Approximate Solutions of Generalized Riemann Problems for Hyperbolic Conservation Laws and Their Application to High Order Finite Volume Schemes

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Introduction

A conservation law in one spatial dimension is a first-order partial differential equation of the form

$$\frac{\partial}{\partial t}u(x,t) + \frac{\partial}{\partial x}f\left(u(x,t)\right) = 0, \qquad (x,t) \in \mathbb{R} \times [0,\infty),\ u \in \mathbb{R}^m,\ f : \mathbb{R}^m \to \mathbb{R}^m.$$

While some conservation laws are elliptic, those arising from the classical problems in continuum physics (and most importantly gas or fluid dynamics) are typically hyperbolic. This characterisation is associated with a finite speed of propagation. The second outstanding feature of nonlinear hyperbolic conservation laws is that, even for arbitrarily smooth initial data, classical solutions may break down in finite time and shocks develop spontaneously. It is therefore mandatory to work with weak solutions and the lack of regularity renders many tools from modern analysis inapplicable. Studying hyperbolic conservation laws thus often requires the use of highly technical and very specialized methods.

Conservation laws arise in many applications, ranging form areas as elasticity or multi-component chromatography to the simulation of traffic flow or tsunami waves. To solve real-world problems, there is a growing demand for high order accurate numerical methods that are capable of capturing shocks. Questions of efficient implementation, parallelization and grid-adaptivity are becoming more and more important but still there comes a great benefit for numerical tools from a rigorous study of the underlying analytical problems, and vice versa. This is especially true in the field of hyperbolic conservation laws, where the connection between classical analysis and numerical methods is even closer than in other areas. Indeed, Glimm's celebrated existence result [35] is based on the convergence of an approximation method that can be implemented as a useful numerical scheme.

A fundamental building block in both the analytical and the numerical treat-

ment of hyperbolic conservation laws is the Riemann problem, which is a Cauchy problem with piecewise constant but discontinuous initial data. Modern results concerning existence, uniqueness and continuous dependence on the initial data for nonlinear systems of hyperbolic conservation laws rely on front tracking approximations, which need to solve local Riemann problems. In his ground breaking work, Godunov [36] proposed to solve conservation laws numerically by approximating the solution by a piecewise constant function and then solving the resulting local Riemann problems exactly. The natural extension of this approach is the use of piecewise smooth approximations instead of piecewise constant functions. The resulting initial value problem with piecewise smooth but discontinuous initial data is called the generalized Riemann problem.

A state of the art variant of this generalized Godunov approach is the ADER (*Arbitrary DERivatives*) scheme. Since the generalized Riemann problem usually cannot be solved analytically, Toro and Titarev [98] proposed to approximate the solution by a Taylor series expansion and compute the coefficients in that expansion by solving a sequence of classical Riemann problems. This method (which we call the Toro-Titarev solver) has performed very well in a wide range of applications, but very few rigorous analysis of this technique has been reported so far. Moreover, it was reported that while ADER schemes do achieve the designed order of accuracy in regions where the solution is smooth, the Toro-Titarev solver encounters severe difficulties when dealing with large jumps in the initial data ([19], [65]). Currently there seems to be no explanation for this effect. It is this need for a better analytical understanding of the Toro-Titarev solver that has motivated this thesis.

Our main tool for analysing the Toro-Titarev solver is a comparison with an asymptotic expansion for the solution of a generalized Riemann problem that was constructed by LeFloch and Raviart [56]. This expansion, when truncated, coincides with a local Taylor series approximation. Coefficients in this expansion are computed through the Rankine-Hugoniot conditions.

We show that both the LeFloch-Raviart expansion and the Toro-Titarev solver formally construct the same truncated Taylor series expansion. The only difference is the way spatial derivatives at the origin are found. It turns out that the way those derivatives are computed is the same in both methods when they are ap-

plied to scalar problems. For nonlinear systems the two methods no longer produce the same results. While the Toro-Titarev solver uses linear Riemann problems to compute spatial derivatives at the origin, in the LeFloch-Raviart expansion those derivatives are found by solving a linear system of algebraic equations obtained through the Rankine-Hugoniot conditions. These problems do not have the same solution. We show, however, that their distance is small when the jump in the initial states is small. This explains for the first time why the Toro-Titarev solver produces very good results in regions where the solution is smooth but has difficulties in the presence of large shocks.

In Chapter 1 we give the necessary definitions of entropy weak solutions and give a rough overview of existence, uniqueness and regularity results. We review the classical theory of Riemann problems and discuss the local properties of generalized Riemann problems. In Chapter 2 we present the basic framework for generalized Godunov schemes and the ADER scheme with the Toro-Titarev solver in particular. We give a detailed review of the LeFloch-Raviart expansion in Chapter 3. For illustration we provide the full analysis for 2×2 systems of conservation laws up to quadratic terms, which seems like it has not been available in the literature so far.

The main results of this thesis are contained in Chapter 4. We first show that the LeFloch-Raviart expansion and the Toro-Titarev solver both formally construct the same Taylor approximation, the only difference being the way spatial derivatives are found. Theorem 4.2 states that both methods agree for scalar problems. We then use this new insight that the ADER numerical flux is indeed a truncated Taylor expansion to extend a stability result that previously required an exact solution of the generalized Riemann problem (Theorem 4.5). Finally, we show that the Toro-Titarev solver no longer reproduces the Taylor approximation in the case of systems (Theorem 4.6). We explain why this error is small when the jump in the initial data is small.

In Chapter 5 we illustrate our results by providing theoretical examples (for Burgers equation and a system from two component chromatography) as well as numerical test cases (for shallow water equations and for an Aw-Rascle type model for traffic flow). We summarize our work and give an outlook on open problems in Chapter 6.

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- C. R. Goetz and A. Iske, *Approximate solutions of generalized Riemann problems: The Toro-Titarev solver and the LeFloch-Raviart expansion*, Numerical Methods for Hyperbolic Equations: Theory and Applications, M. E. Vásques-Cedón and A. Hidalgo and P. García-Navarro and L. Cea, eds., Taylor-Francis Group, 2013, pp. 267 - 275

- C. R. Goetz and A. Iske, *Approximate Solutions of Generalized Riemann problems for Nonlinear Systems of Hyperbolic Conservation Laws*, Hamburger Beiträge zur Angewandten Mathematik 2013-02 (*preprint, submitted*)

1 Preliminaries

1.1 Analytical framework

In this chapter we set the analytical framework for our study of generalized Riemann problems and give the necessary definition of entropy weak solutions. We understand solutions to the Cauchy problem (i.e., the initial value problem with data on the whole real line) for nonlinear hyperbolic conservation laws as limits obtained by a front tracking approximation (see below) in the space BV of functions with bounded variation. We present the basic results concerning existence, uniqueness and continuous dependence on the initial data. Since front tracking and the numerical schemes to be discussed in later chapters rely on the solution of Riemann problems, we present the classical theory of Riemann problems in some detail. Moreover, it turns out that the solution of a generalized Riemann problem has a wave structure that is similar to that of an underlying classical Riemann problem. We discuss this local structural stability.

There are many texts that cover the basic theory of hyperbolic conservation laws, our main sources for this chapter being the monographs by Bressan [14], LeFloch [55] and Serre [75], where the notation mostly follows [55]. Another rich source for examples, applications and references is the book by Dafermos [29].

Out of a long list of references concerning the numerical treatment of hyperbolic conservation laws, we mention the books by LeVeque [58] and Toro [91], the lecture notes by Tadmor [82] and the review article by Morton and Sonar [66].

1.1.1 Functions of bounded variation

The space BV of functions with bounded variation plays an important role in the theory of hyperbolic conservation laws. Though L^1_{loc} is the natural space to look

for solutions and a fine L^1- and L^∞-theory has been developed for scalar problems, existence, uniqueness and continuous dependence on the initial data for systems of hyperbolic conservation laws have thus far been proven in the space BV with small BV initial data. For a detailed discussion of BV functions and their properties we refer to the monographs by Ziemer [108] and Evans and Gariepy [33]. The major contribution to the treatment of conservation laws in the space BV is due to Volpert [101].

Recall that a function $u : D \to \mathbb{R}^m$, where $D \subseteq \mathbb{R}^d$, is said to have *bounded variation* if its distributional derivatives are Radon measures. In the case $D = (a, b) \subset \mathbb{R}$, where (a, b) may be unbounded, this can be expressed as the condition

$$TV(u;(a,b)) = \sup_{\substack{D_n(a,b)\\ n\in\mathbb{N}}} \left\{ \sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)| \right\} < \infty,$$

where the supremum is taken over all sets $D_n(a, b) = \{x_1, x_2, \ldots, x_n\}$ such that $a < x_1 < x_2 < \cdots < x_n < b$. Then, for smooth u we have

$$TV(u;(a,b)) = \|\partial_x u\|_{L^1((a,b);\mathbb{R}^m)}$$

and if u has bounded variation, we get

$$TV(u;(a,b)) = \sup_{h>0} \frac{1}{h} \|u(\cdot + h) - u(\cdot)\|_{L^1((a,b);\mathbb{R}^m)}.$$

The space $BV((a, b); \mathbb{R}^m)$ of all functions with bounded variation equipped with the norm

$$\|u\|_{BV((a,b);\mathbb{R}^m)} = \|u\|_{L^\infty((a,b);\mathbb{R}^m)} + TV(u;(a,b))$$

is a Banach space.

Although the existence of distributional derivatives as Radon measures is the weakest differentiability condition in a measure theoretical context, it turns out that BV functions have a fine geometrical structure. The following regularity result is essential to establish many desired properties of solutions of conservation laws.

Theorem 1.1. *Given a function $u \in BV(\mathbb{R} \times (0,T), \mathbb{R}^m)$, there exists a representative of u, which differs from u only on a set of Lebesgue measure zero and is still denoted by u, together with a decomposition*

$$\mathbb{R} \times (0,T) = \mathcal{C}(u) \cup \mathcal{J}(u) \cup \mathcal{I}(u)$$

such that:

1. *$\mathcal{C}(u)$ is the set of points of approximate L^1-continuity (x,t) in the sense that for a ball $B_r(x,t)$ of radius r centred at (x,t) we have*

$$\lim_{r \to 0} \frac{1}{r^2} \int_{B_r(x,t)} |u(y,s) - u(x,t)| \, dyds = 0.$$

2. *$\mathcal{J}(u)$ is the set of points of approximate jump discontinuity (x,t), which means that at those points there exists a propagation speed $\lambda^u(x,t)$ and left- and right approximate limits $u_-(x,t)$, $u_+(x,t)$, respectively, such that*

$$\lim_{r \to 0} \frac{1}{r^2} \int_{B_r^{\pm}(x,t)} |u(y,s) - u_{\pm}(x,t)| \, dyds = 0,$$

where

$$B_r^{\pm}(x,t) = B_r(x,t) \cap \{\pm(y - \lambda^u(x,t)s) \geq 0\}.$$

Moreover, the set $\mathcal{J}(u)$ is the union of at most countable arcs in the plane, and we have

$$\int_{\mathcal{J}(u)} |u_+ - u_-| \, d\mathcal{H}_1 < \infty,$$

where $\mathcal{H}_1$ denotes the one-dimensional Hausdorff measure.

3. *Finally, the set of interaction points $\mathcal{I}(u)$ has one-dimensional Hausdorff measure zero:*

$$\mathcal{H}_1(\mathcal{I}(u)) = 0.$$

■

See [33, Chapter 5.9] and [108, Chapter 5.9] for a proof.

We will see in Theorem 1.15 that we can even improve on this when u is a solution (in a suitable sense) of a hyperbolic conservation law instead of an arbitrary BV function. As for compactness in the space BV, we have the following famous result:

Theorem 1.2 (Helly's compactness theorem in BV)**.** *Let $u_h : (a,b)\times[0,\infty) \to \mathbb{R}^m$ be a sequence of Lebesgue measurable functions such that*

$$\begin{aligned} \|u_h(\cdot,t)\|_{L^\infty((a,b);\mathbb{R}^m)} + TV(u_h(\cdot,t);(a,b)) &\le C, \qquad t \ge 0, \\ \|u_h(\cdot,t_1) - u_h(\cdot,t_2)\|_{L^1((a,b);\mathbb{R}^m)} &\le C|t_1 - t_2|, \qquad t_1, t_2 \ge 0, \end{aligned}$$

for some constant $C > 0$.

Then there exists a subsequence, still labelled u_h, and a function $u : (a,b) \times [0,\infty) \to \mathbb{R}^m$ with:

$$\begin{aligned} u_h(x,t) &\to u(x,t) \qquad \text{for a.e. } (x,t) \in (a,b) \times [0,\infty), \\ u_h(\cdot,t) &\to u(\cdot,t) \qquad \text{in } L^1_{loc}. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \|u(\cdot,t)\|_{L^\infty((a,b);\mathbb{R}^m)} + TV(u(\cdot,t);(a,b)) &\le C, \qquad t \ge 0, \\ \|u(\cdot,t_1) - u(\cdot,t_2)\|_{L^1((a,b);\mathbb{R}^m)} &\le C|t_1 - t_2|, \qquad t_1, t_2 \ge 0. \end{aligned}$$

■

A proof of Helly's Theorem can be found in [14, Theorem 2.3]. Helly's theorem is, of course, an important ingredient in the convergence analysis for numerical schemes. Even more, results on existence of solutions are based on the construction of approximate solutions and Helly's theorem is the usual way to establish convergence of these approximations. Typically, establishing the bounds on the total variation is the hard part.

Note that because of to the nonlinearity of the flux, constructing a weakly-$*$ convergent sequence of approximate solutions (say, by uniform L^∞-bounds and Alaoglu's theorem, see [103]) is not enough. In that case the limit will, in general, not be a solution of the differential equation. For more on the troubles caused by a

lack of strong convergence, see [32]. Another approach to circumvent this problem without relying on TV bounds is the concept of *measure-valued solutions*, due to DiPerna [31]. For a detailed introduction to this subject, see the monograph [62]. Convergence of numerical schemes in the context of measure-valued solutions is discussed in [25, 50, 81].

1.1.2 Weak solutions and entropy

Let $\mathcal{U} \subset \mathbb{R}^m$ be an open and convex subset, $f : \mathcal{U} \to \mathbb{R}^m$ a smooth function and $\hat{u} : \mathbb{R} \to \mathcal{U}$. We consider the *Cauchy problem*:

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0, \qquad x \in \mathbb{R},\ t > 0. \tag{1.1}$$

$$u(x, 0) = \hat{u}(x), \qquad x \in \mathbb{R},\ t = 0. \tag{1.2}$$

It is well known that even for smooth initial data $\hat{u}$, classical (that is, C^1) solutions to (1.1), (1.2) may break down in finite time. See the classical results by John [45]. Therefore, we have to work with weak solutions. Denote the space of all C^∞-functions with compact support contained inside $\mathbb{R}\times[0,\infty)$ by $C_0^\infty(\mathbb{R}\times[0,\infty))$.

Definition 1.3. *Let $\hat{u} \in L^\infty(\mathbb{R},\mathcal{U})$. Then a measurable function $u : \mathbb{R} \times [0,\infty) \to \mathcal{U}$ is a* distributional solution *to the Cauchy problem* (1.1),(1.2)*, if we have*

$$\int_{\mathbb{R}} \int_0^\infty \left(u \frac{\partial}{\partial t} \varphi + f(u) \frac{\partial}{\partial x} \varphi \right) \, dt dx + \int_{\mathbb{R}} \varphi(\cdot, 0) \hat{u} \, dx = 0$$

for every test function $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$. Moreover, if u is a distributional solution such that the mapping $t \mapsto u(\cdot, t)$ is continuous from $[0,\infty) \to L^1_{loc}$, we say that u is a weak solution*.*

Weak solutions are, in general, not unique. To overcome this problem, additional admissibility conditions are required. We first note that every *classical solution* satisfies an additional accompanying conservation law: If $Q : \mathcal{U} \to \mathbb{R}$ and $F : \mathcal{U} \to \mathbb{R}$ are smooth functions such that $\nabla F(u)^T = \nabla Q(u)^T \ Df(u)$, then every classical

solution satisfies

$$\frac{\partial}{\partial t}Q(u) + \frac{\partial}{\partial x}F(u) = 0.$$

This motivates the following definition:

Definition 1.4 (Entropy condition)**.** *Let $Q : \mathcal{U} \to \mathbb{R}$ be continuously differentiable. Then Q is called an* entropy *for the system* (1.1) *with* entropy flux $F : \mathcal{U} \to \mathbb{R}$*, if*

$$\nabla F(u)^T = \nabla Q(u)^T \; Df(u), \qquad u \in \mathcal{U}.$$

We say that a weak solution of (1.1) *satisfies the* entropy condition*, if the inequality*

$$\frac{\partial}{\partial t}Q(u) + \frac{\partial}{\partial x}F(u) \leq 0 \tag{1.3}$$

holds in the distributional sense for every convex entropy Q with corresponding entropy flux F. That is, for every test function $\varphi \in C_0^\infty(\mathbb{R}^m \times [0,\infty))$ with $\varphi \geq 0$, we have

$$\int_{\mathbb{R}^m}\int_0^\infty \left(Q(u)\frac{\partial}{\partial t}\varphi + F(u)\frac{\partial}{\partial x}\varphi \right) \; dtdx \geq 0.$$

We refer to a weak solution that satisfies the entropy condition as an *entropy weak solution.*

We can interpret the entropy condition as a generalization of L^2 energy estimates that we encounter frequently for linear problems. For the role of entropy conditions in the stability analysis for numerical methods, see the survey article by Tadmor [83] and references therein.

1.1.3 Theory for scalar problems

For scalar problems one may use the famous one-parameter family of entropy-entropy flux pairs of Kruzkov [51]:

Definition 1.5 (Entropy for scalar problems)**.** *Let $u : \mathbb{R} \times [0,\infty) \to \mathcal{U} \subseteq \mathbb{R}$ be a weak solution to the scalar problem* (1.1)*. Then u satisfies the entropy condition, if for every constant $k \in \mathbb{R}$ and every test function $\varphi \in C_0^\infty(\mathbb{R} \times [0,\infty))$ with $\varphi \geq 0$,*

the following holds:

$$\int_{\mathbb{R}}\int_0^\infty \operatorname{sgn}(u-k)\left((u-k)\frac{\partial}{\partial t}\varphi + (f(u)-f(k))\,\frac{\partial}{\partial x}\varphi\right)\,dtdx \geq 0.$$

Weaker than the Kruzkov entropy condition, but still a very useful admissibility criterion to work with are the *Lax shock inequalities*: Assume that a weak solution to (1.1) is piecewise constant, taking values u_- and u_+, respectively, in two regions separated by a curve $x = \gamma(t)$ with speed $\dot{\gamma}(t) = \sigma(t)$. Then it follows from the entropy condition that

$$f'(u_-) \geq \sigma(t) \geq f'(u_+). \tag{1.4}$$

Note that for a strictly convex flux f, condition (1.4) reduces to $u_- > u_+$. Moreover, it is well know that such a function u is a weak solution if and only if it satisfies the *Rankine-Hugoniot conditions*:

$$f(u_+) - f(u_-) = \dot{\sigma}(t)(u_+ - u_-).$$

We then have the following result:

Theorem 1.6. *Let $f \in C^1(\mathbb{R})$ (not necessarily convex) and $\hat{u} \in L^\infty(\mathbb{R})$. Then (1.1),(1.2) has a unique entropy weak solution defined for all $(x,t) \in \mathbb{R} \times [0,\infty)$. The solution has the properties:*

1. *(Max-Min-Principle) For every $t > 0$,*

$$\sup_{x\in\mathbb{R}} u(x,t) \leq \sup_{x\in\mathbb{R}} \hat{u}(x), \qquad \inf_{x\in\mathbb{R}} u(x,t) \geq \inf_{x\in\mathbb{R}} \hat{u}(x);$$

2. *(L^1_{loc}-continuity) For every compact set $\Omega \subset \mathbb{R}$, the mapping $t \mapsto u(\cdot,t)$ is continuous from $[0,\infty) \to L^1(\Omega)$. Further,*

$$\int_\Omega |u(x,t) - \hat{u}(x)|\,dx \to 0, \quad \text{as } t \to 0;$$

3. *(L^1-contraction) If $\hat{u}, \hat{v} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ and u, v are the corresponding weak*

solutions, then, for every $t \geq 0$,

$$\int_{\mathbb{R}} |u(x,t) - v(x,t)| \, dx \leq \int_{\mathbb{R}} |\hat{u}(x) - \hat{v}(x)| \, dx.$$

■

For a proof of these statements, see [43]. If, in addition, f is strictly convex, we get some sharper results. The following Theorem summarizes the well known results of Lax [53], Oleinik [68] and Dafermos [28].

Theorem 1.7. *Let* f *be strictly convex in* $[a,b]$ *(where we allow* $a = -\infty$ *or* $b = \infty$*) and* $\hat{u}$ *be measurable and bounded with* $a \leq \hat{u} \leq b$. *Let*

$$L(x,t;u) = t(uf'(u) - f(u)) + \int_0^{x - f'(u)t} \hat{u}(y) \, dy.$$

Then the function $G(x,t) = \min_{u \in [a,b]} L(x,t;u)$ *exists and has the following properties:*

1. G *is continuous on* $\mathbb{R} \times [0,\infty)$;
2. $\partial G(x,t)/\partial x = u(x,t)$ *exists on* $\mathbb{R} \times [0,\infty)$ *except for some set* Γ, *which is the union of an at most countable set of Lipschitz continuous curves;*
3. $\partial G(x,t)/\partial t = -f'(u(x,t))$ *for all* $(x,t) \in (\mathbb{R} \times [0,\infty)) \setminus \Gamma$;
4. *The function* u *given by* $\partial G(x,t)/\partial x = u(x,t)$ *is the unique weak solution to* (1.1), (1.2) *that satisfies the Lax shock inequalities;*
5. $u(x\pm,t)$ *exists for all* $t > 0$;
6. *For any fixed point* (x_0,t_0) *with* $t_0 > 0$, *we have for all* $0 < t < t_0$: $u(x_0\pm,t_0) = u(x_0 - f'(u(x\pm,t_0))(t_0 - t),t)$;
7. u *is locally Lipschitz continuous on* $(\mathbb{R} \times [0,\infty)) \setminus \overline{\Gamma}$;
8. *If* $\hat{u}$ *is piecewise monotone, so is* $u(\cdot,t)$.

■

Here, $u(x+,t)$ denotes the limit $\lim_{y \to x, y > x} u(y,t)$.

1.1.4 The Riemann problem

The *Riemann problem* is the Cauchy problem with piecewise constant initial data, say

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = 0, \qquad x \in \mathbb{R}, t > 0, \tag{1.5}$$

$$u(x,0) = \begin{cases} \hat{u}_L, & x < 0, \\ \hat{u}_R, & x > 0, \end{cases} \tag{1.6}$$

where $\hat{u}_L, \hat{u}_R \in \mathcal{U}$. First formulated and solved by Riemann for problems of linearised gas dynamics [72], it is the building block of many numerical methods, starting with the classic Godunov scheme [36], as well as an important ingredient in several theoretical results, such as Glimm's existence and uniqueness theorem for small BV data, which will be discussed in the next section. See [92] for an overview of the history of the Riemann problem in computational science.

We describe the solution of the Riemann problem by means of composite wave mappings, in the framework considered by Lax [53]. While Lax' assumptions (given in the following definitions) in his ground breaking work still form the basis for many advanced results today, Riemann problems have been solved under a variety of weaker assumptions on the conservation law. See [29, Chapter 9] for an extensive list of references concerning Riemann problems under different settings.

Definition 1.8. *The conservation law* (1.1) *is called* strictly hyperbolic *if the Jacobian $A(u) = Df(u)$ has m distinct real eigenvalues*

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_m(u) \quad \textit{for all } u \in \mathcal{U}. \tag{1.7}$$

We choose bases of left eigenvectors, $\{\ell_1(u), \ldots, \ell_m(u)\}$, and right eigenvectors, $\{r_1(u), \ldots, r_m(u)\}$. So for all $u \in \mathcal{U}$ we have

$$A(u)r_i(u) = \lambda_i(u)r_i(u), \quad \ell_i(u)^T A(u) = \lambda_i(u)\ell_i(u)^T, \quad i = 1, \ldots, m.$$

We normalize the eigenvectors to

$$|r_i(u)| = 1, \quad \ell_j(u) \cdot r_i(u) = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \text{ for all } u \in \mathcal{U}. \tag{1.8}$$

We assume f to be a smooth function and thereby all λ_i, r_i, ℓ_i have the same regularity.

Definition 1.9. *We say that the ith characteristic field is* genuinely nonlinear *if*

$$\nabla\lambda_i(u) \cdot r_i(u) \neq 0 \quad \textit{for all } u \in \mathcal{U}, \tag{1.9}$$

or linearly degenerate *if*

$$\nabla\lambda_i(u) \cdot r_i(u) \equiv 0. \tag{1.10}$$

Note that in the scalar case, $f : \mathcal{U} \to \mathbb{R}$, this corresponds to f being strictly convex, $f'' > 0$ (or strictly concave, $f'' < 0$), or f being linear, $f(u) = au$, $a = const.$, respectively.

It is worth noting that the Riemann problem is invariant under the transformation $(x, t) \mapsto (\alpha x, \alpha t)$, $\alpha > 0$. Thus, we naturally look for self-similar solutions of the form $u(x, t) = q(\xi)$, where $\xi = x/t$. Then it is easy to see that such a solution of the Riemann problem must satisfy the following boundary value problem for an ordinary differential equation:

$$-\xi \frac{d}{d\xi} q + \frac{d}{d\xi} f(q) = 0, \qquad q(-\infty) = \hat{u}_L, \quad q(+\infty) = \hat{u}_R. \tag{1.11}$$

We proceed by giving more precise results on *simple wave* solutions, that is on shock waves, rarefaction waves and contact discontinuities. For the rest of this section let $V \subset \mathcal{U}$ denote an open subset.

Shock waves. Shock waves are solutions that consist of only two constant states $u_-, u_+ \in V$ and have the form

$$u(x, t) = \begin{cases} u_-, & x < \sigma t, \\ u_+, & x > \sigma t. \end{cases}$$

Then σ is called the shock speed and it is well known that a shock wave is a weak

solution if and only if it satisfies the *Rankine-Hugoniot condition*

$$f(u_+) - f(u_-) = \sigma(u_+ - u_-). \tag{1.12}$$

For a fixed $u_- \in V$ the set of all states $u_+ \in V$ that satisfy (1.12) for some σ is called the *Hugoniot set* of u_-. For a proof of the following Theorem, see [55, Chapter 4, Theorem 1.1].

Theorem 1.10. *There exists an open subset $W \subset V$ and a $s > 0$ such that for every $u_- \in W$, the Hugoniot set can be decomposed into m curves $\varepsilon \mapsto S_i(\varepsilon)(u_-)$, $i = 1, \dots, m$ for $\varepsilon \in (-s, s)$. These curves depend smoothly on ε and u_-. Moreover, we have*

$$S_i(\varepsilon)(u_-) = u_- + \varepsilon r_i(u_-) + \mathcal{O}(\varepsilon^2). \tag{1.13}$$

The shock speed σ_i satisfies

$$\sigma_i = \lambda_i(u_-) + \frac{\varepsilon}{2}(\nabla\lambda_i \cdot r_i)(u_-) + \mathcal{O}(\varepsilon^2).$$

■

The Lax shock inequalities in this case become

$$\lambda_i(u_-) \geq \sigma_i \geq \lambda_i(u_+).$$

Moreover, the shock speed σ_i is the ith eigenvalue of the averaged Jacobian of the flux. That is, σ_i is the ith eigenvalue of

$$\int_0^1 A(\theta u_+ + (1-\theta)u_-)\, d\theta.$$

Rarefaction waves. A rarefaction wave is a smooth and self-similar solution of (1.1), so a solution $u(x,t) = q(x/t)$ that satisfies

$$(A(q) - \xi)\frac{d}{d\xi}q = 0.$$

This means that when $\frac{d}{d\xi}q \neq 0$ there exists a scalar $\kappa(\xi)$ such that

$$\frac{d}{d\xi}q(\xi) = \kappa(\xi) r_i\left(q(\xi)\right), \qquad \xi = \lambda_i\left(q(\xi)\right),$$

for some $i \in \{1, \ldots, m\}$. This allows us to define an integral curve R_i by

$$\frac{d}{d\varepsilon}R(\varepsilon)(q) = r_i\left(R_i(\varepsilon)(q)\right), \qquad R_i(0)(q) = q_0.$$

Theorem 1.11. *There exists and open subset $W \subset V$ and a $s > 0$ such that for every $u_- \in W$ the following holds. For each $i = 1, \ldots, m$ the curve $\varepsilon \mapsto R_i(\varepsilon)(u_-)$ is defined for $\varepsilon \in (-s, s)$ and depends smoothly on ε and u_-. It satisfies*

$$R_i(\varepsilon)(u_-) = u_- + \varepsilon r_i(u_-) + \mathcal{O}(\varepsilon^2)$$

and

$$\lambda_i\left(R_i(\varepsilon)(u_-)\right) = \lambda_i(u_-) + \varepsilon \nabla\lambda_i(u_-) \cdot r_i(u_-) + \mathcal{O}(\varepsilon^2).$$

■

See [55, Chapter 4, Theorem 1.2] for a proof. An *i-rarefaction wave* connecting u_- and u_+ is a solution that has the form

$$u(x,t) = \begin{cases} u_-, & x/t < \lambda_i(u_-), \\ R_i(\varepsilon)(u_-), & x/t \in [\lambda_i(u_-), \lambda_i(u_+)], \;\; x/t = \lambda_i(R_i(\varepsilon)(u_-)), \\ u_+, & x/t > \lambda_i(u_+). \end{cases} \tag{1.14}$$

The mapping $\varepsilon = \varepsilon(\xi)$ can be defined only in the case that $\nabla\lambda_i \cdot r_i \neq 0$ and when $\nabla\lambda_i \cdot r_i > 0$, only the part of the integral curve with $\varepsilon > 0$ can be used in the rarefaction wave. When $\lambda_i(u_-) < 0 < \lambda_i(u_+)$, we say that an i-rarefaction wave is a *transonic* wave.

Contact discontinuities. Shock waves associated with linearly degenerate fields are called *contact discontinuities*.

Theorem 1.12. *There exists an open subset $W \subset V$ and a $s > 0$ such that for every $u_- \in W$ the following holds. Suppose that the ith characteristic field is*

linearly degenerate,

$$\nabla\lambda_i \cdot r_i \equiv 0.$$

Then the i-Hugoniot curve and the i-integral curve coincide. That is, for $\varepsilon \in (-s, s)$ *we have*

$$S_i(\varepsilon)(u_-) = R_i(\varepsilon)(u_-).$$

Moreover, along that curve the shock speed and the characteristic speed coincide.

■

A proof is given in [55, Chapter 4, Theorem 1.3]. If two states u_-, u_+ are connected by a i-contact discontinuity the shock speed σ_i satisfies

$$\sigma_i = \lambda_i(u_-) = \lambda_i(u_+).$$

Finally, we can construct a solution for the Riemann problem by means of composite wave mappings (see [55, Chapter 4, Theorem 1.6]):

Theorem 1.13. *Assume that in* $\mathcal{U}$ *the system* (1.1) *is strictly hyperbolic and that all characteristic fields are either genuinely nonlinear or linearly degenerate. Then there exist* $\delta > 0$ *and* $s > 0$ *such that for a ball* $V \subset \mathcal{U}$ *with radius* $\delta > 0$ *the following holds. For any* $u_- \in V$ *and any* $i = 1, \ldots, m$*, we can define a map* $\psi_i : (-s, s) \times V \to \mathbb{R}^m$ *by*

$$\psi_i(\varepsilon)(u_-) = \begin{cases} S_i(\varepsilon)(u_-), & \varepsilon \in (-s, 0), \\ R_i(\varepsilon)(u_-), & \varepsilon \in [0, s). \end{cases} \tag{1.15}$$

Then for any $\hat{u}_L,\ \hat{u}_R \in V$*, the Riemann problem* (1.5),(1.6) *admits a unique self-similar solution that consists of* $m + 1$ *constant states*

$$\hat{u}_L = u_0,\ u_1,\ \ldots,\ u_m = \hat{u}_R$$

separated by simple waves. The intermediate states satisfy

$$u_i = \psi_i(\varepsilon_i)(u_{i-1})$$

for some $\varepsilon_i \in (-s, s)$. If the i-th field is linearly degenerated, u_{i-1} and u_i are connected by a contact discontinuity. Else, if $\varepsilon_i \geq 0$, by a rarefaction wave and if $\varepsilon_i < 0$ by a shock wave that satisfies the entropy condition and the Lax shock inequalities. ■

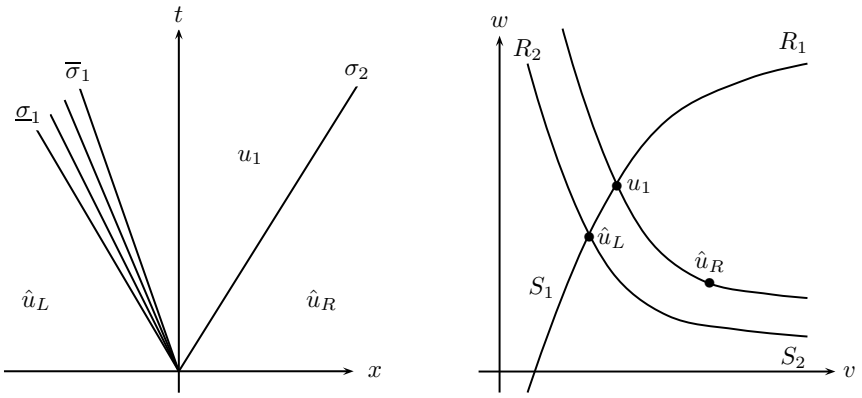


Figure 1.1: Typical solution of the Riemann problem for a 2×2 system. Left: Waves and states. Right: Hugoniot curves and integral curves.

Actually computing the solution of the Riemann problem for a nonlinear system of hyperbolic conservation laws, however, can still be a very difficult task. But for many system of practical interest, exact Riemann solvers are available. See, e.g., [80, Chapter 18] for a complete explicit solution of the Riemann problem for the Euler equation of gas dynamics (with ideal gas or polytropic gas equations of state), [63] for shallow water equations, [74] for a Bear-Nunziato-type model for two-face flow and [34] for relativistic magnetohydrodynamics.

1.1.5 Theory for systems

For systems of conservation laws, existence theory generally is developed along two different lines of strategy. The first is the *vanishing viscosity* method, where solutions of the hyperbolic problem are understood as limits of a parabolic problem, say $u = \lim_{\mu \to 0} u^\mu$, where u^μ is the solution of

$$\frac{\partial}{\partial t} u^\mu + \frac{\partial}{\partial x} f(u^\mu) = \mu \Delta u^\mu.$$

We refer to [16] for a survey on this approach.

The second route to existence results is the study of piecewise constant approximations to the exact solution. This is the approach taken by Glimm in his seminal work [35]. His method relies on the exact solutions of Riemann problems and a random restarting procedure. Later, a deterministic version of the method was presented by Liu [60]. A detailed description of the Glimm scheme can be found in the textbook by Smoller [80, Chapter 19]. It is hardly surprising that Glimm's method, being somewhat similar to the Godunov scheme, can be turned into a numerical tool. In this case, it is usually called the random choice method (RCM). See [22] and [23] for early work in this direction and [91, Chapter 7] for a description of the numerical details and further references. We will use the RCM method in Chapter 5 to compute numerical reference solutions for various test problems.

In the last 15 years tremendous advances in the theory of nonlinear systems of conservation laws in one spatial dimension were made employing a *wave front tracking* approach. Roughly speaking, wave front tracking works as follows: Solutions are approximated by a piecewise constant function, that function is evolved in time by solving Riemann problems until waves emerging at the jumps interact. Since the interaction of rarefaction waves are quite complicated to resolve, simplified Riemann problems with only shocks as solutions are considered. Showing that this approximation actually converges requires some very delicate wave interaction estimates. We refer to [14, 15] and [41] for a detailed explanation of the used techniques.

Theorem 1.14. *Consider the Cauchy problem* (1.1), (1.2) *for a strictly hyperbolic system with smooth flux. Assume that each characteristic field is either genuinely nonlinear or linearly degenerate. Then for $\delta > 0$ sufficiently small, the following holds. For every initial condition $u(x, 0) = \hat{u}(x)$ with*

$$TV(\hat{u}) < \delta, \qquad \|\hat{u}\|_{L^\infty} < \delta,$$

the Cauchy problem has a weak solution, defined for all times $t \geq 0$, obtained as limit of front tracking approximations.

Moreover, the solution is unique and satisfies the Lax shock inequalities. Adopt-

ing a semigroup notation, this solution can be written as

$$u(x,t) = \mathcal{E}(t)\hat{u}(x).$$

The semigroup $\mathcal{E} : [0,\infty) \times \mathcal{U} \to \mathcal{U}$ *can be defined on a closed domain* $\mathcal{U} \subset L^1$ *containing all functions with sufficiently small total variation. It is uniformly Lipschitz continuous w.r.t. both time and the initial data:*

$$\|\mathcal{E}(t)\hat{u} - \mathcal{E}(s)\hat{v}\|_{L^1} \leq L\|\hat{u} - \hat{v}\|_{L^1} + M|t-s|.$$

■

Since the semigroup $\mathcal{E}$ is constructed by solving Riemann problems, it is often called the standard *Riemann semigroup*. Finally, we give a result by Bressan and LeFloch [18] which establishes that admissible BV solutions possess better regularity properties than general BV functions. See [14] for a detailed explanation of the underlying approach:

Theorem 1.15. *Let* u *be the solution of the Cauchy problem* (1.1), (1.2)*, obtained by the front tracking algorithm. Then* $\mathbb{R} \times [0,\infty)$ *can be partitioned into the union* $\mathbb{R} \times [0,\infty) = \mathcal{C}(u) \cup \mathcal{J}(u) \cup \mathcal{I}(u)$*, such that:*

(i) u *is continuous at each point* $(x,t) \in \mathcal{C}(u)$*;*

(ii) $\mathcal{I}(u)$ *is (at most) countable;*

(iii) $\mathcal{J}(u)$ *is the union of (at most) countably many Lipschitz arcs*

$$\{(x,t) \mid t \in (a_j, b_j),\ x = \gamma_j(t)\},\ j = 1,2,\ldots.$$

Whenever $x_0 = \gamma_j(t_0)$ *and* $(x_0,t_0) \notin \mathcal{I}(u)$*, then* u *is continuous at* (x_0,t_0) *relative to* $\{(x,t) \mid t \in (a_j,b_j),\ x < \gamma_j(t)\}$ *and also relative to* $\{(x,t) \mid t \in (a_j,b_j),\ x > \gamma_j(t)\}$*, with corresponding distinct limits* u_- *and* u_+*. Furthermore,* γ_j *is differentiable at* t_0 *with derivative* $\sigma = \dot{\gamma}_j(t_0)$ *and* u_-, u_+ *and* σ *satisfy the Rankine-Hugoniot condition.*

■

This is a crucial result for us, since we are going to discuss numerical methods relying on *point values* of the solution. Solutions merely lying in BV or some L^p space without these further continuity properties would not allow this.

1.2 The generalized Riemann problem

In this section we discuss the Cauchy problem for (1.1) with initial data

$$u(x,0) = \begin{cases} \hat{u}_L(x), & x < 0, \\ \hat{u}_R(x), & x > 0. \end{cases} \tag{1.16}$$

Assume $\hat{u}_L,\ \hat{u}_R : \mathbb{R} \to \mathcal{U}$ to be smooth but that (1.16) is discontinuous at $x = 0$. The Cauchy problem (1.1), (1.16) is called the *generalized Riemann problem*. We let

$$\hat{u}_L^0 = \hat{u}_L(0) \quad \text{and} \quad \hat{u}_R^0 = \hat{u}_R(0).$$

We assume the system (1.1) to be strictly hyperbolic such that every characteristic field is either genuinely nonlinear or linearly degenerate. Then it is well-known (see [59, 84]) that for sufficiently small $|\hat{u}_R^0 - \hat{u}_L^0| > 0$, there exists a neighbourhood around the origin in which (1.1), (1.16) has a unique entropy weak solution.

Moreover, for sufficiently small $T > 0$, the strip $\mathbb{R} \times [0,T)$ can be partitioned into $m+1$ open *domains of smoothness* $\mathfrak{D}_i,\ i = 0,\dots,m$, separated by smooth curves $\gamma_j(t)$ passing through the origin, or by rarefaction zones with boundaries $\underline{\gamma}_j(t),\ \overline{\gamma}_j(t)$, where $\underline{\gamma}_j(t),\ \overline{\gamma}_j(t)$ are smooth characteristic curves passing through the origin. More precisely: We have curves $\gamma_j(t)$ and *rarefaction zones* $\mathfrak{R}_j$:

$$\mathfrak{R}_j = \left\{ (x,t) \in \mathbb{R} \times [0,T) \;\middle|\; \underline{\gamma}_j(t) < x < \overline{\gamma}_j(t) \right\}.$$

For $\gamma_j(t)$, we let $\underline{\gamma}_j(t) = \overline{\gamma}_j(t) = \gamma_j(t)$ for all $t \in [0,T)$. Then we can write

$$\begin{aligned}
\mathfrak{D}_0 &= \left\{ (x,t) \in \mathbb{R} \times [0,T) \mid x < \underline{\gamma}_1(t) \right\}, \\
\mathfrak{D}_i &= \left\{ (x,t) \in \mathbb{R} \times [0,T) \mid \overline{\gamma}_i(t) < x < \underline{\gamma}_{i+1}(t) \right\}, \qquad i = 1,\dots,m-1, \\
\mathfrak{D}_m &= \left\{ (x,t) \in \mathbb{R} \times [0,T) \mid \overline{\gamma}_m(t) < x \right\}.
\end{aligned}$$

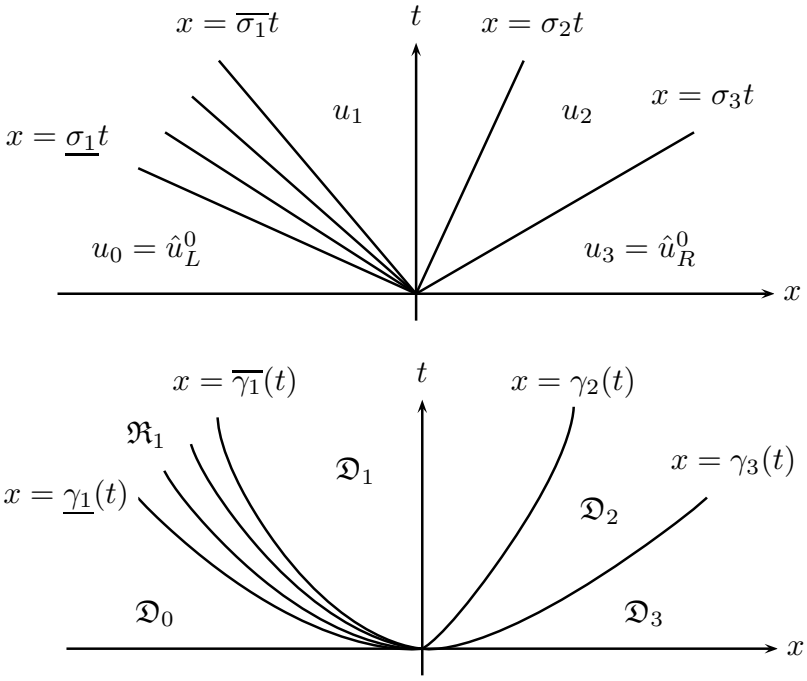


Figure 1.2: Corresponding wave patterns for a classical and a generalized Riemann problem.

The solution u is smooth inside each domain $\mathfrak{D}_i$ and inside each rarefaction zone $\mathfrak{R}_j$. Moreover, u has a shock or contact discontinuity across each curve $x = \gamma_j(t)$ and is continuous across the characteristic curves $x = \underline{\gamma}_j(t)$, $x = \overline{\gamma}_j(t)$.

The solution of the generalized Riemann problem and the solution of the corresponding classical Riemann problem with the initial states $\hat{u}_L^0 = \hat{u}_L(0)$ and $\hat{u}_R^0 = \hat{u}_R(0)$ have a similar wave structure, at least for small time $t > 0$. That is, if the j-wave in the solution of the classical Riemann problem is a shock wave, a contact discontinuity or a rarefaction wave, the corresponding j-wave in the generalized Riemann problem is of the same respective family. A typical configuration of corresponding wave patterns for a 3×3 system is shown in Figure 1.2. More precisely: Denote the constant states in the solution of the Riemann problem with initial data $\hat{u}_L^0,\ \hat{u}_R^0$ by $u_i^0,\ i = 0, \ldots, m$, and the wave speeds by $\underline{\sigma}_j^0, \overline{\sigma}_j^0\ j = 1, \ldots, m$, where we set $\underline{\sigma}_j^0 = \overline{\sigma}_j^0 = \sigma_j^0$ if the j-wave is a shock wave or a contact discontinuity.

Then the curves $\underline{\gamma}_j$, $\overline{\gamma}_j$ satisfy for $j = 1, \ldots, m$:

$$\underline{\gamma}_j(0) = \overline{\gamma}_j(0) = 0 \qquad \lim_{t\to 0} \dot{\underline{\gamma}}_j(t) = \underline{\sigma_j}^0, \qquad \lim_{t\to 0} \dot{\overline{\gamma}}_j(t) = \overline{\sigma}_j^0,$$

The solution u of the generalized Riemann problem satisfies within each domain of smoothness $\mathfrak{D}_i$:

$$\lim_{\substack{t\to 0 \\ (x,t)\in\mathfrak{D}_i}} u(x,t) = u_i^0, \qquad i = 0, \ldots, m.$$

The generalized Riemann problem has been the subject of ongoing research. Special emphasis has been put on the question of global existence and structural stability of self-similar solutions. We refer to [20, 21, 48, 49, 76] and references therein for an up to date account on the generalized Riemann problem. However, for the analysis of the numerical schemes under consideration in this thesis, we can work with results on local existence and local structural stability.

The generalized Riemann problem is highly complicated so we should not expect to solve it exactly. Several approaches towards constructing a solution by means of asymptotic series expansion have been reported. The main source for our work is the expansion constructed by LeFloch and Raviart [56]. We discuss this methods in great detail in Chapter 3. See [13] for an application of this technique to the Euler equations of gas dynamics and [42] for its use in building a version of Glimm's method using piecewise linear data. Related approaches are also discussed in [37] and [57]. Another somewhat different approach to asymptotic expansion for the Euler equations was given by Men'shov [64] and an approach via Riemann invariants is due to Ben-Artzi and Li [10].

2 Generalized Godunov schemes

In this chapter we discuss the basic ideas for constructing high order accurate finite volume schemes, based on a generalisation of the Godunov scheme. In Section 2.1 we outline the general framework and in Section 2.2 we review the basics of the ADER (*Arbitrary DERivatives*) scheme, where we focus on the Toro-Titarev solver for the generalized Riemann problem.

Generalized Godunov schemes contain a reconstruction step at each time level, in which a high order accurate piecewise smooth approximation to the exact solution is computed from the cell averages. Although high order reconstruction is an important research topic that has received a lot of attention lately, a detailed treatment of this topic is out of the scope of this thesis. We refer to [79] for a survey of WENO (*weighted essentially non-oscillatory*) reconstruction and, e.g., [105, 106, 107] for recent results concerning the trade-off between maintaining high accuracy near smooth extrema and achieving a TVD-property or maximum-principle.

2.1 Reconstruct - Evolve - Average

To solve the Cauchy problem

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{2.1}$$

$$u(x,0) = \hat{u}(x), \qquad x \in \mathbb{R},\ t = 0 \tag{2.2}$$

numerically, we want to extend the ideas of the classic Godunov finite volume scheme [36]. Starting with the pioneering work of Kolgan [47] and van Leer [99],

piecewise linear reconstruction in space has become a commonly used tool for improving accuracy over the Godunov scheme. An early example for the use of higher order polynomials is the piecewise parabolic method (PPM) of Colella and Woodward [24] and indeed, the numerical flux originally proposed by Harten, Engquist, Osher and Chakravarthy in their seminal work on ENO methods [39] can be interpreted in a generalized Godunov framework. However, the scheme that has the most in common from a conceptual point of view with what we are going to discuss in this thesis is the Generalized Riemann Problem (GRP) scheme of Ben-Artzi and Falcovitz [7, 8]. See [9] and [11, 102] for recent developments on the GRP scheme.

We work on a grid

$$x_i = i\Delta x, \quad x_{i+1/2} = \left(i + \frac{1}{2}\right)\Delta x, \quad i \in \mathbb{Z}, \qquad t^n = n\Delta t, \quad n \in \mathbb{N},$$

with $\Delta x,\ \Delta t > 0$ and consider the cells (control volumes)

$$\mathcal{I}_i \times [t^n, t^{n+1}], \quad \mathcal{I}_i = [x_{i-1/2}, x_{i+1/2}], \qquad i \in \mathbb{Z},\ n \in \mathbb{N}.$$

For the sake of simplicity we assume uniform grids.

A generalized Godunov scheme consists of the following steps (compare Figure 2.1): Start with cell averages

$$\bar{u}_i^0 = \frac{1}{\Delta x}\int_{\mathcal{I}_i} \hat{u}(x)\ dx. \tag{2.3}$$

Then for any time step $n = 0, 1, \ldots$, given the values $\{\bar{u}_i^n\}_{i\in\mathbb{Z}}$, do the following:

- Construct a piecewise constant function

$$\bar{u}^n(x) = \sum_{i\in\mathbb{Z}} \bar{u}_i^n \chi_{\mathcal{I}_i}(x), \tag{2.4}$$

  where $\chi_{\mathcal{I}_i}$ denotes the characteristic function of the ith cell.

- Find a *piecewise smooth* reconstruction with the help of a nonlinear *conservative reconstruction* operator $\mathcal{R} : L^1_{\text{loc}} \to L^1_{\text{loc}}$. That is, compute a function

$g^n = \mathcal{R}\bar{u}^n$ such that for all $i \in \mathbb{Z}$ we have:

$$g_i^n = g^n|_{\mathcal{I}_i} \text{ is smooth and } \frac{1}{\Delta x}\int_{\mathcal{I}_i} g^n(x)\ dx = \bar{u}_i^n. \tag{2.5}$$

- Use the function g^n as initial data, i.e. pose the Cauchy problem

$$\begin{aligned} &\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = 0, && x \in \mathbb{R},\ t > 0, \\ &u(x,0) = g_i^n(x), && x \in [x_{i-1/2}, x_{i+1/2}],\ i \in \mathbb{Z}. \end{aligned}$$

  Solve this problem *exactly* and evolve the data for one time step. Denote the exact entropy evolution operator associated with (2.1) (that is, the semigroup introduced in Theorem 1.14) by $\mathcal{E}$ and define a function $\tilde{u}^{n+1}$ by

$$\tilde{u}^{n+1} = \mathcal{E}(\Delta t-)g^n, \tag{2.6}$$

  where $\Delta t- = \lim_{t\to\Delta t, t<\Delta t} t$.

- Update the cell averages by

$$\bar{u}_i^{n+1} = \mathcal{A}_i\tilde{u}^{n+1}, \tag{2.7}$$

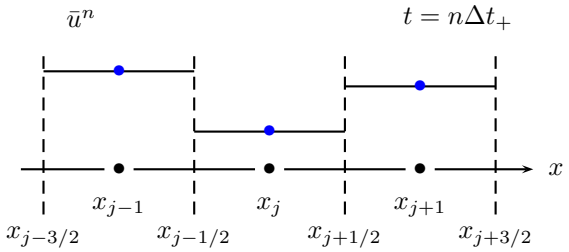
  Where

$$\mathcal{A}_i u = \frac{1}{\Delta x}\int_{\mathcal{I}_i} u(x)\ dx$$

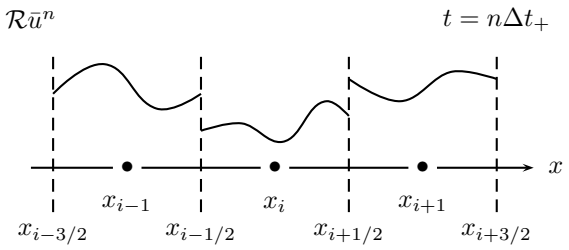
  denotes the cell-averaging operator of the ith cell.

Remark. *(a)* Note that both $\mathcal{R}$ and $\mathcal{E}$ are nonlinear operators. However, by a slight abuse of notation, we simply write $\mathcal{R}\bar{u}^n$ and $\mathcal{E}(\tau)\mathcal{R}\bar{u}^n$. We avoid 'Godunov's accuracy barrier' [36] by using nonlinear reconstruction.

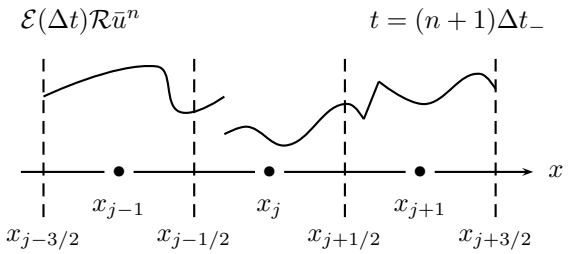
(b) From a computational point of view, the definition of the piecewise constant function $\bar{u}^n$ is not necessary. We might as well consider a reconstruction operator $\mathcal{R}$ acting on discrete values rather than on L^1_{loc} functions and arrive at exactly the same scheme. In that case, we only need three steps: Reconstruction, evolution and averaging. Thus, generalized Godunov schemes fall into LeVeque's notion of REA-schemes [58].



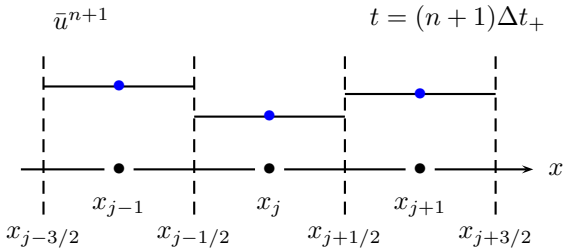
(a) Piecewise constant data.



(b) Piecewise smooth reconstruction.



(c) Exact evolution.



(d) New cell averages.

Figure 2.1: Steps in a generalized Godunov scheme

However, we do include the definition of a piecewise constant function in our scheme for purposes of convergence analysis. When talking about the convergence of a scheme, we mean the convergence of the function $\bar{u}^n$ as $\Delta x \to 0$ in the strong L^1_{loc} topology.

(c) If we consider averaging and construction of a piecewise constant function as one step, we can define an operator

$$\mathcal{P}\bar{u}^{n+1} = \frac{1}{\Delta x} \sum_{i \in \mathbb{Z}} \chi_{\mathcal{I}_i}(\cdot) \int_{\mathcal{I}_i} \mathcal{E}(\Delta t -) \mathcal{R} \bar{u}^n(x) \, dx.$$

Clearly, $\mathcal{P}$ is a projection onto the space of piecewise constant functions.

In a finite volume framework we can accomplish the evolution and averaging at once by using the update

$$\bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x} \left(\bar{f}_{i+1/2}^n - \bar{f}_{i-1/2}^n \right). \tag{2.8}$$

Here, $\bar{f}_{i+1/2}^n$ is the exact averaged flux through the cell interface $x_{i+1/2}$ during one time step:

$$\bar{f}_{i+1/2}^n = \frac{1}{\Delta t} \int_0^{\Delta t} f\left(\mathcal{E}(\tau) \mathcal{R} \bar{u}^n(x_{i+1/2}) \right) d\tau, \tag{2.9}$$

and $\tau = t - t^n$ is the local time.

A vast list of literature on generalized Godunov schemes is available. See [26, 52, 67, 70, 100, 105] for studies on stability and convergence and [106, 107] for schemes satisfying a maximum principle. We remark that the usual way of deriving desired properties is tuning the reconstruction step (say, by applying suitable slope limiters) and then choosing a numerical flux that keeps the desired properties. In many cases this means assuming that the generalized Riemann problem can indeed be solved exactly (in a pointwise sense or in an averaged flux sense as in (2.9)). This is, however, only possible in some special cases. The effects of using approximate solutions of the generalized Riemann problem in schemes for which qualities such as entropy stability or a TVD property can be proven in the case of exact solvers for the generalized Riemann problem, seem to be largely unexplored territory. We give an example of analysis in this direction in section 4.2.

2.2 The ADER scheme and the Toro-Titarev solver

A state of the art variant of the generalized Godunov approach is the ADER scheme [87, 95]. In order to solve the generalized Riemann problems numerically, Toro and Titarev [98] proposed to build a Taylor approximation of the solution whose coefficients are computed by solving a sequence of classical Riemann problems.

We focus on hyperbolic systems in conservation form in one spatial dimension, but the ADER approach can be extended to a much broader set of problems, see e.g. [1, 40, 46, 73, 88, 94, 96]. Stability and the order of accuracy can be verified numerically, see [89] and references therein. Questions of efficient implementation are discussed in [4, 5].

However, it was reported by Castro and Toro [19] that the solver of Toro and Titarev encounters some difficulties for nonlinear systems when dealing with large jumps in the initial data. For more on this problem, as well as alternative approaches for solving the generalized Riemann problem numerically, see also [65]. As we are going to see later on, this phenomenon can be explained by Theorem 4.6. We provide several related theoretical and numerical examples in Chapter 5.

To define a numerical flux we follow a *state expansion* approach. That is, we use a Taylor expansion in time of the solution around $t = 0$ right at the cell-interface $x_{i+1/2}$:

$$u(x_{i+1/2}, \tau) \approx u(x_{i+1/2}, 0+) + \sum_{k=1}^{r-1} \frac{\partial^k u}{\partial t^k}(x_{i+1/2}, 0+) \frac{\tau^k}{k!}. \tag{2.10}$$

Here, $r > 1$ is a given integer and $u(x_{i+1/2}, 0+)$ denotes the value of the solution of the generalized Riemann problem

$$\begin{aligned} &\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0, && x \in \mathbb{R}, t > 0, \\ &u(x, 0) = \begin{cases} \hat{u}_L(x) = g_i^n(x), & \text{if } x < x_{i+1/2}, \\ \hat{u}_R(x) = g_{i+1}^n(x), & \text{if } x > x_{i+1/2}. \end{cases} \end{aligned}$$

right at the cell-interface for time $t = 0+$. The functions g_i^n, g_{i+1}^n are smooth functions in cell $\mathcal{I}_i$ (left of $x_{i+1/2}$) and $\mathcal{I}_{i+1}$ (right of $x_{i+1/2}$) .

Note that although the solution u may be discontinuous, the function $u(x_{i+1/2}, \cdot)$ for a fixed point in space as a function of the time variable will be smooth, provided

$t > 0$ is small enough. So the Taylor expansion (2.10) is well defined only for small $\tau > 0$. That means we have to restrict the time step Δt, to make sure that no other waves issuing from other cell-boundaries $x_{j+1/2}$ reach the line $x = x_{i+1/2}$ within one time step. This can be guaranteed by imposing the usual CFL-Condition (from Courant-Friedrichs-Lewy [27])

$$\Delta t S_{\max} \leq \Delta x,$$

where $S_{\max}$ is the maximal wave-speed. However, for non-linear f new shocks might be formed during one time step. Moreover, in the solution of a generalized Riemann problem, a curve of discontinuity can change its direction and pass the line $x = x_{i+1/2}$ for some $\tau > 0$. We hence assume further that Δt is sufficiently small, such that no shock wave crosses the line $x = x_{i+1/2}$ for $0 < \tau < \Delta t$.

If we can solve the generalized Riemann problem and give a meaning to the time derivatives in (2.10), the easiest way to define a numerical flux is to approximate the time-integral in (2.9) by a Gaussian quadrature:

$$f^n_{i+1/2} = \sum_{\gamma=1}^{N} \omega_\gamma f(u(x_{i+1/2}, \tau_\gamma)), \tag{2.11}$$

where $\omega_\gamma, \tau_\gamma$ are suitable weights and nodes and N is the number of nodes, which is chosen according to the desired accuracy. The values $u(x_{i+1/2}, \tau_\gamma)$ are computed by (2.10). For a discussion of different numerical fluxes in the ADER context, see [91, 97].

The ADER-approach now consists of the following steps:

- Find a piecewise polynomial reconstruction g^n from the cell-averages with possible discontinuities at the points $x_{i+1/2}$. In each cell $\mathcal{I}_i$, the function $g^n_i = g^n|_{\mathcal{I}_i}$ is a polynomial of degree $r - 1$, where $r \geq 1$ is a given integer. Usually this is done by a WENO-reconstruction [6, 44, 61, 77, 79];

- Use the piecewise smooth function from the reconstruction step to set up generalized Riemann problems at the cell-interfaces. Solve these problems approximately, using the GRP-solver described below;

- Use the solution of these generalized Riemann problems to compute the inter-cell flux and update the cell-averages via (2.8),(2.10), (2.11).

We now describe how to compute the coefficients in (2.10), according to Toro and Titarev [98]. The key idea is to reduce the solution of the generalized Riemann problem to a sequence of classical Riemann problems. To find the value $u(x_{i+1/2}, 0+)$ we take the *extrapolated values*

$$\hat{u}_L^0 = \lim_{x \to x_{i+1/2},-} \hat{u}_L(x), \qquad \hat{u}_R^0 = \lim_{x \to x_{i+1/2},+} \hat{u}_R(x),$$

and solve a classical Riemann problem

$$\frac{\partial}{\partial t} u + \frac{\partial}{\partial x} f(u) = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{2.12}$$

$$u(x,0) = \begin{cases} \hat{u}_L^0 & \text{if } x < x_{i+1/2}, \\ \hat{u}_R^0 & \text{if } x > x_{i+1/2}. \end{cases} \tag{2.13}$$

This problem has a similarity solution that we denote by $\phi^0((x - x_{i+1/2})/t)$. The leading term of the expansion (2.10) is then given by $u(x_{i+1/2}, 0+) = \phi^0(0)$. We call this the *Godunov state* of (2.12), (2.13). For nonlinear systems of conservation laws, computing the complete solution of the Riemann problem can be a quite difficult task, so we might need to employ a numerical (approximative) Riemann solver (see [91]) to compute the leading term. However, for the moment we are mainly interested in the analytical aspects of the scheme, so we assume that all states and wave speeds in the solution of the Riemann problem (2.12), (2.13) can be computed exactly.

For higher order terms we perform a Cauchy-Kovalevskaya-type procedure to express each time derivative of order $k \geq 1$ as a function of the spatial derivatives up to order k. That is, we find a recursive mapping

$$\frac{\partial^k u}{\partial t^k} = \mathcal{C}^k \left(u, \frac{\partial u}{\partial x}, \ldots, \frac{\partial^k u}{\partial x^k} \right), \quad k = 0, \ldots, r-1. \qquad \mathcal{C}^0(u) = u.$$

Roughly, the Cauchy-Kovalevskaya theorem states that if flux and initial data are analytic, then there exists (locally) a unique analytical solution. See [104] for an

overview on the subject and [78] for a proof of the Cauchy-Kovalevskaya theorem.

For illustration of the Cauchy-Kovalevskaya procedure consider a scalar problem with a smooth solution, so that we have $\partial_t u = -f'(u)\partial_x u$. Differentiating this with respect to t yields

$$\frac{\partial^2 u}{\partial t^2} = -f''(u)\frac{\partial u}{\partial t}\frac{\partial u}{\partial x} - f'(u)\frac{\partial}{\partial t}\frac{\partial}{\partial x}u.$$

Now interchange differentiation with respect to time and space and use the differential equation again to obtain

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= -f''(u)\left(-f'(u)\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial x} - f'(u)\frac{\partial}{\partial x}\left(-f'(u)\frac{\partial u}{\partial x}\right)\\ &= 2f'(u)f''(u)\left(\frac{\partial u}{\partial x}\right)^2 + \left(f'(u)\right)^2\frac{\partial^2 u}{\partial x^2}\\ &= \mathcal{C}^2\left(u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}\right).\end{aligned}$$

Note however, that while the Cauchy-Kovalevskaya procedure does not involve any sophisticated techniques (in fact, we only need differentiation using the chain rule), it still involves a heavy load of symbolic manipulation. While this can be done using computer algebra systems, the terms can quickly become rather cumbersome.

Lax and Wendroff [54] used a similar idea to express spatial derivatives as functions of time derivatives to solve boundary-value problems (in which case time derivatives at the boundary are known) and so this strategy is sometimes called Lax-Wendroff procedure.

Clearly, for only piecewise smooth initial data the classic Cauchy-Kovalevskaya theorem does not apply. To make the formal derivation of the equations for the spatial derivatives rigorous will take some effort. But to illustrate the basic ideas, assume u was smooth. In that case, the equations in the following can be obtained by simple manipulations of derivatives.

With the help of the map $\mathcal{C}^k$, we can compute the expansion (2.10) provided we can find the spatial derivatives

$$\phi^k(0) = \lim_{\substack{x\to x_{i+1/2}\\ t\to 0_+}} \frac{\partial^k u}{\partial x^k}(x,t).$$

To do so, we use the *one-sided derivatives*

$$\hat{u}_L^k = \lim_{x \to x_{i+1/2,-}} \frac{\partial^k \hat{u}_L}{\partial x^k}(x), \qquad \hat{u}_R^k = \lim_{x \to x_{i+1/2,+}} \frac{\partial^k \hat{u}_R}{\partial x^k}(x).$$

These values are then used as initial conditions for classical Riemann problems. For the evolution equations for the spatial derivatives we have the following lemma.

Lemma 2.1. *Let f be a smooth function and let u be a smooth solution of (2.1). For $k \geq 1$, denote the kth spatial derivative of u by ϕ^k. Then all ϕ^k satisfy a semilinear hyperbolic equation of the form*

$$\frac{\partial}{\partial t}\phi^k + A(u)\frac{\partial}{\partial x}\phi^k = S^k(\phi^0, \ldots, \phi^k), \tag{2.14}$$

where $A(u) = Df(u)$ is the Jacobian of the flux and the function S^k depends only on $\phi^0, \ldots, \phi^k$.

Proof. We only prove the scalar case. Denoting $\phi^k = \frac{\partial^k u}{\partial x^k}$ we proceed in Cauchy-Kovalevskaya fashion:

$$\begin{aligned}
\frac{\partial}{\partial t}\phi^k &= \frac{\partial^k}{\partial x^k}\frac{\partial u}{\partial t} = \frac{\partial^k}{\partial x^k}\left(-f'(u)\frac{\partial u}{\partial x}\right) \\
&= \frac{\partial^{k-1}}{\partial x^{k-1}}\left(-f''(u)\left(\frac{\partial u}{\partial x}\right)^2 - f'(u)\frac{\partial^2 u}{\partial x^2}\right) \\
&= \frac{\partial^{k-2}}{\partial x^{k-2}}\left(-f'''(u)\left(\frac{\partial u}{\partial x}\right)^3 - 3f''(u)\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - f(u)\frac{\partial^3 u}{\partial x^3}\right) \\
&= \ldots \\
&= -f^{(k)}(u)\left(\frac{\partial u}{\partial x}\right)^k - \cdots - f''(u)\frac{\partial u}{\partial x}\frac{\partial^k u}{\partial x^k} - f'(u)\frac{\partial^{k+1} u}{\partial x^{k+1}}.
\end{aligned}$$

And so we have

$$\frac{\partial}{\partial t}\phi^k + f'(u)\frac{\partial}{\partial x}\phi^k = -f^{(k)}(\phi^0)\left(\phi^0\right)^k - \cdots - f''(\phi^0)\phi^0\phi^k = S^k(\phi^0, \ldots, \phi^k).$$

□

Moreover, we see that in the linear case (i.e. $A(u) = A = const.$ matrix), the function S^k vanishes identically.

It is important to stress that while we can derive (2.14) wherever the solution is smooth, we do not have a rigorous analysis whether these equations also can be used for discontinuous solutions yet.

Toro and Titarev then suggested to simplify the problem (2.14) in two ways: Firstly, neglect the source terms and secondly, linearise the equations. We solve:

$$\frac{\partial}{\partial t}\phi^k + A_{LR}\frac{\partial}{\partial x}\phi^k = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{2.15}$$

$$\phi^k(x,0) = \begin{cases} \hat{u}_L^k, & \text{if } x < x_{i+1/2}, \\ \hat{u}_R^k, & \text{if } x > x_{i+1/2}. \end{cases} \tag{2.16}$$

Here $A_{LR} = A(u(x_{i+1/2}, 0_+))$. Then the self-similar solutions ϕ^k of these *linear* problems can be easily computed. Note that for all k we have the same A_{LR}.

Finally, approximate the solution u along the t-axis by the truncated Taylor expansion

$$u(x_{i+1/2}, \tau) \approx \phi^0(0) + \sum_{k=1}^{r-1} \mathcal{C}^k\left(\phi^0, \phi^1, \ldots, \phi^k\right)(0)\frac{\tau^k}{k!}.$$

These simplifications seem natural and have been used in practical applications. However, to our best knowledge, no theoretical justification of the simplifying steps has been reported so far. In Chapter 4 we will see that for a scalar problem with strictly convex flux this method of solution indeed reproduces the first r terms from a truncated Taylor series expansion, see Theorem 4.2. Using characteristic decomposition it is straightforward to show that (2.14), (2.16) and (2.15), (2.16) have the same Godunov state. However, it turns out that in the case of nonlinear systems Riemann problems are not suitable for describing the behaviour of spatial derivatives across discontinuities of the solution. This result is formulated more precisely in Theorem 4.6.

3 The LeFloch-Raviart expansion

Because there are, in general, no exact solutions available for the generalized Riemann problem, we turn our attention to approximate solutions based on asymptotic expansion. LeFloch and Raviart [56] have shown that a local asymptotic series expansion (which, if truncated, agrees with a local Taylor approximation) for the solution of generalized Riemann problem can be constructed. We present the key steps in the construction in Section 3.1. The main Theorem in [56] states that all coefficients in this series expansion can be found explicitly by solving linear systems of algebraic equations. We review how to compute these coefficients in Section 3.2.

We illustrate this technique by providing the full detail for scalar problems and for systems of two equations up to quadratic terms. First order terms were presented in [13] for the Euler equations, but there seems to be no explicit computation of higher order terms available in the literature.

3.1 Asymptotic expansion of the solution to the generalized Riemann problem

3.1.1 Basics

We construct a local series expansion for the solution of the generalized Riemann problem

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = 0, \qquad x \in \mathbb{R},\ t > 0, \tag{3.1}$$

$$u(x,0) = \begin{cases} \hat{u}_L(x), & x < 0, \\ \hat{u}_R(x), & x > 0, \end{cases} \tag{3.2}$$

closely following the lines of [56].

As described in Section 1.2, the Cauchy problem (3.1), (3.2) permits a unique entropy weak solution. The solution is piecewise smooth in (at most) $m+1$ open domains $\mathfrak{D}_i,\ 0 \le i \le m$. These domains of smoothness are either, in the case of shock waves or contact discontinuities, separated by smooth curves $x = \gamma_j(t)$, or by rarefaction zones $\mathfrak{R}_j$ with smooth boundaries $\underline{\gamma}_j(t), \overline{\gamma}_j(t),\ 1 \le j \le m$.

We want to find an asymptotic expansion of the form

$$u(x,t) = \sum_{k\ge 0} t^k q^k(\xi) \tag{3.3}$$

with $\xi = x/t$. This is possible in any domain of smoothness $\mathfrak{D}_i$, by simply taking a Taylor expansion. So every q^k is a polynomial of degree k. We return to this at the beginning of Chapter 4.

Inside a rarefaction zone $\mathfrak{R}$ the solution u is singular at the origin. However, such a series expansion can also be constructed inside a rarefaction zone, see [56, Lemma 7].

Roughly speaking, the construction can be summarized as follows: Take a Taylor expansion wherever the solution u is smooth and then carefully investigate the jump conditions at the boundaries of the domains of smoothness.

For a 2×2 system of conservation laws,

$$u = \begin{pmatrix} v \\ w \end{pmatrix} \in \mathcal{U} \subset \mathbb{R}^2, \qquad f : \mathcal{U} \to \mathbb{R}^2, \quad f(u) = \begin{pmatrix} f_1(v,w) \\ f_2(v,w) \end{pmatrix},$$

we denote the expansion by

$$v(x,t) = \sum_{k\ge 0} t^k v^k(\xi), \qquad w(x,t) = \sum_{k\ge 0} t^k w^k(\xi), \qquad q^k(\xi) = \begin{pmatrix} v^k(\xi) \\ w^k(\xi) \end{pmatrix}.$$

As we are looking for an expansion in terms of self-similar functions, it is useful to change the variables and work with $\xi = x/t$. We set $\tilde{u}(\xi,t) = u(\xi t, t)$ and check that

$$\frac{\partial}{\partial x} = \frac{1}{t}\frac{\partial}{\partial \xi}, \qquad \frac{\partial u}{\partial t} = \frac{\partial \tilde{u}}{\partial t} - \frac{\xi}{t}\frac{\partial \tilde{u}}{\partial \xi}. \tag{3.4}$$

3.1.2 Step I: Derivation of the differential equations

To derive an explicit construction of the functions q^k in (3.3), we will first derive a series of ordinary differential equations satisfied by these functions. Using (3.4), the conservation law becomes

$$t\frac{\partial}{\partial t}\tilde{u} - \xi\frac{\partial}{\partial \xi}\tilde{u} + \frac{\partial}{\partial \xi}f(\tilde{u}(\xi,t)) = 0.$$

Observe that the expansion

$$\tilde{u}(\xi,t) = \sum_{k\geq 0} t^k q^k(\xi)$$

gives

$$t\frac{\partial \tilde{u}}{\partial t} - \xi\frac{\partial \tilde{u}}{\partial \xi} = -\xi\frac{dq^0}{d\xi} + \sum_{k\geq 1} t^k\left(kq^k - \xi\frac{dq^k}{d\xi}\right). \tag{3.5}$$

Inserting this expansion into the flux function f leads to

$$f(\tilde{u}(\xi,t)) = f(q^0) + \sum_{k\geq 1} t^k\left(A(q^0)u^k + f^k(Q^{k-1})\right). \tag{3.6}$$

Here, the function f^k depends only on the previous terms $Q^{k-1} = (q^0,\ldots,q^{k-1})$. We get f^k by a Taylor expansion of the flux in powers of t, such that f^k accounts for all terms in that expansion belonging to t^k that do not depend on q^k, i.e., all but $A(q^0)q^k$. This will be our standard trick in the following analysis, so we discuss the method in more detail.

At first, consider the expansion of the flux around $t = 0$ for the scalar case:

$$\begin{aligned} &f\left(\sum_{k\geq 0} t^k q^k(\xi)\right) \\ &\quad = f(q^0) + t\frac{\partial}{\partial t}\, f\left(\sum_{k\geq 0} t^k q^k(\xi)\right)\Bigg|_{t=0} + \frac{t^2}{2}\frac{\partial^2}{\partial t^2}\, f\left(\sum_{k\geq 0} t^k q^k(\xi)\right)\Bigg|_{t=0} + \ldots \end{aligned}$$

We compute

$$
\begin{aligned}
&f\left(\sum_{k\geq 0} t^k q^k(\xi)\right) \\
&\quad = f(q^0) + tf'\left(\sum_{k\geq 0} t^k q^k(\xi)\right)\Bigg|_{t=0} \left(\sum_{k\geq 0} k t^{k-1} q^k(\xi)\right)\Bigg|_{t=0} \\
&\qquad + \frac{t^2}{2}\left\{ f''\left(\sum_{k\geq 0} t^k q^k(\xi)\right)\Bigg|_{t=0} \left(\sum_{k\geq 0} k t^{k-1} q^k(\xi)\right)^2\Bigg|_{t=0}\right\} \\
&\qquad + \frac{t^2}{2}\left\{ f'\left(\sum_{k\geq 0} t^k q^k(\xi)\right)\Bigg|_{t=0} \left(\sum_{k\geq 0} k(k-1) t^{k-2} q^k(\xi)\right)\Bigg|_{t=0}\right\} + \dots \ .
\end{aligned}
$$

So we have

$$
f(\tilde{u}(\xi,t)) = f(q^0) + tf'(q^0)q^1 + t^2\left\{ f'(q^0)q^2 + \frac{1}{2}f''(q^0)(q^1)^2 \right\} + \mathcal{O}(t^3)
$$

as $t \to 0$, and we see that

$$
f^1(q^0) = 0, \qquad f^2(q^0, q^1) = \frac{1}{2}f''(q^0)(q^1)^2.
$$

For a 2×2 system, we have for $p = 1, 2$:

$$
\begin{aligned}
\frac{\partial^2 f_p(u)}{\partial t^2}\Bigg|_{t=0} = \Bigg(&\frac{\partial^2 f_p(q^0)}{\partial v^2}\left(v^1\right)^2 + 2\frac{\partial f_p(q^0)}{\partial v}v^2 + 2\frac{\partial^2 f_p(q^0)}{\partial v \partial w}v^1 w^1 \\
&+ \frac{\partial^2 f_p(q^0)}{\partial w^2}\left(w^1\right)^2 + 2\frac{\partial f_p(q^0)}{\partial w}w^2 \Bigg),
\end{aligned}
$$

and thus

$$
\frac{\partial^2 f_p(u)}{\partial t^2} = 2A(q^0)q^2 + \left(\frac{\partial^2 f_p}{\partial v^2}(q^0)\left(v^1\right)^2 + 2\frac{\partial^2 f_p}{\partial v \partial w}(q^0)v^1 w^1 + \frac{\partial^2 f_p}{\partial w^2}(q^0)\left(w^1\right)^2 \right)_{p=1,2}.
$$

So we can write

$$f(u(x,t)) = f(q^0) + tA(q^0)q^1 + t^2\left\{A(q^0)q^2 + f^2(q^0,q^1)\right\} + \mathcal{O}(t^3),$$

as $t \to 0$, with

$$\begin{aligned} f^2(q^0,q^1) &= \frac{1}{2}\left(\frac{\partial^2 f_p}{\partial v^2}(q^0)\left(v^1\right)^2 + 2\frac{\partial^2 f_p}{\partial v \partial w}(q^0)v^1w^1 + \frac{\partial^2 f_p}{\partial w^2}(q^0)\left(w^1\right)^2\right)_{p=1,2} \\ &= \frac{1}{2}\begin{pmatrix} q^1 \cdot H_1(q_0^0)q^1 \\ q^1 \cdot H_2(q_0^0)q^1 \end{pmatrix}, \end{aligned} \tag{3.7}$$

where $H_p = D^2 f_p$ denotes the Hessian matrix of $f_p,\ p = 1, 2$.

By this Taylor expansion of f in powers of t it is also easy to check that if every q^s is a polynomial (in ξ) of degree at most $s,\ 0 \le s \le k-1$, then f^k is a polynomial of degree at most k.

Next, we combine (3.5) and (3.6) to find

$$-\xi\frac{dq^0}{d\xi} + \frac{d}{d\xi}f(q^0) + \sum_{k\ge 1} t^k\left(kq^k - \xi\frac{dq^k}{d\xi} + \frac{d}{d\xi}\left(A(q^0)u^k + f^k\right)\right) = 0.$$

So we have for $k = 0$:

$$-\xi\frac{dq^0}{d\xi} + \frac{d}{d\xi}f(q^0) = 0, \tag{3.8}$$

and for $k \ge 1$:

$$kq^k - \xi\frac{dq^k}{d\xi} + \frac{d}{d\xi}\left(A(q^0)u^k + f^k\right) = 0.$$

Setting

$$h^k(\xi) = -\frac{d}{d\xi}f^k(q^0,\ldots,q^{k-1})$$

this becomes

$$kq^k - \xi\frac{dq^k}{d\xi} + \frac{d}{d\xi}(A(u^0)q^k) = h^k. \tag{3.9}$$

We emphasize that in (3.9) the coefficient $A(q^0)$ depends on q^0, *but not* on q^k. Thus, (3.9) is a *semi-linear* equation. Moreover, recall that f^k is a polynomial in ξ of degree at most k, so h^k is a polynomial of degree at most $k-1$.

3.1.3 Step II: Jump conditions

The above construction is valid wherever u is smooth. So next we need to investigate the jump conditions satisfied by q^k at the boundaries of the domains of smoothness. So take a curve $x = \gamma(t)$ that separates two domains of smoothness of u. Because these curves are all smooth, we can use a Taylor expansion to write

$$\gamma(t) = \sigma^0 t + \sigma^1 t^2 + \cdots + \sigma^{k-1} t^k + \dots \ . \tag{3.10}$$

It follows in (3.3) that

$$u(\gamma(t), t) = \sum_{k\geq 0} t^k q^k \left(\frac{\gamma(t)}{t}\right) = \sum_{k\geq 0} t^k q^k \left(\sum_{\ell \geq 0} t^\ell \sigma^\ell\right).$$

In fact, the solution u is smooth, not only in $\mathfrak{D}_i$, but also in the closure $\overline{\mathfrak{D}}_i$, see [59]. So we can use, again, a Taylor expansion in powers of t around the origin to obtain

$$u(\gamma(t), t) = q^0(\sigma^0) + \sum_{k\geq 1} t^k \left(q^k(\sigma^0) + \sigma^k \frac{dq^0}{d\xi}(\sigma^0)\right) + \sum_{k\geq 0} t^k z^k(\Sigma^{k-1}, Q^{k-1}). \tag{3.11}$$

Similar to the f^k in (3.6), the functions z^k depend only on $\Sigma^{k-1} = (\sigma^0, \dots, \sigma^{k-1})$ and Q^{k-1}. Again, we plug all higher order terms in an Taylor expansion into this z^k. In particular, $z^1 = 0$ and

$$z^2(\Sigma^1, Q^1) = \frac{1}{2}(\sigma^1)^2 \frac{d^2 q^0}{d\xi^2}(\sigma^0) + \sigma^1 \frac{dq^1}{d\xi}(\sigma^0). \tag{3.12}$$

We denote the jump of a function q at a point ξ_0 by

$$[[q]](\xi_0) = u(\xi_0+) - x(\xi_0-),$$

so that in the case where u is continuous across the curve $x = \gamma(t)$ we simply get from (3.11) for $k = 0$

$$\left[[q^0]\right](\sigma^0) = 0, \tag{3.13}$$

and for $k \geq 1$

$$\left[\left[q^k + \sigma^k \frac{dq^0}{d\xi} + z^k(\Sigma^{k-1}, Q^{k-1})\right]\right](\sigma^0) = 0. \tag{3.14}$$

So we see that q^0 is continuous at the point σ^0, whereas q^k will in general be discontinuous at this point for $k \geq 1$. Recall that σ^0 is the point we get if we look at $\xi = \gamma(t)/t$ and then let $t \to 0$.

Now let u have a jump across the curve $x = \gamma(t)$. Then, by the Rankine-Hugoniot conditions, we have

$$\dot{\gamma}(t)[[u]](x) = [[f(u)]](x), \quad x = \gamma(t).$$

To derive the correct jump conditions satisfied by the functions q^k, we will take a Taylor expansion for $f(u)$ and for $\dot{\gamma}u$ along $x = \gamma(t)$, respectively. We start with the flux along that curve of discontinuity: By a Taylor expansion around $t = 0$ we get

$$\begin{aligned}
f\left(u(\gamma(t), t)\right) &= f\left(\sum_{k\geq 0} t^k \left\{q^k(\sigma^0) + \sigma^k \frac{dq^0}{d\xi}(\sigma^0) + z^k(\Sigma^{k-1}, Q^{k-1})\right\}\right) \\
&= f\left(q^0(\sigma^0)\right) + tA\left(q^0(\sigma^0)\right)\left(q^1(\sigma^0) + \sigma^1 \frac{dq^0}{d\xi}(\sigma^0)\right) \\
&\quad + t^2\left\{A\left(q^0(\sigma^0)\right)\left(q^2(\sigma^0) + \sigma^2 \frac{dq^0}{d\xi}(\sigma^0)\right) + a^2(\Sigma^1, Q^1)\right\} + \dots \\
&\quad + t^k\left\{A\left(q^0(\sigma^0)\right)\left(q^k(\sigma^0) + \sigma^k \frac{dq^0}{d\xi}(\sigma^0)\right) + a^k(\Sigma^{k-1}, Q^{k-1})\right\} + \dots,
\end{aligned}$$

where for the 2×2 system we can give a^2 explicitly:

$$\begin{aligned}
a^2(\Sigma^1, Q^1) = \frac{1}{2}\Bigg(&\frac{\partial f_p}{\partial v}(q^0(\sigma^0))\left\{(\sigma^1)^2 \frac{d^2 v^0}{d\xi^2}(\sigma^0) + \sigma^1 \frac{dv^1}{d\xi}(\sigma^0)\right\} \\
&+ \frac{\partial f_p}{\partial w}(q^0(\sigma^0))\left\{(\sigma^1)^2 \frac{d^2 w^0}{d\xi^2}(\sigma^0) + \sigma^1 \frac{dw^1}{d\xi}(\sigma^0)\right\} \\
&+ \frac{\partial^2 f_p}{\partial v^2}(q^0(\sigma^0))\left\{v^1(\sigma^0) + \sigma^1 \frac{dv^0}{d\xi}(\sigma^0)\right\}^2 + \frac{\partial^2 f_p}{\partial w^2}(q^0(\sigma^0))\left\{w^1(\sigma^0) + \sigma^1 \frac{dw^0}{d\xi}(\sigma^0)\right\}^2 \\
&+ 2\frac{\partial^2 f_p}{\partial v \partial w}(q^0(\sigma^0))\left\{v^1(\sigma^0) + \sigma^1 \frac{dv^0}{d\xi}(\sigma^0)\right\}\left\{w^1(\sigma^0) + \sigma^1 \frac{dw^0}{d\xi}(\sigma^0)\right\}\Bigg)_{p=1,2}.
\end{aligned} \tag{3.15}$$

Further, we have

$$\begin{aligned}
&\dot\gamma(t)u(\gamma(t),t)\\
&\quad= \sigma^0 q^0(\sigma^0) + t\left\{\sigma^0\left(q^1(\sigma^0)+\sigma^1\frac{dq^0}{d\xi}(\sigma^0)\right)+2\sigma^1 q^0(\sigma^0)\right\}\\
&\qquad+ t^2\left\{\sigma^0\left(q^2(\sigma^0)+\sigma^2\frac{dq^0}{d\xi}(\sigma^0)\right)+3\sigma^2 q^0(\sigma^0)+b^2(\Sigma^1,Q^1)\right\}+\dots\\
&\qquad+ t^k\left\{\sigma^0\left(q^k(\sigma^0)+\sigma^k\frac{dq^0}{d\xi}(\sigma^0)\right)+(k+1)\sigma^k q^0(\sigma^0)+b^k(\Sigma^{k-1},Q^{k-1})\right\}+\dots
\end{aligned}$$

with

$$b^2(\Sigma^1,Q^1) = 2\sigma^1\left(q^1(\sigma^0)+\sigma^1\frac{dq^0}{d\xi}(\sigma^0)\right)+\sigma^0 z^2(\Sigma^1,Q^1). \tag{3.16}$$

In summary, at $\xi=\sigma^0$ the jump conditions are for $k=0$:

$$\sigma^0[[q^0]] = [[f(q^0)]] \quad \text{at } \sigma^0, \tag{3.17}$$

and for $k\geq 1$ we get

$$\begin{aligned}
\big[\big[(A(q^0)-\sigma^0)q^k\big]\big] + \sigma^k\left[\left[(A(q^0)-\sigma^0)\frac{dq^0}{d\xi}\right]\right]&\\
-\sigma^k\big[\big[(k+1)q^0\big]\big]+\big[\big[c^k\big]\big] = 0 \quad \text{at } \sigma^0,&
\end{aligned} \tag{3.18}$$

with a function

$$c^k(\Sigma^{k-1},Q^{k-1}) = a^k(\Sigma^{k-1},Q^{k-1}) - b^k(\Sigma^{k-1},Q^{k-1}).$$

Finally, we remark that for $|\xi|$ large enough, say $|\xi|\geq\xi_0$,

$$q^0(\xi) = \begin{cases} \hat u_R^0, & \xi>\xi_0,\\ \hat u_L^0, & \xi<-\xi_0. \end{cases} \tag{3.19}$$

We now can summarize the above construction:

Lemma 3.1. *The function q^0 satisfies the relations* (3.8),(3.13), (3.17) *and* (3.19), *which characterize the piecewise continuous self-similar entropy solution $q^0(x,t) = q^0(\xi)$ of the Riemann problem for* (3.1) *with initial data $\hat{u}_L^0,\ \hat{u}_R^0$.* □

Therefore, the Toro-Titarev solver indeed sets up 'the right problem' for computing the leading term of the expansion.

3.1.4 Step III: Higher order terms

Recall that the solution q^0 of the classical Riemann problem for the leading term has the form

$$q^0(\xi) = \begin{cases} q_0^0 = \hat{u}_L^0, & \xi \in (-\infty, \underline{\sigma}_1^0), \\ q_i^0, & \xi \in (\overline{\sigma}_i^0, \underline{\sigma}_{i+1}^0), \quad i = 1, \dots, m-1, \\ q_n^0 = \hat{u}_R^0, & \xi \in (\overline{\sigma}_m^0, \infty). \end{cases}$$

Where in the case of an i-shock we have $\underline{\sigma}_i^0 = \overline{\sigma}_i^0 = \sigma_i^0$ and

$$\lambda(q_i^0) \geq \sigma_i^0 \geq \lambda(q_{i+1}^0),$$

and for an i-contact discontinuity

$$\sigma_i^0 = \lambda(q_i^0) = \lambda(q_{i+1}^0).$$

Now consider the domains in which q^0 takes the constant value q_i^0,

$$\mathfrak{D}_i^0 = \{\xi \in \mathbb{R} \mid \underline{\sigma}_i^0 < \xi < \overline{\sigma}_{i+1}^0\}, \qquad i = 0, \dots, m.$$

As a convention, we let $\underline{\sigma}_0^0 = -\infty$, $\overline{\sigma}_{m+1}^0 = +\infty$. Then equation (3.9) in $\mathfrak{D}_i^0$ becomes

$$k q^k + \left(A(q_i^0) - \xi\right) \frac{d}{d\xi} q^k = h^k. \tag{3.20}$$

Recall that h^k is a polynomial of degree at most $k-1$. It is then straightforward to show (see Lemma 2 in [56]) that the general solution of (3.20) is given by

$$q^k(\xi) = \left(\xi - A(q_i^0)\right)^k q_i^k + p_i^k(\xi), \tag{3.21}$$

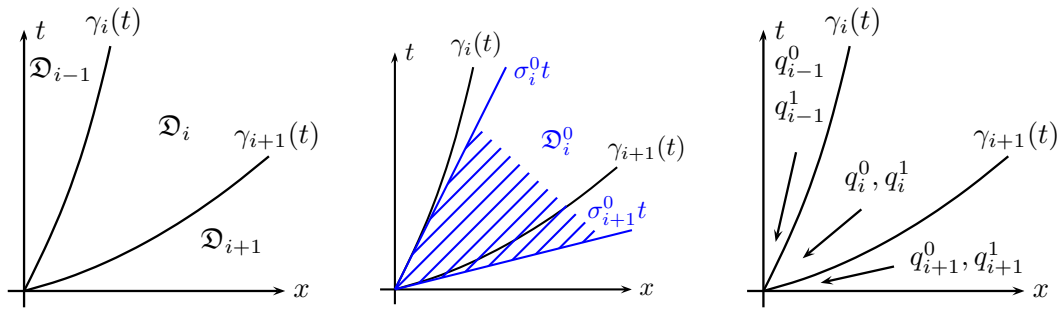


Figure 3.1: Domains of smoothness and corresponding coefficients

where $q_i^k \in \mathbb{R}^m$ is an arbitrary vector and $p_i^k : \mathbb{R} \to \mathbb{R}^m$ is a polynomial of degree at most $k-1$ with coefficients that depend only on $q^0, \ldots, q^{k-1}$. Compare Figure 3.1.

More precisely, $(\xi - A(q_i^0)^k q_i^k$ is a solution of the homogeneous part of (3.20) and p_i^k is a particular solution of (3.20). Since $f^1 = 0$, we have $h^1 = 0$, and therefore, $p^1 = 0$. For the 2×2 system, this means that

$$\begin{pmatrix} v_i^1(\xi) \\ w_i^1(\xi) \end{pmatrix} = \begin{pmatrix} \xi q_{i,1}^1 - \frac{\partial f_1}{\partial v}(q_i^0) q_{i,1}^1 - \frac{\partial f_1}{\partial w}(q_i^0) q_{i,2}^1 \\ \xi q_{i,2}^1 - \frac{\partial f_2}{\partial v}(q_i^0) q_{i,1}^1 - \frac{\partial f_2}{\partial w}(q_i^0) q_{1,2}^1 \end{pmatrix},$$

where we denote $q_i^1 = (q_{i,1}^1,\ q_{i,2}^1)^T$. Then we get

$$\begin{aligned} h_i^2(\xi) = &- v_i^1(\xi) \left(\frac{\partial^2 f_p}{\partial v^2}(q_i^0) q_{i,1}^1 + \frac{\partial^2 f_p}{\partial v \partial w}(q_i^0) q_{i,2}^1 \right)_{\ell=1,2} \\ &- w_i^1(\xi) \left(\frac{\partial^2 f_p}{\partial v \partial w}(q_i^0) q_{i,1}^1 + \frac{\partial^2 f_p}{\partial w^2}(q_i^0) q_{i,2}^1 \right)_{\ell=1,2}. \end{aligned}$$

We rewrite the above expression as

$$h_i^2(\xi) = -\xi \begin{pmatrix} q_i^1 \cdot (H_1(q_i^0) q_i^1) \\ q_i^1 \cdot (H_2(q_i^0) q_i^1) \end{pmatrix} + \begin{pmatrix} (A(q_i^0) q_i^1) \cdot (H_1(q_i^0) q_i^1) \\ (A(q_i^0) q_i^1) \cdot (H_2(q_i^0) q_i^1) \end{pmatrix}.$$

In general, writing

$$h_i^k(\xi) = \sum_{\ell=0}^{k-1} \beta_i^\ell \xi^\ell \qquad \text{and} \qquad p_i^k(\xi) = \sum_{\ell=0}^{k-1} \theta_i^\ell \xi^\ell,$$

the coefficients θ_i^ℓ of the polynomial p_i^k can be obtained as follows (see Lemma 2 in [56]):

$$\theta_i^{k-1} = \beta_i^{k-1}, \tag{3.22}$$

$$(\ell+1)A(q_i^0)\theta_i^{\ell+1} + (k-\ell)\theta_i^\ell = \beta_i^\ell, \qquad 0 \le \ell \le k-2. \tag{3.23}$$

The fact that the function q^0 is piecewise constant allows us to simplify some of the above expressions. Let u have a jump across the curve $x = \gamma_i(t)$, then we have

$$q^0(\sigma_i^0-) = q_{i-1}^0, \quad q^0(\sigma_i^0+) = q_i^0, \qquad \frac{dq^0}{d\xi}(\sigma_i^0-) = \frac{dq^0}{d\xi}(\sigma_i^0+) = 0,$$

and thus, using the solution formula (3.21), we get from (3.12) for the 2×2 system

$$\begin{aligned} z^2(\Sigma_i^1, Q^1)(\sigma_i^0+) &= \sigma_i^1 \frac{dq^1}{d\xi}(\sigma^0) = \sigma_i^1 q_i^1, \\ z^2(\Sigma_i^1, Q^1)(\sigma_i^0-) &= \sigma_i^1 q_{i-1}^1. \end{aligned} \tag{3.24}$$

This gives

$$\begin{aligned} b^2(\Sigma_i^1, Q^1)(\sigma_i^0+) &= \sigma_i^1 \left(2q^1(\sigma_i^0+) + \sigma_i^0 q_i^1\right), \\ b^2(\Sigma_i^1, Q^1)(\sigma_i^0-) &= \sigma_i^1 \left(2q^1(\sigma_i^0-) + \sigma_i^0 q_{i-1}^1\right). \end{aligned}$$

By (3.21), we have

$$q^1(\sigma_i^0+) = \left(\sigma_i^0 - A(q_i^0)\right) q_i^1, \qquad q^1(\sigma_i^0-) = \left(\sigma_i^0 - A(q_{i-1}^0)\right) q_{i-1}^1,$$

and therefore

$$b^2(\Sigma_i^1, Q^1)(\sigma_i^0+) = \sigma_i^1 \left(3\sigma_i^0 - A(q_i^0)\right) q_i^1,$$
$$b^2(\Sigma_i^1, Q^1)(\sigma_i^0-) = \sigma_i^1 \left(3\sigma_i^0 - A(q_{i-1}^0)\right) q_{i-1}^1.$$

Moreover (3.15), reduces to

$$a^2(\Sigma^1, Q^1) = \frac{1}{2}\left(\sigma^1 \left\{ \frac{\partial f_p}{\partial v}(q^0(\sigma^0))\frac{dv^1}{d\xi}(\sigma^0) + \frac{\partial f_p}{\partial w}(q^0(\sigma^0)\frac{dw^1}{d\xi}(\sigma^0) \right\} \right.$$
$$+ \frac{\partial^2 f_p}{\partial v^2}(q^0(\sigma^0))\left(v^1(\sigma^0)\right)^2 + \frac{\partial^2 f_p}{\partial w^2}(q^0(\sigma^0))\left(w^1(\sigma^0)\right)^2$$
$$\left. +2\frac{\partial^2 f_p}{\partial v \partial w}(q^0(\sigma^0))v^1(\sigma^0)w^1(\sigma^0)\right)_{p=1,2}.$$

And thus we can write

$$a^2(\Sigma_i^1, Q^1)(\sigma_i^0+) = \frac{1}{2}\left(\sigma_i^1 A(q_i^0)q_i^1 + \begin{pmatrix} q^1(\sigma_i^0+)\cdot(H_1(q_i^0)q^1(\sigma_i^0+)) \\ q^1(\sigma_i^0+)\cdot(H_2(q_i^0)q^1(\sigma_i^0+)) \end{pmatrix}\right),$$

$$a^2(\Sigma_i^1, Q^1)(\sigma_i^0-) = \frac{1}{2}\left(\sigma_i^1 A(q_{i-1}^0)q_{i-1}^1 + \begin{pmatrix} q^1(\sigma_i^0-)\cdot(H_1(q_{i-1}^0)q^1(\sigma_i^0-)) \\ q^1(\sigma_i^0-)\cdot(H_2(q_{i-1}^0)q^1(\sigma_i^0-)) \end{pmatrix}\right).$$

Now assume that the solution q^0 contains an i-rarefaction wave. This case is slightly more complicated because the function q^0 is no longer constant inside a rarefaction zone. The speeds $\underline{\sigma}_i^0$, $\overline{\sigma}_i^0$ are given by

$$\underline{\sigma}_i^0 = \lambda_i(q_{i-1}^0), \qquad \overline{\sigma}_i^0 = \lambda_i(q_i^0).$$

Moreover, for $\xi \in \mathfrak{R}_i^0 = (\underline{\sigma}_i^0, \overline{\sigma}_i^0)$, we have

$$\lambda_i(q^0(\xi)) = \xi, \qquad \frac{d}{d\xi}q^0(\xi) = \kappa(\xi)r_i(q^0), \tag{3.25}$$

with a scalar $\kappa(\xi)$. We determine the function q^k inside $\mathfrak{R}_i^0$ through the functions

$\alpha_j^k,\ j = 1, \ldots, m$, with

$$q^k(\xi) = \sum_{j=1}^{m} \alpha_j^k(\xi) r_j\left(q^0(\xi)\right). \tag{3.26}$$

Then (3.9) becomes

$$k \sum_{k=1}^{m} \alpha_j^k r_j(q^0) - \xi \frac{d}{d\xi}\left(\sum_{k=1}^{m} \alpha_j^k r_j(q^0)\right) + \frac{d}{d\xi}\left(\sum_{k=1}^{m} \lambda_j(q^0)\alpha_j^k r_j(q^0)\right) = h^k. \tag{3.27}$$

For the 2×2 system, the function h^2 inside $\mathfrak{R}_i^0$ can be obtained from (3.7):

$$\begin{aligned} h^2\left(q^0, q^1\right) &= -\frac{d}{d\xi} f^2(q^0, q^1) \\ &= -\frac{1}{2}\left(\left\{\frac{\partial^3 f_p(q^0)}{\partial v^3}\frac{dv^0}{d\xi} + \frac{\partial^3 f_p(q^0)}{\partial v^2 \partial w}\frac{dw^0}{d\xi}\right\}(v^1)^2 + 2\frac{\partial^2 f_p(q^0)}{\partial v^2} v^1 \frac{dv^1}{d\xi}\right. \qquad (3.28) \\ &\quad + 2\left\{\frac{\partial^3 f_p(q^0)}{\partial v^2 \partial w}\frac{dv^0}{d\xi}\frac{\partial^3 f_p(q^0)}{\partial v \partial w^2}\frac{dw^0}{d\xi}\right\} v^1 w^1 + 2\frac{\partial^2 f_p(q^0)}{\partial v \partial w}\left\{v^1\frac{dw^1}{d\xi} + w^1\frac{dv^1}{d\xi}\right\} \\ &\quad + \left.\left\{\frac{\partial^3 f_p(q^0)}{\partial v \partial^2 w}\frac{dv^0}{d\xi} + \frac{\partial^3 f_p(q^0)}{\partial w^3}\frac{dw^0}{d\xi}\right\}(w^1)^2 + 2\frac{\partial^2 f_p(q^0)}{\partial w^2} w^1 \frac{dw^1}{d\xi}\right)_{p=1,2}. \qquad (3.29) \end{aligned}$$

We have

$$\frac{d}{d\xi}\left(\sum_{k=1}^{m} \alpha_j^k r_j(q^0)\right) = \sum_{j=1}^{m}\left\{\frac{d\alpha_j^k}{d\xi} r_j(q^0) + \alpha_j^k Dr_j(q^0)\frac{dq^0}{d\xi}\right\}$$

and

$$\begin{aligned} \frac{d}{d\xi}\left(\sum_{k=1}^{m} \lambda_j(q^0)\alpha_j^k r_j(q^0)\right) &= \sum_{j=1}^{m}\left\{\nabla\lambda_j(q^0)\cdot\frac{dq^0}{d\xi} + \lambda_j(q^0)\frac{d\alpha_j^k}{d\xi}\right\} \\ &\quad + \sum_{j=1}^{m} \lambda_j(q^0)\alpha_j^k Dr_j(q^0)\frac{dq^0}{d\xi}, \end{aligned}$$

so after introducing the notation

$$h^k = \sum_{j=1}^{m} \mu_j^k r_j(q^0), \quad \text{and} \quad Dr_j(q^0)\frac{d}{d\xi}q^0 = \sum_{p=1}^{m} \omega_{ijp} r_p(q^0),$$

and multiplying (3.27) by $\ell_j(q^0),\ 1 \le j \le m$ we find:

$$\begin{aligned}\left(\lambda_j(q^0) - \xi\right)\frac{d}{d\xi}\alpha_j^k + \left(k + \nabla\lambda_j(q^0)\cdot\frac{d}{d\xi}q^0\right)\alpha_j^k \\ + \sum_{\substack{p=1\\p\neq i}} \left(\lambda_p(q^0) - \xi\right)\omega_{ipj}\alpha_p^k = \mu_j^k. \end{aligned} \tag{3.30}$$

In particular, this gives for $i = j$:

$$\left(k + \nabla\lambda_i(q^0)\cdot\frac{d}{d\xi}q^0\right)\alpha_i^k + \sum_{\substack{p=1\\p\neq i}} \left(\lambda_p(q^0) - \xi\right)\omega_{ipi}\alpha_p^k = \mu_i^k. \tag{3.31}$$

Note that $\lambda_j(q^0) - \xi \neq 0$ for all $j \neq i$. Then (3.30) is uniquely solvable, provided we can give meaningful initial data.

3.2 Finding the unknown coefficients in the LeFloch-Raviart expansion

With the construction from the previous section we can build an approximation to the exact solution, provided we can find the unknown coefficients q_i^k in (3.21) and equip (3.30) with suitable initial data. To determine the vectors q_i^k, we first describe q_0^k and q_m^k. Using the notation from section 2.2, we can write for the initial data

$$\hat{u}_L(x) = \hat{u}_L^0 + \sum_{k=1}^{r-1} \frac{\hat{u}_L^k}{k!}x^k, \qquad \hat{u}_R(x) = \hat{u}_R^0 + \sum_{k=1}^{r-1} \frac{\hat{u}_R^k}{k!}x^k.$$

In $\mathfrak{D}_0^1$, the solution is given by the functions

$$q^k(\xi) = (\xi - A(q_0^0))^k q_0^k + p_0^k(\xi).$$

Since p_0^k is a polynomial of degree at most $k-1$, we find

$$\lim_{\substack{t\to 0\\(x,t)\in\mathfrak{D}_0}} t^k q^k\left(\frac{x}{t}\right) = x^k q_0^k$$

Hence, it follows that

$$u(x,0) = \lim_{\substack{t\to 0\\(x,t)\in\mathfrak{D}_0}} u(x,t) = q_0^0 + \sum_{k=1}^{r-1} q_0^k x^k.$$

Therefore, for all $k \geq 1$ we have

$$q_0^k = \frac{\hat{u}_L^k}{k!}, \qquad \text{and} \qquad q_m^k = \frac{\hat{q}_R^k}{k!},$$

where the second statement is derived analogously. To find the remaining q_i^k we write each coefficient q_i^k in the form

$$q_i^k = \sum_{j=1}^{m} \alpha_{ij}^k r_j(q_i^0).$$

Note that the coefficients $\alpha_{0,j}^k$, $\alpha_{m,j}^k$, $j = 1, \ldots, m$ are known from the initial data. To characterize the coefficients α_{ij}^k, $i = 1, \ldots, m-1$, we consider the case of a jump in the solution first.

Lemma 3.2 (Lemma 4 in [56]). *Assume that the ith wave is a shock wave or a contact discontinuity. Then for all $k \geq 1$, there exists a $s_i^k = s_i^k(\Sigma^{k-1}, Q^{k-1}) \in \mathbb{R}^m$, such that*

$$\begin{aligned}\left(A(q_i^0) - \sigma_i^0\right)^{k+1} q_i^k = \left(A(q_{i-1}^0) - \sigma_i^0\right)^{k+1} q_{i-1}^k \\ + (-1)^k (k+1)\sigma_i^k (q_i^0 - q_{i-1}^0) + s_i^k. \qquad (3.32)\end{aligned}$$

More precisely, (3.32) *holds with*

$$s_i^k = (-1)^{k+1} \left(\left(A(q_i^0) - \sigma_i^0 \right) p_i^k(\sigma_i^0) - \left(A(q_{i-1}^0) - \sigma_i^0 \right) p_{i-1}^k(\sigma_i^0) \right) + c_i^k(\sigma_i^0+) - c_i^k(\sigma_i^0-). \tag{3.33}$$

■

Lemma 3.3 (Corollary from Theorem 1 in [56])**.** *Assume the ith wave is a shock wave or a contact discontinuity. Then we have for $i \neq j$:*

$$\sum_{p=1}^{m} \left(\lambda_p(q_i^0) - \sigma_i^0 \right)^{k+1} \ell_j(q_{i-1}^0) \cdot r_p(q_i^0) \alpha_{ip}^k - \left(\lambda_j(q_{i-1}^0) - \sigma_i^0 \right)^{k+1} \alpha_{i-1,j}^k = \frac{\ell_j(q_{i-1}^0) \cdot \left(q_i^0 - q_{i-1}^0 \right)}{\ell_i(q_{i-1}^0) \cdot \left(q_i^0 - q_{i-1}^0 \right)} \left\{ \sum_{p=1}^{m} \left(\lambda_p(q_i^0) - \sigma_i^0 \right)^{k+1} \ell_i(q_{i-1}^0) \cdot r_p(q_i^0) \alpha_{ip}^k - \left(\lambda_i(q_{i-1}^0) - \sigma_i^0 \right)^{k+1} \alpha_{i-1,i}^k - \ell_i(q_{i-1}^0) \cdot s_i^k \right\} + \ell_j(q_{i-1}^0) \cdot s_i^k. \tag{3.34}$$

Moreover,

$$\sigma_i^k = \frac{(-1)^k}{(k+1)\ell_i(q_{i-1}^0) \cdot \left(q_i^0 - q_{i-1}^0 \right)} \left\{ \sum_{p=1}^{m} \left(\lambda_p(q_i^0) - \sigma_i^0 \right)^{k+1} \ell_i(q_{i-1}^0) \cdot r_p(q_i^0) \alpha_{ip}^k - \left(\lambda_i(q_{i-1}^0) - \sigma_i^0 \right)^{k+1} \alpha_{i-1,i}^k - \ell_i(q_{i-1}^0) \cdot s_i^k \right\} \tag{3.35}$$

■

This gives $m-1$ linear algebraic equations for every shock or contact discontinuity in the solution. Note that by Theorem 1.10 we have

$$q_{i+1}^0 = q_i^0 + \varepsilon r_i(q_i^0) + \mathcal{O}(\varepsilon^2)$$

and so we have

$$\ell_i(q_i^0) \cdot \left(q_{i+1}^0 - q_i^0 \right) \neq 0.$$

Both (3.32) and (3.33) follow from the jump relation (3.18), by noting that

$$q^0(\sigma_i^0-) = q_{i-1}^0, \qquad q^0(\sigma_i^0+) = q_i^0, \qquad \frac{d}{d\xi} q^0(\sigma_i^0 \pm) = 0$$

and using the solution formula (3.21):

$$\begin{aligned} q^k(\sigma_i^0+) &= \left(\sigma_i^0 - A(q_i^0)\right)^k q_i^k + p_i^k(\sigma_i^0+), \\ q^k(\sigma_i^0-) &= \left(\sigma_i^0 - A(q_{i-1}^0)\right)^k q_{i-1}^k + p_{i-1}^k(\sigma_i^0-). \end{aligned}$$

If the solution contains an i-rarefaction wave, we get by the condition (3.14):

$$\left[\left[q^k\right]\right] + \underline{\sigma}_i^0 \left[\left[\frac{dq^0}{d\xi}\right]\right] + \left[\left[\underline{z}_i^k\right]\right] = 0 \quad \text{at } \underline{\sigma}_i^0$$

and

$$\left[\left[q^k\right]\right] + \overline{\sigma}_i^0 \left[\left[\frac{dq^0}{d\xi}\right]\right] + \left[\left[\overline{z}_i^k\right]\right] = 0 \quad \text{at } \overline{\sigma}_i^0,$$

where $\underline{z}_i^k$, $\overline{z}_i^k$ depend only on Q^{k-1} and $\underline{\Sigma}_i^k = (\underline{\sigma}_i^0, \dots, \underline{\sigma}_i^k)$ or $\overline{\Sigma}_i^k = (\overline{\sigma}_i^0, \dots, \overline{\sigma}_i^k)$, respectively.

Lemma 3.4 (Lemma 5 in [56])**.** *Assume that q^0 contains an i-rarefaction wave. Then for all $k \geq 1$, there exist vectors $\underline{s}_i^k = \underline{s}_i^k(Q^{k-1}, \underline{\Sigma}_i^{k-1})$ and $\overline{s}_i^k = \overline{s}_i^k(Q^{k-1}, \overline{\Sigma}_i^{k-1})$ such that*

$$q^k(\underline{\sigma}_i^0+) + \underline{\sigma}_i^k \frac{d}{d\xi} q^0(\underline{\sigma}_i^0+) = \left(\underline{\sigma}_i^0 - A(q_{i-1}^0)\right)^k q_{i-1}^k + \underline{s}_i^k, \tag{3.36}$$

$$q^k(\overline{\sigma}_i^0-) + \overline{s}_i^k = \left(\overline{\sigma}_i^0 - A(q_i^0)\right)^k q_i^k - \overline{\sigma}_i^k \frac{d}{d\xi} q^0(\overline{\sigma}_i^0-). \tag{3.37}$$

More precisely, (3.36) *and* (3.37) *hold with*

$$\underline{s}_i^k = p_{i-1}^k(\underline{\sigma}_i^0) - \left[\left[\underline{z}_i^k\right]\right](\underline{\sigma}_i^0)$$

and

$$\overline{s}_i^k = -p_i^k(\overline{\sigma}_i^0) - \left[\left[\overline{z}_i^k\right]\right](\overline{\sigma}_i^0),$$

respectively. ■

Recall that for $\xi \in (\underline{\sigma}_i^0, \overline{\sigma}_i^0)$ we have

$$\frac{d}{d\xi} q^0(\xi) = \kappa(\xi) r_i \left(q^0(\xi)\right)$$

with as scalar $\kappa(\xi)$. So by multiplying (3.36) by $\ell_j(q_{i-1}^0)^T,\ 1 \leq j \leq m$ we find

$$\alpha_i^k(\underline{\sigma}_i^0+) + \kappa(\underline{\sigma}_i^0)\underline{\sigma}_i^k = \ell_i(q_{i-1}^0) \cdot \underline{s}_i^k, \tag{3.38}$$

and for $j \neq i$:

$$\alpha_j^k(\underline{\sigma}_i^0+) = \left(\underline{\sigma}_i^0 - \lambda_j(q_{i-1}^0)\right)^k \alpha_{i-1,j}^k + \ell_j(q_{i-1}^0)^T \underline{s}_i^k. \tag{3.39}$$

Similarly, we have

$$\alpha_i^k(\overline{\sigma}_i^0-) + \kappa(\overline{\sigma}_i^0)\overline{\sigma}_i^k = \ell_i(q_i^0) \cdot \overline{s}_i^k, \tag{3.40}$$

and for $j \neq i$:

$$\alpha_j^k(\overline{\sigma}_i^0-) = \left(\overline{\sigma}_i^0 - \lambda_j(q_i^0)\right)^k \alpha_{ij}^k - \ell_j(q_i^0) \cdot \overline{s}_i^k. \tag{3.41}$$

Note that for $i \neq j$, we always have $\overline{\sigma}_i^0 \neq \lambda_j(q_i^0)$. So by (3.41) we get explicitly :

$$\alpha_{ij}^k = \frac{\alpha_j^k(\overline{\sigma}_i^0-) + \ell_j(q_i^0) \cdot \overline{s}_i^k}{\left(\overline{\sigma}_i^0 - \lambda_j(q_i^0)\right)^k}. \tag{3.42}$$

To find the value $\alpha_j^k(\overline{\sigma}_i^0-)$, we solve the ordinary differential equation (3.30) with initial data given by (3.39) and evaluate the solution at $\overline{\sigma}_i^0$. When we have found all $\alpha_j^k,\ j \neq i$, we get α_i^k by (3.31) and $\underline{\sigma}_i^k,\ \overline{\sigma}_i^k$ are obtained through (3.38) and (3.40), respectively.

4 Connecting the Toro-Titarev solver and the LeFloch-Raviart expansion

With the help of the asymptotic expansion constructed in Chapter 3, we now have all the tools we need to analyse the Toro-Titarev solver. It turns out that both the LeFloch-Raviart expansion and the approximation built with the Toro-Titarev solver formally construct the same truncated Taylor series expansion. The only possible difference is the way spatial derivatives at the origin are found. This is the same in both methods for scalar problems (see Theorem 4.2). In Section 4.2 we use this result to extend a recent stability result for a numerical scheme with exact flux computation to the case of the ADER flux. In Section 4.3 we show (see Theorem 4.6), that both methods no longer agree in the case of nonlinear systems.

4.1 Formal comparison of the resulting approximations

Let us take a look at the Taylor expansion that we used to define the functions q^k: We consider the domains

$$\mathfrak{D}_i = \left\{ \xi \in \mathbb{R} \;\middle|\; \frac{\gamma_{i-1}(t)}{t} < \xi < \frac{\gamma_i(t)}{t} \right\}.$$

Since we have $\gamma_i(0) = 0, \dot{\gamma}_i(0) = \sigma_i^0$ the domains remain close to the domains $\mathfrak{D}_i^0$ in which u^0 is constant, for small $t > 0$. Although the Toro-Titarev solver only builds an approximation to the exact solution along the line-segment $\{x = 0\} \times [0, \Delta t]$,

it provides all the information that we need to construct a Taylor approximation in space and time. Moreover, if we do not consider only the Godunov state in the leading term Riemann problem, but all states in that Riemann problem, we can construct such a Taylor expansion in each domain of smoothness. It is then a simple observation that the Toro-Titarev solver and the LeFloch-Raviart expansion formally construct the same approximation.

Lemma 4.1. *Let $r > 1$ be a given integer and consider the functions q^k, $k = 0, \dots, r-1$ in the LeFloch-Raviart expansion. Then the coefficients q_i^k in (3.21) are given by*

$$q_i^k = \lim_{\substack{(x,t)\to(0,0+)\\(x,t)\in\mathfrak{D}_i}} \frac{1}{k!}\frac{\partial^k u}{\partial x^k}(x,t)$$

for $i = 0, \dots, m$ and $k = 0, \dots, r-1$.

Proof. In every domain of smoothness $\mathfrak{D}_i$ we can take some (x_0, t_0) close to the origin and write

$$u(x,t) = u(x_0,t_0) + \sum_{k=1}^{\infty}\sum_{\ell=0}^{k} \frac{\partial^\ell}{\partial x^\ell}\frac{\partial^{k-\ell}}{\partial t^{k-\ell}} u(x_0,t_0)\frac{(x-x_0)^\ell (t-t_o)^{k-\ell}}{\ell!(k-\ell)!}.$$

We may take the limit $(x_0, t_0) \to (0, 0_+)$ inside $\mathfrak{D}_i$ and thus the Taylor expansion around the origin gives

$$u(x,t) = q_i^0 + \sum_{k=1}^{\infty} t^k \sum_{\ell=0}^{k} \frac{\partial^\ell}{\partial x^\ell}\frac{\partial^{k-\ell}}{\partial t^{k-\ell}} \frac{u(0,0+)}{\ell!(k-\ell)!}\left(\frac{x}{t}\right)^\ell.$$

This means that the function q^k in the LeFloch-Raviart expansion is given by

$$q^k\left(\frac{x}{t}\right) = \sum_{\ell=0}^{k} \frac{\partial^\ell}{\partial x^\ell}\frac{\partial^{k-\ell}}{\partial t^{k-\ell}} \frac{u(0,0+)}{\ell!(k-\ell)!}\left(\frac{x}{t}\right)^\ell.$$

Thus, the values of q_i^k in (3.21), which gives the leading coefficient of this polynomial, is given by the value $\partial_x^k u(0,0+)/k!$. □

Given that in both the Toro-Titarev solver and the LeFloch-Raviart expansion we formally compute the same truncated Taylor expansion, the only difference can lie in the way spatial derivatives at the origin are found. To illustrate this, we check that for a scalar problem both methods formally construct the expansion up to quadratic terms. Assume that $\sigma_i^0 < 0 < \sigma_{i+1}^0$ and consider the Taylor approximation inside $\mathfrak{D}_i$:

$$\begin{aligned} u(x,t) \approx u(0,0+) + t\left\{\frac{\partial u(0,0+)}{\partial x}\frac{x}{t} + \frac{\partial u(0,0+)}{\partial t}\right\} \\ + t^2\left\{\frac{1}{2}\frac{\partial^2 u(0,0+)}{\partial x^2}\left(\frac{x}{t}\right)^2 + \frac{\partial^2 u(0,0+)}{\partial x \partial t}\left(\frac{x}{t}\right) + \frac{1}{2}\frac{\partial^2 u(0,0+)}{\partial t^2}\right\}, \end{aligned}$$

where function evaluations and derivatives at $(0,0+)$ are considered as limits $(x,t) \in \mathfrak{D}_i,\ (x,t) \to (0,0+)$. The Cauchy-Kovalevskaya procedure now gives:

$$\begin{aligned} \frac{\partial u}{\partial t} &= -f'(u)\frac{\partial u}{\partial x}, \\ \frac{\partial^2 u}{\partial x \partial t} &= -f''(u)\left(\frac{\partial u}{\partial x}\right)^2 - f'(u)\frac{\partial^2 u}{\partial x^2}, \\ \frac{\partial^2 u}{\partial t^2} &= 2f'(u)f''(u)\left(\frac{\partial u}{\partial x}\right)^2 + (f'(u))^2\frac{\partial^2 u}{\partial x^2}. \end{aligned}$$

Inserting this into our Taylor approximation yields

$$\begin{aligned} u(x,t) \approx u(0,0+) + t\left\{\left(\frac{x}{t} - f'\left(u(0,0+)\right)\right)\frac{\partial u}{\partial x}(0,0+)\right\} \\ + t^2\left\{\left(\frac{x}{t} - f'\left(u(0,0+)\right)\right)^2\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(0,0+) - f''\left(u(0,0+)\right)\left(\frac{\partial u}{\partial x}(0,0+)\right)^2\left(\frac{x}{t}\right)\right. \\ \left. + f'\left(u(0,0+)\right)f''\left(u(0,0+)\right)\left(\frac{\partial u}{\partial x}(0,0+)\right)^2\right\}. \end{aligned} \tag{4.1}$$

Now let us compute us the terms up to q^2 in the LeFloch-Raviart expansion. We

have

$$q_i^1(\xi) = \left(\xi - f'(q_i^0)\right) q_i^1.$$

For q^2, we first compute

$$\begin{aligned} h^2(\xi) &= -\frac{d}{d\xi} f^2\left(q_i^0(\xi), q_i^1(\xi)\right) = -\frac{1}{2}\frac{d}{d\xi}\left(f''(q_i^0)\left(q_i^1(\xi)\right)^2\right) \\ &= -f''(q_i^0)\left(\xi - f'(q_i^0)\right)\left(q_i^0\right)^2 = \beta_i^1 \xi + \beta_i^0, \end{aligned}$$

where

$$\beta_i^1 = -f''(q_i^0)\left(q_i^1\right)^2, \qquad \beta_i^0 = f'(q_i^0) f''(q_i^0)\left(q_i^1\right)^2.$$

Writing

$$p_i^2(\xi) = \theta_i^1 \xi + \theta_i^0,$$

we get

$$\theta_i^1 = \beta_i^1, \qquad \theta_i^0 = \frac{1}{2}\left(\beta_i^0 - f'(q_i^0)\theta_i^1\right) = f'(q_i^0) f''(q_i^0)\left(q_i^1\right)^2.$$

Thus, we have

$$q_i^2(\xi) = \left(\xi - f'(q_i^0)\right)^2 q_i^2 - f''(q_i^0)\left(q_i^1\right)^2 \xi + f'(q_i^0) f''(q_i^0)\left(q_i^1\right)^2.$$

Recall that $q_i^k = \partial_x^k u(0, 0+)/k!$ and thus $q^0 + tq^1 + t^2 q^2$ agrees with (4.1). For the scalar case, this immediately leads to the following result.

Theorem 4.2 (Main result for scalar problems)**.** *Consider the generalized Riemann problem for a scalar, nonlinear hyperbolic conservation law in one spatial dimension with strictly convex flux. Let the initial data consist of piecewise polynomials of degree $r-1$. Assume the solution does not contain a transonic wave.*

Then the Toro-Titarev solver and the LeFloch-Raviart expansion yield the same truncated Taylor expansion in time at $x = 0$,

$$\sum_{k=0}^{r-1} \mathcal{C}^k\left(\phi^0, \ldots, \phi^k\right)(0)\frac{\tau^k}{k!} = \sum_{k=0}^{r-1} q^k(0)\tau^k = \mathcal{E}(\tau)\hat{u}(0) + \mathcal{O}(\Delta t^r)$$

for $0 < \tau < \Delta t$ as $\Delta t \to 0+$.

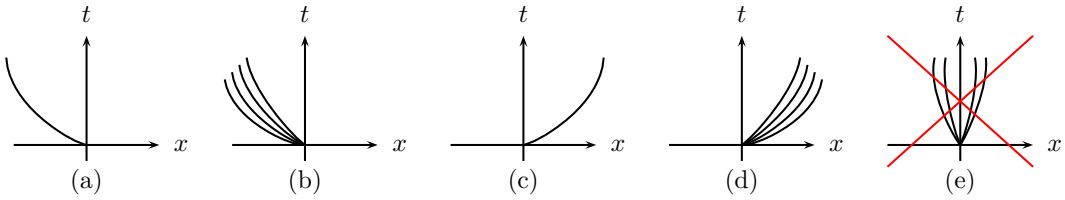


Figure 4.1: Domains of smoothness for scalar problems

Proof. For a strictly convex flux, $f'' > 0$, we only have two domains of smoothness, compare Figure 4.1. In that case, all coefficients $q_i^k, i = 0, 1$, and $k = 1, \ldots, r-1$ are uniquely determined by the initial data and its derivatives. Assuming that there is no transonic wave (i.e., case (e) in Figure 4.1 does not occur), solving linear Riemann problems merely means picking the left or the right side, depending on the sign of the coefficient in the evolution equation. Thus, to build the expansion, we first have to solve one nonlinear Riemann problem to determine which domain of smoothness contains the line segment $\{x = 0\} \times [0, \Delta t]$. Then only use the data from that side, which is equivalent to solving linear Riemann problems. So the solver of Toro and Titarev reproduces the first r terms of the expansion of LeFloch and Raviart exactly.

□

4.2 Application of Theorem 4.2: A TVD-result

Theorem 4.2 has some very pleasant implications. It helps us to extend results concerning exact GRP solvers to the ADER case. We demonstrate this for a recent stability result developed in [52].

We denote the normalized convolution of a function u with the characteristic function of an interval $[-\delta/2, \delta/2]$ by $[u]_\delta$,

$$[u]_\delta(x,t) = \frac{1}{\delta}(u(\cdot,t) * \chi_{[-\delta/2,\delta/2]}(\cdot))(x) = \frac{1}{\delta}\int_{\mathbb{R}} u(x,t)\chi_{[-\delta/2,\delta/2]}(x-y)\, dy$$
$$= \frac{1}{\delta}\int_{x-\delta/2}^{x+\delta/2} u(y,t)\, dy.$$

Let $u \in L^\infty(\mathbb{R} \times [0,\infty))$ be the unique entropy solution to (3.1) with initial data $\hat{u} \in L^\infty(\mathbb{R})$. Wet set $c_1 = \inf_{x\in\mathbb{R}} f'(\hat{u}(x))$, $c_2 = \sup_{x\in\mathbb{R}} f'(\hat{u}(x))$ and denote the *domain of dependence* of a point $(x,t) \in \mathbb{R} \times [0,\infty)$ by

$$D_u(x,t) = [x - c_2 t, x - c_1 t].$$

Given a $\delta > 0$, the following minimum-maximum principle for convoluted solutions holds:

Theorem 4.3 (Lagoutière [52, Theorem 2.1])**.** *The convoluted entropy solution* $[u]_\delta$ *satisfies:*

$$\min_{y\in D_u(x,t)} [u]_\delta(y,0) \le [u]_\delta(x,t) \le \max_{y\in D_u(x,t)} [u]_\delta(y,0) \qquad \textit{for all } (x,t) \in \mathbb{R} \times [0,\infty).$$

Moreover, for every $(x,t) \in \mathbb{R} \times \mathbb{R}^+$*, there exists a* $y \equiv y(x,t) \in D_u(x,t)$*, such that*

$$[u]_\delta(x,t) = [u]_\delta(y(x,t),0).$$

■

Recall that a numerical scheme is said to be TVD (*total variation diminishing*) if for each $n \in \mathbb{N}$ the values $\bar{u}_i^n$ satisfy

$$\sum_{i\in\mathbb{Z}} |\bar{u}_i^{n+1} - \bar{u}_{i-1}^{n+1}| \le \sum_{i\in\mathbb{Z}} |\bar{u}_i^n - \bar{u}_{i-1}^n|,$$

and it is L^∞*-decreasing* if for each $n \in \mathbb{N}$ we have

$$\sup_{i\in\mathbb{Z}} |\bar{u}_i^{n+1}| \le \sup_{i\in\mathbb{Z}} |\bar{u}_i^n|.$$

By a classical argument of Harten [38], a sufficient condition for a scheme to be TVD and L^∞-decreasing is that for each $i \in \mathbb{Z},\ n \in \mathbb{N}$ we have

$$\min(\bar{u}_{i-1}^n, \bar{u}_i^n) \le \bar{u}_i^{n+1} \le \max(\bar{u}_{i-1}^n, \bar{u}_i^n).$$

Let $g^n(x) = \mathcal{R}\bar{u}^n$ be a piecewise polynomial function that is reconstructed from

the cell averages $\{\bar{u}_i^n\}_{i\in\mathbb{Z}}$ such that

$$\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} g^n(x)\,dx = \bar{u}_i^n, \qquad \text{and therefore } \int_{\mathbb{R}} g^n(x)dx = \Delta x \sum_{i\in\mathbb{Z}} \bar{u}_i^n.$$

Theorem 4.4 (Lagoutière [52, Proposition 3.2])**.** *Consider a generalized Godunov scheme with exact numerical flux*

$$\bar{f}_{i+1/2}^n = \frac{1}{\Delta t}\int_0^{\Delta t} f\left(\mathcal{E}(\tau)g^n(x_{i+1/2})\right)\,d\tau,$$

and a suitable CFL-condition. Assume that $f' > 0$. Then the scheme is TVD and L^∞-decreasing (and thus convergent), if the following holds for every $i \in \mathbb{Z},\ n \in \mathbb{N}$:

$$\min(\bar{u}_{i-1}^n, \bar{u}_i^n) \le [g^n]_{\Delta x}((i-\theta)\Delta x) \le \max(\bar{u}_{i-1}^n, \bar{u}_i^n) \qquad \textit{for all } \theta \in [0,1]. \quad (4.2)$$

Proof (As in [52])**.** Assume that condition (4.2) holds. Note that there are no transonic waves, since we assumed $f'(u) > 0$. Conservativity of the reconstruction implies that for all $i \in \mathbb{Z},\ n \in \mathbb{N}$, we have

$$\bar{u}_i^n = [\mathcal{R}\left(\{\bar{u}_j^n\}_{j\in\mathbb{Z}}\right)]_{\Delta x}(x_i) = [g^n]_{\Delta x}(x_i).$$

In that case, condition (4.2) reads

$$\begin{aligned}\min\left([g^n]_{\Delta x}(x_{i-1}),\ [g^n]_{\Delta x}(x_i)\right) \ &\le\ [g^n]_{\Delta x}(x_i - \theta\Delta x)) \\ &\le \max\left([g^n]_{\Delta x}(x_{i-1}),\ [g^n]_{\Delta x}(x_i)\right) \qquad \text{for all } \theta \in [0,1].\end{aligned}$$

Under the CFL-condition, the domain of dependence of the point $(x,t) = (x_i, \Delta t)$ is included in the interval $[x_{i-1}, x_i]$. We can view this interval as the set of convex combinations of x_{i-1} and x_i, so the maximum principle for convoluted solutions gives

$$\begin{aligned}\min_{\theta\in[0,1]}[g^n]_{\Delta x}(\theta x_{i-1} + (1-\theta)x_i) &\le [\mathcal{E}(\Delta t)g^n]_{\Delta x}(x_i) \\ &\le \max_{\theta\in[0,1]}[g^n]_{\Delta x}(\theta x_{i-1} + (1-\theta)x_i).\end{aligned}$$

So, by condition (4.2), we get

$$\min(\bar{u}_{i-1}^n, \bar{u}_i^n) \leq [\mathcal{E}(\Delta t) g^n]_{\Delta x}(x_i) \leq \max(\bar{u}_{i-1}^n, \bar{u}_i^n).$$

Finally, we note that

$$[\mathcal{E}(\Delta t) g^n]_{\Delta x}(x_i) = \mathcal{A}_j \mathcal{E}(\Delta t) g^n(x_j) = \bar{u}_i^{n+1},$$

and thus

$$\min(\bar{u}_{i-1}^n, \bar{u}_i^n) \leq \bar{u}_i^{n+1} \leq \max(\bar{u}_{i-1}^n, \bar{u}_i^n).$$

□

What is remarkable about condition (4.2) is that it imposes a restriction on the total variation of the *sliding averages* of the reconstruction,

$$[g^n]_{\Delta x}(x_i - \theta \Delta x) = \int_{x_{i-1/2} - \theta \Delta x}^{x_{i+1/2} - \theta \Delta x} g^n(x)\ dx, \qquad \theta \in [0, 1],$$

rather than on the total variation of the reconstruction itself.

A crucial ingredient in the line of reasoning in the above proof is that the new cell-averages are computed as *exact averages* of the *evolved initial data.* Equivalently, this requires that the numerical flux

$$\bar{f}_{j+1/2}^n = \frac{1}{\Delta t} \int_0^{\Delta t} f(\mathcal{E}(\tau) g^n(x_{i+1/2}))\ d\tau$$

can be computed *exactly.* However, this usually is not the case. Instead we assume that the numerical flux $f_{i+1/2}^n$ satisfies

$$|\bar{f}_{i+1/2}^n - f_{i+1/2}^n| = \mathcal{O}(\Delta t^r), \quad \Delta t \to 0, \qquad i \in \mathbb{Z},\ n \in \mathbb{N}.$$

As we have seen, the ADER flux has this property. As an extension of Lagoutière's Theorem, we state the following stability result for ADER schemes:

Theorem 4.5. *Let the assumptions of Lagoutière's Theorem 4.4 be satisfied but instead of the exact averaged flux consider the ADER numerical flux. Assume further that the reconstruction has the property for all $i \in \mathbb{Z}$:*

$$\operatorname{sgn}(\bar{u}_{i-1}^n - \bar{u}_i^n) = \operatorname{sgn}(g^n(x_{i-1/2}) - g^n(x_{i+1/2})). \tag{4.3}$$

Then, for Δt sufficiently small, the ADER scheme is TVD and L^∞-decreasing.

Proof. We note that in the case of exact flux computation, the condition

$$\min(\bar{u}_{i-1}^n, \bar{u}_i^n) \leq [\mathcal{E}(\Delta t) g^n]_{\Delta x}(x_i) \leq \max(\bar{u}_{i-1}^n, \bar{u}_i^n).$$

may equivalently be expressed as

$$\min(\bar{u}_{i-1}^n, \bar{u}_i^n) \leq \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\bar{f}_{i+1/2}^n - \bar{f}_{i-1/2}^n\right) \leq \max(\bar{u}_{i-1}^n, \bar{u}_i^n).$$

Now let $J := \{k, k+1, \ldots, k+\ell\} \subseteq \mathbb{Z},\ \ell \geq 2$ for some $k \in \mathbb{Z}$ be an index set such that

$$\bar{u}_{i-1}^n \leq \bar{u}_i^n \leq \bar{u}_{i+1}^n, \qquad k+1 \leq i \leq k+\ell-1.$$

The following holds analogously if we alter all the inequalities. We have the condition

$$\bar{u}_{i-1}^n \leq \bar{u}_i^{n+1} = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\bar{f}_{i+1/2}^n - \bar{f}_{i-1/2}^n\right) \leq \bar{u}_i^n. \tag{4.4}$$

We want to analyse whether such an estimate still holds if we replace the exact flux $\bar{f}_{j+1/2}^n$ by the ADER flux $f_{i+1/2}^{n,ADER}$. We have to check four cases:

a) Both inequalities in (4.4) *hold with equality.* In this case, the condition

$$\bar{u}_{i-1}^n = [g^n]_{\Delta x}(x_i - \theta \Delta x) = \bar{u}_i^n, \qquad \text{for all } \theta \in [0,1]$$

leads to $g^n \equiv \bar{u}_i^n$ on $[x_{j-3/2}, x_{j+1/2}]$ and thus

$$f_{j+1/2}^{n,ADER} - f_{j-1/2}^{n,ADER} = 0.$$

b) Both inequalities in (4.4) *hold strictly.* We get the condition

$$\bar{u}_{i-1}^n \leq \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(f_{j+1/2}^{n,ADER} - f_{j-1/2}^{n,ADER}\right) \leq \bar{u}_i^n,$$

which can be written as

$$\bar{u}_{i-1}^n \leq \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\bar{f}_{i+1/2}^n - \bar{f}_{i-1/2}^n + \mathcal{O}(\Delta t^r)\right) \leq \bar{u}_i^n, \tag{4.5}$$

and since both inequalities in (4.4) hold strictly, (4.5) holds for Δt sufficiently small.

c) The left inequality in (4.4) *holds with equality.* We have

$$\bar{u}_{i-1}^n = \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\bar{f}_{i+1/2}^n - \bar{f}_{i-1/2}^n\right) < \bar{u}_i^n,$$

which implies $\bar{f}_{i+1/2}^n - \bar{f}_{i-1/2}^n > 0$, but small. Then (4.5) holds if $f_{i+1/2}^{n,ADER} - f_{i-1/2}^{n,ADER} \geq 0$ but small. Recall that the ADER flux is given by

$$f_{j+1/2}^{n,ADER} = \frac{1}{\Delta t}\int_0^{\Delta t} f\left(u(x_{i+1/2}, n\Delta t_+) + \sum_{k=1}^{r-1}\frac{\partial^k u}{\partial t^k}(x_{j+1/2}, n\Delta t_+)\frac{\tau^k}{k!}\right)\, d\tau.$$

By a Taylor expansion of f we see that for sufficiently small Δt the condition

$$f_{i+1/2}^{n,ADER} - f_{i-1/2}^{n,ADER} \geq 0$$

is satisfied, provided we have $f(u(x_{i+1/2}, n\Delta t_+)) \geq f(u(x_{i-1/2}, n\Delta t_+))$. Since we assumed $f' > 0$, this reduces to

$$u(x_{i+1/2}, n\Delta t_+) \geq u(x_{i-1/2}, n\Delta t_+).$$

Recall that the value $u(x_{i+1/2}, n\Delta t_+)$ is given by the Godunov state of the associated classical Riemann problem with initial data from the extrapolated boundary values of the reconstruction. Denote the reconstruction in the cell $[x_{i-1/2}, x_{i+1/2}]$ by g_i^n. Since $f' > 0$, the Godunov states are given by

$$u(x_{i+1/2}, n\Delta t_+) = g_i^n(x_{i+1/2}), \qquad u(x_{j-1/2}, n\Delta t_+) = g_{i-1}^n(x_{i-1/2}).$$

Thus, a sufficient condition for (4.5) is

$$g_i^n(x_{i+1/2}) \geq g_{i-1}^n(x_{i-1/2}).$$

d) The right inequality in (4.4) *holds with equality.* The condition

$$\bar{u}_{i-1}^n < \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(\bar{f}_{i+1/2}^n - \bar{f}_{j-1/2}^n\right) = \bar{u}_i^n$$

implies $\bar{f}_{i+1/2}^n - \bar{f}_{i-1/2}^n = 0$. However, we only need

$$f_{i+1/2}^{n,ADER} - f_{i-1/2}^{n,ADER} \geq 0, \text{ but small,}$$

to get

$$\bar{u}_{i-1}^n \leq \bar{u}_i^n - \frac{\Delta t}{\Delta x}\left(f_{i+1/2}^{n,ADER} - f_{i-1/2}^{n,ADER}\right) \leq \bar{u}_i^n.$$

Again, this only requires $g_{i-1}^n(x_{i-1/2}) < g_i^n(x_{i+1/2})$ and Δt small. □

There seems to be no numerical scheme available that uses a high order reconstruction method which satisfies (4.2) and (4.3). A good starting point could be data-bounded ENO interpolation (BENO), as developed in [12]. But whether further slope-limiting to enforce conditions (4.2) and (4.3) leads to a loss of accuracy, still needs to be analysed and tested numerically.

4.3 Systems

Naturally, the question arises whether the result of Theorem 4.2 can be extended to the case of systems. What we have to compare are the coefficient q_{i*}^k and the Godunov state for the kth spatial derivative in the Toro-Titarev solver. It turns out that for nonlinear systems the two methods do not yield the same approximation:

Theorem 4.6 (Main result for systems)**.** *Consider the generalized Riemann problem for a strictly hyperbolic $m \times m$ system of conservation laws, such that every characteristic field is either genuinely nonlinear or linearly degenerate. Let the initial data consist of polynomials $\hat{u}_L$, $\hat{u}_R$ of degree $r-1$ with $|\hat{u}_L(0) - \hat{u}_R(0)| > 0$ sufficiently small. Assume that the solution contains only shock waves and contact discontinuities.*

Then for $k \geq 1$ the coefficients q_i^k in the LeFloch-Raviart expansion and the states u_i^k in the linear Riemann problems of the Toro-Titarev solver satisfy the relation

$$q_i^k = \frac{1}{k!} u_i^k \qquad \text{for } i = 0 \text{ and for } i = m.$$

This does, in general, not hold for $1 \leq i \leq m-1$.

Proof. The statement that

$$q_0^k = \frac{1}{k!} u_0^k, \qquad q_m^k = \frac{1}{k!} u_m^k, \qquad k = 1, \ldots, r-1,$$

was already shown at the beginning of Section 3.1.4. Now take (3.18) for $k = 1$, in which case we have $c^1 = 0$ and $s_i^1 = 0$, so (3.18) becomes

$$\left[\left[\left(A(q^0) - \sigma_i^0\right) q^1\right]\right] - \sigma_i^1 \left[\left[\left(A(q^0) - \sigma_i^0\right) \frac{d}{d\xi} q^0 - 2q^0\right]\right] = 0 \quad \text{at } \sigma_i^0. \tag{4.6}$$

We note that

$$q^0(\sigma_i^0-) = q_{i-1}^0, \qquad q^0(\sigma_i^0+) = q_i^0, \qquad \frac{d}{d\xi} q^0(\sigma_i^0 \pm) = 0,$$

and by (3.21)

$$q^1(\sigma_i^0+) = \left(A(q_i^0) - \sigma_i^0\right) q_i^1, \qquad q^1(\sigma_i^0-) = \left(A(q_{i-1}^0) - \sigma_i^0\right) q_{i-1}^1.$$

Thus, the jump condition (4.6) becomes

$$\left(A(q_i^0) - \sigma_i^0\right)^2 q_i^1 - \left(A(q_{i-1}^0) - \sigma_i^0\right)^2 q_{i-1}^1 - 2\sigma_i^1 \left(q_i^0 - q_{i-1}^0\right) = 0. \tag{4.7}$$

Now consider the Toro-Titarev solver. We denote the solution of the linearised Riemann problem (2.16) by ϕ^k and let ϕ_i^k, $i = 0, \ldots, m$ be the constant states in that solution. If ϕ_{i*}^0 is the Godunov state for ϕ^0, then solving the Riemann problems linearised around ϕ_{i*}^0 is equivalent to imposing the jump conditions

$$\left(A(\phi_{i*}^0) - \lambda_i(u_{i*}^0)\right)\left(\phi_i^1 - \phi_{i-1}^1\right) = 0, \qquad i = 1, \ldots, m. \tag{4.8}$$

Clearly, (4.7) and (4.8) do not have the same solution. An analogous argument holds in the case $k > 1$. □

We remark, however, that when all states q_i^0 are close, the solutions of (4.7) and (4.8) are close. This depends only on the leading term q^0, but not on higher order terms. Thus, when the jump in the initial data $|\hat{u}_L^0 - \hat{u}_R^0|$ is small we expect (4.8) to give a good approximation to (4.7). This explains why the ADER method does achieve the designed order of accuracy in regions where the solution is smooth, since in those regions the WENO reconstruction leads to initial data with only very small jumps.

5 Examples and applications

To illustrate the rather technical results form the previous chapters, we provide several examples. Firstn we solve the generalized Riemann problem for Burgers equation with piecewise quadratic initial data. In this case the exact solution can be computed by the method of characteristics and we see that both the Toro-Titarev solver and the LeFloch-Raviart expansion reproduce the quadratic Taylor approximation around the origin. Next, we study a system that is inspired by a model for two component-chromatography. Due to the special eigenstructure of the system, we can carry out all the computations for first order terms in the LeFloch-Raviart expansion explicitly. The next two examples are less theoretically motivated and show that the LeFloch-Raviart expansion can be used in actual numerical computations. We first present numerical tests for shallow water equations and finally, we study a system from the modelling of traffic flow which has only linearly degenerate characteristic fields.

5.1 Burgers equation

Consider Burgers equation,

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0,$$

with initial data

$$u(x,0) = \begin{cases} \hat{u}_L(x) = x^2 + 2x + 1, & x < 0, \\ \hat{u}_R(x) = 2x^2 - 4x + 2, & x > 0. \end{cases} \qquad (x,t) \in \mathbb{R}\times[0,\infty), \quad u(x,t) \in \mathbb{R}.$$

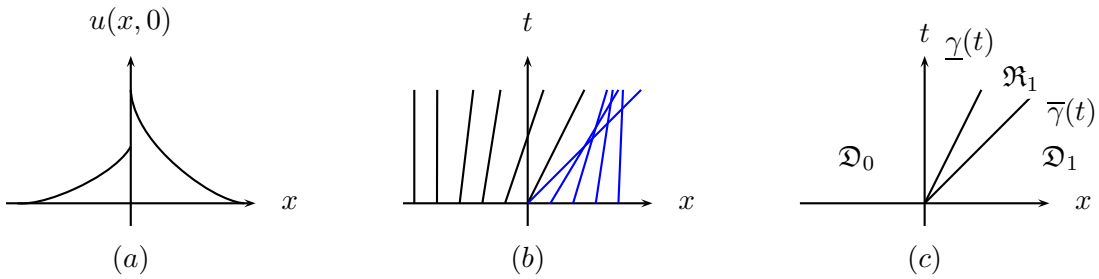


Figure 5.1: (a): Initial data, (b) characteristics, (c) domains of smoothness

We have $\hat{u}_L^0 = 1 < 2 = \hat{u}_R^0$. Therefore, the classical Riemann problem for the leading term contains a rarefaction wave. The solution of the generalized Riemann problem thus is

$$u(x,t) = \begin{cases} u_0(x,t), & x/t \le \underline{\gamma}(t)/t, \\ x/t, & \underline{\gamma}(t)/t < x/t < \overline{\gamma}(t)/t \\ u_1(x,t), & \overline{\gamma}(t)/t \le x/t. \end{cases}$$

Using the method of characteristics, we can explicitly compute

$$u_0(x,t) = \frac{2t(x+1)+1-\sqrt{4t(x+1)+1}}{2t^2},$$
$$u_1(x,t) = \frac{4t(x-1)+1-\sqrt{8t(x-1)+1}}{4t^2}.$$

The boundaries of the rarefaction zone are given by the head-characteristic $\overline{\gamma}(t) = 2t$ and the tail-characteristic $\underline{\gamma}(t) = t$.

The deceasing initial data $\hat{u}_R$ will lead to the formation of a new shock. It can be easily checked that characteristics cross for time larger than $t_F = 1/4$. For sufficiently small time $t > 0$ we approximate the solution along the t-axis by

$$u(0,t) \approx u(0,0+) + \frac{\partial}{\partial t}u(0,0+)t + \frac{\partial^2}{\partial t^2}u(0,0+)\frac{t^2}{2}.$$

Note that the t-axis is contained in the domain $\mathfrak{D}_0$, so in the subsequent analysis

we only need to concern the function u_0. We have

$$\frac{\partial}{\partial t}u_0(x,t) = \frac{1 - t(x+1)\left(-3 + \sqrt{4t(x+1)+1}\right) - \sqrt{4t(x+1)+1}}{t^3\sqrt{4t(x+1)+1}},$$

from which we get

$$\lim_{t\to 0+}\frac{\partial}{\partial t}u_0(x,t) = -2(x+1)^3, \qquad \frac{\partial}{\partial t}u_0(0,0+) = -2, \tag{5.1}$$

and

$$\begin{aligned}\frac{\partial^2}{\partial t^2}u_0(x,t) = &\\ &\frac{2t^2(x+1)^2\left(4\sqrt{4t(x+1)+1} - 15\right) + 2t(x+1)\left(7\sqrt{4t(x+1)+1} - 10\right)}{t^4(4t(x+1)+1)^{3/2}}\\ &+ \frac{3\left(\sqrt{4t(x+1)+1} - 1\right)}{t^4(4t(x+1)+1)^{3/2}}.\end{aligned}$$

Therefore

$$\lim_{t\to 0+}\frac{\partial^2}{\partial t^2}u_0(x,t) = 10(x+1)^4, \qquad \frac{\partial^2}{\partial t^2}u_0(0,0+) = 10. \tag{5.2}$$

Thus, the Taylor expansion of the exact solution gives

$$u(0,t) \approx 1 - 2t + 5t^2.$$

We now compute the terms of the LeFloch-Raviart expansion up to the function $q^2(\xi)$. So we construct the approximation

$$\begin{aligned}u(x,t) \approx & u(0,0+) + t\left(\frac{\partial}{\partial t}u(0,0+) + \frac{\partial}{\partial x}u(0,0+)\left(\frac{x}{t}\right)\right)\\ &+ t^2\left(\frac{1}{2}\frac{\partial^2}{\partial t^2}u(0,0+) + \frac{\partial^2}{\partial x\partial t}u(0,0+)\left(\frac{x}{t}\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}u(0,0+)\left(\frac{x}{t}\right)^2\right).\end{aligned}$$

Then, q^0 is the solution of the Riemann problem

$$\frac{\partial}{\partial t}q^0 + \frac{\partial}{\partial x}\left(\frac{(q^0)^2}{2}\right) = 0 \qquad x \in \mathbb{R},\ t > 0,$$

$$q^0(x,0) = \begin{cases} 1, & x < 0, \\ 2, & x > 0, \end{cases}$$

in which case $q^0(\xi) = 1$ for all $\xi \in \mathfrak{D}_0$. The function q^1 satisfies the ordinary differential equation

$$q_0^1 + (1-\xi)\frac{d}{d\xi}q_0^1 = 0,$$

and the solution consistent with the initial data is

$$q_0^1(\xi) = 2(\xi - 1).$$

Therefore, we have

$$h^2(\xi) = -\frac{d}{d\xi}\frac{f''(q_0^0)}{2}(q^1(\xi))^2 = -4(\xi - 1).$$

The inhomogeneous equation for q^2,

$$2q_0^2 + (1-\xi)\frac{d}{d\xi}q_0^2 = -4(\xi - 1),$$

has the solution $q_0^2(\xi) = (\xi-1)^2 - 4(\xi-1)$. Thus,

$$\begin{aligned} \tilde{q}(\xi,t) &\approx 1 + 2(\xi-1)t + ((\xi-1)^2 - 4(\xi-1))t^2, \\ q(x,t) &\approx x^2 + 2x - 6xt + 5t^2 - 2t + 1, \end{aligned}$$

and in particular

$$q(0,t) \approx 1 - 2t + 5t^2.$$

Now we use the solver of Toro and Titarev, where we find the leading term $u(0,0+)$ by solving

$$\frac{\partial}{\partial t}\phi^0 + \frac{\partial}{\partial x}\left(\frac{(\phi^0)^2}{2}\right) = 0 \quad \text{for } x \in \mathbb{R},\ t > 0,$$

$$\phi^0(x,0) = \begin{cases} 1, & x < 0, \\ 2, & x > 0, \end{cases}$$

This gives $\phi^0(0,0+)=1$. Next, the Cauchy-Kovalevskaya procedure leads to

$$\begin{aligned}\frac{\partial u}{\partial t}=-u\frac{\partial u}{\partial x},\qquad \frac{\partial^2 u}{\partial t}&=2\left(\frac{\partial u}{\partial x}\right)^2+u\frac{\partial^2 u}{\partial x^2},\\ \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right)&=-\left(\frac{\partial u}{\partial x}\right)^2-u\frac{\partial^2 u}{\partial x^2},\\ \frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial x^2}\right)&=-3\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t}-u\frac{\partial^3 u}{\partial x^3}.\end{aligned}\tag{5.3}$$

For $\phi^1=\partial_x u$ and $\phi^2=\partial_x^2 u$, we have the evolution equations

$$\frac{\partial \phi^1}{\partial t}+\phi^0\frac{\partial \phi^1}{\partial x}=-\left(\phi^1\right)^2$$

and

$$\frac{\partial \phi^2}{\partial t}+\phi^0\frac{\partial \phi^2}{\partial x}=-3\phi^1\phi^2$$

with the initial conditions

$$\phi^1(x,0)=\begin{cases}\hat{u}_L^1=2, & x<0,\\ \hat{u}_R^1=-4, & x>0,\end{cases}\tag{5.4}$$

and

$$\phi^2(x,0)=\begin{cases}\hat{u}_L^2=2 & x<0,\\ \hat{u}_R^2=4 & x>0.\end{cases}\tag{5.5}$$

We drop the source terms and linearise around $\phi^0(0,0+)=1$, so that we have

$$\frac{\partial \phi^k}{\partial t}+\phi^0\frac{\partial \phi^k}{\partial x}=0\quad \text{for } k=1,2,$$

together with the initial conditions (5.4) and (5.5), respectively. These *linear* problems are readily solved, where we find the Godunov states

$$\phi^1(0,0+)=2\quad \text{and}\quad \phi^2(0,0+)=2.$$

By the Cauchy-Kovalevskaya procedure (5.3) we get

$$\frac{\partial u}{\partial t}(0,0+) = -\phi^0(0,0+)\phi^1(0,0+) = -2,$$
$$\frac{\partial^2 u}{\partial t^2}(0,0+) = 2\left(\phi^1(0,0_+)\right)^2 + \phi^0(0,0_+)\phi^2(0,0_+) = 10,$$

and thus again we find

$$u(0,t) \approx 1 - 2t + 5t^2.$$

5.2 Two-component chromatography

Consider the system

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} v(1+v+w)^{-1} \\ w(1+v+w)^{-1} \end{pmatrix} = 0, \qquad v, w > 0 \tag{5.6}$$

and denote $u = (v,w)^T$, $u \in \mathcal{U} = (0,\infty) \times (0,\infty) \subset \mathbb{R}^2$. It is inspired by the analysis of two-component chromatography, as described by Temple [86]. Note that the chromatography system contains a constant $\kappa \in (0,1)$ in the second equation. Since we are more interested in the analysis and less in the modelling, we have set $\kappa = 1$ to simplify matters without changing the structure of the solution. The solution of the Riemann problem for (5.6) can be found in [14], a thorough discussion of mathematical models for chromatography is presented in [71]. This system has a very nice eigenstructure that allows us to carry out all the computations we need to determine q^1 explicitly. We demonstrate analytically that the difference between ϕ^1_* and q^1_* only depends on the size of the jump in the initial states, not on the size of the jump in the derivatives of the initial data.

The Jacobian of the flux is given by

$$A(v,w) = \frac{1}{(1+v+w)^2}\begin{pmatrix} 1+w & -v \\ -w & 1+v \end{pmatrix}$$

and it has the eigenvalues and (normalized) eigenvectors

$$\lambda_1(u) = \frac{1}{(1+v+w)^2}, \qquad r_1(u) = \frac{1}{\sqrt{v^2+w^2}} \begin{pmatrix} -v \\ -w \end{pmatrix}$$
$$\lambda_2(u) = \frac{1}{1+v+w}, \qquad r_2(u) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The left eigenvectors, normalized to $\ell_i(v,w) \cdot r_j(v,w) = \delta_{ij}$, are

$$\ell_1(v,w) = -\frac{\sqrt{v^2+w^2}}{v+w} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad \ell_2(v,w) = -\frac{\sqrt{2}}{v+w} \begin{pmatrix} -w \\ v \end{pmatrix}.$$

Since we have

$$\nabla\lambda_1(u) \cdot r_1(u) = \frac{2}{\sqrt{v^2+w^2}} \cdot \frac{v+w}{(1+v+w)^3} > 0,$$
$$\nabla\lambda_2(u) \cdot r_2(u) = 0,$$

the first characteristic field is genuinely nonlinear and the second characteristic field is linearly degenerate.

For this system, shock and rarefaction curves coincide in the sense that each point in the i-Hugoniot set ($i = 1, 2$) of a given point u_- lies on the integral curve of r_i through u_-. Due to the simple nature of the eigenvectors, the integral curves here are straight lines in the space of conserved variables. We can directly compute

$$R_1(\varepsilon)(u) = u + \varepsilon r_1(u), \qquad R_2(\varepsilon)(u) = u + \varepsilon r_2(u). \tag{5.7}$$

Both properties - coinciding shock and rarefaction curves and straight line integral curves - are of particular interest in the analysis of nonlinear hyperbolic systems, because the simplified structure of the solution allows answers to questions still open for general hyperbolic system. See the work of Temple [85, 86] for a discussion of coinciding shock and rarefaction curves. For systems with straight line curves, Arora and Roe [2] observed only small postshock oscillations in numerical schemes and conjectured that for system that do not have the straight line property, Godunov and Roe schemes will produce postshock oscillations. Somewhat similar,

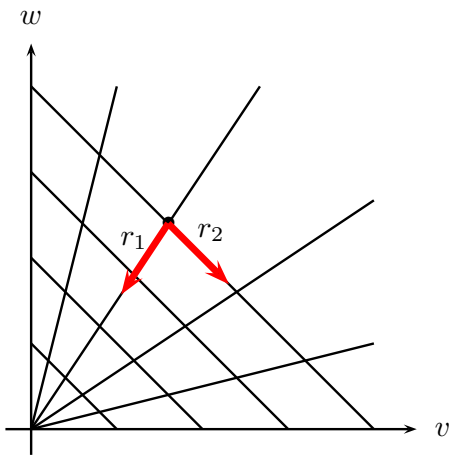


Figure 5.2: Integral curves for a system from chromatography

though more theoretically motivated, is a result by Bressan and Jenssen [17] who were able to use the straight line property do establish BV bounds necessary to prove convergence for the Godunov scheme.

Now consider the Riemann problem with initial data $u_L = (v_L, w_L)^T$, $u_R = (v_R, w_R)^T$. Then the Riemann solution contains an intermediate state $u_* = (v_*, w_*)$, such that

$$\begin{pmatrix} v_* \\ w_* \end{pmatrix} = \begin{pmatrix} v_L \\ w_L \end{pmatrix} + \frac{\varepsilon_1}{\sqrt{v_L^2 + w_L^2}} \begin{pmatrix} -v_L \\ -w_L \end{pmatrix}, \tag{5.8}$$

$$\begin{pmatrix} v_R \\ w_R \end{pmatrix} = \begin{pmatrix} v_* \\ w_* \end{pmatrix} + \frac{\varepsilon_2}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \tag{5.9}$$

for some $\varepsilon_1, \varepsilon_2$.

Because the second field is linearly degenerate, we have $\lambda_2(u_*) = \lambda_2(u_R)$ and therefore

$$v_* + w_* = v_R + w_R. \tag{5.10}$$

Further, it follows from (5.8) that

$$v_L w_* = v_* w_L \tag{5.11}$$

and that

$$\varepsilon_1 = \sqrt{(v_L)^2 + (w_L)^2} - \sqrt{(v_*)^2 + (w_*)^2}.$$

Combing the two conditions (5.10) and (5.11), we can explicitly compute

$$v_* = \frac{v_L(v_R + w_R)}{v_L + w_L}, \qquad w_* = \frac{w_L(v_R + w_R)}{v_L + w_L}.$$

Therefore, the wave strength ε_1 is

$$\varepsilon_1 = \left(1 - \frac{v_R^2 + w_R^2}{v_L^2 + w_L^2}\right) \sqrt{v_L^2 + w_L^2}.$$

Recall that the type of wave associated with the first characteristic family depends on the sign of ε_1: We get a 1-shock for $\varepsilon \leq 0$ and a 1-rarefaction for $\varepsilon_1 > 0$ (the second wave is always a contact discontinuity, independent of the sign of ε_2).

Thus, if $v_R^2 + w_R^2 \leq v_L^2 + w_L^2$, the solution contains a 1-shock, a 2-contact discontinuity and is given by

$$u(x,t) = \begin{cases} u_L, & x/t < \sigma_1, \\ u_*, & \sigma_1 < x/t < \sigma_2, \\ u_R, & \sigma_2 < x/t, \end{cases}$$

where $\sigma_2 = \lambda_2(u_*) = \lambda_2(u_R)$ and the shock speed σ_1 can be computed from the Rankine-Hugoniot conditions:

$$\sigma_1 = \int_0^1 \lambda_1(\theta u_* + (1-\theta)u_L)\, d\theta = \frac{1}{(1 + v_L + w_L)(1 + v_* + w_*)}.$$

Now consider the generalized Riemann problem with *piecewise linear* initial data:

$$u(x,0) = \hat{u}(x) = \begin{cases} \hat{u}_L(x), & x > 0, \\ \hat{u}_R(x), & x < 0, \end{cases}$$

where

$$\hat{u}_L(x) = \begin{pmatrix} \hat{v}_L(x) \\ \hat{w}_L(x) \end{pmatrix} = \begin{pmatrix} \hat{v}_L^0 \\ \hat{w}_L^0 \end{pmatrix} + x \begin{pmatrix} \hat{v}_L^1 \\ \hat{w}_L^1 \end{pmatrix}, \tag{5.12}$$

and

$$\hat{u}_R(x) = \begin{pmatrix} \hat{v}_R(x) \\ \hat{w}_R(x) \end{pmatrix} = \begin{pmatrix} \hat{v}_R^0 \\ \hat{w}_R^0 \end{pmatrix} + x \begin{pmatrix} \hat{v}_R^1 \\ \hat{w}_R^1 \end{pmatrix}. \tag{5.13}$$

Assume that the solution contains a 1-shock. Denote $\hat{u}_a^k = (v_a^k,\ w_a^k)^T$ for $a = L, R,\ k = 0, 1$ and let ϕ^0 be the solution of the classical Riemann problem for (5.6) with initial data $\hat{u}_L^0,\ \hat{u}_R^0$. Denote the intermediate state in that solution by ϕ_*^0. Note that $\lambda_1(u) > 0$, so the Godunov state is $\hat{u}_L^0$.

The simplified problem in the Toro-Titarev solver for the spatial derivatives is then given by

$$\begin{aligned} &\frac{\partial}{\partial t}\phi^1 + A_{LR}\frac{\partial}{\partial x}\phi^1 = 0, \qquad x \in \mathbb{R},\ t > 0, \\ &\phi^1(x,0) = \begin{pmatrix} \hat{v}(x) \\ \hat{w}(x) \end{pmatrix} = \begin{cases} \hat{u}_L^1, & x < 0, \\ \hat{u}_R^1, & x > 0. \end{cases} \end{aligned}$$

Here, $A_{LR} = A(\hat{u}_L^0)$. This linear problem has the solution

$$\phi^1(x,t) = \hat{v}\left(x - \lambda_1(\hat{u}_L^0)t\right) r_1(\hat{u}_L^0) + \hat{w}\left(x - \lambda_2(\hat{u}_L^0)t\right) r_2(\hat{u}_L^0).$$

We express the vectors $\hat{u}_L^1,\ \hat{u}_R^1$ in terms of the basis $\{r_1(\hat{u}_L^0), r_2(\hat{u}_L^0)\}$, i.e.

$$\hat{u}_L^1 = \alpha_{L,1}^1 r_1(\hat{u}_L^0) + \alpha_{L,2}^1 r_2(\hat{u}_L^0), \qquad \hat{u}_L^1 = \alpha_{R,1}^1 r_1(\hat{u}_L^0) + \alpha_{R,2}^1 r_2(\hat{u}_L^0).$$

We have

$$\begin{aligned} \alpha_{L,1}^1 =& -\left(\hat{v}_L^1 + \hat{w}_L^1\right) \frac{\sqrt{(\hat{v}_L^0)^2 + (\hat{w}_L^0)^2}}{\hat{v}_L^0 + \hat{w}_L^0}, \\ \alpha_{L,2}^1 =& \sqrt{2}\left(\left(\hat{v}_L^1 + \hat{w}_L^1\right) \frac{\hat{w}_L^0}{\hat{v}_L^0 + \hat{w}_L^0} - \hat{w}_L^1\right). \end{aligned}$$

Then the intermediate state ϕ_*^1 can be computed as

$$\begin{aligned} \phi_*^1 &= \alpha_{L,1}^1 r_1(\hat{u}_L^0) + \alpha_{R,2}^1 r_2(\hat{u}_L^0) \\ &= \begin{pmatrix} (\hat{v}_R^1 + \hat{w}_R^1)\frac{\hat{v}_L^0}{\hat{v}_L^0 + \hat{w}_L^0} + (\hat{v}_L^1 + \hat{w}_L^1)\frac{\hat{w}_L^0}{\hat{v}_L^0 + \hat{w}_L^0} - \hat{w}_L^1 \\ (\hat{v}_R^1 + \hat{w}_R^1 - \hat{v}_L^1 - \hat{w}_L^1)\frac{\hat{w}_L^0}{\hat{v}_L^0 + \hat{w}_L^0} + \hat{w}_L^1 \end{pmatrix}. \end{aligned} \tag{5.14}$$

Now we compare ϕ_*^1 with the vector q_1^1, which the defines the function $q^1(\xi)$ in the LeFloch-Raviart approximation

$$u(x,t) = q^0(\xi) + tq^1(\xi) + \dots \ .$$

As above, we find q^0 by solving the classical Riemann problem with initial states $\hat{u}_L^0,\ \hat{u}_R^0$ and we denote the constant states in that solution by $\hat{u}_L^0 = q_0^0,\ q_1^0,\ q_2^0 = \hat{u}_R^0$. The function $q^1(\xi)$ is given in each domain $\mathfrak{D}_i^0$ by

$$q^1(\xi) = \big(\xi - A(q_i^0)\big)\, q_i^1, \qquad 0 \le i \le 2,$$

where we express the unknown vectors q_i^1 as

$$q_i^1 = \alpha_{i,1}^1 r_1(q_i^0) + \alpha_{i,2}^1 r_2(q_i^0).$$

The coefficients $\alpha_{0,j}^1,\ \alpha_{2,j}^1$ for $j = 1, 2$ are determined by the initial data $\hat{u}_L^1$ and $\hat{u}_R^1$ respectively, while the remaining coefficients $\alpha_{1,1}^1,\ \alpha_{1,2}^2$ are found by solving a linear 2-by-2 system of algebraic equations that can be derived from the jump conditions. We arrive at the system (see Lemma 3.3))

$$\begin{aligned}
&\big(\lambda_1(q_1^0) - \sigma_1^0\big)^2 \ell_2(q_0^0)\cdot r_1(q_1^0)\alpha_{1,1}^1 + \big(\lambda_2(q_1^0) - \sigma_1^0\big)^2 \ell_2(q_0^0)\cdot r_2(q_1^0)\alpha_{1,2}^1 \\
&\qquad - \big(\lambda_2(q_0^0) - \sigma_1^0\big)^2 \alpha_{0,2}^1 \\
&= \frac{\ell_2(q_0^0)\cdot(q_1^0 - q_0^0)}{\ell_1(q_0^0)\cdot(q_1^0 - q_0^0)} \Big\{ \big(\lambda_1(q_1^0) - \sigma_1^0\big)^2 \ell_1(q_0^0)\cdot r_1(q_1^0)\alpha_{1,1}^1 \\
&\qquad + \big(\lambda_2(q_1^0) - \sigma_1^0\big)^2 \ell_1(q_0^0)\cdot r_2(q_1^0)\alpha_{1,2}^1 + \big(\lambda_1(q_0^0) - \sigma_1^0\big)^2 \alpha_{0,1}^1 \qquad \Big\} \qquad (5.15)
\end{aligned}$$

$$\begin{aligned}
&\left(\lambda_1(q_2^0)-\sigma_2^0\right)^2\ell_1(q_1^0)\cdot r_1(q_2^0)\alpha_{2,1}^1+\left(\lambda_2(q_2^0)-\sigma_2^0\right)^2\ell_1(q_1^0)\cdot r_2(q_2^0)\alpha_{2,2}^1\\
&\qquad-\left(\lambda_1(q_1^0)-\sigma_2^0\right)^2\alpha_{1,1}^1\\
&=\frac{\ell_1(q_1^0)\cdot(q_2^0-q_1^0)}{\ell_2(q_1^0)\cdot(q_2^0-q_1^0)}\Big\{\left(\lambda_1(q_2^0)-\sigma_2^0\right)^2\ell_2(q_1^0)\cdot r_1(q_2^0)\alpha_{2,1}^1\\
&\qquad+\left(\lambda_2(q_2^0)-\sigma_2^0\right)^2\ell_2(q_1^0)\cdot r_2(q_2^0)\alpha_{2,2}^1+\left(\lambda_2(q_1^0)-\sigma_2^0\right)^2\alpha_{1,2}^1\quad\Big\}
\end{aligned}\tag{5.16}$$

Now recall that q_0^0, q_1^0, q_2^0 are the constant states in the solution of a classical Riemann problem. Then, by the relations (5.8), (5.9) for the intermediate state q_1^0, we find

$$q_1^0-q_0^0=\varepsilon_1 r_1(q_0^0),\qquad q_2^0-q_1^0=\varepsilon_2 r_2(q_1^0).$$

Therefore, by the orthonormality of the eigenvectors, we get

$$\begin{aligned}
\ell_2(q_0^0)\cdot(q_1^0-q_0^0)&=0,\qquad \ell_1(q_0^0)\cdot(q_1^0-q_0^0)=\varepsilon_1,\\
\ell_1(q_1^0)\cdot(q_2^0-q_1^0)&=0,\qquad \ell_2(q_1^0)\cdot(q_2^0-q_1^0)=\varepsilon_2.
\end{aligned}$$

Noting further that $\lambda_2(q_1^0)=\lambda_2(q_2^0)=\sigma_2^0$ and that $\ell_1\cdot r_2=0$, the system (5.15), (5.16) reduces to

$$\begin{aligned}
&\left(\lambda_1(q_1^0)-\sigma_1^0\right)^2\ell_2(q_0^0)\cdot r_1(q_1^0)\alpha_{1,1}^1+\left(\lambda_2(q_1^0)-\sigma_1^0\right)^2\ell_2(q_0^0)\cdot r_2(q_1^0)\alpha_{1,2}^1\\
&\qquad-\left(\lambda_2(q_0^0)-\sigma_1^0\right)^2\alpha_{0,2}^1=0
\end{aligned}\tag{5.17}$$

$$\left(\lambda_1(q_2^0)-\sigma_2^0\right)^2\ell_1(q_1^0)\cdot r_1(q_2^0)\alpha_{2,1}^1-\left(\lambda_1(q_1^0)-\sigma_2^0\right)^2\alpha_{1,1}^1=0\tag{5.18}$$

Using condition (5.10), $v_1^0+w_1^0=v_2^0+w_2^0$, we have

$$\left(\frac{\lambda_1(q_2^0)-\sigma_2^0}{\lambda_1(q_1^0)-\sigma_2^0}\right)^2=1,$$

and thus we get from (5.18) that

$$\alpha_{1,1}^1=\ell_1(q_1^0)\cdot r_1(q_2^0)\alpha_{2,1}^1=-\left(v_2^1+w_2^1\right)\frac{\sqrt{(v_1^0)^2+(w_1^0)^2}}{v_2^0+w_2^0}.\tag{5.19}$$

Also, we can see that by (5.11), we have

$$\ell_2(q_0^0) \cdot r_1(q_1^0) = -\frac{\sqrt{2}}{(v_0^0 + w_0^0)\sqrt{(v_1^0)^2 + (w_1^0)^2}}(w_0^0 v_1^0 - w_1^0 v_0^0) = 0.$$

Moreover, it is straightforward to show that

$$\frac{\lambda_2(q_0^0) - \sigma_1^0}{\lambda_2(q_1^0) - \sigma_1^0} = \frac{v_1^0 + w_1^0}{v_0^0 + w_0^0} = \frac{v_2^0 + w_2^0}{v_0^0 + w_0^0}.$$

Finally, this yields

$$\alpha_{1,2}^1 = \left(\frac{v_2^0 + w_2^0}{v_0^0 + w_0^0}\right)^2 \alpha_{0,2}^1 = \sqrt{2}\left(\frac{v_2^0 + w_2^0}{v_0^0 + w_0^0}\right)^2 \left((v_0^1 + w_0^1)\frac{w_0^0}{v_0^0 + w_0^0} - w_0^1\right). \tag{5.20}$$

Thus, the coefficient q_1^1 is given by

$$\begin{aligned} q_1^1 &= \alpha_{1,1}^1 r_1(q_1^0) + \alpha_{1,2}^1 r_2(q_1^0) \\ &= \begin{pmatrix} (v_2^1 + w_2^1)\frac{v_0^0}{v_0^0+w_0^0} + \left(\frac{v_2^0+w_2^0}{v_0^0+w_0^0}\right)^2 \left((v_0^1 + w_0^1)\frac{w_0^0}{v_0^0+w_0^0} - w_0^1\right) \\ (v_2^1 + w_2^1)\frac{v_0^0}{v_0^0+w_0^0} - \left(\frac{v_2^0+w_2^0}{v_0^0+w_0^0}\right)^2 \left((v_0^1 + w_0^1)\frac{w_0^0}{v_0^0+w_0^0} - w_0^1\right) \end{pmatrix}. \end{aligned} \tag{5.21}$$

Comparing (5.14) and (5.21), we see that the only difference is the factor

$$\left(\frac{v_2^0 + w_2^0}{v_0^0 + w_0^0}\right)^2 = \left(\frac{|\hat{u}_R^0|_1}{|\hat{u}_L^0|_1}\right)^2,$$

whose size depends on the size of the jump in the initial data, but not on higher order terms.

5.3 Shallow water equations

A widely used model for many environmental and engineering problems are the shallow water equations, which we present here in their simplest one-dimensional form:

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ hu \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \end{pmatrix} = 0.$$

Here h denotes the water hight, u the velocity and g is the gravitational constant. There is an immense list of literature on shallow water equations, we merely mention [58, Chapter 13] for a discussion of the Riemann problem and [90, 93] for an overview on numerical techniques.

Using the notation from above and introducing $v = h,\ w = uh$, this can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} v \\ w \end{pmatrix} + \frac{\partial}{\partial x}\begin{pmatrix} w \\ \frac{w^2}{v} + \frac{1}{2}gv^2 \end{pmatrix} = 0.$$

Note that v denotes the water height, *not* the velocity. This deviation from standard notation might be slightly inconvenient, but we are trying to be consistent with the notation introduced in Chapter 3.

The Jacobian of the flux is

$$A(v,w) = \begin{pmatrix} 0 & 1 \\ -\left(\frac{w}{v}\right)^2 + gv & 2\frac{w}{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -u^2 + gh & 2u \end{pmatrix},$$

with eigenvalues

$$\lambda_1 = u - \sqrt{gh}, \qquad \lambda_2 = u + \sqrt{gh},$$

and corresponding eigenvectors

$$r_1 = \begin{pmatrix} 1 \\ u - \sqrt{gh}, \end{pmatrix} \qquad r_2 = \begin{pmatrix} 1 \\ u + \sqrt{gh}, \end{pmatrix}.$$

Left eigenvectors, normalized to $\ell_j \cdot r_i = \delta_{ij}$, are

$$\ell_1 = \frac{1}{2\sqrt{gh}}\begin{pmatrix} u + \sqrt{gh} \\ -1 \end{pmatrix}, \qquad \ell_2 = -\frac{1}{2\sqrt{gh}}\begin{pmatrix} u - \sqrt{gh} \\ -1 \end{pmatrix}.$$

We consider the generalized Riemann problem with initial data $\hat{h}_L = \hat{h}_R = 1$ and

$$\hat{u}_L(x) = a_L x^2 + b_L x + c_L, \qquad \hat{u}_R(x) = a_R x^2 + b_R x + c_R.$$

When $c_R < 0 < c_L$, this data leads to a solution with two shock waves.

We compare the resulting approximations up to quadratic terms obtained by the LeFloch-Raviart expansion and the Toro-Titarev solver, respectively. Reference solutions are obtained by a random choice method (RCM) on a very fine grid using an exact Riemann solver and van der Corput pseudo random numbers (see [91, Chapter 7]). We perform three series of tests:

(i) Large jumps in the initial data, fixed derivatives. We fix $a_L = 0.02, a_R = -0.01$ and $b_L = 0.4,\ b_R = -0.2$. We solve the GRP for $c_R = -1$ and $c_L = 2,\ 4,\ 6$, respectively. Results are shown in Figure 5.3. Denoting $v = h$ and $w = hu$, the plots show the reference solution (thick black line), the LeFloch-Raviart approximation (blue circles) and the Toro-Titarev approximation (red crosses) along the line $\xi = 0$ for times $0 \leq t \leq 0.05$. The difference in the two approximations increases with the size of the jump. We observe that the LeFloch-Raviart approximation is almost identical to the reference solution.

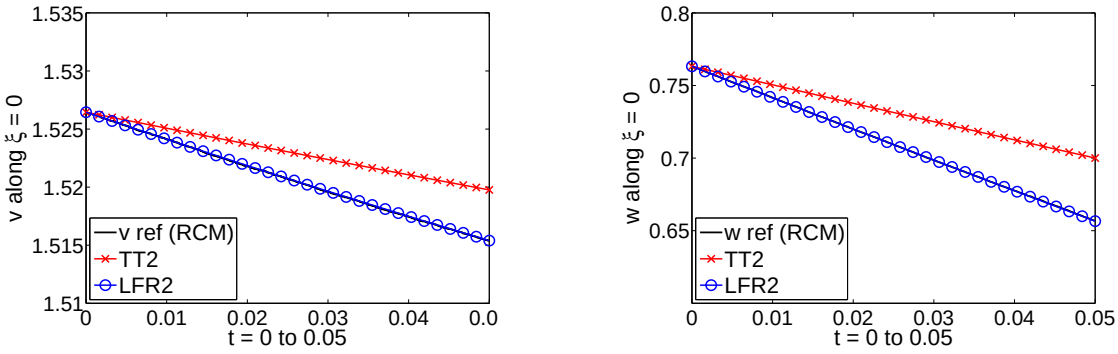
(ii) Large jumps in first derivatives, fixed jump in states. We fix $a_L = 0.02,\ a_R = -0.01$ and $c_L = 0.2,\ c_R = -0.1$, so we have a fixed jump $|\hat{u}_L^0 - \hat{u}_R^0| = 0.3$. We let $b_R = -1$ and test for $b_L = 2,\ 4,\ 6$, see Figure 5.4. For all test cases both approximations are very close to the reference solution.

(iii) Large jumps in second derivatives, fixed jump in states. We take $b_L = 0.4,\ b_R = 0.1$ and $c_L = 0.2,\ c_R = -0.1$. We choose $a_R = -1$ and solve for $a_L = 2,\ 4,\ 6$, respectively. Results are shown in figure 5.5. Again, for all test cases both approximations are very close to the reference solution.

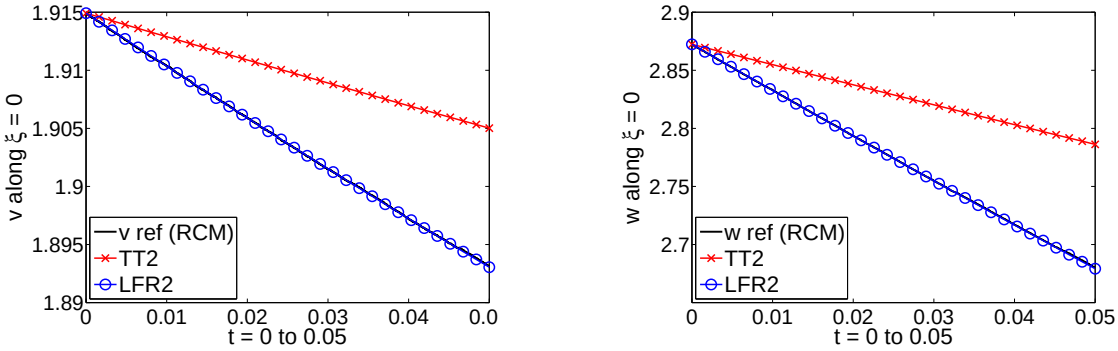
Let us compare the resulting L_1 errors, compared to the reference solution (see Table 5.3, Table 5.3 shows the relative L^1-errors):

For large jumps in the initial states we note that the Toro-Titarev approximation has a L^1-error that is two orders larger than the L^1-error of the LeFloch-Raviart approximation. If the jump in the initial states is small, the order of the size of

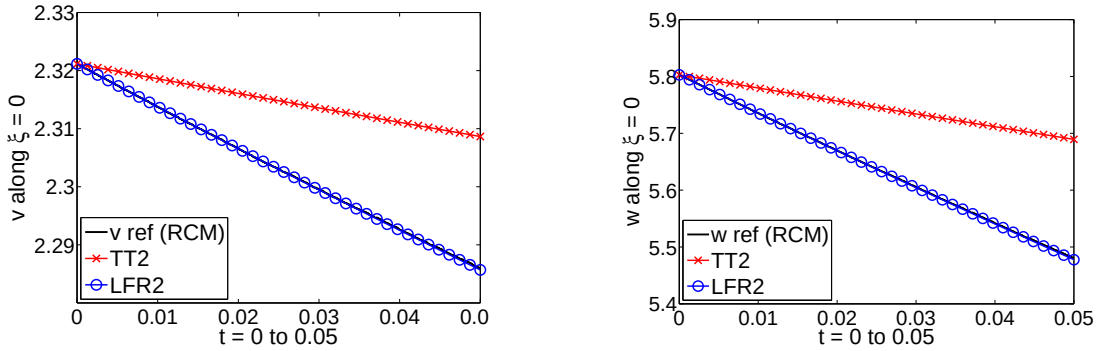
$$\hat{u}_L(x) = 0.02x^2 + 0.4x + c_L, \qquad \hat{u}_R(x) = -0.01x^2 - 0.2x - 1$$



(a) $c_L = 2, \quad |\hat{u}_L^0 - \hat{u}_R^0| = 3$



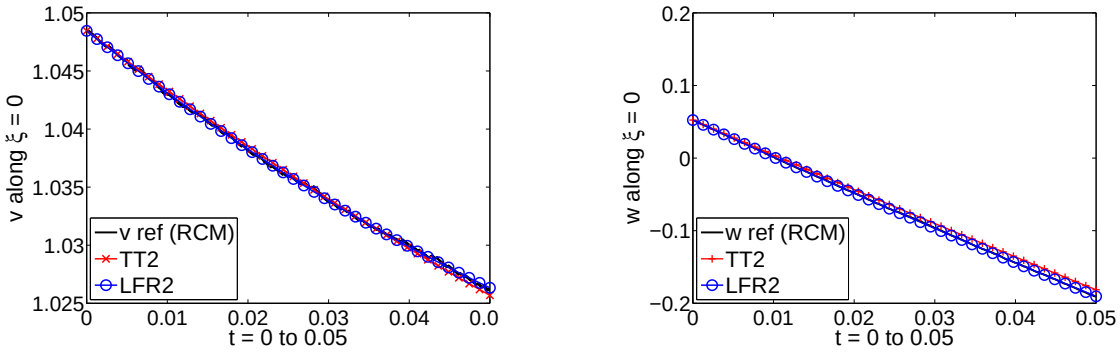
(b) $c_L = 4, \quad |\hat{u}_L^0 - \hat{u}_R^0| = 5$



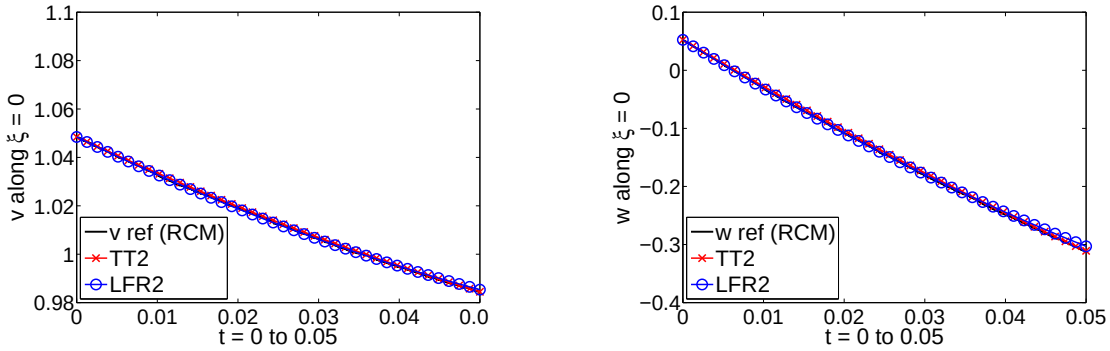
(c) $c_L = 6, \quad |\hat{u}_L^0 - \hat{u}_R^0| = 7$

Figure 5.3: Two-shock solution for 1-d shallow water: Jumps in the initial states

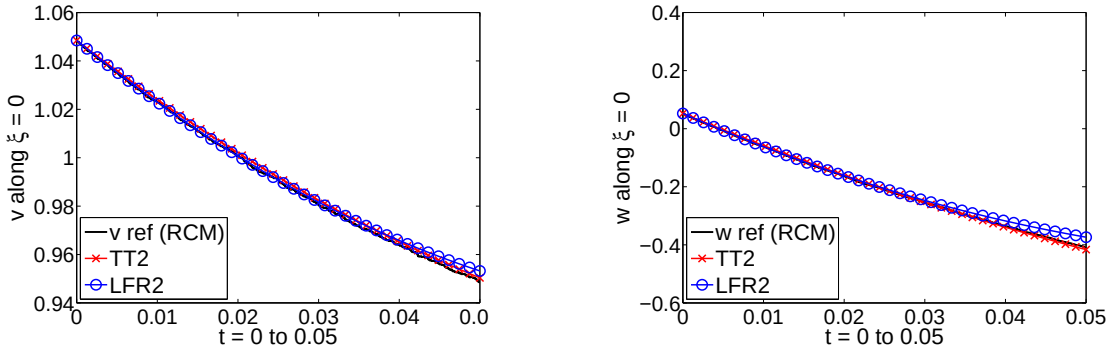
$$\hat{u}_L(x) = 0.02x^2 + b_L x + 0.2, \qquad \hat{u}_R(x) = -0.01x^2 - x - 0.1$$



(a) $b_L = 2, \quad |\hat{u}_L^1 - \hat{u}_R^1| = 3$



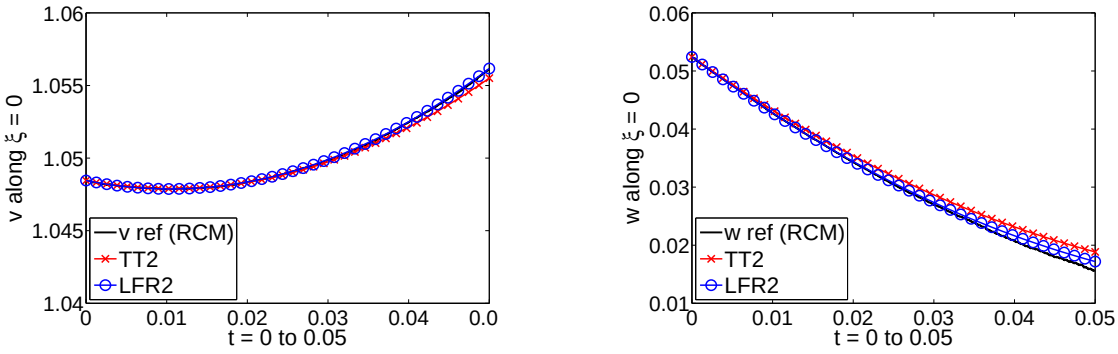
(b) $b_L = 4, \quad |\hat{u}_L^1 - \hat{u}_R^1| = 5$



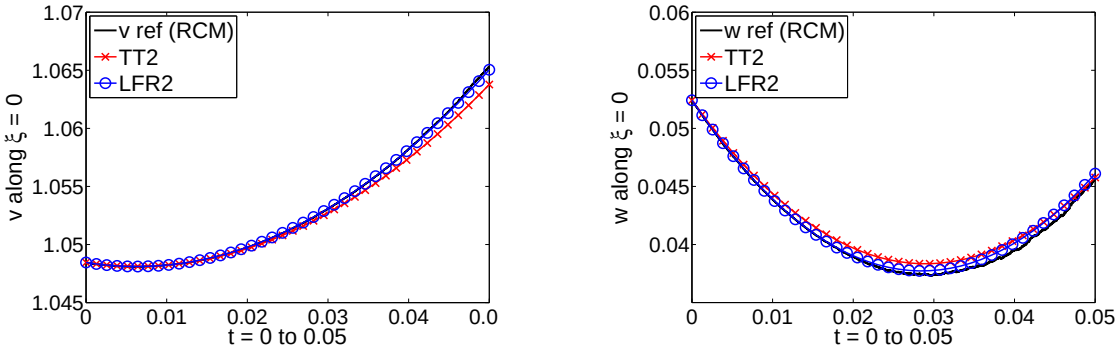
(c) $b_L = 6, \quad |\hat{u}_L^1 - \hat{u}_R^1| = 7$

Figure 5.4: Two-shock solution for 1-d shallow water: Jumps in first derivatives

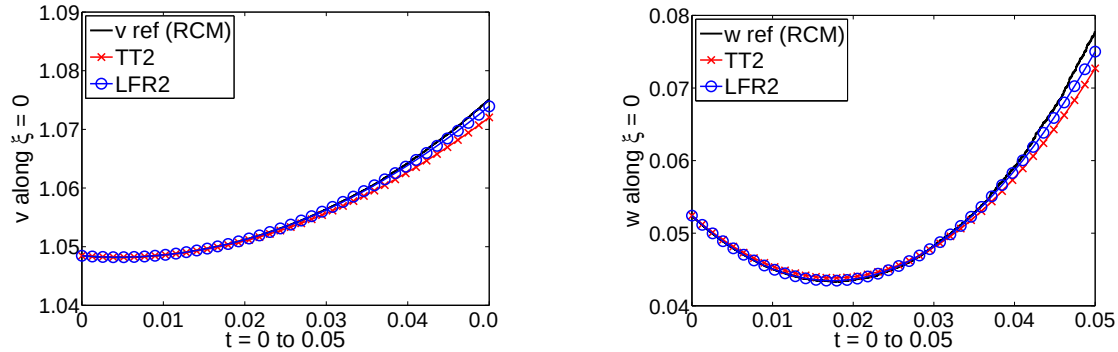
$$\hat{u}_L(x) = a_L x^2 + 0.4x + 0.2, \qquad \hat{u}_R(x) = -1x^2 - 0.2x - 0.1$$



(a) $a_L = 2, \quad |\hat{u}_L^2 - \hat{u}_R^2| = 3$



(b) $a_L = 4, \quad |\hat{u}_L^2 - \hat{u}_R^2| = 5$



(c) $a_L = 6, \quad |\hat{u}_L^2 - \hat{u}_R^2| = 7$

Figure 5.5: Two-shock solution for 1-d shallow water: Jumps in second derivatives

	$\|v - v_{LFR}\|_1$	$\|w - w_{LFR}\|_1$	$\|v - v_{TT}\|_1$	$\|w - w_{TT}\|_1$
$[[\hat{u}^0]] = 3$	1.893×10^{-6}	7.709×10^{-6}	1.144×10^{-4}	1.095×10^{-3}
$[[\hat{u}^0]] = 5$	2.380×10^{-6}	1.967×10^{-5}	3.059×10^{-4}	2.703×10^{-3}
$[[\hat{u}^0]] = 7$	5.061×10^{-6}	4.920×10^{-5}	5.818×10^{-4}	5.340×10^{-3}
$[[\hat{u}^1]] = 3$	5.935×10^{-6}	2.493×10^{-5}	1.362×10^{-5}	2.923×10^{-4}
$[[\hat{u}^1]] = 5$	1.347×10^{-5}	1.073×10^{-4}	3.136×10^{-5}	1.795×10^{-4}
$[[\hat{u}^1]] = 7$	6.397×10^{-5}	4.727×10^{-4}	8.043×10^{-5}	1.995×10^{-4}
$[[\hat{u}^2]] = 3$	1.670×10^{-6}	2.333×10^{-5}	7.824×10^{-6}	7.538×10^{-5}
$[[\hat{u}^2]] = 5$	2.169×10^{-6}	1.395×10^{-5}	2.005×10^{-6}	3.314×10^{-5}
$[[\hat{u}^2]] = 7$	1.012×10^{-5}	1.987×10^{-5}	3.882×10^{-5}	4.919×10^{-5}

Table 5.1: L^1-errors for two shock shallow water

	$\frac{\|v-v_{LFR}\|_1}{\|v\|_1}$	$\frac{\|w-w_{LFR}\|_1}{\|w\|_1}$	$\frac{\|v-v_{TT}\|_1}{\|v\|_1}$	$\frac{\|w-w_{TT}\|_1}{\|w\|_1}$
$[[\hat{u}^0]] = 3$	2.499×10^{-5}	2.180×10^{-4}	1.509×10^{-3}	3.097×10^{-2}
$[[\hat{u}^0]] = 5$	2.507×10^{-5}	1.421×10^{-4}	3.221×10^{-3}	1.953×10^{-2}
$[[\hat{u}^0]] = 7$	4.404×10^{-5}	1.749×10^{-4}	5.062×10^{-3}	1.898×10^{-2}
$[[\hat{u}^1]] = 3$	1.150×10^{-4}	6.077×10^{-3}	2.639×10^{-4}	7.123×10^{-2}
$[[\hat{u}^1]] = 5$	2.663×10^{-4}	1.467×10^{-2}	6.201×10^{-4}	2.454×10^{-2}
$[[\hat{u}^1]] = 7$	1.292×10^{-3}	4.627×10^{-2}	1.624×10^{-3}	1.953×10^{-2}
$[[\hat{u}^2]] = 3$	6.225×10^{-5}	1.054×10^{-1}	2.280×10^{-4}	1.417×10^{-1}
$[[\hat{u}^2]] = 5$	9.696×10^{-5}	7.471×10^{-2}	4.529×10^{-4}	8.483×10^{-2}
$[[\hat{u}^2]] = 7$	9.665×10^{-5}	5.911×10^{-2}	4.514×10^{-4}	6.712×10^{-2}

Table 5.2: Relative L^1-errors for two shock shallow water

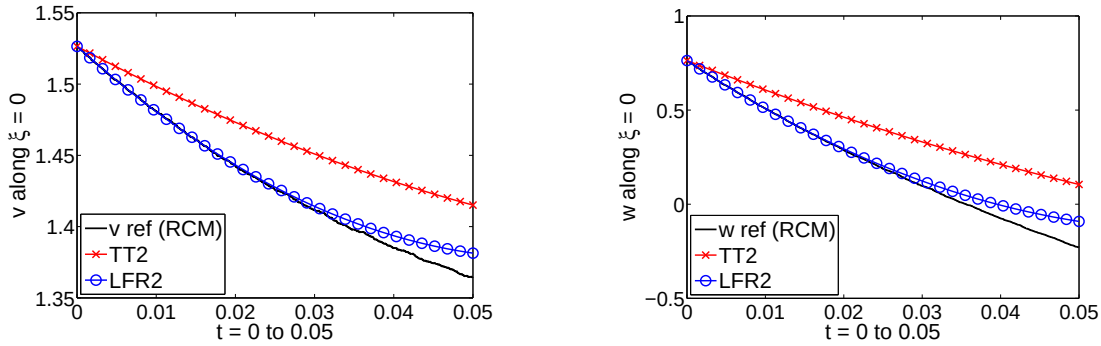


Figure 5.6: Two-shock solution for 1-d shallow water: Mixed example

the error is the same for both methods.

Now consider an example with large jumps in the states, as well as in first and second derivatives (see Figure 5.3). Take $\hat{u}_L(x) = 2x^2 + 6x + 2$, $\hat{u}_R(x) = -x^2 - x - 1$. We can see that the behaviour we observed for large jumps in the initial states dominates the behaviour for the approximate solution. The LeFloch-Raviart approximation is more accurate, although the is a clearly visible error for larger output times.

Next we turn to a generalized dam-break problem, in which the 1-wave is a rarefaction wave. Denote the by $q_0^0 = (v_0^0,\ w_0^0)$ the left state in the solution of the classical Riemann problem for the leading term and let

$$u_0 = \frac{w_0^0}{v_0^0}, \qquad a_0 = \sqrt{g v_0^0}.$$

Then solution of the classical Riemann problem inside the 1-rarefaction wave is given by

$$q^0(\xi) = \begin{pmatrix} v^0(\xi) \\ w^0(\xi) \end{pmatrix} = \frac{1}{9g} \begin{pmatrix} (u_0 + 2a_0 - \xi)^2 \\ \frac{1}{3}(u_0 + 2a_0 + 2\xi)(u_0 + 2a_0 - \xi)^2 \end{pmatrix}.$$

This gives

$$\frac{d}{d\xi} q^0(\xi) = -\frac{2}{9g}(u_0 + 2a_0 - \xi) \begin{pmatrix} 1 \\ \xi \end{pmatrix}, \qquad \xi \in (\underline{\sigma}_1^0, \overline{\sigma}_1^0).$$

For $\xi \in (\underline{\sigma}_1^0, \overline{\sigma}_1^0)$ we have $\xi = \lambda_1(q^0(\xi))$ and so it follows that

$$\begin{aligned}\frac{d}{d\xi}q^0(\xi) &= -\frac{2}{9g}(u_0 + a_0 - \xi)\begin{pmatrix} 1 \\ \frac{w^0(\xi)}{v^0(\xi)} - \sqrt{gv^0(\xi)} \end{pmatrix} \\ &= -\frac{2}{9g}(u_0 + 2a_0 - \xi) r_1\left(q^0(\xi)\right).\end{aligned}$$

Next we can compute

$$\nabla\lambda_1\left(q^0(\xi)\right) \cdot \frac{dq^0(\xi)}{d\xi} = \frac{1}{3g}(u_0 + 2a_0 - \xi)\frac{\sqrt{gv^0(\xi)}}{v^0(\xi)}.$$

Moreover, we get

$$\begin{aligned}Dr_2\left(q^0(\xi)\right)\frac{dq^0(\xi)}{d\xi} &= -\frac{2}{9g}(u_0 + 2a_0 - \xi) Dr_2\left(q^0(\xi)\right) r_1\left(q^0(\xi)\right) \\ &= \frac{1}{9g}(u_0 + 2a_0 - \xi)\begin{pmatrix} 0 \\ \frac{\sqrt{gv^0(\xi)}}{v^0(\xi)} \end{pmatrix}.\end{aligned}$$

Thus, writing

$$Dr_2\left(q^0(\xi)\right)\frac{dq^0(\xi)}{d\xi} = \omega_{1,2,1} r_1\left(q^0(\xi)\right) + \omega_{1,2,2} r_2\left(q^0(\xi)\right),$$

we first see that $\omega_{1,2,1} = -\omega_{1,2,2}$ and we compute

$$\omega_{1,2,2} = \frac{u_0 + 2a_0 - \xi}{18gv^0(\xi)}. \tag{5.22}$$

From (3.30) we get

$$\frac{d}{d\xi}\alpha_2^k = \mu_2^k - \left(\frac{k + \nabla\lambda_1(q^0) \cdot \frac{dq^0}{d\xi}}{\lambda_2(q^0) - \xi)} + \omega_{1,2,2}\right)\alpha_2^k = \mu_2^k - \frac{3k+2}{2(u_0 + 2a_0 - \xi)}\alpha_2^k.$$

For $k = 1$ we have $\mu_2^1 = 0$ and with initial data given by (3.39) the ordinary

differential equation for α_2^1 can be solved exactly:

$$\alpha_2^1(\xi) = \left(\underline{\sigma}_1^0 - \lambda_2(q_0^0)\right) \alpha_{0,2}^1 \left(\frac{u_0 + 2a_0 - \xi}{3a_0}\right)^{5/2}.$$

From (3.31) we get

$$\alpha_1^1(\xi) = -\frac{\lambda_2(q^0) - \xi}{1 + \nabla\lambda_1(q^0) \cdot \frac{dq^0}{d\xi}} \alpha_2^1(\xi),$$

and we first compute

$$\nabla\lambda_1(q^0) \cdot \frac{dq^0}{d\xi} = \frac{1}{3g}(u_0 + 2a_0 - \xi)\frac{\sqrt{gv}}{v}.$$

Next, recall that for $\xi \in \mathfrak{R}_1^0$ we have $\xi = w^0/v^0 - \sqrt{gv^0}$ and thus

$$\lambda_2(q^0) - \xi = 2\sqrt{gv^0}.$$

Using the fact that $\omega_{1,2,1} = -\omega_{1,2,2}$ we find by (5.22):

$$\alpha_1^1(\xi) = \frac{u_0 + 2a_0 - \xi}{9\sqrt{gv^0} + 3(u_0 + 2a_0 - \xi)} \alpha_2^1(\xi).$$

Finally, note that inside $\mathfrak{R}_1^0$ we get

$$\sqrt{gv^0(\xi)} = \frac{1}{3}(u_0 + 2a_0 - \xi)$$

and thus

$$\alpha_1^1(\xi) = \frac{1}{6}\alpha_2^1(\xi).$$

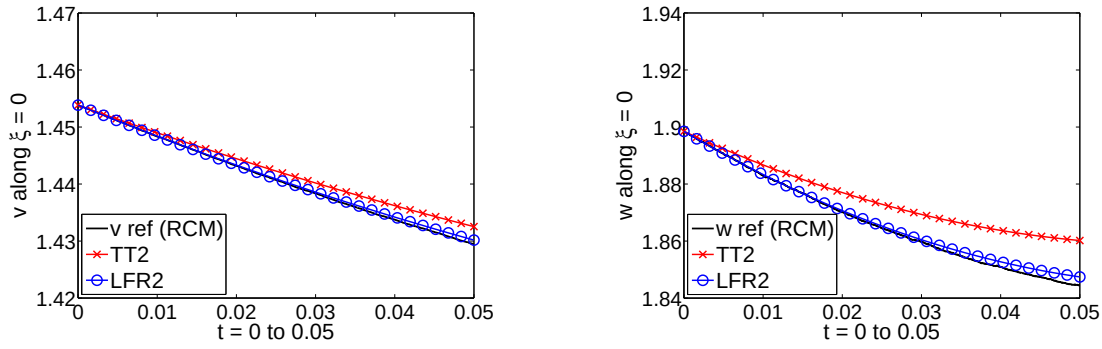
For the quadratic terms, we first note that for $f_1(v, w) = w$ all second or higher

order derivatives of f_1 vanish. Then we get by (3.29) that $\mu_1^1 = -\mu_2^2$ and

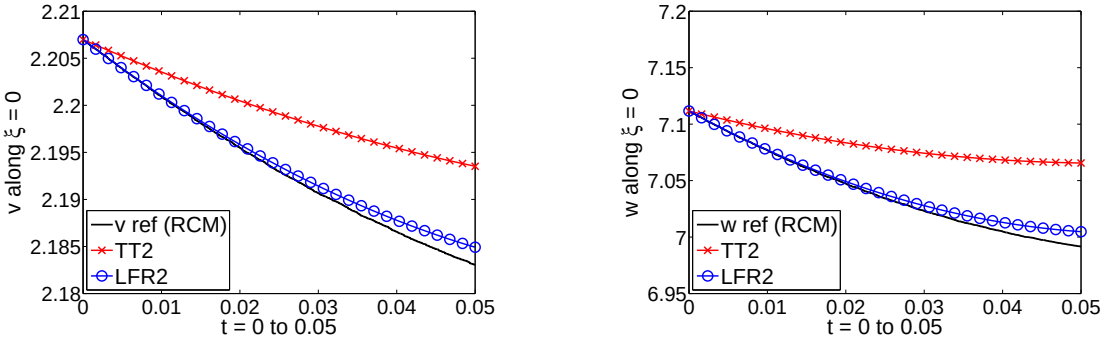
$$\begin{aligned}
&\mu_2^2\left(q^0, q^1\right) \\
&\quad = -\frac{1}{4\sqrt{gv^0}}\left(\left\{\frac{\partial^3 f_2(q^0)}{\partial v^3}\frac{dv^0}{d\xi} + \frac{\partial^3 f_2(q^0)}{\partial v^2 \partial w}\frac{dw^0}{d\xi}\right\}(v^1)^2 + 2\frac{\partial^2 f_2(q^0)}{\partial v^2}v^1\frac{dv^1}{d\xi}\right. \\
&\qquad + 2\left\{\frac{\partial^3 f_2(q^0)}{\partial v^2 \partial w}\frac{dv^0}{d\xi}\frac{\partial^3 f_2(q^0)}{\partial v \partial w^2}\frac{dw^0}{d\xi}\right\}v^1 w^1 + 2\frac{\partial^2 f_2(q^0)}{\partial v \partial w}\left\{v^1\frac{dw^1}{d\xi} + w^1\frac{dv^1}{d\xi}\right\} \\
&\qquad \left. + \left\{\frac{\partial^3 f_2(q^0)}{\partial v \partial^2 w}\frac{dv^0}{d\xi} + \frac{\partial^3 f_2(q^0)}{\partial w^3}\frac{dw^0}{d\xi}\right\}(w^1)^2 + 2\frac{\partial^2 f_2(q^0)}{\partial w^2}w^1\frac{dw^1}{d\xi}\right).
\end{aligned}$$

Again, we run three test cases with large jump in the initial states, first derivatives and second derivatives, respectively. In all three cases we take $\hat{u}_L = \hat{u}_R = 0$ and solve the generalized Riemann problem for different piecewise quadratic $\hat{h}$. The ordinary differential equation for α_2^2 is solved numerically, using a classical fourth order Runge-Kutta method. Results are shown in Figures 5.7-5.9 and the resulting L^1-errors are documented in Tables 5.3 and 5.4.

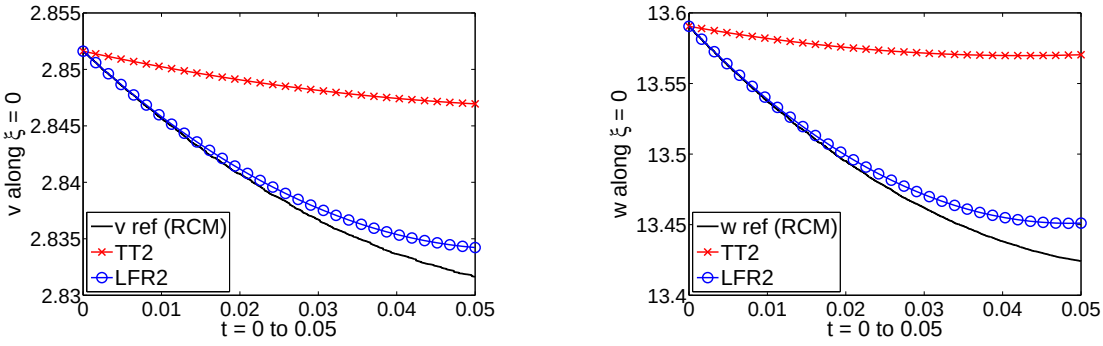
$$\hat{h}_L(x) = 0.2x^2 + 0.2x + c_L, \qquad \hat{h}_R(x) = -0.1x^2 - 0.1x + 1$$



(a) $c_L = 2, \quad |\hat{h}_L^0 - \hat{h}_R^0| = 1$



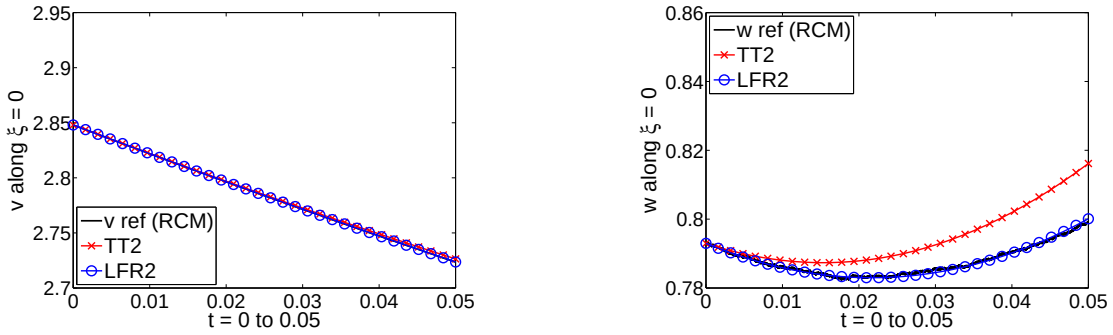
(b) $c_L = 4, \quad |\hat{h}_L^0 - \hat{h}_R^0| = 3$



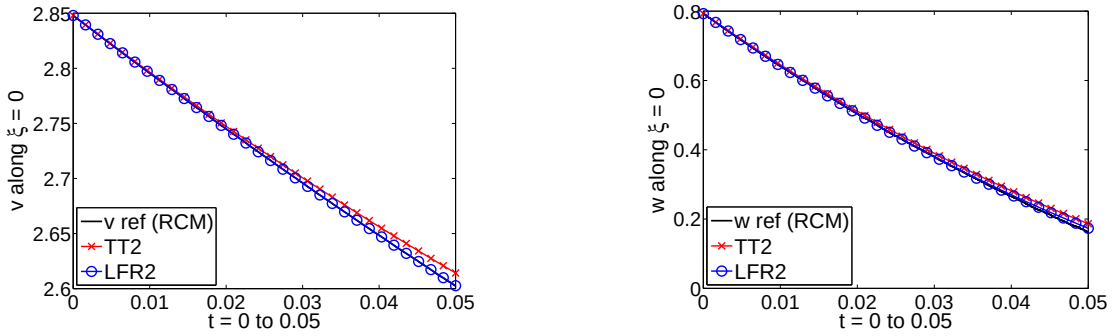
(c) $c_L = 6, \quad |\hat{h}_L^0 - \hat{h}_R^0| = 5$

Figure 5.7: Generalized dam-break problem: Jumps in the initial states

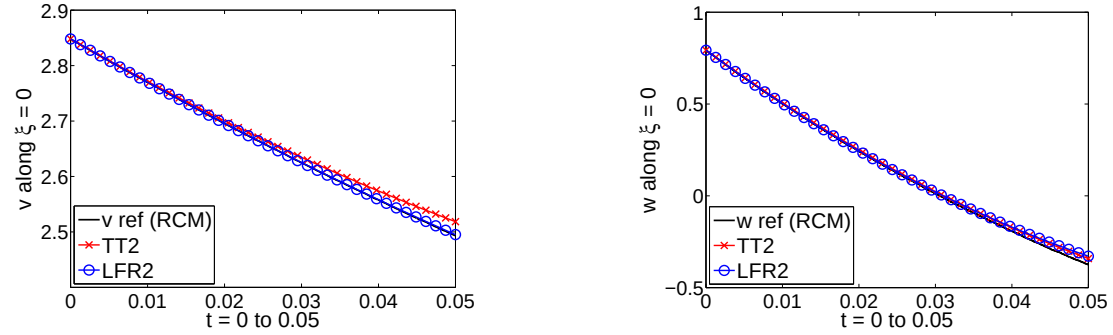
$$\hat{h}_L(x) = 0.2x^2 + b_L x + 3, \qquad \hat{h}_R(x) = -0.1x^2 - 0.5x + 2.7$$



(a) $b_L = 0.5, \quad |\hat{h}_L^1 - \hat{h}_R^1| = 1$



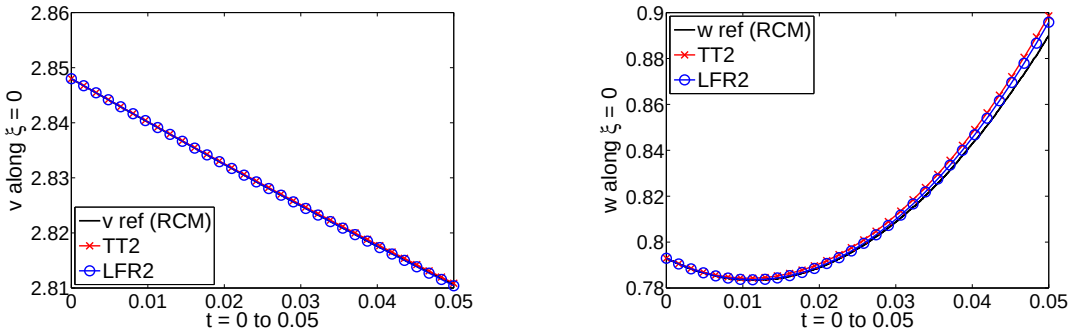
(b) $b_L = 1.5, \quad |\hat{h}_L^1 - \hat{h}_R^1| = 2$



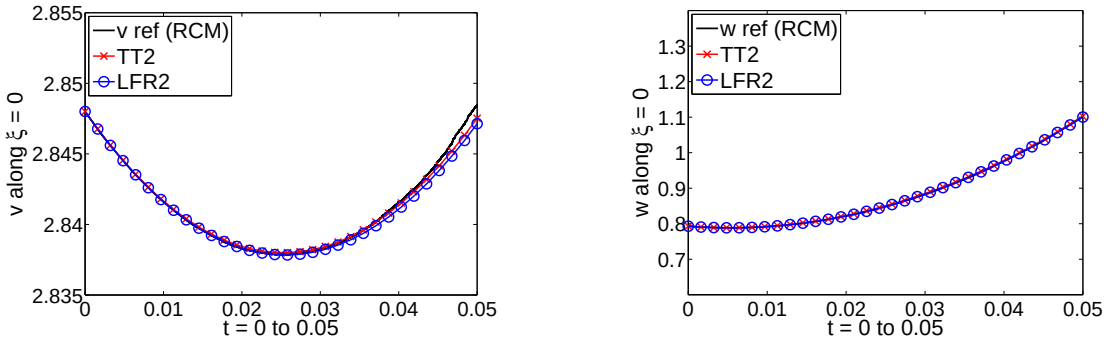
(c) $b_L = 2.5, \quad |\hat{h}_L^1 - \hat{h}_R^1| = 3$

Figure 5.8: Generalized dam-break problem: Jumps in first derivatives

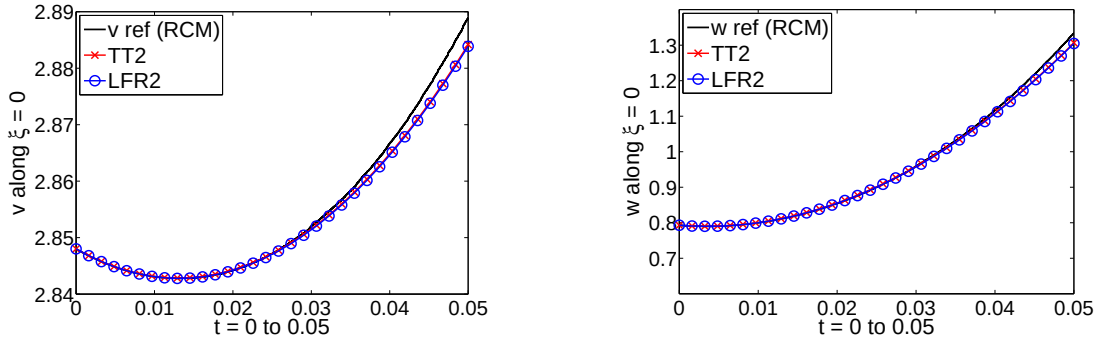
$$\hat{h}_L(x) = a_L x^2 + 0.1x + 3, \qquad \hat{h}_R(x) = -0.5x^2 - 0.1x + 2.7$$



(a) $a_L = 0.5, \quad |\hat{h}_L^2 - \hat{h}_R^2| = 1$



(b) $a_L = 1.5, \quad |\hat{h}_L^2 - \hat{h}_R^2| = 2$



(c) $a_L = 2.5, \quad |\hat{h}_L^2 - \hat{h}_R^2| = 3$

Figure 5.9: Generalized dam-break problem: Jumps in second derivatives

	$\|v - v_{LFR}\|_1$	$\|w - w_{LFR}\|_1$	$\|v - v_{TT}\|_1$	$\|w - w_{TT}\|_1$
$[[\hat{h}^0]] = 1$	1.640×10^{-5}	5.962×10^{-5}	8.152×10^{-5}	4.197×10^{-4}
$[[\hat{h}^0]] = 2$	3.479×10^{-5}	2.256×10^{-4}	2.909×10^{-4}	2.098×10^{-3}
$[[\hat{h}^0]] = 3$	4.685×10^{-5}	4.590×10^{-4}	4.617×10^{-4}	4.431×10^{-3}
$[[\hat{h}^1]] = 1$	6.352×10^{-5}	2.338×10^{-5}	1.118×10^{-4}	3.359×10^{-4}
$[[\hat{h}^1]] = 2$	1.189×10^{-4}	4.043×10^{-4}	3.292×10^{-4}	7.958×10^{-4}
$[[\hat{h}^1]] = 3$	1.841×10^{-4}	1.077×10^{-3}	6.072×10^{-4}	1.122×10^{-3}
$[[\hat{h}^2]] = 1$	1.370×10^{-5}	4.299×10^{-5}	2.208×10^{-5}	1.028×10^{-4}
$[[\hat{h}^2]] = 2$	2.963×10^{-5}	3.071×10^{-4}	2.484×10^{-5}	2.456×10^{-4}
$[[\hat{h}^2]] = 3$	8.847×10^{-5}	7.332×10^{-4}	8.216×10^{-5}	6.776×10^{-4}

Table 5.3: L^1-errors for the generalized dam-break problem

	$\frac{\|v-v_{LFR}\|_1}{\|v\|_1}$	$\frac{\|w-w_{LFR}\|_1}{\|w\|_1}$	$\frac{\|v-v_{TT}\|_1}{\|v\|_1}$	$\frac{\|w-w_{TT}\|_1}{\|w\|_1}$
$[[\hat{u}^0]] = 3$	2.284×10^{-4}	6.410×10^{-4}	1.135×10^{-3}	4.513×10^{-3}
$[[\hat{u}^0]] = 5$	3.179×10^{-4}	6.424×10^{-4}	2.658×10^{-3}	5.947×10^{-3}
$[[\hat{u}^0]] = 7$	3.306×10^{-4}	6.818×10^{-4}	3.258×10^{-3}	6.583×10^{-3}
$[[\hat{u}^1]] = 3$	4.570×10^{-4}	5.943×10^{-4}	8.042×10^{-4}	8.538×10^{-3}
$[[\hat{u}^1]] = 5$	8.752×10^{-4}	1.813×10^{-2}	2.424×10^{-3}	3.569×10^{-2}
$[[\hat{u}^1]] = 7$	1.386×10^{-3}	7.217×10^{-2}	4.571×10^{-3}	7.515×10^{-2}
$[[\hat{u}^2]] = 3$	9.701×10^{-5}	1.042×10^{-3}	1.563×10^{-4}	2.534×10^{-3}
$[[\hat{u}^2]] = 5$	2.088×10^{-4}	6.927×10^{-3}	1.750×10^{-4}	5.539×10^{-3}
$[[\hat{u}^2]] = 7$	3.273×10^{-3}	2.027×10^{-1}	3.298×10^{-3}	2.040×10^{-1}

Table 5.4: Relative L^1-errors for the generalized dam-break problem

We observe the same qualitative behaviour as before: When there is a large jump in the initial data, the Toro-Titarev solver leads to an error which grows as the jump in the data increases and this error is larger than the error of the LeFloch-Raviart expansion. If the jump in the initial states is small, the two approximation methods lead to an error of the same order of magnitude.

5.4 A system from traffic flow

The following system arises in a macroscopic model of traffic flow that was proposed by Aw and Rascle [3]. It was designed to cure some of the major flaws that previous models suffered from when attempting to introduce ideas from fluid dynamics to the world of traffic modelling, see [30]. A detail introduction to conservation law models in macroscopic traffic flow and further references are found in [69]. We consider the system

$$\begin{aligned} \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho s) &= 0 \\ \frac{\partial}{\partial t}(s + p(\rho)) + s\frac{\partial}{\partial x}(s + p(\rho)) &= 0. \end{aligned} \tag{5.23}$$

Here, ρ is the density of cars, s the velocity. The function p acts like a pressure term in fluid dynamics, but is understood as an 'anticipation factor'. Global existence and smoothness of the solution to a generalized Riemann problem for (5.23) are discussed in [84]. We assume $\rho > 0$ (although the case $\rho = 0$ can be handled) and we also assume $p'(\rho) > 0$, where the specific choice of p will be discussed later.

We can write the system in quasi-linear form,

$$\frac{\partial}{\partial t}z + A(z)\frac{\partial}{\partial x}z = 0,$$

where $z = (\rho, s)^T$ and

$$A(z) = \begin{pmatrix} s & \rho \\ 0 & s - \rho p'(\rho) \end{pmatrix}.$$

To derive the correct shock-conditions, we need to rewrite the system in conservative form. Setting

$$w = \rho(s + p(\rho)),$$

the system can be written as

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho s) = 0,$$
$$\frac{\partial}{\partial t}w + \frac{\partial}{\partial x}(ws) = 0.$$

Noting that $s = w/\rho - p(\rho)$ we let $u = (\rho, w)^T$ and arrive at

$$\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}f(u) = 0, \qquad f(u) = \begin{pmatrix} w - \rho p(\rho) \\ w^2/\rho - wp(\rho) \end{pmatrix}.$$

Thus, we have

$$Df(u) = \begin{pmatrix} -p(\rho) - \rho p'(\rho) & 1 \\ -(w/\rho)^2 - wp'(\rho) & 2w/\rho - p(\rho). \end{pmatrix}$$

Then the eigenvalues and eigenvectors are

$$\lambda_1(\rho, w) = \frac{w}{\rho} - p(\rho) - \rho p'(\rho), \qquad r_1(\rho, w) = \begin{pmatrix} 1 \\ \frac{w}{\rho} \end{pmatrix},$$
$$\lambda_2(\rho, w) = \frac{w}{\rho} - p(\rho) = s, \qquad r_2(\rho, w) = \begin{pmatrix} 1 \\ \frac{w}{\rho} + \rho p'(\rho) \end{pmatrix}.$$

After normalizing to $l_i(\rho, w) \cdot r_j(\rho, w) = \delta_{ij}$, the left eigenvectors are

$$\ell_1(\rho, w) = \frac{1}{\rho p'(\rho)} \begin{pmatrix} \frac{w}{\rho} + \rho p'(\rho) \\ -1 \end{pmatrix}, \qquad \ell_2(\rho, w) = \frac{1}{\rho p'(\rho)} \begin{pmatrix} -\frac{w}{\rho} \\ 1 \end{pmatrix}.$$

It is easy to see that the second field is always linearly degenerate, while the first field is linearly degenerate, if and only if

$$p(\rho) = a - \frac{b}{\rho}, \qquad a, b > 0.$$

Under this assumption, both waves are contact discontinuities. For actually modelling traffic flow, one usually takes p in the form $p(\rho) = \rho^\gamma,\ \gamma > 0$. However, for purposes of analysis and less of modelling, we are interested in a nonlinear system

for which all fields are linearly degenerate. With this choice of p we get

$$\lambda_1(\rho, w) = \frac{w}{\rho} - a, \qquad \lambda_2(\rho, w) = \frac{w+b}{\rho} - a$$

and

$$r_1(\rho, w) = \begin{pmatrix} 1 \\ \frac{w}{\rho} \end{pmatrix}, \qquad r_2(\rho, w) = \begin{pmatrix} 1 \\ \frac{w+b}{\rho} \end{pmatrix},$$

$$\ell_1(\rho, w) = \frac{1}{b} \begin{pmatrix} w+b \\ -\rho \end{pmatrix}, \qquad \ell_2(\rho, w) = \frac{1}{b} \begin{pmatrix} w \\ -\rho \end{pmatrix}.$$

The value of w is given by

$$w = (a+s)\rho - b.$$

Since both waves are contact discontinuities, we can compute the Godunov state from the conditions

$$\lambda_1(q_0^0) = \lambda_1(q_1^0), \qquad \lambda_2(q_1^0) = \lambda_2(q_2^0),$$

which gives

$$q_1^0 = \begin{pmatrix} \rho_1^0 \\ s_1^0 \end{pmatrix} = \begin{pmatrix} \frac{\rho_0^0 b}{\rho_0^0(s_2^0 - s_0^0) - b} \\ s_2^0 \end{pmatrix}. \tag{5.24}$$

Next, we present some numerical results considering the generalized Riemann problem with piecewise constant density $\hat{\rho}_L$ and $\hat{\rho}_R$, and piecewise quadratic velocity. We fix $a = 0.5,\ b = 1$.

As in the previous section, we compare the resulting approximations up to quadratic terms obtained by the LeFloch-Raviart expansion and the Toro-Titarev solve, respectively. Reference solutions are, again, obtained by a random choice method (RCM) on a very fine grid using an exact Riemann solver and van der Corput pseudo random numbers. We perform three series of tests:

(i) Large jumps in the initial data, fixed derivatives. We take

$$\hat{\rho}_L(x) = 0.8, \qquad \hat{\rho}_R(x) = 0.6$$

and solve the generalized Riemann problem for different choices of $\hat{s}_L,\ \hat{s}_R$. Results

along the line $\xi = 0$ for $0 \leq t \leq 0.01$ are shown in Figure 5.10.

(ii) Large jumps in first derivatives, fixed jump in states Again we take $\hat{\rho}_L(x) = 0.62,\ \hat{\rho}_R(x) = 0.6$ and choose different $\hat{s}_L,\ \hat{s}_R$ such that we have small jumps in the initial states and large jumps in first derivatives, see Figure 5.11.

(iii) Large jumps in second derivatives, fixed jump in states. We fix $\hat{\rho}_L(x) = 0.62,\ \hat{\rho}_R(x) = 0.6$ and solve the generalized Riemann problem with different $\hat{s}_L$ and $\hat{s}_R$ such that increasing jumps in second derivatives. The results are shown in Figures and 5.12.

Tables 5.5 and 5.6 show the L^1 -errors and relative L^1 -errors, respectively. Note that the errors are measured on the interval $[0, 0.01]$, whereas we took $[0, 0.05]$ for the shallow water case. Further numerical experiments indicate that the Toro-Titarev approximation quickly becomes less accurate for larger output times. Moreover, we need a jump in the initial states of size in the order of 0.01 to observe the same behaviour as before. If we choose a jump of size 0.1 (as we did in the shallow water examples), the Toro-Titarev approximations produce an error two orders larger than the error for the LeFloch-Raviart approximation. We observe that the accuracy of the Toro-Titarev approximation is very sensitive towards jumps in the initial data. On the other hand, the LeFloch-Raviart approximations are in all cases very accurate. This can be explained by the special nature of the system: Both characteristic fields are linearly degenerated, so the problem is in some sense "almost linear". Therefore, piecewise quadratic data leads to an "almost quadratic" solutions and a quadratic Taylor approximation is very accurate. But it is still a nonlinear system, so the Toro-Titarev approximation is not a Taylor approximation. There are no compression phenomena (as would be the case for shock waves), and the lack of this stabilizing effect explains the sensitivity problems.

	$\|v - v_{LFR}\|_1$	$\|w - w_{LFR}\|_1$	$\|v - v_{TT}\|_1$	$\|w - w_{TT}\|_1$
$[[\hat{u}^0]] = 3$	4.822×10^{-9}	5.965×10^{-8}	2.921×10^{-6}	1.961×10^{-5}
$[[\hat{u}^0]] = 5$	2.846×10^{-9}	3.922×10^{-8}	3.191×10^{-6}	3.202×10^{-5}
$[[\hat{u}^0]] = 7$	1.739×10^{-9}	3.921×10^{-8}	3.274×10^{-6}	4.378×10^{-5}
$[[\hat{u}^1]] = 3$	1.411×10^{-6}	8.004×10^{-7}	1.319×10^{-6}	2.597×10^{-6}
$[[\hat{u}^1]] = 5$	1.881×10^{-6}	4.712×10^{-7}	1.713×10^{-6}	2.936×10^{-6}
$[[\hat{u}^1]] = 7$	2.866×10^{-6}	4.271×10^{-7}	2.187×10^{-6}	3.286×10^{-6}
$[[\hat{u}^2]] = 3$	1.710×10^{-7}	2.266×10^{-7}	2.057×10^{-7}	4.839×10^{-7}
$[[\hat{u}^2]] = 5$	8.510×10^{-8}	1.291×10^{-7}	1.823×10^{-7}	4.045×10^{-7}
$[[\hat{u}^2]] = 7$	6.574×10^{-8}	9.976×10^{-8}	1.618×10^{-7}	3.737×10^{-7}

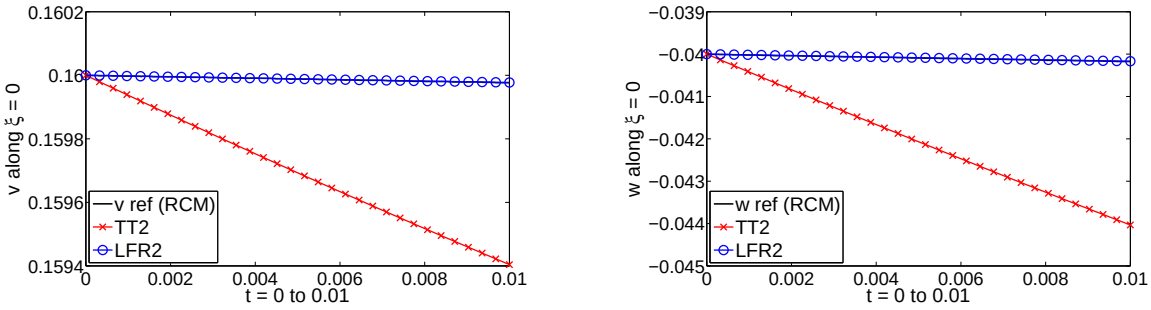
Table 5.5: L^1-errors for traffic flow

	$\frac{\|v-v_{LFR}\|_1}{\|v\|_1}$	$\frac{\|w-w_{LFR}\|_1}{\|w\|_1}$	$\frac{\|v-v_{TT}\|_1}{\|v\|_1}$	$\frac{\|w-w_{TT}\|_1}{\|w\|_1}$
$[[\hat{u}^0]] = 3$	3.038×10^{-6}	1.500×10^{-4}	1.840×10^{-3}	4.930×10^{-2}
$[[\hat{u}^0]] = 5$	2.741×10^{-6}	1.507×10^{-4}	3.074×10^{-3}	1.230×10^{-1}
$[[\hat{u}^0]] = 7$	2.254×10^{-6}	2.028×10^{-4}	4.244×10^{-3}	2.264×10^{-1}
$[[\hat{u}^1]] = 3$	2.400×10^{-4}	4.658×10^{-2}	2.243×10^{-4}	1.511×10^{-1}
$[[\hat{u}^1]] = 5$	3.176×10^{-4}	2.747×10^{-2}	2.891×10^{-4}	1.712×10^{-1}
$[[\hat{u}^1]] = 7$	4.810×10^{-4}	2.474×10^{-2}	3.670×10^{-4}	1.903×10^{-1}
$[[\hat{u}^2]] = 3$	2.914×10^{-5}	6.927×10^{-2}	3.505×10^{-5}	1.479×10^{-1}
$[[\hat{u}^2]] = 5$	1.448×10^{-5}	3.959×10^{-2}	3.103×10^{-5}	1.240×10^{-1}
$[[\hat{u}^2]] = 7$	1.119×10^{-5}	3.058×10^{-2}	2.752×10^{-5}	1.146×10^{-1}

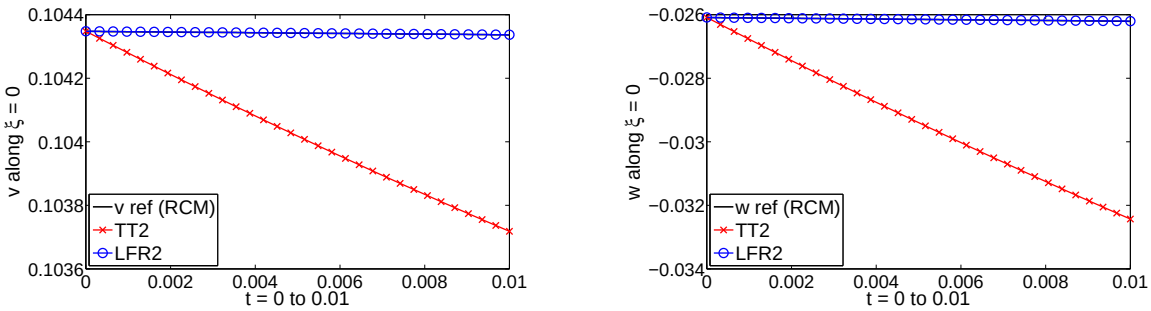
Table 5.6: Relative L^1-errors for traffic low

From what we have a seen for all three examples (shallow water with two shocks and shallow water with rarefaction and shock, traffic flow with two contact discontinuities) we conclude that the size of the error in the Toro-Titarev approximation depends mainly on the jump in the initial states. This is consistent with our analytical results, see the remark after Theorem 4.6.

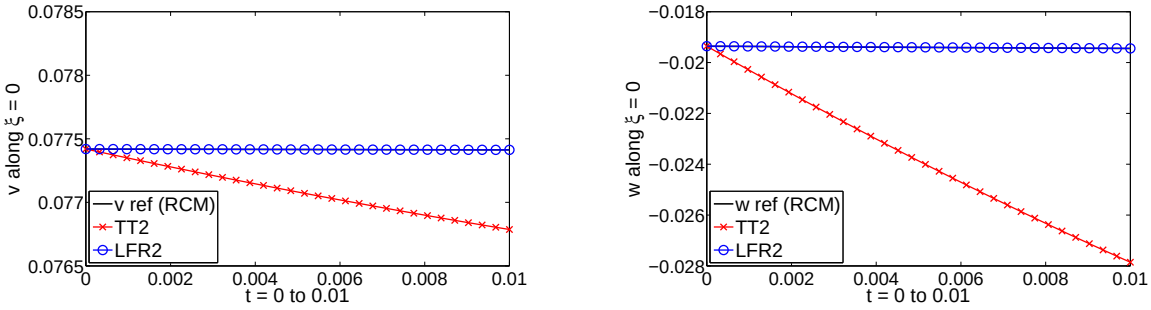
$$\hat{s}_L(x) = 1/10x^2 + 1/10x + 1/2, \qquad \hat{s}_R(x) = -1/10x^2 - 1/10x + c_R$$



(a) $c_R = 33/6, \quad |\hat{u}_L^0 - \hat{u}_R^0|_1 = 3$



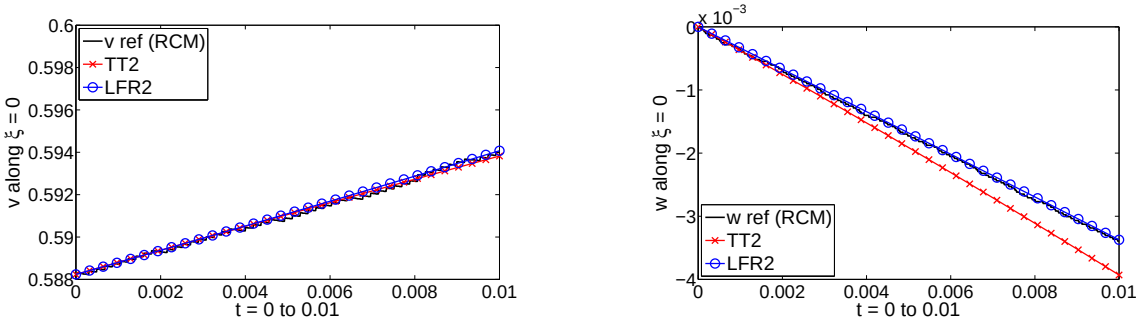
(b) $c_R = 53/6, \quad |\hat{u}_L^0 - \hat{u}_R^0|_1 = 5$



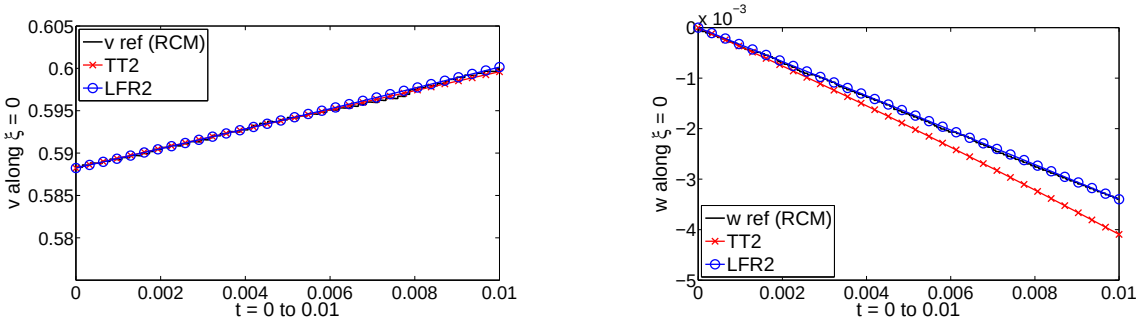
(c) $c_R = 73/6, \quad |\hat{u}_L^0 - \hat{u}_R^0|_1 = 7$

Figure 5.10: Jumps in the states

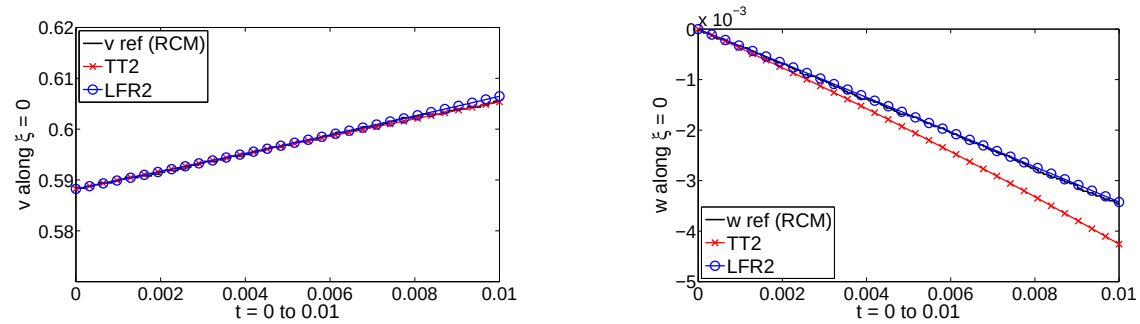
$$\hat{s}_L(x) = 0.5x^2 + 0.1x + 1.1129, \qquad \hat{s}_R(x) = -0.1x^2 + b_R x + 1.2$$



(a) $b_R = -4.833, \quad |\hat{u}_L^1 - \hat{u}_R^1|_1 = 3$



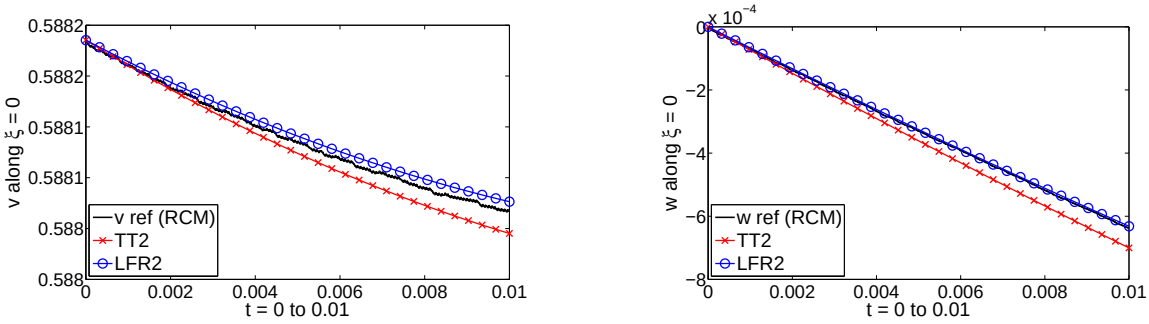
(b) $b_R = -7.8167, \quad |\hat{u}_L^1 - \hat{u}_R^1|_1 = 5$



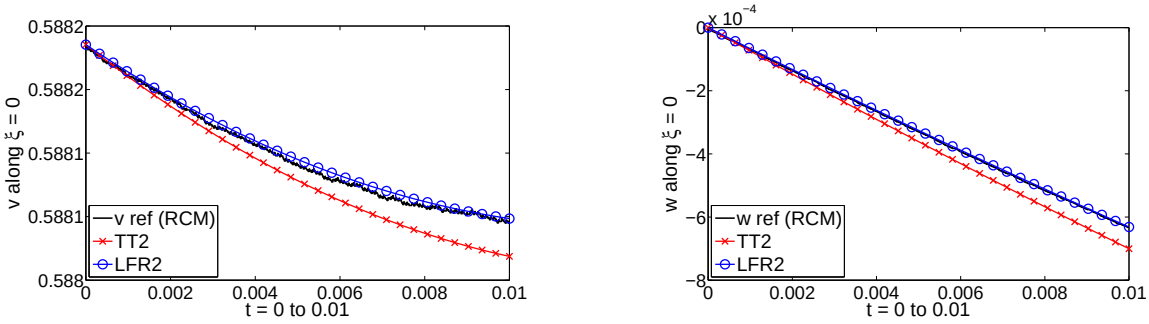
(c) $b_R = -11.15, \quad |\hat{u}_L^1 - \hat{u}_R^1|_1 = 7$

Figure 5.11: Jumps in first derivatives

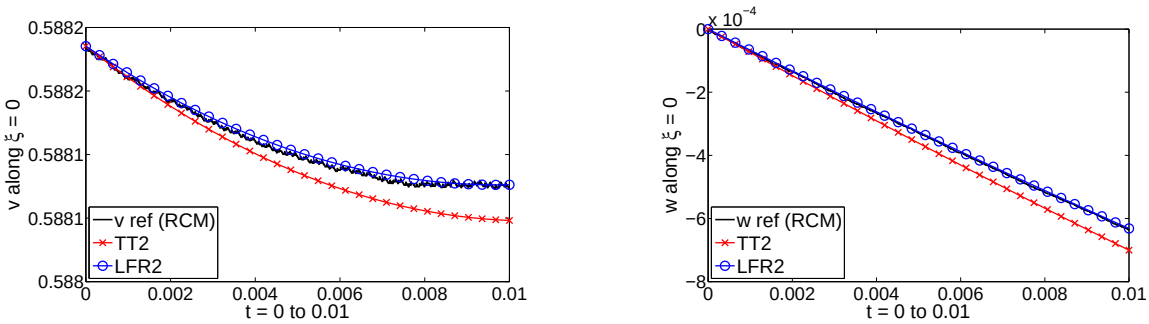
$$\hat{s}_L(x) = 0.5x^2 + 0.1x + 1.1129, \qquad \hat{s}_R(x) = a_R x^2 - 0.1x + 1.2$$



(a) $a_R = -4.833, \quad |\hat{u}_L^2 - \hat{u}_R^2|_1 = 3$



(b) $a_R = -7.8167, \quad |\hat{u}_L^2 - \hat{u}_R^2|_1 = 5$



(c) $a_R = -11.15, \quad |\hat{u}_L^2 - \hat{u}_R^2|_1 = 7$

Figure 5.12: Jumps in second derivatives

6 Conclusion and outlook

We have shown that the Toro-Titarev solver computes a truncated Taylor-Series expansion when applied to a scalar problem. We used this insight to extend a stability result that used exact flux computation to the case of the ADER numerical flux. It seems likely that more results in this direction are possible. Moreover, if we take any entropy - entropy flux pair Q, F and define a numerical entropy flux by $F_{i+1/2} = \int_0^{\Delta t} F(u(x_{i+1/2}, t))\ dt$, this approximation result could play a mayor role in proving entropy stability of the numerical scheme.

For nonlinear systems we have shown that the Toro-Titarev solver does not compute a Taylor expansion. The resulting approximation does, in general, not satisfy the Rankine-Hugoniot conditions. However, the error is small, provided the jump in the initial states is small. Whether the LeFloch-Raviart expansion, which gives more accurate approximations in the case of large jumps, could be used with higher orders at a competitive computational cost, remains to be tested.

We based our analysis on the assumption that the system is strictly hyperbolic with only genuinely nonlinear or linearly degenerate characteristic fields. While this assumption does cover many systems of practical interest, it leaves out a broad set of problems. The Riemann problem can be solved for a lot of systems that are not strictly hyperbolic or for which some characteristic fields are neither genuinely nonlinear nor linearly degenerate. If we view the generalized Riemann problem locally as a perturbation of a classical Riemann problem, it seems reasonable that the generalized Riemann problem should also have a solution in these cases and a similar asymptotic analysis as in our case is possible. However, if we want to do this there is still a large amount of highly technical work ahead of us.

We did not analyse the effects of using approximate Riemann solvers for the leading term Riemann problem. It is one of the advantages of the Toro-Titarev solver that an approximate solution of the generalized Riemann problem can be

computed once the (approximate) Godunov state of the leading term Riemann problem is known. On the other hand, the LeFloch-Raviart expansion requires a full Riemann solver. Approximate solutions of the leading term Riemann problem could be used, but the method needs all intermediate states and all waves speeds in the classical Riemann problem.

Problems in several spatial dimensions are most important in applications but extremely hard to analyse. Local generalised Riemann problems at the cell-boundaries are essentially one-dimensional and can be handled using the approach described by Harabetian [37]. However, at the corners of the cells, where more than two cells meet, a very rich variety of wave interactions is to be expected. This is already true for the classical Riemann in two dimension and it seems like there are no results on the piecewise smooth case available yet.

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Zusammenfassung

Wir untersuchen die analytischen Eigenschaften von Methoden zur näherungsweisen Lösung von verallgemeinerten Riemann-Problemen. Dabei konzentrieren wir uns auf den Toro-Titarev-Löser [98], der den zentralen Baustein zur Flussberechnung in ADER Finite Volumen Methoden zur numerischen Lösung von hyperbolischen Erhaltungsgleichungen darstellt. Toro und Titarev schlagen vor, die Lösung des verallgemeinerten Riemann-Problems durch eine Taylor-Reihe zu approximieren. Die Koeffizienten der Taylor-Polynome werden in der Methode von Toro und Titarev mit Hilfe einer Cauchy-Kowalewskaya-Prozedur und einer Folge von klassischen Riemann-Problemen berechnet.

Diese Strategie zur Berechnung von Näherungslösungen für verallgemeinerte Riemann-Probleme erzielt in einer Vielzahl von Anwendungen sehr gute numerische Resultate, eine gründliche analytische Untersuchung des Verfahrens steht allerdings noch aus. Insbesondere wurde beobachtet, dass ADER-Verfahren für glatte Lösungen die erwartete Genauigkeitsordnug erreichen, bei großen Sprüngen in den Anfangsdaten aber auf Probleme stoßen [19, 65]. Dieses Phänomen konnte bisher nicht erklärt werden.

Wir untersuchen den Toro-Titarev-Löser, indem wir ihn mit einer asymptotischen Reihenentwicklung für die Lösung des verallgemeinerten Riemann-Problems vergleichen, die von LeFloch und Raviart konstruiert wurde [56]. Es stellt sich heraus, dass beide Methoden formell die selbe Taylor-Approximation konstruieren und sich nur in der Berechnung der Ortsableitungen im Ursprung unterscheiden.

Wir zeigen, dass beide Methoden für skalare 1D Probleme zur selben Näherungslösung führen. Für nichtlineare Systeme von hyperbolischen Erhaltungsgleichungen ergibt sich ein Unterschied. Wir zeigen analytisch, dass dieser Unterschied direkt von der Größe des Sprungs in den Anfangsdaten abhängt und dass der Unterschied klein ist, wenn der Sprung in den Anfangsdaten klein ist.

Wir illustrieren diesen Sachverhalt, indem wir analytische Resultate für Burgers Gleichung und für ein System aus der Zwei-Komponenten Chromatographie bereitstellen. Weitere numerische Resultate für Flachwassergleichungen und für ein System aus der Verkehrsflussmodellierung bestätigen die analytischen Resultate.

Abstract

We study the analytical properties of approximate solvers for the generalized Riemann problem. We focus on the Toro-Titarev solver [98], which is the heart of the flux computation in ADER finite volume methods for solving hyperbolic conservation laws. Toro and Titarev suggested to approximate the solution of the generalized Riemann problem by a truncated Taylor series expansion. Coefficients in this expansion are found using a Cauchy-Kovalevskaya procedure and a sequence of classical Riemann problems.

This method for approximately solving the generalized Riemann problem has been applied successfully to a wide range of problems, but few rigorous analysis of this strategy has been reported so far. It was observed that the ADER scheme achieves the designed order of accuracy in regions where the solution is smooth, but can encounter difficulties if the initial data contains large jumps [19, 65]. This phenomenon has thus far not been explained.

We study the solver of Toro and Titarev by comparing it to a local asymptotic series expansion for the solution of the generalized Riemann problem that was constructed by LeFloch and Raviart [56]. It turns out that both methods formally construct the same truncated Taylor series expansion. The only difference is the way spatial derivatives at the origin are found.

We show that both methods lead to the same truncated Taylor series expansion when they are applied to scalar problems. For systems of hyperbolic conservation laws, there is a difference. We show that this difference can be clearly traced back to the jump in the initial data. Moreover, we show that when the jump in the initial data is small, the two resulting approximations are close.

We illustrate this by giving analytical results for Burgers equation and a system from two-component chromatography. Numerical results for shallow water equations and for a system from traffic flow further support the analytical results.

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