Exact Results for $\mathcal{N} = 1$ Theories of Class $\mathcal{S}_k$

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Abstract

A large class of four dimensional $\mathcal{N} = 1$ theories can be obtained by twisted compactification along a Riemann surface $\mathcal{C}$ of six dimensional theories with $\mathcal{N} = (1, 0)$ supersymmetry. When the $(1, 0)$ theory is that of the theory on M5-branes on a transverse $\mathbb{C}^2/\mathbb{Z}_k$ singularity the resulting four dimensional theories are said to be of class $\mathcal{S}_k$. These are orbifolds of class $\mathcal{S}$ theories and as such, much structure is inherited from the class $\mathcal{S}$ mother theory, for which a large number of results have been computed and understood.

In this thesis we investigate exact results, to which we can obtain, for theories of class $\mathcal{S}_k$. We describe and define, via the orbifold inheritance structure, Higgs and Coulomb branches of the moduli space of supersymmetric vacua for theories of class $\mathcal{S}_k$. This is in contrast to 'generic' $\mathcal{N} = 1$ theories where no such distinction can be made. The Hilbert series for these branches is computed for a variety of examples. Certain distinguished limits, that would again not exist for generic $\mathcal{N} = 1$ theories, of the superconformal index are defined and exact results can be written in terms of a matrix integral. In some cases the integrals can be explicitly performed. The relation to dimensional deconstruction of $\mathcal{N} = (1, 1)$ LST is reviewed and we are able to match the $\frac{1}{2}$-BPS partition functions of the class $\mathcal{S}_k$ theory and the LST, providing another check of the dimensional deconstruction proposal of Arkani-Hamed, Cohen, Kaplan, Karch and Motl. Holomorphic $\mathcal{N} = 1$ curves encoding the low energy superpotential on the Coulomb branch are derived for a variety of class $\mathcal{S}_k$ theories, following the methodology of Seiberg and Intriligator. The $\mathcal{N} = 1$ analogue of Nekrasov's partition function of instantons is computed and is matched to $\mathcal{W}_{kN}$ conformal blocks.

We also investigate $\mathcal{N} = 3$ theories that arise from discrete gauging of an enhanced discrete symmetry which emerges at strong coupling of $\mathcal{N} = 4$ SYM. We study their moduli spaces, focusing on the Higgs and Coulomb branches. The Hilbert series are computed. Moreover, in some cases we can also compute the superconformal index. We compare some of the properties to the S-fold theories of Garcìa-Etxebarria and Regalado.

Lastly, BPS-strings in 6d $\mathcal{N} = (1, 0)$ SCFTs on $\mathbb{R}^4 \times T^2$ in the presence of surfaces defects are examined. The $(1, 0)$ theories in question are the same ones used to engineer class $\mathcal{S}_k$ theories however, in this case, the surface defects lie along an $\mathbb{R}^2 \subset \mathbb{R}^4$ as opposed to on the Riemann surface $\mathcal{C} = T^2$ such that we get an $\mathcal{N} = 2$ theory with a defect upon compactification. The strings wrap the T^2. The

elliptic genus partition function of the strings is computed.

Zusammenfassung

Eine umfangreiche Klasse der vierdimensionalen $\mathcal{N} = 1$ Theorien kann mithilfe einer getwisteten Kompaktifizierung einer sechsdimensional Theorie mit $\mathcal{N} = (1,0)$ Supersymmetrie entland einer riemannschen Fläche konstruiert werden. In dem falle, dass die $(1,0)$ Theorie einer Theorie auf der transversen $\mathbb{C}^2/\mathbb{Z}_k$ Singularität auf M5-Branen entspricht, wird die resultierende Theorie in vier Dimensionen als der Klasse $\mathcal{S}_k$ zugehörig bezeichnet. Diese Theorien sind Orbifaltigkeiten von Theorien der Klasse $\mathcal{S}$ und dementsprechend wird einiges der Struktur der Muttertheorie, für die sehr viele Ergebnisse berechnet und verstanden wurden, von der Tochtertheorie übernommen.

In dieser Dissertation untersuchen wir exakte Resultate von Theorien der Klasse $\mathcal{S}_k$, soweit wir diese erziehlen könnnen. Unter Anwednung der Vererbungsstruktur der Orbifaltigkeiten beschreiben und definieren wir den Higgs- sowie den Coulombzweig des Modulraums der supersymmetrischen Vakua der Theorien der Klasse $\mathcal{S}_k$. Dies steht im Kontrast zu generischen $\mathcal{N} = 1$ Theorien, in denen eine solche Aufteilung nicht möglich ist. Die Hilbertreihe wird für eine Vielzahl dieser Zweige beispielhaft berechnet. Einige bedeutende Grenzwerte des superkonformen Indexes, die für generische $\mathcal{N} = 1$ Theorien ebensowenig existieren würden, werden definiert und die exakten Ergebnisse können zudem in Matrixintegralen ausgedrückt werden. In manchen Fällen lassen sich die Integral explizit berechnen. Der Zusammenhang mit der dimensionalen Dekonstruktion von $\mathcal{N} = (1,1)$ LST wird besprochen und wir gleichen die $\frac{1}{2}$-BPS Zustandsfunktionen der Theorien der Klasse $\mathcal{S}_k$ mit jenen der LST ab, was einen zusätzlichen Test des dimensionalen Dekonstruktionsvorschlags von Arkani-Hamed, Cohen, Kaplan, Korch und Motl darstellt. Holomorphe $\mathcal{N} = 1$ Kurven, die das Niederenergiesuperpotential auf dem Coulombzweig kodieren, werden nach der Methodik von Seiberg und Intriligator für einige der Theorien der Klasse $\mathcal{N}_k$ hergeleitet. Das $\mathcal{N} = 1$ Analog der nekrasovschen Zustandsfunktion der Instantons wird berechnet und mit den konformen $\mathcal{W}_{kN}$ Blocks verglichen.

Wir untersuchen des Weiteren $\mathcal{N} = 3$ Theorien, die aus einer diskreten Eichung einer erhöhten diskreten Symmetrie, die sich aus der starken Kopplung von $\mathcal{N} = 4$ SYM ergibt, entstehen. Wir analysieren deren Modulräume, wobei wir uns auf den Higgs- und den Coulombzweig fokussieren. Die Hilbertreihe wird berechnet. Darüber hinaus ist es uns in einigen Fällen m—'oglich auch den superkonformen Index zu berechnen. Wir vergleichen eine Reihe von Eigenschaften hiervon mit den S-fold Theorien von Garcia-Etxebarria und Regalado.

Letztlich untersuchen wir BPS-Strings in sechs dimensionalen $\mathcal{N} = (1,0)$ superkonformen Feldtheorien auf $\mathbb{R}^4 \times T^2$ in der Anwesenheit von Oberflächendefekten. Die betrachteten $(1,0)$ Theorien sind dieselben wie bei der Konstruktion der Theorien der Klasse $\mathcal{S}_k$, wobei, in diesem Fall, der Oberflächendefekt, entgegen der zuvor verwendeten riemannschen Oberfläche $\mathcal{C} = T^2$ entlang eines $\mathbb{R}^2 \subset \mathbb{R}^4$ liegt, sodass wir nach der Kompaktifizierung eine $\mathcal{N} = 2$ Theorie mit einem Defekt erhalten. Der String umwickelt den T^2. Die Zustandsfunktion des elliptischen Genus des Strings wird berechnet.

This Thesis is Based Upon:

- Bourton, T. & Pomoni, E *Instanton Counting in Class $\mathcal{S}_k$* `arXiv:1712.01288`.
- Bourton, T., Pini, A. & Pomoni, E. *4d $\mathcal{N}=3$ Indices via Discrete Gauging*, J. High Energ. Phys. (2018) 2018: 131. `doi.org/10.1007/JHEP10(2018)131` .
- Bourton, T., Pini, A. & Pomoni, E. *The Higgs and Coulomb Branches of Theories of Class $\mathcal{S}_k$*, To Appear

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Notation	Explanation
$\mathbb{N} = \{0, 1, 2 \dots\}$	The natural numbers
$\mathbb{Z}^+ = \{1, 2 \dots\}$	The positive integers
$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$	The integers
$\mathbb{K}$	Generic field
$C_n = \mathbb{Z}/n\mathbb{Z} = \{0, 1, \dots, n-1\}$	Group of integers under addition modulo n
$\mathbb{Z}_n$	Group of integers under multiplication modulo n
S_N	Permutation group on N objects
$\widetilde{X}$	Universal cover of X
$X^{\mathbb{C}} = X \otimes_{\mathbb{R}} \mathbb{C}$	Complexification of X
$\mathrm{Aut}(X)$	Automorphism group of X
$\mathrm{Out}(X)$	Outer automorphism group of X
$\mathrm{Inn}(X)$	Inner automorphism group of X
$\mathrm{End}(X)$	Endomorphisms of X
$\mathrm{Sym}^N(X) = X^N/S_N$	N^{th}-Symmetric product of X
$C^m(X)$	Space of functions on X with m continuous derivatives
d	Exterior derivative
$\Omega^d(X)$	Space of d-forms on X
$\star$	Hodge star
TX	Tangent bundle of manifold X
T^*X	Cotangent bundle of manifold X
$\mathrm{i}^2 = -1$	Unit imaginary number
$\Re z$, $\Im z$	Real and imaginary parts of $z \in \mathbb{C}$
G	Real matrix Lie group
$T(G)$	Maximal torus of G
$d\mu_G$	Haar measure for G
$\operatorname{rank} G = \dim_{\mathbb{R}} T(G)$	Rank of G
$\mathfrak{g} = Lie(G)$	Lie algebra of G
$C_G(S) = \{g \in G \| \operatorname{Ad}_g s = s \,\forall\, s \in S\}$	Centraliser of S in G
$\mathfrak{h}$, $\mathfrak{t}$	Cartan subalgebra of $\mathfrak{g}$
$\operatorname{Ad}_g(X) = gXg^{-1}$	Adjoint action $g \in G$, $X \in \mathfrak{g}$
$\operatorname{ad}_X Y = [X, Y]$	Adjoint action of X on $\mathfrak{g}$, $X, Y \in \mathfrak{g}$
M, **F**, **HB**, **CB**	Affine varieties
M, F, HB, CB	Coordinate rings of **M**, **F**, **HB**, **CB**
R	Rings
I, J	Ideals
$\mathrm{HS}(\tau; R)$	Hilbert series for the ring R

$\mathcal{M}_d$	d-dimensional Riemannian manifold
$\mathcal{C}$, $\mathcal{X}$	Riemann surfaces
$\mathbb{S}^d$	Round d-dimensional sphere
$T^d = (\mathbb{S}^1)^d$	d-dimensional torus
AdS	Anti-de-Sitter space
$\mathcal{I}$	Supersymmetric index
HL	HL limit of the 4d supersymmetric index
Ell	Elliptic genus (2d supersymmetric index)
Z	Other types of partition functions
$\mathrm{Q}, \widetilde{\mathrm{Q}}$	2d Euclidean Poincaré supercharges
$\mathcal{Q}, \widetilde{\mathcal{Q}}$	4d Euclidean Poincaré supercharges
$\mathtt{Q}, \widetilde{\mathtt{Q}}$	5d Euclidean Poincaré supercharges
$\mathsf{Q}, \widetilde{\mathsf{Q}}$	6d Euclidean Poincaré supercharges
Φ, Q, $\widetilde{Q}$	4d $\mathcal{N} = 1$ chiral multiplets
$\overline{\Phi}$, $\overline{Q}$, $\overline{\widetilde{Q}}$	4d $\mathcal{N} = 1$ anti-chiral multiplets

Chapter 1

Introduction

1.1 Recent Advances and Exact Results

In recent decades there has been huge interest towards the study and classification of quantum field theories in various dimensions. One motivation for this study is the desire to understand the 'landscape' of all permissible quantum field theories. Some regions of this 'landscape' are somewhat understood, for example Yang-Mills theories at high energies (UV). This is largely due to the framework for organising the perturbation theory introduced by Feynman in the 1940's. The importance of perturbation theory cannot be overstressed; high precision tests of the electron g-factor from the LHC agree with the theoretical predicitions of the standard model up to 14 digits of precision [7]. Conversely, relatively little is known about about the low energy (IR) behaviour of such theories because said perturbative techniques breaks down. This is because because in Yang-Mills theories the coupling is a function of the energy scale and the coupling is said to 'run' with the energy scale. Yang-Mills theories are weakly coupled in the UV but as we flow to the IR the coupling grows and non-perturbative phenomena, such as colour confinement and instantons, become relevant; they are therefore said to be *asymptotically free*. Indeed, the perturbative expansion is an asymptotic expansion in the coupling and convergence of the series requires the inclusion of non-perturbative effects. Moreover, it has become clear that the landscape of quantum field theories also contains a vast number of theories that cannot be understood via the standard local Lagrangian/path integral formulation found in introductory QFT textbooks.

If we restrict our attention to theories with a certain amount of *supersymmetry* much more can be understood. This may be seen as a lowering of ambition (all though still an incredibly lofty undertaking) because supersymmetry is either yet to be discovered or may not exist in nature. Nevertheless, supersymmetric theories offer examples of consistent quantum field theories. Supersymmetric theories are far from non-trivial and also offer many of the interesting phenomena expected in non-supersymmetric theories, such as confinement, but in a more constrained and controllable framework.

Supersymmetry is a fermionic spacetime symmetry which pairs bosons and fermions together. The presence of this additional symmetry constrains the dynamics of the theory and can often mean that supersymmetric theories are far more tractable to study. In comparison to non-supersymmetric theories, where relatively little is known outside of the perturbative regime, for supersymmetric theories there are by now a large number of results and many of them will be reviewed and discussed within this thesis.

As previously discussed, for non-supersymmetric theories, we, so far, have a rather limited toolkit with which to study such theories; the main tool being perturbation theory.[1] But, when we restrict our attention to QFTs with some amount of supersymmetry, we can employ a plethora of new tools with which we can access and study non-perturbative effects. An overview of these tools can be found in [8, 9, 10, 11]. They can often allow us to compute *exact results* for these theories. In other words, observables for the theory which are valid even for the non-perturbative regime. Much of the computation and study of exact results has so far been dedicated to theories with eight or more supercharges, in four dimensions this is $\mathcal{N} \geqslant 2$ supersymmetry. Additionally, aided by string/M/F-theoretic techniques, physicists have been able to construct a vast number of so-called *non-Lagrangian* theories. As we previously mentioned they do not admit a path integral description and usually live as isolated conformal theories. They can nevertheless sometimes be partially understood using string/M/F-theoretic reasoning and often are related to Lagrangian theories via strong/weak dualities.

1.2 Seiberg-Witten Theory

In 1994 Seiberg & Witten provided one of the first computations of an exact result in a four dimensional gauge theory. They were able to compute the low-energy effective

[1] For very special (highly symmetrical) non-supersymmetric QFTs, such as topological, conformal or integrable QFTs we have other tools and can often compute some non-perturbative results or even ‘solve’ the theory.

action on the Coulomb branch for four-dimensional $\mathcal{N}=2$ gauge theories based upon a Lie-algebra $\mathfrak{g}$ [12, 13] with Cartan subalgebra $\mathfrak{h}$.[2] These theories are comprised of $\mathcal{N}=2$ vector multiplets and a collection of hypermultiplets. The on-shell field content of a $\mathcal{N}=2$ vector multiplet is the same as for a $\mathcal{N}=1$ chiral multiplet Φ (containing a complex scalar ϕ and complex Weyl fermion ψ), anti-chiral multiplet $\overline{\Phi}$ and $\mathcal{N}=1$ vector multiplet V (Weyl fermion λ, anti-Weyl fermion $\overline{\lambda}$ and one-form gauge potential A) all in the adjoint representation of the gauge algebra. The field content of the hypermultiplet in a representation $\mathcal{R}\oplus\overline{\mathcal{R}}$, for $\mathcal{R}$ pseudo-real, is two half-hypermultiplets $(Q,\widetilde{Q})$ in the representations $\mathcal{R}$ and $\overline{\mathcal{R}}$. A half-hypermultiplet is a $\mathcal{N}=1$ chiral multiplet. The decomposition $\mathfrak{g}=\oplus_i\mathfrak{g}_i$, with $\mathfrak{g}_i$ either simple or $\mathfrak{u}(1)$, induces the decomposition $\mathcal{R}=\oplus\mathcal{R}_{i,a}$ with $\mathcal{R}_{i,a}$ irreducible. The one-loop beta function associated to the factor $\mathfrak{g}_i$ is then

$$\beta_i = E\frac{d}{dE}g_i = -\frac{g^3}{(4\pi)^2}b_i\,, \quad b_i = 2h^\vee(\mathfrak{g}_i) - \sum_a c_2(\mathcal{R}_{i,a})\prod_{j\neq i}\dim_{\mathbb{R}}\mathcal{R}_{j,a}\,. \tag{1.1}$$

Here $h^\vee(\mathfrak{g})$ denotes the dual Coxeter number of $\mathfrak{g}$ and $c_2(\mathcal{R})$ the quadratic Casimir (Dynkin index) of the representation $\mathcal{R}$.

Due to powerful non-renomalisation theorems [15, 16, 17, 18] this is an exact statement valid to all orders in perturbation theory. In order for the theory to be UV-complete we must have that $b_i \geqslant 0$. If each $b_i = 0$ then the theory is (perturbatively) conformal.

Such a theory has a moduli space of supersymmetric vacua $\mathbf{M}$. This is the space of zero-energy configurations of the theory and is schematically defined as

$$\mathbf{M} := \{\phi, q, \widetilde{q} | V(\phi, q, \widetilde{q}) = 0\}/G\,, \tag{1.2}$$

here $\phi, q, \widetilde{q}$ denote the scalar components of $\Phi, Q, \widetilde{Q}$, respectively and $V(\phi, q, \widetilde{q})$ denotes the scalar potential. The Coulomb branch $\mathbf{CB}\subset\mathbf{M}$ for a four dimensional $\mathcal{N}=2$ theory is a particular submanifold reached by allowing constant vevs for $\langle\phi\rangle = a\in\mathfrak{h}$ while setting to zero vevs for $\langle Q\rangle = \langle\widetilde{Q}\rangle = 0$. The Coulomb-branch is therefore parametrised by the generators $u_{i=1,2,\dots\,\mathrm{rank}\,\mathfrak{g}}$ of the ring of $\mathfrak{g}$-invariant polynomials in the ϕ and so $\dim_{\mathbb{C}}\mathbf{CB} = \mathrm{rank}\,\mathfrak{g}$. By gauge transformation we can take a to be diagonal and therefore the gauge algebra $\mathfrak{g}$ is broken down to the sta-

[2] Their original computations were based off of $\mathfrak{g}=\mathfrak{su}(2)$ with $N_f = 0,1,\dots,4$ fundamental hypermultiplets or a single adjoint hypermultiplet ($\mathcal{N}=2^*$). Over the years this has since been generalised to a large number of $\mathcal{N}=2$ theories, see Section 4.5 of [14] for an exhaustive list.

biliser subalgebra of $\mathfrak{g}$ with respect to a. This is simply the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e. the Lie algebra of $C_G(a)$. Therefore, in the IR, we will have an abelian gauge theory based on gauge group $\exp \mathfrak{h} \cong U(1)^{\operatorname{rank} \mathfrak{g}}$. Due to $\mathcal{N} = 2$ supersymmetry the form of the effective action of this theory is highly constrained and depends only on a single holomorphic function $\mathcal{F}(a)$ called the *prepotential* [19]. The effective action in $\mathcal{N} = 1$ language reads

$$S_{\text{eff}} = \frac{1}{4\pi} \Im \left[\int d^4\theta a_i^D \overline{a}_i + \frac{1}{2} \int d^2\theta \tau_{ij} \, W_\alpha^i W^{\alpha j} \right] , \tag{1.3}$$

$$\tau_{ij}(a) = \frac{\partial^2 \mathcal{F}(a)}{\partial a^i \partial a^j} , \quad a_i^D(a) = \frac{\partial \mathcal{F}(a)}{\partial a^i} , \tag{1.4}$$

$$i, j \in \{1, 2, \ldots, \operatorname{rank} \mathfrak{g}\} . \tag{1.5}$$

The task is then to compute $\mathcal{F}(a)$. The solution of Seiberg-Witten is that the prepotential is encoded in a pair $(\mathcal{X}_{\{u_i\}}, \lambda_{\text{SW}})$, where $\mathcal{X}_{\{u_i\}}$ denotes a Riemann surface of genus $g \geqslant \operatorname{rank} \mathfrak{g}$ called the Seiberg-Witten curve and λ_{SW} a meromorphic one-form on $\mathcal{X}_{\{u_i\}}$. There are a family of Riemann surfaces $\mathfrak{X} = \{\mathcal{X}_{\{u_i\}}\}$ realised as a fibration over the moduli space of Coulomb branch vacua **CB**, which is parametrised by the u_i. At a generic point on **CB** specified by the u_i the fiber is $\mathcal{X}_{\{u_i\}}$ and is given in terms of a polynomial $y^2 = F(x; u_i, q)$ in two auxiliary variables x, y. We will now sketch the derivation of this result for the case $\mathfrak{g} = \mathfrak{su}(2)$ theory without flavour, as first derived in [12]. The first point is to notice that S_{eff} possesses an $SL(2, \mathbb{Z})$ S-duality group

$$SL(2, \mathbb{Z}) \cong \langle S, T | S^4 = 1, (ST)^3 = 1 \rangle \cong C_4 * C_3 , \tag{1.6}$$

where C_n is the cyclic group $|C_n| = n$ and $*$ is the free product. An explicit realisation is

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{1.7}$$

An element of $SL(2, \mathbb{Z})$ acts on the vector $(a_D \ a)^{\mathrm{T}}$ in the fundamental representation while it acts on the effective gauge coupling via fractional linear transformation. In particular S sends $\tau(a) \to \tau_D(a_D) = -1/\tau(a)$. In the weak coupling one can easily see the appearance of non-trivial monodromies. The prepotential is [19]

$$\mathcal{F}(a) = \frac{\mathrm{i}}{2\pi} a^2 \log \frac{a^2}{\Lambda^2} + \sum_{i=1}^{\infty} c_k \frac{\Lambda^{4i}}{a^{4i}} a^2 , \tag{1.8}$$

where Λ^4 is the holomorphic dynamically generated scale. In the weak coupling $a \gg \Lambda$ and so, (after normalising $u = \operatorname{tr}\phi^2/\Lambda^2$) looping around infinity on the u-plane $u \mapsto e^{2\pi \mathrm{i}}u$ means

$$\begin{aligned}\begin{pmatrix} a^D \\ a \end{pmatrix} = \begin{pmatrix} \partial\mathcal{F}/\partial a \\ a \end{pmatrix} \to M_\infty \begin{pmatrix} a^D \\ a \end{pmatrix} &= \begin{pmatrix} -1 & 2 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a^D \\ a \end{pmatrix} \\ &= S^2 T^{-2} \begin{pmatrix} a^D \\ a \end{pmatrix} .\end{aligned} \tag{1.9}$$

Due to the R-symmetry $u \mapsto -u$ Seiberg and Witten conjectured that the u-plane has three singular points, at $u = \infty, +1, -1$. The monodromy at $u = +1$ can be determined by appealing to the $SL(2,\mathbb{Z})$ duality. By applying an S-transformation a^D becomes the good coordinate in which to write down the effective action in terms of. Therefore, around the point $u = +1$, a^D is a good coordinate and $a^D \sim b(u-1)$ for a constant b. Around this point the one-loop beta function shows $\tau^D \simeq -\frac{\mathrm{i}}{\pi}\log a^D$ and so

$$\begin{aligned} a(u) = -\int \tau^D \, da^D &\simeq a(u=1) + \frac{\mathrm{i}}{\pi} a^D \log a^D \\ &\simeq a(u=1) + \frac{\mathrm{i}}{\pi} b(u-1)\log(u-1) . \end{aligned} \tag{1.10}$$

The monodromy around $u = 1$ is therefore

$$\begin{pmatrix} a^D \\ a \end{pmatrix} \to M_1 \begin{pmatrix} a^D \\ a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} a^D \\ a \end{pmatrix} = ST^2S^{-1} \begin{pmatrix} a^D \\ a \end{pmatrix} . \tag{1.11}$$

The monodromy at $u = -1$ can now easily be computed by demanding

$$M_{-1} = M_1^{-1} M_\infty . \tag{1.12}$$

If one supplements M_1, M_∞ with $-\mathbb{I}$ (u is invariant under $a \to -a$) then it is not too hard to show that those matrices generate the group $\Gamma(2) \subset SL(2,\mathbb{Z})$ where

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) \middle| a, d = 1 \bmod n \,, b, c = 0 \bmod n \right\} , \tag{1.13}$$

is the principal congruence subgroup of level n in $SL(2,\mathbb{Z})$. The solution of the model is therefore to specify the the modular curve $\mathfrak{X} = X(2) = \mathbb{H}/\Gamma(2)$ which can be explicitly written down as as the set of a 1-parameter family of curves $\mathfrak{X} = \{\mathcal{X}_u\}$

$$\mathcal{X}_u: \quad y^2 = F(x;u) = (x-u)(x-1)(x+1)\,. \tag{1.14}$$

Recall that we have normalised $u = \operatorname{tr}\phi^2/\Lambda^2$. Choosing a basis $\{A, B\}$ for the $2\operatorname{rank}\mathfrak{g} = 2$ independent one-cycles the effective action is computed via the integrals

$$a^i = \int_{A_i} \lambda_{\mathrm{SW}}\,, \quad a_i^D = \int_{B_i} \lambda_{\mathrm{SW}}\,. \tag{1.15}$$

See also Appendix D.3 for more information. As pointed out in [13, 20] the Seiberg-Witten technology may also, in part, be applied to certain theories with only $\mathcal{N} = 1$ supersymmetry. In particular, if the $\mathcal{N} = 1$ theory has a Coulomb phase in which the effective action is that of an abelian gauge theory the Seiberg-Witten technology may be applied in order to compute the effective superpotential W. On the other hand the Käher potential (which is fixed in terms of $\mathcal{F}(a)$ for $\mathcal{N} = 2$ theories, $K = \Im\partial_{a^i}\mathcal{F}\overline{a}_i$) is left unfixed. We investigate this in Chapter 3.

1.3 Instantons

In the last decades much progress has been made towards understanding the non-perturbative (exact) aspects of Yang-Mills theories. One particular class of solutions to the Yang-Mills equations are called *instantons*. Yang-Mills instantons are, by definition, solutions of the classical field equations on a Riemannian manifold with finite action. Reviews of Yang-Mills instantons can be found in [21, 22, 23, 24] while a more mathematical treatment may be found in [25].

We will consider pure Yang-Mills theory in $d = 4$ dimensions based on a Lie algebra $\mathfrak{g}$ on an orientable Riemannian manifold $\mathcal{M}$ with metric g and volume form $\omega \in \Omega^d(\mathcal{M})$. This is a theory of vector bundles $V \to \mathcal{M}$ equipped with connection ∇ with curvature $F = \operatorname{ad}_\nabla \nabla \in \mathfrak{g} \otimes \Omega^2(\mathcal{M})$ associated to principal G-bundles $P \to \mathcal{M}$ where G is a connected Lie group of rank r with Lie algebra $\mathfrak{g}$. This theory has action

$$S = -\frac{1}{2g^2}\int_{\mathcal{M}} \operatorname{tr} F \wedge \star F = -\frac{1}{2g^2}\int_{\mathcal{M}} \operatorname{tr}\langle F, F\rangle\,\omega\,, \tag{1.16}$$

here $\star$ denotes the Hodge star on $\mathcal{M}$. The *Yang-Mills equations* are then

$$\nabla \star F = 0\,. \tag{1.17}$$

Recalling that $\star^2 = (-1)^{p(d-p)}$ on p-forms we can write

$$\mathrm{tr}\langle F, F\rangle = \frac{1}{2}\,\mathrm{tr}\langle F \pm \star F, F \pm \star F\rangle \mp \mathrm{tr}\langle F, \star F\rangle \geqslant \mp\,\mathrm{tr}\langle F, \star F\rangle\,, \tag{1.18}$$

we therefore have that the action minimising configurations satisfy $F = \pm \star F$.

In a local trivialisation of V we write $\nabla = d + A$, $A \in \mathfrak{g} \otimes \Omega^1(\mathcal{M})$. We define a G-framed Yang-Mills instanton as a solution to the (anti-)self-dual Yang-Mills equations

$$F = \pm \star F \tag{1.19}$$

such that A approaches pure gauge at infinity. Using the Bianchi identity $\nabla F = 0$ we can immediately see that any solution to (1.19) automatically solves the Yang-Mills equations (1.17). In a given instanton bundle such solutions admit a topological invariant - the instanton number

$$k = \int_{\mathcal{M}} c_1(F)^2 = \frac{1}{16\pi^2}\int_{\mathcal{M}} \mathrm{tr}\, F \wedge F = \frac{1}{16\pi^2}\int_{\mathcal{M}} \mathrm{tr}\langle F, \star F\rangle\,\omega \in \mathbb{Z}\,. \tag{1.20}$$

Here the trace is always normalised such that k is integral, $c_1^2(F) \in H^4(\mathcal{M}, \pi_3(G))$ is the square of the first Chern class and $\pi_3(G) \cong \mathbb{Z}$ for any connected compact simple Lie group. For generic $G, \mathcal{M}$ there may be other topological invariants besides (1.20), for instance when the second Steifel-Whitney class $w_2(V) \in H^2(\mathcal{M}, \mathbb{Z}_2)$ is non-trivial there may be G bundles which do not lift to $\widetilde{G}$ bundles where $\widetilde{G}$ denotes the universal cover of G [26]. For example it was shown in [27] that when $G = SO(n)$ and $\mathcal{M}$ is any 4-complex the associated bundles V are classified by a pair $(w_2(V), c_2(F))$. Unless otherwise stated we will specialise to $G = SU(N)$ and $\mathcal{M} = \mathbb{R}^4$ (possibly with Ω-deformation parameters ϵ_1, ϵ_2 turned on). Since $H^{1,2,3}(\mathbb{R}^4, \mathbb{Z})$ (and also $H^{1,2,3}(\mathbb{S}^4, \mathbb{Z})$) are all trivial such bundles are classified by (1.20) alone.

1.3.1 Instanton Moduli Space

We identify solutions to the self-dual instanton equations (1.19) if they are related by gauge transformations $g(x) \in G$. We therefore have a moduli space $\mathbf{M}$ of solutions called *instanton moduli space*. Because of the topologically invariant instanton number (1.20) the instanton moduli space formally admits a decomposition

$$\mathbf{M} \cong \bigoplus_{k=1}^{\infty} q^k \mathbf{M}_k\,. \tag{1.21}$$

$\mathbf{M}_k$ is called the *k-instanton moduli space*. The k-instanton moduli space is parametrised by so-called *collective coordinates*. To compute its dimension we can consider the case when the k-instanton solution looks like k 1-instanton solutions who's centres are well-separated and then superimpose them while adding corrections to satisfy (1.19) (assuming that the dimension does not change as we do this). Let us describe an explicit example, namely the moduli space of one $\mathfrak{g}$ instanton on $\mathcal{M} = \mathbb{R}^4$. In that case collective coordinates are:

- Four parameters labelling the center of the instanton $(X^1, X^2, X^3, X^4) \in \mathbb{R}^4$
- One parameter for the size of the instanton $\rho^2 \in \mathbb{R}_{\geqslant 0}$
- $4h^\vee(\mathfrak{g}) - 5$ embedding parameters $\mathfrak{su}(2) \to \mathfrak{g}$ of the 1-instanton $\mathfrak{su}(2)$ solution into $\mathfrak{g}$

Therefore, the dimension of the k-instanton moduli space is $\dim_{\mathbb{R}} \mathbf{M}_k = 4kh^\vee(\mathfrak{g})$. For the case $G = SU(N)$ and $\mathcal{M}$ a general compact 4-manifold the dimension is [28]

$$\dim_{\mathbb{R}} \mathbf{M}_k = 4Nk - (N^2-1)\frac{\chi(\mathcal{M}) + \sigma(\mathcal{M})}{2}\,. \tag{1.22}$$

Choosing local coordinates x^μ on $\mathbb{R}^4$ the solution $A = A(X^\mu, \rho, g, x^\mu)$ to (1.19) for $G = SU(2)$ is

$$A_\mu dx^\mu = \sum_{i=1}^{3} \frac{\rho^2 (x-X)^\nu \bar{\eta}^{i\nu}_\mu}{(x-X)^2((x-X)^2+\rho^2)} (g(x)\sigma^i g(x)^\dagger) dx^\mu\,. \tag{1.23}$$

Explicit solutions for general N are harder to explicitly write down and we will come back to these in the next subsection.

Denoting the collective coordinates by $\{X^a\}$, $a = 1, \ldots, \dim_{\mathbb{R}} \mathbf{M}_k$ and a solution $A = A(X^a, x^\mu)$ to (1.19). The moduli space inherits a metric h given by

$$h = \left(\int_{\mathcal{M}} \operatorname{tr} \delta_a A \wedge \star \delta_b A\right) dX^a \otimes dX^b \tag{1.24}$$

where $\delta_a A = \partial A / \partial X^a$. One can also show that h defined this way is *hyperKähler*.

1.3.2 ADHM Construction

Due to Atiyah, Drinfeld, Hitchin and Manin [29] there is a powerful method to solve the self-dual Yang-Mills equations for $G = SU(N)$ on $\mathbb{R}^4$. The idea is to realise $\mathbf{M}_k$

as a hyperKähler quotient of a complex vector space. In order to specify a solution one gives the so-called ADHM data. This is made up of a pair of complex vector spaces V, W with $\dim_{\mathbb{C}} V = k$, $\dim_{\mathbb{C}} W = N$ and a set of linear maps

$$B_1, B_2 : V \to V, \quad I : V \to W, \quad J : W \to V. \tag{1.25}$$

Subject to the ADHM (moment map) equations

$$\mu_{\mathbb{C}} = [B_1, B_2] + JI = 0, \quad \mu_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + JJ^\dagger - I^\dagger I = 0. \tag{1.26}$$

Moreover there are natural $U(k) : V \to V$, $U(N) : W \to W$ actions which act by

$$(B_1, B_2, I, J) \mapsto (gB_1g^\dagger, gB_2g^\dagger, Ig^\dagger, gJ)\,, \quad g \in U(k) \tag{1.27}$$

$$(B_1, B_2, I, J) \mapsto (B_1, B_2, \tau I, J\tau^\dagger)\,, \quad \tau \in U(N) \tag{1.28}$$

which respects the hyperKähler structure. The moduli space is

$$\mathbf{M}_{k,\zeta} := \{B_1, B_2, I, J \,|\, \mu_{\mathbb{C}} = 0\,, \mu_{\mathbb{R}} = \zeta \mathbb{I}_k\} \,/U(k)\,. \tag{1.29}$$

Note that here we have added a non-zero FI-term ζ. Mathematically; allowing for non-zero ζ, means are actually considering the space obtained from the original instanton moduli space by a series of blowups (resolutions of singularities) which are smooth and we consider not only bundles but torsion free sheaves [30, 25, 31, 22, 32, 33]. Physicially this corresponds to studying instantons on non-commutative $\mathbb{R}^4$ [33, 34] defined by

$$[z_1, \bar{z}_1] = [z_2, \bar{z}_2] = -\frac{\zeta}{2}\,, \quad (z_1, z_2) = (x^1 + \mathrm{i}x^2, x^3 + \mathrm{i}x^4)\,. \tag{1.30}$$

In practical applications it is often more convenient to instead work holomorphically with an equivalent form where we drop the real constraint and quotient out by complexified gauge transformations

$$\widetilde{\mathbf{M}}_{k,\zeta} := \{B_1, B_2, I, J \,|\, \mu_{\mathbb{C}} = 0,\ \text{stability}(\zeta)\} \,/GL(k, \mathbb{C})\,. \tag{1.31}$$

The main point of [29] is that they were able to prove that

$$\mathbf{M}_k \cong \mathbf{M}_{k,\zeta=0} \cong \widetilde{\mathbf{M}}_{k,\zeta=0}\,. \tag{1.32}$$

To construct the self-dual solutions explicitly we consider the $(N+2k)\times 2k$ matrix

$$Y = \begin{pmatrix} J^\dagger & I \\ B_2^\dagger + z_2^\dagger & -B_1 - z_1 \\ B_1^\dagger + z_1^\dagger & B_2 + z_2 \end{pmatrix} . \tag{1.33}$$

We then have

$$Y^\dagger Y = \begin{pmatrix} f^{-1} & \mu_{\mathbb{C}} \\ \mu_{\mathbb{C}}^\dagger & -\mu_{\mathbb{R}} + f^{-1} \end{pmatrix} = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix} \tag{1.34}$$

with $f^{-1} := JJ^\dagger + B_2B_2^\dagger + B_2z_2^\dagger + B_2^\dagger z_2 + B_1^\dagger z_1 + B_1 z_1^\dagger + z_1^\dagger z_1 + z_2^\dagger z_2$ and therefore Y factorises and is invertible (the inverse is simply $Y^{-1} = fY^\dagger$). Now, the basis vectors for the null-space of $Y^\dagger$ can be assembled into a $(N+2k)\times N$ matrix $U = U(x)$

$$Y^\dagger U = 0\,, \quad U^\dagger U = \mathbb{I}_N\,. \tag{1.35}$$

The k-instanton gauge potential is then given by

$$A = U^\dagger \frac{\partial}{\partial x^\mu} U dx^\mu\,. \tag{1.36}$$

1.3.3 ADHM Construction from Type-II String Theory

For physical purposes one of the most natural ways to view the ADHM construction is within string theory. String theory can be a powerful tool in understanding the dynamics of gauge theories. Indeed one of the most important tools that we have used in Chapter 4 of this thesis is the relationship between the ADHM construction of instantons and D$(p-4)$-branes [35, 36, 37, 24, 38, 23, 39].

Let us suppose that we have a gauge theory in $p+1$ dimensions which can be embedded within Type-II string theory on $\mathbb{R}^{10}$ as a theory living on the worldvolume of a stack of N Dp-branes with coordinates $X^1,\ldots,X^{p+1}$. We now further consider inserting a stack of k D$(p-4)$-branes.

The Type-II string action contains terms of the form

$$\sum_l \int_{\mathbb{R}^{10}} C_l \wedge J_{10-l} \tag{1.37}$$

with $l = 0,2,4,6,8,10$ for IIB and $l = 1,3,5,7,9$ for IIA. Here C_l is the RR l-form that couples (electrically) to the $(10-l)$-form D$(l-1)$-brane current J_{10-l}. The D$(9-l)$-brane is a source for C_l.

If we take the $l = p-3$ term while taking C_{p-3} constant over the $\mathbb{R}^{10-(p-4)}$ transverse to the D$(p-4)$-branes we can write

$$\begin{aligned}\int_{\mathbb{R}^{p+1}} C_{p-3} \wedge \left(\int_{\mathbb{R}^{9-p}} J_{13-p} \right) &= \frac{1}{2g^2} \int_{\mathbb{R}^{p+1}} C_{p-3} \wedge F \wedge F \\ &= \frac{8k\pi^2}{g^2} \int_{\mathbb{R}^{p-3}} C_{p-3} \,.\end{aligned} \tag{1.38}$$

In other words an instanton with topological charge k gives rise to a source for the RR-form. This is nothing other than the source induced by a D$(p-4)$-brane. Therefore we have that

$$|k| \text{ (A)SD instantons in a D}p\text{-brane} \equiv |k| \text{ (anti-)D}(p-4)\text{-branes}\,. \tag{1.39}$$

Demanding that C_{p-3} constant over the transverse $\mathbb{R}^{9-p}$ is equivalent to the statement that the D$(p-4)$-branes are confined to live within the Dp-branes. This is the Higgs-branch of the theory on the D$(p-4)$-branes. Therefore, the moduli space of k-instantons for the gauge theory living on the Dp-branes, $\mathbf{M}_k$, is isomorphic to the Higgs branch of the theory living on the D$(p-4)$-branes

$$\mathbf{M}_k \cong \mathbf{HB}_{k\ \mathrm{D}(p\,-\,4)} \tag{1.40}$$

$$= \left\{ \{\phi\} | V_{p-3} = 0\,, X^{p+2} = X^{p+3} = \cdots = X^{10} \right\} / U(k) \tag{1.41}$$

where $\{\phi\}$ collectively denotes all of the scalars of theory on the Higgs branch and $V_{p-3} = \sum |F|^2 + \frac{1}{2} D^2$ is the scalar potential of the $(p-3)$-dimensional theory living on the worldvolume of the D$(p-4)$-branes. The vanishing of the potential $F = D = 0$ translate into the ADHM constraints [39, 23] as we will explicitly demonstrate for a specific example in the next section.

When supersymmetry is present the Higgs branch is protected from quantum corrections [8] and the fluctuation determinants in the instanton measure cancel [39, 23, **?**]. The action of the theory on the D$(p-4)$-branes is then equivalent to the action of the theory evaluated on the given instanton configuration. Hence the partition function of the theory of k D$(p-4)$-branes is then nothing else but the partition function of k instantons for the theory living on the Dp-branes

$$Z_k(a,\epsilon_i,\dots) = \int_{\mathbf{M}_k} e^{-S^{(k)}_{\text{inst}}(a,\dots)} \tag{1.42}$$

$$= \mathrm{Tr}_{\mathcal{H}_k}(-1)^F e^{a^A h_A + \sum_i \epsilon_i j_i} \tag{1.43}$$

$$= Z_{\text{extra}}(\epsilon_i)\, Z_{\text{Higgs}}(a,\epsilon_i,\dots)\,, \tag{1.44}$$

where $\mathcal{H}_k$ denotes the Hilbert space of the worldvolume theory on k D$(p-4)$-branes. We will more carefully define this quantity in Section 1.4 where we will discuss how it may also be computed using localisation (see equation (1.74)). The factor Z_{extra} is often present due to the fact that the theory on the D$(p-4)$-branes provides a UV completion of the ADHM sigma model [40, 41] and therefore it may contain extra degrees of freedom which do not appear in the ADHM construction (such as a non-trivial Coulomb branch) which should be excluded from the instanton calculus. Those extra degrees of freedom generally decouple from the the ADHM degrees of freedom and the partition function factorises as above.

1.3.4 Example: ADHM for $\mathcal{N}=4$ SYM

Let us now demonstrate the power of the string-theoretic realisation of the ADHM construction. We consider $p=3$ with a stack of N D3-branes along X^1, X^2, X^3, X^4. In the low energy limit quantisation of open D3–D3 strings gives rise to the familiar $G=U(N)$ $\mathcal{N}=4$ SYM theory. The self-dual instantons are realised as $D(-1)$-branes. The number of supercharges preserved by this brane configuration is $32/(2^2)=8$ as required for $\frac{1}{2}$-BPS instantons of $\mathcal{N}=4$ SYM.

Quantisation of open D(-1)–D(-1) strings gives rise to the reduction to zero dimensions of 4d $\mathcal{N}=4$ SYM with gauge group $U(k)$. This has on-shell bosonic field content as a 4d theory of $A_{\mu=7,8,9,10}$, $z=X^1+\mathrm{i}X^2$ $\widetilde{z}=X^3+\mathrm{i}X^4$, $\phi=X^5+\mathrm{i}X^6$ all in the adjoint representation of $\mathfrak{u}(k)$.

D(-1)–D3 strings gives rise to the dimensional reduction of 4d $\mathcal{N}=2$ hypermultiplets $(q,\widetilde{q})$ in the $\mathbf{k}\otimes\overline{\mathbf{N}}$ representation of $U(k)\times U(N)$. The coupling of these bifundamental hypers to the maximally supersymmetric $U(k)$ theory is fixed by demanding $\mathcal{N}=2$ supersymmetry. As a 4d theory the superpotential is

$$W = q\phi\widetilde{q} + \mathrm{tr}\left([\phi,z]\widetilde{z} + W_\alpha W^\alpha\right)\,. \tag{1.45}$$

The bosonic part of the action reduced to zero dimensions is

$$
\begin{aligned}
\mathcal{L}_{bos} = & \frac{1}{2}\,\mathrm{tr}\left([X_\mu, X_\nu]^2 + |[X_\mu, \phi]|^2 + |[X_\mu, z]|^2 + |[X_\mu, \widetilde{z}]|^2\right) \\
& + q^\dagger X^\mu X_\mu q + \widetilde{q}^\dagger X^\mu X_\mu \widetilde{q} + \left(\sum_{f\in\{q,\widetilde{q},z,\widetilde{z},\phi\}} \mathrm{tr}\left(\frac{\partial W}{\partial f} F_f + F_f^2\right) + h.c.\right) \\
& + \mathrm{tr}\left(\frac{1}{2}D^2 + D\left([\phi, \phi^\dagger] + [\widetilde{z}, \widetilde{z}^\dagger] + [z, z^\dagger] + qq^\dagger - \widetilde{q}^\dagger\widetilde{q} - \zeta\right)\right) .
\end{aligned}
\tag{1.46}
$$

Here we have added Fayet-Iliopoulos parameters $\zeta_1 = \zeta_2 = \cdots = \zeta_k = \zeta$ for $U(1)^k \to U(k)$; without loss of generality for the instanton discussion we may take them to be all equal since their precise values will not be important, only that they are non-zero. The F terms are

$$
F_\phi = -q\widetilde{q} - [z, \widetilde{z}]\,, \quad F_q = -\phi\widetilde{q}\,, \quad F_{\widetilde{q}} = -q\phi\,, \tag{1.47}
$$

$$
F_z = [\phi, \widetilde{z}]\,, \quad F_{\widetilde{z}} = [\phi, z]\,, \tag{1.48}
$$

and the D-term is

$$
D = -qq^\dagger + \widetilde{q}^\dagger\widetilde{q} - [z, z^\dagger] - [\widetilde{z}, \widetilde{z}^\dagger] - [\phi, \phi^\dagger] + \zeta\mathbb{I}_k = 0\,. \tag{1.49}
$$

As we mentioned in the previous section, to reach the correspondence with instantons we must move to the Higgs branch of this theory. The Higgs branch is reached by setting $\phi = X_{\mu=7,8,9,10} = 0$ while enforcing that the scalar potential vanishes $D = 0$, and $F_f = 0$

$$
\mathbf{HB}_{k\ \mathrm{D}(-1)} = \{q, \widetilde{q}, z, \widetilde{z} | F_\phi = 0\,, D = 0\}\,/U(k) \tag{1.50}
$$

$$
F_\phi = -q\widetilde{q} - [z, \widetilde{z}] = 0\,, \quad D = -qq^\dagger + \widetilde{q}^\dagger\widetilde{q} - [z, z^\dagger] - [\widetilde{z}, \widetilde{z}^\dagger] - \zeta\mathbb{I}_k = 0 \tag{1.51}
$$

These are precisely the ADHM equations (1.26)

$$
\mu_{\mathbb{C}} = -F_\phi = 0\,, \quad \mu_{\mathbb{R}} - \zeta\mathbb{I}_k = -D = 0 \tag{1.52}
$$

with $z = B_1$, $\widetilde{z} = B_2$, $q = J$ and $\widetilde{q} = I$. And therefore we have

$$
\mathbf{HB}_{k\ \mathrm{D}(-1)} \cong \mathbf{M}_{k,\zeta}\,. \tag{1.53}
$$

1.4 Partition Functions and Supersymmetric Localisation

In any quantum mechanical or statistical model a basic quantity one would like to compute is the *partition function.* The partition function is a counting device which counts the number of microstates weighted by *fugacities.* Formally the canonical partition function of a thermodynamical system of temperature T and of fixed volume and number of particles N may be defined as

$$Z(\beta; N) = \operatorname{Tr} e^{-\beta H} = \sum_s e^{-\beta E_s} \tag{1.54}$$

here H denotes the Hamiltonian and $\beta = \frac{1}{k_B T}$ is the conjugate variable to the Hamiltionian. Tr denotes the summation/integral over all microstates, the space of all microstates $\{s\}$ can be discrete/continuous. The partition function is a useful quantity to compute because it encodes many quantities of the system. In particular, the probability P_s of the system to be in a microstate of energy E_s is given by $P_s = Z^{-1} e^{-\beta E_s}$. Then the expectation value of the energy and the entropy are respectively given by

$$\langle E \rangle := \sum_s E_s P_s = -\partial_\beta \log Z\,, \tag{1.55}$$

$$S := -k_B \sum_s P_s \log P_s = k_B(\log Z + \beta \langle E \rangle)\,. \tag{1.56}$$

So $Z = e^{-\beta F}$ with $F := \langle E \rangle - TS$ the Helmholtz free energy.

The system of interest may have a set of observables $\{H_i\}$, $[H_i, H_j] = 0$ of interest. In those cases one may compute a generalised version of (1.54)

$$\mathtt{Z}(\beta_1, \beta_2, \dots) = \operatorname{Tr} e^{-\sum_i \beta_i H_i}\,. \tag{1.57}$$

Note that the $\{H_i\}$ need not be conserved for the above definition to hold. An example of this is when the particle number of the system is not fixed. In that case, the appropriate quantity to compute is the *grand partition function*

$$\mathtt{Z}(\beta, z) = \operatorname{Tr} z^N e^{-\beta H}\,, \quad z = e^{\beta\mu} \tag{1.58}$$

where N is the particle number operator (which is not conserved) and μ is the chemical potential conjugate to N. z is called the fugacity. The grand partition function may be written as a formal sum

$$\mathtt{Z}(\beta, z) = \sum_n z^n Z(\beta; n) \tag{1.59}$$

where Z is the canonical partition function (1.54).

We are ultimately interested, not in quantum mechanics or statistical systems but rather, in QFTs. In QFTs it is possible to define a large number of partition functions by considering placing the QFT on various manifolds. The direct analogue of the partition function (1.54) is the Euclidean partition function on $\mathbb{S}^1 \times \mathbb{R}^3$

$$Z(\beta; \mathbb{S}^1 \times \mathbb{R}^3) = \mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H} = \int\limits_{\mathbf{C}(\mathbb{S}^1 \times \mathbb{R}^3)} [\mathcal{D}\phi(\tau, \vec{x})]\, e^{-S[\phi(\tau,\vec{x})]}\,, \tag{1.60}$$

here $\mathbf{C}(\mathbb{S}^1 \times \mathbb{R}^3)$ denotes the space of field configurations on $\mathbb{S}^1 \times \mathbb{R}^3$ over which we choose to integrate. For a theory containing only scalars a sensible choice could be

$$\mathbf{C}(\mathbb{S}^1 \times \mathbb{R}^3) = \{\phi(\tau, \vec{x}) \in C^\infty(\mathbb{S}^1 \times \mathbb{R}^3) | \phi(\tau, \vec{x}) = \phi(\tau + \beta, \vec{x}), \dots\} \tag{1.61}$$

here the dots indicate other conditions which we may like to impose (such as certian integrability conditions, etc). This space is an infinite dimensional Banach space and it is a theorem that such spaces have no analogue of a Lebesgue measure [42, 43]. In particular the $[\mathcal{D}\phi]$ in the above expression must be carefully defined, indeed the measure contains essentially all of the quantum physics of the system, we will come back to discuss this point shortly. Note that in (1.60) β is unrelated to the temperature (the system is at strictly zero temperature) but rather β is the radius of the circle. The trace is taken over the Fock space of the theory. Note that the second equality is only valid when the QFT of interest has a Lagrangian description while the first equality is valid regardless. Here Z is formally divergent but nevertheless can be assigned physical meaning after removing the overall infinite constant.

Another useful quantity is the *generating function of connected correlators* which is defined for a Lagrangian QFT on a manifold $\mathcal{M}$ as

$$W[J_1, J_2, \dots] = \log \int\limits_{\mathbf{C}(\mathcal{M})} [\mathcal{D}\phi]\, e^{-S[\phi] + \sum_i \int_{\mathcal{M}} \mathcal{O}_i \wedge \star J_i} \tag{1.62}$$

here we collectively denote the local operators of the theory by $\mathcal{O}_i = \mathcal{O}_i(x)$ and

$\mathbf{C}(\mathcal{M})$ denotes the space field configurations over which we choose to integrate. J_i is a source for $\mathcal{O}_i$. Connected correlation functions can then be computed using

$$\langle \prod_{j=1}^{n} \mathcal{O}_{i_j}(x_j) \rangle^{\text{conn.}} = \delta_{J_{i_1}} \dots \delta_{J_{i_n}} W[J_1, J_2, \dots] \Big|_{J_j=0} . \tag{1.63}$$

These are related to the usual correlation functions

$$\langle \prod_{j=1}^{n} \mathcal{O}_{i_j}(x_j) \rangle = \frac{\langle\langle \prod_{j=1}^{n} \mathcal{O}_{i_j}(x_j) \rangle\rangle}{\langle\langle 1 \rangle\rangle} , \tag{1.64}$$

$$\langle\langle f(\phi) \rangle\rangle := \int\limits_{\mathbf{C}(\mathcal{M})} [\mathcal{D}\phi] \, e^{-S[\phi]} f(\phi) , \tag{1.65}$$

via,

$$\langle\langle \exp\left(\prod_{i_j} \mathcal{O}_{i_j} z_{i_j} \right) \rangle\rangle = \prod_{n_{i_j}} \exp\left(\langle \prod_{i_j} \mathcal{O}_{i_j}^{n_{i_j}} \rangle^{\text{conn.}} \prod_{i_j} \frac{z_{i_j}^{n_{i_j}}}{n_{i_j}!} \right) . \tag{1.66}$$

For a generic interacting QFT partition functions such as (1.60) & (1.62) are generally very hard to compute exactly and one must resort to evaluating the functional integrals in a perturbative series. Again, in these cases, a rigorous definition of the measure $[\mathcal{D}\phi]$ is lacking in all but a few special examples. In some cases of supersymmetric QFTs it is possible to employ powerful localisation techniques [44, 45] in order to avoid these issues and compute partition functions and correlators of certain operators exactly. The basic idea of supersymmetric localisation goes as follows. Consider a theory with the existence of an anomaly free fermionic symmetry generator $\mathfrak{Q}$ such that the action $S[\phi]$ obeys $\mathfrak{Q}S[\phi] = 0$ and $\mathfrak{Q}$ squares to a collection of bosonic symmetry generators. Depending on the manifold $\mathcal{M}$ upon which the theory lives, it may or may not be possible to find such a symmetry [46]. We then consider a deformation to the action by a $\mathfrak{Q}$ exact term $S[\phi] \to S[\phi] + t\mathfrak{Q}V[\phi]$ where $\mathfrak{Q}^2 V[\phi] = 0$ and $t \in \mathbb{R}$. The partition function is then defined as

$$Z(t) = \langle\langle 1 \rangle\rangle_t , \quad \langle\langle f(\phi) \rangle\rangle_t := \int\limits_{\mathbf{C}(\mathcal{M})} [\mathcal{D}\phi] \, e^{-S[\phi] - t\mathfrak{Q}V[\phi]} f(\phi) . \tag{1.67}$$

The point is that, under reasonable assumptions, $Z(t)$ and more generally the correlation functions of any operators $\{\mathcal{O}_i\}$ obeying $\mathfrak{Q}\mathcal{O}_i = 0$ is actually independent of the parameter t

$$\frac{d}{dt}\langle\langle\prod_{j=1}^{n}\mathcal{O}_{i_j}\rangle\rangle_t=\langle\langle V\mathfrak{Q}\left(\prod_{j=1}^{n}\mathcal{O}_{i_j}\right)\rangle\rangle_t-\langle\langle\mathfrak{Q}\left(V\prod_{j=1}^{n}\mathcal{O}_{i_j}\right)\rangle\rangle_t=0 \tag{1.68}$$

The first term vanishes identically due to $\mathfrak{Q}\mathcal{O}_i=0$ while for the second we assume that the path integral measure can be defined in such a way that it is $\mathfrak{Q}$-invariant. We therefore have that the result is independent of t and so

$$\langle\langle\prod_{j=1}^{n}\mathcal{O}_{i_j}\rangle\rangle_{t=0}=\langle\langle\prod_{j=1}^{n}\mathcal{O}_{i_j}\rangle\rangle_t=\lim_{t\to\infty}\int_{\mathbf{C}(\mathcal{M})}[\mathcal{D}\phi]\,e^{-S[\phi]-t\mathfrak{Q}V[\phi]}\prod_{j=1}^{n}\mathcal{O}_{i_j}\,. \tag{1.69}$$

Writing $\phi=\phi_0+t^{-1/2}\delta\phi$ we can expand the action for large t

$$S[\phi]+t\mathfrak{Q}V[\phi]=S[\phi_0]+\left.\frac{\delta^2\mathfrak{Q}V[\phi]}{\delta\phi^2}\right|_{\phi=\phi_0}(\delta\phi)^2+\mathcal{O}(t^{-1/2})\,. \tag{1.70}$$

Integrating out the fluctuations by performing the Gaussian integrals over $\delta\phi$ we are schematically left with

$$\langle\langle\prod_{j=1}^{n}\mathcal{O}_{i_j}\rangle\rangle_t=\int_{\mathbf{C}_{\mathrm{BPS}}(\mathcal{M})}[\mathcal{D}\phi_0]\,\frac{e^{-S[\phi_0]}}{\mathrm{SDet}\left[\frac{\delta^2\mathfrak{Q}V[\phi_0]}{\delta\phi_0^2}\right]}\prod_{j=1}^{n}\mathcal{O}_{i_j}|_{\phi=\phi_0}\,, \tag{1.71}$$

and the path integral *localises* onto a reduced space of field configurations

$$\mathbf{C}_{\mathrm{BPS}}(\mathcal{M})=\{\phi_0\in\mathbf{C}(\mathcal{M})|\mathfrak{Q}V[\phi_0]=0\}\,. \tag{1.72}$$

In many cases the space $\mathbf{C}_{\mathrm{BPS}}(\mathcal{M})$ is finite dimensional and the functional integral (1.69) reduces to a normal integral over a finite number of variables. Note that this technique bypasses the main two issues we mentioned earlier. Firstly, for the localisation above, the actual definition of the path integral measure was not required, one need only that there exists any definition obeying the desired properties. Moreover, when $\mathbf{C}_{\mathrm{BPS}}(\mathcal{M})$ is finite dimensional, one may simply use the standard Lebesgue measure to perform the remaining integrations. Secondly, because of the reduced space of field configurations, the path integral can sometimes be performed either exactly or numerically allowing us to extract non-pertubative results about the theory.

1.4.1 Nekrasov Partition Function

In [47, 32] Nekrasov was able to give a UV derivation of the results of Seiberg & Witten. Nekrasov was able to successfully perform the localisation procedure for (topolgically twisted) $\mathcal{N}=2$ gauge theories on a deformed version of $\mathbb{R}^4$ called the Ω-background. The Ω-background preserves a $U(1)\times U(1)\subset Spin(4)$ which acts by

$$U(1)\times U(1):(z_1,z_2)\to(e^{i\epsilon_1}z_1,e^{i\epsilon_2}z_2)\,,\qquad \begin{aligned} z_1&=x^1+\mathrm{i}x^2\\ z_2&=x^3+\mathrm{i}x^4\end{aligned}\,. \tag{1.73}$$

The (anti-)topologically twisted theory [48] in this background preserves atleast a single scalar supercharge $\mathcal{Q}$. Nekrasov was able to show that the path integral localises to a set of finite dimensional integrals over (anti-)self-dual instanton moduli spaces. The partition function is then given by

$$\begin{aligned} Z_{\text{Nek}}(a,q;\epsilon_1,\epsilon_2)&=\sum_{k\geqslant 0}q^k\int_{\mathbf{M}_{k,\zeta}}e^{-S^{(k)}_{\text{inst}}(a,\dots)}=\sum_{k\geqslant 0}q^k\int_{\mathbf{M}_{k,\zeta}}1\\ &=Z_{\text{pert.}}(a,q;\epsilon_1,\epsilon_2)\sum_{k\geqslant 0}q^kZ_k(a;\epsilon_1,\epsilon_2)\\ &=Z_{\text{pert.}}(a,q;\epsilon_1,\epsilon_2)Z_{\text{inst.}}(a,q;\epsilon_1,\epsilon_2)\end{aligned} \tag{1.74}$$

here $\mathbf{M}_{k,\zeta}$ is (a smooth version of) the moduli space of k framed instantons, which we have defined and reviewed in Section 1.3. The Z_k's appearing in this quantity are precisely equal to those we presented in (1.42). The Coulomb branch parameter(s) $a=\lim_{x\to\infty}\phi(x)$ where $\phi(x)$ is the scalar field in the $\mathcal{N}=2$ vector multiplet(s). $S^{(k)}_{\text{inst}}$ denotes the action of the theory evaluated on the instanton configuration, i.e. by plugging the k-instanton solution (1.36) into the action S of the theory. For the topologically twisted theory $S^{(k)}_{\text{inst}}$ (and likewise S) turns out to be a $\mathcal{Q}$-exact quantity and is hence cohomologically equivalent to the identity. Finally q is the instanton counting parameter of the theory. We demonstrate various ways to compute the Z_k's, in addition to the example that we will shortly present, for various theories in Chapters 4 and 6 and Appendix I. Nekrasov was able to prove that

$$\mathcal{F}(a,q)=-\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2 Z_{\text{Nek}}(a,q;\epsilon_1,\epsilon_2) \tag{1.75}$$

where $\mathcal{F}$ is the prepotential of the previous section.

Example: Instanton Counting for 5d $\mathcal{N} = 1^*$ Theory

Let us perform an example computation of Nekrasov's computation using the localisation method. We will perform a slightly more general theory from Nekrasov's original computations by considering the 5d gauge theory on $\mathbb{S}^1_\beta \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$. Taking the limit that the circle radius $\beta \to 0$ in the end will return us to the expressions for 4d theories. For simplicity we will consider the 5d $\mathcal{N} = 1^*$ theory with gauge algebra $(\mathfrak{s})\mathfrak{u}(N)$. This is the theory of a $\mathcal{N} = 1$ vector multiplet and hypermultiplet in the adjoint representation of the gauge algebra.

The instanton partition function is computed from equivariant integration over the moduli space $\mathbf{M}_k$ of k $(S)U(N)$ instantons. $\mathbf{M}_k$ carries a torus action $T := T^3_{\epsilon_1,\epsilon_2,m} \times T(U(N)) \circlearrowright \mathbf{M}_k$ where $T(G)$ denotes a maximal torus of G. The instanton moduli space $\mathbf{M}_k$ may be described as an algebraic variety using the ADHM construction, as we discussed in Section 1.3. Let V, W be vector spaces of dimension $\dim_{\mathbb{C}} V = k$ and $\dim_{\mathbb{C}} W = N$. Let us introduce linear maps

$$B_l : V \to V\,, \quad J : W \to V\,, \quad I : V \to W\,. \tag{1.76}$$

for $l = 1, 2, 3, 4$. The ADHM equations are

$$\mu^{(1)}_{\mathbb{C}} := [B_1, B_2] + [B_3^\dagger, B_4^\dagger] + JI\,, \tag{1.77}$$

$$\mu^{(2)}_{\mathbb{C}} := [B_1, B_3] - [B_2^\dagger, B_4^\dagger] \tag{1.78}$$

$$\mu^{(3)}_{\mathbb{C}} := [B_1, B_4] + [B_2^\dagger, B_3^\dagger]\,, \tag{1.79}$$

$$\mu_{\mathbb{R}} := \sum_{l=1}^{4}[B_l, B_l^\dagger] + JJ^\dagger - I^\dagger I\,. \tag{1.80}$$

Note that these are slightly modified with respect to (1.26) due to the additional matter content of the theory. The moduli space is given by

$$\mathbf{M}_k := \left\{ B_l\,,\, J\,,\, I \,\middle|\, \mu^{(i)}_{\mathbb{C}} = \mu_{\mathbb{R}} = 0 \right\} / U(k)\,, \tag{1.81}$$

where the $g \in U(k)$ acts by

$$(B_l, J, I) \mapsto \left(gB_lg^{-1}, gJ, Ig^{-1}\right)\,. \tag{1.82}$$

The torus action $T^3_{\epsilon_1,\epsilon_2,m}$ acts on the ADHM data by

$$(B_l, J, I) \mapsto (e^{\epsilon_l} B_l, J, e^{2\epsilon_+} I)\,, \tag{1.83}$$

where $\epsilon_3 := m - \epsilon_+$, $\epsilon_4 := -m - \epsilon_+$ and $2\epsilon_\pm = \epsilon_1 \pm \epsilon_2$. The fixed points of the torus action are labelled by N-tuples of Young diagrams $\vec{\mu}$ such that $|\vec{\mu}| = k$ (see Appendix A.1.4 for our conventions). Choosing bases

$$W = \mathrm{span}_{\mathbb{C}} \left\{ w_I | I = 1, 2, \dots, N \right\}\,, \tag{1.84}$$

$$V = \mathrm{span}_{\mathbb{C}} \left\{ v_I^{(i,j)} \middle| I = 1, 2, \dots, N, s = (i,j) \in \mu_I \right\}\,. \tag{1.85}$$

The torus action T acts by

$$w_I \mapsto e^{\alpha_I} w_I\,, \quad v_I^{(i,j)} \mapsto e^{(1-i)\epsilon_1 + (1-\epsilon_2)} v_I^{(i,j)}\,. \tag{1.86}$$

Then

$$B_1 v_I^{(i,j)} = v_I^{(i+1,j)}\,, \quad B_2 v_I^{(i,j)} = v_I^{(i,j+1)}\,, \quad J w_I = v_I^{(1,1)}\,, \tag{1.87}$$

$$I = B_3 = B_4 = 0\,. \tag{1.88}$$

The tangent space $T\mathbf{M}_k$ at the fixed point labelled by $\vec{\mu}$ is then most conveniently desribed by the cohomology $\mathrm{Ker}\, d_2 / \mathrm{Im}\, d_1$ of the following complex specified by the data $(B_1, B_2, 0, 0, J, 0)$ [49, 50, 51, 52]

$$\mathrm{Hom}(V,V) \xrightarrow{d_1} \begin{array}{c} \mathrm{Hom}(V,V) \otimes T_{\epsilon_1} \\ \oplus \\ \mathrm{Hom}(V,V) \otimes T_{\epsilon_2} \\ \oplus \\ \mathrm{Hom}(W,V) \\ \oplus \\ \mathrm{Hom}(V,W) \otimes T_{\epsilon_1} \otimes T_{\epsilon_2} \end{array} \xrightarrow{d_2} \mathrm{Hom}(V,V) \otimes T_{\epsilon_1} \otimes T_{\epsilon_2}$$

Here

$$d_1(x) = \begin{pmatrix} [x, B_1] \\ [x, B_2] \\ xJ \\ 0 \end{pmatrix}\,, \quad d_2 \begin{pmatrix} C_1 \\ C_2 \\ P \\ Q \end{pmatrix} = [B_1, C_2] + [C_1, B_2] + JQ\,. \tag{1.89}$$

The character of the tangent space at fixed point $\vec{\mu}$ is given by

$$\begin{aligned}\chi^{\text{Vec}}_{\vec{\mu}} &= \text{Tr}_{\text{Ker}\, d_2/\text{Im}\, d_1}\, e^{\epsilon_1 T_{\epsilon_1}} e^{\epsilon_2 T_{\epsilon_2}} e^{m T_m} \prod_{I=1}^{N} e^{\alpha_I t_I} \\ &= \left(W^* V + e^{2\epsilon_+} V^* W - (1-e^{\epsilon_1})(1-e^{\epsilon_2}) V^* V\right) \sum_{t\in\mathbb{Z}} e^{\frac{2\pi t}{r}} .\end{aligned} \tag{1.90}$$

Here we abused notation an identified the vector spaces with their characters:

$$V = \sum_{I=1}^{N} \sum_{(i,j)\in\mu_I} e^{\alpha_I + (1-i)\epsilon_1 + (1-j)\epsilon_2} , \quad W = \sum_{I=1}^{N} e^{\alpha_I} , \tag{1.91}$$

where the conjugation flips the sign of the exponents. We also added the dressing by momentum factors along the $\mathbb{S}^1$. Using the identity (A.32) it can be shown that

$$\chi^{\text{Vec}}_{\vec{\mu}} = \sum_{t\in\mathbb{Z}} e^{\frac{2\pi t}{r}} \sum_{I,J=1}^{N} \sum_{s\in\mu_J} \left(e^{E_{IJ}(s)} + e^{2\epsilon_+ - E_{IJ}(s)}\right) , \tag{1.92}$$

where,

$$E_{IJ}(s) := \alpha_I - \alpha_J - (\mu^{\text{T}}_{J;j} - i)\epsilon_1 + (\mu_{I;i} - j + 1)\epsilon_2 . \tag{1.93}$$

There is also a contribution from the adjoint hypermultiplet matter which is given by

$$\chi^{\text{Hyp}}_{\vec{\mu}} = -e^{m-\epsilon_+} \chi^{\text{Vec}}_{\vec{\mu}} . \tag{1.94}$$

Writing,

$$\chi_{\vec{\mu}} := \chi^{\text{Vec}}_{\vec{\mu}} + \chi^{\text{Hyp}}_{\vec{\mu}} , \tag{1.95}$$

the contribution of the fixed point $\vec{\mu}$ to the instanton partition is obtained from the character by

$$\chi_{\vec{\mu}} := \sum_i n_i e^{w_i} \quad \rightarrow \quad z_{\vec{\mu}} = \prod_i w_i^{-n_i} . \tag{1.96}$$

Using the Euler infinite product representation for the sine function (A.21) the instanton partition function is given by a weighted sum over all possible N-tuples $\vec{\mu}$

$$Z_{\text{inst}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r, g^2_{\text{YM}}\right) = \sum_{\vec{\mu}} q^{|\vec{\mu}|} z_{\vec{\mu}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r\right) , \tag{1.97}$$

$$z_{\vec{\mu}}\left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r\right) = \prod_{I,J=1}^{N} \prod_{s\in\mu_J} \frac{\sin\frac{r(E_{IJ}(s)+m-\epsilon_+)}{2} \sin\frac{r(E_{IJ}(s)-m-\epsilon_+)}{2}}{\sin\frac{rE_{IJ}(s)}{2} \sin\frac{r(E_{IJ}(s)-2\epsilon_+)}{2}} , \tag{1.98}$$

where $q := e^{-4\pi^2 r/g^2_{\rm YM}} = e^{-2\pi r/\beta}$ and $2\epsilon_\pm = \epsilon_1 \pm \epsilon_2$. Note that, $(m, \epsilon_1, \epsilon_2)$ are periodic in $\frac{2\pi}{r}$ shifts.

1.4.2 $\mathbb{S}^d$ Partition Function

Applying the localisation procedure Pestun was able to compute a large number of exact results for $\mathcal{N} = 2$ gauge theories on the round $\mathcal{M} = \mathbb{S}^4$ [53]. These results we generalised to the squashed four-sphere $\mathbb{S}^4_{\epsilon_1,\epsilon_2}$ in [54] where

$$\mathbb{S}^{2n}_{\epsilon_1,\dots,\epsilon_n} := \{\vec{x} \in \mathbb{R}^{2n+1} | x_1^2 + \sum_{i=1}^{n} \epsilon_i^2 (x_{2i}^2 + x_{2i+1}^2) = 1\} \,, \tag{1.99}$$

is the squashed $d = 2n$-sphere and the round $\mathbb{S}^{2n} = \mathbb{S}^{2n}_{1,\dots,1}$. These results have been generalised to other theories for the cases of $d = 2, 3, 4, 5$ a comprehensive list of references can be found in [10].

Quantities which lend themselves accessible via localisation are the partition function, certain correlation functions and expectation values of certain extended operators, such as Wilson loops. Wilson loops are defined as

$$W = \mathrm{tr}_R \, \mathcal{P} \exp \left[\oint_C (\mathrm{i}\dot{x}^\mu A_\mu + |\dot{x}|\phi) \right] \,, \tag{1.100}$$

where $\mathcal{P}\exp$ denotes the path-ordered exponential. In the Nekrasov form the partition function for a four dimensional $\mathcal{N} = 2$ theory takes the form [55]

$$Z_{\mathbb{S}^4} = \int [da] \, |Z_{\text{Nek.}}(a, q; \epsilon_1, \epsilon_2)|^2 \,. \tag{1.101}$$

It is comprised of two copies of the Nekrasov partition function (1.74), this corresponds to the fact that the $\mathbb{S}^4_{\epsilon_1,\epsilon_2}$ partition function localises onto self-dual instanton configurations at the north pole and anti-self-dual configurations at the south pole. The integration over a is the integration over the scalar zero modes of the vector muliplet scalars ϕ. The Ω-deformation parameters $\epsilon_{1,2}$ play the role of the squashing parameters of the $\mathbb{S}^4_{\epsilon_1,\epsilon_2}$.

	r	ϵ_1	ϵ_2	m
$Z^{(1)}_{\text{Nek}}$	$\frac{2\pi}{\omega_1}$	ω_2	ω_3	$\mu+\frac{3}{2}\omega_1$
$Z^{(2)}_{\text{Nek}}$	$\frac{2\pi}{\omega_2}$	ω_3	ω_1	$\mu+\frac{3}{2}\omega_2$
$Z^{(3)}_{\text{Nek}}$	$\frac{2\pi}{\omega_3}$	ω_1	ω_2	$\mu+\frac{3}{2}\omega_3$

Table 1.1: *Arguments for the Nekrasov partition functions associated to the three localisation circles.*

Example: $\mathcal{N}=1^*$ $\mathbb{S}^5$ Partition Function

The $\mathbb{S}^5$ partition function of $\mathcal{N}=1^*$ with gauge algebra $\mathfrak{g}=\mathfrak{u}(N)$ can be expressed as [56, 57, 58, 59, 60, 61]

$$Z_{\mathbb{S}^5}=\int[d\alpha]e^{\frac{2\pi^2(\vec{\alpha},\vec{\alpha})}{\beta\omega_1\omega_2\omega_3}}\prod_{i=1}^{3}Z^{(i)}_{\text{Nek}}\left(\vec{\alpha},\mu+\frac{3\omega_i}{2},\omega_{i+1},\omega_{i+2},\frac{2\pi}{\omega_i},2\pi\beta\right)\,, \tag{1.102}$$

where the index i is taken modulo 3 and $(\cdot,\cdot)$ denotes the standard metric on $\mathfrak{t}^*$ where $\mathfrak{t}\subset\mathfrak{g}$ is a Cartan subalgebra. The domain of integration is $\mathrm{i}\mathbb{R}^N$ and the integration measure is given by

$$[d\alpha]=\frac{\mathrm{i}^N}{N!}\prod_{I=1}^{N}d\alpha_I\,. \tag{1.103}$$

The partition function is expressed as three copies of the K-theoretic Nekrasov partition function $Z^{(i)}_{\text{Nek}}(\vec{\alpha},m,\epsilon_1,\epsilon_2,r,g^2_{\text{YM}})$ on $\mathbb{S}^1_r\times\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ where the radius is r. The Nekrasov partition function includes both the perturbative and instanton contributions

$$\begin{aligned}Z^{(i)}_{\text{Nek}}(\vec{\alpha},m,\epsilon_1,\epsilon_2,r,g^2_{\text{YM}})=&Z^{(i)}_{\text{pert}}(\vec{\alpha},m,\epsilon_1,\epsilon_2,r)\\&\times Z^{(i)}_{\text{inst}}(\vec{\alpha},m,\epsilon_1,\epsilon_2,r,g^2_{\text{YM}}).\end{aligned} \tag{1.104}$$

Each of the three factors corresponds to the fact that the Localisation locus $\mathcal{L}=\coprod_{i=1}^{3}\mathbb{S}^1_{2\pi/\omega_i}$ on $\mathbb{S}^5$ factorises into three fixed circles of the $U(1)^3$ action (1.120). We collect all of the perturbative factors $Z_{\text{pert}}=\prod_{i=1}^{3}Z^{(i)}_{\text{pert}}$. The perturbative piece factorise into a contribution from the vector multiplet and adjoint hypermultiplet

$$Z_{\text{pert}}\left(\vec{\alpha},\omega_1,\omega_2,\omega_3,\mu\right)=Z_{\text{vec}}\left(\vec{\alpha},\omega_1,\omega_2,\omega_3\right)Z_{\text{hyp}}\left(\vec{\alpha},\omega_1,\omega_2,\omega_3,\mu\right)\,. \tag{1.105}$$

They are given by [56, 57, 60]

$$Z_{\text{vec}}\left(\vec{\alpha}, \omega_1, \omega_2, \omega_3, \mu\right) = \prod_{I,J=1}^{N} S_3'\left(\alpha_I - \alpha_J|\vec{\omega}\right) \,, \tag{1.106}$$

$$Z_{\text{hyp}}\left(\vec{\alpha}, \omega_1, \omega_2, \omega_3, \mu\right) = \prod_{I,J=1}^{N} \frac{1}{S_3\left(\alpha_I - \alpha_J + \mu + \frac{3}{2}|\vec{\omega}\right)} \,, \tag{1.107}$$

where $S_3(z|\vec{\omega})$ is the triple sine function defined in (A.25). The prime indicates that when $I = J$ the $n_1 = n_2 = n_3 = 0$ term in the infinite product product of the triple sine function should be removed. The instanton contribution is given in (1.97).

1.4.3 $\mathbb{S}^1 \times \mathcal{M}_{d-1}$ Partition Function

The localisation procedure can also be applied when the manifold is of the form $\mathcal{M} = \mathbb{S}^1 \times \mathcal{M}_{d-1}$. The partition function for a supersymmetric theory with a global group symmetry G (which can be also be conformal & spacetime symmetries, R-symmetries and/or other global or discrete symmetries) with algebra $\mathfrak{g}$ can be expressed as a trace [62, 63, 64, 65]

$$Z_{\mathbb{S}^1\times\mathcal{M}_{d-1}} = e^{-\beta E_{\text{Casimir}}}\mathcal{I}_{\mathcal{M}_{d-1}} \,, \quad \mathcal{I}_{\mathcal{M}_{d-1}} := \text{Tr}_{\mathcal{M}_{d-1}}(-1)^F e^{-\beta H} \prod_{i=1}^{\text{rank}\, G} z_i^{h_i} \,. \tag{1.108}$$

Here β can be identified with the radius of the $\mathbb{S}^1$ and E_{Casimir} is the Casimir energy of the theory. The trace is taken over the Hilbert space of the theory on $\mathcal{M}_{d-1}$, F is the fermion number operator and $H = \{\mathfrak{Q}, \mathfrak{Q}^\dagger\}$ where $\mathfrak{Q}$ is a distinguished supersymmetry generator of the theory. Finally, in analogy with (1.54)-(1.58), the z_i are called fugacities and parametrise the maximal torus $(z_1, \ldots, z_{\text{rank}\, G}) \in T(G)$ and $h_i \in \mathfrak{h} \subset \mathfrak{g}$ are elements of the Cartan subalgebra of $\mathfrak{g}$ and must also commute $[\mathfrak{Q}, h_i] = 0$. In this way the index $\mathcal{I}_{\mathcal{M}_{d-1}}$ can be thought of as a mapping from a supersymmetric QFT onto the group G. When the theory has a Lagrangian description one can equivalently write (1.107) as a path integral with twisted boundary conditions around the $\mathbb{S}^1$.

$\mathcal{I}_{\mathcal{M}_{d-1}}$ counts (with signs) cohomology classes of the distinguished supercharge $\mathfrak{Q}$. For unitary theories $H \geqslant 0$, additionally it is not too difficult to show that states with $H \neq 0$ always come in pairs with opposite values for $(-1)^F$ and they cancel in the trace. Therefore (1.107) receives non-zero contribution only from those states with $H = 0$ and $\mathcal{I}_{\mathcal{M}_{d-1}}$ is therefore formally independent of β. It is a general result due to [65, 48] that partition functions on such manifolds are independent of continuous deformations of the theory.[3] In particular this implies that $\mathcal{I}_{\mathcal{M}_{d-1}}$ is independent of the energy scale at which the theory is defined. This fact often makes these type of partition functions particularly simple to compute since, if the theory of interest has a region of parameter space in which the theory becomes free, it is possible to treat the computation of $\mathcal{I}_{\mathcal{M}_{d-1}}$ simply as a free field combinatorics problem.

Of particular importance is the case $\mathcal{M}_{d-1} = \mathbb{S}^{d-1}$. In this case $\mathcal{I} := \mathcal{I}_{\mathbb{S}^{d-1}}$ is called the *supersymmetric index* (or *superconformal index*) of the theory. This was first defined and computed in [62, 64] and in-depth reviews of the current state-of-the-art can be found in [5, 66].

Since $\mathcal{I}$ is independent of the energy scale and also because $\mathbb{S}^1 \times \mathbb{S}^{d-1}$ is conformally equivalent to $\mathbb{R}^d$ the quantity $\mathcal{I}$ has the interpretation of counting operators in the CFT (or family of CFT's if there is a conformal manifold) obtained at the conformal fixed point of the theory. More formally, the Hilbert space of the theory on $\mathbb{S}^1 \times \mathbb{S}^3$ carries a unitary representation of the superconformal algebra

$$\mathcal{H}(\tau) \cong \bigoplus_i n_i(\tau) \mathcal{R}_i \tag{1.109}$$

where $\mathcal{R}_i$ are representations of the superconformal algebra. Note that this Hilbert space depends on the parameters τ of the theory. Among the representations $\mathcal{R}_i$ some are generic, a.k.a. long representations $\mathcal{A}$ while others are non-generic, a.k.a. short representations $\mathcal{S}$ and they contain non-generic states which saturate $H \geqslant 0$. By construction, $\mathcal{I}$ counts states with $H = 0$, they must therefore belong to short representations of the corresponding conformal algebra. In a CFT, the precise spectrum of short representations can change upon deformation of the theory. When the unitary bound is saturated a long multiplet $\mathcal{A}$ decomposes

$$\mathcal{A} \cong \bigoplus_i \mathcal{S}_i \,. \tag{1.110}$$

By construction $\mathcal{I}$ counts the short representations modulo those that can combine into long representations of the above CFT and the space of those representations is independent of the continuous parameters of the theory.

[3] By this we mean, for example, mass parameters $m \to m + \delta m$, coupling constants $\tau \to \tau + \delta\tau$, etc. I.e. those associated to conserved currents ($T_{\mu\nu}$, j^μ, etc). In the case of discrete parameters the above arguments can be bypassed since they can not be associated to conserved currents.

The index $\mathcal{I}$ then has a formal supercharacter expansion

$$\mathcal{I}(z_i) = \chi_{\mathcal{H}(\tau)}(z_i) = \sum_i n_i(\tau)\chi_{\mathcal{R}_i}(z_i) = \sum_{\{\mathcal{S}\}\text{ mod recomb.}} \chi_{\mathcal{S}}(z_i) \tag{1.111}$$

where $\chi_{\mathcal{R}}(z_i)$ denotes a certain supercharacter of the representation $\mathcal{R}$.

In the following we present two example computations. We provide many other explicit computations of $\mathcal{I}$ for various theories in Chapters 2, 4, 5 and 6.

Example: Index of 4d $\mathcal{N}=1$ Gauge Theories

To demonstrate let us compute the right-handed superconformal index computed with respect to $\widetilde{\mathcal{Q}}_{\dot{-}}$ of a 4d $\mathcal{N}=1$ gauge theory with matter with gauge group U and global flavour symmetry group Y. This contains a vector multiplet in the adjoint of U and trivial representation of G and a collection of chiral multiplets Φ_i in representations $\mathcal{P}_i$ of U and $\mathcal{R}_i$ of Y. This index can be written as

$$\mathcal{I}(p,q,\mathbf{y}) = \mathrm{Tr}(-1)^F p^{j_1+j_2+\frac{r}{2}} q^{-j_1+j_2+\frac{r}{2}} e^{-\beta\widetilde{\delta}_{\dot{-}}} \prod_{n=1}^{\mathrm{rank}\, Y} y_n^{h_n} \,. \tag{1.112}$$

Here j_1, j_2 are Cartans for $\mathfrak{so}(4) \cong \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2$, r is the generator for the $\mathcal{N}=1$ $\mathfrak{u}(1)_r$ R-symmetry. F is the fermion number operators and, by the spin-statistics theorem, can be taken to be $F = 2j_1 + 2j_2$. The superconformal index (1.111) receives contributions only from those states satisfying

$$\widetilde{\delta}_{\dot{-}} = 2\{\widetilde{\mathcal{Q}}_{\dot{-}}, \widetilde{\mathcal{S}}^{\dot{-}}\} = E - 2j_2 - \frac{3}{2}r = 0\,, \tag{1.113}$$

where E is the conformal energy generator. The single letter contributions of $\mathcal{N}=1$ chiral multiplets and vector multiplets may be computed by enumerating all letters with $\widetilde{\delta}_{\dot{-}} = 0$. These are listed in Table 1.2. The single letter contribution of a chiral multiplet $\Phi = \{\phi, \psi, F\}$ and its conjugate $\overline{\Phi} = \{\overline{\phi}, \overline{\psi}, \overline{F}\}$ is given by

$$i_\Phi(p,q) = \frac{(pq)^{\frac{r_\Phi}{2}}\chi_{\mathcal{P}_i}(\mathbf{u})\chi_{\mathcal{R}_i}(\mathbf{y}) - (pq)^{\frac{2-r_\Phi}{2}}\chi_{\overline{\mathcal{P}_i}}(\mathbf{u})\chi_{\overline{\mathcal{R}_i}}(\mathbf{y})}{(1-p)(1-q)}\,, \tag{1.114}$$

	$E_{\rm UV}$	j_1	j_2	$r_{\rm UV}$	$r_{\rm IR}$	Index
ϕ	1	0	0	$\frac{2}{3}$	r_Φ	$(pq)^{r_\Phi/2}$
$\overline{\psi}_{\dot{+}}$	3/2	0	$+\frac{1}{2}$	$\frac{1}{3}$	$-r_\Phi+1$	$-(pq)^{1-r_\Phi/2}$
$\widetilde{F}_{\dot{+}\dot{+}}$	2	0	1	0	0	pq
$\lambda_\pm$	3/2	$\pm\frac{1}{2}$	0	1	1	$-p,\ -q$
$\partial\lambda=0$	5/2	0	$+\frac{1}{2}$	1	1	pq
$\partial_{\pm\dot{+}}$	1	$\pm\frac{1}{2}$	$+\frac{1}{2}$	0	0	$p,\ q$

Table 1.2: *Letters with $\delta=0$ of a free chiral multiplet Φ, its conjugate $\overline{\Phi}$ and free vector multiplet V. For the vector multiplet we must take into account the equation of motion $\partial\lambda = \partial_{+\dot{+}}\lambda_- + \partial_{-\dot{+}}\lambda_+ = 0$.*

here, $r_\Phi := r_{\rm IR}[\Phi]$. Similarly, the single letter contribution for a vector multiplet $V = \{\lambda_\alpha, \overline{\lambda}_{\dot\alpha}, F_{\alpha\beta}, \widetilde{F}_{\dot\alpha\dot\beta}, D, \overline{D}\}$ is given by

$$i_V(p,q) = -\left(\frac{p}{1-p}+\frac{q}{1-q}\right)\chi_{\rm adj.}(\mathbf{u})\,. \tag{1.115}$$

The vector multiplet also comes with an integral over the gauge group (A.61). The index can then be expressed as a matrix integral

$$\mathcal{I}(p,q,\mathbf{y}) = \oint d\mu_U(\mathbf{u})\,\mathrm{PE}\left[i_V(p,q)+\sum_i i_{\Phi_i}(p,q)\right]\,, \tag{1.116}$$

here $d\mu_U(\mathbf{u})$ denotes the Haar measure of the group U and PE denotes the *Plethystic Exponential*, defined in (A.1).

For concreteness, let us write down the index of $\mathcal{N}=4$ SYM based on gauge group $U=SU(N)$. Considered as an $\mathcal{N}=1$ theory this is the theory of a $\mathcal{N}=1$ Vector multiplet and three chiral multiplets in the $(\Phi,Q,\widetilde{Q})$ in the $\mathbf{3}$ of the $Y=SU(3)\subset SU(4)$ flavour symmetry, the adjoint of U and which all have $r_{\rm IR}=r_{\rm UV}=2/3$. The index is therefore explicitly given by

$$\begin{aligned}
&\mathcal{I}(p,q,\mathbf{y}) = \\
&\oint d\mu_{SU(N)}(\mathbf{u})\,\mathrm{PE}\left[\frac{-p-q+2pq+(pq)^{\frac{1}{3}}\chi_{\mathbf{3}}(\mathbf{y})-(pq)^{\frac{2}{3}}\chi_{\overline{\mathbf{3}}}(\mathbf{y})}{(1-p)(1-q)}\chi_{\rm adj.}(\mathbf{u})\right] \\
&= \frac{\kappa}{N!}\oint\limits_{|u_A|=1}\prod_{A=1}^{N-1}\frac{du_A}{2\pi\mathrm{i}u_A}\frac{\prod_{A,B=1}^{N}\prod_{b=1}^{3}\Gamma_e\left((pq)^{\frac{1}{3}}y_b\frac{u_A}{u_B}\right)}{\prod_{A\neq B}\Gamma_e\left(\frac{u_A}{u_B}\right)}
\end{aligned} \tag{1.117}$$

The measure $d\mu_{SU(N)}$ is defined in (A.61), the characters in (A.59), $\kappa = (p;p)^{N-1}(q;q)^{N-1}$ and Γ_e denotes the Elliptic Gamma function (A.22). The fugacities obey $\prod_{A=1}^{N} u_A = \prod_{b=1}^{3} y_b = 1$.

Example: Index of the 6d $\mathcal{N} = (2,0)$ Theory

Let us apply the above disucssion to theories with $\mathcal{N} = (2,0)$ supersymmetry in six dimensions. This will demonstrate the remarkable power of the supersymmetric index, because the $\mathcal{N} = (2,0)$ theories have no known Lagrangian description, nevertheless we will still be able to define the index and even compute it explicitly in certain limits.

The 6d $\mathcal{N} = (2,0)$ theories are strongly coupled, isolated SCFTs and (at the level of the local operator spectrum) are in one to one correspondence with the finite subgroups of $SU(2)$ [67, 68, 69, 70, 71, 72, 71, 73, 74]. It is therefore common to label them of type $\mathfrak{g} = ADE$. We will focus on the theory of type $\mathfrak{g} = A_{N-1}$. This theory can be realised as the worldvolume theory on a stack of N parallel and coincident M5-branes.

The $\mathcal{N} = (2,0)$ superconformal algebra is $\mathfrak{osp}(8|4)$. Representations of $\mathfrak{osp}(8|4)$ are labelled by the Cartans of the maximal bosonic subalgebra

$$\mathfrak{so}(1,7) \oplus \mathfrak{usp}(4) \supset \mathfrak{u}(1)_E \oplus \mathfrak{so}(6) \oplus \mathfrak{usp}(4)\,. \tag{1.118}$$

The Cartans are $(E, h_1, h_2, h_3, (R_1 + R_2)/2, (R_1 - R_2)/2)$, where E corresponds to dilatations, (h_1, h_2, h_3) to 2-plane rotations in $\mathbb{R}^6$ and R_1, R_2 to $\mathfrak{so}(5) \cong \mathfrak{usp}(4)$ Cartans. The theory has 16 Poincaré supercharges $\mathsf{Q}^{R_1 R_2}_{h_1 h_2 h_3}$ with $-8h_1h_2h_3 = 2E = 1$ and 16 conformal supercharges with $-8h_1h_2h_3 = 2E = -1$.

We will define the superconformal index with respect to the supercharge $\mathsf{Q} := \mathsf{Q}^{++}_{---}$ and its conjugate $\mathsf{Q}^\dagger = \mathsf{Q}^{--}_{+++}$ which have $R_1 + R_2 = 2E = -2h_i = 1$ and $R_1 + R_2 = 2E = -2h_i = -1$ respectively. The superconformal index is then defined to be [63, 56, 62, 57, 64, 75]

$$\mathcal{I}\left(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}\right) = \mathrm{Tr}_{\mathbb{S}^5}(-1)^F e^{-\beta\delta} \mathfrak{q}_1^{h_1 + \frac{R_1+R_2}{2}} \mathfrak{q}_2^{h_2 + \frac{R_1+R_2}{2}} \mathfrak{q}_3^{h_3 + \frac{R_1+R_2}{2}} \mathfrak{p}^{R_2 - R_1} \tag{1.119}$$

The trace is taken over the Hilbert space on $\mathbb{S}^5$. We also defined

$$\mathfrak{q}_i := e^{-\beta(a_i+1)} = e^{-\beta\omega_i}\,, \quad \mathfrak{p} := e^{-\beta\mu/2}\,. \tag{1.120}$$

Here the $a_i = \omega_i - 1$ are related to $U(1)^3 \circlearrowright \mathbb{C}^3$ corresponding to

$$(z_1, z_2, z_3) \mapsto (e^{\mathrm{i}a_1} z_1, e^{\mathrm{i}a_2} z_2, e^{\mathrm{i}a_3} z_3)\,. \tag{1.121}$$

The index (1.118) receives contribution only from states satisfying

$$\delta := 2\left\{\mathsf{Q}, \mathsf{Q}^\dagger\right\} = E - h_1 - h_2 - h_3 - 2R_1 - 2R_2 = 0\,. \tag{1.122}$$

Since (1.118) receives contribution only from states with $\delta = 0$ it is; considered as a formal power series in the $\mathfrak{q}_i$ & $\mathfrak{p}$, independent of β.

Let us now discuss a few useful limits of the $\mathcal{N} = (2,0)$ supersymmetric index. Firstly, perhaps the most simple limit of the index, is called the $\frac{1}{2}$-*BPS limit of the index.* It is given by taking

$$\mathfrak{q}_i \to 0,\, \mathfrak{p} \to 0 \text{ with } \mathsf{x} = \frac{\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}}{\mathfrak{p}} \text{ held fixed}\,. \tag{1.123}$$

In this limit the index becomes

$$\mathcal{I}^{\frac{1}{2}\text{-BPS}}(\mathsf{x}) := \lim_{\mathfrak{q}_i \to 0} \mathcal{I}(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \frac{\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}}{\mathsf{x}}) = \mathrm{Tr}'(-1)^F \mathsf{x}^{E-R_1}\,. \tag{1.124}$$

here Tr' denotes the restriction of $\mathrm{Tr}_{\mathbb{S}^5}$ to states with $h_{i=1,2,3} + R_2 = 0$. $\mathcal{I}^{\frac{1}{2}\text{-BPS}}$ then receives extra superconformal shortening from the supercharges with $\delta = E - R_1 = 0$. All in all $\mathcal{I}^{\frac{1}{2}\text{-BPS}}$ counts states annihilated by eight supercharges

$$\mathsf{Q}\,, \quad \mathsf{Q}^\dagger\,, \quad \mathsf{Q}^{+-}_{\substack{-++\\+-+\\++-}}\,, \quad \mathsf{Q}^{-+}_{\substack{-+-\\--+\\+--}}\,. \tag{1.125}$$

One therefore expects $\mathcal{I}^{\frac{1}{2}\text{-BPS}}$ to be a $\frac{1}{4}$-BPS object, however as we will shortly demonstrate it is infact a $\frac{1}{2}$-BPS object.

A slightly less restrictive limit is the so-called chiral algebra limit. This is obtained by considering the limit

$$\mathfrak{p} \to \sqrt{\frac{\mathfrak{q}_1\mathfrak{q}_2}{\mathfrak{q}_3}}\,. \tag{1.126}$$

The index becomes

$$\mathcal{I}^S(\mathfrak{q}_3, \mathfrak{s}) = \mathcal{I}\left(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \sqrt{\frac{\mathfrak{q}_1\mathfrak{q}_2}{\mathfrak{q}_3}}\right) = \mathrm{Tr}_{\mathbb{S}^5}(-1)^F \mathfrak{q}_3^{E-R_1} \mathfrak{s}^{h_1+R_2}\,, \tag{1.127}$$

where we have defined $\mathfrak{s} = \mathfrak{q}_1/\mathfrak{q}_2$. In this limit there is enhanced supersymmetry by a second supercharge and $\mathcal{I}^S$ counts states annihilated by

$$\mathsf{Q}\,, \quad \mathsf{Q}^\dagger\,, \quad \mathsf{Q}^{+-}_{++-}\,, \quad \mathsf{Q}^{-+}_{--+}\,. \tag{1.128}$$

The index, in this limit, can be interpreted as the vacuum character of the associated chiral algebra on the h_3 two-plane [76]. Indeed, using the existence of the chiral algebra correspondence the authors of [76] were able to provide a conjecture for this limit of the index

$$\mathcal{I}^S(\mathfrak{q}_3, \mathfrak{s}) = \mathrm{PE}\left[\sum_{n \in \mathrm{exponents}(\mathfrak{g})} \frac{\mathfrak{q}_3^{n+1}}{1-\mathfrak{q}_3}\right]\,, \tag{1.129}$$

where exponents($\mathfrak{g}$) denotes the set of exponents for the Lie algebra $\mathfrak{g}$, for example exponents$(A_{N-1}) = \{1, 2, \ldots, N-1\}$; see also Table 5.2. Having defined the index for the $\mathcal{N} = (2,0)$ theory we would now like to compute it. However, because very little is understood about these theories; precisely because there is no Lagrangian description of them, we are forced to either restrict ourselves to to special cases, limits as well as 'indirect' computation.

Let us begin with the most simple case, which is the free theory.

The Index of the $\mathfrak{g} = \mathfrak{u}(1)$ Free Tensor Multiplet Since this theory is free we may use letter counting. The free $(2,0)$ tensor multiplet contains a scalar ϕ in the fundamental representation of $\mathfrak{so}(5)$, sixteen chiral fermions $\lambda^{R_1R_2}_{h_1h_2h_3}$ with $8h_1h_2h_3 = -1$ and a self-dual threeform H. The free tensor multiplet index can then be written down as

$$\mathcal{I}(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}) = \mathrm{PE}\left[i(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p})\right]\,. \tag{1.130}$$

The plethysic exponent i is the single letter index and is computed by enumerating the single letters, these are listed in Table 1.3. The single letter index is given by

$$i(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p}) = \frac{\left(\mathfrak{p} + \mathfrak{p}^{-1}\right)\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3} - \mathfrak{q}_1\mathfrak{q}_2 - \mathfrak{q}_1\mathfrak{q}_3 - \mathfrak{q}_2\mathfrak{q}_3 + \mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}{(1-\mathfrak{q}_1)(1-\mathfrak{q}_2)(1-\mathfrak{q}_3)}\,. \tag{1.131}$$

Let us write down the $\frac{1}{2}$-BPS index (1.122) for this theory. The single letter index is $i \to \mathsf{x}$ this is simply counting the top $\mathfrak{so}(5)$ component $\phi = \phi^1$ with $(R_1, R_2) = (1, 0)$.

Letters	E	h_1	h_2	h_3	R_1	R_2	$i(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \mathfrak{p})$
ϕ^1	2	0	0	0	1	0	$\mathfrak{p}^{-1}\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}$
ϕ^3	2	0	0	0	0	1	$\mathfrak{p}\sqrt{\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3}$
λ^{++}_{++-}	5/2	1/2	1/2	$-1/2$	1/2	1/2	$-\mathfrak{q}_1\mathfrak{q}_2$
λ^{++}_{+-+}	5/2	1/2	$-1/2$	1/2	1/2	1/2	$-\mathfrak{q}_1\mathfrak{q}_3$
λ^{++}_{-++}	5/2	$-1/2$	1/2	1/2	1/2	1/2	$-\mathfrak{q}_2\mathfrak{q}_3$
$\partial\lambda = 0$	7/2	1/2	1/2	1/2	1/2	1/2	$\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3$
$\partial_{z_{i=1,2,3}}$	1	$1,0,0$	$0,1,0$	$0,0,1$	0	0	$\mathfrak{q}_1,\ \mathfrak{q}_2,\ \mathfrak{q}_3$

Table 1.3: *Letters with $\delta = 0$ of the abelian $\mathcal{N} = (2,0)$ free tensor multiplet.*

ϕ constitutes the entire $\frac{1}{2}$-BPS ring for the $\mathfrak{u}(1)$ $(2,0)$ theory. The index becomes

$$\mathcal{I}^{\frac{1}{2}\text{-BPS}} = \mathrm{PE}[\mathsf{x}] = \frac{1}{1-\mathsf{x}}\,. \tag{1.132}$$

The $\frac{1}{2}$-BPS ring for the $\mathfrak{g} = \mathfrak{u}(N)$ theory (this is the same as the $\mathfrak{g} = A_{N-1}$ theory only with an additional decoupled free $\mathfrak{u}(1)$ tensor multiplet) may be obtained as the N^{th} symmetric product of the $N = 1$ case. Therefore, the $\frac{1}{2}$-BPS index can be obtained as

$$\begin{aligned}\mathcal{I}^{\frac{1}{2}\text{-BPS}} &= \frac{1}{N!}\frac{\partial^N}{\partial v^N}\,\mathrm{PE}\left[\frac{v}{1-\mathsf{x}}\right]\Bigg|_{v=0}\\ &= \frac{1}{N!}\frac{\partial^N}{\partial v^N}\sum_{n=0}^{\infty}\mathrm{PE}\left[\sum_{i=1}^{n}\mathsf{x}^i\right]v^n\Bigg|_{v=0} = \mathrm{PE}\left[\sum_{i=1}^{N}\mathsf{x}^i\right] = (\mathsf{x};\mathsf{x})_N^{-1}\,.\end{aligned} \tag{1.133}$$

Indeed, the $\frac{1}{2}$-BPS ring is simply given by $\mathbb{C}[\phi]^{\mathfrak{g}} = \mathbb{C}[\operatorname{tr}\phi, \operatorname{tr}\phi^2, \dots, \operatorname{tr}\phi^N]$. We can also discuss the chiral algebra limit for the $\mathfrak{u}(1)$ theory. It is

$$\mathcal{I}^{S} = \mathrm{PE}\left[i(\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3, \sqrt{\mathfrak{q}_1\mathfrak{q}_2/\mathfrak{q}_3})\right] = \mathrm{PE}\left[\frac{\mathfrak{q}_3}{1-\mathfrak{q}_3}\right] = \frac{1}{(\mathfrak{q}_3;\mathfrak{q}_3)}\,. \tag{1.134}$$

This index is simply counting the letters $\partial^n_{z_3}\phi$, i.e. ϕ and it's h_3-plane descendants.

Large N Limit To compute the index in the large N limit we can appeal to the AdS/CFT correspondence [77]. The $\mathfrak{g} = A_N$ $(2,0)$ theory is conjectured to be dual to M-theory on $AdS_7 \times \mathbb{S}^4$ with N units of four-form flux on $\mathbb{S}^4$. In the strict large N limit the of the dual theory is a gas of free supergravitions in the 11d SUGRA on $AdS_7 \times \mathbb{S}^4$ [78, 79, 77, 80]. The protected spectrum of the A_∞ $(2,0)$ theory is therefore dual to the former. The full supergravition spectrum is listed in [63] in

	E	$\mathfrak{so}(6)$	$\mathfrak{so}(5)$	$(-1)^F$
$p \geqslant 1$	$2p$	$(0,0,0)$	$(p,0)$	$+$
$p \geqslant 1$	$2p+\frac{1}{2}$	$(\frac{1}{2},\frac{1}{2},\frac{1}{2})$	$(\frac{2p-1}{2},\frac{1}{2})$	$+$
$p \geqslant 2$	$2p+1$	$(1,0,0)$	$(p-1,1)$	$+$
$p \geqslant 3$	$2p+\frac{3}{2}$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\frac{2p-3}{2},\frac{3}{2})$	$+$
$p = 1$	$\frac{7}{2}$	$(\frac{1}{2},\frac{1}{2},-\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	$-$

Table 1.4: *$AdS_6 \times \mathbb{S}^4$ supergravition spectrum that contain $\delta = 0$ states.*

Table 1.4 we list only those containing $\delta = 0$ states. The primaries of the spectrum are labelled by $E = R_1 = 2p$, $R_2 = h_1 = h_2 = h_3 = 0$ for p a positive integer. In the notation of [1] these are $D_1[0,0,0]_{2p}^{(\frac{p}{2},\frac{p}{2})}$ multiplets. The single particle index is then simply a formal sum of supercharacters for said representations

$$
\begin{aligned}
i_{\text{s.p.}}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p}) &= \sum_{p=1}^{\infty} \chi_{D_1[0,0,0]_{2p}^{(\frac{p}{2},\frac{p}{2})}}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p}) \\
&= \frac{\mathfrak{q}\chi_1(\mathfrak{p}) - \mathfrak{q}^2\sum_{i=1}^{3}(\mathfrak{q}_i^{-1} - \mathfrak{q}_i) - \chi_1(\mathfrak{p})\mathfrak{q}^3}{(1-\mathfrak{p}\mathfrak{q})(1-\mathfrak{p}^{-1}\mathfrak{q})(1-\mathfrak{q}_1)(1-\mathfrak{q}_2)(1-\mathfrak{q}_3)}
\end{aligned}
\tag{1.135}
$$

where $\mathfrak{q} := (\mathfrak{q}_1\mathfrak{q}_2\mathfrak{q}_3)^{1/2}$. The full index is then given by

$$
\mathcal{I}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p}) = \mathrm{PE}[i_{\text{s.p.}}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p})]\,. \tag{1.136}
$$

Indeed, if one considers the $\frac{1}{2}$-BPS limit we simply have that $i_{\text{s.p.}} \to \frac{\mathsf{x}}{1-\mathsf{x}} = \sum_{i=1}^{N=\infty}\mathsf{x}^i$, in agreement with (1.132). Moreover, in the chiral algebra limit (1.125) we have $i_{\text{s.p.}} \to \mathfrak{q}_3/(1-\mathfrak{q}_3)^2$.

Casimir Energy As we wrote in equation (1.107) the superconformal index (1.118) is expected to be equal to the $\mathbb{S}^1_\beta \times \mathbb{S}^5$ partition function $Z_{\mathbb{S}^1_\beta\times\mathbb{S}^5}$ up to an overall Casimir energy factor [81, 82, 60, 56], namely,

$$
Z_{\mathbb{S}^1_\beta\times\mathbb{S}^5}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p}) = e^{-\beta E(\mathfrak{g})}\mathcal{I}(\mathfrak{q}_1,\mathfrak{q}_2,\mathfrak{q}_3,\mathfrak{p})\,. \tag{1.137}
$$

It is conjectured that the Casimir energy $E(\mathfrak{g})$ is equal to an equivariant integral of the anomaly polynomial of the $\mathcal{N} = (2,0)$ theory of type $\mathfrak{g}$ [82]

$$
E(\mathfrak{g}) = -\int A_8(\mathfrak{g})\,. \tag{1.138}
$$

For $\mathfrak{g} = \mathfrak{u}(1)$ (we use this notation to denote the free theory on a single M5-brane) and $\mathfrak{g} = ADE$ the anomaly polynomial is respectively given by [83, 84, 85]

$$A_8(\mathfrak{u}(1)) = \frac{1}{48}\left[p_2(N\mathcal{M}) - p_2(T\mathcal{M}) + \frac{1}{4}(p_1(N\mathcal{M}) - p_1(T\mathcal{M}))^2\right], \tag{1.139}$$

$$A_8(\mathfrak{g}) = \operatorname{rank}\mathfrak{g} A_8(\mathfrak{u}(1)) + \dim\mathfrak{g} h^\vee(\mathfrak{g})\frac{p_2(N\mathcal{M})}{24}. \tag{1.140}$$

Here $N\mathcal{M}$ and $T\mathcal{M}$ denote the normal and tangent bundles to the six-manifold $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^5$ and $p_j(V)$ denotes the j^{th} Pontryagin class of the bundle V. The integral (1.137) may be promoted to an equivariant integral with respect to the $U(1)^4 \circlearrowright \mathbb{S}^1 \times \mathbb{S}^5$ action [82]

$$\begin{aligned} E(\mathfrak{u}(1)) &= -\frac{-1}{16\beta} \\ &- \frac{1}{48\omega_1\omega_2\omega_3}\left[\frac{1}{8}\prod_{l_1,l_2\in\{1,-1\}}(\omega_1 + l_1\omega_2 + l_2\omega_3) + \mu^2(2\mu^2 - \sum_{i=1}^{3}\omega_i^2)\right], \end{aligned} \tag{1.141}$$

$$E(\mathfrak{g}) = \operatorname{rank}\mathfrak{g}\, E(\mathfrak{u}(1)) - \dim\mathfrak{g}\, h^\vee(\mathfrak{g})\frac{\left(\frac{9}{4} - \beta^2\mu^2\right)^2}{24\beta}. \tag{1.142}$$

Relation to the 5d $\mathcal{N} = 2$ SYM So far we have only been able to discuss the index in a few special cases, primarily the $\mathfrak{g} = \mathfrak{u}(1), \mathfrak{u}(\infty)$ cases. We would like to have an expression that is valid for any N. To go someway to achieving this it is possible to appeal to the following conjecture [73, 86, 61, 87, 88, 56, 89, 90]:

Compactification of the theory of type $\mathfrak{g}$ on a circle with radius $\beta \to 0$ gives rise to the 5d $\mathcal{N} = 2$ SYM theory with gauge algebra $\mathfrak{g}$. It is conjectured that (atleast at the level of the BPS spectrum) that the 6d $\mathcal{N} = (2,0)$ theory of type $\mathfrak{g}$ on $\mathbb{S}^1_\beta \times \mathcal{M}_5$ is equivalent to the 5d $\mathcal{N} = 2$ theory with gauge algebra $\mathfrak{g}$ and gauge coupling $g^2_{\text{YM}} = 2\pi\beta$ on the compact 5-manifold $\mathcal{M}_5$. Roughly speaking, the instantonic configurations of the 5d SYM should reproduce the Kaluza-Klein modes around the $\mathbb{S}^1$ of the 6d theory.

It is therefore believed that the $\mathcal{M}_5$ partition function for the 5d theory should reproduce the $\mathbb{S}^1_\beta \times \mathcal{M}_5$ partition function for the $\mathcal{N} = (2,0)$ theory. For the case $\mathcal{M}_5 = \mathbb{S}^5$ with $\mathfrak{g} = \mathfrak{u}(1), \mathfrak{u}(\infty)$, this conjecture has been supported for [56, 61] by matching the $\mathbb{S}^5$ partition functions to the superconformal index computations known from free field theory and $AdS/$CFT, respectively. In summary,

$$Z_{\mathbb{S}^1_\beta\times\mathcal{M}_5}\left(\text{6d }(2,0)\text{ of type }\mathfrak{g}\right) = Z_{\mathcal{M}_5}\begin{pmatrix}\text{5d }\mathcal{N}=2\text{ SYM}\\ \text{gauge algebra}=\mathfrak{g}\\ g^2_{\text{YM}}=2\pi\beta\end{pmatrix}. \tag{1.143}$$

For example, let us revist the $\mathbb{S}^5$ partition function for the 5d theory that we discussed in the previous subsection. We want to consider the chiral algebra limit (1.125) $\mu = \frac{1}{2}(\omega_1+\omega_2-\omega_3)$. This has been studied in in [58, 76, 61]. In this limit one can see that the perturbative piece simply becomes

$$Z_{\text{pert}}\left(\vec{\alpha},\omega_1,\omega_2,\omega_3,\omega_1+\omega_2-\frac{3}{2}\right) = \prod_{I>J} 4\sinh\frac{\alpha_{IJ}}{\omega_1}\sinh\frac{\alpha_{IJ}}{\omega_2}\,, \tag{1.144}$$

with $\alpha_{IJ} := \alpha_I - \alpha_J$. For the instanton pieces, in this limit, using the periodicity of the variables, Table 1.1 reduces to that of Table 1.5. From Table 1.5, in the 5d

	r	ϵ_1	ϵ_2	m
$Z^{(1)}_{\text{Nek}}$	$\frac{2\pi}{\omega_1}$	ω_2	ω_3	$2\omega_1+\frac{\omega_2-\omega_3}{2}\sim\frac{\omega_2-\omega_3}{2}=\epsilon_-$
$Z^{(2)}_{\text{Nek}}$	$\frac{2\pi}{\omega_2}$	ω_3	ω_1	$2\omega_2+\frac{\omega_1-\omega_3}{2}\sim\frac{\omega_1-\omega_3}{2}=-\epsilon_-$
$Z^{(3)}_{\text{Nek}}$	$\frac{2\pi}{\omega_3}$	ω_1	ω_2	$\omega_3+\frac{\omega_1+\omega_2}{2}\sim\frac{\omega_1+\omega_2}{2}=\epsilon_+$

Table 1.5: *Unrefined limits of the Nekrasov partition functions.*

description, the chiral algebra limits correspond to studying the Instanton partition function in the cases $m=\pm\epsilon_-$ and $\epsilon_+=m$. In these limits

$$z_\mu(\vec{\alpha},m=\pm\epsilon_-,\epsilon_1,\epsilon_2,r,g^2_{\text{YM}}) \equiv 0\,, \tag{1.145}$$

$$z_\mu(\vec{\alpha},m=\pm\epsilon_+,\epsilon_1,\epsilon_2,r,g^2_{\text{YM}}) \equiv 1\,. \tag{1.146}$$

So, in the former cases the partition function gets non-zero contributions only from the zero instanton sector. In the latter case, the instanton partition function is simply counting coloured young diagrams with a single fugacity q for the number of boxes:

$$Z_{\text{inst}}\left(\vec{\alpha},m=\epsilon_+,\epsilon_1,\epsilon_2,r,g^2_{\text{YM}}\right) = \sum_{\vec{\mu}} q^{|\vec{\mu}|} = q^{-N/24}(q;q)^{-N} = \eta(q)^{-N}\,. \tag{1.147}$$

So, in total, in this limit (1.101) becomes

$$Z_{\mathbb{S}^5} = \frac{1}{\eta(e^{-\frac{4\pi^2}{\beta\omega_3}})^N} \int [d\alpha] e^{\frac{2\pi^2(\vec{\alpha},\vec{\alpha})}{\beta\omega_1\omega_2\omega_3}} \prod_{I>J} 2\sin\frac{\alpha_I-\alpha_J}{\omega_1} 2\sin\frac{\alpha_I-\alpha_J}{\omega_2} . \tag{1.148}$$

This integral was computed in [61] and in total reads

$$Z_{\mathbb{S}^5} = e^{-\beta E_{p=1}(\mathfrak{u}(N))} \prod_{I=1}^{N} \frac{1}{(\mathfrak{q}_3^I;\mathfrak{q}_3)} = e^{-\beta E_{p=1}(\mathfrak{u}(N))}\,\mathrm{PE}\left[\sum_{I=1}^{N} \frac{\mathfrak{q}_3^I}{1-\mathfrak{q}_3}\right] . \tag{1.149}$$

We can see that this is in agreement with the chiral algebra limit for the $N = 1, \infty$ cases discussed in the preceding subsection aswell as the conjecture coming from the chiral algebra (1.128).

There are also conjectures stating that a similar correspondence also holds between 6d $\mathcal{N} = (1, 0)$ theories and 5d $\mathcal{N} = 1$ theories. For instance the $(1, 0)$ theory on N M5-branes on transverse ADE singularities and $5d\,\mathcal{N} = 1$ affine ADE shaped quivers with $U(N)$ gauge groups.

1.5 Hilbert Series and Operator Counting

We now move to describing another type of partition function that will be of interest in this thesis. The partition function of interest is known as the *Hilbert(-Poincaré) series*. In Section 1.2 we have already seen an example of the interplay between geometry and supersymmetric QFTs. The Hilbert series is a mathematical tool that can be used firstly to count operators and secondly to better understand the geometry of the moduli spaces of supersymmetric vacua (1.2). The relevant mathematical definitions and theorems about algebraic geometry have been collected in Appendix B. Mathematical reviews can be found in [91, 92, 93, 94] while physics oriented reviews can be found in [95, 96, 97].

In this thesis we are mostly interested in four dimensional theories and hence in this subsection we will mainly focus on four dimensional $\mathcal{N} = 1$ theories. The computation of the Hilbert series for four dimensional theories often reduces simply to classical operator counting. This is to be compared to three dimensional $\mathcal{N} = 2$ theories, which in recent years have attracted a lot of attention, where one must include ‘t Hooft monopole operators which are purely quantum operators that sit in chiral multiplets.

Let us first review some basic facts about $\mathcal{N} = 1$ supersymmetric gauge theories.

An in-depth review can be found in [98]. The $\mathcal{N}=1$ supersymmetry algebra is an extension of the Poincaré algebra $\mathfrak{io}(4)$. The $\mathcal{N}=1$ supersymmetry algebra has a bosonic subalgebra $\mathfrak{io}(4)\oplus\mathfrak{u}(1)_r$. The Poincaré algebra on Euclidean space is generated by the standard generators $M^{\mu\nu}$, P^μ with $\mu,\nu\in\{1,2,3,4\}$ and $\mathfrak{u}(1)_r$ is generated by r. The generators $\mathcal{Q}_\alpha$, $\widetilde{\mathcal{Q}}^{\dot\alpha}$ have the following relations

$$[\mathcal{Q}_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\beta}\mathcal{Q}_\beta\,, \quad [\widetilde{\mathcal{Q}}^{\dot\alpha}, M^{\mu\nu}] = (\widetilde{\sigma}^{\mu\nu})_{\dot\beta}^{\dot\alpha}\widetilde{\mathcal{Q}}^{\dot\beta}\,, \tag{1.150}$$

$$[\mathcal{Q}_\alpha, P^\mu] = [\widetilde{\mathcal{Q}}^{\dot\alpha}, P^\mu] = 0\,, \quad [r,\mathcal{Q}_\alpha] = -\mathcal{Q}_\alpha\,, \quad [r,\widetilde{\mathcal{Q}}^{\dot\alpha}] = \widetilde{\mathcal{Q}}^{\dot\alpha}\,, \tag{1.151}$$

$$\{\mathcal{Q}_\alpha, \widetilde{\mathcal{Q}}^{\dot\alpha}\} = 2(\sigma_\mu)_\alpha^{\dot\alpha}P^\mu\,, \quad \{\mathcal{Q}_\alpha,\mathcal{Q}_\beta\} = \epsilon_{\alpha\beta}C\,, \quad \{\widetilde{\mathcal{Q}}^{\dot\alpha},\widetilde{\mathcal{Q}}^{\dot\beta}\} = \epsilon^{\dot\alpha\dot\beta}\widetilde{C}\,, \tag{1.152}$$

$C,\widetilde{C}$ are central elements which, for simplicity, we take to vanish $C=\widetilde{C}=0$.

The fields of any $\mathcal{N}=1$ field theory sit in representations of the above algebra. The local operator content of an $\mathcal{N}=1$ gauge theory can be specified in terms of the following data [99, 100, 101]:

- Gauge group G with lie algebra $\mathfrak{g}=Lie(G)$ in which a vector multiplet sits in the adjoint representation.
- Representation $\mathcal{R}$ of G to which chiral multplets $X^{i=1,\dots,\dim\mathcal{R}}$ belong to.
- G-invariant holomorphic polynomial $W(X^i)$ of R-charge $r[W]=2$.

The scalar potential is then

$$V(X,\overline{X}) = \sum_{i=1}^{\dim\mathcal{R}} |F_i(X)|^2 + \frac{g^2}{2}\sum_{a=1}^{\operatorname{rank}\mathfrak{g}} D^a(X,\overline{X})^2\,. \tag{1.153}$$

The moduli space of supersymmetric vacua is then

$$\mathbf{M} = \left\{X^i,\overline{X}^i\middle|V(X,\overline{X})=0\right\}/G \cong \left\{X^i\middle|F_j(X)=0\right\}/G^{\mathbb{C}} := \mathbf{F}/G^{\mathbb{C}}\,, \tag{1.154}$$

for the second equality we used the fact that $\mathbf{M}$ has an equivalent holomorphic description obtained by dropping the real D-term constraints and quotienting instead by complexified gauge transformations [102, 103, 104]. $\mathbf{F}$ is often referred to as *master space* [105, 106, 96, 107, 108]. $\mathbf{F}$ is a complex affine variety written in terms of the coordinates X^i defined by the vanishing of the F-terms $F_j(X)\in\mathbb{C}[X^1,X^2,\dots]$.

Four-dimensional $\mathcal{N}=1$ supersymmetry implies that $\mathbf{M}$ is Kähler [109]. We can see this by first noting that, almost by construction, $\mathbf{M}$ is a complex manifold with complex structure $I = -\mathrm{i}\sigma_2$, $I^2 = -1$ provided by the supersymmetry transformations

$$\overline{\mathcal{Q}}_{\dot\alpha}\mathcal{Q}_\alpha \begin{pmatrix} \Re X^i \\ \Im X^i \end{pmatrix} = \sigma^\mu_{\dot\alpha\alpha}\partial_\mu I \begin{pmatrix} \Re X^i \\ \Im X^i \end{pmatrix} \tag{1.155}$$

and Kähler metric $\partial^2 K/\partial X^i \partial \overline{X}^j$. Note that if the theory in question is an $\mathcal{N} = 2$ supersymmetric theory without vector multiplets (for example the Higgs branch $\mathbf{HB} \subset \mathbf{M}$ of an $\mathcal{N} = 2$ gauge theory) the moduli space (or appropriate subvariety) is hyperKähler. This is because we now have three complex structures $I^{(c=1,2,3)}$ provided by

$$\overline{\mathcal{Q}}^{(c)}_{\dot\alpha}\mathcal{Q}^{(c)}_\alpha \begin{pmatrix} \Re X^i \\ \Im X^i \end{pmatrix} = \sigma^\mu_{\dot\alpha\alpha}\partial_\mu I^{(c)} \begin{pmatrix} \Re X^i \\ \Im X^i \end{pmatrix}, \quad \begin{aligned} \mathcal{Q}^{(c)}_\alpha &= c_1 \mathcal{Q}^1_\alpha + c_2 \mathcal{Q}^2_\alpha \,, \\ |c_1|^2 + |c_2|^2 &= 1 \,. \end{aligned} \tag{1.156}$$

One can show that $(I^{(1)}, I^{(2)}, I^{(3)}) = (I, J, K)$ satisfy $I^2 = J^2 = K^2 = IJK = -1$ and thus form a quaternionic algebra; giving the hyperKähler structure.

Directly studying $\mathbf{M}$ as an affine variety can often be rather complicated. Instead, it is much easier to work with its coordinate ring. The coordinate ring of $\mathbf{F}$ is

$$F = \mathbb{C}[X^1, X^2, \dots]/I \,, \quad I = \langle F_1, F_2, \dots \rangle \equiv \mathfrak{I}(\mathbf{F}) \,, \tag{1.157}$$

here $\mathfrak{I}(\mathbf{V})$ is the map that associates to an affine variety $\mathbf{V} \subset \mathbb{K}^n$ an ideal $\mathfrak{I}(\mathbf{V}) \subset \mathbb{K}[x_1, \dots, x_n]$; see (B.8) for the full definition. From F we can easily construct the coordinate ring of $\mathbf{M}$ as

$$M = (F)^G = \mathbb{C}[\mathcal{O}_1, \mathcal{O}_2, \dots]/\tilde{I} \,, \tag{1.158}$$

$(F)^G$ simply means to take the G-invariant polynomials in F. Mathematically speaking the ideal is [110]

$$\tilde{I} = \mathrm{Im}\left(F \xrightarrow{\rho} \mathbb{C}[\mathcal{O}_1, \mathcal{O}_2, \dots]\right) \,, \tag{1.159}$$

where ρ defines a *ring map*

$$\rho : \mathbb{C}[X^1, X^2, \dots] \to \mathbb{C}[\mathcal{O}_1, \mathcal{O}_2, \dots] \,. \tag{1.160}$$

Under some conditions it is then possible to reconstruct $\mathbf{M}$ from M in a one-to-one fashion, see Appendix B, by the map

$$\mathbf{M} = \mathfrak{V}(\tilde{I}) \,, \tag{1.161}$$

where $\mathfrak{V}$ is defined in (B.9).

Here the $\mathcal{O}_i$ span a basis for the gauge invariant polynomials in the X^i, operators of this type belong to the $\mathcal{N}=1$ $\frac{1}{2}$-*BPS chiral ring* [111]. This is the scalar sector of the $\widetilde{\mathcal{Q}}^{\dot{1}} \cap \widetilde{\mathcal{Q}}^{\dot{2}}$-Cohomology of the theory[4] and consists of local operators satisfying[5]

$$\widetilde{\mathcal{Q}}^{\dot{\alpha}}\mathcal{O}_i(x) = M^{\mu\nu}\mathcal{O}_i(x) = 0\,. \tag{1.162}$$

Note that, from (1.151), one can see that

$$\{\mathcal{Q}_\alpha, \widetilde{\mathcal{Q}}^{\dot{\alpha}}\} = 2\mathrm{i}(\sigma^\mu)^{\dot{\alpha}}_{\alpha}\partial_\mu \tag{1.163}$$

and therefore spacetime-derivatives of the $\mathcal{O}_i$ are $\widetilde{\mathcal{Q}}^{\dot{\alpha}}$-exact. This means that we can identify $\mathcal{O}_i(x) \equiv \mathcal{O}_i$ as long as we are in the cohomology. Moreover this implies that the correlation function of a product of chiral operators decomposes into a product of one-point functions

$$\langle\prod_{i=1}^{n}\mathcal{O}_i(x_i)\rangle = \prod_{i=1}^{n}\langle\mathcal{O}_i(x_i)\rangle = \prod_{i=1}^{n}\langle\mathcal{O}_i\rangle\,. \tag{1.164}$$

These operators therefore have a commutative product

$$\mathcal{O}_i\mathcal{O}_j = \mathcal{O}_j\mathcal{O}_i = c^l_{ij}\mathcal{O}_l + \widetilde{\mathcal{Q}}^{\dot{\alpha}}(\ldots)_{\dot{\alpha}}\,, \tag{1.165}$$

by passing to the $\widetilde{\mathcal{Q}}^{\dot{1}} \cap \widetilde{\mathcal{Q}}^{\dot{2}}$-Cohomology it is easy to see that this product satisfies all of the properties of Definition 1 required in order to have a graded commutative ring. The grading is provided by the $\mathfrak{u}(1)_r$ charge normalised such that it is integral (for superconformal theories this is always possible).

Let us now move to the main object we wish to compute. This is the Hilbert series for the ring M. A mathematical treatment of this quantity can be found in Appendix B.3. For the physical purpose is suffices to write

$$\mathrm{HS}(\tau; M) := \mathrm{Tr}_M\, \tau^E \tag{1.166}$$

namely we can consider the Hilbert series as a partition function over the moduli

[4]The full $\mathcal{N}=1$ $\frac{1}{2}$-BPS chiral ring, in the sense of [111], is the one obtained by relaxing the scalar condition and taking the $\widetilde{\mathcal{Q}}^{\dot{1}} \cap \widetilde{\mathcal{Q}}^{\dot{2}}$-Cohomology. This contains glueball operators such as $\mathrm{tr}\, W_\alpha W_\beta$ amongst others.

[5]Here we are abusing the notation where we now pass to a representation of the algebra as differential operators on $C^\infty(\mathbb{R}^4)$.

space; it is simply counting all operators parametrising M. For a gauge theory the general form for the Hilbert series for $M = (F)^G$ is then given as an integral over the Hilbert series for the master-space $\mathbf{F}$

$$\mathrm{HS}(\tau; M) = \oint d\mu_G(z)\, \mathrm{HS}(\tau, z; F)\,, \tag{1.167}$$

here, with respect to (1.165), (B.16) we take $E \propto r$ with the proportionality fixed by demanding $E \geqslant 1$ be integral. $d\mu_G(z)$ denotes the Haar measure for the gauge group G and it projects onto G-invariants. In the case when the polynomials F_j generating the ideal $I = \langle F_1, F_2, \dots \rangle$ ($F = R/I$) form what is known as a *regular sequence* (roughly speaking this is a statement about the algebraic independence of the F_j, see Definition 14) the Hilbert series $\mathrm{HS}(\tau, z; F)$ is particularly simple to compute since one can apply letter counting techniques much like one does for the computation of the supersymmetric index of the previous section (1.115). In the case when the F_j do not form a regular sequence the computation generally requires the use of *Gröbner-Bases* and one must resort to using an algebraic-geometry computer package, such as *Macaulay2* [112].

Let us see how the above works in practise for gauge theories. We consider a simple model that will capture many of the ideas we have presented so far within a few simple examples. The simplest model one can write down is the $4d$ $G = U(1)_z$ $\mathcal{N} = 4$ SYM theory. In $\mathcal{N} = 1$ notation this theory has three chiral multiplets X, Y, Z in the adjoint of $\mathfrak{u}(1)_z$ and vanishing superpotential $W = [X, [Y, Z]] = 0$. For $U(1)$ this theory is free. Therefore the quotient by $G^{\mathbb{C}}$ is trivial and moduli space is simply

$$\mathbf{M} = \mathbf{F}/GL(1, \mathbb{C}) = \mathbf{F} = \{X, Y, Z\} \cong \mathbb{C}^3 \leftrightarrow \mathbb{C}[X, Y, Z]\,. \tag{1.168}$$

We wrote the Hilbert series for $\mathbb{C}^{n=3}$ in Appendix B, keeping track of the three independent gradings for X, Y, Z it reads

$$\mathrm{HS}(\tau_1, \tau_2, \tau_3; \mathbb{C}[X, Y, Z]) = \frac{1}{(1-\tau_1)(1-\tau_2)(1-\tau_3)}\,. \tag{1.169}$$

We discuss the Hilbert series for the non-abelian $\mathcal{N} = 4$ SYM theories in Chapter 5. Another simple model we consider is a $U(1)$ gauge theory with three chiral multiplets X, Y, Z with superpotential $W = XYZ$. We give X, Y, Z charges $+1, -1, 0$ under the $U(1)_z$ gauge group and $E = \frac{3}{2} r = +1, +1, +1$ under the R-symmetry. This is similar to the previous $\mathcal{N}=4$ model but now we allow for different $U(1)_z$ representations to appear. The master space is given by

$$\mathbf{F}=\{X,Y,Z|XY=XZ=YZ=0\}\leftrightarrow F=\frac{\mathbb{C}[X,Y,Z]}{\langle XY,XZ,YZ\rangle}\,. \tag{1.170}$$

The Hilbert series for $\mathbf{F}$ can easily be computed using *Macaulay2* [112]

$$\mathrm{HS}(\tau_1,\tau_2,\tau_3,z;F)=\frac{1-\tau_1\tau_2-z\tau_1\tau_3-z^{-1}\tau_2\tau_3+2\tau_1\tau_2\tau_3}{(1-z\tau_1)\,(1-z^{-1}\tau_2)\,(1-\tau_3)}\,, \tag{1.171}$$

here we refined the standard expression by keeping track of the powers of X,Y,Z using τ_1,τ_2,τ_3. One can see the denominator encodes the three generators X,Y,Z of the ring. Denoting $F_1=YZ$, $F_2=XZ$, $F_3=XY$ the minus signs in the numerator implements the F-term relations $F_1=F_2=F_3=0$ while the term $+2\tau_1\tau_2\tau_3$ accounts for the syzygies (relations-between-relations) $XF_1=YF_2=ZF_3$. The Hilbert series for $\mathbf{M}$ is then given by

$$\mathrm{HS}(\tau_1,\tau_2,\tau_3;M)=\oint\limits_{|z|=1}\frac{dz}{2\pi\mathrm{i}z}\mathrm{HS}(\tau_1,\tau_2,\tau_3,z;F)=\frac{1}{1-\tau_3}\,, \tag{1.172}$$

recall that $0\leqslant\tau_i\leqslant 1$ and therefore only the poles $z=0,\tau_2$ are picked up. The interpretation of the result is straightforward: the only gauge invariant operators are those of the form $(XY)^pZ^q$ with $p,q\geqslant 0$ integer. But the F-term $F_3=XY=0$ demands $p=0$ leaving only arbitrary powers of Z. This implies that $\mathbf{M}\cong\mathbb{C}$. In Sections 2 & 5 we provide more examples of the Hilbert series applied to a variety of $\mathcal{N}\geqslant 1$ theories in four dimensions.

1.6 Compactifications of Six Dimensional SCFTs

As we have touched upon, over the last decade much effort has be devoted towards understanding the ‘space’ or ‘landscape’ of all consistent, UV complete, quantum field theories. It is not clear how to begin to answer, or to even discuss the well-posedness of, the question. Along the same lines one can ask, for example, what is the complete set of data which uniquely specifies any QFT? Trying to answer these types of question for *all* QFTs is a goal that seems vastly out of reach. However some progress has been made in understanding ‘subspaces’ of the ‘space’ of all QFTs. In particular those QFTs with some amount of supersymmetry. See Figure 1.1 for

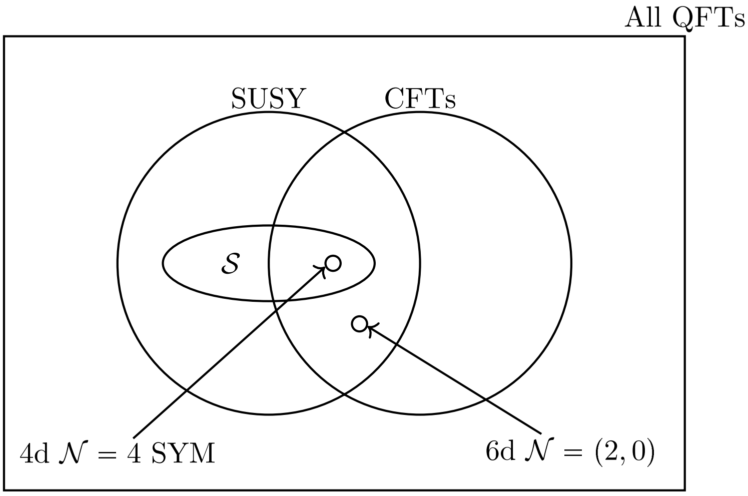


Figure 1.1: *Schematic overview of the 'landscape' of quantum field theories.*

a schematic overview. For example, four dimensional theories with $\mathcal{N} = 4$ supersymmetry are believed to be completely specified by the following data: $(G, \tau, \{\theta_i\})$ where G denotes the gauge group, τ the complexified gauge coupling and $\{\theta_i\}$ a set of discrete θ-angle like parameters [26]. This data is an over-parametrisation and one must also quotient by S-duality [113, 114], see Section 5.2.2.

A more ambitious goal that has been initiated is to classify/construct all possible four dimensional $\mathcal{N} = 2$ theories [115, 116, 14, 117, 118, 119, 120].

1.6.1 Class $\mathcal{S}$

A particularly useful tool to construct a large class of 4d $\mathcal{N} = 2$ theories has been via compatification of the 6d $\mathcal{N} = (2, 0)$ theories [121, 122]. We discussed these theories in Section 1.4.3 of this chapter. Again we focus on the theories of A_{N-1} type; realised as the worldvolume theory of a stack of N parallel and coincident M5-branes.

The 4d $\mathcal{N} = 2$ theories engineered by compactifying this theory on a Riemann surface $\mathcal{C}$ are said to be of 'class $\mathcal{S}$'. Note that in order to preserve supersymmetry we must partially twist the $(2, 0)$ theory. This is achieved as follows. The superconformal algebra is [123, 124]

$$\mathfrak{osp}(8^*|4) \supset \mathfrak{so}(6) \oplus \mathfrak{so}(5)_R \,, \tag{1.173}$$

in particular, the supercharges Q transform in the $(\mathbf{4}, \mathbf{4})$ representation of the subalgebra. Because we put the 6d theory on a product manifold $\mathbb{R}^4 \times \mathcal{C}$ we decompose $\mathfrak{so}(6) \supset \mathfrak{so}(4) \oplus \mathfrak{u}(1)_s$. For the purposes of the twisting we also decompose $\mathfrak{so}(5)_R \supset \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r$. Under the decomposition $\mathfrak{so}(4) \oplus \mathfrak{u}(1)_s \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r$ the supercharges transform as

$$\mathsf{Q} \in (\mathbf{2}, \frac{1}{2}, \mathbf{2}, \frac{1}{2}) \oplus (\mathbf{2}, \frac{1}{2}, \overline{\mathbf{2}}, -\frac{1}{2}) \oplus (\overline{\mathbf{2}}, -\frac{1}{2}, \mathbf{2}, \frac{1}{2}) \oplus (\overline{\mathbf{2}}, -\frac{1}{2}, \overline{\mathbf{2}}, -\frac{1}{2})\,. \tag{1.174}$$

The twisting procedure then involves replacing the holonomy algebra with the diagonal subgroup $\mathfrak{u}(1)_s \to \mathfrak{u}(1)_{s'} \subset \mathfrak{u}(1)_r \oplus \mathfrak{u}(1)_s$. After the twisting, under $\mathfrak{so}(4) \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r \oplus \mathfrak{u}(1)_{s'}$,

$$\mathsf{Q} \in (\mathbf{2}, \mathbf{2}, 1, \frac{1}{2}) \oplus (\mathbf{2}, \overline{\mathbf{2}}, 0, -\frac{1}{2}) \oplus (\overline{\mathbf{2}}, \mathbf{2}, 0, \frac{1}{2}) \oplus (\overline{\mathbf{2}}, \overline{\mathbf{2}}, -1, -\frac{1}{2})\,, \tag{1.175}$$

in particular we identify the eight $\mathfrak{u}(1)_{s'}$ scalar supercharges with the $\mathcal{N} = 2$ supercharges. Therefore, protected quantities of the 4d theory are independent of the choice of metric on $\mathcal{C}$ and depend only on the complex structure moduli. More rigorously, the space of UV gauge couplings is identified with the space of complex structure deformations $\mathcal{E}$ of the underlying surface $\mathcal{C}$. We have an isomorphism

$$\mathcal{E} \cong Teich(\mathcal{C})/MCG(\mathcal{C}) \tag{1.176}$$

where $Teich(\mathcal{C})$ is the Teichmüller space of $\mathcal{C}$ and is parametrised by the same cross ratios q appearing in the Seiberg-Witten curve. $MCG(\mathcal{C})$ is the mapping class group of $\mathcal{C}$, in physics terms this is the 'generalised S-duality group'. Moreover, the Seiberg-Witten curve for the class $\mathcal{S}$ theory is then obtained as a rank $\mathfrak{g}$ cover of the underlying Riemann surface $\mathcal{X}_u \xrightarrow{\text{rank}\,\mathfrak{g}:1} \mathcal{C}$. In addition, many protected quantities, such as partition functions and correlation functions of BPS operators, in class $\mathcal{S}$ SCFTs may be computed as observables of a 2d theory which lives on $\mathcal{C}$ [125, 126]. One manifestation of this 4d/2d relation is that the partition function on an ellipsoid $\mathbb{S}^4_{\epsilon_1,\epsilon_2}$ (1.100) is equal to correlators in Liouville/Toda CFT [125, 127]. In particular, the 2d Virasoro/W-algebra conformal blocks are mapped to Nekrasov's instanton partition function. Another manifestation of a 4d/2d relation is the partition function on $\mathbb{S}^3 \times \mathbb{S}^1$ (1.107) which can be recast as a correlator of a 2d TQFT living on $\mathcal{C}$ [128, 129].

Using the above compactification procedure we can generate an infinite number of 4d $\mathcal{N} = 2$ theories labelled by compact Riemann surfaces $\mathcal{C}$. These Riemann surfaces also carry marked punctured, corresponding to inserting a variety (while still

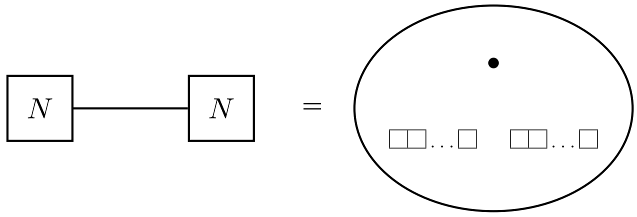


Figure 1.2: $\mathcal{N} = 2$ *hypermultiplets in the class $\mathcal{S}$ description. The dot represents the $N = (N-1)+1$ minimal puncture.*

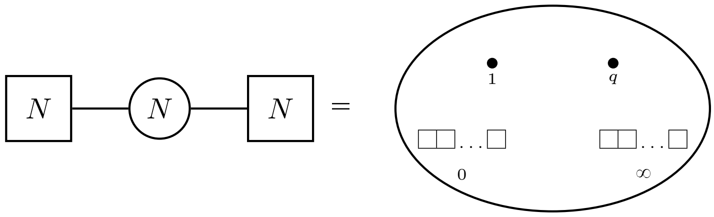


Figure 1.3: $\mathcal{N} = 2$ *SQCD with $N_f = 2N$ hypermultiplets in the class $\mathcal{S}$ description.*

preserving eight supercharges) of defect operators of the 6d theory at the punctures.

For the case of $\mathfrak{g} = A_{N-1}$ these $\frac{1}{2}$-BPS defect operators are labelled by Young tableaux [122, 121] with N boxes. The two main punctures that we will discuss are 'maximal' punctures labelled by tableaux of type $N = 1+1+\cdots+1$ and 'minimal' punctures of type $N = (N-1)+1$.

For example, the $\mathcal{N} = 2$ hypermultiplet is obtained by compactification of the $(2,0)$ theory on a sphere with a minimal puncture and two maximal punctures, this is pictured in Figure 1.2. From this basic building block we can construct linear and affine quivers with n gauge nodes, corresponding to spheres with $n+1$ minimal plus two maximal punctures and tori with n minimal punctures, respectively. For example, conformal $\mathcal{N} = 2$ $N_f = 2N$ SQCD can be made by gluing two of these three-punctured spheres together with a tube connecting the punctures, interpreted as gauging the flavour symmetry. This is pictured in Figure 1.3. The punctures can be mapped, via conformal transformation, to $z = 0, 1, q, \infty$ and q is identified with the complexified gauge coupling via $q = e^{2\pi \mathrm{i} \tau_{\mathrm{YM}}}$.

Within this class the majority of theories do not admit known Lagrangian descriptions. Indeed, with generic defect operators inserted, any other type of description of the resulting 4d theory is often lacking. Examples of non-Lagrangian theories that may be engineered within class $\mathcal{S}$ are the so-called T_N SCFTs [121]. These correspond to spheres with three maximal punctures. For example, the T_3 theory. This

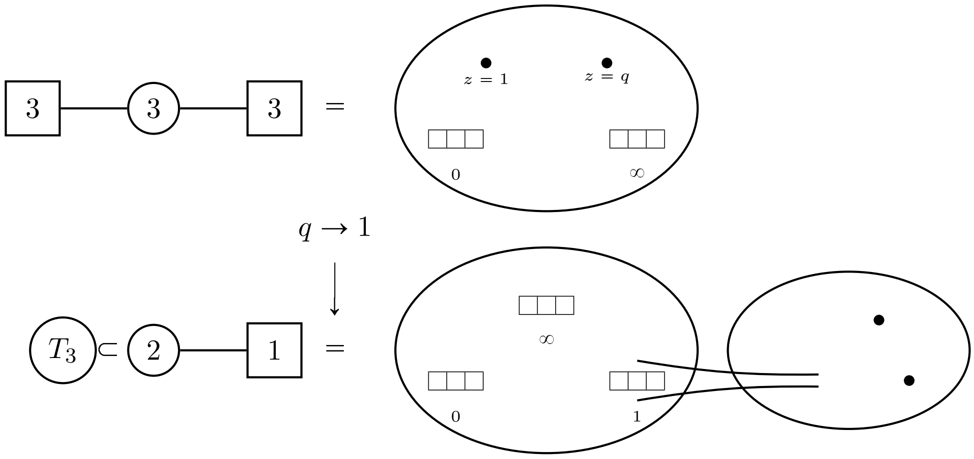


Figure 1.4: *Description of the $T_3 = MN(E_6)$ SCFT in class $\mathcal{S}$ description.*

theory is the same as the rank one E_6 SCFT of Minihan and Nemeschansky [130], $T_3 = MN(E_6)$. To find this SCFT we begin with the $N = 3$, $N_f = 2N = 6$ superconformal QCD, described in Figure 1.3. According to Argyres-Seiberg duality [131] in the $\tau_{\mathrm{YM}} \to 1$ region of this theory it admits a dual description in terms of the $MN(E_6)$ SCFT weakly coupled to a single $SU(2)$ hypermultiplet. This is neatly illustrated within class $\mathcal{S}$ as we begin with the four punctured sphere and consider the limit where we collide the two minimal punctures at $z = 1, q$ together, see Figure 1.4. These types of manipulations may be used to also find the class $\mathcal{S}$ descriptions of the $MN(E_7)$ & $MN(E_8)$ SCFTs [132]. Moreover, applying the same logic new strong-weak dual descriptions are able to be conjectured, see e.g. [8] for more information on these points.

1.6.2 Class $\mathcal{S}_k$

$\mathcal{N} = 1$ variations of the class $\mathcal{S}$ construction have been proposed in [133, 134, 135, 136, 137, 138, 139, 140, 141, 142, 143, 144, 145, 146]. The idea is to obtain 4d $\mathcal{N} = 1$ theories via (twisted) compactification of 6d $(1,0)$ SCFTs again on punctured Riemann surfaces $\mathcal{C}$. $\frac{1}{2}$-BPS defect operators of the $(1,0)$ theory can then be localised at the punctures on $\mathcal{C}$ to create a wide variety of $\mathcal{N} = 1$ theories.

The landscape of 6d $(1,0)$ SCFTs is far richer than that of $(2,0)$ theories; in addition to their respective superconformal symmetries $(2,0)$ theories can have only discrete global symmetries while $(1,0)$ theories can also have continuous flavour symmetry groups (this is analagous to the difference in the possible 4d $\mathcal{N} = 4$ versus

$\mathcal{N} = 2$ theories). An F-theoretic classification of $(1,0)$ theories has been explored in [147, 148, 149, 150] but a complete field theoretic understanding is currently lacking.

Let us outline the twisting procedure for the $(1,0)$ theories. The R-symmetry is now $\mathfrak{su}(2)$ under which the supercharges are in the fundamental representation and again in the spinor represenation of $\mathfrak{so}(6)$. In this case one chooses a $\mathfrak{so}(4) \oplus \mathfrak{u}(1)_s \subset \mathfrak{so}(6)$ and $\mathfrak{u}(1)_{r'} \subset \mathfrak{su}(2)_R$. Under $\mathfrak{so}(4) \oplus \mathfrak{u}(1)_s \oplus \mathfrak{u}(1)_{r'}$

$$\mathsf{Q} \in (\mathbf{2}, \frac{1}{2}, \frac{1}{2}) \oplus (\mathbf{2}, \frac{1}{2}, -\frac{1}{2}) \oplus (\overline{\mathbf{2}}, -\frac{1}{2}, \frac{1}{2}) \oplus (\overline{\mathbf{2}}, -\frac{1}{2}, -\frac{1}{2})\,. \tag{1.177}$$

Now replacing $\mathfrak{u}(1)_s \to \mathfrak{u}(1)_{s'} \subset \mathfrak{u}(1)_{r'} \oplus \mathfrak{u}(1)_s$ we have four $\mathfrak{u}(1)_{s'}$-scalar supercharges which are identified with the supercharges of the 4d $\mathcal{N} = 1$ superalgebra.

An interesting subset of 6d $(1,0)$ theories are those which may be engineered within M-theory by considering the low energy theory living on N coincident and parallel M5-branes at the tip of a transverse $\Gamma = ADE$ singularity. These $(1,0)$ theories are orbifolds of the $\mathfrak{g} = A_{N-1}$ $(2,0)$ theory, which we denote by $(1,0)_\Gamma$. 4d $\mathcal{N} = 1$ theories obtained in this way are said to lie within 'class $\mathcal{S}_\Gamma$' [133, 135]. The 4d theories of type $\Gamma = A_{k-1}$ will be one of the primary subjects of this thesis; theories of this type will be designated to be that of 'class $\mathcal{S}_k$'.

In contrast to the $(2,0)$ theories (who's $\frac{1}{2}$-BPS defect operators are simply classified by Young's tableaux) the permissible defect operators for the $(1,0)_\Gamma$ theories have a very complicated classification [135]. Moreover, descriptions of the resulting 4d $\mathcal{N} = 1$ theories upon compactification are currently lacking for all but a few configurations of punctures.

In [133] the superconformal index was computed for a variety of class $\mathcal{S}_k$ theories and most strikingly it was recast as a correlator of a 2d TQFT establishing the first 2d/4d relation for class $\mathcal{S}_k$. Subsequently, in [151, 152], the index and its TQFT description was also computed in the presence of $\frac{1}{2}$-BPS surface defects. Additionally, in [137], SW curves were computed and some of their properties explored. In [153], guided by the SW curves, the existence of an AGT-like correspondence for the class $\mathcal{S}_k$, denoted AGT$_k$ correspondence, was conjectured. Furthermore, the instanton partition function was proposed via a relation to $\mathcal{W}_{kN}$ conformal blocks.

4d $\mathcal{N} = 1$ Class $\mathcal{S}_k$ Quiver Gauge Theories

In this section we review some basic details of the 'core theories' in class $\mathcal{S}_k$ introduced in [133]. We will mostly follow the notation of [154].

As we have discussed, we realise the theories in the class $\mathcal{S}_k$ as twisted compactifications along a Riemann surface of genus g with ℓ punctures of the $(1,0)_{A_{k-1}}$ SCFT. The 6d theory has a $\mathfrak{u}(1)_t \oplus \mathfrak{su}(k)_\beta \oplus \mathfrak{su}(k)_\gamma$ global symmetry. The compactifications typically preserve only a Cartan subalgebra $\mathfrak{u}(1)_t \oplus \mathfrak{u}(1)_\beta^{\oplus k-1} \oplus \mathfrak{u}(1)_\gamma^{\oplus k-1}$ which are 'intrinsic' global symmetries carried by all class $\mathcal{S}_k$ theories. There is also the $\mathcal{N}=1$ $\mathfrak{u}(1)_r$ R-symmetry. The punctures carry additional data associated to inserting a variety of defect operators localised at the punctures in six dimensions [133, 144, 135], in this paper we will focus only on maximal and minimal punctures.

Maximal punctures are labelled with a colour $c \in \{1, 2, \ldots, k\}$, a sign $\sigma = \pm 1$ and an orientation $o = l/r$. We will label these maximal punctures with the notation $s_c^{o,\sigma}$. Maximal punctures also carry an associated $\mathfrak{su}(N)^{\oplus k}$ flavour symmetry algebra. Minimal punctures carry a $\mathfrak{u}(1)$ symmetry under which the baryonic operators of the form $\det Q_i$, $\det \widetilde{Q}_i$ and $\det \Phi_i$ are charged while mesonic operators are uncharged.

The core theories that we will focus on are associated to spheres with $\ell - 2$ minimal punctures and two maximal punctures $s_{c_l}^{l,+}$ & $s_{c_r}^{r,+}$ with $c_r = (c_l + \ell - 2) \bmod k$, and $\frac{1}{2k}$ unit of flux for $U(1)_t$. These admit a quiver description, as shown in FIgure 1.8.

This theory admits a weakly coupled Lagrangian description associated to a pair of pants decomposition of the Riemann surface into a chain of $n = 0, 1, \ldots, \ell - 3$ spheres with one minimal puncture and two maximal punctures $s_{c_l+n}^{l,+}$ & $s_{c_l+n+1}^{r,+}$. These three punctured spheres correspond to quiver theories of bifundamental chiral multiplets called the *free trinion* and is pictured in Figure 1.5. The maximal punctures of equal colour, opposite orientation and equal sign of the n^{th} and $(n+1)^{\text{th}}$ three punctured spheres are then glued with tubes associated to spheres with two maximal punctures $s_{c_l+n+1}^{l,+}$ & $s_{c_l+n+1}^{r,+}$. This gluing corresponds to gauging the diagonal $\mathfrak{su}(N)_{\mathbf{z}}^{\oplus k} \subset \mathfrak{su}(N)_{\mathbf{z}_l}^{\oplus k} \oplus \mathfrak{su}(N)_{\mathbf{z}_r}^{\oplus k}$ of the two maximal punctures with free $\mathcal{N}=1$ vector multiplets and k bifundamental chiral multiplets Φ_i. This type of gluing is called Φ-gluing and is pictured in Figure 1.6. This leads to quiver theories of the type pictured in Figure 1.8. The representations of the fields under the global symmetries are summarised in Table 2.3 and will be used later.

From these theories it is possible to construct theories associated to tori with $\ell - 2$ minimal punctures by Φ-gluing the 'open' maximal punctures $s_{c_l}^{l,+}$ & $s_{c_r}^{r,+}$, these are $\mathbb{Z}_k \times \mathbb{Z}_{\ell-2}$ orbifold theories of $\mathcal{N}=4$ SYM. Unless $c_r - c_l = 0 \bmod k$ (or equivilantly $\ell - 2 \bmod k = 0$) this procedure breaks the $\mathfrak{u}(1)_\beta^{\oplus k-1} \oplus \mathfrak{u}(1)_\gamma^{\oplus k-1}$ symmetry. The quiver diagram of such theories can be found in Figure 1.7. This leads to quiver theories of this type shown in Figure 1.7.

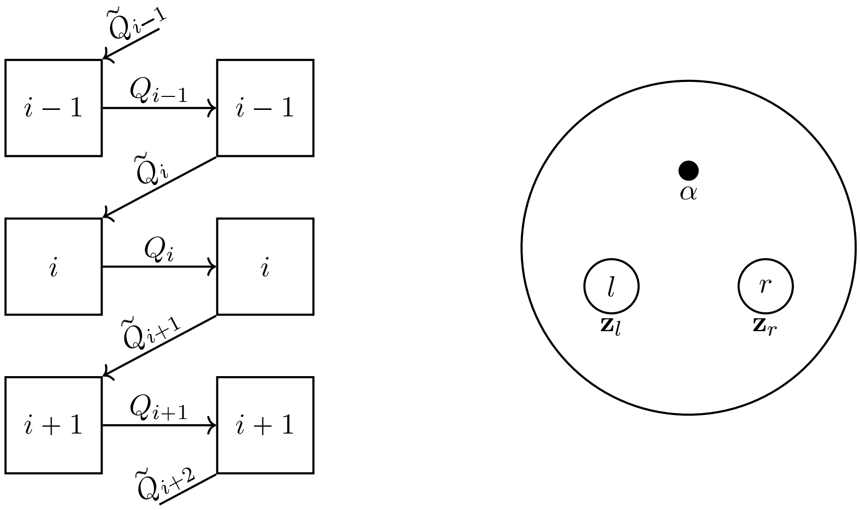


Figure 1.5: *Quiver diagram for the free trinion theory of bifundamental chiral multiplets, associated to a sphere with one minimal puncture and two maximal punctures* $s_c^{l,+}$ *and* $s_{c+1}^{r,+}$.

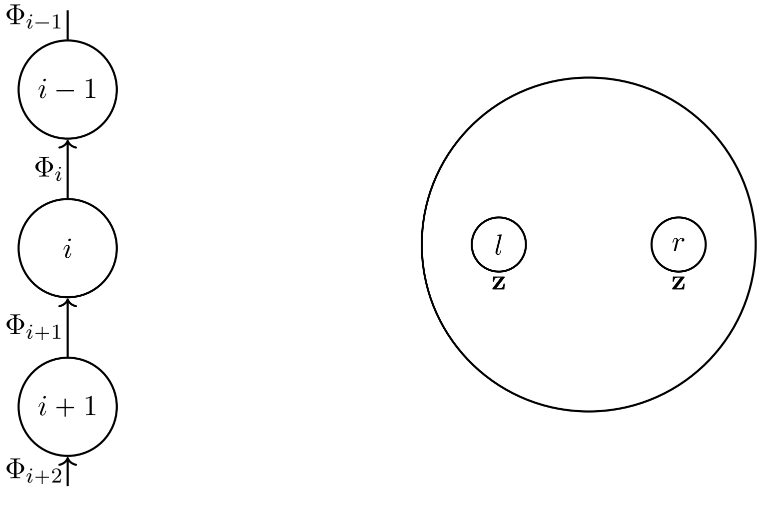


Figure 1.6: *Quiver associated to* Φ*-gluing.*

Finally, a general Lagrangian theory in class $\mathcal{S}_k$, made using the above ingredients, associated to a genus g Riemann surface with ℓ punctures has superpotential

$$\begin{aligned} W_{\mathcal{S}_k} = &\sum_{n=1}^{3g-3+\ell}\sum_{i=1}^{k} \operatorname{tr}\left(\widetilde{Q}_{(i,n-1)}Q_{(i,n-1)}\Phi_{(i,n)} - Q_{(i-1,n)}\widetilde{Q}_{(i,n)}\Phi_{(i,n)}\right) \\ &+ \sum_{n=1}^{3g-3+\ell}\sum_{i=1}^{k} \frac{\tau_{YM,ni}}{8\pi \mathrm{i}} \operatorname{tr} W_{ni}^{\alpha} W_{ni,\alpha}\,. \end{aligned} \tag{1.178}$$

One may also generate other Lagrangian theories associated to spheres with one minimal puncture and two maximal puncture with a variety of different configurations $s_c^{o,\sigma}$ by, for example, turning on flux for $U(1)_t$. We will not consider these theories in this thesis.

Type-II Description

In this section we present the brane setups which we use to 'engineer' Lagrangian theories in class $\mathcal{S}_k$. We begin with the toroidal $\mathcal{N} = 1$ quiver theories in class $\mathcal{S}_k$ which are obtained using Type-IIB string theory with N D3 branes probing a $\mathbb{Z}_\ell \times \mathbb{Z}_k$ orbifold singularity (Table 1.6). They are examples of $\mathcal{N} = 1$ orbifold daughters of $\mathcal{N} = 4$ SYM [155, 156] and were extensively studied in the early days of AdS/CFT. After T-duality we land on Type-IIA string theory with N D4 and ℓ NS5 branes in the presence of a $\mathbb{Z}_k$ orbifold singularity, which was used in [133], this is the Hanany-Witten picture. From the Type-IIA setup we can consider a decoupling limit upon which we arrive at the 'core theories' corresponding to spheres with two maximal punctures and a collection of minimal punctures.

Type-IIB Consider Type-IIB string theory on $\mathbb{R}^4 \times \mathbb{R}^6/\Gamma$ with $\Gamma = \mathbb{Z}_\ell \times \mathbb{Z}_k$ with $\ell, k \in \mathbb{Z}^+$. Our goal is to engineer the class $\mathcal{S}_k$ theories corresponding to a torus with ℓ minimal punctures within Type-IIB string theory. Hence, we add a set of N parallel and coincident D3 branes along the $\mathbb{R}^4$ as described in Table 1.6. We parametrise the worldvolume of the D3 branes with four real coordinates X^1, X^2, X^3, X^4, which arrange themselves into the vector representation of $Spin(4) \cong SU(2)_\alpha \times SU(2)_{\dot\alpha}$. The Cartans j_1, j_2, of $\mathfrak{su}(2)_\alpha, \mathfrak{su}(2)_{\dot\alpha}$ are defined such that lower $\alpha = 1, 2$ have $j_1 = +\frac{1}{2}, -\frac{1}{2}$ and $\dot\alpha = \dot 1, \dot 2$ have $j_2 = +\frac{1}{2}, -\frac{1}{2}$. The $\mathbb{R}^6 \cong \mathbb{C}^3$ is parametrised by six real coordinates $X^5, X^6, X^7, X^8, X^9, X^{10}$ and the isomorphism is made by the

	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9	X^{10}
N D3	$-$	$-$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$A_{\ell-1}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\times$	$\times$
A_{k-1}	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\cdot$

Table 1.6: *Type-IIB setup engineering Lagrangian 4d SCFTs in class $\mathcal{S}_k$.*

choice of arrangement into the complex coordinates

$$z_1 := \frac{X^5 + \mathrm{i}X^6}{\sqrt{2}} = \Phi|_{\theta=0}, \quad z_2 := \frac{X^7 + \mathrm{i}X^{10}}{\sqrt{2}} = Q|_{\theta=0}, \tag{1.179}$$

$$z_3 := \frac{X^8 + \mathrm{i}X^9}{\sqrt{2}} = \widetilde{Q}|_{\theta=0} \tag{1.180}$$

and their hermitian conjugates. The orbifold Γ acts on those coordinates (1.178) as

$$(z_1, z_2, z_3) \mapsto \left(\omega_k z_1, \omega_\ell z_2, \omega_\ell^{-1}\omega_k^{-1} z_3\right), \quad \omega_k^k = \omega_\ell^\ell = 1\,. \tag{1.181}$$

Before the orbifold action, fundamental strings stretching between D3 branes give rise to the $SU(N)$ $\mathcal{N} = 4$ SYM theory on their worldvolume; with R-symmetry $Spin(6)_R \cong SU(4)_R$, the rotation group of the transverse $\mathbb{R}^6$. In $\mathcal{N} = 1$ superspace the theory contains a vector multiplet $\mathcal{V}$ and three chiral superfields in the adjoint of the gauge group: $\left(\Phi, Q, \widetilde{Q}\right)$ transforming in the $\mathbf{3}$ of $SU(3)_R \subset SU(4)_R$. The superpotential is given by

$$W_{\mathcal{N}=4} = \operatorname{tr} \Phi[Q, \widetilde{Q}] + \frac{\tau_{YM}}{8\pi \mathrm{i}} \operatorname{tr} W^\alpha W_\alpha\,. \tag{1.182}$$

The chiral superfields are identified with transverse coordinates (1.178) hence the action of Γ on $\mathbb{C}^3$ lies diagonally inside $SU(3)_R$ in the form

$$M := \begin{pmatrix} \omega_k & 0 & 0 \\ 0 & \omega_\ell & 0 \\ 0 & 0 & \omega_\ell^{-1}\omega_k^{-1} \end{pmatrix} \in SU(3)_R\,. \tag{1.183}$$

Note that Γ also has an action inside the gauge group, $SU(N)$ [157]. Its action can be conjugated to an element h of the maximal torus $T(SU(N)) = U(1)^{N-1}$. After scaling $N \to |\Gamma|N = \ell k N$ this action breaks $G \to \prod_{n=1}^{\ell} \prod_{i=1}^{k} SU(N_{ni})$ specified by a partition of $\ell k N = \sum_{n,i} N_{ni}$ into ℓk integers. Note we always take the orbifold indices to be $n, m = 1, \ldots, \ell$ and $i, j = 1, \ldots, k$ and we impose orbifold periodicity

$n \sim n + \ell$, $i \sim i + k$. To obtain the correct SCFT in class $\mathcal{S}_k$; we choose the action of Γ such that $N_{ni} = N$ for all n, i. Hence h may be written as

$$h = \operatorname{diag}\left(\omega_\ell \omega_k^{-1}\mathbb{I}, \ldots, \omega_\ell \omega_k^{-k}\mathbb{I}, \ldots, \omega_\ell^\ell \omega_k^{-1}\mathbb{I}, \ldots, \omega_\ell^\ell \omega_k^{-k}\mathbb{I}\right) \tag{1.184}$$

where $\mathbb{I}$ denotes the $N \times N$ identity matrix. Quotienting by Γ imposes the identifications

$$\mathcal{V} \sim h^\dagger \mathcal{V} h \,, \qquad \begin{pmatrix} \Phi \\ Q \\ \widetilde{Q} \end{pmatrix} \sim M \begin{pmatrix} h^\dagger \Phi h \\ h^\dagger Q h \\ h^\dagger \widetilde{Q} h \end{pmatrix} . \tag{1.185}$$

After performing these identifications the resulting theory is an $\mathcal{N} = 1$ torodial quiver gauge theory with gauge group $SU(N)^{\ell k}$ and superpotential given by (1.177). The fields now transform as $\Phi_{(n,i)} \in (N_{ni}, \overline{N}_{n(i-1)})$, $Q_{(n,i)} \in (N_{ni}, \overline{N}_{(n+1)i})$ and $\widetilde{Q}_{(n,i)} \in (N_{n(i-1)}, \overline{N}_{(n-1)i})$ under the gauge group $\prod_{n,i} SU(N_{ni}) = SU(N)^{\ell k}$. We summarise the field content in the quiver diagram of Figure 1.7. Horizontal lines between node (i, n) and $(i, n+1)$ denote $Q_{(i,n)}$ fields. Vertical lines between node (i, n) and $(i-1, n)$ denote $\Phi_{(i,n)}$ fields. Diagonal lines between $(i-1, n+1)$ and (i, n) denote $\widetilde{Q}_{(i,n)}$ fields. See also Section 2.2. The quiver should be periodically identified in both directions, such that it has the topology of a tessellation of the torus. The individual couplings for each gauge node $g^2_{YM,ni}$ are given by integration of a non-zero B-field flux over the two-cycles C_{ni} of the space obtained by resolving the $\mathbb{C}^3/\Gamma$ singularities

$$\int_{C_{ni}} B = \frac{4\pi^2}{g^2_{YM,ni}} \,, \qquad \sum_{n,i} \frac{1}{g^2_{YM,ni}} = \frac{1}{g^2_{YM}} \,. \tag{1.186}$$

These are precisely the same class of $\mathcal{N} = 1$ SCFTs which we expect to describe, at low energies, the 4d theory obtained by placing N M5-branes at the tip of an A_{k-1} singularity), compactified on T^2 with ℓ punctures and complex structure $\tau_{YM} = \frac{4\pi \mathrm{i}}{g^2_{YM}} + \frac{\theta}{2\pi}$ [133].

Type-IIA By performing a T-duality along, say, X^7 to the setup of Table 1.6 we may obtain the Hanany-Witten description of the above class $\mathcal{S}_k$ theories in Type-IIA as described in [133]. To perform the T-duality we may partially resolve the $\mathbb{C}^3/\Gamma$ singularity. Resolving the $A_{\ell-1}$ singularity gives rise to an ALE space which

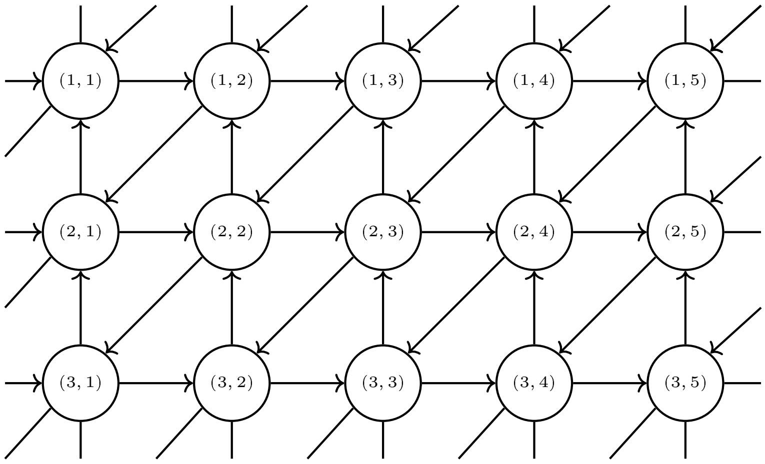


Figure 1.7: *Section of the quiver diagram of the $\mathbb{Z}_k \times \mathbb{Z}_\ell$ orbifold theory of $\mathcal{N} = 4$ SYM. Circular nodes denote $(S)U(N)$ vector multiplets and directed arrows denote chiral multiplets.*

is equivalent to the $\lambda \to \infty$ limit of the ℓ-centred Taub-Nut space TN_ℓ with metric

$$ds^2 = V^{-1}\left(d\Theta + \vec{A}\cdot d\vec{x}\right)^2 + Vd\vec{x}^2, \quad V = \sum_{n=1}^{\ell} \frac{1}{|\vec{x} - \vec{x}_n|} + \frac{1}{\lambda^2} \tag{1.187}$$

subject to the condition $\vec{\nabla}V = -\vec{\nabla} \times \vec{A}$. The underlying geometry is that of an $\mathbb{S}^1$ fibered over an $\mathbb{R}^3$ base. To perform the T-duality we hence replace $\mathbb{C}^3/\Gamma$ by $(\mathbb{C} \times \mathrm{TN}_\ell)/\mathbb{Z}_k$ where $\mathbb{C}$ is parametrised by $z_1, \overline{z}_1$ and TN_ℓ is parametrised by $\Theta = X^7$, $\vec{x} = \left(X^8, X^9, X^{10}\right)$. We may then T-dualise along the TN_ℓ circle (which is invariant under the $\mathbb{Z}_k$ action). We hence obtain the Hanany-Witten description shown in Table 1.7. Under the T-duality the ℓ centers of TN_ℓ become ℓ NS5-branes fixed at positions Θ_n and $\vec{x}_n$ in the transverse directions.

	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9	X^{10}
N D4	–	–	–	–	·	·	–	·	·	·
ℓ NS5	–	–	–	–	–	–	·	·	·	·
A_{k-1}	·	·	·	·	×	×	·	×	×	·

Table 1.7: *The Type-IIA setup obtained by a T-duality along X^7 to Table 1.6.*

By performing a decoupling limit $\tau_{\ell i} = 0$ we can 'open' the quiver and arrive at the theories corresponding to a sphere with $\ell - 2$ minimal punctures and two maximal punctures. This corresponds to replacing the $\mathbb{S}^1$ parametrised by X^7 with a finite

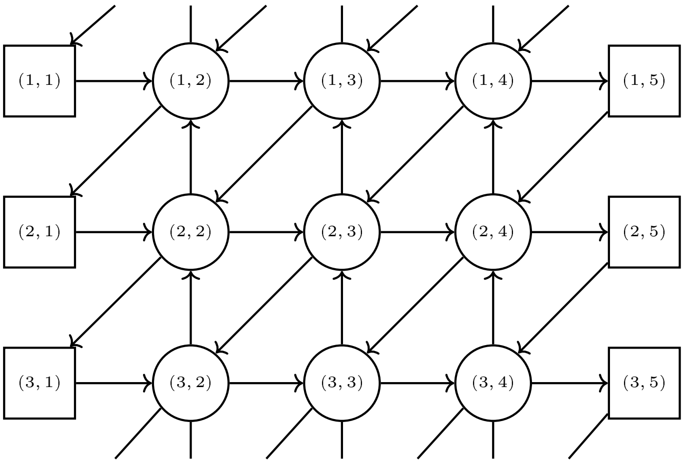


Figure 1.8: *Open quiver corresponding to a sphere with two maximal punctures and* $\ell - 2 = 4$ *minimal punctures. The quiver is still periodically identified in the '*k*'-direction.*

interval and terminating the open NS5-branes with D6-branes. This gives quivers of the type shown in Figure 1.8.

1.7 Thesis Overview

This thesis begins a programme towards the computation of exact results for $\mathcal{N} = 1$ supersymmetric theories in four dimensions, particularly those of class $\mathcal{S}_k$.

In Chapter 2 we analyse the moduli spaces $\mathbf{M}$ of theories of class $\mathcal{S}_k$. In particular we demonstrate that $\mathcal{N} = 1$ analogues of Higgs and Coulomb branches may be defined. This definition relies on the orbifold structure. As for $\mathcal{N} = 2$ theories these branches often have special properties, making them particularly nice to study. For example, the Coulomb branch chiral ring is freely generated [154]. The main tool that we use to characterise these branches is the Hilbert series. We compute the Hilbert series for a variety of theories in class $\mathcal{S}_k$. In some cases the Hilbert series for the Higgs branch is equal to a certain limit of the superconformal index which, in $\mathcal{N} = 2$ nomenclature, we refer to as the Hall-Littlewood index.

In Chapter 3 we focus on the Coulomb branch for certain theories within class $\mathcal{S}_k$. Following Seiberg and Intriligator [20] we derive the $\mathcal{N} = 1$ holomorphic curves that encode the low energy superpotential for theory on the Coulomb branch. The curves contain information regarding the effective coupling matrix as well as potentially knowing about various dualities, as for class $\mathcal{S}$ theories [121].

Chapter 4 is dedicated to instantons in a certain subset of class $\mathcal{S}_k$. These theories are those living on a stack of N D3-branes on transverse $\mathbb{C}^3/(\mathbb{Z}_k \times \mathbb{Z}_\ell)$ in Type-IIB sring theory. Using the correspondence between instantons and D(-1)-branes we describe the ADHM construction for these theories as a matrix model. We then compute the partition function of this matrix model. This partition function is identified with the partition function over instanton configurations for this $\mathcal{N} = 1$ gauge theory; the analogue of the Nekrasov partition function. This computation is performed by uplifting the matrix model to a two dimensional gauge theory in the presence of a defect; the Elliptic genus (2d supersymmetric index) of the 2d theory with defect is computed. The matrix model partition function is then obtained as the zero area limit of the Elliptic genus. This partition function is then identified with conformal blocks of the $\mathcal{W}_{kN}$ algebra, verifying a conjecture given in [153].

In Chapter 5 we change our attention to four dimensional $\mathcal{N} = 3$ theories. This is motivated by the recent discovery [158] of genuine $\mathcal{N} = 3$ supersymmetric theories engineered using a special quotient geometry within F-theory dubbed S-fold. The existence of yet more new $\mathcal{N} = 3$ theories was conjectured in [159], by means of gauging a discrete symmetry of $\mathcal{N} = 4$ that emerges at strong coupling. These constructions bypasses the no-go theorems stating that every CPT-complete $\mathcal{N} = 3$ theory automatically enhances to $\mathcal{N} = 4$ supersymmetry; by means of the theories having no Lagrangian description. We focus out attention on the theories constructed as in [159] and their higher rank generalisations. We review the details regarding both the construction of these theories as well as their various universal properties. We then focus on the computation of two type of exact results for these theories; being the Coulomb limit of the supersymmetric index and the Hilbert series for the Higgs branch. These quantities encode several important features of the moduli space of supersymmetric vacua for these theories. In certain special cases we are also able to compute the fully refined supersymmetric index.

Chapter 6 is dedicated to $\frac{1}{2}$-BPS defects in six dimensional $\mathcal{N} = (1,0)$ SCFTs. The SCFTs that we focus on, when compactified on a punctured Riemann surface $\mathcal{C}$, are precisely those which give rise to theories of class $\mathcal{S}_k$, namely the $(1,0)_{A_{k-1}}$ theories of type $\mathfrak{g} = A_{N-1}$. The $\frac{1}{2}$-BPS defects are amongst those which may be inserted at the punctures to engineer various $\mathcal{S}_k$ theories. In this chapter we focus on the self-dual (tensionless) strings of the 6d theories in the presence of the defect. In the class $\mathcal{S}_k$ picture wrapping these strings on $\mathcal{C}$ gives rise to point-like BPS states in the 4d theory. The strings admit a dual effective 2d-gauge theory description living on their worldvolume. We describe this gauge theory and compute its Elliptic genus. These strings provide the main contribution to the $T^2 \times \mathbb{R}^4$ BPS-partition function for the 6d theory and it can be written in a expansion over Elliptic genera. We also perform a similar computation for the supersymmetric index of 5d $\mathcal{N} = 1$ theories in the presence of defects.

Appendix A collects the various definitions, identities and special functions used throughout the thesis. Appendix B provides a mathematical overview of the relevant statements & theorems algebraic geometry that are necessary to understand affine varieties, coordinate rings and Hilbert series. Appendices C-G are chapter specific appendices. Appendix H provides additional details regarding the relationship between the Hall-Littlewood index and the Higgs branch Hilbert series for theories of class $\mathcal{S}$. Finally, Appendix I contains a generalisation of the ramified instanton counting performed in Section 6.

Chapter 2

The Higgs and Coulomb Branches of Theories of Class $\mathcal{S}_k$

The work in this chapter is based on [160].

2.1 Introduction

We have seen that supersymmetric theories have a moduli space $\mathbf{M}$ (1.153) of supersymmetric vacua parametrised by vevs for gauge invariant, scalar, chiral operators. Moreover, for theories with eight or more supercharges we can define two interesting sub-branches of $\mathbf{M}$, namely the Coulomb branch $\mathbf{CB}$ (in which the gauge group is typically broken to an abelian subgroup $G \to U(1)^{\dim \mathbf{CB}}$) and the Higgs branch $\mathbf{HB}$ (in which the gauge group is typically completely broken $G \to \{1\}$). For concreteness let us specialise to four dimensions. Then the Coulomb branch is reached by allowing the scalars Φ in vector multiplets to acquire vevs while setting to zero hypermultiplet vevs $(Q, \widetilde{Q})$, hence the operators on $\mathbf{CB}$ satisfy $E = -r_{\mathcal{N}=2}$ and $R_{\mathcal{N}=2} = 0$. On the other hand the Higgs branch is reached by the opposite; setting $\phi = 0$ while allowing $(Q, \widetilde{Q})$ to be non-zero, these have $r_{\mathcal{N}=2} = 0$ and $E = 2R_{\mathcal{N}=2}$. As we have seen, $\mathbf{HB}$ defined in this way turns out to be a hyperKähler manifold.

Now, for generic 4d $\mathcal{N} = 1$ theories, there is typically no sensible definition for analogues of $\mathbf{CB}$ and $\mathbf{HB}$. This is because, with only $\mathcal{N} = 1$ supersymmetry, there is only one type of chiral multiplet who's lowest components is a scalar, whereas for $\mathcal{N} = 2$ there are two (vector multiplets and hypermultiplets). Put another way, $\mathcal{N} = 2$ supersymmetry has a rank $(\mathfrak{su}(2)_{R_{\mathcal{N}=2}} \oplus \mathfrak{u}(1)_{r_{\mathcal{N}=2}}) = 2$ R-symmetry algebra with which to select operators; $\mathcal{N} = 1$ theories have only rank $\mathfrak{u}(1)_r = 1$.

As we will show in this chapter, if we consider a class of non-generic $\mathcal{N} = 1$ theories; namely those of class $\mathcal{S}_k$, it is possible to define analogues of **CB** and **HB**. Because theories of class $\mathcal{S}_k$ arise as orbifolds of $\mathcal{N} = 2$ theories we can exploit the orbifold structure to define them.

2.2 Moduli Space of Supersymmetric Vacua

Gauge theories with $\mathcal{N} = 1$ supersymmetry can often have a moduli space of supersymmetric vacua $\mathbf{M}$ given by the space of solutions to the F-term and D-term constraints modulo gauge transformations by gauge group G. It is well known [104], at the level of the moduli space, that setting the D-terms to zero and modding out by G is equivalent to dropping the D-term constraints and modding out by complexified gauge transformations $G^{\mathbb{C}}$. Therefore the moduli space can be described by the following symplectic quotient between the *master space* $\mathbf{F}$ and the complexified gauge group

$$\mathbf{M} \simeq \mathbf{F}/G^{\mathbb{C}}\,, \quad \mathbf{F} = \{(v_1, v_2, \dots)|F_1 = F_2 = \dots = 0\}\,, \tag{2.1}$$

here v_i denotes the scalar vevs and $F_i = \partial W/\partial v_i$ are the F-term constraints, where W is the superpotential. $\mathbf{F}$ is a complex algebraic variety. It can be characterised in terms of a quotient ring R/I where $R = \mathbb{C}[v_1, v_2, \dots]$ is the polynomial ring in the v_i and $I = \langle F_1, F_2, \dots \rangle$ is the ideal generated by the F-terms (see Appendix B for more details regarding the *algebra-geometry dictionary*) .

In generic $\mathcal{N} = 1$ gauge theories, the superpotential can receive quantum corrections up to 1-loop in perturbation theory, or from non-perturbative effects encoded in $W_{\text{n.p.}}$; for example the ADS-superpotential [161]. The full superpotential can then be written as

$$W = W_{\text{Classical}} + W_{\text{1-loop}} + W_{\text{n.p.}}\,. \tag{2.2}$$

The definition (2.1) can also be applied to define the moduli space of the classical theory

$$\mathbf{M}_{\text{Classical}} \simeq \mathbf{F}_{\text{Classical}}/G^{\mathbb{C}}\,, \tag{2.3}$$

$$\mathbf{F}_{\text{Classical}} = \left\{ (v_1, v_2, \dots) \middle| \frac{\partial W_{\text{Classical}}}{\partial v_1} = \frac{\partial W_{\text{Classical}}}{\partial v_2} = 0 \right\}\,. \tag{2.4}$$

In general $\mathbf{M} \ncong \mathbf{M}_{\text{Classical}}$.

2.2.1 The Moduli Space M and its Branches

In [96] and [162, 105, 106] the moduli spaces of a large class of 4d $\mathcal{N} \geqslant 1$ have been studied. The authors of these articles considered QFTs living on a stack of N D3-branes with transverse space $\mathcal{X}$ a Calabi-Yau three-fold. They focused their attention on the IR physics of this system, where all the abelian symmetry decouples and the gauge symmetry is completely non-Abelian. Therefore all the abelian factors are not gauged and appear only as global symmetry of the QFT taken into account.

In general the full moduli space $\mathbf{M}$ can have several branches, such as mesonic and baryonic branches or Coulomb and Higgs branches. Moreover these branches do not necessarily appear as irreducible components of the full moduli space but in general could be non-trivially merged into each other. Nevertheless, even if in general the baryonic and the mesonic branch are mixed, it is still makes sense to define the *mesonic moduli space* ${}^{\mathrm{mes}}\mathbf{M}$ as a sub-branch of $\mathbf{M}$.

In order to do this, following [96], let's begin considering the case of just one D3-brane probing the Calabi-Yau three-fold $\mathcal{X}$. The corresponding IR theory is free, therefore the moduli space $\mathbf{M}$ coincides with the master space $\mathbf{F}$. The mesonic moduli space is obtained imposing the $U(1)_D$ D-term constraints. It is given by the symplectic quotient

$$ {}^{\mathrm{mes}}\mathbf{M} := \mathbf{M} // U(1)_D \, . \tag{2.5} $$

From a physical point of view the above quotient corresponds to eliminating from the spectrum all the gauge invariants operators that are charged under the set of $U(1)_D$ symmetries. Therefore the branches that are eliminated taking the above quotient correspond to baryonic directions. Clearly, for the case of just one D3-brane, it's not possible to talk about baryons and the above directions are interpreted as turning on VEVs for Fayet-Iliopoulos background fields.

For the $N = 1$ case of a single D3-brane the mesonic moduli space is simply the Calabi-Yau itself ${}^{\mathrm{mes}}\mathbf{M} \cong \mathcal{X}$. The $N > 1$ result is then obtained as the N^{th}-symmetric product of the $N = 1$ case [163], i.e.

$$ {}^{\mathrm{mes}}\mathbf{M} \cong \mathrm{Sym}^N(\mathcal{X}) \ . \tag{2.6} $$

The approach used in [96] is based on the so called *geometry-algebra dictionary.* One of the main outcomes is that, at least for the $N = 1$ case, the master space always has an irreducible component that turns out to be a mesonic branch. Moreover, on this branch, the scalar fields arising from all $\mathcal{N} = 1$ chiral multiplet can have a

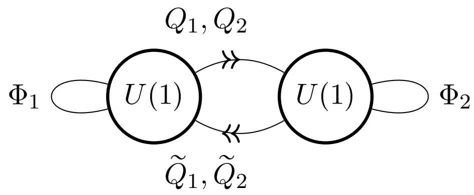


Figure 2.1: *Quiver for the $\mathcal{X} = \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$ theory.*

non-trivial VEV.

This algebraic approach has the advantage of providing a systematic procedure to decompose the master space of a theory into irreducible components. However, from our perspective, this procedure suffers of two main limitations:

- The decomposition of the master space $\mathbf{F}$ into irreducible components does not necessarily provide a decomposition of the moduli space $\mathbf{M}$ into irreducible components (an irreducible decomposition of $\mathbf{F}$ does not necessarily descend to one for the quotient $\mathbf{F}/G^{\mathbb{C}}$).

- The decomposition of the master space does not necessarily provide a clear identification of the Higgs branch and Coulomb branch of the theory as defined in Section 2.2.2.

An Example: $\mathcal{X} = \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ Theory

In order to exemplify the above statements and see how the algebraic procedure works let's review the $\mathcal{X} = \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_2$ theory. This theory has enhanced $\mathcal{N} = 2$ supersymmetry. The quiver diagram for this theory can be found in Figure 2.1. The corresponding superpotential reads

$$W = \Phi_1(Q_1\widetilde{Q}_1 - Q_2\widetilde{Q}_2) + \Phi_2(\widetilde{Q}_2Q_2 - \widetilde{Q}_1Q_1)\,. \tag{2.7}$$

After primary decomposition the F-terms ideal can be rewritten as

$$I = J_1 \cap J_2\,, \quad J_1 = \langle \Phi_1 - \Phi_2,\ Q_1\widetilde{Q}_1 - Q_2\widetilde{Q}_2 \rangle\,, \quad J_2 = \langle Q_1, Q_2, \widetilde{Q}_1, \widetilde{Q}_2 \rangle\,, \tag{2.8}$$

one can check that J_1, J_2 are *prime ideals*. Therefore the master space $\mathbf{F}$ reads

$$\mathbf{F} = \mathfrak{V}(J_1) \cup \mathfrak{V}(J_2)\,, \tag{2.9}$$

where $\mathfrak{V}$ is the map that associates to each prime ideal J the corresponding irreducible variety $\mathfrak{V}(J)$, defined in (B.9). $\mathfrak{V}(J_1) = \mathbb{C}\times\mathcal{C}$, is a the trivial bundle between a conifold (parametrized by $\{Q_{i=1,2}, \widetilde{Q}_{i=1,2}\}$) and a $\mathbb{C}$-line (defined by $\Phi_1 = \Phi_2$). On the other hand $\mathfrak{V}(J_2)$ is a $\mathbb{C}^2$ space parametrized by $\{\Phi_1, \Phi_2\}$. Let's now consider the moduli space $\mathbf{M}$. After integration over the gauge group we observe that the variety $\mathfrak{V}(J_1)$ becomes $\mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}$, where we can take the factor $\mathbb{C}^2/\mathbb{Z}_2$ to be parametrized by the gauge invariant combinations $\{x = Q_1\widetilde{Q}_1, y = Q_1\widetilde{Q}_2, z = Q_2\widetilde{Q}_1\}$ that satisfy the relation $x^2 = yz$. On the other hand the $\mathbb{C}$-line factor is parametrized by $v = \Phi_1 = \Phi_2$. While the second variety $\mathfrak{V}(J_2)$ does not change after the projection to gauge invariant operators. Therefore ${}^{\text{mes}}\mathbf{M}$ is given by the union of two branches that intersect in a non-trivial way. Therefore, even in this simple $\mathcal{N} = 2$ example, there is **not** a clear separation between the Higgs branch (where only the scalars inside the $\mathcal{N} = 2$ hypermultiplets are taking a VEV) and Coulomb branch of the theory (that is parametrized by the VEV of the scalars inside the $\mathcal{N} = 2$ vector multiplet).

Therefore in this chapater we plan to provide a definition of the Higgs branch of a particular class of $\mathcal{N} = 1$ theories without refering to the algebra-geometry dictionary. We will then outline the relation between the Higgs branch and the the mesonic branch (2.5), at least in the case of the abelian theories discussed in [96].

2.2.2 Higgs and Coulomb Branches for Class $\mathcal{S}_k$

Now let us specialise our discussion to theories in Class $\mathcal{S}_k$. The full moduli space for these theories is typically very rich. They share many similarities with $\mathcal{N} = 2$ theories, in particular they possess Coulomb, Higgs and mixed phases. On the other hand our 'core theories' (see Section 1.6.2) possess the nice property that their quantum moduli space $\mathbf{M}$ (2.1) coincides with the classial moduli space $\mathbf{M}_{\text{Classical}}$ (2.3) in the sense that, althought there may be dependence on the quantum scale Λ it is possible always to find an isomorphism $\mathbf{M} \cong \mathbf{M}_{\text{Classical}}$. This can be seen as follows: from the point of view of the $(i,n)^{\text{th}}$ gauge node, see the quivers diagrams of Figures 1.8 & 1.7, the theory is SQCD with $N_f = 3N_c$. It is known that the quantum moduli space of SQCD with $N_f \geqslant N_c + 1$ coincides with the classical one [164]. Therefore we have

$$\mathbf{M} \cong \mathbf{M}_{\text{Classical}}\,, \tag{2.10}$$

this also follows for all sub-branches, hence from now on we will refer only to $\mathbf{M}$.

The Coulomb moduli spaces of these theories have been studied in [137, 154].

Here we will mainly focus on two truncations of the full moduli space which, in $\mathcal{N}=2$ nomenclature, we will refer to as *Higgs* and *Coulomb* branches.

The Higgs Branch

Let us first present the definition for the Higgs branch for theories of class $\mathcal{S}_k$. This definition is valid for theories either with a Lagrangian description or those related to theories with a Lagrangian description by dualities and can be made by restricting to scalar operators that have

$$E = 2R_{\mathcal{N}=2} = r + \frac{2q_t}{3}\,, \quad r_{\mathcal{N}=2} = \frac{2q_t}{3} - \frac{r}{2} = 0\,, \tag{2.11}$$

where q_t and r denote the generators of the $\mathfrak{u}(1)_t$ and $\mathcal{N}=1$ $\mathfrak{u}(1)_r$ R-symmetry respectively. Provided that the R-symmetry is not broken (which is true for any $\mathcal{N}=1$ SCFT) and that $\mathfrak{u}(1)_t$ is also non-broken we can always decompose the ring $\mathcal{R}$ under the $\mathfrak{u}(1)_r \oplus \mathfrak{u}(1)_t$ grading.

For our basic core theories corresponding to spheres with two maximal and a collection of minimal punctures (obtained by Φ-gluing collections of three punctured spheres) this definition coincides with turning on generic diagonal vevs for scalars in the chiral multiplets associated to the free trinion while setting to zero vevs for the scalars in the chiral multiplets coming from Φ-gluing. Those choices of vevs completely breaks the $SU(N)$ gauge symmetry at each node. Hence, we refer to this sub-branch of $\mathbf{M}$ as the *Higgs branch*

$$\mathbf{HB} = \mathbf{F_H}/G^{\mathbb{C}}\,, \quad \mathbf{F_H} = \left\{ Q_{(i,n)}, \widetilde{Q}_{(i,n)} \,\middle|\, F_{(j,m)} = 0 \right\}\,. \tag{2.12}$$

where $Q_{(i,n)}, \widetilde{Q}_{(i,n)}$ are those scalar vevs of the theory which have $r_{\mathcal{N}=2} = 0$. The only non-trivial F-terms in a Lagrangian theory using the building blocks we described in Section 1.6.2 associated to a Riemann surface of genus g with ℓ punctures on this branch are therefore just $F_{(i,n)} = \partial W_{\mathcal{S}_k}/\partial \Phi_{(i,n)}$ where $W_{\mathcal{S}_k}$ is given in (1.177). The space $\mathbf{HB}$ is a Kähler manifold, as we have explained in Section 1.5.

The coordinate ring of $\mathbf{F_H}$ can be described as the quotient ring $F_H = R_H/I_H$ where

$$R_H = \mathbb{C}[Q_{(i,n)}, \widetilde{Q}_{(i,n)}]\,, \tag{2.13}$$

$$I_H = \langle F_{(1,1)}, \ldots, F_{(k,1)}, \ldots, F_{(1,3g-3+\ell)}, \ldots, F_{(k,3g-3+\ell)} \rangle\,. \tag{2.14}$$

The coordinate ring HB of $\mathbf{HB}$ is given by taking the G-invariant polynomials $HB = (R_H/I_H)^G$.

We are now in a position to define the Higgs branch Hilbert series for our class $\mathcal{S}_k$ theories. The grading on the ring R_H can be paramterised by a fugacity τ for the generator $E = 2q_t = \frac{3}{2}r$, from the conditions (2.11), as well as the fugacities for $U(1)_\beta^{k-1} \times U(1)_\gamma^{k-1}$ and any other global symmetries. The Higgs branch Hilbert series is then defined to be [107, 108]

$$\mathrm{HS}(\tau; HB) := \mathrm{Tr}_{HB}\, \tau^E e^{-\mathcal{J}}\,, \tag{2.15}$$

where Tr_{HB} denotes the trace over the space of operators parametrising the Higgs branch (2.12) and $e^{-\mathcal{J}}$ collectively denotes the fugacities for the remaining intrinsic $U(1)_\gamma^{k-1} \times U(1)_\beta^{k-1}$ and any other global symmetries. See also [110, 165] for a detailed analysis of the Hilbert series for $\mathcal{N} = 1$ SQCD. For gauge theories (2.15) takes the general form of a integral over the gauge group G of the Hilbert series of the master space $\mathbf{F_H}$ which we denote by $f_H^\flat$

$$\mathrm{HS}(\tau; HB) = \oint d\mu_G(\mathbf{z}) f_H^\flat(\tau, \mathbf{z}, \dots)\,, \tag{2.16}$$

where $d\mu_G(\mathbf{z})$ denotes the Haar measure of G.

Examining Table C.3, where short representations of the $\mathcal{N} = 1$ superconformal algebra are listed, it is clear that the Hilbert series (2.15) counts the top components of $\overline{\mathcal{D}}_{(0,0)}$ and $\overline{\mathcal{B}}_{r,(0,0)}$ multiplets of the $\mathcal{N} = 1$ superconformal algebra. These multiplets have $E = \frac{3}{2}r$ and $j_1 = j_2 = 0$. We can therefore see that, for $k = 1$, this definition does indeed coincide with the usual Higgs branch definition for $\mathcal{N} = 2$ theories.

The Coulomb Branch

We can also make a similar definition for the Coulomb branch. Namely, analogously to $\mathcal{N} = 2$ theories, we may define a consistent truncation of the moduli space by restricting those operators which have

$$E = -r_{\mathcal{N}=2} = -\frac{2q_t}{3} + \frac{r}{2}\,, \quad R_{\mathcal{N}=2} = \frac{r}{2} + \frac{q_t}{3} = 0\,. \tag{2.17}$$

For Lagrangian theories this coincides with setting the scalars arising from the free trinions to zero while giving the scalar in the chiral multiplets coming from the Φ-

gluing generic diagonal vevs. On this branch the gauge symmetry is broken down to the stabiliser subgroup of $SU(N)^{k(3g-3+\ell)}$ with respect to the vevs $\Phi_{(i,n)}$ which is given by $U(1)^{(k-1+\delta_{k,1})(3g-3+\ell)}$ where ℓ is the number of punctures. We have that

$$\mathbf{CB} = \mathbf{F_C}/G^{\mathbb{C}}\,, \quad \mathbf{F_C} = \left\{\Phi_{(i,n)}\right\}\,, \tag{2.18}$$

where the $\Phi_{(i,n)}$ are those scalar vevs of the theory that have $R_{\mathcal{N}=2} = 0$. On this branch all of the F-terms in a Lagrangian theory are trivial because they are all proportional to either a Q or $\widetilde{Q}$. Therefore $\mathbf{F_C}$ is simply associated to the freely generated ring $R_C = \mathbb{C}[\Phi_{(i,n)}]$. Consequently $CB = (R_C)^G$. Similarly, we may also define the Hilbert series for the Coulomb branch

$$\mathrm{HS}(T;CB) := \mathrm{Tr}_{CB}\, T^E e^{-\mathcal{J}}\,, \tag{2.19}$$

where Tr_{CB} denotes the trace over the space of operators parametrising (2.18). In other words, the space of scalar operators of the theory satisfying (2.17) ($E = \frac{3}{2}r = -q_t$). Because the F-terms are trivial, for Lagrangian theories the F-flat Hilbert series (2.19) may be computed by multiplying the contribution coming from each Φ-multiplet and integrating over the gauge group. One extra simplification that arises is the fact that, because $Q = \widetilde{Q} = 0$, the $(i,n)^{\mathrm{th}}$ node in the quiver is not coupled to the $(j, m \neq n)^{\mathrm{th}}$. Therefore (2.19) reduces to a product of factors associated to the Φ-gluing of colour c and positive sign $\sigma = +$

$$\mathrm{HS}(T;CB) = \prod_{\Phi_c^+} h_{\Phi_c^+}\,, \quad h_{\Phi_c^+} = \oint d\mu\, f_{\Phi_c^+}\,, \tag{2.20}$$

$$f_{\Phi_c^+} = \prod_{i=1}^{k} \frac{(1-T)^{\delta_{1,k}}}{\prod_{A,B=1}^{N}\left(1 - T\frac{\gamma_i}{\beta_{i+c-2}}\frac{z_{i,A}}{z_{i-1,B}}\right)}\,, \tag{2.21}$$

the z's denote fugacities for the product gauge group, which are integrated over using the invariant measure $d\mu$. We notice that the factor $f_{\Phi_c^+} = \mathcal{I}^{\mathrm{C}}_{\Phi_c^+}$ is precisely that of the Coulomb limit of the index of the Φ-gluing factor (2.42). Therefore, we can conclude, for these Lagrangian theories that $\mathrm{HS}(T;CB) = \mathcal{I}^{\mathrm{C}}$ where the Coulomb branch index is defined in (2.41). The integrals $h_{\Phi_c^+}$ have been computed in [154]

$$h_{\Phi_c^+} = \mathrm{PE}\left[\sum_{i=1}^{k}\frac{\gamma_i^N}{\beta_{i+c-2}^N}T^N + \sum_{A=1}^{N-1}T^{Ak} - \delta_{1,k}T\right]\,, \tag{2.22}$$

where PE is defined in (A.1).

2.3 The Superconformal Index and Unrefined Limits

The right-handed $\mathcal{N} = 1$ superconformal index computed with respect to $\widetilde{\mathcal{Q}}_{\dot{-}}$ is given by [62, 64]

$$\begin{aligned} \mathcal{I}\,(t,p,q,\dots) &= \mathrm{Tr}(-1)^F p^{j_1+j_2+\frac{r}{2}-\frac{2q_t}{3}} q^{-j_1+j_2+\frac{r}{2}-\frac{2q_t}{3}} t^{q_t} e^{-\mathcal{J}} e^{-\beta\widetilde{\delta}_{\dot{-}}} \\ &= \mathrm{Tr}(-1)^F \sigma^{\frac{1}{2}\delta_{1+}} \rho^{\frac{1}{2}\delta_{1-}} \tau^{\frac{1}{2}\widetilde{\delta}_{2\dot{+}}} e^{-\mathcal{J}} e^{-\beta'\widetilde{\delta}_{\dot{-}}} \end{aligned} \tag{2.23}$$

where Tr denotes the trace over the Hilbert space on $\mathbb{S}^3$ in the radial quantisation, (E, j_1, j_2, r) denote the Cartans of the maximal compact bosonic subalgebra $\mathfrak{u}(1)_E \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus \mathfrak{u}(1)_r \subset \mathfrak{su}(2,2|1)$, q_t denotes the generator for the 'intrinsic' global $\mathfrak{u}(1)_t$ symmetry and we have defined

$$\delta_{1\pm} = E \pm 2j_1 - \frac{r}{2} - \frac{4q_t}{3}\,, \quad \widetilde{\delta}_{2\dot{\pm}} = E \pm 2j_2 + \frac{r}{2} + \frac{4q_t}{3}\,, \tag{2.24}$$

$$p = \tau\sigma\,, \quad q = \tau\rho\,, \quad t = \tau^2\,. \tag{2.25}$$

The superconformal index (2.23) receives contributions only from those states satisfying

$$\widetilde{\delta}_{\dot{-}} = \widetilde{\delta}_{1\dot{-}} = 2\{\widetilde{\mathcal{Q}}_{\dot{-}}, \widetilde{\mathcal{S}}^{\dot{-}}\} = E - 2j_2 - \frac{3}{2}r = 0\,. \tag{2.26}$$

Special attention should be paid to the fugactity t; when $k = 1$ the combinations

$$R_{\mathcal{N}=2} = \frac{r}{2} + \frac{q_t}{3}\,, \quad r_{\mathcal{N}=2} = \frac{2q_t}{3} - \frac{r}{2}\,, \tag{2.27}$$

and (2.24) are elements of the enhanced $\mathfrak{su}(2,2|2)$ superconformal algebra, see Table 2.1. When $k \geqslant 2$ there is generically no $\mathcal{N} = 2$ enhancement and q_t generates a global $U(1)_t$ symmetry of the corresponding theory. Finally, $e^{-\mathcal{J}}$ collectively denotes the fugacities for the remaining intrinsic $U(1)_\gamma^{k-1} \times U(1)_\beta^{k-1}$ and any other global symmetries. Note that if the lowest component of a chiral superfield is given by f then the fermion which also contributes to the index has

$$\delta_{1\pm}[\widetilde{\mathcal{Q}}_{\dot{+}}\overline{f}] = 2 - \delta_{1\pm}[f]\,, \quad \widetilde{\delta}_{2\dot{+}}[\widetilde{\mathcal{Q}}_{\dot{+}}\overline{f}] = 4 - \widetilde{\delta}_{2\dot{+}}[f]\,. \tag{2.28}$$

$\mathcal{Q}$	j_1	j_2	$R_{\mathcal{N}=2}$	$r_{\mathcal{N}=2}$	r	q_t	$\delta = 2\{\mathcal{Q},\mathcal{S}\}$
$\mathcal{Q}_{1\pm}$	$\pm\frac{1}{2}$	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$+\frac{1}{3}$	$+1$	$\delta_{1\pm} = E \pm 2j_1 - \frac{r}{2} - \frac{4q_t}{3}$
$\mathcal{Q}_{\pm} = \mathcal{Q}_{2\pm}$	$\pm\frac{1}{2}$	0	$-\frac{1}{2}$	$+\frac{1}{2}$	-1	0	$\delta_{\pm} = \delta_{2\pm} = E \pm 2j_1 + \frac{3r}{2}$
$\widetilde{\mathcal{Q}}_{\dot\pm} = \widetilde{\mathcal{Q}}_{1\dot\pm}$	0	$\pm\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	$+1$	0	$\widetilde{\delta}_{\dot -} = \widetilde{\delta}_{1\dot\pm} = E \pm 2j_2 - \frac{3}{2}r$
$\widetilde{\mathcal{Q}}_{2\dot\pm}$	0	$\pm\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{3}$	-1	$\widetilde{\delta}_{2\dot\pm} = E \pm 2j_2 + \frac{r}{2} + \frac{4q_t}{3}$

Table 2.1: *Supercharges of the $\mathfrak{su}(2,2|2)$ superalgebra and its $\mathfrak{su}(2,2|1)$ subsuperalgebra. The orbifold breaks $\mathfrak{su}(2,2|2) \to \mathfrak{su}(2,2|1) \oplus \mathfrak{u}(1)_t$ and projects out any supercharge with $q_t \neq 0$. In radial quantisation $\mathcal{S} = \mathcal{Q}^\dagger$.*

	j_1	j_2	r	q_t	Index	δ_{1+}	δ_{1-}	$\widetilde{\delta}_{2\dot+}$
Q_i	0	0	$\frac{2}{3}$	$\frac{1}{2}$	$\sqrt{t}\frac{\beta_{i+c-1}}{\alpha}\chi_{l,i}\overline{\chi}_{r,i}$	0	0	2
$\overline{\psi}_{\dot+i}$	0	$+\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{pq}{\sqrt{t}}\frac{\alpha}{\beta_{i+c-1}}\chi_{r,i}\overline{\chi}_{l,i}$	2	2	2
$\widetilde{Q}_i$	0	0	$\frac{2}{3}$	$\frac{1}{2}$	$\sqrt{t}\frac{\alpha}{\gamma_i}\chi_{r,i-1}\overline{\chi}_{l,i}$	0	0	2
$\overline{\widetilde{\psi}}_{\dot+i}$	0	$+\frac{1}{2}$	$\frac{1}{3}$	$-\frac{1}{2}$	$-\frac{pq}{\sqrt{t}}\frac{\gamma_i}{\alpha}\chi_{l,i}\overline{\chi}_{r,i-1}$	2	2	2
$\partial_{\pm\dot+}$	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	p, q	0, 2	2, 0	2, 2

Table 2.2: *Letters satisfying the BPS condition* (2.26) *for the free trinion theory associated to a sphere with one minimal puncture (with associated $U(1)$ valued fugacity α) and two maximal punctures $s_c^{l,+}$ and $s_{c+1}^{r,+}$. Here $\chi_{o,i} \equiv \chi_{(1,0,\ldots,0)}(\mathbf{z}_{o,i})$, $\overline{\chi}_{o,i} \equiv \chi_{(0,0,\ldots,1)}(\mathbf{z}_{o,i})$ are shorthand for the characters of the fundamental and anti-fundamental representations of $SU(N)$, defined in* (A.59). *We importantly note that $\delta_{1\pm}, \widetilde{\delta}_{2\dot+} \geqslant 0$.*

This implies that, for any chiral superfield f, the condition that each state contributing to the index has $\delta_{1\pm}, \widetilde{\delta}_{2\dot+} \geqslant 0$ is equivalent to

$$0 \leqslant \delta_{1\pm}[f] \leqslant 2\,, \quad 0 \leqslant \widetilde{\delta}_{2\dot+}[f] \leqslant 4\,. \tag{2.29}$$

We reviewed the construction of the basic Lagrangian theories in class $\mathcal{S}_k$ in Section 1.6.2. The letters of the free trinion of Figure 1.5 that contribute to the index are listed in Table 2.2. The free trinion contributes to the index a factor

$$\mathcal{I}_{s_c^{l,+},s_{c+1}^{r,+}} = \prod_{i=1}^{k}\prod_{A,B=1}^{N}\Gamma_e\left(\frac{\sqrt{t}\beta_{i+c-1}}{\alpha}\frac{z_{l,i,B}}{z_{r,i,A}}\right)\Gamma_e\left(\frac{\sqrt{t}\alpha}{\gamma_i}\frac{z_{r,i-1,B}}{z_{l,i,A}}\right)\,, \tag{2.30}$$

where $\Gamma_e(z)$ denotes the Elliptic Gamma function, defined in (A.22) and with $\prod_{i=1}^{k}\gamma_i = \prod_{i=1}^{k}\beta_i = 1$. We will also sometimes adopt the notation $\mathcal{I}_{s_c^{l,+},s_{c+1}^{r,+}} =$

	j_1	j_2	r	q_t	Index	δ_{1+}	δ_{1-}	$\widetilde{\delta}_{2\dot{+}}$
Φ_i	0	0	$\frac{2}{3}$	-1	$\frac{pq}{t}\frac{\gamma_i}{\beta_{i+c-2}}\chi_i\overline{\chi}_{i-1}$	2	2	0
$\overline{\lambda}_{\dot{+}i}$	0	$+\frac{1}{2}$	$+\frac{1}{3}$	$+1$	$-t\frac{\beta_{i+c-2}}{\gamma_i}\chi_{i-1}\overline{\chi}_i$	0	0	4
$\lambda_{\pm i}$	$\pm\frac{1}{2}$	0	1	0	$-p\chi_i^{\text{adj.}}$, $-q\chi_i^{\text{adj.}}$	0, 2	2, 0	2, 2
$\widetilde{F}_{\dot{+}\dot{+}i}$	0	1	0	0	$pq\chi_i^{\text{adj.}}$	2	2	4
$\partial\lambda_i = 0$	0	$+\frac{1}{2}$	1	0	$pq\chi_i^{\text{adj.}}$	2	2	4
$\partial_{\pm\dot{+}}$	$\pm\frac{1}{2}$	$\frac{1}{2}$	0	0	p, q	0, 2	2, 0	2, 2

Table 2.3: *Letters satisfying the BPS condition* (2.26) *of the free* $\mathcal{N} = 1$ *theory corresponding to a tube which implements the* Φ*-gluing of two punctures of equal colour, opposite orientation and sign* $\sigma = +$*. For the vector multiplet piece we must take into account the equation of motion* $\partial\lambda = \partial_{+\dot{+}}\lambda_- + \partial_{-\dot{+}}\lambda_+ = 0$*. Here* $\chi_i \equiv \chi_{(1,0,\ldots,0)}(\mathbf{z}_i)$*,* $\overline{\chi}_i \equiv \chi_{(0,0,\ldots,1)}(\mathbf{z}_i)$ *and* $\chi_i^{adj.} \equiv \chi_{(1,0,\ldots,1)}(\mathbf{z}_i)$ *are shorthand for the characters of the fundamental, anti-fundamental and adjoint representations of* $SU(N)$*, defined in* (A.59)*. We importantly note that* $\delta_{1\pm}, \widetilde{\delta}_{2\dot{+}} \geqslant 0$*.*

$\mathcal{I}_{\mathbf{z}_l\alpha}{}^{\mathbf{z}_r}$, leaving implicit the colour of the punctures. The contribution from the Φ-gluing of two maximal punctures of the same sign $\sigma = +$, of colour c and opposite orientation is listed in Table 2.3. Enumerating those letters gives[1]

$$\mathcal{I}_{\Phi_c^+} = \prod_{i=1}^{k} \frac{\kappa \prod_{A,B=1}^{N} \Gamma_e\left(\frac{pq}{t}\frac{\gamma_i}{\beta_{i+c-2}}\frac{z_{i,B}}{z_{i-1,A}}\right)}{\Gamma_e\left(\frac{pq}{t}\right)^{\delta_{k,1}} \Delta(\mathbf{z}_i) \prod_{A\neq B} \Gamma_e\left(\frac{z_{i,A}}{z_{i,B}}\right)} . \tag{2.31}$$

The factors κ and Δ are given by

$$\Delta(\mathbf{z}) = \prod_{A\neq B}\left(1 - \frac{z_A}{z_B}\right) , \quad \kappa := (p;p)^{N-1}(q;q)^{N-1} . \tag{2.32}$$

Note that in the above, and throughout, products and sums over i shall always be taken modulo k, i.e. $i + k \sim i$ unless otherwise stated. For instance, we denote the Φ-gluing of two three punctured spheres to obtain the theory associated to a sphere with two minimal and two maximal punctures at the level of the index by

$$\mathcal{I}_{s_c^{l,+}, s_{c+2}^{r,+}} = \oint \prod_{i=1}^{k} d\mu_i \, I_{s_c^{l,+}, s_{c+1}^{r,+}} \mathcal{I}_{\Phi_{c+1}^+} I_{s_{c+1}^{l,+}, s_{c+2}^{r,+}} \equiv \mathcal{I}_{\mathbf{u}\alpha\delta}{}^{\mathbf{v}} , \tag{2.33}$$

[1]The factor $\delta_{k,1}$ is to account for the fact that for $k = 1$ Φ sits in the adjoint of $\mathfrak{su}(N)$ adj. $\cong \mathbf{N}\otimes\overline{\mathbf{N}} - 1$, while for $k > 1$ Φ sits in bifundamenal representations.

For example, setting $k = N = c + 1 = 2$, we can expand (2.33) in terms of $\mathcal{N} = 1$ index equivalence classes $I_{[\tilde{r},j_1]_\pm}$ (C.15) [166, 4, 167] as

$$\begin{aligned}
\mathcal{I}_{\mathbf{u}\alpha\delta}{}^{\mathbf{v}} = {} & 1 + \frac{1}{t'^2}\left[1 + \frac{\beta^2}{\gamma^2} + \frac{\gamma^2}{\beta^2}\right]\mathcal{I}_{[-\frac{2}{3},0]_-} + \sum_{i=1}^{2} t'\left[\beta_i^2\left(\frac{1}{\alpha^2} + \frac{1}{\delta^2}\right)\right. \\
& + \gamma_i^2\left(\alpha^2 + \delta^2\right) + \left(\frac{\alpha}{\delta} + \frac{\delta}{\alpha}\right)\frac{\beta_i}{\gamma_i}\chi_1(u_i)\chi_1(v_{i+1}) + \beta_i\gamma_i\chi_1(v_i)\chi_1(v_{i+1}) \\
& \left. + \beta_i\gamma_i\chi_1(u_i)\chi_1(u_{i+1}) + \left(\alpha\delta + \frac{1}{\alpha\delta}\right)\chi_1(u_i)\chi_1(v_i)\right]\mathcal{I}_{[-\frac{2}{3},0]_-} \\
& + \sum_{i=1}^{2}\left[\left(\frac{\alpha\delta}{\beta_i\gamma_i} + \frac{\beta_i\gamma_i}{\alpha\delta}\right)\chi_1(u_i)\chi_1(v_{i+1})\right. \\
& \left. + \chi_2(u_i) + \chi_2(v_i) + 4\right]\mathcal{I}_{[0,0]_+} + \sum_{i=1}^{2}\left[\left(\frac{\beta_i^3}{\alpha\delta\gamma_i} + \frac{\alpha\delta\beta_i}{\gamma_i^3}\right)\chi_1(u_i)\chi_1(v_{i+1})\right. \\
& \left. + \frac{\gamma_i^2}{\beta_i^2}\left(\chi_2(u_{i+1}) + \chi_2(v_i) + 1\right)\right]\mathcal{I}_{[0,0]_-} + \mathcal{O}((pq)^{4/3})
\end{aligned} \tag{2.34}$$

where $\beta = \beta_1 = \beta_2^{-1}$ and $\gamma = \gamma_1 = \gamma_2^{-1}$; we have defined $t' := t/(pq)^{2/3}$ so that the expansion is made using the free R-symmetry. The equivalence class $[\frac{1}{3}, \frac{1}{2}]_-$, which has only a single representative $\hat{\mathcal{C}}_{(\frac{1}{2},0)}$, contains a spin 3/2 current and contributes to the index a factor proportional to $+(pq)^{2/3}(p + q)/(1 - p)(1 - q)$, is absent. This implies that either the theory has no supersymmetry enhancement, or that it contains a number of $\overline{\mathcal{B}}_{\frac{7}{3}(\frac{1}{2},0)}$ multiplets (which is the single representative of the $[\frac{1}{3}, \frac{1}{2}]_+$ equivalence class) [167]. Note that all of the equivalence classes in the above contain only a single representative. In particular one can replace $\mathcal{I}_{[-\frac{2}{3},0]_-} = \mathcal{I}_{\overline{\mathcal{B}}_{\frac{4}{3},(0,0)}}$, $\mathcal{I}_{[0,0]_+} = \mathcal{I}_{\hat{\mathcal{C}}_{(0,0)}}$ and $\mathcal{I}_{[0,0]_-} = \mathcal{I}_{\overline{\mathcal{B}}_{2,(0,0)}}$. The *net degeneracy* [4], defined in (C.16), of the $[0, 0]_\pm$ equivalence classes counts

$$\begin{aligned}
\mathrm{ND}[0,0] = {} & \#[0,0]_+ - \#[0,0]_- = \#\overline{\mathcal{B}}_{2,(0,0)} - \#\hat{\mathcal{C}}_{(0,0)} \\
= {} & \#\text{marginal operators} - \#\text{conserved currents} \\
= {} & 30 - 36\,.
\end{aligned} \tag{2.35}$$

When $k = 1$ the index (2.23) admits various interesting limits involving the three fugacities p, q, t (or ρ, σ, τ) in which the index recieves contribution only from states annihilated by two or more $\mathcal{N} = 2$ Poincaré supercharges (one of them, of course, always being $\widetilde{\mathcal{Q}}_{\dot{-}}$) [168].

Because $\widetilde{\delta}_{\dot{-}}[\mathcal{Q}_\alpha] \neq 0$ and $\widetilde{\delta}_{\dot{-}}[\widetilde{\mathcal{Q}}_{\dot{+}}] \neq 0$ the index of a generic $\mathcal{N} = 1$ SCFT admits no non-trivial limits in which the states contributing to it are annihilated by more than one supercharge. However, when the $\mathcal{N}=1$ SCFT has flavour symmetry, we may consider taking limits also involving the flavour fugacities. For generic theories there is no guarantee that such limits are well defined. Moreover, the index in certain limits; although not leading to extra superconformal shortening, can often admit drastic simplifications. Similar ideas have also been deployed in studying 'non-generic' $\mathcal{N}=1$ SCFTs in e.g. [4, 133, 5].

2.3.1 The Hall-Littlewood Limit of the Index

We can study the limit which, for $\mathcal{N}=2$ theories, is equivalent to the so-called Hall-Littlewood limit of the index [168] [2]

$$\sigma \to 0\,, \quad \rho \to 0\,, \quad \tau \text{ fixed,} \tag{2.36}$$

or, equivalently, $p, q \to 0$ with t held fixed. From (2.30) and (2.31) we see that the Hall-Littlewood limit of the indices for the Lagrangian building blocks is well defined, we can therefore write

$$\mathrm{HL}(\tau, \dots) := \lim_{\sigma,\rho\to 0} \mathcal{I} = \mathrm{Tr}_{\mathrm{HL}}(-1)^F \tau^{2q_t} e^{-\beta\tilde{\delta}_{\dot{-}}} e^{-\mathcal{J}}\,, \tag{2.37}$$

here $\mathrm{Tr}_{\mathrm{HL}}$ denotes the restriction of Tr to the states satisfying $\delta_{1\pm}=0$, i.e. $2q_t = \frac{3}{2}r + 3j_2 = E + j_2$ and $j_1 = 0$. The indices for the building blocks (2.30) and (2.31) become

$$\mathrm{HL}_{s_c^{l,+},s_{c+1}^{r,+}} = \prod_{i=1}^{k}\prod_{A,B=1}^{N} \frac{1}{\left(1-\tau\frac{\beta_{i+c-1}}{\alpha}\frac{z_{l,i,B}}{z_{r,i,A}}\right)\left(1-\tau\frac{\alpha}{\gamma_i}\frac{z_{r,i-1,B}}{z_{l,i,A}}\right)} \tag{2.38}$$

$$\mathrm{HL}_{\Phi_c^+} = \frac{\prod_{i=1}^{k}\prod_{A,B=1}^{N}\left(1-\tau^2\frac{\beta_{i+c-2}}{\gamma_i}\frac{z_{i-1,A}}{z_{i,B}}\right)}{(1-\tau^2)^{\delta_{k,1}}}\,. \tag{2.39}$$

The existence of this limit is equivalent to the fact that each letter contributing to the basic building building blocks have $\delta_{1\pm} \geqslant 0$ (see Tables 2.2 and 2.3), as is the case for all $\mathcal{N}=2$ theories.

As pointed out in [168, 170], for class $\mathcal{S}$ theories at genus $g=0$, the Hall-Littlewood limit of the index coincides with the Hilbert series of the Higgs branch. For Lagrangian theories this can be explicitly proved and can be argued to extend

[2] This limit has also been considered for $\mathcal{N}=1$ theories in [169, 5]. Geometrically it corresponds to collapsing the $\mathbb{S}^3$ to a point.

to theories related to Lagrangian theories by S-duality [171]. To the best of the authour's knowledge a full proof that extends to all class $\mathcal{S}$ theories is currently lacking. In Section 2.4 we will demonstrate that the same property also holds for Lagrangian theories made using Φ-gluing in class $\mathcal{S}_k$ at genus $g=0$.

2.3.2 The Coulomb Limit of the Index

The Coulomb limit of the index for $\mathcal{N}=2$ theories is given by

$$\tau \to 0\,, \quad \rho, \sigma \text{ fixed}, \tag{2.40}$$

or, equivalently, $t, p, q \to 0$ with $T := pq/t = \sigma\rho$ and $V := p/q = \sigma/\rho$ held fixed. An extensive study of this limit of the index was given in [154]. For generic $\mathcal{N}=1$ theories we would have no reason to believe that this limit exists since $\widetilde{\delta}_{2\dot{+}} \geqslant 0$ is no longer guaranteed. However, let us assume that it does. In this limit the index would take the form

$$\mathcal{I}^{\mathrm{C}}(T, V, \dots) = \mathrm{Tr}_{\mathrm{C}}(-1)^F T^{E+j_2} V^{j_1} e^{-\mathcal{J}} e^{-\beta' \widetilde{\delta}_{\dot{-}}} \tag{2.41}$$

Here Tr_{C} denotes the restriction of Tr to states with $\widetilde{\delta}_{2\dot{+}} = 0$. Indeed, we can see at the level of the Lagrangian building blocks that the limit does exist and, moreover, is conjectured to exist for all theories in class $\mathcal{S}_k$ of type A_{N-1} [154]. The indices for the Lagrangian building blocks (2.30) and (2.31) become

$$\mathcal{I}^{\mathrm{C}}_{s_c^{l,+}, s_{c+1}^{r,+}} = 1\,, \quad \mathcal{I}^{\mathrm{C}}_{\Phi_c^+} = \prod_{i=1}^{k} \frac{(1-T)^{\delta_{k,1}}}{\prod_{A,B=1}^{N} \left(1 - T \frac{\gamma_i}{\beta_{i+c-2}} \frac{z_{i,B}}{z_{i-1,A}}\right)}\,. \tag{2.42}$$

For Lagrangian theories made with Φ gluing the interpretation of this limit of the index is clear. The Coulomb limit of the index is simply counting the possible gauge invariants that can be made from the bifundamental scalar fields in the chiral multiplets Φ. These operators are the top components of the $\frac{1}{2}$-BPS multiplets $\overline{\mathcal{D}}^{(-1)}_{(0,0)}$ and $\overline{\mathcal{B}}^{(-\frac{2r}{3})}_{r,(0,0)}$ which simultaneously have $E = \frac{3}{2}r = -q_t$ and $j_1 = j_2 = 0$ (see Appendix C.1). Indeed we will demonstrate in Section 2.2.2 that, for those theories, $\mathcal{I}^{\mathrm{C}}$ can be given the interpretation of a Hilbert series constructed to count the above $\frac{1}{2}$-BPS multiplets on the Coulomb branch.

2.3.3 The Schur & Madonald Limits of the Index

For completeness of our discussion we can also define analogues of the Schur and Macdonald limits of the index of [168]. The analogue of the Macdonald index is obtained by taking $\sigma \to 0$, which is well defined for Lagrangian theories because each letter has $\delta_{1+} \geqslant 0$, while holding ρ, τ fixed

$$\mathcal{I}^M(q,t,\dots) = \mathrm{Tr}_M(-1)^F q^{-j_1+j_2+\frac{r}{2}-\frac{2q_t}{3}} t^{q_t} e^{-\mathcal{J}} e^{-\beta\tilde{\delta}_{\dot{-}}}\,, \tag{2.43}$$

where Tr_M denotes the restriction of Tr to states with $\delta_{1+} = 0$. The Schur index is defined by setting $\rho = \tau$, or $q = t$

$$\mathcal{I}^S(q,p,\dots) = \mathrm{Tr}(-1)^F p^{j_1+j_2+\frac{r}{2}-\frac{2q_t}{3}} q^{-j_1+j_2+\frac{r}{2}+\frac{q_t}{3}} e^{-\mathcal{J}} e^{-\beta\tilde{\delta}_{\dot{-}}}\,. \tag{2.44}$$

Note that, unlike for $\mathcal{N} = 2$ theories, the Schur index (2.44) is dependent on $\sigma = p/\sqrt{q}$. However, for Lagrangian theories we have $\delta_{1+} \geqslant 0$ and therefore we can consider a further limit, which we will call the reduced Schur index

$$\mathcal{I}^{RS}(q,\dots) := \lim_{p\to 0} \mathcal{I}^S = \mathrm{Tr}_M(-1)^F q^{E-\frac{r}{2}-\frac{q_t}{3}} e^{-\mathcal{J}} e^{-\beta\tilde{\delta}_{\dot{-}}} \equiv \mathcal{I}^M|_{q=t}\,. \tag{2.45}$$

In particular, for the index $\mathcal{I}^{RS}$, the Lagrangian building blocks become

$$\mathcal{I}^{RS}_{s_c^{l,+},s_{c+1}^{r,+}} = \prod_{i=1}^{k}\prod_{A,B=1}^{N} \frac{1}{\left(\sqrt{q}\frac{\beta_{i+c-1}}{\alpha}\frac{z_{l,i,B}}{z_{r,i,A}};q\right)\left(\sqrt{q}\frac{\alpha}{\gamma_i}\frac{z_{r,i-1,B}}{z_{l,i,A}};q\right)}\,, \tag{2.46}$$

$$\mathcal{I}^{RS}_{\Phi_c^+} = (q;q)^{k-\delta_{k,1}} \prod_{i=1}^{k}\prod_{A,B=1}^{N} \left(q\frac{\beta_{i+c-2}}{\gamma_i}\frac{z_{i-1,A}}{z_{i,B}};q\right)\left(q\frac{z_{i,A}}{z_{i,B}};q\right)\,. \tag{2.47}$$

For the $k = N = 2$ theory associated to a sphere with two minimal punctures and two maximal punctures $s_1^{l,+}$, $s_3^{r,+}$ the reduced Schur index can be expanded as

$$
\begin{aligned}
\mathcal{I}^{RS} &= 1 + \Big[\chi_1(m^2)\left(a^2[0,1,0;0,0,0] + \frac{1}{a^2}[0,0,0;0,1,0]\right) \\
&\quad + [1,0,0;0,0,1] + [0,0,1;1,0,0]\Big]q \\
&\quad + \Big[\frac{1}{a^2}\chi_1(m^2)\left([0,0,1;1,1,0] + [1,0,0;0,1,1]\right) \\
&\quad + a^2\chi_1(m^2)\left([1,1,0;0,0,1] + [0,1,1;1,0,0]\right) + [0,1,0;0,1,0] \\
&\quad + [1,0,1;1,0,1] + [2,0,0;0,0,2] + [0,0,2;2,0,0] \\
&\quad + \chi_2(m^2)\left(a^4[0,2,0;0,0,0] + \frac{1}{a^4}[0,0,0;0,2,0] + [0,1,0;0,1,0]\right)\Big]q^2 \\
&\quad + \mathcal{O}(q^3)\,.
\end{aligned}
\tag{2.48}
$$

Note that here, we used the symmetry enhancement of this theory (2.60), which we will discuss in more detail in Section 2.4.3. Here $[d_1, d_2, d_3; d_1', d_2', d_3']$ denotes the character of the enhanced $SU(2N)^k = SU(4)^2$ symmetry.

For theories with $\mathcal{N} \geqslant 2$ supersymmetry the quantity $\mathcal{I}^S$ (which, for $\mathcal{N} \geqslant 2$ supersymmetry, equals $\mathcal{I}^{RS}$) plays a pivotal role in the chiral algebra 2d/4d correspondence [172]. In particular $\mathcal{I}^S$ is identified with the vacuum character of the associated chiral algebra.

The stress tensor of the associated chiral algebra is identified with the top component of the $\mathfrak{su}(2)_{R_{\mathcal{N}=2}}$ current, namely j^μ_{11}. This current lives in the stress tensor multiplet $\hat{\mathcal{C}}_{0(0,0)}$ which contains conserved $\mathfrak{su}(2)_{R_{\mathcal{N}=2}}$ and $\mathfrak{u}(1)_{r_{\mathcal{N}=2}}$ currents $j^\mu_{(IJ)}$ and j^μ with $I, J = 1, 2$ $\mathfrak{su}(2)_{R_{\mathcal{N}=2}}$ indices. This current enters the Schur index $\mathcal{I}^S$ with a factor $\mathcal{I}^S_{\hat{\mathcal{C}}_{0(0,0)}} = q^2/(1-q)$. Under the decomposition $\mathfrak{su}(2,2|2) \to \mathfrak{su}(2,2|1) \oplus \mathfrak{u}(1)_t$ the $\mathcal{N} = 2$ stress tensor multiplet decomposes as

$$
\hat{\mathcal{C}}_{0(0,0)} \cong (\hat{\mathcal{C}}_{(0,0)}, 0) \oplus (\hat{\mathcal{C}}_{(\frac{1}{2},0)}, +1) \oplus (\hat{\mathcal{C}}_{(0,\frac{1}{2})}, -1) \oplus (\hat{\mathcal{C}}_{(\frac{1}{2},\frac{1}{2})}, 0)\,. \tag{2.49}
$$

Correspondingly, the index can be written as

$$
\begin{aligned}
\mathcal{I}_{\hat{\mathcal{C}}_{0(0,0)}} =& \frac{-pq}{(1-p)(1-q)} + \frac{t(p+q)}{(1-p)(1-q)} + \frac{t^{-1}p^2q^2}{(1-p)(1-q)} \\
&+ \frac{-pq(p+q)}{(1-p)(1-q)}\,;
\end{aligned}
\tag{2.50}
$$

see Figure 2.2. Since, the stress tensor multiplet of the mother SCFT sits in trivial representations of any flavour symmetries (e.g. it sits in a trivial $SU(N_f)$ represen-

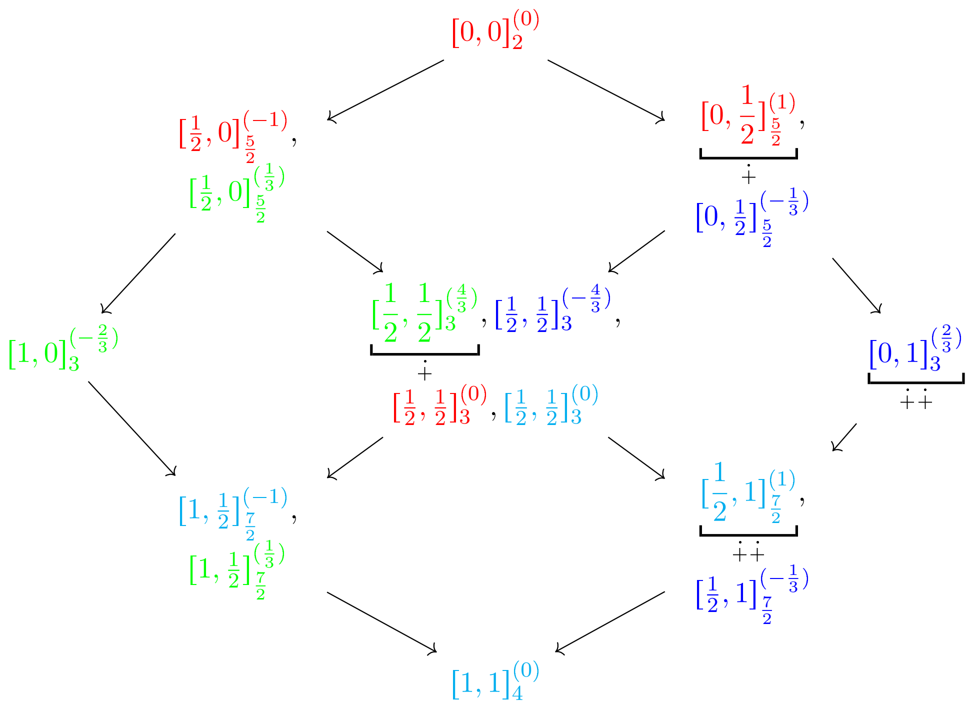


Figure 2.2: *Branching of* $\hat{\mathcal{C}}_{0(0,0)} \cong \hat{\mathcal{C}}_{(0,0)} \oplus \hat{\mathcal{C}}_{(\frac{1}{2},0)} \oplus \hat{\mathcal{C}}_{(0,\frac{1}{2})} \oplus \hat{\mathcal{C}}_{(\frac{1}{2},\frac{1}{2})}$. *Underlined are those states, with given* j_2*, which have* $\widetilde{\delta}_{\dot{-}} = E - 2j_2 - \frac{3}{2}r = 0$ *and thus can contribute to the right-handed index* (2.23).

tation) under the $\mathbb{Z}_k$ orbifold the projection of the stress tensor should simply be $\hat{\mathcal{C}}_{0(0,0)} \xrightarrow{\mathbb{Z}_k} \hat{\mathcal{C}}_{(0,0)} \oplus \hat{\mathcal{C}}_{(\frac{1}{2},\frac{1}{2})}$. Indeed the multiplets $\hat{\mathcal{C}}_{(\frac{1}{2},0)}$ and $\hat{\mathcal{C}}_{(0,\frac{1}{2})}$ contain additional supersymmetry currents which would lead to enhanced $\mathcal{N} \geqslant 2$ supersymmetry if present.

We can identify in the decomposition the $\hat{\mathcal{C}}_{(0,0)}$ as the $\mathfrak{u}(1)_t$ flavour current multiplet while $\hat{\mathcal{C}}_{(\frac{1}{2},\frac{1}{2})}$ is of course the $\mathcal{N}=1$ stress tensor multiplet whose lowest component is the $\mathfrak{u}(1)_r$ current. They are built from linear combinations of $j^\mu_{12} = j^\mu_{21}$ and j^μ in accordance with (2.27). .

The currents j^μ_{11} and j^μ_{22} belong to $\hat{\mathcal{C}}_{(\frac{1}{2},0)}$ and $\hat{\mathcal{C}}_{(0,\frac{1}{2})}$, respectively and are expected to be projected out by the orbifold. In particular we note that a factor $q^2/(1-q)$ in a trivial flavour symmetry representation does not appear in the expansion (2.48).

2.4 Genus Zero Theories

In this section we consider class $\mathcal{S}_k$ theories at genus $g = 0$. We show that, for this subclass of theories, the Hall-Littlewood limit of the index coincides with the corresponding Higgs branch Hilberts series. Then we provide a closed form expression for the Higgs branch Hilbert series of some Lagrangian genus zero theories, namely the free trinion and the interacting SCFT associated to a sphere with two maximal and two minimal punctures.

2.4.1 Hilbert Series and the Hall-Littlewood Limit of the Index

We are now in a position to show that the Hall-Littlewood limit of the index coincides with the Higgs branch Hilbert series at genus $g = 0$ for Lagrangian theories made using Φ-gluing. For the theory corresponding to a sphere with $\ell - 2$ minimal punctures and two maximal punctures the relevant F-terms for the Higgs branch are

$$F_{(i,n)} := \frac{\partial W_{\mathcal{S}_k}}{\partial \Phi_{(i,n)}} = \widetilde{Q}_{(i,n-1)} Q_{(i,n-1)} - Q_{(i-1,n)} \widetilde{Q}_{(i,n)} = 0\,, \tag{2.51}$$

for $n = 1, \ldots, -3+\ell$ and $i+k \sim i = 1, \ldots, k$. For genus $g = 0$ these constitute $k(-3+\ell)$ independent constraints on the $Q_{(i,n)}$ and $\widetilde{Q}_{(i,n)}$. More precisely the ideal I_H comprised of the list of the $F_{(i,n)}$ forms a regular sequence in $R_H = \mathbb{C}[Q_{(i,n)}, \widetilde{Q}_{(i,n)}]$, see Appendix B. This means that the variety whose coordinate ring is given by the quotient ring R_H/I_H is a complete intersection and we may apply letter counting techniques to compute the Hilbert series for the master space $\mathbf{F_H}$.

The Hilbert series of the Higgs branch of the theory associated to the a sphere with $\ell - 2$ minimal punctures and two maximal punctures $s_1^{l,+}$, $s_{\ell-1}^{r,+}$ is precisely given by

$$\mathrm{HS}(\tau, \ldots; HB) = \oint \prod_{i=1}^{k} \prod_{n=2}^{\ell-2} d\mu_{(i,n)} f_H^\flat(\tau, \ldots) \tag{2.52}$$

$$f_H^\flat = \frac{\mathrm{PE}\left[\sum_{n=1}^{\ell-2} \sum_{i=1}^{k} \sum_{A,B=1}^{N} \left(\tau \frac{\beta_{i+n-1}}{\alpha_n} \frac{z_{(i,n),B}}{z_{(i,n+1),A}} + \tau \frac{\alpha_n}{\gamma_i} \frac{z_{(i-1,n+1),B}}{z_{(i,n),A}}\right)\right]}{\mathrm{PE}\left[\sum_{n=2}^{\ell-2} \sum_{i=1}^{k} \sum_{A,B=1}^{N} \tau^2 \frac{\beta_{i+n-2}}{\gamma_i} \frac{z_{(i-1,n),A}}{z_{(i,n),B}} + (3-\ell)\tau^2 \delta_{k,1}\right]} \tag{2.53}$$

and we see that $f_H^\flat = \prod_{n=1}^{\ell-2} \mathrm{HL}_{s_n^{l,+}, s_{n+1}^{r,+}} \prod_{n=2}^{\ell-2} \mathrm{HL}_{\Phi_n^+}$ and therefore $\mathrm{HS} \equiv \mathrm{HL}$ for this class of genus zero theories. The contribution of the $\overline{\lambda}_{\dot{+}(i,n)}$ to the Hall-Littlewood index coming from Φ-gluing precisely plays the role of the F-term constraints (2.51) in the Higgs-branch Hilbert series.

In other words, we see that for this class of theories $\mathrm{Tr}_{\mathrm{HL}} = \mathrm{Tr}_{HB}$. Indeed, one can see that if one further introduces the condition $j_2 = 0$ into those defining the Hall-Littlewood limit of the index $\widetilde{\delta}_{\dot{-}} = \delta_{1\pm} = 0$ we have $E = 2q_t = \frac{3}{2}r$, $j_1 = j_2 = 0$ which are precisely the conditions defining the Higgs branch (2.12).

2.4.2 The Free Trinion

Let us consider the Hall-Littlewood index/Hilbert series of the $\mathfrak{g} = A_{N-1}$ theory associated to a sphere with one minimal puncture with fugacity α and two maximal punctures $s_1^{l,+}$ and $s_2^{r,+}$ a.k.a. the free trinion. The expression for the Higgs branch Hilbert series was given in (2.38) and it reads

$$\mathrm{HS}_{s_1^{l,+},s_2^{r,+}} = \prod_{i=1}^{k}\prod_{A,B=1}^{N} \frac{1}{\left(1-\tau\frac{\beta_i}{\alpha}\frac{u_{i,A}}{v_{i,B}}\right)\left(1-\tau\frac{\alpha}{\gamma_i}\frac{v_{i-1,B}}{u_{i,A}}\right)} \tag{2.54}$$

note that we set $z_l = u$ and $z_r = v$ with respect to (2.38). We checked for various low values of N in expansion around $\tau = 0$ that the identity

$$\begin{aligned}\prod_{A,B=1}^{N}\frac{1}{1-a\tau\frac{u_A}{v_B}} &= \sum_{l\geqslant 0}\sum_{\lambda}(a\tau)^{Nl+\sum_{A=1}^{N-1}A(\lambda_A-\lambda_{A+1})}s_\lambda(\mathbf{u})s_{\overline{\lambda}}(\mathbf{v})\\ &= \sum_{\{n_1,\dots,n_N\}\geqslant 0}(a\tau)^{\sum_{A=1}^{N}An_A}\chi_{(n_1,n_2\dots,n_{N-1})}(\mathbf{u})\chi_{(n_{N-1},n_{N-2},\dots,n_1)}(\mathbf{v})\end{aligned} \tag{2.55}$$

holds, with $\prod_{A=1}^{N}u_A = \prod_{A=1}^{N}v_A = 1$ and where the $SU(N)$ characters are defined in (A.59). In the second line we rewrote the expression in terms of Schur polynomials, the relevant definitions and identities can be found in Appendix A. Therefore, we can write (2.54) as

$$\begin{aligned}\mathrm{HS}_{s_1^{l,+},s_2^{r,+}} = \sum_{l^{(j)},m^{(j)}\geqslant 0}\sum_{\lambda^{(j)},\mu^{(j)}}\prod_{i=1}^{k}\Bigg\{\\ \frac{s_{\lambda^{(i)}}(\mathbf{u}_i)s_{\overline{\mu}^{(i)}}(\mathbf{u}_i)s_{\overline{\lambda}^{(i)}}(\mathbf{v}_i)s_{\mu^{(i)}}(\mathbf{v}_{i-1})}{\prod_{A=1}^{N-1}\left(\frac{\beta_i\tau}{\alpha}\right)^{A(\lambda_{A+1}^{(i)}-\lambda_A^{(i)})-Nl^{(i)}}\left(\frac{\alpha\tau}{\gamma_i}\right)^{A(\mu_{A+1}^{(i)}-\mu_A^{(i)})-Nm^{(i)}}}\Bigg\}.\end{aligned} \tag{2.56}$$

Applying (A.48)

$$\begin{aligned}&\mathrm{HS}_{s_1^{l,+},s_2^{r,+}} = \\ &\sum_{\lambda^{(j)},\mu^{(j)},\nu^{(j)},\eta^{(j)}} \prod_{i=1}^{k} \frac{c^{\eta^{(i)}}_{\lambda^{(i)}\overline{\mu}^{(i)}} c^{\nu^{(i)}}_{\mu^{(i+1)}\overline{\lambda}^{(i)}} s_{\eta^{(i)}}(\mathbf{u}_i) s_{\nu^{(i)}}(\mathbf{v}_i)\, \mathrm{PE}\left[\frac{\alpha^N}{\gamma_i^N}\tau^N + \frac{\beta_i^N}{\alpha^N}\tau^N\right]}{\prod_{A=1}^{N-1}\left(\left(\frac{\beta_i\tau}{\alpha}\right)^{A(\lambda^{(i)}_{A+1}-\lambda^{(i)}_A)}\left(\frac{\alpha\tau}{\gamma_i}\right)^{A(\mu^{(i)}_{A+1}-\mu^{(i)}_A)}\right)}\,. \end{aligned} \tag{2.57}$$

Specialising to the case $\mathfrak{g} = A_1$, where we explicitly know the Littlewood-Richardson coefficients $c^{\nu}_{\lambda\mu}$, we have

$$\begin{aligned}\mathrm{HS}_{s_1^{l,+},s_2^{r,+}} =& \prod_{i=1}^{k} \sum_{n_i,n_i'\geqslant 0} \sum_{d_i=|n_i-n_i'|}^{n_i+n_i'} \sum_{d_i'=|n_i-n_{i+1}'|}^{n_i+n_{i+1}'} \Bigg\{ \\ & \frac{\chi_{d_i}(u_i)\chi_{d_i'}(v_i)}{\left(1-\frac{\beta_i^2}{\alpha^2}\tau^2\right)\left(1-\frac{\alpha^2}{\gamma_i^2}\tau^2\right)} \left(\frac{\beta_i\tau}{\alpha}\right)^{n_i}\left(\frac{\alpha\tau}{\gamma_i}\right)^{n_i'}\Bigg\}\,. \end{aligned} \tag{2.58}$$

2.4.3 Core Interacting Theories

Let us first consider the Hall-Littlewood index/Higgs-branch Hilbert series for the interacting SCFT associated to a sphere with two minimal punctures and two maximal punctures $s_1^{l,+}, s_3^{r,+}$. It is given by

$$\begin{aligned}\mathrm{HL}_{s_1^{l,+},s_3^{r,+}} &= \mathrm{HL}_{\mathbf{u}\delta\alpha}{}^{\mathbf{v}} = \prod_{i=1}^{k}\oint d\mu_i\, \mathrm{HL}_{s_1^{l,+},s_2^{r,+}} \mathrm{HL}_{\Phi_2^+} \mathrm{HL}_{s_2^{l,+},s_3^{r,+}} \\ &= \prod_{i=1}^{k}\oint d\mu_i \frac{\mathrm{PE}\left[\sum_{A,B=1}^{N} \frac{\tau\beta_i}{\delta}\frac{u_{i,A}}{z_{i,B}} + \frac{\tau\alpha}{\gamma_i}\frac{v_{i-1,A}}{z_{i,B}} + \frac{\tau\beta_{i+1}}{\alpha}\frac{z_{i,B}}{v_{i,A}} + \frac{\tau\delta}{\gamma_i}\frac{z_{i-1,B}}{u_{i,A}}\right]}{\mathrm{PE}\left[-\tau^2\delta_{k,1} + \sum_{A,B=1}^{N}\tau^2\frac{\beta_i}{\gamma_i}\frac{z_{i-1,A}}{z_{i,B}}\right]}\,. \end{aligned} \tag{2.59}$$

An important observation that will allow us to write down the Highest Weight Generating (HWG) function [173] for the Hilbert series for this theory is the fact that there is, at the level of the Lagrangian, a symmetry enhancement

$$\begin{aligned}& SU(N)^{2k} \times U(1)_\gamma^{k-1} \times U(1)_\beta^{k-1} \times U(1)_\delta \times U(1)_\alpha \\ & \to SU(2N)^k \times U(1)_a^{k-1} \times U(1)_m\,, \end{aligned} \tag{2.60}$$

which we can make manifest in (2.59) by writing

$$\mathbf{x}_i = \left(\sqrt{\frac{\gamma_i \beta_i}{\delta\alpha}} \mathbf{u}_i, \sqrt{\frac{\delta\alpha}{\gamma_i\beta_i}} \mathbf{v}_{i-1} \right) , \quad a_i = \sqrt{\frac{\beta_i}{\gamma_i}} , \quad m = \sqrt{\frac{\alpha}{\delta}} , \tag{2.61}$$

corresponding to writing $q_i = (Q_i^L, \widetilde{Q}_{i-1}^R)$, $\widetilde{q}_i = (\widetilde{Q}_i^L, Q_{i-1}^R)$ in the decomposition $\mathbf{2N} \to (\mathbf{N}, \mathbf{1})_1 \oplus (\mathbf{1}, \mathbf{N})_{-1}$ under $SU(2N) \to SU(N) \times SU(N) \times U(1)$ and $Q_i^L, \widetilde{Q}_i^L$ are the bifundamental chiral multiplets of the first free trinion and $Q_i^R, \widetilde{Q}_i^R$ the second. Then the Hall-Littlewood index / Hilbert series can be written as

$$\begin{aligned} &\mathrm{HS}_{s_1^{l,+}, s_3^{r,+}} = \\ &\prod_{i=1}^{k} \oint \prod_{A=1}^{N-1} \frac{dz_{i,A}}{2\pi \mathrm{i} z_{i,A}} \frac{\prod\limits_{1 \leqslant A < B \leqslant N} \left(1 - \frac{z_{i,B}}{z_{i,A}}\right) \prod_{A,B=1}^{N} \left(1 - \tau^2 a_i^2 \frac{z_{i-1,A}}{z_{i,B}}\right)}{(1-\tau^2)^{\delta_{k,1}} \prod\limits_{A=1}^{N} \prod\limits_{B=1}^{2N} \left(1 - \tau a_i m \frac{x_{i,B}}{z_{i,A}}\right) \left(1 - \tau \frac{a_i}{m} \frac{z_{i-1,A}}{x_{i,B}}\right)} . \end{aligned} \tag{2.62}$$

The symmetry enhancement (2.60) allows us to conjecture the following expression for the Highest Weight Generating (HWG) function for the Hilbert series as

$$\begin{aligned} \mathrm{HWG}_{s_1^{l,+}, s_3^{r,+}} = \mathrm{PE} \Bigg[& \delta_{1,k} \tau^2 + \sum_{i=1}^{k} \sum_{A=1}^{N-1} \mu_A^{(i)} \mu_{2N-A}^{(i+1)} a_i^A a_{i+1}^A \tau^{2A} \\ & + \sum_{i=1}^{k} \mu_N^{(i)} a_i^N \left(m^N + \frac{1}{m^N} \right) \tau^N \Bigg] , \end{aligned} \tag{2.63}$$

where $\{\mu_A^{(i)}\}$ denotes a set of highest weights for the i^{th} flavour node. From the HWG one then obtains the Hilbert series by replacing

$$\prod_{A=1}^{2N-1} {\mu_A^{(i)}}^{d_A} \leftrightarrow [d_1, d_2, \ldots, d_{2N-1}]_{\mathbf{x}_i} \tag{2.64}$$

where $[d_1, d_2, \dots, d_{2N-1}]_{\mathbf{x}} \equiv \chi_{(d_1,d_2,\dots,d_{2N-1})}(\mathbf{x})$ denotes the character of $SU(2N)$, defined in (A.59). The Hilbert series is given by

$$\begin{aligned}\mathrm{HS}_{s_1^{l,+},s_3^{r,+}} = &\left(1-\tau^2\right)^{-\delta_{1,k}} \sum_{\{n_A^{(i)},p^{(i)},l^{(i)}\geqslant 0\}} \prod_{i=1}^{k} \Bigg\{ \\ &\left[n_1^{(i)}, n_2^{(i)}, \dots, n_{N-1}^{(i)}, p^{(i)} + l^{(i)}, n_{N-1}^{(i+1)}, n_{N-2}^{(i+1)}, \dots, n_1^{(i+1)}\right]_{\mathbf{x}_i} \\ &m^{Np^{(i)}-Nl^{(i)}} a_i^{Np^{(i)}+Nl^{(i)}+\sum_{A=1}^{N-1} An_A^{(i)}} a_{i+1}^{\sum_{A=1}^{N-1} An_A^{(i+1)}} \tau^{Np^{(i)}+Nl^{(i)}+2\sum_{A=1}^{N-1} An_A^{(i)}} \Bigg\}.\end{aligned} \tag{2.65}$$

We checked in an expansion around $\tau = 0$ that (2.65) agrees with (2.62) for $(N, k) = \{(2, 2), (2, 3), (3, 2)\}$. For theories associated to ℓ-punctured spheres with two maximal punctures and $\ell - 2 > 2$ minimal punctures there is no symmetry enhancement (2.60) and the number of monomials in the PLog of the HWG is not finite. This means that it is not practically possible to write down a closed form expression for the HWG.

We can however perform the expansion of the integral (2.52) around $\tau = 0$. For example, for $k = N = 2$, $\ell - 2 = 3$ we have

$$\begin{aligned}\mathrm{HS} = 1 + &\Bigg[\chi_2^L \chi_1^L \left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right) \\ &+ \chi_2^R \chi_1^R \left(\beta\gamma + \frac{1}{\beta\gamma}\right) + \sum_{n=1}^{3}\sum_{i=1}^{2}\left(\frac{\beta_i^2}{\alpha_n^2} + \gamma_i^2 \alpha_n^2\right)\Bigg]\tau^2 \\ &+ \Bigg[\left(\frac{\alpha_3\alpha_2}{\alpha_1} + \frac{\alpha_1\alpha_2}{\alpha_3} + \frac{\alpha_1\alpha_3}{\alpha_2} + \frac{1}{\alpha_1\alpha_3\alpha_2}\right)\sum_{i=1}^{2}\beta_i \chi_i^L \chi_i^R \\ &+ \left(\alpha_2\alpha_3\alpha_1 + \frac{\alpha_1}{\alpha_2\alpha_3} + \frac{\alpha_3}{\alpha_2\alpha_1} + \frac{\alpha_2}{\alpha_3\alpha_1}\right)\sum_{i=1}^{2}\gamma_{i+1}\chi_i^L \chi_{i+1}^R\Bigg]\tau^3 + \mathcal{O}(\tau^4)\end{aligned} \tag{2.66}$$

where $\beta_1 = \frac{1}{\beta_2} = \beta$, $\gamma_1 = \frac{1}{\gamma_2} = \gamma$ and $\chi_i^L = \chi_1(z_{(i \bmod k,1)})$, $\chi_i^R = \chi_1(z_{(i \bmod k,4)})$ denotes the characters of the fundamental representation of the corresponding $SU(2)$'s.

2.5 Genus One Theories

In this section we consider some theories that are closely related to class $\mathcal{S}_k$ theories at genus one. The theories that we study are the $\mathcal{N} = 1$ $\mathfrak{u}(N)^{\oplus \ell k}$ toroidal quiver gauge theories realised as the $l_s \to 0$ limit of the worldvolume theory on a stack of

N D3-branes probing a transverse $\mathbb{C}^3/(\mathbb{Z}_\ell \times \mathbb{Z}_k)$ singularity where the quotient acts by

$$(z_1, z_2, z_3) \mapsto (\omega_k z_1, \omega_\ell z_2, \omega_\ell^{-1}\omega_k^{-1} z_3)\,, \quad \omega_k^k = \omega_\ell^\ell = 1\,. \tag{2.67}$$

These are orbifolds of $\mathcal{N}=4$ SYM; we discussed them at length in Section 1.6.2. In the case where the $\mathfrak{u}(1)^{\oplus \ell k} \subset \mathfrak{u}(N)^{\oplus \ell k}$ is not gauged, these are precisely the theories in class $\mathcal{S}_k$ associated to Riemann surfaces of genus one with ℓ punctures.

The action (2.67) is an element of the $SO(6)_R$ R-symmetry group of $\mathcal{N}=4$ SYM. If we denote the $\mathcal{N}=4$ supercharges by $\mathcal{Q}_\alpha^{q_1q_2q_3}$ with $8q_1q_2q_3 = -1$ and $\overline{\mathcal{Q}}_{\dot\alpha}^{q_1q_2q_3}$ with $8q_1q_2q_3 = +1$ where the $q_i = \pm\frac{1}{2}$. The orbifold acts by

$$\mathcal{Q}_\alpha^{q_1q_2q_3} \mapsto \omega_k^{q_1-q_3}\omega_\ell^{q_2-q_3}\mathcal{Q}_\alpha^{q_1q_2q_3}\,, \quad \overline{\mathcal{Q}}_{\dot\alpha}^{q_1q_2q_3} \mapsto \omega_k^{q_1-q_3}\omega_\ell^{q_2-q_3}\overline{\mathcal{Q}}_{\dot\alpha}^{q_1q_2q_3}\,. \tag{2.68}$$

The surviving supercharges are those with $q_1 = q_2 = q_3$

$$\mathcal{Q}_\alpha^{---} = \mathcal{Q}_\alpha\,, \quad \overline{\mathcal{Q}}_{\dot\alpha}^{+++} = \overline{\mathcal{Q}}_{\dot\alpha}\,. \tag{2.69}$$

The Cartans q_1, q_2, q_3 of $\mathfrak{so}(6)_R$ are related to the more natural $\mathcal{N}=1$ symmetries by

$$r = \frac{2}{3}(q_1+q_2+q_3)\,, \quad q_t = -q_1 + \frac{1}{2}(q_2+q_3)\,, \quad q_b = -q_2+q_3\,, \tag{2.70}$$

there is also the overall $U(1)$ generated by b.

It will later be useful to have to hand the Hilbert series for this quotient space. This is given by the *Molien series*

$$\begin{aligned} M(\tau_1,\tau_2,\tau_3;\mathbb{C}^3/(\mathbb{Z}_\ell\times\mathbb{Z}_k)) &= \frac{1}{\ell k}\sum_{\substack{g\in\mathbb{Z}_\ell\\ h\in\mathbb{Z}_k}} \mathrm{PE}\left[h\tau_1 + g\tau_2 + g^{-1}h^{-1}\tau_3\right] \\ &= \frac{\sum_{r=0}^{LCM(\ell,k)-1}(\tau_1\tau_2\tau_3)^r\tau_1^{-\lfloor\frac{r}{k}\rfloor k}\tau_2^{-\lfloor\frac{r}{\ell}\rfloor\ell}}{(1-\tau_1^k)(1-\tau_2^\ell)(1-\tau_3^{LCM(\ell,k)})}\,, \end{aligned} \tag{2.71}$$

where τ_1, τ_2, τ_3 are related to the $SO(6)_R \supset U(1)^3 \circlearrowleft \mathbb{C}^3$ toric action generated by q_1, q_2, q_3. Moreover $LCM(\ell,k)$ denotes the Lowest Common Multiple of ℓ and k. For $\ell = k$ these varieties are complete intersections since

$$M(\tau_1,\tau_2,\tau_3;\mathbb{C}^3/(\mathbb{Z}_k\times\mathbb{Z}_k)) = \mathrm{PE}\left[\sum_{n=1}^3 \tau_n^k + \tau_1\tau_2\tau_3 - \tau_1^k\tau_2^k\tau_3^k\right]\,. \tag{2.72}$$

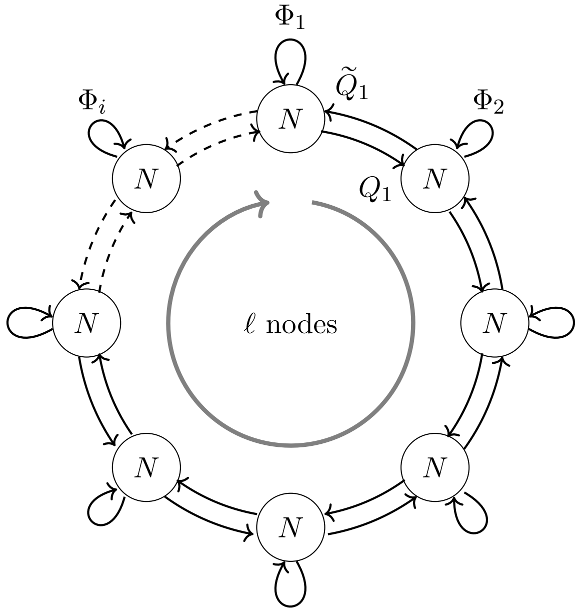


Figure 2.3: *The circular quiver with gauge group $U(N)^\ell$*

This space can be realised as $\mathbb{C}^3/(\mathbb{Z}_k \times \mathbb{Z}_k) \hookrightarrow \mathbb{C}^4$ defined by the equation

$$w_1 w_2 w_3 - w_4^k = 0\,, \tag{2.73}$$

with $w_i = z_i^k$, $w_4 = z_1 z_2 z_3$. As a warm up we will now review the computation for $k = 1$.

2.5.1 Class $\mathcal{S}$

We reviewed this theory and it's moduli space $\mathbf{M}$ for $\ell = 2$ in detail in Section 2.2.1. The quiver diagram for this theory is given in Figure 2.3.

Hilbert Series

We begin with the computation of the mesonic moduli space ${}^{\text{mes}}\mathbf{M}$. Let us begin with the case of $N = 1$. The master space for $\mathbf{M}$ is then associated to R/I with

$$R = [Q_1, \ldots, Q_\ell, \widetilde{Q}_1, \ldots, \widetilde{Q}_\ell, \Phi_1, \ldots, \Phi_\ell]\,, \tag{2.74}$$

$$I = \langle F_{Q_1}, \ldots, F_{Q_\ell}, F_{\widetilde{Q}_1}, \ldots, F_{\widetilde{Q}_\ell}, F_{\Phi_1}, \ldots, F_{\Phi_\ell} \rangle\,, \tag{2.75}$$

$$F_{Q_n} = \widetilde{Q}_n(\Phi_{n+1} - \Phi_n)\,, \quad F_{\widetilde{Q}_n} = Q_n(\Phi_{n+1} - \Phi_n)\,, \tag{2.76}$$

$$F_{\Phi_n} = Q_{n-1}\widetilde{Q}_{n-1} - Q_n\widetilde{Q}_n\,, \tag{2.77}$$

where $n \sim n + \ell$. We perform a primary decomposition of the above ideal and we select the prime ideal corresponding to a mesonic branch, the corresponding Hilbert series is

$${}^{\text{mes}}f^{\flat} = \text{PE}\left[\sum_{n=1}^{\ell}\left(\tau_1 + \frac{z_n}{z_{n+1}}\tau_2 + \frac{z_{n+1}}{z_n}\tau_3\right) - \sum_{n=1}^{\ell-1}(\tau_2\tau_3 + \tau_1)\right]. \tag{2.78}$$

The τ_i are the same as those in (2.71). Explicitly, in terms of the parameters appearing in (2.23)

$$\tau_1 = \frac{pq}{t} = T\,, \quad \tau_2 = \frac{\sqrt{t}}{b} = \frac{\tau}{b}\,, \quad \tau_3 = b\sqrt{t} = b\tau \tag{2.79}$$

where $b^\ell = \prod_{n=1}^{\ell}\alpha_n$ is the product of all of the fugacities for minimal punctures. So, the Hilbert series for the coordinate ring of ${}^{\text{mes}}\mathbf{M}$ is given by

$$\begin{aligned} \text{HS}(\tau_1,\tau_2,\tau_3;{}^{\text{mes}}M) &= \prod_{n=1}^{\ell}\oint_{|z_n|=1}\frac{dz_n}{2\pi \mathrm{i} z_n}\,{}^{\text{mes}}f^{\flat} \\ &= \frac{1-\tau_2^\ell\tau_3^\ell}{(1-\tau_1)(1-\tau_2\tau_3)(1-\tau_2^\ell)(1-\tau_3^\ell)} \end{aligned} \tag{2.80}$$

We observe that $\text{HS}(\tau_1,\tau_2,\tau_3;{}^{\text{mes}}M) = M(\tau_1,\tau_2,\tau_3;\mathbb{C}\times\mathbb{C}^2/\mathbb{Z}_\ell)$. The Higgs branch of this theory is purely mesonic, hence the Hilbert series for the Higgs branch can easily obtained by considering the $\tau_1 \to 0$ limit

$$\text{HS}(\tau_2,\tau_3;HB) = \lim_{\tau_1\to 0}\text{HS}(\tau_1,\tau_2,\tau_3;{}^{\text{mes}}M) \tag{2.81}$$

$$= \text{PE}\left[\tau_2\tau_3 + \tau_2^\ell + \tau_3^\ell - \tau_2^\ell\tau_3^\ell\right]. \tag{2.82}$$

As we can see, taking the PLog of (2.82), the Higgs branch is generated by one generator $m = Q_1\widetilde{Q}_1 = \cdots = Q_\ell\widetilde{Q}_\ell$ of dimension 2 and two generators $B = \prod_{n=1}^{\ell}Q_n$, $\widetilde{B} = \prod_{n=1}^{\ell}\widetilde{Q}_n$ of dimension ℓ. They satisfy the following relation at order 2ℓ

$$m^\ell = B\widetilde{B}\,, \tag{2.83}$$

this corresponds to the variety $\mathbb{C}^2/\mathbb{Z}_\ell$. The Coulomb branch is simply a copy of $\mathbb{C}^\ell$ parametrised by $\{\Phi_1, \ldots, \Phi_\ell\}$

$$\mathrm{HS}(\tau_1; CB) = \lim_{\tau_1 \to 0} \mathrm{HS}(\tau_1, \tau_2, \tau_3; M) = \mathrm{PE}\left[\ell \tau_1\right] . \tag{2.84}$$

The mesonic Coulomb branch is the subvariety defined by $\Phi_1 = \Phi_2 = \cdots = \Phi_\ell$.

For general N the mesonic moduli space can be obtained as the N^{th} symmetric product of the $N = 1$ case [107, 108, 74, 174, 96] and

$$^{\text{mes}}\mathbf{M} = \mathrm{Sym}^N \left(\mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_\ell\right) \tag{2.85}$$

and the Hilbert series is given by

$$\mathrm{HS}\left(\tau_1, \tau_2, \tau_3; {}^{\text{mes}}M\right) = \frac{1}{N!} \frac{\partial^N}{\partial \nu^N} \mathrm{PE}\left[\nu M \left(\tau_1, \tau_2, \tau_3; \mathbb{C} \times \mathbb{C}^2/\mathbb{Z}_\ell\right)\right]\Bigg|_{\nu=0} . \tag{2.86}$$

Hall-Littlewood Index

Let's now move to the computation of the Hall-Littlewood index of the theory in Figure 2.3. Let's begin the case with $N = 1$ and generic ℓ the computation can be explicitly performed and we get

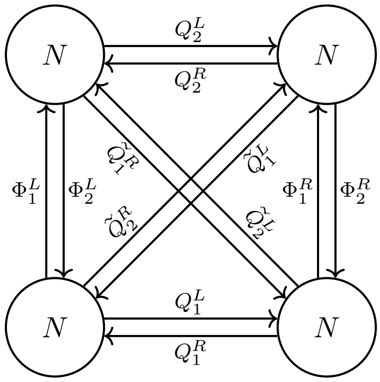
$$\begin{aligned}\mathrm{HL}(\tau_2, \tau_3) &= \prod_{n=1}^{\ell} \oint_{|z_n|=1} \frac{dz_n}{2\pi \mathrm{i} z_n} \mathrm{PE}\left[\sum_{n=1}^{\ell} \left(\alpha_n^{-1} \frac{z_n}{z_{n+1}} \tau + \alpha_n \frac{z_{n+1}}{z_n} \tau - \tau^2\right)\right] \\ &= \mathrm{PE}[\tau_2^\ell + \tau_3^\ell - \tau_2^\ell \tau_3^\ell] ,\end{aligned} \tag{2.87}$$

therefore the ratio between the Higgs branch Hilbert series (2.82) and the above index reads

$$\frac{\mathrm{HL}(\tau_2, \tau_3)}{\mathrm{HS}(\tau_2, \tau_3; HB)} = \mathrm{PE}[-\tau_2 \tau_3] = \mathrm{PE}[-\tau^2] = \mathrm{PE}[\mathrm{HL}_{\mathcal{D}_{0,(0,0)}}] , \tag{2.88}$$

here $\mathrm{HL}_{\mathcal{D}_{0,(0,0)}}$ denotes the Hall-Littlewood index of the free $\mathcal{N} = 2$ vector multiplet [168, 2]. We now want to compute the expression of the Hall-Littlewood index for a generic value of N. Since $\mathcal{D}_{0,(0,0)}$ is a free field multiplet it naturally decouples from the theory.

We would like to conjecture that the Hall-Littlewood index for general N can be



$$
\begin{array}{ll}
Q_1^L : \frac{\beta_1}{\alpha_1}\tau & Q_1^R : \frac{\beta_2}{\alpha_2}\tau \\
Q_2^L : \frac{\beta_2}{\alpha_1}\tau & Q_2^R : \frac{\beta_1}{\alpha_2}\tau \\
\tilde{Q}_1^L : \frac{\alpha_1}{\gamma_1}\tau & \tilde{Q}_1^R : \frac{\alpha_2}{\gamma_1}\tau \\
\tilde{Q}_2^L : \frac{\alpha_1}{\gamma_2}\tau & \tilde{Q}_2^R : \frac{\alpha_2}{\gamma_2}\tau \\
\Phi_i^L : \frac{\gamma_{i-1}}{\beta_i}\rho\sigma & \Phi_i^R : \frac{\gamma_{i-1}}{\beta_{i-1}}\rho\sigma
\end{array}
$$

Figure 2.4: *Quiver diagram of the $k = 2$ theory associated to a torus with $\ell = 2$ minimal punctures.*

obtained as the coefficient HL_N^ℓ of the following expansion

$$
\mathrm{PE}[\nu \mathrm{HL}_{N=1}^\ell(\tau_2, \tau_3)] = \sum_{N=1}^{\infty} \nu^N \mathrm{HL}_N^\ell(\tau_2, \tau_3)\,, \tag{2.89}
$$

$$
\mathrm{HL}_{N=1}^\ell(\tau_2, \tau_3) = \mathrm{PE}[\tau_2^\ell + \tau_3^\ell - \tau_2^\ell \tau_3^\ell]\,. \tag{2.90}
$$

We verified this this conjecture for various low values N and ℓ, that is to say for $(\ell, N) = \{(1,2), (2,2), (3,2), (1,3), (2,3), (1,4)\}$.

2.5.2 Class $\mathcal{S}_k$

Let us move to the case of general values of k, ℓ, while again focusing on $N = 1$. Let's firstly consider $k = \ell = 2$. The quiver diagram is given in Figure 2.4. Using *Macaulay2* we can compute the Hilbert series for R/I. The full moduli space $\mathbf{M}$ is then given as a projection onto gauge invariants

$$
\mathrm{HS}(\tau_1, \tau_2, \tau_3; M) = \oint \prod_{A=1}^{4} \left(\frac{dz_A}{2\pi \mathrm{i} z_A} \right) f^\flat(\tau_1, \tau_2, \tau_3, \alpha, \delta, \gamma, \beta; z_A)\,, \tag{2.91}
$$

of the Hilbert series for $\mathbf{F}$

$$
\begin{aligned}
f^\flat = &\frac{z_1 z_2 z_3 z_4 P(\tau_1, \tau_2, \tau_3, \alpha, \delta, \gamma, \beta; z_A)}{(\beta\tau_1 z_1 - \gamma z_2)(\beta z_1 - \gamma\tau_1 z_2)(\alpha\beta z_1 - \tau_3 z_3)(\beta\tau_3 z_1 - \delta z_3)} \\
&\times \frac{1}{(\alpha\tau_2 z_1 - \gamma z_4)(z_1 - \gamma\delta\tau_2 z_4)(\alpha\gamma\tau_2 z_2 - z_3)(\gamma z_2 - \delta\tau_2 z_3)} \\
&\times \frac{1}{(\tau_1 z_3 - \beta\gamma z_4)(z_3 - \beta\gamma\tau_1 z_4)(\alpha z_2 - \beta\tau_3 z_4)(\tau_3 z_2 - \beta\delta z_4)}
\end{aligned} \tag{2.92}
$$

with P polynomial in the z_A and $\tau_{1,2,3}$ and we have set $\gamma = \gamma_1$, $\beta = \beta_1$, $\alpha = \alpha_1$ and $\delta = \alpha_2$. After performing the integrals, we have

$$\begin{aligned}
&\mathrm{HS}(\tau_1,\tau_2,\tau_3;M) = \\
&\mathrm{PE}\left[2\tau_1^2+2\tau_2^2+2\tau_3^2\right]\big(\tau_2^2\tau_3^4\tau_1^4+\tau_2^2\tau_1^4+\tau_2^4\tau_3^2\tau_1^4-4\tau_2^2\tau_3^2\tau_1^4+\tau_3^2\tau_1^4 \\
&-\tau_2^3\tau_3^3\tau_1^3+\tau_2\tau_3^3\tau_1^3+\tau_2^3\tau_3\tau_1^3-\tau_2\tau_3\tau_1^3+\tau_2^4\tau_1^2+\tau_2^4\tau_3^4\tau_1^2-4\tau_2^2\tau_3^4\tau_1^2 \\
&+\tau_3^4\tau_1^2-3\tau_2^2\tau_1^2-4\tau_2^4\tau_3^2\tau_1^2+11\tau_2^2\tau_3^2\tau_1^2-3\tau_3^2\tau_1^2+\tau_2^3\tau_3^3\tau_1-\tau_2\tau_3^3\tau_1 \\
&-\tau_2^3\tau_3\tau_1+\tau_2\tau_3\tau_1+\tau_2^2\tau_3^4+\tau_2^4\tau_3^2-3\tau_2^2\tau_3^2+1\big)\,.
\end{aligned} \tag{2.93}$$

Note that the final expression is independent of $\frac{\alpha}{\delta}$ and both γ and β. This is because for the quivers with $U(N)$ gauge groups the $U(1)$ transformations generated by $q_{\alpha_n \bmod b}, q_{\gamma_i}, q_{\beta_i}$ are isomorphic to gauge transformations, while for $SU(N)$ gauge groups they are global symmetries. As before the Higgs branch is reached by considering the $\tau_1 \to 0$ limit

$$\begin{aligned}
\mathrm{HS}(\tau_2,\tau_3;HB) &= \lim_{\tau_1\to 0}\mathrm{HS}(\tau_1,\tau_2,\tau_3;M) = \frac{1-3\tau_3^2\tau_2^2+\tau_3^2\tau_2^4+\tau_3^4\tau_2^2}{\left(1-\tau_2^2\right)^2\left(1-\tau_3^2\right)^2} \\
&= \mathrm{PE}[\tau_2^2+\tau_3^2]+\mathrm{PE}[2\tau_2^2]+\mathrm{PE}[2\tau_3^2]-\mathrm{PE}[\tau_2^2]-\mathrm{PE}[\tau_3^2]\,.
\end{aligned} \tag{2.94}$$

We notice that the Hilbert series splits into that for $\mathbb{C}^2/(\mathbb{Z}_2\times\mathbb{Z}_2)$ (the mesonic Higgs branch moduli space), and two copies of $(\mathbb{C}/\mathbb{Z}_2)^2$, minus the two common $\mathbb{C}/\mathbb{Z}_2$-line intersections. From the Plethystic Logarithm

$$\mathrm{PLog}[\mathrm{HS}(\tau_2,\tau_3;HB)] = 2\tau_2^2+2\tau_3^2-3\tau_3^2\tau_2^2+\tau_3^2\tau_2^4+\tau_3^4\tau_2^2+\dots\,, \tag{2.95}$$

we recognise the generators as $B_i = Q_i^L Q_i^R$, $\widetilde{B}_i = \widetilde{Q}_i^L \widetilde{Q}_{i+1}^R$. At the next order we have the relations $B_1\widetilde{B}_1 = B_1\widetilde{B}_2 = B_2\widetilde{B}_1 = B_2\widetilde{B}_2$ thanks to the F-terms $\widetilde{Q}_i^L Q_i^L = Q_i^R \widetilde{Q}_{i-1}^R$. The higher order terms are Hilbert syzygies (a.k.a. relations between relations). This should be compared with $\mathrm{PLog}[M(0,\tau_2,\tau_3;\mathbb{C}^2/(\mathbb{Z}_2\times\mathbb{Z}_2))] = \tau_2^2+\tau_3^2$ which is counting only a particular subset of allowed operators.

We can also consider the Coulomb limit

$$\lim_{\tau_2\to 0}\lim_{\tau_3\to 0}\mathrm{HS}(\tau_1,\tau_2,\tau_3;M) = \mathrm{HS}(T;CB) = \mathrm{PE}\left[2T^2\right] \tag{2.96}$$

corresponding to the operators $\prod_{i=1}^k \Phi_i^L$, $\prod_{i=1}^k \Phi_i^R$ which indeed have dimension two and have fugacity $\prod_{i=1}^k \frac{\gamma_i}{\beta_{i-1}} = 1$ and $\prod_{i=1}^k \frac{\gamma_i}{\beta_i} = 1$ under the intrinsic symmetries.

For higher values of k, ℓ the computations of the Hilbert series become increasingly complex, due to the requirement of making primary decomposition in a larger number of variables using *Macaulay2*. We were able to compute the Higgs-branch Hilbert series for $k = \ell = 3$.

$$\mathrm{HS}(\tau_2, \tau_3; HB) = \mathrm{PE}[\tau_2^3 + \tau_3^3] + \mathrm{PE}[3\tau_2^3] + \mathrm{PE}[3\tau_3^3] - \mathrm{PE}[\tau_2^3] - \mathrm{PE}[\tau_3^3] \tag{2.97}$$

we note that γ_i, β_i do not appear and α_n enters only via $\alpha_1\alpha_2\alpha_3 = b^3$ in the Hilbert series for the Higgs Branch. Moreover, we conjecture that the form of the Hilbert series for arbitrary $k = \ell$ reads

$$\mathrm{HS}(\tau_2, \tau_3; HB) = \mathrm{PE}[\tau_2^\ell + \tau_3^\ell] + \mathrm{PE}[\ell\tau_2^\ell] + \mathrm{PE}[\ell\tau_3^\ell] - \mathrm{PE}[\tau_2^\ell] - \mathrm{PE}[\tau_3^\ell]\,. \tag{2.98}$$

This form for the Hilbert series implies that the moduli space is made up of a copy $\mathbb{C}^2/(\mathbb{Z}_\ell \times \mathbb{Z}_\ell)$, two copies of $(\mathbb{C}/\mathbb{Z}_\ell)^\ell$ minus $\mathbb{C}/\mathbb{Z}_\ell$ common intersections. Importantly, we have that our definition for the Higgs branch, in general, *does not* coincide with the mesonic moduli space $\mathbb{C}^2/(\mathbb{Z}_k \times \mathbb{Z}_\ell)$. Therefore, in general one cannot obtain the result for general N by simply taking the N^{th} symmetric product of the $N = 1$ result.

Hall-Littlewood Index

We can also compute the corresponding Hall-Littlewood index for our theories. The general expression is given by

$$\begin{aligned}\mathrm{HL}(\tau_2, \tau_3) = &\oint \prod_{n=1}^{\ell}\prod_{i=1}^{k}\left\{\frac{dz_{n,i}}{2\pi\mathrm{i}z_{n,i}}\right.\\ &\left.\times\,\mathrm{PE}\left[\left(\frac{\beta_{i+n-1}}{\alpha_n}\frac{z_{n,i}}{z_{n+1,i}}\tau + \frac{\alpha_n}{\gamma_i}\frac{z_{n+1,i-1}}{z_{n,i}}\tau - \frac{\beta_{i+n-1}}{\gamma_i}\frac{z_{n+1,i-1}}{z_{n+1,i}}\tau^2\right)\right]\right\}.\end{aligned} \tag{2.99}$$

For example, for $k = \ell = 2$, $N = 1$ we have

$$\mathrm{HL}(\tau_2, \tau_3) = \frac{1 - 4\tau_2^2\tau_3^2 + 2\tau_2^2\tau_3^4 + 2\tau_2^4\tau_3^2 - \tau_2^4\tau_3^4}{(1-\tau_2^2)^2(1-\tau_3^2)^2}\,. \tag{2.100}$$

For general $\ell = k$ with $N = 1$ we were propose the following conjecture, which we checked for various low values of $\ell = k$,

$$\mathrm{HL}(\tau_2, \tau_3) = \mathrm{PE}[\ell\tau_2^\ell] + \mathrm{PE}[\ell\tau_3^\ell] - 1\,. \tag{2.101}$$

The ratio HL/HS is then counting, with signs, the protected operators of the theory which have $j_1 + j_2 \geqslant \frac{1}{2}$ and $r + 2j_2 = \frac{4}{3}q_t$.

It is possible compute the Hall-Littlewood index for the class $\mathcal{S}_k$ theories with $SU(N)$ gauge groups. For $k = \ell = 2$ with $SU(2)^{k\ell}$ gauge group this reads

$$\begin{aligned}&\mathrm{HL}_{\alpha_1\alpha_2} = \\ &\oint \prod_{n=1}^{\ell}\prod_{i=1}^{k}\left(\frac{dz_{n,i}}{4\pi \mathrm{i} z_{n,i}} \frac{\left(1 - z_{n,i}^{\pm 2}\right)\left(1 - \frac{\beta_{i+n-1}}{\gamma_i}\frac{z_{n+1,i-1}^{\pm1}}{z_{n+1,i}^{\pm1}}\tau^2\right)}{\left(1 - \frac{\beta_{i+n-1}}{\alpha_n}\frac{z_{n,i}^{\pm1}}{z_{n+1,i}^{\pm1}}\tau\right)\left(1 - \frac{\alpha_n}{\gamma_i}\frac{z_{n+1,i-1}^{\pm1}}{z_{n,i}^{\pm1}}\tau\right)}\right).\end{aligned} \tag{2.102}$$

Expanding and then taking the PLog we have

$$\begin{aligned}\mathrm{PLog}\left[\mathrm{HL}_{\alpha_1\alpha_2}\right] = &\sum_{i=1}^{2}\left[\sum_{n=1}^{2}\left(\frac{\beta_i^2}{\alpha_n^2} + \gamma_i^2\alpha_n^2\right) + \left(\frac{1}{\alpha_1\alpha_2} + \alpha_1\alpha_2\right)\right]\tau^2 \\ &+ \sum_{n=1}^{2}\left[\frac{\alpha_n^2}{\alpha_{n-1}^2} - \sum_{i=1}^{2}\frac{\alpha_n}{\alpha_{n-1}}\left(\beta_i^2 + \frac{1}{\gamma_i^2}\right) - \sum_{i=1}^{2}\frac{\beta_{i+n-1}^2}{\gamma_{i-1}^2}\right]\tau^4 + \mathcal{O}(\tau^6)\,,\end{aligned} \tag{2.103}$$

recall we also have $\gamma_2 = \gamma_1^{-1}$, $\beta_2 = \beta_1^{-1}$ and the sums over i, n are taken modulo $k = 2$ and $\ell = 2$, respectively. The operators with $q_t = 1$ correspond to 'baryonic type' $\det Q_i^o$, $\det \widetilde{Q}_i^o$, with $o = L/R$ and 'mesonic type' $\operatorname{tr} Q_i^L Q_i^R$ and $\operatorname{tr} \widetilde{Q}_i^L \widetilde{Q}_i^R$. At the next order we have bosonic operators $\operatorname{tr} \widetilde{Q}_1^o Q_1^o \widetilde{Q}_2^o Q_2^o$ and fermionic operators of the form $\det Q_i^R \widetilde{Q}_{i-1}^L \overline{\lambda}_j^L$, $\det Q_i^L \widetilde{Q}_{i-1}^R \overline{\lambda}_j^R$ and finally $\operatorname{tr} Q_i^o \widetilde{Q}_{i-1}^o \overline{\lambda}_i^o$. Here $\overline{\lambda}_i^o = \overline{\lambda}_{i\dot{+}}^o$ denotes fermion in the superfield expansion $\overline{\Phi}_i^o = \overline{\Phi}_i^o + \overline{\theta}^{\dot{\alpha}}\overline{\lambda}_{i\dot{\alpha}}^o + \ldots$.

The unrefined $\alpha_n = \gamma_i = \beta_i = 1$ limit can be computed exactly and reads

$$\mathrm{HL}_{\delta\alpha} = \frac{1 + 6\tau^2 + 11\tau^4 - 12\tau^8 - 4\tau^{10}}{(1-\tau^2)^6}\,. \tag{2.104}$$

2.5.3 Deconstruction Limit

There have been many precision tests [175, 176, 177, 178] of the deconstruction proposal [179]. Of most interest to us in this subsection is the fact that [175, 97] were able to show that the $\frac{1}{2}$-BPS partition function of the $\mathcal{N} = (2,0)$ LST is equal to the Higgs branch Hilbert series of the corresponding 4d $\mathcal{N} = 2$ theory in the deconstruction limit. This naturally leads one to expect that a similar story should also exist for the $\mathcal{N} = (1,1)$ LST. The $\mathcal{N} = (1,1)$ LST arises as the worldvolume theory on a stack of N parallel NS5-branes in type IIB string theory [180, 181, 70, 182, 183]. The basic deconstruction proposal is that the 6d $\mathcal{N} = (1,1)$ LST may be effectively described by considering the $\mathcal{N} = 1$ $\mathfrak{u}(N)^{\oplus \ell k}$ toroidal quiver gauge theory realised as the $l_s \to 0$ limit of the worldvolume theory on a stack of N D3-branes probing a transverse $\mathbb{C}^3/(\mathbb{Z}_\ell \times \mathbb{Z}_k)$ singularity where the quotient acts as in (2.67). We then go to the point in parameter space where the vevs $v_5 = \langle Q_{(i,n)} \rangle$, $v_6 = \langle \Phi_{(i,n)} \rangle$ (this is a mixed branch in our language) and couplings $G = g^{\mathrm{YM}}_{(i,n)}$ are equal for all nodes in the quiver and then take the limit[3]

$$\ell \to \infty\,, \quad k \to \infty\,, \quad G \to \infty\,, \quad v_{5,6} \to \infty\,, \tag{2.105}$$

while holding $2\pi R_5 G v_5 = \ell$ and $2\pi R_6 G v_6 = k$ fixed. The main point is that, in this limit, the transverse $\mathbb{C}^3/(\mathbb{Z}_\ell \times \mathbb{Z}_k)$ can be approximated by $T^2 \times \mathbb{R}^4$ where the radii of the torus are $r_5 = v_5/k$ and $r_6 = v_6/\ell$. Performing T-duality along the two circles and then S-duality gives the rank N $\mathcal{N} = (1,1)$ LST on a torus with radii R_5, R_6.

Representations of the $\mathcal{N} = (1,1)$ supersymmetry algebra may be decomposed into a finite sum of representations of the bosonic subalgebra given by the sum of Lorentz algebra $\mathfrak{so}(6)$ and R-symmetry algebra $\mathfrak{so}(4) \cong \mathfrak{su}(2)_{R_1} \oplus \mathfrak{su}(2)_{R_2}$. The supercharges sit in the representations $\mathsf{Q} \in [0,1,0]^{(\frac{1}{2},0)}_{\frac{1}{2}}$ and $\widetilde{\mathsf{Q}} \in [0,0,1]^{(0,\frac{1}{2})}_{\frac{1}{2}}$ with representations labelled by $[h_1,h_2,h_3]^{(R_1,R_2)}_E$. In other words $\mathsf{Q}^a_{h_1h_2h_3}$ with $h_i = \pm\frac{1}{2}$ such that $8h_1h_2h_3 = -1$ and $\widetilde{\mathsf{Q}}^{\dot{a}}_{h_1h_2h_3}$ with $8h_1h_2h_3 = +1$. They obey

$$\{\mathsf{Q}^a, \mathsf{Q}^b\} = \epsilon^{ab}\eta^\mu p_\mu\,, \quad \{\widetilde{\mathsf{Q}}^{\dot{a}}, \widetilde{\mathsf{Q}}^{\dot{b}}\} = \epsilon^{\dot{a}\dot{b}}\widetilde{\eta}^\mu p_\mu\,, \quad \{\mathsf{Q}^a, \widetilde{\mathsf{Q}}^{\dot{b}}\} = 0\,, \tag{2.106}$$

where $\eta^\mu, \widetilde{\eta}^\mu$, $\mu = 1,2,\ldots,6$ denote the 't Hooft symbols which intertwine between $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$. The 4d $\mathcal{N} = 1$ supersymmetry algebra plus the residual global $\mathfrak{u}(1)_t \oplus \mathfrak{u}(1)_b$ symmetry algebras can be embedded into the 6d $\mathcal{N} = (1,1)$ algebra with the relations[4]

$$2j_1 = h_2 + h_3\,, \quad 2j_2 = h_2 - h_3\,, \tag{2.107}$$

$$\frac{3}{2}r = 2h_1 + R_1\,, \quad 2q_t = -h_1 + R_1 - 3R_2\,, \quad q_b = h_1 - R_1 - R_2\,. \tag{2.108}$$

[3] Our parameters are related to those of [179] by $N_5 = k$, $N_6 = \ell$.

[4] These can be obtained in the following way: the first two relations are simply the identifications one would make between the Cartans for $\mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \cong \mathfrak{so}(4) \subset \mathfrak{so}(6)$. When compactifying the 6d theory on T^2 the spinor label h_1 $h_1 = H = H_R + H_L$ on T^2 and $2H_L = q_1$, $2H_R = q_2$. Finally $2R_2 = q_1 - q_2$ is fixed by demanding that R_2 evaluates to zero on the $\mathcal{N} = 1$ subsector. This then fixes $q_3 = R_1$. The identifications (2.70) then give (2.108).

6d $\mathcal{N}=(1,1)$	(h_1,h_2,h_3,R_1,R_2)	(j_1,j_2)	r	q_t	q_b	4d $\mathcal{N}=1$
$\mathsf{Q}^1_{+\pm\mp}$	$(+\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},0)$	$(0,\pm\frac{1}{2})$	$+1$	0	0	$\widetilde{\mathcal{Q}}_{\pm}$
$\mathsf{Q}^1_{-\pm\pm}$	$(-\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2},+\frac{1}{2},0)$	$(\pm\frac{1}{2},0)$	$-\frac{1}{3}$	1	-1	
$\mathsf{Q}^2_{+\pm\mp}$	$(+\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2},-\frac{1}{2},0)$	$(0,\pm\frac{1}{2})$	$+\frac{1}{3}$	-1	$+1$	
$\mathsf{Q}^2_{-\pm\pm}$	$(-\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},0)$	$(\pm\frac{1}{2},0)$	-1	0	0	$\mathcal{Q}_{\pm}$
$\widetilde{\mathsf{Q}}^1_{+\pm\pm}$	$(+\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},0,+\frac{1}{2})$	$(\pm\frac{1}{2},0)$	$+\frac{2}{3}$	-1	0	
$\widetilde{\mathsf{Q}}^1_{-\pm\mp}$	$(-\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2},0,+\frac{1}{2})$	$(0,\pm\frac{1}{2})$	$-\frac{2}{3}$	$-\frac{1}{2}$	-1	
$\widetilde{\mathsf{Q}}^2_{+\pm\pm}$	$(+\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},0,-\frac{1}{2})$	$(\pm\frac{1}{2},0)$	$+\frac{2}{3}$	$+\frac{1}{2}$	$+1$	
$\widetilde{\mathsf{Q}}^2_{-\pm\mp}$	$(-\frac{1}{2},\pm\frac{1}{2},\mp\frac{1}{2},0,-\frac{1}{2})$	$(0,\pm\frac{1}{2})$	$-\frac{2}{3}$	$+1$	0	

Table 2.4: *One choice of embedding of the 4d $\mathcal{N}=1$ superalgebra into the 6d $\mathcal{N}=(1,1)$ superalgebra.*

The relationship between the 4d and 6d supercharges is given in Table 2.4.

Let us now move to our candidate set of operators on the 4d side that will ultimately reproduce the $\frac{1}{2}$-BPS scalar operators in the $(1,1)$ LST. Let's assume that, after primary decomposition, we can always identify a irreducible branch ${}^{\text{mes}}\mathbf{M} = \mathbf{M}//U(1)_D$ inside the full moduli space $\mathbf{M}$. ${}^{\text{mes}}\mathbf{M}$ will be our candidate for reproducing the 6d $\frac{1}{2}$-BPS ring. For any $\mathcal{N}=1$ theory, the operators parametrising $\mathbf{M}$ are themselves $\frac{1}{2}$-BPS with respect to the $\mathcal{N}=1$ supersymmetry algebra, namely they are annihilated by $\widetilde{\mathcal{Q}}_{\dot{\alpha}}$.

One may wonder why we have picked the subvariety ${}^{\text{mes}}\mathbf{M}$ as opposed to the full moduli space $\mathbf{M}$. Shortly we will prove that that, for the theory with $U(N)$ gauge groups, in the $k,\ell\to\infty$ limit that ${}^{\text{mes}}\mathbf{M}$ coincides with $\mathbf{M}$.

Let us first focus on the $N=1$ case. We can set, without loss of generality in this limit, $k=\ell$. The possible gauge invariant operators parametrising $\mathbf{M}$ for the theory with $U(1)$ gauge groups are in correspondence with the number of closed, directed paths that one can draw on the quiver diagram. There are four main types of operators. There are those which involve an equal number $p\leqslant k=\ell$ of Q, $\widetilde{Q}$ and Φ fields. Such an operator enters the Hilbert series with fugacity $\tau_1^p\tau_2^p\tau_3^p$. Operators of this form correspond to picking a base node, say $(i=1,n=1)$, and drawing a closed loop involving p vertical steps, p horizontal steps and p diagonal steps. However, recall the F-terms set

$$F_{\Phi_{(i,n)}} = \widetilde{Q}_{(i,n-1)}Q_{(i,n-1)} - Q_{(i-1,n)}\widetilde{Q}_{(i,n)} = 0\,, \tag{2.109}$$

$$F_{Q_{(i,n)}} = \Phi_{(i,n+1)}\widetilde{Q}_{(i,n)} - \widetilde{Q}_{(i+1,n)}\Phi_{(i+1,n)} = 0\,, \tag{2.110}$$

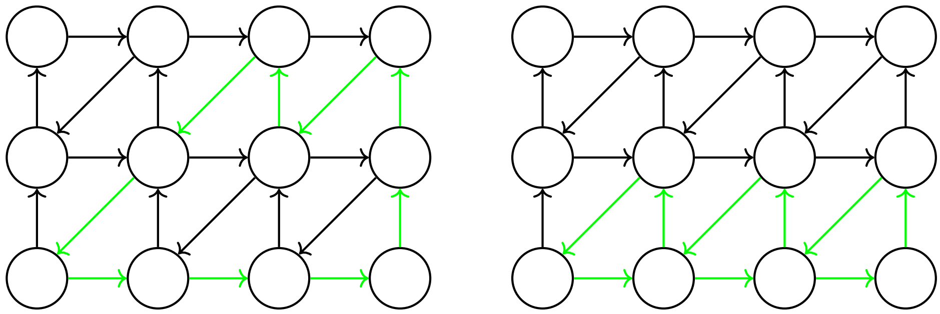


Figure 2.5: *Visualisation for the example of* $p = 3$.

$$F_{\widetilde{Q}_{(i,n)}} = Q_{(i,n)}\Phi_{(i,n+1)} - \Phi_{(i,n)}Q_{(i-1,n)} = 0\,. \tag{2.111}$$

In terms of the quiver diagram Figure 1.7, this means that the operations of moving right, up or diagonally all commute. In other words this means that the associated operator is independent of the choice of base node and of the specific path chosen, it depends only on the length $3p$. See Figure 2.5 for a diagrammatic example for $p = 3$. In that example on the left we have the operator $Q_{(i,n)}Q_{(i,n+1)}Q_{(i,n+2)}\Phi_{(i,n+3)}\Phi_{(i-1,n+3)}\widetilde{Q}_{(i-1,n+2)}\Phi_{(i-1,n+2)}\widetilde{Q}_{(i-1,n+1)}\widetilde{Q}_{(i,n)}$ which is reduced to $\prod_{m=0}^{2} Q_{(i,n+m)}\Phi_{(i,n+m+1)}\widetilde{Q}_{(i,n+m)}$. Again further applying the F-terms means this operator can be written simply as $m^p = m^3$ with, say, $m = \widetilde{Q}_{(1,\ell)}Q_{(1,\ell)}\Phi_{(1,1)}$. We can therefore write these operators as m^p. The other types of operators involve only Q's, $\widetilde{Q}$'s or Φ's. They can be written as powers of

$$B_{\Phi_n} = \prod_{i=1}^{\ell} \Phi_{(i,n)}\,, \quad B_{Q_i} = \prod_{n=1}^{\ell} Q_{(i,n)}\,, \quad B_{\widetilde{Q}_i} = \prod_{j=1}^{\ell} \widetilde{Q}_{(i+j,i-j)}\,, \tag{2.112}$$

they enter the Hilbert series with fugacity τ_1^ℓ, τ_2^ℓ and τ_3^ℓ, respectively. In degree $\geqslant \min(k,\ell)$ there can be complicated relations and higher syzgies between m, B_{Q_i}, $B_{\widetilde{Q}_i}$ and B_{Φ_n}, leading to a complicated structure for $\mathbf{M}$. However, the main point is that in the $\ell = k \to \infty$ limit the B_{Q_i}, $B_{\widetilde{Q}_i}$ and B_{Φ_n} operators all become infinity heavy and their dimensions $E = k = \ell$ tend to infinity, in particular $\lim_{\ell\to\infty} \tau_{1,2,3}^\ell \to 0$ and their contribution to the Hilbert series vanishes. This also implies that the corresponding ring becomes freely generated, since there are no relations between the operators in degree smaller than $\min(k,\ell) \to \infty$. Only the dimension of m remains finite in the limit. m is a purely mesonic operator and therefore, in this limit, one indeed expects ${}^{\mathrm{mes}}\mathbf{M}$ to coincide with $\mathbf{M}$.

The above arguments can be simply extended to the case $N \geqslant 2$ by taking traces. Each path with an equal number p of Q, $\widetilde{Q}$ and Φ's now corresponds to $A = 1, \ldots, N$ operators of dimension $3pA$ corresponding to operators schematically of the form $\operatorname{tr}(Q^p \widetilde{Q}^p \Phi^p)^A$. As long as we consider the traces we can apply the same rules that we did for $N = 1$, namely that the operations of moving right, up or diagonally commute. This again means any path of length $3p$ can be written as the loop given by m^p with, say, $m = \widetilde{Q}_{(1,\ell)} Q_{(1,\ell)} \Phi_{(1,1)}$ which is now a $N \times N$ matrix. The N operators corresponding to one of these closed paths can then be written as $m^p_{(A)}$ with

$$m_{(A)} := \operatorname{tr} m^A = \operatorname{tr} \left(\widetilde{Q}_{(1,\ell)} Q_{(1,\ell)} \Phi_{(1,1)} \right)^A . \tag{2.113}$$

As before the $m_{(A)}$ can have complicated relations with the operators which are of the form $\operatorname{tr} B^A_{Q_i}$, $\operatorname{tr} B^A_{\widetilde{Q}_i}$ and $\operatorname{tr} B^A_{\Phi_i}$ but the dimensions of the latter are $Ak = A\ell \to \infty$.

We therefore arrive at the conclusion that, in the deconstruction $k, \ell \to \infty$ limit $\mathbf{M}$ coincides with ${}^{\text{mes}}\mathbf{M}$ for all N.

For the $N = 1$ case we can identify ${}^{\text{mes}}\mathbf{M} = \mathbb{C}^3/(\mathbb{Z}_\ell \times \mathbb{Z}_k)$, so the Hilbert series for the $N = 1$ case is given by (2.71). Therefore, the Hilbert series for general N is given by

$$N!\mathrm{HS}(\tau_1, \tau_2, \tau_3; {}^{\text{mes}}M) = \frac{\partial^N}{\partial \nu^N} \mathrm{PE}\left[\nu M(\tau_1, \tau_2, \tau_3; \mathbb{C}^3/(\mathbb{Z}_\ell \times \mathbb{Z}_k))\right]\Big|_{\nu=0} . \tag{2.114}$$

Let us now consider the deconstruction limit. Taking the limit on the $N = 1$ result, recalling that $|\tau_{1,2,3}| < 1$, gives

$$\begin{aligned} &\lim_{k,\ell\to\infty} M(\tau_1, \tau_2, \tau_3; \mathbb{C}^3/(\mathbb{Z}_\ell \times \mathbb{Z}_k)) \\ &= \lim_{k\to\infty} \mathrm{PE}\left[\tau_1\tau_2\tau_3 + \tau_1^k + \tau_2^k\tau_3^k - \tau_1^k\tau_2^k\tau_3^k\right] = \mathrm{PE}\left[\tau_1\tau_2\tau_3\right] = \mathrm{PE}[pq] . \end{aligned} \tag{2.115}$$

Therefore, for general N, we have

$$\lim_{\ell,k\to\infty} \mathrm{HS}(\tau_1, \tau_2, \tau_3; {}^{\text{mes}}M) = \mathrm{PE}\left[\sum_{p=1}^{N} (\tau_1\tau_2\tau_3)^p\right] = \frac{1}{(\tau_1\tau_2\tau_3; \tau_1\tau_2\tau_3)_N} . \tag{2.116}$$

The coordinate-ring of $\mathbf{M}$ for this theory in the $k, \ell \to \infty$ limit is therefore simply $M = \mathbb{C}[m_{(1)}, m_{(2)}, \ldots, m_{(N)}]$.

Computation of the $\frac{1}{2}$-BPS Partition Function for $(1,1)$ LST

Let us now move to the computation of the 6d quantity that we would like to match to the 4d quantity (2.116). We can use the fact that, at low energies, the $\mathcal{N}=(1,1)$ LST admits an effective description as 6d maximally supersymmetric SYM theory with gauge group $U(N)$. The on-shell degrees of freedom of the $(1,1)$ SYM theory contains a 2-form gauge field strength $F \in [0,1,1]_2^{(0,0)}$, scalars $X \in [0,0,0]_2^{(\frac{1}{2},\frac{1}{2})}$ and fermions $\lambda \in [0,1,0]_{\frac{5}{2}}^{(0,\frac{1}{2})}$, $\widetilde{\lambda} \in [0,0,1]_{\frac{5}{2}}^{(\frac{1}{2},0)}$ all in the adjoint of $\mathfrak{g}=\mathfrak{u}(N)$. The supersymmetry transformations of interest to us are

$$\mathsf{Q}^a X^{b\dot{b}} = \frac{1}{\sqrt{2}}\epsilon^{ab}\lambda^{\dot{b}}\,, \quad \widetilde{\mathsf{Q}}^{\dot{a}} X^{b\dot{b}} = \frac{1}{\sqrt{2}}\epsilon^{\dot{a}\dot{b}}\widetilde{\lambda}^{b}\,. \tag{2.117}$$

We can therefore construct $\frac{1}{2}$-BPS multiplets whose highest weight state is annihilated by both Q^1 and $\widetilde{\mathsf{Q}}^{\dot{1}}$. The supersymmetric primary of these multiplets is given by

$$\mathcal{O}_A := \mathrm{tr}\left(X^{1\dot{1}}\right)^A \tag{2.118}$$

which are independent for $A=1,\ldots,N$. These have

$$h_1 = h_2 = h_3 = 0 \text{ and } E = 4R_1 = 4R_2 = 2A\,. \tag{2.119}$$

By acting with all possible supersymmetries $\mathsf{Q},\widetilde{\mathsf{Q}}$ and $\mathfrak{so}(6)\oplus\mathfrak{su}(2)_{R_1}\oplus\mathfrak{su}(2)_{R_2}$ generators we can generate the entire $\frac{1}{2}$-BPS multiplet by acting on the highest weight state $\mathcal{O}_A$. We can define a $\frac{1}{2}$-BPS partition function (Hilbert series) by passing to the scalar sector of the $\mathsf{Q}^1 \cap \widetilde{\mathsf{Q}}^{\dot{1}}$ -cohomology

$$Z^{(1,1)}_{\frac{1}{2}\text{-BPS}} = \mathrm{Tr}_{\mathcal{H}}\, x^{2R_1}\,, \tag{2.120}$$

$$\mathcal{H} = \left\{\mathbb{C}[\mathcal{O}]^{\mathfrak{g}} \middle| E = 4R_1 = 4R_2\,, h_1 = h_2 = h_3 = 0\right\}\,. \tag{2.121}$$

With this definition we have constructed an object which is counting only the gauge invariant words comprised of scalar component $X^{1\dot{1}}$ in the $(1,1)$ SYM theory. Using letter counting we can compute

$$Z^{(1,1)}_{\frac{1}{2}\text{-BPS}} = \oint d\mu_{U(N)}\,\mathrm{PE}\left[x\,\chi_{\mathrm{adj}}(\mathbf{z})\right] = \frac{1}{(x;x)_N}\,, \tag{2.122}$$

this is the Hilbert series for $\mathbb{C}[\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_N]$. Identifying $x = \tau_1\tau_2\tau_3 = pq$ we have matched $Z^{(1,1)}_{\frac{1}{2}\text{-BPS}}$ with (2.116) and the map between the operators is just

$$m_{(A)} \overset{\text{4d:6d}}{\longleftrightarrow} \mathcal{O}_A\,. \tag{2.123}$$

These operators can be expected to be found as a subset of those counted by the $\mathbb{S}^1 \times \mathbb{S}^5$ partition function. By the Nahm classification [124], there is no superconformal algebra associated to $\mathcal{N} = (1,1)$ SUSY in 6d so there is no superconformal index associated to this theory nevertheless we can define the $\mathbb{S}^1 \times \mathbb{S}^5$ partition function of the theory

$$Z^{(1,1)}_{\mathbb{S}^1\times\mathbb{S}^5}(x, y, p_1, p_2) \propto \mathrm{Tr}_{\mathbb{S}^5}(-1)^F e^{-\beta H} x^{E-R_1} y^{R_2} p_1^{h_1-h_3} p_2^{h_2-h_3} \tag{2.124}$$

where $H = \{\mathsf{Q}, \mathsf{Q}^\dagger\} = E - h_1 - h_2 - h_3 - 4R_1$ with $\mathsf{Q} := \mathsf{Q}^1_{---}$. The partition function (2.124) can be computed using the elliptic genus method [184, 185, 186] or using the refined topological string [187, 57]. $Z^{(1,1)}_{\mathbb{S}^1\times\mathbb{S}^5}$ is expected to receive both perturbative contributions aswell as non-perturbative contributions from 6d SYM instanton string states.

If we consider taking $xy = p_1 = p_2 = 1$ then

$$Z^{(1,1)}_{\mathbb{S}^1\times\mathbb{S}^5}(x, x^{-1}, 1, 1) \propto \mathrm{Tr}_{\mathbb{S}^5}(-1)^F e^{-\beta H} x^{E-R_1-R_2} \tag{2.125}$$

receives extra shortening and is annihilated by Q and $\widetilde{\mathsf{Q}}^{\dot{1}}_{+--}$,$\widetilde{\mathsf{Q}}^{\dot{1}}_{-+-}$, $\widetilde{\mathsf{Q}}^{\dot{1}}_{--+}$. Therefore the unrefined limit (2.125) receives non-zero contributions only from states with $h_1 = h_2 = h_3 = E - 4R_2$ and $E + 2R_1 = 6R_2$. $\mathcal{H}$ is clearly contained as a subset of those states when $E = 4R_2$.

We also note that (2.120) is equal to the $\frac{1}{2}$-BPS limit of the index of the $\mathcal{N} = (2,0)$ theory of type $\mathfrak{g} = \mathfrak{u}(N)$, which we have computed in (1.132). Our result therefore falls into the general result that, in all known examples, the $\frac{1}{2}$-BPS partition function seems to be a universal quantity in all maximally supersymmetric theories in 3, 4 and 6 dimensions [59]. Additionally, the $(1,1)$ and $(2,0)$ LSTs are related by T-duality. T-duality exchanges winding and momentum modes along the temporal $\mathbb{S}^1$. Since, by definition, $Z^{(1,1)}_{\frac{1}{2}\text{-BPS}}$ and the analogous quantity $Z^{(2,0)}_{\frac{1}{2}\text{-BPS}}$ [188, 61] defined for the $(2,0)$ theory count operators only in the zero winding and zero momentum sectors one naturally expects $Z^{(1,1)}_{\frac{1}{2}\text{-BPS}} = Z^{(2,0)}_{\frac{1}{2}\text{-BPS}}$. It would be interesting to further investigate if it is possible to obtain the more general partition

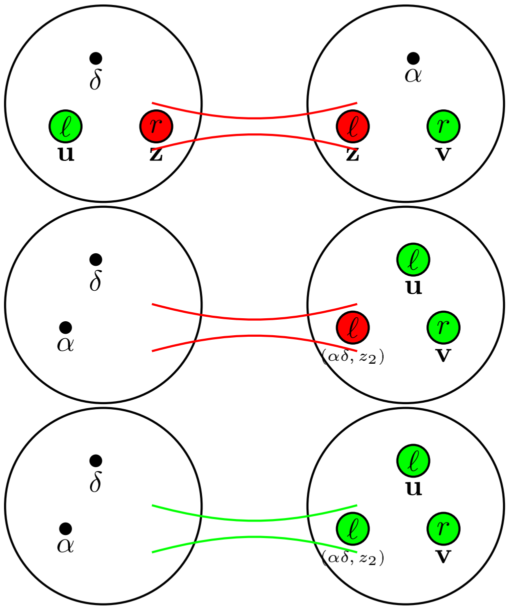


Figure 2.6: *Three 'S-dual' frames of the basic A_1 four punctured sphere in class $\mathcal{S}_2$. On the top is the canonical weakly coupled frame with fluxes for $U(1)_\beta \times U(1)_\gamma \times U(1)_t$ given by $\mathcal{F} = (0, 0, 1)$. In the middle is the S-dual frame obtained by gluing a quiver tail to the T_B theory with flux $\mathcal{F}_B = (-\frac{1}{4}, \frac{1}{4}, 1)$. On the bottom is the S-dual frame obtained by gluing a quiver tail to the T_A theory with fluxes $\mathcal{F}_A = (\frac{1}{4}, \frac{1}{4}, 1)$. Green (red) circles indicate maximal punctures of colour $c = 0$ ($c = 1$). $o = l, r$ indicates the orientation of a maximal puncture. Dots indicate minimal punctures.*

functions, such as (2.124) and (2.125), from 4d class $\mathcal{S}_k$ quantities using deconstruction [175, 176, 177].

2.6 Interacting Trinion SCFTs

As pictured in Figure 2.6 the basic A_1 four punctured sphere in class $\mathcal{S}_2$ admits three 'S-dual' descriptions. The first is the most familiar, being the standard Lagrangian frame, and is given in Figure 2.7. The other descriptions involve strongly interacting SCFTs, associated to spheres with three maximal punctures, with an $SU(2)$ gauging to a quiver tail. These SCFTs, denoted T_A & T_B in [136] carry global symmetries of, at least $\mathfrak{su}(2)^2_z \oplus \mathfrak{su}(2)^2_v \oplus \mathfrak{su}(2)^2_u \oplus \mathfrak{u}(1)_\gamma \oplus \mathfrak{u}(1)_\beta \oplus \mathfrak{u}(1)_t$. The field content of the B-type quiver tails is listed in Table 2.5. The superpotential for that theory is

$$W_B \supset \sum_{a=\pm} \left(q^{(a)} \Phi'^{(+)} B_{1,a+} + q^{(a)} \Phi'^{(-)} B_{2,a-} \right) + q^{(+)} q^{(-)} T_0 \,, \tag{2.126}$$

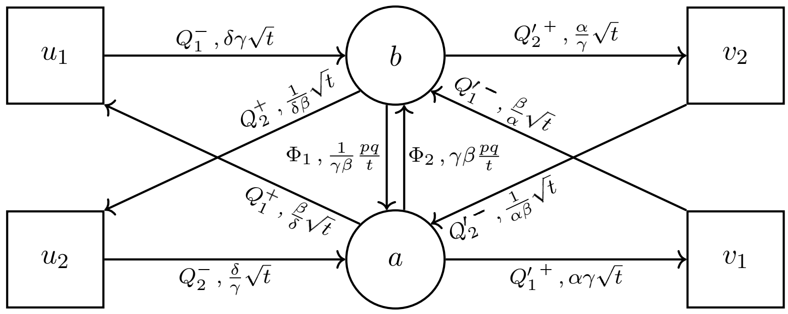


Figure 2.7: *Quiver diagram of the theory associated to a sphere with two minimal punctures and two maximal punctures of colour $c_l = c_r = 0$ in the canonical weakly coupled frame.*

Similarly, the field content of the A-type quiver tails can be found in Table 2.6.

Field(s)	$\mathfrak{su}(2)_{z_2}$	$\mathfrak{u}(1)_\delta$	$\mathfrak{u}(1)_\alpha$	$\mathfrak{u}(1)_r$	$\mathfrak{u}(1)_\beta$	$\mathfrak{u}(1)_\gamma$	$\mathfrak{u}(1)_t$	$\delta_{1\pm}$	$\widetilde{\delta}_{2\dot{+}}$
$q^{(\pm)}$	$\mathbf{2}$	± 1	∓ 1	0	1	-1	0	0	0
$\Phi'^{(\pm)}$	$\mathbf{2}$	∓ 1	∓ 1	$2/3$	∓ 1	∓ 1	-1	2	0
$B_{2,\pm -}$	$\mathbf{1}$	-1 ∓ 1	-1 ± 1	$4/3$	-2	0	1	0	4
$B_{1,\pm +}$	$\mathbf{1}$	1 ∓ 1	1 ± 1	$4/3$	0	2	1	0	4
T_0	$\mathbf{1}$	0	0	2	-2	2	0	2	4

Table 2.5: *Field content associated to the type B 'quiver tail' in class $\mathcal{S}_2$ corresponding to a sphere with two minimal punctures α, δ and a 'pinched' maximal puncture $(z_1 = \alpha\delta, z_2)$ which can be glued to a maximal puncture to convert it to two minimal punctures. Each field is an $\mathcal{N} = 1$ chiral multiplet. In the final columns we firstly list the value of $\delta_{1\pm} = r + 2j_2 - \frac{4}{3}q_t \pm 2j_1$ and $\widetilde{\delta}_{2\dot{+}} = 4j_2 + 2r + \frac{4}{3}q_t$ of the corresponding field.*

The superpotential is

$$W_A \supset \sum_{a,b=\pm} q^{(a)}\Phi'^{(b)}B_{1,ab} + q^{(+)}q^{(-)}T_0\,. \tag{2.127}$$

The respective T_B & T_A SCFTs have mesonic operators $M^u_\pm, M^v_\pm, M^z_\pm$ which are in the bifundamental representation of the corresponding $\mathfrak{su}(2)^{\oplus 2}$ and in the $+1$ representation of $\mathfrak{u}(1)_t$. In the case of the T_B theory $M^u_\pm, M^v_\pm$ have charge ± 1 under $\mathfrak{u}(1)_\gamma$ and ∓ 1 $\mathfrak{u}(1)_\beta$ while $M^z_\pm$ has ± 1, ± 1. In the case of the T_A theory $M^u_\pm, M^v_\pm, M^z_\pm$ all have charge ± 1 under $\mathfrak{u}(1)_\gamma$ and ∓ 1 $\mathfrak{u}(1)_\beta$. The tails couple to their respective

Field(s)	$\mathfrak{su}(2)_{z_2}$	$\mathfrak{u}(1)_\delta$	$\mathfrak{u}(1)_\alpha$	$\mathfrak{u}(1)_r$	$\mathfrak{u}(1)_\beta$	$\mathfrak{u}(1)_\gamma$	$\mathfrak{u}(1)_t$	$\delta_{1\pm}$	$\widetilde{\delta}_{2\dot{+}}$
$q^{(\pm)}$	**2**	±1	∓1	0	-1	-1	0	0	0
$\Phi'^{(\pm)}$	**2**	∓1	∓1	2/3	±1	∓1	-1	2	0
$B_{1,\pm\pm}$	**1**	0	±2	4/3	1 ∓ 1	1 ± 1	1	0	4
$B_{1,\mp\pm}$	**1**	±2	0	4/3	1 ∓ 1	1 ± 1	1	0	4
T_0	**1**	0	0	2	2	2	0	2	4

Table 2.6: *Field content associated to the type A 'quiver tail' in class $\mathcal{S}_2$ corresponding to a sphere with two minimal punctures α, δ and a 'pinched' maximal puncture $(z_1 = \alpha\delta, z_2)$ which can be glued to a maximal puncture to convert it to two minimal punctures. Each field is an $\mathcal{N} = 1$ chiral multiplet. In the final columns we firstly list the value of $\delta_{1\pm} = r + 2j_2 - \frac{4}{3}q_t \pm 2j_1$ and $\widetilde{\delta}_{2\dot{+}} = 4j_2 + 2r + \frac{4}{3}q_t$ of the corresponding field.*

trinions through superpotentials

$$\Delta W_B = \sum_{a=\pm} \Phi'^{(a)} M_a^z\,, \quad \Delta W_A = \sum_{a=\pm} \Phi'^{(a)} M_a^z\,. \tag{2.128}$$

Ultimately we have the following identity between the indices

$$\mathcal{I}_{\mathbf{u}\delta\alpha}{}^{\mathbf{v}} = \mathcal{I}_{\mathbf{u}\delta}{}^{\mathbf{z}} \cdot \mathcal{I}_{\mathbf{z}\alpha}{}^{\mathbf{v}} = \mathcal{I}_{\delta\alpha}^{(B)\mathbf{z}} \cdot \mathcal{I}_{\mathbf{zu}}^{(T_B)\mathbf{v}} = \mathcal{I}_{\delta\alpha}^{(A)\mathbf{z}} \cdot \mathcal{I}_{\mathbf{zu}}^{(T_A)\mathbf{v}} \tag{2.129}$$

and we set $\mathbf{z} = (\alpha\delta, z_2)$. The expression for $\mathcal{I}_{\mathbf{u}\delta\alpha}{}^{\mathbf{v}}$ is given in (2.33). The final two expressions read [133, 136]

$$\begin{aligned}
\mathcal{I}_{\delta\alpha}^{(B)\mathbf{z}} \cdot \mathcal{I}_{\mathbf{zu}}^{(T_B)\mathbf{v}} &= \oint \frac{\kappa dz_2}{4\pi \mathrm{i} z_2} \delta\left(z_2, s; \frac{\beta}{\gamma}\right) \frac{\Gamma_e\left(t\frac{\gamma}{\beta}(\beta\gamma z_1)^{\pm1} s^{\pm1}\right)}{\Gamma_e\left(t(\gamma\beta z_1)^{\pm1} z_2^{\pm1}\right)} \mathcal{I}_{\mathbf{zu}}^{(T_B)\mathbf{v}}\,, \\
\mathcal{I}_{\delta\alpha}^{(A)\mathbf{z}} \cdot \mathcal{I}_{\mathbf{zu}}^{(T_A)\mathbf{v}} &= \oint \frac{\kappa dz_2}{4\pi \mathrm{i} z_2} \delta\left(z_2, s; \frac{1}{\beta\gamma}\right) \frac{\Gamma_e\left(t\beta\gamma \left(\frac{z_1\gamma}{\beta}\right)^{\pm1} s^{\pm1}\right)}{\Gamma_e\left(t\left(\frac{z_1\gamma}{\beta}\right)^{\pm1} z_2^{\pm1}\right)} \mathcal{I}_{\mathbf{zu}}^{(T_A)\mathbf{v}}\,,
\end{aligned} \tag{2.130}$$

where we have defined $s = \frac{\delta}{\alpha}$ and $z_1 = \alpha\delta$ and the function $\delta(x, y; T)$ is defined in (A.54). We now need to apply the Spiridonov-Warnaar inversion formula [189] to invert the above integrals.

2.6.1 T_B Theory

Hall-Littlewood and Coulomb Indices

Applying (A.56) the index for the T_B theory is

$$\mathcal{I}_{\mathbf{zu}}^{(T_B)\mathbf{v}} = \Gamma_e\left(t(\gamma\beta z_1)^{\pm 1} z_2^{\pm 1}\right) \oint_{C_{z_2}} \frac{\kappa ds}{4\pi \mathrm{i} s} \frac{\delta\left(s, z_2; \frac{\gamma}{\beta}\right) \mathcal{I}_{\mathbf{u}\sqrt{s z_1}\sqrt{\frac{z_1}{s}}}{}^{\mathbf{v}}}{\Gamma_e\left(t\frac{\gamma}{\beta}(\beta\gamma z_1)^{\pm 1} s^{\pm 1}\right)}, \tag{2.131}$$

where C_{z_2} is a deformation of the unit circle which encloses $s = \frac{\gamma}{\beta} z_2^{\pm 1}$ and excludes $s = \frac{\beta}{\gamma} z_2^{\pm 1}$. The same expression was also given in (A.1) and (5.20) of [136] so we will not analyse the full expression. But rather let us focus on the Hall-Littlewood limit $\frac{p}{\sqrt{t}}, \frac{q}{\sqrt{t}} \to 0$ of (2.131). We know that this limit is well defined for the original Lagrangian four punctured sphere theory. We can also see from Table 2.5 that each letter for the tail has $\delta_{1\pm} \geqslant 0$, meaning that the limit is also well defined for the B-type tail theory; implying that the Hall-Littlewood limit of the index for the T_B theory is also expected to be well defined. In this limit the three punctured sphere index becomes

$$\begin{aligned} \mathrm{HL}_{\mathbf{zu}}^{(T_B)\mathbf{v}} =& \frac{\left(1 - \frac{\gamma^2}{\beta^2}\right)}{\left(1 - \tau^2 (\gamma\beta z_1)^{\pm 1} z_2^{\pm 1}\right)} \\ & \times \oint_{C_{z_2}} \frac{ds}{4\pi \mathrm{i} s} \left(1 - s^{\pm 2}\right) \frac{\left(1 - \tau^2 \frac{\gamma}{\beta}(\beta\gamma z_1)^{\pm 1} s^{\pm 1}\right)}{\left(1 - \frac{\gamma}{\beta} s^{\pm 1} z_2^{\pm 1}\right)} \mathrm{HL}_{\mathbf{u}\sqrt{s z_1}\sqrt{\frac{z_1}{s}}}{}^{\mathbf{v}} \end{aligned} \tag{2.132}$$

where $\mathrm{HL}_{\mathbf{u}\sqrt{s z_1}\sqrt{\frac{z_1}{s}}}{}^{\mathbf{v}}$ is given in (2.59). The global symmetry of this theory does not enhance from the expected one and is given by $\mathfrak{g}^{(T_B)} = \mathfrak{su}(2)^2_{\mathbf{z}} \oplus \mathfrak{su}(2)^2_{\mathbf{v}} \oplus \mathfrak{su}(2)^2_{\mathbf{u}} \oplus \mathfrak{u}(1)_\gamma \oplus \mathfrak{u}(1)_\beta \oplus \mathfrak{u}(1)_t$. In expansion around $\tau = 0$ and $\frac{\gamma}{\beta} = 0$

$$\mathrm{HL}_{\mathbf{zu}}^{(T_B)\mathbf{v}} = 1 + P_1 \tau^2 + P_2 \tau^4 + \mathcal{O}(\tau^6, \frac{\gamma^6}{\beta^6}) \tag{2.133}$$

with

$$\begin{aligned} P_1 =& f_1^-([0;0;0;0;1;1] + [0;0;1;1;0;0]) + f_1^+[1;1;0;0;0;0] \\ & + \gamma^2 [0;1;1;0;1;0] + [1;0;0;1;1;0] + [1;0;1;0;0;1] \\ & + \frac{1}{\beta^2}[0;1;0;1;0;1], \end{aligned} \tag{2.134}$$

$$
\begin{aligned}
P_2 = {} & 1 + f_2^-([0;0;0;0;2;2] + [0;0;2;2;0;0]) + f_2^+[2;2;0;0;0;0] \\
& + \frac{\beta^2}{\gamma^2} + \left(f_2^- + 1\right)[0;0;1;1;1;1] + f_1^+ f_1^-[1;1;1;1;0;0] \\
& + f_1^+ f_1^-[1;1;0;0;1;1] + f_1^-([1;0;1;0;1;0] + [1;0;0;1;0;1]) \\
& + f_1^+([0;1;1;0;0;1] + [0;1;0;1;1;0]) + [2;0;1;1;1;1] \\
& + \frac{\gamma^2}{\beta^2}[0;2;1;1;1;1] + \frac{1}{\beta^2}[1;1;1;1;0;2] + \gamma^2[1;1;1;1;2;0] \\
& + \gamma^2[1;1;2;0;1;1] + \frac{1}{\beta^2}[1;1;0;2;1;1] + \gamma^2 f_1^-[0;1;2;1;1;0] \\
& + f_1^-([1;0;1;0;1;2] + [1;0;0;1;2;1] + [1;0;1;2;1;0]) \\
& + f_1^-[1;0;2;1;0;1] + \gamma^2 f_1^-[0;1;1;0;2;1] + \frac{1}{\beta^2} f_1^-[0;1;1;2;0;1] \\
& + \frac{1}{\beta^2} f_1^- + [0;1;0;1;1;2] + f_1^+([2;1;1;0;0;1] + [2;1;0;1;1;0]) \\
& + \gamma^2 f_1^+[1;2;1;0;1;0] + \frac{1}{\beta^2} f_1^+[1;2;0;1;0;1] + \gamma^4[2;2;2;2;2;2] \\
& + [2;0;2;0;0;2] + \frac{1}{\beta^4}[0;2;0;2;0;2] + ([2;0;0;0;0;0] + \text{perms.})
\end{aligned}
\tag{2.135}
$$

where we write the character of $SU(2)_{z_1} \times SU(2)_{z_2} \times SU(2)_{v_1} \times SU(2)_{v_2} \times SU(2)_{u_1} \times SU(2)_{u_2}$ as $[n_1;n_2;n_3;n_4;n_5;n_6]$. In order to condense the notation we have also defined $f_n^\pm \equiv \sum_{i=0}^n \gamma^{n-2i}\beta^{\pm(n-2i)}$. Note the appearance of the Mesonic generators $M^u_\pm, M^v_\pm, M^z_\pm$ in the first line of (2.134). We also have additional generators, in the second line of (2.134), in trifundamental representations of the corresponding $SU(2)^3$'s. The conformal R-symmetry of the T_B theory is [136]

$$
r_c = r + 0.0985(q_\gamma - q_\beta) - 0.043523 q_t\,, \tag{2.136}
$$

where r denotes the free R-symmetry, for the Hall-Littlewood index this can be easily computed by using $r = -2j_2 + \frac{4}{3}q_t$ from (2.37). Operators contributing to the Hall-Littlewood limit of the index therefore have conformal dimension

$$
E_c = 2j_2 + \frac{3}{2}r_c = -j_2 + 0.14775(q_\gamma - q_\beta) + 1.93472 q_t\,. \tag{2.137}
$$

Recall for unitarity that $E_c \geqslant 1$, this implies that the (bosonic) operators with $q_t = 1$ appearing in (2.134) must have $j_2 = 0$. This, in particular, implies that they all are the top components of $\overline{\mathcal{B}}_{r_c,(0,0)}$ multiplets and therefore are Higgs branch operators, in the definition of Section 2.2.2. Therefore, the Higgs branch of the T_B theory

	E_c	$\mathfrak{su}(2)^2_z$	$\mathfrak{su}(2)^2_v$	$\mathfrak{su}(2)^2_u$	q_γ	q_β	q_t
M^u_+	2.23022	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{2})$	$+1$	-1	1
M^u_-	1.63922	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{2})$	-1	$+1$	1
M^v_+	2.23022	$(\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{2})$	$(\mathbf{1},\mathbf{1})$	$+1$	-1	1
M^v_-	1.63922	$(\mathbf{1},\mathbf{1})$	$(\mathbf{2},\mathbf{2})$	$(\mathbf{1},\mathbf{1})$	-1	$+1$	1
$M^z_\pm$	1.93472	$(\mathbf{2},\mathbf{2})$	$(\mathbf{1},\mathbf{1})$	$(\mathbf{1},\mathbf{1})$	± 1	± 1	1
B_{211}	2.23022	$(\mathbf{1},\mathbf{2})$	$(\mathbf{2},\mathbf{1})$	$(\mathbf{2},\mathbf{1})$	2	0	1
B_{121}	1.93472	$(\mathbf{2},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{2},\mathbf{1})$	0	0	1
B_{112}	1.93472	$(\mathbf{2},\mathbf{1})$	$(\mathbf{2},\mathbf{1})$	$(\mathbf{1},\mathbf{2})$	0	0	1
B_{222}	2.23022	$(\mathbf{1},\mathbf{2})$	$(\mathbf{1},\mathbf{2})$	$(\mathbf{1},\mathbf{2})$	0	-2	1

Table 2.7: *(Subset of) Higgs branch operators of the T_B SCFT. We write* $\mathfrak{su}(2)^2_{z,v,u} = \mathfrak{su}(2)_{z_1,v_1,u_1} \oplus \mathfrak{su}(2)_{z_2,v_2,u_2}$.

contains at least the operators listed in Table 2.7. It is tempting to conjecture that Table 2.7 is actually the complete list of operators on the Higgs branch. Expansion of the Plethystic Logarithm $\mathrm{PLog}\,\mathrm{HL}^{(T_B)} = 1 + L_1\tau^2 + L_2\tau^4 + \ldots$, $P_1 = L_1$ shows that every coefficient in L_2 is negative implying those operators are all fermionic and there are no new scalar operators appearing at order τ^4. This of course does not rule out the possibility of the existence of additional scalar operators that are cancelled by fermionic operators due to the factor $(-1)^F$ or new operators appearing with $q_t \geqslant 3$. We can consider an unrefined $\mathbf{z} = \mathbf{u} = \mathbf{v} = \mathbf{1}$, $\gamma\beta = 1$ limit. In this limit we were able to cast $\mathrm{HL}_{\mathbf{1}\sqrt{s}\sqrt{\frac{1}{s}}}{}^{\mathbf{1}}$ into a relatively closed form and the three punctured sphere index becomes

$$\mathrm{HL}^{(T_B)\mathbf{1}}_{\mathbf{11}}\Big|_{\gamma\beta=1} = \oint_{C_1} \frac{ds}{2\pi \mathrm{i}} \left\{ \frac{1}{\left(\frac{\beta}{\gamma} - s\right)^2 \left(\frac{s\beta}{\gamma} - 1\right)^2} \times \frac{P(s,\gamma\beta^{-1},\tau)}{(s-\tau^2)^6 (s\tau^2 - 1)^6 \left(\frac{\tau^2\beta}{\gamma} - s\right)^2 \left(\frac{s\tau^2\beta}{\gamma} - 1\right)^2} \right\} \tag{2.138}$$

where $P(s,\gamma\beta^{-1},\tau)$ is a polynomial in s of degree 17. We then have to collect the residues of the poles in the interior of C_1; located at $s = \frac{\gamma}{\beta}, \tau^2, \frac{\tau^2\beta}{\gamma}$. The final result is given by

$$\mathrm{HL}^{(T_B)\mathbf{1}}_{\mathbf{11}}\Big|_{\gamma\beta=1} = \frac{\tau^{19}\gamma^2}{\beta^2} \frac{Q_B\left(\frac{\beta}{\gamma},\frac{1}{\tau}\right) - Q_B\left(\frac{\gamma}{\beta},\tau\right)}{\left(1-\frac{\beta}{\gamma}\tau^2\right)^7 \left(1-\frac{\gamma}{\beta}\tau^2\right)^9 (1-\tau^2)^{11}}\,, \tag{2.139}$$

where $Q_B(\gamma\beta^{-1},\tau)$ is a degree 19 polynomial in τ. For brevity we relegate the full expression of $Q_B(\gamma\beta^{-1},\tau)$ to Appendix C.2 and quote only the result for $\gamma=\beta=1$

$$\begin{aligned}\mathrm{HL}^{(T_B)\mathbf{1}}_{\mathbf{11}}\Big|_{\gamma=\beta=1} &= \frac{1}{(1-\tau^2)^{18}}\big(\tau^{20}+38\tau^{18}+474\tau^{16}+2582\tau^{14} \\ &+6895\tau^{12}+9516\tau^{10}+6895\tau^{8}+2582\tau^{6}+474\tau^{4}+38\tau^{2}+1\big)\ .\end{aligned} \tag{2.140}$$

Note that the numerator of (2.140) possesses a palindromic symmetry. We can also compute the Coulomb limit of the index ($p,q,t\to 0$, $\frac{pq}{t}\to T$); this was originally given in [154]. We have that

$$\begin{aligned}\mathcal{I}^{C\,(T_B)\mathbf{v}}_{\mathbf{zu}} &= \left(1-T(\gamma\beta z_1)^{\pm1}z_2^{\pm1}\right) \\ &\times \oint_{C_{z_2}} \frac{ds}{4\pi\mathrm{i}s}\frac{\left(1-\frac{\gamma^2}{\beta^2}\right)(1-s^{\pm2})}{\left(1-\frac{\gamma}{\beta}z_2^{\pm1}s^{\pm1}\right)}\frac{\mathcal{I}^{C}{}_{\mathbf{u}\sqrt{sz_1}\sqrt{\frac{z_1}{s}}}{}^{\mathbf{v}}}{\left(1-T\frac{\beta}{\gamma}(\beta\gamma z_1)^{\pm1}s^{\pm1}\right)}\ .\end{aligned} \tag{2.141}$$

The Coulomb index for the 4-punctured sphere was given in (2.22) and is given by

$$\mathcal{I}^{C}{}_{\mathbf{u}\sqrt{sz_1}\sqrt{\frac{z_1}{s}}}{}^{\mathbf{v}} = \mathrm{PE}\left[\left(\gamma^2\beta^2+\frac{1}{\gamma^2\beta^2}+1\right)T^2\right]\ , \tag{2.142}$$

notice that it is independent of s and therefore we can easily compute the above integral by collecting the residues of the poles inside C_{z_2} at
$s=T\beta^2 z_1, \frac{T}{\gamma^2 z_1}, \frac{\gamma}{\beta}z_2^{\pm1}$. The final result is

$$\mathcal{I}^{C\,(T_B)\mathbf{v}}_{\mathbf{zu}} = \mathrm{PE}\left[\left(\gamma^2\beta^2+\frac{1}{\gamma^2\beta^2}+\frac{\beta^2}{\gamma^2}\right)T^2\right]\ . \tag{2.143}$$

Higgs and Coulomb Branch Hilbert Series

We would also like to apply the duality conjecture to compute the Hilbert series for the T_B SCFT. We can write an expression similar to that of (2.129), the duality implies that

$$\begin{aligned}\mathrm{HS}(\tau;HB_{\text{4-punc}}) &= \oint\frac{dz_2}{4\pi\mathrm{i}}(1-z_2^{\pm2})\,g_{z_2}\,\mathrm{HS}(\tau;HB_B)\,\mathrm{HS}(\tau;HB_{T_B}) \\ &= \mathrm{HL}_{\mathbf{u}\sqrt{sz_1}\sqrt{\frac{z_1}{s}}}{}^{\mathbf{v}}\ .\end{aligned} \tag{2.144}$$

Here, (2.144) is the Hilbert series for the 4-punctured sphere theory which, as discussed in Section 2.4.1, is equal to the Hall-Littlewood limit of the index for that theory. $\mathrm{HS}(\tau; HB_B)$, $\mathrm{HS}(\tau; HB_{T_B})$ denote the Higgs branch Hilbert series for the B-type quiver tail and the T_B theory, respectively. Finally g_{z_2} denotes a possible gluing factor which takes into account the coupling between the tail and T_B through the superpotential (2.128).

Firstly, we can compute the Coulomb/Higgs branch Hilbert series for the B-type quiver tail, this is particularly simple, we can simply enumerate all of the letters which satisfy the corresponding shortening condition from Table 2.5, taking into account the F-terms generated by the superpotential W_B. Considering first the Hilbert series for the Higgs branch, we have

$$HB_B = \frac{\mathbb{C}[q^{(\pm)}, B_{2,\pm-}, B_{1,\pm+}]}{\langle B_{1,-+}q^{(-)} + B_{1,++}q^{(+)}, B_{2,--}q^{(-)} + B_{2,+-}q^{(+)}, q^{(+)}q^{(-)}\rangle}\,. \tag{2.145}$$

It's easy to check with *Macaulay2* that the above list is not a regular sequence. We then need to compute

$$\mathrm{HS}(\tau, \ldots; HB_B) := \mathrm{Tr}_{HB_B}\, \tau^{2q_t} z_2^{j_{z_2}} \delta^{q_\delta} \alpha^{q_\alpha} \beta^{q_\beta} \gamma^{q_\gamma}\,. \tag{2.146}$$

The above quantity can be computed using *Macaulay2* and the full expression reads

$$\begin{aligned}\mathrm{HS}(\tau, \ldots; HB_B) = \mathrm{PE}\left[e\frac{\beta}{\gamma}[1;1;0] + e\frac{\gamma}{\beta}[0;1;1]\tau^2\right]\Big\{-\frac{\beta}{\gamma}\tau^4 e^5[1;1;0] \\ +\frac{\beta}{\gamma}\tau^2 e^3[0;1;1] + \tau^4 e^4[2;0;0] - \tau^2 e^2[1;0;1] + \frac{\beta^2}{\gamma^2}\tau^4 e^6 - \frac{\beta^2}{\gamma^2}e^2 + 1\Big\}\end{aligned} \tag{2.147}$$

where, as before, $s = \delta/\alpha$ and $z_1 = \alpha\delta$ and here $[n; m; l] \equiv \chi_n(z_2)\chi_m(s)\chi_l(\beta\gamma z_1)$. Note that we also made explicit the extra fugacity e for the UV energy generator E under which each of the letters of (2.145) have charge $+1$. This fugacity is simply counting the number of 'letters' making up a given 'word' and the above should be strictly speaking always be evaluated on $e = 1$. Observe that

$$\begin{aligned}\mathrm{HL}^{(B)\mathbf{z}}_{\delta\alpha} =& \mathrm{PE}\left[\frac{\beta}{\gamma}[1;1;0] + \frac{\gamma}{\beta}[0;1;1]\tau^2 - \frac{\beta^2}{\gamma^2} - [1;0;1]\tau^2\right] \\ \neq& \mathrm{HS}(\tau; HB_B)\,,\end{aligned} \tag{2.148}$$

namely that for the B-type tail, associated to a sphere with two minimal and one maximal puncture, it Higgs-branch Hilbert series does not coincide with the Hall-Littlewood limit of its superconformal index.

Now, we come across two problems in trying to fix the Hilbert series for the T_B

theory. The first problem that arises is that the form of (2.144) does not lend itself to apply the elliptic inversion formula. Secondly, it is unclear how to compute the gluing factor g_{z_2}. In the, analagous to our case, Class $\mathcal{S}$ T_3 theory the gluing factor, on the Higgs branch, is simply taken into account by adding in the new F-term relations for ϕ arising due to the $\mathfrak{su}(2)$ gauging [190]. These set a relation between the $U(1)$ hypermultiplet and T_3 mesonic operators $Q\widetilde{Q} \sim M$. In our current case, due to having only $\mathcal{N} = 1$ supersymmetry, the superpotentials (2.126), (2.128) are more complicated.

However, we can try to follow the prescription that isolates the spectrum of T_B theory within the 4-punctured sphere theory, given in [136, 133]. This prescription was demonstrated to work at the level of the index. The steps, starting with the 4-punctured sphere theory, are as follows:

- Add matter:- we add the following chiral fields to the 4-punctured sphere theory with superpotential (2.149) where

Field(s)	$\mathfrak{su}(2)_{z_2}$	$\mathfrak{u}(1)_\delta$	$\mathfrak{u}(1)_\alpha$	$\mathfrak{u}(1)_r$	$\mathfrak{u}(1)_\beta$	$\mathfrak{u}(1)_\gamma$	$\mathfrak{u}(1)_t$	$\delta_{1\pm}$	$\widetilde{\delta}_{2\dot{+}}$
$\widetilde{q}^{(\pm)}$	$\mathbf{2}$	±1	∓1	0	-1	1	0	0	0
$b_{1,\pm-}$	$\mathbf{1}$	1 ± 1	1 ∓ 1	2/3	2	0	-1	2	0
$b_{2,\pm+}$	$\mathbf{1}$	-1 ± 1	-1 ∓ 1	2/3	0	-2	-1	2	0
t_0	$\mathbf{1}$	0	0	2	2	-2	0	2	4
$\phi'^{(\pm)}$	$\mathbf{2}$	±1	±1	4/3	±1	±1	1	0	4

  $b_1 = (b_{1,+-}, b_{1,--})$, $b_2 = (b_{2,++}, b_{2,-+})$, $\widetilde{q} = (\widetilde{q}^{(+)}, \widetilde{q}^{(-)})$ are $SU(2)_s$ doublets. We will use the notation $\det_s, \mathrm{tr}_s$ to distinguish between $\det, \mathrm{tr}$ taken over the original 4-punctured sphere gauge groups. It is clear that the only gauge invariant operators in the Higgs branch sector are those made up of $Q_l^+, {Q'}_l^+, \widetilde{q}^{(\pm)}, \phi'^{(\pm)}$.

- Go to the infinite coupling limit and gauge the enhanced $\mathfrak{su}(2)_s$, $s = \frac{\delta}{\alpha}$ symmetry (using the integration contour C_{z_2})

$$\Delta W = b_2 \cdot \begin{pmatrix} \det {Q'}_1^+ \\ \det Q_1^- \end{pmatrix} + b_1 \cdot \begin{pmatrix} \det Q_2^+ \\ \det {Q'}_2^- \end{pmatrix} + t_0 \det_s \widetilde{q} + \phi'^{(+)} \widetilde{q} b_2 + \phi'^{(-)} \widetilde{q} b_1 \qquad (2.149)$$

By examining (2.131) one can see that, at the level of the index, this prescription is precisely mirroring that of the Elliptic inversion formula and therefore this prescription is correct, atleast at the level of the protected spectrum of the theory. Since, by construction, our Hilbert series' are $\frac{1}{2}$-BPS objects, we also naively expect that this prescription should also yield the correct result for the Hilbert series. Therefore, all in all, the coordinate ring of the Higgs branch of the T_B theory can be expected to be described as the following quotient

$$HB_{T_B} = (F_{HT_B})^G\,, \quad F_{HT_B} = R/I\,, \tag{2.150}$$

$$G = SU(2)_a \times SU(2)_b \times SU(2)_s\,, \quad R = \mathbb{C}[Q_l^{\pm}, Q'^{\pm}_l, \widetilde{q}^{(\pm)}, \phi'^{(\pm)}]\,, \tag{2.151}$$

$$\begin{aligned} I = \langle & Q_1^+ Q_1^- - Q'^+_1 Q'^-_1, Q'^+_2 Q'^-_2 - Q_2^+ Q_2^-, \widetilde{q}^{(-)}\widetilde{q}^{(+)}, \\ & \det Q_1^- + \phi'^{(+)}\widetilde{q}^{(+)}, \det Q'^+_1 + \phi'^{(+)}\widetilde{q}^{(-)}, \\ & \det Q_2^+ + \phi'^{(-)}\widetilde{q}^{(-)}, \det Q'^-_2 + \phi'^{(-)}\widetilde{q}^{(+)}\rangle\,. \end{aligned} \tag{2.152}$$

The F-terms coming from the new fields have fugacities $\frac{\gamma^2}{\beta^2}, \frac{z_1}{s}\gamma^2\tau, z_1 s\gamma^2\tau$. We therefore have to compute

$$\mathrm{HS}(\tau, \ldots; HB_{T_B}) = \oint_{C_{z_2}} d\mu_s \oint d\mu_a \oint d\mu_b\, \mathrm{HS}(\tau, \ldots; F_{HT_B})\,, \tag{2.153}$$

where the Hilbert series for $\mathbf{F_{HT_B}}$ is

$$\mathrm{HS}(\tau, \ldots; F_{HT_B}) = \mathrm{Tr}_{R/I}\, \tau^{2q_t} a^{2j_a} b^{2j_b} z_2^{2j_{z_2}} \delta^{q_\delta} \alpha^{q_\alpha} \beta^{q_\beta} \gamma^{q_\gamma} \prod_{i=1}^{2} u_i^{2j_{u_i}} v_i^{2j_{v_i}}\,. \tag{2.154}$$

The expression for $\mathrm{HS}(\tau; F_{HT_B})$ is rather complicated, however, it can be computed using *Macaulay2*. In a series expansion around $\tau = 0$, $\frac{\gamma}{\beta} = 0$ it reads

$$\begin{aligned} \mathrm{HS}(\tau; HB_{T_B}) = 1 + \Bigg[& \left(\frac{\beta}{\gamma} + \frac{\gamma}{\beta}\right)([0;0;0;0;1;1] + [0;0;1;1;0;0]) \\ + & \left(\beta\gamma + \frac{1}{\beta\gamma}\right)[1;1;0;0;0;0] + \gamma^2[0;1;1;0;1;0] + [1;0;0;1;1;0] \\ + & [1;0;1;0;0;1] + \frac{1}{\beta^2}[0;1;0;1;0;1]\Bigg]\tau^2 + \mathcal{O}(\tau^4, \frac{\beta^2}{\gamma^2})\,. \end{aligned} \tag{2.155}$$

At order τ^2 our expression for the Hilbert series matches that of the Hall-Littlewood index (2.132), but the two expressions begin to differ at $\tau^{n\geqslant 4}$. In particular, fractional coefficients begin to appear in the expansion for $\tau^{n\geqslant 4}$. For example $+\frac{7}{2}\frac{\beta}{\gamma}[1;0;1;0;1;0]\tau^4$. This can be attributed to the fact that $\mathrm{HS}(\tau, \ldots; F_{HT_B})$ does not actually exhibit the $SU(2)_s$ enhancement, namely that it is not invariant under the action of the Weyl group of $\mathfrak{su}(2)_s$ which acts by exchanging $s \to s^{-1}$ and therefore can not be expanded in $SU(2)_s$ characters. Thus the prescription, while true for the index, gives an incorrect result for the Hilbert series (where we would expect only positive integer coefficients to appear). One possible reason for this is that the index is insensitive to the precise form for the superpotentials, whereas the Hilbert series is highly dependent on such choices.

We can perform a similar analysis for the Coulomb branch. We count only scalar operators ($j_1 = j_2 = 0$) and we impose the conditions $\widetilde{\delta}_{1\dot{-}} = \widetilde{\delta}_{2\dot{+}} = 0$, therefore we have $r = -\frac{2}{3}q_t$. Let us first discuss the B-type tail theory. From Table 2.5 the Coulomb ring is simply $\mathbb{C}[q^{(\pm)}, \Phi'^{\pm}]$, the F-terms ideal is

$$\langle q^{(+)}\Phi'^{(+)}, q^{(+)}\Phi'^{(-)}, q^{(-)}\Phi'^{(+)}, q^{(-)}\Phi'^{(-)}, q^{(+)}q^{(-)}\rangle \,. \tag{2.156}$$

The Hilbert series can be computed using *Macaulay2* and, using the notation $[n;m;l] \equiv \chi_n(z_2)\chi_m(s)\chi_l(\beta\gamma z_1)$, reads

$$\begin{aligned}\mathrm{HS}(T; CB_B) =& \,\mathrm{PE}\left[\frac{\gamma}{\beta}[1;1;0]e + [1;0;1]eT\right]\left\{-\frac{\beta^3 e^5}{\gamma^3}[1;1;0]T^2\right.\\ &+ 1 - \frac{\beta^2 e^2}{\gamma^2} - \frac{\beta e^2 T}{\gamma}[0;1;1]\\ &\left.+\frac{\beta^4}{\gamma^4}e^6T^2 + \frac{\beta^2 e^4 T^2}{\gamma^2}[0;2;0] + \frac{\beta^2 e^3 T}{\gamma^2}[1;0;1]\right\}\end{aligned} \tag{2.157}$$

Using the prescription outlined in Section 2.6.1 we can also compute the Coulomb branch Hilbert series for the T_B theory. In total the coordinate ring is

$$CB_{T_B} = (F_{CT_B})^G \,, \quad F_{CT_B} = \frac{\mathbb{C}[\Phi_i, \widetilde{q}, b_1, b_2]}{\langle \det_s \widetilde{q}, \widetilde{q}b_2, \widetilde{q}b_1\rangle} \,, \tag{2.158}$$

The Hilbert series for F_C can be easily computed using *Macaulay2* and

$$\begin{aligned}\mathrm{HS}(T; CB_{T_B}) =& \oint_{C_{z_2}} d\mu_s \oint d\mu_{g_1} \oint d\mu_{g_2} \mathrm{HS}(T; F_{CT_B}) && (2.159)\\ =& \,\mathrm{PE}\left[\left(1 + \beta^2\gamma^2 + \frac{1}{\beta^2\gamma^2} + \frac{\beta^2}{\gamma^2}\right)T^2\right] && (2.160)\\ =& \,\mathrm{PE}[T^2]\,\mathcal{I}^{C(T_B)^{\mathbf{v}}}_{\mathbf{zu}} \,. && (2.161)\end{aligned}$$

The generators corresponding to $\left(1 + \beta^2\gamma^2 + \frac{1}{\beta^2\gamma^2}\right)T^2$ may be thought of as comping from the original 4-punctured sphere theory $\operatorname{tr}\Phi_1\Phi_2$, $\det\Phi_1$, $\det\Phi_2$. The the

other possible G-invariant generators are $\det_s \widetilde{q}$, $\mathrm{tr}_s \widetilde{q}b_1$, $\mathrm{tr}_s \widetilde{q}b_2$, $\det_s b_1$, $\det_s b_2$ and $\mathrm{tr}_s b_1 b_2$. The first three are set to zero by the F-terms while $\det_s b_1$, $\det_s b_2$ are not selected by the contour. Therefore
$CB_{T_B} = \mathbb{C}[\mathrm{tr}\,\Phi_1\Phi_2, \det\Phi_1, \det\Phi_2, \mathrm{tr}_s\, b_1 b_2]$.

2.6.2 T_A Theory

Hall-Littlewood and Coulomb Indices

Applying (A.56) the index for the T_A theory is

$$\begin{aligned}\mathcal{I}^{(T_A)\mathbf{v}}_{\mathbf{zu}} =&\Gamma_e\left(t\left(\frac{\gamma z_1}{\beta}\right)^{\pm 1} z_2^{\pm 1}\right)\kappa \\ &\times \oint_{C_{z_2}} \frac{ds}{4\pi i s}\delta\left(s, z_2; \beta\gamma\right) \frac{\mathcal{I}_{\mathbf{u}\sqrt{s z_1}\sqrt{\frac{z_1}{s}}}{}^{\mathbf{v}}}{\Gamma_e\left(t\beta\gamma\left(\frac{z_1\gamma}{\beta}\right)^{\pm 1} s^{\pm 1}\right)},\end{aligned} \tag{2.162}$$

where now C_{z_2} encloses $s = \beta\gamma z_2^{\pm 1}$ and excludes $s = \frac{1}{\beta\gamma} z_2^{\pm 1}$. The same expression was also given in (B.3) and (5.17) of [136]. As pointed out in [136] the flavour symmetry enhances $\mathfrak{su}(2)_z^2 \oplus \mathfrak{su}(2)_v^2 \oplus \mathfrak{su}(2)_u^2 \oplus \mathfrak{u}(1)_\gamma \oplus \mathfrak{u}(1)_\beta \oplus \mathfrak{u}(1)_t \rightarrow \mathfrak{g}^{(T_A)} = \mathfrak{so}(8)_w \oplus \mathfrak{su}(2)_{z_2} \oplus \mathfrak{su}(2)_{v_1} \oplus \mathfrak{su}(2)_{u_1} \oplus \mathfrak{u}(1)_{\gamma\beta} \oplus \mathfrak{u}(1)_t$. The enhancement to $\mathfrak{so}(8)$ is made manifest by identifying $\mathbf{w} = (\frac{\gamma}{\beta}u_2, \frac{\beta}{\gamma}\frac{1}{u_2}, \frac{\beta}{\gamma}u_2, \frac{\gamma}{\beta}\frac{1}{u_2}, v_2 z_1, \frac{1}{v_2 z_1}, \frac{v_2}{z_1}, \frac{z_1}{v_2})$ as the $SO(8)$ fugacities. We can again take the Hall-Littlewood limit. In an expansion around $\tau = 0$, $\beta\gamma = 0$ the Hall-Littlewood index reads

$$\begin{aligned}&\mathrm{HL}^{(T_A)\mathbf{v}}_{\mathbf{zu}} = \\ &1 + \Big[[1,0,0,0;0;0;1] + [0,0,1,0;0;1;0] + [0,0,0,1;1;0;0] \\ &+ \gamma^2\beta^2[0,0,0,0;1;1;1]\Big]\tau^2 + \Big[\gamma^4\beta^4[0,0,0,0;2;2;2] \\ &+ \gamma^2\beta^2([1,0,0,0;1;1;2] + [0,0,1,0;1;2;1] + [0,0,0,1;2;1;1]) \\ &+ [2,0,0,0;0;0;2] + [0,0,2,0;0;2;0] + [0,0,0,2;2,0,0] \\ &+ [1,0,1,0;0;1;1] + [1,0,0,1;1;0;1] + [0,0,1,1;1;1;0] \\ &+ [0,1,0,0;0;0;0] - \gamma^2\beta^2\Big]\tau^4 + \mathcal{O}(\tau^6, \beta^6\gamma^6)\end{aligned} \tag{2.163}$$

where we denote the character of $SO(8)_{\mathbf{w}} \times SU(2)_{z_2} \times SU(2)_{v_1} \times SU(2)_{u_1}$ by $[d_1, d_2, d_3, d_4; d_5; d_6; d_7]$. Notice the $-\gamma^2\beta^2\tau^4$ coefficient in the expansion of $\mathrm{HL}^{(T_A)\mathbf{v}}_{\mathbf{zu}}$, we can see from Table C.3, Table C.4 and equations (C.18)-(C.22) that this multiplet (since the T_A theory is interacting it can not have higher spin free fields) must be a $\overline{\mathcal{C}}_{\frac{2}{3}(0,0)}$ multiplet with $\mathfrak{u}(1)_t$ charge $q_t = 2$ and $q_\beta = q_\gamma = 2$. In particular the negative coefficient appearing in the expansion of the Hall-Littlewood index means that the Hall-Littlewood index for generic genus zero class $\mathcal{S}_k$ theories is, in general, *not* equal to the Hilbert series for its Higgs branch (as per the definition of Section 2.2.2) whose expansion coefficients must all be positive.

The expansion of the Hall-Littlewood limit of the index of the four punctured sphere theory has all positive coefficients, in particular it does not contain a $\overline{\mathcal{C}}_{\frac{2}{3}(0,0)}$ multiplet with $q_t = q_\beta = q_\gamma = 2$. As we approach the complete decoupling limit where the $SU(2)_{z_2}$ is ungauged the four punctured sphere theory decomposes into the T_A SCFT plus the A-type tail theory. As we approach that limit a long multiplet $\mathcal{A}^3_{\frac{2}{3},(0,0)}$ of the four-punctured sphere theory decomposes via the recombination rule (C.4)

$$\mathcal{A}^3_{\frac{2}{3},(0,0)} \cong \overline{\mathcal{C}}_{\frac{2}{3},(0,0)} \oplus \overline{\mathcal{B}}_{\frac{8}{3},(0,0)} \ . \tag{2.164}$$

The $\overline{\mathcal{B}}_{\frac{8}{3},(0,0)}$ with $q_t = q_\beta = q_\gamma = 2$ lives in the A-type quiver theory. Going through the list of Hall-Littlewood ($\delta_{1\pm} = 0$) operators with $(-1)^F = +1$, $q_t = q_\beta = q_\gamma = 2$ in the tail theory Table 2.6 gives an operator in the $\mathbf{2} \otimes \mathbf{2} \cong \mathbf{3} \oplus \mathbf{1}$ of the enhanced $SU(2)_{s=\delta/\alpha}$

$$B_{IJ} := \begin{pmatrix} B_{1,++}B_{1,+-} & B_{1,++}B_{1,--} \\ B_{1,-+}B_{1,+-} & B_{1,--}B_{1,+-} \end{pmatrix}_{IJ} = B_{(IJ)} + B_{[IJ]} \tag{2.165}$$

with $I, J = 1, 2$ $SU(2)_s$ indices. Since the T_A theory contains only $SU(2)_s$ singlets we are instructed to take the $\mathbf{1}$ in the decomposition and we conclude that the operator

$$B_{[IJ]} = \frac{1}{2}(B_{1,++}B_{1,--} - B_{1,-+}B_{1,+-})\epsilon_{IJ} \tag{2.166}$$

is the top component of the $\overline{\mathcal{B}}_{\frac{8}{3},(0,0)}$ multiplet in the above recombination rule. The conformal R-symmetry of the T_A theory is [136]

$$r_c = r + 0.0689(q_\gamma + q_\beta) - 0.044777 q_t \, . \tag{2.167}$$

Therefore, operators contributing to the Hall-Littlewood limit of the index have conformal energy

$$E_c = 2j_2 + \frac{3}{2} r_c = -j_2 + 1.93283 q_t + 0.10335(q_\gamma + q_\beta) \, . \tag{2.168}$$

	E_c	$\mathfrak{so}(8)_{\mathbf{w}}$	$\mathfrak{su}(2)_{z_2}$	$\mathfrak{su}(2)_{v_1}$	$\mathfrak{su}(2)_{u_1}$	$q_\gamma + q_\beta$	q_t
M^v	1.93283	$\mathbf{8}_v$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	0	1
M^c	1.93283	$\mathbf{8}_c$	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{1}$	0	1
M^s	1.93283	$\mathbf{8}_s$	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	0	1
B	2.34623	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	4	1

Table 2.8: *(Subset of) Higgs branch operators of the T_A SCFT.*

By the same arguments as the ones made for the T_B theory the operators with $q_t = 1$ appearing in the expansion (2.163) have $j_2 = 0$. Therefore the operators of Table 2.8 comprise, at least a subset of, the possible operators on the Higgs branch of the T_A SCFT. In the notation of Table 2.7 we have combined $M^v = \{M^u_+, M^u_-, B_{121}\}$, $M^c = \{M^v_+, M^v_-, B_{112}\}$, $M^s = \{M^z_+, M^z_-, B_{222}\}$ and $B = \{B_{211}\}$. The expansion of $\mathrm{PLog}\,\mathrm{HL}^{(T_A)} = \tilde{L}_1\tau^2 + \tilde{L}_2\tau^4 + \dots$ with the coefficients of $\tilde{L}_2$ all being negative again does not rule out the possibility that Table 2.8 is in fact the complete list of Higgs branch generators for the T_A theory and that new generators do not appear at higher orders. We were able to obtain a closed expression in the unrefined $\mathbf{z} = \mathbf{v} = \mathbf{u} = \mathbf{1}$, $\gamma\beta^{-1} = 1$ limit:

$$\begin{aligned}
\mathrm{HL}^{(T_A)\mathbf{1}}_{\mathbf{11}}\Big|_{\frac{\gamma}{\beta}=1} =& \oint_{C_1} \Bigg\{ \frac{1}{(s-\beta\gamma)^2(\beta\gamma s-1)^2} \\
&\times \frac{\widetilde{P}(s,\gamma\beta,\tau)ds}{(\beta\gamma s-\tau^2)^5\,(s\tau^2-\beta\gamma)^5\,(s-\beta\gamma\tau^2)^3\,(\beta\gamma s\tau^2-1)^3} \\
=&\beta^5\gamma^5\tau^{15}\frac{Q_A(\tau,\beta\gamma)-Q_A(\tau^{-1},\beta^{-1}\gamma^{-1})}{(1-\tau^2)^{19}(1-\gamma^2\beta^2\tau^2)^4}
\end{aligned} \tag{2.169}$$

where $\widetilde{P}(s,\gamma\beta,\tau)$ is a polynomial in s and in the second line we took the residues at $s = \beta\gamma, (\beta\gamma)^{\pm 1}\tau^2$ and $Q_A(\tau,\beta\gamma)$ is a polynomial of degree 15 in τ. We list the full expression for it in (C.27) and quote here only the result for $\gamma = \beta = 1$ which reads

$$\begin{aligned}
\mathrm{HL}^{(T_A)\mathbf{1}}_{\mathbf{11}}\Big|_{\gamma=\beta=1} =& \frac{1}{(1-\tau^2)^{18}}\big(\tau^{20} + 38\tau^{18} + 474\tau^{16} + 2582\tau^{14} \\
&+6895\tau^{12} + 9516\tau^{10} + 6895\tau^8 + 2582\tau^6 + 474\tau^4 + 38\tau^2 + 1\big)\,,
\end{aligned} \tag{2.170}$$

again, note the palindromic structure of the numerator. Also note that in the fully unrefined limit $\mathrm{HL}^{(T_A)\mathbf{1}}_{\mathbf{11}}\Big|_{\gamma=\beta=1} = \mathrm{HL}^{(T_B)\mathbf{1}}_{\mathbf{11}}\Big|_{\gamma=\beta=1}$. This is expected since the T_A & T_B SCFTs differ only by choices for fluxes for $U(1)_\gamma$ and $U(1)_\beta$. This equality no longer holds with refinement turned on, since T_A and T_B have different global symmetry algebras.

The Coulomb index for the T_A theory can also be computed, the computation is identical to the T_B case and it reads

$$\mathcal{I}^{C(T_A)\mathbf{v}}_{\mathbf{zu}} = \mathrm{PE}\left[\left(\beta^2\gamma^2 + \frac{2}{\beta^2\gamma^2}\right)T^2\right]. \tag{2.171}$$

Higgs and Coulomb Branch Hilbert Series

We can perform the analagous computation to the one that we performed for the T_B theory. Let us compute the Higgs branch Hilbert series for the A-type tail theory. This coordinate ring can be described as

$$HB_A = \frac{\mathbb{C}[q^{(\pm)}, B_{1,\pm\pm'}]}{\langle B_{1,-+}q^{(-)} + B_{1,++}q^{(+)}, B_{1,--}q^{(-)} + B_{1,+-}q^{(+)}, q^{(+)}q^{(-)}\rangle}. \tag{2.172}$$

At the level of the tail theory can can see that (2.172) and (2.145) are isomorphic as graded rings, where the isomorphism is made by $B_{1,--} \to B_{2,--}$ $B_{1,+-} \to B_{2,+-}$ while redefining the grading $\beta \to \frac{1}{\beta}$. Therefore, we have

$$\mathrm{HS}(\tau, \beta, \gamma, s, z_1; HB_A) = \mathrm{HS}(\tau, \beta^{-1}, \gamma, s, z_1; HB_B). \tag{2.173}$$

To obtain an expression for the Hilbert series for the T_A SCFT we again follow the prescription of [136]

- Add matter:- we add the following chiral fields to the 4-punctured sphere theory with superpotential (2.174) where $b'_1 = (b_{1,++}, b_{1,-+})$,

Field(s)	$\mathfrak{su}(2)_{z_2}$	$\mathfrak{u}(1)_\delta$	$\mathfrak{u}(1)_\alpha$	$\mathfrak{u}(1)_r$	$\mathfrak{u}(1)_\beta$	$\mathfrak{u}(1)_\gamma$	$\mathfrak{u}(1)_t$	$\delta_{1\pm}$	$\widetilde{\delta}_{2\dot{+}}$
$\widetilde{q}^{(\pm)}$	$\mathbf{2}$	±1	∓1	0	1	1	0	0	0
$b_{1,\pm\pm}$	$\mathbf{1}$	0	∓2	2/3	-1 ± 1	-1 ∓ 1	-1	2	0
$b_{1,\mp\pm}$	$\mathbf{1}$	∓2	0	2/3	-1 ± 1	-1 ∓ 1	-1	2	0
t_0	$\mathbf{1}$	0	0	2	-2	-2	0	2	4
$\phi'^{(\pm)}$	$\mathbf{2}$	±1	±1	4/3	∓1	±1	1	0	4

  $b_1 = (b_{1,+-}, b_{1,--})$, $\widetilde{q} = (\widetilde{q}^{(+)}, \widetilde{q}^{(-)})$ are $SU(2)_s$ fundamental doublets. We again use the notation $\det_s, \mathrm{tr}_s$ to distinguish between det, tr taken over the original 4-punctured sphere gauge groups. It is clear that the only gauge invariant operators in the Higgs branch sector are those made up of $Q_l^+, Q'^+_l, \widetilde{q}, \phi'^{(\pm)}$.

- Go to the infinite coupling limit and gauge the enhanced $\mathfrak{su}(2)_s$, $s = \frac{\delta}{\alpha}$ symmetry (using the integration contour C_{z_2})

$$\Delta W = b_1' \cdot \begin{pmatrix} \det Q'^{+}_1 \\ \det Q^{-}_1 \end{pmatrix} + b_1 \cdot \begin{pmatrix} \det Q^{+}_1 \\ \det Q'^{-}_1 \end{pmatrix} + t_0 \det_s \widetilde{q} + \phi'^{(+)} \widetilde{q} b_1' + \phi'^{(-)} \widetilde{q} b_1 \tag{2.174}$$

Therefore, the coordinate ring of Higgs branch of the T_A SCFT can be expected to be described as the following quotient

$$HB_{T_A} = (F_{HT_A})^G\,, \quad F_{HT_A} = R/I\,, \tag{2.175}$$

$$G = SU(2)_a \times SU(2)_b \times SU(2)_s\,, \quad R = \mathbb{C}[Q^{\pm}_l, Q'^{\pm}_l, \widetilde{q}, \phi'^{(\pm)}]\,, \tag{2.176}$$

$$\begin{aligned} I = \langle & Q^{+}_1 Q^{-}_1 - Q'^{+}_1 Q'^{-}_1, Q'^{+}_2 Q'^{-}_2 - Q^{+}_2 Q^{-}_2, \widetilde{q}^{(-)} \widetilde{q}^{(+)}, \\ & \det Q'^{+}_1 + \phi'^{(+)} \widetilde{q}^{(-)}, \det Q^{-}_1 + \phi'^{(+)} \widetilde{q}^{(+)}, \\ & \det Q^{+}_1 + \phi'^{(-)} \widetilde{q}^{(-)}, \det Q'^{-}_1 + \phi'^{(-)} \widetilde{q}^{(+)} \rangle\,. \end{aligned} \tag{2.177}$$

We can perform a similar analysis for the Coulomb branch. The computation is the same as that for the T_B theory of the previous section. The Coulomb ring for the A-type tail is the same as for the B-type tail simply with $\beta \to \frac{1}{\beta}$ while leaving everything else fixed. So,

$$\mathrm{HS}(T, \gamma, \beta, s, z_1; CB_A) = \mathrm{HS}(T, \gamma, \beta^{-1}, s, z_1; CB_B)\,. \tag{2.178}$$

Using the prescription outlined in the previous section we can also compute the Coulomb branch Hilbert series for the T_A theory. In total

$$CB_{T_A} = (F_{CT_A})^G\,, \quad F_{CT_A} = \frac{\mathbb{C}[\Phi_i, \widetilde{q}, b_1, b_1']}{\langle \det_s \widetilde{q}, \widetilde{q} b_1, \widetilde{q} b_1' \rangle}\,. \tag{2.179}$$

It is not too difficult to see that $CB_{T_A} \cong CB_{T_B}$ as graded rings. Here, the isomorphism is made by $b_1' \to b_2$, however, we also have to exchange the grading $\beta \to \frac{1}{\beta}$ of $\widetilde{q}$, b_1, b_1' while keeping the gradings of the four-punctured sphere fields Φ_i fixed. Therefore, we simply have that
$CB_{T_A} = \mathbb{C}[\operatorname{tr} \Phi_1 \Phi_2, \det \Phi_1, \det \Phi_2, \operatorname{tr}_s b_1 b_1']$ and the Hilbert series is

$$\mathrm{HS}(T, \ldots; CB_{T_A}) = \mathrm{PE}\left[\left(1 + \gamma^2 \beta^2 + \frac{2}{\beta^2 \gamma^2}\right) T^2\right] = \mathrm{PE}[T^2]\, I^{C(T_A)\mathbf{v}}_{\mathbf{zu}}\,. \tag{2.180}$$

2.7 Outlook and Open Problems

In this chapter we have defined Higgs and Coulomb branches for $\mathcal{N}=1$ theories of class $\mathcal{S}_k$. As we have stressed, for generic $\mathcal{N}=1$ theories these branches generally cannot be distinguished and are merged into one another. On the other hand, theories of class $\mathcal{S}_k$ inherit the Higgs and Coulomb branch definitions from the $\mathcal{N}=2$ theory from which they arise as orbifolds from. We were also able to compute the Hilbert series for the Higgs and Coulomb branches for a large variety of class $\mathcal{S}_k$ theories. Such quantities have been integral to the study the reductions to three dimensions of class $\mathcal{S}$ theories, in particular, when one dimensionally reduces theories of class $\mathcal{S}$ on a circle a three dimensional $\mathcal{N}=4$ theory is obtained. These theories often lack a Lagrangian description, however the 3d mirror is a Lagrangian theory [191]. Under 3d mirror symmetry the Higgs and Coulomb branches are exchanged. Since the Higgs branch is invariant under circle reduction the Higgs branch Hilbert series for the 4d class $\mathcal{S}$ theory is equal to Coulomb branch Hilbert series of the 3d mirror, hence the Hilbert series can be a very useful tool to check the mirror theory. It would be very interesting to investigate this story for theories of class $\mathcal{S}_k$.

Additionally, we have also been able to define analogues of the Hall-Littlewood, Coulomb, Macdonald and Schur limits of the superconformal index for theories of class $\mathcal{S}_k$. For theories of class $\mathcal{S}$ the TQFT structure constants in the given limit are diagonalised by the corresponding polynomials [168]. For class $\mathcal{S}_k$ this is no longer true, and an explicit description for the polynomials that diagonalise the structure constants is not known. It would be very useful to have such an explicit description, knowing this would allow one to write down the superconformal index for any class $\mathcal{S}_k$ theory, in the given limit.

It would be interesting to further study the theories of $\mathcal{S}_{\Gamma=D,E}$. However, in this case, a direct generalisation of the work carried out in this chapter is not immediate. The basic reason is that, integral to our work defining the Higgs and Coulomb branches and the unrefined limits of the superconformal index for the $\Gamma=A$ theories was the existence of the intrinsic $\mathfrak{u}(1)_t$ symmetry; in addition to the $\mathfrak{u}(1)_r$ R-symmetry. For the case $\Gamma=D,E$ such symmetry typically is not present. The geometric interpretation is that the space $\mathbb{C}^2/A_{k-1}$ has an additional $U(1)$ holomorphic isometry (for $k=1,2$ this enhances to $SU(2)$). This can be seen from the defining equations $xy=w^k$ and we see that we can give $U(1)$ charges $+1,-1,0$ to x,y,w. This symmetry is identified with $U(1)_t$. On the other hand, when $\Gamma=DE$ we have no such $U(1)$. For example the defining equations for $\mathbb{C}^2/D_{k+2}$ is $x^2+y^2w=w^{k+1}$

and has empty isometry group.

Chapter 3

Seiberg-Witten Curves for Theories of Class $\mathcal{S}_k$

3.1 Introduction

Shortly after the seminal work by Seiberg and Witten on $\mathcal{N} = 2$ gauge theories [12, 13] Seiberg and Intriligator pointed out that similar techniques may be deployed to study $\mathcal{N} = 1$ gauge theories that possess a pure abelian Coulomb branch [20]. When supersymmetry is unbroken the theory may often possess several inequivalent ground states for any given values of the moduli $\{u_i\}$ of the theory. The ground states are generically in different 'branches' or 'phases' of the theory. These phases may be Coulomb, Higgs, Confining or, more generally, a mixture between them. In many cases the moduli space $\mathbf{M}$ is a manifold parametrised by the u_i. In particular, they were able to write down a family of holomorphic curves $\mathfrak{X}$ encoding the low energy superpotential on the Coulomb branch. It is useful to think of the total space $\mathfrak{X}$ as a fibration of holomorphic curves fibered over the moduli space base

$$\mathfrak{X} = \begin{array}{c} \mathcal{X} \\ \downarrow \\ \mathbf{CB} \end{array} \tag{3.1}$$

here $\mathcal{X}$ is a $\{u_i\}$ dependent Riemann surface specified by a polynomial in two auxiliary variables x, y and the various coupling constants q and masses,

$$\mathcal{X}: \quad y^2 = F(x; u_i, q, m)\,. \tag{3.2}$$

The low energy effective coupling matrix then is computed by inverting

$$\int_{B_j} \omega_i = \sum_{l=1}^{g} \tau_{il} \int_{A_l} \omega_j \,, \tag{3.3}$$

where $\omega \in \Omega^{1,0}(\mathcal{X})$ with $\Omega^{p,q}(\mathcal{X})$ is the space of closed (p,q)-forms on the genus g curve $\mathcal{X}$ at fixed u_i.

In contrast to $\mathcal{N} = 2$ gauge theory, where the techniques of Seiberg and Witten allows one to compute the low energy Wilsonian effective action and the masses of the lightest single particle states exactly, computing the Seiberg-Intriligator curve for an is not enough to 'solve' the Coulomb branch of $\mathcal{N} = 1$ theory. In particular, in the case of a generic $\mathcal{N} = 1$ gauge theory, supersymmetry does not relate the superpotential to the Kähler part of the action. Furthermore, the lack of a BPS bound means that there is no direct way to compute the masses of the lightest states.

On the contrary, the $\mathcal{N} = 1$ curves still encode a large amount of information and may be particularly instructive in, for example, a Gaiotto type classification for theories of class $\mathcal{S}_k$ [121, 122].

3.2 Review of $\mathcal{N} = 1$ Seiberg-Intriligator Curves

In this section we wish to provide a brief review of a few rank one examples of the $\mathcal{N} = 1$ curves proposed by Seiberg and Intriligator [20]. Guided by the following basic assumptions:

- The low energy effective action has an $SL(2,\mathbb{Z})$ duality group meaning that τ is not a single valued function but rather a section of an $SL(2,\mathbb{Z})$ bundle over **CB** which can be identified with the period matrix of a genus g Riemann surface $\mathcal{X}_u$ which is given by a ratio of the periods $\tau = \int_B \omega / \int_A \omega$ where A, B are independent 1-cycles, the so called electric and magnetic cycles respectively.

- Because τ is a global holomorphic section in the light fields and the various coupling constants, over **CB**, the curve $\mathcal{X}$ should be holomorphic in them too.

- The curve should respect the global symmetries.

- The curve should be described by a polynomial of the form $y^2 = F(x; u, q, m)$.

they were able to write down the family of holomorphic curves $\mathfrak{X}$ encoding the low energy superpotential on the Coulomb branch.

3.2.1 Examples

$\mathcal{N}=2 \to \mathcal{N}=1$ Mass Deformation of Pure $SU(2)$ $\mathcal{N}=2$ Gauge Theory

The first example they considered is the mass deformation of pure $SU(2)$ $\mathcal{N}=2$ gauge theory. The $\mathcal{N}=2 \to \mathcal{N}=1$ deformation is given by adding $W \to W + mu$ where $u = \frac{1}{2}\operatorname{tr}\phi^2$ and ϕ is the adjoint chiral multiplet of the $\mathcal{N}=2$ vector multiplet. The curve of this $\mathcal{N}=1$ theory is given by

$$y^2 = x^3 - mux^2 + \Lambda_d^6 x. \tag{3.4}$$

$\mathcal{N}=1$ $SU(2)\times SU(2)$ Gauge Theory

Perhaps the most relevant example for us that was studied in [20] is $\mathcal{N}=1$ gauge theory with $G = SU(2)_1 \times SU(2)_2$ with two chiral superfields $\Phi_{ia_1a_2}$ in the $(\mathbf{2},\mathbf{2})$ representation of G, labelled by $a_1, a_2 = 1, 2$ respectively. The theory has a $SU(2)_F$ flavour symmetry $\Phi_i \to f_{ij}\Phi_j$ where $f_{ij} \in SU(2)_F$. The theory can, on one hand, be understood as a $\mathbb{Z}_2$ orbifold of the above $\mathcal{N}=2$ pure $SU(2)$ gauge theory. Equivalently, the theory can be obtained by compactifying the 6d $(1,0)_{A_1}$ theory on a punctured Riemann surface $\mathcal{C}$ of genus zero in the presence of a certain set of defect operators associated to so called 'wild' punctures. As in $\mathcal{N}=2$ theories the Seiberg-Intriligator curve $\mathcal{X}$ is a double covering of $\mathcal{C}$. The theory has a three complex dimensional moduli space of supersymmetric vacua parametrised by the gauge invariant quantities $M_{ij} := \det \Phi_i\Phi_j$ where the determinant is taken over G.

This theory has an one complex-dimensional abelian coulomb branch where $SU(2)_1 \times SU(2)_2 \to U(1)_D \subset SU(2)_D$ and $SU(2)_D$ denotes the diagonal subgroup of $SU(2)_1 \times SU(2)_2$. The moduli space **CB** of gauge inequivalent coulomb vacua is parametrised by a single gauge and flavour singlet $u = \det_{ij} M_{ij}$. The holomorphic gauge coupling in this phase is $\tau_D = \tau_D\left(u, \Lambda_1^4, \Lambda_2^4\right)$ where $\Lambda_i^4 := \mu^4 e^{2\pi i \tau_i}$ are the instanton parameters. We may deduce the form of the curve describing the moduli space by considering a few limits. Firstly consider the limit where Φ_1 acquires a large diagonal vev but the vev of Φ_2 is vanishing. The gauge group is broken to $SU(2)_D$. Φ_2 decomposes into $\mathbf{2}\otimes\mathbf{2} \to \mathbf{3}\oplus\mathbf{1}$ of $SU(2)_D$. There is also the heavy singlet field M_{11}. If we assume that the singlets decouple then the theory is approximately pure

$\mathcal{N}=2$ $SU(2)_D$ gauge theory - the curve of that theory is

$$y^2 = x^3 - u_D x^2 + \frac{1}{4}\Lambda_D^4 x\,, \quad \text{for large } u \tag{3.5}$$

where

$$u_D = \operatorname{tr}\phi^2 = \frac{2u}{M_{11}}\,, \quad \Lambda_D^4 = \frac{16\Lambda_1^4\Lambda_2^4}{M_{11}}\,. \tag{3.6}$$

After rescaling x, y, by the $\mathrm{Aut}(Quiver) \cong \mathbb{Z}_2$ symmetry acting by $1 \leftrightarrow 2$, agreement at large u with (3.5), analyticity and dimensional analysis the most general form of the $\mathcal{N}=1$ curve is

$$y^2 = x^3 - \left(u - \alpha(\Lambda_1^4+\Lambda_2^4)\right)x^2 + \Lambda_1^4\Lambda_2^4 x\,. \tag{3.7}$$

To fix α it is useful to focus on one of the two gauge nodes, say the $SU(2)_2$ node, from that point of view it sees $SU(2)$ SQCD with $N_f=2$ with holomorphic scale given by Λ_2 .[1,2] For SQCD we have the following quantum constraint

$$\operatorname{Pf} V = u + E\widetilde{u} = \Lambda_2^4 \tag{3.8}$$

with E some energy scale and $\widetilde{u} = \operatorname{tr}\widetilde{\phi}^2$, $\widetilde{\phi} = \frac{1}{E}\left(\Phi_1\Phi_2 - \frac{1}{2}\operatorname{tr}\Phi_1\Phi_2\right)$. To then fix α consider the limit where the second gauge group is very strongly coupled $\Lambda_2 \to \infty$, $\Lambda_2 \gg \Lambda_1$. On one hand, in this limit, the theory is approximately $SU(2)_1$ gauge theory coupled to the adjoint field $\widetilde{\phi}$ and singlets $\det\Phi_1^2$, $\det\Phi_2^2$, $\operatorname{tr}\Phi_1\Phi_2$. Assuming that the singlets decouple, the theory is approximately pure $\mathcal{N}=2$ gauge theory with holomorphic scale Λ_1 which has curve

$$y^2 = x^3 - \widetilde{u}x^2 + \frac{1}{4}\Lambda_1^4 x\,. \tag{3.9}$$

and is singular at

$$\widetilde{u} = \frac{1}{E}\left(\Lambda_2^4 - u\right) = \pm\Lambda_1^2\,. \tag{3.10}$$

On the other hand (3.7) is singular at $u = \alpha\left(\Lambda_1^4+\Lambda_2^4\right) \pm 2\Lambda_1^2\Lambda_2^2 \approx \alpha\Lambda_2^4 \pm 2\Lambda_1^2\Lambda_2^4$. Comparing with (3.10) implies that $\alpha = 1$ and therefore curve $\mathcal{X}_u$ of this theory is

[1]By $SU(N)$ SQCD with N_f flavours to be the theory consisting of an $\mathcal{N}=1$ $SU(N)$ vector multiplet coupled to N_f chiral multiplets Q_i in the $\mathbf{N}$ of $SU(N)$ and N_f chiral multiplets $\widetilde{Q}^i$ in the $\bar{\mathbf{N}}$ possibly with non-trivial superpotential.

[2]Recall that the $\mathbf{2}$ of $SU(2)$ is pseudo-real and therefore isomorphic to the $\bar{\mathbf{2}}$.

given by

$$y^2 = x^3 + (-u + \Lambda_1^4 + \Lambda_2^4)x^2 + \Lambda_1^4\Lambda_2^4 x\,. \tag{3.11}$$

Interestingly, the solution of this $SU(2) \times SU(2)$ $\mathcal{N} = 1$ gauge theory is isomorphic to that of $SU(2)$ $\mathcal{N} = 2$ gauge theory with the isomorphism $\mathfrak{X} \cong \mathfrak{X}_{\mathcal{N}=2} \cong \mathbb{H}/\Gamma_0(4)$ provided by the mapping which shifts $f : u_{\mathcal{N}=2}, \Lambda_{\mathcal{N}=2} \mapsto u - \Lambda_1^4 - \Lambda_2^4, \Lambda_1\Lambda_2$.

3.3 Core Theories in Class $\mathcal{S}_k$

In this section we derive the $\mathcal{N} = 1$ Seiberg-Intriligator curves for the 'core theories' in class $\mathcal{S}_k$, namely those associated to spheres with two maximal and $\ell - 2$ minimal punctures.

3.3.1 Classical Analysis

The theory admits a rather intricate phase structure however, we may restrict our attention to the Coulomb branch defined by giving non zero vevs to Φ's while $Q, \widetilde{Q}$ have vanishing vevs. For generic vevs the gauge group is broken from $SU(N)^{k\ell} \to U(1)^{(N-1)\ell}$. The branches don't mix because they may be differentiate using the $U(1)_t$ symmetry, as we have shown in Chapter 2.

We first define $\Phi_n := \prod_{i=1}^k \Phi_{(i,n)}$, the Coulomb branch may be parametrised by the following N gauge invariants

$$u_{lk,n} := \begin{cases} \frac{1}{l}\operatorname{tr}\left(\Phi_n - \frac{1}{N}\operatorname{tr}\Phi_n\right)^l & 2 \leqslant l \leqslant N \\ \operatorname{tr}\Phi_n & l = 1 \end{cases} \tag{3.12}$$

of dimension lk and k 'baryonic' gauge invariant objects

$$B_{i,n} := \frac{1}{(N!)^2}\epsilon_{a_1\ldots a_N}\epsilon^{b_1\ldots b_N}\Phi^{a_1}_{(i,n)b_1}\cdots\Phi^{a_N}_{(i,n)b_N} = \det\Phi_{(i,n)}\,, \tag{3.13}$$

of dimension N. Classically there is a relation

$$\det M_{i+1,n} - B_{i,n}B_{i+1,n} = 0\,, \quad (M_{i+1,n})^a_c := \Phi^a_{(i,n)b}\Phi^b_{(i+1,n)c}\,, \tag{3.14}$$

this relation is modified quantum mechanically [164]

$$\det M_{i+1,n} - B_{i,n}B_{i+1,n} = \Lambda^{2N}_{i+1,n} \tag{3.15}$$

here $\Lambda_{i+1,n}^{2N}$ is the effective holomorphic scale of the $(i+1,n)^{\text{th}}$ gauge group. It can be written in terms of the gauge couplings $q_{(j,n)}$ and the masses for hypermultiplets which we will discuss in detail in the next section.

Note that there is an over-parametrisation since the $u_{lk,n}$ and $B_{i,n}$ are not all independent but rather related by the applying the Cayley-Hamilton theorem to the matrix Φ_n:

$$p(\Phi_n) = \sum_{l=1}^{N} c_{l,n}\Phi_n^l + (-1)^N \det \Phi_n \mathbb{I}_N = 0 \tag{3.16}$$

where

$$c_{N-l,n} = \frac{(-1)^l}{l!} BE_l\left(\mathfrak{u}_{k,n}, -1!\mathfrak{u}_{2k,n}, \dots, (-1)^{l-1}(l-1)!\mathfrak{u}_{lk,n}\right) , \tag{3.17}$$

with $c_N = 1$ and BE_l is the l^{th} complete exponential Bell polynomial and we defined $\mathfrak{u}_{lk,n} := \operatorname{tr} \Phi_n^l$. The $u_{lk,n}$ may be expressed in terms of the $\mathfrak{u}_{lk,n}$ as $u_{lk,n} = \frac{1}{l}\sum_{p=0}^{l}\binom{l}{p}\left(-\frac{1}{N}\mathfrak{u}_{k,n}\right)^p \mathfrak{u}_{(l-p)k,n}$. Taking the trace of (3.16) implies, for generic $\Phi_{(i,c)}$, a single relation between (3.12) and (3.13)

$$\operatorname{tr} p(\Phi_n) = \sum_{l=1}^{N} c_{l,n}\mathfrak{u}_{lk,n} + (-1)^N N \prod_{i=1}^{k} B_{i,n} = 0\,. \tag{3.18}$$

In particular, this implies that $u_{Nk,n}$ can be completely written in terms of the $B_{i,n}$ and the $u_{lk,n}$ $1 \leqslant l \leqslant N-1$. Hence, the coordinate ring of the Coulomb branch for $k \geqslant 2$ is expected to be a freely generated ring of dimension $(3g-3+\ell)(k+N-1)$

$$CB = \mathbb{C}[u_{lk,n}, B_{i,n}]\,, \quad \begin{array}{l} l \in \{1,2,\dots,N-1\}\,, \\ i \in \{1,2,\dots,k\}\,, \\ n \in \{1,2,\dots,3g-3+\ell\}\,. \end{array} \tag{3.19}$$

3.3.2 Mass Parameters

We may regard the masses corresponding to the flavour symmetries for $SU(N)_{(i,0)} = SU(N)_{(i,L)}$ and $SU(N)_{(i,\ell+1)} = SU(N)_{(i,R)}$ as expectation values for background superfields $\Phi_{(i,0)} = M_i^L$, $\Phi_{(i,\ell+1)} = M_i^R$. Hence we may construct flavour invariant combinations of masses in the same fashion, to that end we define

$$\mu_{lk}^{L} := \mathrm{tr}\left(\prod_{i=1}^{k} M_i^L\right)^l , \quad \mu_{lk}^{R} := \mathrm{tr}\left(\prod_{i=1}^{k} M_i^R\right)^l , \tag{3.20}$$

$$J_i^L := \det M_i^L , \quad J_i^R := \det M_i^R , \tag{3.21}$$

for $1 \leqslant l \leqslant N$. Moreover, $M_i^{L/R}$ may be diagonalised by $SU(N)$ transformations such that they take they form

$$M_i^{L/R} = \mathrm{diag}\left(m_{i,1}^{L/R}, \ldots, m_{i,N}^{L/R}\right) . \tag{3.22}$$

Additionally, when $\ell = 1$, the $SU(N)^{2k}$ flavour symmetry enhances to $SU(2N)^k$ and in that case we find it convenient to instead combine the mass parameters as

$$M_i := M_i^L \oplus M_i^R := \mathrm{diag}\left(m_{(i),1}, \ldots, m_{(i),2N}\right) , \tag{3.23}$$

and to write the invariants as

$$\mu_{lk} := \mathrm{tr}\left(\prod_{i=1}^{k} M_i\right)^l , \quad J_i := \det M_i , \tag{3.24}$$

where now $1 \leqslant l \leqslant 2N$. After giving vevs in the UV each $U(1)$ factor is decoupled and the corresponding coupling matrix $\tau_{ab,n}^{\mathrm{UV}}$ is diagonal and they are related to the $SU(N)_{(i,n)}$ gauge couplings by

$$\tau_{ab,n}^{\mathrm{UV}} = 2\delta_{a,b} \sum_{i=1}^{k} \tau_{(i,n)} \tag{3.25}$$

3.3.3 Curves for $N = k = 2$

We will now compute the curve for the simplest of our theories, namely the $k = 2$ quiver with $N = 2$ and $\ell = 1$. We will drop the 'n' index for compactness, e.g. here $\Phi_{(i,n)} = \Phi_{(i,1)} := \Phi_{(i)}$.

Diagonal Limit

Initially let us consider the limit where, say, $\Phi_{(1)}$ gets a large diagonal vev $\left\langle \Phi_{(1)} \right\rangle = \mathrm{diag}(a, a)$, by examining the Lagrangian one sees that the gauge group is broken down to a $SU(2)_D$ diagonal subgroup just as in [20], under which both bifundamentals decompose into an adjoint and a singlet, of which the adjoint associated to $\Phi_{(1)}$ becomes the longitudinal modes of the massive $(SU(2)_{(1)} \times SU(2)_{(2)})/SU(2)_D$ gauge bosons, while the (anti)fundamentals of the $SU(2)$'s decompose into (anti)fundamentals of the $SU(2)_D$.

The uneaten adjoint is $\phi = \Phi_{(2)} - \frac{1}{2}\operatorname{tr}\Phi_{(2)}$. Below the scale a the quarks $Q_{(1,0)}$, $Q_{(2,1)}$, $\tilde{Q}_{(1,0)}$, $\tilde{Q}_{(1,1)}$ can be integrated out, the super potential (without the singlets) is then [3] In general there will also be the singlet $u_2 = \operatorname{tr}\Phi_{(1)}\Phi_{(2)}$ of which there be an associated mass deformation to the diagonal superpotential W_D

$$W_D \to W_D + m_s u_2 \tag{3.26}$$

Then below m_s the singlet u_2 can be integrated out.[4] then below the scale a the theory flows to $\mathcal{N} = 2$ QCD with $N_f = 4$ flavors [194] which was studied in [12]. The curve for that theory in quartic form[5] is [195, 13]

$$y^2 = (x^2 - u_D)^2 - \frac{4q_D}{(1+q_D)^2}\prod_{j=1}^{4}\left(x + \tilde{\mu}_j - \frac{q_D}{2(1+q_D)}\sum_{m=1}^{4}\tilde{\mu}_m\right) \tag{3.27}$$

where $\tilde{\mu}_j = m_{(1),j}m_{(2),j}/a$, $u_D := \frac{1}{2}\operatorname{tr}\phi^2 = u_4/a^2 = (\mathfrak{u}_4 - \mathfrak{u}_2^2)/2a^2$ and $q_D = e^{2\pi i\tau_D} = q_{(1)}q_{(2)}$ is associated to the coupling for the $SU(2)_D$. After rescaling $x \to x/a$, $y \to y/a^2$ and substituting in the above relations we have

$$y^2 = (x^2 - u_4)^2 - \frac{4q}{(1+q)^2}\prod_{j=1}^{4}\left(x + m_{(1),j}m_{(2),j} - \frac{q}{2(1+q)}\mu_2\right) \tag{3.28}$$

where $q := q_{(1)}q_{(2)}$ and μ_2 is defined as in (3.24). Now consider integrating out all of the flavours from this $\mathcal{N} = 2$ curve, leaving us with pure $SU(2)$ $\mathcal{N} = 2$ gauge theory; [195] tells us to hold fixed, in our conventions, the relation

$$\Lambda_D^4 = \frac{4q_D}{(1+q_D)^2}\prod_{j=1}^{4}\tilde{\mu}_j = \frac{4q_{(1)}q_{(2)}}{(1+q_{(1)}q_{(2)})^2}\prod_{j=1}^{4}\frac{m_{(1),j}m_{(2),j}}{a} \tag{3.29}$$

[3] The "diagonal" quark masses can be calculated by solving the F-term's for $Q_{(1,0)}$, $Q_{(2,1)}$, $\tilde{Q}_{(1,0)}$, $\tilde{Q}_{(1,1)}$.

[4] In [20, 192, 193] the assumption that the singlet does not enter the gauge dynamics appears to be consistent assumption, atleast below the scales m_s where the singlet can be integrated out.

[5] The interested reader may consult Appendix D.1 for the explicit details regarding the change of variables in order to move between quartic and cubic forms.

On the other hand, from [20] we have that [6]

$$\Lambda_D^4 = 4\frac{\Lambda_{(1)}^4\Lambda_{(2)}^4}{a^4} \tag{3.30}$$

should be held fixed. Equating them implies that

$$\Lambda_{(1)}^4\Lambda_{(2)}^4 = \frac{q_D}{(1+q_D)^2}\prod_{j=1}^4 m_{(1),j}m_{(2),j} = \frac{q_{(1)}q_{(2)}}{(1+q_{(1)}q_{(2)})^2}J_1J_2 \tag{3.31}$$

should be held fixed under the limit, with J_i defined in (3.24). By the $\mathrm{Aut}(Quiver)$ symmetry we must have that the matching condition is

$$\Lambda_{(i)}^4 = \pm\frac{q_{(i)}}{1+q_{(1)}q_{(2)}}J_i. \tag{3.32}$$

Positivity of $\Re\Lambda_{(i)}^4$ demands that we take the positive sign.

Hence, the most general form of the curve which is both polynomial in masses and Coulomb moduli, is invariant under all of the symmetries and reproduces (3.28) in the $\mathcal{N}=2$ limit is

$$\begin{aligned} y^2 &= \left(x^2 - u_4 + a_{12}J_1 + a_{21}J_2 + b\mu_2^2 + c\mu_2u_2 + d\mu_4\right)^2 \\ &\quad - \frac{4q_{(1)}q_{(2)}}{(1+q_{(1)}q_{(2)})^2}\prod_{j=1}^4\left(x + m_{(1),j}m_{(2),j} - \frac{q_{(1)}q_{(2)}}{2(1+q_{(1)}q_{(2)})}\mu_2\right) \end{aligned} \tag{3.33}$$

where $a_{12} := a(q_{(1)}, q_{(2)})$, $a_{21} := a(q_{(2)}, q_{(1)})$ and b, c, d are all symmetric functions in $q_{(1)}, q_{(2)}$. Note however that we may immediately restrict the dependence of b, c, d on $q_{(1)}, q_{(2)}$ by demanding agreement with the curves [196, 20] upon integrating out some of the flavours. To have a well defined limit we must have that b, c, d are functions of only the product $q_{(1)}q_{(2)}$ e.g. $b = b(q_{(1)}q_{(2)})$ and, moreover, that as power series in $q_{(1)}q_{(2)}$ the leading terms of b, c, d are proportional to $q_{(1)}^n q_{(2)}^n$ for $n \geqslant 1$, e.g. $b(q_{(1)}q_{(2)}) = \sum_{n\geqslant 0} b_n q_{(1)}^n q_{(2)}^n$ and $b_0 \equiv 0$.

$q_{(2)} \gg q_{(1)}$ Limit

We can now consider a similar limit as taken in [20], namely where, say $q_{(2)} \gg q_{(1)}$. Below the scale set by the matching condition $\Lambda_2^4 = \frac{q_{(2)}}{1+q_{(1)}q_{(2)}}J_2$, from the point

[6] Note that to save on various factors of $\sqrt{2}$ we prefer to use $u_D = \frac{1}{2}\operatorname{tr}\phi$ instead of $u_D = \operatorname{tr}\phi$, accounting for factor 4 instead of 16

of view of $SU(2)_1$ the theory is approximately described by a single $SU(2)$ gauge theory with an adjoint field $\tilde{\phi} = \frac{1}{E}(\Phi_1\Phi_2 - \frac{1}{2}\operatorname{tr}\Phi_1\Phi_2)$ and the singlets b_1, b_2 and u_2. There are also singlets involving fundamental $Q, \widetilde{Q}$'s which do not appear in the curve for reasons discussed earlier. If we assume that the singlets do not enter into the gauge dynamics in the Coulomb phase then in the above limit the theory is approximately a $\mathcal{N} = 2$ $SU(2)_1$ $N_f = 4$ gauge theory with instanton parameter $q_{(1)}$ and with mass matrix given by M_1.

The fields are constained, implementing the matching relation, by the quantum result [164]

$$\operatorname{Pf} V = u - E^2\tilde{u} = \frac{q_{(2)}}{1 + q_{(1)}q_{(2)}} J_2\,, \tag{3.34}$$

with $\tilde{u} = \frac{1}{2}\operatorname{tr}\tilde{\phi}^2$ and E some energy scale.

To fix a_{12}, a_{21} we need only consider the mass configurations $M_1 = 0$, $M_2 \sim E\mathbb{I}_4$. Then the $\mathcal{N} = 2$ theory has an order 4 singularity, associated to the quarks becoming massless, when $\tilde{u} = 0$. For these mass configurations the discriminant of (3.33) has a point of vanishing order 4 at

$$u_4 = \frac{q_{(2)}}{1 + q_{(1)}q_{(2)}} J_2 \tag{3.35}$$

which implies that $a_{21} \equiv \frac{q_{(2)}}{1+q_{(1)}q_{(2)}}$ and therefore $a_{12} \equiv \frac{q_{(1)}}{1+q_{(1)}q_{(2)}}$.

Following this we may appeal to a simple argument to fix the remaining functions b, c, d: Consider the limit of integrating out all of flavours coupled to, say, the $SU(2)_2$ by taking $q_{(2)} \to 0$ $m_{(2),j} \to \infty$ while holding fixed the matching condition (3.32). However, since $q_{(1)}$ is an independent parameter we are free to also take $q_{(1)} \to \infty$ while also holding $q_{(1)}q_{(2)} \propto 1$ fixed. We expect the resulting curve to describe a $\mathcal{N} = 1$ $SU(2)_1 \times SU(2)_2$ quiver theory with instanton parameters $q_{(1)}, \Lambda^4_{(2)}$ with 4 chiral multiplets in $(\mathbf{2}, \mathbf{1})$ and 2 in $(\mathbf{2}, \mathbf{2})$. However, in order for the resulting curve to we well defined we require that $b = c = d = 0$. [7] The curve is therefore:

$$
\begin{aligned}
y^2 = &\left(x^2 - u_4 + \frac{q_{(1)}}{1+q_{(1)}q_{(2)}}J_1 + \frac{q_{(2)}}{1+q_{(1)}q_{(2)}}J_2\right)^2 \\
&- \frac{4q_{(1)}q_{(2)}}{(1+q_{(1)}q_{(2)})^2}\prod_{j=1}^{4}\left(x + m_{(1),j}m_{(2),j} - \frac{q_{(1)}q_{(2)}}{2(1+q_{(1)}q_{(2)})}\mu_2\right) ,
\end{aligned}
\tag{3.36}
$$

$$
J_i = \det M_i = \prod_{j=1}^{4} m_{(i),j} , \quad \mu_2 = \operatorname{tr} M_1 M_2 = \sum_{j=1}^{4} m_{(1),j}m_{(2),j} . \tag{3.37}
$$

Checks

It can be immediately verified that our curve (3.36) reproduces those of [196, 20] Let us again consider the $q_{(2)} \gg q_{(1)}$ limit and check consistency for a few mass configurations:

$M_i = \mathrm{diag}(m, -m, m, -m)$, $M_2 \sim E\mathbb{I}_4$ In this configuration the adjoint field of $SU(2)_1$ is singular at the order 4 quark singularity $\tilde{u} = m^2$ whilst the corresponding vanishing order 4 point of (3.36) is at

$$
\begin{aligned}
u_4 &= m^2E^2 + \frac{q_{(1)}}{1+q_{(1)}q_{(2)}}m^4 + \frac{q_{(2)}}{1+q_{(1)}q_{(2)}}E^4 \\
&\approx m^2E^2 + \frac{q_{(2)}}{1+q_{(1)}q_{(2)}}E^4
\end{aligned}
\tag{3.38}
$$

which agrees nicely with (3.34).

$M_1 = \mathrm{diag}(m, m, m, 0)$, $M_2 \sim E\mathbb{I}_4$ The $\mathcal{N} = 2$ curve has an order 3 quark singularity at $4\tilde{u} = m^2(2 - q_{(1)})^2/(1 + q_{(1)})^2 = 4m^2 + \mathcal{O}(q_{(1)})$. On the other hand (3.36) has a vanishing order 3 point at

$$
\begin{aligned}
u_4 &= \frac{q_{(2)}}{1+q_{(1)}q_{(2)}}E^4 + E^2m^2 + \frac{3(1-q_{(1)}q_{(2)})}{1+q_{(1)}q_{(2)}} \\
&\approx \frac{q_{(2)}}{1+q_{(1)}q_{(2)}}E^4 + 4E^2m^2 + \mathcal{O}(q_{(1)})
\end{aligned}
\tag{3.39}
$$

again, in agreement with (3.34).

[7] For example consider the term $\lim b(q_{(1)}q_{(2)})\mu_4 \sim b(1)\infty^2 \sim \infty^2$ hence $b \equiv 0$ in order to get a well defined result.

$M_1 = \mathrm{diag}(m, m, m, m)$, $M_2 \sim E\mathbb{I}_4$ The $\mathcal{N} = 2$ curve has an order 4 quark singularity at $\tilde{u} = m^2(1 - q_{(1)})^2/(1 + q_{(1)})^2 = m^2 + \mathcal{O}(q_{(1)})$. The discriminant of (3.36) has a zero of degree 4 located at

$$\begin{aligned} u_4 =& \frac{q_{(1)}}{1 + q_{(1)}q_{(2)}} m^4 + \frac{q_{(2)}}{1 + q_{(1)}q_{(2)}} E^4 + \frac{(1 - q_{(1)}q_{(2)})^2}{(1 + q_{(1)}q_{(2)})^2} m^2 E^2 \\ \approx& \frac{q_{(2)}}{1 + q_{(1)}q_{(2)}} E^4 + m^2 E^2 \end{aligned} \tag{3.40}$$

again, in agreement with (3.34).

3.3.4 Curve for $k = 2$ General N

The generalisation to $SU(N)$ is rather straightforward. We again consider the diagonal limit.

Diagonal Limit

The same diagonal limit is reached by giving diagonal vev $\langle \Phi_{(1)} \rangle = a\mathbb{I}_N$. In this limit the theory is again approximately $\mathcal{N} = 2$ $SU(N)$ gauge theory with $N_f = 2N$ flavours and the adjoint field is $\phi = \Phi_{(2)} - \frac{1}{N} \operatorname{tr} \Phi_{(2)}$. The curve of that theory is given by [195]

$$\begin{aligned} y^2 =& \left(x^N - \sum_{l=2}^{N} u_{D,l} x^{N-l} \right)^2 \\ &- \frac{4q_D}{(1 + q_D)^2} \prod_{j=1}^{2N} \left(x + \tilde{\mu}_j - \frac{q_D}{N(1 + q_D)} \sum_{m=1}^{2N} \tilde{\mu}_m \right) \end{aligned} \tag{3.41}$$

where $\tilde{\mu}_j = m_{(1),j} m_{(2),j}/a$, $u_{D,l} := \frac{1}{l} \operatorname{tr} \phi^l = u_{2l}/a^l$ and $q_D = q_{(1)}q_{(2)}$ is associated to the coupling for the $SU(2)_D$. After rescaling $x \to x/a$, $y \to y/a^{2N}$ and substituting in the above relations we have

$$\begin{aligned} y^2 =& \left(x^N - \sum_{l=2}^{N} u_{2l} x^{N-l} \right)^2 \\ &- \frac{4q_{(1)}q_{(2)}}{(1 + q_{(1)}q_{(2)})^2} \prod_{j=1}^{2N} \left(x + m_{(1),j} m_{(2),j} - \frac{q_{(1)}q_{(2)}}{N(1 + q_{(1)}q_{(2)})} \mu_2 \right) . \end{aligned} \tag{3.42}$$

Now consider integrating out all of the flavours from this $\mathcal{N}=2$ curve, leaving us with pure $SU(2)$ $\mathcal{N}=2$ gauge theory; [195] tells us to hold fixed the relation

$$\Lambda_D^{2N} = \frac{4q_D}{(1+q_D)^2}\prod_{j=1}^{2N}\tilde{\mu}_j = \frac{4q_{(1)}q_{(2)}}{(1+q_{(1)}q_{(2)})^2}\prod_{j=1}^{2N}\frac{m_{(1),j}m_{(2),j}}{a} \tag{3.43}$$

On the other hand, from [192] we have that

$$\Lambda_D^{2N} = 4\frac{\Lambda_{(1)}^{2N}\Lambda_{(2)}^{2N}}{a^{2N}} \tag{3.44}$$

should be held fixed. Equating them implies that

$$\Lambda_{(1)}^{2N}\Lambda_{(2)}^{2N} = \frac{q_{(1)}q_{(2)}}{(1+q_{(1)}q_{(2)})^2}J_1J_2 \tag{3.45}$$

should be held fixed under the limit, with J_i defined in (3.24). By the $\mathrm{Aut}(Quiver)$ symmetry we must have that

$$\Lambda_{(i)}^{2N} = \pm\frac{q_{(i)}}{1+q_{(1)}q_{(2)}}J_i\,. \tag{3.46}$$

Positivity of $\Re\Lambda_{(i)}^{2N}$ demands that we take the positive sign.

We have have then the most general ansatz for the curve

$$\begin{aligned} y^2 = {} & \left(x^N - \sum_{l=2}^{N}\left[u_{2l} + f_l\left(\mathfrak{u}_n,\mu_m;q_{(1)}q_{(2)}\right)\right]x^{N-l} + a_{12}J_1 + a_{21}J_2\right)^2 \\ & - \frac{4q_{(1)}q_{(2)}}{(1+q_{(1)}q_{(2)})^2}\prod_{j=1}^{2N}\left(x + m_{(1),j}m_{(2),j} - \frac{q_{(1)}q_{(2)}}{N(1+q_{(1)}q_{(2)})}\mu_2\right) \end{aligned} \tag{3.47}$$

where $f_l\left(u_{2n},\mu_m;q_{(1)}q_{(2)}\right)$ is a function with mass dimension (or equivalently R-charge) $2l$ which satisfies $\lim_{\mu_m\to 0} f_l\left(u_{2n},\mu_m;q_{(1)}q_{(2)}\right) = 0$. In other words the f_l are always subdominant compared to u_{2l} in the large u_{2l} limit. For the same reasons as before f_l is also a function only of $q_{(1)}q_{(2)}$. On the other hand, the same steps as for the $N=2$ case may now be simply run.

$q_{(2)} \gg q_{(1)}$ Limit

We may again consider the $q_{(2)} \gg q_{(1)}$ limit. In that limit, from the point of view of the $SU(N)_1$ the theory is approximately $\mathcal{N}=2$ $SU(N)_1$ gauge theory with $N_f = 2N$ and instanton parameter $q_{(1)}$. There is the adjoint field $\widetilde{\phi} = \frac{1}{E}\left(\Phi_{(1)}\Phi_{(2)} - \frac{1}{N}\operatorname{tr}\Phi_{(1)}\Phi_{(2)}\right)$. There is also a quantum modified constraint on moduli space [164]

$$\det \Phi_{(1)}\Phi_{(2)} - B_1 B_2 = \frac{q_{(2)}}{1+q_{(1)}q_{(2)}} J_2 \,. \tag{3.48}$$

This is implemented on the curves by

$$\widetilde{u}_l = \begin{cases} \frac{u_{2l}}{E^l} & l \neq N \\ \frac{1}{E^l}\left(u_{2N} + \frac{q_{(2)}}{1+q_{(1)}q_{(2)}} J_2\right) & l = N \end{cases} \tag{3.49}$$

Now we may fix $a_{12}(q_{(1)}, q_{(2)}) = a_{21}(q_{(2)}, q_{(1)})$. The massless $M_1 = 0$ $\mathcal{N} = 2$, $N_f = 2N$ theory is singular when $\widetilde{u}_l = 0$. On the other hand, our curve (3.47) with $M_1 = 0$ is singular when

$$u_{2l} = \begin{cases} 0 & l \neq N \\ a_{21} J_2 & l = N \end{cases} \tag{3.50}$$

Comparison with (3.49) implies $a_{21} = \frac{q_{(2)}}{1+q_{(1)}q_{(2)}}$. We may then run the argument that we used below equation (3.35) to immediately set all of the $f_l \equiv 0$. Hence the curve for $k = 2$ is

$$\begin{aligned} y^2 = &\left(x^N - \sum_{l=2}^{N} u_{2l} x^{N-l} + \frac{q_{(1)}}{1+q_{(1)}q_{(2)}} J_1 + \frac{q_{(2)}}{1+q_{(1)}q_{(2)}} J_2\right)^2 \\ &- \frac{4 q_{(1)} q_{(2)}}{(1+q_{(1)}q_{(2)})^2} \prod_{j=1}^{2N}\left(x + m_{(1),j} m_{(2),j} - \frac{q_{(1)}q_{(2)}}{N(1+q_{(1)}q_{(2)})}\mu_2\right) \,. \end{aligned} \tag{3.51}$$

3.3.5 Curve for general N & k

The generalisation to arbitrary k is largely the same. It is the extension of [192] to include flavours. The only new phenomena is the implementation of the quantum relation (3.48) for the quiver. We will simply state the result. The curve may be written as

$$\begin{aligned} y^2 = &\left(\sum_{l=1}^{N} c_l \mathfrak{u}_{lk} x^{N-l} + (-1)^N \prod_{i=1}^{k} B_i + \left(B_i B_{i+1} \to \frac{q_{i+1}}{1+q} J_{i+1}\right)\right)^2 \\ &- \frac{4q}{(1+q)^2} \prod_{j=1}^{2N}\left(x + m_j - \frac{q}{N(1+q)}\mu_k\right) , \end{aligned} \tag{3.52}$$

here the c_l are defined by (3.16), $q := \prod_{i=1}^{k} q_{(i)}$, $m_j := \prod_{i=1}^{k} m_{(i),j}$. Finally the brackets mean to replace the pairs $B_i B_{i+1}$ appearing in $(-1)^N \prod_{i=1}^{k} B_i$ with the corresponding mass condition in all possible ways, for example at $k = 4$ the brackets should be read as

$$\begin{aligned}(-1)^N \left(B_i B_{i+1} \to \Lambda_{i+1}^{2N}\right) =& \Lambda_2^{2N} B_3 B_4 + B_1 \Lambda_3^{2N} B_4 + B_1 B_2 \Lambda_4^{2N} \\ &+ \Lambda_1^{2N} B_2 B_3 + \Lambda_2^{2N} \Lambda_4^{2N} + \Lambda_1^{2N} \Lambda_3^{2N} \,,\end{aligned} \tag{3.53}$$

for shorthand we use $\Lambda_i^{2N} = q_i J_i/(1+q)$.

3.4 Comparison to M-Theory Curves

We can place our curves into Gaiotto form. We would like to match with [137]. Let us begin by review the change of variables needed for the class $\mathcal{S}$ case.

3.4.1 Review of $k = 1$ Class $\mathcal{S}$

The Seiberg-Witten curve in the original form reads [195]

$$y^2 = \left(x^N - \sum_{l=2}^{N} u_l x^{N-l}\right)^2 - \frac{4q}{(1+q)^2} \prod_{j=1}^{2N} (x - \mathfrak{m}_j) \tag{3.54}$$

where we have defined $\mathfrak{m}_j := -m_j + \frac{q}{N(1+q)}\mu_1$. (x, y) have dimension (equal to minus $U(1)_{r_{\mathcal{N}=2}}$ charge) $(1, N)$. Now we make the change of variables

$$y = -\frac{2t}{1+q} \prod_{j=1}^{N} (x - \mathfrak{m}_j) + \left(x^N - \sum_{l=2}^{N} u_l x^{N-l}\right) \,. \tag{3.55}$$

Which results in

$$\prod_{i=1}^{N} (x - \mathfrak{m}_i^L) t^2 - (1+q) \left(x^N - \sum_{l=2}^{N} u_l x^{N-l}\right) t + q \prod_{i=1}^{N} (x - \mathfrak{m}_i^R) = 0 \tag{3.56}$$

and matches the computation from Type-IIA side [197]. This curve can be placed in ‘Gaiotto form’

$$z^N + \sum_{i=1}^{N} z^{N-l} \phi_l(t) = 0 \,, \tag{3.57}$$

where $\phi_l(t)dt^l$ are degree l differentials on the Riemann surface $\mathcal{C} \xleftarrow{1:N} \mathcal{X}$. t is a local coordinate on $\mathcal{C}$ and (z,t) on $T^*\mathcal{C}$. We write $x = tz$. Considering first the massless $m_i = 0$ case the curve is

$$z^N + \sum_{l=2}^{N} z^{N-l} \frac{(1+q)u_l}{t^{l-1}(t-1)(t-q)} = 0\,. \tag{3.58}$$

In the massive case, expanding $\prod_{l=1}^{N}(x - \mathfrak{m}_l^{L/R}) = x^N + \sum_{l=1}^{N} x^{N-l} f_l(\mathfrak{m}_l^{L/R})$ we have

$$z^N + \sum_{l=1}^{N} z^{N-l} \frac{t^2 f_l(\mathfrak{m}_p^L) + (1+q)tu_l + qf_l(\mathfrak{m}_p^R)}{t^l(t-1)(t-q)} = 0\,. \tag{3.59}$$

So, the differentials are

$$\phi_l(t)dt^l = \frac{t^2 f_l(\mathfrak{m}_p^L) + (1+q)tu_l + qf_l(\mathfrak{m}_p^R)}{t^l(t-1)(t-q)} dt^l\,. \tag{3.60}$$

Notice that at $t = 0, \infty$ (recall that t is a coordinate on $\mathcal{C}$) ϕ_l has order l poles. These are interpreted as maximal punctures. On the other hand, at $t = 1, q$ ϕ_l has simple poles, these are the locations of the minimal punctures.

3.4.2 Class $\mathcal{S}_2$

Let us perform the same manipulations to the $k = 2$ case. The curve (3.51) is

$$\begin{aligned} y^2 = &\left(x^N - \sum_{l=2}^{N} u_{2l} x^{N-l} + \frac{q_{(1)}}{1 + q_{(1)}q_{(2)}} J_1 + \frac{q_{(2)}}{1 + q_{(1)}q_{(2)}} J_2 \right)^2 \\ &- \frac{4q_{(1)}q_{(2)}}{(1 + q_{(1)}q_{(2)})^2} \prod_{j=1}^{2N} \left(x + m_{(1),j} m_{(2),j} - \frac{q_{(1)}q_{(2)}}{N(1 + q_{(1)}q_{(2)})} \mu_2 \right)\,. \end{aligned} \tag{3.61}$$

The dimensions of (x, y) is now (k, Nk). We perform the change of variables

$$y = -\frac{2t}{1 + q_{(1)}q_{(2)}} \prod_{j=1}^{N} (x - \mathfrak{m}_j) + \left(x^N - \sum_{l=2}^{N} u_{2l} x^{N-l} + \sum_{i=1}^{2} \frac{q_{(i)}}{1 + q_{(1)}q_{(2)}} J_i \right) \tag{3.62}$$

with $\mathfrak{m}_j := -m_{(1),j}m_{(2),j} + \frac{q_{(1)}q_{(2)}}{N(1+q_{(1)}q_{(2)})}\mu_2$; as before $q := \prod_{i=1}^k q_{(i)}$. Then the curve becomes

$$\begin{aligned} 0 = &\prod_{i=1}^N (x - \mathfrak{m}_i^L)t^2 - (1+q)\left(x^N - \sum_{l=2}^N u_{2l}x^{N-l} + \sum_{i=1}^2 \frac{q_{(i)}}{1+q}J_i\right)t \\ &+ q\prod_{i=1}^N (x - \mathfrak{m}_i^R)\,. \end{aligned} \tag{3.63}$$

If we would like to again have the interpretation of (z, t) being canonical coordinates on $T^*\mathcal{C}$ we require that (z, t) have dimension $(1, 0)$. Therefore we must write write $x = tz^k$. Plugging this in we find

$$z^{kN} + \sum_{l=1}^N z^{kN-kl}\phi_{kl}(t) = 0\,, \tag{3.64}$$

$$\phi_{kl}(t) = \begin{cases} \frac{t^2 f_1(\mathfrak{m}_p^L) + q f_1(\mathfrak{m}_p^R)}{t(t-1)(t-q)} & l = 1\,, \\ \frac{t^2 f_l(\mathfrak{m}_p^L) + q f_l(\mathfrak{m}_p^R) + (1+q)tu_{2l}}{t^l(t-1)(t-q)} & 1 < l < N\,, \\ -\frac{q_{(1)}J_1 + q_{(2)}J_2}{t^{N-1}(t-1)(t-q)} & l = N\,. \end{cases} \tag{3.65}$$

Notice that ϕ_{kl} again has order l poles at $t = 0, \infty$ and simple poles at $t = 0, q$. These signify the locations of the maximal and minimal punctures, respectively. If we consider the massless case the answer simplifies to

$$z^{kN} + \frac{\sum_{l=2}^N (1+q)z^{k(N-l)}u_{2l}}{t^{l-1}(t-1)(t-q)} = 0\,. \tag{3.66}$$

3.5 Conclusions

In this chapter we have derived curves, à la Seiberg and Intriligator, encoding the low energy behaviour of a certain class $\mathcal{S}_k$ theories on the Coulomb branch. We are able to place these curves in 'Gaiotto form' and in that case we can give an interpretation as a curve embedded in $T^*\mathcal{C}$ where $\mathcal{C}$ is the Riemann surface that gives rise to the class $\mathcal{S}_k$ theory from the 6d $(1, 0)_{A_{k-1}}$ SCFT. We have matched the curves to those computed from M-theory [137].

The curves, while not encoding enough information to 'solve' the low energy effective theory due to the Kähler part of the action being unconstrained by holomorphicity, are believed to still encode a large amount of information regarding the theory. In particular, they may prove invaluable for deriving new theories and dualities, as was performed for class $\mathcal{S}$ in [121].

Chapter 4

Instanton Counting for Theories of Class $\mathcal{S}_k$

The work in this chapter is based on [198].

4.1 Introduction

Using the localisation procedure outlined in Section 1.4.1 Nekrasov was able to verify the results of Seiberg & Witten [12, 13] from a purely field theoretic perspective and derive the instanton partition function for $\mathcal{N} = 2$ gauge theories by performing the integration over instanton moduli space [47, 32]. From his work it is possible to write down the instanton partition functions for general Lagrangian theories in class $\mathcal{S}$.

While the localisation procedure used in Nekrasov's computation relies on $\mathcal{N} = 2$ supersymmetry (in order to preserve a supersymmetry via topological twisting on Ω-background) one can also consider the analagous quantity for theories with less (or no) supersymmetry. Indeed, the idea of expanding the path integral around instanton configurations and integrating over instanton moduli space is an idea that dates back to 't Hooft [199]. However, for non-supersymmetric theories the fluctations around the instanton background at strong coupling are very hard to perform; while for supersymmetric theories the fluctations cancel. See also [22, 23] for reviews.

In this chapter we will compute the instanton partition functions and verify the proposal of [153] for a subset of class $\mathcal{S}_k$ theories with a Lagrangian description; we take the compactification surface $\mathcal{C}$ to be either an ℓ-punctured torus $T^2\backslash\{p_1, \ldots, p_\ell\}$ or an ℓ+2-punctured sphere $\mathbb{CP}^1\backslash\{p_1, \ldots, p_{\ell+2}\}$. We discussed these theories in detail in Section 1.6.2. These theories are conformal and have weakly coupled Lagrangian descriptions in terms of toroidal or cylindrical $\mathcal{N}=1$ quiver theories.

Because of their orbifold constructions class $\mathcal{S}_k$ theories suggest an ideal starting point to begin to look for exact results and 2d/4d relations in 4d $\mathcal{N}=1$ theories which, so far, have been largely unexplored. Often the orbifolded daughter theory possesses many similarities with their mother theory [200, 201].

We engineer the toroidal $\mathcal{N}=1$ quiver theories of class $\mathcal{S}_k$ in Type-IIB string theory as $\mathbb{Z}_\ell \times \mathbb{Z}_k$ orbifolds of D3-branes; as in Section 1.6.2. We can then study the dynamics of K instantons on the Dp-branes via the correspondence between the ADHM construction [29] and D(-1)-branes within D3-branes; explained in Section 1.3.3. The instanton moduli space on D3-branes is isomorphic to the Higgs branch of the theory on D(-1)-branes. The partition function of the supersymmetric matrix model theory on the worldvolume of K D(-1)-branes is then equal to the K instanton partition function of the corresponding class $\mathcal{S}_k$ theory.

Instantons of orbifold daughters of $\mathcal{N}=4$ SYM were intensively studied in the early days of AdS/CFT [202, 203, 204, 205, 206, 207, 208] and recent computations in theories with eight or more supercharges in various dimensions [40, 89, 185, 209, 210] have been possible due to significant improvement of the old techniques.

Using T-duality on the setup of D3-branes in the presence of a $\mathbb{Z}_\ell \times \mathbb{Z}_k$ orbifold singularity we land on D5-branes in the presence of a $\mathbb{Z}_\ell \times \mathbb{Z}_k$ orbifold which engineers a 6d (elliptic) uplift of the 4d theories we are interested in. The matrix model describing pointlike instantons of the 4d theory living on D3-branes is lifted to a 2d gauge theory, the superconformal index of which computes the instanton partition function of the corresponding 6d gauge theory on T^2, living on the worldvolume of the D5-branes. The 2d superconformal calculation is very well studied [211, 212, 213, 214, 215, 216]. Taking the 4d limit of the 6d instanton partition function we obtain the instanton partition function of the 4d class $\mathcal{S}_k$ theory.

4.2 Brane Setup

In this paper our primary interest will be that of the theory living on the D(-1)/D0-branes in the setups of Tables 1.6 and 1.7. In both cases these are supersymmetric matrix models invariant under at least two supercharges. We could choose to work directly with these matrix models, however we find it more convenient to work instead with the two dimensional uplift of those matrix models. Hence, we instead work with the brane setup of Table 4.1. Before performing the T-duality we also

	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9	X^{10}
N D5	$-$	$-$	$-$	$-$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$A_{\ell-1}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\times$	$\times$
A_{k-1}	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\times$	$\times$	$\cdot$	$\times$	$\times$	$\cdot$
K D1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table 4.1: *Type IIB setup engineering a 6d uplift of the 4d $\mathcal{N} = 1$ theories.*

assume that X^5 may be safely decompactified such that it parametrises a space with the topology of $\mathbb{R}$. The T-duality may again be performed by replacing $A_{\ell-1}$ with TN_ℓ , we then T-dualise along the TN_ℓ circle, landing us on the setup of Table 4.1 . The $Spin(6)_R$ R-symmetry group has been broken to a subgroup $U(1)_{56} \times Spin(4)_R$ which acts by rotations along $\mathbb{R}^2$, $\mathbb{R}^4$ parametrised by X^5, X^6 and X^7, X^8, X^9, X^{10} respectively. The $\mathbb{Z}_\ell$ orbifold further breaks the $Spin(4)_R$ R-symmetry group down to a subgroup $SU(2)_R$ corresponding to the isometry group of TN_ℓ, while, the $\mathbb{Z}_k$ orbifold breaks the $U(1)_{56} \times SU(2)_R$ down to the maximal torus of $SU(2)_R$.

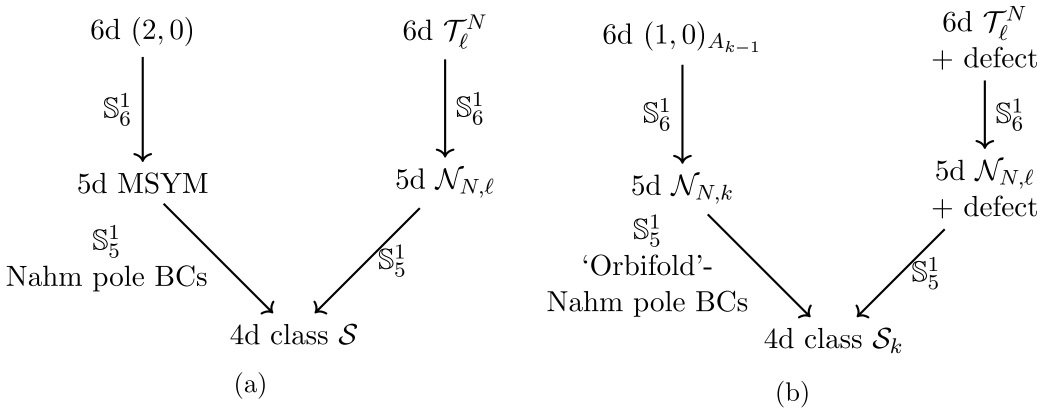


Figure 4.1: Left: *Schematic overview for $k = 1$ of two alternate ways to obtain the 4d $\mathcal{N} = 2$ $\tilde{A}_{\ell-1}$ circular quivers with $SU(N)^\ell$ gauge group in class $\mathcal{S}$ from compactifications of 6d theories.* Right: *A schematic overview of the $k > 1$ generalisations of 6d compactifications. The resulting 4d SCFTs are $\mathcal{N} = 1$ $\tilde{A}_{\ell-1} \times \tilde{A}_{k-1}$ torodial quivers in class $\mathcal{S}_k$ with gauge group $SU(N)^{\ell k}$.*

We can discuss the 6d theory on D5-branes. Let us begin with the $k = 1$ case. Without the $A_{\ell-1}$ transverse orbifold the theory on N D5-branes is simply the 6d $\mathcal{N} = (1,1)$ SYM theory. Upon performing the $\mathbb{Z}_\ell$ quotient we have a 6d $\mathcal{N} = (1,0)$ quiver theory of vector multiplets with bifundamental hypermultiplets, which we will call $\mathcal{T}^N_\ell$, see Figure 4.2. Compactifying on $\mathbb{S}^1_6$ gives 5d $\mathcal{N} = 1$ circular quivers

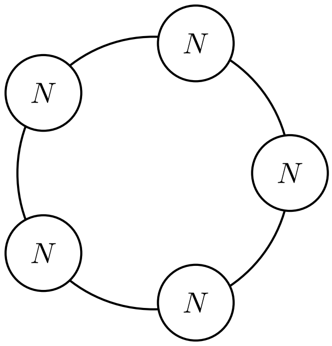


Figure 4.2: *6d circular (necklace) quiver $\mathcal{T}_\ell^N$ for $\ell = 5$. Circular nodes denote $\mathcal{N} = (1,0)$ vector multiplets and solid lines connecting them denote bifundamental $\mathcal{N} = (1,0)$ hypermultiplets. $\mathbb{S}^1/T^2$ reduction of $\mathcal{T}_\ell^N$ results in the 5d/4d circular $\tilde{A}_{\ell-1}$ affine quiver theories.*

$\mathcal{N}_{N,\ell}$ with ℓ nodes denoting $SU(N)$ gauge groups and ℓ links denoting bifundamental hypermultiplets. Further compactification on $\mathbb{S}^1_5$ results in 4d $\mathcal{N} = 2$ $\tilde{A}_{\ell-1}$ circular quiver theories. The $\tilde{A}_{\ell-1}$ theory may also be realised via the well known class $\mathcal{S}$ construction obtained by compactifying the A_{N-1} $(2,0)$ theory on the ℓ punctured torus with certain $\frac{1}{2}$-BPS Nahm pole boundary conditions specified at the punctures [217, 218, 219], see Figure 4.1a.

When $k > 1$ the resulting 6d theory corresponds to $\mathcal{T}_\ell^N$ in the presence of a codimension-two Gukov-Witten [220, 221] surface operator associated to ℓ copies of the partition

$$kN = N_1 + \cdots + N_k = N + \cdots + N \tag{4.1}$$

for the factors of $SU(kN)^\ell$. Finally, KK reducing along the circle leads to the 5d $\mathcal{N} = 1$ circular/necklace quiver gauge theory $\mathcal{N}_{N,\ell}$ on $\mathbb{R}^4 \times \mathbb{S}^1_5$ in the presence of the defect along the circle (see Figure 4.1b). Analogously, there is also the class $\mathcal{S}_k$ construction obtained by compactification of the $(1,0)_{A_{k-1}}$ SCFT associated to N M5-branes on transverse A_{k-1} singularity on a torus with ℓ punctures with 'orbifold' Nahm pole boundary conditions specified at each puncture [133, 135, 144].

4.2.1 Supersymmetry of the D1/D5 System

Type IIB string theory has 32 supersymmetries parametrised by two 32 component spinors ϵ_L , ϵ_R of positive chirality $\Gamma^{11}\epsilon_{L/R} = +\epsilon_{L/R}$ where $\Gamma^{11} = \Gamma^1 \dots \Gamma^{10}$ and Γ^M are the 32×32 Gamma matrices. The D5/D1 system preserves 1/4 of the 32 supersymmetries. Between them they preserve only those supersymmetries of the form

$$\epsilon_L = \Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\Gamma^6\epsilon_R\,, \quad \epsilon_L = \sigma\Gamma^5\Gamma^6\epsilon_R\,, \tag{4.2}$$

with $\sigma = \pm 1$ corresponding to whether we choose to insert D1- or anti-D1-branes. The theory living on the (anti-)D1-branes then possesses (p,q) supersymmetry with $p+q = 32/4 = 8$. By choosing an explicit representation for the Gamma matrices it can be shown that $p = q = 4$ and that the preserved supercharges are

$$\mathrm{Q}_+^{\alpha a}, \quad \mathrm{Q}_-^{\alpha\dot{a}}, \quad \text{if } \sigma = +1 \tag{4.3}$$

$$\widetilde{\mathrm{Q}}_+^{\dot{\alpha}\dot{a}}, \quad \widetilde{\mathrm{Q}}_-^{\dot{\alpha}a}, \quad \text{if } \sigma = -1 \tag{4.4}$$

where $a, \dot{a} = 1, 2$ are indices of $Spin(4) \cong SU(2)_a \times SU(2)_{\dot{a}}$ and the subscript $\pm$ on fermions denotes the $\pm\frac{1}{2}$ representation under the $U(1)_{56}$ which acts as the Lorentz group of the D1-brane worldvolume theory.

The $SU(2)_a \times SU(2)_{\dot{a}}$ rotates the two planes of the $\mathbb{C}^2$ parametrised by z_2, z_3 into one another. The Cartans of $\mathfrak{su}(2)_a, \mathfrak{su}(2)_{\dot{a}}$ j_L, j_R may be expressed in terms of the generators J_{710} and J_{89} of $U(1)$ rotations in their respective planes as

$$j_L = \frac{1}{2}\left(J_{710} - J_{89}\right)\,, \quad j_R = \frac{1}{2}\left(J_{710} + J_{89}\right)\,, \tag{4.5}$$

which are defined such that lower $a = 1, 2$ have $j_L = +\frac{1}{2}, -\frac{1}{2}$ and lower $\dot{a} = \dot{1}, \dot{2}$ have $j_R = +\frac{1}{2}, -\frac{1}{2}$. Hence the Γ action on the supercharges is

$$\Gamma : \left(\mathrm{Q}_+^{\alpha a}, \mathrm{Q}_-^{\alpha\dot{a}}\right) \mapsto \omega_\ell^{2j_L}\omega_k^{J_{56}+j_L-j_R}\left(\mathrm{Q}_+^{\alpha a}, \mathrm{Q}_-^{\alpha\dot{a}}\right)\,, \tag{4.6}$$

$$\Gamma : \left(\widetilde{\mathrm{Q}}_+^{\dot{\alpha}\dot{a}}, \widetilde{\mathrm{Q}}_-^{\dot{\alpha}a}\right) \mapsto \omega_\ell^{2j_L}\omega_k^{J_{56}+j_L-j_R}\left(\widetilde{\mathrm{Q}}_+^{\dot{\alpha}\dot{a}}, \widetilde{\mathrm{Q}}_-^{\dot{\alpha}a}\right). \tag{4.7}$$

Hence, the supercharges which survive the orbifold action are $\mathrm{Q}_-^{\alpha\dot{1}}$ for $\sigma = +1$ or $\widetilde{\mathrm{Q}}_+^{\dot{\alpha}\dot{2}}$ for $\sigma = -1$. We will use one of them to compute the superconformal index in the next section.

4.3 Instantons for 4d $\mathcal{N} = 2^*$ SYM

In this section we warm up for our main calculation that we perform in the next section by reproducing the well known instanton partition function of $\mathcal{N} = 2^*$ via a 2d superconformal index calculation. We parameterise our partition function and use a supercharge that survives the orbifold projection (4.6) (4.7) so that we are well prepared for the next section.

As discussed in Section 1.3, there is a correspondence between the ADHM construction of instantons [29] and D$(p-4)$-branes [35, 36, 37, 24, 38]. Since the class $\mathcal{S}_k$ gauge theories of interest in this chapter may be realised within Type-II string theory as a theory living on the worldvolume of Dp-branes we may use this correspondence to derive its ADHM construction. The case that interests us is the case $p = 5$, i.e. D5-branes on $\mathbb{R}^4 \times T^2$, thus we have to compute the partition function of the 2d gauge theory living on the world volume D1-branes wrapping a T^2. This partition function is the 2d superconformal index a.k.a. flavoured elliptic genus.

4.3.1 D1 Worldvolume Theory

Before discussing the supersymmetric index we must first discuss the worldvolume theory living on the D1-branes in the low energy limit in the presence of the D5s.

D1-D1 The theory arising from quantising open strings stretching between K parallel and coincident Dp-branes is given by $p+1$ dimensional Yang-Mills theory with 16 supercharges, for $p = 1$ that is the well known $\mathcal{N} = (8, 8)$ SYM theory. In terms of multiplets under the $\mathcal{N} = (4, 4)$ subalgebra given by $\widetilde{\mathrm{Q}}_{+}^{\dot{\alpha}\dot{a}}, \widetilde{\mathrm{Q}}_{-}^{\dot{\alpha}a}$ they form a $\mathcal{N} = (4, 4)$ vector multiplet V and hypermultiplet H, which can be thought of as the reduction to 2d of a 4d $\mathcal{N} = 2$ vector multiplet and hypermultiplet respectively. V contains a 2d gauge field $A_{\pm}$, four scalars degrees of freedom $Y_{a\dot{a}}$, right moving fermions $\overline{\lambda}_{+}^{\dot{\alpha}a}$ and left moving fermions $\overline{\xi}_{-}^{\dot{\alpha}\dot{a}}$. H contains scalars $X_{\alpha\dot{\alpha}}$, right moving fermions $\xi_{+}^{\alpha\dot{a}}$ and left moving fermions $\lambda_{-}^{\alpha a}$.

D1-D5 Open D1-D5 strings preserve $\mathcal{N} = (4, 4)$ supersymmetry and gives rise to a $\mathcal{N} = (4, 4)$ hypermultiplet U in the bifundamental representation of $U(K) \times SU(N)$. U contains two complex scalars $\phi^{\dot{\alpha}}$ and their conjugates $\phi^{\dagger}_{\dot{\alpha}}$, and fermions $\chi_{+}^{\dot{a}}$, ψ_{-}^{a} plus their conjugates $\chi^{\dagger}_{+\dot{a}}$, $\psi^{\dagger}_{-a}$. Finally, the field content may be conveniently summarised in the quiver diagram of Figure 4.3. There, using $\mathcal{N} = (4, 4)$ notation, solid lines denote hypermultiplets, while the circular node denotes the $U(K)$ vector multiplet.

ADHM Equations We can easily obtain the BPS equations on the Higgs-branch of this system. As we elucidated in Section 1.3 these translate to the ADHM equations (1.26) $F = \mu_{\mathbb{C}} = 0$, $D = \mu_{\mathbb{R}} - \zeta \mathbb{I}_K = 0$. The Higgs branch is reached by setting $Y_{a\dot{a}} = 0$.

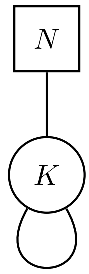


Figure 4.3: *The $\mathcal{N} = (4,4)$ 2d quiver of the gauge theory on K D1-branes in the presence of N D5-branes.*

Before the $\mathbb{Z}_k \times \mathbb{Z}_\ell$ orbifolding the non-zero F- and D-term equations on the Higgs-Branch read

$$D + \zeta \mathbb{I}_K = \mu_{\mathbb{R}} = JJ^\dagger - I^\dagger I + [B_1, B_1^\dagger] + [B_2, B_2^\dagger] = \zeta \mathbb{I}_K \,, \tag{4.8}$$

$$F = \mu_{\mathbb{C}} = JI + [B_1, B_2] = 0 \,. \tag{4.9}$$

where $J = \phi^{\dot{1}}$, $I = \overline{\phi}^{\dot{1}}$, $B_1 = X^{1\dot{1}}$ and $B_2 = X^{2\dot{1}}$. The orbifolding acts rather simply on those fields since they all have $j_L = J_{56} = 0$. The only non-trivial action is due to the gauge (4.64) and flavour (1.183) holonomies. We have

$$D_{ni} + \zeta \mathbb{I}_{K_{ni}} = \mu_{\mathbb{R}}^{(ni)} = J_{ni} J_{ni}^\dagger - I_{ni}^\dagger I_{ni} + [B_{1,ni}, B_{1,ni}^\dagger] + [B_{2,ni}, B_{2,ni}^\dagger] \,, \tag{4.10}$$

$$F_{ni} = \mu_{\mathbb{C}}^{(ni)} = I_{ni} J_{ni} + [B_{1,ni}, B_{2,ni}] \,. \tag{4.11}$$

The Higgs-branch, or equivalently the $\{K_{11}, \ldots, K_{\ell k}\}$-instanton moduli space is then

$$\begin{aligned} &\mathbf{M}_{\{K_{ni}\}} \cong \mathbf{HB} \\ &= \left\{ B_{1,ni} \,, B_{2,ni} \,, I_{ni} \,, J_{ni} \,|\, \mu_{\mathbb{R}}^{(ni)} = \zeta \mathbb{I}_{K_{ni}} \,, \mu_{\mathbb{C}}^{(ni)} = 0 \right\} / \left(\prod_{i,n} U(K_{ni}) \right) \,. \end{aligned} \tag{4.12}$$

4.3.2 Elliptic Genus Computation

We now turn to the computation of the supersymmetric index a.k.a flavoured elliptic genus partition function for our $\mathcal{N} = (4,4)$ theory. The supersymmetric index can be understood as the Witten index of the theory quantised on $\mathbb{S}^1 \times \mathbb{R}$ refined by fugacities which keep track of further relevant quantum numbers and it is independent of the coupling constants of the theory. Since our theory admits a free field limit, computing the index is equivalent to enumerating all gauge invariant operators of the theory on $\mathbb{R}^2$ [211, 222, 212, 213, 214, 215, 216]. For theories with a Lagrangian description the index can also be obtained using localisation techniques [223, 224] and explicitly performing the path integral of the 2d theory on T^2, however for simplicity we will follow the former approach.

We also choose to view our $\mathcal{N} = (4,4)$ theory as an $\mathcal{N} = (0,2)$ theory with additional flavour symmetry. We choose the $\mathcal{N} = (0,2)$ supercharges to be

$$\mathrm{Q} := \widetilde{\mathrm{Q}}_{+}^{\dot{2}\mathrm{i}}\,, \quad \widetilde{\mathrm{Q}} := \widetilde{\mathrm{Q}}_{+}^{\mathrm{i}\dot{2}}\,. \tag{4.13}$$

2d $\mathcal{N} = (0,2)$ theories have a single right moving $U(1)_{\mathfrak{R}}$ R-symmetry. The $\mathcal{N} = (0,2)$ IR R-symmetry for this model was computed in [225] and it is given by[1]

$$\mathfrak{R}_{\mathrm{IR}} = -2j_2\,, \tag{4.14}$$

under which $\mathfrak{R}_{\mathrm{IR}}[\mathrm{Q}] = -1$ and $\mathfrak{R}_{\mathrm{IR}}[\widetilde{\mathrm{Q}}] = +1$. We will compute the index which counts cohomology classes of $\widetilde{\mathrm{Q}}$. Since $\widetilde{\mathrm{Q}}$ and its conjugate $\widetilde{\mathrm{Q}}^\dagger = \widetilde{\mathrm{S}}$ commutes with $SU(2)_\alpha$ and $SU(2)_a$ we may include fugacities v and w for their Cartans. Furthermore, they also commute with the diagonal subgroup $SU(2)_D \subset SU(2)_{\dot{\alpha}} \times SU(2)_{\dot{a}}$ hence we also include a fugacity z for its Cartan $j_D = j_2 + j_R$. Recall that[2] the Cartans of $\mathfrak{su}(2)_\alpha, \mathfrak{su}(2)_{\dot{\alpha}}, \mathfrak{su}(2)_a$ and $\mathfrak{su}(2)_{\dot{a}}$ all commute with the orbifold and the fugacities v, w and z that we introduced here will still be meaningful for our calculation in the next section. We also include fugacities x_A for the Cartans f_A of $\mathfrak{su}(N)$ and y_I for the Cartans g_I of $\mathfrak{u}(K)$. The index is then defined as

$$Z_K^{6d}(q,v,w,z,x_A) = \mathrm{Tr}(-1)^F q^{H_-} \overline{q}^{\delta} v^{2j_1} w^{2j_L} z^{2j_2+2j_R} \prod_{A=1}^{N} x_A^{f_A} \prod_{I=1}^{K} y_I^{g_I} \tag{4.15}$$

where $F = F_- + F_+$ is the fermion number and H_-, H_+ are the left and right moving Hamiltonians, respectively. In Euclidean signature we define $2H_\pm = H \mp \mathrm{i}P$ and $q := e^{2\pi\mathrm{i}\tau}$ with τ the complex structure of the T^2 is generated by $\omega \sim \omega + 1 \sim \omega + \tau$. Explicitly, we will work with the square torus with complex structure $\tau = \mathrm{i}\beta_6/\beta_5$ with β_5, β_6 the radii of the two $\mathbb{S}^1$ factors. In radial quantisation the conformal map from the plane to the cylinder is $z_1 = e^{2\pi\mathrm{i}\omega}$, where $\omega := \sigma + \mathrm{i}t$ and Lorentz

[1] Our D5/D1 setup is precisely that of [225] with $Q_5^+ = N$, $Q_5^- = 0$, $R^- = -2j_2$ and $R^+ = -2j_R$.

[2] The Cartans of $\mathfrak{su}(2)_a$ and $\mathfrak{su}(2)_{\dot{a}}$ denoted as j_L and j_R can be written in terms of the generators J_{710} and J_{89} which are the $U(1)$ rotations in the respective planes (4.5).

transformations $z_1 \mapsto e^{\mathrm{i}\theta} z_1$ are then mapped to translations around the $\mathbb{S}^1$ factor of the cylinder

$$(\sigma, t) \mapsto \left(\sigma + \frac{\theta}{2\pi}, t\right) \tag{4.16}$$

generated by $P = iJ_{56}$.

One of the crucial properties of the quantity (4.15) is that it receives contributions only from those states which satisfy

$$\delta = \left\{\widetilde{\mathrm{Q}}, \widetilde{\mathrm{S}}\right\} = H_+ - \frac{1}{2}\mathfrak{R} = 0 \tag{4.17}$$

where $\mathfrak{R}$ is the $\mathcal{N} = (0, 2)$ R-symmetry; the index is independent of $\overline{q}$. Furthermore the index (4.15) is also independent of all continuous parameters such as coupling constants and Fayet-Iliopoulos parameters [65, 62], hence we can compute the index in the free field limit where it reduces to a counting problem.

In the free field limit we have a $\mathcal{N} = (0, 2)$ superconformal theory with $\mathfrak{Vir} \oplus \overline{\mathfrak{sVir}}_{NS}$ symmetry where $\mathfrak{Vir}$ is the standard ($\mathcal{N} = 0$) left-moving Virasoro algebra generated by $\{L_n, c\}$, $\overline{\mathfrak{sVir}}_{NS}$ is the $\mathcal{N} = 2$ super-Virasoro algebra in the NS sector generated by $\left\{\overline{L}_n, \overline{G}^{\pm}_r, \overline{J}_n, \overline{c}\right\}$ and $n, r + \frac{1}{2} \in \mathbb{Z}$. Our choice of the Neveu-Schwarz basis over the Ramond basis is purely for calculational convenience and the index is independent of this choice up to an overall factor [216]. We will require the following brackets of the $\overline{\mathfrak{sVir}}_{\mathcal{N}=2,NS}$ algebra:

$$\left\{\overline{G}^+_r, \overline{G}^-_s\right\} = \overline{L}_{r+s} + \frac{1}{2}(r - s)\overline{J}_{r+s} + \frac{\overline{c}}{6}\left(r^2 - \frac{1}{4}\right)\delta_{r+s,0}\,, \tag{4.18}$$

$$\left[\overline{L}_0, \overline{G}^{\pm}_r\right] = -r\overline{G}^{\pm}_r\,, \quad \left[\overline{J}_0, \overline{G}^{\pm}_r\right] = \pm\overline{G}^{\pm}_r\,. \tag{4.19}$$

In the free field limit, we identify

$$H_- = L_0, \quad H_+ = \overline{L}_0\,, \quad \mathfrak{R} = \overline{J}_0\,, \tag{4.20}$$

$$\mathrm{Q} = \overline{G}^-_{-\frac{1}{2}}\,, \quad \mathrm{S} = \overline{G}^+_{+\frac{1}{2}}, \quad \widetilde{\mathrm{Q}} = \overline{G}^+_{-\frac{1}{2}}\,, \quad \widetilde{\mathrm{S}} = \overline{G}^-_{+\frac{1}{2}}\,. \tag{4.21}$$

Away from the free limit, the theory is not conformal and we have an RG flow from the free UV fixed point to an IR fixed point. This IR fixed point is expected to be precisely the ADHM sigma model with target space the instanton moduli space (4.12). The R-charge assignments generally change along RG flow. Nonetheless, the index is RG invariant and we can evaluate the index at the IR fixed point by using the non-anomalous R-symmetry assignment in the IR which, in our case, is (4.14).

At the UV fixed point the shortening condition (4.17) can be written as

$$\delta = \left\{\overline{G}^{+}_{-\frac{1}{2}}, \overline{G}^{-}_{+\frac{1}{2}}\right\} = \overline{L}_0 - \frac{1}{2}\overline{J}_0 = 0\,. \tag{4.22}$$

and the states contributing to the index must have $\overline{J}_0 = 2\overline{L}_0$ in the UV. The IR R-symmetry (4.14) is then taken into account by shifting $q^{L_0} \to q^{L_0 - \overline{L}_0 + \frac{1}{2}\mathfrak{R}_{\mathrm{IR}}}$ in the index.

Letter Counting

As stressed earlier the index may be computed in the free field limit. This is done by identifying all 'letters' with $\delta = 0$. The single letter partition functions for the $\mathcal{N} = (4,4)$ multiplets may be easily read off from Tables E.1, E.2, E.3 in Appendix E.1. They are given by

$$i_V(q,w,z,y_I) = \left[\frac{\left(w+w^{-1}\right)\left(z+qz^{-1}\right) - qz^{-2} - z^2 - 2q}{1-q}\right]\sum_{I,J=1}^{K} y_I y_J^{-1}\,, \tag{4.23}$$

$$i_H(q,v,w,z,y_I) = \left[\frac{q^{\frac{1}{2}}\left(v+v^{-1}\right)\left(z+z^{-1}-w^{-1}-w\right)}{1-q}\right]\sum_{I,J=1}^{K} y_I y_J^{-1}\,, \tag{4.24}$$

$$i_U(q,w,z,x_A,y_I) = \left[\frac{q^{\frac{1}{2}}\left(z+z^{-1}-w^{-1}-w\right)}{1-q}\right]\sum_{I=1}^{K}\sum_{A=1}^{N}\left(\frac{y_I}{x_A}+\frac{x_A}{y_I}\right)\,. \tag{4.25}$$

The full index is then by enumerating all possible 'words' and then projecting onto gauge singlets by integrating over the Haar measure $d\mu_G$ of the $G = U(K)$ gauge group, given in (A.38). The full index is then given by

$$Z_K^{6d}(q,v,w,z,x_A) = \oint d\mu_{U(K)}\, Z^{(0)} \prod_{P=V,H,U} Z_P \tag{4.26}$$

where $Z^{(0)}$ is the Casimir contribution which may, apriori, depend on all fugacities. It is given by [81, 226, 227, 5, 82, 228, 229, 56]

$$Z^{(0)} \equiv Z^{(0)}(q,v,w,z) = q^{\frac{1}{2}E_{\mathrm{Casimir}}}\,, \quad E_{\mathrm{Casimir}} = \underset{q\to 1}{\mathrm{Finite}}\left[\sum_P \frac{\partial i_P}{\partial \log q}\right]\,. \tag{4.27}$$

and

$$Z_P \equiv Z_P(q,v,w,z,x_A,y_I) := \mathrm{PE}\left[i_P(q,v,w,z,x_A,y_I)\right]\,, \tag{4.28}$$

where PE denotes the Plethystic exponential; defined in (A.1) and i_P are the single letter partition functions (4.23), (4.24) and (4.25). Explicitly:

$$Z_V(q,w,z,y_I) = \prod_{I,J=1}^{K} \frac{\left(q\frac{y_I}{y_J};q\right)^2 \theta\left(qz^{-2}\frac{y_I}{y_J};q\right)}{\theta\left(wz\frac{y_I}{y_J};q\right)\theta\left(qwz^{-1}\frac{y_I}{y_J};q\right)}\,, \tag{4.29}$$

$$Z_H(q,v,w,z,y_I) = \prod_{I,J=1}^{K} \frac{\theta\left(q^{\frac{1}{2}}vw\frac{y_I}{y_J};q\right)\theta\left(q^{\frac{1}{2}}v^{-1}w\frac{y_I}{y_J};q\right)}{\theta\left(q^{\frac{1}{2}}vz^{-1}\frac{y_I}{y_J};q\right)\theta\left(q^{\frac{1}{2}}v^{-1}z^{-1}\frac{y_I}{y_J};q\right)}\,, \tag{4.30}$$

$$Z_U(q,w,z,x_A,y_I) = \prod_{I=1}^{K}\prod_{A=1}^{N} \frac{\theta\left(q^{\frac{1}{2}}w\frac{x_A}{y_I};q\right)\theta\left(q^{\frac{1}{2}}w\frac{y_I}{x_A};q\right)}{\theta\left(q^{\frac{1}{2}}z^{-1}\frac{x_A}{y_I};q\right)\theta\left(q^{\frac{1}{2}}z^{-1}\frac{y_I}{x_A};q\right)}\,, \tag{4.31}$$

where $\theta(x;q)$ and $(x;q)$ are the q-theta function and q-Pochammer symbol; defined in (A.13) and (A.12) respectively. Finally, we conclude that the full index is given by

$$\begin{aligned} Z_K^{6d} =& \frac{(q;q)^{2K}}{K!} \oint_{T(G)} \prod_{I=1}^{K}\frac{dy_I}{2\pi \mathrm{i} y_I} \prod_{I=1}^{K}\prod_{A=1}^{N} \frac{\theta\left(q^{\frac{1}{2}}w\frac{x_A}{y_I};q\right)\theta\left(q^{\frac{1}{2}}w\frac{y_I}{x_A};q\right)}{\theta\left(q^{\frac{1}{2}}z^{-1}\frac{x_A}{y_I};q\right)\theta\left(q^{\frac{1}{2}}z^{-1}\frac{y_I}{x_A};q\right)} \\ &\times Z^{(0)} \prod_{I,J=1}^{K} \frac{\theta\left(\frac{y_I}{y_J};q\right)'\theta\left(qz^{-2}\frac{y_I}{y_J};q\right)\theta\left(q^{\frac{1}{2}}vw\frac{y_I}{y_J};q\right)}{\theta\left(wz\frac{y_I}{y_J};q\right)\theta\left(qwz^{-1}\frac{y_I}{y_J};q\right)\theta\left(q^{\frac{1}{2}}vz^{-1}\frac{y_I}{y_J};q\right)} \\ &\times \prod_{I,J=1}^{K} \frac{\theta\left(q^{\frac{1}{2}}v^{-1}w\frac{y_I}{y_J};q\right)}{\theta\left(q^{\frac{1}{2}}v^{-1}z^{-1}\frac{y_I}{y_J};q\right)} \end{aligned} \tag{4.32}$$

where the prime means to remove those factors $\theta(1;q)$ and we used the identity $(x;q) = (1-x)\,(qx;q)$. It also useful to assemble the grand partition function

$$Z^{6d}(q,v,w,z,x_A;\mathbf{q}_{6d}) := \sum_{K\geqslant 0} \mathbf{q}_{6d}^K Z_K^{6d}(q,v,w,z,x_A) \tag{4.33}$$

with $\mathbf{q}_{6d}$ a formal dimensionless parameter. When considering our 6d theory on $\mathbb{R}^4\times\mathbb{S}^1_5\times\mathbb{S}^1_6$ as a 5d theory on $\mathbb{R}^4\times\mathbb{S}^1_5$ dressed by KK modes along $\mathbb{S}^1_6$ we may regard $\mathbf{q}_{6d}$ as a fugacity for the topological $U(1)$ global symmetry associated to the conserved current $\star_{5d}J = \frac{1}{8\pi^2}\,\mathrm{tr}\,F\wedge F$.

4.3.3 The 6d Instanton Partition Function

The countour integrals (4.32) may be computed via the Jefferey-Kirwan residue prescription [230, 223, 224]. Using (A.17), we can perform the residue prescription. The integrand of (4.33) has simple poles at

$$y_I = y_J\,(zw)^{\pm 1}\,, \quad y_I = y_J\left(\frac{z}{qw}\right)^{\pm 1}\,, \tag{4.34}$$

$$y_I = y_J\left(\frac{vz}{q^{1/2}}\right)^{\pm 1}\,, \quad y_I = y_J\left(\frac{z}{vq^{1/2}}\right)^{\pm 1}\,, \quad y_I = x_A\left(\frac{z}{q^{1/2}}\right)^{\pm 1}\,. \tag{4.35}$$

As explained in [89, 41] only residues arising from the poles (4.35) should be kept. We assume that the x_A's are sufficiently generic and furthermore we close the contour such that we collect residues coming from poles with the positive sign exponents. The solutions to (4.35) may be classified by N-coloured Young's diagrams $\vec{\mu} = \{\mu_1, \ldots, \mu_N\}$ with each diagram μ_A containing $|\mu_A|$ boxes such that $|\vec{\mu}| := \sum_A |\mu_A| = K$. Given a Young's diagram μ_A a box s is labelled by coordinates (l, p) and the corresponding pole is given by

$$y(s) = x_A\left(\frac{z}{q^{1/2}}\right)^{l+p-1} v^{l-p}\,. \tag{4.36}$$

The residue for a fixed coloured Young diagram is then

$$Z^{6d}_{\vec{\mu}} = Z^{(0)} \prod_{A,B=1}^{N} \prod_{s\in\mu_A} \frac{\theta\left(q^{-1}zw^{-1}\mathcal{E}_{AB};q\right)\theta\left(zw\mathcal{E}_{AB};q\right)}{\theta\left(\mathcal{E}_{AB};q\right)\theta\left(q^{-1}z^2\mathcal{E}_{AB};q\right)}\,, \tag{4.37}$$

where we defined

$$\mathcal{E}_{BA} := \frac{x_B}{x_A}\left(\frac{vz}{q^{1/2}}\right)^{L_A(s)}\left(\frac{q^{1/2}v}{z}\right)^{A_B(s)+1} \tag{4.38}$$

and where $L_B(s)$ and $A_B(s)$ denote the distance from the box s to the right end and the bottom of the Young diagram μ_B respectively. $Z^{(0)} = q^{\frac{1}{2}E_{\text{Casimir}}}$ is the Casimir contribution (4.27). To compute it one is forced to specify the q-dependence of the fugacities, we hence define

$$v := q^{\frac{\beta_5\epsilon_-}{2\mathrm{i}\pi}}\,, \quad wq^{\frac{1}{2}} := q^{\frac{\beta_5 m}{\mathrm{i}\pi}}\,, \quad zq^{-\frac{1}{2}} := q^{\frac{\beta_5\epsilon_+}{2\mathrm{i}\pi}}\,, \quad x_A := q^{\frac{\beta_5 a_A}{\mathrm{i}\pi}}\,, \tag{4.39}$$

where we used the shorthand notation

$$\epsilon_\pm := \epsilon_1 \pm \epsilon_2 \,. \tag{4.40}$$

$Z^{(0)}$ is then a constant and is given by

$$Z^{(0)} = q^{\frac{\beta_5^2 NK}{\pi^2}\left(\frac{\epsilon_+}{2} - m + \frac{\mathrm{i}\pi}{\beta_5}\right)\left(\frac{\epsilon_+}{2}+m\right)} \,. \tag{4.41}$$

The K instanton partition function (4.32) is then given by summing over all coloured Young diagrams $\vec{\mu}$. Hence, equation (4.33) finally reads

$$Z^{6d} := \sum_{K\geqslant 0} \mathbf{q}_{6d}^{K} \sum_{\substack{\vec{\mu} \\ |\vec{\mu}|=K}} Z^{6d}_{\vec{\mu}} = \sum_{\vec{\mu}} \mathbf{q}_{6d}^{|\vec{\mu}|} Z^{6d}_{\vec{\mu}} \,. \tag{4.42}$$

4.3.4 Reduction to Five and Four Dimensions

Reducing the D5/D1 system on $\mathbb{S}^1_6$ by taking $\beta_6 \to 0$ results in 5d $\mathcal{N} = 2$ SYM with gauge group $SU(N)$ at Chern-Simons level $\kappa = 0$. Hence by either taking the 5d limit directly to (4.42) or taking the limit directly the contour integral (4.32) we expect to obtain the instanton partition function for the 5d theory. Here we take the first approach but we detail the limit of the contour integral expression in Appendix E.2.1.

Recall that $q = e^{2\pi \mathrm{i}\tau}$ and $\tau = \mathrm{i}\beta_6/\beta_5$ therefore this limit corresponds to taking

$$q \to 1 \,. \tag{4.43}$$

Further note that, by definition (4.27),

$$\lim_{q\to 1} Z^{(0)} = q^{\frac{1}{2}E_{\text{Casimir}}} = 1 \,. \tag{4.44}$$

Applying (A.20) to (4.42) yields

$$Z^{5d} = \sum_{\vec{\mu}} \mathbf{q}_{5d}^{|\vec{\mu}|} Z^{5d}_{\vec{\mu}} \tag{4.45}$$

$$Z^{5d}_{\vec{\mu}} = \prod_{A,B=1}^{N} \prod_{s\in\mu_A} \frac{\sinh\beta_5\left(E_{AB} + \frac{\epsilon_+}{2} - m\right)\sinh\beta_5\left(E_{AB} + \frac{\epsilon_+}{2} + m\right)}{\sinh\beta_5\left(E_{AB}\right)\sinh\beta_5\left(E_{AB} + \epsilon_+\right)} \tag{4.46}$$

where we have defined

$$E_{BA} = a_B - a_A + \epsilon_1 L_A(s) - \epsilon_2 \left(A_B(s) + 1\right) \,, \tag{4.47}$$

such that $\mathcal{E}_{AB} = q^{\frac{E_{AB}}{\mathrm{i}\pi}}$ and we take $\mathbf{q}_{5d} = \lim_{q\to 1} \mathbf{q}_{6d}$. This reproduces the instanton partition for the mass deformed 5d $\mathcal{N}=2$ theory (a.k.a. $\mathcal{N}=1^*$) on $\mathbb{R}^4 \times \mathbb{S}^1_5$ in the Ω-background, which was computed via localisation of the path integral of the ADHM quantum mechanics in e.g. [40, 89, 231]. Hence we indeed identify

$$Z^{5d} = Z^{5d}_{\text{inst},\mathcal{N}=1^*} \,. \tag{4.48}$$

Armed with the above, the equivalence of Z^{5d} in the 4d limit ($\beta_5 \to 0$) with the instanton partition function $Z^{4d}_{\text{inst},\mathcal{N}=2^*}$ for the 4d $\mathcal{N}=2^*$ theory is essentially trivial to prove. Taking the $\beta_5 \to 0$ limit of (4.46) or equivalently evaluating the contour integrals (E.21) we obtain

$$Z^{4d} = \sum_{\vec{\mu}} \mathbf{q}_{4d}^{|\vec{\mu}|} Z^{4d}_{\vec{\mu}} = Z^{4d}_{\text{inst},\mathcal{N}=2^*} \tag{4.49}$$

$$Z^{4d}_{\vec{\mu}} = \prod_{A,B=1}^{N} \prod_{s\in\mu_A} \frac{\left(E_{AB} + \frac{\epsilon_+}{2} - m\right)\left(E_{AB} + \frac{\epsilon_+}{2} + m\right)}{E_{AB}\left(E_{AB} + \epsilon_+\right)} \tag{4.50}$$

we then identify m as the hypermultiplet mass in the Ω-background [232] and set $\mathbf{q}_{4d} = \lim_{\beta_5 \to 0} \mathbf{q}_{5d}$. It may be shown [52, 233, 234] that the vector multiplet contribution to the instanton partition function from the fixed point labelled by $\vec{\mu}$ is given by

$$z_{\text{vec}}(a, \vec{\mu}) = \prod_{A,B=1}^{N} \prod_{s\in\mu_A} \frac{1}{E_{AB}\left(E_{AB} + \epsilon_+\right)} \tag{4.51}$$

and the contribution from the adjoint hypermultiplet is

$$z_{\text{adj}}(a, m, \vec{\mu}) = \prod_{A,B=1}^{N} \prod_{s\in\mu_A} \left(E_{AB} + \frac{\epsilon_+}{2} - m\right)\left(E_{AB} + \frac{\epsilon_+}{2} + m\right) \tag{4.52}$$

also note that

$$z_{\text{adj}}\left(a, \frac{\epsilon_+}{2}, \vec{\mu}\right) = \frac{1}{z_{\text{vec}}(a, \vec{\mu})} . \tag{4.53}$$

4.4 Orbifolding to 4d $\mathcal{N} = 1, 2$ Periodic Quivers

The main goal of this chapter is to compute the 2d index in the presence of the $\Gamma = \mathbb{Z}_\ell \times \mathbb{Z}_k$ orbifold before reducing to the zero dimensional matrix model partition function which is expected to be equal to the partition function of instantons for the class $\mathcal{S}_k$ theory.

In principle we could work directly with the 2d orbifolded theory by working out the projections and writing down the Lagrangian and computing it's partition function. However we prefer instead to work with the SCI interpreted as a counting device to which we implement projection onto Γ-invariant states.

We take the same approach, as one takes when computing, e.g. the supersymmetric Lens space $L(1,r) \times \mathbb{S}^1$ indices [81, 235, 236, 237, 238].

4.4.1 Orbifolding the Supersymmetric Index

If we denote some (mother) theory by M we may obtain a new (daughter) theory $D = M/\Gamma$ by quotienting out by an orbifold group Γ which is generically embedded inside both the global symmetry group F_M and gauge group G_M of M. We collectively denote the generators of Γ by γ. If M is a supersymmetric theory with a supercharge $\mathfrak{Q}$, then it is possible to count cohomology classes of $\mathfrak{Q}$, i.e. to compute the supersymmetric index of M for the supercharge $\mathfrak{Q}$ which, providing that M admits a suitable free field limit such that standard letter counting techniques can be applied, is schematically defined to be

$$\mathcal{I}_M(a) = \mathrm{Tr}_{\mathcal{H}}(-1)^F e^{-\beta\{\mathfrak{Q},\mathfrak{Q}^\dagger\}} a^{f_M} \tag{4.54}$$

where f_M collectively denotes the subset of linearly independent generators of F_M such that $[\mathfrak{Q}, f_M] = 0$ and a their fugacities. If we assume that Γ is abelian and furthermore commutes with both $\mathfrak{Q}$ and its conjugate

$$[\mathfrak{Q}, \gamma] = [\mathfrak{Q}^\dagger, \gamma] = 0 \tag{4.55}$$

then the theory D will also generically possesses at least one supersymmetry, namely $\mathfrak{Q}$. D also has reduced global and gauge symmetry groups $F_D = C_{F_M}(\Gamma)$, $G_D = C_{G_M}(\Gamma)$ where $C_G(S)$ denotes the centraliser of S in G which, of course, depends on the choice of embedding $\Gamma \hookrightarrow F_M \times G_M$. One may then obtain the supersymmetric index of D for the supercharge $\mathfrak{Q}$ by means of projection:

$$\mathcal{I}_D(a) = \sum_\rho \mathrm{Tr}_{\mathcal{H}_\rho} \int [d\Gamma] \varepsilon^\gamma (-1)^F e^{-\beta\{\mathfrak{Q},\mathfrak{Q}^\dagger\}} a^f \,. \tag{4.56}$$

The 'integral' over the invariant Haar measure of the group Γ implements the projection onto Γ-invariant states. When Γ is discrete and abelian the Haar measure is simply given by summing over all elements of the group and dividing by the number of elements of the group

$$\int [d\Gamma] = \frac{1}{|\Gamma|}\sum_{\varepsilon\in\Gamma} \,. \tag{4.57}$$

Since $\mathcal{H}$ is a Hilbert space with grading by global symmetries, it may be decomposed $\mathcal{H} = \oplus_\rho \mathcal{H}_\rho$ according to the Γ action. To include states which can also be twisted in the 'time' direction we must also sum over different vacuua $\mathcal{H}_\rho$. This definition automatically receives contributions from both untwisted sectors as well as sectors which may be twisted by global or gauge symmetries. Note that in the computation of $\mathcal{I}_D$, since G_M was gauged, one should 'integrate' over all independent (up to $G_M/C_{G_M}(\Gamma)$ gauge transformations) embeddings $\Gamma \hookrightarrow G_M$. Note also that in some cases one may also choose to instead use a weighted sum over embeddings, for example discrete theta angles [236]. On the other hand F_M is not gauged and hence one should fix a particular embedding $\Gamma \hookrightarrow F_M$ which in turn specifies the global symmetry of the daughter theory D.

Toy Example - Orbifold Index of a Free Fermi Multiplet

Let us proceed with a simple toy example. Staying in 2d, since that is the most relevant for us, we let M be the theory of a free $\mathcal{N} = (0,2)$ Fermi multiplet $\psi_-, \overline{\psi}_-$. Both left moving fermions have $\overline{L}_0 = \overline{J}_0 = 0$ and hence both satisfy the BPS condition $\delta = \left\{\mathfrak{Q}, \mathfrak{Q}^\dagger\right\} = 0$ (4.22) furthermore they both have $L_0 = \frac{1}{2}$. The free Fermi multiplet admits a $U(1)_f$ flavour symmetry generated by f under which ψ_-, $\overline{\psi}_-$ have charges $f = +1, -1$. The total global symmetry is then $F_M = U(1)_f \times \mathrm{Iso}(T^2) \times U(1)_{\mathfrak{R}}$. In particular $\mathbb{R}^2 \subset \mathfrak{iso}(T^2)$ is the algebra of translations around the two cycles of the torus, where $\overline{L}_0 - L_0$ generates radial translations and $\overline{L}_0 + L_0$ generate time translations. $U(1)_{\mathfrak{R}}$ is the R-symmetry generated by $\overline{J}_0 = \mathfrak{R}$ under which $\mathfrak{Q}, \mathfrak{Q}^\dagger$ have charges $\mathfrak{R} = -1, +1$ as before. Enumerating the letters of Table

Letter	L_0	$\overline{L}_0$	$\overline{J}_0$	f	Index	Orb Index
ψ_-	1/2	0	0	+1	$q^{\frac{1}{2}}a$	$-q^{1/2}a$
$\overline{\psi}_-$	1/2	0	0	−1	$q^{1/2}a^{-1}$	$-\varepsilon^{-1}q^{1/2}a^{-1}$
∂_-	1	0	0	0	q	$\varepsilon^{-1}q$

Table 4.2: *Letters of the Fermi multiplet.*

4.2 we obtain

$$\mathcal{I}_M(a,q) = \mathrm{Tr}_{\mathcal{H}}(-1)^F q^{L_0} a^f = \mathrm{PE}\left[-\frac{q^{\frac{1}{2}}\left(a+a^{-1}\right)}{1-q}\right] = \theta\left(aq^{\frac{1}{2}};q\right)\,, \tag{4.58}$$

this is simply counting, with signs, all operators of the form $\partial_-^n\psi_-^m\overline{\psi}_-^l$. We consider the theory obtained by the $\mathbb{Z}_k$ orbifold $D = M/\mathbb{Z}_k$ where $\mathbb{Z}_k$ has an action inside all three factors of F_M. To preserve the supercharge $\mathfrak{Q}$ we require that $\mathbb{Z}_k$ translation in $\mathrm{Iso}(T^2)$ and rotation by $U(1)_{\mathfrak{R}}$ acts by equal but opposite amounts. Hence the $\mathbb{Z}_k$ group elements are of the form

$$e^{\frac{2\pi \mathrm{i}}{k}\gamma} \in \mathbb{Z}_k\,, \quad \gamma = \frac{f}{2} + \left(\overline{L}_0 - L_0\right) + \frac{\mathfrak{R}}{2}\,. \tag{4.59}$$

Since the quotient acts in the radial direction only $\mathcal{I}_D$ is simply obtained by taking the Pleythistic exponent of the orbifolded single letter index

$$\begin{aligned} i(a,q) =& \frac{1}{k}\sum_{\varepsilon\in\mathbb{Z}_k}\left[-\left(\varepsilon^{-\frac{1}{2}}q^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}a + \varepsilon^{-\frac{1}{2}}q^{\frac{1}{2}}\varepsilon^{-\frac{1}{2}}a^{-1}\right)\right]\sum_{n\geqslant 0}\varepsilon^{-n}q^n & (4.60)\\ =& \frac{1}{k}\sum_{\varepsilon\in\mathbb{Z}_k}\left[-q^{\frac{1}{2}}a - q^{-\frac{1}{2}}a^{-1}\right]\sum_{\tilde{n}\geqslant 0}q^{k\tilde{n}}\sum_{j=1}^{k-1}\varepsilon^{-j}q^j + q^{-\frac{1}{2}}a^{-1} & (4.61)\\ =& -\frac{aq^{\frac{1}{2}} + a^{-1}q^{k-\frac{1}{2}}}{1-q^k} & (4.62)\end{aligned}$$

where we used the basic fact that $\sum\limits_{\varepsilon\in\mathbb{Z}_k}\varepsilon = 0$. The index for the theory D for the supercharge $\mathfrak{Q}$ is then

$$\mathcal{I}_D(a,q) = \mathrm{PE}\left[i(a,q)\right] = \mathrm{PE}\left[-\frac{aq^{\frac{1}{2}} + a^{-1}q^{k-\frac{1}{2}}}{1-q^k}\right] = \theta\left(aq^{\frac{1}{2}};q^k\right)\,. \tag{4.63}$$

The counted operators are simply $\partial_-^{nk}\psi_-^m(\partial_i\overline{\psi}_-)^l$. We now move to apply this general discussion to the D1-brane worldvolume theory.

4.4.2 Computing the Orbifolded Elliptic Genus

The orbifold acts on the coordinates $X^1, \ldots, X^{10}$ as in (1.180). The orbifold action is also embedded in the $SU(\ell kN)$ flavour group as in (1.183), where we also scaled $N \to \ell kN$ with respect to the previous section. Furthermore, the orbifold also has an action inside the $U(K)$ gauge group of the D1 worldvolume theory breaking $U(K) \to \prod_{n=1}^{\ell}\prod_{i=1}^{k} U(K_{ni})$, $\sum_{n,i} K_{ni} = K$ and $U(0)$ is defined to be the trivial group. As in equation (1.183) the action may be conjugated to an element of the maximal torus $g \in T(U(K))$

$$g = \operatorname{diag}\left(\omega_\ell \omega_k^{-1}\mathbb{I}_{K_{11}}, \ldots, \omega_\ell\omega_k^{-k}\mathbb{I}_{K_{1k}}, \ldots, \omega_\ell^{\ell}\omega_k^{-1}\mathbb{I}_{K_{\ell 1}}, \ldots, \omega_\ell^{\ell}\omega_k^{-k}\mathbb{I}_{K_{\ell k}}\right). \tag{4.64}$$

The only difference is that we do not fix K_{ni} but rather 'integrate' over all possible K_{ni} satisfying $\sum_{n,i} K_{ni} = K$ which indeed are in one-to-one correspondence with embeddings $\Gamma \hookrightarrow U(K)$ up to gauge transformations. In the language of the discussion in Section 4.4.1 we consider $G_M = U(K)$, $F_M = SU(\ell kN) \times \mathrm{Iso}\left(T^2\right) \times Spin(4)_R$ and $\Gamma = \mathbb{Z}_\ell \times \mathbb{Z}_k$. Then $C_{SU(\ell kN)}(\Gamma)$[3] coincides with $SU(N)^{\ell k}$. On the other hand $C_{G_M}(\Gamma) = \prod_{n,i} U(K_{ni})$ corresponding to the (unordered) partition $K = K_{11} + \cdots + K_{\ell k}$ but since G_M was gauged we sum over all partitions.

For convenience we choose to split the Cartans f_A, $A = 1, \ldots, \ell kN$ of $\mathfrak{su}(\ell kN)$ into Cartans $f_{ni,A}$, $A = 1, \ldots, N$ of $\oplus_{ni}\mathfrak{su}(N)$ we also do the same with the $\mathfrak{u}(K)$ Cartan generators g_I, $I = 1, \ldots, K$ into Cartans $g_{ni,I}$, $I = 1, \ldots, K_{ni}$ of $\oplus_{ni}\mathfrak{u}(K_{ni})$. Following the above discussion, recalling that the supercharge $\widetilde{\mathrm{Q}}$ (given in (4.13)) and its conjugate $\widetilde{\mathrm{S}}$ commute with the orbifold action, we compute

$$\begin{aligned} Z_K^{6d,\ell,k}(q,v,w,z,x_{ni,A}) = \operatorname{Tr}\Bigg[& \frac{1}{\ell k}\sum_{\substack{\varepsilon_\ell\in\mathbb{Z}_\ell\\ \varepsilon_k\in\mathbb{Z}_k}} \varepsilon_\ell^{\gamma_\ell}\varepsilon_k^{\gamma_k}\prod_{n,i}\prod_{A=1}^{N}\left(\varepsilon_\ell^n\varepsilon_k^i\right)^{f_{ni,A}} \\ & \times\prod_{I=1}^{K_{ni}}\left(\varepsilon_\ell^n\varepsilon_k^i\right)^{g_{ni,I}}(-1)^F q^{H_-}v^{2j_1}w^{2j_L}z^{2j_R+2j_2}\prod_{n,i}\prod_{A=1}^{N}x_{ni,A}^{f_{ni,A}}\prod_{I=1}^{K_{ni}}y_{ni,I}^{g_{ni,I}}\Bigg], \end{aligned} \tag{4.65}$$

where the first line corresponds to the projection operator $\int[d\Gamma]\,\varepsilon^\gamma$ of (4.56) implementing the Douglas-Moore orbifold procedure with:

$$\gamma_\ell = J_{710} - J_{89} := 2j_L\,, \quad \gamma_k := J_{56} - J_{89} = J_{56} + j_L - j_R\,. \tag{4.66}$$

[3]Note that here $C_{SU(\ell kN)}(\Gamma)$ is equivalent to the Levi subgroup $\mathbb{L}$ specified by ℓ copies of (4.1).

Recall that in computing the index previously we mapped the plane to the cylinder $z_1 = e^{2\pi\mathrm{i}(\sigma+\mathrm{i}t)}$ hence, rotations of the plane $Z_1 \mapsto e^{\mathrm{i}\theta} z_1$ are mapped to translations $\sigma \mapsto \sigma + \frac{\theta}{2\pi}$ (4.16). Hence, quotienting out by rotations on the plane corresponds, after the conformal map, to quotienting out translations generated by $\overline{L}_0 - L_0$ on the torus.

We assume that the IR R-symmetry $\mathfrak{R}_{\mathrm{IR}}$ does not change under the orbifold; in which case, relegating the explicit derivation to Appendix E.1, we obtain the orbifolded single letter indices which are denoted by $i_{\Gamma V}$, $i_{\Gamma H}$, $i_{\Gamma U}$ and are given by equations (E.11), (E.12), (E.13) respectively. In addition, for a fixed partition $\{K_{ni}\}$ the Haar measure becomes

$$\prod_{n=1}^{\ell}\prod_{i=1}^{k}\frac{1}{K_{ni}!}\oint_{T(U(K_{ni}))}\prod_{I=1}^{K_{ni}}\frac{dy_{ni,I}}{2\pi\mathrm{i}y_{ni,I}}\prod_{I\neq J}\left(1-\frac{y_{ni,I}}{y_{ni,J}}\right)=\oint\prod_{n,i}d\mu_{U(K_{ni})} \tag{4.67}$$

which coincides with the Haar measure for the product group $\prod_{n,i} U(K_{ni})$. The 'orbifolded' index for a fixed partition $\{K_{ni}\}$ is then

$$Z^{6d,\ell,k}_{\{K_{ni}\}}(q,v,w,z,x_{ni,A}) := \oint\prod_{n,i}d\mu_{U(K_{ni})}Z^{(0)}_{\{K_{ni}\}}(q,\dots)\prod_{P=V,H,U}\mathcal{Z}_{\Gamma P}(q,\dots)\,. \tag{4.68}$$

The Casimir contribution $Z^{(0)}_{\{K_{ni}\}}$ is defined in the same way as (4.27). The 'orbifolded' single letter partition function are defined as in (4.28) and are explicitly given by

$$\begin{aligned}
Z_{\Gamma V} = &\left(q^k;q^k\right)^{2K}\prod_{n,i}\prod_{I\neq J}\frac{\theta\left(\frac{y_{ni,I}}{y_{ni,J}};q^k\right)}{\left(1-\frac{y_{ni,I}}{y_{ni,J}}\right)}\prod_{i\neq j}\prod_{I,J=1}^{K_{ni},K_{nj}}\theta\left(q^{L_{ij}}\frac{y_{ni,I}}{y_{nj,J}};q^k\right)\\
&\times\prod_{n,i,j}\frac{\prod_{I,J=1}^{K_{ni},K_{nj}}\theta\left(z^{-2}q^{L_{ij}+1}\frac{y_{ni,I}}{y_{nj,J}};q^k\right)}{\prod_{I,J=1}^{K_{ni},K_{(n+1)j}}\theta\left(wzq^{L_{ij}}\frac{y_{ni,I}}{y_{(n+1)j,J}};q^k\right)\theta\left(z^{-1}wq^{L_{ij}+1}\frac{y_{ni,I}}{y_{(n+1)j,J}};q^k\right)}\,,
\end{aligned} \tag{4.69}$$

$$Z_{\Gamma H} = \prod_{n,i,j}\frac{\prod_{I,J=1}^{K_{ni},K_{(n+1)j}}\theta\left(wvq^{L_{ij}+\frac{1}{2}}\frac{y_{ni,I}}{y_{(n+1)j,J}};q^k\right)\theta\left(\frac{w}{v}q^{L_{ij}+\frac{1}{2}}\frac{y_{ni,I}}{y_{(n+1)j,J}};q^k\right)}{\prod_{I,J=1}^{K_{ni},K_{nj}}\theta\left(\frac{v}{z}q^{L_{ij}+\frac{1}{2}}\frac{y_{ni,I}}{y_{nj,J}};q^k\right)\theta\left(\frac{q^{L_{ij}+\frac{1}{2}}}{vz}\frac{y_{ni,I}}{y_{nj,J}};q^k\right)}\,, \tag{4.70}$$

$$Z_{\Gamma U} = \prod_{n,i,j}\prod_{A,I=1}^{N,K_{ni}}\frac{\theta\left(wq^{L_{ij}+\frac{1}{2}}\frac{y_{ni,I}}{x_{(n+1)j,A}};q^k\right)\theta\left(w^{-1}q^{k-L_{ji}-\frac{1}{2}}\frac{y_{ni,I}}{x_{(n-1)j,A}};q^k\right)}{\theta\left(z^{-1}q^{L_{ij}+\frac{1}{2}}\frac{y_{ni,I}}{x_{nj,A}};q^k\right)\theta\left(zq^{k-L_{ji}-\frac{1}{2}}\frac{y_{ni,I}}{x_{nj,A}};q^k\right)}\,. \tag{4.71}$$

Here

$$L_{ij} := \{\# \in \mathbb{Z} \,|\, 0 \leqslant \# \leqslant k-1 \text{ and } \# = i-j \bmod k\} \tag{4.72}$$

is a unique integer and is equivalent to the $L_{ij} = [[i-j]]$ as defined in [81]. It satisfies the important relation

$$L_{ij} = \begin{cases} k - L_{ji} & i-j \neq 0 \bmod k\,, \\ 0 & i-j = 0 \bmod k\,. \end{cases} \tag{4.73}$$

It also satisfies $L_{(i+k)j} = L_{ij} = L_{i(j+k)}$ allowing us to consistently abuse the orbifold condition $i \sim i+k$ within products, etc.

We also consider a rewriting of (4.68) in which many simplifications become manifest. We change integration variables and define shifted variables

$$y_{ni,I} \to q^{k-i} y_{n,\mathcal{I}}\,, \quad x_{ni,A} := q^{k-i} \tilde{x}_{n,\mathcal{A}}\,, \tag{4.74}$$

we also combine the indices such that $\mathcal{A} = A + N(i-1) = 1, \ldots, kN$ and $\mathcal{I} = I + K_n(i-1) = 1, \ldots, K_n$ where $K_n := \sum_{i=1}^k K_{ni}$.

The shifts may be interpreted in the following way: since the effect of a non-trivial holonomy may always be locally removed by a gauge transformation, the shift of the fugacities x can be thought of as a gauge transformation on the D5 theory. On the other hand, the shift of the y's may always be made by a change of integration variables.

Those variables allow us to make rewritings of the form

$$\prod_{i,j=1}^{k} \theta\left(q^{L_{ij}-i+j} x; q^k\right) = \theta\left(x; q^k\right)^k \prod_{i>j} \theta\left(x; q^k\right) \prod_{j>i} \theta\left(q^k x; q^k\right) \tag{4.75}$$

$$= \prod_{i,j=1}^{k} \theta\left(x; q^k\right) \prod_{j>i} \left(\frac{-1}{x}\right) \tag{4.76}$$

where the second line follows by applying the definition (4.72) and the relation (4.73) while the third line is courtesy of the identity $-x\theta\,(qx; q) = \theta\,(x; q)$. In terms of those variables we have

$$
\begin{aligned}
&Z^{6d,\ell,k}_{\{K_{ni}\}}(q,v,w,z,x_{n,iA}) = \\
&Z^{(0)}_{\{K_{ni}\}}\prod_{n=1}^{\ell}\left(\prod_{i=1}^{k}\frac{\left(q^k;q^k\right)^{2K_{ni}}}{K_{ni}!}\right)\prod_{n=1}^{\ell}\prod_{j>i}\prod_{A=1}^{N}\left(\frac{\tilde{x}_{n+1,jA}\tilde{x}_{n-1,jA}}{\tilde{x}^2_{n,jA}}\right)^{K_{ni}} \\
&\times\prod_{n=1}^{\ell}\oint\prod_{\mathcal{I}=1}^{K_n}\frac{dy_{n,\mathcal{I}}}{2\pi\mathrm{i}y_{n,\mathcal{I}}}\prod_{\mathcal{A},\mathcal{I}=1}^{kN,K_n}\frac{\theta\left(wq^{\frac{1}{2}}\frac{y_{n,\mathcal{I}}}{\tilde{x}_{n+1,\mathcal{A}}};q^k\right)\theta\left(w^{-1}q^{-\frac{1}{2}}\frac{y_{n,\mathcal{I}}}{\tilde{x}_{n-1,\mathcal{A}}};q^k\right)}{\theta\left(z^{-1}q^{\frac{1}{2}}\frac{y_{n,\mathcal{I}}}{\tilde{x}_{n,\mathcal{A}}};q^k\right)\theta\left(zq^{-\frac{1}{2}}\frac{y_{n,\mathcal{I}}}{\tilde{x}_{n,\mathcal{A}}};q^k\right)} \\
&\times\prod_{n=1}^{\ell}\prod_{\mathcal{I},\mathcal{J}=1}^{K_n}\frac{\theta\left(\frac{y_{n,\mathcal{I}}}{y_{n,\mathcal{J}}};q^k\right)\theta\left(z^{-2}q\frac{y_{n,\mathcal{I}}}{y_{n,\mathcal{J}}};q^k\right)}{\theta\left(vz^{-1}q^{\frac{1}{2}}\frac{y_{n,\mathcal{I}}}{y_{n,\mathcal{J}}};q^k\right)\theta\left(v^{-1}z^{-1}q^{\frac{1}{2}}\frac{y_{n,\mathcal{I}}}{y_{n,\mathcal{J}}};q^k\right)} \\
&\times\prod_{\mathcal{I},\mathcal{J}=1}^{K_n,K_{n+1}}\frac{\theta\left(wvq^{\frac{1}{2}}\frac{y_{n,\mathcal{I}}}{y_{n+1,\mathcal{J}}};q^k\right)\theta\left(wv^{-1}q^{\frac{1}{2}}\frac{y_{n,\mathcal{I}}}{y_{n+1,\mathcal{J}}};q^k\right)}{\theta\left(wz\frac{y_{n,\mathcal{I}}}{y_{n+1,\mathcal{J}}};q^k\right)\theta\left(z^{-1}wq\frac{y_{n,\mathcal{I}}}{y_{n+1,\mathcal{J}}};q^k\right)},
\end{aligned}
\tag{4.77}
$$

the prime means that the $\mathcal{I}=\mathcal{J}$ term should be excluded from the product. The full ‘orbifolded’ index (4.65) is then given by summing over all partitions $\{K_{ni}\}$ of K. However, in analogy with (4.33) from the 5d point of view we expect to have a $U(1)^{\ell k}$ topological symmetry associated to the currents $\star_{5d}J_{ni}=\frac{1}{8\pi^2}\operatorname{tr}F_{ni}\wedge F_{ni}$ for the $(i,n)^{\text{th}}$ gauge node in the quiver. Since the D-instantons serve as sources for those currents and the associated instanton number K_{ni} is related to the partition $\{K_{ni}\}$ following our discussion in Section 4.4.1 we may weight each contribution with fugacities $\mathbf{q}_{6d,ni}$ for each current and assemble the grand partition function

$$
Z^{6d,\ell,k}(q,\ldots;\mathbf{q}_{6d,ni}) = \sum_{\{K_{ni}\}}\left(\prod_{n,i}\mathbf{q}^{K_{ni}}_{6d,ni}\right)Z^{6d,\ell,k}_{\{K_{ni}\}}(q,\ldots) \tag{4.78}
$$

the sum over all possible partitions of K is equivalent to summing over all embeddings $\Gamma\hookrightarrow U(K)$ up to gauge transformation.

4.4.3 The 6d Orbifolded Instanton Partition Function

In this section we check that for $k=1$ our partition function indeed reproduces the known result for the partition function for $\{K_1,\ldots,K_\ell\}$ D1-branes in the presence of ℓN D5-branes on $A_{\ell-1}$ singularity as computed in [239]. Furthermore we show that the $\mathbb{Z}_k$ orbifolded index may be obtained from the partition function without orbifold (up to an overall shift in the Casimir contribution) with the substitution rule $x \to \tilde{x}$ and $q \to q^k$.

We must first evaluate the contour integrals (4.68). It bares many resemblances with the unorbifolded ($\ell = k = 1$) partition function (4.33) discussed in Section 4.3. In particular, the poles are now located at

$$y_{n,\mathcal{I}} = y_{n-1,\mathcal{J}}\,(zw)\,, \quad y_{n,\mathcal{I}} = y_{n+1,\mathcal{J}}\left(\frac{z}{qw}\right), \tag{4.79}$$

$$y_{n,\mathcal{I}} = v^{\pm' 1} y_{n,\mathcal{J}}\left(\frac{vz}{q^{1/2}}\right)^{\pm 1}, \quad y_{n,\mathcal{I}} = \tilde{x}_{n,\mathcal{A}}\left(\frac{z}{q^{1/2}}\right)^{\pm 1}. \tag{4.80}$$

Proceeding in an analogous way to the unorbifolded index we again assume that the $\tilde{x}$'s can be made sufficiently generic and that the correct residues to collect are again those coming from the poles (4.80). Solutions to (4.80) are then in fact classified by ℓ lots of kN-coloured Young's diagrams which we label by $\vec{\mu}_n = \{\mu_{n,\mathcal{A}}\} = \{\mu_{n,1}, \ldots, \mu_{n,kN}\}$ again with each diagram $\mu_{n,\mathcal{A}}$ containing $|\mu_{n,\mathcal{A}}|$ boxes such that $|\vec{\mu}_n| := \sum_{\mathcal{A}} |\mu_{n,\mathcal{A}}| = K_n$ where $K_n := \sum_{i=1}^{k} K_{ni}$ as before. Given a Young's diagram $\mu_{n,\mathcal{A}}$ a box s is labelled by coordinates (l, p) and the corresponding pole is given by

$$y_n(s) = \tilde{x}_{n,\mathcal{A}}\left(\frac{z}{q^{1/2}}\right)^{l+p-1} v^{l-p} \tag{4.81}$$

Hence, for a fixed partition $\{K_{ni}\}$ the residue of (4.68) for a fixed set of kN-tuples $\vec{\mu}_1, \ldots, \vec{\mu}_\ell$ is

$$\begin{aligned} Z^{6d,\ell,k}_{\{K_{ni}\},\{\vec{\mu}_n\}} =& Z^{(0)}_{\{K_{ni}\},\{\vec{\mu}_n\}} \prod_{n=1}^{\ell} \Bigg\{ \left(\frac{K_n!}{\prod_{j=1}^{k} K_{ni}!}\right) \prod_{j>i} \prod_{A=1}^{N} \left(\frac{\tilde{x}_{n+1,jA}\tilde{x}_{n-1,jA}}{\tilde{x}^2_{n,jA}}\right)^{K_{ni}} \\ &\times \prod_{\mathcal{A},\mathcal{B}=1}^{kN} \frac{\prod_{s\in\mu_{n+1,\mathcal{A}}} \theta\left(q^{-1}zw^{-1}\mathcal{E}_{(n+1)n,\mathcal{A}\mathcal{B}}; q^k\right)}{\prod_{s\in\mu_{n,\mathcal{A}}} \theta\left(\mathcal{E}_{nn,\mathcal{A}\mathcal{B}}; q^k\right)} \\ &\times \prod_{\mathcal{A},\mathcal{B}=1}^{kN} \frac{\prod_{s\in\mu_{n,\mathcal{A}}} \theta\left(zw\mathcal{E}_{n(n+1),\mathcal{A}\mathcal{B}}; q^k\right)}{\prod_{s\in\mu_{n,\mathcal{A}}} \theta\left(q^{-1}z^2\mathcal{E}_{nn,\mathcal{A}\mathcal{B}}; q^k\right)} \Bigg\} \end{aligned} \tag{4.82}$$

where we defined

$$\mathcal{E}_{nm,AB} := \frac{\tilde{x}_{n,\mathcal{A}}}{\tilde{x}_{m,\mathcal{B}}}\left(\frac{vz}{q^{1/2}}\right)^{L_{m,\mathcal{B}}(s)}\left(\frac{q^{1/2}v}{z}\right)^{\left(A_{n,\mathcal{A}}(s)+1\right)} \tag{4.83}$$

where $L_{n,\mathcal{A}}(s)$ and $A_{n,\mathcal{A}}(s)$ denote the distance from the box s to the right end and the bottom of the Young diagram $\mu_{n,\mathcal{A}}$ respectively. Furthermore, we also notice the multinomial coefficient $K_n!/\prod_{i=1}^{k} K_{ni}! = \binom{\sum_i K_{ni}}{K_{n1},\dots,K_{nk}}$.

We are still yet to describe the Casimir contribution. Again we must specify the q-dependence of the fugacities. We write

$$v := q^{\frac{\beta_5 k\epsilon_-}{2\mathrm{i}\pi}}\,, \quad wq^{\frac{1}{2}} := q^{\frac{\beta_5 km}{\mathrm{i}\pi}}\,, \quad zq^{-\frac{1}{2}} := q^{\frac{\beta_5 k\epsilon_+}{2\mathrm{i}\pi}}\,, \quad \tilde{x}_{n,\mathcal{A}} := q^{\frac{\beta_5 k\tilde{a}_{n,\mathcal{A}}}{\mathrm{i}\pi}}\,. \tag{4.84}$$

The Casimir contribution is explicitly given in equation (E.14) and after evaluating its residue for a fixed kN-tuples $\{\vec{\mu}_n\}$ is

$$\begin{aligned} Z^{(0)}_{\{K_{ni}\},\{\vec{\mu}_n\}} = &\prod_{n=1}^{\ell}\prod_{\mathcal{A}=1}^{kN} q^{\frac{kK_n\beta_5^2}{2\pi^2}\left[2\tilde{a}^2_{n,\mathcal{A}}-\tilde{a}^2_{n-1,\mathcal{A}}-\tilde{a}^2_{n+1,\mathcal{A}}+2m\left(\tilde{a}_{n+1,\mathcal{A}}-\tilde{a}_{n-1,\mathcal{A}}\right)\right]} \\ &\times q^{\frac{-k^2KN\beta_5^2}{\mathrm{i}\pi}\left(\frac{\epsilon_+}{2}+m\right)\left(\frac{\epsilon_+}{2}-m+\frac{1}{\beta_5\mathrm{i}\pi}\right)} \\ &\times\prod_{n=1}^{\ell}\prod_{\mathcal{A},\mathcal{B}=1}^{kN}\prod_{s\in\mu_{n,\mathcal{B}}} q^{\frac{\beta_5^2}{2\pi^2}\left(\phi_n(s)+\frac{\epsilon_+}{2}\right)\left(\tilde{a}_{n+1,\mathcal{A}}+\tilde{a}_{n-1,\mathcal{A}}-2\tilde{a}_{n,\mathcal{A}}\right)}\,, \end{aligned} \tag{4.85}$$

where as before we must use the definitions (4.84). For a box $s \in \mu_{n,\mathcal{A}}$ the function $\phi_n(s)$ is given by

$$\phi_n(s) = \tilde{a}_{n,\mathcal{A}} + (l-1)\epsilon_1 + (p-1)\epsilon_2\,. \tag{4.86}$$

Equation (4.77) is then obtained by summing over all $\vec{\mu}_n$

$$Z^{6d,\ell,k}_{\{K_{ni}\}}(q,v,w,z,x_{ni,A}) = \sum_{\substack{\vec{\mu}_1,\dots,\vec{\mu}_\ell\\|\vec{\mu}_n|=K_n}} Z^{6d,\ell,k}_{\{K_{ni}\},\{\vec{\mu}_n\}}(q,v,w,z,x_{ni,A})\,. \tag{4.87}$$

4.4.4 Five Dimensional Limit

We can again take the 5d $\beta_6 \to 0$ ($q \to 1$) limit. For $k = 1$ the resulting 5d theories are the $\mathcal{N} = 1$ circular quivers denoted by $\mathcal{N}_{N,\ell}$ on $\mathbb{R}^4 \times \mathbb{S}^1_5$. For $k > 1$ we expect the resulting 5d theory is $\mathcal{N}_{kN,\ell}$ with k codimension 1 defects which fill the $\mathbb{R}^4$ and are located at points $\Theta = \Theta_{j=1,\dots,k}$ where $\Theta \sim \Theta + 2\pi$ is the coordinate of $\mathbb{S}^1_5$.

Following the same procedure as before and making the identifications (4.84), we find

$$\begin{aligned} Z^{5d,\ell,k}_{\{K_{ni}\}} = &\sum_{\substack{\vec{\mu}_1,\vec{\mu}_2,\dots,\vec{\mu}_\ell\\|\vec{\mu}_n|=K_n}}\prod_{n=1}^{\ell}\binom{K_n}{K_{n1},\dots,K_{nk}}\prod_{\mathcal{A},\mathcal{B}=1}^{kN}\prod_{s\in\mu_{n,\mathcal{A}}} \\ \times\ &\frac{\sinh\beta_5\left(E_{n(n-1),\mathcal{AB}}+\frac{\epsilon_+}{2}-m\right)\sinh\beta_5\left(E_{(n-1)n,\mathcal{AB}}+\frac{\epsilon_+}{2}+m\right)}{\sinh\beta_5\left(E_{nn,\mathcal{AB}}\right)\sinh\beta_5\left(E_{nn,\mathcal{AB}}+\epsilon_+\right)} \end{aligned} \tag{4.88}$$

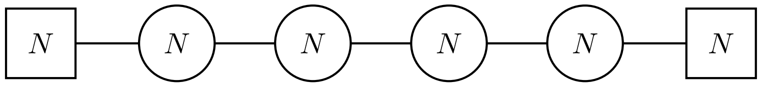


Figure 4.4: *The 5d $\mathcal{N}_{N,\ell}$ quiver for $\ell = 5$ after taking the decoupling limit obtained by sending one of the couplings to zero. Circle reduction gives 4d $\mathcal{N} = 2$ linear $A_{\ell-1}$ quivers.*

again $\lim_{q\to 1} Z^{(0)}_{\{K_{ni}\}} = 1$ and the function $E_{nm,\mathcal{AB}}$ is defined as

$$E_{nm,\mathcal{AB}} := \tilde{a}_{n,\mathcal{A}} - \tilde{a}_{m,\mathcal{B}} + \epsilon_1 L_{n,\mathcal{A}}(s) - \epsilon_2 \left(A_{n,\mathcal{B}}(s) + 1\right) . \tag{4.89}$$

As before the instantons partition function of the resulting 5d theory is given by

$$Z^{5d,\ell,k} = \sum_{\{K_{ni}\}} \left(\prod_{n,i} \mathbf{q}_{5d,ni}^{K_{ni}} \right) Z^{5d,\ell,k}_{\{K_{ni}\}} . \tag{4.90}$$

4.4.5 Four Dimensional Limit

Finally, we take the 4d $\beta_5 \to 0$ limit. We expect that by taking the 4d limit we land on the 4d torodial quiver SCFTs in class $\mathcal{S}_k$. In particular, we want to compare our expression in this limit with the expression proposed in [153]. Applying the 4d limit to (4.88) yields:

$$\begin{aligned} Z^{4d,\ell,k}_{\{K_{ni}\}} = & \sum_{\substack{\vec{\mu_1},\vec{\mu_2},\ldots,\vec{\mu_\ell} \\ |\vec{\mu_n}|=K_n}} \prod_{n=1}^{\ell} \binom{K_n}{K_{n1},\ldots,K_{nk}} \prod_{n=1}^{\ell} \prod_{\mathcal{A},\mathcal{B}=1}^{kN} \\ \times & \frac{\prod_{s\in\mu_{n+1,\mathcal{A}}} \left(E_{(n+1)n,\mathcal{AB}} + \frac{\epsilon_+}{2} - m\right) \prod_{s\in\mu_{n,\mathcal{B}}} \left(E_{n(n+1),\mathcal{AB}} + \frac{\epsilon_+}{2} + m\right)}{\prod_{s\in\mu_{n,\mathcal{A}}} \left(E_{nn,\mathcal{AB}}\right) \prod_{s\in\mu_{n,\mathcal{A}}} \left(E_{nn,\mathcal{AB}} + \epsilon_+\right)} \end{aligned} \tag{4.91}$$

and the partition function of instantons reads

$$Z^{4d,\ell,k} = \sum_{\{K_{ni}\}} \left(\prod_{n,i} \mathbf{q}_{4d,ni}^{K_{ni}} \right) Z^{4d,\ell,k}_{\{K_{ni}\}} = Z^{4d}_{\mathrm{inst},\tilde{A}_{\ell-1}\times\tilde{A}_{k-1}} . \tag{4.92}$$

For $k = 1$ (4.91) may be compared with the partition function of instantons for the 4d $\mathcal{N} = 2$ $\tilde{A}_{\ell-1}$ circular quiver theories.

4.4.6 From Necklace/Toroidal to Linear/Cylindrical Quivers

In this subsection we want to briefly explain, for the sake of the non-expert reader, how we can obtain the instanton partition functions for linear $\mathcal{N}=2$ (generic ℓ, $k=1$) or cylindrical $\mathcal{N}=1$ quivers (generic ℓ and k) from the formulas we have just derived that are for necklace $\mathcal{N}=2$ (generic ℓ, $k=1$) and toroidal $\mathcal{N}=1$ (generic ℓ and k) quivers respectively. Firstly, for the $\mathcal{N}=2$ theories with generic ℓ and $k=1$, we choose a fugacity $\mathbf{q}_{6d,n}$ for one n corresponding to one coupling constant of the n^{th} gauge node and send it to zero in equation (4.78). This corresponds to ungauging this gauge factor and breaks the necklace at this node. See Figure 4.2 and Figure 4.4. It is useful for notational clarity to ungauge the node with $n=\ell$. Then the hypermultiplets that couple to this node from the left and from the right become fundamental and the Coulomb branch parameters $a_1=m_L$ and $a_\ell=m_R$ are interpreted as anti-fundamental and fundamental masses, respectively. Moving on to the toroidal $\mathcal{N}=1$ quivers with generic ℓ and k, we can obtain cylindrical $\mathcal{N}=1$ quivers via ungauging all of the k-nodes with $n=\ell$, setting in (4.78) $\mathbf{q}_{6d,\ell i}=0$ for all $i=1,\dots,k$. See Figure 1.7 and Figure 1.8. Finally, let us stress that this ungauging procedure can be done for all 6d, 5d and 4d instanton partition functions.

4.5 Conclusions

In this chapter we have computed the instanton partition function of 4d $\mathcal{N}=1$ theories in class $\mathcal{S}_k$ and a 5d and 6d uplift of them, which correspond to 5d $\mathcal{N}=1$ and 6d $(1,0)$ theories in the presence of a $\frac{1}{2}$-BPS defect. We further observed that class $\mathcal{S}_k$ instanton partition functions can be obtained from the 4d $\mathcal{N}=2$ theories in class $\mathcal{S}$ and their 5d and 6d uplifts: the 5d $\mathcal{N}=1$ necklace quiver $\mathcal{N}_{N,\ell}$ and the 6d $(1,0)$ theory $\mathcal{T}_\ell^N$ (without the defect) via imposing the 'orbifold condition' on the Coulomb moduli and mass parameters as

$$Z_{\text{inst}}^{\mathcal{S}_k,SU(N)}(a_A)=Z_{\text{inst}}^{\mathcal{S},SU(kN)}(a_{\mathcal{A}})\quad\text{with}\quad a_{\mathcal{A}}\to a_A e^{2\pi \mathrm{i} j/k} \tag{4.93}$$

with $\mathcal{A}=A+N(j-1)$ being an $SU(kN)$ fundamental index, $A=1,\dots,N$ an $SU(N)$ index and $j=1,\dots,k$ counting the number of the mirror images.

It is important to stress that our result for the class $\mathcal{S}_k$ instanton partition functions match with the prediction of [153] coming from a calculation of a completely different type. In [153] based on the anticipation of an AGT type correspondence for theories in class $\mathcal{S}_k$, and the comparison of the spectral curves of theories in class

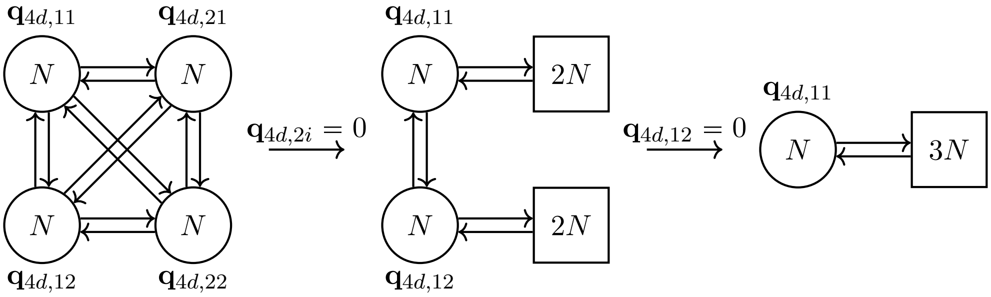


Figure 4.5: Left: $\tilde{A}_1 \times \tilde{A}_1$ *quiver.* Middle: *Decoupling limit giving SCQCD*$_2$*.* Right: *Further decoupling limit giving* $\mathcal{N} = 1$ *SQCD with* $N_f = 3N$ *theory.*

$\mathcal{S}_k$ with 2d CFT blocks, the 2d CFT symmetry algebra and its representations that should underlie AGT$_k$ were identified. These conformal blocks led to a prediction for the instanton partition functions of the 4d $\mathcal{N} = 1$ SCFTs of class $\mathcal{S}_k$ which we precisely reproduce here. $\mathcal{N} = 1$ SQCD with $N_f = 3N$ can be obtained from the $\mathbb{Z}_2 \times \mathbb{Z}_2$ class $\mathcal{S}_k$ theory depicted in Figure 4.5 in the limit where three of the coupling constants go to zero as shown in the figure. It would be very interesting to learn how to isolate the instantons of the $\mathcal{N} = 1$ SQCD with $N_f = 3N$. Moreover, we would like to bring to the attention of the reader the fact that the $\mathcal{N} = 1$ instanton partition function we derived is a product of an orbifolded vector and a bifundamental hyper multiplet contributions

$$Z_{\text{inst}} = \prod_{\text{quiver}} z_{\text{vec}}^{\text{orb}}\, z_{\text{bif}}^{\text{orb}} \tag{4.94}$$

directly arising from their $\mathcal{N} = 2$ mother theory construction. An important question is if the instanton partition function may be further reorganised, for the $\mathcal{S}_k$ theory, as

$$Z_{\text{inst}} \stackrel{?}{=} \prod_{\text{quiver}} z_{\text{vector}}^{\mathcal{N}=1}\, z_{\text{chiral}}^{\mathcal{N}=1}\,. \tag{4.95}$$

At this stage it is unclear to us if this is even possible, however we believe that carefully studying the SW curves away from the orbifold point will be illuminating.

A comment concerning our strategy is in order. For the computation of the instanton partition function we have decided, instead of computing the matrix model path integral of the 0d theory that lives on the D(-1)-branes, to compute the 2d superconformal index of the gauge theory which lives on the K D1-branes for the following reasons. First of all, the superconformal index computation is very well studied and understood and is rather more tractable than directly localising the matrix model. Secondly, along the way to the instanton partition function of the 4d theories of interest we also computed the partition functions for certain 6d $(1,0)$ theories in the presence of half-BPS surface operators, which is a very interesting result in its own right. It would be very interesting to attempt localisation and compute the 2d partition function that we have computed, in analogy with the localisation computation of the Lens space index [81, 237]. Another very interesting alternative to our strategy is the Origami approach of Nekrasov [240].

Our strategy may be applied to several other interesting cases such as computing partition functions of $(1,0)$ theories in the presence of a surface operator lying along $\mathbb{C} \subset \mathbb{C}^2_{\epsilon_1\epsilon_2}$. Some results along these lines already exist in the literature [49, 241, 242, 243, 244]. We will have more to say about this in Chapter 6. Finally, one could also try to compute partition functions in class $\mathcal{S}_k$ in the presence of defects via combining the two orbifold constructions.

In an orthogonal direction, and in connection with [153], it would be instructive to try to repeat the method of [245], who, starting from the $(2,0)$ theory in 6d, were able to obtain a direct derivation of the AGT correspondence, for the $\mathcal{N}=1$ theories of class $\mathcal{S}_k$.

Finally, using our results and taking the large N limit, one could learn about the gravity dual of $\mathcal{N}=1$ theories in class $\mathcal{S}_k$ following the work of [202, 203].

Chapter 5

$\mathcal{N} = 3$ SCFTs from Discrete Gauging

The work in this chapter is based on [246].

5.1 Introduction

Very recently García-Etxebarria and Regalado have proposed a construction of genuine 4d $\mathcal{N} = 3$ supersymmetric theories [158]. Much progress has been made towards understanding the properties of 4d $\mathcal{N} = 1, 2, 4$ theories but the $\mathcal{N} = 3$ case had been long ignored. This is due to the fact that, preceding their work, no example of an $\mathcal{N} = 3$ SCFT that did not enhance to $\mathcal{N} = 4$ supersymmetry was known.

One reason for this is the fact that the only multiplet of $\mathcal{N} = 3$ supersymmetry that one can use to write down a renormalisable, local Lagrangian density is the free vector multiplet which, after imposing CPT invariance, has identical field content to the $\mathcal{N} = 4$ vector multiplet. Thus, there are no free genuinely $\mathcal{N} = 3$ theories and all genuinely $\mathcal{N} = 3$ theories have to be strongly coupled.

García-Etxebarria and Regalado considered $\mathcal{N} = 3$ theories embedded in type IIB string theory (F-theory) by generalizing the well known orientifold construction to $\mathcal{N} = 3$ preserving S-folds. The S-fold includes a $\mathbb{Z}_k$ projection on both the R-symmetry directions as well as the $SL(2, \mathbb{Z})$ S-duality group of Type-IIB (the torus of F-theory). As pointed out in [247], via a classification of different variants of S-folds, one may create a large variety of $\mathcal{N} = 3$ theories further distinguished by an analogue of discrete torsion. More new $\mathcal{N} = 3$ SCFTs were also proposed [159] via gauging a discrete subgroup of the global symmetry group of $\mathcal{N} = 4$ SYM.

The first to seriously consider the consequences of $\mathcal{N} = 3$ supersymmetry was [248] who, by studying the $\mathcal{N} = 3$ superconformal algebra representations, were able to reveal several basic properties that consistent $\mathcal{N} = 3$ QFTs should possess; were they to exist. These properties include:

- Conformal anomalies: $a = c$ (as is the case for $\mathcal{N} = 4$ theories, while generic $\mathcal{N} = 2$ theories have $a \neq c$).[1]
- Global symmetries: $SU(2, 2|3) \times H$, where H is discrete.
- Trivial conformal manifold: No marginal couplings, they are strongly coupled isolated fixed points; no Lagrangian description.
- No $\mathcal{N} = 3$ preserving relevant deformations (because there are no continuous flavour symmetries).
- Viewed as $\mathcal{N} = 2$ theories there is a $U(1)_f$ flavour symmetry $SU(2, 2|2) \times U(1)_f \subset SU(2, 2|3)$.

Because $\mathcal{N} = 3$ SCFTs do not have a Lagrangian description they can be studied only with certain tools. Representation theory alone has taken us very far [248, 249, 167]. String-, F- and M-theoretic constructions provide the primary way that we have to study $\mathcal{N} = 3$ SCFTs [158, 250]. The Type-IIB description allows for an AdS gravity dual description which can be used to examine the properties of $\mathcal{N} = 3$ theories in the large N limit [158, 247, 251]. Moreover, $\mathcal{N} = 3$ theories have Seiberg-Witten solutions [13, 12] which encode the low energy effective action of the theory on the Coulomb branch. Various aspects of the Coulomb branches for these theories have been studied in [252, 116, 115, 253, 247, 248, 159, 254]. Another powerful tool is the superconformal bootstrap which has been employed in [255]. The bootstrap can also be suplemented with chiral algebra techniques [172] and has been studied in [256, 255]. Further techniques have been developed in [257, 258].

In this chapter we will focus on the moduli spaces of those $\mathcal{N} = 3$ SCFTs that may be obtained by gauging a discrete symmetry of $\mathcal{N} = 4$ SYM. We will bestow particular emphasis towards the Hilbert series for the moduli spaces of these theories aswell as certain limits of the superconformal index.

[1] Recall that these are the coefficients appearing in the 1-point function of the trace of the stress-tensor

$$\langle \mathrm{tr}\, T \rangle = \frac{c}{16\pi^2} C^2 - \frac{a}{2} |\det g| \star e \tag{5.1}$$

where C is the Weyl tensor and $e = \frac{1}{4\pi^2}\, \mathrm{Pf}(Riem)$ is the $d = 4$-form Euler density, with $Riem$ the Riemann Tensor of the Levi-Civita connection associated to metric g.

This chapter is organized as follows: In Section 5.2 we review the possible constructions of $\mathcal{N}=3$ SCFTs; both S-folding and discrete gauging. This gives us the opportunity to discuss in detail the symmetries of both the mother and the daughter theories and to embed the discrete subgroup that we want to gauge in the $SU(4)$ R-symmetry group and the $SL(2,\mathbb{Z})$ S-duality group of $\mathcal{N}=4$ SYM. In Section 5.3 we gather some facts about the representation theory of $\mathfrak{su}(2,2|\mathcal{N})$ superconformal algebras that we will need for the index computation and interpretation. In Section 5.4 we introduce the refined version of the superconformal index, its Coulomb branch limit and the Higgs branch Hilbert series; the discrete gauging prescription is presented. Section 5.5 is devoted to rank one examples. Section 5.6 deals with higher rank examples. We focus on the Coulomb and Higgs branches. Our higher rank computations allow us to make new predictions for these theories. Finally, in Section 5.7 we compute the single trace index in the large N limit and compare to the AdS/CFT result of [251].

5.2 Constructions of $\mathcal{N}=3$ SCFTs

5.2.1 S-Folds

One possible way to realise $\mathcal{N}=3$ SCFTs is via S-folds. S-folds were originally introduced in [158] and are non-perturbative generalisations of the standard orientifolds in string theory. The construction introduced in [158] goes as follows: consider F-theory on $\mathbb{R}^4 \times \left(\mathbb{R}^6 \times T^2\right)/\mathbb{Z}_k$. The $\mathbb{Z}_k \subset Spin(6) \times SL(2,\mathbb{Z})$ and we denote its generator in $\mathfrak{su}(4) \cong \mathfrak{so}(6)$ to be r_k which acts on the coordinates X^i, $i=1,\ldots,6$ of $\mathbb{R}^6$ by rotation corresponding to

$$R_k = e^{\frac{2\pi \mathrm{i}}{k}(q_1+q_2-q_3)} = \begin{pmatrix} \hat{R}_k & 0 & 0 \\ 0 & \hat{R}_k & 0 \\ 0 & 0 & \hat{R}_k^{-1} \end{pmatrix} \in SO(6)\,, \tag{5.2}$$

where $\hat{R}_k$ denotes rotation by $2\pi/k$ in the corresponding 2-plane. q_1, q_2, $q_3 \in \mathfrak{so}(6)$ denote the Cartan generators of $\mathfrak{so}(6)$. The corresponding element in $SU(4) \cong$

$Spin(6)$ is just

$$\widetilde{R}_k = e^{\frac{2\pi \mathrm{i}}{k} r_k} = e^{\frac{2\pi \mathrm{i}}{k}\left(\frac{R_1}{2}+R_2+\frac{3R_3}{2}\right)} = \begin{pmatrix} e^{\mathrm{i}\pi/k} & 0 & 0 & 0 \\ 0 & e^{\mathrm{i}\pi/k} & 0 & 0 \\ 0 & 0 & e^{\mathrm{i}\pi/k} & 0 \\ 0 & 0 & 0 & e^{-3\mathrm{i}\pi/k} \end{pmatrix} \in SU(4)\,. \tag{5.3}$$

We choose a basis for the Cartans R_1, R_2, $R_3 \in \mathfrak{su}(4)$ given by[2]

$$R_1 = \mathrm{diag}(1,-1,0,0)\,, \quad R_2 = \mathrm{diag}(0,1,-1,0)\,, \tag{5.4}$$

$$R_3 = \mathrm{diag}(0,0,1,-1)\,. \tag{5.5}$$

On the other hand the quotient on the torus acts as an involution of the torus only for $k = 1,2,3,4,6$. Moreover $k = 3,4,6$ require fixed complex structure of $\tau = e^{\mathrm{i}\pi/3}, \mathrm{i}, e^{\mathrm{i}\pi/3}$ respectively. In that case we denote the generator of $\mathbb{Z}_k \subset SL(2,\mathbb{Z})$ by s_k. s_k acts on the coordinate $x+\tau y$ of the T^2 corresponding to $S_k \in SL(2,\mathbb{Z})$ with

$$S_2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\,, \quad S_3 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}\,, \tag{5.6}$$

$$S_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\,, \quad S_6 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\,. \tag{5.7}$$

The elements of $\mathbb{Z}_k \subset Spin(6) \times SL(2,\mathbb{Z})$ are of the form $e^{\frac{2\pi \mathrm{i}}{k}(r_k+s_k)}$, corresponding to the combined action (5.3) and (5.6).

After taking the Type-IIB limit of F-theory the singular geometry can be probed with a stack of N D3-branes. The resulting low energy theory on the D3-branes (for $k = 3,4,6$) is a strongly interacting $\mathcal{N} = 3$ SCFT. In Appendix F.1 we explicitly show the supercharges that are preserved by the $\mathbb{Z}_k$ quotient.

A careful analysis [247] of the discrete global symmetries indicates, as for the $(k=2)$ $O3^{\pm}, \widetilde{O3}^{\pm}$ perturbative orientifolds, the $k = 3,4,6$ S-folds are characterised by different $\mathbb{Z}_p \subset \mathbb{Z}_k$ global symmetries. The S-fold variants are then labelled by $k, \ell = k/p$. We denote the theory of N D3-branes by $S^N_{k,\ell}$ and it has Coulomb branch

[2]We use the same conventions as in [62] and the $\mathfrak{so}(6)$ Dynkin labels (q_1,q_2,q_3) are related to the $\mathfrak{su}(4)$ Dynkin labels (R_1,R_2,R_3) by

$$q_1 = \frac{R_1}{2} + R_2 + \frac{R_3}{2}\,, \quad q_2 = \frac{R_1}{2} + \frac{R_3}{2}\,, \quad q_3 = \frac{R_1}{2} - \frac{R_3}{2}\,.$$

operators of dimension

$$k\,,\ 2k\,,\ \ldots\,,\ (N-1)k\,;\ N\ell\,, \tag{5.8}$$

corresponding to Coulomb branch operators $\left(\sum_{i=1}^{N} z_i^{jk}\right)$, $j = 1, \ldots, N-1$, and the Pfaffian-like operator $(z_1 z_2 \ldots z_N)^\ell$ where z_i denote the positions of the D3-branes in $\mathbb{C}/\mathbb{Z}_k$. Consequently the theory has central charge given by [259, 260, 247]

$$a_{k,\ell} = c_{k,\ell} = \frac{kN^2 + (2\ell - k - 1)N}{4}\,. \tag{5.9}$$

The theory $S^N_{k,\ell}$ associated to each value of k, ℓ has a global symmetry of (at least) $\mathbb{Z}_p = \mathbb{Z}_{k/\ell}$ which acts on the Pfaffian-like operator
$(z_1 z_2 \ldots z_N)^\ell \mapsto (e^{2\pi \mathrm{i}/k} z_1 z_2 \ldots z_N)^\ell = e^{2\pi \mathrm{i}/p}(z_1 z_2 \ldots z_N)^\ell$ while acting trivially on every other Coulomb branch operator. By gauging $\mathbb{Z}_{p'} \subset \mathbb{Z}_p \subset \mathbb{Z}_k$ discrete symmetry we obtain further theories

$$S^N_{k,\ell} \xrightarrow{\mathbb{Z}_{p'}\text{ gauging}} S^N_{k,\ell,p'}\,, \tag{5.10}$$

which, since they arise as discrete gauging of a 'parent' theory, have central charge (5.9) and the theory $S^N_{k,\ell,p'}$ has Coulomb branch operators of dimension

$$k\,,\ 2k\,,\ \ldots\,,\ (N-1)k\,;\ Np'\,. \tag{5.11}$$

Since the $\mathbb{Z}_{p'}$ acts non-trivially only on a single operator quotienting by $\mathbb{Z}_{p'}$ does not introduce relations and the corresponding ring is freely generated.

5.2.2 Discrete Gauging

In this paper we use a different construction to the one described in Section 5.2.1. Consider instead $\mathcal{N} = 4$ SYM with gauge group G. The theory has an exactly marginal gauge coupling τ. $\mathcal{N} = 4$ SYM (on $\mathbb{R}^4$) has an S-duality group generated by [113, 114, 261, 262, 263, 264, 265]

$$(\tau\,, G) \mapsto (\tau + 1\,, G) \quad \text{and} \quad (\tau\,, G) \mapsto \left(-\frac{1}{\lambda_q^2 \tau}\,, {}^L G\right)\,, \tag{5.12}$$

where $\lambda_q = 2\cos\frac{\pi}{q}$ and ${}^L G$ denotes the Langlands dual of G. The action on τ forms a group known as a Hecke group $H(\lambda_q) \subset SL(2, \mathbb{Z}[\lambda_q])$. $H(\lambda_q)$ can be represented by

$$H(\lambda_q) \cong \langle S, T | S^4 = 1, (ST)^{2q} = 1\rangle \cong C_4 \star C_{2q}\,. \tag{5.13}$$

An explicit matrix representation is given by

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -\lambda_q^{-1} \\ \lambda_q & 0 \end{pmatrix}. \tag{5.14}$$

For $q = 3$ $H(\lambda_3) = H(1) = SL(2,\mathbb{Z})$. Let $\mathfrak{g} = Lie(G)$. When $\mathfrak{g} = ADE$ (or $\mathfrak{u}(N)$) $q = 3$ while for $\mathfrak{g} = BCF$ $q = 4$ and for $\mathfrak{g} = G_2$ $q = 6$. We define the *self-duality group* of the theory with gauge group G to be the subset of transformations $\tau \mapsto \tau'$ in $H(\lambda_q)$ which map the theory to itself. When one considers non-local operators this subset of transformations is generally a subgroup of $H(\lambda_q)$ due to the fact that $G \mapsto {}^LG$ clearly changes the global structure of the theory and therefore the spectrum of non-local operators. However, at the level of local operators, when $\mathfrak{g} = ADE$ (or $\mathfrak{u}(N)$) we have $\mathfrak{g} = {}^L\mathfrak{g}$ and then at the level of local operators the *self-duality group* is simply the full $SL(2,\mathbb{Z})$. On the other hand when $\mathfrak{g} = BCFG$ then $\mathfrak{g} \neq {}^L\mathfrak{g}$. In particular $B_N \neq {}^LB_N \cong C_N$, $F_4 \neq {}^LF_4 \cong F_4$ and $G_2 \neq {}^LG_2 \cong G_2$[3] and the *self-duality group* even at the level of local operators is reduced to a subgroup of $H(\lambda_q)$.[4] In this paper we will discuss only the cases when $\mathfrak{g} = {}^L\mathfrak{g}$.

Let us now discuss the possible symmetry enhancements. $H(\lambda_q)$ has finite cyclic subgroups given by

$$H(\lambda_q) \supset \mathbb{Z}_n \text{ for } n = \{2, 4, q, 2q\} \tag{5.15}$$

$$\text{fixes } \tau = \left\{ \text{any}, \frac{\mathrm{i}}{2\cos\frac{\pi}{q}}, -\frac{1}{2} + \frac{\mathrm{i}}{2}\tan\frac{\pi}{q}, -\frac{1}{2} + \frac{\mathrm{i}}{2}\tan\frac{\pi}{q} \right\}, \tag{5.16}$$

where we take only those fixed points with $\mathrm{Im}\,\tau \geqslant 0$. For $q = 3$ these of course reduce to the usual order $n = 4, 3, 6$ $SL(2,\mathbb{Z})$ fixed points at $\tau = \mathrm{i}, e^{\mathrm{i}\pi/3}, e^{\mathrm{i}\pi/3}$.

At a generic point on the conformal manifold the global symmetry group of the theory is at least $PSU(2,2|4)$. On the other hand, for τ fixed as in (5.15), the global symmetry group (acting on local operators) has a $\mathbb{Z}_n$ enhancement for $n = 3, 4, 6$ where the $\mathbb{Z}_n$ is generated by (5.6). We use the notation $\mathbb{Z}_n$ since n should generally be considered unrelated to the parameters k, ℓ, p' appearing in the

[3]S-duality transformations for non-simply-laced Lie algebras are more complicated. In the particular cases of G_2 and F_4 Lie algebras we can perform a rotation on the root system of the corresponding Langland dual algebras ${}^LG_2 = G_2'$ and ${}^LF_4 = F_4'$ such that these turn out to be isomorphic to the initial one [262].

[4]The Langlands dual algebra is obtained by exchanging $\alpha \mapsto \alpha^\vee = \frac{2}{(\alpha\cdot\alpha)}\alpha$. For simply laced algebras we have $\alpha^\vee = \alpha$ and $\mathfrak{g} = {}^L\mathfrak{g}$. On the other hand, when $\mathfrak{g}$ is not simply laced $\alpha^\vee \neq \alpha$ if α is a long root and $\mathfrak{g} \neq {}^L\mathfrak{g}$.

S-fold construction of the previous section. From now one we restrict out attention to $q = 3$. We therefore have a discrete global symmetry

$$\mathbb{Z}_n \subset SU(4) \times SL(2, \mathbb{Z}) \tag{5.17}$$

generated by $r_n + s_n$. We may consider gauging the $\mathbb{Z}_n$ (or in the case when n is not prime, subgroups of the $\mathbb{Z}_n$) global symmetry [159]. Doing so results in a new theory with a different spectrum of local and non-local operators, but, with equivalent local dynamics and therefore the same values for the a and c anomaly coefficients. One may also worry that this $\mathbb{Z}_n$ may have non-vanishing 't Hooft anomalies. We give an argument that this is not the case (at least for the topologically twisted theory on $\mathbb{S}^1 \times \mathbb{S}^3$) in Appendix F.4. The action (5.17) preserves the same supercharges as the $\mathbb{Z}_k$ S-fold, i.e. the $n = 3, 4, 6$ discrete gaugings preserves four dimensional $\mathcal{N} = 3$ supersymmetry. Therefore, the theories we will construct are to be labelled by the parent $\mathcal{N} = 4$ theory and the discrete group to be gauged. The possible parent theories are labelled by a choice of gauge group G. We will only consider parent theories where G is connected.

Moreover, since we will eventually be interested in computing quantities sensitive only to the local operator spectrum, the global form of the gauge group will not play a role in the computations[5] and therefore the theories can rather be labelled by the Lie algebra $\mathfrak{g}$ of G.

Coulomb Branch

Let us now briefly compare with the construction in the previous subsection. Considered as an $\mathcal{N} = 2$ theory we have algebraically independent (over $\mathbb{C}$) Coulomb branch operators u_j, $1 \leqslant j \leqslant N$, $N := \operatorname{rank} \mathfrak{g}$, of dimension $E(u_j)$. In the notation of [266] the u_j's are the highest weight states of the chiral $\mathcal{E}_{r,(0,0)}$ multiplets, with conformal dimension $E(u_j) = r(u_j)$ where $r(u_j)$ is the charge under the $\mathfrak{u}(1)_r$ of the $\mathcal{N} = 2$ superconformal algebra (see Table F.1). They are built up out of $\mathfrak{g}$-invariant combinations of the scalar $X \in \mathfrak{h}$ in the $\mathcal{N} = 2$ vector multiplet, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, while setting the adjoint hypermultiplet scalars $Y = Z = 0$. Let us now go to a point on the conformal manifold where we have an enhanced $\mathbb{Z}_n$ global symmetry generated by $r_n + s_n$. In comparison with the discussion (5.9)-(5.11) this $\mathbb{Z}_n$ global symmetry acts non-trivially on multiple Coulomb branch operators of the

[5] Since $\pi_1(\mathbb{S}^3) = \pi_2(\mathbb{S}^3) = \{1\}$ the superconformal index ($\mathbb{S}^3 \times \mathbb{S}^1$ partition function) is sensitive only to the spectrum of local operators i.e., for connected groups, a choice of Lie algebra $\mathfrak{g}$.

parent theory, namely

$$\mathbb{Z}_n : u_j \mapsto e^{\frac{2\pi \mathrm{i}}{n} E(u_j)} u_j \,. \tag{5.18}$$

It is clear that this $\mathbb{Z}_n$ action does not generically generate a complex reflection group $G(N,m,n)$[6] on the Coulomb branch chiral ring

$$CB_{\mathfrak{g}} := \mathbb{C}\left[u_1, u_2, \ldots, u_N\right] \,, \quad N := \operatorname{rank}\mathfrak{g} = \dim_{\mathbb{C}} CB_{\mathfrak{g}} \,. \tag{5.19}$$

Therefore, by the Chevalley-Shephard-Todd theorem [267, 247], the resulting quotient ring generically has relation(s). Hence, when $\operatorname{rank}\mathfrak{g} \geqslant 2$ and $n \geqslant 2$, the quotient of the Coulomb branch of the parent theory $\mathbf{CB}_{\mathfrak{g}}$ by (5.18)

$$\mathbf{CB}_{\mathfrak{g},n} := \mathbf{CB}_{\mathfrak{g}}/\mathbb{Z}_n \tag{5.20}$$

generally has a non-planar topology. We will see that the structure of the coordinate ring can be often be deduced by studying the Coulomb branch index. Some properties of non-freely generated Coulomb branch chiral rings have been described in [268]. We would also like to point out that in [247] discrete gauging which results in non-freely generated Coulomb branches was explicitly not considered. They considered discrete gauging of the parent theories $S^N_{k,\ell}$ of only $\mathbb{Z}_{p'} \subset \mathbb{Z}_{k/\ell}$ discrete symmetry which acts non-trivially only on a single Coulomb branch operator. However these theories may have larger discrete symmetry groups which may act non-trivially on multiple Coulomb branch operators. Upon gauging such discrete symmetries one can obtain theories with non-freely generated Coulomb branches. Because the discrete gauging does not change the values of a and c we expect them to be equal to those of the $\mathcal{N}=4$ parent theory. If the Coulomb branch operators of the $\mathcal{N}=4$ parent theory have dimension $E(u_i)$ then the a and c anomaly coefficients are given by [259, 260]

$$a = c = \sum_{i=1}^{\operatorname{rank}\mathfrak{g}} \frac{2E(u_i)-1}{4} \,. \tag{5.21}$$

Higgs Branch

Considered as a $\mathcal{N}=2$ theory the Higgs branch is reached by setting $X=0$ and by giving diagonal vevs to the adjoint hypermultiplet scalars $Y, Z \in \mathfrak{h}$. The Higgs branch $\mathbf{HB}_{\mathfrak{g}}$ is then parametrised by $\mathfrak{g}$-invariant combinations $W_i^{(f)}$ of the Y, Z that transform in the f-representation of $U(1)_f$. Where $U(1)_f$ is the flavour symmetry

[6] See equation (2.10) of [247] for a definition.

that all $\mathcal{N} = 3$ theories have when seen as $\mathcal{N} = 2$ theories; we will review this in Section 5.3. In the notation of [266] the $W_i^{(f)}$ are the highest weight states of $\hat{\mathcal{B}}_R$ multiplets and have $E = 2R$ and $r = 0$, where R is the Cartan of the $\mathfrak{su}(2)_R$ R-symmetry of the $\mathcal{N} = 2$ superconformal algebra (see Table F.1). When $\mathfrak{g}$ is non-abelian the Higgs branch-chiral ring $HB_{\mathfrak{g}}$ (the coordinate ring of the variety $\mathbf{HB}_{\mathfrak{g}}$) is generically non-freely generated. Since Y, Z have $s_n = 0$ and $r_n = r + f = f$ the $\mathbb{Z}_n$ acts by

$$\mathbb{Z}_n : W_i^{(f)} \mapsto e^{\frac{2\pi\mathrm{i}}{n} f} W_i^{(f)} . \tag{5.22}$$

Therefore, after the discrete gauging, the Higgs branch is given by the quotient of the Higgs branch of the parent theory $HB_{\mathfrak{g}}$ by the $\mathbb{Z}_n$ action (5.22)

$$\mathbf{HB}_{\mathfrak{g},n} := \mathbf{HB}_{\mathfrak{g}}/\mathbb{Z}_n . \tag{5.23}$$

In Section 5.4 we discuss how to compute the Hilbert series of (5.23).

5.3 $\mathfrak{su}(2,2|\mathcal{N})$ Representation Theory

In this section we will describe some basic facts about representations of (the complexification of) $\mathfrak{su}(2,2|\mathcal{N})$ and their decompositions into subalgebras.

5.3.1 $\mathfrak{psu}(2,2|4) \to \mathfrak{su}(2,2|3)$ Decomposition

The superconformal symmetry algebra of 4d $\mathcal{N} = 4$ SYM is given by $\mathfrak{su}(2,2|4)$. We are interested in unitary representations of $\mathfrak{psu}(2,2|\mathcal{N})$ that can be induced from representations of a bosonic subalgebra $\mathfrak{b}$. Unitary representations of $\mathfrak{psu}(2,2|\mathcal{N})$ are necessarily non-compact. the representations are labelled by $(E, j_1, j_2, R_1, R_2, R_3)$ which label representations under the maximal bosonic subalgebra

$$\mathfrak{b} = \mathfrak{u}(1)_E \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus \mathfrak{su}(4) \subset \mathfrak{psu}(2,2|4) . \tag{5.24}$$

Here E labels the conformal dimension, j_1, j_2 label spin representations and R_1, R_2, R_3 are the Dynkin labels of $\mathfrak{su}(4)$.

As we discussed in Section 5.2, upon the $\mathbb{Z}_n$ discrete gauging $\mathfrak{psu}(2,2|4)$ superconformal symmetry is broken down to $\mathfrak{su}(2,2|3)$ (for $n = 3, 4, 6$). Representations of this algebra are labelled by $(E, j_1, j_2, R_1, R_2, r_{\mathcal{N}=3})$ of the maximal compact bosonic subalgebra

$$\mathfrak{u}(1)_E \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus \mathfrak{su}(3) \oplus \mathfrak{u}(1)_{r_{\mathcal{N}=3}} \subset \mathfrak{su}(2,2|3)\,. \tag{5.25}$$

In particular $\mathfrak{su}(4) \to \mathfrak{su}(3) \oplus \mathfrak{u}(1)_{r_{\mathcal{N}=3}}$. The surviving supercharges are simply given by $\mathcal{Q}^{I=1,2,3}_{\alpha}$, $\widetilde{\mathcal{Q}}_{\dot{\alpha}I=1,2,3}$ and their conjugates. The Cartans of $\mathfrak{su}(3)$ are given by R_1, R_2 and $\mathfrak{u}(1)_{r_{\mathcal{N}=3}}$ is generated by

$$r_{\mathcal{N}=3} = \frac{R_1}{3} + \frac{2R_2}{3} + R_3 \tag{5.26}$$

under which the $\mathcal{Q}^{I=1,2,3}_{\alpha}$ have $r_{\mathcal{N}=3} = \frac{1}{3}$ and $\widetilde{\mathcal{Q}}_{\dot{\alpha}I=1,2,3}$ have $r_{\mathcal{N}=3} = -\frac{1}{3}$.

One of the most important multiplets of $\mathfrak{psu}(2,2|4)$ are the half-BPS multiplets called $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,R_2,0]}$ in the language of [266]. These multiplets obey maximal shortening given by $R_2 = E$. The superconformal primaries of these multiplets are given by single trace operators of the form $\operatorname{tr}\phi^{(I_1J_1}\dots\phi^{I_mJ_m)}$ (see Table 5.1 for conventions) with $(E, j_1, j_2, R_1, R_2, R_3) = (R_2, 0, 0, 0, R_2, 0)$. Under $\mathfrak{psu}(2,2|4) \to \mathfrak{su}(2,2|3)$ these multiplets decompose as

$$\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,R_2,0]} \cong \bigoplus_{i=0}^{R_2} \hat{\mathcal{B}}_{[R_2-i,i]}\,. \tag{5.27}$$

Note that this is a simple consequence of the branching of $\mathfrak{su}(4) \to \mathfrak{su}(3) \oplus \mathfrak{u}(1)_{r_{\mathcal{N}=3}}$

$$\mathbf{[0,R_2,0]} \to \bigoplus_{i=0}^{R_2} \mathbf{[R_2-i,i]}_{\frac{4i}{3}-\frac{2R_2}{3}}\,, \tag{5.28}$$

where the subscript denotes the $\mathfrak{u}(1)_{r_{\mathcal{N}=3}}$ charge. The multiplets $\hat{\mathcal{B}}_{[R_1,R_2]}$ obey the shortening condition $E = R_1 + R_2$, $r_{\mathcal{N}=3} = \frac{2}{3}(R_2 - R_1)$. The superconformal primary of these multiplets is given by an operator with $(E, j_1, j_2, R_1, R_2, r_{\mathcal{N}=3}) = \left(R_1 + R_2, 0, 0, R_1, R_2, \frac{2R_2-2R_1}{3}\right)$ corresponding to the decomposition of $\operatorname{tr}\phi^{(I_1J_1}\dots\phi^{I_mJ_m)}$ under the branching (5.28) .

5.3.2 $\mathfrak{su}(2,2|3) \to \mathfrak{su}(2,2|2)$ Decomposition

For practical applications, rather than dealing with $\mathfrak{su}(2,2|3)$ representations, it is often convenient to choose a $\mathfrak{su}(2,2|2) \subset \mathfrak{su}(2,2|3)$ subalgebra. Representations of this algebra are labelled by (E, j_1, j_2, R, r) under the maximal bosonic subalgebra

$$\mathfrak{u}(1)_R \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus \mathfrak{su}(2)_R \oplus \mathfrak{u}(1)_r \subset \mathfrak{su}(2,2|2)\,. \tag{5.29}$$

There are essentially three different choices of such subalgebras. Throughout this paper we will require only one and we choose it to contain $\mathcal{Q}_{\alpha}^{I=1,2}$ and $\widetilde{\mathcal{Q}}_{\dot{\alpha}I=1,2}$ as the $\mathcal{N}=2$ supercharges. This corresponds to $\mathfrak{su}(3)\oplus\mathfrak{u}(1)_{r_{\mathcal{N}=3}}\to\mathfrak{su}(2)_R\oplus\mathfrak{u}(1)_r\oplus\mathfrak{u}(1)_f$. The Cartan of $\mathfrak{su}(2)_R$ is given by R and we take[7]

$$r=\frac{R_1}{2}+R_2+\frac{R_3}{2}\,,\quad R=\frac{R_1}{2}\,,\quad f=R_3\,. \tag{5.30}$$

Let us now list the branching of the multiplets $\hat{\mathcal{B}}_{[R_1,R_2]}$ under $\mathfrak{su}(2,2|3)\to\mathfrak{su}(2,2|2)\oplus\mathfrak{u}(1)_f$. For $\mathfrak{su}(2,2|2)$ multiplets we use the notation of [266]. See also [1, 255] for more general $\mathcal{N}=3\to\mathcal{N}=2$ multiplet decompositions. We have, valid for $R_1R_2\neq 0$,

$$\begin{aligned}\hat{\mathcal{B}}_{[R_1,R_2]}\simeq\;&\hat{\mathcal{B}}^{(R_2-R_1)}_{\frac{R_1+R_2}{2}}\oplus\mathcal{D}^{(R_2-R_1-1)}_{\frac{R_1+R_2-1}{2}(0,0)}\oplus\overline{\mathcal{D}}^{(R_2-R_1+1)}_{\frac{R_1+R_2-1}{2}(0,0)}\oplus\hat{\mathcal{C}}^{(R_2-R_1)}_{\frac{R_1+R_2-2}{2}(0,0)}\\&\oplus\bigoplus_{i=0}^{R_2-2}\left(\mathcal{B}^{(i-R_1)}_{\frac{R_1+i}{2},R_2-i(0,0)}\oplus\mathcal{C}^{(i-R_1+1)}_{\frac{R_1-1+i}{2},R_2-i-1(0,0)}\right)\\&\oplus\bigoplus_{i=0}^{R_1-2}\left(\overline{\mathcal{B}}^{(R_2-i)}_{\frac{R_2+i}{2},i-R_1(0,0)}\oplus\overline{\mathcal{C}}^{(R_2-i-1)}_{\frac{R_2+i-1}{2},i+1-R_1(0,0)}\right)\,,\end{aligned} \tag{5.31}$$

here the superscript lists the $\mathfrak{u}(1)_f$ charge. Moreover, the above is written with the understanding that any multiplet labelled with a negative value of R is set to zero. The stress-tensor is contained in (5.31) for $R_1=R_2=1$. We also stress that the $\simeq$ symbol means that the decomposition (5.31) holds only modulo long multiplets which begin to appear in the decomposition for $R_1R_2\geqslant 4$. For $R_2=0$ the decomposition is

$$\hat{\mathcal{B}}_{[R_1,0]}\cong\hat{\mathcal{B}}^{(-R_1)}_{\frac{R_1}{2}}\oplus\overline{\mathcal{D}}^{(1-R_1)}_{\frac{R_1-1}{2}(0,0)}\oplus\overline{\mathcal{E}}^{(0)}_{-R_1(0,0)}\oplus_{i=1}^{R_1-2}\overline{\mathcal{B}}^{(i-R_1+1)}_{\frac{R_1-i-1}{2},-i-1,(0,0)}\,, \tag{5.32}$$

while its conjugate with $R_1=0$ is given by

$$\hat{\mathcal{B}}_{[0,R_2]}\cong\hat{\mathcal{B}}^{(R_2)}_{\frac{R_2}{2}}\oplus\mathcal{D}^{(R_2-1)}_{\frac{R_2-1}{2}(0,0)}\oplus\mathcal{E}^{(0)}_{R_2(0,0)}\oplus_{i=1}^{R_2-2}\mathcal{B}^{(R_2-i-1)}_{\frac{R_2-i-1}{2},i+1,(0,0)}\,, \tag{5.33}$$

and contains $\mathcal{N}=2$ Coulomb branch operators. We stress that here we use the symbol $\cong$ to indicate that the decompositions (5.32) and (5.33) are exact. It is interesting to note that, simply by examining (5.31)-(5.33), we realize that once we know the Higgs branch ($\hat{\mathcal{B}}_R$ multiplets) we can predict the Coulomb branch ($\mathcal{E}_{r,(0,0)}$ multiplets) but not vice-versa. Note that, as a check, our above syntheses

[7] Our conventions for r,R,f are chosen to match those of [256].

and decompositions in terms of $\mathfrak{su}(2,2|3)$ representations are compatible with the decomposition [266]:

$$\begin{aligned}
\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,R_2,0]} \cong\,& (R_2+1)\hat{\mathcal{B}}_{\frac{R_2}{2}} \oplus \mathcal{E}_{R_2,(0,0)} \oplus \overline{\mathcal{E}}_{-R_2,(0,0)} + (R_2-1)\hat{\mathcal{C}}_{\frac{R_2-2}{2},(0,0)} \\
&\oplus R_2\mathcal{D}_{\frac{R_2-1}{2},(0,0)} \oplus R_2\overline{\mathcal{D}}_{\frac{R_2-1}{2},(0,0)} \\
&\oplus \bigoplus_{i=1}^{R_2-2} (i+1)\left(\mathcal{B}_{\frac{i}{2},R_2-i,(0,0)} \oplus \overline{\mathcal{B}}_{\frac{i}{2},i-R_2,(0,0)}\right) \\
&\oplus \bigoplus_{i=0}^{R_2-3} (i+1)\left(\mathcal{C}_{\frac{i}{2},R_2-i-2,(0,0)} \oplus \overline{\mathcal{C}}_{\frac{i}{2},i-R_2+2,(0,0)}\right) \\
&\oplus \bigoplus_{i=0}^{R_2-4}\bigoplus_{j=0}^{R_2-i-4} (i+1)\mathcal{A}^{R_2}_{\frac{i}{2},R_2-i-4-2j,(0,0)}\,.
\end{aligned} \tag{5.34}$$

5.4 Indices and the Discrete Gauging Prescription

Let us introduce the various quantities that we plan to discuss in this paper.

5.4.1 The Superconformal Index

The superconformal index for $\mathcal{N}=4$ SYM with respect to the supercharge $\mathcal{Q}=\mathcal{Q}^{I=1}_{-}$ is defined as [62, 64]

$$\begin{aligned}
\mathcal{I}^{\mathfrak{g}}\left(t,y,p,q\right) &= \mathrm{Tr}_{\mathbb{S}^3}(-1)^F t^{2(E+j_1)} y^{2j_2} p^{R_2} q^{R_2+2R_3} \\
&= \mathrm{Tr}_{\mathbb{S}^3}(-1)^F t^{2(E+j_1)} y^{2j_2} (pq)^{r-R} u_f^f
\end{aligned} \tag{5.35}$$

in the second line, since we often wish to treat $\mathcal{N}=4$ SYM as an $\mathcal{N}=2$ theory, we used (5.30) to write the generators in $\mathcal{N}=2$ language. We also defined $u_f^2=q^3/p$. The trace is taken over the Hilbert space of $\mathcal{N}=4$ SYM with gauge algebra $Lie(G)=\mathfrak{g}$ in the radial quantisation. In radial quantisation we can take $\mathcal{Q}^\dagger=\mathcal{S}$. The index (5.35) receives contributions only from those states satisfying

$$\delta := 2\left\{\mathcal{Q},\mathcal{S}\right\} = E-2j_1-\frac{1}{2}(3R_1+2R_2+R_3) \tag{5.36}$$

$$= E-2j_1-2R-r=0\,. \tag{5.37}$$

Letters	E	j_1	j_2	R_1	R_2	R_3	$i(t,y,p,q)$
F_{++}	2	1	0	0	0	0	t^6
$\bar{\lambda}^{I=1}_{\dot{\pm}}$	$\frac{3}{2}$	0	$\pm\frac{1}{2}$	1	0	0	$-t^3(y+y^{-1})$
$\lambda_{+I=2,3,4}$	$\frac{3}{2}$	$\frac{1}{2}$	0	$1,0,0$	$-1,1,0$	$0,-1,1$	$-t^4\left(\frac{1}{pq}+\frac{p}{q}+q^2\right)$
X,Y,Z	1	0	0	$0,1,1$	$1,-1,0$	$0,1,-1$	$t^2\left(pq+\frac{q}{p}+\frac{1}{q^2}\right)$
$\partial\bar{\lambda}^1=0$	$\frac{5}{2}$	$\frac{1}{2}$	0	1	0	0	t^6
$\partial_{+\dot{\pm}}$	1	$\frac{1}{2}$	$\pm\frac{1}{2}$	0	0	0	$t^3y\ ,\ t^3y^{-1}$

Table 5.1: *Letters with $\delta=0$ of the free $\mathcal{N}=4$ vector multiplet.*

The superconformal index is independent under continuous deformation of the corresponding QFT. In particular

$$\frac{\partial}{\partial\tau}\mathcal{I}^{\mathfrak{g}}\left(t,y,p,q\right)=0\,, \tag{5.38}$$

that is to say (5.35) is independent of the gauge coupling τ of $\mathcal{N}=4$ SYM. Following (5.38) the superconformal index (5.35) may be computed in the free theory by enumerating all of the components of the $\mathcal{N}=4$ field strength multiplet that obey (5.36) and then projecting onto gauge invariants. The projection onto gauge invariants is implemented by integration over the gauge group G. The index (5.35) then takes the form

$$\mathcal{I}^{\mathfrak{g}}(t,y,p,q)=\int d\mu_G(\mathbf{z})\,\mathrm{PE}\left[i(t,y,p,q)\chi^G_{\mathrm{adj}}(\mathbf{z})\right]\,, \tag{5.39}$$

$d\mu_G$ denotes the Haar measure of the gauge group G and χ^G_{adj} the character of its adjoint representation. The Plethystic exponential is defined in (A.1). The single letter index $i(t,y,p,q)$ may be computed by enumerating all letters with $\delta=0$, listed in Table 5.1. The on-shell degrees of freedom of the $\mathcal{N}=4$ field strength multiplet are $F_{\alpha\beta}$, $\widetilde{F}_{\dot{\alpha}\dot{\beta}}$, $\lambda_{\alpha I}$, $\bar{\lambda}^I_{\dot{\alpha}}$, ϕ^{IJ} with $I=1,2,3,4$ and ϕ^{IJ} is in the $[\mathbf{0},\mathbf{1},\mathbf{0}]$ of $\mathfrak{su}(4)$. We define $X=\phi^{12}$, $Y=\phi^{13}$ and $Z=\phi^{14}$. $\partial\bar{\lambda}^1$ denotes the equation of motion $\partial_{+\dot{+}}\bar{\lambda}^1_{\dot{-}}+\partial_{+\dot{-}}\bar{\lambda}^1_{\dot{+}}=0$ which enters with opposite statistics. Equivalently, one can evaluate the index of the $\mathfrak{psu}(2,2|4)$ multiplet $\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,1,0]}$, which is the free $\mathcal{N}=4$ vector multiplet plus conformal descendents. It is given by

$$\begin{aligned} i(t,y,p,q) =& \frac{\left(p^{-1}q+pq+q^{-2}\right)t^2-\left(q^2+p^{-1}q^{-1}+pq^{-1}\right)t^4}{(1-t^3y)(1-t^3y^{-1})} \\ & -\frac{t^3y}{1-t^3y}-\frac{t^3y^{-1}}{1-t^3y^{-1}}\,, \end{aligned} \tag{5.40}$$

where $\chi_{2j_2}(y)$ denotes the $SU(2)$ character given by

$$\chi_s(y)\equiv\chi_s=y^s+y^{s-2}+\cdots+y^{-s}\,. \tag{5.41}$$

The index (5.35) counts short representations of the $\mathfrak{su}(2,2|4)$ superconformal algebra, modulo recombination. Meaning that all short multiplets, see (F.14)-(F.19), contribute to the index, however, when they satisfy the recombination rules (F.8)-(F.13) they sum to zero. Recombination happens when a long multiplet $\mathcal{A}^{E}_{[R_1,R_2,R_3],(j_1,j_2)}$ hits the unitary bound and decomposes into semi-direct sums of short representations. We list the possible recombination rules, viewing as an $\mathcal{N}=2$ theory, in equations (F.8)-(F.13). The index (5.35) can therefore be expanded in the following form

$$\mathcal{I}^{\mathfrak{g}}(t,y,p,q)=\sum_{S_{\mathcal{N}=4}\in\text{shorts}}\mathcal{I}_{S_{\mathcal{N}=4}}(t,y,p,q)\,, \tag{5.42}$$

where the sum is taken over the short multiplets of the theory, modulo those that can recombine into long multiplets. We list the indices of multiplets of an $\mathfrak{su}(2,2|2)\subset\mathfrak{psu}(2,2|4)$ subalgebra in Appendix F.2. As we discussed in Section 5.2 at $\tau=e^{\pi\mathrm{i}/3},\mathrm{i},e^{\pi\mathrm{i}/3}$ the global symmetry group (at the level of local operators) of the theory has a $\mathbb{Z}_n$ enhancement. Correspondingly the Hilbert space carries an extra $\mathbb{Z}_n$ grading at those values of the coupling. Therefore one may define a further refined version of the superconformal index given by

$$\mathcal{I}^{\mathfrak{g}}\left(t,y,p,q,\epsilon\right)=\mathrm{Tr}_{\mathbb{S}^3}(-1)^F t^{2(E+j_1)}y^{2j_2}(pq)^{r-R}u_f^f\epsilon^{r_n+s_n}\,, \tag{5.43}$$

where we introduced the $\mathbb{Z}_n$-valued fugacity ϵ in order to keep track of the discrete symmetry. We stress that the $\mathbb{Z}_n$ is a global symmetry only at $\tau=e^{\pi\mathrm{i}/3},\mathrm{i},e^{\pi\mathrm{i}/3}$. As we showed in Appendix F.1 the $\mathbb{Z}_n$ commutes with the supercharges $\mathcal{Q}^{I=1}_{-}$ and $\mathcal{S}^{-}_{I=1}$ that we used to compute the index (5.35) with respect to. Moreover, we also demonstrated that the $\mathbb{Z}_n$ preserves a $\mathfrak{su}(2,2|3)\subset\mathfrak{psu}(2,2|4)$ subalgebra and it therefore preserves the recombination rules (F.8)-(F.13). Therefore, the refined index (5.43) can again be expanded

$$\begin{aligned}\mathcal{I}^{\mathfrak{g}}(t,y,p,q,\epsilon) &= \sum_{\mathcal{S}_{\mathcal{N}=4}\in\text{shorts}} \mathcal{I}_{\mathcal{S}_{\mathcal{N}=4}}(t,y,p,q,\epsilon) \\ &= \sum_{\bigoplus_i \mathcal{S}^{(i)}_{\mathcal{N}=3}\in\text{shorts}} \epsilon^{r_n(\mathcal{S}_{\mathcal{N}=3})+s_n(\mathcal{S}_{\mathcal{N}=3})}\mathcal{I}_{\mathcal{S}_{\mathcal{N}=3}}(t,y,p,q)\,.\end{aligned} \tag{5.44}$$

In the final equality, we firstly used the fact that any short multiplet $\mathcal{S}_{\mathcal{N}=4}$ of $\mathfrak{psu}(2,2|4)$ can be decomposed into multiplets of a $\mathfrak{su}(2,2|3)$ subalgebra $\mathcal{S}_{\mathcal{N}=4} \simeq \bigoplus_i \mathcal{S}^{(i)}_{\mathcal{N}=3}$. Secondly we used the fact that the action of $r_n + s_n$ preserves the $\mathfrak{su}(2,2|3) \subset \mathfrak{psu}(2,2|4)$ subalgebra and by $r_n(\mathcal{S}_{\mathcal{N}=3}) + s_n(\mathcal{S}_{\mathcal{N}=3})$ we mean the generator of $\mathbb{Z}_n$ evaluated on the given multiplet. For example, using (5.27), the refined index on the free $\mathcal{N}=4$ vector multiplet is given by

$$\mathcal{I}_{\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,1,0]}}(t,y,p,q,\epsilon) = \epsilon^{-1}\,\mathcal{I}_{\hat{\mathcal{B}}_{[1,0]}}(t,y,p,q) + \epsilon\,\mathcal{I}_{\hat{\mathcal{B}}_{[0,1]}}(t,y,p,q) \tag{5.45}$$

Where

$$\mathcal{I}_{\hat{\mathcal{B}}_{[1,0]}}(t,y,p,q) = \frac{q^{-2}t^2 - \left(p^{-1}q^{-1} + pq^{-1}\right)t^4 + t^6}{(1-t^3y)(1-t^3y^{-1})} \tag{5.46}$$

$$\mathcal{I}_{\hat{\mathcal{B}}_{[0,1]}}(t,y,p,q) = \frac{p^{-1}qt^2 + pqt^2 - (y+y^{-1})t^3 - q^2t^4 + t^6}{(1-t^3y)(1-t^3y^{-1})} \tag{5.47}$$

We may then gauge the discrete $\mathbb{Z}_n$ symmetry by making the projection

$$\mathcal{I}^{\mathfrak{g}}_{\mathbb{Z}_n}(t,y,p,q) := \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n} \mathcal{I}^{\mathfrak{g}}(t,y,p,q,\epsilon)\,. \tag{5.48}$$

The discrete gauging restricts each contribution, in terms of either $\mathfrak{su}(2,2|3)$ or $\mathfrak{su}(2,2|2)$ multiplets, to satisfy

$$r_n + s_n = r + f + s_n = 0 \bmod n\,. \tag{5.49}$$

We demonstrate in Section 5.7 that (5.44) reproduces the *refined* superconformal index (5.43) at large N by matching with the KK supergraviton index (5.137) obtained by the AdS/CFT computation of [251]. We would like stress that the final expression for KK supergraviton index is *not equal* to the index of a theory obtained through an S-fold projection. Since, as shown in Section 5.7, this last expression is obtained implementing the orbifold projection at the level of the single particle index.

5.4.2 Coulomb Branch Limit

The graded index (5.43) may be rewritten as [168]

$$\mathcal{I}^{\mathfrak{g}}\left(t, y, p, q, \epsilon\right) = \mathrm{Tr}_{\mathbb{S}^3}(-1)^F \tau^{\frac{1}{2}\delta^2_+} \sigma^{\frac{1}{2}\widetilde{\delta}_{\dot{+}2}} \rho^{\frac{1}{2}\widetilde{\delta}_{\dot{-}2}} u_f^f \epsilon^{r_n + s_n}\,, \tag{5.50}$$

with

$$\tau := \frac{t^2}{\sqrt{pq}}\,, \quad \sigma := ty\sqrt{pq}\,, \quad \rho := \frac{t\sqrt{pq}}{y}\,, \quad u_f := \sqrt{\frac{q^3}{p}}\,, \tag{5.51}$$

and

$$\delta^2_\pm := 2\left\{\mathcal{Q}^2_\pm, \left(\mathcal{Q}^2_\pm\right)^\dagger\right\} = E \pm 2j_1 + 2R - r\,, \tag{5.52}$$

$$\widetilde{\delta}_{\dot{\pm}2} := 2\left\{\widetilde{\mathcal{Q}}_{\dot{\pm}2}, \left(\widetilde{\mathcal{Q}}_{\dot{\pm}2}\right)^\dagger\right\} = E \pm 2j_2 - 2R + r\,. \tag{5.53}$$

In the parametrisation (5.50) the Coulomb branch limit of the superconformal index is defined to be [168]

$$\tau \to 0\,, \quad \rho\,, \sigma \ \text{ fixed,} \tag{5.54}$$

which is well defined since $\delta^2_+ \geqslant 0$. In this limit the index is then given by

$$\mathcal{I}^{\mathfrak{g}}_{\mathrm{CB}}\left(\rho, \sigma, u_f, \epsilon\right) = \mathrm{Tr}_{\mathbb{S}^3|\delta^2_+=0}(-1)^F \sigma^{\frac{1}{2}\widetilde{\delta}_{\dot{+}2}} \rho^{\frac{1}{2}\widetilde{\delta}_{\dot{-}2}} u_f^f \epsilon^{r_n + s_n}\,. \tag{5.55}$$

Defining

$$\rho\sigma = x\,, \quad \rho/\sigma = v\,, \tag{5.56}$$

the single letter index (5.40) in the Coulomb branch limit becomes

$$i_{\mathrm{CB}}(x) = x\,. \tag{5.57}$$

In our $\mathcal{N} = 2$ decomposition this is simply the contribution of the single letter X described in Table 5.1. Since, for our theories, it is independent of both the ratio $v = \rho/\sigma$ and u_f then, due to $\left(\widetilde{\delta}_{\dot{+}2} + \widetilde{\delta}_{\dot{-}2}\right)\mathcal{Q}^1_\alpha = \left(\widetilde{\delta}_{\dot{+}2} + \widetilde{\delta}_{\dot{-}2}\right)\mathcal{Q}^2_+ = 0$, (5.55) is further shortened and preserves $\mathcal{Q}^1_\pm$, $\mathcal{Q}^2_+$. This allows us to write

$$E = r\,, \quad j_1 = j_2 = f = R = 0\,, \tag{5.58}$$

$\mathfrak{g}$	exponents($\mathfrak{g}$)
$\mathfrak{u}(N)$	$0, 1, 2, \ldots, N-1$
A_N	1, 2, 3, ..., N
D_N	1, 3, 5, ..., $2N-3$; $N-1$
E_6	1, 4, 5, 7, 8, 11
E_7	1, 5, 7, 9, 11, 13, 17
E_8	1, 7, 11, 13, 17, 19, 23, 29

Table 5.2: *Exponents of the Lie algebra* $\mathfrak{g}$.

these are the highest weight states of $\mathcal{E}_{r,(0,0)}$ multiplets that generate the $\mathcal{N}=2$ Coulomb branch chiral ring. Therefore the only non-zero contributions to (5.55) are from

$$\mathcal{I}_{\mathcal{E}_{r,(0,0)}}(x, \epsilon = 1) = x^r \,. \tag{5.59}$$

Hence, following (5.18), the prescription (5.44) can be implemented simply by

$$x \to \epsilon x \,, \tag{5.60}$$

and the index can be written as

$$\mathcal{I}^{\mathfrak{g}}_{\mathrm{CB}}\,(x, \epsilon) = \mathrm{Tr}_{\mathbb{S}^3|\delta^2_+=0}\left[\epsilon^r x^r\right] = \int d\mu_G(\mathbf{z})\,\mathrm{PE}\left[i_{\mathrm{CB}}(\epsilon x)\,\chi^G_{\mathrm{adj}}(\mathbf{z})\right] \,. \tag{5.61}$$

We would like to stress that the $\mathcal{E}_{r,(0,0)}$ multiplets do not recombine [266] and therefore turning on the refinement ϵ for the discrete symmetry commutes with the integration over G. Let G be connected then, as pointed out in [168], (5.61) may be explictly evaluated thanks to Macdonald's constant-term identities [269, 270]

$$\mathcal{I}^{\mathfrak{g}}_{\mathrm{CB}}\,(x, \epsilon) = \mathrm{PE}\left[\sum_{j \in \mathrm{exponents}(\mathfrak{g})} \epsilon^{j+1} x^{j+1}\right] \,, \tag{5.62}$$

where exponents($\mathfrak{g}$) denotes the set of exponents of the Lie algebra $\mathfrak{g} = Lie(G)$. The elements of exponents($\mathfrak{g}$) are in one-to-one correspondence with the degrees of the generators of the ring of $\mathfrak{g}$-invariant polynomials. We list the elements of exponents($\mathfrak{g}$) for $\mathfrak{g} = ADE$ and $\mathfrak{u}(N)$ in Table 5.2. According to (5.48), upon the discrete gauging, we then have

$$\mathcal{I}^{\mathfrak{g}}_{\mathbb{Z}_n,\mathrm{CB}}(x) = \frac{1}{|\mathbb{Z}_n|} \sum_{\epsilon \in \mathbb{Z}_n} \mathcal{I}^{\mathfrak{g}}_{\mathrm{CB}}(x, \epsilon) \; . \tag{5.63}$$

Since, in all known examples, (5.63) counts only gauge invariant chiral operators with $j_1 = j_2 = 0$, it is equal to the Coulomb branch Hilbert series for the discretely gauged theory

$$\mathcal{I}^{\mathfrak{g}}_{\mathbb{Z}_n,\mathrm{CB}}(x) \equiv \mathrm{HS}(x; CB_{\mathfrak{g},n}) \tag{5.64}$$

Therefore the rank, i.e. the complex dimension of the Coulomb branch chiral ring is equal to [95]

$$\dim_{\mathbb{C}} CB_{\mathfrak{g},n} = \Big(\text{Order of pole at } x = 1 \text{ of } \mathcal{I}^{\mathfrak{g}}_{\mathbb{Z}_n,\mathrm{CB}}(x)\Big) \; . \tag{5.65}$$

We of course expect that $\dim_{\mathbb{C}} CB_{\mathfrak{g},n} = \mathrm{rank}\,\mathfrak{g}$. In the following sections we analyse some examples.

5.4.3 Higgs Branch Hilbert Series

In general the Hilbert series [108, 107] counts gauge invariant chiral operators graded by their charges under a maximally commuting subalgebra of the global symmetry algebra. We will be interested in computing the Hilbert series for the Higgs branch $HB_{\mathfrak{g}}$ of $\mathcal{N} = 4$ SYM (using the $\mathcal{N} = 2$ decomposition (5.30)). This is given by

$$\mathrm{HS}(\tau, u_f, \epsilon; HB_{\mathfrak{g}}) \equiv \mathrm{HS}^{\mathfrak{g}}(\tau, u_f, \epsilon) := \mathrm{Tr}_{HB_{\mathfrak{g}}}\, \tau^{2R} u_f^f \epsilon^{r_n + s_n} \; , \tag{5.66}$$

where $HB_{\mathfrak{g}} = \{\mathcal{O}_i | \widetilde{\mathcal{Q}}^I_{\dot\alpha} \mathcal{O}_i = 0\, , M_{\mu\nu}\mathcal{O}_i = 0\, , r\mathcal{O}_i = 0\}$ is the Higgs branch chiral ring, i.e. the space of scalar, $\mathfrak{g}$-invariant chiral operators that parametrize the Higgs branch moduli space of vacua. In the language of the previous section (5.66) is counting $\hat{\mathcal{B}}_R$ operators with $E = 2R$ and $r = j_1 = j_2 = 0$. We stress that there is no recombination rule (F.8)-(F.13) involving only $\hat{\mathcal{B}}_R$ operators.

The Hilbert series for the Higgs branch is closely related to a certain limit of the superconformal index known as the *Hall-Littlewood limit* [168]. In fact, for genus zero theories in class $\mathcal{S}$ they are conjectured to be equal, with τ of (5.66) the same as for (5.50). For genus one (this is the relevant case for $\mathcal{N} = 4$ SYM) and greater the Hall-Littlewood operators contain the Higgs branch operators as a subset. See Appendix H for more discussion on this point.

In the $\mathcal{N}=2$ decomposition that we used in Section 5.3, the Higgs branch for our theories is reached by setting equal to zero the scalar field X in the $\mathcal{N}=2$ vector multiplet. Therefore there is only one relevant F-term that we must take into account

$$\partial_X W = [Y, Z] = 0\,, \tag{5.67}$$

where W is the superpotential for $\mathcal{N}=4$ SYM. Unfortunately, due to the fact that the gauge group is not completely broken on the Higgs branch, letter counting techniques cannot be used to compute (5.66). Instead, in order to compute (5.66), we use the package *Macaulay2* [112]. By inputting the ring of polynomials $R=\mathbb{C}[Y,Z]$ and the ideal $I=\langle [Y,Z]\rangle$ given by (5.67), *Macaulay2* can compute the Hilbert series for R/I.

Since both r and s_n act trivially on the fields Y, Z; on the Higgs branch $r_n+s_n=f$. Therefore the extra grading may be implemented by $u_f \to \epsilon u_f$. The Higgs branch Hilbert series then takes the form

$$\mathrm{HS}^{\mathfrak{g}}(\tau, u_f, \epsilon) = \int d\mu_G(\mathbf{z}) f^{\flat}(\tau, \epsilon u_f, \mathbf{z})\,, \tag{5.68}$$

where $f^{\flat}(\tau, u_f, \mathbf{z}) := \mathrm{HS}(\tau, u_f, \mathbf{z}; R/I)$ denotes the Hilbert series for the ring R/I. The discrete gauged Higgs branch Hilbert series reads

$$\mathrm{HS}(\tau, u_f; HB_{\mathfrak{g},n}) := \mathrm{HS}^{\mathfrak{g}}_{\mathbb{Z}_n}(\tau, u_f) = \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n} \mathrm{HS}^{\mathfrak{g}}(\tau, u_f, \epsilon)\,. \tag{5.69}$$

One important piece of information carried by (5.66) is the dimension of the Higgs branch

$$\dim_{\mathbb{C}} HB_{\mathfrak{g},n} = \left(\text{Order of pole at } \tau=1 \text{ of } \mathrm{HS}^{\mathfrak{g}}_{\mathbb{Z}_n}(\tau,1)\right)\,. \tag{5.70}$$

A particularly useful quantity is the Plethystic logarithm of the Hilbert series

$$h^{\mathfrak{g}}_{\mathbb{Z}_n}(\tau, u_f) := \mathrm{PLog}\left[\mathrm{HS}^{\mathfrak{g}}_{\mathbb{Z}_n}(\tau, u_f)\right]\,. \tag{5.71}$$

The Plethystic logarithm is defined in (A.2). The Plethystic logarithm of the Hilbert series satisfies [108, 271]:

- When the moduli space is a complete intersection variety $h^{\mathfrak{g}}_{\mathbb{Z}_n}$ is a polynomial of finite degree. When it is not the $h^{\mathfrak{g}}_{\mathbb{Z}_n}$ is an infinite series in τ.
- It has been conjectured in [107, 108] that when the moduli space is a complete

intersection variety the first coefficients with positive sign in the polynomial $h^{\mathfrak{g}}_{\mathbb{Z}_n}$ encode the generators of the variety. Negative coefficients encode relations. When the moduli space is not a complete intersection the generators of the moduli space are generally still captured by the first positive terms. However, in this last case, most of the contributions in the PLog expansion represent Hilbert syzygies.

5.5 Rank One Theories

Having introduced the main quantities that we wish to compute we will now go ahead and compute them for the possible $\mathcal{N}=3$ rank one theories that can be obtained via discrete gauging of $\mathcal{N}=4$ SYM. As we mentioned previously, if we restrict to connected groups then, from the point of view of the superconformal index there are only two distinct possibilities, labelled by the two choices of Lie algebras of rank one i.e. $\mathfrak{g}=\mathfrak{u}(1)$ and $\mathfrak{g}=\mathfrak{su}(2)$.

5.5.1 $\mathfrak{g}=\mathfrak{u}(1)$

Let us begin with the $\mathbb{Z}_n$ gauging of $\mathfrak{g}=\mathfrak{u}(1)$ $\mathcal{N}=4$ SYM.

Superconformal Index

Since the $\mathfrak{u}(1)$ $\mathcal{N}=4$ theory is free the index for the discrete gauging can be computed explicitly. Using (5.45), it is given by

$$\begin{aligned}\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_n}(t,y,p,q) =&\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_n}(t,y,p,q,\epsilon)\\ =&\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\mathrm{PE}\left[\epsilon^{-1}\,\mathcal{I}_{\hat{\mathcal{B}}_{[1,0]}}+\epsilon\,\mathcal{I}_{\hat{\mathcal{B}}_{[0,1]}}\right]\,.\end{aligned}\tag{5.72}$$

The index may be equivalently expressed in terms of Elliptic Gamma functions [272]

$$\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_n}=\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\frac{\Gamma\left(\frac{\epsilon q t^2}{p};t^3y,\frac{t^3}{y}\right)\Gamma\left(\frac{t^2}{\epsilon q^2};t^3y,\frac{t^3}{y}\right)\Gamma\left(\epsilon p q t^2;t^3y,\frac{t^3}{y}\right)}{(\epsilon t^3y;t^3y)^{-1}\left(\frac{t^3y}{\epsilon};t^3y\right)^{-1}\Gamma\left(\frac{t^3}{\epsilon y};t^3y,\frac{t^3}{y}\right)}\,.\tag{5.73}$$

The Elliptic Gamma function and q-Pochammer symbol are defined in equations (A.22) and (A.12) respectively. When $n=2$ the expression (5.72) is exactly the index for the $G=O(2)$ $\mathcal{N}=4$ theory which matches the expectation that the this theory is nothing but the usual $O3^-$ orientifold theory. We can perform several other checks of our expression (5.72) by studying the various limits that we outlined in Section 5.4.

Coulomb Branch Limit

After taking the Coulomb branch limit (5.54) we find, for the discrete gauging of the $\mathfrak{g} = \mathfrak{u}(1)$ theory,

$$\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_n,\text{CB}}(x) = \frac{1}{n}\sum_{\epsilon\in\mathbb{Z}_n} \text{PE}\left[\epsilon x\right] = \text{PE}\left[x^n\right] . \tag{5.74}$$

This implies that the Coulomb branch is freely generated by $\widetilde{u} = u^n$ with $u = X$ the parent Coulomb branch parameter. Therefore $E(\widetilde{u}) = r(\widetilde{u}) = n$ which implies that the $\widetilde{u}$ is the superconformal primary of the $\mathfrak{su}(2,2|2)$ multiplet $\mathcal{E}_{n,(0,0)} \subset \hat{\mathcal{B}}_{[0,n]}$. The topology is simply $CB_{\mathfrak{u}(1),n} = \mathbb{C}[\widetilde{u}]$ and so $\mathbf{CB}_{\mathfrak{u}(1),n} \cong \mathbb{C}$. We wish to point out that (5.74) is in perfect agreement with the expected spectrum of Coulomb operators (5.11) coming from the S-fold analysis [247] and the Seiberg-Witten curve analysis for the quotient of the I_0 geometry in the discussion below equation (2.8) of [159].

Higgs Branch Hilbert Series

We now compute the Higgs branch Hilbert Series for these theories. For $\mathfrak{g} = \mathfrak{u}(1)$ the superpotential (5.67) is trivial and we may actually use letter counting. We find that $f^\flat(\tau, u_f, \epsilon) = \text{PE}\left[\epsilon u_f \tau + \epsilon^{-1} u_f^{-1}\tau\right]$. The integration over the gauge group is trivially performed and we get

$$\begin{aligned}\text{HS}^{\mathfrak{u}(1)}_{\mathbb{Z}_n}(\tau, u_f) &= \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n} f^\flat(\tau, \epsilon u_f) = \text{PE}\left[\tau^2 + (u_f^n + u_f^{-n})\tau^n - \tau^{2n}\right] \\ &= \text{Hilbert Series of } \mathbb{C}^2/\mathbb{Z}_n\,.\end{aligned} \tag{5.75}$$

The generators are simply given by

$$W^+ = Y^n\,, \quad W^- = Z^n\,, \quad J = YZ\,. \tag{5.76}$$

They satisfy the relation $W^+W^- = J^n$. For $n = 1, 2$ (5.75) can be expanded in $\mathfrak{su}(2)_{u_f}$ characters (in agreement with the fact that these theories have $\mathcal{N} = 4$ supersymmetry) while for $n = 3, 4, 6$ there is only $\mathfrak{u}(1)_{u_f}$. In terms of $\mathfrak{su}(2,2|3)$

multiplets this is equivalently expressed as

$$\hat{\mathcal{B}}_{[n,0]}\hat{\mathcal{B}}_{[0,n]} \sim \left(\hat{\mathcal{B}}_{[1,1]}\right)^{n}. \tag{5.77}$$

The topology of the moduli space and relation are in perfect agreement with equations (2.1) and (2.16), respectively, of [256, 255].

Schur Limit

The Schur limit of the index given by setting [168]

$$t = ypq\,. \tag{5.78}$$

Hence, in the Schur limit:

$$\mathcal{I}^{\mathfrak{g}}_{\mathbb{Z}_k,\text{Schur}}(\tau, u_f) = \text{Tr}_{\mathbb{S}^3}(-1)^F \tau^{2E-2R} u_f^f\,, \tag{5.79}$$

where $\tau = \frac{t^2}{\sqrt{pq}}$. Writing $\tau^2 = \mathbf{q} = e^{\beta}$ it may be argued that [273, 229, 274, 275]

$$\lim_{\tau\to 1} \log \mathcal{I}^{\mathfrak{g}}_{\mathbb{Z}_k,\text{Schur}}(\tau, 1) \sim \frac{4\pi^2(a-c)}{\beta} + \mathcal{O}(\beta^0)\,, \tag{5.80}$$

where c, a denote the central charge and weyl anomaly coefficient for the 4d SCFT. For any 4d $\mathcal{N} \geqslant 3$ SCFT [248]

$$a = c\,. \tag{5.81}$$

We find

$$\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_k,\text{Schur}}(\tau, u_f) = \frac{1}{k}\sum_{\epsilon\in\mathbb{Z}_k} \frac{\left(\epsilon^{-1}\tau^2;\tau^2\right)\left(\epsilon\tau^2;\tau^2\right)}{\left(\epsilon^{-1}u_f^{-1}\tau;\tau^2\right)\left(\epsilon u_f\tau;\tau^2\right)}\,. \tag{5.82}$$

We can now check $a \stackrel{?}{=} c$ which we of course expect to hold for any $\mathcal{N}=3$ theory. It can be shown that

$$\log\left(\mathbf{q}^x;\mathbf{q}\right) = -\mathcal{L}_{\mathbf{q}}(-1, x) \tag{5.83}$$

where

$$\mathcal{L}_{\mathbf{q}}(s, x) = \sum_{j=1}^{\infty} \frac{j^s \mathbf{q}^{jx}}{1-\mathbf{q}^j} \tag{5.84}$$

is the Lambert series. For $s=-1$ it may be expressed as [276]

$$\mathcal{L}_{\mathbf{q}}(-1,x) = -\sum_{n=0}^{\infty}\log(n+x) + B_1(x)\log(-\beta) - \sum_{\substack{n\geqslant 0\\ n\neq 1}}^{\infty}\frac{\zeta(2-n)}{n!}B_n(x)\beta^{n-1} \tag{5.85}$$

where $B_n(x)$ is the n^{th} Bernoulli polynomial. $B_n(x)$ satisfies $\lim_{x\to\infty}B_n(x)=x^n$. Let us suppose $x=(\sigma+\eta\beta)/\beta$ with σ, η c-numbers independent of β then the leading divergence is

$$\lim_{\beta\to 0}\mathcal{L}_{\mathbf{q}}\left(-1,\frac{\sigma+\eta\beta}{\beta}\right) = -\frac{\zeta(2)}{\beta} - \frac{1}{\beta}\sum_{n=2}^{\infty}\frac{\zeta(2-n)}{n!}a^n + \ldots \tag{5.86}$$

where the dots indicate less singular terms. In particular, this implies that

$$\begin{aligned}\lim_{\beta\to 0}\log\frac{\left(\mathbf{q}^{\frac{\sigma+\beta\eta}{\beta}};\mathbf{q}\right)}{\left(\mathbf{q}^{\frac{\gamma+\beta\rho}{\beta}};\mathbf{q}\right)} &= \lim_{\beta\to 0}\left[\mathcal{L}_{\mathbf{q}}\left(-1,\frac{\gamma+\rho\beta}{\beta}\right) - \mathcal{L}_{\mathbf{q}}\left(-1,\frac{\sigma+\eta\beta}{\beta}\right)\right] \\ &\sim \frac{1}{\beta}\sum_{n=2}^{\infty}\frac{\zeta(2-n)}{n!}(\sigma^n-\gamma^n)+\ldots\,.\end{aligned} \tag{5.87}$$

Using (5.84) to rewrite (5.82) at $u_f=1$ as

$$\frac{1}{k}\sum_{l=1}^{k} e^{-\mathcal{L}_{\mathbf{q}}\left(-1,\frac{\beta-\pi i l/k}{\beta}\right)-\mathcal{L}_{\mathbf{q}}\left(-1,\frac{\beta+\pi i l/k}{\beta}\right)+\mathcal{L}_{\mathbf{q}}\left(-1,\frac{\beta/2-\pi i l/k}{\beta}\right)+\mathcal{L}_{\mathbf{q}}\left(-1,\frac{\beta/2+\pi i l/k}{\beta}\right)}\,. \tag{5.88}$$

Taking the logarithm and applying the formula (5.86), (5.87) we find

$$\lim_{\tau\to 0}\log\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_k,\text{Schur}}(\tau,1) \sim \frac{0}{\beta}+\ldots\,, \tag{5.89}$$

where the dots indicate less divergent terms. Applying (5.80) implies that indeed $a=c$ for this theory.

5.5.2 $\mathfrak{g}=\mathfrak{su}(2)$

For $\mathfrak{g}=\mathfrak{su}(2)$ it is very difficult to compute (5.35) in closed form. For this reason we will instead study only the Coulomb branch limit of the index and the Higgs branch Hilbert series.

Coulomb Branch Limit

Let us now study the Coulomb branch limit (5.63). The corresponding computation can be easily performed and we get

$$\mathcal{I}^{\mathfrak{su}(2)}_{\mathbb{Z}_n,\text{CB}}(x)=\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\text{PE}\left[\epsilon^2x^2\right]=\begin{cases}\text{PE}\left[x^2\right] & n=1\\ \text{PE}\left[x^n\right] & n=2,4,6\\ \text{PE}\left[x^6\right] & n=3\end{cases}. \tag{5.90}$$

In each case is $CB_{\mathfrak{su}(2),n}=\mathbb{C}\left[\widetilde{u}\right]$ and so $\mathbf{CB}_{\mathfrak{su}(2),n}\cong\mathbb{C}$. For $n=2,4,6$ the Coulomb branch ring of the discretely gauged theory, $CB_{\mathfrak{su}(2),n}$, is generated by $\widetilde{u}=u^{n/E(u)}$ where $u=\frac{1}{2}\operatorname{tr}X^2$ is the Coulomb branch parameter of the parent theory. Therefore $E(\widetilde{u})=r(\widetilde{u})=n$ which belong to $\mathcal{E}_{n,(0,0)}\subset\hat{\mathcal{B}}_{[0,n]}$ for $n=2,4,6$. This matches with the discussion below equation (2.8) of [159] for the I_4-series I_0^* geometries. The $n=3$ case is slightly different since $E(u)=2$ is not a divisor of $n=3$ and $CB_{\mathfrak{su}(2),3}$ is generated by $\widetilde{u}=u^n=u^3$. Nevertheless this is in perfect agreement with the discussion below equation (A.7) of [159] for the I_2-series I_0^* geometries. These parent theories do not come from S-folds and so do not fall into the considerations of [247].

Higgs Branch Hilbert Series

Let us now compute the Higgs branch Hilbert series (5.66). For the case at hand the gauge group is not completely broken and we cannot use letter counting. Therefore we compute the F-flat Hilbert series using *Macaulay2*. We obtain

$$\begin{aligned}f^\flat(\tau,\epsilon u_f,z)&=\left(1-\chi_2(z)\tau^2+\left(\epsilon u_f+\frac{1}{\epsilon u_f}\right)\tau^3\right)\\&\quad\times\text{PE}\left[\tau\left(\epsilon u_f+\frac{1}{\epsilon u_f}\right)\chi_2(z)\right].\end{aligned} \tag{5.91}$$

Note that the same result was already found, for $n=1$, in [277]. After the integration over the $SU(2)$ gauge group we get

$$\begin{aligned}\text{HS}^{\mathfrak{su}(2)}_{\mathbb{Z}_n}(\tau,u_f)&=\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\int d\mu_{SU(2)}(z)f^\flat(\tau,\epsilon u_f,z)\\&=\frac{1}{n}\sum_{\epsilon\in\mathbb{Z}_n}\text{PE}\left[\left(1+u_f^2\epsilon^2+\frac{1}{u_f^2\epsilon^2}\right)\tau^2-\tau^4\right].\end{aligned} \tag{5.92}$$

Summing over the possible values of ϵ we get

$$\mathrm{HS}^{\mathfrak{su}(2)}_{\mathbb{Z}_n}(\tau, u_f) = \begin{cases} \mathrm{PE}\left[\left(1 + u_f^2 + u_f^{-2}\right)\tau^2 - \tau^4\right] & n = 1 \\ \mathrm{PE}\left[\tau^2 + \left(u_f^n + u_f^{-n}\right)\tau^n - \tau^{2n}\right] & n = 2, 4, 6 \\ \mathrm{PE}\left[\tau^2 + \left(u_f^6 + u_f^{-6}\right)\tau^6 - \tau^{12}\right] & n = 3 \end{cases} . \tag{5.93}$$

We again define the generators

$$W^+ = \frac{1}{2}\operatorname{tr} Y^n\,, \quad W^- = \frac{1}{2}\operatorname{tr} Z^n\,, \quad J = \frac{1}{2}\operatorname{tr} YZ\,. \tag{5.94}$$

For $n \in \{2, 4, 6\}$ we have $W^+W^- = J^n$ and the topology of the Higgs branch is $\mathbb{C}^2/\mathbb{Z}_n$. The $n = 1$ case is the same as $n = 2$. The $n = 3$ case is also the same as $n = 6$. In terms of $\mathfrak{su}(2,2|3)$ multiplets, after discarding the $n = 3$ case, we again have

$$\hat{\mathcal{B}}_{[n,0]}\hat{\mathcal{B}}_{[0,n]} \sim \left(\hat{\mathcal{B}}_{[1,1]}\right)^n . \tag{5.95}$$

We again find agreement with [256, 255].

5.6 Higher Rank Theories

Having studied in detail the rank one theories we now turn our attention to $\mathbb{Z}_n$ discrete gauging of higher rank theories. We limit most of our attention to the cases of $\mathfrak{g} = AD$ and $\mathfrak{g} = \mathfrak{u}(N)$ where the S-duality group (5.12) acting on local operators is given by $SL(2, \mathbb{Z})$. In general the computation of the full discretely gauged index (5.48) for $\mathfrak{g} = \mathfrak{u}(N), A, D$ is very difficult to perform. Therefore, also for this class of theories, we decide to focus our attention only on the Coulomb branch limit of the index (5.63) and on the Higgs branch Hilbert series (5.69). For the Hilbert series we only explicitly present the rank 2 cases. In the final subsection we will discuss the Coulomb branch index for the cases $\mathfrak{g} = E_6, E_7, E_8$.

5.6.1 $\mathfrak{g} = \mathfrak{u}(N)$

Coulomb Branch Limit

Let us study the Coulomb branch limit (5.54). Applying (5.62) we find

$$\mathcal{I}^{\mathfrak{u}(N)}_{\mathbb{Z}_n,\mathrm{CB}}(x) = \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n} \mathrm{PE}\left[\sum_{j=1}^{N} \epsilon^j x^j\right] . \tag{5.96}$$

We list a few cases for low rank. We define for $n=1$ the generators of $CB_{\mathfrak{u}(N)}$ to be $u_j = \frac{1}{j}\,\mathrm{tr}\, X^j$. For $N=2$ we collate the results for the Coulomb branch index below

$\mathcal{I}^{\mathfrak{u}(2)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$	n	Generators	Relation
$\mathrm{PE}\left[x+x^2\right]$	1	u_1, u_2	$\diagup$
$\mathrm{PE}\left[2x^2\right]$	2	$\widetilde{u}_1 = u_1^2,\, u_2$	$\diagup$
$\mathrm{PE}\left[2x^3+x^6-x^9\right]$	3	$\widetilde{u}_1 = u_1^3,\, \widetilde{u}_2 = u_1 u_2,\, \widetilde{u}_3 = u_2^3$	$\widetilde{u}_1\widetilde{u}_3 = \widetilde{u}_2^3$
$\mathrm{PE}\left[3x^4-x^8\right]$	4	$\widetilde{u}_1 = u_1^4,\, \widetilde{u}_2 = u_2^2,\, \widetilde{u}_3 = u_1^2 u_2$	$\widetilde{u}_1\widetilde{u}_2 = \widetilde{u}_3^2$
$(1+2x^6)\,\mathrm{PE}\left[2x^6\right]$	6	Not complete intersection	

By $\diagup$ we mean that the corresponding variety is freely generated with no relation. For $n=3,4$ $CB_{\mathfrak{u}(2),n}$ is not freely generated. Moreover for $n=6$ we find that Coulomb branch is not a complete intersection. This is in agreement with the expectation that we outlined above (5.20). The dimension of Coulomb branch, as a complex manifold, is given by applying (5.65) and $\dim_{\mathbb{C}} CB_{\mathfrak{u}(2),n} = 2$ in each case. For the case when $\mathbf{CB}_{\mathfrak{u}(N),n}$ is non-planar but a complete intersection one can easily read off the generators and relation. Conversely when it is not a complete intersection some more effort is required. The expansion of the Plethystic logarithm of the $n=6$ Coulomb branch index reads

$$\mathrm{PLog}\left[\mathcal{I}^{\mathfrak{u}(2)}_{\mathbb{Z}_6,\mathrm{CB}}(x)\right] = 4x^6 - 3x^{12} + 2x^{18} + \mathcal{O}(x^{24}) . \tag{5.97}$$

The generators at x^6 are

$$\widetilde{u}_1 = u_2^3 , \quad \widetilde{u}_2 = u_1^6 , \quad \widetilde{u}_3 = u_2 u_1^4 , \quad \widetilde{u}_4 = u_2^2 u_1^2 , \tag{5.98}$$

they are primaries of the multiplets $\mathcal{E}_{6,(0,0)}$. There are three relations at x^{12}

$$I_1 : \widetilde{u}_1\widetilde{u}_2 - \widetilde{u}_3\widetilde{u}_4 = 0 , \quad I_2 : \widetilde{u}_4^2 - \widetilde{u}_3\widetilde{u}_1 = 0 , \quad I_3 : \widetilde{u}_3^2 - \widetilde{u}_2\widetilde{u}_4 = 0 . \tag{5.99}$$

However these relations are not all independent; at x^{18} we have syzygies

$$\widetilde{u}_3 I_1 + \widetilde{u}_2 I_2 + \widetilde{u}_4 I_3 \equiv 0\,, \quad \widetilde{u}_4 I_1 + \widetilde{u}_3 I_2 + \widetilde{u}_1 I_3 \equiv 0\,. \tag{5.100}$$

Generally the moduli space should be characterised by (5.98), (5.99) and (5.100). For $N = 3$ the Coulomb branch index is given by

$\mathcal{I}^{\mathfrak{u}(3)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$	n	Generators	Relation
$\mathrm{PE}\left[x + x^2 + x^3\right]$	1	u_1, u_2, u_3	⁄
$\mathrm{PE}\left[2x^2 + x^4 + x^6 - x^8\right]$	2	$\widetilde{u}_1 = u_1^2,\ u_2,$ $\widetilde{u}_2 = u_1 u_3,\ \widetilde{u}_3 = u_3^2$	$\widetilde{u}_1\widetilde{u}_3 = \widetilde{u}_2^2$
$\mathrm{PE}\left[3x^3 + x^6 - x^9\right]$	3	$\widetilde{u}_1 = u_1^3,\ \widetilde{u}_2 = u_1 u_2,$ $u_3,\ \widetilde{u}_3 = u_2^3$	$\widetilde{u}_1\widetilde{u}_3 = \widetilde{u}_2^3$
$\frac{(1+x^4)(1+x^4+2x^8)}{(1-x^4)^3(1+x^4+x^8)}$	4	Not complete intersection	
$(1 + 4x^6 + x^{12})\,\mathrm{PE}\left[3x^6\right]$	6	Not complete intersection	

For $N = 4$ the Coulomb branch index is given by

$\mathcal{I}^{\mathfrak{u}(4)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$	n	Generators	Relation
$\mathrm{PE}\left[x + x^2 + x^3 + x^4\right]$	1	u_1, u_2, u_3, u_4	⁄
$\mathrm{PE}\left[2x^2 + 2x^4 + x^6 - x^8\right]$	2	$u_2,\ \widetilde{u}_1 = u_1^2,\ u_4,$ $\widetilde{u}_2 = u_1 u_3,\ \widetilde{u}_3 = u_3^2$	$\widetilde{u}_1\widetilde{u}_3 = \widetilde{u}_2^2$
$\frac{(1+x^3+x^6)(1+2x^6)}{(1-x^3)^4(1+x^3)^2(1+x^6)}$	3	Not complete intersection	
$\frac{(1+x^4)(1+x^4+2x^8)}{(1-x^4)^4(1+x^4+x^8)}$	4	Not complete intersection	
$\frac{(1+2x^6)(1+4x^6+x^{12})}{(1-x^6)^4(1+x^6)}$	6	Not complete intersection	

We would like to point out that the dimension formula (5.65) is in perfect agreement with the above results. We checked up to $N = 60$ and order x^{70} that $\mathbf{CB}_{\mathfrak{u}(N),n}$ for $n \geqslant 2$ is not a complete intersection for all $N \geqslant 5$. In principle the analysis that we performed (5.97) - (5.100) can be repeated for each case, however doing so is beyond the current scope of this article. Further note that for each N and $n \geqslant 3$ we do not have Coulomb branch operators of dimension one or two, implying that we indeed have genuine $\mathcal{N} = 3$ supersymmetry [248].

Higgs Branch Hilbert Series

Let us now analyse the Higgs branch for these theories. We restrict our attention to the case $\mathfrak{g}=\mathfrak{u}(2)$. Using *Macaulay2* and performing the integration over $U(2)$ gauge group the Higgs branch Hilbert series reads

$$\begin{aligned}&\mathrm{HS}^{\mathfrak{u}(2)}_{\mathbb{Z}_n}(\tau,u_f)=\\&\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\mathrm{PE}\left[\left(\epsilon u_f+\epsilon^{-1}u_f^{-1}\right)\tau+\left(1+\epsilon^2u_f^2+\epsilon^{-2}u_f^{-2}\right)\tau^2-\tau^4\right].\end{aligned}\tag{5.101}$$

After performing the sum over $\mathbb{Z}_n$ (5.101) becomes

$$\begin{aligned}&\mathrm{HS}^{\mathfrak{u}(2)}_{\mathbb{Z}_n}(\tau,u_f)=\\&\begin{cases}\mathrm{PE}\left[(u_f+u_f^{-1})\tau+(1+u_f^2+u_f^{-2})\tau^2-\tau^4\right] & n=1\\ \mathrm{PE}\left[2(1+u_f^2+u_f^{-2})\tau^2-2\tau^4\right] & n=2\\ \frac{\left(\tau^6(u_f^6+u_f^{-6})+\tau^3(1+\tau^2)(1+\tau^2+\tau^4)(u_f^3+u_f^{-3})+3\tau^4+3\tau^8+\sum_{a=0}^{6}\tau^{2a}\right)}{(1+\tau^2)^{-1}\left(1+\tau^6-\tau^3(u_f^3+u_f^{-3})\right)^2\left(1+\tau^6+\tau^3(u_f^3+u_f^{-3})\right)} & n=3\\ \frac{(1+\tau^2)^2(u_f^4+\tau^4(1+(4+\tau^4)u_f^4+u_f^8))^2}{(1+\tau^8-\tau^4(u_f^4+u_f^{-4}))} & n=4\\ \frac{(1+\tau^2)^2(2\tau^6(1+\tau^4)+(1+4\tau^4+9\tau^8+4\tau^{12}+\tau^{16})u_f^6+2\tau^6(1+\tau^4)u_f^{12})}{(1+\tau^{12}-\tau^6(u_f^6+u_f^{-6}))^2} & n=6\end{cases}\end{aligned}\tag{5.102}$$

When $n=1,2$ we get a complete intersection. Moreover, in an expansion around τ the dependence on u_f in (5.101) arranges itself into characters of $SU(2)$ implying that the $U(1)_f$ isometry of the Higgs branch is enhanced to $SU(2)_f$ for these theories. This is of course due to the fact that supersymmetry is enhanced to $\mathcal{N}=4$ for $n=1,2$. For $n=3,4,6$ we do not have complete intersections, nonetheless we may identify the first generators and their relation. Moreover, for each n, by applying the dimension formula (5.70) we find that the Higgs branch is a manifold of complex dimension four. We define

$$W_{j,m}=\frac{1}{j+m}\,\mathrm{tr}\,Y^jZ^m\,.\tag{5.103}$$

For $n=1$, by taking the Plethystic logarithm of (5.101), we find that the Higgs branch is generated by the $W_{j,0}$, $W_{0,j}$ for $j=1,2$ and $W_{1,1}$. There is a relation of dimension 4 between them given by $2W_{1,1}(2W_{1,1}-W_{0,1}W_{1,0})+W_{0,1}^2W_{2,0}+W_{1,0}^2W_{0,2}=$

0. The variety is $\mathbf{HB}_{\mathfrak{u}(2),1} \cong \mathrm{Sym}^2\left(\mathbb{C}^2\right)$ [31, 107].

For $n = 2$ the Higgs branch is generated by $W_{0,2}$, $W_{2,0}$ $W_{1,1}$, $\widetilde{W} = W_{1,0}{}^2$, $\widetilde{V} = W_{0,1}{}^2$ and $\widetilde{J} = W_{1,0}W_{0,1}$ and there are two relations of dimension 4. The variety is $\mathbf{HB}_{\mathfrak{u}(2),2} \cong \left(\mathrm{Sym}^2\left(\mathbb{C}^2\right)\right)/\mathbb{Z}_2 \cong \mathbb{C}^2/\mathbb{Z}_2 \times \mathbb{C}^2/\mathbb{Z}_2$.

At $n = 3$ we do not get a complete intersection, nevertheless we can expand the Plethystic logarithm of (5.101)

$$\begin{aligned}\mathrm{PLog}\left[\mathrm{HS}^{\mathfrak{u}(2)}_{\mathbb{Z}_3}(\tau, u_f)\right] =& 2\tau^2 + 2\tau^3\left(u_f^3 + u_f^{-3}\right) + 2\tau^4 + \tau^5\left(u_f^3 + u_f^{-3}\right)\\ &+ \tau^6\left(u_f^6 + u_f^{-6} - 4\right) + \mathcal{O}\left(\tau^7\right) .\end{aligned} \tag{5.104}$$

The generators are $W_{1,1}$, $\widetilde{J}_1 = W_{0,1}W_{1,0}$, $\widetilde{W}_1 = W_{1,0}{}^3$, $\widetilde{V}_1 = W_{0,1}{}^3$, $\widetilde{W}_2 = W_{2,0}W_{1,0}$, $\widetilde{V}_2 = W_{0,2}W_{0,1}$, $\widetilde{J}_2 = W_{2,0}W_{0,2}$, $W_{2,2}$, $\widetilde{W}_3 = W_{2,0}^2W_{0,1}$, $\widetilde{V}_3 = W_{0,2}{}^2W_{1,0}$, $\widetilde{W}_4 = W_{0,2}{}^3$ and $\widetilde{V}_4 = W_{2,0}{}^3$. There are four relations of dimension six between them. In terms of $\mathfrak{su}(2,2|3)$ mutiplets these have the correct quantum numbers to be

$$\begin{aligned}&\hat{\mathcal{B}}_{[1,1]}\,, \quad \hat{\mathcal{B}}_{[1,1]}\,, \quad \hat{\mathcal{B}}_{[3,0]}\,, \quad \hat{\mathcal{B}}_{[0,3]}\,, \quad \hat{\mathcal{B}}_{[3,0]}\,, \quad \hat{\mathcal{B}}_{[0,3]}\,,\\ &\hat{\mathcal{B}}_{[2,2]}\,, \quad \hat{\mathcal{B}}_{[2,2]}\,, \quad \hat{\mathcal{B}}_{[4,1]}\,, \quad \hat{\mathcal{B}}_{[1,4]}\,, \quad \hat{\mathcal{B}}_{[6,0]}\,, \quad \hat{\mathcal{B}}_{[0,6]}\,.\end{aligned} \tag{5.105}$$

Note that, using (5.31)-(5.33), it is easily checked that (5.105) agrees with the spectrum of Coulomb branch operators that we found for the $\mathfrak{u}(2)$ $\mathbb{Z}_{n=3}$ theory (5.96). Note that in (5.104) two generators appear which have the correct quantum numbers to belong to $\hat{\mathcal{B}}_{[1,1]}$ multiplets. This implies that the theory contains two conserved spin two currents (which lie inside $\hat{\mathcal{C}}_{0,(0,0)}$ multiplets in $\mathcal{N} = 2$ language).

At $n = 4, 6$ we again do not get complete intersection varieties. One can perform a similar analysis for those cases as we did for $n = 3$.

We note that for $\mathfrak{u}(N)$ it is possible, in principal, to obtain the result for any value of N. Since, for $\mathfrak{u}(N)$ $\mathcal{N} = 4$ SYM the Higgs branch is purely mesonic, it can be obtained as the N^{th} symmetric product of the $\mathfrak{u}(1)$ case [107, 108, 74, 174, 96]. That is to say

$$\mathbf{HB}_{\mathfrak{u}(N),1} = \mathrm{Sym}^N\left(\mathbf{HB}_{\mathfrak{u}(1),1}\right) = \mathrm{Sym}^N\left(\mathbb{C}^2\right) . \tag{5.106}$$

It therefore follows that the result for the $\mathbb{Z}_n$ discrete gauging is simply

$$\mathbf{HB}_{\mathfrak{u}(N),n} = \mathbf{HB}_{\mathfrak{u}(N),1}/\mathbb{Z}_n = \left(\mathrm{Sym}^N\left(\mathbb{C}^2\right)\right)/\mathbb{Z}_n\,. \tag{5.107}$$

The Hilbert series for the variety $\mathbf{M} = \mathrm{Sym}^N(\mathbf{V})$ can be obtained by the following formula [31, 108, 107]

$$\mathrm{HS}(\tau;M)=\frac{1}{N!}\frac{\partial^N}{\partial\nu^N}\,\mathrm{PE}\left[\nu\mathrm{HS}(\tau;V)\right]\big|_{\nu=0}\,. \tag{5.108}$$

Applying the formula (5.108) we therefore have

$$\begin{aligned}\mathrm{HS}^{\mathfrak{u}(N)}_{\mathbb{Z}_n}(\tau,u_f)=&\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\frac{1}{N!}\frac{\partial^N}{\partial\nu^N}\,\mathrm{PE}\left[\nu\mathrm{HS}^{\mathfrak{u}(1)}_{\mathbb{Z}_1}(\tau,\epsilon u_f)\right]\Big|_{\nu=0} &(5.109)\\ =&\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\frac{1}{N!}\frac{\partial^N}{\partial\nu^N}\,\frac{1}{\prod_{i,j\geqslant 0}(1-\nu u_f^{i-j}\epsilon^{i-j}\tau^{i+j})}\Bigg|_{\nu=0}\,. &(5.110)\end{aligned}$$

For generic large values of N evaluating the derivative is rather tricky. For $N=2$ we explicitly checked (5.110) against (5.101) finding precise agreement. For higher values of N explicit evaluation of the Hilbert series using *Macaulay2* is no longer possible. But the agreement for $N=2$ provides strong evidence that (5.107) & (5.110) are true for general N. We also discuss the case of $N=\infty$ in Section 5.7.

5.6.2 $\mathfrak{g}=\mathfrak{su}(N+1)$

Coulomb Branch Limit

Let us study the Coulomb branch limit. From (5.62) we have

$$\mathcal{I}^{\mathfrak{su}(N+1)}_{\mathbb{Z}_n,\mathrm{CB}}(x)=\frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\mathrm{PE}\left[\sum_{j=2}^{N+1}\epsilon^j x^j\right]\,. \tag{5.111}$$

Let us examine a few cases for low rank. We define the generators of $CB_{\mathfrak{su}(N+1)}$ for the parent theory to be given by $u_j=\frac{1}{j}\operatorname{tr}X^j$. For $N+1=3$ we have

$\mathcal{I}^{\mathfrak{su}(3)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$	n	Generators	Relation
$\mathrm{PE}\left[x^2+x^3\right]$	1	$u_2,\ u_3$	/
$\mathrm{PE}\left[x^2+x^6\right]$	2	$u_2,\ \tilde{u}_1=u_3^2$	/
$\mathrm{PE}\left[x^3+x^6\right]$	3	$u_3,\ \tilde{u}_1=u_2^3$	/
$\mathrm{PE}\left[x^4+x^8+x^{12}-x^{16}\right]$	4	$\tilde{u}_1=u_2^2,\ \tilde{u}_2=u_2u_3^2,\ \tilde{u}_3=u_3^4$	$\tilde{u}_1\tilde{u}_3=\tilde{u}_2^2$
$\mathrm{PE}\left[2x^6\right]$	6	$\tilde{u}_1=u_2^3,\ \tilde{u}_2=u_3^2$	/

When $n=1,2,3,6$ $CB_{\mathfrak{su}(3),n}$ is freely generated, in agreement with our discussion above (5.20). For $N+1=4$ we have

$\mathcal{I}^{\mathfrak{su}(4)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$	n	Generators	Relation
$\mathrm{PE}\left[x^2+x^3+x^4\right]$	1	$u_2,\,u_3,\,u_4$	$\diagup$
$\mathrm{PE}\left[x^2+x^4+x^6\right]$	2	$u_2,\,u_4,\,\widetilde{u}_1=u_2^3$	$\diagup$
$\mathrm{PE}\left[x^3+2x^6+x^{12}-x^{18}\right]$	3	$u_3,\,\widetilde{u}_1=u_2^3,$ $\widetilde{u}_2=u_2u_4,\,\widetilde{u}_3=u_4^3$	$\widetilde{u}_1\widetilde{u}_3=\widetilde{u}_2^3$
$\mathrm{PE}\left[2x^4+x^8+x^{12}-x^{16}\right]$	4	$\widetilde{u}_1=u_2^2,\,u_4,$ $\widetilde{u}_2=u_2u_3^2,\,\widetilde{u}_3=u_3^4$	$\widetilde{u}_1\widetilde{u}_3=\widetilde{u}_2^2$
$\mathrm{PE}\left[3x^6+x^{12}-x^{18}\right]$	6	$\widetilde{u}_1=u_2^3,\,\widetilde{u}_2=u_3^2,$ $\widetilde{u}_3=u_2u_4,\,\widetilde{u}_4=u_4^3$	$\widetilde{u}_1\widetilde{u}_4=\widetilde{u}_3^3$

For $N+1=5$ we have

$\mathcal{I}^{\mathfrak{su}(5)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$	n	Generators	Relation
$\mathrm{PE}\left[\sum_{A=2}^{5}x^A\right]$	1	u_2,u_3,u_4,u_5	$\diagup$
$\mathrm{PE}\left[\sum_{A=1}^{5}x^{2A}-x^{16}\right]$	2	$u_2,\,u_4,\,\widetilde{u}_1=u_3^2,$ $\widetilde{u}_2=u_3u_5,\,\widetilde{u}_3=u_5^2$	$\widetilde{u}_3\widetilde{u}_1=\widetilde{u}_2^2$
$\frac{1+x^6+2x^9+2x^{12}+x^{15}+2x^{18}}{(1-x^3)^4(1+x^3)^2(1+(x^3+x^6+x^9)(1+x^3+x^9))}$	3	Not complete intersection	
$\frac{(1+x^8)(1+x^8+x^{12}+x^{16})}{(1-x^4)^4(1+(x^4+x^8)(2+2x^4+2x^8+x^{12}+x^{16}))}$	4	Not complete intersection	
$\frac{1+x^6+4x^{12}+4x^{18}+3x^{24}+3x^{30}+2x^{36}}{(1-x^6)^4(1+2x^6+2x^{12}+2x^{18}+2x^{24}+2x^{30})}$	6	Not complete intersection	

Out of the theories with $N+1>5$ we find that, apart from $n=2$, $N+1=6$, the Coulomb branch for $n=2,3,4,6$ is never a complete intersection. We checked this up to $N=60$ and x^{70}. In each case the dimension formula (5.65) holds and is equal to N as expected. In the cases where the moduli space is not a complete intersection variety the analysis that we demonstrated (5.97) - (5.100) can, in principle, be repeated. Again, for each N, with $n\geqslant 3$ we do not have Coulomb branch operators of dimension one or two implying genuine $\mathcal{N}=3$ supersymmetry [248].

Higgs Branch Hilbert Series

Let us turn to analysing the Higgs branch for these theories. We restrict ourselves only to the case $\mathfrak{g}=\mathfrak{su}(3)$. Using *Macaluay2* and performing the integration over the gauge group the Higgs branch Hilbert series reads

$$\begin{aligned}&\mathrm{HS}^{\mathfrak{su}(3)}_{\mathbb{Z}_n}(\tau,u_f)=\\&\frac{1}{n}\sum_{\epsilon\in\mathbb{Z}_n}\frac{1+\tau^2+\left(u_f\epsilon+\frac{1}{u_f\epsilon}\right)\tau^3+\tau^4+\tau^6}{\left(1-\frac{\tau^2}{u_f^2\epsilon^2}\right)\left(1-\tau^2u_f^2\epsilon^2\right)\left(1-\frac{\tau^3}{u_f^3\epsilon^3}\right)\left(1-\tau^3u_f^3\epsilon^3\right)}\,.\end{aligned}\tag{5.112}$$

We find that, for all n, the corresponding moduli space is never a complete intersection. Moreover, applying (5.70), we find that in each case the Higgs branch is of complex dimension four. A complete analysis of the Higgs branches of these theories is beyond the scope of this paper. However, as we did for the $\mathfrak{u}(2)$ case we would like to demonstrate with an example. The generators of the parent ($n=1$) theory are $W_{j,0}$ and $W_{0,j}$ for $j\in\{2,3\}$, $W_{1,1}$, $W_{2,1}$ and $W_{1,2}$ where, as before,

$$W_{j,m}=\frac{1}{j+m}\operatorname{tr}Y^jZ^m\,.\tag{5.113}$$

As an example let us expand the Plethystic logarithm of (5.112) for $n=3$

$$\begin{aligned}\mathrm{PLog}\left[\mathrm{HS}^{\mathfrak{su}(3)}_{\mathbb{Z}_3}(\tau,u_f)\right]=&\tau^2+(u_f^{-3}+u_f^3)\tau^3+\tau^4+(u_f^{-3}+u_f^3)\tau^5\\&+(u_f^{-6}+u_f^6)\tau^6-\tau^8-(u_f^{-6}+u_f^6)\tau^9\\&-(1+u_f^{-6}+u_f^6)\tau^{10}+\mathcal{O}(\tau^{11})\,.\end{aligned}\tag{5.114}$$

The generators are $W_{1,1}$, $W_{3,0}$, $W_{0,3}$, $\widetilde{J}=W_{2,0}W_{0,2}$, $\widetilde{W}_1=W_{2,1}W_{2,0}$, $\widetilde{V}_1=W_{1,2}W_{0,2}$, $\widetilde{W}_2={W_{2,0}}^3$ and $\widetilde{V}_2={W_{0,2}}^3$. In terms of $\mathfrak{su}(2,2|3)$ multiplets these have the correct quantum numbers to correspond to

$$\hat{\mathcal{B}}_{[1,1]}\,,\quad\hat{\mathcal{B}}_{[3,0]}\,,\quad\hat{\mathcal{B}}_{[0,3]}\,,\quad\hat{\mathcal{B}}_{[2,2]}\,,\quad\hat{\mathcal{B}}_{[4,1]}\,,\quad\hat{\mathcal{B}}_{[1,4]}\,,\quad\hat{\mathcal{B}}_{[6,0]}\,,\quad\hat{\mathcal{B}}_{[0,6]}\,.\tag{5.115}$$

We can again write down the result for any value of N. The Higgs branch moduli space for $\mathfrak{su}(N+1)$ $\mathcal{N}=4$ SYM can though of as a hypersurface living inside the $\mathfrak{u}(N+1)$ case defined by the equations $W_{0,1}=W_{1,0}=0$. Therefore we have [74, 96]

$$\mathbf{HB}_{\mathfrak{su}(N+1)}\cong\mathbb{C}^{2N}/S_{N+1}\subset\mathbf{HB}_{\mathfrak{u}(N+1)}\tag{5.116}$$

The Hilbert series for this variety is simply

$$\mathrm{HS}^{\mathfrak{su}(N+1)}_{\mathbb{Z}_1}(\tau,u_f)=(1-u_f\tau)(1-u_f^{-1}\tau)\,\mathrm{HS}^{\mathfrak{u}(N+1)}(\tau,u_f)\tag{5.117}$$

with $\mathrm{HS}^{\mathfrak{u}(N+1)}_{\mathbb{Z}_1}$ defined in (5.110). By definition

$$\mathbf{HB}_{\mathfrak{su}(N+1),n} \cong \left(\mathbb{C}^{2N}/S_{N+1}\right)/\mathbb{Z}_n\,. \tag{5.118}$$

And the Hilbert series is

$$\mathrm{HS}^{\mathfrak{su}(N+1)}_{\mathbb{Z}_n}(\tau,u_f) = \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\mathrm{HS}^{\mathfrak{su}(N+1)}_{\mathbb{Z}_1}(\tau,\epsilon u_f)\,. \tag{5.119}$$

For $N=2$ we explicitly checked (5.119) against (5.112) finding precise agreement. For higher values of N explicit evaluation of the Hilbert series using *Macaulay2* is no longer possible.

5.6.3 $\mathfrak{g} = \mathfrak{so}(2N)$

Coulomb Branch Limit

The Coulomb branch limit (5.62) reads

$$\mathcal{I}^{\mathfrak{so}(2N)}_{\mathbb{Z}_n,\mathrm{CB}}(x) = \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\mathrm{PE}\left[\epsilon^N x^N + \sum_{j=1}^{N-1}\epsilon^{2j}x^{2j}\right]\,. \tag{5.120}$$

We would like to discuss firstly the $n=2$ case where there are two distinct cases. Namely when $N=2M$ or $N=2M-1$ for $M\in\mathbb{Z}$. Let the us choose a basis for the Coulomb branch chiral ring given by

$$u_{2j} = \mathrm{tr}\,X^{2j}\,,\quad 1\leqslant j\leqslant N-1 \quad \text{and} \quad \hat{u}_N = \mathrm{Pf}\,X\,, \tag{5.121}$$

where Pf denotes the Pfaffian. The dimensions of the above operator are $E(u_j)=2j$, $E(\hat{u}_N)=N$. When $\mathfrak{g}=\mathfrak{so}(4M)$ we can write
$X=\mathrm{diag}\,(x_1\sigma_2,x_2\sigma_2,\ldots,x_{2N}\sigma_2)$ then the $\mathbb{Z}_2$ acts by $r_2+s_2: X\mapsto -X = g^{-1}Xg$ with $g=\mathrm{diag}\,(\sigma_3,\sigma_3,\ldots,\sigma_3)\in SO(4M)$, where σ_i denotes the Pauli matrices, and thus $r_2\cdot s_2$ is isomorphic to a gauge transformation and therefore the $n=2$ case with $N=2M$ should lead to exactly the same theory as the $n=1$ case. This is to be compared to the case when $\mathfrak{g}=\mathfrak{so}(4M-2)$. Writing $X=\mathrm{diag}\,(x_1\sigma_2,x_2\sigma_2,\ldots,x_{2N-1}\sigma_2)$ as before we have $r_2\cdot s_2 : X\mapsto -X = g^{-1}Xg$ now with $g=\mathrm{diag}\,(\sigma_3,\sigma_3,\ldots,\sigma_3)\notin SO(4N-2)$, infact, $g\in O(4N-2)$ and in this case the $\mathbb{Z}_2$ does generate a genuine global symmetry which, when gauged, will lead to a distinct theory. Indeed we find

that for $N=2M$

$$\mathcal{I}^{\mathfrak{so}(4M)}_{\mathbb{Z}_2,\mathrm{CB}}(x)=\mathrm{PE}\left[x^{2M}+\sum_{j=1}^{2M-1}x^{2j}\right]=\mathcal{I}^{\mathfrak{so}(4M)}_{\mathbb{Z}_1,\mathrm{CB}}(x)\,. \tag{5.122}$$

On the other hand, for $N=2M-1$

$$\mathcal{I}^{\mathfrak{so}(4M-2)}_{\mathbb{Z}_2,\mathrm{CB}}(x)=\mathrm{PE}\left[x^{4M-2}+\sum_{j=1}^{2M-2}x^{2j}\right]\,, \tag{5.123}$$

and the new Coulomb branch operators are simply given by $u_2,u_4,\dots,u_{4M-4}$ and $\widetilde{u}=(\hat{u}_{2M-1})^2=\det X$. Let us now turn on the cases $n=3,4,6$ for different values of N. In the following we collate the results that we found.
For $N=2$ we have

$\mathcal{I}^{\mathfrak{so}(4)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$	n	Generators	Relation
$(1+2x^6)\,\mathrm{PE}\left[1-2x^6\right]$	3	Not complete intersection	
$\mathrm{PE}\left[3x^4-x^8\right]$	4	$\widetilde{u}_1=u_2\hat{u}_2,\ \widetilde{u}_2=u_2^2,\ \widetilde{u}_3=\hat{u}_2^2$	$\widetilde{u}_1^2=\widetilde{u}_2\widetilde{u}_3$
$(1+2x^6)\,\mathrm{PE}\left[1-2x^6\right]$	6	Not complete intersection	

Note that, since $\mathfrak{so}(4)\cong\mathfrak{su}(2)\oplus\mathfrak{su}(2)$, for $n=1,2$ we have
$\mathcal{I}^{\mathfrak{so}(4)}_{\mathbb{Z}_{n=1,2},\mathrm{CB}}=\left(\mathcal{I}^{\mathfrak{su}(2)}_{\mathbb{Z}_{n=1,2},\mathrm{CB}}\right)^2$. On the other hand, for $n\geqslant 3$,
$\mathcal{I}^{\mathfrak{so}(4)}_{\mathbb{Z}_{n=3,4,6},\mathrm{CB}}\neq\left(\mathcal{I}^{\mathfrak{su}(2)}_{\mathbb{Z}_{n=3,4,6},\mathrm{CB}}\right)^2$. Since $\mathfrak{so}(6)\cong\mathfrak{su}(4)$ the Coulomb branch index for $N=3$ is the same as for the Coulomb branch index for the $\mathfrak{g}=\mathfrak{su}(4)$ theory (5.111) and therefore
$\mathcal{I}^{\mathfrak{so}(6)}_{\mathbb{Z}_n,\mathrm{CB}}(x)=\mathcal{I}^{\mathfrak{su}(4)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$.

For $N=4$ we find

$\mathcal{I}^{\mathfrak{so}(8)}_{\mathbb{Z}_n,\mathrm{CB}}(x)$	n	Generators	Relation
$\frac{1+2x^6+5x^{12}+x^{18}}{(1-x^6)^4(1+x^6)^2}$	3	Not complete intersection	
$\mathrm{PE}\left[3x^4+x^8+x^{12}-x^{16}\right]$	4	$u_4,\ \hat{u}_4,\ \widetilde{u}_1=u_2^2,$ $\widetilde{u}_2=u_2u_6,\ \widetilde{u}_3=u_6^2$	$\widetilde{u}_2^2=\widetilde{u}_1\widetilde{u}_3$
$\frac{1+2x^6+5x^{12}+x^{18}}{(1-x^6)^4(1+x^6)^2}$	6	Not complete intersection	

Out of the theories with $N>4$ we find that, apart from the $n=2$ cases, which we discussed separately, the Coulomb branch for $n=3,4,6$ is a not a complete intersection. We again checked this up to $N=60$ and x^{70}. In each case the dimension formula (5.65) holds and the dimension is equal to N as expected. With $n \geqslant 3$ we do not have Coulomb branch operators of dimension one or two, implying that we indeed have genuine $\mathcal{N} = 3$ supersymmetry [248].

Higgs Branch Hilbert Series

Using the software *Macaulay2* we did the computation of the Higgs branch Hilbert series for the theory with Lie algebra $\mathfrak{g} = \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$. After the integration over the gauge group we get

$$\mathrm{HS}^{\mathfrak{so}(4)}_{\mathbb{Z}_n}(\tau, u_f) = \frac{1}{|\mathbb{Z}_n|} \sum_{\epsilon \in \mathbb{Z}_n} \mathrm{PE}[2\tau^2 + 2(\epsilon^2 u_f^2 + \epsilon^{-2} u_f^{-2})\tau^2 - 2\tau^4] \,. \tag{5.124}$$

We observe that the above Hilbert series has a pole of order four at $\tau = 1$ and therefore, by (5.70), the complex dimension of the Higgs branch is four. For $n = 1, 2$ we get a complete intersection variety with Hilbert series

$$\mathrm{HS}^{\mathfrak{so}(4)}_{\mathbb{Z}_{1,2}}(\tau, u_f) = \mathrm{PE}[2(1 + u_f^2 + u_f^{-2})\tau^2 - 2\tau^4] = \left(\mathrm{HS}^{\mathfrak{su}(2)}_{\mathbb{Z}_1}(\tau, u_f)\right)^2 \,. \tag{5.125}$$

At $n = 1, 2$ it is clear that the Higgs branch moduli space is equal to two copies of the $\mathfrak{su}(2)$ case. We discussed that in Section 5.5.2. The topology of the moduli space is therefore $\left(\mathbb{C}^2/\mathbb{Z}_2\right)^2$. For $n = 3, 4, 6$ we observe that the corresponding Hilbert series is not a complete intersection. Moreover the Hilbert series for $n = 3, 6$ are equal.

5.6.4 $\mathfrak{g} = E_N$

In this subsection, since we can make use of (5.62), we focus on the Coulomb branch limit of the index for E_6, E_7 and E_8.

$\mathfrak{g} = E_6$

The Coulomb branch index reads

$$\mathcal{I}^{E_6}_{\mathbb{Z}_n,\mathrm{CB}}(x) = \frac{1}{|\mathbb{Z}_n|} \sum_{\epsilon \in \mathbb{Z}_n} \mathrm{PE}\left[\sum_{i \in \{2,5,6,8,9,12\}} \epsilon^i x^i\right] . \tag{5.126}$$

For $n=2$ the Coulomb branch is no longer freely generated. The Coulomb branch index reads

$$\mathcal{I}^{E_6}_{\mathbb{Z}_2,\mathrm{CB}}(x) = \mathrm{PE}\left[x^2 + x^{18} + \sum_{j=3}^{7} x^{2j} - x^{28}\right]. \tag{5.127}$$

The generators and relation are

$$u_2,\, u_6,\, u_8,\, \widetilde{u}_1 = u_5^2,\, u_{12},\, \widetilde{u}_2 = u_5 u_9,\, \widetilde{u}_3 = u_9^2\,; \quad \widetilde{u}_2^2 = \widetilde{u}_1 \widetilde{u}_3\,, \tag{5.128}$$

where the u_j are E_6-invariant polynomials of degree j. For $n=3,4,6$ the variety is not a complete intersection. To save on lengthy formulas we will list, as an example, only the case of $n=6$. In that case the Coulomb branch index reads

$$\mathcal{I}^{E_6}_{\mathbb{Z}_6,\mathrm{CB}}(x) = \frac{\left(1 - x^6 + 3x^{12} + 3x^{24} + 3x^{36} - x^{42} + x^{48}\right)\mathrm{PE}\left[6x^6\right]}{\left(1 + 11x^{24} + 11x^{30} + x^{54} + \sum_{a=1}^{3} 3a(x^{6a} + x^{54-6a})\right)}\,. \tag{5.129}$$

$\mathfrak{g} = E_7$

By applying (5.62) we have

$$\mathcal{I}^{E_7}_{\mathbb{Z}_n,\mathrm{CB}}(x) = \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n} \mathrm{PE}\left[\sum_{i\in\{2,6,10,12,14,18\}} \epsilon^i x^i\right]. \tag{5.130}$$

Clearly $\mathcal{I}^{E_7}_{\mathbb{Z}_1,\mathrm{CB}}(x) = \mathcal{I}^{E_7}_{\mathbb{Z}_2,\mathrm{CB}}(x)$. This is to be expected since $\mathrm{Out}(E_7)$ is trivial. For $n=\{3,4,6\}$ we do not get a complete intersection.

$\mathfrak{g} = E_8$

The Coulomb branch index reads

$$\mathcal{I}^{E_8}_{\mathbb{Z}_n,\mathrm{CB}}(x) = \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n} \mathrm{PE}\left[\sum_{i\in\{2,8,12,14,18,20,24,30\}} \epsilon^i x^i\right]. \tag{5.131}$$

We observe that $\mathcal{I}^{E_8}_{\mathbb{Z}_1,\mathrm{CB}}(x) = \mathcal{I}^{E_8}_{\mathbb{Z}_2,\mathrm{CB}}(x)$ again, this is to be expected due to the fact that $\mathrm{Out}(E_8)=1$. While it's easy to check that for $n=3,4,6$ the space is no longer freely generated.

5.7 Large N Limit

The large N limit of the index of $G = U(N)$ SYM may be written as [62]

$$\mathcal{I}^{\mathrm{u}(\infty)}(t, y, p, q) = \mathrm{PE}\left[Z^{\mathrm{S.T.}}(t, y, p, q)\right] \tag{5.132}$$

where

$$Z^{\mathrm{S.T.}}(t, y, p, q) = \sum_{R_2=1}^{\infty} \mathcal{I}_{\mathcal{B}^{\frac{1}{2},\frac{1}{2}}_{[0,R_2,0]}}(t, y, p, q) = \sum_{R_2=1}^{\infty} \sum_{i=0}^{R_2} \mathcal{I}_{\hat{\mathcal{B}}_{[R_2-i,i]}}(t, y, p, q)\,, \tag{5.133}$$

where, in the second line we made us of (5.27). By applying $r_n + s_n$ given in (5.3) and (5.6) we can write the single letter index corresponding to the refined index (5.43), it is given by

$$Z^{\mathrm{S.T.}}(t, y, p, q, \epsilon) = \sum_{R_2=1}^{\infty} \sum_{i=0}^{R_2} \epsilon^{R_2-2i}\, \mathcal{I}_{\hat{\mathcal{B}}_{[R_2-i,i]}}(t, y, p, q)\,, \tag{5.134}$$

where we used that

$$(r_k + s_k)\hat{\mathcal{B}}_{[R_1,R_2]} = (R_2 - R_1)\hat{\mathcal{B}}_{[R_1,R_2]}\,. \tag{5.135}$$

The refined index, at large N is then given by

$$\mathcal{I}^{\mathrm{u}(\infty)}(t, y, p, q, \epsilon) = \mathrm{PE}\left[Z^{\mathrm{S.T.}}(t, y, p, q, \epsilon)\right]\,. \tag{5.136}$$

The KK supergraviton index graded by ϵ for $r_k + s_k$, as computed from AdS/CFT, reads [251]

$$\begin{aligned}\mathcal{I}^{\mathrm{KK}}\left(t, p, q, y, \epsilon\right) = &-\frac{1-\epsilon^{-1}t^6}{(1-t^3y)\,(1-t^3/y)} \\ &+\frac{\left(1-\epsilon^{-1}t^3y\right)\left(1-\epsilon^{-1}t^3/y\right)\left(1-t^4\left(\frac{\epsilon}{pq}+\frac{\epsilon p}{q}+\frac{q^2}{\epsilon}\right)+(1+\epsilon)\,t^6\right)}{(1-t^3y)\,(1-t^3/y)\,(1-t^2pq/\epsilon)\,(1-t^2q/p\epsilon)\,(1-t^2\epsilon/q^2)}\,.\end{aligned} \tag{5.137}$$

Expansion around $t = 0$ (we checked up to order t^{20}) verifies that

$$Z^{\mathrm{S.T.}}(t, y, p, q, \epsilon) = \mathcal{I}^{\mathrm{KK}}\left(t, p, q, y, \epsilon\right)\,. \tag{5.138}$$

The index for the $\mathbb{Z}_n$ discrete gauging of the $\mathfrak{u}(N=\infty)$ theory is therefore

$$\mathcal{I}^{\mathfrak{u}(\infty)}_{\mathbb{Z}_n}(t,y,p,q) = \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n} \mathrm{PE}\left[Z^{\text{S.T.}}(t,y,p,q,\epsilon)\right] . \tag{5.139}$$

On the other hand, as computed in [251], we may also obtain the index for the $k=1,2,3,4,6$ $\mathcal{N}=3$ S-fold SCFTs at large N, $S^{\infty}_{k,\ell}$, by implementing the projection at the level of the single letter index. The spectrum of protected single trace operators in the S-fold $S^{\infty}_{k,\ell}$ theory is given by

$$Z^{\text{S.T}}_{k}(t,y,p,q) := \frac{1}{|\mathbb{Z}_k|}\sum_{R_2=1}^{\infty}\sum_{i=0}^{R_2}\sum_{\epsilon\in\mathbb{Z}_k}\epsilon^{R_2-2i}\mathcal{I}_{\hat{\mathcal{B}}_{[R_2-i,i]}}(t,y,p,q)\,, \tag{5.140}$$

for $k=1,2,3,4,6$. Note that, at large N, the index does not distinguish between theories with different values of ℓ [251]. One advantage of (5.140) is that it manifestly organises expression into multiplets of $\mathfrak{su}(2,2|3)$. The index for the S-fold at large N is then given by

$$\mathcal{I}^{N=\infty}_{\mathbb{Z}_k\ \text{S-fold}}(t,p,q,y) = \mathrm{PE}\left[Z^{\text{S.T}}_{k}(t,p,q,y)\right] . \tag{5.141}$$

Note that the procedure (5.140) is a S-fold and not a discrete gauging since it is implemented at the level of the single particle index. It is clear that

$$\mathcal{I}^{\mathfrak{u}(\infty)}_{\mathbb{Z}_{n=k}}(t,y,p,q) \neq \mathcal{I}^{N=\infty}_{\mathbb{Z}_k\ \text{S-fold}}(t,p,q,y)\,. \tag{5.142}$$

We can also discuss the Higgs branch Hilbert series of this these theories. The result for $\mathfrak{u}(N)$ $\mathcal{N}=4$ SYM in the $N\to\infty$ limit can be obtained by application of the formula of [108] which claims that

$$\mathrm{HS}^{\mathfrak{u}(\infty)}_{\mathbb{Z}_1}(\tau,u_f) = \mathrm{PE}\left[f_\infty(\tau,u_f)\right] , \tag{5.143}$$

$$f_\infty(\tau,u_f) = \mathrm{HS}^{\mathfrak{u}(1)}_{\mathbb{Z}_1}(\tau,u_f) = \mathrm{PE}\left[(u_f+u_f^{-1})\tau\right] . \tag{5.144}$$

Therefore, for the $\mathbb{Z}_n$-gauged theory the Hilbert series is given by

$$\mathrm{HS}^{\mathfrak{u}(\infty)}_{\mathbb{Z}_n}(\tau,u_f) = \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\exp\left(\sum_{j=1}^{\infty}\frac{f_\infty(\tau,\epsilon u_f)-f_\infty(0,0)}{j}\right) \tag{5.145}$$

$$= \frac{1}{|\mathbb{Z}_n|}\sum_{\epsilon\in\mathbb{Z}_n}\prod_{i,j\geqslant 0}\frac{(1-u_f^{i-j}\epsilon^{i-j}\tau^{i+j+2})}{(1-u_f^{i-j-1}\epsilon^{i-j-1}\tau^{i+j+1})(1-u_f^{i-j+1}\epsilon^{i-j+1}\tau^{i+j+1})}\,, \tag{5.146}$$

which is the Hilbert series for $\left(\mathrm{Sym}^\infty(\mathbb{C}^2)\right)/\mathbb{Z}_n$. For example, for $n=4$ we have

$$\begin{aligned}\mathrm{HS}^{\mathfrak{u}(\infty)}_{\mathbb{Z}_4}(\tau,u_f) =& 1+2\tau^2+(4+5b_4)\tau^4+(12+19b_4)\tau^6\\ &+(32+55b_4+22b_8)\tau^8+\mathcal{O}(\tau^9)\end{aligned} \tag{5.147}$$

where $b_{np}=\sum_{i=0}^{2p}u_f^{n(p-i)}=u_f^{np}+u_f^{np-n}+\cdots+u_f^{-np}$. Conversely, for the $S^\infty_{k,\ell}$ S-fold theory the Hilbert series reads

$$\begin{aligned}\mathrm{HS}^{\mathfrak{u}(\infty)}_{\mathbb{Z}_k\text{ S-fold}}(\tau,u_f) =& \mathrm{PE}\left[\frac{1}{k}\sum_{\epsilon\in\mathbb{Z}_k}f_\infty(\tau,\epsilon u_f)\right]\\ =& \mathrm{PE}\left[\frac{1-\tau^{2k}}{(1-\tau^2)(1-u_f^k\tau^k)(1-u_f^{-k}\tau^k)}\right]\\ =& \prod_{i,j,l\geqslant 0}\left[\frac{\left(1-u_f^{k(i-j)}\tau^{2l+k(i+j+2)}\right)^2}{\left(1-u_f^{k(i-j)}\tau^{2+2l+k(j+i)}\right)\left(1-u_f^{k(i-j)}\tau^{2+2l+k(j+i+2)}\right)}\right.\\ &\left.\times\frac{\left(1-u_f^{k(i-j-1)}\tau^{2+2l+k(j+i+1)}\right)\left(1-u_f^{k(i-j+1)}\tau^{2+2l+k(j+i+1)}\right)}{\left(1-u_f^{k(i-j-1)}\tau^{2l+k(j+i+1)}\right)\left(1-u_f^{k(i-j+1)}\tau^{2l+k(j+i+1)}\right)}\right]\end{aligned} \tag{5.148}$$

which is the Hilbert series for $\mathrm{Sym}^\infty(\mathbb{C}^2/\mathbb{Z}_k)$. For $k=4$ this has the expansion

$$\begin{aligned}\mathrm{HS}^{\mathfrak{u}(\infty)}_{\mathbb{Z}_4\text{ S-fold}}(\tau,u_f) =& 1+\tau^2+(1+b_4)\tau^4+(1+2b_4)\tau^6\\ &+(2+2b_4+2b_8)\tau^8+\mathcal{O}(\tau^9)\,.\end{aligned} \tag{5.149}$$

5.8 Conclusions

In this chapter we have presented a prescription on how to implement the discrete gauging of a four dimensional $\mathcal{N}=4$ mother theory, resulting in a $\mathcal{N}=3$ daughter theory, at the level of the superconformal index.

We explicitly computed the Coulomb branch limit of the index as well as the

Higgs branch Hilbert series for a number of theories based on simply laced groups. For rank one theories, the Coulomb branch index and Higgs branch Hilbert series we computed reproduce precisely all known results, while, for higher rank theories we make concrete predictions for the Coulomb and Higgs branches of these $\mathcal{N} = 3$ theories. Most strikingly we find that, in general, the higher rank theories, have non-freely generated Coulomb branches. In a few cases the Coulomb branch is a complete intersection variety and, using the Coulomb branch index, we were able to read off the topology of the corresponding space. Generally the Coulomb branch of the theory after the discrete gauging is not a complete intersection and the topology becomes harder to extract. It would be interesting to further study this aspect in the future.

Since the superconformal index of the $\mathfrak{u}(1)$ theory is easily reorganised into $\mathcal{N} = 3$ multiplets we were able to compute the full superconformal index for the discrete gauging of it, given in equation (5.73). Moreover, in the large N limit, a similar reorganisation happens meaning that it is also possible to compute the superconformal index for the discrete gauging, given in equation (5.139). For general rank the computation of the full index, or other more refined limits such as the Schur limit, is much more difficult and we leave it for future work. However, we can easily compute the Coulomb branch limit of the superconformal index and the Higgs branch Hilbert series. Other, more refined, limits contain more types of short multiplets, which of course contain more interesting information. In particular the Schur index is related to the vacuum character of chiral algebras. The latter allows for the computation of correlation functions in a protected sector [172]. For $\mathcal{N} = 3$ theories the study of chiral algebras was initiated in [256, 255] and it would be very interesting to further pursue. With the help of the superconformal index, we can construct and analyse the corresponding chiral algebras for the discrete gauging that we studied in this paper.

It is important to note that the spectrum of non-local operators may reduce the possible $\mathbb{Z}_n$'s that can enhance to symmetries of the theory and therefore be gauged. The standard superconformal index that we studied in this paper can say nothing about the non-local operator spectrum. It captures only the spectrum of protected local operators. Using our current tools we only claim that if a theory exists we can compute its index, but we have no way of deciding if a theory actually exists. To break this impasse, a very interesting quantity to compute for the theories obtained via discrete gauging is the Lens space index [81, 237, 235, 236]. It is a generalisation of the standard superconformal index that has a representation as a path integral on $\mathbb{S}^1 \times \mathbb{S}^3/\mathbb{Z}_r$. For $r = 1$ this reduces to the usual superconformal index, however, since $\pi_1\left(\mathbb{S}^3/\mathbb{Z}_r\right) = \mathbb{Z}_r$ the Lens space index has the advantage that it is sensitive to the spectrum of line operators of the theory. Our construction can be immediately generalized for $r \neq 1$.

In a similar spirit it is also possible to compute the index in the presence of certain extended operators [278, 279]. These should also shed light to the possible discrete gaugings allowed for a given theory. Computing such quantities may be able to teach us more about, the currently mysterious, 'new' $\mathcal{N} = 4$ theories [159] and discrete gaugings thereof.

Finally, our procedure can also be applied to discrete gauging that preserves $\mathcal{N} = 2$ or $\mathcal{N} = 1$ superconformal symmetry as in [159] to generate new novel theories that inherit a large amount of structure from their $\mathcal{N} = 4$ mother theory.

Chapter 6

$\mathcal{N} = (1, 0)$ Strings and Surface Operators

6.1 Introduction

The $(1,0)_\Gamma$ SCFTs, associated to M5-branes on a transverse $\mathbb{C}^2/\Gamma$ singularity, admit a wide variety of $\frac{1}{2}$-BPS surface defects. As we have discussed in detail throughout this thesis, this theory and those defects are used to engineer the 4d theories within class $\mathcal{S}_\Gamma$. Amongst the permissible $\frac{1}{2}$-BPS, codimension 2 surface defects are those of of Gukov-Witten type [221, 122, 121]. Closely related are defects in 6d $\mathcal{N} = (2,0)$ theories which have been studied in, to name a few, [217, 280, 281, 76, 122, 58].

The $\mathcal{N} = (1,0)$ theories that we will focus on are $(1,0)_{A_{N-1}}$ theories; the world-volume theory on M M5-branes probing an A_{N-1} singularity. The surface defects may be realised in M-theory by additional M5-brane stacks intersecting codimension 2 surfaces of the original M5-brane stack, or, by an orbifold construction. In this chapter we compute the elliptic genus partition function of the 2d theory describing the tensionless BPS strings of these theories in the presence of the defect by studying a dual Type-IIB geometry. The M-string elliptic genus is expected to be equal to the $T^2 \times \mathbb{R}^4$ partition function of the 6d theory, up to a perturbative contribution (see [282] for a complete review and list of references). We also compute the $\mathbb{S}^1 \times \mathbb{S}^4$ partition function of 5d $\mathcal{N} = 1$ circular quiver theories in the presence of defects operators inserted at north and south poles.

6.2 Self-Dual Strings with Surface Operators

In this section we describe the details of the self dual strings in the presence of the surface operators for the $(1,0)$ theories of interest. In particular we derive a dual 2d UV gauge theory description. We then describe the computation of the Elliptic genus and of the BPS partition function.

6.2.1 M-Theory Description

To describe the the surface defects it is useful to first deform the $(1,0)_{A_{N-1}}$ theory by moving away from the conformal point to a 'Coulomb' branch. We separate the M5-branes in, say, the X^7 direction so that the n^{th} brane lies at position a_n. The dynamics is now described by a theory of strings which couple to the 2-form potentials B for the self-dual field strengths $dB = H = \star H$ in the $M-1$ $\mathcal{N} = (1,0)$ tensor multiplets. The strings arise as M2-branes which are of codimension 4 with respect to the M5-branes and which sit in one direction transverse to the M5-branes. The tension between the n^{th} and m^{th} string is then [283]

	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}_{\epsilon_2}$		T^2		$\mathbb{S}^1$	Γ-ALF			
	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9	X^{10}	X^{11}
M $M5$	–	–	–	–	–	–	·	·	·	·	·
K $M2$	·	·	·	·	–	–	–	·	·	·	·
k $M5'$	·	·	–	–	–	–	·	–	–	·	·

Table 6.1: *M-theory description of the surface defects.*

$$t_{n,m} = \left| \int_{a_m}^{a_n} dx \wedge \frac{dz}{z} \right| = \left| (a_n - a_m) \frac{dz}{z} \right| , \tag{6.1}$$

where z denotes the coordinate on the T^2 and $x = X_{7(11)}$ where

$$X_{ij} := \frac{X^i + \mathrm{i}X^j}{\sqrt{2}} , \quad i, j \in \{1, 2, \dots, 11\} . \tag{6.2}$$

a_n denotes the position of the n^{th} brane in the x direction. At the conformal point $a_n = a_m$ and the strings become tensionless.

We wish to study these strings in the presence of $\frac{1}{2}$-BPS codimension 2 defects. We often use the phrases 'defect', 'surface operator' and 'surface defect' interchangeably. The defect can be realised by inserting k M5'-branes as in Table 6.1. The data specifying the defect is given by a homomorphism $\rho = \bigoplus_{A=1}^{N} \rho_A : \mathfrak{su}(2)^N \to A_{M-1}^N$ labelled by choices of partition of M_A into k integers for $A = 1, \ldots, N$

$$\rho = [\rho_1, \rho_2, \ldots, \rho_N] , \quad \rho_A = [M_{A1}, M_{A2}, \ldots, M_{Ak}] , \tag{6.3}$$

and $M_A = \sum_{A=1}^{N} M_{A,i}$. This describes M_{Ai} M5-branes ending on the A^{th} centre of the Γ-ALF space and intersecting the i^{th} M5'-brane.

So called 'minimal' defects of type $\rho_A = [1, 1, \ldots, 1, (M_A - n)]$ have been studied from the point of view of topological strings in [284, 285, 286, 287]. The relation between M-strings and the refined topological string partition function in the presence of a defect of type $2 = 1 + 1$ has been studied in [284]. We review this in Appendix G.3. In this chapter we wish to present a method to compute the partition functions for any choice of ρ, by using the Elliptic genus method.

Before inserting the surface operator there is a $Spin(4) \cong SU(2)_\alpha \times SU(2)_{\dot\alpha}$ symmetry acting on $\mathbb{C}^2 \cong \mathbb{R}^4$ parametrised by X_{12} and X_{34}. The $\mathbb{C}^2/\Gamma$ singularity admits a resolution to a Γ-ALF space. This is parametrised by X_{89}, $X_{(10)(11)}$ and has a $SU(2) \times U(1)_b$ isometry, where $U(1)_b$ is generated by $J_{89} + J_{(10)(11)}$ and J_{ij} denotes the generator of $U(1)_{ij}$ rotations in the two-plane parametrised by X_{ij}. When $\Gamma = A$ the Γ-ALF space is the multi-centred Tau-Nut space TN_N and there is a also $U(1)_f$ holomorphic isometry generated by $J_{89} - J_{(10)(11)}$.

We introduce the Ω-background parameters by fibering non-trivially the Γ-ALF space and the $\mathbb{R}^4 \cong \mathbb{C}^2$ over the $\mathbb{S}^1$ parametrised X^5 such that as we go around the $\mathbb{S}^1$ we rotate by

$$U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} : \begin{array}{l} (X_{12}, X_{34}) \mapsto \left(e^{2\pi i \epsilon_1} X_{12}, e^{2\pi i \epsilon_2} X_{34}\right) , \\ (X_{89}, X_{(10)(11)}) \mapsto \left(e^{-\frac{\epsilon_1+\epsilon_2}{2}} X_{89}, e^{-\frac{\epsilon_1+\epsilon_2}{2}} X_{(10)(11)}\right) . \end{array} \tag{6.4}$$

When $\Gamma = A$ we may also fiber non-trivially the TN_N over the $\mathbb{S}^1$ parametrised by X^6 such that we rotate by

$$U(1)_f = U(1)_m : (X_{89}, X_{(10)(11)}) \mapsto (e^{2\pi i m} X_{89}, e^{-2\pi i m} X_{(10)(11)}) \tag{6.5}$$

upon going around the circle.

With the surface operator the geometry preserves $32/16 = 2$ real supercharges. In particular, the 2d theory describing self-dual strings has $\mathcal{N} = (0, 2)$ supersymmetry.

An Alternative M-Theory Realisation of the Surface Operator from Orbifolding

When $\Gamma = A$ there is an equivalent description of the surface operator of interest. It relies on the additional $U(1)_f$ isometry of the A-type ALF space TN_N. In particular we can define the surface operator as a orbifold singularity at $X_{12} = 0$. We begin with the M-theory setup of Table 6.2. In particular, the space $\mathbb{C}^3/(\mathbb{Z}_N \times \mathbb{Z}_k)$ admits

	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}_{\epsilon_2}$		T^2		$\mathbb{S}^1$	TN_N			
	X^1	X^2	X^3	X^4	X^5	X^6	X^9	X^7	X^8	X^{10}	X^{11}
M M5	–	–	–	–	–	–	·	·	·	·	·
K M2	·	·	·	·	–	–	–	·	·	·	·
$\mathbb{Z}_k$	×	×	·	·	·	·	·	×	·	×	·

Table 6.2: *Orbifold description of the surface operator.*

a partial resolution as $(\mathbb{C} \times \mathrm{TN}_N)/\mathbb{Z}_k$. At the origin TN_N may be parametrised as $AB = C^N$ with $A = {X_{7(10)}}^N$, $B = {X_{8(11)}}^N$ and $C = X_{8(11)}X_{7(10)}$.[1] Clearly both $U(1)_{7(10)}$ and $U(1)_{8(11)}$ leave this description invariant and therefore the orbifold action

$$\mathbb{Z}_k : (X_{12}, X_{7(10)}) \mapsto \gamma^{J_{12}-J_{7(10)}}(X_{12}, X_{7(10)}) = (\gamma X_{12}, \gamma^{-1} X_{7(10)}) \,, \tag{6.6}$$

$\gamma = e^{2\pi \mathrm{i}/k}$, is well defined.

One can see that (atleast at the level of the BPS strings of the 6d theory) that the descriptions of Table 6.1 and Table 6.2 are equivalent in the following sense: We first obtain a Type-IIA description by taking the setup of Table 6.1 and compactify by taking the M-theory circle to be the TN_N circle X^{11}. We then perform a T-duality along X^7 to find the Type-IIB description of Table 6.3. Whilst we are unaware of

	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}_{\epsilon_2}$		T^2		$\mathbb{C}^2$			
	X^1	X^2	X^3	X^4	X^5	X^6	X^7	X^8	X^9	X^{10}
$\mathbb{Z}_M$	·	·	·	·	·	·	×	×	×	×
K D1	·	·	·	·	–	–	·	·	·	·
N D5	–	–	–	–	–	–	·	·	·	·
$\mathbb{Z}_k$	×	×	·	·	·	·	×	·	·	×

Table 6.3: *Type-IIB setup obtained by compactifying the M-theory geometry of Table 6.1 on T^2 parametrised by X^7 and X^{11}.*

[1] Descriptions of the orbifolds $\mathbb{C}^2/\Gamma$ can be found in 3.9 of [108].

an explicit metric of the CY_3 geometry dual to the NS5-NS5' system, if we allow ourselves to decompactify the $\mathbb{S}^1$ parameterised by X^7 (which is irrelevant from the point of view of the theory living on $X^1, \ldots, X^6$ provided we keep only the X^7 zero modes) then the dual geometry admits a description as the orbifold $\mathbb{C}^3/(\mathbb{Z}_k \times \mathbb{Z}_M)$ [288]. The orbifold action is given by

$$\begin{aligned} \mathbb{Z}_M : \left(X_{12}, X_{7(10)}, X_{89}\right) &\mapsto \omega^{J_{7(10)}-J_{89}} \left(X_{12}, X_{7(10)}, X_{89}\right) \,, \quad \omega = e^{2\pi \mathrm{i}/M} \,, \\ \mathbb{Z}_k : \left(X_{12}, X_{7(10)}, X_{89}\right) &\mapsto \gamma^{J_{12}-J_{7(10)}} \left(X_{12}, X_{7(10)}, X_{89}\right) \,, \quad \gamma = e^{2\pi \mathrm{i}/k} \,. \end{aligned} \tag{6.7}$$

Alternatively, if one takes the description of the surface operator of Table 6.2 and compactifies on X^{11} and then T-dualises on X^9 one obtains the same Type-IIB geometry as Table 6.3.

Surface Operator Data in the Type II Description

One may worry that the data (6.3) specifying the surface operator has been lost upon compactifying to a Type-II description. However it can be recovered by tracking the T-duality carefully. Let us first describe the case without surface operator, i.e. the $k = 1$ case (when $k = 1$ we can pull out the M5'-brane of Table 6.1).

Before the reduction to the Type-II description we have M M5-branes on N-centred Taub-Nut space. We have $M_A = M/N$ fractional branes ending on the A^{th} center $\vec{x}_A$ with $M = \sum_{A=1}^{N} M_A$. After moving to the Type-II description the TN_N space becomes N D$(p = 5, 6)$-branes with the A^{th} brane located at $\vec{x}_A$ in the transverse space. In the Type-IIA description, the M M5-branes have become M NS5-branes of which M_A are fixed at $\vec{x}_A$ in the transverse space. In the Type-IIB description the M5-branes have become a transverse TN_M space of which M_A centres have been collided. Therefore, in the Type-II description the original brane setup is now described by a partition of N into M integers

$$\tilde{\rho} = [\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_M] \tag{6.8}$$

$$= \Bigg[\underbrace{\frac{N}{M_1}, \ldots, \frac{N}{M_1}}_{M_1 \text{ times}}, \underbrace{\frac{N}{M_2}, \ldots, \frac{N}{M_2}}_{M_2 \text{ times}}, \ldots, \underbrace{\frac{N}{M_N}, \ldots, \frac{N}{M_N}}_{M_N \text{ times}}\Bigg] \,. \tag{6.9}$$

The case with surface operator is essentially the same. The surface operator described by k M5'-branes. M_{Ai} M5-branes end on the $\vec{x}_{A^{\text{th}}}$ center and i^{th} M5'-brane. After moving to the IIA description we have N_i, $N = \sum_{i=1}^{k} N_i$ fractional D6-branes ending on a stack of intersecting M_{Ai} NS5-branes, therefore the surface operator data may be recast in the following form:

$$\begin{aligned}\tilde{\rho}_n &= [\tilde{\rho}_{n1}, \tilde{\rho}_{n2}, \ldots, \tilde{\rho}_{nk}] \\ &= \Bigg[\underbrace{\frac{N}{M_{A1}}, \ldots, \frac{N}{M_{A1}}}_{M_{A1}\ \text{times}}, \underbrace{\frac{N}{M_{A2}}, \ldots, \frac{N}{M_{A2}}}_{M_{A2}\ \text{times}}, \ldots, \underbrace{\frac{N}{M_{Ak}}, \ldots, \frac{N}{M_{Ak}}}_{M_{Ak}\ \text{times}}\Bigg],\end{aligned} \tag{6.10}$$

with $\tilde{\rho} = [\tilde{\rho}_1, \tilde{\rho}_2, \ldots, \tilde{\rho}_M]$.

6.2.2 2d Gauge Theory

The goal of this section is to find a description of the CFT describing the low energy theory of self-dual strings in the $(1,0)_A$ theories in the presence of the surface operator.

The CFTs describing the dynamics of self-dual strings tend to be rather complicated NLSMs [282]. However, because we are ultimately interested in computing the torus partition function (elliptic genus) of said CFTs and because that quantity is constant under RG flow it suffices to consider, if it exists, a UV description that flows to the CFT of interest in the IR.

In this case, the CFT of interest is not an isolated fixed point and can be realised as the IR fixed point of the 2d gauge theory on D1-branes in the Type-IIB setup of Table 6.3. The elliptic genus of that 2d gauge theory is then expected to be equal to the elliptic genus for the IR CFT describing dynamics of self-dual strings.

Before computing the elliptic genus we must first discuss the worldvolume theory living on the D1-branes in the low energy limit. Let us first discuss the supersymmetries preserved by the 2d theory in the presence of the surface operator.

Type-IIB string theory has 32 supersymmetries parametrised by a 32 component spinors $\epsilon = \epsilon_L + \epsilon_R$ of positive chirality $\Gamma^{11}\epsilon_{L/R} = +\epsilon_{L/R}$ where $\Gamma^{11} = \Gamma^1 \ldots \Gamma^{10}$ and $\Gamma^1, \ldots, \Gamma^{10}$ are the 32×32 Gamma matrices. The D5/D1 system preserves 1/4 of the 32 supersymmetries. Between them they preserve only those supersymmetries of the form

$$\epsilon_L = \Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5\Gamma^6\epsilon_R, \quad \epsilon_L = \sigma\Gamma^5\Gamma^6\epsilon_R, \tag{6.11}$$

with $\sigma = \pm 1$ corresponding to whether we have D1- or $\overline{\text{D1}}$-branes. The theory living on the ($\overline{\text{D1}}$-)D1-branes then possesses (p, q) supersymmetry with $p + q = 32/4 = 8$. By choosing an explicit representation for the Gamma matrices it can be shown that

$p = q = 4$ and that the preserved supercharges are

$$\mathrm{Q}_+^{\alpha a},\ \mathrm{Q}_-^{\alpha\dot{a}} \text{ if } \sigma = +1\,, \qquad \widetilde{\mathrm{Q}}_+^{\dot{\alpha}\dot{a}},\ \widetilde{\mathrm{Q}}_-^{\dot{\alpha}a} \text{ if } \sigma = -1\,, \tag{6.12}$$

where $\alpha, \dot{\alpha} = 1, 2$ are indices for $Spin(4) \cong SU(2)_\alpha \times SU(2)_{\dot{\alpha}}$ and $a, \dot{a} = 1, 2$ are indices of $Spin(4)^R \cong SU(2)_a \times SU(2)_{\dot{a}}$ and the subscript $\pm$ denotes the spin-$\pm\frac{1}{2}$ representation under $U(1)_{56}$ which acts as the Lorentz group of the D1-brane worldvolume theory.

The $Spin(4) \cong SU(2)_\alpha \times SU(2)_{\dot{\alpha}}$ rotates the two planes of the $\mathbb{C}^2$ parametrised by (X_{12}, X_{34}) into one another. The Cartans J_L, J_R of $\mathfrak{su}(2)_\alpha, \mathfrak{su}(2)_{\dot{\alpha}}$ may be expressed in terms of the generators J_{12} and J_{34} of $U(1)$ rotations in their respective planes as

$$J_L = \frac{1}{2}\left(-J_{12} + J_{34}\right), \quad J_R = -\frac{1}{2}\left(J_{12} + J_{34}\right), \tag{6.13}$$

which are defined such that lower $\alpha = 1, 2$ have $J_L = +\frac{1}{2}, -\frac{1}{2}$ and lower $\dot{\alpha} = \dot{1}, \dot{2}$ has $J_R = +\frac{1}{2}, -\frac{1}{2}$.

On the other hand $Spin(4)^R \cong SU(2)_a \times SU(2)_{\dot{a}}$ rotates the two planes of the $\mathbb{C}^2$ parametrised by $\left(X_{7(10)}, X_{89}\right)$ into one another. The Cartans J_L^R, J_R^R of $\mathfrak{su}(2)_a, \mathfrak{su}(2)_{\dot{a}}$ may be expressed in terms of the generators J_{710} and J_{89} of $U(1)$ rotations in their respective planes as

$$J_L^R = \frac{1}{2}\left(J_{710} - J_{89}\right), \quad J_R^R = \frac{1}{2}\left(J_{710} + J_{89}\right), \tag{6.14}$$

which are defined such that lower $a = 1, 2$ have $J_L^R = +\frac{1}{2}, -\frac{1}{2}$ and lower $\dot{a} = \dot{1}, \dot{2}$ has $J_R^R = +\frac{1}{2}, -\frac{1}{2}$. We can then write $J_{12} - J_{710} = -\left(J_R + J_L + J_R^R + J_L^R\right)$. Hence the total preserved supersymmetries are

$$\mathrm{Q}_-^{1\dot{2}},\ \mathrm{Q}_-^{2\dot{1}} \text{ if } \sigma = +1\,, \qquad \widetilde{\mathrm{Q}}_+^{\dot{1}\dot{2}},\ \widetilde{\mathrm{Q}}_+^{\dot{2}1} \text{ if } \sigma = -1\,. \tag{6.15}$$

Because parity exchanges D1 $\leftrightarrow$ $\overline{\text{D1}}$ we may take $\sigma = -1$ without loss of generality.

D1-Worldvolume Theory Without Surface Operator

Let us first discuss the worldvolume theory without surface operator. In the Type-IIB description the theory can be understood as a $\mathbb{Z}_M$ orbifold of the well known D5/D1-system.

The theory arising from quantising open strings stretching between K parallel

$(4,4)$	$(0,2)$	$U(K)$	$SU(N)$	J_{56}	J_L	J_R	J_L^R	J_R^R	On-shell D.O.F.
V	Υ	adj.	$\mathbf{1}$	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	$-\frac{1}{2}$	$\zeta_-^{\dot{2}\dot{1}}$, F_{+-}
	ζ	adj.	$\mathbf{1}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	$\zeta_-^{\dot{1}\dot{1}}$
	Y	adj.	$\mathbf{1}$	0	0	0	$+\frac{1}{2}$	$+\frac{1}{2}$	$Y^{2\dot{2}}$, $\varepsilon_+^{\dot{2}2}$
	$\widetilde{Y}$	adj.	$\mathbf{1}$	0	0	0	$-\frac{1}{2}$	$+\frac{1}{2}$	$Y^{1\dot{2}}$, $\varepsilon_+^{\dot{2}1}$
H	X	adj.	$\mathbf{1}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	$X^{1\dot{1}}$, $\xi_+^{1\dot{1}}$
	$\widetilde{X}$	adj.	$\mathbf{1}$	0	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0	$X^{2\dot{1}}$, $\xi_+^{2\dot{1}}$
	λ	adj.	$\mathbf{1}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	λ_-^{12}
	$\widetilde{\lambda}$	adj.	$\mathbf{1}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	λ_-^{11}
U	ϕ	$\square$	$\overline{\square}$	0	0	$-\frac{1}{2}$	0	0	$\phi^{\dot{1}}$, $\chi_+^{\dot{1}}$
	$\widetilde{\phi}$	$\overline{\square}$	$\square$	0	0	$-\frac{1}{2}$	0	0	$\overline{\phi}^{\dot{1}}$, $\overline{\chi}_+^{\dot{1}}$
	ψ	$\square$	$\overline{\square}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	ψ_-^{1}
	$\widetilde{\psi}$	$\overline{\square}$	$\square$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0	$\overline{\psi}_-^{1}$

Table 6.4: *Gauge covariant field content with $\delta = 0$ of the $\mathcal{N}=(4,4)$ vector multiplet V. The charges are displayed for the lowest component of the relevant multiplet.*

and coincident Dp-branes is given by $p+1$ dimensional Yang-Mills theory with 16 supercharges, for $p=1$ that is the well known $\mathcal{N}=(8,8)$ SYM theory. In terms of multiplets under the $\mathcal{N}=(4,4)$ subalgebra given by $\widetilde{\mathrm{Q}}_+^{\dot{\alpha}\dot{a}}, \widetilde{\mathrm{Q}}_-^{\dot{\alpha}a}$ they form a $\mathcal{N}=(4,4)$ field strength multiplet V and hypermultiplet H, which can be thought of as the reduction to 2d of a 4d $\mathcal{N}=2$ field strength multiplet and hypermultiplet respectively. V contains a 2d field strength $F_{-+}=-F_{-+}$, four scalars degrees of freedom $Y^{a\dot{a}}$, right moving fermions $\varepsilon_+^{\dot{\alpha}a}$ and left moving fermions $\zeta_-^{\dot{\alpha}\dot{a}}$. H contains scalars $X_{\alpha\dot{\alpha}}$, right moving fermions $\xi_+^{\alpha\dot{a}}$ and left moving fermions $\lambda_-^{\alpha a}$.

Open D1-D5 strings preserves that $\mathcal{N}=(4,4)$ supersymmetry and gives rise to a $\mathcal{N}=(4,4)$ hypermultiplet U in the bifundamental representation of $U(K)\times SU(N)$. U contains two complex scalars $\phi^{\dot{\alpha}}$ and their conjugates $\overline{\phi}_{\dot{\alpha}}$, and fermions $\chi_+^{\dot{a}}$, ψ_-^{a} plus their conjugates $\overline{\chi}_{+\dot{a}}$, $\overline{\psi}_{-a}$.

Since the surface operator and $\mathbb{Z}_M$ orbifold eventually picks out a favoured $\mathcal{N}=(0,2)$ algebra (6.15) it is convenient to describe the field content in terms of that $\mathcal{N}=(0,2)$ subalgebra given by the supercharges

$$\mathrm{Q} := \widetilde{\mathrm{Q}}_+^{\dot{2}\dot{1}}\,, \quad \widetilde{\mathrm{Q}} := \widetilde{\mathrm{Q}}_+^{\dot{1}\dot{2}}\,. \tag{6.16}$$

We list the field content (minus conjugates) under this $\mathcal{N}=(0,2)$ subalgebra in Table 6.4. In words: V splits into a $\mathcal{N}=(0,2)$ field strength Fermi multiplet Υ, a

Fermi multiplet ζ and their conjugates $\overline{\Upsilon}$, $\overline{\zeta}$ and finally chiral multiplets Y, $\widetilde{Y}$ and their anti-chiral conjugates $\overline{Y}$, $\overline{\widetilde{Y}}$. H decomposes into chiral multiplets X, $\widetilde{X}$ and their conjugates $\overline{X}$, $\overline{\widetilde{X}}$ and a pair of Fermi multiplet λ, $\widetilde{\lambda}$ and their conjugates $\overline{\lambda}$, $\overline{\widetilde{\lambda}}$. The bi-fundamental U decomposes into two chiral multiplets ϕ, $\widetilde{\phi}$ in bi-fundamental representations $K \otimes \overline{N}$, $N \otimes \overline{K}$ of $U(K) \times SU(N)$ aswell as their conjugates $\overline{\phi}$, $\overline{\widetilde{\phi}}$ in conjugate representations. Furthermore there are Fermi multiplets ψ, $\widetilde{\psi}$ and their conjugates $\overline{\psi}, \overline{\widetilde{\psi}}$ in bi-fundamental representations of $U(K) \times SU(N)$.

Note that $\overline{}$ on $\mathcal{N} = (0,2)$ superfields means conjugation with respect to the algebra (6.16), namely it acts by conjugation on all of the quantum numbers barring J_{56} in Table 6.4.

$\mathbb{Z}_M$ **Orbifold** To obtain the effective theory of M-strings for the $(1,0)_A$ theory, without surface operator, we perform the $\mathbb{Z}_M$ orbifold on the above D1-brane world-volume theory via the standard Douglas-Moore orbifold procedure [157]. For a field f in representations $\mathcal{R} : U(K) \times SU(N) \to \operatorname{End} \mathcal{V}$ where $\mathcal{V}$ corresponds to the Chan-Paton space, then the $\mathbb{Z}_M$ orbifold acts by identification

$$f \sim \omega^{2J_L^R} \mathcal{R}(\tau, g)\, f \tag{6.17}$$

where τ, g are elements of $\mathbb{Z}_M$ subgroups of $U(K)$ and $SU(N)$ respectively. The partitions $\tilde{\kappa} = [K_1, K_2, \ldots, K_M]$ and (6.8) determine the embedding of $\mathbb{Z}_M$ into the respective groups. Note that, to get the correct M-string description we must set $K_M = 0$ [209]. By conjugation with elements of $U(K)/\prod_n U(K_n)$ and $SU(N)/S\left(\prod_n U(N_n)\right)$ we may take them to be block diagonal, explicitly:

$$\tau = \operatorname{diag}\left(\omega \mathbb{I}_{K_1}, \ldots, \omega^n \mathbb{I}_{K_n}, \ldots, \omega^M \mathbb{I}_{K_M}\right) \in U(K)\,, \tag{6.18}$$

$$g = \operatorname{diag}\left(\omega \mathbb{I}_{N_1}, \ldots, \omega^n \mathbb{I}_{N_n}, \ldots, \omega^M \mathbb{I}_{N_M}\right) \in SU(N)\,. \tag{6.19}$$

In particular

$$\Upsilon \sim \tau \Upsilon \tau^\dagger\,, \quad \zeta \sim \tau \zeta \tau^\dagger\,, \quad X \sim \tau X \tau^\dagger\,, \quad \widetilde{X} \sim \tau \widetilde{X} \tau^\dagger\,, \tag{6.20}$$

$$Y \sim \omega \tau Y \tau^\dagger\,, \quad \widetilde{Y} \sim \omega^{-1} \tau \widetilde{Y} \tau^\dagger\,, \quad \lambda \sim \omega \tau \lambda \tau^\dagger\,, \quad \widetilde{\lambda} \sim \omega^{-1} \tau \widetilde{\lambda} \tau^\dagger\,, \tag{6.21}$$

$$\phi \sim \tau \phi g^\dagger\,, \quad \widetilde{\phi} \sim g \widetilde{\phi} \tau^\dagger\,, \quad \psi \sim \omega^{-1} \tau \psi g^\dagger\,, \quad \widetilde{\psi} \sim \omega^{-1} g \widetilde{\psi} \tau^\dagger\,. \tag{6.22}$$

The field content of the orbifolded theory is then given by solving the equations (6.20), (6.21) and (6.22). The field content is summarised, after setting $k = 1$, in

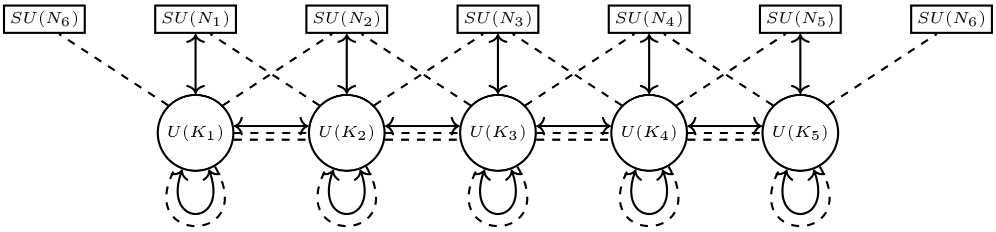


Figure 6.1: *Quiver diagram in* $\mathcal{N} = (0,2)$ *notation describing M-strings for the* $(1,0)_{A_{M-1}}$ *theory with* $M = 6$, $\tilde{\rho} = [N_1, N_2, \dots, N_6]$ *and* $\tilde{\kappa} = [K_1, K_2, K_3, K_4, K_5, K_6]$, $K_6 = 0$.

Table 6.5. From (6.20) we end up with $U(K_n)$, $n = 1, \dots, M$ vector multiplets with field strengths Υ_n with chiral multiplets $X_n, \widetilde{X}_n$ and Fermi multiplet ζ_n all in the adjoint representation of $\mathfrak{u}(K_n)$. Between each $U(K_n)$ vector multiplet we have chiral multiplets Y_n, $\widetilde{Y}_n$ in representations $\mathbf{K}_n \otimes \overline{\mathbf{K}}_{n+1}$, $\mathbf{K}_n \otimes \overline{\mathbf{K}}_{n-1}$ of $\prod_n U(K_n)$ and Fermi multiplets $\lambda_n, \widetilde{\lambda}_n$ also in representations $\mathbf{K}_n \otimes \overline{\mathbf{K}}_{n+1}$, $\mathbf{K}_n \otimes \overline{\mathbf{K}}_{n-1}$ respectively. From (6.22) there are Chiral multiplets ϕ_n, $\widetilde{\phi}_n$ in representations $\mathbf{K}_n \otimes \overline{\mathbf{N}}_n$, $\mathbf{N}_n \otimes \overline{\mathbf{K}}_n$ respectively. There are also Fermi multiplets ψ_n, $\widetilde{\psi}_n$ in representations $\mathbf{K}_n \otimes \overline{\mathbf{N}}_{n-1}$, $\mathbf{N}_{n+1} \otimes \overline{\mathbf{K}}_n$. Note that we identify $n \sim n + M$ with the understanding that $K_M = 0$. This can be conveniently summarised in the quiver diagram of Figure 6.1 where circular nodes denote vector multiplets, arrows denote chiral multiplets and dashed lines denote Fermi multiplets.

D1 Worldvolume Theory & 'Chain-Saw' Quiver

The theory of M-strings in the presence of the defect is then given by performing a further $\mathbb{Z}_k$ orbifold of the above quiver theory of Figure 6.1. Since the $\mathbb{Z}_k$ orbifold action preserves the $\mathcal{N} = (0,2)$ supercharges Q, $\widetilde{\mathrm{Q}}$ we may proceed as before. For a $\mathcal{N} = (0,2)$ superfield f in representation $\mathcal{R} : \prod_n U(K_n) \times S \prod_n U(N_n) \to \operatorname{End} \mathcal{V}$ the orbifold identifies

$$f \sim \gamma^{J_{12} - J_{710}} \mathcal{R} \left(\tau_1 \otimes \cdots \otimes \tau_M, g_1 \otimes \cdots \otimes g_M \right) f \,, \tag{6.23}$$

where

$$\tau_n = \operatorname{diag}\left(\gamma \mathbb{I}_{K_{n1}}, \dots, \gamma^i \mathbb{I}_{K_{ni}}, \dots, \gamma^k \mathbb{I}_{K_{nk}} \right) \in U(K_n) \,, \tag{6.24}$$

$$g_n = \gamma^{\frac{1}{2}} \operatorname{diag}\left(\gamma \mathbb{I}_{N_{n1}}, \dots, \gamma^i \mathbb{I}_{N_{ni}}, \dots, \gamma^k \mathbb{I}_{N_{nk}} \right) \in U(N_n) \,, \tag{6.25}$$

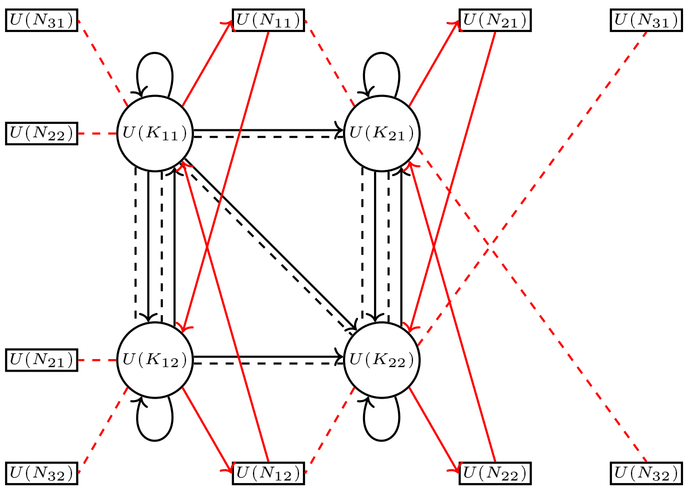


Figure 6.2: *Quiver diagram in $\mathcal{N} = (0,2)$ notation describing M-strings for the $(1,0)_{A_{M-1}}$ theory with $M = 3$, $k = 2$ $MkN = \sum_{n,i} N_{ni}$ and $\tilde{\kappa}_{n=1,2} = [K_{n1}, K_{n2}]$, $\tilde{\kappa}_3 = [0]$.*

corresponding to $\tilde{\rho}_n = [N_{n1}, \dots, N_{nk}]$ and $\tilde{\kappa}_n = [K_{n1}, \dots, K_{nk}]$.

Note that we also shifted the $\mathbb{Z}_k$ holonomy in $SU(N_n)$ as in [58, 49]. Explicitly

$$\Upsilon_n \sim \tau_n \Upsilon_n \tau_n^\dagger\,, \quad \bar{\xi}_n \sim \gamma\tau_n\bar{\xi}_n\tau_n^\dagger\,, \quad X_n \sim \gamma\tau_n X_n \tau_n^\dagger\,, \quad \tilde{X}_n \sim \tau_n \tilde{X}_n \tau_n^\dagger\,, \tag{6.26}$$

$$Y_n \sim \gamma^{-1}\tau_n Y_n \tau_{n+1}^\dagger\,, \quad \tilde{Y}_n \sim \tau_n \tilde{Y}_n \tau_{n-1}^\dagger\,, \quad \lambda_n \sim \tau_n \lambda_n \tau_{n+1}^\dagger\,, \tag{6.27}$$

$$\phi_n \sim \gamma^{\frac{1}{2}}\tau_n \phi_n g_n^\dagger\,, \quad \tilde{\lambda}_n \sim \gamma^{-1}\tau_n\tilde{\lambda}_n\tau_{n-1}^\dagger\,, \tag{6.28}$$

$$\tilde{\phi}_n \sim \gamma^{\frac{1}{2}} g_n \tilde{\phi}_n \tau_n^\dagger\,, \quad \psi_n \sim \gamma^{\frac{1}{2}}\tau_n\psi g_{n-1}^\dagger\,, \quad \tilde{\psi}_n \sim \gamma^{\frac{1}{2}} g_{n+1}\tilde{\psi}\tau_n^\dagger\,. \tag{6.29}$$

Solving the above equations (6.26)-(6.29) we end up with the field content of Table 6.5. Namely we have a set of $U(K_{ni})$ field strength multiplets Υ_{ni}. At each node we have a chiral superfield $\widetilde{X}_{ni}$ in the adjoint. On the other hand along the '$\mathbb{Z}_k$' direction of the quiver nodes are connected by chiral multiplets X_{ni} and Fermi multiplets ζ_{ni} in $\mathbf{K}_{ni} \otimes \overline{\mathbf{K}}_{n(i+1)}$. In addition there are also the chiral multiplets Y_{ni}, $\widetilde{Y}_{ni}$ in $\mathbf{K}_{ni} \otimes \overline{\mathbf{K}}_{(n+1)(i-1)}$, $\mathbf{K}_{ni} \otimes \overline{\mathbf{K}}_{(n-1)i}$ aswell as Fermi multiplets λ_{ni}, $\widetilde{\lambda}_{ni}$ in $\mathbf{K}_{ni} \otimes \overline{\mathbf{K}}_{(n+1)i}$, $\mathbf{K}_{ni} \otimes \overline{\mathbf{K}}_{(n-1)(i-1)}$. Coupling to $SU(N_{ni})$ global symmetries we have chirals ϕ_{ni}, $\widetilde{\phi}_{ni}$ in representations $\mathbf{K}_{ni} \otimes \overline{\mathbf{N}}_{ni}$, $\mathbf{N}_{n(i-1)} \otimes \overline{\mathbf{K}}_{ni}$ and Fermi multiplets ψ_{ni}, $\widetilde{\psi}_{ni}$ in $\mathbf{K}_{ni} \otimes \overline{\mathbf{N}}_{(n-1)i}$, $\mathbf{N}_{(n+1)(i-1)} \otimes \overline{\mathbf{K}}_{ni}$ representations. The field content can be summarised in the quiver diagram of Figure 6.2. This is a generalisation of the chainsaw quiver of [49]. Circular nodes denote vector multiplets, arrows denote

$(0,2)$	$\prod_{n,i} U(K_{ni})$	$\prod_{n,i} SU(N_{ni})$	J_{56}	J_L	J_R	J_L^R	J_R^R
Υ_{ni}	adj_{ni}	$\mathbf{1}$	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0	$-\frac{1}{2}$
ζ_{ni}	$\square_{ni}\otimes\overline{\square}_{n(i+1)}$	$\mathbf{1}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
Y_{ni}	$\square_{ni}\otimes\overline{\square}_{(n+1)(i-1)}$	$\mathbf{1}$	0	0	0	$+\frac{1}{2}$	$+\frac{1}{2}$
$\widetilde{Y}_{ni}$	$\square_{ni}\otimes\overline{\square}_{(n-1)i}$	$\mathbf{1}$	0	0	0	$-\frac{1}{2}$	$+\frac{1}{2}$
X_{ni}	$\square_{ni}\otimes\overline{\square}_{n(i+1)}$	$\mathbf{1}$	0	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\widetilde{X}_{ni}$	adj_{n}	$\mathbf{1}$	0	$+\frac{1}{2}$	$-\frac{1}{2}$	0	0
λ_{ni}	$\square_{ni}\otimes\overline{\square}_{(n+1)i}$	$\mathbf{1}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$+\frac{1}{2}$	0
$\widetilde{\lambda}_{ni}$	$\square_{ni}\otimes\overline{\square}_{(n-1)(i-1)}$	$\mathbf{1}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0
ϕ_{ni}	$\square_{ni}$	$\overline{\square}_{ni}$	0	0	$-\frac{1}{2}$	0	0
$\widetilde{\phi}_{ni}$	$\overline{\square}_{ni}$	$\square_{n(i-1)}$	0	0	$-\frac{1}{2}$	0	0
ψ_{ni}	$\square_{ni}$	$\overline{\square}_{(n-1)i}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0
$\widetilde{\psi}_{ni}$	$\overline{\square}_{ni}$	$\square_{(n+1)(i-1)}$	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	0

Table 6.5: *Gauge covariant field content with $\delta=0$ of the $\mathcal{N}=(4,4)$ vector multiplet V.*

chiral multiplets and dashed lines denote Fermi multiplets. Coloured in red are those, and only those, fields which couple to $U(N_{ni})$ global symmetries. Again we always identify $n\sim n+M$, $i\sim i+k$ and we take $K_{Mi}\equiv 0$.

BPS Equations

For completeness we list the BPS equations of the 2d gauge system. In other words, the vanishing of the F- and D-term equations. The BPS equations are

$$\phi\overline{\phi}-\overline{\widetilde{\phi}}\widetilde{\phi}+[X,\overline{X}]+[\widetilde{X},\overline{\widetilde{X}}]=\mathfrak{z}\mathbb{I}_K\,,\quad \phi\widetilde{\phi}+[X,\widetilde{X}]=0\,, \tag{6.30}$$

where here we abuse the notation such that, for example, X denotes the scalar in the X chiral multiplet. We also added in the Fayet-Iliopoulos deformation $\mathfrak{z}$. These are nothing but the ADHM equations for K $U(N)$ instantons. See also Section 1.3. By implementing the projections (6.20), (6.21), (6.22), (6.26), (6.27) and (6.29) to (6.30) we obtain the BPS equations of the system in the presence of the surface operator

$$\phi_{ni}\overline{\phi}_{ni}-\overline{\widetilde{\phi}}_{ni}\widetilde{\phi}_{ni}+X_{ni}\overline{X}_{n(i+1)}-\overline{X}_{n(i-1)}X_{n(i-1)}+[\widetilde{X}_{ni},\overline{\widetilde{X}}_{ni}]=\mathfrak{z}\mathbb{I}_{K_{ni}}\,, \tag{6.31}$$

$$\phi_{ni}\widetilde{\phi}_{n(i+1)}+X_{ni}\widetilde{X}_{n(i+1)}-\widetilde{X}_{ni}X_{ni}=0\,. \tag{6.32}$$

6.2.3 BPS Partition Function on $T^2 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$

The $T^2 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ partition function for the $(1,0)_A$ theories with surface operator labelled by ρ (or equivilently $\tilde{\rho}$) can be defined as the following trace over the Hilbert space $\mathcal{H}$ of the theory [185, 63, 184]

$$\begin{aligned} Z^\rho = \mathrm{Tr}_{\mathcal{H}} \Bigg[& (-1)^F Q_\tau^{H_-} \overline{Q_\tau}^{H_+} (\mathfrak{q}\mathfrak{t})^{-J_L} \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{J_R + J_R^R} \mathfrak{c}^{2J_L^R} \\ & \times \prod_{n=1}^{M} \prod_{i=1}^{k} \left(w_{ni}^{K_{ni}} \prod_{A=1}^{N} x_{ni,A}^{f_{ni,A}} \right) \Bigg] . \end{aligned} \tag{6.33}$$

Here $F = F_- + F_+$ is the fermion number, $H_\pm = \frac{1}{2}(H \pm P)$ are the right/left-moving Hamiltonians on T^2 and $Q_\tau = e^{2\pi \mathrm{i}\tau}$ is the modular parameter of the torus. Using the 6d supersmmetry generators $\mathsf{Q}^{\alpha\dot{a}}_{+-}$, $\widetilde{\mathsf{Q}}^{\dot{\alpha}\dot{a}}_{++}$. We identify

$$\mathsf{Q} := \widetilde{\mathsf{Q}}^{\dot{1}\dot{2}}_{++} = \mathrm{Q}\,, \quad \widetilde{\mathsf{Q}} := \widetilde{\mathsf{Q}}^{\dot{2}\dot{1}}_{++} = \widetilde{\mathrm{Q}}\,. \tag{6.34}$$

Then, the right-moving Hamiltonian can be written as $H_+ = \{\mathsf{Q}, \widetilde{\mathsf{Q}}\}$ which is also preserved by the surface operator. Since Q, $\widetilde{\mathsf{Q}}$ commute with $SU(2)_\alpha$ and $SU(2)_a$ we may include fugacities for their Cartans. Furthermore they commute with the diagonal subgroup $SU(2)_D \subset SU(2)_{\dot{\alpha}} \times SU(2)_{\dot{a}}$ hence we also include a fugacity for its Cartan $J_D = J_R + J_R^R$. We also include fugacities $x_{ni,A}$ for the Cartans $f_{ni,A}$ of $\mathfrak{u}(N_{ni})$. We also include fugacity w_{in} for the string winding number K_{ni}. Moreover, $\mathfrak{q}, \mathfrak{t}, \mathfrak{c}$ are related to the torus action $U(1)_{\epsilon_1} \times U(1)_{\epsilon_2} \times U(1)_m$ of (6.4), (6.5) on $\mathbb{C}^3$ by

$$\mathfrak{q} = e^{2\pi \mathrm{i}\epsilon_1}\,, \quad \mathfrak{t} = e^{-2\pi \mathrm{i}\epsilon_2}\,, \quad \mathfrak{c} = e^{2\pi \mathrm{i} m}\,. \tag{6.35}$$

The partition function Z counts states annihilated by both Q and $\widetilde{\mathsf{Q}}$. In particular it factorises as

$$Z^\rho = Z^\rho_{\text{pert}} Z^\rho_{\text{string}}\,, \quad Z^\rho_{\text{string}} = \sum_{\substack{K_{ni} \geqslant 0 \\ K_{Mi}=0}} \left(\prod_{n=1}^{M} \prod_{i=1}^{k} w_{ni}^{K_{ni}} \right) \mathrm{Ell}^{\tilde{\rho}}_{\vec{\kappa}=[K_{11},\ldots,K_{Mk}]} \tag{6.36}$$

where Z_{pert} is the contribution from the BPS particles. Due to the Ω-background the BPS strings are localised to sit at the origin of $\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ and wrap the T^2. This gives rise to the effective 2d description that we described in the previous section. $\mathrm{Ell}^{\tilde{\rho}}_{\vec{\kappa}}$ is the Elliptic genus partition function, which will be defined in Section 6.2.4, of

		$\mathfrak{su}(2)_L$	$\mathfrak{su}(2)_R$	$\mathfrak{su}(2)_R^R$	$\mathfrak{u}(1)_{H_-}$	i
Half-Hyper	ϕ	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	0	$\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}+\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}$
	ψ	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	0	$-\sqrt{\mathfrak{q}\mathfrak{t}}-\sqrt{\frac{1}{\mathfrak{q}\mathfrak{t}}}$
Tensor	σ	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	1
	λ	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{2}$	0	$-\mathfrak{q}-\mathfrak{t}-\frac{1}{\mathfrak{q}}-\frac{1}{\mathfrak{t}}$
	B	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	0	$\mathfrak{q}\mathfrak{t}+1+\frac{1}{\mathfrak{t}\mathfrak{q}}$
	$\partial_{\alpha\dot{\alpha}}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{1}$	0	$\mathfrak{q},\mathfrak{t},\frac{1}{\mathfrak{q}},\frac{1}{\mathfrak{t}}$
	∂_-	$\mathbf{2}$	$\mathbf{1}$	$\mathbf{1}$	1	Q_τ

Table 6.6: *Letters contributing to* Z^{ρ}_{pert}.

said 2d theory and $\mathrm{Ell}^{\tilde{\rho}}_{[0,0,\ldots,0]}\equiv 1$. The perturbative piece is given by enumerating all letters of Table 6.6 [185, 184], for the case without defect $\rho=\varnothing$ it reads

$$Z^{\rho=\varnothing}_{\text{pert}}=\mathrm{PE}\left[i\right]\,,\quad i=\frac{\left(i_{\text{hyp}}+i_{\text{Tensor}}\right)\mathfrak{q}\mathfrak{t}}{(1-Q_\tau)(1-\mathfrak{q})^2(1-\mathfrak{t})^2} \tag{6.37}$$

$$i_{\text{hyp}}=\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}+\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}-\sqrt{\mathfrak{q}\mathfrak{t}}-\sqrt{\frac{1}{\mathfrak{q}\mathfrak{t}}}\right)\left(\mathfrak{c}+\mathfrak{c}^{-1}\right)\sum_{n=1}^{M}\sum_{A=1}^{N}x_{n,A}x^{-1}_{n+1,B} \tag{6.38}$$

$$i_{\text{Tensor}}=NM\left(2+\mathfrak{q}\mathfrak{t}+\frac{1}{\mathfrak{t}\mathfrak{q}}-\mathfrak{q}-\mathfrak{t}-\frac{1}{\mathfrak{q}}-\frac{1}{\mathfrak{t}}\right)\,. \tag{6.39}$$

We can also compute the perturbative piece for the $T^2\times\mathbb{R}^4_{\epsilon_1,\epsilon_2}$ partition function for the theory in the presence of the defect by performing the orbifolding procedure to the letters of Table 6.6.

$$Z^{\rho}_{\text{pert}}=\mathrm{PE}\left[i^{\text{orb}}\right]\,,\quad i^{\text{orb}}=\frac{1}{|\mathbb{Z}_k|}\sum_{\gamma\in\mathbb{Z}_k}\frac{\left(i^{\text{orb}}_{\text{hyp}}+i^{\text{orb}}_{\text{Tensor}}\right)\gamma\mathfrak{q}\mathfrak{t}}{(1-Q_\tau)(1-\gamma\mathfrak{q})^2(1-\mathfrak{t})^2} \tag{6.40}$$

$$\begin{aligned}i^{\text{orb}}_{\text{hyp}}&=\left(\gamma\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}+\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}-\gamma\sqrt{\mathfrak{q}\mathfrak{t}}-\sqrt{\frac{1}{\mathfrak{q}\mathfrak{t}}}\right)\left(\frac{\mathfrak{c}}{\gamma}+\frac{1}{\mathfrak{c}}\right)\\&\quad\times\sum_{i,j=1}^{k}\sum_{n=1}^{M}\sum_{A=1}^{N_{ni}}\sum_{B=1}^{N_{(n+1)j}}\frac{\gamma^{i-j}x_{ni,A}}{x_{(n+1)j,B}}\end{aligned} \tag{6.41}$$

$$i^{\text{orb}}_{\text{Tensor}}=NM\left(2+\gamma\mathfrak{t}\mathfrak{q}+\frac{\gamma}{\mathfrak{t}\mathfrak{q}}-\gamma\mathfrak{q}-\mathfrak{t}-\frac{\gamma}{\mathfrak{q}}-\frac{1}{\mathfrak{t}}\right)\,. \tag{6.42}$$

6.2.4 M-String Elliptic Genus Without Surface Operator

We now turn towards computing the supersymmetric index a.k.a flavoured elliptic genus partition function for our theory. In this section we begin by reviewing the index computation for the case without surface operator ($k = 1$) i.e K M-strings inside M M5-branes sitting at the tip of a A_{N-1} singularity without surface operator. Following that we compute the M-string elliptic genus partition function with the surface operator.

This index may be viewed either as a path integral of the theory on T^2 in which case it may be computed from localisation techniques [223, 224] or, since our theory admits a free field limit, as a counting problem on T^2 in the radial quantisation in which case it may be computed via 'letter counting' [211, 222, 212, 213, 214, 215, 216, 198].

The elliptic genus for a fixed string number configuration
$\tilde{\kappa} = [K_1, K_2, \ldots, K_{M-1}, K_M = 0]$ and $\tilde{\rho} = [N_1, \ldots, N_M]$ is then defined to be

$$\begin{aligned}&\mathrm{Ell}_{\tilde{\kappa}}^{\tilde{\rho}}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathbf{c}, \vec{x}_n) = \\ &\qquad \mathrm{Tr}_{\mathrm{R}}\left[(-1)^F Q_\tau^{H_-}\,(\mathfrak{q}\mathfrak{t})^{-J_L}\left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{J_R+J_R^R}\mathfrak{c}^{2J_L^R}\prod_{n=1}^{M}\prod_{A=1}^{N} x_{n,A}^{f_{n,A}}\right].\end{aligned} \tag{6.43}$$

Here $F = F_- + F_+$ is the fermion number and the trace is taken over the Hilbert space on $\mathbb{S}^1$ in the radial quantisation. The full index is then given by

$$\mathrm{Ell}_{\tilde{\kappa}}^{\tilde{\rho}}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathbf{c}, \vec{x}_n) = \prod_{n=1}^{M}\frac{1}{K_n!}\oint\prod_{I=1}^{K_n}\frac{dy_{n,I}}{2\pi \mathrm{i} y_{n,I}}\prod_P \mathrm{Ell}_P(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathfrak{c}, \vec{x}_n, \vec{y}_n) \tag{6.44}$$

where the product over P denotes the product over all $\mathcal{N} = (0, 2)$ multiplets of the theory and Ell_P their contributions to the index, which are listed in Appendix G.1.1. It is convenient to rewrite the index as

$$\mathrm{Ell}_{\tilde{\kappa}}^{\tilde{\rho}} = \prod_{n=1}^{M}\frac{1}{K_n!}\oint\prod_{I=1}^{K_n}\frac{dy_{n,I}}{2\pi \mathrm{i} y_{n,I}}\mathrm{Ell}_{\tilde{\kappa}}^{\mathrm{tens}}\mathrm{Ell}_{\tilde{\kappa}}^{\mathrm{hyp}}; \tag{6.45}$$

we have divided up the corresponding contributions from 6d $\mathcal{N} = (1, 0)$ tensor multiplets (uncharged under $\mathfrak{c}$)

$$\begin{aligned}
\mathrm{Ell}^{\mathrm{tens}}_{\tilde{\kappa}} &= \prod_{P\in\{\Upsilon,\zeta,X,\tilde{X},\phi,\tilde{\phi}\}} \mathrm{Ell}_{\mathcal{M}} \\
&= \prod_{n=1}^{M} (\mathrm{i}\eta(Q_\tau))^{3K_n} \frac{\prod_{I\neq J}\theta_1\left(\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)\prod_{I,J=1}^{K_n}\theta_1\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)}{\prod_{I,J=1}^{K_n}\theta_1\left(\mathfrak{t}^{-1}\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)\theta_1\left(\mathfrak{q}\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)} \\
&\times \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \frac{(\mathrm{i}\eta(Q_\tau))^2}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{x_{n,A}}{y_{n,I}};Q_\tau\right)\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{y_{n,I}}{x_{n,A}};Q_\tau\right)}
\end{aligned} \tag{6.46}$$

and $\mathcal{N}=(1,0)$ hypermultiplets (charged under $\mathfrak{c}$)

$$\begin{aligned}
\mathrm{Ell}^{\mathrm{hyp}}_{\tilde{\kappa}} &= \prod_{P\in\{\lambda,\tilde{\lambda},Y,\tilde{Y},\psi,\tilde{\psi}\}} \mathrm{Ell}_{P} \\
&= \prod_{n=1}^{M}\prod_{I=1}^{K_n} \frac{\prod_{J=1}^{K_{n-1}}\theta_1\left(\mathfrak{c}\sqrt{\frac{1}{\mathfrak{q}\mathfrak{t}}}\frac{y_{n-1,J}}{y_{n,I}};Q_\tau\right)\prod_{J=1}^{K_{n+1}}\theta_1\left(\mathfrak{c}^{-1}\sqrt{\frac{1}{\mathfrak{q}\mathfrak{t}}}\frac{y_{n+1,J}}{y_{n,I}};Q_\tau\right)}{\prod_{J=1}^{K_{n-1}}\theta_1\left(\mathfrak{c}^{-1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{n,I}}{y_{n-1,J}};Q_\tau\right)\prod_{J=1}^{K_{n+1}}\theta_1\left(\mathfrak{c}\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{y_{n,I}}{y_{n+1,J}};Q_\tau\right)} \\
&\times \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \frac{\theta_1\left(\mathfrak{c}\frac{x_{n-1,A}}{y_{n,I}};Q_\tau\right)\theta_1\left(\mathfrak{c}\frac{y_{n,I}}{x_{n+1,A}};Q_\tau\right)}{(\mathrm{i}\eta(Q_\tau))^2}\,.
\end{aligned} \tag{6.47}$$

See Appendix G.1 for the definition of the θ_1 function and various properties. We identify $n\sim n+M$ orbifold indices within the products. The countour integrals (6.45) should be computed using the Jefferey-Kirwan residue prescription [230, 223, 224]. Equation (6.43) may be expressed as a sum of residues associated to poles classified by M N-coloured Young's diagrams $\vec{\mu}_n$ such that $|\vec{\mu}_n| := \sum_A |\mu_{n,A}| = K_n$. We choose the $+\sum y$ JK prescription. The pole corresponding to the box $s=(l,p)\in\mu_{n,A}$ is then

$$y_n(s) = x_{n,A}\mathfrak{q}^{l-\frac{1}{2}}\mathfrak{t}^{-p+\frac{1}{2}}\,. \tag{6.48}$$

As explained in [89, 41] only residues arising from these poles should be kept. We also assumed that the x's can be made sufficiently generic. The residue for a fixed coloured Young diagram is then

$$\mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa},\vec{\mu}_n} = \mathrm{Ell}^{\mathrm{tens}}_{\tilde{\kappa},\vec{\mu}_n}\mathrm{Ell}^{\mathrm{hyp}}_{\tilde{\kappa},\vec{\mu}_n} \tag{6.49}$$

where

$$
\begin{aligned}
\mathrm{Ell}^{\mathrm{tens}}_{\tilde{\kappa},\vec{\mu}_n} = \prod_{n=1}^{M}\prod_{A,B=1}^{N}\Bigg\{ & \prod_{\substack{(l_1,p_1)\in\mu_{n,A}\\(l_2,p_2)\in\mu_{n,B}}} \frac{\theta_1\left(\mathfrak{q}^{l_1-l_2}\mathfrak{t}^{-p_1+p_2}\frac{x_{n,A}}{x_{n,B}};Q_\tau\right)}{\theta_1\left(\mathfrak{q}^{l_1-l_2}\mathfrak{t}^{-p_1+p_2-1}\frac{x_{n,A}}{x_{n,A_2}};Q_\tau\right)}\\
& \prod_{\substack{(l_1,p_1)\in\mu_{n,A}\\(l_2,p_2)\in\mu_{n,B}}} \frac{\theta_1\left(\mathfrak{q}^{l_1-l_2}\mathfrak{t}^{-p_1+p_2}\frac{\mathfrak{q}}{\mathfrak{t}}\frac{x_{n,A}}{x_{n,B}};Q_\tau\right)}{\theta_1\left(\mathfrak{q}^{l_1-l_2+1}\mathfrak{t}^{-p_1+p_2}\frac{x_{n,A}}{x_{n,B}};Q_\tau\right)}\\
& \prod_{(l,p)\in\mu_{n,B}} \frac{1}{\theta_1\left(\mathfrak{q}^{-l+1}\mathfrak{t}^{p-1}\frac{x_{n,A}}{x_{n,B}};Q_\tau\right)\theta_1\left(\mathfrak{q}^{l}\mathfrak{t}^{-p}\frac{x_{n,B}}{x_{n,A}};Q_\tau\right)}\Bigg\},
\end{aligned}
\tag{6.50}
$$

and

$$
\begin{aligned}
\mathrm{Ell}^{\mathrm{hyp}}_{\tilde{\kappa},\vec{\mu}_n} = \prod_{n=1}^{M}\prod_{A,B=1}^{N}\Bigg\{ & \prod_{\substack{(l_1,p_1)\in\mu_{n,A}\\(l_2,p_2)\in\mu_{n-1,B}}} \frac{\theta_1\left(\mathfrak{c}^{-1}\mathfrak{q}^{l_1-l_2-\frac{1}{2}}\mathfrak{t}^{-p_1+p_2-\frac{1}{2}}\frac{x_{n,A}}{x_{n-1,B}};Q_\tau\right)}{\theta_1\left(\mathfrak{c}\mathfrak{q}^{-l_1+l_2-\frac{1}{2}}\mathfrak{t}^{p_1-p_2+\frac{1}{2}}\frac{x_{n-1,B}}{x_{n,A}};Q_\tau\right)}\\
& \prod_{\substack{(l_1,p_1)\in\mu_{n,A}\\(l_2,p_2)\in\mu_{n-1,B}}} \frac{\theta_1\left(\mathfrak{c}\mathfrak{q}^{-l_1+l_2-\frac{1}{2}}\mathfrak{t}^{p_1-p_2-\frac{1}{2}}\frac{x_{n-1,B}}{x_{n,A}};Q_\tau\right)}{\theta_1\left(\mathfrak{c}^{-1}\mathfrak{q}^{l_1-l_2-\frac{1}{2}}\mathfrak{t}^{-p_1+p_2+\frac{1}{2}}\frac{x_{n,A}}{x_{n-1,B}};Q_\tau\right)}\\
& \prod_{(l,p)\in\mu_{n,B}} \theta_1\left(\mathfrak{c}\mathfrak{q}^{-l+\frac{1}{2}}\mathfrak{t}^{p-\frac{1}{2}}\frac{x_{n-1,A}}{x_{n,B}};Q_\tau\right)\theta_1\left(\mathfrak{c}\mathfrak{q}^{l-\frac{1}{2}}\mathfrak{t}^{-p+\frac{1}{2}}\frac{x_{n,B}}{x_{n+1,A}};Q_\tau\right)\Bigg\}.
\end{aligned}
\tag{6.51}
$$

Note that the above function contains various factors of $\theta_1(1;Q_\tau)$ albeit which cancel between numerator and denominator, thus rendering (6.49) well defined. Equation (6.49) may be simplified by means of the identity (A.32) where the sums over monomials are translated into products over Jacobi theta functions. Hence we may write (6.49) as

$$
\begin{aligned}
\mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa},\vec{\mu}_n} = \prod_{n=1}^{M}\prod_{A,B=1}^{N}\prod_{(l,p)\in\mu_{n,A}} \Bigg\{ & \frac{\theta_1\left(\mathfrak{c}\mathfrak{q}^{l-\mu^{\mathrm{T}}_{n-1,B;p}-\frac{1}{2}}\mathfrak{t}^{-\mu_{n,A;l}-\frac{1}{2}+p}\frac{x_{n-1,B}}{x_{n,A}};Q_\tau\right)}{\theta_1\left(\mathfrak{q}^{l-\mu^{\mathrm{T}}_{n,B;p}}\mathfrak{t}^{-1-\mu_{n,A;l}+p}\frac{x_{n,A}}{x_{n,B}};Q_\tau\right)}\\
& \times\frac{\theta_1\left(\mathfrak{c}\mathfrak{q}^{\mu^{\mathrm{T}}_{n+1,B;p}-l+\frac{1}{2}}\mathfrak{t}^{-p+\mu_{n,A;l}+\frac{1}{2}}\frac{x_{n,A}}{x_{n+1,B}};Q_\tau\right)}{\theta_1\left(\mathfrak{q}^{1+\mu^{\mathrm{T}}_{n,B;p}-l}\mathfrak{t}^{-p+\mu_{n,A;l}}\frac{x_{n,B}}{x_{n,A}};Q_\tau\right)}\Bigg\}.
\end{aligned}
\tag{6.52}
$$

Summing over all coloured Young diagrams $\vec{\mu}_n$ with exactly K_n boxes means equation (6.43) reads

$$\mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa}}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathfrak{c}, \vec{x}_n) := \sum_{\substack{\vec{\mu}_n \\ |\vec{\mu}_n| = K_n}} \mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa}, \vec{\mu}_n}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathfrak{c}, \vec{x}_n) \,. \tag{6.53}$$

The grand partition function obtained by 'integrating' over all partitions $\tilde{\kappa}$ which we weight by fugacity w_n ($w_M = 0$ to enforce $K_M = 0$) is then the total self dual-string contribution to the partition function:

$$Z^{\rho}_{\mathrm{string}}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathfrak{c}, \vec{x}_n; w_n) := \sum_{\vec{\mu}_n} \left(\prod_{n=1}^{M} w_n^{|\vec{\mu}_n|} \right) \mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa}, \vec{\mu}_n}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathfrak{c}, \vec{x}_n) \,. \tag{6.54}$$

Considering (6.54) as a generating function for the partition functions of a $\tilde{\kappa}$ M-string configurations in which case $w_n = e^{2\pi \mathrm{i} t_{n,n+1}}$ is related to the separation between the n^{th} and $(n+1)^{\mathrm{th}}$ M5-brane (6.1) [209]. It was demonstrated in [209] that (6.54), combined with the perturbative piece (6.37) reproduces the worldvolume theory of M M5-branes on a transverse A_{N-1} singuarity, as computed using the refined topological vertex.

6.2.5 M-String Elliptic Genus With Surface Operator

We now turn to the case with defect. The partition function we compute is equivalent to the K-theoretic version of the one computed in [49] further generalised to quiver theory with flavour. The elliptic genus for the theory of defect M-strings is defined in mostly the same way as before. The only difference in the definition is that we now have fugacities $\vec{x}_{ni}$ for the Cartans f_{ni} of $\mathfrak{u}(N_{ni})$. For a fixed string partition $\tilde{\kappa}$ we have

$$\begin{aligned} &\mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa}}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathfrak{c}, \vec{x}_{ni}) = \\ &\qquad \mathrm{Tr}_{\mathrm{R}} \left[(-1)^F q^{H_-} (\mathfrak{q}\mathfrak{t})^{-J_L} \left(\frac{\mathfrak{q}}{\mathfrak{t}} \right)^{-J_R - J_R^R} \mathfrak{c}^{2J_L^R} \prod_{n,i} \prod_{A=1}^{N_{ni}} x_{ni,A}^{f_{ni,A}} \right] . \end{aligned} \tag{6.55}$$

The elliptic genus then takes the form

$$\begin{aligned} &\mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa}}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathfrak{c}, \vec{x}_{ni}) := \\ &\qquad \prod_{n=1}^{M} \prod_{i=1}^{k} \frac{1}{K_{ni}!} \oint \prod_{I=1}^{K_{ni}} \frac{dy_{ni,I}}{2\pi \mathrm{i} y_{ni,I}} \prod_{P} \mathrm{Ell}^{\mathbb{Z}_k}_{P}(Q_\tau, \mathfrak{q}, \mathfrak{t}, \mathfrak{c}, \vec{x}_{ni}, \vec{y}_{ni}) \,, \end{aligned} \tag{6.56}$$

where the 1-loop determinants $\mathrm{Ell}_P^{\mathbb{Z}_k}$ for each multiplet P is given in Appendix G.1.2. At the level of the elliptic genus the orbifold action (6.7), (6.23) in the IIB description may be traded for an action on the fugacities

$$\mathbb{Z}_k : \mathfrak{q} \mapsto \gamma\mathfrak{q}\,, \quad \mathfrak{c}^2 \mapsto \gamma^{-1}\mathfrak{c}^2\,, \quad \vec{x}_{ni} \mapsto \gamma^{i+\frac{1}{2}}\vec{x}_{ni}\,, \quad \vec{y}_{ni} \mapsto \gamma^{i}\vec{y}_{ni}\,, \tag{6.57}$$

with trivial action on all other fugacties. As before, the terms attributed to 6d $\mathcal{N}=(1,0)$ tensor multiplets are uncharged under the $\mathfrak{c}$ fugacity while those coming from $(1,0)$ hypermultiplets are charged under $\mathfrak{c}$. We may now perform the integrals. After the orbifolding the solutions are again labelled by M N-tuples of Youngs diagrams $\vec{\mu}_n = \{\mu_{ni,A}\}$ with $n = 1,\ldots,M$ $i = 1,\ldots,k$ and $A = 1,\ldots,N_{ni}$. Each N-tuple has $|\vec{\mu}_n| = \sum_{i=1}^{k} K_{ni} = K_n$. For a box $s = (l,p) \in \mu_{ni,A}$ in the $+\sum y$ JK-prescription the corresponding pole is given by

$$y_{nj}(s) = x_{ni,A}\mathfrak{q}^{l-\frac{1}{2}}\mathfrak{t}^{-p+\frac{1}{2}}|_{i+l=j \bmod k} \tag{6.58}$$

i.e. the $\mathbb{Z}_k$ invariant part of (6.48). In [243] the contour integral representation for the partition function of $\mathbb{C}\times\mathbb{C}/\mathbb{Z}_k$ instantons for 4d $\mathcal{N}=2$ theories was analysed in detail. It was shown that to evaluate the contour integral is equivalent to projecting onto $\mathbb{Z}_k$ invariant terms in the original Nekrasov expression. As in the case without surface operator, the partition function (6.56) may be thought of as the elliptic uplift of [243]. Hence our integrals may be computed by projecting onto the $\mathbb{Z}_k$ invariant parts of (6.49). The string number K_{ni} for each factor in $\prod_{n,i} U(N_{ni})$ is hence given by the height of the l^{th} row in a given $\mu_{ni,A}$ and then summing over all contributions

$$K_{nj} = K_{nj}(\vec{\mu}_{ni}) = \sum_{l=0}^{\infty}\sum_{A=1}^{N_{n(i+1-l)}} \mu^{\mathrm{T}}_{n(i+1-l),A;l}\,. \tag{6.59}$$

The $\mathbb{Z}_k$ invariant part of (6.53) is [243, 49] [2]

$$\mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa},\vec{\mu}_{ni}} = \mathrm{Ell}^{\text{tens}}_{\tilde{\kappa},\vec{\mu}_{ni}}\mathrm{Ell}^{\text{hyp}}_{\tilde{\kappa},\vec{\mu}_{ni}} \tag{6.60}$$

[2]Upon taking a decoupling limit where we discard all θ_1 which depend on $\mathfrak{c}$ and then take, for example $N_{(n=1)i} = N_i$, $N_{(n\neq 1)i} = 0$ (6.61) is to be compared with the elliptic version of equation (2.43) of [49]. Sending back $\epsilon_2^{\text{them}}/k \mapsto \epsilon_2^{\text{them}}$ and $\epsilon_1^{\text{them}} \leftrightarrow \epsilon_2^{\text{them}}$ then our parameters are related (up to various factors of 2πi) by $x_{1i,A} \sim e^{i\epsilon_1^{\text{them}} - a^{\text{them}}_{A,i}}$, $\mathfrak{t}^{-1} \sim e^{\epsilon_1^{\text{them}}}$, $\mathfrak{q} \sim e^{\epsilon_2^{\text{them}}}$.

where the contribution from tensor multiplets is

$$
\begin{aligned}
\mathrm{Ell}^{\mathrm{tens}}_{\tilde{\kappa},\vec{\mu}_{ni}} = & \prod_{n=1}^{M} \prod_{u,i,j=1}^{k} \prod_{\substack{A_1,A_2=1 \\ (kl_1+i,p_1) \\ (kl_2+j,p_2)}}^{N_{n(u-i+1)},N_{n(u-j+1)}} \frac{\theta_1\left(\mathfrak{q}^{\gamma_{ij}}\,\mathfrak{t}^{-p_1+p_2}\,\frac{x_{n(u-i+1),A_1}}{x_{n(u-j+1),A_2}};Q_\tau\right)}{\theta_1\left(\mathfrak{q}^{\gamma_{ij}}\,\mathfrak{t}^{-p_1+p_2-1}\,\frac{x_{n(u-i+1),A_1}}{x_{n(u-j+1),A_2}};Q_\tau\right)} \\
& \times \prod_{n=1}^{M} \prod_{u,i,j=1}^{k} \prod_{\substack{A_1,A_2=1 \\ (kl_1+i,p_1) \\ (kl_2+j,p_2)}}^{N_{n(u-i)},N_{n(u-j+1)}} \frac{\theta_1\left(\mathfrak{q}^{\gamma_{(i+1)j}}\,\mathfrak{t}^{-p_1+p_2-1}\,\frac{x_{n(u-i),A_1}}{x_{n(u-j+1),A_2}};Q_\tau\right)}{\theta_1\left(\mathfrak{q}^{\gamma_{(i+1)j}}\,\mathfrak{t}^{-p_1+p_2}\,\frac{x_{n(u-i),A_1}}{x_{n(i-j+1),A_2}};Q_\tau\right)} \\
& \times \prod_{n=1}^{M} \prod_{i,j=1}^{k} \prod_{A=1}^{N_{ni}} \frac{1}{\prod\limits_{\substack{B=1 \\ (lk+j,p)}}^{N_{n(i-j+1)}} \theta_1\left(\mathfrak{q}^{-\gamma_j}\,\mathfrak{t}^{-1+p}\,\frac{x_{ni,A}}{x_{n(i-j+1),B}};Q_\tau\right)} \\
& \times \prod_{n=1}^{M} \prod_{i,j=1}^{k} \prod_{A=1}^{N_{ni}} \frac{1}{\prod\limits_{\substack{B=1 \\ (kl+j,p)}}^{N_{n(i-j)}} \theta_1\left(\mathfrak{q}^{\gamma_{j+1}}\,\mathfrak{t}^{-p}\,\frac{x_{n(i-j),B}}{x_{ni,A}};Q_\tau\right)} \,.
\end{aligned}
$$

and, from hypers,

$$\mathrm{Ell}^{\mathrm{hyp}}_{\tilde{\kappa},\vec{\mu}_{ni}} = \prod_{n=1}^{M} \prod_{i,j=1}^{k} \left\{ \prod_{A=1}^{N_{(n-1)i}} \prod_{\substack{B=1 \\ (kl+j,p)}}^{N_{n(i-j)}} \theta_1 \left(\mathfrak{c}\mathfrak{q}^{-\gamma_{j+1}+\frac{1}{2}} \mathfrak{t}^{p-\frac{1}{2}} \frac{x_{(n-1)i,A}}{x_{n(i-j),B}}; Q_\tau \right) \right.$$

$$\prod_{u=1}^{k} \frac{\displaystyle\prod_{\substack{A_1,A_2=1 \\ (kl_1+i,p_1) \\ (kl_2+j,p_2)}}^{N_{(n-1)(u-i+1)},N_{n(u-j)}} \theta_1 \left(\mathfrak{c}\mathfrak{q}^{\gamma_{i(j+1)}+\frac{1}{2}} \mathfrak{t}^{-p_1+p_2-\frac{1}{2}} \frac{x_{(n-1)(u-i+1),A_1}}{x_{n(u-j),A_2}}; Q_\tau \right)}{\displaystyle\prod_{\substack{A_1,A_2=1 \\ (kl_1+i,p_1) \\ (kl_2+j,p_2)}}^{N_{n(u-i+1)}N_{(n-1)(u-j+1)}} \theta_1 \left(\mathfrak{c}^{-1}\mathfrak{q}^{\gamma_{ij}-\frac{1}{2}} \mathfrak{t}^{-p_1+p_2+\frac{1}{2}} \frac{x_{n(u-i+1),A_1}}{x_{(n-1)(u-j+1),A_2}}; Q_\tau \right)}$$

$$\times \prod_{u=1}^{k} \frac{\displaystyle\prod_{\substack{A_1,A_2=1 \\ (kl_1+i,p_1) \\ (kl_2+j,p_2)}}^{N_{(n+1)(u-i+1)},N_{n(u-j+1)}} \theta_1 \left(\mathfrak{c}\mathfrak{q}^{\gamma_{ij}-\frac{1}{2}} \mathfrak{t}^{-p_1+p_2-\frac{1}{2}} \frac{x_{n(u-i+1),A_1}}{x_{(n+1)(u-j+1),A_2}}; Q_\tau \right)}{\displaystyle\prod_{\substack{A_1,A_2=1 \\ (kl_1+i,p_1) \\ (kl_2+j,p_2)}}^{N_{n(u-i+1)},N_{(n+1)(u-j)}} \theta_1 \left(\mathfrak{c}\mathfrak{q}^{\gamma_{i(j+1)}+\frac{1}{2}} \mathfrak{t}^{-p_1+p_2+\frac{1}{2}} \frac{x_{n(u-i+1),A_1}}{x_{(n+1)(u-j),A_2}}; Q_\tau \right)}$$

$$\left. \times \prod_{A=1}^{N_{(n+1)i}} \prod_{\substack{B=1 \\ (kl+j,p)}}^{N_{n(i-j+1)}} \theta_1 \left(\mathfrak{c}\mathfrak{q}^{\gamma_j+\frac{1}{2}} \mathfrak{t}^{-p+\frac{1}{2}} \frac{x_{n(i-j+1),B}}{x_{(n+1)i),A}}; Q_\tau \right) \right\},$$

and we understand

$$\gamma_j := -k \left\lfloor \frac{i-j}{k} \right\rfloor + j - 1 + kl\,, \tag{6.61}$$

$$\gamma_{ij} := -k \left\lfloor \frac{u-i}{k} \right\rfloor + k \left\lfloor \frac{u-j}{k} \right\rfloor + i - j + k(l_1 - l_2)\,, \tag{6.62}$$

within the products, where $\lfloor \cdot \rfloor$ is the floor function. For the sake of compactness we write

$$\prod_{\substack{A,B=1 \\ (l_1,p_1)\in\mu_{ni,A} \\ (l_2,p_2)\in\mu_{mj,B}}}^{N_{ni},N_{mj}} \equiv \prod_{A=1}^{N_{ni}} \prod_{B=1}^{N_{mj}} \prod_{(l_1,p_1)\in\mu_{ni,A}} \prod_{(l_2,p_2)\in\mu_{mj,B}}\,. \tag{6.63}$$

As before we may then define the grand partition function

$$Z^{\rho}_{\text{string}}\left(Q_\tau, \ldots; w_{ni}\right) := \sum_{\vec{\mu}_{ni}} \left(\prod_{n,i} w_{ni}^{K_{ni}(\vec{\mu}_{ni})} \right) \mathrm{Ell}^{\tilde{\rho}}_{\tilde{\kappa},\vec{\mu}_{ni}}\left(Q_\tau, \ldots\right)\,, \tag{6.64}$$

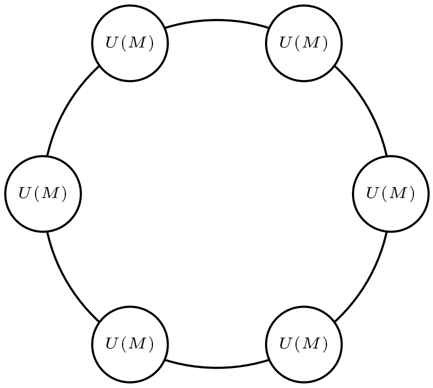


Figure 6.3: *5d $\mathcal{N} = 1$ necklace quiver $\mathcal{N}_{M,N}$ for $N = 6$.*

with K_{ni} determined by (6.59).

Combining (6.64) with the perturbative piece (6.40) is then expected to compute the $T^2 \times \mathbb{R}^4_{\epsilon_1,\epsilon_2}$ partition function for the worldvolume theory of M M5-branes on a transverse A_{N-1} singularity in the presence of k M5'-branes intersecting the original stack along co-dimension 2 planes.

6.3 5d Theories With Surface Operators

We now turn our focus towards 5d gauge theories with surface operators. We focus on the theories which can be obtained by reducing the $(1,0)_{A_{N-1}}$ theories, described in the previous section, with GW surface operator on a circle. Taking instead the M-theory circle of Table 6.2 to be X^6 we obtain the brane setup of Table 6.7. The theory on D4-branes without surface operator admits a description in terms

	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}_{\epsilon_2}$		T^2		TN_N			
	X^1	X^2	X^3	X^4	X^5	X^9	X^7	X^8	X^{10}	X^{11}
M D4	–	–	–	–	–	·	·	·	·	·
K F1	·	·	·	·	–	–	·	·	·	·
$\mathbb{Z}_k$	×	×	·	·	·	·	×	·	×	·

Table 6.7: *Type-IIA description of the 5d $\mathcal{N}_{M,N}$ quivers with a codimension two surface operator.*

of a 5d elliptic quiver theory which we denote by $\mathcal{N}_{M,N}$. The surface operator is characterised by the partition ρ (6.3) which specifies the embedding into each factor $\mathbb{Z}_k \hookrightarrow U(M = M_A)$. We called this resulting theory $\mathcal{N}^{\rho}_{M,N}$. We first briefly review the case without defect before turning to the computation with the defect present.

6.3.1 Review of $\mathbb{S}^1 \times \mathbb{S}^4$ Partition Function for $\mathcal{N}_{M,N}$

We first review the computation for the case without defect, i.e. $k = 1$. The partition function is defined as

$$Z(s, p, v, \mathbf{q}_A) = \mathrm{Tr}_{\mathbb{S}^4}\left[(-1)^F e^{-\beta\delta} s^{-2J_R - 2J_R^R} p^{-2J_L} v^{2J_L^R} \prod_{A=1}^{N} \mathbf{q}_A^{K_A}\right] \tag{6.65}$$

where

$$s := e^{\mathrm{i}\beta\epsilon_+/2}\,, \quad p := e^{\mathrm{i}\beta\epsilon_-/2}\,, \quad v := e^{\mathrm{i}\beta m}\,, \tag{6.66}$$

with $\epsilon_\pm = \epsilon_1 \pm \epsilon_2$ as before. The trace is taken over the Hilbert space of the theory on $\mathbb{S}^4$ in the radial quantisation. The $\mathcal{N}_{N,M}$ theories have 5d $\mathcal{N} = 1$ supersymmetry with supercharges $\mathtt{Q}^{\alpha\dot{a}}$, $\widetilde{\mathtt{Q}}^{\dot{\alpha}\dot{a}}$. We compute the partition function with respect to $\widetilde{\mathtt{Q}} := \widetilde{\mathtt{Q}}^{\dot{1}\dot{2}}$ and in the radial quantisation $\widetilde{\mathtt{S}} = \widetilde{\mathtt{Q}}^\dagger$ its conjugate. Z is formally independent of β since it receives contributions only from those states which satisfy

$$\delta := \left\{\widetilde{\mathtt{Q}}, \widetilde{\mathtt{S}}\right\} = E - 2J_R - 3J_R^R = 0\,, \tag{6.67}$$

$\mathbf{q}_A$ are fugacities for the topological $U(1)$'s associated to the conserved currents

$$*_5\, J_A = \frac{1}{4\pi^2} F_A \wedge F_A \tag{6.68}$$

corresponding to each gauge node. Z can be computed from localisation, which we review in Appendix G.2, and takes the following form

$$Z = \oint \prod_{A=1}^{N} [da_A]\, Z_{\text{south}} Z_{\text{north}} = \oint \prod_{A=1}^{N} [da_A]\, |Z_{\text{Nek}}|^2 \tag{6.69}$$

where

$$M_A![da_A] = \Delta(a_A) \prod_{n=1}^{M_A} \frac{da_{A,n}}{2\pi} = \prod_{n=1}^{M_A} \frac{da_{A,n}}{2\pi} \prod_{m\neq n} 2\sin\frac{a_{A,n} - a_{A,m}}{2} \tag{6.70}$$

is the Haar measure for $U(M_A = M)$. Here Z_{Nek} is the Nekrasov partition function for self-dual instantons

$$Z_{\text{Nek}} = Z_{\text{south}} = Z_{\text{south}}^{\text{1-loop}} Z_{\text{inst}}\,. \tag{6.71}$$

1-Loop Contributions

The one-loop contributions factor into contributions from vector and hypermultiplets

$$Z^{\text{1-loop}}_{\text{south}} = Z^{\text{1-loop}}_{\text{south,vec}} Z^{\text{1-loop}}_{\text{south,hyp}} \,. \tag{6.72}$$

Let us begin with the vector multiplets.

Vector Multiplet Contributions The 1-loop contribution for the vectors given by expansion of the equivariant index (G.48). At the south pole we should take $|e^{\mathrm{i}\epsilon_1}|, |e^{\mathrm{i}\epsilon_2}| > 1$ hence

$$\begin{aligned} \tilde{Z} = \prod_{p\in\mathbb{Z}} \prod_{\mu,\nu=0}^{\infty} \prod_{A=1}^{N} \prod_{n\neq m} \Bigg\{ & \left[\frac{2\pi p}{\beta} + \mu\epsilon_1 + \nu\epsilon_2 + \frac{a_{A,n}-a_{A,m}}{\beta}\right]^{\frac{1}{2}} \\ & \times \left[\frac{2\pi p}{\beta} + (\mu+1)\epsilon_1 + (\nu+1)\epsilon_2 + \frac{a_{A,n}-a_{A,m}}{\beta}\right]^{\frac{1}{2}} \Bigg\} \,. \end{aligned} \tag{6.73}$$

The partition function is defined only up to an overall constant and $\sim$ denotes that the divergent factor, which is independent of all chemical potentials, has been removed.

$$\begin{aligned} \tilde{Z} \sim & \prod_{\mu,\nu=0}^{\infty} \prod_{A=1}^{N} \prod_{n\neq m} \Bigg\{ \sin\left[\frac{a_{A,n}-a_{A,m}-\mu\beta\epsilon_1-\nu\beta\epsilon_2}{2}\right]^{\frac{1}{2}} \\ & \times \sin\left[\frac{a_{A,n}-a_{A,m}-(\mu+1)\beta\epsilon_1-(\nu+1)\beta\epsilon_2}{2}\right]^{\frac{1}{2}} \Bigg\} \\ = & e^{\frac{\beta\epsilon_+ E_0}{2}} \left(\prod_{A=1}^{N} \Delta(a_A)^{\frac{1}{2}}\right) \mathrm{PE}\left[-\frac{s(p+p^{-1})}{2(1-sp)(1-sp^{-1})} \sum_{A=1}^{N} \sum_{n\neq m} \frac{x_{A,n}}{x_{A,m}}\right] \\ \equiv & \left(\prod_{A=1}^{N} \Delta(a_A)^{\frac{1}{2}}\right) Z^{\text{1-loop}}_{\text{south,vec}} \,. \end{aligned} \tag{6.74}$$

In the final line E_0 is the Casimir energy. We also defined $x_{A,n} := e^{\mathrm{i}a_{A,n}}$ and then factored out the Vandermonde determinants

$$\Delta(a_A) = \exp\left(-\sum_{l=1}^{\infty} \frac{1}{l} \sum_{n\neq m} \left(e^{\mathrm{i}a_{A,n}-\mathrm{i}a_{A,m}}\right)^l\right) = \mathrm{PE}\left[-\sum_{n\neq m} x_{A,n} x_{A,m}^{-1}\right] \,, \tag{6.75}$$

Letter	E	J_R	J_R^R	J_L	J_L^R	Index
$\varphi^{\dot{2}}$	$3/2$	0	$+1/2$	0	$+1/2$	sv
$\bar{\varphi}^{\dot{2}}$	$3/2$	0	$+1/2$	0	$-1/2$	sv^{-1}
$\partial_{1\dot{1}}$	1	$+1/2$	0	$+1/2$	0	sp
$\partial_{2\dot{1}}$	1	$+1/2$	0	$-1/2$	0	sp^{-1}

Table 6.8: *The letters of the Hypermultiplet with $\delta = 0$.*

which arises from the zero modes of the vector multiplets. Moreover, the vector contribution is invariant under conjugation so

$$Z^{\text{1-loop}}_{\text{north,vec}} = \left(Z^{\text{1-loop}}_{\text{south,vec}}\right)^* = Z^{\text{1-loop}}_{\text{south,vec}}\,. \tag{6.76}$$

Hypermultiplet Contributions Here we compute the contributions from hypermultiplets. The hypermultiplet contains complex scalars $\varphi^{\dot{a}}$ and fermions ψ plus their conjugates. Note $\varphi^{\dot{1}}, \varphi^{\dot{2}}$ are identified with $X^\dagger_{89}$, X_{710} and hence have charges $+\frac{1}{2}$ under J^R_L. The multiplets live in bifundamental representations of $SU(M_A) \times SU(M_{A+1})$. Since there are no-zero modes coming from the hypermultiplet its contribution to the index may be computed via letter counting. The contribution to the index from hypermultiplets is given by enumerating all 'letters' listed in Table 6.8. We may identify the contribution localised at the south pole as coming from a half-hypermultiplet (G.50)

$$Z^{\text{1-loop}}_{\text{south,hyp}} = e^{\frac{-\beta\epsilon_+ E_0}{2}}\,\text{PE}\left[\sum_{A=1}^{N}\sum_{n=1}^{M_A}\sum_{m=1}^{M_{A+1}}\frac{sv}{(1-sp)(1-sp^{-1})}x_{A,n}x^{-1}_{A+1,m}\right]. \tag{6.77}$$

At the north pole the anti-half-hypermultiplet contribution is localised

$$Z^{\text{1-loop}}_{\text{north,hyp}} = e^{\frac{-\beta\epsilon_+ E_0}{2}}\,\text{PE}\left[\sum_{A=1}^{N}\sum_{n=1}^{M_A}\sum_{m=1}^{M_{A+1}}\frac{sv^{-1}}{(1-sp)(1-sp^{-1})}x^{-1}_{A,n}x_{A+1,m}\right]. \tag{6.78}$$

Instanton Contributions

Recall from the localisation that (anti-)self-dual instantons ($F^- = 0$) $F^+ = 0$ can be localised at (north)south-pole. The instanton contribution may be computed from equivariant integration over the moduli space $\mathbf{M}_K$ of $K = \sum_A K_A$ $U(M)^N$ instantons. See Sections 1.3 & 1.4.1. $\mathbf{M}_K$ carries a torus action $T := T^3_{\epsilon_1,\epsilon_2,m} \times T(U(N))^M \circlearrowright \mathbf{M}_K$ where $T(G)$ denotes a maximal torus of G. ϵ_1, ϵ_2, m and $\vec{a} \in \mathfrak{t}$

. The instanton moduli space $\mathbf{M}_K$ may be described as a algebraic variety using the ADHM construction [29]. The fixed points of the torus action T on the ADHM data are labelled by N M-tuples of Young diagrams $\vec{\mu}_A$ such that $|\vec{\mu}_A| = K_A$. The character of the tangent space $T\mathbf{M}_K$ at the fixed point labelled by a set of M-tuples $\{\vec{\mu}_A\}$ is then

$$\chi_{\{\vec{\mu}_A\}}\left(T\mathbf{M}_K\right) := \chi^{\text{vec}}_{\{\vec{\mu}_A\}} + \chi^{\text{hyp}}_{\{\vec{\mu}_A\}}\,, \tag{6.79}$$

where

$$\chi^{\text{vec}}_{\{\vec{\mu}_A\}} = \sum_{A=1}^{N} \left(W_A^* V_A + e^{\epsilon_+} V_A^* W_A - (1-e^{\epsilon_1})(1-e^{\epsilon_2}) V_A^* V_A\right) \mathcal{P}\,, \tag{6.80}$$

$$\begin{aligned}\chi^{\text{hyp}}_{\{\vec{\mu}_A\}} = -e^{\tilde{m}} \sum_{A=1}^{N} \big(& W_{A+1}^* V_A + e^{\epsilon_+} V_A^* W_{A-1} \\ & -(1-e^{\epsilon_1})(1-e^{\epsilon_2}) V_A^* V_{A-1}\big) \mathcal{P}\,.\end{aligned} \tag{6.81}$$

Where $\tilde{m} = m - \frac{\epsilon_+}{2}$ and

$$V_A = \sum_{n=1}^{M} \sum_{(l,p)\in\mu_{A,n}} e^{a_{A,n}+(1-l)\epsilon_1+(1-p)\epsilon_2}\,, \quad W_A = \sum_{n=1}^{M} e^{a_{A,n}}\,, \tag{6.82}$$

and the conjugation flips the sign of the exponents. We also added the dressing by momentum factors along the $\mathbb{S}^1$, with radius $r = \beta$

$$\mathcal{P} := \sum_{t\in\mathbb{Z}} e^{\frac{2\pi t}{r}}\,. \tag{6.83}$$

We have arrived at these expressions in precisely the same way we used in Section 1.4.1. Using the identity (A.32) it can be shown that

$$\chi^{\text{vec}}_{\{\vec{\mu}_A\}} = \sum_{A=1}^{N} \sum_{n,n'=1}^{M} \sum_{s\in\mu_{A,n'}} \left(e^{E_{AA,nn'}(s)} + e^{\epsilon_+ - E_{AA,nn'}(s)}\right) \mathcal{P}\,, \tag{6.84}$$

$$\chi^{\text{hyp}}_{\{\vec{\mu}_A\}} = \sum_{A=1}^{N} \sum_{n,n'=1}^{M} \sum_{s\in\mu_{A,n'}} e^{\tilde{m}} \left(e^{E_{A(A-1),nn'}(s)} + e^{\epsilon_+ - E_{A(A+1),nn'}(s)}\right) \mathcal{P}\,, \tag{6.85}$$

where, for a box $s = (l, p)$,

$$E_{AB,nn'}(s) := a_{A,n} - a_{B,n'} - (\mu^{\mathrm{T}}_{B,n';j} - l)\epsilon_1 + (\mu_{A,n;i} - p + 1)\epsilon_2\,. \tag{6.86}$$

The contribution of the fixed point $\{\vec{\mu}_A\}$ to the instanton partition is obtained from the character by

$$\chi_{\{\vec{\mu}_A\}}(T\mathbf{M}_K) := \sum_i n_i e^{w_i} \quad \to \quad z_{\vec{\mu}} = \prod_i w_i^{-n_i} \,. \tag{6.87}$$

Finally, after applying the infinite product (A.21) for the sine function, the instanton partition function is given by a weighted sum over all possible N-tuples $\vec{\mu}_A$

$$Z_{\text{inst}}(\vec{a}_A, m, \epsilon_1, \epsilon_2, \mathbf{q}_A, \beta) = \sum_{\vec{\mu}_A} \left(\prod_{A=1}^{N} \mathbf{q}_A^{|\vec{\mu}_A|} \right) z_{\{\vec{\mu}_A\}}(\vec{a}_A, m, \epsilon_1, \epsilon_2, \beta) \,, \tag{6.88}$$

$$\begin{aligned} z_{\{\vec{\mu}_A\}} = \prod_{A=1}^{N} \prod_{n,n'=1}^{M} \prod_{s\in\mu_{A,n'}} \Bigg\{ \\ \frac{\sin\frac{\beta}{2}\left(E_{A(A-1),nn'}(s)+\tilde{m}\right)\sin\frac{\beta}{2}\left(E_{A(A+1),nn'}(s)-\tilde{m}-\epsilon_+\right)}{e^{\kappa_A\phi_{A,n}(s)}\sin\frac{\beta}{2}E_{AA,nn'}(s)\sin\frac{\beta}{2}\left(E_{AA,nn'}(s)-\epsilon_+\right)} \Bigg\} \,, \end{aligned} \tag{6.89}$$

where $\tilde{m} := m - \frac{\epsilon_+}{2}$ we also allowed for non-zero Chern-Simons levels κ_A and the function $\phi_{A,n}(s)$ is defined to be

$$\phi_{A,n}(s) := a_{A,n} + \left(l - \frac{1}{2}\right)\epsilon_1 + \left(p - \frac{1}{2}\right)\epsilon_2 \,. \tag{6.90}$$

6.3.2 $\mathbb{S}^1 \times \mathbb{S}^4$ Partition Function for $\mathcal{N}^{\rho}_{M,N}$

We now come to the calculation in the presence of the defect. The defect is supported along a codimension 2 hypersurface within $\mathbb{S}^1 \times \mathbb{S}^4$. For simplicity we will consider only the following: choose a local coordinate patch $\mathbb{R}^4 \cong \mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$ parametrised by (z_1, z_2), at the either pole we may choose for a defect of type ρ to be supported along either $\mathbb{S}^1 \times \mathbb{C}_{\epsilon_1}$ or $\mathbb{S}^1 \times \mathbb{C}_{\epsilon_2}$. Requiring that the defect preserves atleast $\widetilde{\mathtt{Q}} = \widetilde{\mathtt{Q}}^{\mathrm{i}\dot{2}}$ means we can only choose the latter. Recall that the surface operator is defined by demanding singular behaviour of the gauge fields, in local coordinates let $z_2 = re^{\mathrm{i}\theta}$ then

$$A_{A,\mu}dx^\mu + A_{A,5}dx^5 \sim \mathrm{diag}\left(\alpha_{A1}, \ldots, \alpha_{Ak}\right)\mathrm{i}d\theta \,, \tag{6.91}$$

which breaks the gauge group

$$\prod_{A=1}^{N} U(M_A) \to \prod_{A=1}^{N}\prod_{i=1}^{k} U(M_{Ai}) \,. \tag{6.92}$$

We then compute

$$Z^{\rho}\left(s,p,v,\mathbf{q}_{Ai}\right)=\mathrm{Tr}_{\mathbb{S}^4}\left[(-1)^F e^{-\beta\delta}s^{-2J_R-2J_R^R}p^{-2J_L}v^{2J_L^R}\prod_{A=1}^{N}\prod_{i=1}^{k}\mathbf{q}_{Ai}^{K_{Ai}}\right]. \tag{6.93}$$

Where

$$\mathbf{q}_{Ai}=\mathbf{q}_A^{\alpha_{Ai}-\alpha_{A(i+1)}}\,,\quad i\sim i+k\,. \tag{6.94}$$

Since the partition function localises on the north and south poles, it is expected that the insertion of a defect at e.g. the south pole does does not affect the contributions localised at the north pole. We hence assume that the index may again be expressed as

$$Z^{\rho}=\int\prod_{A,i}[da_{Ai}]Z^{\rho}_{\text{south}}\left(\vec{a}_{Ai},\epsilon_1,\epsilon_2,m,\mathbf{q}_{Ai}\right)Z^{\rho}_{\text{north}}\left(\vec{a}_{Ai},\epsilon_1,\epsilon_2,m,\overline{\mathbf{q}}_{Ai}\right). \tag{6.95}$$

In the neighbourhood of the defect we may employ the $\mathbb{Z}_k$ orbifold description of the surface operator at the north and south poles. For example, at the south pole the defect may be effectively described by $\mathbb{C}_{\epsilon_1}\times\mathbb{C}_{\epsilon_2}/\mathbb{Z}_k$ acting by

$$\mathbb{Z}_k:(z_1,z_2)\mapsto(z_1,\omega_k z_2) \tag{6.96}$$

in order to preserve supersymmetry we also twist the coordinate $U(1)_{\epsilon_2}$ action on z_2 with a $U(1)\subset SU(2)$ of the R-symmetry group as in (6.7). The $\mathbb{Z}_k$ actions commute with the localising supercharge. We employ the similar description of the surface defect at the north pole.

1-Loop Contributions

To compute the 1-loop factors in the presence of the defect we employ the orbifold description of the defect. Since the $\mathbb{Z}_k$ actions commute with (G.44) the vector multiplet 1-loop contributions at north & south pole may be obtained by computing the equivariant index of the $\mathbb{Z}_k$ invariant part of the complexes (G.46) & (G.47) respectively. The 1-loop contributions from the hypermultiplet may be obtained in a similar fashion, by restricting to the $\mathbb{Z}_k$ invariant part of the Dirac complex (G.49). Equivalently, we may also count only $\mathbb{Z}_k$ invariant letters of Table 6.8.

Vector Multiplets For concreteness let us describe the defect ρ at the south pole. In terms of $\mathbb{C}^2$ local coordinates (z_1, z_2) the orbifold acts by

$$\mathbb{Z}_k : \epsilon_2 \mapsto \epsilon_2 + \frac{2\pi\mathrm{i}}{\beta k} j \,, \quad a_{Ai,n} \mapsto a_{Ai,n} + \frac{2\pi\mathrm{i}}{k} j \left(i + \frac{1}{2} \right) \,. \tag{6.97}$$

Keeping only orbifold invariant factors leads to the modified 1-loop determinant

$$\begin{aligned}
\widetilde{Z}^{\rho}(\epsilon_1, \epsilon_2, \vec{a}_{Ai}) &= \left(\prod_{A=1}^{N} \prod_{i=1}^{k} \Delta(a_{Ai})^{\frac{1}{2}} \right) Z^{\text{1-loop},\rho}_{\text{south,vectors}}(\epsilon_1, \epsilon_2, \vec{a}_{Ai}) \\
= \prod_{\mu,\nu=0}^{\infty} \prod_{A=1}^{N} \Bigg\{ \prod_{i=1}^{k} \prod_{n \neq m} & \Bigg(\sin \left[\frac{a_{Ai,n} - a_{Ai,m} - \mu\beta\epsilon_1 - \nu k \beta \epsilon_2}{2} \right]^{\frac{1}{2}} \\
\times \sin \left[\frac{a_{A,n} - a_{A,m} - (\mu+1)\beta\epsilon_1 - (\nu+1)k\beta\epsilon_2}{2} \right]^{\frac{1}{2}} & \Bigg) \prod_{i \neq j} \prod_{n,m=1}^{M_{Ai}, M_{Aj}} \Bigg(\\
\times \sin \left[\frac{-a_{Ai,n} + a_{Aj,m} - (1+\mu)\beta\epsilon_1 - (\nu k + [[j-i]])\beta\epsilon_2}{2} \right]^{\frac{1}{2}} & \\
\times \sin \left[\frac{-a_{Ai,n} + a_{Aj,m} - \mu\beta\epsilon_1 - (k\nu + 1 + [[j-i-1]])\beta\epsilon_2}{2} \right]^{\frac{1}{2}} & \Bigg) \Bigg\}
\end{aligned} \tag{6.98}$$

where, as before, we dropped the overall, fugacity independent, infinite constant and

$$[[x]] := \{ \# \in \mathbb{Z} \,|\, 0 \leqslant \# \leqslant k-1 \text{ and } \# = x \bmod k \} = \left| x - k \left\lfloor \frac{x}{k} \right\rfloor \right| \,. \tag{6.99}$$

We may also write (6.98) more compactly in terms of the Pletystic exponent

$$\begin{aligned}
& Z^{\text{1-loop},\rho}_{\text{south,vectors}}(\epsilon_1, \epsilon_2, \vec{a}_{Ai}) = e^{\frac{\beta\epsilon + E_0^{\mathbb{Z}_k}}{4}} \times \\
& \mathrm{PE} \left[- \sum_{i,j=1}^{k} \frac{e^{\mathrm{i}\beta\epsilon_1} e^{\mathrm{i}\beta[[j-i]]\epsilon_2} + e^{\mathrm{i}\beta([[j-1-i]]+1)\epsilon_2}}{2(1-e^{\mathrm{i}\beta\epsilon_1})(1-e^{\mathrm{i}\beta k\epsilon_2})} \sum_{A=1}^{N} \sum_{\substack{n,m=1 \\ n \neq m \text{ if } i=j}}^{M_{Ai}, M_{Aj}} \frac{e^{\mathrm{i}a_{Ai,n}}}{e^{\mathrm{i}a_{Aj,m}}} \right] .
\end{aligned} \tag{6.100}$$

At the north pole, the conribution is the same and $Z^{\text{1-loop},\rho}_{\text{south,vec}} = Z^{\text{1-loop},\rho}_{\text{north,vec}}$ for any ρ.

Hypermultiplets For the hypermultiplet, in addition to the action (6.97) there is also the additional action on the mass parameter

$$\mathbb{Z}_k : m \mapsto m - \frac{2\pi\mathrm{i}}{\beta k} \frac{j}{2} \,. \tag{6.101}$$

In the presence of the defect at the south pole we have

$$Z^{\text{1-loop},\rho}_{\text{south,hypers}} \sim \\ \mathrm{PE}\left[\sum_{A=1}^{N}\sum_{i,j=1}^{k}\sum_{n,m=1}^{M_{Ai},M_{(A+1)j}} \frac{e^{\mathrm{i}\beta\frac{\epsilon_+}{2}}e^{\mathrm{i}\beta m}e^{\mathrm{i}\beta[[j-i]]\epsilon_2}e^{\mathrm{i}a_{Ai,n}-\mathrm{i}a_{(A+1)j,m}}}{(1-e^{\mathrm{i}\beta\epsilon_1})(1-e^{\mathrm{i}\beta k\epsilon_2})}\right]. \tag{6.102}$$

On the other hand, at the north pole

$$Z^{\text{1-loop},\rho}_{\text{north,hypers}} \sim \\ \mathrm{PE}\left[\sum_{A=1}^{N}\sum_{i,j=1}^{k}\sum_{n,m=1}^{M_{Ai},M_{(A+1)j}} \frac{e^{\mathrm{i}\beta\frac{\epsilon_+}{2}}e^{-\mathrm{i}\beta m}e^{-\mathrm{i}\beta[[j+1-i]]\epsilon_2}e^{-\mathrm{i}a_{Ai,n}+\mathrm{i}a_{(A+1)j,m}}}{(1-e^{\mathrm{i}\beta\epsilon_1})(1-e^{\mathrm{i}\beta k\epsilon_2})}\right]. \tag{6.103}$$

Because the conjugation exchanges $m \to -m$, in the presence of defects $Z^{\text{1-loop},\rho}_{\text{north,hypers}} \neq \left(Z^{\text{1-loop},\rho}_{\text{south,hypers}}\right)^*$.

Instanton Contributions

After the orbifolding the character of the tangent space $T\mathbf{M}_K$ at the fixed point labelled by a set of M-tuples $\{\vec{\mu}_A\}$ with $\{\vec{\mu}_A\} = \{\mu_{Ai,n}\}$ with $A = 1,\ldots,N$, $i = 1,\ldots,k$, $n = 1,\ldots,M_{Ai}$ is given by restricting to the fixed point of the action (6.97), (6.101). It is given by

$$\chi^{\text{orb}}_{\{\vec{\mu}_A\}}(T\mathbf{M}_K) := \chi^{\text{vec,orb}}_{\{\vec{\mu}_A\}} + \chi^{\text{hyp,orb}}_{\{\vec{\mu}_A\}}, \tag{6.104}$$

where,

$$\begin{aligned}\chi^{\text{vec,orb}}_{\{\vec{\mu}_A\}} = \sum_{A=1}^{N}\sum_{i=1}^{k}\big(&W^*_{A,i}V_{A,i} + e^{\epsilon_+}V^*_{A,i-1}W_{A,i} - (1-e^{\epsilon_1})V^*_{A,i}V_{A,i} \\ &+(1-e^{\epsilon_1})e^{\epsilon_2}V^*_{A,i-1}V_{A,i}\big)\,\mathcal{P}\,,\end{aligned} \tag{6.105}$$

$$\begin{aligned}\chi^{\text{hyp,orb}}_{\{\vec{\mu}_A\}} = -\,e^{\tilde{m}}\sum_{A=1}^{N}\sum_{i=1}^{k}\big(&W^*_{A+1,i+1}V_{A,i} + e^{\epsilon_+}V^*_{A,i}W_{A-1,i} \\ -(1-e^{\epsilon_1})V^*_{A,i+1}V_{A-1,i} &+ (1-e^{\epsilon_1})e^{\epsilon_2}V^*_{A,i}V_{A-1,i}\big)\,\mathcal{P}\,.\end{aligned} \tag{6.106}$$

Where

$$V_{A,i} = \sum_{j=1}^{k} \sum_{n=1}^{M_{A(i-j+1)}} \sum_{(l,kp+j)\in\mu_{A(i-j+1),n}} e^{v_{A,ij,n}(l,p)}\,, \quad W_{A,i} = \sum_{n=1}^{M_{Ai}} e^{a_{Ai,n}}\,, \tag{6.107}$$

$$v_{A,ij,n}(l,p) := a_{A(i-j+1),n} + (1-l)\epsilon_1 + (1-pk-j)\epsilon_2 + k\left\lfloor \frac{i-j}{k} \right\rfloor \epsilon_2 \tag{6.108}$$

and the conjugation flips the sign of the exponents. We also added the dressing by momentum factors along the $\mathbb{S}^1$. Converting the character using (6.87) gives

$$z^{\text{orb}}_{\{\vec{\mu}_A\}} = z^{\text{vec,orb}}_{\{\vec{\mu}_A\}} z^{\text{hyp,orb}}_{\{\vec{\mu}_A\}}\,, \tag{6.109}$$

$$\begin{aligned} z^{\text{vec,orb}}_{\{\vec{\mu}_A\}} = \prod_{A,i,j,n,n'} \Bigg\{ & \prod_{(l,kp+j)\in\mu_{A(i-j+1),n}} \sin\frac{r}{2}\left(v_{A,ij,n}(l,p) - a_{Ai,n'}\right) \\ \times & \prod_{(l,kp+j)\in\mu_{A(i-j+1),n}} \sin\frac{r}{2}\left(-v_{A,ij,n}(l,p) + a_{A(i+1),n'} + \epsilon_+\right) \\ \times & \prod_{j'} \prod_{\substack{(l,kp+j)\in\mu_{A(i-j+1),n} \\ (l',kp'+j')\in\mu_{A(i-j'+1),n'}}} \frac{\sin\frac{r}{2}\left(v_{A,ij,n}(l,p) - v_{A,ij',n'}(l',p') + \epsilon_1\right)}{\sin\frac{r}{2}\left(v_{A,ij,n}(l,p) - v_{A,ij',n'}(l',p')\right)} \\ \times & \prod_{j'} \prod_{\substack{(l,kp+j)\in\mu_{A(i-j+1),n} \\ (l',kp'+j')\in\mu_{A(i-j'),n'}}} \frac{\sin\frac{r}{2}\left(v_{A,ij,n}(l,p) - v_{A,(i-1)j',n'}(l',p') + \epsilon_2\right)}{\sin\frac{r}{2}\left(v_{A,ij,n}(l,p) - v_{A,(i-1)j',n'}(l',p') + \epsilon_+\right)} \Bigg\}\,, \end{aligned} \tag{6.110}$$

$$\begin{aligned} z^{\text{hyp,orb}}_{\{\vec{\mu}_A\}} = \prod_{A,i,j,n,n'} \Bigg\{ & \prod_{(l,kp+j)\in\mu_{A(i-j+1),n}} \frac{1}{\sin\frac{r}{2}\left(v_{A,ij,n}(l,p) - a_{(A-1)(i-1),n'} + \tilde{m}\right)} \\ \times & \prod_{(l,kp+j)\in\mu_{A(i-j+1),n}} \frac{1}{\sin\frac{r}{2}\left(-v_{A,ij,n}(l,p) + a_{(A+1)i,n'} + \tilde{m} + \epsilon_+\right)} \\ \times & \prod_{\substack{j' \\ (l,kp+j)\in\mu_{(A-1)(i-j+1),n} \\ (l',kp'+j')\in\mu_{A(i-j'+2),n'}}} \frac{\sin\frac{r}{2}\left(v_{A-1,ij,n}(l,p) - v_{A,(i+1)j',n'}(l',p') + \tilde{m}\right)}{\sin\frac{r}{2}\left(v_{A-1,ij,n}(l,p) - v_{A,(i+1)j',n'}(l',p') + \frac{\epsilon_-}{2} + m\right)} \\ \times & \prod_{\substack{j' \\ (l,kp+j)\in\mu_{(A-1)(i-j+1),n} \\ (l',kp'+j')\in\mu_{A(i-j'+1),n'}}} \frac{\sin\frac{r}{2}\left(v_{A-1,ij,n}(l,p) - v_{A,ij',n'}(l',p') + \tilde{m} + \epsilon_+\right)}{\sin\frac{r}{2}\left(v_{A-1,ij,n}(l,p) - v_{A,ij',n'}(l',p') + m - \frac{\epsilon_-}{2}\right)} \Bigg\}\,. \end{aligned} \tag{6.111}$$

All in all, the instanton partition function with surface operator is given by

$$Z^{\text{orb}}_{\text{inst}}\left(\vec{a}_{Ai},m,\ldots;\mathbf{q}_{Ai}\right)=\sum_{\vec{\mu}_A}\left(\prod_{A=1}^{N}\prod_{i=1}^{k}\mathbf{q}_{Ai}^{K_{Ai}(\vec{\mu}_A)}\right)z^{\text{orb}}_{\{\vec{\mu}_A\}}\left(\vec{a}_{Ai},\ldots\right)\,, \tag{6.112}$$

where

$$K_{Ai}=d_{Ai}(\vec{\mu_A})=\sum_{p=1}^{\infty}\sum_{n=1}^{M_{A(i+1-j)}}\mu_{A(i+1-j),n;p}\,. \tag{6.113}$$

A generalisation of this result for generic orbifold action $(z_1,z_2)\mapsto(\omega_k^{q_1}z_1,\omega_k^{q_2}z_2)$ for the case of 5d $\mathcal{N}=1^*$ theory can be found in Appendix I. It would be interesting to investigate whether this result can be used to study the partition functions of 5d theories in the presence of more general defects.

6.3.3 Unrefined Limits for 5d MSYM

In certain special limits the index without defect (6.65) is known to admit drastic simplifications. We would like to investigate whether this holds true also when one considers the case with defect (6.69).

We restrict ourselves to the case $N=1$ then we have a theory of a single $U(M)$ $\mathcal{N}=1$ vector multiplet with an adjoint hypermultiplet, i.e. the $U(M)$ $\mathcal{N}=2$ theory. This theory has additional supersymmetry, namely 16 supercharges $\mathtt{Q}^{\alpha a}$, $\widetilde{\mathtt{Q}}^{\dot\alpha\dot a}$; aswell as $\mathtt{Q}^{\alpha\dot a}$ and $\widetilde{\mathtt{Q}}^{\dot\alpha a}$ from before. In particular, $\widetilde{\mathtt{Q}}^{\dot 1 a}$ commute with the distinguished supercharge $\widetilde{\mathtt{Q}}=\widetilde{\mathtt{Q}}^{\dot 1\dot 2}$. Moreover, those preserved by the ϵ_2-plane defects are

$$\widetilde{\mathtt{Q}}^{\dot 1\dot 2}\,,\quad \widetilde{\mathtt{Q}}^{\dot 2\dot 1}\,,\quad \widetilde{\mathtt{Q}}^{\dot 1 2}\,,\quad \widetilde{\mathtt{Q}}^{\dot 2 1}\,,\quad \mathtt{Q}^{12}\,,\quad \mathtt{Q}^{21}\,,\quad \mathtt{Q}^{1\dot 2}\,,\quad \mathtt{Q}^{2\dot 1}\,. \tag{6.114}$$

We can consider some unrefined limits of the index (6.65).

$2m=-\epsilon_+$

The first one we consider is $s=v^{-1}$ or $2m=-\epsilon_+$. In this limit the index (6.69) becomes

$$Z^{\rho}\left(s,p,s^{-1},\mathbf{q}_i\right)=\mathrm{Tr}_{\mathbb{S}^4}\left[(-1)^F e^{-\beta\delta}s^{-2J_R-2J_R^R-2J_L^R}p^{-2J_L}\prod_{i=1}^{k}\mathbf{q}_i^{K_i}\right]. \tag{6.115}$$

In this limit the index is, atleast, a $\frac{1}{4}$-BPS object annihilated by both $\widetilde{\mathtt{Q}}$ and $\widetilde{\mathtt{Q}}^{\dot 1 2}$ which also has $E-2J_R-3J_R^L=0$, implying that this limit of the index recieves non-zero contributions only from states with $J_L^R = J_R^R$.

Without Defect Let us first consider the case without defect. We consider the specialisation of to $2m = -\epsilon_+$. In this case, one can check that the perturbative contributions receive large amounts of cancellations and

$$\Delta(a) Z^{\text{1-loop}}_{\text{south,vec}} Z^{\text{1-loop}}_{\text{north,vec}} Z^{\text{1-loop}}_{\text{south,hyp}} Z^{\text{1-loop}}_{\text{north,hyp}} \to 1 \tag{6.116}$$

One can also easily see from (6.89) that the contribution for each instanton fixed point is simply $z_{\vec{\mu}} = 1$. It is particularly simple to see by realising that, in this limit, $\chi^{\text{vec}}_{\{\vec{\mu}_A\}} = -\left(\chi^{\text{hyp}}_{\{\vec{\mu}_A\}}\right)^*$. Therefore the instanton partition function is simply counting the number of fixed points

$$Z_{\text{inst}} \to \sum_{\vec{\mu}} \mathbf{q}^{|\vec{\mu}|} = \prod_{n=1}^{\infty} \frac{1}{(1-\mathbf{q}^n)^M} = \text{PE}\left[M \frac{\mathbf{q}}{1-\mathbf{q}}\right]. \tag{6.117}$$

The total partition function (6.115) is then simply

$$Z(s, p, s^{-1}, \mathbf{q}) = \oint [da] \left| \text{PE}\left[M \frac{\mathbf{q}}{1-\mathbf{q}}\right]\right|^2 = \left|(\mathbf{q};\mathbf{q})^M\right|^2. \tag{6.118}$$

With Defect We now consider the limit in the presence of the defects. It is not too hard to again show that

$$\left(\prod_{i=1}^{k} \Delta(a_i)\right) Z^{\text{1-loop},\rho}(\epsilon_1, \epsilon_2, m = -\frac{\epsilon_+}{2}, \vec{a}_i) = 1 \tag{6.119}$$

$$z^{\text{orb}}_{\{\vec{\mu}\}}\left(\vec{a}_i, m = \pm\frac{\epsilon_+}{2}, \epsilon_1, \epsilon_2, \beta\right) = 1 \tag{6.120}$$

due to cancellations between vector and hypermultiplets. The instanton piece simply counts the number of fixed points

$$\begin{aligned} Z^{\text{orb}}_{\text{inst}}\left(\vec{a}_i, m = -\frac{\epsilon_+}{2}, \epsilon_1, \epsilon_2, \mathbf{q}_i, \beta\right) &= \sum_{\vec{\mu}_A} \left(\prod_{i=1}^{k} \mathbf{q}_i^{d_i(\vec{\mu})}\right) \cdot 1 \\ &= (\mathbf{q};\mathbf{q})^{-M} \prod_{i=1}^{k}\prod_{p=1}^{\infty}\left(1 - \prod_{j=1}^{i+p-1} \mathbf{q}_j\right)^{-M_i} \\ &= \text{PE}\left[M\frac{\mathbf{q}}{1-\mathbf{q}} + \sum_{i=1}^{k}\sum_{j=i+1}^{k} M_i \frac{\prod_{p=i}^{j}\mathbf{q}_p}{1-\mathbf{q}} + \sum_{i=2}^{k}\sum_{j=1}^{i-1} M_i \frac{\mathbf{q}\prod_{p=i}^{j}\mathbf{q}_p}{1-\mathbf{q}}\right], \end{aligned} \tag{6.121}$$

the first equality was given in [49] and the second conjectured in [58]. Therefore, in this limit, the partition function with defects at north and south poles reduces to

$$Z^\rho(s,p,s^{-1},\mathbf{q}_i) = \left|\mathrm{PE}\left[M\frac{\mathbf{q}}{1-\mathbf{q}} + \sum_{i=1}^{k}\sum_{j=i+1}^{k} M_i \frac{\prod_{p=i}^{j}\mathbf{q}_p}{1-\mathbf{q}} + \sum_{i=2}^{k}\sum_{j=1}^{i-1} M_i \frac{\mathbf{q}\prod_{p=i}^{j}\mathbf{q}_p}{1-\mathbf{q}}\right]\right|^2 \tag{6.122}$$

$2m = -\epsilon_-$

We can also consider the limit $v = p^{-1}$ or $2m = -\epsilon_-$. In this limit the index becomes

$$Z^\rho\left(s,p,p^{-1},\mathbf{q}_i\right) = \mathrm{Tr}_{\mathbb{S}^4}\left[(-1)^F e^{-\beta\delta} s^{-2J_R-2J_R^R} p^{-2J_L-2J_L^R}\prod_{i=1}^{k}\mathbf{q}_i^{K_i}\right] \tag{6.123}$$

Without Defect As before, we again consider the case without any defect. For the perturbative piece, is it simple to show that, for $v = p^{-1}$

$$Z_{\text{south}}^{\text{1-loop}} Z_{\text{north}}^{\text{1-loop}} = \mathrm{PE}\left[M\frac{sp+sp^{-1}}{(1-sp)(1-sp^{-1})}\right] \tag{6.124}$$

Moreover, one can show, for the instanton piece that for $|\vec{\mu}| > 0$

$$z_{\{\vec{\mu}\}}\left(\vec{a}_i, m = \pm\frac{\epsilon_-}{2}, \epsilon_1, \epsilon_2, \beta\right) \equiv 0\,. \tag{6.125}$$

Therefore, in this limit, the partition function simply becomes

$$Z^\rho\left(s,p,p^{-1},\mathbf{q}_i\right) = \mathrm{PE}\left[M\frac{sp+sp^{-1}}{(1-sp)(1-sp^{-1})}\right]\,. \tag{6.126}$$

With Defect In the case of inserting the defect, for $m = -2\epsilon_-$ the perturbative piece takes on a similar form

$$Z_{\text{south}}^{\text{1-loop},\rho} Z_{\text{north}}^{\text{1-loop},\rho} = \mathrm{PE}\left[M\frac{sp+s^kp^{-k}}{(1-sp)(1-s^kp^{-k})}\right]\,, \tag{6.127}$$

with $M = \sum_{i=1}^{k} M_i$. It was also shown in [58] that, for the ramified instanton partition function, for $|\vec{\mu}| \neq 0$

$$z^\rho_{\{\vec{\mu}\}}\left(\vec{a}_i, m = \pm\frac{\epsilon_-}{2}, \epsilon_1, \epsilon_2, \beta\right) \equiv 0\,. \tag{6.128}$$

and therefore, the partition function in the precense of the defects is

$$Z^{\rho}\left(s,p,p^{-1},\mathbf{q}_i\right) = \mathrm{PE}\left[M\frac{sp+s^kp^{-k}}{(1-sp)(1-s^kp^{-k})}\right]. \tag{6.129}$$

6.4 Conclusions

We have computed the elliptic genus partition functions of the self-dual strings of $\mathcal{N}=(1,0)$ theories associated to M M5-branes on A_{N-1} singularities in the presense of codimension two surface defects labelled by partitions $\rho_A = [M_{A1},\ldots,M_{Ak}]$. These BPS strings capture (up to a 'perturbative' pre-factor) the states contributing to the $\mathbb{S}^4\times T^2$ partition function of the 6d theory. So called 'minimal' defects of type $\rho=[1,\ldots,1,(M-p)]$ have been studied from the point of view of topological strings in [284, 285, 286, 287]. Our method allows straightforward generalisation to obtain the partition functions for any choice of embedding ρ's, which engineer the generic GW-surface operators. The relation between M-strings and refined topological string partition function in the presence of a defect of type $2=1+1$ has been studied in [284]. It is therefore expected that our M-strings computation captures the information about the refined topological string partiton function for the M-theory background 6.1.

In addition, we have also computed the $\mathbb{S}^4\times\mathbb{S}^1$ partition functions for the 5d $\mathcal{N}_{n,m}$ theories in the presence of $\frac{1}{2}$-BPS codimension two defect operators labelled by ρ inserted at the north and south poles.

Chapter 7

Final Conclusions and Outlook

Over the past decades there has been huge interest in the study of supersymmetric quantum field theories and in particular the computation of exact results. This study is primarily carried out with the hope that a understanding supersymmetric theories will ultimately lead us towards a better understanding of non-supersymmetric theories, such as the standard model of particle physics. This is also of great interest from a mathematical point of view because such study may also lead us towards a mathematically rigorous definition of quantum field theories, similar to that achieved for conformal field theories in two dimensions.

Much of the recent study has been dedicated to theories with $\mathcal{N}=2$ supersymmetry in four dimensions. In this thesis we have initiated a programme towards the study and computation of exact results for certain four dimensional $\mathcal{N}=1$ theories, namely those said to lie within class $\mathcal{S}_k$. They can be described in terms of a twisted compactification of the 6d $(1,0)_{A_{k-1}}$ SCFT on a Riemann surface $\mathcal{C}$. Within this class lies theories which are of similar type to that of $\mathcal{N}=1$ SQCD; namely a theory of $\mathcal{N}=1$ vector multiplets coupled to chiral matter in fundamental representations. $\mathcal{N}=1$ SQCD displays many of the same interesting phenomena as non-supersymmetric QCD, such as confinement. However, class $\mathcal{S}_k$ presents a framework for exploiting new dualities, due to the string and M-theory constructions. Additionally they also inherit many similar properties from their class $\mathcal{S}$ mother theories. Therefore, class $\mathcal{S}_k$ seems like an ideal starting point to begin to compute exact results for $\mathcal{N}=1$ theories, with the hope of being able to understand them as well as we understand $\mathcal{N}=2$ theories.

In this thesis we have, firstly, demonstrated that $\mathcal{N}=1$ analogues of Higgs and Coulomb branches may be defined for theories of class $\mathcal{S}_k$. We have computed the

Hilbert series for them and described many of their properties. We have also defined and discussed many interesting limits of the superconformal index. Following this we have described the said Coulomb branch geometry by deriving explicit forms for the Seiberg and Intrilligator curves for the class $\mathcal{S}_k$ theories corresponding to the Riemann surface $\mathcal{C}$ being spheres with two maximal and two minimal punctures. We have matched them to M-theory predictions. Moreover, they can be placed in 'Gaiotto form', embedded as a surface within $T^*\mathcal{C}$.

The second accomplishment of this thesis is the computation of the partition function of instantons for a certian subset of class $\mathcal{S}_k$ theories. Our work relied on the correspondence between instantons and D(-1)-branes from which we derived the ADHM construction for these theories as a matrix model. This partition function is then identified with conformal blocks of the $\mathcal{W}_{kN}$ algebra [153], pointing towards the possible existence of further 2d/4d relations for class $\mathcal{S}_k$ theories. It would be very interesting to understand this starting from the 6d $(1,0)_\Gamma$ SCFTs and to try to derive this from there, as in [245]. Along this direction we have investigated some $\frac{1}{2}$-BPS defects in six dimensional $\mathcal{N} = (1,0)$ SCFTs. We have computed the elliptic genus of the 2d theory living on the world-volume of the BPS strings of the 6d theories in the presence of the defect. The strings provide the main contribution to the $T^2 \times \mathbb{R}^4$ BPS-partition function for the 6d theory and it can be written in a expansion over Elliptic genera.

We have also studied $\mathcal{N} = 3$ theories, obtained via discrete gauging of $\mathcal{N} = 4$ SYM. The discrete group that we gauge lies within the $SL(2,\mathbb{Z})$ S-duality group, of which the subgroup enhances to a symmetry at certian points of the conformal manifold, aswell as a factor within the $SU(4)$ R-symmetry group. In particular we focused on the structure of their moduli spaces and computed its Hilbert series. In certain special cases we are also able to compute the fully refined supersymmetric index. It would be worth trying to extend this computation to the fully refined superconformal index, other limits of the superconformal index, or even other types of partition function, e.g. the $\mathbb{S}^4$ partition function for both the S-fold theories and discrete gaugings of $\mathcal{N} = 4$ SYM.

Appendix A

Definitions, Identities and Special Functions

The purpose of this Appendix is to collect the various definitions and identities used through this thesis. Formulas and identities that are used only once will not be listed here, but rather in the relevant place where they are used. Conversely, identities used multiple times throughout the thesis will simply be stated in this Appendix and then referred to from the main text.

A.1 Identities and Special Functions

A.1.1 Plethystics

The *Plethystic Exponential* is given by

$$\mathrm{PE}\left[f(x_1,\ldots,x_m)\right] := \exp\left(\sum_{n=1}^{\infty}\frac{f(x_1^n,\ldots,x_m^n)-f(0,\ldots,0)}{n}\right) . \tag{A.1}$$

Its inverse is called the *Plethystic Logarithm*

$$\begin{aligned}\mathrm{PE}^{-1}[f(x_1,\ldots,x_m)] :=& \,\mathrm{PLog}[f(x_1,\ldots,x_m)] \\ =& \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log f(x_1^n,\ldots,x_m^n)\,,\end{aligned} \tag{A.2}$$

where $\mu(n)$ is the Möbius μ function. Some basic identities are

$$\mathrm{PE}[t] = \frac{1}{1-t}\,, \quad \mathrm{PE}[-t] = 1-t\,, \quad \sum_{n=0}^{N-1} t^n = \frac{1-t^N}{1-t} = \mathrm{PE}[t-t^N]\,. \tag{A.3}$$

A.1.2 Modular Forms and Theta Functions

The Dedekind eta function is

$$\eta(q) = q^{1/24}\prod_{a=1}^{\infty}(1-q^a) \tag{A.4}$$

and

$$\begin{aligned}\theta_1(x;q) &= \mathrm{i}q^{1/8}(x^{-1/2}-x^{1/2})\prod_{r=1}^{\infty}(1-q^r)(1-xq^r)(1-x^{-1}q^r)\\ &= -\mathrm{i}q^{1/8}x^{1/2}\prod_{r=1}^{\infty}(1-q^r)(1-xq^r)(1-x^{-1}q^{r-1})\\ &= -\mathrm{i}q^{1/8}x^{1/2}\sum_{r=-\infty}^{\infty}(-1)^r(xq^{\frac{1}{2}})^r q^{\frac{r^2}{2}}\end{aligned} \tag{A.5}$$

is the Jacobi theta function. Clearly

$$\theta_1(x;q) = -\theta_1\left(x^{-1};q\right)\,. \tag{A.6}$$

We often also write

$$\theta_1(z|\tau) \equiv \theta_1(x;q) \tag{A.7}$$

with $x := e^{2\pi\mathrm{i}z}$ and $q := e^{2\pi\mathrm{i}\tau}$. The Jacobi theta function satisfies the properties

$$\theta_1\left(xq^{a+b/\tau};q\right) = (-1)^{a+b}x^{-a}q^{-a^2/2}\theta_1(x;q)\,, \quad \theta_1(x;q) = -\theta_1(x^{-1};q)\,. \tag{A.8}$$

$\theta_1(x;q)$ has simple zeros for $x = q^{a+b/\tau}$ for $a,b\in\mathbb{Z}$ and no poles. Furthermore, to compute residues, note that

$$\frac{\partial}{\partial y}\theta_1(y;q)\,|_{y=1} = 2\pi\eta(q)^3 \tag{A.9}$$

hence the residue is given by

$$\oint_{y=q^{a+b/\tau}}\frac{dy}{2\pi\mathrm{i}y}\frac{1}{\theta(y;q)} = \frac{(-1)^{a+b}q^{a^2/2}}{2\pi\eta(q)^3}\,. \tag{A.10}$$

A.1.3 Other Special Functions

We will often use the shorthand notation

$$f(z^{\pm n}) \equiv f(z^n)f(z^{-n})\,. \tag{A.11}$$

We use the following notation for the q-Pochammer symbols

$$(a;q)_N = \prod_{n=0}^{N}(1-aq^n)\,, \quad (a;q) := (a;q)_\infty = \mathrm{PE}\left[\frac{a}{1-q}\right]\,. \tag{A.12}$$

The q-theta function is

$$\theta\left(x;q\right) := \left(x;q\right)\left(qx^{-1};q\right) = \mathrm{PE}\left[\frac{x+qx^{-1}}{1-q}\right]\,. \tag{A.13}$$

They are related to the Jacobi theta function through

$$\theta_1(x;q) = \mathrm{i}q^{\frac{1}{12}}\eta(q)x^{-\frac{1}{2}}\theta(x;q)\,. \tag{A.14}$$

An obvious but important identity that we often employ is

$$x\theta\left(qx;q\right) = -\theta\left(x;q\right)\,. \tag{A.15}$$

The function $\theta\left(y;q\right)$ has simple zeros for $y = q^{a+b/\tau}$ for $a,b \in \mathbb{Z}$ and no poles. To compute residues note that

$$\frac{\partial}{\partial y}\theta\left(y;q\right)|_{y=1} = -\left(q;q\right)^2\,. \tag{A.16}$$

Using the identity $x\theta\left(qx;q\right) = -\theta\left(x;q\right)$ the residue is given by

$$\oint_{y=q^{a+b/\tau}} \frac{dy}{2\pi\mathrm{i}y}\frac{1}{\theta\left(y;q\right)} = (-1)^{a+1}\left(q;q\right)^{-2}q^{\frac{a}{2}(a-1)}\,. \tag{A.17}$$

We are often interested in the $q \to 1$ limit of the above q-series'. To take the limit, first note that the ratio of q-theta function may be rewritten as

$$\frac{\theta\left(q^a;q\right)}{\theta\left(q^b;q\right)} = \frac{[a]_q}{[b]_q}\prod_{n=1}^{\infty}\frac{[n+a]_q\,[n-a]_q}{[n+b]_q\,[n-b]_q} \tag{A.18}$$

where $[n]_q := (1-q^n)/(1-q)$ is the q-number. The q-number has the property that

$$\lim_{q\to 1}[n]_q = n \tag{A.19}$$

and therefore, for q-independent a, b, we have

$$\lim_{q\to 1}\frac{\theta\left(q^a;q\right)}{\theta\left(q^b;q\right)} = \frac{\sinh i\pi a}{\sinh i\pi b}\,, \tag{A.20}$$

where we have used the Euler infinite product representation for the sine function

$$\sin(x) = x\prod_{t=1}^{\infty}\left(1-\frac{x^2}{\pi^2 t^2}\right)\,. \tag{A.21}$$

The Elliptic Gamma function is defined as

$$\Gamma_e(z) := \Gamma(z;p,q) = \prod_{i,j=0}^{\infty}\frac{1-z^{-1}p^{i+1}q^{j+1}}{1-zp^iq^j} = \mathrm{PE}\left[\frac{z-\frac{pq}{z}}{(1-p)(1-q)}\right]\,. \tag{A.22}$$

An obvious, yet important, identity is

$$\Gamma_e(z)\Gamma_e(pq/z) = 1\,. \tag{A.23}$$

Multiple gamma functions are defined as the regularised infinite products

$$\Gamma_r(z|\vec{\omega}) \sim \prod_{n_1,n_2,\ldots,n_r=0}^{\infty}(\vec{n}\cdot\vec{\omega}+z)^{-1} \tag{A.24}$$

and $\vec{n}\cdot\vec{\omega} = n_1\omega_1 + n_2\omega_2 + n_3\omega_3$. The multiple sine function is defined as the regularised product

$$\begin{aligned} S_r(z|\vec{\omega}) &= \Gamma_r(z|\vec{\omega})^{-1}\Gamma_r(|\vec{\omega}|-z|\vec{\omega})^{(-1)^r} \\ &\sim \prod_{n_1,n_2,\ldots,n_r=0}^{\infty}\left(\vec{n}\cdot\vec{\omega}+|\vec{\omega}|-z\right)\left(\vec{n}\cdot\vec{\omega}+z\right)^{(-1)^{r+1}}\,, \end{aligned} \tag{A.25}$$

with $|\vec{\omega}| = \omega_1 + \cdots + \omega_r$. It has the symmetry property

$$S_r(z|\vec{\omega}) = S_r(|\vec{\omega}|-z|\vec{\omega})^{(-1)^{r+1}}\,. \tag{A.26}$$

A.1.4 Young Diagrams

We use Greek letters η, μ, λ, ν to denote partitions of natural numbers. We denote the empty partition by $\varnothing$. A non-empty partition is a set of integers λ

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \lambda_l \geqslant \cdots \geqslant \lambda_{\ell(\lambda)} > 0\,, \tag{A.27}$$

with $\lambda_A \in \mathbb{N}$ and $\ell(\lambda)$ the number of parts of λ. This definition is also extended to include $\lambda_{l>\ell(\lambda)} \equiv 0$. λ^{T} is the transpose. We denote

$$|\lambda| := \sum_{l=1}^{\ell(\lambda)} \lambda_l\,, \quad ||\lambda||^2 := \sum_{l=1}^{\ell(\lambda)} \lambda_l^2 = \sum_{(l,p)\in\lambda^{\mathrm{T}}} \lambda_l\,. \tag{A.28}$$

We give a box s in the Young diagram coordinates $s = (l, p)$ such that

$$\lambda = \{(l,p)|l = 1, \ldots, \ell(\lambda); p = 1, \ldots, \lambda_l\}\ . \tag{A.29}$$

By definition

$$\prod_{(l,p)\in\lambda} g(l,p) = \prod_{l=1}^{\ell(\lambda)} \prod_{p=1}^{\lambda_l} g(l,p)\,. \tag{A.30}$$

We will also be interested in collections of Young diagrams in which case we will give them labels, for example, λ_A in which case we write $\lambda_{A;l}$ to denote the number of boxes in the l^{th} column of the diagram λ_A. An N-tuple of Young diagrams is then

$$\vec{\mu} = \{\mu_A|I = 1, 2, \ldots, N\}\ . \tag{A.31}$$

We will also make use the identity [31, 289]

$$\begin{aligned}
&\sum_{(i,j)\in\mu} e^{i\epsilon_1+j\epsilon_2} + \sum_{(i',j')\in\mu'} e^{\epsilon_+-(i'\epsilon_1+j'\epsilon_2)} \\
&- (1-e^{\epsilon_1})(1-e^{\epsilon_2}) \sum_{(i,j)\in\mu} \sum_{(i',j')\in\mu'} e^{(i-j')\epsilon_1+(j-j')\epsilon_2} \\
&= \sum_{(i,j)\in\mu} e^{-(\mu'^{\mathrm{T}}_j-i)\epsilon_1+(\mu_i-j+1)\epsilon_2} + e^{2\epsilon_+} \sum_{(i',j')\in\mu'} e^{(\mu^{\mathrm{T}}_{j'}-i')\epsilon_1-(\mu'_{i'}-j'+1)\epsilon_2}\,.
\end{aligned} \tag{A.32}$$

with $\epsilon_\pm = \epsilon_1 \pm \epsilon_2$. We also use the identity

$$\mathcal{N}_{\nu,\mu}(Q;\mathfrak{q},\mathfrak{t}) := \prod_{l,p=1}^{\infty} \frac{1 - Q\mathfrak{q}^{\nu_l - p}\mathfrak{t}^{\mu_p^{\mathrm{T}} - l + 1}}{1 - Q\mathfrak{q}^{-p}\mathfrak{t}^{-l+1}} = \mathcal{N}_{\nu^{\mathrm{T}},\mu^{\mathrm{T}}}(Q;\mathfrak{t},\mathfrak{q}) \tag{A.33}$$

$$= \prod_{(l,p)\in\nu} \left(1 - Q\mathfrak{q}^{\nu_l - p}\mathfrak{t}^{\mu_p^{\mathrm{T}} - l + 1}\right) \prod_{(l,p)\in\mu} \left(1 - Q\mathfrak{q}^{-\mu_l + p - 1}\mathfrak{t}^{-\nu_p^{\mathrm{T}} + l}\right) . \tag{A.34}$$

A.1.5 Schur and Skew-Schur Functions

The Schur functions $s_\mu(x)$ are functions which depend on a given young diagram μ. They form an orthogonal basis for the ring of symmetric functions Λ. The basis is orthogonal with respect to the Hall inner product:

$$\langle s_\eta(x), s_\nu(x)\rangle = \delta_{\eta\nu} \, . \tag{A.35}$$

An explicit representation is given by

$$s_\lambda(\mathbf{x}) = \frac{\det_{AB} x_A^{\lambda_B + N - B}}{\det_{AB} x_A^{N-B}} \, , \tag{A.36}$$

which are orthogonal with respect to the measure

$$\langle s_\lambda, s_\mu\rangle := \oint d\mu(\mathbf{x}) s_\lambda(\mathbf{x}) s_\mu(\mathbf{x}) = \delta_{\lambda,\mu} \, , \tag{A.37}$$

$$\oint d\mu_{U(N)}(\mathbf{z}) = \oint d\mu(\mathbf{z}) = \frac{1}{N!} \oint_{|z_A|=1} \prod_{A=1}^{N} \frac{dz_A}{2\pi \mathrm{i} z_A} \prod_{A\neq B} \left(1 - \frac{z_A}{z_B}\right) . \tag{A.38}$$

Furthermore,

$$s_{\lambda\otimes\mu}(x) = s_\lambda(x) s_\mu(x) = \sum_\eta c^\eta_{\lambda\mu} s_\eta(x) \, , \tag{A.39}$$

where

$$c^\mu_{\nu\eta} = c^{\mu^{\mathrm{T}}}_{\nu^{\mathrm{T}}\eta^{\mathrm{T}}} \tag{A.40}$$

are Littlewood-Richardson coefficients. The skew Schur functions may then be defined by

$$\langle s_{\lambda/\mu}(x), s_\nu(x)\rangle = \langle s_\lambda(x), s_\mu(x) s_\nu(x)\rangle \tag{A.41}$$

and $s_\eta(x) \equiv s_{\eta/\varnothing}(x)$. From (A.39) we have the identity

$$s_{\lambda_1\otimes\cdots\otimes\lambda_n}(x) = \prod_{i=1}^{n} s_{\lambda_i}(x) = \sum_{\eta_1,\ldots,\eta_{n-1}} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2} c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right) s_{\eta_{n-1}}(x)\,. \tag{A.42}$$

Hence,

$$\left\langle s_{(\lambda_1\otimes\cdots\otimes\lambda_n)/\mu}(x), s_\nu(x)\right\rangle = \langle s_{\lambda_1\otimes\cdots\otimes\lambda_n}(x), s_\mu(x)s_\nu(x)\rangle \tag{A.43}$$

$$= \sum_{\eta_1,\ldots,\eta_{n-1},\sigma} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2} c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right) c^{\sigma}_{\mu\nu}\left\langle s_{\eta_{n-1}}(x), s_\sigma(x)\right\rangle \tag{A.44}$$

$$= \sum_{\eta_1,\ldots,\eta_{n-1}} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2} c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right) c^{\eta_{n-1}}_{\mu\nu} \tag{A.45}$$

$$= \sum_{\eta_1,\ldots,\eta_{n-1},\rho} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2} c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right) c^{\eta_{n-1}}_{\mu\rho}\langle s_\rho(x), s_\nu(x)\rangle\,, \tag{A.46}$$

where we understand $\eta_0 := \varnothing$ and $c^{\nu}_{\varnothing\mu} = 1$. By the non-degeneracy of $\langle\cdot,\cdot\rangle$ we have

$$s_{(\lambda_1\otimes\cdots\otimes\lambda_n)/\mu}(x) = \sum_{\eta_1,\ldots,\eta_{n-1},\rho} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2} c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right) c^{\eta_{n-1}}_{\mu\rho} s_\rho(x)\,. \tag{A.47}$$

The skew Schur function may be equivalently expressed as

$$s_{\mu/\nu}(x) = \sum_{\eta} c^{\mu}_{\nu\eta} s_\eta(x)\,. \tag{A.48}$$

Therefore (A.47) may be written as

$$s_{(\lambda_1\otimes\cdots\otimes\lambda_n)/\mu}(x) = \sum_{\eta_1,\ldots,\eta_{n-1}} c^{\eta_1}_{\lambda_1\lambda_2}\left(\prod_{i=1}^{n-2} c^{\eta_{i+1}}_{\eta_i\lambda_{i+2}}\right) s_{\eta_{n-1}/\mu}(x)\,. \tag{A.49}$$

The skew Schur functions satisfy the Cauchy identities

$$\sum_\eta s_{\eta/\lambda}(x)s_{\eta/\mu}(y) = \prod_{l,p=1}^{\infty}(1-x_l y_p)^{-1}\sum_\eta s_{\mu/\eta}(x)s_{\lambda/\eta}(y)\,, \tag{A.50}$$

$$\sum_\eta s_{\eta^{\mathrm{T}}/\lambda}(x)s_{\eta/\mu}(y) = \prod_{l,p=1}^{\infty}(1+x_l y_p)\sum_\eta s_{\mu^{\mathrm{T}}/\eta}(x)s_{\lambda^{\mathrm{T}}/\eta^{\mathrm{T}}}(y)\,, \tag{A.51}$$

$$s_{\mu/\nu}(Qx) = Q^{|\mu|-|\nu|}s_{\mu/\nu}(x)\,. \tag{A.52}$$

For Schur polynomials of type A_1 the Littlewood-Richardson coefficients are, of course, simply

$$s_{2j}s_{2j'} = \sum_{2j''=2|j-j'|}^{2j+2j'} s_{2j''}\,, \tag{A.53}$$

with $2j, 2j', 2j'' \in \mathbb{N}$.

A.1.6 Spiridonov-Warnaar Inversion Formula

Define

$$\delta(x, y; T) := \frac{\Gamma_e(Tx^{\pm 1}y^{\pm 1})}{\Gamma_e(T^2)\Gamma_e(x^{\pm 2})}\,, \tag{A.54}$$

and consider the integral

$$f(z) = \kappa \oint \frac{dw}{4\pi\mathrm{i}w}\delta(w, z; T)\hat{f}(w)\,, \tag{A.55}$$

such that $\max\{|p|, |q|\} < |T|^2 < 1$. A consequence of the Spiridonov-Warnaar theorem is that [189, 290]

$$\hat{f}(w) = \kappa \oint_{C_w} \frac{dz}{4\pi\mathrm{i}z}\delta(z, w; T^{-1})f(z)\,, \tag{A.56}$$

where C_w is a deformation of the unit circle that includes the poles at $z = T^{-1}w^{\pm 1}$ but excludes those at $Tw^{\pm 1}$. Note that, if $\lim_{p,q\to 0}(f, \hat{f}, T) := (\tilde{f}, \tilde{\hat{f}}, T)$ is smooth, (A.56) implies

$$\begin{aligned} \tilde{f}(z) &= \oint \frac{dw}{4\pi\mathrm{i}w}\frac{(1-T^2)(1-w^{\pm 2})}{(1-Tw^{\pm 1}z^{\pm 1})}\tilde{\hat{f}}(w) \\ \Longrightarrow \quad \tilde{\hat{f}}(w) &= \oint_{C_w} \frac{dz}{4\pi\mathrm{i}z}\frac{(1-T^{-2})(1-z^{\pm 2})}{(1-T^{-1}w^{\pm 1}z^{\pm 1})}\tilde{f}(z)\,. \end{aligned} \tag{A.57}$$

A.2 Lie Groups, Lie Algebras and Representations

A.2.1 $SU(N)$

Highest weight, irreducible representations $\mathcal{R}_{(d_1,d_2,\ldots,d_{N-1})}$ of $SU(N)$ may be labelled by a Young diagram λ (A.27) of length $\ell(\lambda) = N$ with λ_N. The relations between the Dynkin labels of the representation and the Young diagram is

$$d_A = \lambda_A - \lambda_{A+1} , \quad \lambda_A = \sum_{i=A}^{N-1} d_i . \tag{A.58}$$

The conjugate representation $\overline{\mathcal{R}_{(d_1,d_2,\ldots,d_{N-1})}} = \mathcal{R}_{(d_{N-1},d_{N-2},\ldots,d_1)}$ is therefore associated to the Young diagram $\overline{\lambda}$ with $\overline{\lambda}_A = \sum_{r=A}^{N-1} d_{N-r} = \sum_{r=A}^{N-1}(\lambda_{N-r} - \lambda_{N-r+1}) = \lambda_1 - \lambda_{N-A+1}$.

The characters for the representation $\mathcal{R}_{(d_1,d_2,\ldots,d_{N-1})}$ are given by Schur polynomials (A.36) for the Young diagram λ that specifies the representation

$$\chi_{(d_1,d_2,\ldots,d_{N-1})}(\mathbf{x}) = s_\lambda(\mathbf{x}) \tag{A.59}$$

with $\lambda_N = 0$ and $\prod_{A=1}^N x_A = 1$. We also often abuse notation and denote these characters simply by their Dynkin labels $\chi_{(d_1,d_2,\ldots,d_{N-1})}(\mathbf{x}) \equiv [d_1, d_2, \ldots, d_{N-1}]$. The representation labelled by λ has dimension

$$|\mathcal{R}_{(d_1,d_2,\ldots,d_{N-1})}| = s_\lambda(\mathbf{1}) = \prod_{1\leqslant A<B\leqslant N} \frac{\lambda_A - \lambda_B - A + B}{-A+B} . \tag{A.60}$$

The characters are orthogonal with respect to the Haar measure of $SU(N)$

$$\oint d\mu_{SU(N)}(\mathbf{z}) = \oint d\mu(\mathbf{z})\delta\left(\prod_{A=1}^N z_A - 1\right) , \tag{A.61}$$

with $d\mu$ defined in (A.38). One fact that we will often use is, that for any class function $f : SU(N) \to \mathbb{C}$ that is also invariant under the Weyl group of $SU(N)$ we can write

$$\oint d\mu_{SU(N)}(\mathbf{z}) f(\mathbf{z}) = \oint_{|z_A|=1} \prod_{A=1}^{N-1} \frac{dz_A}{2\pi \mathrm{i} z_A} \prod_{1\leqslant A<B\leqslant N} \left(1 - \frac{z_A}{z_B}\right) f(\mathbf{z}) . \tag{A.62}$$

A.2.2 $SO(2N)$

We will also sporadically make use of $SO(2N)$ characters. These are given by

$$\hat{s}_\lambda(\mathbf{x}) = \frac{\det_{AB}\left(x_A^{\lambda_B+N-B} - x_A^{-\lambda_B-N+B}\right)}{\det_{AB}(x_A - x_A^{-1})^{N-B}} . \tag{A.63}$$

where the $\lambda_1 \geqslant \lambda_2 \geqslant, \ldots, \geqslant |\lambda_N| \geqslant 0$ is related to the Dynkin labels $(d_1, d_2, \ldots, d_N)$ by

$$\lambda_A = -d_{N-1}\delta_{A,N} + \frac{1}{2}(d_N + d_{N-1}) + \sum_{n=A}^{N-2} d_n\,. \tag{A.64}$$

Appendix B

Algebraic Geometry

The aim of this section is largely to present the relevant definitions that will be required in order to understand the Hilbert series and the relation to supersymmetric theories. Complete proofs and derivations are beyond the scope of this appendix but can be found in [92].

B.1 Definitions

Definition 1 ((Graded) Rings:)**.** *A ring R is an abelian group with a multiplication operation $(a, b) \mapsto ab$ and an identity element 1 satisfying*

- *Associativity:* $a(bc) = (ab)c$
- *Identity:* $1a = a1 = a$
- *Distributivity :* $(b + c)a = ba + ca$

for all $a, b, c \in R$. R is commutative if $ab = ba$ and for the remainder of the text we will always take R to be commutative. A graded ring is a ring R together with a direct sum decomposition

$$R = \oplus_{i=0}^{\infty} R_i \quad \textit{such that } R_i R_j \subset R_{i+j} \textit{ for all } i, j\,. \tag{B.1}$$

Definition 2 (Invertible Elements)**.** *An invertible element in a ring R is an element $a \in R$ such that $ab = 1$ with $b = a^{-1} \in R$ unique.*

Definition 3 (Fields)**.** *A field $\mathbb{K}$ is a ring in which every nonzero element $a \in K$ is invertible. For example $\mathbb{K} = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.*

Definition 4 (Polynomials)**.** *A polynomial f in $x_1,\ldots,x_n$ with coefficients in $\mathbb{K}$ is a finite linear combination of monomials of the form $f = \sum_\alpha f_{\alpha_1\ldots\alpha_n} x_1^{\alpha_1}\ldots x_n^{\alpha_n}$. The set of all such polynomials is denoted by $\mathbb{K}[x_1,\ldots,x_n]$. In particular $\mathbb{K}[x_1,\ldots,x_n]$ forms a $\mathbb{Z}^n$-graded ring where the grading is provided by degree.*

Definition 5 (Ideals)**.** *An ideal in a ring R is an additive subgroup $I \subset R$ such that $ap \in I$ for all $a \in R$ and $p \in I$. An ideal is said to be generated by a subset $S \subset R$ if every element $p \in I$ can be written as*

$$p = \sum_i a_i s_i \tag{B.2}$$

for $a_i \in R$ and $s_i \in S$. In particular, for $f_1,\ldots,f_s \in \mathbb{K}[x_1,\ldots,x_n]$

$$\langle f_1,\ldots,f_s\rangle = \left\{\sum_{i=1}^{s} h_i f_i | h_1,\ldots,h_s \in \mathbb{K}[x_1,\ldots,x_n]\right\} \tag{B.3}$$

is an ideal of $\mathbb{K}[x_1,\ldots,x_n]$ which we call the ideal generated by $f_1,\ldots,f_s$.

Definition 6 (Radical Ideals)**.** *An ideal I is radical if $f^m \in I$ if for some $m \in \mathbb{Z}^+$ implies $f \in I$.*

Definition 7 (Radical of an Ideal)**.** *The radical $\sqrt{I}$ of an ideal $I \in \mathbb{K}[x_1,\ldots,x_n]$ is*

$$\sqrt{I} = \{f | f^m \in I \, for\ some\ m \in \mathbb{Z}^+\}. \tag{B.4}$$

It follows $I \subset \sqrt{I}$ (since $f \in I$ implies that $f^1 \in I$). Moreover the radical of an ideal I is always a radical ideal.

Definition 8 ((Graded) Modules)**.** *Let R be a ring, an R-module M is an abelian group with an action of R, $R \times M \to M$ satisfying*

- *Associativity:* $a(bm) = (ab)m$
- *Identity:* $1m = m1 = m$
- *Distributivity :* $(b+c)m = bm + cm$, $a(m+n) = am + an$

for all $a, b \in R$ and $m, n \in M$ Let $R = \bigoplus_{i=0}^\infty R_i$ be a graded ring. Then a graded module over R is a module M with a decomposition

$$M = \bigoplus_{-\infty}^{\infty} M_i \quad such\ that\ R_i M_j \subset M_{i+j}\ for\ all\ i,j \tag{B.5}$$

Definition 9 (Affine Spaces)**.** *Let* $\mathbb{K}$ *be a field and* $n \in \mathbb{Z}^+$*. The* n*-dimensional affine space over* $\mathbb{K}$ *is*

$$\mathbb{K}^n = \{(a_1, \ldots, a_n) | a_1, \ldots, a_n \in \mathbb{K}\} . \tag{B.6}$$

Definition 10 (Affine Varieties)**.** *Let* $f_1, \ldots, f_s \in \mathbb{K}[x_1, \ldots, x_n]$ *then the affine variety defined by* $f_1, \ldots, f_s$ *is*

$$\begin{aligned} \mathfrak{V}(f_1, \ldots, f_s) =& \{(a_1, \ldots, a_n) \in \mathbb{K}^n | f_i(a_1, \ldots, a_n) = 0 \, \forall \, i \in \{1, \ldots, s\}\} \\ \subset& \mathbb{K}^n \end{aligned} \tag{B.7}$$

Definition 11 (Irreducible Varieties)**.** *An affine variety* $\mathbf{V} \subset \mathbb{K}^n$ *is said to be irreducible if whenever* $\mathbf{V}$ *is written in the form* $\mathbf{V} = \mathbf{V}_1 \cup \mathbf{V}_2$*, where* $\mathbf{V}_1$ *and* $\mathbf{V}_2$ *are affine varieties, then either* $\mathbf{V}_1 = \mathbf{V}$ *or* $\mathbf{V}_2 = \mathbf{V}$*.*

B.2 Algebra-Geometry Correspondence

Given an affine variety $\mathbf{V} \subset \mathbb{K}^n$ (this will play the role of the moduli space of supersymmetric vacua $\mathbf{M}$ (1.2)). To this variety we can associate to it an ideal $\mathfrak{I}(\mathbf{V}) \subset \mathbb{K}[x_1, ..., x_n]$ (this will play the role of an operator algebra, such as chiral ring, etc) using the following map

$$\mathfrak{I}(\mathbf{V}) = \{f \in \mathbb{K}[x_1, ..., x_n] \mid f(a_1, \ldots, a_n) = 0 \ \forall \ (a_1, \ldots, a_n) \in \mathbf{V}\} , \tag{B.8}$$

the proof that this indeed gives an ideal can be found in [91]. On the other hand, given an ideal $I \subset \mathbb{K}[x_1, ...x_n]$ we can associate to it a variety $\mathfrak{V}(I)$ (see Definition 10)

$$\begin{aligned} \mathfrak{V}(f_1, \ldots, f_s) =& \{(a_1, \ldots, a_n) \in \mathbb{K}^n | f_i(a_1, \ldots, a_n) = 0 \, \forall \, i \in \{1, \ldots, s\}\} \\ \subset& \mathbb{K}^n . \end{aligned} \tag{B.9}$$

It can be shown that $\mathfrak{V}(\mathfrak{I}(\mathbf{V})) = \mathbf{V}$. We therefore have two maps which provide a correspondence and affine varieties. However, in general, this correspondence is not one to one. For example let us consider the family of distinct ideals $\langle x^n \rangle$ (with $n \in \mathbb{N}$) in $\mathbb{C}[x]$, then it's easy to see that to the map (B.9) associates to each of them the same variety, namely $\mathfrak{V}(x^n) = \{0\}$.

Theorem 1 (The (Strong) Nullstellensatz)**.** *Let* $\mathbb{K}$ *be algebraically closed and let*

Algebra	$\leftrightarrow$	Geometry
Radical ideal $I = \sqrt{I}$		Affine variety $\mathbf{V} = \mathfrak{V}(I)$
Addition of ideals $I + J$		Intersection of varieties $\mathfrak{V}(I) \cap \mathfrak{V}(J)$
Product of ideals IJ		Union of varieties $\mathfrak{V}(I) \cup \mathfrak{V}(J)$
Prime ideal I		Affine irreducible variety $\mathfrak{V}(I)$
Regular sequence		Complete intersection

Table B.1: *Summary of the algebra-geometry correspondence.*

$I \subset \mathbb{K}[x_1, \ldots, x_n]$ *be an ideal, then*

$$\mathfrak{I}(\mathfrak{V}(I)) = \sqrt{I}\,. \tag{B.10}$$

In particular, this theorem tells us that the maps (B.8)-(B.9) provide us a one to one correspondence between affine varieties and radical ideals. This dictionary can be extended reformulating in algebraic terms geometrical problems, see Table B.1. A particular useful class of ideals is provided by the so called *prime ideals*

Definition 12 (Prime Ideals)**.** *An ideal $I \subset \mathbb{K}[x_1, ..., x_n]$ is a prime ideal if $f, g \in \mathbb{K}[x_1, ...x_n]$ and $fg \in I$, implies either $f \in I$ or $g \in I$.*

It's easy to prove that every prime ideal is also a radical ideal, and that moreover there is a one-to-one correspondence between *irreducible varieties* and prime ideals via the maps (B.8) & (B.9)[91].

Definition 13 (Noetherian Rings)**.** *A ring R is called Noetherian if there is no infinite ascending sequence of left (or right) ideals. Therefore, given any chain of left (or right) ideals,*

$$I_1 \subseteq \ldots I_{k-1} \subseteq I_k \subseteq I_{k+1} \ldots, \tag{B.11}$$

there exists an n such that

$$I_n = I_{n+1} = \ldots\,. \tag{B.12}$$

When R is a *Noetherian ring*, using the *Lasker-Noether theorem*, we can always decompose and ideal of R as an irredundant intersection of a finite set $\{J_i\}$ of primary ideals [91], this procedure is called *primary decomposition.* In particular we can take into account the ideal I and we get

$$I = \bigcap_i J_i \ . \tag{B.13}$$

We can then consider the different radical ideals $\sqrt{J_i}$ associated to each of the primary ideals in (B.13). When the J_i are all prime ideals we have a one to one correspondence between affine irreducible complex variates and the radical ideals $\sqrt{J_i}$. This implies that the corresponding complex variety, can be written as

$$\mathbf{F} = \bigcup_i \mathfrak{V}(\sqrt{J_i}) \,, \tag{B.14}$$

in this way the algebraic approach turns out to be very powerful since it provides a systematic way to decompose the F-flat moduli space $\mathbf{F}$ of a theory into different irreducible branches. Remarkably, we can also establish when the space $\mathbf{F}$ is a complete intersection.

Definition 14 (Regular Sequences)**.** *A sequence of non-constant polynomials $P_1, P_2, ...P_r$ is said to be regular if for all $i = 1, ..., r$, P_i is not a zero divisor modulo the partial ideal $(P_1, ..., P_{r-i})$.*

Given an ideal generated by a regular sequence the following theorem holds [291]

Theorem 2. *Given the ring of polynomials $R = \mathbb{C}[x_1, ...x_n]$ and the ideal $I \subset R$ then the algebraic variety associated to the quotient ring R/I is a complete intersection if and only if I is generated by a regular sequence of homogeneous polynomials.*

Therefore if the ideal is generated by a *regular sequence* of polynomials then we can use letter counting for the computation of the corresponding Hilbert series. For application of the above theorem in a different context see [97], we also refer the interested reader to Appendix A of that paper for a more detailed discussion related to this issue.

B.3 Hilbert Series

We are now in a position to define the Hilbert function and Hilbert series.

Definition 15. *Hilbert function & Hilbert series Let M be a finitely generated graded module over $\mathbb{K}[x_1, \ldots, x_n]$ graded by degree $\deg x_i = 1$ for all $i = 1, \ldots, n$. The*

Hilbert function is defined to be

$$HF_M(s) := \dim_{\mathbb{K}} M_s\,, \tag{B.15}$$

note that HF is finite in every degree. The Hilbert series is then defined in a formal power series as the generating function for the Hilbert function

$$HS(\tau; M) := \sum_{s=0}^{\infty} HF_M(s)\tau^s = \mathrm{Tr}_M\, \tau^E \tag{B.16}$$

and here $E : M_s \to \mathbb{Z}$ stands for the operator which computes the degree of an element $y \in M_s$ $E(y) = s$. If M is generated by d homogeneous elements of degrees $E_1, \ldots, E_d$ the Hilbert series takes the form

$$HS(\tau; M) = \frac{P(\tau)}{\prod_{i=1}^{d} (1 - \tau^{E_i})} \tag{B.17}$$

where $P(\tau)$ is a polynomial in the formal parameter τ with integer coefficients.

For example, let $M = \mathbb{K}[x_1, \ldots, x_n]$ be a polynomial ring with degrees $E(x_i) = 1$. The Hilbert series is simply

$$\mathrm{HS}(\tau; M) = \frac{1}{(1-\tau)^n}\,. \tag{B.18}$$

The Hilbert function is therefore

$$\mathrm{HF}_M(s) = \frac{\prod_{i=0}^{s-1} (n+i)}{s!}\,. \tag{B.19}$$

The results (B.16) & (B.17) may also be generalised to the case of rings with $\mathbb{Z}^b$ grading $M = \bigoplus_{s_1, s_2, \ldots, s_b} M_{s_1 s_2 \ldots s_b}$ in which case the result takes the general form

$$\mathrm{HS}(\tau_1, \tau_2, \ldots, \tau_b; M) = \mathrm{Tr}_M \prod_{l=1}^{b} \tau_l^{E^{(l)}} = \frac{P(\tau_1, \ldots, \tau_b)}{\prod_{i=1}^{d} \left(1 - \prod_{l=1}^{b} \tau_l^{E_i^{(l)}}\right)}\,. \tag{B.20}$$

If we let $\mathbf{M} = \mathfrak{V}(I)$ be the algebraic variety defined as the set of zeros of the ideal $I \subset \mathbb{K}[x_1, \ldots, x_n]$ then the Hilbert series of $R = \mathbb{K}[x_1, \ldots, x_n]/I$ (as written in the form (B.17)) allows us to compute the dimension of $\mathbf{M}$ and the degree of $\mathbf{M}$ (the number of intersection points of $\mathbf{M}$ with $d = \dim_{\mathbb{K}} \mathbf{M}$ generic hyperplanes) $\deg \mathbf{M}$

as

$$\dim_{\mathbb{K}} \mathbf{M} = \dim_{\mathbb{K}} M = \text{Order of pole at } \tau = 1 \text{ of } \mathrm{HS}(\tau; M) = d\,, \tag{B.21}$$

$$\deg \mathbf{M} = P(1)\,. \tag{B.22}$$

B.4 Example Computations with *Macaulay2*

Let us present some example code for *Macaulay2* [112]. We firstly focus on the computations of the Hilbert series performed in the introduction 1.5.

Note that *Macaulay2* represents elements of $\mathbb{C}$ via floating point approximations. $\mathbb{C}$ is an example of an *inexact field*. *Macaulay2* uses Gröbner bases and the algorithms it uses do not work over $\mathbb{C}$. Therefore when performing practical computations we must instead work over $\mathbb{Q}$ (or any other exact field of characteristic zero will do) while using *Macaulay2* before tensoring any final result with $\mathbb{C}$. Tensoring with $\mathbb{C}$ can destroy some properties which may hold over $\mathbb{Q}$. For example an ideal that is prime over $\mathbb{Q}$ may fail to be prime after tensoring with $\mathbb{C}$.[1] Hence the *Macaulay2* output should always be read with the above caveat in mind; nevertheless, our main object of interest, the Hilbert series is expected to be independent of the above.

The first example that we presented there was the most simple case of $\mathcal{N} = 4$ SYM with gauge group $G = U(1)$ (free theory). This computation can of course simply be performed by hand. Nevertheless one can also use *Macaulay2* by inputting $F = R/I$ with $R = \mathbb{C}[X, Y, Z]$ and I trivial.

```
i1 : R=QQ[X,Y,Z,Degrees=>{{1,0,0,0},{0,1,0,0},{0,0,1,0}}]


i2 : hilbertSeries(R)
```

This of course outputs the Hilbert series

```
                         1
o2 = ------------------------
     (1 - T )(1 - T )(1 - T )
           2        1        0
```

One then takes the output and sets $\tau_{1,2,3} = T_{0,1,2}$.

[1]See e.g. [292].

For the second case presented in the introduction of the $\mathcal{N} = 1$ $G = U(1)$ gauge theory with $F = R/I$, $R = \mathbb{C}[X,Y,Z]$ and $I = \langle XY, XZ, YZ \rangle$ an example *Macaulay2* input is

```
i1 : R=QQ[X,Y,Z,Degrees=>{{1,0,0,1},{0,1,0,-1},{0,0,1,0}}]

i2 : I=ideal(X*Y,X*Z,Y*Z)

i3 : hilbertSeries(R/I)

i4 : primaryDecomposition I

i5 : radical I

i6 : isPrime I
```

This outputs

```
                             -1
     1 - T T  - T T T  - T T T   + 2T T T
          0 1    0 2 3    1 2 3      0 1 2
o3 = ------------------------------------
                        -1
         (1 - T )(1 - T T  )(1 - T T )
               2       1 3        0 3

o4 = {ideal (X, Y), ideal (X, Z), ideal (Y, Z)}

o5 = monomialIdeal (X*Y, X*Z, Y*Z)

o6 = false
```

Setting $\tau_{1,2,3} = T_{0,1,2}$ and $z = T_3$ identifies $o3$ with (1.170).

We can also compute for more complicated non-abelian cases. Of course, the computations become more and more complex with $\dim G$. Let us take the case of $\mathcal{N} = 4$ SYM with gauge group $G = SU(2)$. In this case $F = R/I$ with $R =$

$\mathbb{C}[X,Y,Z]$ where now $X,Y,Z \in \mathfrak{su}(2)$

$$X = \begin{pmatrix} X_1 & X_2 \\ X_3 & -X_1 \end{pmatrix}, \quad Y = \begin{pmatrix} Y_1 & Y_2 \\ Y_3 & -Y_1 \end{pmatrix}, \quad Z = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1 \end{pmatrix} \tag{B.23}$$

The ideal is $I = \langle [X,Y],[X,Z],[Y,Z] \rangle$ coming from the superpotential $\operatorname{tr} X[Y,Z]$. We input

```
i1 : R=QQ[X1, X2, X3, Y1, Y2, Y3, Z1, Z2, Z3,
Degrees=>{{1, 0, 0, 0}, {1, 0, 0, -2}, {1, 0, 0, 2},
{0, 1, 0, 0}, {0, 1, 0, -2}, {0, 1, 0, 2},
{0, 0, 1, 0}, {0, 0, 1, -2}, {0, 0, 1, 2}}]

i2 : I=ideal(-*Y3*Z2 + *Y2*Z3, *Y3*Z1 - *Y1*Z3, -*Y2*Z1 + *Y1*Z2,
*X3*Z2 -*X2*Z3, -*X3*Z1 + *X1*Z3, *X2*Z1 - *X1*Z2,
 -*X3*Y2 + *X2*Y3, *X3*Y1 - *X1*Y3, -*X2*Y1 + *X1*Y2)

i3 : hilbertSeries(R/I)

i4 : primaryDecomposition I

i5 : radical I

i6 : isPrime I
```

This outputs (here we have suppressed the output for the Hilbert series $o3$ in the raw form to save on space, we present it in the simple form below)

```
o4 = {ideal (Y3*Z2 - Y2*Z3, X3*Z2 - X2*Z3, Y3*Z1 - Y1*Z3,
Y2*Z1 - Y1*Z2, X3*Z1 - X1*Z3, X2*Z1 - X1*Z2, X3*Y2 - X2*Y3,
X3*Y1 - X1*Y3, X2*Y1 - X1*Y2)}

o5 = ideal (Y3*Z2 - Y2*Z3, X3*Z2 - X2*Z3, Y3*Z1 - Y1*Z3,
Y2*Z1 - Y1*Z2, X3*Z1 - X1*Z3, X2*Z1 - X1*Z2, X3*Y2 - X2*Y3,
X3*Y1 - X1*Y3, X2*Y1 - X1*Y2)

o6 = true
```

After converting the raw output for the Hilbert series as before we have

$$
\begin{aligned}
\mathrm{HS}(\tau_1, \tau_2, \tau_3, z; F) = \mathrm{PE}[\chi_2(z)(\tau_1 + \tau_2 + \tau_3)]\Big\{ & 1 + \tau_1^2\tau_2^2\tau_3^2 \\
& + (\chi_4(z) + \chi_2(z))\tau_1\tau_2\tau_3 + (\tau_1 + \tau_2)(\tau_1 + \tau_3)(\tau_2 + \tau_3) \\
& - (\tau_1\tau_2 + \tau_1\tau_3 + \tau_2\tau_3)\chi_2(z) - \chi_2(z)(\tau_1 + \tau_2 + \tau_3)\tau_1\tau_2\tau_3\Big\} .
\end{aligned}
\tag{B.24}
$$

Appendix C

Appendices for Chapter 2

C.1 Representation Theory of $\mathfrak{su}(2,2|1)$

In this appendix we discuss the representation theory of (the complexification of) the $\mathfrak{su}(2,2|1)$ superalgebra. Unitary representations of $\mathfrak{su}(2,2|1)$ can be decomposed into a finite sum of representations $[j_1, j_2]_E^{(r)}$ of the maximal compact bosonic subalgebra

$$\mathfrak{u}(1)_E \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus \mathfrak{u}(1)_r \subset \mathfrak{su}(2,2|1)\,. \tag{C.1}$$

The $\mathfrak{su}(2,2|1)$ superalgebra has four Poincaré supercharges

$$\mathcal{Q} \in [1/2, 0]_{\frac{1}{2}}^{(-1)}\,, \quad \widetilde{\mathcal{Q}} \in [0, 1/2]_{\frac{1}{2}}^{(1)}\,. \tag{C.2}$$

Note that with respect to [1] we use $j = 2j_1$, $\bar{j} = 2j_2$. The $\mathcal{Q}$ and $\widetilde{\mathcal{Q}}$ shortening conditions are listed in Table C.1 and Table C.2 respectively. We list all possible short unitary multiplets of the $\mathfrak{su}(2,2|1)$ superconformal algebra in Table C.3. Several of these multiplets contain conserved currents; we list all multiplets that contain conserved currents in Table C.4.

Name	Primary	Unitarity bound	$\mathcal{Q}$-Null state
L	$[j_1, j_2]_E^{(r)}$	$E > 2 + 2j_1 - \frac{3}{2}r$	None
A_1	$[j_1 \geqslant 1/2, j_2]_E^{(r)}$	$E = 2 + 2j_1 - \frac{3}{2}r$	$[j_1 - 1/2, j_2]_{E+1/2}^{(r-1)}$
A_2	$[0, j_2]_E^{(r)}$	$E = 2 - \frac{3}{2}r$	$[0, j_2]_{E+1}^{(r-2)}$
B_1	$[0, j_2]_E^{(r)}$	$E = -\frac{3}{2}r$	$[1/2, j_2]_{E+1/2}^{(r-1)}$

Table C.1: *$\mathcal{Q}$-shortening conditions.*

Name	Primary	Unitarity bound	$\widetilde{\mathcal{Q}}$-Null state
$\overline{L}$	$[j_1,j_2]^{(r)}_E$	$E>2+2j_1+\frac{3}{2}r$	None
$\overline{A}_1$	$[j_1,j_2\geqslant 1/2]^{(r)}_E$	$E=2+2j_1+\frac{3}{2}r$	$[j_1,j_2-1/2]^{(r+1)}_{E+1/2}$
$\overline{A}_2$	$[j_1,0]^{(r)}_E$	$E=2+\frac{3}{2}r$	$[j_1,0]^{(r+2)}_{E+1}$
$\overline{B}_1$	$[j_1,0]^{(r)}_E$	$E=\frac{3}{2}r$	$[j_1,1/2]^{(r+1)}_{E+1/2}$

Table C.2: *$\widetilde{\mathcal{Q}}$-shortening conditions.*

CDI-notation	Quantum number relations	DO-notation
$L\overline{L}[j_1,j_2]^{(r)}_E$	$E>2+\max\left\{2j_1-\frac{3}{2}r, 2j_2+\frac{3}{2}r\right\}$	$\mathcal{A}^E_{r,(j_1,j_2)}$
$B_1\overline{L}[0,j_2]^{(r)}_E$	$r<-\frac{2}{3}(j_2+1),\ E=-\frac{3}{2}r$	$\mathcal{B}_{r,(0,j_2)}$
$L\overline{B}_1[j_1,0]^{(r)}_E$	$r>\frac{2}{3}(j_1+1),\ E=\frac{3}{2}r$	$\overline{\mathcal{B}}_{r,(j_1,0)}$
$B_1\overline{B}_1[0,0]^{(r)}_E$	$E=r=0$	$\hat{\mathcal{B}}$
$A_\ell\overline{L}[j_1,j_2]^{(r)}_E$	$r<\frac{2}{3}(j_1-j_2),\ E=2+2j_1-\frac{3}{2}r$	$\mathcal{C}_{r,(j_1,j_2)}$
$L\overline{A}_{\bar{\ell}}[j_1,j_2]^{(r)}_E$	$r>\frac{2}{3}(j_1-j_2),\ E=2+2j_2+\frac{3}{2}r$	$\overline{\mathcal{C}}_{r,(j_1,j_2)}$
$A_\ell\overline{A}_{\bar{\ell}}[j_1,j_2]^{(r)}_E$	$r=\frac{2}{3}(j_1-j_2),\ E=2+j_1+j_2$	$\hat{\mathcal{C}}_{(j_1,j_2)}$
$B_1\overline{A}_{\bar{\ell}}[0,j_2]^{(r)}_E$	$E=-\frac{3}{2}r=1+j_2$	$\mathcal{D}_{(0,j_2)}$
$A_\ell\overline{B}_1[j_1,0]^{(r)}_E$	$E=\frac{3}{2}r=1+j_1$	$\overline{\mathcal{D}}_{(j_1,0)}$

Table C.3: *Unitary representations of the $\mathfrak{su}(2,2|1)$ superconformal algebra. In the above we have $\ell=1$ if $j_1\geqslant\frac{1}{2}$, $\ell=2$ if $j_1=0$, $\bar{\ell}=1$ if $j_2\geqslant\frac{1}{2}$ and $\bar{\ell}=2$ if $j_2=0$. In the first column we ist the notation of [1] and in third column we list the corresponding Dolan & Osborn style notation [2] which was also used in [3, 4, 5, 6].*

Conserved current multiplet(s)	Comment(s)
$\hat{\mathcal{C}}_{(0,0)}$	Flavour current
$\hat{\mathcal{C}}_{(\frac{1}{2},0)}$, $\hat{\mathcal{C}}_{(0,\frac{1}{2})}$	Supersymmetric currents
$\hat{\mathcal{C}}_{(\frac{1}{2},\frac{1}{2})}$	Stress tensor, contains $\mathfrak{u}(1)_r$ current
$\hat{\mathcal{C}}_{(j_1,j_2)}\vert_{j_1+j_2>1}$	Higher spin currents
$\overline{\mathcal{D}}_{(0,0)}$ $(\mathcal{D}_{(0,0)})$	Free (anti-)Chiral field Φ $(\overline{\Phi})$
$\overline{\mathcal{D}}_{(\frac{1}{2},0)}$ $(\mathcal{D}_{(0,\frac{1}{2})})$	Free (anti-)vector superfield W_α $(\overline{W}_{\dot{\alpha}})$
$\overline{\mathcal{D}}_{(j_1\geqslant 1,0)}$, $\mathcal{D}_{(0,j_2\geqslant 1)}$	Higher spin free fields.

Table C.4: *Conserved current multiplets of $\mathfrak{su}(2,2|1)$ superconformal algebra.*

Using

$$\mathcal{C}_{r,(-\frac{1}{2},j_2)} \cong \mathcal{B}_{r-1,(0,j_2)}\,, \quad \overline{\mathcal{C}}_{r,(j_1,-\frac{1}{2})} \cong \overline{\mathcal{B}}_{r+1,(j_1,0)}\,, \tag{C.3}$$

the $\mathcal{N}=1$ recombination rules can be written as

$$\mathcal{A}^{2+2j_1-\frac{3}{2}r}_{r<\frac{2}{3}(j_1-j_2),(j_1,j_2)} \cong \mathcal{C}_{r,(j_1,j_2)} \oplus \mathcal{C}_{r-1,(j_1-\frac{1}{2},j_2)}\,, \tag{C.4}$$

$$\mathcal{A}^{2+2j_2+\frac{3}{2}r}_{r>\frac{2}{3}(j_1-j_2),(j_1,j_2)} \cong \overline{\mathcal{C}}_{r,(j_1,j_2)} \oplus \overline{\mathcal{C}}_{r+1,(j_1,j_2-\frac{1}{2})}\,, \tag{C.5}$$

$$\mathcal{A}^{2+j_1+j_2}_{\frac{2}{3}(j_1-j_2),(j_1,j_2)} \cong \hat{\mathcal{C}}_{(j_1,j_2)} \oplus \mathcal{C}_{\frac{2}{3}(j_1-j_2)-1,(j_1-\frac{1}{2},j_2)} \oplus \overline{\mathcal{C}}_{\frac{2}{3}(j_1-j_2)+1,(j_1,j_2-\frac{1}{2})}\,. \tag{C.6}$$

The multiplets $\overline{\mathcal{D}}_{(j_1,0)}$, $\overline{\mathcal{B}}_{r<\frac{2}{3}j_1+2,(j_1,0)}$ and their complex conjugates have no recombination rules and are therefore absolutely protected. We list below the contribution of each multiplet to the right-handed index (1.111)

$$\mathcal{I}_{\mathcal{A}^E_{r,(j_1,j_2)}} = \mathcal{I}_{\mathcal{B}_{r,(0,j_2)}} = \mathcal{I}_{\mathcal{C}_{r,(j_1,j_2)}} = 0\,, \tag{C.7}$$

$$\mathcal{I}_{\hat{\mathcal{C}}_{(j_1,j_2)}} = \frac{(-1)^{2j_1+2j_2+1}(pq)^{\frac{2}{3}j_2+\frac{1}{3}j_1+1}}{(1-p)(1-q)}\chi_{2j_1}(\sqrt{p/q})\,, \tag{C.8}$$

$$\mathcal{I}_{\overline{\mathcal{C}}_{r,(j_1,j_2)}} = \frac{(-1)^{2j_1+2j_2+1}(pq)^{\frac{r}{2}+j_2+1}}{(1-p)(1-q)}\chi_{2j_1}(\sqrt{p/q})\,, \tag{C.9}$$

$$\mathcal{I}_{\overline{\mathcal{B}}_{r,(j_1,0)}} = \frac{(-1)^{2j_1}(pq)^{\frac{r}{2}}}{(1-p)(1-q)}\chi_{2j_1}(\sqrt{p/q}) \tag{C.10}$$

$$\mathcal{I}_{\mathcal{D}_{(0,j_2)}} = \frac{(-1)^{2j_2+1}(pq)^{\frac{2}{3}j_2+\frac{2}{3}}}{(1-p)(1-q)}\,, \tag{C.11}$$

$$\mathcal{I}_{\overline{\mathcal{D}}_{(j_1,0)}} = \frac{(-1)^{2j_1}(pq)^{\frac{j_1+1}{3}}\left(\chi_{2j_1}(\sqrt{p/q}) - \sqrt{pq}\chi_{2j_1-1}(\sqrt{p/q})\right)}{(1-p)(1-q)}\,, \tag{C.12}$$

where the character of the spin-$\frac{s}{2}$ representation of $SU(2)$ is
$\chi_s(y) = (y^{s+1} - y^{-s-1})/(y - y^{-1})$..

By construction the right-handed index of all multiplets of the type $X\overline{L}[j_1,j_2]^{(r)}_E$ is zero. In computing the above one must carefully deal with equations of motion. If any given $\mathfrak{so}(4,2)$ representation appearing in a multiplet saturates the unitarity bound then there will be a corresponding equation of motion which must enter the index with opposite statistics. We list the the possible null states of $\mathfrak{so}(4,2)$ in Table C.5.

Primary	Unitarity bound	Null state
$[j_1 \geqslant 1/2, j_2 \geqslant 1/2]_E$	$E \geqslant 2 + j_1 + j_2$	$[j_1 - 1/2, j_2 - 1/2]_{E+1}$
$[j_1 \geqslant 1/2, 0]_E$	$E \geqslant j_1 + 1$	$[j_1 - 1/2, 1/2]_{E+1}$
$[0, j_2 \geqslant 1/2]_E$	$E \geqslant j_2 + 1$	$[1/2, j_2 - 1/2]_{E+1}$
$[0, 0]_E$	$E \geqslant 1$	$[0, 0]_{E+2}$
$[0, 0]_E$	$E = 0$	$[1/2, 1/2]_{E+1}$

Table C.5: *Unitary representations of* $\mathfrak{so}(4,2)$*. In the final column we list the associated null state when the unitarity bound is saturated. The null states correspond to equations of motion and as such their contribution must be subtracted from the index.*

C.1.1 $\mathcal{N} = 1$ Index Equivalence Classes

By either examining the recombination rules (C.4)-(C.6), or, by directly observing the contribution to the index from each multiplet (C.7)-(C.12) we can immediately read off the (left-)right-handed index equivalence classes [166, 4, 167]. That is, the set of multiplets with equal contribution to the (left-)right-handed index. Here we focus only on the right-handed index. The equivalence classes (leaving implicit the quantum number inequalities of Table C.3) are

$$[\widetilde{r}, j_1]_+ := \{\overline{\mathcal{C}}_{r,(j_1,\frac{\widetilde{r}-r}{2})} | \widetilde{r} - r \in 2\mathbb{Z}_{\geqslant 0}\} \cup \{\hat{\mathcal{C}}_{(j_1,\frac{3\widetilde{r}-2j_1}{4})} | 3\widetilde{r} - 2j_1 \in 4\mathbb{Z}_{\geqslant 0}\} \tag{C.13}$$

$$\begin{aligned}[\widetilde{r}, j_1]_- :=& \{\overline{\mathcal{C}}_{r,(j_1,\frac{\widetilde{r}-r}{2})} | \widetilde{r} - r \in -1 + 2\mathbb{Z}_{\geqslant 0}\} \\ &\cup \{\hat{\mathcal{C}}_{(j_1,\frac{3\widetilde{r}-2j_1}{4})} | 3\widetilde{r} - 2j_1 \in 2 + 4\mathbb{Z}_{\geqslant 0}\}\end{aligned} \tag{C.14}$$

and their contributions to the index are

$$\mathcal{I}_{[\widetilde{r},j_1]_+} = -\mathcal{I}_{[\widetilde{r},j_1]_-} = \frac{(-1)^{2j_1+1}(pq)^{\frac{\widetilde{r}}{2}+1}}{(1-p)(1-q)} \chi_{2j_1}(\sqrt{p/q})\,. \tag{C.15}$$

The cases in which we can extract the most information regarding the spectrum from the index are those in which the equivalence class contains a small number of representatives. For example, for example if $\widetilde{r} \in (2j_1/3, 4/3 + 2j_1/3]$ then $[\widetilde{r}, j_1]_+$ is empty and $[\widetilde{r}, j_1]_-$ can contain only $\overline{\mathcal{B}}_{\widetilde{r}-1,(j_1,0)}$. The multiplets $\mathcal{D}_{(0,j_2)}$ and $\overline{\mathcal{D}}_{(j_1,0)}$ are free fields and sit in their own equivalence classes. Finally the multiplets $\hat{\mathcal{C}}_{(\frac{1}{2},0)}$ and $\hat{\mathcal{C}}_{(0,\frac{1}{2})}$ are the only representatives within the classes $[\frac{1}{3}, \frac{1}{2}]_+$ and $[\frac{2}{3}, 0]_-$, respectively. $[\frac{1}{3}, \frac{1}{2}]_-$ and $[\frac{2}{3}, 0]_+$ also contain only a single representatives, being $\overline{\mathcal{B}}_{\frac{7}{3}(\frac{1}{2},0)}$ and

Multiplet	q_t bound
$\overline{\mathcal{B}}_{r,(j_1,0)}$	$\frac{r}{2} - \frac{2}{3}q_t \geqslant j_1$
$\overline{\mathcal{C}}_{r,(j_1,j_2)}$	$\frac{r}{2} + j_2 + 1 - \frac{2}{3}q_t \geqslant j_1$
$\hat{\mathcal{C}}_{(j_1,j_2)}$	$1 - \frac{2}{3}q_t \geqslant \frac{2}{3}(j_1 - j_2)$
$\mathcal{D}_{(0,j_2)}$	$j_2 + 1 - q_t \geqslant 0$
$\overline{\mathcal{D}}_{(j_1,0)}$	$-2j_1 + 1 - q_t \geqslant 0$

Table C.6: *Restrictions imposed on the $\mathfrak{u}(1)_t$ representations implied by the existence of the Hall-Littlewood limit of the index. In order that a multiplet contributes to the Hall-Littlewood index it must have $j_1 = 0$ and saturate the bound.*

$\overline{\mathcal{C}}_{\frac{2}{3}(0,0)}$ respectively. We also define the *net degeneracy*

$$\mathrm{ND}[\tilde{r}, j_2] := \#[\tilde{r}, j_2]_+ - \#[\tilde{r}, j_2]_- \,. \tag{C.16}$$

C.1.2 Hall-Littlewood Limit

The indices (C.7)-(C.12) can of course enter into the character expansion of (2.23) with factors of $(\tau/\rho\sigma)^{2q_t/3} = (t/(pq)^{2/3})^{q_t}$. By construction the Hall-Littlewood limit of the index (2.37) counts only those operators with $2j_2 = -r + \frac{4}{3}q_t$ and $j_1 = 0$. Assuming that this limit always exists for theories in class $\mathcal{S}_k$ we can extract bounds for the value of $\mathfrak{u}(1)_t$ charges for given multiplets appearing the character expansion of the index. In particular, using the fact that

$$\lim_{\sigma\to 0}\lim_{\rho\to 0}(\sigma\rho)^a \chi_{2j_1}\left(\sqrt{\sigma/\rho}\right) = \begin{cases} \delta_{j_1,0} & a = j_1\,, \\ 0 & a > j_1\,, \\ \infty & a < j_1\,, \end{cases} \tag{C.17}$$

with $j_1 \geqslant 0$ so that the limit exists only if $a \geqslant j_1$. Therefore, from (C.7)-(C.12) one can see that the the $\mathfrak{u}(1)_t$ charges of the multiplets contributing to the right handed index must obey the constraints of Table C.6. By appying conjugation it is possible to find similar bounds for multiplets appearing in the character expansion of the left-handed index. One may also repeat such an exercise with the Macdonald limit ($\sigma \to 0$) of the index. Defining $\mathrm{HL}_{\mathcal{S}^{(q_t)}} = \lim_{\sigma,\rho\to 0}(\tau/\sigma\rho)^{2q_t/3}\,\mathcal{I}_{\mathcal{S}}$ and assuming that the bounds of Table C.6 are satisfied (such that the limit always always exists) we have

$$\mathrm{HL}_{\mathcal{A}^{E,(q_t)}_{r,(j_1,j_2)}} = \mathrm{HL}_{\mathcal{B}^{(q_t)}_{r,(0,j_2)}} = \mathrm{HL}_{\mathcal{C}^{(q_t)}_{r,(j_1,j_2)}} = 0\,, \tag{C.18}$$

$$\mathrm{HL}_{\overline{\mathcal{B}}^{(q_t)}_{r,(j_1,0)}} = \tau^{2q_t}\delta_{j_1,0}\delta_{\frac{3r}{4},q_t}\,, \tag{C.19}$$

$$\mathrm{HL}_{\overline{\mathcal{C}}^{(q_t)}_{r,(j_1,j_2)}} = (-1)^{2j_2+1}\tau^{2q_t}\delta_{j_1,0}\delta_{q_t,\frac{3}{2}(j_2+1)+\frac{3}{4}r}\,, \tag{C.20}$$

$$\mathrm{HL}_{\hat{\mathcal{C}}^{(q_t)}_{(j_1,j_2)}} = (-1)^{2j_2+1}\tau^{2q_t}\delta_{j_1,0}\delta_{q_t,j_2+\frac{3}{2}}\,, \tag{C.21}$$

$$\mathrm{HL}_{\mathcal{D}^{(q_t)}_{(0,j_2)}} = (-1)^{2j_2+1}\tau^{2q_t}\delta_{q_t,j_2+1}\,,\quad \mathrm{HL}_{\overline{\mathcal{D}}^{(q_t)}_{(j_1,0)}} = \tau\delta_{j_1,0}\delta_{q_t,\frac{1}{2}}\,. \tag{C.22}$$

so the only multiplets that can contribute to the right-handed index in the Hall-Littlewood limit are

$$\overline{\mathcal{B}}^{(\frac{4r}{3})}_{r,(0,0)}\,,\quad \overline{\mathcal{C}}^{(\frac{3}{2}(j_2+1)+\frac{3}{4}r)}_{r,(0,j_2)}\,,\quad \hat{\mathcal{C}}^{(j_2+\frac{3}{2})}_{(0,j_2)}\,,\quad \mathcal{D}^{(j_2+1)}_{(0,j_2)}\,,\quad \overline{\mathcal{D}}^{(\frac{1}{2})}_{(0,0)}\,. \tag{C.23}$$

Equality of the Hall-Littlewood limit of the index with the Hilbert series of the Higgs branch at would imply that in those theories the Hall-Littlewood index receives contribution only from $\overline{\mathcal{D}}^{(\frac{1}{2})}_{(0,0)}$ and $\overline{\mathcal{B}}^{(\frac{4r}{3})}_{r,(0,0)}$ multiplets.

C.2 Unrefined HL Index for Interacting Trinions

C.2.1 T_B Theory

The Hall-Littlewood index for the three-punctured sphere T_B theory in the unrefined $\mathbf{z} = \mathbf{u} = \mathbf{v} = \mathbf{1}$, $\gamma\beta = 1$ limit is given by

$$\mathrm{HL}^{(T_B)\mathbf{1}}_{\mathbf{11}}\Big|_{\gamma\beta=1} = \frac{\gamma^2}{\beta^2}\frac{\tau^{19}\left(Q_B(\gamma^{-1}\beta,\tau^{-1}) - Q_B(\gamma\beta^{-1},\tau)\right)}{\left(1-\frac{\beta}{\gamma}\tau^2\right)^7\left(1-\frac{\gamma}{\beta}\tau^2\right)^9(1-\tau^2)^{11}}\,. \tag{C.24}$$

The polynomial $Q_B(\gamma\beta^{-1},\tau)$ is given by

$$\begin{aligned}
Q_B(\gamma\beta^{-1},\tau) &= -\frac{3\gamma^5\tau^9}{\beta^5}-\frac{15\gamma^5\tau^7}{\beta^5}-\frac{17\gamma^5\tau^5}{\beta^5}-\frac{5\gamma^5\tau^3}{\beta^5}+\frac{21\gamma^4\tau^{11}}{\beta^4}\\
&-\frac{14\gamma^4\tau^7}{\beta^4}-\frac{6\gamma^4\tau^5}{\beta^4}+\frac{201\gamma^4\tau^3}{\beta^4}+\frac{45\gamma^4\tau}{\beta^4}+\frac{\gamma^3\tau^{17}}{\beta^3}-\frac{3\gamma^3\tau^{15}}{\beta^3}\\
&-\frac{55\gamma^3\tau^{11}}{\beta^3}+\frac{305\gamma^3\tau^9}{\beta^3}+\frac{\beta^3\tau^9}{\gamma^3}-\frac{387\gamma^3\tau^7}{\beta^3}-\frac{11\beta^3\tau^7}{\gamma^3}-\frac{885\gamma^3\tau^5}{\beta^3}\\
&+\frac{919\gamma^3\tau^3}{\beta^3}-\frac{265\beta^3\tau^3}{\gamma^3}-\frac{298\gamma^3\tau}{\beta^3}+\frac{818\beta^3\tau}{\gamma^3}+\frac{\gamma^2\tau^{19}}{\beta^2}+\frac{13\gamma^2\tau^{17}}{\beta^2}\\
&-\frac{231\gamma^2\tau^{11}}{\beta^2}+\frac{15\beta^2\tau^{11}}{\gamma^2}+\frac{1421\gamma^2\tau^9}{\beta^2}-\frac{149\beta^2\tau^9}{\gamma^2}-\frac{132\gamma^2\tau^7}{\beta^2}\\
&-\frac{714\beta^2\tau^5}{\gamma^2}+\frac{4332\gamma^2\tau^3}{\beta^2}-\frac{2327\beta^2\tau^3}{\gamma^2}+\frac{1484\gamma^2\tau}{\beta^2}+\frac{4101\beta^2\tau}{\gamma^2}\\
&-\frac{193\gamma\tau^{13}}{\beta}+\frac{52\beta\tau^{13}}{\gamma}-\frac{805\gamma\tau^{11}}{\beta}-\frac{308\beta\tau^{11}}{\gamma}+\frac{2208\gamma\tau^9}{\beta}+\frac{81\beta\tau^9}{\gamma}\\
&+\frac{2933\beta\tau^7}{\gamma}-\frac{8621\gamma\tau^5}{\beta}-\frac{4192\beta\tau^5}{\gamma}+\frac{5807\gamma\tau^3}{\beta}-\frac{2504\beta\tau^3}{\gamma}\\
&+52\tau^{15}-36\tau^{13}-982\tau^{11}+1322\tau^9+3423\tau^7-9605\tau^5+1849\tau^3\\
&+\frac{1224\gamma\tau^7}{\beta}+14421\tau+\frac{12360\beta\tau}{\gamma}-\frac{156\gamma^2\tau^{13}}{\beta^2}+\frac{83\gamma\tau^{15}}{\beta}\\
&+\frac{73\gamma^4\tau^9}{\beta^4}+\frac{36\gamma^2\tau^{15}}{\beta^2}-\frac{67\gamma^3\tau^{13}}{\beta^3}-\frac{3340\gamma^2\tau^5}{\beta^2}+\frac{582\beta^2\tau^7}{\gamma^2}+\frac{67\beta^3\tau^5}{\gamma^3}\\
&+\frac{7024\gamma\tau}{\beta}+\frac{15\gamma\tau^{17}}{\beta}\,.
\end{aligned} \tag{C.25}$$

C.2.2 T_A Theory

The Hall-Littlewood index for the T_A theory in the unrefined limit with $\gamma=\beta$ reads

$$\mathrm{HL}^{(T_A)\mathbf{1}}_{\mathbf{11}}\Big|_{\frac{\gamma}{\beta}=1} = \frac{\beta^5\gamma^5\tau^{15}\left(Q_A(\tau,\beta\gamma)-Q_A(\tau^{-1},\beta^{-1}\gamma^{-1})\right)}{(1-\tau^2)^{19}(1-\gamma^4\beta^4\tau^2)^4} \tag{C.26}$$

where

$$
\begin{aligned}
Q_A(\tau,\beta\gamma) = & -\beta^5\gamma^5\tau^{15} - 29\beta^5\gamma^5\tau^{13} - 274\beta^5\gamma^5\tau^{11} - 1122\beta^5\gamma^5\tau^9 \\
& - 2222\beta^5\gamma^5\tau^5 - 1122\beta^5\gamma^5\tau^3 + \frac{\tau^3}{\beta^5\gamma^5} - 274\beta^5\gamma^5\tau + \frac{29\tau}{\beta^5\gamma^5} \\
& + 525\beta^3\gamma^3\tau^9 + 4216\beta^3\gamma^3\tau^7 + 11048\beta^3\gamma^3\tau^5 - \frac{4\tau^5}{\beta^3\gamma^3} + 12523\beta^3\gamma^3\tau^3 \\
& + 6519\beta^3\gamma^3\tau - \frac{1524\tau}{\beta^3\gamma^3} - \beta\gamma\tau^{11} + 15\beta\gamma\tau^9 - 350\beta\gamma\tau^7 + \frac{6\tau^7}{\beta\gamma} \\
& + \frac{286\tau^5}{\beta\gamma} - 19741\beta\gamma\tau^3 + \frac{3706\tau^3}{\beta\gamma} - 27485\beta\gamma\tau + \frac{15982\tau}{\beta\gamma} \\
& - 19\beta^3\gamma^3\tau^{11} - \frac{144\tau^3}{\beta^3\gamma^3} - 5418\beta\gamma\tau^5 - 4\beta^3\gamma^3\tau^{13} - 2222\beta^5\gamma^5\tau^7\,.
\end{aligned}
\tag{C.27}
$$

Appendix D

Appendices for Chapter 3

D.1 Quartic and Weierstrass Forms

The explicit transformations for transforming from Quartic to Weierstrass forms are given in [293] Section 1.2. Here is an explicit check for our case. Given the curve (3.11) define

$$z = \frac{\tilde{x} + 2\gamma}{4}, \quad w = \frac{\tilde{y}}{8} \quad , \gamma := -u + \Lambda_1^4 + \Lambda_2^4 \tag{D.1}$$

gives

$$w^2 = z^3 + 2\gamma z^2 - 4(\gamma^2 - 4\Lambda_1^4\Lambda_2^4(z+2)) \tag{D.2}$$

Then, let

$$w = \frac{4}{x^3}(\gamma^2 - 4\Lambda_1^4\Lambda_2^4)(y + \sqrt{\gamma^2 - 4\Lambda_1^4\Lambda_2^4} - ux^2) \tag{D.3}$$

$$z = \frac{2}{x^2}\sqrt{\gamma^2 - 4\Lambda_1^4\Lambda_2^4}(y + \sqrt{\gamma^2 - 4\Lambda_1^4\Lambda_2^4}) \tag{D.4}$$

Plugging into (3.11) yields the quadratic form for the curve [192]

$$y^2 = (x^2 - u + \Lambda_1^4 + \Lambda_2^4)^2 - 4\Lambda_1^4\Lambda_2^4 \,. \tag{D.5}$$

D.2 Anomaly Free R-Symmetry

Here we verify that the $\mathfrak{u}(1)_r$ R-symmetry for the core theories in class $\mathcal{S}_k$ are anomaly free. The 1-loop running of the holomorphic UV coupling at the (i,n)th node is

$$\tau_{(i,n)}(E) = \tau_{(i,n),\mathrm{UV}} - \frac{b_{(i,c)}}{2\pi i} \log \frac{E}{\Lambda_{(i,c),\mathrm{UV}}} \,. \tag{D.6}$$

From the point of view of the $(i,n)^{\mathrm{th}}$ gauge group, there are $3N$ chiral multiplets in the fundamental representation $\mathbf{N}$ and $3N$ in the anti-fundamental $\overline{\mathbf{N}}$ and therefore

$$\begin{aligned} b_{(i,n)} =& E\frac{d}{dE}\frac{8\pi^2}{g^2_{(i,n)}} && \text{(D.7)} \\ =& 3c_2(\text{adj.}) - 3Nc_2(\mathbf{N}) - 3Nc_2(\overline{\mathbf{N}}) && \text{(D.8)} \\ =& 0 && \text{(D.9)} \end{aligned}$$

where $c_2(\mathcal{R})$ is the Dynkin index of a representation $\mathcal{R}$ of a compact, semi-simple Lie algebra. For $SU(N)$, $c_2(\mathbf{N}) = c_2(\overline{\mathbf{N}}) = 1/2$ and $c_2(\text{adj.}) = N$.

Suppose that under some combination of the $U(1)$ symmetries of the Lagrangian the gauginos and the fermion parts of matter fields coupled to the gauge group labelled by (i,n) transform as

$$\begin{aligned} \Phi_{(i,n)}|_\theta \to& e^{\mathrm{i}\alpha_{(i,n)}}\Phi_{(i,n)}|_\theta && \text{(D.10)} \\ Q_{(i,n)}|_\theta \to& e^{\mathrm{i}\beta_{(i,n)}}Q_{(i,n)}|_\theta && \text{(D.11)} \\ \widetilde{Q}_{(i,n)}|_\theta \to& e^{\mathrm{i}\gamma_{(i,n)}}\widetilde{Q}_{(i,n)}|_\theta && \text{(D.12)} \\ \lambda^\alpha_{(i,n)} \to& e^{\mathrm{i}\sigma_{(i,n)}}\lambda^\alpha_{(i,n)} && \text{(D.13)} \end{aligned}$$

Where $|_\theta = \frac{\partial}{\partial\theta}|_{\theta=0}$ denotes the projection onto the theta component. Then, the fermionic part of the path integral measure transforms, due to the zero modes, as

$$\begin{aligned} &\prod_{n=1}^{\ell-2}\prod_{i=1}^{k}[\mathcal{D}\lambda^\alpha_{(i,n)}\mathcal{D}\Phi_{(i,n)}|_\theta\mathcal{D}\widetilde{Q}_{(i,n)}|_\theta\mathcal{D}Q_{(i,n)}|_\theta] \to \\ &\qquad \prod_{n=1}^{\ell-2}\prod_{i=1}^{k} e^{2\mathrm{i}f_{(i,n)}K_{(i,n)}}[\mathcal{D}\lambda^\alpha_{(i,n)}\mathcal{D}\Phi_{(i,n)}|_\theta\mathcal{D}\widetilde{Q}_{(i,n)}|_\theta\mathcal{D}Q_{(i,n)}|_\theta] \end{aligned} \tag{D.14}$$

where

$$\begin{aligned} f_{(i,n)} =& \sigma_{(i,n)}c_2(\text{adj.}) + (\alpha_{(i,n)} + \beta_{(i,n-1)} + \gamma_{(i,n)})Nc_2(\mathbf{N}) \\ &+ (\alpha_{(i-1,n)} + \beta_{(i,n-1)} + \gamma_{(i-1,n-1)})Nc_2(\overline{\mathbf{N}}) \end{aligned} \tag{D.15}$$

and

$$K_{(i,n)} = \frac{1}{16\pi^2}\int_{\mathbb{R}^4} \mathrm{tr}\, F_{(i,n)} \wedge F_{(i,n)} \in \mathbb{Z}\,, \tag{D.16}$$

is the instanton number.

Therefore, a generic $U(1)$ symmetry is anomalously broken by a non-zero instanton number. Note that it can be compensated for by shifting the theta angle

$$\theta_{(i,n)} \to \theta_{(i,n)} + 2f_{(i,n)} \,. \tag{D.17}$$

This is only a symmetry when $2f_{(i,n)} \in 2\pi\mathbb{Z}$.

The $U(1)_r$ symmetry is anomaly free since

$$r[\Phi_{(i,n)}] = 0 = 1 + r[\Phi_{(i,n)}|_\theta] \,, \tag{D.18}$$
$$r[\widetilde{Q}_{(i,n)}] = 1 = 1 + r[\widetilde{Q}_{(i,n)}|_\theta] \,, \tag{D.19}$$
$$r[Q_{(i,n)}] = 1 = 1 + r[Q_{(i,n)}|_\theta] \,, \tag{D.20}$$
$$r[\lambda^\alpha_{(i,n)}] = 1 \,, \tag{D.21}$$

hence, plugging these charges into $f_{(i,n)}$ yields

$$f_{(i,n)} = N - \frac{N}{2} - \frac{N}{2} = 0 \,. \tag{D.22}$$

D.3 Some Elliptic Curve Theory

Given a compact Riemann surface $\mathcal{X}$ of genus g. We can consider the space of closed holomorphic 1-forms $\Omega^{1,0}(\mathcal{X})$ [294]. By the maximum principle all exact holomorphic one forms are identically zero. One can show that $\dim \Omega^{1,0}(\mathcal{X}) = \dim \Omega^{0,1}(\mathcal{X}) = g$ where $\Omega^{0,1}(\mathcal{X})$ is the space of antiholomorphic closed one forms, $\Omega^{1,0}(\mathcal{X}) \cap \Omega^{0,1}(\mathcal{X}) = 0$ and consequently that $H^1(\mathcal{X}, \mathbb{C}) = \Omega^{1,0}(\mathcal{X}) \oplus \Omega^{0,1}(\mathcal{X})$

One can also consider the meromorphic 1-forms on $\mathcal{X}$, for any non-zero meromorphic 1-form λ we have that $\sum_{P \in \mathcal{X}} \mathrm{ord}_P(\lambda) = 2g - 2$ as a consequence of the Riemann-Hurwitz formula, also, $\sum_{P \in \mathcal{X}} \mathrm{Res}_P(\omega) = 0$.

Choosing a basis for the homology as $\{A_1, \ldots, A_g, B_1, \ldots, B_g\}$, because the is a natural isomorphism $H^1(\mathcal{X}, \mathbb{C}) = \Omega^{1,0}(\mathcal{X}) \oplus \Omega^{0,1}(\mathcal{X}) \cong \mathrm{Hom}(H_1\mathcal{X}, \mathbb{C})$ provided by integration over one-cycles, there then exists a unique, linearly independent, basis $\{\omega_1, \ldots, \omega_g\}$ for $\Omega^{1,0}(\mathcal{X})$ such that

$$\int_{A_j} \omega_i = \delta_{ji} \,, \quad \int_{B_j} \omega_i = \tau_{ji} \,. \tag{D.23}$$

The period matrix is symmetric $\tau_{ji} = \tau_{ij}$ and obeys $\Im\tau > 0$ (by Riemann's bilinear relations). Explicitly, for a hyperelliptic curve $y^2 = F(x, \dots)$ where, considered as a polynomial in x, $\deg F = 2N = 2g + 2$, an explicit basis is given by $\omega_i = x^{i-1} dx/y$.

Now define a mapping $f : H_1\mathcal{X} \hookrightarrow \Lambda \subset \mathbb{C}^g$ given by $\gamma \mapsto (\int_\gamma \omega_1, \dots, \int_\gamma \omega_g)$ which is an embedding of $H_1\mathcal{X}$ into the lattice Λ.

We now focus our attention to the case $g = 1$. Then $\mathcal{X} = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ and the pair of cycles $A_1 = [0, 1]$ and $B_1 = [0, \tau]$ give a basis for $H_1\mathcal{X}$. Such a Riemann surface can be thought of as a two-sheeted covering of $\mathbb{S}^2$ with 4-branch points, by a Möbius transformation these branch points can be taken to be at $x = 0, 1, \infty, q$. We can then take the curve $y^2 = x(x - 1)(x - q)$ then $\omega_1 = \omega = dx/y$ is the unique (up to a constant multiple) holomorphic 1-form on $\mathcal{X}$ and hence satisfies all of the above discussions. How is q related to the complex structure τ of the torus? To answer this question we rewrite the curve as $y^2 = \prod_{i=1}^{3}(x - e_i)$ because the discriminant $\Delta \neq 0$ then $e_1 + e_2 + e_3 = 0$. Then $q = \frac{e_2 - e_3}{e_1 - e_3}$. After computing the periods (D.23) and inverting one finds that

$$q = \lambda(\tau) = \frac{\theta_2(\tau)^4}{\theta_3(\tau)^4}\,. \tag{D.24}$$

$\lambda(\tau)$ is called the modular lambda function.

For hyperelliptic curves, which can be realised as two-sheeted branch coverings of $\mathbb{S}^2$, they have $2g+2$ branch points and explicit computations of the period matrix are harder to perform.

Relation to gauge theory In the Seiberg-Witten theory of the $\mathcal{N} = 2$ gauge theory we have

- The SW curve $\mathfrak{X} = \mathcal{X}_{\{u_i\}}$ which is a rank $\mathfrak{g}$ cover of a Riemann surface $\mathcal{X}_{\{u_i\}} \xrightarrow{\text{rank}\,\mathfrak{g}:1} \mathcal{C}$. The period matrix of $\mathcal{X}_{\{u_i\}}$, in turn, is identified with the IR $U(1)$ gauge couplings and therefore computes the pre-potential $\mathcal{F}$. For a $SU(N)$ $\mathcal{N} = 2$ gauge theory, we have the following relations

$$\frac{\partial a_i^D}{\partial a^j} = \tau_{ij} \tag{D.25}$$

  we then define a_i and a_i^D as integrals of the A_i and B_i cycles respectively over a certain meromorphic differential $\lambda_{\text{SW}} \sim \lambda_{\text{SW}} + df$ since the addition of an exact form df does not affect the integration under closed one-cycles.

$$a^i = \int_{A_i} \lambda_{\mathrm{SW}}\,, \quad a_i^D = \int_{B_i} \lambda_{\mathrm{SW}}\,. \tag{D.26}$$

and we demand

$$\frac{\partial \lambda_{\mathrm{SW}}}{\partial u_i}|_{\text{fixed } x} = \omega_i + df_i \tag{D.27}$$

furthermore we require that for all $p \in X$ $\mathrm{Res}_p \lambda_{\mathrm{SW}}$ is at most linear in the quark bare masses and that $\sum_{p \in X} \mathrm{Res}_p \lambda_{\mathrm{SW}} = 0$.

- For theories of class $\mathcal{S}$ the space of UV gauge couplings is identified with the space of complex structure deformations $\mathcal{E}$ of the underlying surface $\mathcal{C}$. We have an isomorphism

$$\mathcal{E} \cong Teich(\mathcal{C})/MCG(\mathcal{C}) \tag{D.28}$$

where $Teich(\mathcal{C})$ is the Teichmüller space of $\mathcal{C}$ and is parametrised by the same cross ratios q appearing in the SW curve. $MCG(\mathcal{C})$ is the mapping class group of $\mathcal{C}$, in physics terms this is the 'generalised S-duality group'.

D.3.1 Computation of the Period Matrix

We begin with the SW-curve for the pure $\mathfrak{su}(2)$ $\mathcal{N} = 2$ theory

$$y^2 = (x - \Lambda^2)(x + \Lambda^2)(x - u)\,. \tag{D.29}$$

The curve becomes singular at $u = -\Lambda^2, +\Lambda^2, \infty$. The B-cycle is taken to enclose $-\Lambda^2, +\Lambda^2$ while the A-cycle is taken from $+\Lambda^2, u$. We first change coordinates by $x = \Lambda^2(2x' + 1)$ and $y = (2\Lambda^2)^{3/2} y'$

$$y'^2 = x'(x' - 1)\left(x' - \frac{\Lambda^2 + u}{2\Lambda^2}\right) \tag{D.30}$$

Defining $q = \frac{\Lambda^2 + u}{2\Lambda^2}$ we arrive at the desired form

$$y^2 = x(x - 1)(x - q)\,. \tag{D.31}$$

In these coordinates the singular points of the curve are mapped to $q = 0, 1, \infty$. So we have to compute integrals of the form

$$p(\gamma) = \oint_{\gamma} \omega = \oint_{\gamma} \frac{dx}{\sqrt{x(x-1)(x-q)}} \,. \tag{D.32}$$

The we deform the cycle A to enclose $1, q$ and B to enclose $0, 1$. So we can write

$$p(A) = 2\int_1^q \omega + \oint_{C_1} \omega + \oint_{C_q} \omega \,, \quad p(B) = 2\int_0^1 \omega + \oint_{C_0} \omega + \oint_{C_1} \omega \tag{D.33}$$

where C_{x_0} denotes sphere centred around the point $x = x_0$ with infinitesimal radius ϵ. However, one can immediately see that

$$\oint_{C_p} \omega \sim \int_{-\pi}^{\pi} \frac{\mathrm{i}\sqrt{\epsilon} e^{\mathrm{i}\theta/2} d\theta}{\sqrt{\prod_{p' \in \{0,1,q\} \setminus \{p\}} (p - p')}} = \frac{4\mathrm{i}\sqrt{\epsilon}}{\sqrt{\prod_{p' \in \{0,1,q\} \setminus \{p\}} (p - p')}} \to 0 \tag{D.34}$$

for any $p \in \{0, 1, q\}$. Writing $x = w^2$

$$\int_0^p \frac{dx}{\sqrt{x(x-1)(x-q)}} = \int_0^{\sqrt{p}} \frac{2dw}{\sqrt{(w^2-1)(w^2-q)}} = \frac{2}{\sqrt{q}} F(\sqrt{p}; 1/\sqrt{q}) \tag{D.35}$$

where $F(x; k)$ is the incomplete elliptic integral of the first kind in Jacobi form. We can therefore write

$$p(A) = \frac{4}{\sqrt{q}} \left(F(\sqrt{q}; 1/\sqrt{q}) - F(1; 1/\sqrt{q}) \right) \,, \quad p(B) = \frac{4}{\sqrt{q}} F(1; 1/\sqrt{q}) \,. \tag{D.36}$$

Therefore the period matrix is

$$\tau = \frac{p(A)}{p(B)} = \frac{F(\sqrt{q}; 1/\sqrt{q})}{F(1; 1/\sqrt{q})} - 1 \,. \tag{D.37}$$

Appendix E

Appendices for Chapter 4

E.1 The Elliptic Genus Computation

In this appendix we review the computation of the Elliptic genus via letter counting.

E.1.1 Single Letter Indices

As the Witten index is independent of coupling constants we may compute the index in the free field $g \to 0$ limit. To compute the index we list the gauge covariant field content with $\delta = \overline{L}_0 - \frac{1}{2}\overline{J}_0 = 0$ in the UV. Only the 'letters' with $\delta = 0$ contribute to the index and their quantum numbers are listed in Tables E.1, E.2 and E.3. We denote also the decomposition of $\mathcal{N} = (4,4)$ multiplets into $\mathcal{N} = (0,2)$ multiplets. Since the $\mathcal{N} = (0,2)$ field strength multiplet is not conformal extra care must be taken to take the free field limit. In two dimensions the field strength multiplet Υ is nothing but a Fermi multiplet with auxillary $D - \mathrm{i}F_{+-}$. The R-charge of Υ is fixed to unity everywhere along the flow, i.e. $\mathfrak{R}\,[\Upsilon] = \overline{J}_0\,[\Upsilon] = 1$. Therefore the index of the off-diagonal vector multiplet should be equal to that of a off-diagonal Fermi multiplet of R-charge $\mathfrak{R}_{\mathrm{IR}} = \mathfrak{R}_{\mathrm{UV}} = 1$:

$$Z_{\mathrm{vec}}(y_I \neq y_J, q) = \Delta(y)^{-1} Z_{\mathrm{Fermi}}(y_I \neq y_J, q) = \mathrm{PE}\left[-\frac{2q}{1-q}\sum_{I\neq J}\frac{y_I}{y_J}\right] \tag{E.1}$$

where $\Delta(y)$ is the Vandermonde determinant accounting for the Cartan zero modes.

The single letter indices were given in equations (4.23), (4.24) and (4.25). We

$\mathcal{N}=(0,2)$	Letter	L_0	$\overline{L}_0$	$\overline{J}_0$	$2j_1$	$2j_2$	$2j_L$	$2j_R$	Index
$Y, Y^\dagger$	$Y^{2\dot{2}}$	0	0	0	0	0	$+1$	$+1$	wz
	$Y^{1\dot{1}}$	0	0	0	0	0	-1	-1	$w^{-1}z^{-1}$
	$\overline{\lambda}_+^{\dot{1}1}$	0	$\frac{1}{2}$	$+1$	0	-1	-1	0	$-w^{-1}z^{-1}$
$\tilde{Y}^\dagger, \tilde{Y}$	$Y^{1\dot{2}}$	0	0	0	0	0	-1	$+1$	$w^{-1}z$
	$Y^{2\dot{1}}$	0	0	0	0	0	$+1$	-1	wz^{-1}
	$\overline{\lambda}_+^{\dot{1}2}$	0	$\frac{1}{2}$	$+1$	0	-1	$+1$	0	$-wz^{-1}$
$\overline{\xi}, \overline{\xi}^\dagger$	$\overline{\xi}_-^{\dot{1}\dot{1}}$	$\frac{1}{2}$	0	0	0	-1	0	-1	$-qz^{-2}$
	$\overline{\xi}_-^{\dot{2}\dot{2}}$	$\frac{1}{2}$	0	0	0	$+1$	0	$+1$	$-z^2$
$\Upsilon, \Upsilon^\dagger$	$\overline{\xi}_-^{\dot{2}\dot{1}}$	$\frac{1}{2}$	0	$+1$	0	-1	0	-1	$-q$
	$\overline{\xi}_-^{\dot{1}\dot{2}}$	$\frac{1}{2}$	0	-1	0	$+1$	0	$+1$	$-q$
	$\partial_-\overline{\lambda}_+^{\dot{1}1}$	1	$\frac{1}{2}$	$+1$	0	-1	-1	0	$qw^{-1}z^{-1}$
	$\partial_-\overline{\lambda}_+^{\dot{1}2}$	1	$\frac{1}{2}$	$+1$	0	-1	$+1$	0	qwz^{-1}
	∂_-	1	0	0	0	0	0	0	q

Table E.1: *Gauge covariant field content contributing to the index of the* $\mathcal{N}=(4,4)$ *vector multiplet* V.

$\mathcal{N}=(0,2)$	Letter	L_0	$\overline{L}_0$	$\overline{J}_0$	$2j_1$	$2j_2$	$2j_L$	$2j_R$	Index
$X, X^\dagger$	$X^{1\dot{1}}$	0	0	0	-1	-1	0	0	$q^{\frac{1}{2}}v^{-1}z^{-1}$
	$X^{2\dot{2}}$	0	0	0	$+1$	$+1$	0	0	$q^{-\frac{1}{2}}vz$
	$\xi_+^{2\dot{2}}$	0	$\frac{1}{2}$	$+1$	$+1$	0	0	$+1$	$-q^{-\frac{1}{2}}vz$
$\tilde{X}^\dagger, \tilde{X}$	$X^{2\dot{1}}$	0	0	0	$+1$	-1	0	0	$q^{\frac{1}{2}}vz^{-1}$
	$X^{1\dot{2}}$	0	0	0	-1	$+1$	0	0	$q^{-\frac{1}{2}}v^{-1}z$
	$\xi_+^{1\dot{2}}$	0	$\frac{1}{2}$	$+1$	-1	0	0	$+1$	$-q^{-\frac{1}{2}}v^{-1}z$
$\lambda, \lambda^\dagger$	λ_-^{11}	$\frac{1}{2}$	0	0	-1	0	-1	0	$-q^{\frac{1}{2}}v^{-1}w^{-1}$
	$\lambda_{-\frac{1}{2}}^{22}$	$\frac{1}{2}$	0	0	$+1$	0	$+1$	0	$-q^{\frac{1}{2}}vw$
$\tilde{\lambda}^\dagger, \tilde{\lambda}$	λ_-^{12}	$\frac{1}{2}$	0	0	-1	0	$+1$	0	$-q^{\frac{1}{2}}v^{-1}w$
	λ_-^{21}	$\frac{1}{2}$	0	0	$+1$	0	-1	0	$-q^{\frac{1}{2}}vw^{-1}$
	$\partial_-\xi_+^{2\dot{2}}$	1	$\frac{1}{2}$	$+1$	$+1$	0	0	$+1$	$q^{\frac{1}{2}}vz$
	$\partial_-\xi_+^{1\dot{2}}$	1	$\frac{1}{2}$	$+1$	-1	0	0	$+1$	$q^{\frac{1}{2}}v^{-1}z$
	∂_-	1	0	0	0	0	0	0	q

Table E.2: *Gauge covariant field content with $\delta = 0$ of the $\mathcal{N}=(4,4)$ hypermultiplet H.*

$\mathcal{N}=(0,2)$	Letter	L_0	$\overline{L}_0$	$\overline{J}_0$	$2j_1$	$2j_2$	$2j_L$	$2j_R$	Index
$\phi, \phi^\dagger$	$\phi^{\dot{1}}$	0	0	0	0	-1	0	0	$q^{\frac{1}{2}}z^{-1}$
	$\phi^\dagger_{\dot{1}}$	0	0	0	0	$+1$	0	0	$q^{-\frac{1}{2}}z$
	$\chi^\dagger_{+\dot{1}}$	0	$\frac{1}{2}$	$+1$	0	0	0	$+1$	$-q^{-\frac{1}{2}}z$
$\tilde{\phi}^\dagger, \tilde{\phi}$	$\phi^\dagger_{\dot{2}}$	0	0	0	0	-1	0	0	$q^{\frac{1}{2}}z^{-1}$
	$\phi^{\dot{2}}$	0	0	0	0	$+1$	0	0	$q^{-\frac{1}{2}}z$
	$\chi^{\dot{2}}_+$	0	$\frac{1}{2}$	$+1$	0	0	0	$+1$	$-q^{-\frac{1}{2}}z$
$\psi, \psi^\dagger$	ψ^1_-	$\frac{1}{2}$	0	0	0	0	-1	0	$-q^{\frac{1}{2}}w^{-1}$
	$\psi^\dagger_{-1}$	$\frac{1}{2}$	0	0	0	0	1	0	$-q^{\frac{1}{2}}w$
$\tilde{\psi}^\dagger, \tilde{\psi}$	$-\psi^\dagger_{-2}$	$\frac{1}{2}$	0	0	0	0	-1	0	$-q^{\frac{1}{2}}w^{-1}$
	ψ^2_-	$\frac{1}{2}$	0	0	0	0	1	0	$-q^{\frac{1}{2}}w$
	$\partial_-\chi^\dagger_{+\dot{1}}$	1	$\frac{1}{2}$	$+1$	0	0	0	$+1$	$q^{\frac{1}{2}}z$
	$\partial_-\chi^{\dot{2}}_+$	1	$\frac{1}{2}$	$+1$	0	0	0	$+1$	$q^{\frac{1}{2}}z$
	∂_-	1	0	0	0	0	0	0	q

Table E.3: *Gauge covariant field content with $\delta = 0$ of the $\mathcal{N}=(4,4)$ hypermultiplet U.*

again list them here

$$i_V(q,w,z,y_I) = \left[\frac{\left(w+w^{-1}\right)\left(z+qz^{-1}\right) - qz^{-2} - z^2 - 2q}{1-q}\right] \sum_{I,J=1}^{K} y_I y_J^{-1}\,, \tag{E.2}$$

$$i_H(q,v,w,z,y_I) = \left[\frac{q^{\frac{1}{2}}\left(v+v^{-1}\right)\left(z+z^{-1}-w^{-1}-w\right)}{1-q}\right] \sum_{I,J=1}^{K} y_I y_J^{-1}\,, \tag{E.3}$$

$$i_U(q,w,z,x_A,y_I) = \left[\frac{q^{\frac{1}{2}}\left(z+z^{-1}-w^{-1}-w\right)}{1-q}\right] \sum_{I=1}^{K}\sum_{A=1}^{N}\left(\frac{y_I}{x_A}+\frac{x_A}{y_I}\right). \tag{E.4}$$

Finally, we also list the Casimir energy:

$$E_{\text{Casimir}} = \underset{q\to 1}{\text{Finite}}\left[\sum_{\mathcal{M}} \frac{\partial i_{\mathcal{M}}}{\partial \log q}\right] = \frac{\beta_5^2}{\mathrm{i}\pi} 2NK\left(\frac{\epsilon_+}{2}+m\right)\left(\frac{\mathrm{i}\pi}{\beta_5}+\frac{\epsilon_+}{2}-m\right). \tag{E.5}$$

E.1.2 Orbifolded Single Letter Indices

The single letters for the Γ projected multiplets is given by enumerating all letters in Tables E.1, E.2 and E.3 while also inserting fugacities for the Γ action embedded in the global and gauge symmetries. Recall that

$$\gamma_\ell := 2j_L = J_{710} - J_{89}\,, \quad \gamma_k := J_{56} + j_L - j_R = J_{56} - J_{89}\,. \tag{E.6}$$

The projected single letters are thus given by

$$\begin{aligned} i_{\Gamma V}(q,w,z,y_{ni,I}) = \frac{1}{\ell k} &\sum_{\substack{\varepsilon\in\mathbb{Z}_\ell \\ \varepsilon_k\in\mathbb{Z}_k}} \sum_{r\geqslant 0} q^r \varepsilon_k^{-r} \sum_{n,i}^{\ell}\sum_{m,j}^{k}\sum_{I=1}^{K_{ni}}\sum_{I=1}^{K_{mj}} \varepsilon_\ell^{n-m}\varepsilon_k^{i-j} y_{ni,I} y_{mj,J}^{-1} \\ &\times\left[\left(\varepsilon_\ell w + \varepsilon_\ell^{-1}\varepsilon_k^{-1}w^{-1}\right)\left(z+qz^{-1}\right) - qz^{-2} - \varepsilon_k^{-1}z^2 - (\varepsilon_k^{-1}+1)q\right], \end{aligned} \tag{E.7}$$

$$\begin{aligned} i_{\Gamma H}(q,v,w,z,y_{ni,I}) = \frac{1}{\ell k} &\sum_{\substack{\varepsilon\in\mathbb{Z}_\ell \\ \varepsilon_k\in\mathbb{Z}_k}} \left[\left(z^{-1} + \varepsilon_k^{-1}z - \varepsilon_\ell^{-1}\varepsilon_k^{-1}w^{-1} - \varepsilon_\ell w\right)\right] \\ &\times q^{\frac{1}{2}}\left(v+v^{-1}\right)\sum_{r\geqslant 0} q^r\varepsilon_k^{-r} \sum_{n,m}^{\ell}\sum_{i,i}^{k}\sum_{I=1}^{K_{ni}}\sum_{I=1}^{K_{mj}} \varepsilon_\ell^{n-m}\varepsilon_k^{i-j} y_{ni,I} y_{jm,J}^{-1}\,, \end{aligned} \tag{E.8}$$

$$
\begin{aligned}
i_{\Gamma U}(q,w,z,x_{ni,A},y_{ni,I}) &= \frac{1}{\ell k}\sum_{\substack{\varepsilon\in\mathbb{Z}_\ell\\ \varepsilon_k\in\mathbb{Z}_k}}\left[q^{\frac{1}{2}}\left(\frac{z}{\varepsilon_k}+z^{-1}-\frac{w^{-1}}{\varepsilon_\ell\varepsilon_k}-\varepsilon_\ell w\right)\right]\\
&\times\sum_{r\geqslant 0}q^r\varepsilon_k^{-r}\sum_{n,m}^{\ell}\sum_{i,j}^{k}\sum_{A=1}^{N}\varepsilon_\ell^{n-m}\varepsilon_k^{i-j}\left(\sum_{I=1}^{K_{ni}}\frac{y_{ni,I}}{t_{ni}x_{mj,A}}+\sum_{I=1}^{K_{mj}}\frac{t_{ni}x_{ni,A}}{y_{mj,I}}\right).
\end{aligned}
\tag{E.9}
$$

We now detail how to evaluate the sums over conformal descendants and over the orbifold group. Note that here we rescaled $x_{mj,A}$ (which are fugacities associated with $S\left[U(N)^{k\ell}\right]$ and satisfy $\prod_{i=1}^{k}\prod_{n=1}^{\ell}\prod_{A=1}^{N}x_{ni,A}=1$) to $t_{ni}x_{ni,A}$ which now satisfy satisfying $\prod_{A=1}^{N}x_{ni,A}=\prod_{n=1}^{\ell}\prod_{i=1}^{k}x_{ni,A}=\prod_{n=1}^{\ell}\prod_{i=1}^{k}t_{ni}=1$ corresponding to $S\left[U(N)^{k\ell}\right]\cong\frac{U(1)^{\ell k}}{U(1)}\times SU(N)^{\ell k}$ Firstly, to evaluate the sums over conformal descendants we write to $r:=L_{ij}+\tilde{r}k\geqslant 0$ with L_{ij} defined in (4.72). This enables one to rewrite, for any fixed value $1\leqslant j\leqslant k$, to split the sum

$$
\sum_{r\geqslant 0}q^r\varepsilon_k^{-r}=\sum_{i=1}^{k}q^{L_{ij}}\varepsilon_k^{-L_{ij}}\sum_{\tilde{r}\geqslant 0}q^{\tilde{r}k}=\sum_{i=1}^{k}\frac{q^{L_{ij}}\varepsilon_k^{-L_{ij}}}{1-q^k}\,,
\tag{E.10}
$$

recall that $\varepsilon_k^k=1$. After this rewriting the sums over both $\mathbb{Z}_\ell,\mathbb{Z}_k$ may be simply carried out and is essentially equivalent to demanding that the exponents of ε_ℓ, ε_k vanish modulo ℓ, k in each term. Hence we have after, rearranging and applying the identity (4.73),

$$
\begin{aligned}
&i_{\Gamma V}\left(q,w,z,y_{ni,I}\right)=\\
&\sum_{n=1}^{\ell}\sum_{i,j=1}^{k}\frac{1}{1-q^k}\left[-\sum_{I=1}^{K_{ni}}\sum_{J=1}^{K_{nj}}\left(z^{-2}q^{L_{ij}+1}\frac{y_{ni,I}}{y_{nj,J}}+z^2q^{k-L_{ij}-1}\frac{y_{nj,J}}{y_{ni,I}}\right)\right.\\
&+\sum_{I=1}^{K_{ni}}\sum_{J=1}^{K_{(n+1)j}}\left(wq^{L_{ij}}\frac{y_{ni,I}}{y_{(n+1)j,J}}+w^{-1}q^{k-L_{ij}-1}\frac{y_{(n+1)j,J}}{y_{ni,I}}\right)\left(z+\frac{q}{z}\right)\\
&\left.-\sum_{I=1}^{K_{ni}}\sum_{J=1}^{K_{nj}}\left(q^{L_{ij}}y_{ni,I}y_{nj,J}^{-1}+\left(q^{k-L_{ij}}-(1-q^k)\delta_{L_{ni},0}\right)y_{ni,I}^{-1}y_{nj,J}\right)\right],
\end{aligned}
\tag{E.11}
$$

$$
\begin{aligned}
&i_{\Gamma H}\left(q,v,w,z,y_{ni,I}\right) = \\
&\quad \sum_{n=1}^{\ell}\sum_{i,j=1}^{k}\frac{q^{\frac{1}{2}}\left(v+v^{-1}\right)}{1-q^{k}}\left[\sum_{I=1}^{K_{ni}}\sum_{J=1}^{K_{nj}}\left(z^{-1}q^{L_{ij}}\frac{y_{ni,I}}{y_{nj,J}}+zq^{k-L_{ij}-1}\frac{y_{nj,J}}{y_{ni,I}}\right)\right. \\
&\quad \left.-\sum_{I=1}^{K_{ni}}\sum_{J=1}^{K_{(n+1)j}}\left(wq^{L_{ij}}y_{ni,I}y^{-1}_{(n+1)j,J}+w^{-1}q^{k-L_{ij}-1}y^{-1}_{ni,I}y_{(n+1)j,J}\right)\right],
\end{aligned}
\tag{E.12}
$$

$$
\begin{aligned}
i_{\Gamma U}\left(q,w,z,x_{ni,A},y_{ni,I}\right) &= \sum_{n=1}^{\ell}\sum_{i,j=1}^{k}\sum_{A=1}^{N}\frac{q^{\frac{1}{2}}}{1-q^{k}} \\
&\times\left[z^{-1}q^{L_{ij}}\left(\sum_{I=1}^{K_{ni}}y_{ni,I}x^{-1}_{nj,A}+\sum_{I=1}^{K_{nj}}y^{-1}_{nj,I}x_{ni,A}\right)\right. \\
&+zq^{k-L_{ij}-1}\left(\sum_{I=1}^{K_{ni}}y^{-1}_{ni,I}x_{nj,A}+\sum_{I=1}^{K_{nj}}y_{nj,I}x^{-1}_{ni,A}\right) \\
&-wq^{L_{ij}}\left(\sum_{I=1}^{K_{ni}}y_{ni,I}x^{-1}_{(n+1)j,A}+\sum_{I=1}^{K_{(n+1)j}}y^{-1}_{(n+1)j,I}x_{ni,A}\right) \\
&\left.-w^{-1}q^{k-L_{ij}-1}\left(\sum_{I=1}^{K_{ni}}y^{-1}_{ni,I}x_{(n+1)j,A}+\sum_{I=1}^{K_{(n+1)j}}y_{(n+1)j,I}x^{-1}_{ni,A}\right)\right].
\end{aligned}
\tag{E.13}
$$

In this form the plethystics may be easily performed. For the sake of completeness we also list the contribution from the Casimir energy (4.27)

$$
\begin{aligned}
E_{\text{Casimir}} &= \frac{k\beta_5^2}{\mathrm{i}\pi}\left(2NkK\left(\frac{\epsilon_+}{2}+m\right)\left(\frac{\mathrm{i}\pi}{\beta_5}+\frac{\epsilon_+}{2}-m\right)\right) \\
&+\frac{k\beta_5^2}{\pi^2}\sum_{n=1}^{\ell}\sum_{\mathcal{A}=1}^{kN}\sum_{\mathcal{I}=1}^{K_n}u_{n,\mathcal{I}}\left(\tilde{a}_{n+1,\mathcal{A}}+\tilde{a}_{n-1,\mathcal{A}}-2\tilde{a}_{n,\mathcal{A}}\right) \\
&+\frac{k\beta_5^2}{\pi^2}\sum_{n=1}^{\ell}\sum_{\mathcal{A}=1}^{kN}K_n\left(2\tilde{a}^2_{n,\mathcal{A}}-\tilde{a}^2_{n-1,\mathcal{A}}-\tilde{a}^2_{n+1,\mathcal{A}}+2m\tilde{a}_{n+1,\mathcal{A}}-2m\tilde{a}_{n-1,\mathcal{A}}\right)
\end{aligned}
\tag{E.14}
$$

where we also made the gauge transformation and redefinition (4.74) and used the definitions (4.84).

E.2 4d & 5d Contour Integral Representations

In this appendix we present the contour integral representations for the partition functions for the 5d and 4d theories both in the precense of the orbifold and without. These may be obtained by applying the limit directly to the respective 6d contour integral expression. We follow mostly the prescription presented in [295]. We will firstly take the 5d $\beta_6 \to 0$ ($q \to 1$) limit.

E.2.1 5d Limit of the Contour Integral

Using the identifications (4.39) and setting $y_I = q^{\frac{\beta_5 u_I}{\mathrm{i}\pi}}$ we have that

$$\lim_{q\to 1} \Delta(y_I) Z_V = \prod_{I,J=1}^{K} \frac{\sinh\beta_5\,(u_{IJ})'\sinh\beta_5\,(u_{IJ}-\epsilon_+)}{\sinh\beta_5\,(u_{IJ}-\tilde{m}-\epsilon_+)\sinh\beta_5\,(u_{IJ}+\tilde{m})}\,, \tag{E.15}$$

$$\lim_{q\to 1} Z_H = \prod_{I,J=1}^{K} \frac{\sinh\beta_5\left(u_{IJ}+\frac{\epsilon_-}{2}+m\right)\sinh\beta_5\left(u_{IJ}-\frac{\epsilon_-}{2}+m\right)}{\sinh\beta_5\,(u_{IJ}+\epsilon_1)\sinh\beta_5\,(u_{IJ}+\epsilon_2)}\,, \tag{E.16}$$

$$\lim_{q\to 1} Z_U = \prod_{I=1}^{K}\prod_{A=1}^{N} \frac{\sinh\beta_5\,(u_I-a_A-m)\sinh\beta_5\,(u_I-a_A+m)}{\sinh\beta_5\left(u_I-a_A-\frac{\epsilon_+}{2}\right)\sinh\beta_5\left(u_I-a_A+\frac{\epsilon_+}{2}\right)}\,, \tag{E.17}$$

where $u_{IJ} := u_I - u_J$ and $\tilde{m} = m - \epsilon_+/2$. By definition

$$\lim_{q\to 1} Z^{(0)}(q,v,w,z,x_A,y_I) = 1\,. \tag{E.18}$$

Hence, all that remains is to perform the limit on the integration over the maximal torus of $U(K)$:

$$\lim_{q\to 1} \oint_{T[U(K)]} \prod_{I=1}^{K} \frac{dy_I}{2\pi\mathrm{i}y_I} = \lim_{\beta_6\to 0} (2\tau)^K \int_{-\frac{\mathrm{i}\pi}{2\tau}}^{\frac{\mathrm{i}\pi}{2\tau}} \prod_{I=1}^{K} \frac{du_I}{2\pi\mathrm{i}\beta_5} = \int_{-\infty}^{+\infty} \prod_{I=1}^{K} \frac{du_I}{2\pi\mathrm{i}\beta_5}\,. \tag{E.19}$$

Putting all of the above ingredients together we write

$$
\begin{aligned}
Z_K^{5d} &:= \lim_{q\to 1} Z_K^{6d} = \frac{1}{K!}\int \prod_{I=1}^{K} \frac{du_I}{2\pi \mathrm{i}\beta_5} \\
&\times \prod_{I=1}^{K}\prod_{A=1}^{N} \frac{\sinh\beta_5\left(u_I - a_A - m\right)\sinh\beta_5\left(u_I - a_A + m\right)}{\sinh\beta_5\left(u_I - a_A - \frac{\epsilon_+}{2}\right)\sinh\beta_5\left(u_I - a_A + \frac{\epsilon_+}{2}\right)} \\
&\times \prod_{I,J=1}^{K} \frac{\sinh\beta_5\left(u_{IJ}\right)'\sinh\beta_5\left(u_{IJ} - \epsilon_+\right)}{\sinh\beta_5\left(u_{IJ} + \epsilon_1\right)\sinh\beta_5\left(u_{IJ} + \epsilon_2\right)} \\
&\times \prod_{I,J=1}^{K} \frac{\sinh\beta_5\left(u_{IJ} + \frac{\epsilon_-}{2} + m\right)\sinh\beta_5\left(u_{IJ} - \frac{\epsilon_-}{2} + m\right)}{\sinh\beta_5\left(u_{IJ} - \frac{\epsilon_+}{2} - m\right)\sinh\beta_5\left(u_{IJ} + \frac{\epsilon_+}{2} - m\right)} .
\end{aligned}
\tag{E.20}
$$

E.2.2 4d Limit of the Contour Integral

It is then a straightforward exercise to take the 4d limit $\beta_5 \to 0$. We have

$$
Z_K^{4d} := \lim_{\beta_5\to 0} Z_K^{5d} \tag{E.21}
$$

$$
\begin{aligned}
&= \sum_{K\geqslant 0} \frac{1}{K!}\int \prod_{I=1}^{K}\frac{du_I}{2\pi \mathrm{i}} \prod_{I=1}^{K}\prod_{A=1}^{N} \frac{\left(u_I - a_A - m\right)\left(u_I - a_A + m\right)}{\left(u_I - a_A - \frac{\epsilon_+}{2}\right)\left(u_I - a_A + \frac{\epsilon_+}{2}\right)} \\
&\times \prod_{I\neq J} u_{IJ} \prod_{I,J=1}^{K} \frac{\left(u_{IJ} - \epsilon_+\right)\left(u_{IJ} - \frac{\epsilon_-}{2} - m\right)\left(u_{IJ} + \frac{\epsilon_-}{2} - m\right)}{\left(u_{IJ} - m - \frac{\epsilon_+}{2}\right)\left(u_{IJ} + m - \frac{\epsilon_+}{2}\right)\left(u_{IJ} - \epsilon_1\right)\left(u_{IJ} - \epsilon_2\right)} .
\end{aligned}
\tag{E.22}
$$

E.2.3 5d Limit of the Orbifolded Contour Integral

Taking this limit is largely the same procedure as for the $\ell = k = 1$ case however we instead use the slightly different set of variables (4.84). We are again interested in the $q \to 1$ limit of the partition function (4.77). Setting $y_{i,\mathcal{I}} = q^{\frac{k u_{i\mathcal{I}}}{\mathrm{i}\pi}}$ we have

$$
\lim_{q\to 1} Z^{6d,\ell,k}_{\{K_{ij}\}} := Z^{5d,\ell,k}_{\{K_{ij}\}} \tag{E.23}
$$

$$
\begin{aligned}
=&\prod_{i=1}^{\ell}\left[\frac{1}{\prod_{j=1}^{k}K_{ij}!}\int\prod_{\mathcal{I}=1}^{K_i}\frac{du_{i,\mathcal{I}}}{2\pi\mathrm{i}\beta_5}\prod_{\mathcal{I}\neq\mathcal{J}}\sinh\beta_5\left(u_{i,\mathcal{I}}-u_{i,\mathcal{J}}\right)\right.\\
&\times\prod_{\mathcal{I},\mathcal{J}=1}^{K_i}\frac{\sinh\beta_5\left(u_{i,\mathcal{I}}-u_{i,\mathcal{J}}-\epsilon_+\right)}{\sinh\beta_5\left(u_{i,\mathcal{I}}-u_{i,\mathcal{J}}+\epsilon_2\right)\sinh\beta_5\left(u_{i,\mathcal{I}}-u_{i,\mathcal{J}}+\epsilon_1\right)}\\
&\times\prod_{\mathcal{I}=1}^{K_i}\prod_{\mathcal{J}=1}^{K_{i+1}}\frac{\sinh\beta_5\left(u_{i,\mathcal{I}}-u_{i+1,\mathcal{J}}+m+\frac{\epsilon_-}{2}\right)}{\sinh\beta_5\left(u_{i,\mathcal{I}}-u_{i+1,\mathcal{J}}+m-+\frac{\epsilon_+}{2}\right)}\\
&\times\prod_{\mathcal{I}=1}^{K_i}\prod_{\mathcal{J}=1}^{K_{i+1}}\frac{\sinh\beta_5\left(u_{i,\mathcal{I}}-u_{i+1,\mathcal{J}}+m-\frac{\epsilon_-}{2}\right)}{\sinh\beta_5\left(u_{i,\mathcal{I}}-u_{i+1,\mathcal{J}}+m-\frac{\epsilon_+}{2}\right)}\\
&\times\left.\prod_{\mathcal{A}=1}^{kN}\prod_{\mathcal{I}=1}^{K_i}\frac{\sinh\beta_5\left(u_{i,\mathcal{I}}-\tilde{a}_{i+1,\mathcal{A}}+m\right)\sinh\beta_5\left(u_{i,\mathcal{I}}-\tilde{a}_{i-1,\mathcal{A}}-m\right)}{\sinh\beta_5\left(u_{i,\mathcal{I}}-\tilde{a}_{i,\mathcal{A}}-\frac{\epsilon_+}{2}\right)\sinh\beta_5\left(u_{i,\mathcal{I}}-\tilde{a}_{i,\mathcal{A}}+\frac{\epsilon_+}{2}\right)}\right].
\end{aligned}
\tag{E.24}
$$

E.2.4 4d Limit of the Orbifolded Contour Integral

As before it is straightforward to take the 4d limit $\beta_5\to 0$.

$$
Z^{4d,\ell,k}_{\{K_{ij}\}} := \lim_{\beta_5\to 0} Z^{5d,\ell,k}_{\{K_{ij}\}} \tag{E.25}
$$

$$
\begin{aligned}
=&\prod_{i=1}^{\ell}\left[\frac{1}{\prod_{j=1}^{k}K_{ij}!}\int\prod_{\mathcal{I}=1}^{K_i}\frac{du_{i,\mathcal{I}}}{2\pi\mathrm{i}}\prod_{\mathcal{I}\neq\mathcal{J}}\left(u_{i,\mathcal{I}}-u_{i,\mathcal{J}}\right)\right.\\
&\times\prod_{\mathcal{I},\mathcal{J}=1}^{K_i}\frac{\left(u_{i,\mathcal{I}}-u_{i,\mathcal{J}}-\epsilon_+\right)}{\left(u_{i,\mathcal{I}}-u_{i,\mathcal{J}}+\epsilon_2\right)\left(u_{i,\mathcal{I}}-u_{i,\mathcal{J}}+\epsilon_1\right)}\\
&\times\prod_{\mathcal{I}=1}^{K_i}\prod_{\mathcal{J}=1}^{K_{i+1}}\frac{\left(u_{i,\mathcal{I}}-u_{i+1,\mathcal{J}}+m+\frac{\epsilon_-}{2}\right)\left(u_{i,\mathcal{I}}-u_{i+1,\mathcal{J}}+m-\frac{\epsilon_-}{2}\right)}{\left(u_{i,\mathcal{I}}-u_{i+1,\mathcal{J}}+\frac{\epsilon_+}{2}+m\right)\left(u_{i,\mathcal{I}}-u_{i+1,\mathcal{J}}-\frac{\epsilon_+}{2}+m\right)}\\
&\times\left.\prod_{\mathcal{A}=1}^{kN}\prod_{\mathcal{I}=1}^{K_i}\frac{\left(u_{i,\mathcal{I}}-\tilde{a}_{i+1,\mathcal{A}}+m\right)\left(u_{i,\mathcal{I}}-\tilde{a}_{i-1,\mathcal{A}}-m\right)}{\left(u_{i,\mathcal{I}}-\tilde{a}_{i,\mathcal{A}}-\frac{\epsilon_+}{2}\right)\left(u_{i,\mathcal{I}}-\tilde{a}_{i,\mathcal{A}}+\frac{\epsilon_+}{2}\right)}\right].
\end{aligned}
\tag{E.26}
$$

Appendix F

Appendices for Chapter 5

F.1 Preserved Superconformal Algebra

F.1.1 Even Subalgebra

The even subalgebra of $\mathfrak{psu}(2,2|4)$ is $\mathfrak{b} = \mathfrak{so}(5,1) \oplus \mathfrak{su}(4)$ which we take to be generated by $M^{\mu\nu}, K_\mu, P^\mu, E$ with $\mu, \nu = 1,2,3,4$ and R_I^J, $I, J = 1,2,3,4$. The Cartans of $\mathfrak{su}(4)$ are $R_i = R_i^i - R_{i+1}^{i+1}$ with $i = 1,2,3$. We wish to discuss which generators are preserved by the S-folding/discrete gauging procedure. Recall that $SL(2,\mathbb{Z})$ transformations can be defined such that they commute with the generators of $\mathfrak{b}$ [265]. In particular $[s_k, \mathfrak{b}] = 0$. Hence s_k acts non-trivially only on the fermionic subalgebra which we will discuss momentarily. Hence the subalegbra of $\mathfrak{b}$ preserved by the S-folding/discrete gauging is simply the centraliser of $r_k = \frac{R_1}{2} + R_2 + \frac{3R_3}{2} = \frac{1}{2}\sum_{i=1}^{3} R_i^i - \frac{3}{2}R_4^4$ modulo k in $\mathfrak{b}$. Clearly $[r_k, \mathfrak{so}(5,1)] = 0$. On the other hand, using $\left[R_I^J, R_Q^P\right] = \delta_Q^J R_I^P - \delta_I^P R_Q^J$ it can be shown that

$$[r_k, R_I^J] = \begin{cases} 0 & I, J \in \{1,2,3\}\,, \\ 0 & I = J = 4\,, \\ 2R_I^4 & I \in \{1,2,3\}\,, J = 4\,, \\ -2R_4^J & I = 4\,, J \in \{1,2,3\}\,. \end{cases} \tag{F.1}$$

Therefore, the subalgebra of $\mathfrak{su}(4)$ preserved by $r_{k\geqslant 3}$ are given by the R_I^J with $I, J = 1,2,3$ and R_4^4. These generators span a $\mathfrak{su}(3) \oplus \mathfrak{u}(1)$ algebra. Note however that, since we quotient by $e^{\frac{2\pi i}{k} r_k + s_k}$, when $k = 1,2$ the full $\mathfrak{su}(4)$ is preserved.

F.1.2 Odd Subalgebra

The odd subalgebra of $\mathfrak{psu}(2,2|4)$ is spanned by nilpotent generators (supercharges) which sit in representations of the bosonic subalgebra $\mathfrak{b}$. Any representation of $\mathfrak{b}$ can be decomposed into representations of a maximal compact subalgebra $\mathfrak{u}(1)_E \oplus \mathfrak{su}(2)_1 \oplus \mathfrak{su}(2)_2 \oplus \mathfrak{su}(4)$. The supercharges are then given by

$$\mathcal{Q}^I_\alpha \in \left(\frac{1}{2}, \mathbf{2}, \mathbf{1}, \mathbf{4}\right) , \quad \widetilde{\mathcal{Q}}_{\dot\alpha I} \in \left(\frac{1}{2}, \mathbf{1}, \mathbf{2}, \overline{\mathbf{4}}\right) , \tag{F.2}$$

$$\mathcal{S}^\alpha_I \in \left(-\frac{1}{2}, \overline{\mathbf{2}}, \mathbf{1}, \overline{\mathbf{4}}\right) , \quad \widetilde{\mathcal{S}}^{\dot\alpha I} \in \left(-\frac{1}{2}, \mathbf{1}, \overline{\mathbf{2}}, \mathbf{4}\right) . \tag{F.3}$$

The action on the supercharges is then given by

$$[r_k, \mathcal{Q}^I_\alpha] = \begin{cases} \mathcal{Q}^I_\alpha & I = 1,2,3 \\ -3\mathcal{Q}^4_\alpha & I = 4 \end{cases} , \quad [r_k, \widetilde{\mathcal{Q}}_{\dot\alpha I}] = \begin{cases} -\widetilde{\mathcal{Q}}_{\dot\alpha I} & I = 1,2,3 \\ 3\widetilde{\mathcal{Q}}_{\dot\alpha 4} & I = 4 \end{cases} , \tag{F.4}$$

$$[r_k, \mathcal{S}^\alpha_I] = \begin{cases} -\mathcal{S}^I_\alpha & I = 1,2,3 \\ 3\mathcal{S}^4_\alpha & I = 4 \end{cases} , \quad [r_k, \widetilde{\mathcal{S}}^{\dot\alpha I}] = \begin{cases} \widetilde{\mathcal{S}}_{\dot\alpha I} & I = 1,2,3 \\ -3\widetilde{\mathcal{S}}_{\dot\alpha 4} & I = 4 \end{cases} , \tag{F.5}$$

On the other hand, s_k acts on the supercharges by [265, 158, 250]

$$[s_k, \mathcal{Q}^I_\alpha] = -\mathcal{Q}^I_\alpha , \quad [s_k, \widetilde{\mathcal{Q}}_{\dot\alpha I}] = \widetilde{\mathcal{Q}}_{\dot\alpha I} , \tag{F.6}$$

$$[s_k, \mathcal{S}^\alpha_I] = \mathcal{S}^\alpha_I , \quad [r_k, \widetilde{\mathcal{S}}^{\dot\alpha I}] = -\widetilde{\mathcal{S}}_{\dot\alpha I} . \tag{F.7}$$

Therefore, for $k \geqslant 3$, quotienting by $e^{\frac{2\pi i}{k}(r_k+s_k)} \in \mathbb{Z}_k$ preserves 12 Poincaré supercharges and 12 conformal supercharges giving rise to $\mathcal{N} = 3$ superconformal symmetry in four dimensions. All in all, for $k \geqslant 3$, a full $\mathfrak{su}(2,2|3) \subset \mathfrak{psu}(2,2|4)$ superconformal algebra is preserved.

F.2 Indices of $\mathfrak{su}(2,2|2)$ Multiplets

Long multiplets $\mathcal{A}^E_{R,r,(j_1,j_2)}$ are generic, unitary, modules of the $\mathfrak{su}(2,2|2)$ superconformal algebra. The multiplets are labelled by the values of the highest weight state (superconformal primary) (E,R,r,j_1,j_2) under the maximal bosonic subalgebra (5.29). When the some of representation labels take on certain values the superconformal primary is annihilated by (linear combinations of) some of the supercharges $\mathcal{Q}^I_\alpha$, $\widetilde{\mathcal{Q}}_{\dot\alpha I}$ and the multiplet is said to be shortened. The superconformal index (5.35) counts short multiplets modulo those that can recombine into long multiplets. The recombination rules are given by [266]

$$\mathcal{A}^{2R+r+2j_1+2}_{R,r,(j_1,j_2)} \cong \mathcal{C}_{R,r,(j_1,j_2)} \oplus \mathcal{C}_{R+\frac{1}{2},r+\frac{1}{2},(j_1-\frac{1}{2},j_2)}\,, \tag{F.8}$$

$$\mathcal{A}^{2R-r+2j_2+2}_{R,r,(j_1,j_2)} \cong \overline{\mathcal{C}}_{R,r,(j_1,j_2)} \oplus \overline{\mathcal{C}}_{R+\frac{1}{2},r-\frac{1}{2},(j_1,j_2-\frac{1}{2})}\,, \tag{F.9}$$

$$\begin{aligned}\mathcal{A}^{2R+j_1+j_2+2}_{R,j_1-j_2,(j_1,j_2)} \cong& \hat{\mathcal{C}}_{R,(j_1,j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2},(j_1-\frac{1}{2},j_2)} \oplus \hat{\mathcal{C}}_{R+\frac{1}{2},(j_1,j_2-\frac{1}{2})} \\ &\oplus \hat{\mathcal{C}}_{R+1,(j_1-\frac{1}{2},j_2-\frac{1}{2})}\end{aligned}\,. \tag{F.10}$$

By formally allowing the j_1, j_2 to take on the value $-\frac{1}{2}$ we can write

$$\mathcal{C}_{R,r,(-\frac{1}{2},j_2)} \cong \mathcal{B}_{R+\frac{1}{2},r+\frac{1}{2},(0,j_2)}\,, \quad \overline{\mathcal{C}}_{R,r,(j_1,-\frac{1}{2})} \cong \overline{\mathcal{B}}_{R+\frac{1}{2},r-\frac{1}{2},(j_1,0)}\,, \tag{F.11}$$

$$\hat{\mathcal{C}}_{R,(-\frac{1}{2},j_2)} \cong \mathcal{D}_{R+\frac{1}{2},(0,j_2)}\,, \quad \hat{\mathcal{C}}_{R,(j_1,-\frac{1}{2})} \cong \overline{\mathcal{D}}_{R+\frac{1}{2},(j_1,0)}\,, \tag{F.12}$$

$$\hat{\mathcal{C}}_{R,(-\frac{1}{2},-\frac{1}{2})} \cong \mathcal{D}_{R+\frac{1}{2},(0,-\frac{1}{2})} \cong \overline{\mathcal{D}}_{R+\frac{1}{2},(-\frac{1}{2},0)} \cong \hat{\mathcal{B}}_{R+1}\,, \tag{F.13}$$

for $R \geqslant 0$. Equations (F.8)-(F.13) constitute the most general recombination rules for any unitary $\mathcal{N} = 2$ SCFT. We have that

$$\mathcal{I}_{\mathcal{E}_{r,(0,j_2)}} = t^{2r}(pq)^r \frac{1 - t(pq)^{-1}\chi_1(y) + t^2(pq)^{-2}}{(-1)^{2j_2}(1-t^3y)(1-t^3y^{-1})}\chi_{2j_2}(y) \quad r \geqslant 2\,, \tag{F.14}$$

$$\mathcal{I}_{\mathcal{D}_{0,(0,j_2)}} = \frac{pqt^2\chi_{2j_2}(y) - t^3\chi_{2j_2+1}(y) - t^5pq\chi_{2j_2-1}(y) + t^6\chi_{2j_2}(y)}{(-1)^{2j_2}(1-t^3y)(1-t^3y^{-1})}\,, \tag{F.15}$$

$$\mathcal{I}_{\overline{\mathcal{D}}_{0,(j_1,0)}} = \frac{t^{4j_1+4}}{(pq)^{j_1+1}} \frac{1-(pq)t^2}{(-1)^{2j_1+1}(1-t^3y)(1-t^3y^{-1})}\,, \tag{F.16}$$

$$\mathcal{I}_{\mathcal{C}_{R,r(j_1,j_2)}} = \frac{t^{4+4R+6j_1+2r}}{(pq)^{R+1-r}} \frac{\left(1-t^2pq\right)\left(t^2pq - t^3\chi_1(y) + \frac{t^4}{pq}\right)}{(-1)^{2j_1+2j_2+1}\left(1-t^3y\right)\left(1-t^3y^{-1}\right)}\chi_{2j_2}(y)\,, \tag{F.17}$$

$$\mathcal{I}_{\hat{\mathcal{C}}_{R(j_1,j_2)}} = \frac{t^{6+4R+4j_1+2j_2}}{(pq)^{R+j_1-j_2}} \frac{\left(1-t^2pq\right)\left(\frac{t}{pq}\chi_{2j_2+1}(y) - \chi_{2j_2}(y)\right)}{(-1)^{2j_1+2j_2}(1-t^3y)(1-t^3y^{-1})}\,, \tag{F.18}$$

$$\mathcal{I}_{\overline{\mathcal{E}}_{r,(j_1,0)}} = \mathcal{I}_{\mathcal{E}_{0,(0,0)}} = \mathcal{I}_{\overline{\mathcal{C}}_{R,r(j_1,j_2)}} = \mathcal{I}_{\mathcal{A}^E_{R,r(j_1,j_2)}} = 0\,. \tag{F.19}$$

These may be obtained from [168] by conjugation (exchanging $r \to -r$, $j_1 \leftrightarrow j_2$) and setting $\tau = t^2(pq)^{-1/2}$, $\sigma = ty(pq)^{1/2}$, $\rho = ty^{-1}(pq)^{1/2}$. By applying (5.31)-(5.33) in combination with (F.11)-(F.19) one can compute the contribution to the index of

Shortening Conditions				Multiplet
$\mathcal{B}_1$	$\mathcal{Q}_{1\alpha}\lvert R,r\rangle^{h.w.}=0$	$j_1=0$	$E=2R+r$	$\mathcal{B}_{R,r(0,j_2)}$
$\overline{\mathcal{B}}_2$	$\widetilde{\mathcal{Q}}_{2\dot{\alpha}}\lvert R,r\rangle^{h.w}=0$	$j_2=0$	$E=2R-r$	$\overline{\mathcal{B}}_{R,r(j_1,0)}$
$\mathcal{E}$	$\mathcal{B}_1\cap\mathcal{B}_2$	$R=0$	$E=r$	$\mathcal{E}_{r(0,j_2)}$
$\overline{\mathcal{E}}$	$\overline{\mathcal{B}}_1\cap\overline{\mathcal{B}}_2$	$R=0$	$E=-r$	$\overline{\mathcal{E}}_{r(j_1,0)}$
$\hat{\mathcal{B}}$	$\mathcal{B}_1\cap\overline{\mathcal{B}}_2$	$r=0,j_1,j_2=0$	$E=2R$	$\hat{\mathcal{B}}_R$
$\mathcal{C}_1$	$\epsilon^{\alpha\beta}\mathcal{Q}_{1\beta}\lvert R,r\rangle^{h.w.}_{\alpha}=0$ $(\mathcal{Q}_1)^2\lvert R,r\rangle^{h.w.}=0$ for $j_1=0$		$E=2+2j_1+2R+r$ $E=2+2R+r$	$\mathcal{C}_{R,r(j_1,j_2)}$ $\mathcal{C}_{R,r(0,j_2)}$
$\overline{\mathcal{C}}_2$	$\epsilon^{\dot{\alpha}\dot{\beta}}\widetilde{\mathcal{Q}}_{2\dot{\beta}}\lvert R,r\rangle^{h.w.}_{\dot{\alpha}}=0$ $(\widetilde{\mathcal{Q}}_2)^2\lvert R,r\rangle^{h.w.}=0$ for $j_2=0$		$E=2+2j_2+2R-r$ $E=2+2R-r$	$\overline{\mathcal{C}}_{R,r(j_1,j_2)}$ $\overline{\mathcal{C}}_{R,r(j_1,0)}$
	$\mathcal{C}_1\cap\mathcal{C}_2$	$R=0$	$E=2+2j_1+r$	$\mathcal{C}_{0,r(j_1,j_2)}$
	$\overline{\mathcal{C}}_1\cap\overline{\mathcal{C}}_2$	$R=0$	$E=2+2j_2-r$	$\overline{\mathcal{C}}_{0,r(j_1,j_2)}$
$\hat{\mathcal{C}}$	$\mathcal{C}_1\cap\overline{\mathcal{C}}_2$	$r=j_2-j_1$	$E=2+2R+j_1+j_2$	$\hat{\mathcal{C}}_{R(j_1,j_2)}$
	$\mathcal{C}_1\cap\mathcal{C}_2\cap\overline{\mathcal{C}}_1\cap\overline{\mathcal{C}}_2$	$R=0,r=j_2-j_1$	$E=2+j_1+j_2$	$\hat{\mathcal{C}}_{0(j_1,j_2)}$
$\mathcal{D}$	$\mathcal{B}_1\cap\overline{\mathcal{C}}_2$	$r=j_2+1$	$E=1+2R+j_2$	$\mathcal{D}_{R(0,j_2)}$
$\overline{\mathcal{D}}$	$\overline{\mathcal{B}}_2\cap\mathcal{C}_1$	$-r=j_1+1$	$E=1+2R+j_1$	$\overline{\mathcal{D}}_{R(j_1,0)}$
	$\mathcal{E}\cap\overline{\mathcal{C}}_2$	$r=j_2+1,R=0$	$E=r=1+j_2$	$D_{0,(0,j_2)}$
	$\overline{\mathcal{E}}\cap\mathcal{C}_1$	$-r=j_1+1,R=0$	$E=-r=1+j_1$	$\overline{\mathcal{D}}_{0,(j_1,0)}$

Table F.1: *Shortening conditions and short multiplets* $\mathfrak{su}(2,2|2)$.

the $\mathfrak{su}(2,2|3)$ multiplets of $\hat{\mathcal{B}}_{[R_1,R_2]}$.

F.3 Reduction to Three Dimensions

Let us define $\mathfrak{q} = e^{-\beta}$ where β is the radii of the $\mathbb{S}^1$ factor. Following [295, 296, 5] let us write

$$t = \mathfrak{q}^{1/3}\,, \quad y = \mathfrak{q}^{\eta}\,, \quad p = \mathfrak{q}^{\rho}\,, \quad q = \mathfrak{q}^{\gamma}\,. \tag{F.20}$$

We can rewrite the Elliptic Gamma functions as

$$\Gamma\left(\mathfrak{q}^{\alpha};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right) = \prod_{n,m=0}^{\infty} \frac{[-\alpha+2+n(1+\eta)+m(1-\eta)]_{\mathfrak{q}}}{[\alpha+n(1+\eta)+m(1-\eta)]_{\mathfrak{q}}} \tag{F.21}$$

where $[n]_{\mathfrak{q}} = (1-\mathfrak{q}^n)/(1-\mathfrak{q})$ is the q-number. The q-number satisfies $\lim_{\mathfrak{q}\to 1}[n]_{\mathfrak{q}} = n$. The $\beta \to 0$ limit corresponds to $\mathfrak{q} \to 1$. Therefore

$$\lim_{\mathfrak{q}\to 1}\Gamma\left(\mathfrak{q}^{\alpha};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right) = \prod_{n,m=0}^{\infty} \frac{-\alpha+2+n(1+\eta)+m(1-\eta)}{\alpha+n(1+\eta)+m(1-\eta)}\,. \tag{F.22}$$

We define $\eta = (1-b^2)/(1+b^2)$. We then have

$$\lim_{\mathfrak{q}\to 1}\Gamma\left(\mathfrak{q}^{\alpha};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right) = s_b\left(\frac{\mathrm{i}Q}{2}(1-\alpha)\right)\,. \tag{F.23}$$

where $Q = b + b^{-1}$ and $s_b(x)$ is the double sine function. Let us now discuss the limit applied to the index (5.72). We may rewrite (5.72) as

$$\begin{aligned}
\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_k}(t,y,p,q) &= \frac{1}{k}\sum_{l=0}^{k-1}\Bigg\{\Gamma\left(\mathfrak{q}^{2/3+\rho+\gamma-\frac{2\pi\mathrm{i}l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\\
&\times\Gamma\left(\mathfrak{q}^{2/3+\gamma-\rho-\frac{2\pi\mathrm{i}l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\Gamma\left(\mathfrak{q}^{2/3-2\gamma+\frac{2\pi\mathrm{i}l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\\
&\times\prod_{n,m=0}^{\infty}\frac{\left[-\frac{2\pi il}{k\beta}+(n+1)(1+\eta)+m(1-\eta)\right]_{\mathfrak{q}}}{\left[-\frac{2\pi il}{k\beta}+n(1+\eta)+m(1-\eta)+2\right]_{\mathfrak{q}}}\Bigg\}\\
&\times\prod_{n,m=0}^{\infty}\frac{\left[-\frac{2\pi il}{k\beta}+n(1+\eta)+(m+1)(1-\eta)\right]_{\mathfrak{q}}}{\left[\frac{2\pi il}{k\beta}+n(1+\eta)+m(1-\eta)+2\right]_{\mathfrak{q}}}\Bigg\}\,.
\end{aligned} \tag{F.24}$$

It is useful to consider splitting the sum over $l = 0, 1, \dots, k-1$ in order to isolate the $l = 0$ term as follows

$$
\begin{aligned}
\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_k}(t,y,p,q) &= \frac{1}{k}\sum_{l=1}^{k-1}\Bigg\{\Gamma\left(\mathfrak{q}^{2/3+\gamma-\rho-\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right) \\
&\times \frac{\Gamma\left(\mathfrak{q}^{2/3-2\gamma+\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\Gamma\left(\mathfrak{q}^{2/3+\rho+\gamma-\frac{2\pi i l}{k\beta}};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)}{\prod_{n=0}^{\infty}\left[-\frac{2\pi i l}{k\beta}+n(1-\eta)\right]_{\mathfrak{q}}\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)\right]_{\mathfrak{q}}} \\
&\prod_{n,m=0}^{\infty}\frac{\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)\right]_{\mathfrak{q}}}{\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)+2\right]_{\mathfrak{q}}} \\
&\prod_{n,m=0}^{\infty}\frac{\left[-\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)\right]_{\mathfrak{q}}}{\left[\frac{2\pi i l}{k\beta}+n(1+\eta)+m(1-\eta)+2\right]_{\mathfrak{q}}}\Bigg\} \\
&+\frac{1}{k}\frac{\Gamma\left(\mathfrak{q}^{2/3+\gamma-\rho};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\Gamma\left(\mathfrak{q}^{2/3-2\gamma};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)}{\Gamma\left(\mathfrak{q}^{4/3-\rho-\gamma};\mathfrak{q}^{1+\eta},\mathfrak{q}^{1-\eta}\right)\prod_{n=0}^{\infty}\left[n(1-\eta)\right]_{\mathfrak{q}}\left[n(1+\eta)\right]_{\mathfrak{q}}}\,.
\end{aligned}
\tag{F.25}
$$

Due the form of the denominator in the second line it is clear that in the $\mathfrak{q} \to 1$ limit the factors with $l \neq 0$ vanish. Moreover, when $l \neq 0$ no regularisation is required. On the other hand the product from $l = 0$ requires regularisation. The usual prescription is simply to drop the overall infinite contribution [295], rendering the limit finite. This regularisation is of course independent of k. Moreover, following the presciprition of [295], we identify

$$
\eta = \frac{1-b^2}{1+b^2}\,, \quad \gamma = \frac{1}{12} + \frac{\sigma}{\mathrm{i}Q}\,, \quad \rho = \frac{1}{4} - \frac{\sigma}{\mathrm{i}Q}\,. \tag{F.26}
$$

Applying (F.23) we therefore have that

$$
\lim_{\mathfrak{q}\to 1}\mathcal{I}^{\mathfrak{u}(1)}_{\mathbb{Z}_k}(t,y,p,q) = \frac{1}{k}s_b\left(\frac{\mathrm{i}Q}{4}+\sigma\right)s_b\left(\frac{\mathrm{i}Q}{4}-\sigma\right)\,. \tag{F.27}
$$

F.4 Vanishing of $\mathbb{Z}_n$ Anomalies

One possible obstruction to the ideas that we have discussed in this paper is the potential that the $\mathbb{Z}_n \subset SL(2,\mathbb{Z})$ has 't Hooft-anomaly. Since the symmetry is only emergent at strong coupling checking the $\mathbb{Z}_n$-anomalies is a non-trivial. In [261] Vafa and Witten studied the S-duality conjecture in topologically twisted $\mathcal{N} = 4$

SYM with gauge group G with $Lie(G) = \mathfrak{g}$ simply laced on a four-manifold $\mathcal{M}$. The partition function for the topologically twisted theory is given by [261]

$$Z_G(\tau) = |Z(G)|^{b_1(\mathcal{M})-1} \sum_{v \in H^2(\mathcal{M}, \pi_1(G))} Z_v(\tau)\,, \tag{F.28}$$

with $Z(G)$ the center of G and

$$Z_v(\tau) = e^{-2\pi i \tau s} \sum_{K \in \mathbb{Z} - \frac{1}{2}\langle v,v \rangle} \chi\left(\mathbf{M}_{K,v}\right) e^{2\pi i \tau K}\,, \quad \hat{Z}_v(\tau) := \eta(\tau)^{-w} Z_v(\tau)\,, \tag{F.29}$$

where $v = w_2(P) \in H^2(\mathcal{M}, \pi_1(G))$ is the second Stiefel-Whitney class of the G-bundle P over $\mathcal{M}$, $\langle \cdot, \cdot \rangle$ the intersection form on $H^2(\mathcal{M}, \pi_1(G))$ and $\mathbf{M}_{K,v}$ is the moduli space of rank K anti-self-dual instantons on $\mathcal{M}$. Additionally [261, 297, 298, 299]

$$s = (\operatorname{rank} \mathfrak{g} + 1)\chi(\mathcal{M})/4\,, \quad w = -\chi(\mathcal{M})\,. \tag{F.30}$$

Under modular transformations (5.12) the partition function transforms as

$$\hat{Z}_v(\tau + 1) = e^{-\pi i (2s + w/12 + \langle v,v \rangle)} \hat{Z}_v(\tau)\,, \tag{F.31}$$

$$\hat{Z}_v\left(\frac{-1}{\tau}\right) = \pm \frac{1}{|Z(G)|^{b_2(\mathcal{M})/2}} \sum_{u \in H^2(\mathcal{M}, Z(G))} e^{2\pi i \langle v,u \rangle} \hat{Z}_u(\tau)\,. \tag{F.32}$$

Let $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^3$. Its Poincaré polynomial is given by $P_{\mathbb{S}^1 \times \mathbb{S}^3}(x) = 1 + x + x^3 + x^4$ and therefore $b_2(\mathbb{S}^1 \times \mathbb{S}^3) = \chi(\mathbb{S}^1 \times \mathbb{S}^3) = 0$. By Poincaré duality $H^2(\mathbb{S}^1 \times \mathbb{S}^3) \cong H_2(\mathbb{S}^1 \times \mathbb{S}^3) = \{1\}$ in particular this fixes $v = 0$. Therefore, on $\mathcal{M} = \mathbb{S}^1 \times \mathbb{S}^3$, the partition function (F.28) satisfies

$$Z_G(\tau + 1) = Z_G\left(\frac{-1}{\tau}\right) = Z_G(\tau) = Z_{{}^L G}(\tau)\,. \tag{F.33}$$

Since the partition function (F.28) is fully $SL(2,\mathbb{Z})$ invariant, following the arguments of [300], we can conclude that on $\mathbb{S}^1 \times \mathbb{S}^3$ the $\mathbb{Z}_n \subset SL(2,\mathbb{Z})$ symmetries at τ fixed as in (5.15) of (twisted) $\mathcal{N} = 4$ SYM have vanishing 't Hooft anomaly. Therefore we expect that they can be consistently gauged.

F.5 S-Fold $\stackrel{?}{\equiv}$ Discrete Gauging

In a few cases some of the theories that can be obtained from $\mathbb{Z}_n$ discrete gauging of $\mathcal{N} = 4$ SYM are equivalent to some of the theories $S^N_{k,\ell,p'}$. However, as we will now show, in most cases this is a possibility only when the parent S-fold theory $S^N_{k,\ell}$ has enhanced $\mathcal{N} = 4$ supersymmetry. The strategy is simply to compute the possibilities which allow for (5.9) to be equal to (5.21). A more refined strategy, via the comparison of the $\frac{1}{8}$-BPS partition functions has been employed in [301].

In the following we limit the discussion to only the connected part of the gauge group of the parent theories.

$\mathfrak{g} = \mathfrak{u}(N)$ Equating (5.21) with (5.9) we have

$$N^2 = kN^2 + (2\ell - k - 1)N\,. \tag{F.34}$$

This has solution only for $k = \ell = 1$ with N arbitrary ($S^N_{1,1}$) and $N = \ell = 1$ with k arbitrary ($S^1_{k,1}$). However, in both cases, the S-fold parent theory has an operator of dimension 1 and therefore supersymmetry is automatically enhanced to $\mathcal{N} = 4$ [247, 248]. Therefore, performing a $\mathbb{Z}_{p'}$ gauging to the S-fold parent is automatically equivalent to making a $\mathbb{Z}_n = \mathbb{Z}_{p'}$ discrete gauging to $\mathcal{N} = 4$ SYM with gauge algebra $\mathfrak{u}(1)$ since they are the same theory! In both cases, after the discrete gauging, we have the theory $S^1_{k,1,p'}$.

$\mathfrak{g} = \mathfrak{su}(N+1) = A_N$ Equating (5.21) with (5.9) we have

$$N^2 + 2N = kN^2 + (2\ell - k - 1)N\,. \tag{F.35}$$

The only solutions are $N = 1$ $\ell = 2$ with k arbitrary, $N = 2$ $k = 3$ $\ell = 1$ and $N = 3$ $k = 2$ $\ell = 1$. The parent S-fold theory $S^1_{k,2}$ does not exist as an S-fold for $k \neq 2$ [247]. $S^2_{3,1}$ and $S^3_{2,1}$ do but they automatically enhance to $\mathcal{N} = 4$ supersymmetry and are conjectured to be equivalent to the $\mathfrak{g} = \mathfrak{su}(3)$ and $\mathfrak{g} = \mathfrak{so}(6)$ $\mathcal{N} = 4$ theories with gauge coupling $\tau = e^{\pi i/3}$ and $\tau =$ any [247].

$\mathfrak{g} = \mathfrak{so}(2N) = D_N$ Equating (5.21) with (5.9) gives

$$2N^2 - N = kN^2 + (2\ell - k - 1)N\,. \tag{F.36}$$

Again we have solution for $N = \ell = 1$, k arbitrary as well as for $k = 2$, $\ell = 1$ with N arbitrary. The first case is the same as for $\mathfrak{g} = \mathfrak{u}(1)$. For $k = 2$ we always have enhancement to $\mathcal{N} = 4$, in our language these theories are $S^N_{2,1}$.

$\mathfrak{g} = \mathfrak{so}(2N+1) = B_N$ Equating (5.21) with (5.9)

$$2N^2 + N = kN^2 + (2\ell - k - 1)N\,. \tag{F.37}$$

Which has solution only for $k = \ell = 2$ with N arbitrary, $N = 1$ $\ell = 2$ with k arbitrary, $k = 4$ $\ell = 1$ $N = 2$ and $k = 3$ $\ell = 1$ $N = 3$. In the first three cases $S^N_{2,2}$, $S^1_{k,2}$ and $S^2_{4,1}$ always have enhancement to $\mathcal{N} = 4$ [247]. They have the correct spectrum of Coulomb branch operators to be equivalent to the $\mathfrak{g} = B_N$ or C_N, B_1 and $B_2 \cong C_2$ $\mathcal{N} = 4$ theories respectively. In the final case we find that the S-fold $S^3_{3,1}$ has the same central charges as the $\mathcal{N} = 4$ $\mathfrak{so}(7)$ theory. Clearly they are not the same theory, however due to the matching of central charges we cannot rule out the possibility that the discrete gauging $S^3_{3,1,p'}$ may yield the same theory as a $\mathbb{Z}_n = \mathbb{Z}_{p'}$ gauging of $\mathcal{N} = 4$ $\mathfrak{so}(7)$ theory. Our analysis (5.15) would seem to imply that this is infact not the case however, since the $S^3_{3,1}$ theory has a discrete $\mathbb{Z}_3$ global symmetry while the $\mathfrak{so}(7)$ theory can only have $\mathbb{Z}_2, \mathbb{Z}_4, \mathbb{Z}_4, \mathbb{Z}_8$ discrete symmetry groups.

$\mathfrak{g} = \mathfrak{sp}(N) = C_N$ Since the degree of the Casimir invariants for $\mathfrak{sp}(N)$ are the same as for $\mathfrak{so}(2N+1)$ the discussion is the same as above.

$\mathfrak{g} = E_6$

$$78 = 36k + 6(2\ell - k - 1)\,. \tag{F.38}$$

There is solution for $k = \ell = 2$, the corresponding $S^6_{2,2}$ has a dimension 2 Coulomb branch operator and therefore $\mathcal{N} = 4$ enhancement this S-fold is the standard $\mathfrak{g} = B_6$ or C_6 perturbative orientifold.

$\mathfrak{g} = E_7$

$$133 = 49k + 7(2\ell - k - 1)\,, \tag{F.39}$$

There is solution only for $k = 3$, $\ell = 1$. There is no $\mathcal{N} = 4$ enhancement of the corresponding $S^7_{3,1}$ theory.

$\mathfrak{g} = E_8$

$$248 = 64k + 8(2\ell - k - 1)\,, \tag{F.40}$$

There is solution only for $k = 4$, $\ell = 2$. There is no $\mathcal{N} = 4$ enhancement however $S^8_{4,2}$ does not exist as an S-fold [247].

$\mathfrak{g} = F_4$

$$52 = 16k + 4(2\ell - k - 1)\,, \tag{F.41}$$

There is solution only for $k = 4$, $\ell = 1$. There is no $\mathcal{N} = 4$ enhancement.

$\mathfrak{g} = G_2$

$$14 = 4k + 2(2\ell - k - 1)\,, \tag{F.42}$$

There is solution only for $k = 6$ $\ell = 1$ and $k = 4$, $\ell = 2$. In the first case $S^2_{6,1}$ exists as an S-fold but there is $\mathcal{N} = 4$ enhancement and it is believed to be equal to the G_2 $\mathcal{N} = 4$ SYM theory with fixed gauge coupling. In the second case the S-folds of type $S^2_{4,2}$ do not fall into the classification of [247] and are believed to not exist as S-folds.

Appendix G

Appendices for Chapter 6

G.1 Elliptic Genus Computation

The Elliptic genus for 2d $\mathcal{N} = (0,2)$ theories was computed via localisation in [223, 224]. Writing schematically it is given by

$$\mathrm{Ell}(q,a) = \mathrm{Tr}_{\mathrm{R}}\left[(-1)^F q^{H_-} \prod a^f\right] . \tag{G.1}$$

A $\mathcal{N} = (0,2)$ chiral multiplet Φ in representations R of $G \times F$ contributes to Ell

$$\mathrm{Ell}_{\Phi,R}(q,a) = \prod_{\rho \in R} \mathrm{i}\frac{\eta(q)}{\theta_1(a^\rho;q)} . \tag{G.2}$$

Similarly, a $\mathcal{N} = (0,2)$ Fermi multiplet Ψ contributes

$$\mathrm{Ell}_{\Psi,R}(q,a) = \prod_{\rho \in R} \mathrm{i}\frac{\theta_1(a^\rho;q)}{\eta(q)} . \tag{G.3}$$

Finally we present the formula for the $\mathcal{N} = (0,2)$ Vector multiplet V

$$\mathrm{Ell}_{V,G}(q,a) = (\mathrm{i}\eta(q))^{2\,\mathrm{rank}\,G} \prod_{\alpha \in G} \mathrm{i}\frac{\theta_1(a^\alpha;q)}{\eta(q)} , \tag{G.4}$$

which is that of a Fermi multiplet in the adjoint representation of G. The vector multiplets should be paired with the corresponding integration over the maximal torus of G

$$\frac{1}{|W(G)|}\oint_{T[G]}\prod_{n=1}^{\operatorname{rank}G}\frac{da_n}{2\pi \mathrm{i} a_n}\,, \tag{G.5}$$

which is essentially the Haar measure divided by the Vandermonde determinant. $W(G)$ is the Weyl group of G and the contour is taken over $|a| = 1$.

G.1.1 M-Strings Index without Defect

The partition functions corresponding to each $\mathcal{N} = (0, 2)$ multiplet read

$$\mathrm{Ell}_Y = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{J=1}^{K_{n+1}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{c}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{n,I}}{y_{n+1,J}};Q_\tau\right)}\,, \tag{G.6}$$

$$\mathrm{Ell}_{\tilde{Y}} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{J=1}^{K_{n-1}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\frac{1}{\mathfrak{c}}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{n,I}}{y_{n-1,J}};Q_\tau\right)}\,, \tag{G.7}$$

$$\mathrm{Ell}_\zeta = \prod_{n=1}^{M}\prod_{I,J=1}^{K_n} \mathrm{i}\frac{\theta_1\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{G.8}$$

$$\mathrm{Ell}_\Upsilon = \prod_{n=1}^{M}(\mathrm{i}\eta(Q_\tau))^{2K_n}\prod_{I\neq J} \mathrm{i}\frac{\theta_1\left(\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{G.9}$$

$$\mathrm{Ell}_X = \prod_{n=1}^{M}\prod_{I,J=1}^{K_n} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{q}\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)}\,, \tag{G.10}$$

$$\mathrm{Ell}_{\tilde{X}} = \prod_{n=1}^{M}\prod_{I,J=1}^{K_n} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{t}^{-1}\frac{y_{n,I}}{y_{n,J}};Q_\tau\right)}\,, \tag{G.11}$$

$$\mathrm{Ell}_\lambda = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{J=1}^{K_{n+1}} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}\sqrt{\mathfrak{q}\mathfrak{t}}\frac{y_{n,I}}{y_{n+1,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{G.12}$$

$$\mathrm{Ell}_{\tilde{\lambda}} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{J=1}^{K_{n-1}} \mathrm{i}\frac{\theta_1\left(\frac{\sqrt{\mathfrak{q}\mathfrak{t}}}{\mathfrak{c}}\frac{y_{n,I}}{y_{n-1,J}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{G.13}$$

$$\mathrm{Ell}_\phi = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{y_{n,I}}{x_{n,A}};Q_\tau\right)}\,, \tag{G.14}$$

$$\mathrm{Ell}_{\tilde{\phi}} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{x_{n,A}}{y_{n,I}};Q_\tau\right)}, \tag{G.15}$$

$$\mathrm{Ell}_{\psi} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}^{-1}\frac{y_{n,I}}{x_{n-1,A}};Q_\tau\right)}{\eta(Q_\tau)}, \tag{G.16}$$

$$\mathrm{Ell}_{\tilde{\psi}} = \prod_{n=1}^{M}\prod_{I=1}^{K_n}\prod_{A=1}^{N} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}^{-1}\frac{x_{n+1,J}}{y_{n,I}};Q_\tau\right)}{\eta(Q_\tau)}. \tag{G.17}$$

G.1.2 M-Strings Index with Defects

We now list the result of performing the $\mathbb{Z}_k$ orbifold.

$$\mathrm{Ell}_{Y}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{(n+1)(i-1)}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{c}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{ni,I}}{y_{(n+1)(i-1),J}};Q_\tau\right)}, \tag{G.18}$$

$$\mathrm{Ell}_{\Upsilon}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}(\mathrm{i}\eta(Q_\tau))^{2K_{ni}}\prod_{I\neq J} \mathrm{i}\frac{\theta_1\left(\frac{y_{ni,I}}{y_{ni,J}};Q_\tau\right)}{\eta(Q_\tau)}, \tag{G.19}$$

$$\mathrm{Ell}_{\tilde{Y}}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{(n-1)i}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{c}^{-1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\frac{y_{ni,I}}{y_{(n-1)i,J}};Q_\tau\right)}, \tag{G.20}$$

$$\mathrm{Ell}_{\zeta}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{n(i+1)}} \mathrm{i}\frac{\theta_1\left(\frac{\mathfrak{q}}{\mathfrak{t}}\frac{y_{ni,I}}{y_{n(i+1),J}};Q_\tau\right)}{\eta(Q_\tau)}, \tag{G.21}$$

$$\mathrm{Ell}_{X}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{n(i+1)}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{q}\frac{y_{ni,I}}{y_{n(i+1),J}};Q_\tau\right)}, \tag{G.22}$$

$$\mathrm{Ell}_{\tilde{X}}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I,J=1}^{K_{ni}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\mathfrak{t}^{-1}\frac{y_{ni,I}}{y_{ni,J}};Q_\tau\right)}, \tag{G.23}$$

$$\mathrm{Ell}_{\lambda}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{(n+1)i}} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}\mathfrak{q}\mathfrak{t}\frac{y_{ni,I}}{y_{(n+1)i,J}};Q_\tau\right)}{\eta(Q_\tau)}, \tag{G.24}$$

$$\mathrm{Ell}_{\tilde{\lambda}}^{\mathbb{Z}_k} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{J=1}^{K_{(n-1)(i-1)}} \mathrm{i}\frac{\theta_1\left(\frac{\mathfrak{q}\mathfrak{t}}{\mathfrak{c}}\frac{y_{n,I}}{y_{(n-1)(i-1),J}};Q_\tau\right)}{\eta(Q_\tau)}, \tag{G.25}$$

$$\mathrm{Ell}^{\mathbb{Z}_k}_{\phi} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{A=1}^{N_{ni}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{y_{ni,I}}{x_{ni,A}};Q_\tau\right)}\,, \tag{G.26}$$

$$\mathrm{Ell}^{\mathbb{Z}_k}_{\tilde{\phi}} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{A=1}^{N_{n(i-1)}} \mathrm{i}\frac{\eta(Q_\tau)}{\theta_1\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\frac{x_{n(i-1),A}}{y_{ni,I}};Q_\tau\right)}\,, \tag{G.27}$$

$$\mathrm{Ell}^{\mathbb{Z}_k}_{\psi} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{A=1}^{N_{(n-1)i}} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}^{-1}\frac{y_{ni,I}}{x_{(n-1)i,A}};Q_\tau\right)}{\eta(Q_\tau)}\,, \tag{G.28}$$

$$\mathrm{Ell}^{\mathbb{Z}_k}_{\tilde{\psi}} = \prod_{n=1}^{M}\prod_{i=1}^{k}\prod_{I=1}^{K_{ni}}\prod_{A=1}^{N_{(n+1)(i-1)}} \mathrm{i}\frac{\theta_1\left(\mathfrak{c}^{-1}\frac{x_{(n+1)(i-1),A}}{y_{ni,I}};Q_\tau\right)}{\eta(Q_\tau)}\,. \tag{G.29}$$

Clearly each $\mathrm{Ell}^{\mathbb{Z}_k}_{P}$ is invariant under the action (6.57).

G.2 Review of the 5d Index Localisation

We describe in more detail, following [41], the localisation of the superconformal index

$$Z\left(s,p,v,\mathbf{q}_A\right) = \mathrm{Tr}_{\mathcal{H}_{\mathbb{S}^4}}\left[(-1)^F e^{-\beta\delta} s^{-2J_R-2J_R^R} p^{-2J_L} v^{2J_L^R}\prod_{A=1}^{N}\mathbf{q}_A^{K_A}\right]. \tag{G.30}$$

It receives contributions only from those states which satisfy the BPS condition (6.67). After Wick rotation to Euclidean time $X^5 = \mathrm{i}\tau_E$ the index Z admits a path integral representation on $\mathbb{S}^1\times\mathbb{S}^4$

$$Z(s,p,v,\mathbf{q}_A) = \int_{\mathbf{C}(\mathbb{S}^1\times\mathbb{S}^4)}[\mathcal{D}\Phi]e^{-S_E[\Phi]}\,. \tag{G.31}$$

The insertion of chemical potentials results in twisted boundary conditions upon going around the $\mathbb{S}^1$

$$\Phi(\tau_E+\beta) = (-1)^F e^{-\beta(-2J_R-3J_R^R)} s^{-2J_R-2J_R^R} p^{-2J_L} v^{2J_L^R}\Phi(\tau_E)\,. \tag{G.32}$$

The twisted boundary conditions (G.32) may be taken into account by shifting time derivatives

$$\partial_{\tau_E}\to\hat{\partial}_{\tau_E} = \partial_{\tau_E} + (2-\mathrm{i}\epsilon_+)J_R + (3-\mathrm{i}\epsilon_+)J_R^R - \mathrm{i}\epsilon_- J_L + 2mJ_L^R \tag{G.33}$$

and giving periodic boundary conditions to all fermions to account for the insertion of $(-1)^F$. To perform the localisation we deform the Lagrangian by the $\widetilde{\mathtt{Q}}$ exact term

$$\mathcal{L} \to \mathcal{L} + t\left\{\widetilde{\mathtt{Q}}, V\right\} , \tag{G.34}$$

t may then be taken to infinity in which case the path integral localises around the set of saddle points of $\{\widetilde{\mathtt{Q}}, V\} = 0$. The supersymmetry transformation $\widetilde{\mathtt{Q}} + \widetilde{\mathtt{S}}$ is parametrised by the Killing spinor

$$\varepsilon = \varepsilon_q + \varepsilon_s = e^{\frac{1}{2}\theta_1\gamma^{51}} e^{\frac{1}{2}\theta_2\gamma^{12}} e^{\frac{1}{2}\theta_3\gamma^{23}} e^{\frac{1}{2}\theta_4\gamma^{34}} \left(\varepsilon_0^q + \gamma^5\varepsilon_0^s\right) . \tag{G.35}$$

where ε_0^q, ε_0^s are constant spinors corresponding to $\widetilde{\mathtt{Q}}$, $\widetilde{\mathtt{S}}$. ε satisfies the Killing spinor equation

$$\hat{\nabla}_\mu \varepsilon = \frac{1}{2}\gamma_\mu\gamma^5\tilde{\varepsilon} \tag{G.36}$$

where $\hat{\nabla}$ denotes the covariant derivative on $\mathbb{S}^4 \times \mathbb{S}^1$ with the twisted time derivative (G.33) and $\tilde{\varepsilon} = -\varepsilon_q + \varepsilon_s$. The square of the supercharge $\widetilde{\mathtt{Q}} + \widetilde{\mathtt{S}}$ is then given by

$$\delta_\varepsilon^2 = -\mathrm{i}\mathcal{L}_{\frac{\partial}{\partial\tau_E}} + \mathrm{i}G - \epsilon_+ J_R - \epsilon_+ J_R^R - \epsilon_- J_L + 2mJ_L^R \tag{G.37}$$

where $\mathcal{L}_v$ denotes the Lie derivative and G denotes a gauge transformation. We then choose V such that

$$\{\widetilde{\mathtt{Q}}, V\} = \delta_\varepsilon\left((\delta_\varepsilon\lambda)^\dagger \lambda\right) \tag{G.38}$$

upon taking the $t \to \infty$ limit the path integral for the vector multiplets localises onto the critical points of the potential

$$\begin{aligned}(\delta_\varepsilon\lambda)^\dagger \lambda =& F_{\tau_E\mu}F^{\tau_E\mu} + \cos^2\frac{\theta_1}{2}\left(F_{ij}^- - \omega_{ij}^-\phi\right)^2 + \sin^2\frac{\theta_1}{2}\left(F_{ij}^+ - \omega_{ij}^+\right)^2 \\ &+ \left(\hat{\nabla}_\mu\phi\right)^2 - D^2\end{aligned} \tag{G.39}$$

which is positive semi-definite (recall that in Euclidean signature D is pure imaginary) $F_{ij}^\pm = \frac{1}{2}\left(F_{ij} \mp \star_4 F_{ij}\right)$ and

$$\omega_{ij}^+ = \frac{\mathrm{i}}{2\sin^2\frac{\theta_1}{2}}\bar{\tilde{\varepsilon}}^+\gamma^5\gamma^{ij}\varepsilon^+ , \quad \omega_{ij}^- = \frac{\mathrm{i}}{2\cos^2\frac{\theta_1}{2}}\bar{\tilde{\varepsilon}}^-\gamma^5\gamma^{ij}\varepsilon^- , \tag{G.40}$$

$$\omega_{ij}^+\omega^{+ij} = \omega_{ij}^-\omega^{-ij} = 1 . \tag{G.41}$$

Here $\varepsilon^{\pm} = \frac{1}{2}\left(1 \pm \gamma^5\right)\varepsilon$, $\tilde{\varepsilon}^{\pm} = \frac{1}{2}\left(1 \pm \gamma^5\right)\tilde{\varepsilon}$. The classical saddle points of the potential (G.39) are given by $F_{\tau_E \mu} = D^A = 0$ while ϕ is covariantly constant everywhere on $\mathbb{S}^4$. On the other hand, by the Bianchi identity, the second and third terms of (G.39) imply that away from the north and south poles we have $F_{ij} = \phi = 0$. Note also that (anti-)self-dual instantons ($F^- = 0$) $F^+ = 0$ can be localised at (north)south-pole. The index hence factorises as in (6.69)

$$Z = \int \prod_{A=1}^{N} [da_A]\, Z_{\text{south}}\left(a_{A,n}, \epsilon_1, \epsilon_2, m, \mathbf{q}_A\right) Z_{\text{north}}\left(a_{A,n}, \epsilon_1, \epsilon_2, m, \mathbf{q}_A^{-1}\right) \tag{G.42}$$

After the gauge fixing we have a BRST operator $\widetilde{\mathtt{Q}} + \widetilde{\mathtt{S}} \to \mathfrak{Q}$ and may make a cohomological formulation of the supercharge $\mathfrak{Q}$. The bosonic and fermionic fields may be regarded as differential forms on a supermanifold $\mathcal{X}$ such that they form a $\mathfrak{Q}$-complex

$$\mathfrak{Q}\Phi_{b,f} = \Phi'_{f,b}\,, \quad \mathfrak{Q}\Phi'_{f,b} = \mathfrak{Q}^2\Phi_{b,f} \tag{G.43}$$

and

$$\begin{aligned}\mathfrak{Q}^2 =& \mathcal{L}_{\frac{\partial}{\partial \tau_E}} - \frac{a}{\beta} - \mathrm{i}\epsilon_+(J_R + J_R^R) - \mathrm{i}\epsilon_- J_L + 2\mathrm{i}mJ_L^R \\ =& \mathcal{L}_{\frac{\partial}{\partial \tau_E}} - \frac{a}{\beta} + \mathrm{i}\epsilon_1(J_{12} - J_R^R) + \mathrm{i}\epsilon_2(J_{34} - J_R^R) + 2\mathrm{i}mJ_L^R\,.\end{aligned} \tag{G.44}$$

We now study the $\mathfrak{Q}^2$-equivariant cohomology of $\mathcal{X}$. After expanding the gauge fixed $\mathfrak{Q}$ invariant terms to quadratic order the Gaussian integrals may be evaluated. Due to cancelations due to pairing by the $\mathfrak{Q}$-complex (G.43) the 1-loop contributions take the form [41]

$$Z^{\text{1-loop}} = \sqrt{\frac{\det_{\text{coker}\, D} \mathfrak{Q}^2|_f}{\det_{\ker D} \mathfrak{Q}^2|_b}} \tag{G.45}$$

where D is the quadratic operator in (G.38). $Z^{\text{1-loop}}$ may computed using the equivariant Atiyah-Singer index theorem. The fixed point of the torus action of $\mathfrak{Q}^2$ (G.44) are the north and south poles. In a neighbourhood of the north pole D is isomorphic to the anti-self-dual complex (d, d^-)

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^-} \Omega^{2-}\,, \tag{G.46}$$

while at the south pole D is isomorphic to the self-dual complex (d, d^+)

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^+} \Omega^{2+}\,, \tag{G.47}$$

where (Ω^{2-}) Ω^{2+} denotes the space of (anti-)self-dual 2-forms. For concreteness let us focus on the south pole. At the south pole we may choose local coordinates z_1, z_2 parametrising $\mathbb{C}^2$. The torus action acts on those coordinates by
$(z_1, z_2) \mapsto (e^{\mathrm{i}\epsilon_1} z_1, e^{\mathrm{i}\epsilon_2} z_2)$. Furthermore we expand the elements of the self-dual complex in eigenmodes of the circle momenta $\Phi = \sum_{p\in\mathbb{Z}} \Phi_p e^{\frac{2\pi \mathrm{i} p}{\beta}}$. The equiviarint index for the vectors multiplets is then given by

$$\mathrm{ind}_{\mathrm{vec}}\, D = -\frac{1 + e^{\mathrm{i}\beta\epsilon_+}}{2(1 - \epsilon^{\mathrm{i}\epsilon_1})(1 - \epsilon^{\mathrm{i}\epsilon_2})} \sum_{A=1}^{N} \sum_{n\neq m} e^{\frac{\mathrm{i}}{\beta}(a_{A,n} - a_{A,m})} \sum_{p\in\mathbb{Z}} e^{\frac{2\pi \mathrm{i} p}{\beta}}\,. \tag{G.48}$$

One may compute the 1-loop determinants for the hypermultiplets by localisation. In that case the differential operator D for the hypermultiplet is isomorphic to a Dirac complex

$$\Omega^{\left(\frac{1}{2},0\right)} \xrightarrow{D_{\mathrm{Dirac}}} \Omega^{\left(0,\frac{1}{2}\right)}\,. \tag{G.49}$$

The equivariant index at the south pole reads

$$\mathrm{ind}_{\mathrm{hyp}} = \frac{e^{\mathrm{i}\epsilon_+/2}}{(1 - e^{\mathrm{i}\epsilon_1})(1 - e^{\mathrm{i}\epsilon_2})} \sum_{A=1}^{N} \sum_{m=1}^{M_A} \sum_{n=1}^{M_{A+1}} e^{\frac{\mathrm{i}}{\beta}(a_{A,n} - a_{A+1,m}) + \mathrm{i}m} \sum_{p\in\mathbb{Z}} e^{\frac{2\pi \mathrm{i}}{\beta} p}\,. \tag{G.50}$$

G.3 Refined Topological String Computation

The partition function for M M5-branes on A_{N-1} is equivalent to the refined topological string partition function of certain Calabi-Yau 3-folds $X_{M,N}$

$$Z^{\mathrm{top}}_{\mathrm{refined}}\left(X_{M,N}\right) = Z^{\text{M-theory}} \left((A_{N-1} \times \underbrace{\mathbb{R}^4) \ltimes T^2}_{M \text{ M5-branes}} \times \mathbb{R} \right) \tag{G.51}$$

$$= \left(\prod_{A=1}^{N} Z^{(A)}_{U(1)} \right) Z_M^{A_{N-1}}\,, \tag{G.52}$$

where $Z^{(A)}_{U(1)}$ is the partition function for a single M5-brane. The Calabi-Yau 3-folds arise as the M-theory lift of the dual (p,q)-brane web construction. The (p,q) web is obtained by compactifying the setup of Table 6.1 along X^6. This gives rise to the 5d quiver gauge theories $\mathcal{N}_{M,N}$ supported on the D4-branes with $\mathcal{N} = 1$ supersymmetry. The defects become D4'-branes. T-dualising along the Taub-Nut circle X^8 results the setup of Table G.1 After turning on mass deformation we end

	$\mathbb{C}_{\epsilon_1}$		$\mathbb{C}_{\epsilon_2}$		$\mathbb{S}^1$	$\mathbb{R}$	$\mathbb{S}^1$	$\mathbb{R}^3$		
	X^1	X^2	X^3	X^4	X^5	X^7	X^8	X^9	X^{10}	X^{11}
$N\ NS5$	$-$	$-$	$-$	$-$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$M\ D5$	$-$	$-$	$-$	$-$	$-$	$\cdot$	$-$	$\cdot$	$\cdot$	$\cdot$
$K\ F1$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-$	$-$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$k\ D3'$	$\cdot$	$\cdot$	$-$	$-$	$-$	$\cdot$	$\cdot$	$-$	$\cdot$	$\cdot$

Table G.1: *Type-IIB (p,q)-brane web.*

up with a (p,q)-brane web. We may now lift the IIB setup on $\mathbb{S}^1$ to M-theory on T^2. (p,q)-branes corresponds to the degeneration of the (p,q) cycle of the T^2 as we vary along the X^5, X^7 base. Hence the (p,q)-brane web lifts to M-theory on a non-compact, elliptically fibered CY_3 whose toric diagram is given by the (p,q)-web itself. We compute the refined A-model open string amplitude for the strip geometry using the refined topological vertex formalism [302, 187]. The refined topological vertex is labelled by three Young diagrams and is given by

$$\begin{aligned} &C_{\lambda\mu\nu}(\mathbf{t},\mathbf{q}) = \\ &\mathfrak{t}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}}\mathfrak{q}^{\frac{||\mu||^2+||\nu||^2}{2}}\widetilde{Z}_\nu(\mathfrak{t},\mathfrak{q})\sum_\eta\left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{|\lambda|+|\eta|-|\mu|}{2}} s_{\lambda^{\mathrm{T}}/\eta}(\mathfrak{t}^{-\rho}\mathfrak{q}^{-\nu})s_{\mu/\eta}(\mathfrak{t}^{-\nu^{\mathrm{T}}}\mathfrak{q}^{-\rho}) \end{aligned} \tag{G.53}$$

where $\rho = \{-1/2, -3/2, -5/2, \dots\}$, $s_{\lambda/\eta}(x)$ is the skew Schur function and

$$\widetilde{Z}_\nu(\mathfrak{t},\mathfrak{q}) = \prod_{(l,p)\in\nu}\frac{1}{1-\mathfrak{q}^{\nu_l-p}\mathfrak{t}^{\nu_p^{\mathrm{T}}-l+1}}\,. \tag{G.54}$$

$\mathfrak{q},\mathfrak{t}$ are related torus action $(z_1,z_2)\mapsto(e^{2\pi\mathrm{i}\epsilon_1}z_1, e^{2\pi\mathrm{i}\epsilon_2}z_2)$ on $\mathbb{C}^2$ by

$$\mathfrak{q} = e^{2\pi\mathrm{i}\epsilon_1}\,,\quad \mathfrak{t} = e^{-2\pi\mathrm{i}\epsilon_2}\,. \tag{G.55}$$

The framing factors are

$$f_\nu(\mathfrak{t},\mathfrak{q}) = (-1)^{|\nu|}\mathfrak{t}^{\frac{||\nu^{\mathrm{T}}||^2}{2}}\mathfrak{q}^{\frac{-||\nu||^2}{2}}\,,\quad \widetilde{f}_\nu(\mathfrak{t},\mathfrak{q}) = \left(\frac{\mathfrak{t}}{\mathfrak{q}}\right)^{\frac{|\nu|}{2}} f_\nu(\mathfrak{t},\mathfrak{q})\,. \tag{G.56}$$

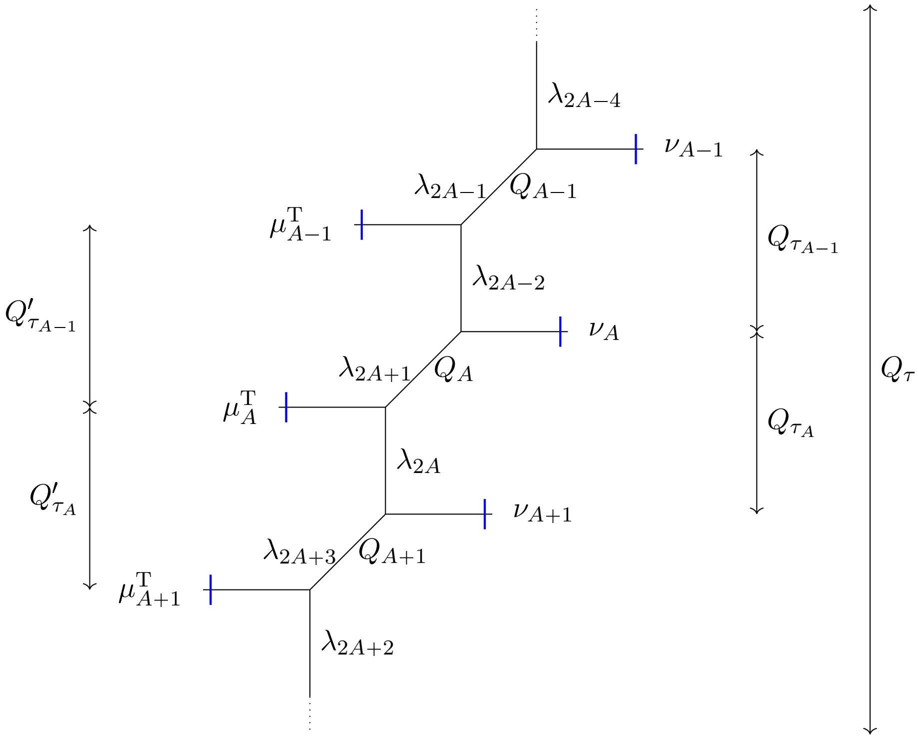


Figure G.1: *Strip geometry which builds the partition function of for the* A_{N-1} *geometry. Blue lines denote the direction of the refined topological vertex. The dotted lines indicate the fact that we are dealing with the partial compactification of the strip geometry.*

G.3.1 Without Defect - M-Strings Review

Applying the standard rules of the refined topological vertex formalism [187] the partition function for the strip geometry Figure G.1 is

$$
\begin{aligned}
Z^{\mu_1,\dots\mu_N}_{\nu_1\dots\nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A}; \mathfrak{t}, \mathfrak{q}) = \sum_{\{\lambda\}} \prod_{A=1}^{N} \Big\{ (-Q_A)^{|\lambda_{2A+1}|} \left(-Q_A^{-1} Q_{\tau_A}\right)^{|\lambda_{2A}|} \\
\times C_{\lambda^{\mathrm{T}}_{2A}\lambda^{\mathrm{T}}_{2A+1}\mu^{\mathrm{T}}_A}\left(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}\right) C_{\lambda_{2A+2}\lambda_{2A+1}\nu_A}\left(\mathfrak{q}^{-1}, \mathfrak{t}^{-1}\right) \Big\},
\end{aligned}
\tag{G.57}
$$

note that, since we partially compactify the strip geometry (dotted lines in the figure) we identify the indices $A \sim A + N$. $Q_\tau = \prod_{A=1}^N Q_{\tau_A} = \prod_{A=1}^N Q'_{\tau_A}$. Inserting the explicit expression for the vertex we have

$$
\begin{aligned}
Z^{\mu_1,\dots\mu_N}_{\nu_1\dots\nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A}; \mathfrak{t}, \mathfrak{q}) = \prod_{A=1}^{N} \Bigg\{ \mathfrak{q}^{-\frac{||\mu_A^{\mathrm{T}}||^2}{2}} \mathfrak{t}^{-\frac{||\nu_A||^2}{2}} \widetilde{Z}_{\mu_A^{\mathrm{T}}}\left(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}\right) \\
\widetilde{Z}_{\nu_A}\left(\mathfrak{q}^{-1}, \mathfrak{t}^{-1}\right) \sum_{\{\lambda\},\{\sigma\}} \Bigg[(-Q_A)^{|\lambda_{2A+1}|} \left(-\frac{Q_{\tau_A}}{Q_A}\right)^{|\lambda_{2A}|} s_{\lambda_{2A}/\sigma_{2A}}\left(\mathfrak{t}^{\rho}\mathfrak{q}^{\mu_A^{\mathrm{T}}}\right) \\
s_{\lambda^{\mathrm{T}}_{2A+1}/\sigma_{2A}}\left(\mathfrak{q}^{\rho+\frac{1}{2}}\mathfrak{t}^{\mu_A-\frac{1}{2}}\right) s_{\lambda^{\mathrm{T}}_{2A+2}/\sigma_{2A+1}}\left(\mathfrak{q}^{\rho}\mathfrak{t}^{\nu_A}\right) \\
s_{\lambda_{2A+1}/\sigma_{2A+1}}\left(\mathfrak{q}^{\nu_A^{\mathrm{T}}-\frac{1}{2}}\mathfrak{t}^{\rho+\frac{1}{2}}\right) \Bigg] \Bigg\}.
\end{aligned}
\tag{G.58}
$$

The method for simplifying this product was given in [210, 209]. Consider

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) := \prod_{A=1}^{N} \Bigg((-Q_A)^{|\lambda_{2A+1}|} \left(\frac{-Q_{\tau_A}}{Q_A}\right)^{|\lambda_{2A}|} \\
\times s_{\lambda_{2A}/\sigma_{2A}}(X_A)\, s_{\lambda^{\mathrm{T}}_{2A+2}/\sigma_{2A+1}}(Y_A)\, s_{\lambda^{\mathrm{T}}_{2A+1}/\sigma_{2A}}(Z_A)\, s_{\lambda_{2A+1}/\sigma_{2A+1}}(W_A)\Bigg)
\end{aligned}
\tag{G.59}
$$

with the products over A are defined modulo N. We now apply repeatedly the identities (A.50), (A.51) and (A.52). We have

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) = \prod_{A=1}^{N} \left(-Q_A^{-1}Q_{\tau_A}\right)^{|\sigma_{2A}|} (-Q_A)^{|\sigma_{2A+1}|} \\
\times\, s_{\sigma^{\mathrm{T}}_{2A}/\lambda_{2A}}(Y_{A-1})\, s_{\sigma^{\mathrm{T}}_{2A-1}/\lambda^{\mathrm{T}}_{2A}}\left(-Q_A^{-1}Q_{\tau_A}X_A\right) s_{\sigma^{\mathrm{T}}_{2A}/\lambda^{\mathrm{T}}_{2A+1}}(-Q_A W_A) \\
\times\, s_{\sigma^{\mathrm{T}}_{2A+1}/\lambda_{2A+1}}(Z_A) \prod_{l,p=1}^{\infty} \left(1 - Q_A^{-1}Q_{\tau_A}X_{A;l}Y_{A-1;p}\right)\left(1 - Q_A Z_{A;l} W_{A;p}\right)
\end{aligned}
\tag{G.60}
$$

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) &= \prod_{A=1}^{N} (-Q_{\tau_A})^{|\lambda_{2A+1}|} \\
&\times \prod_{l,p=1}^{\infty} \frac{\left(1 - Q_A^{-1} Q_{\tau_A} X_{A;l} Y_{A-1;p}\right)\left(1 - Q_A Z_{A;l} W_{A;p}\right)}{\left(1 - Q_{\tau_A} W_{A;l} Y_{A-1;p}\right)\left(1 - Q_A Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1;l} Z_{A;p}\right)} \\
&\times s_{\lambda_{2A+1}^{\mathrm{T}}/\sigma_{2A}^{\mathrm{T}}}\left(Y_{A-1}\right) s_{\lambda_{2A}/\sigma_{2A}^{\mathrm{T}}}\left(-Q_{\tau_A} W_A\right) s_{\lambda_{2A}^{\mathrm{T}}/\sigma_{2A-1}^{\mathrm{T}}}\left(-Q_{A-1} Z_{A-1}\right) \\
&\times s_{\lambda_{2A+1}/\sigma_{2A+1}^{\mathrm{T}}}\left(-Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1}\right)
\end{aligned}
\tag{G.61}
$$

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) &= \prod_{A=1}^{N} (-Q_{\tau_A})^{|\sigma_{2A}|} \\
&\times \prod_{l,p=1}^{\infty} \left\{ \frac{\left(1 - Q_A^{-1} Q_{\tau_A} X_{A;l} Y_{A-1;p}\right)\left(1 - Q_A Z_{A;l} W_{A;p}\right)}{\left(1 - Q_{\tau_A} W_{A;l} Y_{A-1;p}\right)\left(1 - Q_A Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1;l} Z_{A;p}\right)} \right. \\
&\times \left. \left(1 - Q_{A-1} Z_{A-1} Q_{\tau_A} W_A\right)\left(1 - Q_{A+1}^{-1} Q_{\tau_A} Q_{\tau_{A+1}} X_{A+1} Y_{A-1}\right) \right\} \\
&\times s_{\sigma_{2A}/\lambda_{2A+1}^{\mathrm{T}}}\left(-Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1}\right) s_{\sigma_{2A}/\lambda_{2A}}\left(-Q_{\tau_{A-1}} Z_{A-1}\right) \\
&\times s_{\sigma_{2A-1}/\lambda_{2A}^{\mathrm{T}}}\left(Q_{\tau_A} W_A\right) s_{\sigma_{2A+1}/\lambda_{2A+1}}\left(Q_{\tau_A} Y_{A-1}\right)
\end{aligned}
\tag{G.62}
$$

$$
\begin{aligned}
G^{(N)}(X_A, Y_A, Z_A, W_A; Q_A, Q_{\tau_A}) &= \prod_{A=1}^{N} \left(-Q_A^{-1} Q_{\tau_A}\right)^{|\lambda_{2A}|} (-Q_A)^{|\lambda_{2A+1}|} \\
& s_{\lambda_{2A+1}^{\mathrm{T}}/\sigma_{2A}}\left(Q_{A-1} Q_A^{-1} Q_{\tau_A} Z_{A-1}\right) s_{\lambda_{2A}/\sigma_{2A}}\left(Q_{A+1}^{-1} Q_{\tau_{A+1}} Q_A X_{A+1}\right) \\
& s_{\lambda_{2A+1}/\sigma_{2A+1}}\left(Q_{\tau_{A+1}} W_{A+1}\right) s_{\lambda_{2A+2}^{\mathrm{T}}/\sigma_{2A+1}}\left(Q_{\tau_A} Y_{A-1}\right) \\
& \prod_{l,p=1}^{\infty} \left\{ \frac{\left(1 - Q_A^{-1} Q_{\tau_A} X_{A;l} Y_{A-1;p}\right)\left(1 - Q_A Z_{A;l} W_{A;p}\right)}{\left(1 - Q_{\tau_A} W_{A;l} Y_{A-1;p}\right)\left(1 - Q_{A+1}^{-1} Q_{\tau_{A+1}} Q_A X_{A+1;l} Z_{A;p}\right)} \right. \\
& \frac{\left(1 - Q_{A+1}^{-1} Q_{\tau_A} Q_{\tau_{A+1}} Y_{A-1;l} X_{A+1;p}\right)}{\left(1 - Q_{\tau_A} Q_{A-1} Q_{A+1}^{-1} Q_{\tau_{A+1}} X_{A+1;l} Z_{A-1}; p\right)} \\
& \left. \frac{\left(1 - Q_{\tau_A} Q_{A-1} W_{A;l} Z_{A-1;p}\right)}{\left(1 - Q_{\tau_{A+1}} Q_{\tau_A} Y_{A-1;l} W_{A+1;p}\right)} \right\} .
\end{aligned}
\tag{G.63}
$$

Finally, this can be written as

$$
\begin{aligned}
&G^{(N)}(X_A,Y_A,Z_A,W_A;Q_A,Q_{\tau_A}) = \\
&\prod_{A=1}^{N}\prod_{l,p=1}^{\infty}\left\{\frac{\left(1-Q_A^{-1}Q_{\tau_A}X_{A;l}Y_{A-1;p}\right)\left(1-Q_AZ_{A;l}W_{A;p}\right)}{\left(1-Q_{\tau_A}W_{A;l}Y_{A-1;p}\right)\left(1-Q_{A+1}^{-1}Q_{\tau_{A+1}}Q_AX_{A+1;l}Z_{A;p}\right)}\right.\\
&\left.\times\frac{\left(1-Q_{A+1}^{-1}Q_{\tau_A}Q_{\tau_{A+1}}Y_{A-1;l}X_{A+1;p}\right)\left(1-Q_{\tau_A}Q_{A-1}W_{A;l}Z_{A-1;p}\right)}{\left(1-Q_{\tau_A}Q_{A-1}Q_{A+1}^{-1}Q_{\tau_{A+1}}X_{A+1;l}Z_{A-1};p\right)\left(1-Q_{\tau_{A+1}}Q_{\tau_A}Y_{A-1;l}W_{A+1;p}\right)}\right\}\\
&G^{(N)}\left(\frac{Q_{\tau_{A+1}}Q_AX_{A+1}}{Q_{A+1}},Q_{\tau_A}Y_{A-1},\frac{Q_{A-1}Q_{\tau_A}Z_{A-1}}{Q_A},Q_{\tau_{A+1}}W_{A+1};Q_A,Q_{\tau_A}\right)
\end{aligned}
\tag{G.64}
$$

The steps (G.60)-(G.63) may then be iterated $N-1$ more times until one finds

$$
\begin{aligned}
&G^{(N)}(X_A,Y_A,Z_A,W_A;Q_A,Q_{\tau_A}) = \\
&G^{(N)}\left(Q_\tau X_A,Q_\tau Y_A,Q_\tau Z_A,Q_\tau W_A;Q_A,Q_{\tau_A}\right)\\
&\times\prod_{A,B=1}^{N}\prod_{l,p=1}^{\infty}\left\{\frac{\left(1-Q_\tau Q_{AB}^{-1}X_{A;l}Y_{B;p}\right)\left(1-Q_\tau^2Q_{AB}^{-1}X_{A;l}Y_{B;p}\right)}{\left(1-\widetilde{Q}'_{AB}Z_{A;l}X_{B;p}\right)\left(1-Q_\tau\widetilde{Q}'_{AB}Z_{A;l}X_{B;p}\right)}\right.\\
&\left.\frac{\left(1-Q_{AB}Z_{A;l}W_{B;p}\right)\left(1-Q_\tau Q_{AB}Z_{A;l}W_{B;p}\right)}{\left(1-\widetilde{Q}_{AB}Y_{A;l}W_{B;p}\right)\left(1-Q_\tau\widetilde{Q}_{AB}Y_{A;l}W_{B;p}\right)}\right\}
\end{aligned}
\tag{G.65}
$$

here $Q_\tau=\prod_{A=1}^{N}Q_{\tau_A}$. Now we perform (G.60)-(G.65) an infinite number of times and use the fact that

$$
\lim_{r\to\infty}G^{(N)}\left(Q_\tau^rX_A,Q_\tau^rY_A,Q_\tau^rZ_A,Q_\tau^rW_A;Q_A,Q_{\tau_A}\right)=\prod_{r=1}^{\infty}\frac{1}{1-Q_\tau^r}\,, \tag{G.66}
$$

provided $|Q_\tau|<1$. Hence

$$
\begin{aligned}
&G^{(N)}(X_A,Y_A,Z_A,W_A;Q_A,Q_{\tau_A}) = \prod_{r,l,p=1}^{\infty}\left(1-Q_\tau^r\right)^{-N^2}\\
&\prod_{A,B=1}^{N}\prod_{r,l,p=1}^{\infty}\frac{\left(1-Q_\tau^rQ_{AB}^{-1}X_{A;l}Y_{B;p}\right)\left(1-Q_\tau^{r-1}Q_{AB}Z_{A;l}W_{B;p}\right)}{\left(1-Q_\tau^{r-1}\widetilde{Q}'_{AB}Z_{A;l}X_{B;p}\right)\left(1-Q_\tau^{r-1}\widetilde{Q}_{AB}Y_{A;l}W_{B;p}\right)}\,.
\end{aligned}
\tag{G.67}
$$

All in all, the partition function for the strip geometry reads

$$\begin{aligned}
&Z^{\mu_1\dots\mu_N}_{\nu_1\dots\nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A}; \mathfrak{t}, \mathfrak{q}) = \\
&\prod_{A=1}^{N} \mathfrak{q}^{-\frac{||\mu_A^{\mathrm{T}}||^2}{2}} \mathfrak{t}^{\frac{-||\nu_A||^2}{2}} \widetilde{Z}_{\mu_A^{\mathrm{T}}}(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}) \widetilde{Z}_{\nu_A}(\mathfrak{q}^{-1}, \mathfrak{t}^{-1}) \\
&\times \prod_{A,B=1}^{N} \prod_{l,p,r=1}^{\infty} \frac{\left(1 - Q_\tau^{r-1} Q_{AB} \mathfrak{t}^{\mu_{A;l}-p+1/2} \mathfrak{q}^{\nu^{\mathrm{T}}_{B;p}-l+1/2}\right)}{(1-Q_\tau^r)\left(1 - Q_\tau^{r-1} \widetilde{Q}_{BA} \mathfrak{t}^{\nu_{B;l}-p+1} \mathfrak{q}^{\nu^{\mathrm{T}}_{A;p}-l}\right)} \\
&\times \prod_{A,B=1}^{N} \prod_{l,p,r=1}^{\infty} \frac{\left(1 - Q_\tau^{r} Q_{AB}^{-1} \mathfrak{t}^{\nu_{B;l}-p+1/2} \mathfrak{q}^{\mu^{\mathrm{T}}_{A;p}-l+1/2}\right)}{\left(1 - Q_\tau^{r-1} \widetilde{Q}'_{AB} \mathfrak{t}^{\mu_{A;l}-p} \mathfrak{q}^{\mu^{\mathrm{T}}_{B;p}-l+1}\right)}
\end{aligned} \tag{G.68}$$

where we define $Q_\tau := \prod_{A=1}^{N} Q_{\tau_A}$. We may then define the domain wall partition function

$$D^{\mu_1\dots\mu_N}_{\nu_1\dots\nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A}; \mathfrak{t}, \mathfrak{q}) := \frac{Z^{\mu_1\dots\mu_N}_{\nu_1\dots\nu_N}(Q_A, Q_{\tau_A}; \mathfrak{t}, \mathfrak{q})}{Z^{\varnothing\dots\varnothing}_{\varnothing\dots\varnothing}(Q_A, Q_{\tau_A}; \mathfrak{t}, \mathfrak{q})} \tag{G.69}$$

which may be expressed in terms of $\mathcal{N}_{\nu,\mu}(Q; \mathfrak{q}, \mathfrak{t})$ (A.33)

$$\begin{aligned}
&D^{\mu_1\dots\mu_N}_{\nu_1\dots\nu_N}(Q_A, Q_{\tau_A}, Q'_{\tau_A}; \mathfrak{t}, \mathfrak{q}) = \\
&\prod_{A=1}^{N} \mathfrak{q}^{-\frac{||\mu_A^{\mathrm{T}}||^2}{2}} \mathfrak{t}^{\frac{-||\nu_A||^2}{2}} \widetilde{Z}_{\mu_A^{\mathrm{T}}}(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}) \widetilde{Z}_{\nu_A}(\mathfrak{q}^{-1}, \mathfrak{t}^{-1}) \\
&\times \prod_{A,B=1}^{N} \prod_{r=1}^{\infty} \frac{\mathcal{N}_{\mu_A\nu_B}\left(Q_\tau^{r-1} Q_{AB} \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}; \mathfrak{t}, \mathfrak{q}\right) \mathcal{N}_{\nu_B\mu_A}\left(Q_\tau^{r} Q_{AB}^{-1} \sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}; \mathfrak{t}, \mathfrak{q}\right)}{\mathcal{N}_{\mu_A\mu_B}\left(Q_\tau^{r-1} \widetilde{Q}'_{AB}; \mathfrak{t}, \mathfrak{q}\right) \mathcal{N}_{\nu_A\nu_B}\left(Q_\tau^{r-1} \widetilde{Q}_{AB} \frac{\mathfrak{t}}{\mathfrak{q}}; \mathfrak{t}, \mathfrak{q}\right)}
\end{aligned} \tag{G.70}$$

where

$$Q_{AB} = Q_A \prod_{i=1}^{A-1} Q_{\tau_i} \prod_{j=B}^{N} Q_{\tau_j} \bmod Q_\tau\,, \tag{G.71}$$

$$\widetilde{Q}_{AB} = \begin{cases} \prod_{i=B}^{A-1} Q_{\tau_i} & A > B\,, \\ Q_\tau & A = B\,, \\ Q_\tau / \prod_{i=A}^{B-1} Q_{\tau_i} & A < B\,, \end{cases} \tag{G.72}$$

$$\widetilde{Q}'_{AB} = \frac{Q_A}{Q_B} \widetilde{Q}_{AB}\,. \tag{G.73}$$

The M5-brane partition function on A_{N-1} singularity is then given by

$$Z_{\mathrm{M5}}\left(Q_{n,A}, Q_{\tau_{n,A}}, Q_{f,n,A}; \mathfrak{t}, \mathfrak{q}\right) = Z_{\mathrm{rel.}} Z_M^{A_{N-1}}\left(Q_{n,A}, Q_{\tau_{n,A}}, Q_{f,n,A}; \mathfrak{t}, \mathfrak{q}\right) \tag{G.74}$$

where

$$\begin{aligned} & Z_M^{A_{N-1}}\left(Q_{n,A}, Q_{\tau_{n,A}}, Q_{f,n,A}; \mathfrak{t}, \mathfrak{q}\right) = \\ & \sum_{\{\vec{\mu}_n\}}\left(\prod_{n=1}^{M-1}\prod_{A=1}^{N}\left(-Q_{f,n,A}\right)^{|\mu_{n,A}|}\right) Z_{M,\{\vec{\mu}_n\}}^{A_{N-1}}\left(Q_{n,A}, Q_{\tau_{n,A}}; \mathfrak{t}, \mathfrak{q}\right)\,, \end{aligned} \tag{G.75}$$

$$Z_{U(1)}^{(n)} := Z_{\varnothing\ldots\varnothing}^{\varnothing\ldots\varnothing}\left(Q_{n,A}, Q_{\tau_{n,A}}, Q'_{\tau_{n,A}}; \mathfrak{t}, \mathfrak{q}\right)\,, \tag{G.76}$$

$$Z_{\mathrm{rel.}} = \prod_{n=1}^{M} Z_{U(1)}^{(n)}\,, \tag{G.77}$$

and

$$\begin{aligned} & Z_{M,\vec{\mu}_n}^{A_{N-1}}\left(Q_{n,A}, Q_{\tau_{n,A}}; \mathfrak{t}, \mathfrak{q}\right) = D_{\mu_{1,1}\ldots\mu_{1,N}}^{\varnothing\ldots\varnothing}\left(Q_{1,A}, Q_{\tau_{1,A}}, Q'_{\tau_{1,A}}; \mathfrak{t}, \mathfrak{q}\right) \\ & \times D_{\mu_{2,1}\ldots\mu_{2,N}}^{\mu_{1,1}\ldots\mu_{1,N}}\left(Q_{2,A}, Q_{\tau_{2,A}}, Q'_{\tau_{2,A}}; \mathfrak{t}, \mathfrak{q}\right) \\ & \times \cdots \times D_{\varnothing\ldots\varnothing}^{\mu_{M-1,1}\ldots\mu_{M-1,N}}\left(Q_{M,A}, Q_{\tau_{M,A}}, Q'_{\tau_{M,A}}; \mathfrak{t}, \mathfrak{q}\right)\,. \end{aligned} \tag{G.78}$$

Note that the gluing requires

$$Q'_{\tau_{n+1,A}} = Q_{\tau_{n,A}} \implies \widetilde{Q}'_{n+1,AB} = \widetilde{Q}_{n,AB}\,. \tag{G.79}$$

It may be shown that [209] (G.78) may be written as

$$\begin{aligned} & Z_M^{A_{N-1}} = \\ & \sum_{\{\mu_{n,A}\}}\prod_{n=1}^{M-1}\prod_{A=1}^{N}\left(\overline{Q}_{f,n,A}^{|\mu_{n,A}|}\right)\prod_{(l,p)\in\mu_{n,A}}\prod_{B=1}^{N}\frac{\theta_1\left(z_{n,AB}(l,p)|\tau\right)\theta_1\left(w_{n,AB}(l,p)|\tau\right)}{\theta_1\left(u_{n,AB}(l,p)|\tau\right)\theta_1\left(v_{n,AB}(l,p)|\tau\right)} \end{aligned} \tag{G.80}$$

where

$$e^{2\pi\mathrm{i}z_{n,AB}(l,p)} = Q_{n+1,AB}^{-1}\mathfrak{t}^{-\mu_{n,A;l}+p-1/2}\mathfrak{q}^{\mu^{\mathrm{T}}_{n+1,B;p}+l-1/2} \tag{G.81}$$

$$e^{2\pi\mathrm{i}w_{n,AB}(l,p)} = Q_{n,BA}^{-1}\mathfrak{t}^{\mu_{n,A;l}-p+1/2}\mathfrak{q}^{\mu^{\mathrm{T}}_{n-1,B;p}-l+1/2} \tag{G.82}$$

$$e^{2\pi\mathrm{i}u_{n,AB}(l,p)} = \widehat{Q}_{n,BA}^{-1}\mathfrak{t}^{\mu_{n,A;l}-p}\mathfrak{q}^{\mu^{\mathrm{T}}_{n,B;p}-l+1} \tag{G.83}$$

$$e^{2\pi\mathrm{i}v_{n,AB}(l,p)} = \widehat{Q}_{n,AB}^{-1}\mathfrak{t}^{\mu_{n,A;l}+p-1}\mathfrak{q}^{-\mu^{\mathrm{T}}_{n,B;p}+l} \tag{G.84}$$

and

$$\overline{Q}_{f,n,A} = \left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{N-1}{2}} Q_{f,n,A} \left(\prod_{A=1}^{N} Q_{n,A}\right) , \quad \widehat{Q}_{n,AB} = \begin{cases} 1 & A = B\,, \\ \widetilde{Q}_{n,AB} & A \neq B\,. \end{cases} \tag{G.85}$$

The authors of [209] were, remarkably, able to show

$$Z_M^{A_{N-1}} = Z_{\text{string}} \tag{G.86}$$

where Z_{string} is the same as in (6.36).

G.3.2 With Minimal Defect

Let us consider the minimal type defects in the A_0 theory. By minimal we mean the defects of type

$$\rho = [(M-k), \underbrace{1, \ldots, 1}_{k \text{ times}}]\,. \tag{G.87}$$

The relation between the string Elliptic genus and refined topological partition function in the presence of a defect of type $2 = 1 + 1$ has been studied in [284]. We can compute the domain wall partition function (G.70) in the presence of a D3'-brane ending on D5-brane. The effect of the Lagrangian brane is to insert the factor

$$\mathrm{tr}_{\sigma_1^{\mathrm{T}}}(X)\, \mathrm{tr}_{\sigma_2^{\mathrm{T}}}\left(X^{-1}\right) = s_{\sigma_1^{\mathrm{T}}}(x)\, s_{\sigma_2^{\mathrm{T}}}\left(x^{-1}\right)\,. \tag{G.88}$$

Where we assume that the brane has framing factor exponents $p = 1$. The refined open topological string amplitude for the strip geometry with $N = 1$ with a single Lagrangian brane

$$\begin{aligned} &\widehat{Z}_{\nu}^{\mu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) = \\ &\sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2} \Big\{ (-Q_1)^{|\lambda_2|} \left(-Q_1^{-1}Q_\tau\right)^{|\lambda_1|} \left(-\widetilde{Q}^{-1}Q_1^{-1}Q_\tau\right)^{|\sigma_1|} \left(-\widetilde{Q}\right)^{|\sigma_2|} \\ &\times s_{\sigma_1^{\mathrm{T}}}(x)\, s_{\sigma_2^{\mathrm{T}}}\left(x^{-1}\right) C_{(\lambda_1^{\mathrm{T}}\otimes\sigma_1)\lambda_2^{\mathrm{T}}\mu^{\mathrm{T}}}(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}) C_{(\lambda_1\otimes\sigma_2)\lambda_2\nu}(\mathfrak{q}^{-1}, \mathfrak{t}^{-1}) \Big\}\,. \end{aligned} \tag{G.89}$$

From (G.89) after expanding out the topological vertex we have

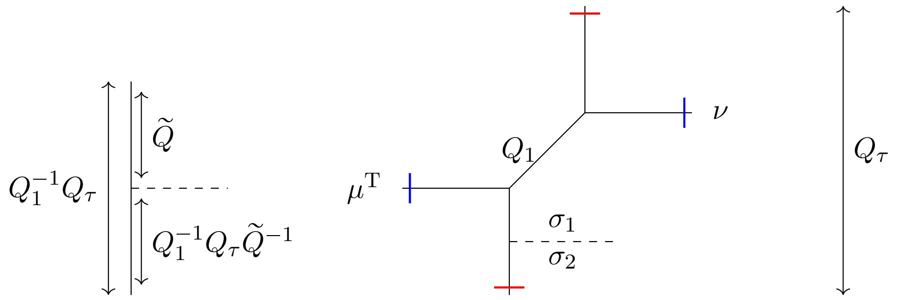


Figure G.2: Left: *Assignment of Kähler parameters for the Lagrangian brane* R*ight: Strip geometry for the A_0 singularity with a single Lagrangian brane corresponding to the defect. The blue lines denote the preferred direction of the refined topological vertex. The red lines denote the direction periodic identification.*

$$
\begin{aligned}
&\widehat{Z}^{\mu}_{\nu}(Q_1,Q_{\tau_1},\widetilde{Q};\mathfrak{t},\mathfrak{q}) = \mathfrak{t}^{-\frac{||\nu||^2}{2}}\mathfrak{q}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}}\widetilde{Z}_{\mu^{\mathrm{T}}}(\mathfrak{t}^{-1},\mathfrak{q}^{-1})\widetilde{Z}_{\nu}(\mathfrak{q}^{-1},\mathfrak{t}^{-1}) \\
&\sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2,\eta_1,\eta_2}\Bigg\{\left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{|\eta_1|-|\eta_2|}{2}}(-Q_1)^{|\lambda_2|}\left(-Q_1^{-1}Q_{\tau_1}\right)^{|\lambda_1|} \\
&\times s_{\sigma_1^{\mathrm{T}}}\left(\widetilde{Q}^{-1}Q_1^{-1}Q_{\tau_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}x\right)s_{\sigma_2^{\mathrm{T}}}\left(\widetilde{Q}\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}x^{-1}\right)s_{(\lambda_1\otimes\sigma_1^{\mathrm{T}})/\eta_2}\left(\mathfrak{t}^{\rho}\mathfrak{q}^{\mu^{\mathrm{T}}}\right) \\
&\times s_{\lambda_2^{\mathrm{T}}/\eta_2}\left(\mathfrak{t}^{\mu}\mathfrak{q}^{\rho}\right)s_{(\lambda_1^{\mathrm{T}}\otimes\sigma_2^{\mathrm{T}})/\eta_1}\left(\mathfrak{q}^{\rho}\mathfrak{t}^{\nu}\right)s_{\lambda_2/\eta_1}\left(\mathfrak{q}^{\nu^{\mathrm{T}}}\mathfrak{t}^{\rho}\right)\Bigg\}\,.
\end{aligned}
\tag{G.90}
$$

Using the identity (A.49) we have

$$
\begin{aligned}
&\widehat{Z}^{\mu}_{\nu}(Q_1,Q_{\tau_1},\widetilde{Q};\mathfrak{t},\mathfrak{q}) = \mathfrak{t}^{-\frac{||\nu||^2}{2}}\mathfrak{q}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}}\widetilde{Z}_{\mu^{\mathrm{T}}}(\mathfrak{t}^{-1},\mathfrak{q}^{-1})\widetilde{Z}_{\nu}(\mathfrak{q}^{-1},\mathfrak{t}^{-1}) \\
&\sum_{\sigma_1,\sigma_2,\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2}\Bigg\{\left(\frac{\mathfrak{q}}{\mathfrak{t}}\right)^{\frac{|\eta_1|-|\eta_2|}{2}}(-Q_1)^{|\lambda_2|}\left(-Q_1^{-1}Q_{\tau_1}\right)^{|\lambda_1|}c^{\gamma_1}_{\lambda_1\sigma_1^{\mathrm{T}}}c^{\gamma_2^{\mathrm{T}}}_{\lambda_1^{\mathrm{T}}\sigma_2^{\mathrm{T}}} \\
&\times s_{\sigma_1^{\mathrm{T}}}\left(\widetilde{Q}^{-1}Q_1^{-1}Q_{\tau_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}x\right)s_{\sigma_2^{\mathrm{T}}}\left(\widetilde{Q}\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}x^{-1}\right)s_{\gamma_1/\eta_2}\left(\mathfrak{t}^{\rho}\mathfrak{q}^{\mu^{\mathrm{T}}}\right) \\
&\times s_{\lambda_2^{\mathrm{T}}/\eta_2}\left(\mathfrak{t}^{\mu}\mathfrak{q}^{\rho}\right)s_{\gamma_2^{\mathrm{T}}/\eta_1}\left(\mathfrak{q}^{\rho}\mathfrak{t}^{\nu}\right)s_{\lambda_2/\eta_1}\left(\mathfrak{q}^{\nu^{\mathrm{T}}}\mathfrak{t}^{\rho}\right)\Bigg\}\,.
\end{aligned}
\tag{G.91}
$$

Now apply the identity (A.48) to obtain

$$\widehat{Z}^{\mu}_{\nu}(Q_1, Q_{\tau_1}, \widetilde{Q}; \mathfrak{t}, \mathfrak{q}) = \mathfrak{t}^{-\frac{||\nu||^2}{2}} \mathfrak{q}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}} \widetilde{Z}_{\mu^{\mathrm{T}}}(\mathfrak{t}^{-1}, \mathfrak{q}^{-1}) \widetilde{Z}_{\nu}(\mathfrak{q}^{-1}, \mathfrak{t}^{-1})$$
$$\sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2} \Bigg\{ (-Q_1)^{|\lambda_2|} \left(\frac{-Q_{\tau_1}}{Q_1}\right)^{|\lambda_1|} s_{\gamma_1/\lambda_1}\left(\frac{Q_{\tau_1}}{\widetilde{Q}Q_1}\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}x\right) s_{\gamma_1/\eta_2}\left(\mathfrak{t}^{\rho}\mathfrak{q}^{\mu^{\mathrm{T}}}\right) \tag{G.92}$$
$$s_{\gamma_2^{\mathrm{T}}/\lambda_1^{\mathrm{T}}}\left(\widetilde{Q}\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}x^{-1}\right) s_{\lambda_2^{\mathrm{T}}/\eta_2}\left(\sqrt{\frac{\mathfrak{q}}{\mathfrak{t}}}\mathfrak{t}^{\mu}\mathfrak{q}^{\rho}\right) s_{\gamma_2^{\mathrm{T}}/\eta_1}\left(\mathfrak{q}^{\rho}\mathfrak{t}^{\nu}\right) s_{\lambda_2/\eta_1}\left(\sqrt{\frac{\mathfrak{t}}{\mathfrak{q}}}\mathfrak{q}^{\nu^{\mathrm{T}}}\mathfrak{t}^{\rho}\right)\Bigg\} .$$

Therefore let us consider

$$G(X,Y,Z,W,A,B;a,b) = \sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2} \Big\{ (a)^{|\lambda_1|} (b)^{|\lambda_2|} s_{\gamma_1/\lambda_1}(A)$$
$$s_{\gamma_2^{\mathrm{T}}/\lambda_1^{\mathrm{T}}}(B)\, s_{\gamma_1/\eta_2}(X)\, s_{\lambda_2^{\mathrm{T}}/\eta_2}(Y)\, s_{\gamma_2^{\mathrm{T}}/\eta_1}(Z)\, s_{\lambda_2/\eta_1}(W)\Big\} \tag{G.93}$$

$$G(X,Y,Z,W,U,A,B;a,b) =$$
$$\sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2} \Big\{ (a)^{|\gamma_1|} (b)^{|\gamma_1|} s_{\lambda_1/\gamma_1}(aX)\, s_{\lambda_1^{\mathrm{T}}/\gamma_2}(Z)\, s_{\eta_2^{\mathrm{T}}/\lambda_2^{\mathrm{T}}}(W) \tag{G.94}$$
$$s_{\eta_1^{\mathrm{T}}/\lambda_2}(bY)\, s_{\eta_1/\gamma_2}(B)\, s_{\eta_2/\gamma_1}(bA)\Big\} \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)}{(1-A_lX_p)(1-B_lZ_p)}$$

$$G(X,Y,Z,W,U,A,B;a,b) = \sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2} \Big\{ (a)^{|\eta_2|} (b)^{|\lambda_1|} s_{\gamma_1^{\mathrm{T}}/\lambda_1}(bZ)$$
$$\times\ s_{\gamma_2^{\mathrm{T}}/\lambda_1^{\mathrm{T}}}(aX)\, s_{\lambda_2/\eta_2^{\mathrm{T}}}(bA)\, s_{\gamma_1^{\mathrm{T}}/\eta_2}(aW)\, s_{\lambda_2^{\mathrm{T}}/\eta_1^{\mathrm{T}}}(B)\, s_{\gamma_2^{\mathrm{T}}/\eta_1}(bY)\Big\} \tag{G.95}$$
$$\times \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)(1+aX_lZ_p)(1+bA_lW_p)(1+bY_lB_p)}{(1-A_lX_p)(1-B_lZ_p)}$$

$$G(X,Y,Z,W,U,A,B;a,b) = \sum_{\lambda_1,\lambda_2,\eta_1,\eta_2,\gamma_1,\gamma_2} \Big\{ (a)^{|\lambda_2|} (b)^{|\lambda_1|} s_{\lambda_1/\gamma_1^{\mathrm{T}}}(aW)$$
$$\times\ s_{\lambda_1^{\mathrm{T}}/\gamma_2^{\mathrm{T}}}(bY)\, s_{\eta_2/\lambda_2}(aB)\, s_{\eta_2/\gamma_1^{\mathrm{T}}}(bZ)\, s_{\eta_1/\lambda_2^{\mathrm{T}}}(bA)\, s_{\eta_1/\gamma_2^{\mathrm{T}}}(aX)\Big\} \tag{G.96}$$
$$\times \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)(1+aX_lZ_p)(1+bA_lW_p)(1+bY_lB_p)(1+bB_lA_p)}{(1-A_lX_p)(1-B_lZ_p)(1-abW_lZ_p)(1-abY_lX_p)}$$

Which is, again, rather similar to our original expressions:

$$G(X,Y,Z,W,A,B;a,b) = G(bZ,aW,aX,bY,aB,bA;a,b)$$
$$\times \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)(1+aX_lZ_p)(1+bA_lW_p)(1+bY_lB_p)(1+bB_lA_p)}{(1-A_lX_p)(1-B_lZ_p)(1-abW_lZ_p)(1-abY_lX_p)} \tag{G.97}$$

So, repeating steps (G.94) to (G.96) again we have that

$$
\begin{aligned}
G(X,Y,Z,W,A,B;a,b) &= G\left(Q_\tau X, Q_\tau Y, Q_\tau Z, Q_\tau W, Q_\tau A, Q_\tau B; a,b\right) \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+bY_lW_p)(1+aX_lZ_p)(1+bA_lW_p)(1+bY_lB_p)(1+bB_lA_p)}{(1-A_lX_p)(1-B_lZ_p)(1-abW_lZ_p)(1-abY_lX_p)} \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+Q_\tau bY_lW_p)(1+Q_\tau aX_lZ_p)(1+Q_\tau bA_lW_p)}{(1-Q_\tau A_lX_p)(1-Q_\tau B_lZ_p)(1-Q_\tau^2 W_lZ_p)} \\
&\times \prod_{l,p=1}^{\infty} \frac{(1+bQ_\tau Y_lB_p)(1+Q_\tau bB_lA_p)}{(1-Q_\tau^2 Y_lX_p)}
\end{aligned}
\tag{G.98}
$$

where $Q_\tau := ab$ so again iterating the steps (G.94) to (G.98) an infinite number of times and using

$$
\lim_{r\to\infty} G\left(Q_\tau^r X, Q_\tau^r Y, Q_\tau^r Z, Q_\tau^r W, Q_\tau^r A, Q_\tau^r B; a,b\right) = \prod_{r=1}^{\infty} \frac{1}{1-Q_\tau^r}
\tag{G.99}
$$

we arrive at

$$
\begin{aligned}
G(X,Y,Z,W,A,B;a,b) &= \prod_{r,l,p=1}^{\infty} \left\{ \frac{\left(1+Q_\tau^{r-1}bY_lW_p\right)\left(1+Q_\tau^{r-1}aX_lZ_p\right)}{(1-Q_\tau^r)\left(1-Q_\tau^{r-1}A_lX_p\right)} \right. \\
&\left. \times \frac{\left(1+Q_\tau^{r-1}bA_lW_p\right)\left(1+bQ_\tau^{r-1}Y_lB_p\right)\left(1+Q_\tau^{r-1}bB_lA_p\right)}{\left(1-Q_\tau^{r-1}B_lZ_p\right)(1-Q_\tau^r W_lZ_p)(1-Q_\tau^r Y_lX_p)} \right\}
\end{aligned}
\tag{G.100}
$$

$$
\begin{aligned}
\widehat{Z}_\nu^\mu(Q_1,Q_\tau,\widetilde{Q},x;\mathfrak{t},\mathfrak{q}) &= \mathfrak{t}^{-\frac{||\nu||^2}{2}}\mathfrak{q}^{-\frac{||\mu^{\mathrm{T}}||^2}{2}}\widetilde{Z}_{\mu^{\mathrm{T}}}(\mathfrak{t}^{-1},\mathfrak{q}^{-1})\widetilde{Z}_\nu(\mathfrak{q}^{-1},\mathfrak{t}^{-1}) \\
&\times \prod_{r,l,p=1}^{\infty} \left\{ \frac{\left(1-Q_1^{-1}Q_\tau^r\mathfrak{q}^{\mu_l^{\mathrm{T}}-p+1/2}\mathfrak{t}^{\nu_p-l+1/2}\right)}{\left(1-Q_\tau^r\mathfrak{t}^{\mu_l-p}\mathfrak{q}^{\mu_p^{\mathrm{T}}-l+1}\right)\left(1-Q_\tau^r\mathfrak{q}^{\nu_l^{\mathrm{T}}-p}\mathfrak{t}^{\nu_p-l+1}\right)} \right. \\
&\frac{\left(1-Q_\tau^r\widetilde{Q}^{-1}\mathfrak{q}^{\nu_l^{\mathrm{T}}}\mathfrak{t}^{-l+1/2}x_p\right)\left(1-Q_\tau^{r-1}\widetilde{Q}Q_1\mathfrak{t}^{\mu_l}\mathfrak{q}^{-l+1/2}x_p^{-1}\right)}{\left(1-Q_1^{-1}\widetilde{Q}^{-1}Q_\tau^r\mathfrak{q}^{\mu_l^{\mathrm{T}}+1/2}\mathfrak{t}^{-l}x_p\right)\left(1-\widetilde{Q}Q_\tau^{r-1}\mathfrak{q}^{-l}\mathfrak{t}^{\nu_l+1/2}x_p^{-1}\right)} \\
&\left. \frac{(1-Q_\tau^r x_l x_p^{-1})\left(1-Q_1Q_{\tau_1}^{r-1}\mathfrak{t}^{\mu_l-p+1/2}\mathfrak{q}^{\nu_p^{\mathrm{T}}-l+1/2}\right)}{(1-Q_\tau^r)} \right\}.
\end{aligned}
\tag{G.101}
$$

As before, we define the domain wall partition function; it turns out that it factorises in the following fashion

$$\begin{aligned}\widehat{D}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) &:= \frac{\widehat{Z}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}; \mathfrak{t}, \mathfrak{q})}{\widehat{Z}^{\varnothing}_{\varnothing}(Q_1, Q_\tau, \widetilde{Q}; \mathfrak{t}, \mathfrak{q})} \\ &= D^{\mu}_{\nu}(Q_1, Q_\tau; \mathfrak{t}, \mathfrak{q}) \widehat{d}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) \,.\end{aligned} \tag{G.102}$$

Where D is given by (G.70) and

$$\begin{aligned}&\widehat{d}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{t}, \mathfrak{q}) \\ &= \prod_{r,p=1}^{\infty} \left\{ \prod_{l=1}^{\ell(\nu^{\mathrm{T}})} \frac{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{q}^{\nu_l^{\mathrm{T}}} \mathfrak{t}^{-l+1/2} x_p\right)}{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{t}^{-l+1/2} x_p\right)} \prod_{l=1}^{\ell(\nu)} \frac{\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{1/2} x_p^{-1}\right)}{\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{\nu_l + 1/2} x_p^{-1}\right)} \right. \\ &\left. \prod_{l=1}^{\ell(\mu)} \frac{\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{t}^{\mu_l} \mathfrak{q}^{-l+1/2} x_p^{-1}\right)}{\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{q}^{-l+1/2} x_p^{-1}\right)} \prod_{l=1}^{\ell(\mu^{\mathrm{T}})} \frac{\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{1/2} \mathfrak{t}^{-l} x_p\right)}{\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{\mu_l^{\mathrm{T}} + 1/2} \mathfrak{t}^{-l} x_p\right)} \right\}\end{aligned} \tag{G.103}$$

$$\begin{aligned}&= \prod_{r,p=1}^{\infty} \left\{ \prod_{(l,q)\in\nu} \frac{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{q}^{l} \mathfrak{t}^{-q+1/2} x_p\right)\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{q-1/2} x_p^{-1}\right)}{\left(1 - Q_\tau^r \widetilde{Q}^{-1} \mathfrak{q}^{l-1} \mathfrak{t}^{-q+1/2} x_p\right)\left(1 - \widetilde{Q} Q_\tau^{r-1} \mathfrak{q}^{-l} \mathfrak{t}^{q+1/2} x_p^{-1}\right)} \right. \\ &\left. \prod_{(l,q)\in\mu} \frac{\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{t}^{q} \mathfrak{q}^{-l+1/2} x_p^{-1}\right)\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{l-1/2} \mathfrak{t}^{-q} x_p\right)}{\left(1 - Q_\tau^{r-1} \widetilde{Q} Q_1 \mathfrak{t}^{q-1} \mathfrak{q}^{-l+1/2} x_p^{-1}\right)\left(1 - Q_1^{-1} \widetilde{Q}^{-1} Q_\tau^r \mathfrak{q}^{l+1/2} \mathfrak{t}^{-q} x_p\right)} \right\} .\end{aligned} \tag{G.104}$$

is the contribution of the defect to the partition function. It does not quite assemble into a nice form in terms of θ_1 functions. However, in the unrefined $\mathfrak{q} = \mathfrak{t}$ limit:

$$\begin{aligned}\widehat{d}^{\mu}_{\nu}(Q_1, Q_\tau, \widetilde{Q}, x; \mathfrak{q}, \mathfrak{q}) &= \prod_{p=1}^{\infty} \left\{ \mathfrak{q}^{\frac{|\mu|-|\nu|}{2}} \prod_{(l,q)\in\nu} \frac{\theta_1\left(\widetilde{Q}^{-1} \mathfrak{q}^{l-q+1/2} x_p; Q_\tau\right)}{\theta_1\left(\widetilde{Q}^{-1} \mathfrak{q}^{l-q-1/2} x_p; Q_\tau\right)} \right. \\ &\left. \times \prod_{(l,q)\in\mu} \frac{\theta_1\left(Q_1^{-1} \widetilde{Q}^{-1} \mathfrak{q}^{l-1/2-q} x_p; Q_\tau\right)}{\theta_1\left(Q_1^{-1} \widetilde{Q}^{-1} \mathfrak{q}^{l+1/2-q} x_p; Q_\tau\right)} \right\}\end{aligned} \tag{G.105}$$

We may then compute the M5-brane partition function in the presence of the defect labelled by the partition (G.87) by gluing together k domain wall partitions of type (G.102) with $M - k$ of type (G.70). This builds the theory labelled by partition

$\rho = [M-k, \underbrace{1,\ldots,1}_{k}]$. Hence we write

$$Z_{\mathrm{M5},\rho} = \left(\prod_{n=1}^{M-k} Z_{U(1)}^{(n)}\right)\left(\prod_{n=M-k+1}^{M} Z_{U(1),\rho}^{(n)}\right) Z_{\rho}^{A_0} \tag{G.106}$$

where,

$$Z_{U(1),\rho}^{(n)}(Q_{n,1}, Q_\tau, \widetilde{Q}_n, x_n; \mathfrak{t}, \mathfrak{q}) = \widehat{Z}_{\varnothing}^{\varnothing}(Q_{n,1}, Q_\tau, \widetilde{Q}_n, x_n; \mathfrak{t}, \mathfrak{q}) \tag{G.107}$$

$$\begin{aligned} Z_{\rho}^{A_0}\left(Q_{f,n,1}, Q_\tau, \widetilde{Q}_n, x_n; \mathfrak{t}, \mathfrak{q}\right) &:= \sum_{\{\mu_n\}}\left(\prod_{n=1}^{M-1}(-Q_{f,n,1})^{|\mu_n|}\right) \\ &\times D_{\mu_1}^{\varnothing}(Q_{1,1}, Q_\tau; \mathfrak{t}, \mathfrak{q}) \times D_{\mu_2}^{\mu_1}(Q_{2,1}, Q_\tau; \mathfrak{t}, \mathfrak{q}) \\ &\times \cdots \times D_{\mu_{M-k}}^{\mu_{M-k-1}}(Q_{M-k,1}, Q_\tau; \mathfrak{t}, \mathfrak{q}) \\ &\times \widehat{D}_{\mu_{M-k+1}}^{\mu_{M-k}}(Q_{M-k+1,1}, Q_\tau, \widetilde{Q}_{M-k+1}, x_{M-k+1}; \mathfrak{t}, \mathfrak{q}) \\ &\times \cdots \times \widehat{D}_{\varnothing}^{\mu_{M-1}}(Q_{M,1}, Q_\tau, \widetilde{Q}_M, x_M; \mathfrak{t}, \mathfrak{q})\,. \end{aligned} \tag{G.108}$$

By the factorisation (G.103) it is clear that

$$\begin{aligned} Z_{\rho}^{A_0}\left(Q_{f,n,1}, Q_{n,1}, Q_\tau, \widetilde{Q}_n, x_n; \mathfrak{t}, \mathfrak{q}\right) &:= \sum_{\{\mu_n\}}\left(\prod_{n=1}^{M-1}(-Q_{f,n,1})^{|\mu_n|}\right) \\ &\times Z_{M,\{\vec{\mu}_n\}}^{A_0}(Q_{n,1}, Q_\tau; \mathfrak{t}, \mathfrak{q}) \times \widehat{d}_{\mu_{M-k+1}}^{\mu_{M-k}}(Q_{M-k+1,1}, Q_\tau, \widetilde{Q}_{M-k+1}, x_{M-k+1}; \mathfrak{t}, \mathfrak{q}) \\ &\times \cdots \times \widehat{d}_{\varnothing}^{\mu_{M-1}}(Q_{M,1}, Q_\tau, \widetilde{Q}_M, x_M; \mathfrak{t}, \mathfrak{q})\,. \end{aligned} \tag{G.109}$$

Appendix H

Hall-Littlewood Index and Hilbert Series

In this Appendix we discuss the relationship between the Hall-Littlewood limit of the index and the Higgs-branch Hilbert series. For $\mathfrak{g} = A_{N-1}$ Class $\mathcal{S}$ theories associated to genus $g = 0$ theories it is conjectured that these two quantities are equal. For $g \geqslant 1$ it is known to no longer hold.

Here we will compare these two quantities, restricting most of our attention to $\mathfrak{g} = A_{N-1}$ class $\mathcal{S}$ SCFTs associated to $g = 1$ Riemann surfaces with a collection of minimal punctures.

H.1 The Higgs Branch of Class $\mathcal{S}$ Theories

For $\mathcal{N} \geqslant 2$ SCFTs the Higgs branch is reached by giving zero vev to operators with $r \neq 0$ while allowing vevs for those operators with $r = j_1 = j_2 = 0$. For theories with $\mathcal{N} > 2$ this depends on a choice of embedding $\mathfrak{su}(2,2|2) \hookrightarrow \mathfrak{su}(2,2|\mathcal{N})$. The Higgs branch is protected from quantum corrections and thus, when the theory has a Lagrangian description can be described as a purely classical object. The coordinate ring of the Higgs-branch is known as the Higgs-branch chiral ring. By abuse of notation we will identify the Higgs-branch as a complex affine variety with it's chiral ring. The Higgs branch (chiral ring) is given by

$$HB = \{\mathcal{O}_i | \widetilde{\mathcal{Q}}^I_{\dot{\alpha}} \mathcal{O}_i = 0\,, M_{\mu\nu}\mathcal{O}_i = 0\,, r\mathcal{O}_i = 0\}\,. \tag{H.1}$$

For superconformal theories the Higgs branch is parametrised by the top components of $\hat{\mathcal{B}}_R$ multiplets which have $E = 2R$ and $r = j_1 = j_2 = 0$, where R is the Cartan of the $\mathfrak{su}(2)_R$ R-symmetry of the $\mathcal{N} = 2$ superconformal algebra. There is no recombination rule (F.8)-(F.13) involving only $\hat{\mathcal{B}}_R$ operators. For gauge theories based on a gauge group G with a collection of hypermultiplets whose scalars are collectively denoted by $Q, \widetilde{Q}$ the ring HB has a rather simple description. Firstly, one constructs the coordinate ring associated to the *master space* (restricted to the Higgs branch) which is

$$F_H = R/I\,, \quad R := \mathbb{C}[Q, \widetilde{Q}]\,, \quad I := \langle \partial_\Phi W \rangle \tag{H.2}$$

here W denotes the superpotential of the theory, and Φ collectively denotes the vector multiplet scalars. Finally, to obtain HB one takes the G-invariant part of F_H

$$HB = (F_H)^G\,. \tag{H.3}$$

The Hilbert series counts gauge invariant chiral operators graded by their charges under a maximally commuting subalgebra of the global symmetry algebra. It is given by

$$\mathrm{HS}(\tau, u_i; HB) \equiv \mathrm{HS}(\tau, u_i) := \mathrm{Tr}_{HB}\, \tau^{2R} \prod_i u_i^{f_i}\,. \tag{H.4}$$

For the case of $\mathcal{N} = 2$ gauge theories the Hilbert series for the Higgs branch takes the form

$$\mathrm{HS}(\tau, u_i) = \int d\mu_G(\mathbf{z}) \mathrm{HS}_F(\tau, u_i, \mathbf{z})\,, \tag{H.5}$$

where $\mathrm{HS}_F(\tau, u_i, \mathbf{z})$ denotes the Hilbert-series for $F = R/I$, defined as

$$\mathrm{HS}_F(\tau, u_i, \mathbf{z}) = \mathrm{HS}(\tau, u_i, \mathbf{z}; F) = \mathrm{Tr}_{R/I}\, \tau^{2R} \prod_i u_i^{f_i} \prod_{a=1}^{\mathrm{rank}\, \mathfrak{g}} z_a^{g_a}\,. \tag{H.6}$$

In the case of genus $g = 0$ class $\mathcal{S}$ theories gauge theories one can show that the set of F-terms generating I form a regular sequence. This implies that the affine variety $\mathbf{F}$ whose coordinate ring is $F = R/I$ is a complete intersection, which further means that it's Hilbert series can be written as $\mathrm{HS}_F = \mathrm{PE}[p(\tau, u_i, z_a)]$ with p a polynomial in τ. This implies that for those theories one can use letter counting in order to compute HS_F.

For genus $g \geqslant 1$ this fails to be the case and letter counting, in general, cannot be used. In that case one must use an algebraic geometry package such as

Macaulay2 [112]. By inputting the ring of polynomials R and the ideal I *Macaulay2* can compute the Hilbert series for $F = R/I$.

H.2 The Hall-Littlewood Index

The superconformal index for a class $\mathcal{S}$ theory is defined as [62, 64]

$$\mathcal{I}(\rho, \sigma, \tau, u_i) = \mathrm{Tr}_{\mathbb{S}^3}(-1)^F \rho^{\frac{1}{2}\delta_{1-}} \sigma^{\frac{1}{2}\delta_{1+}} \tau^{\frac{1}{2}\tilde{\delta}_{2\dot{+}}} \prod u_i^{f_i} \,. \tag{H.7}$$

The trace is taken over the Hilbert space of the theory in the radial quantisation. The index (H.7) receives contributions only from those states satisfying

$$\delta = \tilde{\delta}_{1\dot{-}} := 2\left\{\tilde{\mathcal{Q}}_{1\dot{-}}, \tilde{\mathcal{S}}^{1\dot{-}}\right\} = E - 2j_2 - 2R + r = 0\,. \tag{H.8}$$

We also have

$$\delta_{1\pm} = E \pm 2j_1 - 2R - r\,, \quad \tilde{\delta}_{\dot{\pm}} = E \pm 2j_2 + 2R + r\,. \tag{H.9}$$

The superconformal index is independent under continuous deformation of the corresponding QFT. That means that, if the theory admits a free-field limit, (H.7) may be computed in the free theory by enumerating all of the free fields that satisfy $\delta = 0$ and then projecting onto gauge invariants. The projection onto gauge invariants is implemented by integration over the gauge group G. The index (H.7) for a gauge theory then takes the form

$$\mathcal{I}(\rho, \sigma, \tau, u_i) = \int d\mu_G(\mathbf{z})\, \mathrm{PE}\left[i(\rho, \sigma, \tau, u_i, \mathbf{z})\right]\,, \tag{H.10}$$

$d\mu_G$ denotes the Haar measure of the gauge group G and $\mathrm{PE}\,[f(x)]$ denotes the Plethystic exponential of a function $f(x)$, defined in (A.1). The single letter index i may be computed by enumerating all free field 'letters' with $\delta = 0$. The more powerful statement, however, is that, for class $\mathcal{S}$ theories, the index (H.7) has the interpretation as a partition function of a 2d TQFT living on the Riemann surface C. Given a pair of pants decomposition of C into a collection of three-punctured spheres (where each puncture carries an associated representation of A_n) and tubes. The index of any class $\mathcal{S}$ theory can then be written in terms of the indices of the elementary three point functions (three-punctured sphere indices), expanded in a basis

$$\mathcal{I}_{\mathbf{abc}} = \sum_{\alpha,\beta,\chi} C_{\alpha\beta\gamma} f^{\alpha}(\mathbf{a}) f^{\beta}(\mathbf{b}) f^{\chi}(\mathbf{c}) , \tag{H.11}$$

and propagators (indicies of tubes)

$$\mathcal{I}_{\mathbf{ab}} = \eta_{\mathbf{ab}} = \int \mu_{SU(N)}(\mathbf{a}) \Delta(\mathbf{a}) \mathcal{I}^V(\mathbf{a}) \delta(\mathbf{a}, \mathbf{b}^{-1}) . \tag{H.12}$$

The index (H.7) counts short representations of the $\mathfrak{su}(2,2|\mathcal{N})$ superconformal algebra, modulo recombination. Recombination happens when a long multiplet hits the unitary bound and decomposes into semi-direct sums of short representations. We list the possible $\mathcal{N}=2$ recombination rules in equations (F.8)-(F.13).

The Hall-Littlewood index is defined as

$$\mathrm{HL}(\tau, u_i) = \lim_{\rho,\sigma\to 0} \mathcal{I}\left(\rho, \sigma, \tau, u_i\right) = \mathrm{Tr}_{\mathbb{S}^3|_{\delta_{1\pm}=0}} (-1)^F \tau^{2R+2j_2} \prod_i u_i^{f_i} . \tag{H.13}$$

This limit is always well defined since superconformal symmetry implies $\delta_{1\pm} \geqslant 0$. This limit of the index counts a restricted number of operators, namely those with

$$j_1 = 0 , \quad j_2 = r , \quad E = 2R + j_2 . \tag{H.14}$$

Note that the only superconformal multiplets contributing to the index in this limit are

$$\mathrm{HL}_{\hat{\mathcal{B}}_R}(\tau) = \tau^{2R} , \quad \mathrm{HL}_{\mathcal{D}_{R(0,j_2)}}(\tau) = (-1)^{2j_2+1} \tau^{2+2R+2j_2} . \tag{H.15}$$

We notice also that the Higgs branch chiral ring HB is contained as a subset of Hall-littlewood operators. For genus zero theories it is conjectured that these two rings are equal.

We plan to consider the quantity

$$\frac{\mathrm{HL}(\tau, u_i)}{\mathrm{HS}(\tau, u_i)} = \begin{array}{l} \text{Partition function of operators} \\ \text{with } j_1 = 0,\ j_2 = r \geqslant \frac{1}{2},\ E = 2R + j_2 \end{array} . \tag{H.16}$$

Equivalently the ratio HL/HS has an expansion in terms of $\mathcal{D}_{R(0,j_2)}$ multiplet indices

$$\frac{\mathrm{HL}(\tau, u_i)}{\mathrm{HS}(\tau, u_i)} = \sum_{R,j_2 \in \mathbb{N}/2} p_{R,j_2}(u_i) \mathrm{HL}_{\mathcal{D}_{R(0,j_2)}}(\tau) , \tag{H.17}$$

with p_{R,j_2} K-symmetric polynomials in the u_i with positive integer coefficients, where K is the global symmetry group of the theory. Note however that the Hall-Littlewood index can distinguish only equivalence classes of multiplets, namely

$$[\tilde{R}]_+ = \hat{\mathcal{B}}_{\tilde{R}} \cup \{\mathcal{D}_{\tilde{R}-j_2-1(0,j_2)} | \tilde{R} - j_2 - 1 \geqslant 0, 2j_2 \in 2\mathbb{N} + 1 \in\} \tag{H.18}$$

$$[\tilde{R}]_- = \{\mathcal{D}_{\tilde{R}-j_2-1(0,j_2)} | \tilde{R} - j_2 - 1 \geqslant 0, 2j_2 \in 2\mathbb{N}\} \tag{H.19}$$

and

$$\mathrm{HL}_{[\tilde{R}]_+} = -\mathrm{HL}_{[\tilde{R}]_-} = \tau^{2\tilde{R}}\,. \tag{H.20}$$

Note that the following multiplets contain only a single representative: $[1/2]_+ = \hat{\mathcal{B}}_{\frac{1}{2}}$, $[1]_+ = \hat{\mathcal{B}}_1$, $[1]_- = \mathcal{D}_{0,(0,0)}$, $[3/2]_- = \mathcal{D}_{1/2,(0,0)}$ these correspond to free half-hypers, moment map operator, free vector multiplet (chiral piece) and super-symmetry current. Note that the multiplets $\mathcal{D}_{0(j_1,0)}$ contain free fields. As we just mentioned when $j_1 = 0$ this is a free vector multiplet, when $j_1 \geqslant \frac{1}{2}$ these contain higher-spin free fields.

We will also use the fact that the Plethystic Logarithm counts all single trace operators, in other words

$$\mathrm{PLog}\left[\mathrm{HL}(\tau, u_i)\right] = \begin{array}{l} \text{Partition function of single trace operators} \\ \text{with } j_1 = 0,\ j_2 = r,\ E = 2R + j_2 \end{array}\,. \tag{H.21}$$

H.3 $\mathcal{N} = 4$ SYM Theories

From now we will label quantities by the Class $\mathcal{S}$ data, i.e. type $\mathfrak{g}$ and the Riemann surface data of genus g and n punctures. We will focus much of our attention to the class $\mathcal{S}$ theory associated to a torus with a single puncture, $n = 1$, $g = 1$. This yields the $G = SU(N), U(N)$ MSYM theory (depending on whether we choose to gauge the c.o.m. degree of freedom). This example is particularly tractable because we can compute the Hilbert series for any N. Viewed as an $\mathcal{N} = 2$ theory $\mathcal{N} = 4$ SYM has a $U(1)$ flavour symmetry associated to the puncture which we give fugacity u for, this enhances to $SU(2)$ on the Higgs-branch. As an affine variety, the Higgs branch of this theory is given by

$$\mathbf{HB} = \mathbb{C}^{2r}/W(G) \tag{H.22}$$

where $W(G)$ denotes the Weyl group of G and $r = \operatorname{rank} G$. The chiral ring is obviously just therefore $HB = (\mathbb{C}[x_1, x_2, \ldots, x_{2r}])^{W(G)}$, although this description can be rather cumbersome to work with in practise. The Hilbert series is then simply the Molien series

$$\mathrm{HS}^{G}_{1,1}(\tau,u) = M(\tau,u;\mathbb{C}^{2r}/W(G))\,. \tag{H.23}$$

The Hall-Littlewood index is expressed as the matrix integral

$$\mathrm{HL}^{G}_{1,1} = \oint d\mu_G\,\mathrm{PE}\left[h(\tau,u)\chi_G^{(adj.)}\right]\,, \tag{H.24}$$

$$h(\tau,u) = \chi_1(u)\tau - \tau^2 = \chi_1(u)\mathrm{HL}_{\hat{\mathcal{B}}_{1/2}} + \mathrm{HL}_{\mathcal{D}_{0(0,0)}}\,, \tag{H.25}$$

here $\chi_k(u) \equiv \chi_k = \sum_{i=0}^{k} u^{k-2i}$ is the character of the spin-$k/2$ $SU(2)$ representation. The Hall-Littlewood letters are the top components of half-hypers $X = Q$, $Y = \widetilde{Q}$ which transform in the **2** under the enhanced $SU(2)$ flavour symmetry and $\overline{\lambda} = \overline{\lambda}_{1\dot{+}}$ in the **1** under the $SU(2)$. All the letters are in the adjoint representation of G. Operators appearing in the expansion of the PLog of the Hall-Littlewood index are of the form

$$\mathrm{tr}\, X^n Y^m \overline{\lambda}^k \in \begin{cases} \hat{\mathcal{B}}_{\frac{m+n}{2}} & \text{if } k = 0 \\ \mathcal{D}_{\frac{m+n+k-1}{2}(0,\frac{k-1}{2})} & \text{if } k \geqslant 1 \end{cases} \tag{H.26}$$

due to $SU(2)$ flavour symmetry these must occur symmetrically under $m \leftrightarrow n$.

H.3.1 $G = U(N)$

The Hilbert series for this theory was computed already in Chapter 5 and reads

$$\mathrm{HS}^{U(N)}_{1,1} = \frac{1}{N!}\frac{\partial^N}{\partial\nu^N}\mathrm{PE}\left[\frac{\nu}{(1-u\tau)(1-u^{-1}\tau)}\right]\Bigg|_{\nu=0}\,. \tag{H.27}$$

In particular, the Higgs branch of this theory as a variety is $\mathrm{Sym}^N(\mathbb{C}^2)$.

$N = 1$

We can immediately write down the ratio

$$\frac{\mathrm{HL}^{U(1)}_{1,1}}{\mathrm{HS}^{U(1)}_{1,1}} = \mathrm{PE}[-\tau^2] = \mathrm{PE}[\mathrm{HL}_{[1]_-}] = \mathrm{PE}[\mathrm{HL}_{\mathcal{D}_{0,(0,0)}}]\,. \tag{H.28}$$

I.e. the difference is simply an additional free vector multiplet.

$N = 2$

The Hilbert series reads

$$\mathrm{HS}_{1,1}^{U(2)} = \mathrm{PE}\left[\chi_1\tau + \chi_2\tau^2 - \tau^4\right] . \tag{H.29}$$

The Hall-Littlewood index can be evaluated by means of residues and reads

$$\mathrm{HL}_{1,1}^{U(2)} = (1 + \tau^2 - \chi_1\tau^3)\,\mathrm{PE}\left[\chi_1\tau + \chi_2\tau^2 - 2\tau^2\right] . \tag{H.30}$$

The ratio is

$$\frac{\mathrm{HL}_{1,1}^{U(2)}}{\mathrm{HS}_{1,1}^{U(2)}} = \frac{(1 + \tau^2 - \chi_1\tau^3)(1-\tau^2)^2}{1-\tau^4} = \left(1 - \frac{\chi_1\tau^3}{1+\tau^2}\right)\mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] \tag{H.31}$$

$$= \left(1 + \chi_1 \sum_{n=1}^{\infty} (-1)^n \tau^{2n+1}\right)\mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] \tag{H.32}$$

$$= \left(1 + \chi_1 \sum_{m=1}^{\infty} \left(\mathrm{HL}_{[2m+1/2]_+} + \mathrm{HL}_{[2m-1/2]_-}\right)\right)\mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] \tag{H.33}$$

note that here we have factored out the contribution from a $\mathcal{D}_{0(0,0)}$ multiplet, as we know that this multiplet is always present for the $\mathfrak{u}(N)$ theory and corresponds to the free decoupled $\mathfrak{u}(1)$ in the decomposition $\mathfrak{u}(N) \cong \mathfrak{su}(N) \oplus \mathfrak{u}(1)$. The Plethystic logarithm (spectrum of single-trace operators) is

$$\mathrm{PLog}\left[\frac{\mathrm{HL}_{1,1}^{U(2)}}{\mathrm{HS}_{1,1}^{U(2)}}\right] = -\tau^2 - (u + u^{-1})\tau^3 + (u + u^{-1})\tau^5 + \mathcal{O}(\tau^6) \tag{H.34}$$

corresponding to $\mathrm{tr}\,\overline{\lambda}$, $\mathrm{tr}\,X\overline{\lambda}$, $\mathrm{tr}\,Y\overline{\lambda}$, $\mathrm{tr}\,X\overline{\lambda}^2$, $\mathrm{tr}\,Y\overline{\lambda}^2$, past this order it is no longer possible to uniquely determine the operators.

$N = 3$

The Hilbert series in this case is

$$\mathrm{HS}_{1,1}^{U(3)} = \left(\chi_1\tau^3 + \sum_{n=0}^{3}\tau^{2n}\right)\mathrm{PE}\left[3\chi_1\tau + \chi_2\tau^2 - \tau^2 + (\chi_1 - \chi_3)\tau^3\right] \tag{H.35}$$

where we used the identity $\sum_{n=0}^{p} x^n = (1 - x^{p+1})/(1-x) = \mathrm{PE}[x - x^{p+1}]$.

For $N = 3$ it is still possible to compute using residues and, after using some identities we arrive at

$$\begin{aligned}\mathrm{HL}_{1,1}^{U(3)} = &\left(-\chi_2\tau^4 - \chi_1\tau^5 - (\chi_2-1)\tau^6 + (\chi_3-\chi_1)\tau^7 + \sum_{n=0}^{4}\tau^{2n}\right) \\ &\times \mathrm{PE}\left[3\chi_1\tau + \chi_2\tau^2 + \chi_1\tau^3 - 2\tau^2 - \chi_3\tau^3\right] .\end{aligned} \tag{H.36}$$

The ratio is

$$\begin{aligned}&\frac{\mathrm{HL}_{1,1}^{U(3)}}{\mathrm{HS}_{1,1}^{U(3)}} = \mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] \\ &\times\left(1 - \frac{\chi_1\tau^3 + \chi_2\tau^4 + \chi_1\tau^5 + (\chi_2-1)\tau^6 - (\chi_3-\chi_1)\tau^7 - \tau^8}{1+\tau^2+\chi_1\tau^3+\tau^4+\tau^6}\right) .\end{aligned} \tag{H.37}$$

$N = \infty$

In Chapter 5 we wrote a simple formula for the $U(\infty)$ $\mathcal{N}=4$ Hilbert series, it reads

$$\mathrm{HS}_{1,1}^{U(\infty)} = \mathrm{PE}\left[\frac{1}{(1-u\tau)(1-u^{-1}\tau)}\right] = \mathrm{PE}\left[\sum_{k\geqslant 0}\chi_k(u)\tau^k\right] . \tag{H.38}$$

The spectrum of single trace Higgs-branch operators is a collection of $\hat{\mathcal{B}}_R$ multiplets in the spin R representation of the $SU(2)$ global symmetry. In the large N limit the Hall-Littlewood index can easily be written down by appealing to AdS/CFT [62] it reads

$$\mathrm{HL}_{1,1}^{U(\infty)} = \mathrm{PE}\left[\mathrm{HL}^{KK}\right] , \quad \mathrm{HL}^{KK} = \frac{\chi_1(u)\tau - 2\tau^2}{(1-u\tau)(1-u^{-1}\tau)} . \tag{H.39}$$

So, the ratio is

$$\frac{\mathrm{HL}_{1,1}^{U(\infty)}}{\mathrm{HS}_{1,1}^{U(\infty)}} = \mathrm{PE}\left[\frac{-\tau^2}{(1-u\tau)(1-u^{-1}\tau)}\right] = \mathrm{PE}\left[\sum_{k\geqslant 0}\chi_k(u)\mathrm{HL}_{[1+k/2]_-}\right] \tag{H.40}$$

H.3.2 $G = SU(N)$

The Hilbert series for this theory was computed already in Chapter 5 and is given by

$$\mathrm{HS}_{1,1}^{SU(N)} = (1-u\tau)(1-u^{-1}\tau)\mathrm{HS}_{1,1}^{U(N)} = \frac{\mathrm{HS}_{1,1}^{U(N)}}{\mathrm{HS}_{1,1}^{U(1)}} . \tag{H.41}$$

The Higgs branch of this theory as a variety is $\mathbb{C}^{2N-2}/S_N \subset \mathrm{Sym}^N(\mathbb{C}^2)$, this is the subvariety of the $U(N)$ case defined by demanding tracelessness of the adjoint representation. We checked via series expansion for various low values of N that

$$\mathrm{HL}_{1,1}^{SU(N)} = \frac{\mathrm{HL}_{1,1}^{U(N)}}{\mathrm{HL}_{1,1}^{U(1)}} \,. \tag{H.42}$$

So, we can apply the results of the previous section using

$$\frac{\mathrm{HL}_{1,1}^{SU(N)}}{\mathrm{HS}_{1,1}^{SU(N)}} = \frac{\mathrm{HL}_{1,1}^{U(N)}}{\mathrm{HL}_{1,1}^{U(1)}}\frac{\mathrm{HS}_{1,1}^{U(1)}}{\mathrm{HS}_{1,1}^{U(N)}} = \mathrm{PE}[\mathrm{HL}_{\mathcal{D}_{0,(0,0)}}]\frac{\mathrm{HL}_{1,1}^{U(N)}}{\mathrm{HS}_{1,1}^{U(N)}} \,. \tag{H.43}$$

H.4 Elliptic Quiver Theories

We now allow for n minimal punctures. For $U(N)$ gauge groups this is the $\mathbb{Z}_n$ orbifold theory of $\mathcal{N}=4$ SYM.

H.4.1 $G = U(N)$

The Higgs branch of this theory as a variety is $\mathrm{Sym}^N(\mathbb{C}^2/\mathbb{Z}_n)$ and the Hilbert series is therefore

$$\mathrm{HS}_{1,n}^{U(N)} = \frac{1}{N!}\frac{\partial^N}{\partial \nu^N}\mathrm{PE}\left[\frac{\nu(1-\tau^{2n})}{(1-\tau^2)(1-u^n\tau^n)(1-u^{-n}\tau^n)}\right]\Bigg|_{\nu=0} \,. \tag{H.44}$$

The Hall-Littlewood index is

$$\mathrm{HL}_{1,n}^{U(N)} = \oint\prod_{i=1}^{n}\left(d\mu_{U(N)_i}\,\mathrm{PE}\left[\left(uf_i\overline{f}_{i+1} + u^{-1}\overline{f}_i f_{i+1}\right)\tau - \chi_{U(N)_i}^{(adj.)}\tau^2\right]\right) \tag{H.45}$$

where $f_i = \sum_{a=1}^N z_{i,a}$, $\overline{f}_i = \sum_{a=1}^N z_{i,a}^{-1}$, $\chi_i^{(adj.)} = \sum_{a,b=1}^N \frac{z_{i,a}}{z_{i,b}}$ we also take $z_{i+n,a} = z_{i,a}$.

$N = 1$

For $N=1$ the results are rather simple

$$\mathrm{HS}_{1,n}^{U(1)} = \frac{1-\tau^{2n}}{(1-\tau^2)(1-u^n\tau^n)(1-u^{-n}\tau^n)} \tag{H.46}$$

and

$$\mathrm{HL}_{1,n}^{U(1)} = \frac{1-\tau^{2n}}{(1-u^n\tau^n)(1-u^{-n}\tau^n)} \tag{H.47}$$

we again have that the two quantities differ by a free vector multiplet

$$\frac{\mathrm{HL}_{1,n}^{\mathfrak{u}(1)}}{\mathrm{HS}_{1,n}^{\mathfrak{u}(1)}} = (1-\tau^2) = \mathrm{PE}\left[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}\right] . \tag{H.48}$$

$N = n = 2$

The Hilbert series is

$$\mathrm{HS}_{1,n}^{U(2)} = \frac{(1-\tau^{2n})\left[(1+\tau^{2n})(1-\tau^{2+2n}) + (u^{-n}+u^n)(\tau^{2+n}-\tau^{3n})\right]}{(1-\tau^2)(1-\tau^4)(1-u^{\pm n}\tau^n)(1-u^{\pm 2n}\tau^{2n})} . \tag{H.49}$$

For $n = 2$ this simplifies to

$$\mathrm{HS}_{1,2}^{U(2)} = \left(\chi_2\tau^4 + \sum_{i=0}^{4}\tau^{2i}\right)\mathrm{PE}\left[(\chi_2-1)\tau^2 + (\chi_4-\chi_2)\tau^4\right] . \tag{H.50}$$

The Hall-Littlewood index is

$$\mathrm{HL}_{1,2}^{U(2)} = \left(1+\tau^4-(\chi_2-1)\tau^6\right)\mathrm{PE}\left[(\chi_2-1)\tau^2 + (\chi_4-\chi_2-1)\tau^4\right] . \tag{H.51}$$

The ratio is

$$\frac{\mathrm{HL}_{1,2}^{U(2)}}{\mathrm{HS}_{1,2}^{U(2)}} = \frac{1-\tau^8-(\chi_2-1)\tau^6(1-\tau^4)}{1+\chi_2\tau^4-\chi_2\tau^6-\tau^{10}}\,\mathrm{PE}[-\tau^2] \tag{H.52}$$

$$= \left(1+\chi_2\mathrm{HL}_{[2]_-} + \mathrm{HL}_{[3]_+} + (\chi_4+\chi_2)\mathrm{HL}_{[4]_+} + \mathcal{O}(\tau^{10})\right)\mathrm{PE}[\mathrm{HL}_{\mathcal{D}_{0(0,0)}}] \tag{H.53}$$

H.4.2 Generic $\mathfrak{g} = A_1$ Class $\mathcal{S}$ Theories

The Hall-Littewood index for the A_1 theory associated to a genus g Riemann surface with n punctures is [168]

$$\mathrm{HL}_{g,n}^{SU(2)} = \frac{(1+\tau^2)^\chi}{(1-\tau^2)^{1-g}}\sum_{\lambda=0}^{\infty}\frac{1}{P_\lambda(\tau,\tau^{-1}|\tau)^\chi}\prod_{I=1}^{n}\frac{P_\lambda(a_I,a_I^{-1}|\tau)}{(1-a_I^2\tau^2)(1-a_I^{-2}\tau^2)} \tag{H.54}$$

where $\chi = 2g - 2 + n$ and the Hall-Littlewood polynomials are

$$P_\lambda(a, a^{-1}|\tau) = \begin{cases} \chi_\lambda(a) - \tau^2 \chi_{\lambda-2}(a) & \lambda \geqslant 1 \\ \sqrt{1+\tau^2} & \lambda = 0 \end{cases} \tag{H.55}$$

with $\chi_\lambda = (a^{1+\lambda} - a^{-1-\lambda})/(a - a^{-1})$ the $SU(2)$ characters. On the other hand, the Hilbert series for the same theory is given by [277]

$$\begin{aligned} \mathrm{HS}_{g,n}^{SU(2)} = & \frac{(1+\tau^2)^\chi}{(1-\tau^2)} \left((1+\tau^2)^{1-2g} \prod_{I=1}^{n} \frac{1}{(1-a_I^2\tau^2)(1-a_I^{-2}\tau^2)} \right. \\ & \left. + \sum_{\lambda=1}^{\infty} \frac{1}{P_\lambda(\tau, \tau^{-1}|\tau)^\chi} \prod_{I=1}^{n} \frac{P_\lambda(a_I, a_I^{-1}|\tau)}{(1-a_I^2\tau^2)(1-a_I^{-2}\tau^2)} \right) . \end{aligned} \tag{H.56}$$

It is immediate that, when $g = 0$ we have $\mathrm{HS}_{0,n}^{SU(2)} = \mathrm{HL}_{0,n}^{SU(2)}$. Let us consider the case of a theory associated to a genus $g \geqslant 1$ surface without punctures, in which case the sums can be performed explicitly

$$\mathrm{HL}_{g,0}^{SU(2)} = \frac{(1-\tau^2)^{\chi/2}(\tau^\chi + (1+\tau^2)^{\chi/2}(1-\tau^\chi))}{(1-\tau^\chi)} \tag{H.57}$$

while the corresponding Hilbert series becomes

$$\mathrm{HS}_{g,0}^{SU(2)} = \mathrm{PE}\left[\tau^4 + \tau^\chi + \tau^{\chi+2} - \tau^{2\chi+4}\right] . \tag{H.58}$$

The ratio then takes the form

$$\begin{aligned} \frac{\mathrm{HL}_{g,0}^{SU(2)}}{\mathrm{HS}_{g,0}^{SU(2)}} = & \left(\tau^{2g-2} + (1+\tau^2)^{g-1}(1-\tau^{2g-2})\right) \\ & \times \mathrm{PE}\left[-(g-1)\tau^2 - \tau^4 - \tau^{2g}(1-\tau^{2g})\right] . \end{aligned} \tag{H.59}$$

Appendix I

Ramified Instanton Partition Functions

In this appendix we discuss how to obtain the instanton partition function for the 5d $\mathcal{N} = 1^*$ theory on $\mathbb{S}^1 \times \mathbb{C}^2/\mathbb{Z}_p$. This is a more general result of the one computed in Section 6.3.2 where we take the more generic $\mathbb{Z}_p$ action on the coordinates (θ, z_1, z_2) of $\mathbb{S}^1 \times \mathbb{C}^2$ to be

$$\mathbb{Z}_p : (\theta, z_1, z_2) \mapsto (\theta, \gamma^{q_1} z_1, \gamma^{q_2} z_2)\,, \tag{I.1}$$

with $\gamma^p = 1$, $q_1, q_2, p \in \mathbb{Z}$ and $gcd(q_1, p) = gcd(q_2, p) = 1$. This is generalisation of the results computed in [49] and of Section 6.3.2 which were made with the specialisation $q_1 = 0$, $q_2 = 1$, $p = k$. In the context of $\frac{1}{2}$-BPS defects this generalisation should give rise to other types of defect configurations.

In the Nekrasov instanton counting the Ω-deformation forces the instantons to sit at the origin $z_1 = z_2 = 0$. The instantons therefore sit at the fixed point of the action (I.1). Note that in order to preserve the supercharges used in the instanton localisation computation we have to turn on a background R-current. We *assume* that the projections do not change the localisation structure. Under that assumption the Nekrasov partition function takes the form of an 'orbifolded' Nekrasov partition function $\widetilde{Z}^{\text{orb}}_{\text{nek}} = Z^{\text{orb}}_{\text{cl}} Z^{\text{orb}}_{\text{inst}}$. We have to project onto states left invariant by the orbifold action

$$\alpha_{A,I} \mapsto \alpha_{A,I} - \frac{2\pi \mathrm{i} A}{p}\,, \quad m \mapsto m - \frac{2\pi\mathrm{i}(q_1 + q_2)}{2p}\,, \tag{I.2}$$

$$\epsilon_1 \mapsto \epsilon_1 + \frac{2\pi\mathrm{i} q_1}{p}\,, \quad \epsilon_2 \mapsto \epsilon_2 + \frac{2\pi\mathrm{i} q_2}{p}\,. \tag{I.3}$$

Using the same conventions as those in Chapter 1.4.3 we decompose the vector spaces W, V (for a fixed momentum mode around the $\mathbb{S}^1$) with respect to their $\mathbb{Z}_p$ grading

$$W = \bigoplus_{A=1}^{p} W_A \,, \quad V = \bigoplus_{A=1}^{p} V_A \,, \tag{I.4}$$

of dimension $\dim_{\mathbb{C}} W_A = N_A$ and $\dim_{\mathbb{C}} V_A = k_A$. Moreover, we also take the index $A = 1, \ldots, p$ modulo p. Under the $\mathbb{Z}_p$ action the ADHM data transforms as

$$B^{(l)} \mapsto \gamma^{q_l} B^{(l)} \,, \quad P \mapsto P \,, \quad Q \mapsto \gamma^{q_1+q_2} Q \,, \tag{I.5}$$

where $q_3 := -q_1 - q_2$, $q_4 := 0$. In order to have a non-trivial result, following [157], we also quotient by a $\mathbb{Z}_p \hookrightarrow U(k)$ corresponding to (1.82) with $g = \mathrm{diag}\left(\gamma \mathbb{I}_{k_1}, \gamma^2 \mathbb{I}_{k_2} \ldots, \right) \in U(k)$. This breaks $U(k) \to \prod_{A=1}^{p} U(k_A)$ with $k = \sum_{A=1}^{p} k_A$. The surviving components are

$$B_A^{(l)} \in \mathrm{Hom}\left(V_A, V_{A+q_l}\right) \,, \quad P_A \in \mathrm{Hom}\left(V_A, V_A\right) \,, \tag{I.6}$$

$$Q_A \in \mathrm{Hom}\left(V_A, V_{A+q_1+q_2}\right) \,. \tag{I.7}$$

The ADHM equations $\mu_{\mathbb{C},A}^{(i)} = \mu_{\mathbb{R},A} = 0$ are given by performing the projections to (1.77) and (1.80). The ramified instanton moduli space is then given by

$$\mathbf{M}_{\{k_A\},\{N_A\}}^{\mathbb{Z}_p} = \left\{ B_A^{(l)} \,, P_A \,, Q_A \middle| \mu_{\mathbb{C},A}^{(i)} = \mu_{\mathbb{R},A} = 0 \right\} \,. \tag{I.8}$$

The fixed points after the $\mathbb{Z}_p$ quotient are still labelled by N-tuples of Young diagrams $\vec{\mu}$ which we now label by $\vec{\mu} = \{\mu_{A,I}\}$. We choose bases

$$W_A = \mathrm{span}_{\mathbb{C}} \left\{ w_{A,I} \middle| I = 1, \ldots, N_A \right\} \tag{I.9}$$

$$V_{A+q_1 i+q_2 j} = \mathrm{span}_{\mathbb{C}} \left\{ v_{A+q_1 i+q_2 j,I}^{(i,j)} \middle| I = 1, \ldots, N_{A+q_1 i+q_2 j}, (i,j) \in \mu_{A,I} \right\} \,. \tag{I.10}$$

The torus action acts by

$$w_{A,I} \mapsto e^{\alpha_{A,I}} w_{A,I} \,, \quad v_{A+q_1 i+q_2 j,I}^{(i,j)} \mapsto e^{(1-i)\epsilon_1 + (1-\epsilon_2)} v_{A+q_1 i+q_2 j,I}^{(i,j)} \,. \tag{I.11}$$

The fixed point configuration is given by the orbifold projection of (1.87), namely

$$B_A^{(1)} v_{A,I}^{(i,j)} = v_{A+q_1,I}^{(i+1,j)} \,, \quad B_A^{(2)} v_{A,I}^{(i,j)} = v_{A+q_2,I}^{(i,j+1)} \,, \quad P_A w_{A,I} = v_{A,I}^{(1,1)} \,, \tag{I.12}$$

$$Q_A = B_A^{(3)} = B_A^{(4)} = 0 \,. \tag{I.13}$$

The dimension of V_B is then given by

$$k_B = k_B(\vec{\mu}) = \dim_{\mathbb{C}} V_B = \sum_{A=1}^{p}\sum_{I=1}^{N_A}\sum_{(i,j)\in y^{(I)}_{A,B}} 1\,, \tag{I.14}$$

where $y^{(I)}_{A,B}$ is given by

$$y^{(I)}_{A,B} = \{(i,j)|(i,j)\in\mu_{A,I}, A + q_1 i + q_2 j = B \bmod p\}\ . \tag{I.15}$$

For $q_1 = 0$ and $q_2 = 1$ equation (I.14) reduces to (2.37) of [49]. We demonstrate an explicit example with $p = 3$, $q_1 = 1$, $q_2 = -2$, $N_1 = N_2 = N_3 = 1$, $\mu_{1,1} = \{4,3,2\}$, $\mu_{2,1} = \{2,2\}$ and $\mu_{3,1} = \{3,2\}$ in Figure I.1. Because $N_A = 1$ we drop the I indices, for example $v^{(i,j)}_{A,1} = v^{(i,j)}_{A}$. $k_1 = 7$, $k_2 = 5$ and $k_3 = 6$; in agreement with (I.14). Finally the character of $T\mathbf{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}}$ at the fixed point $\vec{\mu}$ is given by the $\mathbb{Z}_p$ invariant part of (1.95), namely

$$\chi_{\vec{\mu}}\left(T\mathbf{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}}\right) := \chi^{\text{Vec}}_{\vec{\mu},\mathbb{Z}_p} + \chi^{\text{Hyp}}_{\vec{\mu},\mathbb{Z}_p}\,, \tag{I.16}$$

with

$$\begin{aligned}\chi^{\text{Vec}}_{\vec{\mu},\mathbb{Z}_p} = &\sum_{t\in\mathbb{Z}} e^{\frac{2\pi t}{r}}\sum_{B=1}^{p}\big[W^*_B V_B + e^{2\epsilon_+}V^*_{B+q_1+q_2}W_B - V^*_B V_B\\ &+e^{\epsilon_1}V^*_{B+q_1}V_B + e^{\epsilon_2}V^*_{B+q_2}V_B - e^{2\epsilon_+}V^*_{B+q_1+q_2}V_B\big]\,,\end{aligned} \tag{I.17}$$

$$\begin{aligned}\chi^{\text{Hyp}}_{\vec{\mu},\mathbb{Z}_p} = &-\sum_{t\in\mathbb{Z}} e^{\frac{2\pi t}{r}}e^{m-\epsilon_+}\sum_{B=1}^{p}\big[W^*_{B-q_1-q_2}V_B + e^{2\epsilon_+}V^*_B W_B\\ &-V^*_{B-q_1-q_2}V_B + e^{\epsilon_1}V^*_{B-q_2}V_B + e^{\epsilon_2}V^*_{B-q_1}V_B - e^{2\epsilon_+}V^*_B V_B\big]\,.\end{aligned} \tag{I.18}$$

As before conjugation reverses the signs of the exponents. We also abused the notation and identified the vector spaces and their characters

$$\begin{aligned}V_A = &\\ &\sum_{C,D=1}^{p}\ \sum_{I=1}^{N_{q_1C+q_2D-A}}\ \sum_{\substack{(pi-C+1,pj-D+1)\\ \in y^{(I)}_{q_1C+q_2D-A,A}}} e^{\alpha_{q_1C+q_2D-A,I}+(C-pi)\epsilon_1+(D-pj)\epsilon_2}\,,\end{aligned} \tag{I.19}$$

$$W_A = \sum_{I=1}^{N_{p-A+1}} e^{\alpha_{p-A+1,I}}\,, \tag{I.20}$$

$$
\begin{tikzcd}
 & v_1^{(4,1)} & & \\
 & v_3^{(3,1)} \arrow[u, "B_3^{(1)}"'] \arrow[r, "B_3^{(2)}"] & v_1^{(3,2)} & \\
 & v_2^{(2,1)} \arrow[u, "B_2^{(1)}"'] \arrow[r, "B_2^{(2)}"] & v_3^{(2,2)} \arrow[u, "B_3^{(1)}"'] \arrow[r, "B_3^{(2)}"] & v_1^{(2,3)} \\
 & v_1^{(1,1)} \arrow[u, "B_1^{(1)}"'] \arrow[r, "B_1^{(2)}"] & v_2^{(1,2)} \arrow[u, "B_2^{(1)}"'] \arrow[r, "B_2^{(2)}"] & v_3^{(1,3)} \arrow[u, "B_3^{(1)}"'] \\
w_1 \arrow[ur, "P_1"] & & &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & v_3^{(2,1)} \arrow[r, "B_3^{(2)}"] & v_1^{(2,2)} \\
 & v_2^{(1,1)} \arrow[u, "B_2^{(1)}"'] \arrow[r, "B_2^{(2)}"] & v_3^{(1,2)} \arrow[u, "B_3^{(1)}"'] \\
w_2 \arrow[ur, "P_2"] & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & v_2^{(3,1)} & \\
 & v_1^{(2,1)} \arrow[u, "B_1^{(1)}"'] \arrow[r, "B_1^{(2)}"] & v_2^{(2,2)} \\
 & v_3^{(1,1)} \arrow[u, "B_3^{(1)}"'] \arrow[r, "B_3^{(2)}"] & v_1^{(1,2)} \arrow[u, "B_1^{(1)}"'] \\
w_3 \arrow[ur, "P_3"] & &
\end{tikzcd}
$$

Figure I.1: *Example of the fixed point structure for* $p = 3$*,* $q_1 = 1$*,* $q_2 = -2$*,* $N_1 = N_2 = N_3 = 1$.

under the orbifold $\mathbb{Z}_p : V_A, W_A \mapsto \gamma^A V_A, \gamma^A W_A$. At this point it is very important to stress that we understand A, B, C, D to be taken modulo p when and only when they are considered as indices used to label quantities for example $\alpha_{A,I} = \alpha_{A+p,I}$. These quantities are significantly more complicated than those of the case $q_1 = 0$, $q_2 = 1$ of (6.107).

According to the conversion rule (1.96) we can, in principle, compute the partition function

$$\chi_{\vec{\mu}} \left(T\mathbf{M}^{\mathbb{Z}_p}_{\{k_A\},\{N_A\}} \right) \to z^{\mathbb{Z}_p}_{\vec{\mu}} \left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r \right) \,. \tag{I.21}$$

The instanton partition function then reads

$$Z^{\text{orb}}_{\text{inst}} \left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r; \mathbf{q}_A \right) = \sum_{\vec{\mu}} \left(\prod_{B=1}^{p} q_B^{k_B(\vec{\mu})} \right) z^{\mathbb{Z}_p}_{\vec{\mu}} \left(\vec{\alpha}, m, \epsilon_1, \epsilon_2, r \right) \,. \tag{I.22}$$

In the limit $m = -\epsilon_+$ the supersymmetry of the 5d theory enhances $\mathcal{N} = 1^* \to \mathcal{N} = 2$. Correspondingly the instanton partition drastically simplifies and one can see that

$$\left. \chi^{\text{Vec}}_{\vec{\mu},\mathbb{Z}_p} \right|_{m=-\epsilon_+} = - \left. \left(\chi^{\text{Hyp}}_{\vec{\mu},\mathbb{Z}_p} \right)^* \right|_{m=-\epsilon_+} \,. \tag{I.23}$$

From (1.96) we conclude that

$$z^{\mathbb{Z}_p}_{\vec{\mu}} \left(\vec{\alpha}, -\epsilon_+, \epsilon_1, \epsilon_2, r \right) \equiv 1 \,. \tag{I.24}$$

So,

$$Z^{\text{orb}}_{\text{inst}} \left(\vec{\alpha}, -\epsilon_+, \epsilon_1, \epsilon_2, r; \mathbf{q}_A \right) = \sum_{\vec{\mu}} \left(\prod_{B=1}^{p} q_B^{k_B(\vec{\mu})} \right) \,. \tag{I.25}$$

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Declaration

Eidesstattliche Versicherung / Declaration on oath

Hiermit versichere ich an Eides statt, die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt zu haben.

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Die Dissertation wurde in der vorgelegten oder einer ähnlichen Form nicht schon einmal in einem früheren Promotionsverfahren angenommen oder als ungenügend beurteilt.

Hamburg, den November 6th 2019

Thomas Bourton