{s*} follows from the equivalence of said H^1-norms.

The last assertion is a by-product of the proof of Lemma 3.2.1, compare [DE07]. ■

We need to state one more Lemma concerning continuous time-dependence of the previously defined norms.

Lemma 3.2.3. *Let $s \in [0,T]$. For $v_1 \in H^1(\Gamma(s))$, $v_2 \in L^2(\Gamma(s))$, $v_3 \in H^{-1}(\Gamma(s))$ the following expressions are continuous with respect to $t \in [0,T]$*

$$\|\phi_s^t v_1\|_{H^1(\Gamma(t))}\,, \quad \|\phi_s^t v_2\|_{L^2(\Gamma(t))}\,, \quad \|\phi_t^{s*} v_3\|_{H^{-1}(\Gamma(t))}\,.$$

Proof. By the change of variables formula we have

$$\|\phi_s^t v_1\|^2_{H^1(\Gamma(t))} = \int\limits_{\Gamma(s)} \left(\nabla_\Gamma v_1 (D_{\Gamma(s)}\Phi_t^s)^{-1}(D_{\Gamma(s)}\Phi_t^s)^{-T}\nabla_\Gamma v_1 + v_1^2\right) J_t^s \,\mathrm{d}\Gamma(s)\,, \tag{3.2.4}$$

which is a continuous function due to the regularity of Φ stated in Assumption 3.0.5. Similarly we conclude the continuity of the L^2-norm.

Moreover, since $(D_{\Gamma(s)}\Phi_s^s)^{-1}(D_{\Gamma(s)}\Phi_s^s)^{-T} = \mathrm{id}_{T\Gamma(s)}$, $J_s^s = 1$, and $\Phi_{(\cdot)}^s(\cdot) \in C^2(\Gamma(s)\times[0,T])$ Equation (3.2.4) yields

$$\left|\|\phi_s^t v_1\|^2_{H^1(\Gamma(t))} - \|v_1\|^2_{H^1(\Gamma(s))}\right| \le C|t-s|\|v_1\|^2_{H^1(\Gamma(s))}\,,$$

for all $v \in H^1(\Gamma(s))$. Regarding (3.2.3) this allows us to estimate

$$\frac{1}{(1+C|s-t|)^{\frac{1}{2}}}\|v_3\|_{H^{-1}(\Gamma(s))} \le \|\phi_t^{s*} v_3\|_{H^{-1}(\Gamma(t))} \le \frac{1}{(1-C|s-t|)^{\frac{1}{2}}}\|v_3\|_{H^{-1}(\Gamma(s))}\,.$$

■

The results from Lemma 3.2.2 and 3.2.3 can be extended to apply for $H^k(\Gamma(t))$ given that both Γ_0 and the flow Φ are of class C^{k+1}.

3.3 The space W_Γ

In this section a weak material derivative is proposed that is appropriate for the Equation (3.0.1). Also, function spaces are formulated in order to prove the existence of unique weak solutions of (3.0.1) for quite weak right-hand sides u in the next section.
As far as Lemma 3.2.1 is concerned, for a family of functions $\{y(t)\}_{t\in[0,T]}$, $y(t):\Gamma(t)\to\mathbb{R}$, one can define $\dot{y}$ at $\gamma=\Phi_t^0\gamma_0$ simply by

$$\dot{y}(t)[\gamma]=\phi_0^t\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^0 y(t))[\gamma_0,t]=\phi_0^t\frac{\mathrm{d}}{\mathrm{d}t}[y(t)(\Phi_t^0\gamma_0)]\,. \tag{3.3.1}$$

If $\{y(t)\}$ can be smoothly extended, this is equivalent to (3.2.1). However, observe that for functions whose derivatives lie in L^2 Equation (3.3.1) may be rewritten as

$$\dot{y}(t)=\phi_t^{0*}\left(J_t^0\frac{\mathrm{d}}{\mathrm{d}t}(\phi_t^0 y(t))\right)\,, \tag{3.3.2}$$

which, as opposed to (3.3.1), we can hope to generalize to H^{-1}-derivatives.
The following lemmas lead to the definition of a weak material derivative of y that translates into a weak derivative of the pull-back $\phi_t^0 y(t)$.

Lemma and Definition 3.3.1. *Consider the vector bundle $\mathcal{B}_{L^2}=\bigcup_{t\in[0,T]}L^2(\Gamma(t))\times\{t\}$. The set of sections $y:[0,T]\to\mathcal{B}_{L^2}$, $t\mapsto(v,t)$ inherits a canonical vector space structure from the spaces $L^2(\Gamma(t))$ (addition and multiplications with scalars). Given Assumption 3.0.5, for $s\in[0,T]$ we define*

$$L^2_{L^2(\Gamma)}:=\left\{\bar{v}:[0,T]\to\mathcal{B}_{L^2},\ t\mapsto(v_t,t)\ \big|\ \phi_t^s v_t\in L^2((0,T),L^2(\Gamma(s)))\right\}\,.$$

Abusing notation, now and in the following we identify $\bar{v}(t)=(v_t,t)\in L^2_{L^2(\Gamma)}$ with $v(t)=v_t$. Endowed with the scalar product

$$\langle v,w\rangle_{L^2_{L^2(\Gamma)}}=\int\limits_{[0,T]}\langle v_t,w_t\rangle_{L^2(\Gamma(t))}\,\mathrm{d}t\,.$$

$L^2_{L^2(\Gamma)}$ becomes a Hilbert space.
In the same manner we define the space $L^2_{H^1(\Gamma)}$. For $L^2_{H^{-1}(\Gamma)}$ use $\phi_s^{t\,*}$ instead of ϕ_t^s. All three spaces do not depend on s.
For $\varphi\in\phi_s^{(\cdot)}\mathcal{D}((0,T),H^1(\Gamma(s)))=\left\{\varphi\in L^2_{L^2(\Gamma)}\ \big|\ \phi_t^s\varphi\in\mathcal{D}((0,T),H^1(\Gamma(s))\right\}$, it is clear how to interpret $\dot{\varphi}$, namely $\dot{\varphi}=\phi_s^t(\phi_t^s\varphi)'\in H^1(\Gamma(t))$. We say that $y\in L^2_{H^1(\Gamma)}$ has weak material derivative $\dot{y}(t)\in L^2_{H^{-1}(\Gamma)}$ iff there holds

$$\int\limits_{[0,T]}\langle\dot{y},\varphi\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))}\,\mathrm{d}t=-\int\limits_{[0,T]}\langle y,\dot{\varphi}\rangle_{L^2(\Gamma(t))}\,\mathrm{d}t-\int\limits_{[0,T]}\int\limits_{\Gamma(t)}y\varphi\,\mathrm{div}_{\,\Gamma}V\,\mathrm{d}\Gamma(t)\,\mathrm{d}t \tag{3.3.3}$$

for all $\varphi\in\phi_s^{(\cdot)}\mathcal{D}((0,T),H^1(\Gamma(s)))$, and the definition does not depend on s.

Proof. In order to define the scalar product of $L^2_{L^2(\Gamma)}$, we must ensure measurability of $\langle v, w\rangle_{L^2(\Gamma(t))} : [0,T] \to \mathbb{R}$. Since $\langle v, w\rangle = \frac{1}{2}(\|v+w\|^2 - \|v\|^2 - \|w\|^2)$ it suffices to show measurability of $\|v\|^2_{L^2(\Gamma(t))}$ for all $v \in L^2_{L^2(\Gamma)}$. By definition of the set $L^2_{L^2(\Gamma)}$ we have $\phi^s_t v \in L^2([0,T], L^2(\Gamma(s)))$. Hence, there exists a sequence of measurable simple functions $\tilde{v}_i$ that converge pointwise a.e. to $\phi^s_t v$ in $L^2(\Gamma(s))$. Each $\tilde{v}_i$ is the finite sum of measurable single-valued functions, i.e. $\tilde{v}_i = \sum_{j=1}^{M_i} v_{i,j}\mathbb{1}_{B_j}$, $M_i \in \mathbb{N}$, $v_{i,j} \in L^2(\Gamma(s))$, on measurable disjoint sets $B_j \subset [0,T]$. By Lemma 3.2.3 the function

$$\|\phi^t_s \tilde{v}_i\|_{L^2(\Gamma(t))} = \sum_{j=1}^{M_i} \|\phi^t_s v_{i,j}\|_{L^2(\Gamma(t))}\mathbb{1}_{B_j}$$

is the finite sum of measurable functions and thus measurable. Using the continuity of the operator ϕ^t_s, as stated in Lemma 3.2.2, one infers pointwise convergence a.e. of $\|\phi^t_s \tilde{v}_i\|_{L^2(\Gamma(t))}$ towards $\|v\|_{L^2(\Gamma(t))}$ which in turn implies measurability of $\|v\|_{L^2(\Gamma(t))}$.
Again by Lemma 3.2.2 we now conclude integrability of $\|v\|_{L^2(\Gamma(t))}$ and at the same time equivalence of the norms

$$\left(\int\limits_{[0,T]} \|v\|^2_{L^2(\Gamma(t))}\,\mathrm{d}t\right)^{\frac{1}{2}} \quad \text{and} \quad \left(\int\limits_{[0,T]} \|\phi^s_t v\|^2_{L^2(\Gamma(s))}\,\mathrm{d}t\right)^{\frac{1}{2}}.$$

Completeness of $L^2_{L^2(\Gamma)}$ follows, since $L^2_{L^2(\Gamma)}$ and $L^2((0,T), L^2(\Gamma(s)))$ are isomorph. Again because of Lemma 3.2.2, $\phi^s_t v_t \in L^2((0,T), L^2(\Gamma(s)))$ is equivalent to $\phi^r_t v_t \in L^2((0,T), L^2(\Gamma(r)))$, thus the definition does not depend on the choice of s. For $L^2_{H^1(\Gamma)}$ and $L^2_{H^{-1}(\Gamma)}$ we proceed similarly.
We show that the definition of the weak material derivative does not depend on $s \in [0,T]$. On $\Gamma(s)$ Equation (3.3.3) reads

$$\begin{aligned} &\int\limits_{[0,T]} \langle \phi^{t\,*}_s \dot{y}, \tilde{\varphi}\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))}\,\mathrm{d}t = \\ &-\int\limits_{[0,T]}\int\limits_{\Gamma(s)} \left(\phi^s_t y \tilde{\varphi}'(t) + \phi^s_t\left(y \operatorname{div}_{\Gamma(t)} V\right)\tilde{\varphi}\right) J^s_t \,\mathrm{d}\Gamma(s)\,\mathrm{d}t \end{aligned} \tag{3.3.4}$$

for all $\tilde{\varphi} \in \mathcal{D}([0,T], H^1(\Gamma(s)))$. For $r \in [0,T]$, we now transform the relation into one on $\Gamma(r)$, using ϕ^r_s, $(\phi^s_r)^*$ and $\phi^r_t = \phi^r_s \circ \phi^s_t$

$$\begin{aligned} &\int\limits_{[0,T]} \langle \phi^{t\,*}_r \dot{y}, \phi^r_s\tilde{\varphi}\rangle_{H^{-1}(\Gamma(r)),H^1(\Gamma(r))}\,\mathrm{d}t = \\ &-\int\limits_{[0,T]}\int\limits_{\Gamma(r)} \left(\phi^r_t y \left(\phi^r_s\tilde{\varphi}(t)\right)' + \phi^r_t\left(y \operatorname{div}_{\Gamma(t)} V\right)\phi^r_s\tilde{\varphi}\right) J^r_t \,\mathrm{d}\Gamma(r)\,\mathrm{d}t, \end{aligned}$$

and because $\phi^r_s : H^1(\Gamma(s)) \to H^1(\Gamma(r))$ is a linear homeomorphism, it also defines an isomorphism between $\mathcal{D}([0,T], H^1(\Gamma(s)))$ and $\mathcal{D}([0,T], H^1(\Gamma(r)))$. ■

Remark 3.3.2. Strictly speaking the elements of $L^2_{X(\Gamma)}$ are equivalence classes of sections coinciding a.e. in $[0,T]$, just like the elements of $L^2((0,T), X(\Gamma(s)))$.

The definition of the weak derivative of $y \in L^2_{H^1(\Gamma)}$ in (3.3.3) translates into weak derivatives of the pullback $\phi_t^s y$. In order to make the connection between the two, we state the following

Lemma 3.3.3. *Let $w \in W_s(0,T)$ and $f \in C^1([0,T] \times \Gamma(s))$. Then fw also lies in $W_s(0,T)$ and*

$$(fw)' = \underbrace{\partial_t f w}_{\in L^2([0,T], L^2(\Gamma(s)))} + fw' \,,$$

where fw' is to be understood as $\langle fw', \varphi\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))} = \langle w', f\varphi\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))}$.

Proof. We show that for $\varphi \in \mathcal{D}\big((0,T), H^1(\Gamma(s))\big)$ the function $f\varphi$ lies in $W_s(0,T)$. The claim then follows by integration by parts in $W_s(0,T)$.
1. Because $f \in C([0,T]\times\Gamma(s))$ and the strong surface gradient $\nabla_{\Gamma(s)} f \in (C([0,T] \times \Gamma(s)))^{n+1}$ are continuous and thus uniformly continuous on the compact set $[0,T] \times \Gamma(s)$, we infer $f \in C([0,T], C^1(\Gamma(s)))$. Note that $\operatorname{dist}((t,\gamma),(t+k,\gamma)) = k$ in the metric of $[0,T] \times \Gamma(s)$, compare Appendix 1.A. Let $\epsilon > 0$, then for sufficiently small $k_\epsilon > k > 0$ one has

$$\|f(t+k,\cdot) - f(t,\cdot)\|_\infty + \sum_{i=1}^{n+1} \|\nabla^i_{\Gamma(s)} f(t+k,\cdot) - \nabla^i_{\Gamma(s)} f(t,\cdot)\|_\infty \le \epsilon \,.$$

2. As to the distributional derivative of $f\varphi$, we show that $f \in C^1([0,T], C(\Gamma(s)))$. Observe that the uniform continuity of the strong derivative $\partial_t f$ on $[0,T] \times \Gamma(s)$ allows us to estimate

$$\|f(t+k,\cdot) - f(t,\cdot) - \partial_t f(t,\cdot)k\|_\infty = \|k \int_{[0,1]} \partial_t f(t+\tau k,\cdot) - \partial_t f(t,\cdot)\,\mathrm{d}\tau\|_\infty \le k\epsilon$$

for $k_\epsilon > k > 0$ sufficiently small. Again by uniform continuity of $\partial_t f$ we conclude $\partial_t f \in C([0,T], C(\Gamma(s)))$. All told, taking into account the continuity of the pointwise multiplication between the respective spaces, we showed

$$f\varphi \in C([0,T], H^1(\Gamma(s))) \cap C^1([0,T], L^2(\Gamma(s))) \subset W_s(0,T)\,.$$

3. Consider now an arbitrary $w \in W_s(0,T)$. Since $f\varphi \in W_s(0,T)$, by integration by parts as in Lemma 3.1.1[*3.*] it follows

$$\begin{aligned}\int_{[0,T]} \langle w', f\varphi\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(t))_s}\,\mathrm{d}t &= -\int_{[0,T]} \langle w, (f\varphi)'\rangle_{H^1(\Gamma(t))_s,H^{-1}(\Gamma(s))}\,\mathrm{d}t \\ &= -\int_{[0,T]} \langle w, \partial_t f\varphi\rangle_{L^2(\Gamma(s))}\,\mathrm{d}t - \int_{[0,T]} \langle w, f\varphi'\rangle_{L^2(\Gamma(s))}\,\mathrm{d}t\,.\end{aligned}$$

Reordering gives

$$\int_{[0,T]} \langle fw, \varphi'\rangle_{L^2(\Gamma(s))}\,\mathrm{d}t = -\int_{[0,T]} \langle \partial_t f w + fw', \varphi\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(t))_s}\,\mathrm{d}t$$

for any $\varphi \in \mathcal{D}\big((0,T), H^1(\Gamma(s))\big)$. Hence condition (3.1.2) holds for fw. Using the density property stated in Lemma 3.1.1[*2.*], we can approximate w and thus fw by continuous $H^1(\Gamma(s))$-valued functions and infer $fw \in L^2((0,T), H^1(\Gamma(s)))$. The same argument yields $\partial_t fw + fw' \in L^2((0,T), H^{-1}(\Gamma(s)))$. ■

Finally we can define our solution space.

Lemma and Definition 3.3.4. *The solution space* W_Γ *is defined as follows*

$$W_\Gamma = \left\{ v \in L^2_{H^1(\Gamma)} \;\middle|\; \dot v \in L^2_{H^{-1}(\Gamma)} \right\} .$$

W_Γ *is Hilbert with the canonical scalar product* $\int_0^T \langle \cdot, \cdot \rangle_{H^1(\Gamma(t))} + \langle \dot{(\cdot)}, \dot{(\cdot)} \rangle_{H^{-1}(\Gamma(t))} \mathrm{d}t$. *Also* $y \in W_\Gamma$ *iff* $\phi_t^s y \in W_s(0,T)$ *for (every)* $s \in [0,T]$. *In particular one has* $\phi_t^s y \in C([0,T], L^2(\Gamma(s)))$. *For all* $\tilde\varphi \in \mathcal{D}((0,T), H^1(\Gamma(s)))$ *there holds*

$$\int\limits_{[0,T]} \langle {\phi_s^t}^* \dot y, \tilde\varphi \rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))} \,\mathrm{d}t = \int\limits_{[0,T]} \langle ((\phi_t^s y)', J_t^s \tilde\varphi \rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))} \,\mathrm{d}t \,. \tag{3.3.5}$$

One has

$$c_W \|\phi_t^s y\|_{W_s(0,T)} \le \|y\|_{W_\Gamma} \le C_W \|\phi_t^s y\|_{W_s(0,T)} \,,$$

and $c_W, C_W > 0$ *do not depend on* $s \in [0,T]$.

Proof. For $y \in W_\Gamma$, observe that $J_t^s \phi_t^s y \in L^2([0,T], H^1(\Gamma(s))$ and rewrite (3.3.4) as

$$\begin{aligned} \int\limits_{[0,T]} \left\langle J_t^s \phi_t^s y, \tilde\varphi' \right\rangle_{L^2(\Gamma(s))} \,\mathrm{d}t = &- \int\limits_{[0,T]} \langle {\phi_s^t}^* \dot y, \tilde\varphi \rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))} \,\mathrm{d}t \ldots \\ &- \int\limits_{[0,T]} \langle \partial_t J_t^s \phi_t^s y, \tilde\varphi \rangle_{L^2(\Gamma(s))} \,\mathrm{d}t \,, \end{aligned} \tag{3.3.6}$$

for $\tilde\varphi \in \mathcal{D}((0,T), H^1(\Gamma(s)))$. Hence $J_t^s \phi_t^s y \in W_s(0,T)$, and from Lemma 3.3.3 it follows that also $\phi_t^s y \in W_s(0,T)$, because $\frac{1}{J_t^s} \in C^1([0,T] \times \Gamma(s))$. Note that we used $\partial_t J_t^s = \phi_t^s(\operatorname{div}_{\Gamma(t)} V) J_t^s$, see Lemma 3.2.2. On the other hand, for any $\tilde y \in W_s(0,T)$ one has $J_t^s \tilde y \in W_s(0,T)$ and thus $y = \phi_s^t \tilde y \in W_\Gamma$. Hence $\phi_{(\cdot)}^s$ constitutes an isomorphism between W_Γ and $W_s(0,T)$.
Apply Lemma 3.3.3 a second time to obtain $(J_t^s \tilde\varphi)' = \partial_t J_t^s \tilde\varphi + J_t^s \tilde\varphi'$ and because of $\tilde\varphi(0) = \tilde\varphi(T) = 0 \in H^1(\Gamma(s))$ by integration by parts there follows from (3.3.6)

$$\int\limits_{[0,T]} \langle {\phi_s^t}^* \dot y, \tilde\varphi \rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))} \,\mathrm{d}t = \int\limits_{[0,T]} \langle ((\phi_t^s y)', J_t^s \tilde\varphi \rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))} \,\mathrm{d}t \,,$$

compare Lemma 3.1.1[*3.*]. This proves the second claim.
The claim of W_Γ being Hilbert now follows. Observe that point-wise multiplication with J_t^s constitutes a linear homeomorphism in $H^1(\Gamma(s))$ whose inverse is the multiplication by $\frac{1}{J_t^s}$.

One easily checks $\|J_t^s\varphi\|_{H^1(\Gamma(s))} \leq c\|J_t^s\|_{C^1(\Gamma(s))}\|\varphi\|_{H^1(\Gamma(s))} \leq C\|\varphi\|_{H^1(\Gamma(s))}$. This together with Lemma 3.2.2 yields the equivalence of the two norms on W_Γ

$$\int\limits_{[0,T]} \|y\|^2_{H^1(\Gamma(t))} + \|\dot{y}\|^2_{H^{-1}(\Gamma(t))}\,\mathrm{d}t \quad \text{and} \quad \int\limits_{[0,T]} \|\phi_t^s y\|^2_{H^1(\Gamma(s))} + \|(\phi_t^s y)'\|^2_{H^{-1}(\Gamma(s))}\,\mathrm{d}t\,.$$

Completeness of $W_s(0,T)$ then implies completeness of W_Γ. ■

Remark 3.3.5. Formula (3.3.5) can be seen as a generalization of the following relation. Assume $\phi_t^s y \in \mathcal{D}((0,T), H^1(\Gamma(s)))$. Then

$$\int\limits_{[0,T]} \langle \phi_s^{t\,*}\dot{y}, \tilde{\varphi}\rangle_{H^{-1}(\Gamma(s)),H^1(\Gamma(s))}\,\mathrm{d}t = \int\limits_{[0,T]} \langle \dot{y}, \phi_s^t\tilde{\varphi}\rangle_{L^2(\Gamma(t))}\,\mathrm{d}t = \int\limits_{[0,T]} \langle (\phi_t^s y)'\,, J_t^s\tilde{\varphi}\rangle_{L^2(\Gamma(s))}\,\mathrm{d}t\,,$$

which is the situation indicated in (3.3.2).

Using Lemma 3.3.3 and 3.1.1, it is now easy to proof

Lemma 3.3.6. *For two functions $v, w \in W_\Gamma$ the expression $\langle v_t, w_t\rangle_{L^2(\Gamma(t))}$ is absolutely continuous with respect to $t \in [0,T]$ and*

$$\frac{1}{\mathrm{d}t}\int\limits_{\Gamma(t)} vw\,\mathrm{d}\Gamma(t) = \langle \dot{v}, w\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))} + \dots$$
$$\langle v, \dot{w}\rangle_{H^1(\Gamma(t)),H^{-1}(\Gamma(t))} + \int\limits_{\Gamma(t)} vw\ \mathrm{div}_{\,\Gamma(t)}V\,\mathrm{d}\Gamma(t)\,,$$

a.e. in $(0,T)$, and there holds the integration by parts formula

$$\int\limits_{[s,t]} \langle \dot{v}, w\rangle_{H^{-1}(\Gamma(\tau)),H^1(\Gamma(\tau))}\,\mathrm{d}\tau = \langle v, w\rangle_{L^2(\Gamma(t))} - \langle v, w\rangle_{L^2(\Gamma(s))} \dots$$
$$- \int\limits_{[s,t]} \Big[\langle v, \dot{w}\rangle_{H^1(\Gamma(\tau)),H^{-1}(\Gamma(\tau))} + \int\limits_{\Gamma(\tau)} vw\ \mathrm{div}_{\,\Gamma}V\,\mathrm{d}\Gamma(\tau)\Big]\,\mathrm{d}\tau\,.$$

3.4 Weak solutions

We can now give a weak formulation of (3.0.1). Let

$$\mathbf{b} = \phi_0^t\tilde{\mathbf{b}}\,, \quad \tilde{\mathbf{b}} \in C^1([0,T]\times\Gamma_0)\,.$$

We look for solutions $y \in W_\Gamma$ that satisfy $y(0) = y_0 \in L^2(\Gamma_0)$ and for $u \in L^2_{H^{-1}(\Gamma)}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\Gamma(t)} y\,\varphi\,\mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} \nabla_\Gamma y\nabla_\Gamma\varphi + \mathbf{b}y\varphi\,\mathrm{d}\Gamma(t) = \langle\dot{\varphi}, y\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))}\cdots$$
$$+\langle u, \varphi\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))}\,, \tag{3.4.1}$$

for all $\varphi \in W_\Gamma$ and a.e. $t \in (0,T)$. One may equivalently write (3.4.1) as

$$\dot{y} + \Delta_{\Gamma(t)} y + y \left(\operatorname{div}_{\Gamma(t)} V + \mathbf{b} \right) = u \quad \text{in } H^{-1}(\Gamma(t))$$

for a.e. $t \in (0,T)$. We apply known existence and uniqueness results for the pulled-back equation to prove

Theorem 3.4.1. *Let $u \in L^2_{H^{-1}(\Gamma)}$, $y_0 \in L^2(\Gamma_0)$. There exists a unique $y \in W_\Gamma$, such that (3.4.1) is fulfilled for all $\phi \in W_\Gamma$ and a.e. $t \in (0,T)$. There holds*

$$\|y\|_{W_\Gamma} \leq C \left(\|y_0\|_{L^2(\Gamma_0)} + \|u\|_{L^2_{H^{-1}(\Gamma)}} \right) .$$

Proof. Let us relate equation (3.4.1) to the fixed domain $\Gamma(s)$ via

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma(s)} \tilde{y}\, \tilde{\varphi} J_t^s \,\mathrm{d}\Gamma(s) + \int\limits_{\Gamma(s)} \left(\nabla_\Gamma \tilde{y} (D_{\Gamma(s)} \Phi_t^s)^{-1} (D_{\Gamma(s)} \Phi_t^s)^{-T} \nabla_\Gamma \tilde{\varphi} + \tilde{\mathbf{b}} \tilde{y} \tilde{\varphi} \right) J_t^s \,\mathrm{d}\Gamma(s) \ldots$$
$$= \langle \tilde{\varphi}', J_t^s \tilde{y} \rangle_{H^{-1}(\Gamma(s)), H^1(\Gamma(s))} + \langle \tilde{u}, J_t^s \tilde{\varphi} \rangle_{H^{-1}(\Gamma(s)), H^1(\Gamma(s))} ,$$

with $\tilde{y} = \phi_t^s y$, $\tilde{\mathbf{b}} = \phi_t^s b$, and $\tilde{u} = \frac{1}{J_t^s} \phi_s^{t\,*} u \in L^2((0,T), H^{-1}(\Gamma(s))$ and for all $\phi_t^s \varphi = \tilde{\varphi} \in W_s(0,T)$. This again is equivalent to

$$\langle \tilde{y}', \tilde{\varphi} J_t^s \rangle_{H^{-1}(\Gamma(s)), H^1(\Gamma(s))} + \int\limits_{\Gamma(s)} \tilde{y}\, \tilde{\varphi} \left(\phi_t^s (\operatorname{div}_{\Gamma(t)} V) + \tilde{\mathbf{b}} \right) J_t^s \,\mathrm{d}\Gamma(s) + \ldots$$
$$+ \int\limits_{\Gamma(s)} \nabla_\Gamma \tilde{y} (D_{\Gamma(s)} \Phi_t^s)^{-1} (D_{\Gamma(s)} \Phi_t^s)^{-T} \nabla_\Gamma \tilde{\varphi} J_t^s \,\mathrm{d}\Gamma(s) = \langle \tilde{u}, J_t^s \tilde{\varphi} \rangle_{H^{-1}(\Gamma(s)), H^1(\Gamma(s))} .$$

With $\psi = J_t^s \tilde{\varphi}$ one gets for all $\psi \in W_s(0,T)$

$$\langle \tilde{y}', \psi \rangle_{H^{-1}(\Gamma(s)), H^1(\Gamma(s))} + a_s(\tilde{y}, \psi; t) = \langle \tilde{u}, \psi \rangle_{H^{-1}(\Gamma(s)), H^1(\Gamma(s))} , \tag{3.4.2}$$

with a bilinear form

$$a_s(\tilde{y}, \psi; t) = \int\limits_{\Gamma(s)} \nabla_\Gamma \tilde{y} (D_{\Gamma(s)} \Phi_t^s)^{-1} (D_{\Gamma(s)} \Phi_t^s)^{-T} \nabla_\Gamma \psi \,\mathrm{d}\Gamma(s) \ldots$$
$$+ \int\limits_{\Gamma(s)} \tilde{y} \left(\phi_t^s (\operatorname{div}_{\Gamma(t)} V) + \tilde{\mathbf{b}} \right) \psi \,\mathrm{d}\Gamma(s) - \int\limits_{\Gamma(s)} \nabla_\Gamma \tilde{y} (D_{\Gamma(s)} \Phi_t^s)^{-1} (D_{\Gamma(s)} \Phi_t^s)^{-T} \nabla_\Gamma J_t^s \frac{\psi}{J_t^s} \,\mathrm{d}\Gamma(s) .$$

By Assumption 3.0.5 the bilinear form $(D_{\Gamma(s)} \Phi_t^s)^{-1}[\gamma] (D_{\Gamma(s)} \Phi_t^s)^{-T}[\gamma]$ is positive definite on the tangential space $T_\gamma \Gamma(s)$ uniformly in $s, t \in [0,T]$ and $\gamma \in \Gamma(s)$. Thus, there exists $c > 0$ such that for some $k_0 \geq 0$ one has $a_s(\psi, \psi; t) + k_0 \|\psi\|_{L^2(\Gamma(s))} \geq c \|\psi\|_{H^1(\Gamma(s))}$. We are now in the position to apply for example [Lio71, Ch. III, Thm. 1.2], to obtain a unique solution $\tilde{y} \in W_s(0,T)$ to equation (3.4.2) for initial data $\phi_0^s y_0 \in L^2(\Gamma(s))$. Moreover the solution map is continuous

$$\|\tilde{y}\|_{W_s(0,T)} \leq C \left(\|\tilde{u}\|_{L^2((0,T), H^{-1}(\Gamma(s)))} + \|\phi_0^s y_0\|_{L^2(\Gamma(s))} \right)$$

Note again that $\|\tilde{u}\|_{L^2((0,T),H^{-1}(\Gamma(s)))} \le C\|u\|_{L^2_{H^{-1}(\Gamma)}}$, since the multiplication with J_t^s is a globally bounded linear homeomorphism in $H^1(\Gamma(s))$, as stated in the proof os Lemma 3.3.4. The transformation of (3.4.1) into (3.4.2) works both ways, hence the uniqueness of $y \in W_\Gamma$. The norms can be estimated as in Lemma 3.2.2 and Lemma 3.3.4 and the theorem follows. ■

With regard to order-optimal convergence estimates, sometimes a slightly higher regularity than $y \in W_\Gamma$ is required. Assuming $u \in L^2_{L^2(\Gamma)}$, $y_0 \in H^1(\Gamma_0)$, and both Γ_0 and Φ of class C^3, one can apply a Galerkin approximation argument, see [DE07, Thms. 4.4 and 4.5] for manifolds or [Eva98] for open sets, to obtain

$$\|\dot{y}\|^2_{L^2_{L^2(\Gamma)}} + \sup_{t\in[0,T]} \|\nabla_{\Gamma(t)} y\|^2_{L^2(\Gamma(t))} + \int_{[0,T]} \|y\|^2_{H^2(\Gamma(t))} \,\mathrm{d}t \le C \left(\|y\|^2_{H^1(\Gamma(0))} + \|u\|^2_{L^2_{L^2(\Gamma)}} \right) . \quad (3.4.3)$$

Note that from [LM68, Ch. I,Thm. 3.1] it then follows that $\phi_t^s y \in C([0,T], H^1(\Gamma(s)))$.

3.5 Triangulation of the moving surface

We now discretize Γ using an approximation Γ_0^h of Γ_0 which is globally of class $C^{0,1}$ just like in Chapter 2. For the sake of convenience let us assume $n = 2$, i.e. $\Gamma(t)$ is a hypersurface in $\mathbb{R}^3$. Nevertheless, our results hold for $n = 1, 2, 3$.
Following [Dzi88] and [DE07], we consider $\Gamma_0^h = \bigcup_{i\in I_h} T_h^i$ consisting of triangles T_h^i with corners on Γ_0, whose maximum diameter is denoted by h. With FEM error bounds in mind we assume the family of triangulations $\{\Gamma_0^h\}_{h>0}$ to be regular in the usual sense that the angles of all triangles are bounded away from zero uniformly in h.
As detailed in [DE10] and [DE07] an evolving triangulation $\Gamma^h(t)$ of $\Gamma(t)$ is obtained by subjecting the vertices of Γ_0^h to the flow Φ. Hence, the nodes of $\Gamma^h(t)$ reside on $\Gamma(t)$ for all times $t \in [0,T]$, the triangles T_h^i being deformed into triangles $T_h^i(t)$ by the movement of the vertices. Let m_h denote the number of vertices $\{X_j^0\}_{j=1}^{m_h}$ in Γ_0^h. Now $X_j(t)$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t} X_j(t) = V(X_j(t), t) , \quad X_j(0) = X_j^0 . \quad (3.5.1)$$

Consider the finite element space

$$Y_h(t) = \left\{ \varphi \in L^2(\Gamma^h(t)) \;\middle|\; \varphi \in C(\Gamma^h(t)) \text{ and } \forall i \in I_h : \varphi\big|_{T_h^i(t)} \in \Pi^1(T_h^i(t)) \right\}$$

of piecewise linear, globally continuous functions on $\Gamma^h(t)$, and its nodal basis functions $\{\varphi_j(t)\}_{j=1}^{m_h}$ that are one at exactly one vertex $X_i(t)$ of $\Gamma^h(t)$ and zero at all others. While on $\Gamma^h(t)$ the notion of the space H^1 is a little bit more involved than in the smooth case, compare Chapter 2, we can still provide $Y_h(t)$ with an appropriate norm, i.e., for $\varphi \in Y_h(t)$ let

$$\|\varphi\|^2_{Y_h(t)} = \int_{\Gamma^h(t)} \nabla_{\Gamma^h}\varphi \nabla_{\Gamma^h}\varphi + \varphi^2 \,\mathrm{d}\Gamma^h(t) .$$

For the finite element approach, it is crucial for the triangles $T_h^i(t)$ not to degenerate while $\Gamma^h(t)$ evolves, which leads us to the following assumption.

Assumption 3.5.1. The angles of the triangles $T_h^i(t)$ are bounded away from zero, uniformly w.r.t. h, i and t. Also assume $\Pi_{\Gamma(t)}(\Gamma^h(t)) = \Gamma(t)$, with the restriction of $\Pi_{\Gamma(t)}$ to $\Gamma^h(t)$ being a homeomorphism between $\Gamma^h(t)$ and $\Gamma(t)$. Denote by

$$h = h_{max} = \max_{t\in[0,T],i\in I_h} \operatorname{diam}\left(T_h^i(t)\right) \text{ and } h_{min} = \min_{t\in[0,T],i\in I_h} \operatorname{diam}\left(T_h^i(t)\right)$$

the largest and the smallest diameter of all faces of the Γ^h. There holds $\frac{h_{max}}{h_{min}} \leq C$ and $C > 0$.

In order to ensure optimal approximation properties of the discretization of the surface, we require some additional regularity for d.

Assumption 3.5.2. $d \in C^3(\mathcal{N}_\epsilon)$. Let $h > 0$ sufficiently small such that one has $\Gamma^h(t) \subset \mathcal{N}_\epsilon$ for all $t \in [0, T]$.

This can be achieved if Φ and Γ_0 are of class C^3. For the rest of this chapter we assume that Assumption 3.5.1 and 3.5.2 hold.
Let us summarize some basic properties of the family $\{\Gamma^h(t)\}_{t\in[0,T]}$.

Lemma and Definition 3.5.3. *Let $\Phi^s_{\cdot,h} : \Gamma^h(s) \times [0, T] \to \mathbb{R}^3$ denote the flow of Γ^h, i.e. the unique continuous map, such that $\Phi^s_{t,h}(T^i_h(s)) = T^i_h(t)$ and $\Phi^s_{t,h}$ is affine linear on each $T^i_h(s)$. There holds $\Phi^r_{t,h} = \Phi^s_{t,h} \circ \Phi^r_{s,h}$ and thus $\Phi^t_{s,h} \circ \Phi^s_{t,h} = \mathrm{id}_{\Gamma^h(s)}$. The velocity $V_h = \partial_t \Phi^0_{t,h}$ is the piecewise linear interpolant of V on each triangle $T^i_h(t)$.*
As in Lemma 3.2.2 we define the pull-back $\phi^s_{t,h} : L^2(\Gamma^h(t)) \to L^2(\Gamma^h(s))$, $\phi^s_{t,h} v = v \circ \Phi^t_{s,h}$.
The piecewise constant Jacobian $J^s_{t,h}$ of $\Phi^s_{t,h}$ satisfies for all $s, t \in [0, T]$

$$\frac{1}{C_J^h} \leq \min_{\gamma\in\Gamma(s)} J^s_{t,h}(\gamma) \leq \max_{\gamma\in\Gamma(s)} J^s_{t,h}(\gamma) \leq C_J^h\,, \tag{3.5.2}$$

for some constant $C_J^h > 0$ that does not depend on $h > 0$.
Moreover $J^s_{t,h}$ and $D_{\Gamma^h(s)}\Phi^s_{t,h} : T\Gamma^h(s) \to T\Gamma^h(t) \subset \mathbb{R}^3$ are differentiable with respect to time in the interior of each $T^i_h(s)$.
The nodal basis functions have the transport property

$$\dot{\varphi}_i = \phi^t_{0,h} \frac{\mathrm{d}}{\mathrm{d}t} \phi^0_{t,h} \varphi_i \equiv 0\,, \quad 1 \leq i \leq m_h\,. \tag{3.5.3}$$

Let $\nu^h(t)$ denote the normals of $\Gamma^h(t)$, defined on each $T^i_h(t)$.

Proof. Consider a Triangle $T^i_h(s)$, $s \in [0, T]$. W.l.o.g. let $X_1(s), X_2(s), X_3(s)$ denote its vertices. Then, using matrices $X^i(t) = (X_2(t) - X_1(t), X_3(t) - X_1(t))$, we can write $\gamma \in T^i_h(s)$ in reduced barycentric coordinates as $\lambda_\gamma(s) = (X^i(s)^T X^i(s))^{-1} X^i(s)^T (\gamma - X_1(s))$. On $T^i_h(s)$ the transformation $\Phi^s_{t,h}$ is uniquely defined by $\lambda_{\Phi^s_{t,h}\gamma}(t) = \lambda_\gamma(s)$ and thus

$$\Phi^s_{t,h}(\gamma) = X^i(t)(X^i(s)^T X^i(s))^{-1} X^i(s)^T (\gamma - X_1(s)) + X_1(t)\,.$$

In the relative interior of $T_h^i(s)$ the map $\Phi_{t,h}^s : T_h^i(s) \to T_h^i(t)$ is differentiable and its derivative $D_{T_h^i(s)}\Phi_{t,h}^s : \mathbb{R}^3 \supset TT_h^i(s) \to TT_h^i(t) \subset \mathbb{R}^3$ can be represented in terms of the standard basis of $\mathbb{R}^3$ by the matrix $D_{s,t}^i = X^i(t)(X^i(s)^T X^i(s))^{-1} X^i(s)^T$.
A short computation shows that the angle condition in Assumption 3.5.1 ensures the existence of $c > 0$ such that

$$\lambda^T X^i(s)^T X^i(s)\lambda \geq c \min(\|X_2(s) - X_1(s)\|^2, \|X_3(s) - X_1(s)\|^2)\|\lambda\|^2 ,$$

for all $\lambda \in \mathbb{R}^2$, $s \in [0, T]$.
Hence, $\|(X^i(s)^T X^i(s))^{-1}\|_2 \leq \left(c \min(\|X_2(s) - X_1(s)\|^2, \|X_3(s) - X_1(s)\|^2)\right)^{-1}$, and since $\|X^i(t)\|_2^2, \|X^i(s)^T\|_2^2 \leq 2 \max_{\tau\in[0,T]}(\|X_2(\tau) - X_1(\tau)\|^2, \|X_3(\tau) - X_1(\tau)\|^2)$ we get

$$\|D_{T_h^i(s)}\Phi_{t,h}^s\|_2 \leq C\frac{h_{max}}{h_{min}}$$

Using again Assumption 3.5.1 one concludes that the quotient of edge lengths is uniformly bounded.
Also, one easily verifies for $r, t \in [0, T]$

$$\Phi_{t,h}^r\gamma = (\Phi_{t,h}^s \circ \Phi_{s,h}^r)\gamma \text{ and } \Phi_{s,h}^t\Phi_{t,h}^s = \mathrm{id}_{\Gamma^h(s)} . \tag{3.5.4}$$

We have $J_{t,h}^s\Big|_{T_h^i(s)} = \sqrt{\det\left(\mathfrak{B}(s)^T D_{T_h^i(s)}\Phi_{t,h}^s)^T D_{T_h^i(s)}\Phi_{t,h}^s\mathfrak{B}(s)\right)}$ on the triangle $T_h^i(s)$, where the derivative is represented with respect to an orthonormal basis $\mathfrak{B}(s)$ of $TT_h^i(s)$. As per above considerations the spectral radius of $D_{T_h^i(s)}\Phi_{t,h}^s$ is uniformly bounded. Hence, there exists $C_J^h > 0$ such that $J_{t,h}^s \leq C_J^h$. Because we can switch s and t and since by (3.5.4) we have $(\Phi_{t,h}^s)^{-1} = \Phi_{s,h}^t$ and thus $\frac{1}{J_{t,h}^s} = J_{s,h}^t \leq C_J^h$ we conclude

$$\forall s, t \in [0, T] : \; \forall\gamma \in \Gamma_s^h \; \frac{1}{C_J^h} \leq J_{t,h}^s(\gamma) \leq C_J^h .$$

The trajectories $\Phi_{t,h}^s\gamma$, $\gamma \in \Gamma^h(s)$, the Jacobians $J_{t,h}^s$, and the entries of $D_{T_h^i(s)}\Phi_{t,h}^s$ are differentiable for t, because the trajectories $X_j(t)$, $1 \leq j \leq m_h$ are, compare (3.5.1). Hence also $D_{\Gamma^h(s)}\Phi_{t,h}^s$ is differentiable as a map into $\mathbb{R}^3$. The velocity $V_h(\gamma, s) = \partial_t\Phi_{t,h}^s\gamma$ equals V at the vertices and depends linearly on the coordinates λ_γ. As for the transport property (3.5.3), it is a consequence of the piecewise linear transformations of the piecewise linear Ansatz functions φ_i which implies $\phi_{t,h}^0\varphi_i(t) = \varphi_i(0)$, compare [DE07, Prop. 5.4]. ■

In order to compare functions defined on $\Gamma^h(t)$ with functions on $\Gamma(t)$, for sufficiently small $h > 0$ we proceed like in Chapter 2. The projection $\Pi_{\Gamma(t)}$ from (3.0.2) allow us to lift a function $y \in L^2(\Gamma^h(t))$ to $\Gamma(t)$

$$y^l(\Pi_{\Gamma(t)}(x)) = y(x) \quad \forall x \in \Gamma^h(t) ,$$

and for $y \in L^2(\Gamma(t))$ we define the inverse lift

$$y_l(x) = y(\Pi_{\Gamma(t)}(x)) \quad \forall x \in \Gamma^h(t) .$$

For small mesh parameters h the lift operation $(\cdot)_l : L^2(\Gamma(t)) \to L^2(\Gamma^h)$ defines a linear homeomorphism with inverse $(\cdot)^l$. Moreover, there exists $c_{\text{int}} > 0$ such that

$$1 - c_{\text{int}}h^2 \leq \|(\cdot)_l\|^2_{\mathcal{L}(L^2(\Gamma(t)),L^2(\Gamma^h(t)))}, \|(\cdot)^l\|^2_{\mathcal{L}(L^2(\Gamma^h(t)),L^2(\Gamma(t)))} \leq 1 + c_{\text{int}}h^2\,, \tag{3.5.5}$$

as shows the following lemma which generalizes Lemma 3.5.4.

Lemma and Definition 3.5.4. *The restriction of $\Pi_{\Gamma(t)}$ to $\Gamma^h(t)$ is a piecewise diffeomorphism. Denote by δ_h the Jacobian of $\Pi_{\Gamma(t)}|_{\Gamma^h(t)} : \Gamma^h(t) \to \Gamma(t)$, i.e. $\delta_h = \frac{\mathrm{d}\Gamma}{\mathrm{d}\Gamma^h} = |\det(M)|$ where $M \in \mathbb{R}^{2\times 2}$ represents the Derivative $D\Pi_{\Gamma(t)}(\gamma) : T_\gamma\Gamma^h(t) \to T_{\Pi_{\Gamma(t)}(\gamma)}\Gamma(t)$ with respect to arbitrary orthonormal bases of the respective tangential space. For small $h > 0$ there holds*

$$\sup_{t\in[0,T]}\sup_{\Gamma(t)} |1 - \delta_h| \leq Ch^2\,.$$

In particular $\Pi_{\Gamma(t)}|_{\Gamma^h(t)}$ is a diffeomorphism on each triangle $T_h^i(t)$. Now $\frac{1}{\delta_h} = \frac{\mathrm{d}\Gamma^h}{\mathrm{d}\Gamma} = |\det(M^{-1})|$, so that by the change of variable formula

$$\left| \int_{\Gamma^h(t)} v_l \,\mathrm{d}\Gamma^h(t) - \int_{\Gamma(t)} v \,\mathrm{d}\Gamma(t) \right| = \left| \int_{\Gamma(t)} v\frac{1}{\delta_h^l} - v \,\mathrm{d}\Gamma(t) \right| \leq c_{\text{int}}h^2\|v\|_{L^1(\Gamma)}\,.$$

Also there exist constants $C > 0$, independent of $t \in [0,T]$, such that

1. $\sup_{t\in[0,T]} \|\dot{\delta}_h(t)\|_{L^\infty(\Gamma^h(t))} \leq Ch^2$*, where the material derivative is to be understood in the sense of $\Phi^0_{t,h}$ and*
2. $\sup_{t\in[0,T]} \|\mathcal{P}(I - \mathcal{R}_h^l)\mathcal{P}\|_{L^\infty(\Gamma(t))} \leq Ch^2$*, where*

$$\mathcal{R}_h = \frac{1}{\delta_h^l}\left(I - d\mathcal{H}^d\right)\mathcal{P}^h\left(I - d\mathcal{H}^d\right)\,,$$

with the Hessian $\mathcal{H}^d_{ij} = \partial_{x_ix_j}d$, and the orthogonal projection $\mathcal{P} = \{\delta_{ij} - \nu_i\nu_j\}_{i,j=1}^{n+1}$ and $\mathcal{P}^h = \{\delta_{ij} - \nu_i^h\nu_j^h\}_{i,j=1}^{n+1}$ onto the respective tangential space.

The property from Lemma 3.5.4[*2.*] is important when it comes to comparing H^1-norms on $\Gamma(t)$ and $\Gamma^h(t)$.

Proof. We summarize the proof given in [DE07, Lemma 5.1] to extend it for the *1.* assertion. A similar proof can be found in [DE10, Lemma 5.4]. Following [DE07], we use local coordinates on a triangle $e := T_h^i(s)$. W.l.o.g. one can assume $e \in \mathbb{R}^2 \times \{0\}$. Since both d and $\dot{d} = \frac{\mathrm{d}}{\mathrm{d}t}\phi^s_{t,h}d$ equal zero at the corners, the linear interpolates I_hd, $I_h\dot{d}$ vanish on e thus, using standard finite element approximation results, we get

$$\|d\|_{L^\infty(e)} = \|d - I_hd\|_{L^\infty(e)} \leq ch^2\|d\|_{H^{2,\infty}(e)} \leq ch^2\|d\|_{C^{1,1}(\mathcal{N}_\epsilon)}$$

and similarly $\|\dot{d}\|_{L^\infty(e)} \leq ch^2\|d\|_{C^{2,1}(\mathcal{N}_\epsilon)}$. Also one has

$$\|\partial_{x_i}d\|_{L^\infty(e)} \leq ch\|d\|_{C^{1,1}(\mathcal{N}_\epsilon)} \text{ and } \|\partial_{x_i}\dot{d}\|_{L^\infty(e)} \leq ch\|d\|_{C^{2,1}(\mathcal{N}_\epsilon)} \tag{3.5.6}$$

for $i = 1, 2$ at any point $(x_1, x_2, 0) \in e$.
Consider the basis $\mathfrak{B}(t) = \{\partial_{x_1}\Phi^s_{t,h}, \partial_{x_2}\Phi^s_{t,h}, \nu^h(t)\}$ of $\mathbb{R}^3$, whose first two members span the tangential space of $T^i_h(t)$. Let $(\nu_1(t), \nu_2(t), \nu_3(t))^T$ represent $\nu_l(t) = \nabla d(\cdot, t)$ with respect to $\mathfrak{B}(t)$. We have

$$(\nu_1(t), \nu_2(t))^T = M_t^{-1}(D_{(x_1,x_2)}\Phi^s_{t,h})^T \nabla d\,,$$

with the uniformly positive definite matrix $M_t = (D_{(x_1,x_2)}\Phi^s_{t,h})^T D_{(x_1,x_2)}\Phi^s_{t,h}$. Using this relation and the estimates from (3.5.6) we deduce

$$\begin{aligned}\mathcal{O}(h) = D_{(x_1,x_2)}\dot{d} = D_{(x_1,x_2)}\frac{\mathrm{d}}{\mathrm{d}t}\phi^s_{t,h}d =& \frac{\mathrm{d}}{\mathrm{d}t}D_{(x_1,x_2)}\phi^s_{t,h}d = \frac{\mathrm{d}}{\mathrm{d}t}(\phi^s_{t,h}\nabla d^T D_{(x_1,x_2)}\Phi^s_{t,h})\\ =& \frac{\mathrm{d}}{\mathrm{d}t}(\phi^s_{t,h}(\nu_1,\nu_2)M_t) = (\dot{\nu}_1,\dot{\nu}_2)M_t + \underbrace{(\nu_1,\nu_2)\frac{\mathrm{d}}{\mathrm{d}t}M_t}_{\mathcal{O}(h)}\,,\end{aligned}$$

because $\nu_i(\gamma, s) = \partial_{x_i}d(\gamma, s)$. We subsume

$$\|\nu_i\|_{L^\infty(e)},\ \|\dot{\nu}_i\|_{L^\infty(e)} \le ch\|d\|_{C^{2,1}(\mathcal{N}_T)}\,.$$

One has

$$D\Pi_{\Gamma(t)} = \mathrm{Id} - \nabla d(\nabla d)^T - d\nabla^2 d$$

and with $\nabla d(\cdot, s) = (\nu_1(s), \nu_2(s), \nu_3(s))^T$ we compute (see [DE07])

$$\begin{aligned}\delta_h &= \|\partial_{x_1}\Pi_{\Gamma(t)} \times \partial_{x_2}\Pi_{\Gamma(t)}\|\\ &= |\nu_3| + dR(\nu, \partial_{x_1}\nu, \partial_{x_2}\nu) = \sqrt{1 - \nu_1^2 - \nu_2^2} + dR(\nu, \partial_{x_1}\nu, \partial_{x_2}\nu) = 1 + \mathcal{O}(h^2)\end{aligned}$$

with some smooth remainder function R and thus, using $|d|, |\dot{d}| \le Ch^2$ and $|\nu_i|, |\dot{\nu}_i| \le Ch$, $i = 1, 2$ one gets

$$\|\dot{\delta}_h\|_{L^\infty(e)} = \left\|\frac{-\nu_1\dot{\nu}_1 - \nu_2\dot{\nu}_2}{\sqrt{1 - \nu_1^2 - \nu_2^2}} + \mathcal{O}(h^2)\right\|_{L^\infty(e)} \le Ch^2\,,$$

For a proof of *2.* see [DE07, Lemma 5.1]. ■

The next Lemma concerns the continuity of the lift operations between $L^2_{L^2(\Gamma^h)}$ and $L^2_{L^2(\Gamma)}$.

Lemma and Definition 3.5.5. *Using the pull-back $\phi^s_{t,h}$ we can define $L^2_{L^2(\Gamma^h)}$ as in Lemma 3.3.1. For sufficiently small $h > 0$ the lift operation $(\cdot)^l$ constitutes a continuous isomorphism between $L^2_{L^2(\Gamma)}$ and $L^2_{L^2(\Gamma^h)}$ with inverse $(\cdot)_l$. There holds*

$$\left|\langle v_l, w_l\rangle_{L^2_{L^2(\Gamma^h)}} - \langle v, w\rangle_{L^2_{L^2(\Gamma)}}\right| \le Tc_{\mathrm{int}}h^2|\langle v, w\rangle_{L^2_{L^2(\Gamma)}}|\,.$$

Proof. Let $L^2_{L^2(T^i_h)}$, according to the flow $\Phi^s_{t,h}$ as defined in Lemma 3.3.1. We define $L^2_{L^2(\Gamma^h)} = \bigcup_{i\in I_h} L^2_{L^2(T^i_h)}$ with the scalar product $\int_0^T \langle\cdot,\cdot\rangle_{L^2(\Gamma^h(t))}\mathrm{d}t$.

Let $\Psi_t = \Phi_0^t \circ \Pi_{\Gamma(t)} \circ \Phi_{t,h}^0$ denote the mapping between Γ_0^h and Γ_0 induced by the projection $\Pi_{\Gamma(t)}$. By Assumption 3.5.2 and by the construction of Φ_t^0 and $\Phi_{t,h}^0$ is follows that $\Psi_t : \Gamma_0^h \to \Gamma_0$ is a diffeomorphism on each triangle $T_h^i(0)$ and by Assumption 3.5.1 it is also globally one-to-one and onto. Also Ψ_t and its spatial derivatives are continuous w.r.t. time t.
We will show that $\bar\Psi : \Gamma_0^h \times [0,T] \to \Gamma_0 \times [0,T]$, $(\gamma, t) \mapsto (\Psi_t(\gamma), t)$ is a piecewise diffeomorphism whose Jacobian is bounded away from zero. We already have that $\bar\Psi$ is globally one-to-one because Ψ_t is. Together this implies that the pull-back with $\bar\Psi$ constitutes an isomorphism between $L^2(\Gamma_0 \times [0,T])$ and $L^2(\Gamma_0^h \times [0,T])$. This again means that

$$\phi_{t,h}^0 v_l \in L^2([0,T], L^2(\Gamma_0^h)) \Leftrightarrow \phi_t^0 v \in L^2([0,T], L^2(\Gamma_0)) .$$

As to $\bar\Psi$ being al local diffeomorphism, the sets $\bar T_h^i = \bigcup_{t\in[0,T]} T_h^i(t)$ are a partition of $\Gamma_0^h \times [0,T]$. In the interior of each $\bar T_h^i$ the map $\bar\Psi$ is a diffeomorphism. In fact, let $\gamma \in \operatorname{int}(T_h^i)$ for some $1 \le i \le m_h$. Compute

$$D_{\Gamma_0^h \times [0,T]} \bar\Psi(\gamma) = \begin{pmatrix} D_{\Gamma_0^h} \Psi_t(\gamma) & \partial_t \Psi_t(\gamma) \\ 0 & 1 \end{pmatrix} ,$$

in an appropriate (orthogonal) basis. We have $D_{\Gamma_0^h}\Psi_t = D_{\Gamma(t)}\Phi_0^t \circ D_{\Gamma^h(t)}\Pi_{\Gamma(t)} \circ D_{\Gamma_0^h}\Phi_{t,h}^0$. Its Jacobian is the product of the Jacobians J_0^t, δ_h, and $J_{t,h}^0$ that are each bounded away from zero, uniformly in γ and t, compare (3.5.2), and the Lemmas 3.5.4 and 3.2.2. Hence the Jacobian of $\bar\Psi$ is bounded away from zero.
As to continuity of $(\cdot)_l$, by Lemma 3.5.4 we have that

$$\left| \langle v_l, w_l \rangle_{L^2_{L^2(\Gamma^h)}} - \langle v, w \rangle_{L^2_{L^2(\Gamma)}} \right| = \left| \int\limits_{[0,T]} \int\limits_{\Gamma(t)} vw\left(\frac{1}{\delta_h^l} - 1\right) \mathrm{d}\Gamma(t)\, \mathrm{d}t \right| \le T c_{\text{int}} h^2 |\langle v, w \rangle_{L^2_\Gamma}| .$$

■

3.6 Finite element discretization in space

Now, instead of dealing with Problem (3.4.1) directly, w.l.o.g. we consider the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma(t)} y\, \varphi \, \mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} \nabla_\Gamma y \nabla_\Gamma \varphi + \mu y \varphi \, \mathrm{d}\Gamma(t) = \langle \dot\varphi, y \rangle_{L^2(\Gamma(t))} + \langle u, \varphi \rangle_{L^2(\Gamma(t))} , \tag{3.6.1}$$

with $\bar\mu \in \mathbb{R}$ large enough to ensure $\mu := \mathbf{b} + \bar\mu \ge 1$. Note that y solves (3.6.1) iff $e^{\bar\mu t} y$ solves (3.4.1) with right-hand side $e^{\bar\mu t} u$. The restriction on μ is a technical one. All results of this section and the next apply for $\mu \in \mathbb{R}$ as shows Remark 3.7.10.
In order to formulate the space-discretization of (3.6.1), consider the trial space

$$H^1_{Y_h} = \left\{ \sum_{i=1}^{m_h} y_h^i(t) \varphi_i(t) \in L^2_{L^2(\Gamma^h)} \mid y_h^i \in H^1([0,T]) \right\} \simeq H^1([0,T])^{m_h} .$$

The following definition of weak material derivatives for functions in $H^1_{Y_h}$ exploits the fact that $H^1_{Y_h}$ is isomorph to $H^1([0,T])^{m_h}$. It thus avoids the issue of extending the theory from Section 3.4 for the smooth surfaces $\Gamma(t)$ to our Lipschitz approximations $\Gamma^h(t)$.

Lemma and Definition 3.6.1. *The weak material derivative of* $v = \sum_{i=1}^{m_h} \bar{v}_i(t)\varphi_i(t) \in H^1_{Y_h}$ *is* $\dot{v} = \phi^t_{0,h}(\phi^0_{t,h}v)' = \sum_{i=1}^{m_h} \bar{v}_i'(t)\varphi_i(t)$. *Let further* $w \in H^1_{Y_h}$, *then* $\langle v, w\rangle_{L^2(\Gamma^h(t))}$ *is absolutely continuous and*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma^h(t)} vw \,\mathrm{d}\Gamma^h(t) = \int\limits_{\Gamma^h(t)} \dot{v}w + v\dot{w} + vw \ \mathrm{div}_{\Gamma_h} V_h \,\mathrm{d}\Gamma^h(t)\,.$$

Proof. Observe $\dot{v} = \phi^t_{0,h}(\phi^0_{t,h}v)' = \phi^t_{0,h}\left(\sum_{i=1}^{m_h} \bar{v}_i(t)\varphi_i(0)\right)' = \phi^t_{0,h}\left(\sum_{i=1}^{m_h} \bar{v}_i'(t)\varphi_i(0)\right)$ because $(\phi^0_{t,h}\varphi(t))'(\gamma) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi_i(0)(\gamma) = 0$ for all $\gamma \in \Gamma^h_0$, as in (3.5.3).
Apply Lemma 3.2.1 on each triangle to see that $\langle \varphi_i(t), \varphi_j(t)\rangle_{L^2(\Gamma^h(t))}$ is smooth and

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \varphi_i(t), \varphi_j(t)\rangle_{L^2(\Gamma^h(t))} = \int\limits_{\Gamma^h(t)} \varphi_i\varphi_j \ \mathrm{div}_{\Gamma_h} V_h \,\mathrm{d}\Gamma^h(t)\,.$$

Now

$$\langle v, w\rangle_{L^2(\Gamma^h(t))} = \sum_{i,j=1}^{m_h} \bar{v}_i(t)\bar{w}_j(t)\langle \varphi_i(t), \varphi_j(t)\rangle_{L^2(\Gamma^h(t))}$$

and the second assertion follows, since $\bar{v}_i, \bar{w}_j \in H^1([0,T])$, $1 \le i,j \le m_h$. ∎

We approximate (3.6.1) by the following semi-discrete Problem. Consider a piecewise smooth, globally Lipschitz approximation μ_h of μ_l, such that $\mu_h \ge 1$. Find $y \in H^1_{Y_h}$ such that for all $\varphi \in H^1_{Y_h}$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma^h(t)} y_h\,\varphi \,\mathrm{d}\Gamma^h(t) + \int\limits_{\Gamma^h(t)} \nabla_{\Gamma^h} y_h \nabla_{\Gamma^h}\varphi + \mu_h y_h \varphi \,\mathrm{d}\Gamma^h(t) =& \langle \dot{\varphi}, y_h\rangle_{L^2(\Gamma^h(t))} \cdots \\ &+ \langle u_h, \varphi\rangle_{L^2(\Gamma^h(t))}\,,\end{aligned} \tag{3.6.2}$$

and $y_h(0) = y^h_0 \in Y_h(0)$. One possible choice would be $\mu_h = \mu_l$, $u_h = u_l$ and $y^h_0 = P^h_0((y_0)_l)$ with P^h_0 the $L^2(\Gamma^h_0)$-orthogonal projection onto $Y_h(0)$.
First of all let us state that (3.6.2) admits a unique solution in $H^1_{Y_h}$. This is because for $y_h = \sum_{i=1}^{m_h} y^i_h \varphi_i$ we can rewrite (3.6.2) as a smooth linear ODE with non-smooth inhomogeneity for the coefficient vector $\mathbf{y}_h = \{y^i_h\}_{i=1}^{m_h} \in H^1([0,T])^{m_h}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(M(t)\mathbf{y}_h(t)\right) + (A_{\mu_h}(t))\mathbf{y}_h(t) = U(t)\,, \quad y_h(0) = y^h_0\,, \tag{3.6.3}$$

with smooth mass and stiffness matrices

$$M(t) = \{\langle \varphi_i, \varphi_j\rangle_{L^2(\Gamma^h(t))}\}_{i,j=1}^{m_h} \quad \text{and} \quad A_{\mu_h}(t) = \Big\{ \int\limits_{\Gamma^h(t)} \nabla_{\Gamma^h}\varphi_i \nabla_{\Gamma^h}\varphi_j + \mu_h \varphi_i\varphi_j \,\mathrm{d}\Gamma^h(t)\Big\}_{i,j=1}^{m_h}\,,$$

and right-hand side $U(t) = \{\langle u_l, \varphi_i\rangle_{L^2(\Gamma^h(t))}\}_{i=1}^{m_h} \in L^2([0,T], \mathbb{R}^{m_h})$, compare also [DE07]. Observe that we used the continuity of the coefficients $y^i_h \in H^1([0,T])$ as well as $\dot{\varphi}_i = 0$. Existence of a solution $\mathbf{y}_h \in H^1([0,T])^{m_h}$ of (3.6.3) can be argued by variation of constants or, more generally, one can apply an existence result by Carathéodory, compare [CL55, Thms. 1.1+1.3]. Uniqueness of y_h is a consequence of the following lemma.

Lemma 3.6.2 (Stability)**.** *Let $y_0 \in L^2(\Gamma_0)$ and $u \in L^2_{L^2(\Gamma)}$, and let y_h solve* (3.6.2) *with $y_0^h \in Y_h(0)$ and $u_h = u_l$. There exists $C > 0$, such that for sufficiently small $h > 0$ the solution satisfies*

$$\|y_h\|^2_{L^2(\Gamma^h(T))} + \int_0^T \int\limits_{\Gamma^h(t)} (\nabla_{\Gamma^h} y_h)^2 + \mu_h y_h^2 \,\mathrm{d}\Gamma^h(t)\mathrm{d}t \le C \left(\|y_0^h\|^2_{L^2(\Gamma_0^h)} + \|u\|^2_{L^2_{L^2(\Gamma)}} \right) ,$$

as well as

$$\|\dot y_h\|^2_{L^2_{L^2(\Gamma^h)}} + \operatorname*{ess\,sup}_{t\in[0,T]} \int\limits_{\Gamma^h(t)} (\nabla_{\Gamma^h} y_h)^2 + \mu_h y_h^2 \,\mathrm{d}\Gamma^h(t) \le C \left(\|y_0^h\|^2_{Y_h(0)} + \|u\|^2_{L^2_{L^2(\Gamma)}} \right) .$$

Proof. From the definition of M and A_{μ_h} using Lemma 3.2.1 on each triangle $T_h^i(t)$ there follows $M'(t) = \{ \int_{\Gamma^h(t)} \varphi_i\varphi_j \operatorname{div}_{\Gamma^h(t)} V_h \,\mathrm{d}\Gamma^h(t)\}_{i,j=1}^{m_h}$ and

$$(\frac{\mathrm{d}}{\mathrm{d}t} A_{\mu_h})_{ij} = \int\limits_{\Gamma^h(t)} -\nabla_{\Gamma^h}\varphi_i \left(D_{\Gamma^h} V_h + D_{\Gamma^h} V_h^T\right) \nabla_{\Gamma^h}\varphi_j + \dot\mu_h \varphi_i\varphi_j + \dots$$
$$+ (\nabla_{\Gamma^h}\varphi_i \nabla_{\Gamma^h}\varphi_j + \mu_h\varphi_i\varphi_j) \operatorname{div}_{\Gamma^h} V_h \,\mathrm{d}\Gamma^h(t) \,.$$

Multiply (3.6.3) by $\mathbf{y}_h'$ to obtain

$$\overbrace{\mathbf{y}_h' M \mathbf{y}_h'}^{\|\dot y_h\|^2_{L^2(\Gamma^h(t))}} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{y}_h A_{\mu_h} \mathbf{y}_h) = -\mathbf{y}_h' M' \mathbf{y}_h + \frac{1}{2}\mathbf{y}_h A'_{\mu_h} \mathbf{y}_h + U \mathbf{y}_h'$$
$$\le C\Big(\underbrace{\|y_h\|^2_{H^1(\Gamma(t))}}_{\le \mathbf{y}_h A_{\mu_h} \mathbf{y}_h} + \|u_l\|^2_{L^2(\Gamma^h(t))} \Big) + \frac{1}{2}\|\dot y_h\|^2_{L^2(\Gamma^h(t))} ,$$

and a Gronwall argument yields the second estimate. Multiply (3.6.3) by $\mathbf{y}_h$ and proceed similarly to prove the first. ∎

Obviously the material derivative depends on the evolution of the surface, i.e. different derivatives arise according to whether ϕ_t^s or $\phi_{t,h}^s$ is applied to pull back a function to a fixed domain. In order to compare $\dot z_h^l$ with $(\dot z_h)^l$ we need the following lemma.

Lemma 3.6.3. *Let $y = \sum_{i=1}^{m_h} y_h^i \varphi_i \in H^1_{Y_h}$. The lift y^l lies in W_Γ with $\dot y^l \in L^2_{L^2(\Gamma)}$, and for a.e. $t \in [0,T]$ there holds*

$$\left| \dot y^l - (\dot y)^l \right| \le C h^2 \|\nabla_{\Gamma(t)} y^l\|_{\mathbb{R}^{n+1}} ,$$

a.e. on $\Gamma(t)$.

Proof. We start by computing the material derivatives of $\bar\varphi_i(x,t) : \mathcal{N}_\epsilon \to \mathbb{R}$, $\bar\varphi_i(x,t) = \varphi_i^l(\Pi_{\Gamma(t)}(x),t)$, i.e. the constant extension of the trial function φ_i, $1 \le i \le m_h$, along the normal field of $\Gamma(t)$, compare the proof of [DE07, Thm. 6.2]. Observe that φ_i^l is not smooth along the edges of patches $\Pi_{\Gamma(t)}(T_h^j(t))$. However, φ_i^l is smooth in the (relative) interior of all $\Pi_{\Gamma(t)}(T_h^j(t))$.

Differentiate $\bar{\varphi}_i$ at $\gamma \in \text{relint}(T_h^j(t))$ to obtain

$$\begin{aligned}\nabla\bar{\varphi}_i(\gamma,t) &= \nabla\bar{\varphi}_i(\Pi_{\Gamma(t)}(\gamma),t)\left(\text{Id}_{\mathbb{R}^{n+1}} - \nabla d(\gamma,t)\nabla d(\gamma,t)^T - d(\gamma,t)\nabla^2 d(\gamma,t)\right) \\ \partial_t\bar{\varphi}_i(\gamma,t) &= \partial_t\bar{\varphi}(\Pi_{\Gamma(t)}(\gamma),t) + \nabla\bar{\varphi}_i(\Pi_{\Gamma(t)}(\gamma),t)\left(-\partial_t d(\gamma,t)\nabla d(\gamma,t) - d(\gamma,t)\partial_t\nabla d(\gamma,t)\right).\end{aligned} \tag{3.6.4}$$

By construction of $\bar{\varphi}_i$ we have $\nabla\bar{\varphi}_i(\Pi_{\Gamma(t)}(\gamma))\nabla d(\gamma,t) = \nabla_\Gamma\varphi_i^l(\Pi_{\Gamma(t)}(\gamma))\nu(\Pi_{\Gamma(t)}(\gamma),t) = 0$ since $\bar{\varphi}_i$ is constant along orthogonal lines through Γ. Also, from $d(\Phi_t^0(\gamma),t) \equiv 0$ it follows $\partial_t d = -\nabla dV$. The (strong) material derivatives do not depend on the extension $\bar{\varphi}_i$, but only on the values on Γ and Γ^h, respectively. One gets

$$\begin{aligned}\dot{\varphi}_i^l(\Pi_{\Gamma(t)}(\gamma),t) &= \partial_t\bar{\varphi}_i(\Pi_{\Gamma(t)}(\gamma),t) + \nabla\bar{\varphi}_i(\Pi_{\Gamma(t)}(\gamma),t)V(\Pi_{\Gamma(t)}(\gamma),t) \\ \dot{\varphi}_i(\gamma,t) &= \partial_t\bar{\varphi}_i(\gamma,t) + \nabla\bar{\varphi}_i(\gamma,t)V_h(\gamma,t)\,,\end{aligned}$$

which together with (3.6.4) leads us to

$$\dot{\varphi}_i^l = (\dot{\varphi}_i)^l + \left(V - V_h + d((\nabla^2 d)V_h + \partial_t\nabla d)\right)\nabla_{\Gamma(t)}\varphi_i^l\,, \tag{3.6.5}$$

in the relative interior of the patches $\Pi_{\Gamma(t)}(T_h^j(t))$, $j \in I_h$.
In order to prove that the pull-back $\tilde{\varphi} := \phi_t^0\varphi_i^l$ lies in $C^1([0,T],L^2(\Gamma_0)) \cap C([0,T],H^1(\Gamma_0))$ for all $1 \le i \le m_h$ we proceed in four steps.
1. We show that $\tilde{\varphi}$ is globally Lipschitz on $\Gamma_0 \times [0,T]$. Observe, that (3.6.4) implies that all derivatives of $\tilde{\varphi}$ exist and are bounded on the interior of patches $P_h^i(t) = \Phi_0^t(\Pi_{\Gamma(t)}(T_h^i(t)))$. Since $\Psi_t = \Phi_0^t \circ \Pi_{\Gamma(t)} \circ \Phi_{t,h}^0 : \Gamma_0^h \times [0,T] \to \Gamma_0$ smoothly maps the edges of Γ_0^h into Γ_0 the domains $\bigcup_{t\in[0,T]} P_h^i(t) \times \{t\} \subset \Gamma_0 \times [0,T]$ have piecewise C^2-boundaries and fulfill the assumptions on the partition in Lemma 1.3.9. Also, $\tilde{\varphi}$ is continuous and we are in the situation to apply Corollary 1.A.1 to obtain $\tilde{\varphi} \in W^{1,\infty}(\Gamma_0 \times [0,T])$ and $\tilde{\varphi} \in C^{0,1}(\Gamma_0 \times [0,T])$.
2. Now as to the time derivative, fix $\epsilon > 0$ and $t \in (0,T)$. Let $L > 0$ denote the global Lipschitz constant of $\tilde{\varphi}$ on $\Gamma_0 \times [0,T]$ and choose $\eta > 0$ sufficiently small such that

$$\sum_{i\in I_h} \text{meas}(P_h^i(t) \setminus P_{h,\eta}^i(t)) \le \epsilon^2/8L^2\,,$$

where $P_{h,\eta}^i(t) = \left\{\gamma \in P_h^i(t) \;\middle|\; B_\eta(\gamma) \subset P_h^i(t)\right\}$, the balls $B_\eta(\gamma)$ being taken with respect to the metric of Γ_0. Now, as stated above, the patches $P_h^i(t) = \Psi(t)(T_h^i)$ move continuously across Γ_0, and we can choose K sufficiently small such that for all $i \in I_h$ and $k \in (-K,K)$ we have $P_{h,\eta}^i(t) \subset P_h^i(t+k)$. The derivative $\partial_t\tilde{\varphi}(\gamma,t) = \phi_t^0\dot{\varphi}_i^l$ which is defined a.e. on $\Gamma_0 \times [0,T]$ then is continuous on the compact set $\mathcal{K}_\eta = \bigcup_{i\in I_h}\overline{P_{h,\eta}^i(t)} \times [t-K,t+K]$ and we have

$$\begin{aligned}\frac{1}{k^2}\int_{\Gamma_0}(\tilde{\varphi}(t+k) - \tilde{\varphi}(t) - \partial_t\tilde{\varphi}(t)k)^2\,\mathrm{d}\Gamma_0 = \frac{1}{k^2}\sum_{i\in I_h}\Bigg(&\int_{P_{h,\eta}^i}(\tilde{\varphi}(t+k) - \tilde{\varphi}(t) - \partial_t\tilde{\varphi}(t)k)^2\,\mathrm{d}\Gamma_0 \\ &+ \int_{P_h^i\setminus P_{h,\eta}^i}(\tilde{\varphi}(t+k) - \tilde{\varphi}(t) - \partial_t\tilde{\varphi}(t)k)^2\,\mathrm{d}\Gamma_0\Bigg)\,.\end{aligned}$$

Substituting $\tilde{\varphi}(\gamma, t+k) - \tilde{\varphi}(\gamma, t) = \partial_t \varphi(\gamma, t) k + \int_0^1 (\partial_t \varphi(\gamma, t+\tau k) - \partial_t \tilde{\varphi}(\gamma, t)) k \mathrm{d}\tau$ on $P^i_{h,\eta}$ like in the proof of Lemma 3.3.3 we choose k small enough for

$$\sup_{\tau \in [0,1]} \|\partial_t \varphi(t+\tau k) - \partial_t \tilde{\varphi}(t)\|_\infty^2 \leq \frac{\epsilon^2}{2\text{meas}(\Gamma_0)}\,, \tag{3.6.6}$$

which is possible by uniform continuity of $\partial_t \tilde{\varphi}$ on $\mathcal{K}_\eta$. Estimating the second addend by $(2Lk)^2 \sum_{i \in I_h} \text{meas}(P^i_h \setminus P^i_{h,\eta}) \leq \epsilon^2 k^2/2$ yields

$$\limsup_{k \to 0} \frac{1}{k} \|\tilde{\varphi}(t+k) - \tilde{\varphi}(t) - \partial_t \tilde{\varphi}(t) k\|_{L^2(\Gamma_0)} \leq \epsilon\,.$$

for every $\epsilon > 0$. Hence $\tilde{\varphi}$ is differentiable into $L^2(\Gamma_0)$ with derivative $\partial_t \tilde{\varphi}$.
3. Thus in order to show $\tilde{\varphi} \in C^1([0,T], L^2(\Gamma_0))$ it remains to prove that $\partial_t \tilde{\varphi} : [0,T] \to L^2(\Gamma_0)$ is continuous. By (3.6.4) $\partial_t \tilde{\varphi}$ is essentially bounded on $\Gamma_0 \times [0,T]$. Let $M = \|\partial_t \tilde{\varphi}\|_{L^\infty(\Gamma_0 \times [0,T])}$. For $\epsilon > 0$ choose $\eta > 0$ sufficiently small such that $\sum_{i \in I_h} \text{meas}(P^i_h \setminus P^i_{h,\eta}) \leq \epsilon^2/8M^2$. As above, choose $K > 0$ and $\mathcal{K}_\eta$ accordingly. Now, choosing $k > 0$ small enough such that (3.6.6) holds one arrives at

$$\|\partial_t \tilde{\varphi}(t+k) - \partial_t \tilde{\varphi}(t)\|^2_{L^2(\Gamma_0)} = \sum_{i \in I_h} \Bigg(\int_{P^i_{h,\eta}} (\partial_t \tilde{\varphi}(t+k) - \partial_t \tilde{\varphi}(t))^2 \,\mathrm{d}\Gamma_0 \ldots$$
$$+ \int_{P^i_h \setminus P^i_{h,\eta}} (\partial_t \tilde{\varphi}(t+k) - \partial_t \tilde{\varphi}(t))^2 \,\mathrm{d}\Gamma_0 \Bigg) \leq \epsilon^2\,.$$

4. Continuity of $\tilde{\varphi} : [0,T] \to H^1(\Gamma_0)$ follows similarly. In fact, the spatial partial derivatives of $\tilde{\varphi}$ exhibit the same piecewise smooth structure as $\partial_t \tilde{\varphi}$.
Finally, $\tilde{\varphi} = \phi^0_t \varphi^l_i \in C^1([0,T], L^2(\Gamma_0)) \cap C([0,T], H^1(\Gamma_0))$ implies $y^i_h \phi^0_t \varphi^l_i \in W_0(0,T)$, and we conclude $y^l \in W_\Gamma$ as well as $\dot{y} \in L^2_{L^2(\Gamma)}$. The estimate now is a consequence of (3.6.5). ■

3.6.1 Convergence results

Before we proceed to the main result of this section, let us summarize some properties of the approximation of elliptic equations on $\Gamma(t)$ by finite elements on $\Gamma^h(t)$. The discretization of the elliptic problem was investigated in detail in [Dzi88]. There, the following interpolation estimate for continuous functions was proved for the H^1-norm; a slight modification also yields an estimate on the approximation error in the L^2-norm.

Lemma and Definition 3.6.4 (Interpolation, [Dzi88, Lem. 5])**.** *Let $I_h : C(\Gamma(t)) \to (Y_h(t))^l$ denote the nodal interpolation. For a continuous function $f \in C(\Gamma(t))$ the interpolant $I_h f$ is uniquely defined by $I_h f(X_j(t)) = f(X_j(t))$, for $1 \leq j \leq m_h$ and $f_l \in Y_h(t)$. There holds*

$$\|I_h f - f\|_{L^2(\Gamma(t))} + h \|I_h f - f\|_{H^1(\Gamma(t))} \leq C h^2 \|f\|_{H^2(\Gamma(t))}\,.$$

This is where the restriction to $n = 1, 2, 3$ arises in order to ensure the embedding $H^2(\Gamma(t)) \subset C(\Gamma(t))$. Also, hidden in the proof of the Interpolation Lemma 3.6.4 there lies the reason for

our regularity Assumption 3.5.2, since a uniform bound for the third spatial derivatives of d is used.
The next Lemma is a refined version of the convergence estimate 2.1.2 which, in particular, states that the approximation is uniform with respect to time. It was proved in [Dzi88]. We included a version of the proof which emphasizes the time-independence of the constants.

Lemma 3.6.5. *For $t \in [0,T]$ and $y \in L^2(\Gamma(t))$, $y_h \in L^2(\Gamma^h(t))$ consider*

$$a(z,\varphi;t) := \int_{\Gamma(t)} \nabla_\Gamma z \nabla_\Gamma \varphi + \mu z \varphi \, \mathrm{d}\Gamma(t) = \langle y, \varphi \rangle_{L^2(\Gamma(t))}\,, \quad \forall \varphi \in H^1(\Gamma(t)) \tag{3.6.7}$$

and

$$a_h(z_h,\varphi;t) := \int_{\Gamma^h(t)} \nabla_{\Gamma^h} z_h \nabla_{\Gamma^h} \varphi + \mu_l z_h \varphi \, \mathrm{d}\Gamma^h(t) = \langle y_h, \varphi \rangle_{L^2(\Gamma^h(t))}\,, \quad \forall \varphi \in Y_h(t) \tag{3.6.8}$$

with unique solutions $z \in H^1(\Gamma(t))$ and $z_h \in Y_h(t)$. The solution operators $S(t) : L^2(\Gamma(t)) \to L^2(\Gamma(t))$, $y \mapsto z$ and $S_h(t) : L^2(\Gamma^h(t)) \to V_h \subset L^2(\Gamma^h(t))$, $y_h \mapsto z_h$ are self-adjoint. There exists C independent of $t \in [0,T]$ such that

1. $\forall \varphi, \psi \in Y_h(t) : \, |\langle \varphi^l, \psi^l \rangle_{H^1(\Gamma(t))} - \langle \varphi, \psi \rangle_{Y_h(t)}| \le Ch^2 \|\varphi^l\|_{H^1(\Gamma(t))} \|\psi^l\|_{H^1(\Gamma(t))} < \infty$ *as well as*
2. $\|(\cdot)^l S_h(t) (\cdot)^{l*} - S(t)\|_{\mathcal{L}(L^2(\Gamma(t)),L^2(\Gamma(t)))} \le Ch^2$ *and*
3. $\|(\cdot)^l S_h(t) (\cdot)^{l*} - S(t)\|_{\mathcal{L}(L^2(\Gamma(t)),H^1(\Gamma(t)))} \le Ch.$

Proof. The operators being well-defined and self-adjoint follows by standard arguments. Since φ^l and ψ^l are continuous and piecewise smooth on $\Gamma(t)$ and thus lie in $H^1(\Gamma(t))$ (compare Lemma 1.3.9) with

$$a_h(\varphi,\psi;t) = a(\varphi^l,\psi^l;t) + \int_{\Gamma(t)} \nabla_\Gamma \varphi^l \left(\mathcal{R}_h^l - \mathrm{Id}\right) \nabla_\Gamma \psi^l + \mu \varphi^l (\frac{1}{\delta_h^l} - 1) \psi^l \, \mathrm{d}\Gamma(t)\,. \tag{3.6.9}$$

Assertion *1.* now follows from Lemma 3.5.4[*2.*] and by setting $\mu \equiv 1$. In order to prove *3.* consider the intermediate quantity $\tilde{z}_h \in (Y_h(t))^l \subset H^1(\Gamma(t))$ which is the unique solution of

$$a(\tilde{z}_h,\varphi^l;t) = \langle y, \varphi^l \rangle_{L^2(\Gamma(t))}\,, \quad \forall \varphi \in Y_h(t)\,, \tag{3.6.10}$$

which in [DE10] is referred to as Riesz projection of z. We split the error into two components, the dominant part $\|\tilde{z}_h - z\|_{H^1(\Gamma(t))}$ and the asymptotically minor $\|z_h^l - \tilde{z}_h\|_{H^1(\Gamma(t))}$.
For (3.6.10) there holds Cea's Lemma, and using the interpolation Lemma 3.6.4 and the uniform stability Lemma 1.B.1 we get

$$\begin{aligned}\|\tilde{z}_h - z\|_{H^1(\Gamma(t))}^2 &\le a(\tilde{z}_h - z, \tilde{z}_h - z; t) = a(\tilde{z}_h - z, I_h z - z; t) \\ &\le Ch \|\tilde{z}_h - z\|_{H^1(\Gamma(t))} \|z\|_{H^2(\Gamma(t))} \le Ch \|\tilde{z}_h - z\|_{H^1(\Gamma(t))} \|y\|_{L^2(\Gamma(t))}\,.\end{aligned}$$

Now set $y_h = \delta_h y_l = (\cdot)^{l^*} y$ and test (3.6.10) and (3.6.8) with $(\tilde{z}_h)_l - z_h$. Subtract both to get

$$a(\tilde{z}_h - z_h^l, \tilde{z}_h - z_h^l; t) + a(z_h^l, \tilde{z}_h - z_h^l; t) - a_h(z_h, (\tilde{z}_h)_l - z_h; t) = 0$$

which in view of (3.6.9) and Lemma 1.B.1 gives

$$\|z_h^l - \tilde{z}_h\|_{H^1(\Gamma(t))} \leq Ch^2 \|z_h^l\|_{H^1(\Gamma(t))} \leq Ch^2 \|y\|_{L^2(\Gamma(t))}.$$

Assertion *2.* is proved by a duality argument. For this let $w \in H^2(\Gamma(t))$ solve

$$a(\varphi, w; t) = \langle z_h^l - z, \varphi \rangle_{L^2(\Gamma(t))}, \quad \forall \varphi \in H^1(\Gamma(t)),$$

and estimate

$$\begin{aligned}
\|z_h^l - z\|^2_{L^2(\Gamma(t))} &= a(z_h^l - z, w; t) = a(z_h^l - z, w; t) + a(z, I_h w; t) - a_h(z_h, (I_h w)_l; t) \\
&= a(z_h^l - z, w - I_h w; t) + a(z_h^l, I_h w; t) - a_h(z_h, (I_h w)_l; t) \\
&\leq C\|z_h^l - z\|_{H^1(\Gamma(t))} \|w - I_h w\|_{H^1(\Gamma(t))} + Ch^2 \|z_h^l\|_{H^1(\Gamma(t))} \|I_h w\|_{H^1(\Gamma(t))} \\
&\leq Ch^2 \|z\|_{H^2(\Gamma(t))} \|w\|_{H^2(\Gamma(t))} + Ch^2 \|z_h^l\|_{H^1(\Gamma(t))} \|w\|_{H^2(\Gamma(t))}
\end{aligned}$$

Finally, using Lemma 1.B.1 again we end up with $\|w\|_{H^2(\Gamma(t))} \leq C\|z_h^l - z\|_{L^2(\Gamma(t))}$ and $\|z\|_{H^2(\Gamma(t))} \leq C\|y\|_{L^2(\Gamma(t))}$. Also one has $\|z_h\|_{Y_h(t)} \leq \|y_h\|_{L^2(\Gamma^h(t))} = \|y\|_{L^2(\Gamma(t))}$. Using (3.6.9) again we conclude $\|z_h^l\|_{H^1(\Gamma(t))} \leq C\|y\|_{L^2(\Gamma(t))}$ and the Lemma is proved. ■

Theorem 3.6.6. *Let Assumption 3.0.5, 3.5.1 and 3.5.2 hold and let $y \in W_\Gamma$ solve (3.6.1) for some $u \in L^2_{L^2(\Gamma)}$, $y_0 \in H^1(\Gamma_0)$, such that (3.4.3) holds. Let y_h solve (3.6.2) with $\mu_h = \mu_l$ and $u_h = u_l$ and some approximation y_0^h of $(y_0)_l$. There exists $C > 0$ independent of y and h such that*

$$\|y_h^l - y\|^2_{L^2_{L^2(\Gamma)}} \leq C\left(\|y_h^l(0) - y(0)\|^2_{H^{-1}(\Gamma_0)} + h^4\left(\|y_0\|^2_{H^1(\Gamma_0)} + \|y_0^h\|^2_{Y_h(0)} + \|u\|^2_{L^2_{L^2(\Gamma)}}\right)\right).$$

Proof. Define $z = S(t)\left(y_h^l - y\right)$ and $z_h = S_h(t)\left(\delta_h\left(y_h - y_l\right)\right)$ with $S(t)$ and $S_h(t)$ as in Lemma 3.6.5. Now $\delta_h\left(y_h - y_l\right) = (\cdot)^{l^*}\left(y_h^l - y\right)$ and hence it follows from Lemma 3.6.5[*2.*] that

$$\|z_h^l - z\|_{L^2(\Gamma(t))} = \|((\cdot)^l S_h (\cdot)^{l^*} - S)(y_h^l - y)\|_{L^2(\Gamma(t))} \leq Ch^2 \|y_h^l - y\|_{L^2(\Gamma(t))}, \tag{3.6.11}$$

Observe now for $z_h = \sum_{i=1}^{m_h} \bar{z}_i \varphi_i$ using Lemma 3.6.3 we get

$$Y = \{\langle y_h^l - y, \varphi_i^l \rangle_{L^2(\Gamma(t))}\}_{i=1}^{m_h} \in H^1([0,T])^{m_h}, \text{ and thus } \bar{z} = (A_{\mu_h})^{-1} Y \in H^1([0,T])^{m_h}.$$

Hence $z_h \in H^1_{Y_h}$ and again by Lemma one has 3.6.3 $z_h^l \in W_\Gamma$ as well as $\dot{z}_h^l(t) \in L^2(\Gamma(t))$. We can now test (3.6.1) with z_h^l, using (3.6.7) in the process, to obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle y, z_h^l \rangle_{L^2(\Gamma(t))} + \langle y, y_h^l - y \rangle_{L^2(\Gamma(t))} = \langle \dot{z_h^l}, y \rangle_{L^2(\Gamma(t))} + \langle u, z_h^l \rangle_{L^2(\Gamma(t))} \cdots \\
+ \langle -\Delta_\Gamma y + \mu y, z - z_h^l \rangle_{L^2(\Gamma(t))},
\end{aligned} \tag{3.6.12}$$

and testing (3.6.2) with z_h gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y_h, z_h \rangle_{L^2(\Gamma^h(t))} + \langle y_h^l, y_h^l - y \rangle_{L^2(\Gamma(t))} = \langle \dot{z}_h, y_h \rangle_{L^2(\Gamma^h(t))} + \langle u_l, z_h \rangle_{L^2(\Gamma^h(t))}. \tag{3.6.13}$$

Now, since the strong material derivative $\dot{\delta}_h$ exists and is continuous on each triangle $T_h^i(t)$, the scalar products $\langle \varphi_i, \varphi_j \delta_h \rangle_{L^2(\Gamma^h(t))}$, $1 \le i,j \le m_h$, are differentiable with

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \varphi_i, \varphi_j \delta_h \rangle_{L^2(\Gamma^h(t))} = \int\limits_{\Gamma^h(t)} \delta_h \varphi_i \varphi_j \ \mathrm{div}_{\Gamma^h} V_h + \dot{\delta}_h \varphi_i \varphi_j \ \mathrm{d}\Gamma^h(t)$$

and we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle y_h^l, z_h^l \rangle_{L^2(\Gamma(t))} =& \frac{\mathrm{d}}{\mathrm{d}t} \langle y_h, z_h \delta_h \rangle_{L^2(\Gamma^h(t))} \\ =& \frac{\mathrm{d}}{\mathrm{d}t} \langle y_h, z_h \rangle_{L^2(\Gamma^h(t))} + \langle y_h, \dot{z}_h(\delta_h - 1) \rangle_{L^2(\Gamma^h(t))} + \langle y_h, z_h \dot{\delta}_h \rangle_{L^2(\Gamma^h(t))} \cdots \\ &+ \langle \dot{y}_h, z_h(\delta_h - 1) \rangle_{L^2(\Gamma^h(t))} + \langle y_h, z_h \ \mathrm{div}_{\Gamma^h} V_h(\delta_h - 1) \rangle_{L^2(\Gamma^h(t))} \,. \end{aligned}$$

Hence, we can rewrite (3.6.13) by means of the $L^2(\Gamma(t))$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle y_h^l, z_h^l \rangle_{L^2(\Gamma(t))} + \langle y_h^l, y_h^l - y \rangle_{L^2(\Gamma(t))} = \langle (\dot{z}_h)^l, y_h^l \rangle_{L^2(\Gamma(t))} + \langle u, z_h^l \rangle_{L^2(\Gamma(t))} + R^h \,, \tag{3.6.14}$$

with

$$\begin{aligned} R^h =& \langle y_h, z_h \dot{\delta}_h \rangle_{L^2(\Gamma^h(t))} + \langle \dot{y}_h, z_h(\delta_h - 1) \rangle_{L^2(\Gamma^h(t))} + \langle y_h, z_h \ \mathrm{div}_{\Gamma^h} V_h(\delta_h - 1) \rangle_{L^2(\Gamma^h(t))} \cdots \\ &+ \langle u_l, z_h(1 - \delta_h) \rangle_{L^2(\Gamma^h(t))} \,. \end{aligned}$$

Subtracting (3.6.12) from (3.6.14) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle y_h^l - y, z_h^l \rangle_{L^2(\Gamma(t))} + \|y_h^l - y\|^2_{L^2(\Gamma(t))} = \langle (\dot{z}_h)^l - \dot{z}_h^l, y \rangle_{L^2(\Gamma(t))} + \langle \dot{z}_h, (y_h - y_l)\delta_h \rangle_{L^2(\Gamma^h(t))} \cdots \\ + R^h + \langle -\Delta_\Gamma y + \mu y, z_h^l - z \rangle_{L^2(\Gamma(t))} \,. \end{aligned}$$

From (3.6.8) we know $\langle \dot{z}_h, \delta_h(y_h - y_l) \rangle_{L^2(\Gamma^h(t))} = \mathbf{z}_h' A_{\mu_h} \mathbf{z}_h = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{z}_h A_{\mu_h} \mathbf{z}_h) - \frac{1}{2} \mathbf{z}_h A_{\mu_h}'(t) \mathbf{z}_h$, in the notation of (3.6.3). Now, using (3.6.11) and

$$|R^h| \le C h^2 \|z_h\|_{L^2(\Gamma^h(t))} \Big(\|y_h\|_{L^2(\Gamma^h(t))} + \|\dot{y}_h\|_{L^2(\Gamma^h(t))} + \|u_l\|_{L^2(\Gamma^h(t))} \Big)$$

we can estimate

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (\mathbf{z}_h A_{\mu_h}(t) \mathbf{z}_h) + \|y_h^l - y\|^2_{L^2(\Gamma(t))} \le& C \Big(h^2 \|y\|_{L^2(\Gamma(t))} \|\nabla_{\Gamma^h(t)} z_h\|_{(L^2(\Gamma^h(t)))^n} \cdots \\ &+ \|z_h\|^2_{Y_h(t)} + h^2 \|y\|_{H^2(\Gamma(t))} \|y_h - y_l\|_{L^2(\Gamma^h(t))} \Big) + |R^h| \\ \le& \frac{1}{2} \|y_h - y_l\|^2_{L^2(\Gamma^h(t))} + C \Bigg(\mathbf{z}_h A_{\mu_h}(t) \mathbf{z}_h \cdots \\ &+ h^4 \Big(\|y_h\|^2_{L^2(\Gamma^h(t))} + \|\dot{y}_h\|^2_{L^2(\Gamma^h(t))} + \|u_l\|^2_{L^2(\Gamma^h(t))} + \|y\|^2_{H^2(\Gamma(t))} \Big) \Bigg) \,. \end{aligned}$$

We can now apply Gronwall's lemma for

$$\begin{aligned} &[\mathbf{z}_h A_{\mu_h}(t) \mathbf{z}_h]_0^T + \int\limits_{[0,T]} \|y_h^l - y\|^2_{L^2(\Gamma(t))} \, \mathrm{d}t \\ &\le C h^4 \int\limits_{[0,T]} \|y_h\|^2_{L^2(\Gamma^h(t))} + \|\dot{y}_h\|^2_{L^2(\Gamma^h(t))} + \|u_l\|^2_{L^2(\Gamma^h(t))} + \|y\|^2_{H^2(\Gamma(t))} \, \mathrm{d}t \,, \end{aligned} \tag{3.6.15}$$

and with the stability estimate (3.4.3) and the Lemmas 3.6.2 and 3.5.5 we finally arrive at

$$\int_{[0,T]} \|y_h^l - y\|^2_{L^2(\Gamma(t))}\,\mathrm{d}t \leq C\Bigg(\overbrace{\int_{\Gamma_0^h} \left(\nabla_{\Gamma_0^h} z_h\right)^2 + \mu_h z_h^2\,\mathrm{d}\Gamma_0^h}^{=\langle y_h^l(0)-y(0), z_h^l\rangle_{L^2(\Gamma_0)}} \dots \tag{3.6.16}$$
$$+ h^4\left(\|y_0\|^2_{H^1(\Gamma_0)} + \|y_0^h\|^2_{Y_h(0)} + \|u\|^2_{L^2_{L^2(\Gamma)}}\right)\Bigg).$$

Apply again (3.6.11) to prove the lemma. ■

Remark 3.6.7. Depending on the regularity of y_0, possible choices of y_0^h yielding $\mathcal{O}(h^2)$-convergence of y_h^l comprehend the interpolation $(I_h y_0)_l$, the Ritz projection or either the $L^2(\Gamma_0^h)$-orthogonal or the $L^2(\Gamma_0)$-orthogonal projection of $(y_0)_l$ onto $Y_h(0)$. For the latter, the term involving z_h in (3.6.16) vanishes completely, but it's $H^1(\Gamma_0)$-stability requires further investigation.

The order of convergence is lower, if the solution of (3.6.1) does not satisfy the additional regularity estimate (3.4.3).

Theorem 3.6.8. *Let Assumption 3.0.5, 3.5.1 and 3.5.2 hold and let $y \in W_\Gamma$ solve* (3.6.1) *for $u \equiv 0$, and $y_0 \in L^2(\Gamma_0)$. There exists $C > 0$ independent of y and h such that for the solution y_h of* (3.6.2) *with $y_0^h = P_0^h((y_0)_l)$ and $u_h \equiv 0$ there holds*

$$\|y_h^l - y\|^2_{L^2_{L^2(\Gamma)}} \leq C\left(h^2 + \sup_{t\in[0,T]} \|{\mu_h}^l - \mu\|^2_{L^\infty(\Gamma(t))}\right)\|y_0\|^2_{L^2(\Gamma_0)}.$$

Proof. We proceed as in the proof of Theorem 3.6.6 up to (3.6.12) which now reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y, z_h^l\rangle_{L^2(\Gamma(t))} + \langle y, y_h^l - y\rangle_{L^2(\Gamma(t))} = \langle \dot{z}_h^l, y\rangle_{L^2(\Gamma(t))} + \langle -\Delta_\Gamma y + \mu y, z - z_h^l\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))},$$

Analogously to (3.6.11) we can apply Lemma 3.6.5[*3.*] and estimate the last term through

$$\begin{aligned}|\langle -\Delta_\Gamma y + \mu y, z_h^l - z\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))}| &\leq \|-\Delta_\Gamma y + \mu y\|_{H^{-1}(\Gamma(t))}\|z_h^l - z\|_{H^1(\Gamma(t))} \dots \\ &\leq \|-\Delta_\Gamma y + \mu y\|_{H^{-1}(\Gamma(t))} Ch\|y_h^l - y\|_{L^2(\Gamma(t))}.\end{aligned}$$

On the other hand (3.6.14) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y_h^l, z_h^l\rangle_{L^2(\Gamma(t))} + \langle y_h^l, y_h^l - y\rangle_{L^2(\Gamma(t))} = \langle (\dot{z}_h)^l, y_h^l\rangle_{L^2(\Gamma(t))} + \langle (\mu_l - \mu_h) y_h, z_h\rangle_{L^2(\Gamma^h(t))} + R^h.$$

Continue as in the proof of Theorem 3.6.6 to finally arrive at the analogue of (3.6.15)

$$[\mathbf{z}_h A_{\mu_h}(t)\mathbf{z}_h]_0^T + \int_{[0,T]} \|y_h^l - y\|^2_{L^2(\Gamma(t))}\,\mathrm{d}t \leq Ch^2 \int_{[0,T]} \|y\|^2_{H^1(\Gamma(t))} + h^2\|\dot{y}_h\|^2_{L^2(\Gamma^h(t))}\,\mathrm{d}t \dots$$
$$+ C(h^4 + \sup_{t\in[0,T]} \|{\mu_h}^l - \mu\|^2_{L^\infty(\Gamma(t))}) \int_{[0,T]} \|y_h\|^2_{L^2(\Gamma^h(t))}\,\mathrm{d}t,$$

Note that due to Lemma 3.5.4

$$|\mathbf{z}_h(0)A_{\mu_h}(0)\mathbf{z}_h(0)| = |\langle y_h^l(0) - y(0), z_h^l\rangle_{L^2(\Gamma_0)}| \leq \overbrace{|\langle y_h(0) - y_l(0), z_h\rangle_{L^2(\Gamma_0^h)}|}^{=0 \text{ since } y_0^h = P_0^h((y_0)_l)} + Ch^2\|y_0\|^2_{L^2(\Gamma_0)}\,.$$

In view of Lemma 3.6.2 it remains to bound $\int_0^T h^2\|\dot{y}_h\|^2_{L^2(\Gamma^h(t))}\mathrm{d}t$. Again thanks to Lemma 3.6.2 we have

$$\int_0^T \|\dot{y}_h\|^2_{L^2(\Gamma^h(t))}\mathrm{d}t \leq C\|y_0^h\|^2_{Y_h(0)}\,.$$

But an inverse estimate, compare for example [CL91, Thm. 17.2], yields $\|y_0^h\|_{Y_h(0)} \leq \frac{C}{h}\|y_0^h\|_{L^2(\Gamma_0^h)}$, and because of the continuity of the lift $(\cdot)_l$ and of the L^2-projection P_0^h the theorem follows. ■

3.7 Implicit Euler dG discretization in time

In order to solve (3.4.1) we apply a vertical method of lines. Accordingly, having discretized the space-dependency we now further discretize the semi-discrete scheme (3.6.2) with respect to time. For the time-discretization we choose a discontinuous Galerkin (dG) - implicit Euler approach in $L^2_{L^2(\Gamma^h)}$.

Consider an equidistant partition $I_m = (t_{m-1}, t_m]$, $1 \leq m \leq M$, of $(0,T]$ with $M \in \mathbb{N}$, $k = \frac{T}{M}$, and $t_m = mk$. The trial space for the discontinuous Galerkin method is the space of functions that are piecewise constant in the following sense

$$W_k^h = \left\{ v \in L^2_{L^2(\Gamma^h)} \;\middle|\; \forall 1 \leq m \leq M :\; \exists v^m \in Y_h(t_m) :\; v \equiv \phi^t_{t_m,h} v^m \text{ on } I_m \right\}\,.$$

In the following we will omit the operators $\phi^s_{t,h}$ when dealing with functions $w \in W_k^h$. Also, to further simplify notation let again

$$a_h(\psi,\varphi;t) = \int\limits_{\Gamma^h(t)} \nabla_{\Gamma^h}\psi\nabla_{\Gamma^h}\varphi + \mu_h\psi\varphi\,\mathrm{d}\Gamma^h(t)$$

as well as $\langle\cdot,\cdot\rangle_m = \langle\cdot,\cdot\rangle_{L^2(\Gamma^h(t_m))}$.

To motivate the dG method insert the Ansatz

$$y_{h,k}(t) = \sum_{m=1}^{M} \phi^t_{t_m,h} y^m_{h,k} \mathbb{1}_{I_m} \in W_k^h$$

with $y^m_{h,k} \in Y_h(t_m)$ into (3.6.2).

If one understands the time-derivative in (3.6.2) in a distributional sense, integration over time formally yields

$$\langle y^m_{h,k} - y^{m-1}_{h,k}, \varphi\rangle_{m-1} + \int\limits_{I_m} a_h(y^m_{h,k},\varphi;t) + \langle y^m_{h,k}\,\mathrm{div}_{\,\Gamma^h} V_h, \varphi\rangle_{L^2(\Gamma^h(t))}\,\mathrm{d}t = \int\limits_{I_m} \langle u_h,\varphi\rangle_{L^2(\Gamma^h(t))}\,\mathrm{d}t\,,$$

for smooth test functions φ. Instead, apply test functions $\varphi \in W_k^h$ and use $\dot{y}^m = \dot{\varphi}^m = 0$ to obtain

$$\int_{I_m} \langle y^m \operatorname{div}_{\Gamma^h} V_h, \varphi \rangle_{L^2(\Gamma^h(t))} \,\mathrm{d}t = \langle y^m, \varphi^m \rangle_m - \langle y^m, \varphi^m \rangle_{m-1} .$$

Finally, to arrive at a computable scheme, lump the Integral over $a_h(\cdot, \cdot; t)$ and replace the right-hand side appropriately. For arbitrary parameters $y_0^h \in Y_h(0)$ and $u_h \in L^2_{L^2(\Gamma^h)}$ we rewrite the scheme for $y_{h,k} \in W_k^h$ as

$$\begin{aligned} &y_{h,k}^0 = y_0^h , \quad \forall \varphi \in W_k^h , \ 1 \le m \le M : \\ &\qquad \langle y_{h,k}^m, \varphi^m \rangle_m - \langle y_{h,k}^{m-1}, \varphi^m \rangle_{m-1} + k\mathfrak{a}_m(y_{h,k}^m, \varphi^m) = \int_{I_m} \langle \phi_{t,h}^{t_m} u_h, \varphi^m \rangle_m \,\mathrm{d}t , \end{aligned} \tag{3.7.1}$$

where y_0^h, u_h, and μ_h are the same as in (3.6.2). For the approximation of the integral $\mathfrak{a}_m$ we assume $\mathfrak{a}_m(\psi, \varphi) = a_h(\phi_{t,h}^{t_m}\psi, \phi_{t,h}^{t_m}\varphi; t_m) + \mathfrak{r}_m(\psi, \varphi)$, with a remainder term

$$|\mathfrak{r}_m(\psi, \varphi)| \le C_{\mathfrak{r}} k \|\psi\|_{Y_h(t_m)} \|\varphi\|_{Y_h(t_m)} . \tag{3.7.2}$$

One possible choice is $\mathfrak{r}_m \equiv 0$ for $1 \le m \le M$, but we will want to choose $\mathfrak{r}$ more freely, see Remark 3.7.10.

In order to proof convergence of the scheme (3.7.1) in $L^2_{L^2(\Gamma^h)}$ we make use of stability properties of the adjoint scheme for $z \in W_k^h$

$$\begin{aligned} &z^{M+1} = z_T , \quad \forall \varphi \in W_k^h , \ 1 \le m \le M : \\ &\qquad \langle z^m, \varphi^m \rangle_m - \langle z^{m+1}, \varphi^m \rangle_m + k\mathfrak{a}_m(\varphi^m, z^m) = \int_{I_m} \langle \phi_{t,h}^{t_m} v_h, \varphi^m \rangle_m \,\mathrm{d}t . \end{aligned} \tag{3.7.3}$$

with $v_h \in L^2_{L^2(\Gamma^h)}$, $z_T \in Y_h(T)$. In Section 4.2 it will be important that given snapshots $\{\Gamma^h(t_m)\}_{m=1}^M$ of the surface both (3.7.1) and (3.7.3) can be evaluated without any further discretization errors for certain right-hand sides u_h and v_h, e.g. $v_h \in W_k^h$.

Let us introduce the mean value of a function $y \in L^2_{L^2(\Gamma^h)}$ over an interval I_m.

Lemma and Definition 3.7.1. *Let $\phi_{t,h}^s$ denote the pullback operator associated with the flow $\Phi_{t,h}^s$ as in Lemma 3.2.2 and let $s \in [0, T]$. The mean value of a function $y \in L^2_{L^2(\Gamma^h)}$ is defined as $\bar{y}^m(s) = \frac{1}{k} \int_{I_m} \phi_{t,h}^s y \,\mathrm{d}t$ for $s \in I_m$.*

The mean value $\bar{y}^m$ does not depend on $s \in I_m$ in the sense that

$$\bar{y}^m(s) = \int_{I_m} \phi_{t,h}^s y \,\mathrm{d}t = \int_{I_m} \phi_{r,h}^s \phi_{t,h}^r y \,\mathrm{d}t = \phi_{r,h}^s \int_{I_m} \phi_{t,h}^r y \,\mathrm{d}t = \phi_{r,h}^s \bar{y}^m(r) .$$

Similarly one could define the mean value of $y \in L^2_{L^2(\Gamma)}$ if one were to investigate a horizontal method-of-lines approach.

Now for $y_0 \equiv 0$, $z_T \equiv 0$ the schemes are adjoint in the sense

$$k \sum_{m=1}^M \langle \bar{u}_h^m, z(v_h) \rangle_m = k \sum_{m=1}^M \langle \bar{v}_h^m, y_{h,k}(u_h) \rangle_m ,$$

i.e., the discrete solution operators $u_h \mapsto y_{h,k}(u_h)$ and $v_h \mapsto z(v_h)$ are adjoint as operators from $(L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k})$ into itself, where $L^2_{L^2(\Gamma^h)}$ is equipped with the scalar product

$$\langle v, w\rangle_{h,k} = k \sum_{m=1}^{M} \int_{I_m} \langle(\phi^{t_m}_{t,h} v), (\phi^{t_m}_{t,h} w)\rangle_m \, \mathrm{d}t \,. \tag{3.7.4}$$

The discretization of the scalar product is important as to the implementable adjoint scheme (3.7.3). The adjoint with respect to the $L^2_{L^2(\Gamma)}$-scalar product looks like (3.7.3) but with the slightly different right-hand side $\int_{I_m} \langle v_h, \varphi^m\rangle_{L^2(\Gamma(t))} \mathrm{d}t$ which is not, in general, computable even for very simple functions v_h such as $v_h \in W^h_k$.

3.7.1 Convergence results

We now proceed to prove convergence of the scheme (3.7.1). First of all, let us check the consistency of the discretized scalar product from (3.7.4).

Lemma and Definition 3.7.2. *Let $\|\cdot\|_{h,k}$ denote the norm induced by $\langle\cdot,\cdot\rangle_{h,k}$. The norms $\|\cdot\|_{L^2_{L^2(\Gamma^h)}}$ and $\|\cdot\|_{h,k}$ on $L^2_{L^2(\Gamma^h)}$ are equivalent and there holds*

$$\left|\langle v, w\rangle_{h,k} - \langle v, w\rangle_{L^2_{L^2(\Gamma^h)}}\right| \le Ck \left|\langle v, w\rangle_{L^2_{L^2(\Gamma^h)}}\right| \,.$$

Proof. The result follows from the identity

$$\int_{[0,T]} \int_{\Gamma^h(t)} vw \, \mathrm{d}\Gamma^h(t) \, \mathrm{d}t = \sum_{m=1}^{M} \int_{I_m} \int_{\Gamma^h(t_m)} (\phi^{t_m}_{t,h} v)(\phi^{t_m}_{t,h} w) J^t_{t_m,h} \, \mathrm{d}\Gamma^h(t_m) \, \mathrm{d}t \,,$$

and $J^{t_m}_{t,h}$ being Lipschitz with $J^{t_m}_{t_m,h} \equiv 1$. ∎

Note also that for $z \in W^h_k$, since $\dot{z}^m = 0$ on I_m, we can apply the mean value theorem to obtain for some $t \in I_m$

$$\left|\|z^m\|^2_{L^2(\Gamma^h(t))} - \|z^m\|^2_m\right| = k\left|\langle z^m \operatorname{div}_{\Gamma^h(\Theta_m)} V_h, z^m\rangle_{L^2(\Gamma^h(\Theta_m))}\right| \le \mathbf{V} k \|z^m\|^2_{L^2(\Gamma^h(\Theta_m))} \,, \tag{3.7.5}$$

with $\Theta_m \in (t, t_m)$ and

$$\mathbf{V} = \sup_{\tau\in[0,T]} \|\operatorname{div}_{\Gamma^h(\tau)} V_h\|_{L^\infty(\Gamma^h(\tau))} \,.$$

Apply (3.7.5) to itself to obtain for some $\tilde{\Theta}_m \in (\Theta_m, t_m)$

$$\begin{aligned}\left|\|z^m\|^2_{L^2(\Gamma^h(t))} - \|z^m\|^2_m\right| &\le \mathbf{V}k\left(\|z^m\|^2_m + \left(\|z^m\|^2_{L^2(\Gamma^h(\Theta_m))} - \|z^m\|^2_m\right)\right) \\ &\le \mathbf{V}k\left(\|z^m\|^2_m + \mathbf{V}k\|z^m\|^2_{L^2(\Gamma^h(\tilde{\Theta}_m))}\right) \\ &\le \mathbf{V}k\left(1 + C_{L^2(\Gamma^h)}\mathbf{V}k\right)\|z^m\|^2_m \,.\end{aligned} \tag{3.7.6}$$

A similar continuity assertion holds for the $Y_h(t)$-norm, as shows the following lemma.

Lemma 3.7.3. *Let $y, z \in H^1_{Y_h}$, $\tilde{\mu_h} \in C(\Gamma^h(s) \times [0,T])$, and $\mu_h = \phi^t_{s,h}\tilde{\mu_h}$. There exists $C > 0$ such that for every $s \in I_m$*

$$\left| \int_{I_m} a_h(\phi^s_{t,h} y, \phi^s_{t,h} z; s)\mathrm{d}t - \int_{I_m} a_h(y,z;t)\mathrm{d}t \right| \le Ck \int_{I_m} \|\phi^s_{t,h} y\|_{Y_h(s)} \|\phi^s_{t,h} z\|_{Y_h(s)} \mathrm{d}t\,,$$

i.e. for $z \in W^h_k$ we have

$$\left| k a_h(\bar{y}^m, z^m; s) - \int_{I_m} a_h(y,z;t)\mathrm{d}t \right| \le Ck \int_{I_m} \|\phi^s_{t,h} y\|_{Y_h(s)} \|z^m\|_{Y_h(s)} \mathrm{d}t\,.$$

In particular with $\mu_h \equiv 1$ the estimates hold for $a_h(\varphi,\varphi;t) = \|\varphi\|^2_{Y_h(t)}$.

Proof. We abbreviate $\tilde{\Delta}(s,t) = D_{\Gamma^h(s)}\Phi^s_{t,h}(D_{\Gamma^h(s)}\Phi^s_{t,h})^T J^s_{t,h}$. We have

$$\begin{aligned} &|\int_{I_m} a_h(\phi^s_{t,h} y, \phi^s_{t,h} z; s)\mathrm{d}t - \int_{I_m} a_h(y,z;t)\mathrm{d}t| = \dots \\ &= \Big| \int_{I_m} \int_{\Gamma^h(s)} \nabla_{\Gamma^h}\phi^s_{t,h} y \left(\tilde{\Delta}(s,s) - \tilde{\Delta}(s,t)\right) \nabla_{\Gamma^h}\phi^s_{t,h} z + \mu_h \phi^s_{t,h} y (J^s_{s,h} - J^s_{t,h})\phi^s_{t,h} z \,\mathrm{d}\Gamma^h(s)\mathrm{d}t \Big|\,. \end{aligned}$$

The lemma follows from the fact that $\Phi^s_{t,h}$ it linear on each $T^i_h(s)$ and globally Lipschitz in time, as by Lemma 3.5.3. ■

In the process of proving stability estimates we will need the following useful discrete Gronwall inequality, compare [Bee85].

Lemma 3.7.4 (Discrete Gronwall)**.** *Consider sequences $\{a_m\}_{m\in\mathbb{N}}, \{c_m\}_{m\in\mathbb{N}} \subset \mathbb{R}$ and a non-negative sequence $\{b_m\}_{m\in\mathbb{N}} \subset \mathbb{R}^+_0$. The inequality*

$$c_m \le a_m + \sum_{i=1}^{m-1} b_m c_m$$

implies the estimate

$$c_m \le a_m + \sum_{i=1}^{m-1} a_i b_i \prod_{j=i+1}^{m-1} (1+b_j) \le a_m + \sum_{i=1}^{m-1} a_i b_i \, e^{\sum_{j=i+1}^{m-1} b_j}\,.$$

The lower of the two bounds is sharp.

Let us now formulate a crucial stability assertion for the adjoint scheme (3.7.3).

Lemma 3.7.5 (Adjoint stability)**.** *Let $z \in W^h_k$ solve (3.7.3) with right-hand side $v \in L^2_{L^2(\Gamma^h)}$ and final state $z^{M+1} = 0$. For sufficiently small $k > 0$ there exists $C > 0$, depending only on Γ, such that*

$$\frac{1}{k} \sum_{m=1}^{M} \|z^{m+1} - z^m\|^2_m + k \sum_{m=1}^{M} \|z^m\|^2_{Y_h(t_m)} \le C\|v\|^2_{h,k}\,.$$

Proof. Apply (3.7.3) to z^m to obtain

$$\langle z^m - z^{m+1}, z^m\rangle_m + k\mathfrak{a}_m(z^m, z^m) = \int_{I_m} \langle \phi^{t_m}_{t,h} v, z^m\rangle_m \,\mathrm{d}t\,.$$

This leads to

$$\frac{1}{2}\left(\|z^m\|_m^2 + \|z^{m+1} - z^m\|_m^2 - \|z^{m+1}\|_m^2\right) + k\mathfrak{a}_m(z^m, z^m) = \int_{I_m} \langle \phi^{t_m}_{t,h} v, z^m\rangle_m \,\mathrm{d}t$$

$$\leq \int_{I_m} \|\phi^{t_m}_{t,h} v\|_m \,\mathrm{d}t \|z^m\|_m \leq \frac{1}{2k\mathbf{V}}\left(\int_{I_m} \|\phi^{t_m}_{t,h} v\|_m \,\mathrm{d}t\right)^2 + \frac{k\mathbf{V}}{2}\|z^m\|_m^2\,.$$

Sum up, omitting the $\|z^{m+1} - z^m\|_m^2$ term, and use (3.7.6) on $\|z^m\|_m^2 - \|z^m\|_{m-1}^2$ to obtain

$$\left(1 - \frac{\mathbf{V}k}{2}\right)\|z^m\|_m^2 + \sum_{l=m+1}^{M}\left(-\mathbf{V}k\left(1 + \frac{1}{2}C_{L^2(\Gamma^h)}\mathbf{V}k\right)\|z^l\|_l^2 + k\mathfrak{a}_l(z^l, z^l)\right) \leq \frac{1}{2\mathbf{V}}\|v\|_{h,k}^2\,,$$

such that for $0 < k < \min\left(\frac{1}{2C_{\mathfrak{r}}}, \frac{2}{\mathbf{V}C_{L^2(\Gamma^h)}}, \frac{1}{\mathbf{V}}\right)$ we can apply the discrete Gronwall Lemma 3.7.4 with $c_l = \frac{1}{2}\|z^{M+1-l}\|_{M+1-l}^2$ and get

$$\frac{1}{2}\|z^1\|_1^2 \leq \frac{1 + 4\mathbf{V}T}{2\mathbf{V}} e^{4\mathbf{V}T}\|v\|_{h,k}^2 - k\sum_{m=1}^{M} a_h(z^m, z^m; t_m) + \mathfrak{r}_m(z^m, z^m)\,.$$

And because $\mu_h \geq 1$, compare (3.6.1), we conclude

$$\frac{1}{2}k\sum_{m=1}^{M}\|z^m\|_{Y_h(t_m)}^2 \leq \frac{1 + 4\mathbf{V}T}{2\mathbf{V}} e^{4\mathbf{V}T}\|v\|_{h,k}^2 \tag{3.7.7}$$

Now we test (3.7.3) with $z^m - z^{m+1}$ to get

$$\|z^m - z^{m+1}\|_m^2 + \frac{k}{2}\left(\mathfrak{a}_m(z^m, z^m) + \mathfrak{a}_m(z^{m+1} - z^m, z^{m+1} - z^m) - \mathfrak{a}_m(z^{m+1}, z^{m+1})\right) = \dots$$

$$= \int_{I_m} \langle \phi^{t_m}_{t,h} v, z^m - z^{m+1}\rangle_m \,\mathrm{d}t \leq \frac{1}{2}\left(\int_{I_m} \|\phi^{t_m}_{t,h} v\|_m^2 \,\mathrm{d}t\right)^2 + \frac{1}{2}\|z^m - z^{m+1}\|_m^2\,.$$

Summing up and using Lemma 3.7.3 on $\mathfrak{a}$ as well as the estimate (3.7.2) on $\mathfrak{r}$ we arrive at

$$\frac{k}{2}a_h(z^1, z^1; t_1) + \frac{1}{2}\sum_{m=1}^{M}\|z^{m+1} - z^m\|_m^2$$

$$\leq \frac{1}{2}k\|v\|_{h,k}^2 + \frac{k}{2}\sum_{m=2}^{M} a_h(z^m, z^m; t_{m-1}) - a_h(z^m, z^m; t_m) + \mathfrak{r}_{m-1}(z^m, z^m) - \mathfrak{r}_m(z^m, z^m)$$

$$\leq \frac{1}{2}k\|v\|_{h,k}^2 + \frac{k}{2}\sum_{m=2}^{M} Ck\left(\|z^m\|_{Y_h(t_m)}^2 + \|z^m\|_{Y_h(t_{m-1})}^2\right)\,.$$

Combine with (3.7.7) to conclude the lemma. ∎

The following Lemma shows, that it is sufficient to bound the approximation error at the points t_m, $1 \leq m \leq M$ to prove convergence in $L^2_{L^2(\Gamma^h)}$. Note that $H^1([0,T],\mathcal{H}) \subset C([0,T],\mathcal{H})$ for any Hilbert space $\mathcal{H}$, compare [LM68, Thm. 3.1].

Lemma 3.7.6. *Let $r \in H^1([0,T],\mathcal{H})$, $\mathcal{H}$ a separable Hilbert space, then there holds for $\tau \in I_m$*

$$\|r - r(\tau)\|_{L^2(I_m,\mathcal{H})} \leq k\|r'\|_{L^2(I_m,\mathcal{H})} .$$

In our situation this implies for $r \in H^1_{Y_h}$ that

1. $k\|r(\tau) - \bar{r}^m\|^2_{L^2(\Gamma^h(\tau))} \leq Ck^2 \int_{I_m} \|\dot{r}\|^2_{L^2(\Gamma^h(t))} \,\mathrm{d}t\,,$

2. *and* $\int_{I_m} \|r(t) - \bar{r}^m\|^2_{L^2(\Gamma^h(t))} \,\mathrm{d}t \leq Ck^2 \int_{I_m} \|\dot{r}\|^2_{L^2(\Gamma^h(t))} \,\mathrm{d}t\,.$

Proof. For the fist assertion approximate r by $r_i \in \mathcal{D}([0,T],\mathcal{H})$ such that $r_i \overset{H^1([0,T],\mathcal{H})}{\longrightarrow} r$ as $i \to \infty$. Use

$$\begin{aligned}\|r_i - r_i(\tau)\|_{L^2(I_m,\mathcal{H})} &= \left(\int_{I_m} \left\|\int_\tau^t r_i'(\theta)\mathrm{d}\theta\right\|^2_{\mathcal{H}} \mathrm{d}t\right)^{\frac{1}{2}} \\ &\leq \left(\int_{I_m} k \int_{I_m} \left\|r_i'(\theta)\right\|^2_{\mathcal{H}} \mathrm{d}\theta \,\mathrm{d}t\right)^{\frac{1}{2}} \leq k\|r_i'\|_{L^2(I_m,\mathcal{H})}\,,\end{aligned}$$

and the fact that $H^1([0,T],\mathcal{H})$ compactly embeds into $C([0,T],\mathcal{H})$. Hence the first part of the lemma follows by passing to the limit.
In our situation, because of $\phi^\tau_{t,h} r(t) \in H^1([0,T],Y_h(\tau))$, this implies

$$\begin{aligned}\|\bar{r}^m - r(\tau)\|^2_{L^2(\Gamma^h(\tau))} &= \left\|\frac{1}{k}\int_{I_m} \phi^\tau_{t,h} r(t) - r(\tau)\,\mathrm{d}t\right\|^2_{L^2(\Gamma^h(\tau))} \leq \frac{1}{k}\int_{I_m} \left\|\phi^\tau_{t,h} r(t) - r(\tau)\right\|^2_{L^2(\Gamma^h(\tau))} \mathrm{d}t \\ &\leq k\int_{I_m} \|\left(\phi^\tau_{t,h} r(t)\right)'\|^2_{L^2(\Gamma^h(\tau))} \,\mathrm{d}t \leq kC^h_J \int_{I_m} \|\dot{r}\|^2_{L^2(\Gamma^h(t))} \,\mathrm{d}t\,.\end{aligned}$$

This proves *1.*, in order to get *2.* integrate over I_m. ■

We are now prepared to prove the main result of this section.

Theorem 3.7.7. *Let $u \in L^2_{L^2(\Gamma)}$, and let y_h and $y_{h,k}$ solve* (3.6.2) *and* (3.7.1)*, respectively, with $y^h_0 \in L^2(\Gamma^h_0)$ and $u_h = u_l$. There exists a constant $C > 0$ independent of $h, k > 0$ and of u and y^h_0 such that*

$$\|y_h - y_{h,k}\|_{L^2_{L^2(\Gamma^h)}} \leq Ck\left(\|\dot{y}_h\|_{L^2_{L^2(\Gamma^h)}} + \|u\|_{L^2_{L^2(\Gamma)}} + \|y^h_0\|_{L^2(\Gamma^h_0)}\right) .$$

Proof. The proof is inspired by [SD05, Thm. 5.2], compare also [Vie07, Thm 1.2.5] and [MV08a, Thm 5.1]. As a first step, test (3.6.2) with the constant function $\phi^t_{t_m,h}\varphi \in H^1_{Y_h}$, where $\varphi \in Y_h(t_m)$. Integrate over I_m to obtain

$$\langle y_h(t_m), \varphi\rangle_m - \langle y_h(t_{m-1}), \varphi\rangle_{m-1} + \int_{I_m} a_h(y_h, \varphi; t)\,\mathrm{d}t = \int_{I_m} \langle u_l, \varphi\rangle_{L^2(\Gamma^h(t))}\,\mathrm{d}t\,. \tag{3.7.8}$$

Next, solve the adjoint equation (3.7.3) for z with right-hand side $v = \sum_{m=1}^M (\bar{y}_h^m - y_{h,k}^m)\mathbb{1}_{I_m}$ and final value $z^{M+1} = 0$. Apply the test function $\varphi = v$ to get

$$\int_{I_m} \|\bar{y}_h^m - y_{h,k}^m\|_m^2\,\mathrm{d}t = \langle z^m - z^{m+1}, \bar{y}_h^m - y_{h,k}^m\rangle_m + k\mathfrak{a}_m(\bar{y}_h^m - y_{h,k}^m, z^m)\,. \tag{3.7.9}$$

Subtract (3.7.8) from (3.7.1). Tested with z this yields

$$\begin{aligned}\langle y_{h,k}^m - y_h(t_m), z^m\rangle_m - \langle y_{h,k}^{m-1} - y_h(t_{m-1}), z^m\rangle_{m-1} + k\mathfrak{a}_m(y_{h,k}^m - \bar{y}_h^m, z^m) = \dots\\ = \int_{I_m} a_h(y_h, z^m; t)\,\mathrm{d}t - k\mathfrak{a}_m(\bar{y}_h^m, z^m) + k\langle \bar{u}_l^m, z^m\rangle_m - \int_{I_m}\langle u_l, z^m\rangle_{L^2(\Gamma^h(t))}\,\mathrm{d}t\end{aligned}$$

Let $\bar{y}_h = \sum_{m=1}^M \bar{y}_h^m \mathbb{1}_{I_m}$. Add (3.7.9) and sum up over $1 \le m \le M$ to get

$$\begin{aligned}&\langle u_l, z\rangle_{h,k} - \langle u_l, z\rangle_{L^2_{L^2(\Gamma^h)}} + \sum_{m=1}^M \int_{I_m} \|\bar{y}_h - y_{h,k}\|_m^2\,\mathrm{d}t + \int_{I_m} a_h(y_h, z^m; t)\,\mathrm{d}t - ka_h(\bar{y}_h^m, z^m; t_m) =\\ &\sum_{m=1}^M k\mathfrak{r}_m(\bar{y}_h^m, z^m) + \langle \bar{y}_h^m - y_h(t_m), z^m\rangle_m - \langle y_{h,k}^{m-1} - y_h(t_{m-1}), z^m\rangle_{m-1} - \langle z^{m+1}, \bar{y}_h^m - y_{h,k}^m\rangle_m\\ &= \sum_{m=1}^M k\mathfrak{r}_m(\bar{y}_h^m, z^m) + \langle \bar{y}_h^m - y_h(t_m), z^m - z^{m+1}\rangle_m\,,\end{aligned}$$

where we used $z^{M+1} = 0$ and $y_{h,k}^0 - y_h(t_0) = 0$ to cast the sum in its final shape.
Finally, bringing to bear the estimates from Lemma 3.7.3 for $\mathfrak{a}$, the one from Lemma 3.7.2 for the L^2-norms, and the bound on $\mathfrak{r}$ from (3.7.2), we arrive at

$$\begin{aligned}\|\bar{y}_h - y_{h,k}\|_{h,k}^2 \le& \left(k\sum_{m=1}^M \|\bar{y}_h^m - y_h(t_m)\|_m^2\right)^{\frac{1}{2}} \left(\frac{1}{k}\sum_{m=1}^M \|z^m - z^{m+1}\|_m^2\right)^{\frac{1}{2}} + \dots\\ &+ C\left(k\sum_{m=1}^M\left(\int_{I_m}\|\phi_{t,h}^{t_m} y_h\|_{Y_h(t_m)}\,\mathrm{d}t\right)^2\right)^{\frac{1}{2}}\left(k\sum_{m=1}^M\|z^m\|_{Y_h(t_m)}^2\right)^{\frac{1}{2}} + Ck\|u\|_{L^2_{L^2(\Gamma)}}\underbrace{\|z^l\|_{L^2_{L^2(\Gamma)}}}_{\le C\|z\|_{h,k}}\,.\end{aligned}$$

Hence using Lemma 3.7.5 on z we can divide by $\|\bar{y}_h - y_{h,k}\|_{h,k}$. Thanks to the stability of the space discretization (Lemma 3.6.2) we can estimate the $Y_h(t)$-term, to finally arrive at

$$\|\bar{y}_h - y_{h,k}\|_{L^2_{L^2(\Gamma^h)}} \le C\left(\left(k\sum_{m=1}^M \|\bar{y}_h^m - y_h(t_m)\|_m^2\right)^{\frac{1}{2}} + k\|u\|_{L^2_{L^2(\Gamma)}} + k\|y_0^h\|_{L^2(\Gamma_0^h)}\right)\,. \tag{3.7.10}$$

We now apply Lemma 3.7.6[*2.*] to the error $e_k = y_{h,k} - y_h$ and the averaged error $\bar{e}_k = y_{h,k} - \bar{y}_h$ and sum up to obtain $\|e_k - \bar{e}_k\|_{L^2_{L^2(\Gamma^h)}} \le Ck\|\dot{y}_h\|_{L^2_{L^2(\Gamma^h)}}$. Combine with (3.7.10) and 3.7.6[*1.*] to estimate

$$\|e_k\|_{L^2_{L^2(\Gamma^h)}} \le Ck\|\dot{y}_h\|_{L^2_{L^2(\Gamma^h)}} + \|\bar{e}_k\|_{L^2_{L^2(\Gamma^h)}} \le Ck\left(\|\dot{y}_h\|_{L^2_{L^2(\Gamma^h)}} + \|u\|_{L^2_{L^2(\Gamma)}} + \|y_0^h\|_{L^2(\Gamma_0^h)}\right).$$

■

In view of the stability assertions from (3.4.3) and Lemma 3.6.2 and together with Theorem 3.6.6 we get the following Corollary.

Corollary 3.7.8. *In the situation of Theorem 3.7.7 let in addition $\mu_h = \mu_l$ and $y_0 \in H^2(\Gamma_0)$, and choose y_0^h as the piecewise linear interpolation of $(y_0)_l$. There exists a constant $C > 0$ independent of $h, k > 0$ and of u and y_0 such that*

$$\|y_{h,k}^l - y\|_{L^2_{L^2(\Gamma)}} \le C(h^2 + k)\left(\|y_0\|_{H^2(\Gamma_0)} + \|u\|_{L^2_{L^2(\Gamma)}}\right).$$

As addressed in Remark 3.6.7, it is possible to relax the condition on y_0 into $y_0 \in H^1(\Gamma_0)$ using an L^2- or Ritz-like projection onto $Y_h(t_0)$.
Even in the case of low regularity we still get a uniform estimate.

Corollary 3.7.9. *In the situation of Theorem 3.7.7 let only $y_0 \in L^2(\Gamma_0)$ hold while $u \equiv 0$. Let further $y_0^h = P_0^h((y_0)_l)$. There exists a constant $C > 0$ independent of $h, k > 0$ and of y_0 such that*

$$\|y_{h,k}^l - y\|_{L^2_{L^2(\Gamma)}} \le C\left(h + \sup_{t\in[0,T]} \|{\mu_h}^l - \mu\|_{L^\infty(\Gamma(t))} + \frac{k}{h}\right)\|y_0\|_{L^2(\Gamma_0)}.$$

Proof. Regarding Theorem 3.6.8 and 3.7.7 it remains to bound $\|\dot{y}_h\|_{L^2_{L^2(\Gamma^h)}}$. Like in the proof of Theorem 3.6.8, using Lemma 3.6.2 and an inverse estimate, we arrive at the desired estimate. ■

In particular, for $\kappa > 0$, choose $k = \kappa h^2$ and μ_h such that $\sup_{t\in[0,T]} \|{\mu_h}^l - \mu\|_{L^\infty(\Gamma(t))} \le Ch$ to get an $\mathcal{O}(h)$-convergent scheme.
The restriction $u \equiv 0$ is one of notational convenience. It is easy to allow for $u \in L^2_{H^{-1}(\Gamma)}$.

Remark 3.7.10. Note that our freedom in the choice of $\mathfrak{r}$ now allows us to finally drop the conditions on μ_h and μ, respectively, in (3.6.1) and (3.6.2). Let us assume we want to approximate the solution y of (3.4.1) with, for instance, $\mathbf{b} \equiv 0$, $y_0 = 0$, and $u \in L^2_{L^2(\Gamma)}$.
Now $e^{-\mu_h t}y$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-\mu_h t}y) + \Delta(e^{-\mu_h t}y) + \mu_h e^{-\mu_h t}y = e^{-\mu_h t}u \in L^2_{L^2(\Gamma)},$$

and choosing $\mu_h = \mu \equiv 1$ puts us into the situation of (3.6.1).

The approximation $y_{h,k} \in W_k^h$ of y is the solution of

$$y_{h,k}^0 = y_0^h\,, \quad \forall \varphi \in W_k^h\,,\ 1 \le m \le M\ :$$
$$\langle y_{h,k}^m, \varphi\rangle_m - \langle y_{h,k}^{m-1}, \varphi\rangle_{m-1} + k \int\limits_{\Gamma^h(t_m)} \nabla_{\Gamma^h(t_m)} y_{h,k}^m \nabla_{\Gamma^h(t_m)} \varphi \,\mathrm{d}\Gamma^h(t_m) = k\langle \bar{u}_h^m, \varphi\rangle_m\,,$$

which is equivalent to $y_{h,k,\mu_h} = \sum_{m=1}^M e^{-\mu_h t_m} y_{h,k}^m \mathbb{1}_{I_m} \in W_k^h$ solving

$$y_{h,k,\mu_h}^0 = y_0^h\,, \quad \forall \varphi \in W_k^h\,, 1 \le m \le M\ :$$
$$\langle y_{h,k,\mu_h}^m, \varphi\rangle_m - \langle y_{h,k,\mu_h}^{m-1}, \varphi\rangle_{m-1} + k \int\limits_{\Gamma^h(t_m)} \nabla_{\Gamma^h(t_m)} y_{h,k,\mu_h}^m \nabla_{\Gamma^h(t_m)} \varphi + \mu_h y_{h,k,\mu_h}^m \varphi \,\mathrm{d}\Gamma^h(t_m) \ldots$$
$$+ k\mathfrak{r}_m(y_{h,k,\mu_h}^m, \varphi) = k\langle e^{-\mu_h t_{m-1}} \bar{u}_h^m, \varphi\rangle_m\,,$$

with

$$k\mathfrak{r}_m(\psi, \varphi) = (e^{\mu_h k} - 1 - \mu_h k)\langle \psi, \varphi\rangle_m + k(e^{\mu_h k} - 1) \int\limits_{\Gamma^h(t_m)} \nabla_{\Gamma^h(t_m)} \psi \nabla_{\Gamma^h(t_m)} \varphi \,\mathrm{d}\Gamma^h(t_m)\,.$$

Taking into account that

$$\|e^{-\mu_h t} u_t - \sum_{m=1}^M e^{-\mu_h t_{m-1}} \mathbb{1}_{I_m} u_t\|_{L^2_{L^2(\Gamma)}} \le Ck\|u\|_{L^2_{L^2(\Gamma)}}\,, \tag{3.7.11}$$

we apply Corollary 3.7.8 to y_{h,k,μ_h} and infer $\|y_{h,k,\mu_h}^l - e^{-\mu_h t} y\|_{L^2_{L^2(\Gamma)}} \le C(h^2 + k)\|u\|_{L^2_{L^2(\Gamma)}}$. Apply the argument (3.7.11) again, this time to $y_{h,k}$, to conclude

$$\|y_{h,k}^l - y\|_{L^2_{L^2(\Gamma)}} \le Ce^{\mu_h T}(h^2 + k)\|u\|_{L^2_{L^2(\Gamma)}}\,.$$

Chapter 4

Parabolic optimal control on moving surfaces

We consider control-constrained linear-quadratic optimal control problems on evolving hypersurfaces in $\mathbb{R}^{n+1}$. The results from Section 3.3 and Section 3.4 allow us to formulate well-posed optimal control problems in Section 4.1. We then carry out and prove convergence of the variational discretization of distributed optimal control problems in Section 4.2. It turns out that, due to the time-dependency of the involved norms, discretizing Equation (4.1.1) and taking the adjoint do not commute. We close this work with some numerical examples in Section 4.3.
Basic facts on control constrained parabolic optimal control problems and their discretization can be found for example in [Trö05] and [MV08b], respectively.

4.1 Control constrained optimal control problems

Using the existence result from Section 3.4, we can now formulate all kinds of control-constrained optimal control problems known for stationary domains, see for example [Trö05]. As a first example, given an evolving surface as in Assumption 3.0.5, let $S_T : L^2_{L^2(\Gamma)} \to L^2(\Gamma(T))$ denote the solution operator $u \mapsto y(T)$, where y satisfies for a.e. $t \in [0, T]$

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma(t)} y\,\varphi \,\mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} \nabla_\Gamma y \nabla_\Gamma \varphi \,\mathrm{d}\Gamma(t) = \langle \dot{\varphi}, y \rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))} + \langle u, \varphi \rangle_{L^2_{L^2(\Gamma)}}\,, \qquad (4.1.1)$$

for all $\varphi \in W_\Gamma$, and with $y(0) = 0 \in L^2(\Gamma_0)$. We know, that every function $y \in W_\Gamma$ has a representation in $C([0,T], L^2(\Gamma(s)))$ for any $s \in [0,T]$, compare Lemma 3.1.1, and the inclusion $\phi^s_{(\cdot)} W_\Gamma = W_s(0,T) \subset C([0,T], L^2(\Gamma(s)))$ is continuous (in fact compact). Thus S_T is a continuous linear operator. Consider the Control problem

$$(\mathbb{P}_T) \quad \begin{cases} \min_{u \in L^2_{L^2(\Gamma)}} \mathbf{O}(u) := \frac{1}{2}\|S_T(u) - y_T\|^2_{L^2(\Gamma(T))} + \frac{\alpha}{2}\|u\|^2_{L^2_{L^2(\Gamma)}} \\ \text{s.t. } a \le u \le b\,, \end{cases}$$

with $\alpha, a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$, and $y_T \in L^2(\Gamma(T))$. This is now a well posed problem. By standard arguments, see for example [Trö05, Thm. 3.15], using the weak lower semicontinuity of the objective function $\mathbf{O}(\cdot)$, one can conclude the existence of a unique solution $u \in L^2_{L^2(\Gamma)}$.
For another example let the linear continuous solution operator $S_d : L^2_{L^2(\Gamma)} \to L^2_{L^2(\Gamma)}$, $u \mapsto y$, where y solves (4.1.1), and consider the problem

$$(\mathbb{P}_d) \quad \begin{cases} \min_{u \in L^2_{L^2(\Gamma)}} \mathbf{O}(u) := \frac{1}{2}\|S_d(u) - y_d\|^2_{L^2_{L^2(\Gamma)}} + \frac{\alpha}{2}\|u\|^2_{L^2_{L^2(\Gamma)}} \\ \text{s.t. } a \leq u \leq b\,, \end{cases}$$

with α, a, b as above and $y_d \in L^2_{L^2(\Gamma)}$. Again there exists a unique solution, see [Trö05, Thm. 3.16].
The first order necessary optimality condition for $(\mathbb{P}_d)$ reads (compare also (2.0.2))

$$\langle \nabla_u \mathbf{O}(u), v - u\rangle_{L^2_{L^2(\Gamma)}} = \langle \alpha u + S_d^*(S_d u - y_d), v - u\rangle_{L^2_{L^2(\Gamma)}} \geq 0\,, \quad \forall v \in U_{\text{ad}} \tag{4.1.2}$$

with $U_{\text{ad}} = \left\{v \in L^2_{L^2(\Gamma)} \mid a \leq v \leq b\right\}$. The adjoint operator $S_d^* : L^2_{L^2(\Gamma)} \to L^2_{L^2(\Gamma)}$ maps $v \in L^2_{L^2(\Gamma)}$ onto the solution $p \in W_\Gamma \subset L^2_{L^2(\Gamma)}$ of

$$-\langle \dot{p}, \varphi\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))} + \int_{\Gamma(t)} \nabla_\Gamma p \nabla_\Gamma \varphi \, \mathrm{d}\Gamma(t) = \langle v, \varphi\rangle_{L^2(\Gamma(t))}\,, \tag{4.1.3}$$

for all $\varphi \in W_\Gamma$, and $p(T) = 0 \in L^2(\Gamma(T))$. This follows if one tests (4.1.1) with p and (4.1.3) with y. Integrate over $[0, T]$ and use $y(0) = 0$ and $p(T) = 0$ to arrive at $\langle v, y\rangle_{L^2_{L^2(\Gamma)}} = \langle p, u\rangle_{L^2_{L^2(\Gamma)}}$, for $u, v \in L^2_{L^2(\Gamma)}$ arbitrary.
Note that via the time transform $t' = T - t$ Equation (4.1.3) converts into equation (3.4.1) with $\mathbf{b}_t = \operatorname{div}_{\Gamma(t)} V$ and that therefore the existence result from Section 3.4 applies to (4.1.3). In order to see this consider the following reformulation of (4.1.3)

$$-\frac{\mathrm{d}}{\mathrm{d}t}\langle p, \varphi\rangle_{L^2(\Gamma(t))} + \int_{\Gamma(t)} \nabla_\Gamma p \nabla_\Gamma \varphi + p\varphi \operatorname{div}_{\Gamma(t)} V \, \mathrm{d}\Gamma(t) = -\langle p, \dot{\varphi}\rangle_{H^1(\Gamma(t)),H^{-1}(\Gamma(t))} + \langle v, \varphi\rangle_{L^2(\Gamma(t))}\,,$$

the material derivatives then change sign under the time transform.
The necessary condition (4.1.2) characterizes the optimum u as the orthogonal projection of $-\frac{1}{\alpha}S_d^*(S_d u - y_d)$ onto the admissible set U_{ad}. In our situation this is the pointwise application of the projection $P_{[a,b]} : \mathbb{R} \to [a, b]$.

Lemma 4.1.1. *Let $P_{U_{\text{ad}}}$ denote the $L^2_{L^2(\Gamma)}$-orthogonal projection onto U_{ad}, which is defined by*

$$\langle u - P_{U_{\text{ad}}}(u), v - P_{U_{\text{ad}}}(u)\rangle_{L^2_{L^2(\Gamma)}} \leq 0 \quad \forall v \in U_{\text{ad}}\,. \tag{4.1.4}$$

Then for $u \in L^2_{L^2(\Gamma)}$ one has for a.e. $t \in [0, T]$

$$P_{U_{\text{ad}}}(u_t) = P_{[a,b]}(u_t)\,.$$

Proof. The pointwise projection $P_{[a,b]}$ is a continuous, bounded operator on $L^2(\Gamma(s))$ and we have

$$\phi_t^s P_{[a,b]}(u) = P_{[a,b]}(\phi_t^s u) \in L^2([0,T], L^2(\Gamma(s))).$$

Let $\mathcal{C} = \{t \in [0,T] \mid P_{U_{\text{ad}}}(u_t) \neq P_{[a,b]}(u_t)\}$ and assume $\text{meas}(\mathcal{C}) > 0$. Now test (4.1.4) with $v_t = P_{[a,b]}(u_t)$ to arrive at

$$\int_{[0,T]} \langle u - P_{U_{\text{ad}}}(u), P_{[a,b]}(u) - P_{U_{\text{ad}}}(u)\rangle_{L^2(\Gamma(t))}\, \mathrm{d}t \leq 0\,. \tag{4.1.5}$$

But now for a.e. $t \in [0,T]$ and a.e. $\gamma \in \Gamma(t)$ one has

$$\big(u_t[\gamma] - P_{U_{\text{ad}}}(u_t)[\gamma]\big)\big(P_{[a,b]}(u_t)[\gamma] - P_{U_{\text{ad}}}(u_t)[\gamma]\big) \geq 0\,,$$

because $P_{U_{\text{ad}}}(u_t)[\gamma] \in [a,b]$. Moreover for $t \in \mathcal{C}$ we have

$$\big(u_t[\gamma] - P_{U_{\text{ad}}}(u_t)[\gamma]\big)\big(P_{[a,b]}(u_t)[\gamma] - P_{U_{\text{ad}}}(u_t)[\gamma]\big) > 0\,,$$

on a set of positive measure. Since $\text{meas}(\mathcal{C}) > 0$ this contradicts (4.1.5). ■

Introducing the adjoint state $p_d(u) = S_d^*(S_d u - y_d)$, let us now rewrite (4.1.2) as

$$u = P_{[a,b]}\left(-\frac{1}{\alpha} p_d(u)\right)\,. \tag{4.1.6}$$

Similarly the unique solution u of $(\mathbb{P}_T)$ is characterized by $u = P_{[a,b]}\left(-\frac{1}{\alpha} p_T(u)\right)$, with $p_T(u) = S_T^*(S_T u - y_T)$. Note that however the adjoint state p_T in general is less smooth than p_d. This is because the adjoint equation, i.e. the equation describing $S_T^* : L^2(\Gamma(T)) \to L^2_{L^2(\Gamma)}$, $v \mapsto p$, reads

$$-\langle \dot{p}, \varphi\rangle_{H^{-1}(\Gamma(t)),H^1(\Gamma(t))} + \int_{\Gamma(t)} \nabla_\Gamma p \nabla_\Gamma \varphi \,\mathrm{d}\Gamma(t) = 0\,,$$

for all $\varphi \in W_\Gamma$ and with $p(T) = v \in L^2(\Gamma(T))$. While Theorem 3.4.1 applies, this is not the case for the smoothness assertion (3.4.3), as long as $y_d \in L^2(\Gamma(T)) \setminus H^1(\Gamma(T))$.

4.2 Variational discretization

We now return to problem $(\mathbb{P}_d)$ which has the advantage over $(\mathbb{P}_T)$, that its adjoint equation satisfies the regularity estimate (3.4.3). For $(\mathbb{P}_T)$ this is not the case iff $y_T \in L^2(\Gamma(T)) \setminus H^1(\Gamma(T))$. In the spirit of [Hin05], let us approximate $(\mathbb{P}_d)$ by

$$(\mathbb{P}_d^h) \quad \begin{cases} \min_{u \in L^2_{L^2(\Gamma^h)}} \mathbf{O}(u) := \frac{1}{2}\|S_d^h(u) - (y_d)_l\|_{h,k}^2 + \frac{\alpha}{2}\|u\|_{h,k}^2 \\ \text{s.t. } a \leq u \leq b\,, \end{cases}$$

with $\{\Gamma^h(t)\}_{t\in[0,T]}$ as in Section 3.6 and $S_d^h : (L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k}) \to (L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k})$, $u_h \mapsto y_{h,k}$ is defined through the scheme 3.7.1 with $\mu_h \equiv 0$ and $y_0^h \equiv 0$. We choose the scalar product

$\langle \cdot, \cdot \rangle_{h,k}$ defined in (3.7.4) in order to obtain a computable scheme to evaluate $S_d^{h^*}$, namely (3.7.3) with $z^{M+1} = 0$. Given snapshots $\{\Gamma^h(t_m)\}_{m=0}^M$, the product $\langle \cdot, \cdot \rangle_{h,k}$ can be evaluated exactly for functions $\varphi_h \in W_k^h$ as well as for $P_{[a,b]}(\varphi_h)$.
Let $U_{\text{ad}}^h = \left\{ v \in L^2_{L^2(\Gamma^h)} \mid a \le v \le b \right\}$. As in (4.1.2) the first order necessary optimality condition for an optimum u_h of $(\mathbb{P}_d^h)$ is

$$\langle \alpha u_h + S_d^{h^*}(S_d^h u_h - (y_d)_l), v - u_h \rangle_{h,k} \ge 0 \,, \quad \forall v \in U_{\text{ad}} \,. \tag{4.2.1}$$

First, note that like in the continuous case the $\langle \cdot, \cdot \rangle_{h,k}$-orthogonal projection onto U_{ad}^h coincides with the pointwise projection $P_{[a,b]}(v)$. Similar to (4.1.6) we get

$$u_h = P_{[a,b]} \left(-\frac{1}{\alpha} p_d^h(u) \right) \,, \quad p_d^h(u) = S_d^{h^*} \left(S_d^h u - (y_d)_l \right) \,. \tag{4.2.2}$$

Equation (4.2.2) is amenable to a semismooth Newton method that, while still being implementable, operates entirely in $L^2_{L^2(\Gamma^h)}$. The implementation requires one to resolve the boundary between the inactive set $\mathcal{I}_u(t_m) = \left\{ \gamma \in \Gamma(t_m) \;\middle|\; a < -\frac{1}{\alpha} p_d^h(u)[\gamma] < b \right\}$ and the active set $\mathcal{A}_u(t_m) = \Gamma^h(t_m) \setminus \mathcal{I}_u(t_m)$ for $1 \le m \le M$. For details on the implementation see [Vie07], where the parabolic equations on fixed domains are discussed, or Chapter 2, [HV12a], and [HV11] for elliptic problems. Note that in order to implement S_d^h and $S_d^{h^*}$ according to (3.7.1) and (3.7.3) for right-hand sides in W_k^h, again one only needs to know the snapshots $\{\Gamma^h(t_m)\}_{m=0}^M$. The solution of $(\mathbb{P}_d^h)$ converges towards that of $(\mathbb{P}_d)$ and the order of convergence is optimal in the sense that it is the same as the order of convergence of the operators S_d^h and $S_d^{h^*}$.

Theorem 4.2.1 (Order of Convergence for $(\mathbb{P}_d^h)$)**.** *Denote by $u \in L^2_{L^2(\Gamma)}$ and $u_h \in L^2_{L^2(\Gamma^h)}$ the solutions of $(\mathbb{P}_d)$ and $(\mathbb{P}_d^h)$, respectively. Let $C > 1$. Then for sufficiently small $h, k > 0$ there holds*

$$2\alpha \left\| u_h^l - u \right\|^2_{L^2_{L^2(\Gamma)}} + \left\| y_h^l - y \right\|^2_{L^2_{L^2(\Gamma)}} \le C \Bigg(2 \Big\langle \left((\cdot)^l S_d^{h^*} (\cdot)_l - S_d^* \right) (y - y_d), u - u_h^l \Big\rangle_{L^2_{L^2(\Gamma)}} \cdots$$
$$+ \left\| \left((\cdot)^l S_d^h (\cdot)_l - S_d \right) u \right\|^2_{L^2_{L^2(\Gamma)}} \Bigg) \,,$$

with $y = S_d u$ and $y_h = S_d^h u_h$.

Proof. The proof is a modification of the one from [HPUU09, Thm. 3.4], compare also the proof of Theorem 4.2.1. Let $\mathrm{P}_{U_{\text{ad}}^h}(\cdot)$ denote the $\langle \cdot, \cdot \rangle_{h,k}$-orthogonal projection onto U_{ad}^h. We have

$$u_l = P_{[a,b]} \left(-\frac{1}{\alpha} p_d(u) \right)_l = P_{[a,b]} \left(-\frac{1}{\alpha} p_d(u)_l \right) = \mathrm{P}_{U_{\text{ad}}^h} \left(-\frac{1}{\alpha} p_d(u)_l \right) \,.$$

Since $u_h \in U_{\text{ad}}^h$, from the characterization of $\mathrm{P}_{U_{ad}^h}(\cdot)$ it follows

$$\langle -\frac{1}{\alpha} p_d(u)_l - u_l, u_h - u_l \rangle_{h,k} \le 0 \,.$$

On the other hand we can plug u_l into (4.2.1) and get

$$\langle \alpha u_h + p_d^h(u_h), u_l - u_h\rangle_{h,k} \geq 0\,.$$

Adding these inequalities yields

$$\begin{aligned}\alpha\|u_l - u_h\|_{h,k}^2 \leq& \Big\langle \left(p_d^h(u_h) - p_d(u)_l\right), u_l - u_h\Big\rangle_{h,k} \\ =& \langle p_d^h(u_h) - S_d^{h^*}(y-y_d)_l, u_l - u_h\rangle_{h,k} + \langle S_d^{h^*}(y-y_d)_l - p_d(u)_l, u_l - u_h\rangle_{h,k}\,.\end{aligned}$$

The first addend is estimated via

$$\begin{aligned}\langle p_d^h(u_h) - (S_d^h)^*(y-y_d)_l, u_l - u_h\rangle_{h,k} &= \langle y_h - y_l, S_d^h u_l - y_h\rangle_{h,k} \\ &= -\|y_h - y_l\|_{h,k}^2 + \langle y_h - y_l, S_d^h u_l - y_l\rangle_{h,k} \\ &\leq -\frac{1}{2}\|y_h - y_l\|_{h,k}^2 + \frac{1}{2}\|S_d^h u_l - y_l\|_{h,k}^2\,.\end{aligned}$$

This yields

$$2\alpha\|u_l - u_h\|_{h,k}^2 + \|y_h - y_l\|_{h,k}^2 \leq 2\langle (S_d^{h^*}(\cdot)_l - (\cdot)_l S_d^*)(y-y_d), u_l - u_h\rangle_{h,k} + \|S_d^h u_l - y_l\|_{h,k}^2\,.$$

The claim follows for sufficiently small $h, k > 0$, using the equivalence of the involved scalar products stated in Lemma 3.7.2. ■

For the problem

$$(\mathbb{P}_T^h) \quad \begin{cases} \min_{u\in L^2_{L^2(\Gamma^h)}} \mathbf{O}(u) := \frac{1}{2}\|S_T^h(u) - (y_T)_l\|_{L^2(\Gamma^h(T))}^2 + \frac{\alpha}{2}\|u\|_{L^2_{L^2(\Gamma^h)}}^2 \\ \text{s.t. } a \leq u \leq b\,, \end{cases}$$

one can prove a similar result. Here the operator S_T^h is the map $u_h \to y_{h,k}^M$, according to the scheme (3.7.1) with $\mu_h \equiv 0$.

Theorem 4.2.2 (Order of Convergence for $(\mathbb{P}_T^h)$)**.** *Denote by $u \in L^2_{L^2(\Gamma)}$ and $u_h \in L^2_{L^2(\Gamma^h)}$ the solutions of $(\mathbb{P}_T)$ and $(\mathbb{P}_T^h)$, respectively. Let $C > 1$. Then for sufficiently small $h, k > 0$ there holds*

$$\begin{aligned}2\alpha\left\|u_h^l - u\right\|_{L^2_{L^2(\Gamma)}}^2 + \left\|y_h^l - y\right\|_{L^2(\Gamma(T))}^2 \leq C\Bigg(& 2\Big\langle \left((\cdot)^l S_T^{h^*}(\cdot)_l - S_T^*\right)(y - y_T), u - u_h^l\Big\rangle_{L^2_{L^2(\Gamma)}} \dots \\ &+ \left\|\left((\cdot)^l S_T^h(\cdot)_l - S_T\right)u\right\|_{L^2(\Gamma(T))}^2\Bigg)\,,\end{aligned}$$

with $y = S_T u$ and $y_h = S_T^h u_h$.

Now as to the convergence of $\left((\cdot)^l S_d^{h^*}(\cdot)_l - S_d^*\right)$, note that taking the adjoint does not commute with the discretization. Indeed, apply the scheme (3.7.1) to the adjoint equation (4.1.3), i.e. $\mu_h = (\operatorname{div}_{\Gamma(t)} V)_l$ to get

$$z^{M+1} = 0\,, \quad \forall \varphi \in W_k^h\,,\ 1 \leq m \leq M\ : \tag{4.2.3}$$

$$\begin{aligned}\int_{I_m} \langle \phi_{t,h}^{t_m} v_h, \varphi^m\rangle_m \,\mathrm{d}t =& \langle z^m, \varphi^m\rangle_m - \langle z^{m+1}, \varphi^m\rangle_m - \int_{I_m} \langle \varphi^m \operatorname{div}_{\Gamma^h(t)} V_h, z^{m+1}\rangle_{L^2(\Gamma^h(t))}\,\mathrm{d}t \dots \\ &+ k \int_{\Gamma^h(t_m)} \nabla_{\Gamma^h(t_m)}\varphi^m \nabla_{\Gamma^h(t_m)} z^m + (\operatorname{div}_{\Gamma(t_m)} V)_l \varphi^m z^m \,\mathrm{d}\Gamma^h(t_m)\,,\end{aligned}$$

instead of (3.7.3).
In the situation of $(\mathbb{P}_d^h)$ however, this discrepancy can be remedied by Lemma 3.5.5 which implies

$$\|(\cdot)_l - (\cdot)^{l^*}\|_{\mathcal{L}(L^2_{L^2(\Gamma)}, L^2_{L^2(\Gamma^h)})}, \|(\cdot)^l - (\cdot)_l{}^*\|_{\mathcal{L}(L^2_{L^2(\Gamma^h)}, L^2_{L^2(\Gamma)})} \leq Ch^2 ,$$

compare also (2.1.3) and (2.1.8), and due to Lemma 3.7.2 which allows us to conclude

$$\|(\cdot)_l - (\cdot)^{l^*}\|_{\mathcal{L}(L^2_{L^2(\Gamma)}, (L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k}))}, \|(\cdot)^l - (\cdot)_l{}^*\|_{\mathcal{L}((L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k}), L^2_{L^2(\Gamma)})} \leq C(h^2 + k) , \quad (4.2.4)$$

if we interpret $(\cdot)_l, (\cdot)^l$ as operators into or on $(L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k})$, respectively.
Hence we get the estimate

$$\begin{aligned}\left\|(\cdot)^l S_d^{h^*}(\cdot)_l - S_d^*\right\| &\leq \left\|((\cdot)^l - (\cdot)_l{}^*) S_d^{h^*}(\cdot)_l\right\| + \left\|(\cdot)_l{}^* S_d^{h^*}((\cdot)_l - (\cdot)^{l^*})\right\| + \underbrace{\left\|(\cdot)_l{}^* S_d^{h^*}(\cdot)^{l^*} - S_d^*\right\|}_{\|(\cdot)^l S_d^h(\cdot)_l - S_d\|} \\ &\leq C(k + h^2) ,\end{aligned}$$

in the $\mathcal{L}(L^2_{L^2(\Gamma)}, L^2_{L^2(\Gamma)})$-operator norm.
As opposed to problem $(\mathbb{P}_d^h)$, in the case of $(\mathbb{P}_T)$ it is easier to proof the convergence of S_T^{h*} than that of S_T^h itself. In the sense of (3.7.3), consider the discretization of the adjoint operator S_T^*

$$S_T^{h^*} : L^2(\Gamma^h(T)) \ni z_T \mapsto z \in W_k^h \subset (L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k})$$

according to the primal scheme after the time-transform $t' = T - t$ as in (4.2.3)

$$\begin{aligned} z^{M+1} = z_T , \quad & \forall \varphi \in W_k^h , \ 1 \leq m \leq M : \\ & \langle z^m, \varphi^m\rangle_m - \langle z^{m+1}, \varphi^m\rangle_{m+1} \cdots \\ & + k \int\limits_{\Gamma^h(t_m)} \nabla_{\Gamma^h(t_m)} z^m \nabla_{\Gamma^h(t_m)} \varphi^m + \operatorname{div}_{\Gamma^h(t_m)} V_h z^m \varphi^m \, \mathrm{d}\Gamma^h(t_m) = 0 . \end{aligned}$$

Corollary 3.7.9 applies an yields $\|(\cdot)^l S_T^{h^*}(\cdot)_l - S_T^*\|_{\mathcal{L}(L^2(\Gamma(T)), L^2_{L^2(\Gamma)})} \leq C(h + \frac{k}{h})$.
Now in addition to (4.2.4) we have (by Lemma 2.1.1)

$$\|(\cdot)_l - (\cdot)^{l^*}\|_{\mathcal{L}(L^2(\Gamma(T)), L^2(\Gamma^h(T)))} \leq Ch^2 .$$

We conclude

$$\left\|(\cdot)^l S_T^{h^{**}}(\cdot)_l - S_T\right\|_{\mathcal{L}(L^2_{L^2(\Gamma)}, L^2(\Gamma(T)))} \leq C(h + \frac{k}{h}) .$$

Hence, the operator $S_T^h = S_T^{h^{**}} : (L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k}) \to L^2(\Gamma^h(T)))$ is a discretization of S_T. Also, the mapping $S_T^h : u_h \mapsto y_{h,k}(T)$ is implemented by the scheme

$$\begin{aligned} y^0 = 0 , \quad & \forall \varphi \in W_k^h , \ 1 \leq m \leq M : \\ & \langle y^m, \varphi^m\rangle_m - \langle y^{m-1}, \varphi^m\rangle_m \cdots \\ & + k \int\limits_{\Gamma^h(t_m)} \nabla_{\Gamma^h(t_m)} y^m \nabla_{\Gamma^h(t_m)} \varphi^m + (\operatorname{div}_{\Gamma^h(t_m)} V_h)_l y^m \varphi^m \, \mathrm{d}\Gamma^h(t_m) = k\langle \bar{u}_h^m, \varphi^m\rangle_m , \end{aligned}$$

as shows summation over $1 \leq m \leq M$.

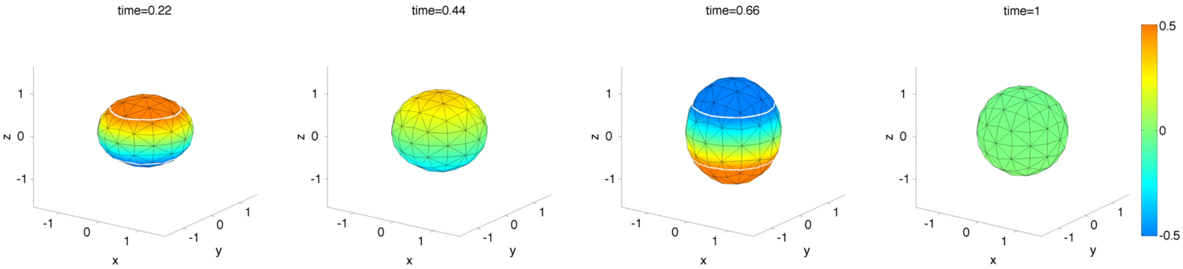

Figure 4.1: Selected time snapshots of u_h computed for Example 4.3.1 on the sphere after four refinements.

Remark 4.2.3. It is also possible to prove convergence of the scheme (3.7.3) towards S_T^*. One way to achieve this is to rewrite Section 3.7 with the roles of the primal and dual schemes reversed. An other involves the remainder

$$\mathfrak{r}_n(\psi,\varphi) = \int_{I_n} \langle \varphi \,\mathrm{div}_{\,\Gamma^h(t)} V_h, \psi\rangle_{L^2(\Gamma^h(t))}\,\mathrm{d}t - k\langle \varphi\,\mathrm{div}_{\,\Gamma^h(t_n)} V_h, \psi\rangle_{L^2(\Gamma^h(t_n))}\,,$$

as well as slight generalizations of Corollary 3.7.9 and Lemma 3.7.5.

If y_T is more regular, such as $y_T \in H^1(\Gamma(T))$, then one might want to apply results from [DE11] that state h^2-convergence of the discretization S_T^h, yet not in the $\mathcal{L}(L^2_{L^2(\Gamma)}, L^2(\Gamma(T)))$-norm. In order to do so, it remains to ensure the regularity assumptions of [DE11, Thm. 4.4] to be met by the optimal control u.

4.3 Numerical examples

Provided the results from [HIK03] and [Ulb03] hold on moving surfaces, compare also Appendix 1.C, Equation (4.2.2) is semismooth due to the smoothing properties of S_d^{h*}. These are due to the stability ensured by Lemma 3.7.5 which a priori holds only in the case $\mu_h \geq 1$, but can easily be extended for arbitrary μ_h, μ for example by rescaling, see Remark 3.7.10. By Lemma 3.7.5 the operator $\phi^s_{\cdot,h} S_d^{h^*}$ continuously maps $(L^2_{L^2(\Gamma^h)}, \langle\cdot,\cdot\rangle_{h,k})$ into

$$L^\infty([0,T], H^1(\Gamma^h(s))) \subset L^p([0,T], L^p(\Gamma^h(s))) \simeq L^p([0,T]\times\Gamma^h(s))$$

for every $2 < p < \infty$. This would imply semismoothness of the operator

$$P_{[a,b]}\left(-\frac{1}{\alpha}\phi^s_{t,h}\left(p_d^h\left(\phi^t_{s,h}(\cdot)\right)\right)\right) : L^2([0,T]\times\Gamma^h(s)) \to L^2([0,T]\times\Gamma^h(s))\,,$$

compare [Ulb11, Thm. 3.49], and thus of equation (4.2.2).
We implemented a semismooth Newton Algorithm for (4.2.2), along the lines of [Vie07].

Example 4.3.1 (High Regularity: Sphere)**.** Consider problem $(\mathbb{P}_d)$ with $\alpha = 1$, $a = -\frac{1}{2}$, $b = \frac{1}{2}$, $T = 1$, and $\Gamma_0 \subset \mathbb{R}^3$ the unit sphere.

R	ERR_{L^2}	EOC_{L^2}	ERR_∞	EOC_∞	R	ERR_{L^2}	EOC_{L^2}	ERR_∞	EOC_∞
0	1.68e-01	-	8.71e-01	-	5	6.78e-03	2.15	1.08e-01	2.01
1	5.40e-02	-	7.88e-01	-	6	3.15e-03	2.09	5.01e-02	1.97
2	4.13e-02	2.45	5.32e-01	0.86	7	1.72e-03	2.03	2.80e-02	1.99
3	2.60e-02	1.78	3.78e-01	1.79	8	7.92e-04	2.02	1.31e-02	1.97
4	1.24e-02	2.21	1.82e-01	1.97					

Table 4.1: L^2-error, L^∞-error and the corresponding EOCs for Example 4.3.1.

Let $\Gamma(t) = \Phi_0^t \Gamma_0$ with $\Phi_0^t(x_1, x_2, x_3) = (x_1, x_2, x_3/\rho_t)^T$ and $\rho_t = e^{\frac{\sin(2\pi t)}{4}}$. In coordinates (x_1, x_2, x_3) of $\mathbb{R}^3$ let

$$u = P_{[-\frac{1}{2},\frac{1}{2}]}(x_3 \sin(2\pi t))$$

and $y_d := S_d u - \tilde{y}_d$ where $\tilde{y}_d = (S_d^*)^{-1} x_3 \sin(2\pi t) = -(x_3 \dot{\sin}(2\pi t)) - \Delta_\Gamma x_3 \sin(2\pi t)$, i.e.,

$$\tilde{y}_d = -x_3 \left(\left(\frac{\pi}{2} \sin(2\pi t) - 2\pi \right) \cos(2\pi t) + \frac{\sin(2\pi t)\rho_t^2}{x_1^2 + x_2^2 + \rho_t^4 x_3^2} \left(\rho_t^2 + 1 - x_3^2 \frac{\rho_t^6 - \rho_t^4}{x_1^2 + x_2^2 + \rho_t^4 x_3^2} \right) \right) .$$

Then u solves $(\mathbb{P}_d)$.

Note that the construction of the desired state y_d in Example 4.3.1 is facilitated by the fact that the adjoint equation in its strong form lacks the $\operatorname{div}_\Gamma V$-term. The same holds for Example 4.3.2.
In order to compute the solution u_h of $(\mathbb{P}_d^h)$ we construct triangulations of Γ_0 from our macro-triangulation R_0^0. In the case of the sphere R_0^0 is a cube whose nodes reside on Γ_0 triangulated into 12 rectangular triangles. We generate R_0^{i+1} from R_0^i through longest edge refinement followed by projecting the inserted vertices onto Γ_0.
Table 4.1 shows the relative error in the $L^2_{L^2(\Gamma^h)}$-norm $ERR_{L^2}^i$ and the relative L^∞-error

$$ERR_\infty^i = \frac{\|\phi_{t,h}^s(u_h - u_l)\|_{L^\infty([0,T]\times\Gamma^h(s))}}{\|\phi_{t,h}^s u_l\|_{L^\infty([0,T]\times\Gamma^h(s))}} ,$$

on the ith refinement of the macro-triangulation as well as the corresponding experimental orders of convergence

$$EOC_{L^2}^i = \ln \frac{ERR_{L^2}^i}{ERR_{L^2}^{i-q}} \left(\ln \frac{H_i}{H_{i-q}} \right)^{-1} , \quad EOC_\infty^i = \ln \frac{ERR_\infty^i}{ERR_\infty^{i-q}} \left(\ln \frac{H_i}{H_{i-q}} \right)^{-1} ,$$

where H_i denotes the maximal edge length of R_0^i, see Table 4.3. Throughout this section we chose $q = 2$ for both EOC_{L^2} and EOC_{L^∞}, and the time step length is $k_i = \frac{1}{20} H_i^2$.
Note that the L^∞-error is estimated by the maximal error at the nodes of the triangulations $\Gamma^h(t_m)$. Since Example 4.3.1 and 4.3.2 admit smooth solutions this procedure is correct of order $\mathcal{O}(h^2)$. The L^2-errors are computed by triangle-wise Gaussian quadrature of order $\mathcal{O}(h^3)$ combined with Gaussian quadrature in time of order $\mathcal{O}(k^3)$.
Figure 4.1 shows the solution of $(\mathbb{P}_d^h)$ at different points in time. Note that the white lines mark the borders between active and inactive sets. On the active parts, the optimal control assumes the value a or b, respectively.

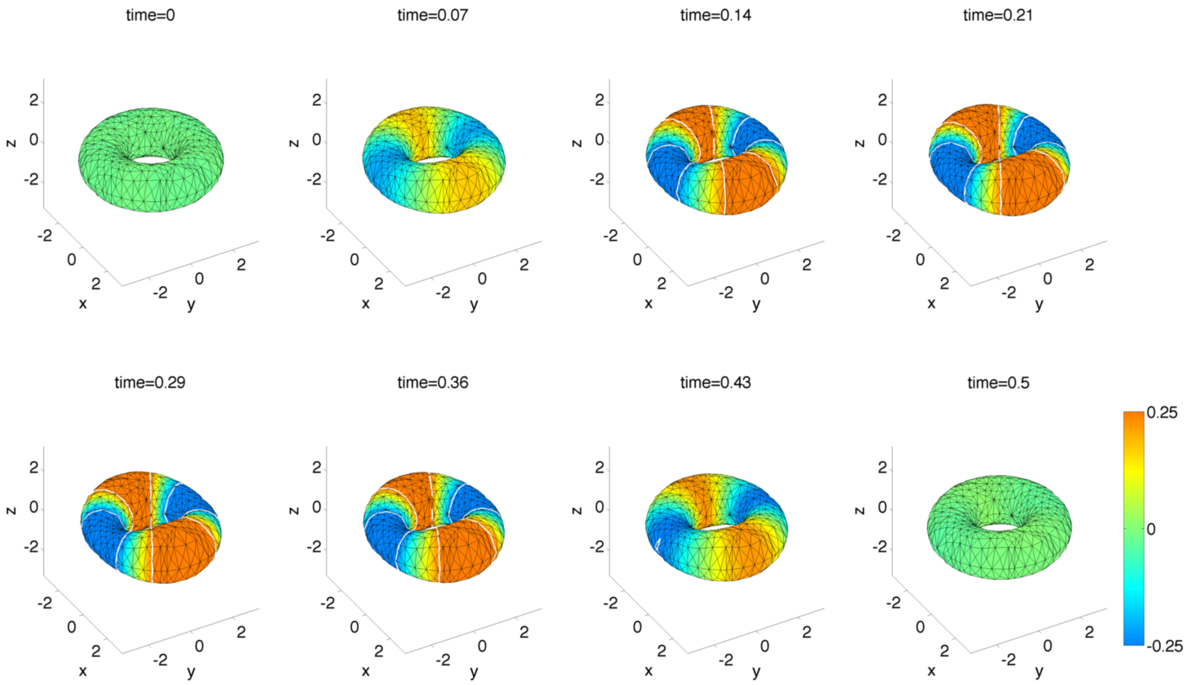

Figure 4.2: Selected time snapshots of u_h computed for Example 4.3.2 on the four times refined torus.

For another Example consider the oscillating torus

$$\Gamma(t) = \mathrm{Im}\left(\begin{pmatrix} \cos(\varphi) \\ \sin(\varphi) \\ 0 \end{pmatrix} + \left(1 + \frac{1}{2}\sin(2\pi t)\cos(2\varphi)\right) \begin{pmatrix} \cos(\psi)\cos(\varphi) \\ \cos(\psi)\sin(\varphi) \\ \sin(\psi) \end{pmatrix} \right), \tag{4.3.1}$$

with coordinates $\varphi, \psi \in [0, 2\pi)$.

Example 4.3.2 (High Regularity: Torus)**.** Consider problem $(\mathbb{P}_d)$ with $\alpha = 100$, $a = -\frac{1}{4}$, $b = \frac{1}{4}$, $T = 1$, and $\Gamma(t)$ as in (4.3.1).
Let

$$u = P_{[-\frac{1}{4},\frac{1}{4}]}\left(\frac{1}{2}\sin(2\pi t)\cos(2\varphi)\right)$$

and the desired state $y_d = (-\partial_t - \Delta_\Gamma)\frac{1}{2}\sin(2\pi t)\cos(2\varphi) + S_d u$.
Then u solves $(\mathbb{P}_d)$.

Table 4.2 and Figure 4.2 illustrate the behavior of the discretization of Example 4.3.2.
Finally, let us conclude with an example for $(\mathbb{P}_T^h)$ with a desired state y_T that just barely lies in $L^2(\Gamma(T))$. In this situation we can only expect $\mathcal{O}(h)$-convergence.

Example 4.3.3 (Low Regularity)**.** Consider problem $(\mathbb{P}_T)$ with $\alpha = 1$, $a = -\infty$, $b = \infty$, $T = 1$ and $\Gamma(t)$ as in Example 4.3.1. Let $y_T = \frac{1}{(x+y)^{0.49}}$.

Since we do not know the exact solution of Example 4.3.3, we estimate the relative error by $ERR_{L^2}^i \simeq \|u_i^{l_h} - u_{i+2}\|_{L^2_{L^2(\Gamma^{h,i+2})}} / \|u_{i+2}\|_{L^2_{L^2(\Gamma^{h,i+2})}}$, where u_i denotes the solution of $(\mathbb{P}_T^h)$ on

R	H	ERR_{L^2}	EOC_{L^2}	ERR_∞	EOC_∞	R	H	ERR_{L^2}	EOC_{L^2}	ERR_∞	EOC_∞
0	2.44	0.3202	-	1.2962	-	5	0.83	0.0157	3.4	0.1425	2.9
1	2.07	0.2335	-	1.2904	-	6	0.61	0.0089	2.1	0.0748	2.0
2	1.60	0.1093	2.6	0.8107	1.1	7	0.52	0.0065	1.8	0.0542	2.0
3	1.12	0.0451	2.7	0.3525	2.1	8	0.46	0.0051	2.0	0.0429	2.0
4	1.00	0.0252	3.1	0.2035	2.9						

Table 4.2: L^2-error, L^∞-error and the corresponding EOCs for Example 4.3.2.

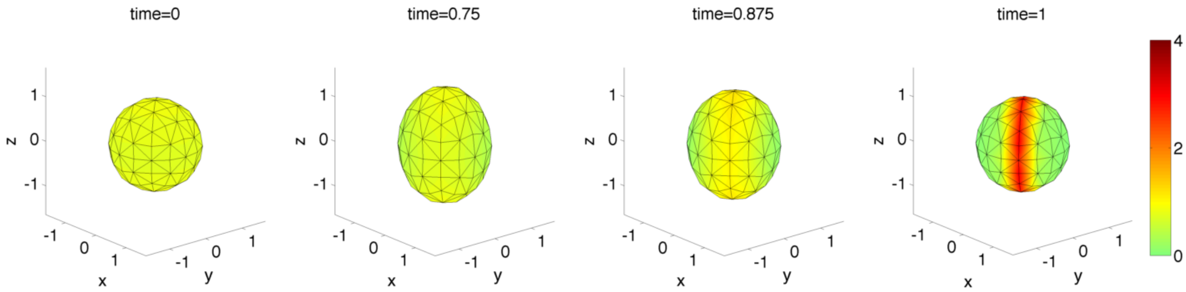

Figure 4.3: Selected time snapshots of u_h computed for Example 4.3.3 on the sphere after four refinements.

R	1	2	3	4	5	6	7	8	9
ERR_{L^2}	0.1899	0.1444	0.1140	0.0701	0.0484	0.0306	0.0215	0.0147	0.0104
EOC_{L^2}	-	-	1.2414	1.3272	1.3709	1.2617	1.2030	1.0781	1.0520
H	1.1547	0.9194	0.7654	0.5333	0.4099	0.2769	0.2085	0.1398	0.1047

Table 4.3: L^2-error and the corresponding EOC for Example 4.3.3. H is the maximal edge length of Γ_0^h (both examples).

the ith refinement $\{\Gamma^{h,i}(t)\}_{t\in[0,T]}$ of $\{\Gamma^h(t)\}_{t\in[0,T]}$. The lift $(\cdot)^{l_h}$ is taken perpendicular to the smooth surface $\Gamma(t)$. Table 4.3 shows the estimated L^2-errors and corresponding EOCs. We computed the $L^2(\Gamma^h(T))$-projection $P_T^h(y_T)_l$ analytically. Otherwise one would need to take measures to prevent the error introduced by the numerical integration of the non-smooth function y_T from dominating the overall numerical error. It helps that all our triangulations resolve the plane $\{x + y = 0\}$.

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Summary

We investigate linear-quadratic parabolic optimal control problems on evolving material hypersurfaces in $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$. In addition, we present a globalized Newton method for elliptic optimal control problems on stationary surfaces.

Following [DE07], we consider parabolic state equations in their weak form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\Gamma(t)} y\,\varphi \,\mathrm{d}\Gamma(t) + \int_{\Gamma(t)} \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{b}\, y\,\varphi \,\mathrm{d}\Gamma(t) = \int_{\Gamma(t)} y\,\dot{\varphi}\,\mathrm{d}\Gamma(t) + \int_{\Gamma(t)} u\,\varphi\,\mathrm{d}\Gamma(t)\,, \quad y(0) = y_0\,,$$

where $\Gamma = \{\Gamma(t)\}^{t\in[0,T]}$ is a family of C^2-smooth, compact n-dimensional surfaces in $\mathbb{R}^{n+1}$, evolving smoothly in time with velocity V, and $\dot{\varphi} = \partial_t \varphi + V\nabla\varphi$ denotes the material derivative of a smooth test function φ. We define unique weak solutions for the state equation under low regularity assumptions on the data u, y_0. In particular we allow for $y_0 \in L^2(\Gamma(0))$. The idea is to introduce distributional material derivatives in the sense of [LM68] and a $W(0,T)$-like solution space.

The stationary diffusion equation on a fixed surface Γ reads

$$\int_\Gamma \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{c}\, y\,\varphi\,\mathrm{d}\Gamma = \int_\Gamma u\,\varphi\,\mathrm{d}\Gamma\,, \forall \varphi \in H^1(\Gamma)\,.$$

Both in the stationary and the instationary case each surface is approximated by a triangulation Γ^h on which a finite element scheme for the state equation is formulated along the lines of [Dzi88] and [DE07], respectively. The approximation error of this discretization of the state equation decomposes into a finite element error, arising from the projection onto a finite dimensional Ansatz space, and a geometrical part which is due to the approximation of Γ by Γ^h. We prove convergence results for the parabolic equations under weak regularity assumptions.

The state equations define linear control-to-state operators. Using these, we formulate control constrained optimal control problems along with their necessary optimality conditions where the adjoint state equations appear. The optimal control problems are subjected to variational discretization, see [Hin05], by replacing Γ and the state equation by their finite dimensional approximations. The variationally discretized problems are amenable to an implementable semismooth Newton algorithm. In both cases we prove convergence of the discretized optimal controls.

In the elliptic case we also discuss in some detail the implementation of a globalized semismooth Newton algorithm for the control problem, involving a new merit function. In the parabolic setting a suitable scalar product is formulated in order to arrive at an easily computable discrete adjoint scheme.

Our analytical findings are complemented with numerical examples.

Zusammenfassung

Wir untersuchen linear-quadratische Optimalsteuerungsprobleme auf sich bewegenden Flächen in $\mathbb{R}^{n+1}$, $n \in \mathbb{N}$. Zusätzlich geben wir ein globalisiertes semiglattes Newtonverfahren für elliptische Optimalsteuerungsprobleme auf stationären Flächen an.
Wie in [DE07] betrachten wir eine parabolische Zustandsgleichung in schwacher Form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{\Gamma(t)} y\,\varphi \,\mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{b}\, y\,\varphi \,\mathrm{d}\Gamma(t) = \int\limits_{\Gamma(t)} y\,\dot{\varphi}\,\mathrm{d}\Gamma(t) + \int\limits_{\Gamma(t)} u\,\varphi\,\mathrm{d}\Gamma(t)\,, \quad y(0) = y_0\,,$$

wobei $\Gamma = \{\Gamma(t)\}^{t\in[0,T]}$ eine Familie von C^2-glatten, kompakten n-dimensionalen Flächen in $\mathbb{R}^{n+1}$ bezeichnet, die sich gemäß des Geschwindigkeitsfeldes V verformen; der Ausdruck $\dot{\varphi} = \partial_t \varphi + V\nabla\varphi$ steht hier für die Materialableitung einer hinreichend glatten Testfunktion φ. Wir definieren eindeutige Lösungen der Zustandsgleichung unter geringen Regularitätsannahmen an die Daten u, y_0. Insbesondere berücksichtigen wir Anfangswerte mit niedriger Regularität $y_0 \in L^2(\Gamma(0))$. Hierzu werden schwache Materialableitungen im Sinne von [LM68] und ein $W(0,T)$-artiger Lösungsraum eingeführt.
Im elliptischen Fall beschreibt die Zustandsgleichung Diffusion auf einer kompakten Hyperfläche Γ

$$\int\limits_{\Gamma} \nabla_\Gamma y \nabla_\Gamma \varphi + \mathbf{c}\, y\,\varphi\,\mathrm{d}\Gamma = \int\limits_{\Gamma} u\,\varphi\,\mathrm{d}\Gamma\,, \forall \varphi \in H^1(\Gamma)\,.$$

Sowohl im parabolischen als auch im elliptischen Fall wird die Zustandsgleichung mittels eines Finite-Element Ansatzes auf Triangulierungen Γ^h der Flächen Γ diskretisiert, siehe [Dzi88] und [DE07]. Der damit verbundene Approximationsfehler zerfällt in einen Finite-Element Anteil, der aus der Projektion auf einen endlichdimensionalen Ansatzraum resultiert, und einen geometrischen Anteil, der der Diskretisierung von Γ durch Γ^h Rechnung trägt. Wir beweisen Konvergenzaussagen für die Diskretisierung der parabolischen Gleichung unter schwachen Regularitätsannahmen.
Mit den Zustandsgleichungen lassen sich kontrollbeschränkte Optimalsteuerungsprobleme formulieren. Diese diskretisieren wir variationell, vergleiche [Hin05], indem wir Γ und die Zustandsgleichung durch ihre jeweiligen diskreten Approximationen ersetzen. Das variationell diskretisierte Problem kann dann mittels eines semiglatten Newtonalgorithmus gelöst werden. In beiden Fällen werden optimale Konvergenzordnung für die Kontrollen bewiesen.
Im elliptischen Fall geben wir zudem eine Globalisierung des Newtonverfahrens an, die auf einer neuen Bewertungsfunktion beruht. Im parabolischen Fall wird das L^2-Skalarprodukt in geeigneter Weise diskretisiert, um einen implementierbaren adjungierten Lösungsoperator zu erhalten.
Die analytischen Betrachtungen werden durch numerische Beispiele ergänzt.

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