Cracking the supergravity correlators

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Abstract

In this thesis, we present the computation of the four-point functions of 1/2-BPS operators in planar $\mathcal{N} = 4$ SYM theory in the strong 't Hooft coupling limit. We perform the calculation for all the operators with weights up to 8. Moreover, we consider several high-weight correlation functions. The results are obtained from the effective type-IIB supergravity action, employing the AdS/CFT correspondence. The computation is done by implementing the newly developed, simplified algorithm, in combination with the harmonic polynomial formalism. These results are additionally used to check the recently conjectured formula for correlators in Mellin space, and we indeed find an agreement.

Zusammenfassung

In dieser Dissertation präsentieren wir die Berechnung der Vierpunkt-Funktionen von 1/2-BPS-Operatoren in der planaren $\mathcal{N} = 4$ SYM-Theorie im Limes starker 't Hooft-Kopplung. Wir führen die Berechnung für alle Operatoren mit Gewichten bis zu 8 durch. Darüber hinaus betrachten wir mehrere Korrelationsfunktionen mit hohem Gewicht. Die Ergebnisse werden erhalten aus der effektiven Supergravitationswirkung von Typ IIB unter Verwendung der AdS/CFT-Korrespondenz. Die Berechnung erfolgt durch Implementierung eines neu entwickelten, vereinfachten Algorithmus in Kombination mit dem Formalismus der harmonischen Polynome. Die Ergebnisse werden außerdem verwendet, um die kürzlich vermutete Formel für Korrelatoren im Mellin-Raum zu überprüfen, und wir finden tatsächlich eine Übereinstimmung.

This thesis is based on the following publications:

[1] Arutyunov, G.; Klabbers, R. and Savin, S., *Four-point functions of 1/2-BPS operators of any weights in the supergravity approximation*, JHEP, 2018, 09, 118, arxiv:1808.06788 [hep-th].

[2] Arutyunov, G.; Klabbers, R. and Savin, S., *Four-point functions of all-different-weight chiral primary operators in the supergravity approximation*, JHEP, 2018, 09, 023, arxiv:1808.06788 [hep-th].

[3] Arutyunov, G.; Klabbers, R. and Savin, S., *Towards 4-point correlation functions of any 1/2 -BPS operators from supergravity*, JHEP, 2017, 04, 005, arxiv:1701.00998 [hep-th].

Other publications by the author:

[4] Arutyunov, G.; Dorigoni, D. and Savin, S., *Resurgence of the dressing phase for* $\mathrm{AdS}_5 \times \mathrm{S}^5$, JHEP, 2017, 01, 055, arxiv:1608.03797 [hep-th].

Contents

Part I

Introduction

Chapter 1

Introduction

An old dream of theoretical physicists is to have a quantitative description of the hadronic spectrum and their excitations. At high energies, quarks interact weakly, and the standard techniques based on Feynmann diagrams can be performed. However, Quantum Chromo-dynamics (QCD) – the theory of strong interactions – fails to obtain analytical results at low energies. At significant separation, the coupling constant becomes too large, and the perturbation theory is not valid anymore. One has to use non-perturbative methods such as numerical calculations on the lattice. Thus, analytically describing Yang-Mills theories at strong coupling regime is a great challenge for theoretical particle physics.

A new perspective in this direction comes with the famous AdS/CFT correspondence conjectured by Maldacena [5]. It proposes the duality between two different physical systems. Namely, certain quantum gauge theories enriched by conformal symmetry can be described in terms of closed strings on a background, containing an Anti-de-Sitter spacetime as a factor (the maximally symmetric space of constant negative curvature). The most well-understood example involves maximally-supersymmetric four-dimensional $\mathcal{N} = 4$ Yang-Mills theory with gauge group $SU(N)$ on the one side and its dual partner – superstring theory of type IIB defined on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background [6]. Though $\mathcal{N} = 4$ SYM is quite similar to QCD, the latter is not conformal. Since there is no string dual to QCD known, $\mathcal{N} = 4$ SYM and its string partner serve as a toy model to understand strongly coupled gauge theories in general. For instance, in selected cases, we can consider any 4-dimensional gauge theory as $\mathcal{N} = 4$ SYM with some particles or interactions added or removed.

In recent years significant progress was made in the spectral problem of the AdS/CFT correspondence. It is a direct consequence of a hidden symmetry – integrability [6, 7]. Indeed, in the planar limit, when the rank of the gauge group $N \to \infty$, one can solve $\mathcal{N} = 4$ SYM exactly. To solve $\mathcal{N} = 4$ SYM theory means to find anomalous dimensions of local operators and to compute their three-point correlation functions. On the string side, the limit $N \to \infty$ corresponds to a free string theory described by a two-dimensional quantum non-linear sigma model. The energy spectrum of the sigma model determines the spectrum of the scaling dimensions of local operators in the dual gauge theory. The fact that the corresponding sigma model is integrable allows one to find the spectrum for strings on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ by using Thermodynamic Bethe Ansatz (TBA) techniques.

However, computing other observables, such as correlation functions or Wilson loops, provides another non-trivial check of the AdS/CFT correspondence and sheds some light on the gauge theories in general. In this thesis, we are interested in computing four-point functions of local 1/2-BPS operators of planar $\mathcal{N} = 4$ SYM theory at strong coupling. In this particular limit, its dual string partner reduces to a classical supergravity theory of type IIB defined on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ spacetime. A precise recipe for computing the four-point functions in the framework of the AdS/CFT duality was proposed by Witten [8] and Gubser, Klebanov, Polyakov [9], and is as follows: one interprets the on-shell supergravity partition function as a generating functional for the four-point functions in SYM theory with sources for the 1/2-BPS operators being the boundary values of the Kaluza-Klein modes of the compactified on the sphere supergravity theory. Thus, computation of an n-point function requires the knowledge of the effective supergravity action - the action expanded up to the n-th order in perturbations of the fields around their background values. Such an action for the case of four-point correlators was found by Arutyunov, Frolov in [10–12] and several results were obtained. The first computed correlators correspond to equal-weight 1/2-BPS operators with (lowest) weight $k = 2$ [13, 14], $k = 3$ [15] and $k = 4$ [16]. The first four-point function featuring different-weight operators was discussed in [17], which considers the correlator of two $k = 2$ and two $k = 3$ operators. This was shortly afterward generalized to the case in which the $k = 3$ operators become arbitrary (but equal) weights [18]. The maximal amount of different operators studied until the present was three in [19], where next-next-to-extremal correlators were studied that generalize those in [18]. Finally, the only other explicitly known result is the equal-weight correlation function of weight $k = 5$, which was computed using a bootstrap approach in [20]. These results, although restricted to 1/2-BPS operators, do in some cases allow for the evaluation of correlation functions of other members of these 1/2-BPS multiplets. For example, for operators belonging to the stress-energy multiplets it was shown in [21] that all their four-point functions can be found from the equal-weight correlator for 1/2-BPS operators of weight 2.

Recently the problem of finding holographic correlators in a planar strong coupling limit has gained renewed interest. In [20, 22] the Mellin space formalism was used to analyze the structure of correlation functions of four 1/2-BPS local operators of arbitrary weights. This culminated in an elegant conjectured closed formula for these correlators in Mellin space, thereby extending the existing expression for equal-weight operators proposed in [23]. The conjecture follows from reformulating the computation of four-point functions as a bootstrap problem by imposing specific properties on the correlators such as superconformal symmetry.

The existence of the simple Mellin-space formula is surprising when one considers the only method known at present to compute these correlation functions explicitly. It includes consideration of the tree-level Witten diagrams for the chosen operators, whose vertices follow from the effective action of the Kaluza-Klein reduction of the type IIB supergravity on the sphere S^5. This computation becomes cumbersome quickly and yields very unwieldy intermediate results, although the final result can usually be expressed in a compact form, in turn suggesting that a simple formula such as the one conjectured in [20] might exist. However, because of the difficulty of their computation, it is equally possible that we have not explored far enough to discover four-point functions whose final expression takes a less appealing form. Indeed, as of yet, all the explicitly computed four-point functions are in some form not completely generic: they are either particularly symmetric (equal-weights) or/and close to extremality condition where the computation simplifies. In fact, not a single non-trivial correlation function has ever been computed between four operators with four different weights. Also, apart from the equal weight case for which an explicit formula has long ago been conjectured [23], no four-point functions which go beyond the next-next-to-extremal case have been considered.

In this thesis, we compute the four-point functions of operators with weights up to 8 plus $\{7, 10, 12, 17\}$ and $\{17, 21, 23, 25\}$. This is a major improvement over the previously available set of correlators. This computation became possible due to simplifications of the algorithm used before, where for a given set of weights one firstly writes the effective supergravity action by hand, evaluates it on-shell and then differentiates the partition function over the sources. Already for known cases, this procedure becomes very unpleasant because of the growing number of terms. However, we show that the procedure can be streamlined to directly obtain the correlation function in terms of contact and exchange Witten diagrams. These simplifications consist of a direct formula for the exchange part and the contact part of the correlation function and can be automatized using, for example, Mathematica.

It is convenient to express the result for the correlators in variables u and v, which are the conformally invariant combinations of the coordinates x_i. As an example, let us present the simplified expression for the $\langle 3456\rangle$ correlator. Any correlator can always be split into the free and the interacting part, where the latter for {3,4,5,6} case takes a form

$$\langle 3456\rangle_{\text{int}} = \frac{\sqrt{10}}{N^2} \frac{\mathcal{R}_{1,2,3,4}\, t_{24} t_{34}^2}{x_{12}^2 x_{14}^2 x_{23}^2 x_{24}^2 x_{34}^6} \left(u\, t_{14} t_{23} \mathcal{F}_1 + uv\, t_{13} t_{24} \mathcal{F}_2 + v\, t_{12} t_{34} \mathcal{F}_3\right).$$

All the notations are given in the next sections, and the dynamical $\mathcal{F}$ functions are written in terms of contact Witten diagrams – the so-called D-functions, which are the known special functions:

$$\begin{aligned}
\mathcal{F}_1 &= -\frac{1}{8} v(6u + 28v + 1)\bar{D}_{2567} - \frac{5}{4} v\bar{D}_{2468} - \frac{7}{8} v\bar{D}_{2477} - \frac{5}{8} v\bar{D}_{2558}\,, \\
\mathcal{F}_2 &= -\frac{3}{2} uv\bar{D}_{2657} - \frac{3}{2} v\bar{D}_{2558} - 3v\bar{D}_{2567}\,, \\
\mathcal{F}_3 &= -\frac{7}{4} uv^2\bar{D}_{2666} - \frac{1}{8} v(10u + 12v + 3)\bar{D}_{2567} - \frac{5}{8} uv\bar{D}_{2657} - \frac{15}{8} v^2\bar{D}_{2576}\,.
\end{aligned}$$

A particular – although not entirely new – feature of the newly computed correlators is the presence of extended operators: it has been known for a long time that in general, the supergravity scalar fields are not dual to the single-trace operators from the 1/2-BPS multiplets, but instead to a linear combination of single- and double-trace operators known as extended operators [24]. Nevertheless, for most purposes, this fact can be ignored as in the supergravity limit it only affects the free part of the correlators containing at least weight-4 operators. The only known correlators for which this effect was seen are those from the family $\langle 22nn\rangle$ for $n > 3$ discussed in [18]. We show the presence of the extended operators by an explicit computation.

Our computations serve multiple goals. First of all, we present simplifications to the algorithm that computes four-point correlation functions from the supergravity action that, given sufficient computing power, should allow for the computation of further non-trivial correlators. This algorithm – being a refined version of the procedure that has been used to compute all the correlators [13–19] – is fully rigorous and does not include any bootstrapping, like the coordinate method proposed in [20]. In particular, the simplifications include an explicit formula for the exchange part in terms of Witten diagrams. Also, we find a drastic simplification of the computation of the vertex building blocks. In particular, we obtain new reduction formulae for different products of tensors appearing in cubic and quartic couplings. These might even serve as a starting point to prove the aforementioned Mellin-space conjecture. Secondly, we use these simplifications to compute a set of new previously unknown non-trivial correlators. Thirdly, all the correlators are used to perform a non-trivial check of the Mellin-space formula for the four-point functions of arbitrary-weight operators conjectured in [20, 22]. We indeed find an agreement with the conjecture. We also discuss how the Mellin-space conjecture can be used to significantly simplify the expressions for the correlators in the coordinate space.

The thesis is set up as follows: in part I we discuss both dual theories, namely, chapter 2 is dedicated to $\mathcal{N} = 4$ SYM theory and the structure of the four-point functions, while chapter 3 is about the AdS/CFT correspondence and the dual supergravity theory. In part II we present the main results. Chapters 1, 2 and 3 contain the aforementioned simplifications of the algorithm. In chapter 4, we discuss the verification of the results and demonstrate the presence of extended operators in the corresponding correlation functions. Finally, in chapter 5, the Mellin-space conjecture is considered.

Chapter 2

Correlators in $\mathcal{N} = 4$ SYM theory

The main goal of this thesis is to compute the four-point functions of 1/2-BPS operators in planar $\mathcal{N} = 4$ SYM theory at strong coupling limit. Before proceeding with the direct computation in the dual supergravity theory, in this chapter, we define our main object of studies. After the discussion of the particle spectrum of $\mathcal{N} = 4$ SYM theory, we describe the representations of its symmetry group $PSU(2, 2|4)$ closely following [25]. 1/2-BPS operators are then those who commute with half of the supercharges. They are the most studied operators, their conformal dimension is protected, i.e., does not receive quantum corrections, and thus, they are essential in testing the AdS/CFT correspondence.

2.1 $\mathcal{N} = 4$ SYM theory

We start by reminding the basic facts about the four dimensional $\mathcal{N} = 4$ SYM theory. It is maximally supersymmetric gauge theory which consists of a covariant derivative $\mathcal{D}_\mu$, constructed from the gauge bosons $\mathcal{A}_\mu$, six massless real scalar fields ϕ^I, four chiral ψ^a_α and four anti-chiral fermions $\overline{\psi}_{\dot{\alpha}\, a}$. Here the spacetime indices μ take four values, while spinor indices $\alpha, \dot{\alpha} = 1, 2$ belong to two independent $\mathfrak{su}(2)$ algebras. Latin indices I, a refer to the global $\mathfrak{su}(4) \simeq \mathfrak{so}(6)$ symmetry, called an R-symmetry: the scalars transform in the $\mathbf{6}$, ψ^a_α in the $\mathbf{4}$ (raised a index) and $\overline{\psi}_{\dot{\alpha}\, a}$ in the $\overline{\mathbf{4}}$ (lowered a index) representations of the R-symmetry algebra. All the elementary fields transform in the adjoint representation of the $SU(N)$ gauge group and are represented by traceless Hermitian $N \times N$ matrices.

The Lagrangian of the theory is unique and given by

$$\begin{aligned}\mathcal{L} \;=\; & Tr\left(\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} + \frac{1}{2}\mathcal{D}^\mu\phi^I\mathcal{D}_\mu\phi_I + \bar{\psi}^a_{\dot{\alpha}}\sigma^{\dot{\alpha}\beta}_\mu\mathcal{D}^\mu\psi_{\beta a}\right. \\ & \left. -\frac{1}{4}g^2[\phi^I, \phi^J][\phi_I, \phi_J] - \frac{1}{2}ig\psi_{\alpha a}\sigma^{ab}_I\epsilon^{\alpha\beta}[\phi^I, \psi_{\beta b}] - \frac{1}{2}ig\bar{\psi}^a_{\dot{\alpha}}\sigma^I_{ab}\epsilon^{\dot{\alpha}\dot{\beta}}[\phi_I, \bar{\psi}^b_{\dot{\beta}}]\right)\,.\end{aligned} \tag{2.1.1}$$

The first three terms are the standard kinetic terms for the gauge field, scalars, and spinors, while the interaction is presented via a quartic coupling of the scalars and a cubic coupling of a scalar and two spinors. The matrices σ^μ and σ^I are the chiral projections of the gamma matrices in four and six dimensions respectively. The symbols ϵ refer to totally antisymmetric tensors of $\mathfrak{su}(2)$. The action for the four-dimensional $\mathcal{N} = 4$ SYM theory can be obtained, for example, as the dimensional reduction on the torus T^6 of the $d = 10$ $\mathcal{N} = 1$ SYM theory.

The theory contains a unique coupling constant, the gauge coupling $g = g_{YM}$. It has the vanishing β-function to all loops; thus, the theory is conformally invariant. The conformal symmetry, the supersymmetry and the R-symmetry of $\mathcal{N} = 4$ SYM are parts of a bigger symmetry group, the superconformal group $PSU(2,2|4)$. The bosonic subgroup of $PSU(2,2|4)$, $SU(2,2)\times SU(4)$, splits into four-dimensional conformal group $SU(2,2) \simeq SO(4,2)$ and the R-symmetry group $SU(4)$.

In the next three sections, we will discuss more on the superconformal group $PSU(2,2|4)$ and its irreducible representations.

2.2 Conformal algebra

Given a metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ with $\mu,\nu = 0,1,2,3$, the conformal algebra includes 10 Poincaré generators: 4 generators of spacetime translations, P_μ, and 6 generators of the Lorentz $SO(3,1)$ transformations, $M_{\mu\nu} = -M_{\nu\mu}$. The others generate dilatations, D, and special conformal transformation, four generators K_μ. In total, 15 generators with the following commutation relations:

$$
\begin{gathered}
[M_{\mu\nu}, P_\lambda] = i(\eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu)\,, \qquad [M_{\mu\nu}, K_\lambda] = i(\eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu)\,, \\
[M_{\mu\nu}, M_{\lambda\rho}] = i(\eta_{\mu\lambda}M_{\nu\rho} - \eta_{\nu\lambda}M_{\mu\rho} - \eta_{\mu\rho}M_{\nu\lambda} + \eta_{\nu\rho}M_{\mu\lambda})\,, \\
[D, P_\mu] = iP_\mu\,, \quad [D, K_\mu] = -iK_\mu\,, \qquad [K_\mu, P_\nu] = -2iM_{\mu\nu} - 2i\eta_{\mu\nu}D\,.
\end{gathered} \tag{2.2.1}
$$

Defining M_{AB}, $A, B = 0,1,\ldots,5$ by

$$
M_{AB} = \begin{pmatrix} M_{\mu\nu} & -\frac{1}{2}(P_\mu - K_\mu) & -\frac{1}{2}(P_\mu + K_\mu) \\ \frac{1}{2}(P_\nu - K_\nu) & 0 & D \\ \frac{1}{2}(P_\nu + K_\nu) & -D & 0 \end{pmatrix}\,, \tag{2.2.2}
$$

one can see that the algebra corresponds to that for $SO(4,2)$ with

$$
[M_{AB}, M_{CD}] = i(\eta_{AC}M_{BD} - \eta_{BC}M_{AD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC})\,, \tag{2.2.3}
$$

and $\eta_{AB} = \mathrm{diag}(-1,1,\ldots,1,-1)$.

The field-theoretic representations of the conformal group are carried by the quasi-primary fields $\mathcal{O}_I(x)$. It is said that $\mathcal{O}_I(x)$ has a dimension Δ, if under the rescaling $x \to \lambda x$ it transforms as $\mathcal{O}(x) \to \lambda^{-\Delta}\mathcal{O}(\lambda x)$. The action of the conformal generators is given by:

$$
\begin{aligned}
&[P_\mu, \mathcal{O}_I(x)] = i\partial_\mu \mathcal{O}_I(x)\,, \qquad [D, \mathcal{O}_I(x)] = i(x\cdot\partial + \Delta)\mathcal{O}_I(x)\,, \\
&[M_{\mu\nu}, \mathcal{O}_I(x)] = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\mathcal{O}_I(x) + \mathcal{O}_J(x)(s_{\mu\nu})^J{}_I\,, \\
&[K_\mu, \mathcal{O}_I(x)] = i(x^2\partial_\mu - 2x_\mu\, x\cdot\partial - 2\Delta x_\mu)\mathcal{O}_I(x) - 2\mathcal{O}_J(x)(s_{\mu\nu})^J{}_I x^\nu\,,
\end{aligned} \tag{2.2.4}
$$

with $(s_{\mu\nu})^I{}_J$ being the spin matrices satisfying the algebra of $M_{\mu\nu}$.

Defining a conformal state as

$$
|\mathcal{O}\rangle_I = \mathcal{O}_I(0)\,|0\rangle\,, \tag{2.2.5}
$$

satisfying

$$
K_\mu\,|\mathcal{O}\rangle_I = 0\,, \qquad D\,|\mathcal{O}\rangle_I = i\Delta\,|\mathcal{O}\rangle_I\,, \qquad M_{\mu\nu}\,|\mathcal{O}\rangle_I = |\mathcal{O}\rangle_J\,(s_{\mu\nu})^J{}_I\,, \tag{2.2.6}
$$

we get the representation space spanned by vectors of the form

$$\prod_n P_{\mu_n} |\mathcal{O}\rangle_I \ . \tag{2.2.7}$$

One can map the field-theoretic representations onto the so-called positive-energy representations for which the norm and, therefore, the notion of unitarity can be defined. Consideration of field-theoretic representations is, however, enough for our purposes.

2.3 Superconformal algebra

The generators of supersymmetry are fermionic and called the supercharges. These are 16 separate supercharges $Q^i{}_\alpha$ and $\bar{Q}_{i\dot\alpha}$, with $i = 1, \ldots, 4$, which in combination with the Poincaré generators define supersymmetry algebra with the following relations:

$$\begin{aligned}
&\{Q^i{}_\alpha, \bar{Q}_{j\dot\alpha}\} = 2\delta^i{}_j \mathrm{P}_{\alpha\dot\alpha}\,, \qquad \{Q^i{}_\alpha, Q^j{}_\beta\} = \{\bar{Q}_{i\dot\alpha}, \bar{Q}_{j\dot\beta}\} = 0\,,\\
&[M_\alpha{}^\beta, Q^i{}_\gamma] = \delta_\gamma{}^\beta Q^i{}_\alpha - \tfrac{1}{2}\delta_\alpha{}^\beta Q^i{}_\gamma\,, \quad [\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{Q}_{i\dot\gamma}] = -\,\delta^{\dot\alpha}{}_{\dot\gamma}\bar{Q}_{i\dot\beta} + \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\bar{Q}_{i\dot\gamma}\,.
\end{aligned} \tag{2.3.1}$$

Here we defined

$$\begin{aligned}
&\mathrm{P}_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha} P_\mu\,, \qquad\qquad \tilde{\mathrm{K}}^{\dot\alpha\alpha} = (\bar\sigma^\mu)^{\dot\alpha\alpha} K_\mu\,,\\
&M_\alpha{}^\beta = -\,\tfrac{1}{4} i (\sigma^\mu\bar\sigma^\nu)_\alpha{}^\beta M_{\mu\nu}\,, \quad \bar{M}^{\dot\alpha}{}_{\dot\beta} = -\tfrac{1}{4} i (\bar\sigma^\mu\sigma^\nu)^{\dot\alpha}{}_{\dot\beta} M_{\mu\nu}\,,
\end{aligned} \tag{2.3.2}$$

and in new notations the commutators for $M_{\mu\nu}$ reduce to

$$[M_\alpha{}^\beta, M_\gamma{}^\delta] = \delta_\gamma{}^\beta M_\alpha{}^\delta - \delta_\alpha{}^\delta M_\gamma{}^\beta\,, \qquad [\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{M}^{\dot\gamma}{}_{\dot\delta}] = -\delta^{\dot\alpha}{}_{\dot\delta}\bar{M}^{\dot\gamma}{}_{\dot\beta} + \delta^{\dot\gamma}{}_{\dot\beta}\bar{M}^{\dot\alpha}{}_{\dot\delta}\,. \tag{2.3.3}$$

Finally, the combination of the supersymmetry with dilatations and special conformal transformations makes a superconformal algebra. The commutators with D are

$$[D, Q^i{}_\alpha] = \tfrac{1}{2} i Q^i{}_\alpha\,, \qquad [D, \bar{Q}_{i\dot\alpha}] = \tfrac{1}{2} i \bar{Q}_{i\dot\alpha}\,. \tag{2.3.4}$$

The special conformal generators do not commute with the supercharges

$$[K_\mu, Q^i{}_\alpha] = \quad -(\sigma_\mu)_{\alpha\dot\alpha}\bar{S}^{i\dot\alpha}\,, \qquad [K_\mu, \bar{Q}_{i\dot\alpha}] = S_i^\alpha(\sigma_\mu)_{\alpha\dot\alpha}\,; \tag{2.3.5}$$

thus, creating new superconformal charges $S^i{}_\alpha$ and $\bar{S}_{i\dot\alpha}$, satisfying the following relations:

$$\begin{aligned}
&\{\bar{S}^{i\dot\alpha}, S_j^\alpha\} = 2\delta^i{}_j \tilde{\mathrm{K}}^{\dot\alpha\alpha}\,, \qquad \{\bar{S}^{i\dot\alpha}, \bar{S}^{j\dot\beta}\} = \{S_i^\alpha, S_j^\beta\} = 0\,,\\
&\{Q^i{}_\alpha, \bar{S}^{j\dot\alpha}\} = \{S_i^\alpha, \bar{Q}_{j\dot\alpha}\} = 0\,,\\
&\{Q^i{}_\alpha, S_j^\beta\} = 4\big(\delta^i{}_j(M_\alpha{}^\beta - \tfrac{1}{2} i\, \delta_\alpha{}^\beta D) - \delta_\alpha{}^\beta R^i{}_j\big)\,,\\
&\{\bar{S}^{i\dot\alpha}, \bar{Q}_{j\dot\beta}\} = 4\big(\delta^i{}_j(\bar{M}^{\dot\alpha}{}_{\dot\beta} + \tfrac{1}{2} i\, \delta^{\dot\alpha}{}_{\dot\beta} D) - \delta^{\dot\alpha}{}_{\dot\beta} R^i{}_j\big)\,,\\
&[M_\alpha{}^\beta, S_i^\gamma] = -\delta_\alpha{}^\gamma S_i^\beta + \tfrac{1}{2}\delta_\alpha{}^\beta S_i^\gamma\,, \qquad [\bar{M}^{\dot\alpha}{}_{\dot\beta}, \bar{S}^{i\dot\gamma}] = \delta^{\dot\gamma}{}_{\dot\beta}\bar{S}^{i\dot\alpha} - \tfrac{1}{2}\delta^{\dot\alpha}{}_{\dot\beta}\bar{S}^{i\dot\gamma}\,,\\
&[D, S_i^\alpha] = -\tfrac{1}{2} i S_i^\alpha\,, \quad [D, \bar{S}^{i\dot\alpha}] = -\tfrac{1}{2} i \bar{S}^{i\dot\alpha}\,,\\
&[P_\mu, \bar{S}^{i\dot\alpha}] = -\,(\bar\sigma_\mu)^{\dot\alpha\alpha} Q^i{}_\alpha\,, \qquad [P_\mu, S_i^\alpha] = \bar{Q}_{i\dot\alpha}(\bar\sigma_\mu)^{\dot\alpha\alpha}\,.
\end{aligned} \tag{2.3.6}$$

Here R_{ij} are the 15 $SU(4)$ R-symmetry generators for which the Lie algebra is

$$[R^i{}_j, R^k{}_l] = \delta^k{}_j R^i{}_l - \delta^i{}_l R^k{}_j\,. \tag{2.3.7}$$

All the conformal generators commute with R_{ij}, while the action on the supercharges is given by

$$[R^i{}_j, Q^k{}_\alpha] = \delta^k{}_j Q^i{}_\alpha - \tfrac{1}{4}\delta^i{}_j Q^k{}_\alpha\,, \qquad [R^i{}_j, \bar{Q}_{k\dot\alpha}] = -\delta^i{}_k \bar{Q}_{j\dot\alpha} + \tfrac{1}{4}\delta^i{}_j \bar{Q}_{k\dot\alpha}\,,$$
$$[R^i{}_j, S_k^{\alpha}] = -\,\delta^i{}_k S_j^{\alpha} + \tfrac{1}{4}\delta^i{}_j S_k^{\alpha}\,, \qquad [R^i{}_j, \bar{S}^{k\dot\alpha}] = \delta^k{}_j \bar{S}^{i\dot\alpha} - \tfrac{1}{4}\delta^i{}_j \bar{S}^{k\dot\alpha}\,. \qquad (2.3.8)$$

Given the structure of the superconformal algebra, we have everything to proceed with building up its representations.

2.4 Superconformal representations and BPS-states

To find the unitary irreducible representations (irreps) of the $PSU(2,2|4)$ group it suffices to analyze the irreps of its bosonic subgroup $SU(2,2)\times SU(4)$. Acting on it by the supercharges Q one gets the irreps of the full superconformal group.

For $SU(4)$ we consider the Chevalley basis, namely H_i, $E_i^{\pm}$, $i=1,2,3$, with the following commutation relations

$$[H_i, H_j] = 0\,, \quad [E_i^+, E_j^-] = \delta_{ij} H_j\,, \quad [H_i, E_j^{\pm}] = \pm K_{ji} E_j^{\pm}\,, \qquad (2.4.1)$$

where the Cartan matrix is

$$K_{ij} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\,. \qquad (2.4.2)$$

Then for any representation space, a convenient basis is given by the eigenvectors of H_i:

$$H_i|\lambda_1,\lambda_2,\lambda_3\rangle = \lambda_i|\lambda_1,\lambda_2,\lambda_3\rangle\,, \qquad (2.4.3)$$

with λ_i integers, and is uniquely determined by its highest weight state

$$E_i^+|\lambda_1,\lambda_2,\lambda_3\rangle^{\rm hw} = 0\,, \qquad \lambda_i \geq 0\,. \qquad (2.4.4)$$

Acting by the lowering operators E_i^-, one obtains the remaining basis vectors.

Further, we write the generators of the Lorentz group as

$$M_\alpha{}^\beta = \begin{pmatrix} J_3 & J_+ \\ J_- & -J_3 \end{pmatrix}\,, \qquad \bar{M}^{\dot\beta}{}_{\dot\alpha} = \begin{pmatrix} \bar{J}_3 & \bar{J}_+ \\ \bar{J}_- & -\bar{J}_3 \end{pmatrix} \qquad (2.4.5)$$

where $J_\pm, J_3$ and $\bar{J}_\pm, \bar{J}_3$ are the standard generators of $SU(2)_J$ and $SU(2)_{\bar{J}}$. The highest-weight state for the superconformal algebra then satisfies

$$E_i^+|\Delta,j,\bar{j};\,\lambda_1,\lambda_2,\lambda_3\rangle^{\rm hw} = J_+|\Delta,j,\bar{j};\,\lambda_1,\lambda_2,\lambda_3\rangle^{\rm hw} = \bar{J}_+|\Delta,j,\bar{j};\,\lambda_1,\lambda_2,\lambda_3\rangle^{\rm hw} = 0\,. \qquad (2.4.6)$$

The supermultiplet is then generated by the action of the supercharges $Q^i{}_\alpha$, $\bar{Q}_{i\dot\alpha}$ on the state with lowest conformal dimension Δ.

Thus, the representations of the superconformal group are labeled by a conformal dimension Δ, two spins $j, \bar{j}$ and three Dynkin labels $[\lambda_1, \lambda_2, \lambda_3]$.

BPS-states
If some of the supercharges annihilate the highest-weight state, we obtain the shortening of the supermultiplet

$$Q^i{}_\alpha |k, p, q; j, \bar{j}\rangle^{\text{hw}} = 0\,, \qquad \alpha = 1, 2\,. \tag{2.4.7}$$

One can obtain the following results

$$\begin{array}{lll} i = 1 & \Delta = \frac{1}{2}(3\lambda_1 + 2\lambda_2 + \lambda_3)\,, & \\ i = 1,2 & \Delta = \frac{1}{2}(2\lambda_2 + \lambda_3)\,, & \lambda_1 = 0\,, \\ i = 1,2,3 & \Delta = \frac{1}{2}\lambda_3\,, & \lambda_1 = \lambda_2 = 0\,, \\ i = 1,2,3,4 & \Delta = 0\,, & \lambda_1 = \lambda_2 = \lambda_3 = 0\,. \end{array} \tag{2.4.8}$$

Equivalently shortening conditions can be obtained for the action of the $\bar{Q}$ supercharges

$$\bar{Q}_{i\dot{\alpha}} |k, p, q; j, \bar{j}\rangle^{\text{hw}} = 0\,, \qquad \dot{\alpha} = 1, 2\,. \tag{2.4.9}$$

In this case we have

$$\begin{array}{lll} i = 4 & \Delta = \frac{1}{2}(\lambda_1 + 2\lambda_2 + 3\lambda_3)\,, & \\ i = 3,4 & \Delta = \frac{1}{2}(\lambda_1 + 2\lambda_2)\,, & \lambda_3 = 0\,, \\ i = 2,3,4 & \Delta = \frac{1}{2}\lambda_1\,, & \lambda_2 = \lambda_3 = 0\,, \\ i = 1,2,3,4 & \Delta = 0\,, & \lambda_1 = \lambda_2 = \lambda_3 = 0\,. \end{array} \tag{2.4.10}$$

In this thesis we are interested only in 1/2-BPS operators, those which commute with half of the supercharges. One can see that intersection of (2.4.8) and (2.4.10) gives representation $[0, p, 0]$ with $\Delta = p$.

2.5 The structure of four-point functions

The central objects of our study are the four-point correlation functions of 1/2-BPS operators in the four-dimensional planar $\mathcal{N} = 4$ SYM theory in the limit when the 't Hooft coupling constant $\lambda = g_{YM}^2 N \to \infty$. The planar limit consists of sending the rank of the gauge group $N \to \infty$ while keeping λ constant. In this limit, all the nonplanar Feynman diagrams are suppressed. We will start with a review of the structure of these correlation functions – due to the presence of superconformal symmetry – here, see, e.g., [26], as it will introduce a lot of our notation.

Recall that the multiplets of 1/2-BPS operators are generated from their highest weight vectors, which we will denote as $\widetilde{\mathcal{O}}_k$ with k indicating the conformal dimension. They can be decomposed as

$$\widetilde{\mathcal{O}}_k = \widetilde{\mathcal{O}}^{i_1 \ldots i_k} t_{i_1} \cdots t_{i_k}, \tag{2.5.1}$$

with $\widetilde{\mathcal{O}}^{i_1 \ldots i_k}$ carrying the dependence on the scalar fields ϕ^i and t being a six-dimensional null vector that keeps track of the SO(6) R-symmetry of the operator. Given four weights

(k_1, k_2, k_3, k_4) with $k_i \geqslant 2$ the four-point correlation function of such operators is given by

$$\langle k_1 k_2 k_3 k_4 \rangle := \langle \widetilde{\mathcal{O}}_{k_1}(x_1, t_1) \cdots \widetilde{\mathcal{O}}_{k_4}(x_4, t_4) \rangle, \tag{2.5.2}$$

where the x_i are spacetime coordinates, and we indicated the dependence on the null vectors $t_1, \ldots, t_4$.[1] The correlator splits into a free part and an interacting part:

$$\langle k_1 k_2 k_3 k_4 \rangle = \langle k_1 k_2 k_3 k_4 \rangle_0 + \langle k_1 k_2 k_3 k_4 \rangle_{\text{int}}, \tag{2.5.3}$$

where the first part $\langle k_1 k_2 k_3 k_4 \rangle_0$ is the Born approximation and can be computed by applying the Wick's theorem. After defining

$$d_{ij} = \frac{t_{ij}}{x_{ij}^2}, \qquad \text{with } t_{ij} = t_i \cdot t_j, \quad x_{ij}^2 = (x_i - x_j)^2 \tag{2.5.4}$$

we can state the form of the free part as:

$$\langle k_1 k_2 k_3 k_4 \rangle_0 = \sum_a c_a \left(\prod_{i,j} d_{ij}^{a_{ij}} \right), \tag{2.5.5}$$

where the sum runs over all the sets $a = \{a_{ij}\}$ and the c_a are constants. The partitions a parametrize all the inequivalent ways one can pair the t vectors, and they can be represented as symmetric 4×4 matrices with nonnegative integer entries with zeroes on the diagonal that satisfy for each i

$$\sum_{\substack{j=1 \\ j \neq i}}^{4} a_{ij} = k_i. \tag{2.5.6}$$

For a given set of weights, these equations have a finite number of solutions. For later convenience we define

$$T_l \equiv \prod_{\substack{i,j=1 \\ i<j}}^{4} t_{ij}^{(a_l)_{ij}}, \tag{2.5.7}$$

where a_l is the lth solution to the equations (2.5.6) in the ordered list of solutions: we order solutions lexicographically as $\{a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}\}$. In fact, the entire correlator can be written in the form (2.5.5), where the constants c_a then become functions of the conformal cross ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{2.5.8}$$

As it turns out, due to supersymmetry the correlator factorizes further: we can write the interacting part as

$$\langle k_1 k_2 k_3 k_4 \rangle_{\text{int}} = \frac{\mathcal{R}_{1,2,3,4}}{x_{13}^2 x_{24}^2} \sum_b \left(\prod_{i,j} d_{ij}^{b_{ij}} \right) \mathcal{F}_b(u, v), \tag{2.5.9}$$

[1] Note that the subscript does not refer to a vector component, but is there to distinguish four different t vectors.

where

$$\begin{aligned}\mathcal{R}_{1,2,3,4} = {} & d_{12}^2 d_{34}^2 x_{12}^2 x_{34}^2 + d_{13}^2 d_{24}^2 x_{13}^2 x_{24}^2 + d_{14}^2 d_{23}^2 x_{14}^2 x_{23}^2 \\ & + d_{12} d_{23} d_{34} d_{14} \left(x_{13}^2 x_{24}^2 - x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2\right) \\ & + d_{12} d_{13} d_{24} d_{34} \left(x_{14}^2 x_{23}^2 - x_{12}^2 x_{34}^2 - x_{13}^2 x_{24}^2\right) \\ & + d_{13} d_{14} d_{23} d_{24} \left(x_{12}^2 x_{34}^2 - x_{14}^2 x_{23}^2 - x_{13}^2 x_{24}^2\right)\end{aligned} \tag{2.5.10}$$

is the fully symmetric general prefactor [27]. This time the partitions b form a symmetric 4×4-matrix with zeroes on the diagonal, but they satisfy the modified equations

$$\sum_{\substack{j=1 \\ j \neq i}}^{4} b_{ij} = k_i - 2. \tag{2.5.11}$$

This reduces the number of partitions a significantly, implying that the correlator can be written as a sum over only a few independent functions $\mathcal{F}_b$ of u and v, which are moreover conformally invariant by construction and carry all the dependence on the 't Hooft coupling.

We will use this decomposition of the correlator for two purposes: firstly to be able to present our computed correlators in a compact form $\{\mathcal{F}_b\}$. Secondly, this decomposition provides a non-trivial check whether these correlators are consistent with superconformal symmetry.

***C*-tensors.** The $\mathcal{N} = 4$ SYM four-point functions can be represented in many forms depending on the context. In particular, we will use the language of C-tensors to compute some of the necessary intermediate objects. The C-tensors can be used to track the SO(6) symmetry instead of the t vectors, by writing (2.5.1) as

$$\widetilde{\mathcal{O}}_k = \widetilde{\mathcal{O}}^{i_1 \dots i_k} C_{i_1, \dots, i_k}, \tag{2.5.12}$$

with the tensor C being totally symmetric and traceless in its indices. The vector space of C-tensors with k indices forms a representation space for the traceless symmetric SO(6) representation with Dynkin labels $[0, k, 0]$. Choosing a basis in this space amounts to selecting a set of C-tensors which we label with an upper index I_k that will appear throughout this paper:

$$C^{I_k}_{i_1, \dots, i_k}, \quad \text{with } I_k = 1, \dots, \dim([0, k, 0]). \tag{2.5.13}$$

The possible contractions of four C-tensors are characterized by the number of connections between the different C-tensors. Denoting the number of connections by a_{ij} they should satisfy $a_{ij} = a_{ji}$, $a_{ii} = 0$ and the sum conditions for a's (2.5.6), showing that the a partitions indeed parametrize the possible tensor structures (2.5.7) for a given correlator (2.5.2). C-tensors will be discussed in more detail in section 3.1 of part II.

Chapter 3

Type IIB supergravity action on $\mathrm{AdS}_5 \times \mathrm{S}^5$

In this chapter, we briefly formulate the relevant part of the AdS/CFT correspondence, proposed by Maldacena in [5], and gather all the necessary ingredients for calculation of the four-point functions. More details can be found in the reviews [28–30].

3.1 The AdS/CFT correspondence

Since the early days of string theory, there have been attempts to derive quantum field theories by taking various limits of string or M-theories. A breakthrough appeared with the paper of Maldacena [5]. He conjectured the duality between certain quantum field theories with exact conformal spacetime symmetry (CFTs) and corresponding string theories defined on a particular background. The latter is a product of an Anti-de-Sitter (AdS) space and a compact manifold. In this formalism, the field theory is formulated on the boundary of the AdS_{d+1} – conformally flat d-dimensional Minkowski spacetime. Thus, one sometimes refers to the CFT as to the boundary theory, and the dual string theory as to the bulk theory.

Most attempts to test this duality were focused on its most symmetric couple: the CFT partner is the four-dimensional $\mathcal{N} = 4$ SYM theory, while on the other side we have type IIB superstrings propagating on the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. $\mathcal{N} = 4$ SYM has two parameters: the 't Hooft coupling $\lambda = g_{YM}^2 N$ and rank N of the gauge group, where g_{YM} is the coupling constant of the theory. The AdS/CFT correspondence relates them to the string coupling constant g_s and the string tension $g = R^2/2\pi\alpha'$, where R is the radius of S^5 and α' is the Regge slope, as follows:

$$\lambda = 4\pi N g_s\,, \qquad g = \sqrt{\lambda}/2\pi\,. \tag{3.1.1}$$

For every string observable at the boundary of AdS_5, there is a corresponding observable in the CFT_4 and vice versa. Their values are supposed to match. This statement, when the conjecture holds for all the values of the theory parameters, is usually referred to as the strong form of the duality. To check Maldacena's conjecture, however, appears to be very difficult in all the regions of the parameter space. Thus, different limits are considered (see fig 3.1 [6]):

- *The 't Hooft limit* [31] consists of keeping the 't Hooft coupling λ fixed while taking the limit $N \to \infty$. In the perturbative expansion of the gauge theories around $\lambda = 0$

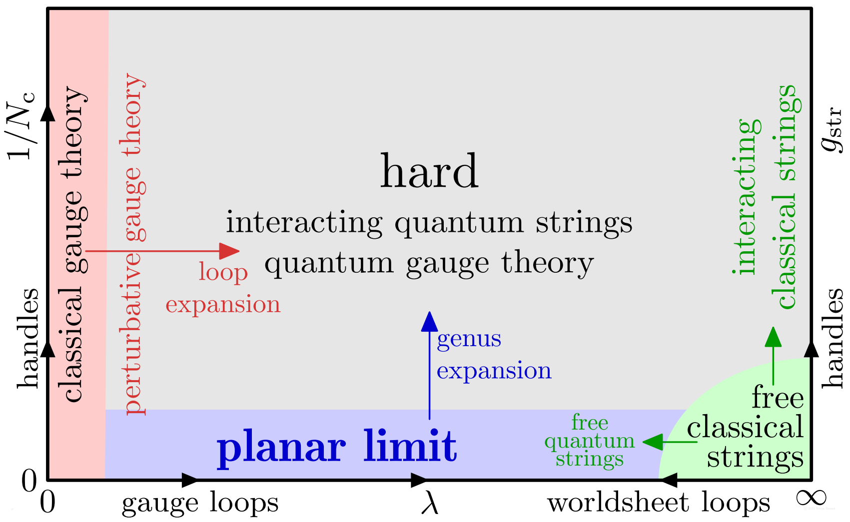

Figure 3.1: Map of the parameter space of $\mathcal{N}=4$ SYM or strings on the $\mathrm{AdS}_5\times\mathrm{S}^5$.

this limit corresponds to a topological $1/N$ expansion of the Feynman diagrams. It appears that only the planar graphs – the diagrams which can be drawn on the plane without crossing lines – survive. That is why this regime is also called the *planar limit*. On the AdS side, since λ is fixed, this limit corresponds to a weakly coupled string theory.

- One can also consider the *large* and *small* λ *limits*. The region of parameter space where λ is small is called the *weak coupling* regime. Here one can apply the Feynman diagrammatic techniques to computations in SYM theory. From the dual string theory side, this regime rather corresponds to the strong coupling, thus, providing a suitable tool to investigate the theory via its dual CFT partner. Another limit is called the *strong coupling* limit, referring to the gauge theory parameter $\lambda\to\infty$. Here, however, the strings are weakly coupled, which lets us study the strong coupling limit of the SYM theory via its string partner.

In this thesis, we consider the strong coupling limit of the planar $\mathcal{N}=4$ SYM theory. In this regime, the dual string theory reduces to type IIB classical supergravity defined on the $\mathrm{AdS}_5\times\mathrm{S}^5$ space. According to the AdS/CFT correspondence, chiral primary operators of four-dimensional $\mathcal{N}=4$ SYM theory are dual to the Kaluza-Klein modes of type IIB supergravity on the $\mathrm{AdS}_5\times\mathrm{S}^5$ compactified on S^5: $\widetilde{\mathcal{O}}_k\longleftrightarrow s_k^{I_k}(z)$, $z\in\mathrm{AdS}_5$. Here, as before, the index I_k runs over the basis of the corresponding SO(6) representation with Dynkin labels $[0,k,0]$. The precise duality between operators was conjectured by Witten and independently by Gubser, Klebanov, Polyakov in [8,9]. The relation is as follows: one computes the on-shell supergravity partition function $\exp(-S_{IIB})$ as a function of the boundary values of the fields, $s_k^{I_k}(\vec{x})$ with $x\in\partial\mathrm{AdS}_5$, and subsequently interprets it as the SYM partition function with sources for $\widetilde{\mathcal{O}}_k$ being $s_k^{I_k}(\vec{x})$. Thus, in practice, the correlation functions of interest can be determined from the expression:

$$\langle\tilde{O}_{k_1}^{I_1}(\vec{x}_1)\tilde{O}_{k_2}^{I_2}(\vec{x}_2)\tilde{O}_{k_3}^{I_3}(\vec{x}_3)\tilde{O}_{k_4}^{I_4}(\vec{x}_4)\rangle=\frac{\delta^4}{\delta s_{k_1}^{I_1}(\vec{x}_1)\delta s_{k_2}^{I_2}(\vec{x}_2)\delta s_{k_3}^{I_3}(\vec{x}_3)\delta s_{k_4}^{I_4}(\vec{x}_4)}\exp(-S_{IIB})\bigg|_{s_k\to0}. \tag{3.1.2}$$

3.2 Effective supergravity action

To compute an n-point function, one has to expand the action up to the n-th order in perturbations of the fields around their background values. This becomes problematic since there is no manifestly Lorentz-invariant action for type IIB supergravity because of the presence of the self-dual 5-form field strength [32], though the covariant equations of motion are known [33,34]. However, later on, it was possible to find the Lorentz-invariant Lagrangian by including additional auxiliary fields [35, 36].

This action was used in [10–12,37] to derive the relevant quadratic effective action for the type IIB supergravity on the $\mathrm{AdS}_5 \times \mathrm{S}^5$. Representing the gravitational field and the four-form potential as

$$G_{MN} = g_{MN} + h_{MN}\,, \quad A_{MNPQ} = \bar{A}_{MNPQ} + a_{MNPQ}\,, \quad F = \bar{F} + f\,, \tag{3.2.1}$$

decomposing the action up to the second-order one gets the quadratic action. After imposing the gauge conditions, elaborating the action term by term, and performing the field redefinitions, one ends up with the covariant quadratic equations of motion which admit lagrangian description.

To obtain cubic and quartic Lagrangian, it is necessary to know only the equations of motion of the 4-form potential and the metric. These are:

$$\begin{aligned} F_{M_1\ldots M_5} &= \frac{1}{5!}\epsilon_{M_1\ldots M_{10}} F^{M_6\ldots M_{10}}\,, \\ R_{MN} &= \frac{1}{3!} F_{MM_1\ldots M_4} F_N{}^{M_1\ldots M_4}\,. \end{aligned} \tag{3.2.2}$$

Here $M, N, ... = 0, ..., 9$ and

$$F_{M_1\ldots M_5} = 5\partial_{[M_1} A_{M_2\ldots M_5]} = \partial_{M_1} A_{M_2\ldots M_5} + 4 \ \text{ terms}\,. \tag{3.2.3}$$

The dual forms are given by:

$$\begin{aligned} &\epsilon_{01\ldots 9} = \sqrt{-G}, \quad \epsilon^{01\ldots 9} = -\frac{1}{\sqrt{-G}}\,, \quad \epsilon^{M_1\ldots M_{10}} = G^{M_1N_1}\cdots G^{M_{10}N_{10}}\epsilon_{N_1\ldots N_{10}}\,, \\ &(F^*)_{M_1\ldots M_k} = \frac{1}{k!}\epsilon_{M_1\ldots M_{10}} F^{M_{k+1}\ldots M_{10}} = \frac{1}{k!}\epsilon^{N_1\ldots N_{10}} G_{M_1N_1}\cdots G_{M_kN_k} F_{N_{k+1}\ldots N_{10}}\,. \end{aligned} \tag{3.2.4}$$

The corresponding $AdS_5 \times S^5$ background solution is as follows:

$$\begin{aligned} &ds^2 = \frac{1}{x_0^2}(dx_0^2 + \eta_{ij}dx^i dx^j) + d\Omega_5^2 = G_{MN}dx^M dx^N\,, \\ &R_{abcd} = -G_{ac}G_{bd} + G_{ad}G_{bc}\,, \quad R_{ab} = -4G_{ab}\,, \\ &R_{\alpha\beta\gamma\delta} = G_{\alpha\gamma}G_{\beta\delta} - G_{\alpha\delta}G_{\beta\gamma}\,, \quad R_{\alpha\beta} = 4G_{\alpha\beta}\,, \\ &\bar{F}_{abcde} = \epsilon_{abcde}; \quad \bar{F}_{\alpha\beta\gamma\delta\epsilon} = \epsilon_{\alpha\beta\gamma\delta\epsilon}\,, \end{aligned} \tag{3.2.5}$$

where the Latin $a, b, c, ...$ indices refer to AdS and the Greek $\alpha, \beta, \gamma, ...$ indices - to the sphere, and η_{ij} is simply the 4-dimensional Minkowski metric.

The gauge symmetry let us impose the de Donder gauge:

$$\nabla^\alpha h_{a\alpha} = \nabla^\alpha h_{(\alpha\beta)} = \nabla^\alpha a_{M_1M_2M_3\alpha} = 0\,, \quad h_{(\alpha\beta)} \equiv h_{\alpha\beta} - \frac{1}{5}g_{\alpha\beta}h_\gamma^\gamma\,. \tag{3.2.6}$$

These conditions imply that the following representation of the 4-form potential components of the form $a_{\alpha\beta\gamma\delta}$ and $a_{a\alpha\beta\gamma}$ can be performed:

$$a_{\alpha\beta\gamma\delta} = \epsilon_{\alpha\beta\gamma\delta\epsilon}\nabla^{\epsilon} b\,, \quad a_{a\alpha\beta\gamma} = \epsilon_{\alpha\beta\gamma\delta\epsilon}\nabla^{\delta}\phi_a^{\epsilon}\,. \tag{3.2.7}$$

To proceed, we expand the fields into spherical components

$$\begin{aligned}
&h_\alpha^\alpha(x,y) = \sum \pi^I(x) Y^I(y)\,, \quad b(x,y) = \sum b^I(x) Y^I(y)\,,\\
&h_{ab}(x,y) = \sum h_{ab}^I(x) Y^I(y)\,, \quad \nabla_\beta^2 Y^k = -k(k+4)Y^k = -f(k)Y^k\,,\\
&h_{a\alpha}(x,y) = \sum h_a^I(x) Y_\alpha^I(y)\,, \quad \phi_{a\alpha}(x,y) = \sum \phi_a^I(x) Y_\alpha^I(y)\,,\\
&(\nabla_\beta^2 - 4) Y_\alpha^k = -(k+1)(k+3) Y_\alpha^k\,,\\
&h_{(\alpha\beta)}(x,y) = \sum \phi^I(x) Y_{(\alpha\beta)}^I(y)\,, \quad (\nabla_\gamma^2 - 10) Y_{(\alpha\beta)}^k = -(k^2+4k+8) Y_{(\alpha\beta)}^k\,,
\end{aligned} \tag{3.2.8}$$

and redefine the fields as follows:

$$\begin{aligned}
&\pi_k = 10 k s_k + 10(k+4) t_k\,, \quad b_k = -s_k + t_k\,,\\
&h_{ab}^k = \varphi_{ab}^k + g_{ab}\eta_k + \nabla_a\nabla_b\zeta_k\,,\\
&A_a^k = h_a^k - 4(k+3)\phi_a^k\,, \quad C_a^k = h_a^k + 4(k+1)\phi_a^k\,,\\
&\zeta_k = \frac{4}{k+1}s_k + \frac{4}{k+3}t_k\,, \quad \eta_k = -\frac{2k(k-1)}{k+1}s_k - \frac{2(k+4)(k+5)}{k+3}t_k\,.
\end{aligned} \tag{3.2.9}$$

Together with (3.2.9) one has to perform other field redefinitions to make the equations of motion lagrangian and to remove higher-derivative terms from quadratic terms in the equations of motion.

This procedure was applied by Arutyunov, Frolov in [10–12, 37] and the explicit expression for the relevant part of the action was obtained:

$$S = \frac{N^2}{8\pi^2}\int [dz](\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4). \tag{3.2.10}$$

Here the quadratic terms are given by

$$
\begin{aligned}
\mathcal{L}_2^{(s)} &= \sum_k \zeta(s_k)\left(-\frac{1}{2}\nabla_\mu s_k \nabla^\mu s_k - \frac{1}{2}m_k^2 s_k^2\right), \\
\mathcal{L}_2^{(t)} &= \sum_k \zeta(t_k)\left(-\frac{1}{2}\nabla_\mu t_k \nabla^\mu t_k - \frac{1}{2}m_{t_k}^2 t_k^2\right), \\
\mathcal{L}_2^{(\phi)} &= \sum_k \zeta(\phi_k)\left(-\frac{1}{2}\nabla_\mu \phi_k \nabla^\mu \phi_k - \frac{1}{2}m_{\phi_k}^2 \phi_k^2\right), \qquad (3.2.11)\\
\mathcal{L}_2^{(A_\mu)} &= \sum_k \zeta(A_{\mu,k})\left(-\frac{1}{4}F_{\mu\nu,k}^2(A_{\mu,k}) - \frac{1}{2}m_A^2 A_{\mu,k}^2\right), \\
\mathcal{L}_2^{(C_\mu)} &= \sum_k \zeta(C_{\mu,k})\left(-\frac{1}{4}F_{\mu\nu,k}^2(C_{\mu,k}) - \frac{1}{2}m_C^2 C_{\mu,k}^2\right), \\
\mathcal{L}_2^{(\varphi_{\mu\nu})} &= \sum_k \zeta(\varphi_{\mu\nu,k})\left(-\frac{1}{4}\nabla_\rho \varphi_{\mu\nu,k}\nabla^\rho \varphi_k^{\mu\nu} + \frac{1}{2}\nabla_\rho \varphi_k^{\rho\mu}\nabla^\lambda \varphi_{\lambda\mu,k} - \frac{1}{2}\nabla_\mu \varphi_{\rho,k}^{\rho}\nabla_\lambda \varphi_k^{\lambda,\mu}\right. \\
&+ \left.\frac{1}{4}\nabla_\rho \varphi_{\mu,k}^{\mu}\nabla^\rho \varphi_{\nu,k}^{\nu} + \frac{1}{4}(2-f_k)\varphi_{\mu\nu,k}\varphi_k^{\mu\nu} + \frac{1}{4}(2+f_k)(\varphi_{\rho,k}^{\rho})^2\right),
\end{aligned}
$$
$$(3.2.12)$$

with the corresponding quadratic couplings being

$$
\begin{aligned}
\zeta(s_k) &= \frac{32(k-1)k(k+2)}{k+1}, \quad \zeta(t_k) = \frac{32(k-2)k(k+1)}{k-1}, \quad \zeta(\phi_k) = \frac{1}{2}, \\
\zeta(A_{\mu,k}) &= \frac{k}{2(k+1)}, \quad z(C_{\mu,k}) = \frac{k}{2(k-1)}, \quad \zeta(\varphi_{\mu\nu,k}) = 1\,. \qquad (3.2.13)
\end{aligned}
$$

The cubic terms may be written as follows

$$
\begin{aligned}
\mathcal{L}_3^{(s)} &= S_{I_1I_2I_3}s^{I_1}s^{I_2}s^{I_3}, \quad \mathcal{L}_3^{(t)} = T_{I_1I_2I_3}s^{I_1}s^{I_2}t^{I_3}, \quad \mathcal{L}_3^{(\phi)} = \Phi_{I_1I_2I_3}s^{I_1}s^{I_2}\phi^{I_3}, \\
\mathcal{L}_3^{(A_\mu)} &= A_{I_1I_2I_3}s^{I_1}\nabla^\mu s^{I_2}A_\mu^{I_3}, \quad \mathcal{L}_3^{(C_\mu)} = C_{I_1I_2I_3}s^{I_1}\nabla^\mu s^{I_2}C_\mu^{I_3}, \quad \mathcal{L}_3^{(\varphi_{\mu\nu})} = G_{I_1I_2I_3}T_{\mu\nu}^{I_1I_2}\varphi^{\mu\nu,I_3},
\end{aligned}
$$
$$(3.2.14)$$

where (...) denotes symmetrization and the stress-energy tensor $T_{\mu\nu}$ has the form

$$T_{\mu\nu} = \nabla_{(\mu}s^{I_1}\nabla_{\nu)}s^{I_2} - \frac{1}{2}g_{\mu\nu}\left(\nabla^\rho s^{I_1}\nabla_\rho s^{I_2} + \frac{1}{2}(m_1^2+m_2^2-f_k)s^{I_1}s^{I_2}\right). \qquad (3.2.15)$$

The corresponding cubic couplings are

$$
\begin{aligned}
S_{I_1 I_2 I_3} &= a_{123}\,\frac{256\,\alpha_1\,\alpha_2\,\alpha_3\,\Sigma\,(\Sigma^2-4)\,(\Sigma^2-1)}{3\,(k_1+1)\,(k_2+1)\,(k_3+1)}\,,\\
T_{I_1 I_2 I_3} &= a_{123}\,\frac{256\,\alpha_1\,\alpha_2\,(\alpha_3-2)\,(\alpha_3-1)\,\alpha_3\,(\alpha_3+1)\,(\alpha_3+2)\,\Sigma}{(k_1+1)\,(k_2+1)\,(k_3-1)}\,,\\
\Phi_{I_1 I_2 I_3} &= p_{123}\,\frac{8\,(\alpha_3-1)\,\alpha_3\Sigma\,(2\Sigma-2)}{(k_1+1)\,(k_2+1)}\,,\\
A_{I_1 I_2 I_3} &= t_{123}\,\frac{4\,\alpha_3\,k_3\,\Sigma\,(2\Sigma-2)(2\Sigma+2)}{(k_1+1)\,(k_2+1)\,(k_3+1)}\,,\\
C_{I_1 I_2 I_3} &= t_{123}\,\frac{16\,(\alpha_3-1)\,\alpha_3\,(\alpha_3+1)\,k_3\,\Sigma}{(k_1+1)\,(k_2+1)\,(k_3-1)}\,,\\
G_{I_1 I_2 I_3} &= a_{123}\,\frac{8\,\alpha_3\,(\alpha_3+1)\,\Sigma\,(2\Sigma+2)}{(k_1+1)\,(k_2+1)}\,.
\end{aligned}
\tag{3.2.16}
$$

Finally, the quartic terms are computed as follows:

$$
\begin{aligned}
&\mathcal{L}_4 = \mathcal{L}_4^{(4)} + \mathcal{L}_4^{(2)} + \mathcal{L}_4^{(0)},\\
&\mathcal{L}_4^{(4)} = (S^{(4)}_{I_1I_2I_3I_4} + A^{(4)}_{I_1I_2I_3I_4})\; s^{I_1}\nabla_\mu s^{I_2}\nabla_\nu^2(s^{I_3}\nabla^\mu s^{I_4}),\\
&\mathcal{L}_4^{(2)} = (S^{(2)}_{I_1I_2I_3I_4} + A^{(2)}_{I_1I_2I_3I_4})\; s^{I_1}\nabla_\mu s^{I_2} s^{I_3}\nabla^\mu s^{I_4},\\
&\mathcal{L}_4^{(0)} = (S^{(0)}_{I_1I_2I_3I_4} + A^{(0)}_{I_1I_2I_3I_4})\; s^{I_1} s^{I_2} s^{I_3} s^{I_4}.
\end{aligned}
\tag{3.2.17}
$$

The corresponding quartic couplings are listed in appendix C. The repeated indices $I_1, \ldots, I_4$ here imply the summation over all possible $[0,k,0]$ representations as well as summation over the basis of the representation space:

$$
g_I s^I \equiv \sum_{k\geqslant 2}\;\sum_{I_k=1}^{\dim([0,k,0])} g_{I_k} s_k^{I_k}. \tag{3.2.18}
$$

We also used the following notations

$$
\begin{aligned}
\alpha_1 = \frac{1}{2}(k_2+k_3-k_1), \quad \alpha_2 &= \frac{1}{2}(k_1+k_3-k_2), \quad \alpha_3 = \frac{1}{2}(k_1+k_2-k_3),\\
\Sigma &= \frac{1}{2}(k_1+k_2+k_3),\\
a_{123} = \int Y^{I_1}Y^{I_2}Y^{I_3}, \quad p_{123} &= \int \nabla^\alpha Y^{I_1}\nabla^\beta Y^{I_2} Y^{I_3}_{(\alpha\beta)}, \quad t_{123} = \int \nabla^\alpha Y^{I_1} Y^{I_2} Y^{I_3}_\alpha\,.
\end{aligned}
\tag{3.2.19}
$$

Computation of the a-, t- and p-tensors for arbitrary weights k presents one of the main difficulties and is discussed in chapter 3 of part II. Knowing them, one can easily compute the necessary couplings and proceed further to computing the on-shell value of the supergravity action. In the next part of the thesis, we present our main results, namely, the simplifications which allow us to compute any four-point function of given weights. This was used to check the recently conjectured analytic closed formula for the four-point functions in the Mellin space for many different non-trivial combinations of the weights.

Part II

Supergravity correlators

Chapter 1

Contact part

Recall that the relevant for our purposes part of the effective action is given as follows:

$$S = \frac{N^2}{8\pi^2} \int [dz](\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4). \tag{1.0.1}$$

where each of the terms was defined in the previous chapter. Here we use the Euclidean AdS metric $[dz] = \frac{d^5 z}{z_0^5}$, which leads to the change of the overall sign for the action. The Lagrangian splits into a quadratic part $\mathcal{L}_2$, a cubic part $\mathcal{L}_3$ and a quartic part $\mathcal{L}_4$. The first two contribute to the so-called *exchange part* of the correlator because they give rise to exchange interactions captured in Witten diagrams such as in fig. 2.1, while $\mathcal{L}_4$ forms the *contact part* as it leads to contact interactions (see fig. 1.1). In this and the next chapter, we will present a streamlined method of how one can directly compute the contribution to the four-point function coming from these two parts for a particular set of weights separately, starting with the latter.

1.1 Simplifying the contact term

Let us remind the expression for the quartic lagrangian (3.2.17):

$$\begin{aligned}
\mathcal{L}_4 &= \mathcal{L}_4^{(4)} + \mathcal{L}_4^{(2)} + \mathcal{L}_4^{(0)}, \\
\mathcal{L}_4^{(4)} &= (S^{(4)}_{I_1 I_2 I_3 I_4} + A^{(4)}_{I_1 I_2 I_3 I_4})\ s^{I_1} \nabla_\mu s^{I_2} \nabla_\nu^2 (s^{I_3} \nabla^\mu s^{I_4}), \\
\mathcal{L}_4^{(2)} &= (S^{(2)}_{I_1 I_2 I_3 I_4} + A^{(2)}_{I_1 I_2 I_3 I_4})\ s^{I_1} \nabla_\mu s^{I_2} s^{I_3} \nabla^\mu s^{I_4}, \\
\mathcal{L}_4^{(0)} &= (S^{(0)}_{I_1 I_2 I_3 I_4} + A^{(0)}_{I_1 I_2 I_3 I_4})\ s^{I_1} s^{I_2} s^{I_3} s^{I_4}.
\end{aligned} \tag{1.1.1}$$

Mention again that the repeated indices $I_1, \ldots, I_4$ here imply the summation over all possible $[0, k, 0]$ representations as well as a summation over the basis of the representation space:

$$g_I s^I \equiv \sum_{k \geqslant 2} \sum_{I_k = 1}^{\dim([0,k,0])} g_{I_k} s_k^{I_k}. \tag{1.1.2}$$

The various couplings S and A, see appendix C, depend on contractions of the previously discussed C-tensors (via the products of a-, t- and p-tensors) which are Clebsch-Gordan coefficients for the tensor product of SO(6) irreducible representations. Their explicit expressions depend on the chosen weights, and their computation becomes complicated quite quickly. Luckily, there is a fast and straightforward approach to find all the necessary terms. For this, we refer the reader to the chapter 3.

The explicit expressions for the quartic couplings, as can be seen from appendix C, take up approximately ten pages. This ends up in expressions with huge rational numbers. However, in all the computed cases [14–19] the authors were able to simplify the final expressions for the contact lagrangian using integration by parts and symmetries of the tensor structures involved. To compute the contribution into the correlation function coming from the contact part, according to (3.1.2), one has to evaluate the contact action on-shell and calculate the variational derivatives. In this chapter, we show how this procedure can be streamlined to get the contact part of the four-point function directly. We found an analytic expression for the contact part for arbitrary weights $k_1, ..., k_4$. Note that this expression does not contain any huge rational numbers when evaluated. However, it will not be presented in this thesis because we were not able to find an analytic expression for the full correlator. Instead, we show how one can obtain the contact part of the four-point function automatically in Mathematica. One can further use the Mellin expression for the correlator to find a significantly simplified answer in the coordinate space, see section 5.3.

The first simplification follows from a closer analysis of the four-derivative terms $\mathcal{L}_4^{(4)}$: it was shown in [24] that these terms can be made to vanish in the extremal $k_1 = k_2+k_3+k_4$ and sub-extremal $k_1 = k_2 + k_3 + k_4 - 2$ cases. Moreover, in appendix B we show that the four-derivative terms vanish for all four-point correlators of 1/2-BPS operators. This observation was of big importance to prove since it is one of the assumptions for finding the expression for four-point functions in the Mellin space. This part will be discussed in chapter 5. To show this, one has to perform integration by parts in the expression for $\mathcal{L}_4^{(4)}$, which produces contributions to lower-derivative terms $\mathcal{L}_4^{(4\to 2)}$ and $\mathcal{L}_4^{(4\to 0)}$. However, in the same manner as in [3], using the reduction formulae (3.1.19) and (3.1.22), one can show that the contribution $\mathcal{L}_4^{(4\to 0)}$ vanishes identically. Thus, the full contact lagrangian can be written as

$$\mathcal{L}_4^{(4\to 2)} + \mathcal{L}_4^{(2)} + \mathcal{L}_4^{(0)}, \tag{1.1.3}$$

where

$$\mathcal{L}_4^{(4\to 2)} = \left(S^{(4)}_{I_1 I_2 I_3 I_4} (m_1^2 + m_2^2 + m_3^2 + m_4^2 - 4) - 4 A^{(4)}_{I_1 I_2 I_3 I_4} \right) s^{I_1} \nabla_\mu s^{I_2} s^{I_3} \nabla^\mu s^{I_4}, \tag{1.1.4}$$

and m_i denotes the AdS mass of the corresponding scalar field.

Now one has to compute the contact part of the on-shell action. For this we solve the equations of motion for scalar fields perturbatively: $s_k^{I_k} = \bar{s}_k^{I_k} + \tilde{s}_k^{I_k}$, where $\bar{s}_k^{I_k}$ is the solution of the linearized equations of motion with fixed boundary conditions and corrections $\tilde{s}_k^{I_k}$ correspond to scalars with vanishing boundary conditions. The solution of the boundary problem is given in [8]:

$$\bar{s}_k^{I_k}(z) = C_k \int d^4\vec{x}\, K_k(z, \vec{x}) s_k^{I_k}(\vec{x}), \tag{1.1.5}$$

where the bulk-to-boundary propagator reads as

$$K_k(z, \vec{x}) = \left(\frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^k. \tag{1.1.6}$$

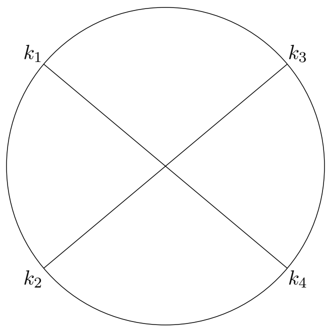

Figure 1.1: A contact Witten diagram.

For now, we will neglect the normalization factors C_k and take them into account when computing the full correlator. According to [38], they are

$$C_2 = \frac{1}{2\pi^2} \quad \text{and} \quad C_k = \frac{\Gamma(k)}{\pi^2\Gamma(k-2)} \quad \text{for } k > 2. \tag{1.1.7}$$

To proceed, we define the so-called D functions[1], or contact Witten diagrams (see fig. 1.1), as integrals over the AdS_5 space, for details see e.g. [13, 18]:

$$\begin{aligned} D_{k_1k_2k_3k_4} &\equiv D_{k_1k_2k_3k_4}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) \\ &= \int [dz]\, K_{k_1}(z, \vec{x}_1) K_{k_2}(z, \vec{x}_2) K_{k_3}(z, \vec{x}_3) K_{k_4}(z, \vec{x}_4). \end{aligned} \tag{1.1.8}$$

and use the identity

$$\begin{aligned} \nabla_\mu K_{k_1}(z, \vec{x}_1) \quad & \nabla^\mu K_{k_2}(z, \vec{x}_2) = \\ & k_1 k_2 \left(K_{k_1}(z, \vec{x}_1) K_{k_2}(z, \vec{x}_2) - 2|\vec{x}_{12}|^2 K_{k_1+1}(z, \vec{x}_1) K_{k_2+1}(z, \vec{x}_2) \right). \end{aligned} \tag{1.1.9}$$

To compute a correlator with specific weights, one needs to compute the quartic couplings corresponding to all the 24 permutations of the weights. Each of them is multiplied by the correspondingly permuted set of D functions, as follows from carefully going through the steps: to compute a correlator with fixed weights $\{k_1, k_2, k_3, k_4\}$ one restricts the infinite sum in (1.1.1) to representations which correspond to these weights. The sums over k's are not ordered; therefore there are 24 nonzero summands corresponding to the 24 permutations of the indices $\{k_1, k_2, k_3, k_4\}$. Even when some of the weights are equal, potential overcounting is compensated by functional differentiation when computing the correlator. Let us illustrate this procedure on the two-derivative term[2] $g_{1234}\, s^1 s^2 \nabla_\mu s^3 \nabla^\mu s^4$, where g_{1234} schematically denotes the corresponding coupling:

$$\int [dz]\, g_{1234}\, s^1 s^2 \nabla_\mu s^3 \nabla^\mu s^4 = \tag{1.1.10}$$
$$\begin{aligned} &\int [dz] \left(g_{1234}\, s^1_{k_1} s^2_{k_2} \nabla_\mu s^3_{k_3} \nabla^\mu s^4_{k_4} + g_{1234}\, s^1_{k_3} s^2_{k_2} \nabla_\mu s^3_{k_1} \nabla^\mu s^4_{k_4} + \ldots \right) = \int d^4\vec{y}_1 d^4\vec{y}_2 d^4\vec{y}_3 d^4\vec{y}_4 \\ &\quad \times \left[g_{1234}\, k_3 k_4 \left(D_{k_1k_2k_3k_4} - 2|\vec{y}_{34}|^2 D_{k_1k_2k_3+1k_4+1} \right) s^1_{k_1}(\vec{y}_1) s^2_{k_2}(\vec{y}_2) s^3_{k_3}(\vec{y}_3) s^4_{k_4}(\vec{y}_4) \right. \\ &\quad \left. + g_{1234}\, k_1 k_4 \left(D_{k_3k_2k_1k_4} - 2|\vec{y}_{14}|^2 D_{k_3k_2k_1+1k_4+1} \right) s^1_{k_3}(\vec{y}_1) s^2_{k_2}(\vec{y}_2) s^3_{k_1}(\vec{y}_3) s^4_{k_4}(\vec{y}_4) + \ldots \right]. \end{aligned}$$

[1] Note, that the set of indices $\{k_1, k_2, k_3, k_4\}$ in the notation for the D-functions is always ordered in such a way that it corresponds to an ordered set of variables $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$

[2] Here for simplicity we write the summation indices $I_1, \ldots, I_4$ as $1, \ldots, 4$. The range of these summation indices depends on the weight k_i.

Here the dots stand for the other 22 terms and we consequently used equations (1.1.5), (1.1.9) and (1.1.8). Now, one differentiates over the boundary values of the fields $s^1_{k_1}(\vec{x}_1), \ldots, s^4_{k_4}(\vec{x}_4)$ and gets

$$g_{1234}\, k_3 k_4 \left(D_{k_1 k_2 k_3 k_4} - 2|\vec{x}_{34}|^2 D_{k_1 k_2 k_3+1 k_4+1}\right)$$
$$+g_{3214}\, k_1 k_4 \left(D_{k_1 k_2 k_3 k_4} - 2|\vec{x}_{14}|^2 D_{k_1+1 k_2 k_3 k_4+1}\right) + \ldots \qquad (1.1.11)$$

Thus, in practice, one can avoid writing down the relevant part of the contact lagrangian altogether and write its contribution to the correlator directly, by simply finding the couplings corresponding to the 24 permutations of the weights and multiplying them by a correspondingly permuted set of D functions as in (1.1.11). This will give the contribution to the four-point function from the contact terms up to a normalization factor, which will be taken into account after computing the exchange part in the next chapter.

Chapter 2

Exchange part

2.1 Simplifying the exchange term

The standard method of computing the exchange part, see, e.g., [14], is again to evaluate the quadratic and cubic terms on-shell and differentiate over the boundary values of the fields. For this one solves the equations of motion perturbatively and substitutes the solution into the full exchange lagrangian. Writing down the latter becomes very unpleasant for higher-weight cases because of the growing number of descendants coupled to the scalars in the cubic terms. However, as for the contact part, the direct procedure can be streamlined, and the contribution to the *non-normalized* four-point correlation function coming from the quadratic and cubic terms can be written as the following sum of the s, t and u channels:

$$\begin{aligned}
\langle k_1k_2k_3k_4\rangle_{\text{Exchange}} &= \Big\{\sum \frac{36}{\zeta(s_k)} S^{I_1I_2I_k}S^{I_3I_4I_k}\mathbf{S}^{k}_{k_1k_2k_3k_4} + \sum \frac{4}{\zeta(t_k)} T^{I_1I_2I_k}T^{I_3I_4I_k}\mathbf{S}^{k+4}_{k_1k_2k_3k_4} \\
&+ \sum \frac{4}{\zeta(\phi_k)} \Phi^{I_1I_2I_k}\Phi^{I_3I_4I_k}\mathbf{S}^{k+2}_{k_1k_2k_3k_4} + \sum \frac{1}{\zeta(A_{\mu,k})} A^{I_1I_2I_k}A^{I_3I_4I_k}\mathbf{V}^{k}_{k_1k_2k_3k_4} \\
&+ \sum \frac{1}{\zeta(C_{\mu,k})} C^{I_1I_2I_k}C^{I_3I_4I_k}\mathbf{V}^{k+2}_{k_1k_2k_3k_4} + \sum \frac{4}{\zeta(\varphi_{\mu\nu,k})} G^{I_1I_2I_k}G^{I_3I_4I_k}\mathbf{T}^{k}_{k_1k_2k_3k_4}\Big\}_{\mathrm{s}} \\
&+ \Big\{\sum \frac{36}{\zeta(s_k)} S^{I_1I_3I_k}S^{I_2I_4I_k}\mathbf{S}^{k}_{k_1k_3k_2k_4} + \sum \frac{4}{\zeta(t_k)} T^{I_1I_3I_k}T^{I_2I_4I_k}\mathbf{S}^{k+4}_{k_1k_3k_2k_4} \\
&+ \sum \frac{4}{\zeta(\phi_k)} \Phi^{I_1I_3I_k}\Phi^{I_2I_4I_k}\mathbf{S}^{k+2}_{k_1k_3k_2k_4} + \sum \frac{1}{\zeta(A_{\mu,k})} A^{I_1I_3I_k}A^{I_2I_4I_k}\mathbf{V}^{k}_{k_1k_3k_2k_4} \\
&+ \sum \frac{1}{\zeta(C_{\mu,k})} C^{I_1I_3I_k}C^{I_2I_4I_k}\mathbf{V}^{k+2}_{k_1k_3k_2k_4} + \sum \frac{4}{\zeta(\varphi_{\mu\nu,k})} G^{I_1I_3I_k}G^{I_2I_4I_k}\mathbf{T}^{k}_{k_1k_3k_2k_4}\Big\}_{\mathrm{t}} \\
&+ \Big\{\sum \frac{36}{\zeta(s_k)} S^{I_1I_4I_k}S^{I_2I_3I_k}\mathbf{S}^{k}_{k_1k_4k_2k_3} + \sum \frac{4}{\zeta(t_k)} T^{I_1I_4I_k}T^{I_2I_3I_k}\mathbf{S}^{k+4}_{k_1k_4k_2k_3} \\
&+ \sum \frac{4}{\zeta(\phi_k)} \Phi^{I_1I_4I_k}\Phi^{I_2I_3I_k}\mathbf{S}^{k+2}_{k_1k_4k_2k_3} + \sum \frac{1}{\zeta(A_{\mu,k})} A^{I_1I_4I_k}A^{I_2I_3I_k}\mathbf{V}^{k}_{k_1k_4k_2k_3} \\
&+ \sum \frac{1}{\zeta(C_{\mu,k})} C^{I_1I_4I_k}C^{I_2I_3I_k}\mathbf{V}^{k+2}_{k_1k_4k_2k_3} + \sum \frac{4}{\zeta(\varphi_{\mu\nu,k})} G^{I_1I_4I_k}G^{I_2I_3I_k}\mathbf{T}^{k}_{k_1k_4k_2k_3}\Big\}_{\mathrm{u}} .
\end{aligned} \tag{2.1.1}$$

Here the exchange Witten diagrams, $\mathbf{S}$, $\mathbf{V}$ and $\mathbf{T}$, are multiplied by the corresponding combination of quadratic, ζ, and cubic, $\{S, T, \Phi, A, C, G\}$, couplings that were derived in [10–12] and were discussed in section 3.2 of part I . The appearing sums run over the

Field	s_k	$A_{\mu,k}$	$C_{\mu,k}$	ϕ_k	t_k	$\varphi_{\mu\nu,k}$
Irrep	$[0,k,0]$	$[1,k-2,1]$	$[1,k-4,1]$	$[2,k-4,2]$	$[0,k-4,0]$	$[0,k-2,0]$
m^2	$k(k-4)$	$k(k-2)$	$k(k+2)$	k^2-4	$k(k+4)$	k^2-4
Δ	k	$k+1$	$k+3$	$k+2$	$k+4$	$k+2$

Table 2.1: KK-modes contributing to the exchange Witten diagrams.

possible set of exchange fields, and the summation convention is the following:

$$\sum \equiv \sum_{k\in\{\text{Exch. fields}\}} \sum_{I_k=1}^{\dim([0,k,0])} .$$

Note that the permutation of the weights $\{k_1,k_2,k_3,k_4\}$ takes place together with the permutation of the coordinates, e.g. $\mathbf{S}^k_{k_1k_3k_2k_4} \equiv \mathbf{S}^k_{k_1k_3k_2k_4}(\vec{x}_1,\vec{x}_3,\vec{x}_2,\vec{x}_4)$. The exchange fields which can show up in an exchange diagram are restricted by the SU(4) selection rule. Namely, these are the fields which appear in the intersection of non-vanishing cubic vertices, see table 2.1. This is determined by the following tensor product decomposition of representations:

$$[0,k_1,0]\otimes[0,k_2,0] = \sum_{r=0}^{\min(k_1,k_2)} \sum_{s=0}^{\min(k_1,k_2)-r} [r,|k_1-k_2|+2s,r]. \tag{2.1.2}$$

An exchange field must occur on the right-hand side of this decomposition for both the ingoing and the outgoing fields.

The exchange Witten diagrams have representation in terms of *exchange integrals*, which are defined as follows:

$$\begin{aligned}
\mathbf{S}^k_{k_1k_2k_3k_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) &= \int [dz][dw]\, K_{k_1}(z,\vec{x}_1)K_{k_2}(z,\vec{x}_2)G^k(z,w)K_{k_3}(w,\vec{x}_3)K_{k_4}(w,\vec{x}_4),\\
\mathbf{V}^k_{k_1k_2k_3k_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) &= \\
&= \int [dz][dw]\, K_{k_1}(z,\vec{x}_1)\overleftrightarrow{\nabla}^{\mu} K_{k_2}(z,\vec{x}_2)G^k_{\mu\nu}(z,w)K_{k_3}(w,\vec{x}_3)\overleftrightarrow{\nabla}^{\nu} K_{k_4}(w,\vec{x}_4),\\
\mathbf{T}^k_{k_1k_2k_3k_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) &= \int [dz][dw]\, T^{\mu\nu}(z,\vec{x}_1,\vec{x}_2)G^k_{\mu\nu\,\mu'\nu'}(z,w)T^{\mu'\nu'}(w,\vec{x}_3,\vec{x}_4),
\end{aligned} \tag{2.1.3}$$

where G^k, $G^k_{\mu\nu}$ and $G^k_{\mu\nu\,\mu'\nu'}$ are the scalar, vector, and the tensor bulk-to-bulk propagators correspondingly. A simple method to compute the exchange integrals was introduced in [39], and further generalizations appeared in [15] and [17]. As it turns out, they can always be represented as a finite sum of D functions.

Further, we derive eq. (2.1.1) and in the next section, discuss how to compute the exchange integrals.

2.1.1 Scalar exchange

Here we restrict ourselves to exchange by a scalar field s_k with the mass m_k in the s channel (see the Witten diagram in fig. 2.1). The contribution from the other two channels is obtained by the corresponding permutation. It can be easily generalized further to an exchange by the scalar fields t and ϕ.

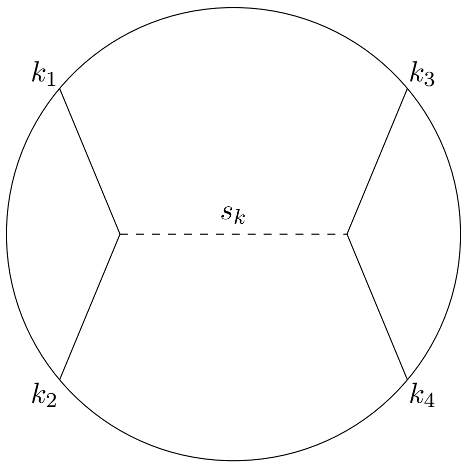

Figure 2.1: The exchange Witten diagram where a scalar of weight k is exchanged in the s channel.

We start with the corresponding quadratic and cubic terms[1], which can be found in [12]:

$$
\begin{aligned}
\mathcal{L}_2^{(s)} &= \sum_k \zeta(s_k)\left(-\frac{1}{2}\nabla_\mu s_k \nabla^\mu s_k - \frac{1}{2} m_k^2 s_k^2\right), \\
\mathcal{L}_3^{(s)} &= S_{I_1 I_2 I_3} s^{I_1} s^{I_2} s^{I_3}.
\end{aligned} \tag{2.1.4}
$$

From here we obtain the equations of motion:

$$
\zeta(s_k)(\nabla_\mu^2 - m_k^2)\, s_k = -\frac{\partial \mathcal{L}_3^{(s)}}{\partial s_k}. \tag{2.1.5}
$$

We solve them perturbatively: $s_k = \bar{s}_k + \tilde{s}_k$, where

$$
\begin{aligned}
(\nabla_\mu^2 - m_k^2)\, \bar{s}_k &= 0, \\
(\nabla_\mu^2 - m_k^2)\, \tilde{s}_k &= -\frac{1}{\zeta(s_k)} \frac{\partial \mathcal{L}_3^{(s)}}{\partial s_k}\bigg|_{s\to\bar{s}},
\end{aligned} \tag{2.1.6}
$$

with fixed boundary conditions for $\bar{s}$ and zero boundary conditions for $\tilde{s}$, and introduce the bulk-to-bulk propagator G^k as the solution of

$$
(\nabla_\mu^2 - m_k^2)\, G^k(z,w) = -\delta(z,w). \tag{2.1.7}
$$

We find

$$
\tilde{s}_k(z) = \frac{1}{\zeta(s_k)} \int [dw] \left(\frac{\partial \mathcal{L}_3^{(s)}(w)}{\partial s_k}\bigg|_{s\to\bar{s}} \right) G^k(z,w). \tag{2.1.8}
$$

Next, we use the equations of motion to get the contribution to the four-point function coming from the quadratic term:

$$
\mathcal{L}_2^{(s_k)} = -\frac{1}{2}\left(\frac{\partial \mathcal{L}_3^{(s)}}{\partial s_k}\bigg|_{s\to\bar{s}} \right) \tilde{s}_k. \tag{2.1.9}
$$

[1] In order not to obstruct the reading, we omit the representation index I_k and write $s_k \equiv s_k^{I_k}$. However, in the expressions similar to that one of $\mathcal{L}_3$, the usual summation convention as for (3.2.18) is assumed.

The cubic term contributing to the four-point correlator must be of the form $\bar{s}\bar{s}\tilde{s}$ and can be computed as follows:

$$\mathcal{L}_3^{(s_k)} = \left(\frac{\partial \mathcal{L}_3^{(s)}}{\partial s_k} \bigg|_{s\to\bar{s}} \right) \tilde{s}_k. \tag{2.1.10}$$

In this way, we obtain the exchange part of the action:

$$S_{\text{Exchange}}(s_k) = \frac{1}{2\zeta(s_k)} \int [dz][dw] \left(\frac{\partial \mathcal{L}_3^{(s)}(z)}{\partial s_k} \bigg|_{s\to\bar{s}} \right) G^k(z,w) \left(\frac{\partial \mathcal{L}_3^{(s)}(w)}{\partial s_k} \bigg|_{s\to\bar{s}} \right). \tag{2.1.11}$$

Writing down the cubic terms of interest explicitly:

$$\mathcal{L}_3^{(s_k)} = 6S_{I_1I_2I_k} s_{k_1}^{I_1} s_{k_2}^{I_2} s_k^{I_k} + 6S_{I_3I_4I_k} s_{k_3}^{I_3} s_{k_4}^{I_4} s_k^{I_k}, \tag{2.1.12}$$

we find

$$\begin{aligned} S_{\text{Exchange}}(s_k) \;=\; & \sum_k \frac{1}{2\zeta(s_k)} \int [dz][dw] \left[6S_{I_1I_2I_k} \bar{s}_{k_1}^{I_1} \bar{s}_{k_2}^{I_2} + 6S_{I_3I_4I_k} \bar{s}_{k_3}^{I_3} \bar{s}_{k_4}^{I_4} \right](z) \cdot G^k(z,w) \\ & \cdot \left[6S_{I_1I_2I_k} \bar{s}_{k_1}^{I_1} \bar{s}_{k_2}^{I_2} + 6S_{I_3I_4I_k} \bar{s}_{k_3}^{I_3} \bar{s}_{k_4}^{I_4} \right](w). \end{aligned} \tag{2.1.13}$$

Finally, using the solution (1.1.5) of the boundary problem and computing variational derivatives, we obtain the first term in (2.1.1).

2.1.2 Vector exchange

Let us now consider the exchange of the vector field $A_{\mu,k}$ with mass $m_A \equiv m_A(k)$. Knowing the quadratic and cubic lagrangian:

$$\begin{aligned} \mathcal{L}_2^{(A_\mu)} &= \sum_k \zeta(A_{\mu,k}) \left(-\frac{1}{4} F_{\mu\nu,k}^2 - \frac{1}{2} m_A^2 A_{\mu,k}^2 \right), \\ \mathcal{L}_3^{(A_\mu)} &= A_{I_1I_2I_3} s^{I_1} \nabla^\mu s^{I_2} A_\mu^{I_3}, \end{aligned} \tag{2.1.14}$$

we obtain the equations of motion for the vectors $A_{\mu,k}$:

$$\zeta(A_{\mu,k}) \left(\nabla^\nu F_{\nu\mu,k} - m_A^2 A_{\mu,k} \right) = -\frac{\partial \mathcal{L}_3^{(A_\mu)}}{\partial A^{\mu,k}}. \tag{2.1.15}$$

We again write the solution as $A_\mu = \bar{A}_\mu + \tilde{A}_\mu$, where $\bar{A}_\mu$ correspond to the linearized equations with fixed boundary conditions and $\tilde{A}_\mu$ are the corrections with vanishing boundary conditions:

$$\begin{aligned} \nabla^\nu \bar{F}_{\nu\mu,k} - m_A^2 \bar{A}_{\mu,k} &= 0, \\ \nabla^\nu \tilde{F}_{\nu\mu,k} - m_A^2 \tilde{A}_{\mu,k} &= -\frac{1}{\zeta(A_{\mu,k})} \frac{\partial \mathcal{L}_3^{(A_\mu)}}{\partial A^{\mu,k}} \bigg|_{s\to\bar{s}}. \end{aligned} \tag{2.1.16}$$

Introducing the vector bulk-to-bulk propagator $G^k_{\mu\nu}$, satisfying the equation

$$\nabla^\rho \left(\nabla_\rho G^k_{\mu\nu} - \nabla_\mu G^k_{\nu\rho}\right) - m_A^2 G^k_{\mu\nu} = -g_{\mu\nu}\delta(z,w)\,, \tag{2.1.17}$$

we find

$$\tilde{A}_{\mu,k} = \frac{1}{\zeta(A_{\mu,k})}\int [dw] \left(\left.\frac{\partial \mathcal{L}_3^{(A_\mu)}}{\partial A^{\nu,k}}\right|_{s\to\bar{s}}\right) G_\mu^{\ \nu,k}(z,w). \tag{2.1.18}$$

With the help of the equations of motion we get the contribution to the four-point function coming from the quadratic term:

$$\mathcal{L}_2^{(A_{\mu,k})} = -\frac{1}{2}\left(\left.\frac{\partial \mathcal{L}_3^{(A_\mu)}}{\partial A_{\mu,k}}\right|_{s\to\bar{s}}\right)\tilde{A}_{\mu,k}. \tag{2.1.19}$$

The cubic term contributing to the four-point correlator must be of the form $\bar{s}\nabla^\mu\bar{s}\tilde{A}_\mu$ and can be computed as follows:

$$\mathcal{L}_3^{(A_{\mu,k})} = \left.\frac{\partial \mathcal{L}_3^{(A_\mu)}}{\partial A_{\mu,k}}\right|_{s\to\bar{s}}\tilde{A}_{\mu,k}. \tag{2.1.20}$$

Summing them up and substituting $\tilde{A}$, one gets the contribution to the exchange part of the action:

$$S_{\text{Exchange}}(A_{\mu,k}) = \frac{1}{2\zeta(A_{\mu,k})}\int [dz][dw] \left(\left.\frac{\partial \mathcal{L}_3^{(A_\mu)}}{\partial A_{\mu,k}}\right|_{s\to\bar{s}}\right) G^k_{\mu\nu}(z,w) \left(\left.\frac{\partial \mathcal{L}_3^{(A_\mu)}}{\partial A_{\nu,k}}\right|_{s\to\bar{s}}\right). \tag{2.1.21}$$

Writing down the cubic terms of interest, containing the exchange vector $A_{\mu,k}$, explicitly

$$\mathcal{L}_3^{(A_{\mu,k})} = A_{I_1I_2I_k} s^{I_1}_{k_1} \overleftrightarrow{\nabla}^\mu s^{I_2}_{k_2} A^{I_k}_{\mu,k} + A_{I_3I_4I_k} s^{I_3}_{k_3} \overleftrightarrow{\nabla}^\mu s^{I_4}_{k_4} A^{I_k}_{\mu,k}, \tag{2.1.22}$$

we find

$$\begin{aligned} S_{\text{Exchange}}(A_{\mu,k}) \;=\; & \sum_k \frac{1}{2\zeta(s_k)}\int [dz][dw] \left[A_{I_1I_2I_k}\bar{s}^{I_1}_{k_1}\overleftrightarrow{\nabla}^\mu \bar{s}^{I_2}_{k_2} + A_{I_3I_4I_k}\bar{s}^{I_3}_{k_3}\overleftrightarrow{\nabla}^\mu \bar{s}^{I_4}_{k_4}\right](z)\,\times \\ & G^k_{\mu\nu}(z,w)\left[A_{I_1I_2I_k}\bar{s}^{I_1}_{k_1}\overleftrightarrow{\nabla}^\nu \bar{s}^{I_2}_{k_2} + A_{I_3I_4I_k}\bar{s}^{I_3}_{k_3}\overleftrightarrow{\nabla}^\nu \bar{s}^{I_4}_{k_4}\right](w). \end{aligned} \tag{2.1.23}$$

Finally, substituting the solution (1.1.5) of the boundary problem and computing variational derivatives, we arrive to the corresponding term in (2.1.1).

2.1.3 Tensor exchange

Finally, consider the exchange of the tensor field $\varphi_{\mu\nu,k}$ with mass squared f_k. Again we start from the quadratic and cubic lagrangians:

$$\begin{aligned} \mathcal{L}_2^{(\varphi_{\mu\nu})} \;=\; & \sum_k \Big(-\frac{1}{4}\nabla_\rho\varphi_{\mu\nu,k}\nabla^\rho\varphi^{\mu\nu}_k + \frac{1}{2}\nabla_\rho\varphi^{\rho\mu}_k\nabla^\lambda\varphi_{\lambda\mu,k} - \frac{1}{2}\nabla_\mu\varphi^\rho_{\rho,k}\nabla_\lambda\varphi^{\lambda,\mu}_k \\ & + \frac{1}{4}\nabla_\rho\varphi^\mu_{\mu,k}\nabla^\rho\varphi^\nu_{\nu,k} + \frac{1}{4}(2-f_k)\varphi_{\mu\nu,k}\varphi^{\mu\nu}_k + \frac{1}{4}(2+f_k)(\varphi^\rho_{\rho,k})^2\Big)\,, \\ \mathcal{L}_3^{(\varphi_{\mu\nu})} \;=\; & G_{I_1I_2I_3}T^{I_1I_2}_{\mu\nu}\varphi^{\mu\nu,I_3}\,, \end{aligned} \tag{2.1.24}$$

where (...) denotes symmetrization and the stress-energy tensor $T_{\mu\nu}$ has the form

$$T_{\mu\nu} = \nabla_{(\mu}s^{I_1}\nabla_{\nu)}s^{I_2} - \frac{1}{2}g_{\mu\nu}\left(\nabla^\rho s^{I_1}\nabla_\rho s^{I_2} + \frac{1}{2}(m_1^2 + m_2^2 - f_k)s^{I_1}s^{I_2}\right). \qquad (2.1.25)$$

From this we obtain the equations of motion

$$\begin{aligned} W_{\mu\nu}^{\ \ \rho\lambda}\varphi_{\rho\lambda,k} &\equiv -\nabla_\rho\nabla^\rho\varphi_{\mu\nu,k} + \nabla_\mu\nabla^\rho\varphi_{\rho\nu,k} + \nabla_\nu\nabla^\rho\varphi_{\rho\mu,k} - \nabla_\mu\nabla_\nu\varphi^\rho_{\rho,k} \\ &\qquad + \Big((f_k - 2)\varphi_{\mu\nu,k} + \frac{1}{3}(6 + f_k)g_{\mu\nu}\varphi^\lambda_{\lambda,k}\Big) \\ &= 2\frac{\partial\mathcal{L}_3^{(\varphi_{\mu\nu})}}{\partial\varphi_k^{\mu\nu}} - \frac{2}{3}g_{\mu\nu}\left(\frac{\partial\mathcal{L}_3^{(\varphi_{\mu\nu})}}{\partial\varphi_k^{\rho\lambda}}g^{\rho\lambda}\right). \end{aligned} \qquad (2.1.26)$$

We again represent the solution in the form $\varphi_{\mu\nu} = \bar{\varphi}_{\mu\nu} + \tilde{\varphi}_{\mu\nu}$ and introduce the bulk-to-bulk propagator $G^k_{\mu\nu\,\rho\lambda}$ as follows:

$$W_{\mu\nu}^{\ \ \rho\lambda}G^k_{\rho\lambda\,\mu'\nu'} = \left(g_{\mu\mu'}g_{\nu\nu'} + g_{\mu\nu'}g_{\nu\mu'} - \frac{2}{3}g_{\mu\nu}g_{\mu'\nu'}\right)\delta(z,w). \qquad (2.1.27)$$

This let us write the correction $\tilde{\varphi}_{\mu\nu}$ in the form

$$\tilde{\varphi}_{\mu\nu,k} = \int[dw]\,G_{\mu\nu}^{\ \ \mu'\nu',k}(z,w)\left(\left.\frac{\partial\mathcal{L}_3^{(\varphi_{\mu\nu})}}{\partial\varphi_k^{\mu'\nu'}}\right|_{s\to\bar{s}}\right). \qquad (2.1.28)$$

After integration by parts $\mathcal{L}_2^{(\varphi_{\mu\nu,k})}$ takes the form:

$$\mathcal{L}_2^{(\varphi_{\mu\nu,k})} = -\frac{1}{4}\left[W_{\mu\nu}^{\ \ \rho\lambda}\varphi_{\rho\lambda,k} + \frac{2}{3}g_{\mu\nu}\left(\frac{\partial\mathcal{L}_3}{\partial\varphi^{\rho\lambda,k}}g^{\rho\lambda}\right)\right]. \qquad (2.1.29)$$

Using the equations of motion, one finds the contribution of the quadratic lagrangian to the four-point function:

$$\mathcal{L}_2^{(\varphi_{\mu\nu,k})} = -\frac{1}{2}\left(\left.\frac{\partial\mathcal{L}_3^{(\varphi_{\mu\nu})}}{\partial\varphi_{\mu\nu,k}}\right|_{s\to\bar{s}}\right)\tilde{\varphi}_{\mu\nu,k}. \qquad (2.1.30)$$

The contribution of the cubic term must be of the form $\bar{T}_{\mu\nu}\tilde{\varphi}^{\mu\nu}$, where $\bar{T}_{\mu\nu} = T_{\mu\nu}|_{s\to\bar{s}}$, and can be computed as follows:

$$\mathcal{L}_3^{(\varphi_{\mu\nu,k})} = \left(\left.\frac{\partial\mathcal{L}_3^{(\varphi_{\mu\nu})}}{\partial\varphi_{\mu\nu,k}}\right|_{s\to\bar{s}}\right)\tilde{\varphi}_{\mu\nu,k}. \qquad (2.1.31)$$

Thus, the corresponding part of the action is given by

$$S_{\mathrm{Exchange}}(\varphi_{\mu\nu,k}) = \frac{1}{2}\int[dz][dw]\left(\left.\frac{\partial\mathcal{L}_3^{(\varphi_{\mu\nu})}}{\partial\varphi_k^{\mu\nu}}\right|_{s\to\bar{s}}\right)G^{\mu\nu\,\mu'\nu',k}(z,w)\left(\left.\frac{\partial\mathcal{L}_3^{(\varphi_{\mu\nu})}}{\partial\varphi_k^{\mu'\nu'}}\right|_{s\to\bar{s}}\right) \qquad (2.1.32)$$

Writing down the cubic terms of interest, containing the exchange tensor $\varphi_{\mu\nu,k}$, explicitly

$$\mathcal{L}_3^{(\varphi_{\mu\nu,k})} = G_{I_1I_2I_k}T^{I_1I_2}_{\mu\nu}\varphi^{\mu\nu,I_k} + G_{I_3I_4I_k}T^{I_3I_4}_{\mu\nu}\varphi^{\mu\nu,I_k}, \qquad (2.1.33)$$

we obtain

$$S_{\text{Exchange}}(\varphi_{\mu\nu,k}) = \int [dz][dw] \left[G_{I_1 I_2 I_k} \bar{T}^{I_1 I_2}_{\mu\nu} + G_{I_3 I_4 I_k} \bar{T}^{I_3 I_4}_{\mu\nu}\right](z) \times G^{k}_{\mu\nu\,\mu'\nu'}(z,w) \left[G_{I_1 I_2 I_k} \bar{T}^{I_1 I_2}_{\mu'\nu'} + G_{I_3 I_4 I_k} \bar{T}^{I_3 I_4}_{\mu'\nu'}\right](w). \tag{2.1.34}$$

Finally, using the solution (1.1.5) of the boundary problem and computing variational derivatives, we arrive to the corresponding term in (2.1.1).

2.2 Computing the exchange integrals

The method of computing the exchange integrals (2.1.3) was developed in [39], and further generalizations appeared in [15] and [17]. If one concentrates only on the z-integrals, then the idea is to use the conformal symmetry to bring the integral into a simple form. Next, based on the conformal invariance, one can propose an ansatz for the z-integral. The action of the wave operator reduces the problem of the direct computation of the z-integrals to solving a system of differential equations. The latter presents no difficulty if one writes a solution in a polynomial form. In some cases, it is possible to write down the closed analytic formula for the exchange integrals, but in general, the computation can be easily automatized, e.g., using Mathematica. Let us illustrate this method in more detail for each case.

2.2.1 Scalar exchange

Here we consider the propagation of the scalar field of weight Δ and mass m. The integrals of interest are given by:

$$\mathbf{S}^{\Delta}_{\Delta_1\Delta_2\Delta_3\Delta_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \int [dz][dw]\, K_{\Delta_1}(z,\vec{x}_1) K_{\Delta_2}(z,\vec{x}_2) G^{\Delta}(z,w) K_{\Delta_3}(w,\vec{x}_3) K_{\Delta_4}(w,\vec{x}_4). \tag{2.2.1}$$

Let us concentrate on computing the z-integral first:

$$A(w,\vec{x_1},\vec{x_2}) = \int \frac{d^{d+1}z}{z_0^{d+1}} G^{\Delta}(u) K_{\Delta_1}(z,\vec{x}_1) K_{\Delta_2}(z,\vec{x}_2)\,. \tag{2.2.2}$$

Firstly, to simplify the integral (2.2.2), one performs the translation $\vec{x}_1 \to 0$, $\vec{x}_2 \to \vec{x}_{21} \equiv \vec{x}_2 - \vec{x}_1$ and the conformal inversion

$$\vec{x}_{12} = \frac{\vec{x}'_{12}}{|\vec{x}'_{12}|^2} \qquad z_\mu = \frac{z'_\mu}{(z')^2} \qquad w_\mu = \frac{w'_\mu}{(w')^2}\,. \tag{2.2.3}$$

The integral thus takes the form

$$A(w,\vec{x}_1,\vec{x}_2) = |\vec{x}_{12}|^{-2\Delta_2}\, I(w' - \vec{x}'_{12}) \tag{2.2.4}$$

where

$$I(w) = \int \frac{d^{d+1}z}{z_0^{d+1}}\, G^{\Delta}(u)\,(z_0)^{\Delta_1} \left(\frac{z_0}{z^2}\right)^{\Delta_2}\,. \tag{2.2.5}$$

Further, note that invariance of $I(w)$ under the scale transformation $w_\mu \to \lambda w_\mu$ and under the d–dimensional Poincare subgroup of $SO(d+1,1)$ implies that $I(w)$ must be of the form

$$I(w) = (w_0)^{\Delta_{12}} X(t) \tag{2.2.6}$$

with t given by

$$t = \frac{w_0^2}{w^2} = \frac{w_0^2}{w_0^2 + |\vec{w}|^2} \tag{2.2.7}$$

and $\Delta_{12} \equiv \Delta_1 - \Delta_2$. Applying the wave operator $(-\Box + m^2)$ to $I(w)$ and use of the equation of motion for the propagator G^Δ leads to the inhomogeneous second order differential equation for the function $X(t)$

$$4(t-1)t^2 X'' + 4t\left[-\Delta_{12} + (\Delta_{12}+1)\,t + 1\right] X' + \left((4-\Delta_{12})\,\Delta_{12} + m^2\right) X = t^{\Delta_2}\,. \tag{2.2.8}$$

It can be reduced further to a system of linear equations for the coefficients a_k in the polynomial expansion

$$X(t) = \sum_k a_k t^k\,, \tag{2.2.9}$$

which can be easily solved by equating the coefficients at equal powers of t. It appears that the series (2.2.9) truncates, and after inverting back to the original coordinates, one finally gets

$$\mathbf{S}^{\Delta}_{\Delta_1\Delta_2\Delta_3\Delta_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \sum_{k_{\min}}^{k_{\max}} a_k\,|\vec{x}_{12}|^{-2\Delta_2+2k} D_{\Delta_1-\Delta_2+k,k,\Delta_3,\Delta_4}\,, \tag{2.2.10}$$

with $k_{\min} = (\Delta - \Delta_{12})/2$ and $k_{\max} = \Delta_2 - 1$.

2.2.2 Vector exchange

The most general Witten diagram, describing the exchange of the vector field from the multiplet $(s_\Delta, ...)$ with mass M, which is coupled to scalar fields s_{Δ_i} of different masses m_{Δ_i}, has the following integral representation:

$$\begin{aligned}\mathbf{V}^{\Delta}_{\Delta_1\Delta_2\Delta_3\Delta_4}&(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \\ &= \int [dz][dw]\, K_{\Delta_1}(z,\vec{x}_1)\overleftrightarrow{\nabla}^{\mu} K_{\Delta_2}(z,\vec{x}_2) G^{\Delta}_{\mu\nu}(z,w) K_{\Delta_3}(w,\vec{x}_3)\overleftrightarrow{\nabla}^{\nu} K_{\Delta_4}(w,\vec{x}_4).\end{aligned} \tag{2.2.11}$$

To simplify the z-integrals

$$A_\mu(w,\vec{x}_1,\vec{x}_2) = \int \frac{d^{d+1}z}{z_0^{d+1}} G^{\Delta}_{\mu\nu}(z,w)\, K_{\Delta_1}(z,\vec{x}_1)\overleftrightarrow{\nabla}^{\nu} K_{\Delta_2}(z,\vec{x}_2),$$

we again use the invariance under translation $A_{\mu\nu}(w,\vec{x}_1,\vec{x}_2) = A_{\mu\nu}(w-\vec{x}_1, 0, \vec{x}_{12})$, where $\vec{x}_{12} \equiv \vec{x}_2 - \vec{x}_1$, and the conformal inversion $w'_\mu = w_\mu/w^2$, $x'_\mu = x_\mu/x^2$. This leads to the following form of the z-integral

$$A_\mu(w,\vec{x}_1,\vec{x}_2) = |\vec{x}_{12}|^{-2\Delta_2} \frac{1}{w^2} J_{\mu\nu}(w) I_\mu(w' - \vec{x}'_{12}), \tag{2.2.12}$$

where $J_{\mu\nu}(w) = \delta_{\mu\nu} - \frac{2}{w^2} w_\mu w_\nu$, and

$$I_\mu(w) = \int \frac{d^{d+1}z}{z_0^{d+1}} G^{\Delta}_{\mu\nu}(w,z) z_0^{\Delta_1} \overleftrightarrow{\nabla}^{\nu} \left(\frac{z_0}{z^2}\right)^{\Delta_2} . \tag{2.2.13}$$

We write the following ansatz, with $\Delta_{12} = \Delta_1 - \Delta_2$ and $t = w_0^2/w^2$:

$$I_\mu(w) = w_0^{\Delta_{12}} \frac{w_\mu}{w^2} X(t) + w_0^{\Delta_{12}} \frac{\delta_{\mu 0}}{w_0} Y(t) . \tag{2.2.14}$$

The next step is to apply the wave operator $-\nabla^\rho \nabla_{[\rho} I_{\mu]} + M^2 I_\mu$ to both expressions for $I_\mu(w)$ and use the equations of motion for the propagator $G^{\Delta}_{\mu\nu}$. This will lead to the coupled system of inhomogeneous differential equations for the functions $X(t)$ and $Y(t)$. We omit the tedious calculations and present the result:

$$\begin{aligned}
&2\Delta_{12} t^2 \left(X' + Y'\right) + t\Delta_{12}^2 X + M^2 Y = -\Delta_{12} t^{\Delta_2}, \\
&4(t-1)t^2 \left(X'' + Y''\right) + 2t \left(\left(\Delta_{12} + 4\right) t - 2\Delta_{12}\right) X' \\
&\quad +2t \left(4t - \Delta_{12}\right) Y' + \big(M^2 + \Delta_{12} \left(-\Delta_{12} + 2t + 2\right)\big) X = -2\Delta_2 t^{\Delta_2} .
\end{aligned} \tag{2.2.15}$$

It can be easily solved if we assume power series expansions

$$X(t) = \sum_k a_k t^k, \qquad Y(t) = \sum_k b_k t^k, \tag{2.2.16}$$

with $k_{min} \leqslant k \leqslant k_{max}$. Substituting this into the system (2.2.15) and equating the coefficients at equal powers of t, one finds the coefficients a_k and b_k. It appears, that the series terminates at

$$k_{min} = \frac{3 - 2\Delta_{12}}{4} + \frac{1}{4}\sqrt{9 + 4M^2}, \quad k_{max} = \Delta_2 - 1. \tag{2.2.17}$$

To recover the vector z-integral in terms of the original coordinates we use

$$\begin{aligned}
w_0' &\to \frac{w_0}{w_0^2 + (\vec{w} - \vec{x}_1)^2}, \\
t = \frac{{w_0'}^2}{(w' - \vec{x}_{21}')^2} &\to q = \vec{x}_{21}^2 \frac{w_0}{w_0^2 + (\vec{w} - \vec{x}_1)^2} \frac{w_0}{w_0^2 + (\vec{w} - \vec{x}_2)^2}, \\
\frac{1}{w^2} J_{\mu\lambda}(w) \frac{(w' - \vec{x}_{21}')_\lambda}{(w' - \vec{x}_{21}')^2} &\to Q_\mu := \frac{(w - \vec{x}_2)_\mu}{(w - \vec{x}_2)^2} - \frac{(w - \vec{x}_1)_\mu}{(w - \vec{x}_1)^2} , \\
\frac{1}{w^2} J_{\mu\lambda}(w) \frac{\delta_{\lambda 0}}{w_0'} &\to R_\mu := \frac{\delta_{\mu 0}}{\omega_0} - 2\frac{(w - \vec{x}_1)_\mu}{(w - \vec{x}_1)^2}.
\end{aligned} \tag{2.2.18}$$

Computation of the remaining w-integral is straightforward.

2.2.3 Tensor exchange

Finally, consider the exchange of the tensor field from the multiplet $(s_\Delta, ...)$ with the mass squared f, which is coupled to scalar fields s_{Δ_i} of different masses m_{Δ_i}

$$\mathbf{T}^{\Delta}_{\Delta_1\Delta_2\Delta_3\Delta_4}(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \int [dz][dw]\, T^{\mu\nu}(z, \vec{x}_1, \vec{x}_2) G^{\Delta}_{\mu\nu\mu'\nu'}(z,w) T^{\mu'\nu'}(z, \vec{x}_3, \vec{x}_4) \tag{2.2.19}$$

and the corresponding z-integral

$$A_{\mu\nu}(w,\vec{x}_1,\vec{x}_2)=\int\frac{d^{d+1}z}{z_0^{d+1}}G^{\Delta}_{\mu\nu\mu'\nu'}(z,w)T^{\mu'\nu'}(z,\vec{x}_1,\vec{x}_2). \tag{2.2.20}$$

Here the stress energy tensor $T_{\mu\nu}$ (2.1.25) is written in terms of the bulk-to-boundary propagators (1.1.5):

$$\begin{aligned}T^{\mu\nu}(z,\vec{x}_1,\vec{x}_2) &= \nabla^{(\mu}K_{\Delta_1}(z,\vec{x}_1)\nabla^{\nu)}K_{\Delta_2}(z,\vec{x}_2)-\frac{1}{2}g^{\mu\nu}\left[\nabla_\rho K_{\Delta_1}(z,\vec{x}_1)\nabla^\rho K_{\Delta_2}(z,\vec{x}_2)\right.\\ &\left.+\frac{1}{2}\left(m^2_{\Delta_1}+m^2_{\Delta_2}-f\right)K_{\Delta_1}(z,\vec{x}_1)K_{\Delta_2}(z,\vec{x}_2)\right].\end{aligned} \tag{2.2.21}$$

To compute the z-integral we again use translation $A_{\mu\nu}(w,\vec{x}_1,\vec{x}_2)=A_{\mu\nu}(w-\vec{x}_1,0,\vec{x}_{12})$ and perform the conformal inversion $w'_\mu=w_\mu/w^2$, $\vec{x}'_\mu=\vec{x}_\mu/x^2$. This gives us

$$A_{\mu\nu}(w,\vec{x}_1,\vec{x}_2)=\frac{1}{2}|\vec{x}_{12}|^{-2\Delta_2}\frac{1}{(w^2)^2}J_{\mu\lambda}(w)J_{\nu\rho}(w)I_{\lambda\rho}(\omega'-\vec{x}'_{12}), \tag{2.2.22}$$

with

$$\begin{aligned}I_{\mu\nu}(w)=\int\frac{d^{d+1}z}{z_0^{d+1}}G^{\Delta\ \mu'\nu'}_{\mu\nu}(w,z)&\left\{\nabla_{(\mu'}z_0^{\Delta_1}\nabla_{\nu')}\left(\frac{z_0}{z^2}\right)^{\Delta_2}\right.\\ &\left.-g_{\mu'\nu'}\left[\nabla_{\rho'}z_0^{\Delta_1}\nabla^{\rho'}\left(\frac{z_0}{z^2}\right)^{\Delta_2}+\frac{1}{2}(m_1^2+m_2^2-f)z_0^{\Delta_1}\left(\frac{z_0}{z^2}\right)^{\Delta_2}\right]\right\}.\end{aligned} \tag{2.2.23}$$

We write the following ansatz

$$I_{\mu\nu}(w)=w_0^{\Delta_{12}}\left[g_{\mu\nu}h(t)+P_\mu P_\nu\phi(t)+\nabla_\mu\nabla_\nu X(t)+\nabla_{(\mu}\left(P_{\nu)}Y(t)\right)\right], \tag{2.2.24}$$

where $P_\mu:=\delta_{\mu0}/w_0$ and $h(t)$, $\phi(t)$, $X(t)$, $Y(t)$ are four unknown scalar functions. Now we act by Ricci operator on both expressions for $I_{\mu\nu}(w)$ and use the wave equation for the propagator $G^{\Delta}_{\mu\nu\mu'\nu'}(\omega,z)$. Equating the coefficients of the different tensor structures appearing in the ansatz for $I_{\mu\nu}(w)$, we, thereby, reduce the problem of computing the z-integral to solving the system of differential equations. We again present only the final expressions, leaving the computation behind:

$$
\begin{aligned}
&\nabla_\mu \nabla_\nu \left[-3h - \phi + \left(-\Delta_{12}^2 + 2\Delta_{12} + f\right) X + 2\Delta_{12} Y\right] = 0\,, \\
&4t(t-1)\phi' + 4\phi + 4\left(\Delta_{12}^2 + \Delta_{12}\right) t(t-1)X' + 2\left(2\Delta_{12} - \Delta_{12}^2\right) X \\
&\qquad + 8\Delta_{12} t(1-t)Y' - 2\left(4\Delta_{12} + f\right) Y + 6\Delta_{12} h = -4\Delta_1 t^{\Delta_2} + c_1\,, \\
&-6t\Delta_{12} h' - 3\left(\Delta_{12} + 1\right)\Delta_{12} h + \left(f - 3\Delta_{12}\right)\phi \\
&\qquad + 6t\Delta_{12}^2 X' + 2t\, f Y' + 2\left(3\Delta_{12}^2 + f\right) Y = 0\,, \\
&4t^2(t-1)h'' + \Big(4\Delta_{12}(t-1)t + 4t(t+1)\Big)h' + \left(-\Delta_{12}^2 + 7\Delta_{12} + \frac{8}{3}(f+3)\right) h \\
&\qquad + 4t(t-1)\phi' + \left(\frac{1}{3}(f+24) - \Delta_{12}\right)\phi \\
&\qquad + 4(1-t)t^2\left(\Delta_{12} + \frac{f}{3}\right) X'' + \left(-\frac{4}{3} f t(t+1) - 8\Delta_{12} t\right) X' \\
&\qquad + 4(1-t)t\left(2\Delta_{12} + \frac{f}{3}\right) Y' + \left(2\Delta_{12}^2 - 16\Delta_{12} - \frac{14}{3} f\right) Y = \frac{2}{3}(m_1^2 + m_2^2 - f)t^{\Delta_2}\,.
\end{aligned}
\tag{2.2.25}
$$

We pick up the trivial solution of the first equation

$$
h = \frac{1}{3}\left(-\phi + (f - \Delta_{12}^2 + 2\Delta_{12})X + 2\Delta_{12} Y\right) \tag{2.2.26}
$$

and, assuming the polynomial form of the solution

$$
X(t) = \sum_k a_k t^k, \qquad Y(t) = \sum_k b_k t^k, \qquad \phi(t) = \sum_k c_k t^k\,, \tag{2.2.27}
$$

we solve the rest by equating the coefficients at equal powers of t.
The last step is again to perform the transformation (2.2.18) to original coordinates and compute the remaining w-integral.

Chapter 3

Couplings

In this chapter, we would like to present two different ways of computing the a-, t- and p-tensors, which are the building blocks for constructing the couplings. In the first section, we describe the straightforward way to compute them using only the definition. One cannot use this method in practice because of the quickly growing complexity. Luckily, there exists a straightforward and quick method – harmonic polynomial formalism – which we review in the second section of this chapter. Though the straightforward method is not practical, it still has some theoretical value. In particular, we obtained some new relations between different products of a-, t- and p-tensors, which allowed to prove a significant fact – the vanishing of the four-derivative lagrangian, see appendix B. This indeed was observed before and was used as an assumption to conjecture the closed analytic formula for the four-point functions in the Mellin space. Optimistically, those new reduction formulae could be used to find a compact formula for the quartic lagrangian and prove the Mellin conjecture analytically.

3.1 C-algebra: the painful way

The quartic couplings, see appendix C, and the product of cubic couplings, see eq. (2.1.1), are represented as sums of products of Clebsch-Gordan coefficients for SO(6) irreps that come in three types: a-, t- and p-tensors. In practice their computation can be cumbersome due to the fact that they contain contractions of C-tensors, which carry the tensor structure of the correlator as described in section 2.5 of part I. Since each C-tensor carries roughly as many indices as the weight of the representation it belongs to this can get unwieldy quickly, explaining why until now almost no correlators were known with weights larger than four.[1] We would firstly like to present here the simplified method which lets us compute the products of a-tensors relatively fast. To find different products of t- and p-tensors, one has to use several reduction relations. We also present new reduction formulae which let us find the couplings for the correlators $\langle 2345\rangle$ and $\langle 3456\rangle$. However, this simplified algorithm cannot be applied for correlators with higher weights. One way to deal with this would be to obtain more reduction relations. Ideally, they could be used to find an analytic expression for the four-point functions in the coordinate space, or at least could help to find significantly simplified expressions for quartic terms - the fact which is believed to be true and was observed in all the computed cases. Luckily,

[1] The only exceptions are the family $\langle 22nn\rangle$ for $n > 1$ in [18] and $\langle 5555\rangle$ in [20], which both feature a large degeneracy allowing for simplifications.

there exists an incredibly simple and efficient method to compute products of a-, t- and p-tensors - harmonic polynomial formalism. We firstly present our first method for the reason for future use. At the end of the chapter, we review the harmonic polynomial formalism, which we used in all our computations.

3.1.1 Definitions

To be more precise, following the notation from [12], we want to compute products of the objects

$$a_{123} = \int Y^{I_1} Y^{I_2} Y^{I_3}, \quad t_{123} = \int \nabla^\alpha Y^{I_1} Y^{I_2} Y^{I_3}_\alpha, \quad p_{123} = \int \nabla^\alpha Y^{I_1} \nabla^\beta Y^{I_2} Y^{I_3}_{(\alpha\beta)}, \quad (3.1.1)$$

where the Y functions are spherical harmonics on the five-sphere and the round brackets indicate traceless symmetrization. More precisely, we need to find expressions for $a_{125}a_{345}$, $t_{125}t_{345}$ and $p_{125}p_{345}$, where $1, 2, 3, 4$ refer to the weights k_1, k_2, k_3, k_4 of the correlator and 5 is an intermediate leg with a weight k_5. The integrals (3.1.1) can be expressed using so-called C-tensors (see for more details [12]). For example, a_{123} is defined as

$$a_{123} = \frac{\prod_{i=1}^{3} \frac{k_i z(k_i)}{\alpha_i!}}{\pi^{\frac{3}{2}} (\sigma+2)!\, 2^{\sigma-1}} \left\langle C^{I_1}_{[0,k_1,0]} C^{I_2}_{[0,k_2,0]} C^{I_3}_{[0,k_3,0]} \right\rangle, \quad (3.1.2)$$

where $\alpha_i = \alpha_i(k_1, k_2, k_3) = \sigma - k_i$ for $i = 1, 2, 3$ and $\sigma = \sigma(k_1, k_2, k_3) = (k_1 + k_2 + k_3)/2$ and where

$$\left\langle C^{I_1}_{[0,k_1,0]} C^{I_2}_{[0,k_2,0]} C^{I_3}_{[0,k_3,0]} \right\rangle = C^{I_1}_{i_1 \ldots i_{\alpha_2} j_1 \ldots j_{\alpha_3}} C^{I_2}_{j_1 \ldots j_{\alpha_3} l_1 \ldots l_{\alpha_1}} C^{I_3}_{l_1 \ldots l_{\alpha_1} i_1 \ldots i_{\alpha_2}} \quad (3.1.3)$$

encodes the tensor structure of the correlator. The number of non-zero tensors is restricted by representation theory and therefore finite.

Although the formulae for $a_{125}a_{345}$, $t_{125}t_{345}$ and $p_{125}p_{345}$ are fully explicit it is not straightforward to compute them. The main obstruction in performing this computation and, therefore, in computing a correlation function from the lagrangian, is in the use of the completeness relations for the C-tensors: for all of the a-, t- and p-tensors one has to evaluate objects of the form

$$\sum_{I_5} \langle C^{I_1}_{k_1} C^{I_2}_{k_2} C^{I_5}_{k_5} \rangle \langle C^{I_3}_{k_3} C^{I_4}_{k_4} C^{I_5}_{k_5} \rangle, \quad (3.1.4)$$

where the sum is over the representation index of the k_5 field. The completeness relations allow us to evaluate this sum, such that only four C-tensors remain that encode the tensor structure of the correlator. For a-tensors a closed formula for the completeness relation exists (given below in (3.1.6)), but its evaluation can become impossible for expressions for higher-weight operators due to computational limits. For t- and p-tensors no such formula exists at present and the best we can do is to determine the completeness relations for a fixed set of weights from the properties of C-tensors. Additionally, we can use so-called reduction relations to reduce the total number of unknown a-, t- and p-tensors. They are discussed in the next subsection 3.1.3. These relations allow us to express the t- and p-tensors with the highest intermediate k_5 weights in terms of a-, t- and p-tensors we already know.

To summarize, the procedure to obtain the a-, t- and p-tensors for a correlator with given weights (k_1, k_2, k_3, k_4) is the following: we compute the necessary a-tensors using the formula (3.1.2) involving a completeness relation that we will get back to in more detail in the next section. Then we use the reduction relations discussed in section 3.1.3 to express the highest weight t- and p-tensors using lower ones and some of the a-tensors we computed. We compute the remaining necessary t- and p-tensors by explicitly finding the completeness relations for the given weights by imposing all the properties of C-tensors on an ansatz. This yields the complete set of needed a-, t- and p-tensors.

3.1.2 Simplifying completeness relations for a-tensors

To compute the a-tensors we need expressions for the completeness relation of C-tensors that allow us to resolve the sum in (3.1.4). Here we will focus only on simplification of the case for a-tensors, since there is an explicit formula available. Ultimately we need to reduce (3.1.4) to a sum of the independent tensor structures of the form $C^{I_1}C^{I_2}C^{I_3}C^{I_4}$ that carry the tensor structure of the correlator only, meaning we should get rid of the sum over the representation index I_5. This can be done in principle as this sum constitutes a completeness condition for the C-tensors, meaning it is expressable as a linear combination of products of Kronecker delta functions carrying the indices, i.e.

$$\sum_I C^I_{i_1\dots i_n} C^I_{i_{n+1}\dots i_{2n}} = \sum_{\sigma\in S_{2n}} A_\sigma \delta_{i_{\sigma(1)}i_{\sigma(2)}} \delta_{i_{\sigma(3)}i_{\sigma(4)}} \dots \delta_{i_{\sigma(2n-1)}i_{\sigma(2n)}}, \tag{3.1.5}$$

where S_n is the symmetric group of n objects and A_σ are coefficients that are to be determined from the properties of the C-tensors: after taking into account the internal symmetries of the product of delta functions, being symmetric under exchange of indices belonging to the same Kronecker delta and permutation of these delta functions, the number of coefficients is $(2n)!/(2^n n!)$. For the case $\langle 3456\rangle$ the largest n one has to consider is $n = 9$, yielding over 34 million terms.

Although this is indeed the whopping number of coefficients that needs to be computed and stored to express the completeness condition (3.1.5) we can consider a simplified version of it to compute the sum in formula (3.1.4), since the other C-tensors have additional symmetries. Our starting point to derive this simplification is the fully explicit formula given in B.5 of [15]:

$$C^I_{i_1,\dots,i_n} C^I_{j_1,\dots,j_n} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \theta_k \sum_{\{l_1\dots l_{2k}\}} \delta_{i_{l_1}i_{l_2}} \dots \delta_{i_{l_{2k-1}}i_{l_{2k}}} \tilde{\delta}^{(n-2k)}_{i_1\dots\hat{i}_{l_1}\dots\hat{i}_{l_{2k}}\dots i_n,(j_{2k+1}\dots j_n} \delta_{j_1j_2} \dots \delta_{j_{2k-1}j_{2k})}, \tag{3.1.6}$$

where the sum over $\{l_1 \dots l_{2k}\}$ runs over all subsets of $\{1, \dots, n\}$ containing $2k$ elements that yield inequivalent products of delta functions, $(\dots)$ stands for symmetrization of the indices and we use the definition $\tilde{\delta}^{(p)}_{i_1,\dots i_p,j_1,\dots j_p} = \delta^{(p)}_{(i_1,\dots i_p),(j_1,\dots j_p)}$ and $\delta^{(p)}_{i_1,\dots i_p,j_1,\dots j_p} = \prod_{k=1}^p \delta_{i_k,j_k}$. The coefficients θ_k are known explicitly as

$$\theta_0 = 1, \quad \text{if } k > 0 : \theta_k = \frac{(-1)^k}{2^k(n+1)\dots(n+2-k)}. \tag{3.1.7}$$

However, the rapidly growing number of summands makes this expression difficult to work with for higher weights, also because it contains symmetrizations that lead to overcounting in the intermediate stages. Since all the C-tensors are traceless and completely symmetric in their indices we can restrict the formula further. In order to provide a succinct derivation we first discuss some theory related to a splitting of the symmetric group.

Decomposing the symmetric group

To simplify the completeness condition it will be convenient to decompose elements of the symmetric group S_n as follows: consider two natural numbers p, q such that $p + q = n$, then S_p and S_q are subgroups of S_n but generically they will not be normal such that the quotient S_n/S_p is not well-defined as a group. Nevertheless, it is possible to decompose any element $\sigma \in S_n$ as a product $\sigma = \rho\sigma_q\sigma_p$, where σ_p only permutes the first p objects, σ_q only permutes the last q objects and ρ is a product of transpositions that each swap one object from the first p with one of the last q. The only complication is that, in the cases in which S_p and/or S_q are not normal subgroups, the set of elements ρ, which we will denote by

$$S_n /\!\!/ (S_p \times S_q) \text{ or simply } S_n^{(p,q)} \tag{3.1.8}$$

is not a subgroup as it is not even closed under multiplication. It is also not uniquely defined, as multiplication by an element from S_p or S_q gives a new set with the same properties. This is ultimately not problematic as any set of representatives will do to perform the sums we are interested in. We will present here one choice for this set.[2]

A characterization of $S_n /\!\!/ (S_p \times S_q)$

First of all, the counting tells us that $S_n^{(p,q)}$ should have $\frac{n!}{p!q!}$ elements which should all be independent, i.e. should not be expressable as a product of other elements in the set. Secondly, since S_p and S_q already take care of all the rotations in the first p and last q objects we can consider only those permutations which are built up from transpositions that lie outside both S_p and S_q. The remaining transpositions are necessarily of the form (ij) with $1 \leqslant i \leqslant p$ and $n \geqslant j > p$. Thirdly, if in a product of transpositions two of them contain the same number i this can be taken away by a suitable transposition from either S_p or S_q. From this we conclude that $S_n^{(p,q)}$ contains all the permutations that can be built from the transpositions (ij) with $i \leqslant p$ and $j > p$ such that all of them are disjoint. Their number is easily computed:

- The identity transposition: 1 element
- All single transpositions: there are pq of them
- All products of two disjoint transpositions: there are $pq(p-1)(q-1)/(2! \cdot 2!)$ of those
- In general the products of r disjoint transpositions are

$$\frac{\prod_{k=0}^{r-1}(p-k)(q-k)}{r! \cdot r!} \tag{3.1.9}$$

  in number.

[2]It seems unlikely that the group theory, decomposition and choice of representatives we present here are all novel, but despite a considerable effort we have not managed to find other accounts. Since it helps to simplify the contraction of symmetric tensors one could nevertheless fathom a number of applications where it could help speed up computations. The authors would welcome seeing any application of these principles, for example an implementation into Form.

This means that we have formed a set containing

$$\sum_{l=0}^{\min(p,q)} \frac{1}{(l!)^2} \prod_{k=0}^{l-1} (p-k)(q-k) = \frac{(p+q)!}{p!q!} \tag{3.1.10}$$

elements. Since the three sets S_p, S_q and $S_n^{(p,q)}$ are all disjoint and independent we see that our characterization results in the correct number of elements $\sigma = \rho\sigma_q\sigma_p$, namely $n!$, implying that this description will work to decompose σ. Finally, we will use the notation $S_n^{(p,q)}(k)$ to denote the part of $S_n^{(p,q)}$ containing those elements built up from exactly k transpositions.

Example

As an example we present the set $S_7^{(4,3)}$, which contains the 35 elements listed in the following table:

part of $S_7^{(4,3)}$	#	elements
$S_7^{(4,3)}(0)$	1	id
$S_7^{(4,3)}(1)$	12	$(1,5)$, $(1,6)$, $(1,7)$, $(2,5)$, $(2,6)$, $(2,7)$, $(3,5)$, $(3,6)$, $(3,7)$, $(4,5)$, $(4,6)$, $(4,7)$
$S_7^{(4,3)}(2)$	18	$(1,5)(2,6)$, $(1,5)(2,7)$, $(1,5)(3,6)$, $(1,5)(3,7)$, $(1,5)(4,6)$, $(1,5)(4,7)$, $(1,6)(2,7)$, $(1,6)(3,7)$, $(1,6)(4,7)$, $(2,5)(3,6)$, $(2,5)(3,7)$, $(2,5)(4,6)$, $(2,5)(4,7)$, $(2,6)(3,7)$, $(2,6)(4,7)$, $(3,5)(4,6)$, $(3,5)(4,7)$, $(3,6)(4,7)$
$S_7^{(4,3)}(3)$	4	$(1,5)(2,6)(3,7)$, $(1,5)(2,6)(4,7)$, $(1,5)(3,6)(4,7)$, $(2,5)(3,6)(4,7)$

Simplifying the completeness condition

We can now simplify the completeness condition using the fact that the indices $\mathcal{I} = (i_1, \dots i_n)$ and $\mathcal{J} = (j_1, \dots j_n)$ are contracted with the indices of traceless symmetric C-tensors in the expression (3.1.4): explicitly we can write (3.1.4) as

$$\sum_{I_5} \sum_{\mathcal{K}_1,\mathcal{K}_2} \sum_{\mathcal{I},\mathcal{J}} C^{I_1}_{\mathcal{I}_2\mathcal{K}_1} C^{I_2}_{\mathcal{K}_1\mathcal{I}_1} C^{I_5}_{\mathcal{I}_1\mathcal{I}_2} C^{I_3}_{\mathcal{J}_2\mathcal{K}_2} C^{I_4}_{\mathcal{K}_2\mathcal{J}_1} C^{I_5}_{\mathcal{J}_1\mathcal{J}_2}, \tag{3.1.11}$$

where $\mathcal{I} = \mathcal{I}_1 \cup \mathcal{I}_2$ and $\mathcal{J} = \mathcal{J}_1 \cup \mathcal{J}_2$ are multi-indices that are summed over with constituents

$$\begin{aligned} \mathcal{I}_1 &= \{i_1, \dots, i_{\alpha_1(k_1,k_2,k_5)}\}, \qquad & \mathcal{J}_1 &= \{j_1, \dots, j_{\alpha_1(k_3,k_4,k_5)}\}, \\ \mathcal{I}_2 &= \{i_{\alpha_2(k_1,k_2,k_5)+1}, \dots, i_n\}, \qquad & \mathcal{J}_2 &= \{j_{\alpha_2(k_3,k_4,k_5)+1}, \dots, j_n\}, \end{aligned} \tag{3.1.12}$$

and $\mathcal{K}_1, \mathcal{K}_2$ are other multi-indices containing $\alpha_3(k_1,k_2,k_5)$ and $\alpha_3(k_3,k_4,k_5)$ indices respectively. In the following we will leave the sums implicit. We will now insert (3.1.6):

$$\begin{aligned} C^{I_1}_{\mathcal{I}_2\mathcal{K}_1} C^{I_2}_{\mathcal{K}_1\mathcal{I}_1} C^{I_3}_{\mathcal{J}_2\mathcal{K}_2} C^{I_4}_{\mathcal{K}_2\mathcal{J}_1} C^{I_5}_{\mathcal{I}_1\mathcal{I}_2} C^{I_5}_{\mathcal{J}_1\mathcal{J}_2} = C^{I_1}_{\mathcal{I}_2\mathcal{K}_1} C^{I_2}_{\mathcal{K}_1\mathcal{I}_1} C^{I_3}_{\mathcal{J}_2\mathcal{K}_2} C^{I_4}_{\mathcal{K}_2\mathcal{J}_1} \times \\ \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \theta_k \sum_{\{l_1 \dots l_{2k}\}} \delta_{i_{l_1} i_{l_2}} \dots \delta_{i_{l_{2k-1}} i_{2k}} \tilde{\delta}^{(n-2k)}_{i_{l_1} \dots \hat{i}_{l_1} \dots \hat{i}_{l_{2k}} \dots i_n, (j_{2k+1} \dots j_n} \delta_{j_1 j_2} \dots \delta_{j_{2k-1} j_{2k})}. \end{aligned} \tag{3.1.13}$$

Let us first note that the first product of delta functions $\delta_{i_{l_1} i_{l_2}} \dots \delta_{i_{l_{2k-1}} i_{2k}}$ is being contracted with two traceless tensors. This implies that whenever l_r, l_s belong to the same index set $\mathcal{I}_1$ or $\mathcal{I}_2$ the contraction will make its contribution vanish. Interestingly, this implies that we can restrict the sum over subsets $\{l_1, \dots, l_{2k}\}$ further using our newly defined subsets $S_n^{(p,q)}(k)$. Let $\tau = (n_1 n_2) \dots (n_{2k-1} n_{2k})$ be a product of k transpositions, then we define

$$\delta_\tau \coloneqq \delta_{n_1,n_2} \dots \delta_{n_{2k-1} n_{2k}}. \tag{3.1.14}$$

Let us further define $\mathcal{I}_\tau = \{i_{n_1}, i_{n_2}, \dots, i_{n_{2k}}\}$ and $\mathcal{I}_\tau^c = \mathcal{I} \setminus \mathcal{I}_\tau$. To simplify the second and third set of deltas we recognize that in the definition of $\tilde{\delta}^p$ the symmetrization over the i indices is unnecessary and the symmetrization over the j can be restricted: splitting the action of S_n on the j indices as $S_n^{2k,n-2k}$ we can write the total contraction as

$$\begin{gathered} C^{I_1}_{\mathcal{I}_2 \mathcal{K}_1} C^{I_2}_{\mathcal{K}_1 \mathcal{I}_1} C^{I_3}_{\mathcal{J}_2 \mathcal{K}_2} C^{I_4}_{\mathcal{K}_2 \mathcal{J}_1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \theta_k \sum_{\tau \in S_n^{(|\mathcal{I}_1|,|\mathcal{I}_2|)}(k)} \delta_\tau \frac{1}{n!} \times \\ \sum_{\substack{\sigma_1 \in S_{n-2k} \\ \rho \in S_n^{(2k,n-2k)}}} \delta^{(n-2k)}_{\mathcal{I}_\tau^c, \sigma_1 \circ \rho(j_{2k+1}) \dots \sigma_1 \circ \rho(j_n)} \sum_{\sigma_2 \in S_{2k}} \delta_{\sigma_2 \circ \rho(j_1) \sigma_2 \circ \rho(j_2)} \dots \delta_{\sigma_2 \circ \rho(j_{2k-1}) \sigma_2 \circ \rho(j_{2k})}, \end{gathered} \tag{3.1.15}$$

Note that the element ρ effectively selects which j indices occur in the delta functions together with the i's. To simplify this expression further we need to identify which indices are contracted: this unfortunately complicates the expression even more, but we will see that the result is simple enough to work with. The idea is to split the elements σ_1 and σ_2 further into products using the decomposition of S_{2k} and S_{n-2k}: depending on ρ the j indices are split into two parts containing the first $2k$ and last $n-2k$ elements, let us denote them as $\mathcal{J}_{2k}$ and $\mathcal{J}_{n-2k}$. The set $\mathcal{J}_{2k}$ describing the third set of delta functions overlaps with indices from $\mathcal{J}_1$ and $\mathcal{J}_2$, each of which forms a completely symmetric set of indices. Therefore symmetrizing indices belonging to $\mathcal{J}_1$ or $\mathcal{J}_2$ is unnecessary and we can restrict the sum over permutations to those which permute $\mathcal{J}_1$ indices to $\mathcal{J}_2$ indices and vice versa. A similar thing is true for the second set of delta functions. However, in this case there are two equivalent ways of doing it, although computationally usually one of the two ways is faster as it will result in less summands: as previously, the indices in $\mathcal{J}_{n-2k}$ overlap with $\mathcal{J}_1$ and $\mathcal{J}_2$, leading to a natural splitting of S_{n-2k}. However, the fact that in these delta functions the j are always paired with an i allows us to exploit the overlap of $\mathcal{I}_\tau^c$ with $\mathcal{I}_1$ and $\mathcal{I}_2$ instead. Since this yields better results for the present case we use this second splitting. This leads to the following rewriting of (3.1.15):

$$C^{I_1}_{\mathcal{I}_2\mathcal{K}_1} C^{I_2}_{\mathcal{K}_1\mathcal{I}_1} C^{I_3}_{\mathcal{J}_2\mathcal{K}_2} C^{I_4}_{\mathcal{K}_2\mathcal{J}_1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \theta_k \sum_{\tau \in S_n^{(|\mathcal{I}_1|,|\mathcal{I}_2|)}(k)} \delta_\tau \frac{1}{n!} \times$$
$$\sum_{\rho \in S_n^{(2k,n-2k)}} \sum_{\substack{\sigma_1^{(1)} \in S_{|\mathcal{I}_1 \cap \mathcal{I}_\tau^c|} \\ \sigma_1^{(2)} \in S_{|\mathcal{I}_2 \cap \mathcal{I}_\tau^c|} \\ \rho_1 \in S_{n-2k}^{(|\mathcal{I}_1 \cap \mathcal{I}_\tau^c|,|\mathcal{I}_2 \cap \mathcal{I}_\tau^c|)}}} \delta^{(n-2k)}_{\mathcal{I}_\tau^c\,,\,\rho_1\circ\sigma_1^{(1)}\circ\sigma_1^{(2)}\circ\rho(j_{2k+1})\ldots\rho_1\circ\sigma_1^{(1)}\circ\sigma_1^{(2)}\circ\rho(j_n)} \times \tag{3.1.16}$$
$$\sum_{\substack{\sigma_2^{(1)} \in S_{|\mathcal{J}_1|} \\ \sigma_2^{(2)} \in S_{|\mathcal{J}_2|} \\ \rho_2 \in S_{2k}^{(|\mathcal{J}_1|,|\mathcal{J}_2|)}}} \delta_{\rho_2\circ\sigma_2^{(1)}\circ\sigma_2^{(2)}\circ\rho(j_1),\rho_2\circ\sigma_2^{(1)}\circ\sigma_2^{(2)}\circ\rho(j_2)} \ldots \delta_{\rho_2\circ\sigma_2^{(1)}\circ\sigma_2^{(2)}\circ\rho(j_{2k-1}),\rho_2\circ\sigma_2^{(1)}\circ\sigma_2^{(2)}\circ\rho(j_{2k})}.$$

Using the symmetry of the index sets $\mathcal{I}_1$, $\mathcal{I}_2$ and $\mathcal{J}_1$ and $\mathcal{J}_2$ we can now simplify this result, as the summands in the sums over the four different types of σ permutations are all the same. This means that the following reduction is possible:

$$C^{I_1}_{\mathcal{I}_2\mathcal{K}_1} C^{I_2}_{\mathcal{K}_1\mathcal{I}_1} C^{I_3}_{\mathcal{J}_2\mathcal{K}_2} C^{I_4}_{\mathcal{K}_2\mathcal{J}_1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \theta_k \sum_{\tau \in S_n^{(|\mathcal{I}_1|,|\mathcal{I}_2|)}(k)} \delta_\tau \frac{1}{n!} \times$$
$$\sum_{\rho \in S_n^{(2k,n-2k)}} \sum_{\rho_1 \in S_{n-2k}^{(|\mathcal{I}_1 \cap \mathcal{I}_\tau^c|,|\mathcal{I}_2 \cap \mathcal{I}_\tau^c|)}} |\mathcal{I}_1 \cap \mathcal{I}_\tau^c|!\,|\mathcal{I}_2 \cap \mathcal{I}_\tau^c|!\,\delta^{(n-2k)}_{\mathcal{I}_\tau^c\,,\,\rho_1\circ\rho(j_{2k+1})\ldots\rho_1\circ\rho(j_n)} \times \tag{3.1.17}$$
$$\sum_{\rho_2 \in S_{2k}^{(|\mathcal{J}_1|,|\mathcal{J}_2|)}} |\mathcal{J}_1|!\,|\mathcal{J}_2|!\,\delta_{\rho_2\circ\rho(j_1),\rho_2\circ\rho(j_2)} \ldots \delta_{\rho_2\circ\rho(j_{2k-1}),\rho_2\circ\rho(j_{2k})}$$

and after noting that the final sum is still overcounting due to symmetries of the delta functions we note that finally we can rewrite our expression as

$$C^{I_1}_{\mathcal{I}_2\mathcal{K}_1} C^{I_2}_{\mathcal{K}_1\mathcal{I}_1} C^{I_3}_{\mathcal{J}_2\mathcal{K}_2} C^{I_4}_{\mathcal{K}_2\mathcal{J}_1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \theta_k \sum_{\tau \in S_n^{(|\mathcal{I}_1|,|\mathcal{I}_2|)}(k)} \delta_\tau \frac{1}{n!} \times$$
$$\sum_{\rho \in S_n^{(2k,n-2k)}} \sum_{\rho_1 \in S_{n-2k}^{(|\mathcal{I}_1 \cap \mathcal{I}_\tau^c|,|\mathcal{I}_2 \cap \mathcal{I}_\tau^c|)}} |\mathcal{I}_1 \cap \mathcal{I}_\tau^c|!\,|\mathcal{I}_2 \cap \mathcal{I}_\tau^c|!\,\delta^{(n-2k)}_{\mathcal{I}_\tau^c\,,\,\rho_1\circ\rho(j_{2k+1})\ldots\rho_1\circ\rho(j_n)} \times \tag{3.1.18}$$
$$\sum_{\tau_2 \in S_{2k}^{(|\mathcal{J}_1|,|\mathcal{J}_2|)}(k)\rho} |\mathcal{J}_1|!\,|\mathcal{J}_2|!\,2^k k!\,\delta_{\tau_2}.$$

Here τ_2 is an element of $S_{2k}^{(|\mathcal{J}_1|,|\mathcal{J}_2|)}(k)\rho$, with which we mean that τ_2 is one of the elements of $S_{2k}^{(|\mathcal{J}_1|,|\mathcal{J}_2|)}(k)$ consisting of k transpositions as it acts on the first $2k$ indices of the index set $\rho\,(\mathcal{J})$. This is our final result, which plays an important role in the computation of the $\langle 3456 \rangle$ correlator: as stated before, the computation of this tensor contraction was unfeasible with the previously known tools, rendering the computation of the $\langle 3456 \rangle$ correlator from the lagrangian infeasible as well. Using our new formula we were able to obtain the most complicated a-tensor for $\langle 3456 \rangle$ with intermediate field with weight 9 in 10 minutes on a standard computer.

3.1.3 Reduction formulae

Formulae

There is no explicit summation formulae as (3.1.6) known for the product of C-tensors corresponding to $[1,k,1]$ and $[2,k,2]$ SO(6) representations to simplify the computation of t- and p-tensors. However, one can generate the completeness conditions for sums involving lower weight representations. For the remaining sums one can obtain a system of linear equations, using the reduction formula (3.1.19), (3.1.20) derived in [12]: writing $f_i \equiv f\left(k_i\right) = k_i(k_i+4)$ they read[3]

$$\begin{aligned}
t_{125}t_{345} &= -\frac{(f_1-f_2)(f_3-f_4)}{4f_5}a_{125}a_{345} - \frac{1}{4}f_5(a_{135}a_{245}-a_{145}a_{235}),\\
f_5\, t_{125}t_{345} &= -\frac{\left(f_1-f_2\right)\left(f_3-f_4\right)\left(f_5-3\right)}{4f_5}a_{125}a_{345}\\
&\quad -\frac{1}{4}\left(f_1+f_2+f_3+f_4-f_5-3\right)f_5\left(a_{135}a_{245}-a_{145}a_{235}\right), \qquad (3.1.19)\\
p_{125}p_{345} &= -\frac{\left(f_1-f_2\right)\left(f_3-f_4\right)}{2\left(f_5-5\right)}t_{125}t_{345} - \frac{1}{20}\left(f_1+f_2-f_5\right)\left(f_3+f_4-f_5\right)a_{125}a_{345}\\
&\quad +\frac{1}{8}\left(f_1+f_3-f_5\right)\left(f_2+f_4-f_5\right)a_{135}a_{245}\\
&\quad +\frac{1}{8}\left(f_1+f_4-f_5\right)\left(f_2+f_3-f_5\right)a_{145}a_{235} - \frac{5}{4\left(f_5-5\right)f_5}d_{125}d_{345}\\
f_5\, p_{125}p_{345} &= -g_{1324}-g_{1423}-\frac{5\left(f_5-8\right)}{4\left(f_5-5\right)f_5}d_{125}d_{345} - \frac{1}{2}\left(f_1-f_2\right)\left(f_3-f_4\right)t_{125}t_{345}\\
&\quad -\frac{1}{20}\left(f_1+f_2-f_5\right)\left(f_3+f_4-f_5\right)\left(f_5+2\right)a_{125}a_{345}\\
&\quad +\frac{1}{8}\left(f_1+f_2-6\right)\left(f_1+f_3-f_5\right)\left(f_2+f_4-f_5\right)a_{135}a_{245}\\
&\quad +\frac{1}{8}\left(f_1+f_2-6\right)\left(f_2+f_3-f_5\right)\left(f_1+f_4-f_5\right)a_{145}a_{235}, \qquad (3.1.20)
\end{aligned}$$

where we denoted

$$\begin{aligned}
g_{1234} &= \frac{1}{4}\left(f_1+f_2-f_5-3\right)\left(f_3+f_4-f_5-3\right)t_{125}t_{345}\\
&\quad -\frac{\left(f_1^2-\left(f_5-f_2\right)^2\right)\left(\left(f_5-f_4\right)^2-f_3^2\right)}{16f_5}a_{125}a_{345},\\
d_{125} &= \frac{1}{20}\left(-5\left(f_1-f_2\right)^2+3f_5^2+2\left(f_1+f_2\right)f_5\right)a_{125}. \qquad (3.1.21)
\end{aligned}$$

[3]Summation over the fifth leg is assumed.

We would also like to attract the reader's attention to the formula, derived in appendix B:

$$
\begin{aligned}
&(f_5-2)^2(t_{135}t_{245}+t_{145}t_{235})= \\
&-\frac{a_{135}a_{245}}{24f_5}\Big(-2f_5^4+2(3f_1+3f_2+3f_3+3f_4-28)f_5^3-2(2f_1^2-(f_2-5f_3-8f_4+28)f_1 \\
&+2f_2^2+2f_4^2+2(f_3-12)(f_3-2)-(f_3+28)f_4+f_2(8f_3+5f_4-28))f_5^2 \\
&+((f_2+3f_3+4f_4-12)f_1^2+(f_2^2+4(f_3+f_4-20)f_2+3(f_3-4)^2+4f_4^2+4(f_3+4)f_4)f_1 \\
&+(3f_2+f_3-12)f_4^2+4((f_3-3)f_2^2+(f_3(f_3+4)+12)f_2-3(f_3-4)f_3)+(3(f_2-4)^2 \\
&+f_3^2+4(f_2-20)f_3)f_4)f_5+96(f_1-f_3)(f_2-f_4)\Big) \\
&-\frac{a_{145}a_{235}}{24f_5}\Big(-2f_5^4+2(3f_1+3f_2+3f_3+3f_4-28)f_5^3-2(2f_1^2-(f_2-8f_3-5f_4+28)f_1 \\
&+2f_2^2+2f_4^2+2(f_3-12)(f_3-2)-(f_3+28)f_4+f_2(5f_3+8f_4-28))f_5^2 \\
&+((f_2+4f_3+3(f_4-4))f_1^2+(f_2^2+4(f_3+f_4-20)f_2+4f_3^2+3(f_4-4)^2+4f_3(f_4+4))f_1 \\
&+(4f_2+f_3-12)f_4^2+3(f_2-4)(f_3-4)(f_2+f_3) \\
&+(4f_2^2+4(f_3+4)f_2+(f_3-80)f_3+48)f_4)f_5+96(f_2-f_3)(f_1-f_4)\Big) \\
&+\frac{a_{125}a_{345}}{24}\Big(-4f_5^3+4(3f_3+3f_4-28)f_5^2-4(2f_4^2+5f_3f_4-28f_4+2(f_3-14)f_3+48)f_5 \\
&+6(f_3-4)(f_4-4)(f_3+f_4)+f_2(5f_3^2+8f_4f_3-64f_3+5f_4^2+12f_5^2-64f_4 \\
&-14(f_3+f_4-8)f_5+96)+f_1(6f_2^2+4(2(f_3+f_4-6)-5f_5)f_2+5f_3^2+5f_4^2+12f_5^2 \\
&-64f_3+8f_3f_4-64f_4-14(f_3+f_4-8)f_5+96)+f_2^2(5f_3+5f_4-8(f_5+3)) \\
&+f_1^2(6f_2+5f_3+5f_4-8(f_5+3))\Big)\,.
\end{aligned}
\tag{3.1.22}
$$

With its help, the *sum* of two quadratic tt's can be completely reduced to different products of a-tensors.

We also obtained new reduction formulae for similar combinations of quadratic pp's. They are a direct consequence of the following identities:

$$
\begin{aligned}
&(f_5-2)^2\,p_{125}p_{345}=P_{1234}+R_{1234}, \\
&R_{1234}=-\frac{(f_1-f_2)\,(f_3-f_4)\,(f_5-7)^2}{2\,(f_5-5)}t_{125}t_{345}-\frac{(-5\,(f_1-f_2)^2+2\,(f_1+f_2)\,f_5+3f_5^2)}{320\,(f_5-5)\,f_5} \\
&\qquad\times(f_5-10)^2(-5\,(f_3-f_4)^2+3f_5^2+2\,(f_3+f_4)\,f_5)\,a_{125}a_{345}, \\
&P_{1234}=2\,(Y_{1234}+Y_{1243})-\frac{1}{20}\,(f_1+f_2-f_5)\,(f_3+f_4-f_5)\,f_5^2\,a_{125}a_{345} \\
&\quad-\frac{(f_2+f_3-f_5)\,(f_1+f_4-f_5)}{16f_5}\Big(\,(f_1+f_2+f_3+f_4-16)\,f_5^2 \\
&\quad-(f_1^2+2\,(f_2-8)\,f_1+(f_2-16)\,f_2+(f_3+f_4-16)\,(f_3+f_4))f_5 \\
&\quad-128f_5+(f_2-f_3)\,(f_1-f_4)\,(f_1+f_2+f_3+f_4-16)\,\Big)\,a_{145}a_{235} \\
&\quad-\frac{(f_1+f_3-f_5)\,(f_2+f_4-f_5)}{16f_5}\Big(\,(f_1+f_2+f_3+f_4-16)\,f_5^2 \\
&\quad-\big(f_1^2+2\,(f_2-8)\,f_1+(f_2-16)\,f_2+(f_3+f_4-16)\,(f_3+f_4)\big)\,f_5 \\
&\quad-128f_5+(f_1-f_3)\,(f_2-f_4)\,(f_1+f_2+f_3+f_4-16)\,\Big)a_{135}a_{245} \\
&\quad-\frac{1}{4}\,(f_1+f_2+f_3+f_4-16)\,(f_1+f_3-f_5-3)\,(f_2+f_4-f_5-3)\,t_{135}t_{245} \\
&\quad-\frac{1}{4}\,(f_1+f_2+f_3+f_4-16)\,(f_2+f_3-f_5-3)\,(f_1+f_4-f_5-3)\,t_{145}t_{235},
\end{aligned}
\tag{3.1.23}
$$

where Y_{1234} is defined by (3.1.42) and has the following symmetry properties:

$$Y_{1234} = Y_{1324} = Y_{2143} = Y_{3412}. \tag{3.1.24}$$

We were not able to reduce Y_{1234}, but rather $Y_{12[34]} \equiv \frac{1}{2}(Y_{1234} - Y_{1243})$:

$$\begin{aligned} Y_{12[34]} &= \frac{1}{8}(f_1+f_2-f_5-5)(f_3+f_4-f_5-3)(f_5+3)\ t_{125}t_{345} \\ &-\frac{1}{8}(f_1+f_3-f_5-3)(f_2+f_4-f_5-3)\,t_{135}t_{245} \\ &+\frac{1}{8}(f_2+f_3-f_5-3)(f_1+f_4-f_5-3)\,t_{145}t_{235} \\ &\frac{1}{32f_5}\left(f_1^2-(f_3-f_5)^2\right)\left((f_4-f_5)^2-f_2^2\right)a_{135}a_{245} \\ &-\frac{1}{32f_5}\left(f_2^2-(f_3-f_5)^2\right)\left((f_4-f_5)^2-f_1^2\right)a_{145}a_{235}. \end{aligned} \tag{3.1.25}$$

Thus, using (3.1.23)-(3.1.25), one is able, similar to (3.1.22), to find an expression only for the *difference* of two quadratic *pp*'s. In fact, it has the following structure (schematically):

$$f_5^2(p_{135}p_{245} - p_{145}p_{235}) = \frac{1}{2}f_5^3\,t_{125}t_{345} + \{...\}, \tag{3.1.26}$$

where $\{...\}$ contains only quadratic tt's and different aa's.

From (3.1.26) it directly follows another useful reduction formula:

$$\begin{aligned} &f_5^3(t_{1234} - t_{1324} + t_{1423}) = \Sigma_{1234} - \Sigma_{1324} + \Sigma_{1423}, \\ &\Sigma_{1234} = (f_1+f_2+f_3+f_4-13)f_5^2\,t_{125}t_{345} \\ &\quad+\Big((10f_3+10f_4-f_1(f_3+f_4-8)-f_2(f_3+f_4-8)-51)f_5 \\ &\qquad+(-5f_1-5f_2+21)(f_3+f_4-3)\Big)t_{125}t_{345} \\ &\quad+\frac{(f_3-f_4)\left((f_2-f_5)^2-f_1^2\right)(f_3+f_4-f_5)}{2f_5}a_{125}a_{345}. \end{aligned} \tag{3.1.27}$$

These new reduction formulae are used as an intermediate check when computing the correlator.

Proof

Let us show how the new formula (3.1.23) can be obtained[4]. To do the calculation we need to use

$$\begin{aligned} &c_{125} = \int \nabla^\alpha\nabla^\beta Y^{I_1}\nabla_\alpha\nabla_\beta Y^{I_2}Y^{I_5} = \frac{1}{4}(f_1+f_2-f_5-8)(f_1+f_2-f_5)\,a_{125}, \\ &\mu_{125} = \frac{f_2-f_1}{f_5-5}t_{125}, \quad \nu_{125} = \frac{5}{4f_5(f_5-5)}d_{125}, \\ &d_{125} = \frac{1}{20}\left(-5(f_1-f_2)^2+3f_5^2+2(f_1+f_2)f_5\right)a_{125}. \end{aligned} \tag{3.1.28}$$

[4]We thank Sergey Frolov for sharing this calculation with us.

which are used in the relation

$$\nabla_{(\alpha}Y^1\nabla_{\beta)}Y^2 = p_{125}Y^5_{(\alpha\beta)} + \mu_{125}\nabla_{(\alpha}Y^5_{\beta)} + \nu_{125}\nabla_{(\alpha}\nabla_{\beta)}Y^5 \,., \tag{3.1.29}$$

where from now on the round bracket denote *traceless* symmetrization. Then we need the formulae

$$\begin{aligned}
&\nabla^2 Y^5 = -f_5 Y^5\,, \quad \nabla^2\nabla_\alpha Y^5 = (4-f_5)\nabla_\alpha Y^5\,,\\
&\nabla^2\nabla_\alpha\nabla_\beta Y^5 = (10-f_5)\nabla_\alpha\nabla_\beta Y^5 + 2g_{\alpha\beta}f_5 Y^5\,,\\
&\nabla^2\nabla_{(\alpha}\nabla_{\beta)}Y^5 = (10-f_5)\nabla_{(\alpha}\nabla_{\beta)}Y^5\,,\\
&\nabla^2 Y^5_\alpha = (1-f_5)Y^5_\alpha\,, \quad \nabla^2\nabla_\alpha Y^5_\beta = (5-f_5)\nabla_\alpha Y^5_\beta + 2\nabla_\beta Y^5_\alpha\,,\\
&\nabla^2\nabla_{(\alpha}Y^5_{\beta)} = (7-f_5)\nabla_{(\alpha}Y^5_{\beta)}\,, \quad \nabla^2 Y^5_{(\alpha\beta)} = (2-f_5)Y^5_{(\alpha\beta)}\,.
\end{aligned} \tag{3.1.30}$$

These formulae and (3.1.29) give

$$(2-f_5)p_{125}Y^5_{(\alpha\beta)} = \nabla^2\big(\nabla_{(\alpha}Y^1\nabla_{\beta)}Y^2\big) - (7-f_5)\mu_{125}\nabla_{(\alpha}Y^5_{\beta)} - (10-f_5)\nu_{125}\nabla_{(\alpha}\nabla_{\beta)}Y^5\,, \tag{3.1.31}$$

and

$$\begin{aligned}
(2-f_5)^2 p_{125}p_{345} = &\int \nabla^2\big(\nabla_{(\alpha}Y^1\nabla_{\beta)}Y^2\big)\nabla^2\big(\nabla_{(\alpha}Y^3\nabla_{\beta)}Y^4\big)\\
&- (7-f_5)^2\mu_{125}\int \nabla_{(\alpha}Y^5_{\beta)}\nabla_{(\alpha}Y^3\nabla_{\beta)}Y^4 - (7-f_5)^2\mu_{345}\int \nabla_{(\alpha}Y^5_{\beta)}\nabla_{(\alpha}Y^1\nabla_{\beta)}Y^2\\
&- (10-f_5)^2\nu_{125}\int \nabla_{(\alpha}\nabla_{\beta)}Y^5\nabla_{(\alpha}Y^3\nabla_{\beta)}Y^4 - (10-f_5)^2\nu_{345}\int \nabla_{(\alpha}\nabla_{\beta)}Y^5\nabla_{(\alpha}Y^1\nabla_{\beta)}Y^2\\
&- \frac{1}{2}(5-f_5)(7-f_5)^2\mu_{125}\mu_{345} - \frac{4}{5}f_5(5-f_5)(10-f_5)^2\nu_{125}\nu_{345}\,,
\end{aligned} \tag{3.1.32}$$

where we took into account that

$$\begin{aligned}
&\int \nabla_{(\alpha}Y^5_{\beta)}\nabla_{(\alpha}Y^5_{\beta)} = -\frac{1}{2}(5-f_5)\,,\\
&\int \nabla_{(\alpha}\nabla_{\beta)}Y^5\nabla_{(\alpha}\nabla_{\beta)}Y^5 = -\frac{4}{5}f_5(5-f_5)\,,\\
&\int \nabla_{(\alpha}Y^5_{\beta)}\nabla_{(\alpha}\nabla_{\beta)}Y^5 = 0\,.
\end{aligned} \tag{3.1.33}$$

Using integration by parts and the definitions (3.1.1) and

$$b_{123} = \int \nabla^\alpha Y^{I_1}\nabla_\alpha Y^{I_2}Y^{I_3} = \tfrac{1}{2}\,(f_1+f_2-f_3)\,a_{123} \tag{3.1.34}$$

we can reduce the second and the third line to t and a contributions:

$$\begin{aligned}
(2-f_5)^2 p_{125}p_{345} &= \int \nabla^2\big(\nabla_{(\alpha}Y^1\nabla_{\beta)}Y^2\big)\nabla^2\big(\nabla_{(\alpha}Y^3\nabla_{\beta)}Y^4\big) + R_{1234},\\
R_{1234} &= -\frac{1}{2}(10-f_5)^2\nu_{125}(f_3b_{453} + f_4b_{354} - f_5b_{345})\\
&\quad - \frac{1}{2}(10-f_5)^2\nu_{345}(f_1b_{251} + f_2b_{152} - f_5b_{125})\\
&\quad + \frac{1}{2}(5-f_5)(7-f_5)^2\mu_{125}\mu_{345} - \frac{4}{5}f_5(5-f_5)(10-f_5)^2\nu_{125}\nu_{345}\\
&\quad - \frac{f_5}{5}(10-f_5)^2(b_{125}\nu_{345} + \nu_{125}b_{345})\,.
\end{aligned} \tag{3.1.35}$$

So, the main problem is to evaluate

$$P_{1234} \equiv \int \nabla^2\big(\nabla_{(\alpha} Y^1 \nabla_{\beta)} Y^2\big) \nabla^2\big(\nabla_{(\alpha} Y^3 \nabla_{\beta)} Y^4\big). \tag{3.1.36}$$

We have

$$\begin{aligned}
P_{1234} &= \frac{1}{2} \int \nabla^2\big(\nabla_\alpha Y^1 \nabla_\beta Y^2\big) \nabla^2\big(\nabla_\alpha Y^3 \nabla_\beta Y^4 + \nabla_\alpha Y^4 \nabla_\beta Y^3 - \frac{2}{5} g_{\alpha\beta} \nabla_\gamma Y^3 \nabla_\gamma Y^4\big) \\
&= \frac{1}{2} \int \nabla^2\big(\nabla_\alpha Y^1 \nabla_\beta Y^2\big) \nabla^2\big(\nabla_\alpha Y^3 \nabla_\beta Y^4 + \nabla_\alpha Y^4 \nabla_\beta Y^3\big) \\
&\quad - \frac{1}{5} \int \nabla^2\big(\nabla_\alpha Y^1 \nabla_\alpha Y^2\big) \nabla^2\big(\nabla_\gamma Y^3 \nabla_\gamma Y^4\big) \\
&= \frac{1}{2} \int \nabla^2\big(\nabla_\alpha Y^1 \nabla_\beta Y^2\big) \nabla^2\big(\nabla_\alpha Y^3 \nabla_\beta Y^4 + \nabla_\alpha Y^4 \nabla_\beta Y^3\big) - \frac{1}{5} f_5^2 b_{125} b_{345}\,, \\
P_{1234} &= \frac{1}{2} \int \Big((8 - f_1 - f_2) \nabla_\alpha Y^1 \nabla_\beta Y^2 + 2 \nabla_\gamma \nabla_\alpha Y^1 \nabla_\gamma \nabla_\beta Y^2\Big) \\
&\qquad \times \Big((8 - f_3 - f_4) \nabla_\alpha Y^3 \nabla_\beta Y^4 + 2 \nabla_\rho \nabla_\alpha Y^3 \nabla_\rho \nabla_\beta Y^4 \\
&\qquad\quad + (8 - f_3 - f_4) \nabla_\alpha Y^4 \nabla_\beta Y^3 + 2 \nabla_\rho \nabla_\alpha Y^4 \nabla_\rho \nabla_\beta Y^3\Big) - \frac{1}{5} f_5^2 b_{125} b_{345}, \\
P_{1234} &= P^{(8)}_{1234} + P^{(6)}_{1234} + P^{(4)}_{1234} - \frac{1}{5} f_5^2 b_{125} b_{345}\,,
\end{aligned} \tag{3.1.37}$$

where we denote

$$\begin{aligned}
P^{(4)}_{1234} &\equiv \frac{1}{2} \int (8 - f_1 - f_2)(8 - f_3 - f_4) \nabla_\alpha Y^1 \nabla_\beta Y^2 \big(\nabla_\alpha Y^3 \nabla_\beta Y^4 + \nabla_\alpha Y^4 \nabla_\beta Y^3\big)\,, \\
P^{(6)}_{1234} &\equiv \int \Big((8 - f_1 - f_2)\big(\nabla_\alpha Y^1 \nabla_\beta Y^2 \nabla_\rho \nabla_\alpha Y^3 \nabla_\rho \nabla_\beta Y^4 + \nabla_\alpha Y^1 \nabla_\beta Y^2 \nabla_\rho \nabla_\alpha Y^4 \nabla_\rho \nabla_\beta Y^3\big) \\
&\quad + (8 - f_3 - f_4)\big(\nabla_\alpha Y^3 \nabla_\beta Y^4 \nabla_\gamma \nabla_\alpha Y^1 \nabla_\gamma \nabla_\beta Y^2 + \nabla_\alpha Y^4 \nabla_\beta Y^3 \nabla_\gamma \nabla_\alpha Y^1 \nabla_\gamma \nabla_\beta Y^2\big)\Big)\,, \\
P^{(8)}_{1234} &\equiv 2 \int \nabla_\gamma \nabla_\alpha Y^1 \nabla_\gamma \nabla_\beta Y^2 \big(\nabla_\rho \nabla_\alpha Y^3 \nabla_\rho \nabla_\beta Y^4 + \nabla_\rho \nabla_\alpha Y^4 \nabla_\rho \nabla_\beta Y^3\big)\,.
\end{aligned} \tag{3.1.38}$$

We immediately find

$$P^{(4)}_{1234} = \frac{1}{2}(8 - f_1 - f_3)(8 - f_2 - f_4)(b_{135} b_{245} + b_{145} b_{235})\,. \tag{3.1.39}$$

To reduce the six-derivative terms we use

$$\begin{aligned}
&\nabla_\beta \nabla_\alpha Y^1 \nabla_\beta Y^2 = t^{(2)}_{125}\, Y^5_\alpha + b^{(2)}_{152}\, \nabla_\alpha Y^5\,, \\
&t^{(2)}_{125} = \frac{1}{2}(f_1 + f_2 - f_5 - 3) t_{125}\,, \\
&b^{(2)}_{152} = \frac{(f_1 + f_2 - f_5)(f_1 - f_2 + f_5)}{4 f_5} a_{125}\,.
\end{aligned} \tag{3.1.40}$$

Then we get

$$\begin{aligned}
P^{(6)}_{1234} &= (8 - f_1 - f_2)\big(t^{(2)}_{135} t^{(2)}_{245} + f_5 b^{(2)}_{351} b^{(2)}_{452} + t^{(2)}_{145} t^{(2)}_{235} + f_5 b^{(2)}_{451} b^{(2)}_{352}\big) \\
&\quad + (8 - f_3 - f_4)\big(t^{(2)}_{135} t^{(2)}_{245} + f_5 b^{(2)}_{153} b^{(2)}_{254} + t^{(2)}_{145} t^{(2)}_{235} + f_5 b^{(2)}_{154} b^{(2)}_{253}\big).
\end{aligned} \tag{3.1.41}$$

Introducing

$$Y_{1234} = \int \nabla_\gamma \nabla_\alpha Y^1 \nabla_\gamma \nabla_\beta Y^2 \nabla_\rho \nabla_\alpha Y^3 \nabla_\rho \nabla_\beta Y^4 \,, \tag{3.1.42}$$

we find the last term

$$P^{(8)}_{1234} = 2(Y_{1234} + Y_{1243}) \,. \tag{3.1.43}$$

It remains to compute $Y_{12[34]} \equiv \frac{1}{2}\,(Y_{1234} - Y_{1243})$. Integration by parts gives:

$$\begin{aligned} Y_{12[34]} = Y_{[12]34} =& \nabla_\alpha \nabla_\beta Y^{[1} \nabla_\gamma \nabla^\beta Y^{2]} \nabla^\alpha \nabla_\delta Y^3 \nabla^\gamma \nabla^\delta Y^4 \\ =& - \nabla_\gamma \nabla_\beta \nabla_\alpha Y^{[1} \nabla^\gamma \nabla^\beta Y^{2]} \nabla^\alpha \nabla_\delta Y^3 \nabla^\delta Y^4 \\ & - \nabla_\alpha \nabla_\beta Y^{[1} \nabla^2 \nabla^\beta Y^{2]} \nabla^\alpha \nabla_\delta Y^3 \nabla^\delta Y^4 \\ & - \nabla_\alpha \nabla_\beta Y^{[1} \nabla_\gamma \nabla^\beta Y^{2]} \nabla^\gamma \nabla^\alpha \nabla_\delta Y^3 \nabla^\delta Y^4 \,. \end{aligned} \tag{3.1.44}$$

With the help of the following identity:

$$[\nabla_\alpha \,,\, \nabla_\beta]\, \xi_\gamma = g_{\alpha\gamma} \xi_\beta - g_{\beta\gamma} \xi_\alpha \,, \tag{3.1.45}$$

one finds the last line in (3.1.44):

$$\begin{aligned} \nabla_\alpha \nabla_\beta Y^{[1} \nabla_\gamma \nabla^\beta Y^{2]} \nabla^\gamma \nabla^\alpha \nabla_\delta Y^3 \nabla^\delta Y^4 &= \frac{1}{2} \nabla_\alpha \nabla_\beta Y^{[1} \nabla_\gamma \nabla^\beta Y^{2]} [\nabla^\gamma, \nabla^\alpha] \nabla_\delta Y^3 \nabla^\delta Y^4 \\ &= \nabla_\alpha \nabla_\beta Y^{[1} \nabla_\gamma \nabla^\beta Y^{2]} \nabla^\alpha Y^{[3} \nabla^\gamma Y^{4]} \,. \end{aligned}$$

Finally, using

$$\begin{aligned} & \nabla_\gamma \nabla_\beta \nabla_\alpha Y^1 \nabla_\gamma \nabla_\beta Y^2 = t^{(4)}_{125}\, Y^5_\alpha + b^{(4)}_{152}\, \nabla_\alpha Y^5 \,, \\ & t^{(4)}_{125} = \frac{1}{4}(f_1 + f_2 - f_5 - 3)(f_1 + f_2 - f_5 - 13) t_{125} + f_1 t_{125} \,, \\ & b^{(4)}_{152} = \frac{(f_1 + f_2 - f_5)(f_1 + f_2 - f_5 - 10)}{4 f_5} b_{152} - \frac{f_1}{f_5} b_{251} \,. \end{aligned} \tag{3.1.46}$$

we find

$$\begin{aligned} Y_{12[34]} =& -\frac{1}{2}(t^{(4)}_{125} - t^{(4)}_{215}) t^{(2)}_{345} - \frac{1}{2} f_5 (b^{(4)}_{152} - b^{(4)}_{251}) b^{(2)}_{354} \\ & - \frac{1}{2}\big((4 - f_2) t^{(2)}_{125} - (4 - f_1) t^{(2)}_{215}\big) t^{(2)}_{345} - \frac{1}{2} f_5 \big((4 - f_2) b^{(2)}_{152} - (4 - f_1) b^{(2)}_{251}\big) b^{(2)}_{354} \\ & - \frac{1}{2}(t^{(2)}_{135} t^{(2)}_{245} - t^{(2)}_{145} t^{(2)}_{235}) - \frac{1}{2} f_5 (b^{(2)}_{153} b^{(2)}_{254} - b^{(2)}_{154} b^{(2)}_{253}) \,. \end{aligned} \tag{3.1.47}$$

We would also like to note, that the method, similar to the one used in appendix B, gives precisely the same relations between cubic *tt*'s, like those derived from the relations for quadratic *pp*'s.

3.2 Harmonic polynomial formalism

However, as was already mentioned, the computation of the *a*-, *p*- and *t*-tensors simplifies incredibly in the *harmonic polynomial* formalism, developed in [40, 41] and applied to compute some supergravity correlators in [18, 19]. It turns out that, after an appropriate normalization, the a-, p- and t-tensors can be expressed as harmonic polynomials $Y_{nm}^{(a,b)}(\sigma,\tau)$ in the new variables σ and τ

$$\sigma \equiv \frac{t_{13}t_{24}}{t_{12}t_{34}} \qquad \text{and} \qquad \tau \equiv \frac{t_{14}t_{23}}{t_{12}t_{34}}, \qquad \text{where} \qquad t_{ij} \equiv t_i \cdot t_j, \tag{3.2.1}$$

which carry the non-trivial dependence on the null vectors t_i. These functions are generalized eigenfunctions of the SO(6) Casimir operator L^2, satisfying

$$L^2 \left(t_{14}^a t_{24}^b Y_{nm}^{(a,b)}(\sigma,\tau) \right) = -2C_{nm} t_{14}^a t_{24}^b Y_{nm}^{(a,b)}(\sigma,\tau), \tag{3.2.2}$$

with C_{nm} being the corresponding eigenvalue. Moreover, one can solve this equation [41] and find that the $Y_{nm}^{(a,b)}$ can be expressed explicitly in terms of Jacobi polynomials $P_n^{(a,b)}$:

$$Y_{nm}^{(a,b)}(\sigma,\tau) = \frac{2(n+1)!(a+b+n+1)!}{(a+1)_m(b+1)_m(a+b+2n+2)!} P_{nm}^{(a,b)}(\sigma,\tau)\,, \tag{3.2.3}$$

where $(\ldots)_m$ is the usual Pochhammer symbol and

$$P_{nm}^{(a,b)}(y,\bar{y}) = \frac{P_{n+1}^{(a,b)}(y)P_m^{(a,b)}(\bar{y}) - P_m^{(a,b)}(y)P_{n+1}^{(a,b)}(\bar{y})}{y-\bar{y}}\,, \tag{3.2.4}$$

which can be related to the original σ and τ variables via

$$\sigma = \frac{1}{4}(y+1)(\bar{y}+1) \qquad \text{and} \qquad \tau = \frac{1}{4}(1-y)(1-\bar{y}). \tag{3.2.5}$$

It was discussed in [19], that the product of C-tensors appearing in the product of scalar $a_{125}a_{345}$, vector $t_{125}t_{345}$ and tensor $p_{125}p_{345}$ harmonics for arbitrary weights with fixed exchange leg k_5 are proportional to these $Y_{nm}^{(a,b)}$:

$$\begin{aligned}
a_{125}a_{345} \sim \;& \langle C_{k_1}^{I_1} C_{k_2}^{I_2} C_{[0,a+b+2m,0]}^{I} \rangle \langle C_{k_3}^{I_3} C_{k_4}^{I_4} C_{[0,a+b+2m,0]}^{I} \rangle = \;& \mathcal{T}\mathcal{B}_a Y_{mm}^{(a,b)}, \\
t_{125}t_{345} \sim \;& \langle C_{k_1}^{I_1} C_{k_2}^{I_2} C_{[1,a+b+2m,1]}^{I} \rangle \langle C_{k_3}^{I_3} C_{k_4}^{I_4} C_{[1,a+b+2m,1]}^{I} \rangle = \;& \mathcal{T}\mathcal{B}_t Y_{m+1,m}^{(a,b)}, \\
p_{125}p_{345} \sim \;& \langle C_{k_1}^{I_1} C_{k_2}^{I_2} C_{[2,a+b+2m,2]}^{I} \rangle \langle C_{k_3}^{I_3} C_{k_4}^{I_4} C_{[2,a+b+2m,2]}^{I} \rangle = \;& \mathcal{T}\mathcal{B}_p Y_{m+2,m}^{(a,b)},
\end{aligned} \tag{3.2.6}$$

where the t-dependent prefactor $\mathcal{T}$ is given by

$$\mathcal{T} = t_{12}^{k_3} t_{13}^{b} t_{14}^{a} t_{34}^{\frac{k_1+k_2+k_3-k_4}{2}} \tag{3.2.7}$$

and k_5 satisfies

$$k_5 = a+b+2m, \quad k_5 = a+b+2m+1 \quad \text{or} \quad k_5 = a+b+2m+2 \tag{3.2.8}$$

for some nonnegative integer m respectively. The proportionality coefficients $\mathcal{B}$ were worked out in [19]: given a set of weights k_1, k_2, k_3, k_4 ordered such that

$$a = \frac{k_1+k_4-k_2-k_3}{2}, \quad b = \frac{k_2+k_4-k_1-k_3}{2} \tag{3.2.9}$$

are nonnegative[5] and an intermediate weight k_5 satisfying (3.2.8) they take the following form:

$$\begin{aligned} \mathcal{B}_a =& \frac{\alpha_{251}!\alpha_{512}!\alpha_{453}!\alpha_{534}!}{a!b!k_5!}, \\ \mathcal{B}_t =& \frac{(k_5+1)(\alpha_{251}-\frac{1}{2})!(\alpha_{512}-\frac{1}{2})!(\alpha_{453}-\frac{1}{2})!(\alpha_{534}-\frac{1}{2})!}{a!b!k_5!}, \\ \mathcal{B}_p =& 2^4 \cdot \frac{\alpha_{251}!\alpha_{512}!\alpha_{453}!\alpha_{534}!}{a!b!k_5!(k_5+1)!}, \end{aligned} \tag{3.2.10}$$

where $\alpha_{123} = \frac{k_1+k_2-k_3}{2}$. This now allows for a straightforward evaluation of $a_{125}a_{345}$, $t_{125}t_{345}$ and $p_{125}p_{345}$ for any weights as all the complicated tensor structure is captured by Jacobi polynomials. To obtain the corresponding tensors in the t and u channel, e.g. $a_{135}a_{245}$ and $a_{145}a_{235}$, one simply reshuffles the t_i.

[5]This might require a shuffle $\{k_1, k_2, k_3, k_4\} \to \{k_3, k_4, k_1, k_2\}$, which is certainly possible in all cases.

Chapter 4

Analyzing the results

4.1 Normalisation

Before passing to discussion of the results, we must mention the final step in computing the *normalized connected* four-point function. One has to multiply the sum of the contact part mentioned above(1.1.3) and the exchange part (2.1.1) by the following overall coefficient:

$$\begin{gathered} \frac{N^2}{8\pi^5} \prod_{k\in\{k_1,\ldots,k_4\}} C_{O_k} C_k, \\ C_{O_2} = \frac{4\sqrt{2}\pi^2}{N}, \quad C_{O_k} = \frac{2\sqrt{2}\pi^2}{N(k-2)\sqrt{(k-1)}}, \quad k > 2, \end{gathered} \tag{4.1.1}$$

where C_k are given in (1.1.7) and the coefficients C_{O_k} provide the canonical normalization for the two-point functions [14], [38]. Note also that the minus sign coming from the Euclidean version of the AdS action compensates the minus sign in (3.1.2).

4.2 Results

The aforementioned simplifications of the algorithm, sections 1 and 2, in combination with the harmonic polynomial formalism, reviewed in section 3.2, allow one to compute supergravity four-point functions of 1/2–BPS operators in (2.5.2) of any reasonably given weights in very little time. We implement the entire algorithm in *Mathematica* and compute all the non-trivial connected four-point functions $\langle k_1 k_2 k_3 k_4 \rangle$ with $2 \leqslant k_1 \leqslant k_2 \leqslant k_3 \leqslant k_4 \leqslant 8$ (94 in total and including 64 previously unknown correlators), which can be found in the database attached to the publication [1]. Additionally, we compute two very high-weight cases, namely $\langle 7\,10\,12\,17 \rangle$ and $\langle 17\,21\,23\,25 \rangle$. The computation of these latter correlation functions takes 1 minute and 40 minutes, respectively, on a standard computer. For each correlation function the database contains a subfolder with the name $k_1\_k_2\_k_3\_k_4$ with up to five plain `txt` files:

- `Fullcorrelator`$k_1\_k_2\_k_3\_k_4$`.txt` contains the full correlator as we compute it directly from the action,

- `Freepart`$k_1\_k_2\_k_3\_k_4$`.txt` contains the free part as we extract it from consistency with superconformal symmetry ,

- Hk_1_k_2_k_3_k_4`.txt` contains a coordinate-space expression for the dynamical function $\mathcal{H}$ in the notation of [20, 22] as it follows from our direct computation,[1]
- HfromMellink_1_k_2_k_3_k_4`.txt` contains a much shorter coordinate-space expression for $\mathcal{H}$ in the notation of [20, 22] and has been derived from its Mellin-space form (the construction of which we discuss in the appendix),
- RZconjk_1_k_2_k_3_k_4`.txt` contains two entries: if this four-point function coincides with the Mellin conjecture the first entry is `yes` and if not it would read `no`. The second entry is the value for the overall scaling function $f(k_1, k_2, k_3, k_4)$.

4.3 Consistency with the structure

The direct results for the correlators are very complicated, warranting the need for good checks of the result. Apart from checking the expressions for the coupling building blocks known as a-, p- and t-tensors using the identities in section 3.1, the most important check of the final result is verifying consistency with the structure explained in section 2.5.

In order to do this, we need to separately compute the free part of the correlator $\langle k_1 k_2 k_3 k_4 \rangle_0$. In principle, this computation is straightforward, but a subtlety concerning the identification of field theory operators with supergravity fields requires a discussion. We will address this subtlety – due to the presence of *extended operators* – in section 4.4.

Assuming we have computed $\langle k_1 k_2 k_3 k_4 \rangle_0$ we can find the interacting part of the correlator. For the interacting part we can check whether it obeys the structure given in (2.5.9). This goes as follows: suppose the interacting part is provided as

$$\sum_{l=1}^{|\{a\}|} \alpha_l \, T_l \tag{4.3.1}$$

and we write the structure in (2.5.9) as

$$\sum_{m=1}^{|\{a\}|} \beta_m \, T_m, \tag{4.3.2}$$

where the β_m follow explicitly from (2.5.9) and depend linearly on the $\mathcal{F}_b$. Since we know that the number of independent functions describing the correlator is $|\{b\}|$, which is always strictly smaller than $|\{a\}|$, equating the expressions (4.3.1) and (4.3.2) yields an overdetermined system of equations for the unknowns $\mathcal{F}_b$. The fact that our results satisfy this system provides a highly non-trivial check of the correlators.

In fact, quite similar to the coordinate space method from [20], this system allows one to determine many of the numbers one has to compute during the computation. For example, in all our cases, this provided an independent check of the symmetry factors in (2.1.1), and with a small modification, we used it to verify the free part of our correlators.

We can further analyze our results by comparing it to the closed Mellin-space formula that was conjectured for any four-point function of 1/2-BPS operators [22]. This conjecture is based on physical arguments and consistency with all the known results and subsequently tested on one new result (the $\langle 5555 \rangle$ correlator) [20]. The computed

[1] This expression has not been simplified and is therefore much longer than what is predicted from the Mellin-space conjecture.

correlators, therefore, provide a new test of this conjecture, in particular, because of their genericness. We also performed this check for all our cases. We obtained a match between the expressions coming from our results and those from the conjecture. See chapter 5 for more details on the Mellin-space correlators.

4.4 Computing the free part: extended operators

In principle the computation of the free part is a straightforward procedure. We can compute it in the field theory picture using the (free) operators dual to the scalar fields $s_k^{I_k}$, simply performing the relevant Wick contractions to obtain the free correlator. For small $k < 4$ the correspondence between fields and operators is simple, namely $s_k^{I_k} \sim \mathcal{O}_k^{I_k}$ where, remembering the decomposition (2.5.1), the field dependence is given by

$$\mathcal{O}^{i_1 \ldots i_k} = \kappa_k \mathrm{Tr}\left(\phi^{i_1} \ldots \phi^{i_k}\right), \tag{4.4.1}$$

where κ_k is the k-dependent normalization determined by demanding canonical two-point functions which in the planar limit can be taken to be $\kappa_k = \sqrt{2^k/(kN^k)}$. The operators with this exact field-dependence are known as single-trace and we distinguish them by omitting the tilde. However, as first noticed in [42] and later further analyzed in [24] this correspondence cannot hold when $k \geqslant 4$: the fact that extremal three-point functions $\langle s_{k_1}^{I_1} s_{k_2}^{I_2} s_{k_3}^{I_3} \rangle$ vanish when computed from the supergravity lagrangian, whereas the quantity

$$\langle \mathcal{O}_{k_1}^{I_1} \mathcal{O}_{k_2}^{I_2} \mathcal{O}_{k_3}^{I_3} \rangle \tag{4.4.2}$$

in the field theory in general is non-vanishing shows that this correspondence cannot continue to hold. The resolution presented in [24] is that the scalar fields are not dual to single-trace, but to so-called *extended* operators[2]:

$$s_{k_1}^{I_1} \sim \widetilde{\mathcal{O}}_{k_1}^{I_1} = \mathcal{O}_{k_1}^{I_1} - \frac{1}{2N} \sum_{\substack{k_2, k_3 \geqslant 2 \\ k_2 + k_3 = k_1}} C^{I_1 I_2 I_3} \mathcal{O}_{k_2}^{I_2} \mathcal{O}_{k_3}^{I_3}, \tag{4.4.3}$$

where $C^{I_1 I_2 I_3} = \sqrt{k_1 k_2 k_3} \langle C^{I_1} C^{I_2} C^{I_3} \rangle$, defined in section 3.1. For convenience we also give the decomposition (2.5.1) for the extended case: decomposing as in (2.5.1) we can write the field-dependence as

$$\widetilde{\mathcal{O}}^{i_1, \ldots, i_{k_1}} = \kappa_{k_1} \mathrm{Tr}\left(\phi^{i_1} \ldots \phi^{i_{k_1}}\right) - \sum_{\substack{k_2, k_3 \geqslant 2 \\ k_2 + k_3 = k_1}} \frac{\lambda_{k_1, k_2, k_3}}{2N} \kappa_{k_2} \kappa_{k_3} \mathrm{Tr}\left(\phi^{i_1} \ldots \phi^{i_{k_2}}\right) \mathrm{Tr}\left(\phi^{i_{k_2+1}} \ldots \phi^{i_{k_1}}\right), \tag{4.4.4}$$

where $\lambda_{k_1, k_2, k_3} = C^{I_1 I_2 I_3}$ and where we omitted symmetrization over all the indices since this is enforced by contraction with the t vectors. Even though the prescription for $\widetilde{\mathcal{O}}_{k_1}^{I_1}$ is completely explicit it can be quite non-trivial to compute the coefficient directly, due to the complexity of the required tensor contractions. Luckily we can circumvent this by noting that the extended operators are required to have vanishing extremal three point functions. Since the number of terms in the summation is exactly equal to the number of

[2] One could alternatively leave the field operators unaltered and modify the supergravity lagrangian instead, by a field redefinition that adds boundary terms such that the three-point correlation functions reflect the field theory result [24].

extremal three-point functions containing $\widetilde{\mathcal{O}}^{I_k}$ one can find the coefficients by demanding vanishing of these three-point functions. For example, we can list the values of the λ for the first few cases:

$$\lambda_{422} = 4, \quad \lambda_{532} = 2\sqrt{30}, \quad \lambda_{642} = 8\sqrt{3}, \quad \lambda_{633} = 3\sqrt{6}, \tag{4.4.5}$$

thereby completely defining the extended operators dual to s_4, s_5 and s_6.

The free part of the supergravity correlators should generically be computed using Wick contractions of the extended operators and then taking the large N limit. However, the leading (planar) order in this computation follows from general considerations of the topology of the diagram combined with some combinatorics. In particular, from these considerations it follows that for connected diagrams the effect of the presence of extended operators was undetectable except for the extremal cases, such that in practice one did not have to consider this complication. Indeed, for the $\langle 4444 \rangle$ correlator, one of the few known correlators for which the weights are high enough to potentially feel this effect, the free part was computed in [16] without explicitly tracking it. Consistency with superconformal symmetry was shown, thereby indicating that the fact that the operators are extended should not play a role. However, the first signs that one should take this effect seriously were presented in the paper [18], that discusses the family of correlators of the form $\langle 22nn \rangle$ for $n \geqslant 2$: already there it was noted that there exists a discrepancy between the free part of the correlator as computed from supergravity as opposed to the result from field theory using non-extended operators when $n > 3$, but its origin remained unexplained. It was argued in [20] that this discrepancy is resolved by computing the free part using extended operators. We confirm this with the explicit computation of the $\langle 22nn \rangle$ correlator for $n = 4, 5, 6$, for which we conclude that the presence of extended operators does play a role.

Moreover, since the computation only concerns the planar diagrams it is possible to prove when it is necessary to take into account that the operators are extended: in the planar limit, only the leading order in N of the free correlator described by a diagram is relevant. From basic observations, we know that, when depicted using the double-line notation, this leading order goes as N^I with I the number of index loops [31]. Extendedness of an operator adds to a diagram a contribution of a second diagram, in which one of the vertices has been split into two parts. As an example, splitting the third vertex looks like

$$A_{k_1,k_2,k_3,k_4} \to A_{k_1,k_2,k_3,k_4} - \frac{\#}{N} A_{k_1,k_2,k_3^{(1)},k_3^{(2)},k_4}, \tag{4.4.6}$$

where $k_3^{(1)} + k_3^{(2)} = k_3$. In terms of the index loops, splitting the vertex will generically reduce the number of index loops by 1, which when combined with the extra $1/N$ in front of the second diagram implies that its contribution is subleading and can be discarded. Only in the special case in which one of the vertices is singly-connected to the rest of the diagram and can be split off completely by one of the extensions – hence yielding a disconnected diagram – can the effect be leading: in that case the number of index loops increases by 1 due to the splitting, which when compensated by the $1/N$ prefactor yields a contribution to the leading term and hence to the planar free correlator. We have illustrated these ideas in fig. 4.1.

In particular, it follows that the presence of extended operators plays a role for all the correlators presented in the previous section. For the more empirically inclined reader we

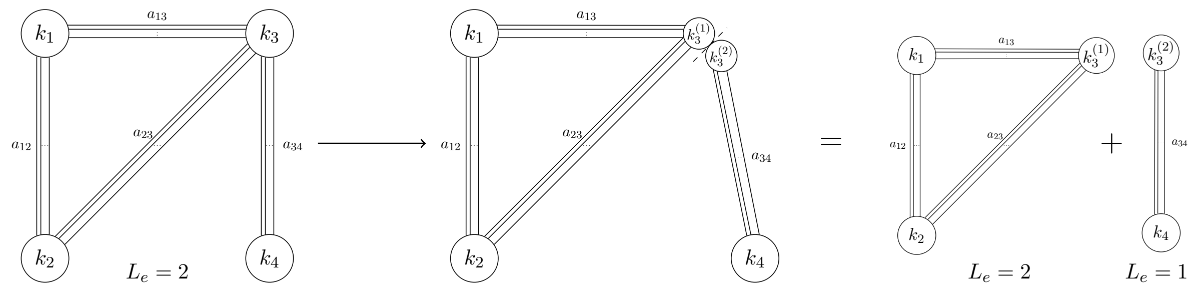

Figure 4.1: An example of a singly-connected graph which has more index loops after splitting: the number of external index loops L_e (loops outside of the lines that connect the nodes) before splitting is 2, whereas after splitting the total number is $2 + 1 = 3$.

provide an overview of the (extended and non-extended) free parts of all the non-trivial four-point functions with weights up to and including 5 in table 4.1. When a difference exists we list both the result for non-extended (upper part) and extended operators (lower part). Although in principle one can use combinatorics to compute the planar limit of the free part[3] we have computed them using a straightforward implementation in Mathematica of Wick contractions between scalars in $\mathcal{N} = 4$ SYM using the formulae in [43], with the exception of $\langle 5555 \rangle$ which we took from [23]. The correlators which are not listed vanish identically. This computation indeed showes that the Kaluza-Klein modes $s_k^{I_k}$ are dual to the extended (4.4.4) rather than to single-trace operators, and together with the procedure discussed in the previous section verifies that the implemented algorithm gives the correct result.

[3] A hint of this fact can be found in the regularity of the appearing numbers: after factoring an overall constant only simple integers remain.

Correlator	Planar free part coefficients $\frac{1}{N^2}\{c_{a_l}\}$
$\langle 2222\rangle$	$\{N^2, 4, N^2, 4, 4, N^2\}$
$\langle 2233\rangle$	$\{0, 12, 0, 6, 6, N^2\}$
$\langle 2244\rangle$	$\{16, 24, 16, 8, 8, N^2\}$ $\{\ 0, 24,\ \ 0, 8, 8, N^2\}$
$\langle 2255\rangle$	$\{30, 40, 30, 10, 10, N^2\}$ $\{\ 0, 40,\ \ 0, 10, 10, N^2\}$
$\langle 2334\rangle$	$\sqrt{2}\{12, 12, 0, 12, 6, 0\}$ $\sqrt{2}\{\ \ 0, 12, 0, 12, 6, 0\}$
$\langle 2345\rangle$	$\sqrt{30}\{6, 6, 4, 4, 2, 0\}$ $\sqrt{30}\{0, 6, 0, 4, 2, 0\}$
$\langle 2444\rangle$	$\sqrt{2}\{16, 16, 16, 16, 16, 16\}$ $\sqrt{2}\{\ \ 0, 16,\ \ 0, 16, 16,\ \ 0\}$
$\langle 2455\rangle$	$\sqrt{2}\{30, 30, 30, 20, 20, 20\}$ $\sqrt{2}\{\ \ 0, 30,\ \ 0, 20, 20,\ \ 0\}$
$\langle 3333\rangle$	$\{N^2, 9, 9, N^2, 9, 18, 9, 9, 9, N^2\}$
$\langle 3344\rangle$	$\{0, 24, 24, 0, 12, 24, 12, 12, 12, N^2\}$
$\langle 3355\rangle$	$\{30, 45, 45, 30, 15, 30, 15, 15, 15, N^2\}$ $\{\ \ 0, 45, 45,\ \ 0, 15, 30, 15, 15, 15, N^2\}$
$\langle 3456\rangle$	$\sqrt{10}\{0, 18, 18, 0, 12, 12, 6, 12, 6, 0\}$
$\langle 3555\rangle$	$\sqrt{15}\{10, 10, 10, 10, 10, 10, 10, 10, 10, 10\}$ $\sqrt{15}\{\ \ 0, 10, 10,\ \ 0, 10, 10, 10, 10, 10,\ \ 0\}$
$\langle 4444\rangle$	$\{N^2, 16, 16, 16, N^2, 16, 32, 32, 16, 16, 32, 16, 16, 16, N^2\}$
$\langle 4455\rangle$	$\{0, 40, 40, 40, 0, 20, 40, 40, 20, 20, 40, 20, 20, 20, N^2\}$
$\langle 5555\rangle$	$\{N^2, 25, 25, 25, 25, N^2, 25, 50, 50, 50, 25, 25, 50, 50, 25, 25, 50, 25, 25, 25, N^2\}$

Table 4.1: Free parts of all non-trivial four-point correlators with weights up to 5 and that of $\langle 3456\rangle$ in the leading order of the planar limit, split to the coefficients in its decomposition (2.5.5) as a list $\frac{1}{N^2}\{c_{a_l}\}$ with $\frac{1}{N^2}$ factored out. If there is a difference between the correlator of non-extended operators and that of the extended ones we list both results with the non-extended correlator appearing first.

Chapter 5

Correlators in the Melling space

5.1 The Mellin amplitude

The idea to represent the CFT correlators in the Mellin space, where their analytic structure becomes transparent, was proposed by Mack in [44], and further studied in [45–51] in the framework of the AdS/CFT. As was observed, in the planar limit, the four-point Mellin amplitudes are simple meromorphic functions. Their poles and residues are entirely determined by two and three-point functions of single-trace operators. Further, based on symmetry, physical assumptions and the known results [13–19] , the authors of [20, 22] conjectured the general formula for the holographic correlators in the Mellin space. In this section, we sketch their work and present their results.

We first reorder the full correlator $\langle k_1 k_2 k_3 k_4\rangle$ such that the weights satisfy $k_1 \geqslant k_2 \geqslant k_3 \geqslant k_4$, and distinguish it by writing $G_{k_1k_2k_3k_4}$ instead. Then using the invariance under the conformal and R-symmetry groups, we write the correlator via the conformal, u and v, and R-symmetry, σ and τ, cross ratios:

$$G_{k_1k_2k_3k_4}(\vec{x},\vec{t}\,) = \prod_{i<j}\left(\frac{t_{ij}}{x_{ij}^2}\right)^{\gamma_{ij}^0}\left(\frac{t_{12}t_{34}}{x_{12}^2x_{34}^2}\right)^{L}\mathcal{G}_{k_1k_2k_3k_4}(u,v,\sigma,\tau)\,, \tag{5.1.1}$$

where the exponents γ_{ij}^0 are

$$\begin{gathered}\gamma_{12}^0 = \tfrac{1}{2}(k_1+k_2-k_3-k_4), \quad \gamma_{13}^0 = \tfrac{1}{2}(k_1+k_3-k_2-k_4)\,,\\ \gamma_{34}^0=\gamma_{24}^0=0, \quad \gamma_{14}^0 = k_4 - L\,,\\ \gamma_{23}^0 = k_4 - L - \tfrac{1}{2}(k_1+k_4-k_2-k_3)\,,\end{gathered} \tag{5.1.2}$$

and L is defined as follows:

$$L = \begin{cases} k_4, & \text{if } k_1+k_4 \leqslant k_2+k_3\\ \tfrac{1}{2}(k_2+k_3+k_4-k_1), & \text{if } k_1+k_4 > k_2+k_3\,.\end{cases} \tag{5.1.3}$$

The rest is defined as usual:

$$\begin{gathered} x_{ij} = x_i - x_j\,, \qquad t_{ij} = t_i\cdot t_j\,,\\ u=\tfrac{(x_{12})^2(x_{34})^2}{(x_{13})^2(x_{24})^2}, \quad v = \tfrac{(x_{14})^2(x_{23})^2}{(x_{13})^2(x_{24})^2}\,,\\ \sigma = \tfrac{t_{13}t_{24}}{t_{12}t_{34}}, \quad \tau = \tfrac{t_{14}t_{23}}{t_{12}t_{34}}\,.\end{gathered} \tag{5.1.4}$$

Next, one uses the invariance under the full superconformal group, which implies the superconformal Ward identity [27, 41]. It splits the correlator into the free (the result in free SYM theory) and dynamical part:

$$\mathcal{G}(u,v,\sigma,\tau) = \mathcal{G}_{\text{free}}(u,v,\sigma,\tau) + R\,\mathcal{H}(u,v,\sigma,\tau)\,, \tag{5.1.5}$$

where

$$R = \tau + (1-\sigma-\tau)v + (-\tau-\sigma\tau+\tau^2)u + (\sigma^2-\sigma-\sigma\tau)uv + \sigma v^2 + \sigma\tau u^2\,. \tag{5.1.6}$$

From now on we consider only the *connected* part of the four-point function, $\mathcal{G}_{\text{conn}}$, and define the Mellin amplitudes of $\mathcal{G}_{\text{conn}}$ and its dynamical part $\mathcal{H}$ as follows:

$$\mathcal{M}(s,t,\sigma,\tau) = \frac{M(s,t,\sigma,\tau)}{\Gamma_{k_1k_2k_3k_4}} \quad \text{and} \quad \widetilde{\mathcal{M}}(s,t,\sigma,\tau) = \frac{\widetilde{M}(s,t,\sigma,\tau)}{\widetilde{\Gamma}_{k_1k_2k_3k_4}}\,, \tag{5.1.7}$$

where

$$\begin{aligned} M(s,t,\sigma,\tau) &= \int_0^\infty\int_0^\infty du\,dv\, v^{-\frac{t}{2}+\frac{\min\{k_1+k_4,k_2+k_3\}}{2}-1} u^{-\frac{s}{2}+\frac{k_3+k_4}{2}-L-1}\,\mathcal{G}_{\text{conn}}(u,v,\sigma,\tau)\,, \\ \widetilde{M}(s,t,\sigma,\tau) &= \int_0^\infty\int_0^\infty du\,dv\, v^{-\frac{t}{2}+\frac{\min\{k_1+k_4,k_2+k_3\}}{2}-1} u^{-\frac{s}{2}+\frac{k_3+k_4}{2}-L-1}\,\mathcal{H}(u,v,\sigma,\tau)\,, \end{aligned} \tag{5.1.8}$$

and $\widetilde{\Gamma}_{k_1k_2k_3k_4}$ is obtained by replacing $u \equiv k_1+k_2+k_3+k_4-s-t \to \widetilde{u} = u-4$ in $\Gamma_{k_1k_2k_3k_4}$:

$$\begin{aligned} \Gamma_{k_1k_2k_3k_4} &= \Gamma[\frac{k_1+k_2-s}{2}]\Gamma[\frac{k_3+k_4-s}{2}]\Gamma[\frac{k_2+k_3-t}{2}] \\ &\quad \Gamma[\frac{k_1+k_4-t}{2}]\Gamma[\frac{k_1+k_3-u}{2}]\Gamma[\frac{k_2+k_4-u}{2}]\,. \end{aligned} \tag{5.1.9}$$

The authors of [20, 22] have analyzed the properties of the Mellin amplitudes:

- *Superconformal symmetry*, as was mentioned, implies the Ward identity. The latter translated to the Mellin space turns to the following identity the Mellin amplitudes must satisfy to:

$$\mathcal{M}(s,t,\sigma,\tau) = \widehat{R}\circ\widetilde{\mathcal{M}}(s,t,\sigma,\tau)\,, \tag{5.1.10}$$

where $\widehat{R}$ is the difference operator analogue of R, defined in (5.1.6):

$$\widehat{R} = \tau\,1 + (1-\sigma-\tau)\,\widehat{v} + (-\tau-\sigma\tau+\tau^2)\,\widehat{u} + (\sigma^2-\sigma-\sigma\tau)\,\widehat{uv} + \sigma\widehat{v^2} + \sigma\tau\,\widehat{u^2}\,, \tag{5.1.11}$$

with the following action of monomials

$$\begin{aligned} \widehat{U^mV^n}\circ\widetilde{\mathcal{M}}(s,t,\sigma,\tau) = \widetilde{\mathcal{M}}&(s-2m,t-2n,\sigma,\tau)\;\times \\ &\left(\tfrac{k_1+k_2-s}{2}\right)_m \left(\tfrac{k_1+k_3-u}{2}\right)_{2-m-n} \left(\tfrac{k_1+k_4-t}{2}\right)_n \\ &\left(\tfrac{k_2+k_3-t}{2}\right)_n \left(\tfrac{k_2+k_4-u}{2}\right)_{2-m-n} \left(\tfrac{k_3+k_4-s}{2}\right)_m\,, \end{aligned} \tag{5.1.12}$$

and $(a)_n = \frac{\Gamma[a+n]}{\Gamma[a]}$ denotes the Pochhammer symbol.

- *Bose symmetry* requires the Mellin amplitude $\mathcal{M}$ to be invariant under the permutation of the Mandelstam variables s, t and u when the external quantum numbers are also permuted accordingly. By construction, the amplitude $\widetilde{M}$ shares the same property under the permutation of the shifted Mandelstam variables s, t, and $\tilde{u}$. In the case of equal weights, $k_1 = ... = k_4$ this transforms to usual crossing symmetry relations.

- *Asymptotics*. In addition, the authors *assume* that $\mathcal{M}$ grows linearly at large values of the Mandelstam variables:

$$\mathcal{M}(\beta s, \beta t, \sigma, \tau) \sim O(\beta) \quad \text{for } \beta \to \infty \,. \tag{5.1.13}$$

  On the language of dual string theory, this assumption means that holographic correlators can get contributions from vertices with at most two derivatives, i.e., the four-derivative contact terms must vanish. This was observed in all the previously known examples, and we prove this statement in full generality in Appendix B.

- *Analytic structure*. $\mathcal{M}$ has a finite number of simple poles in s, t, u, which are located at

$$\begin{aligned} s_0 &= s_M - 2a\,, \quad s_0 \geqslant 2 \\ t_0 &= t_M - 2b\,, \quad t_0 \geqslant 2 \\ u_0 &= u_M - 2c\,, \quad u_0 \geqslant 2\,, \end{aligned} \tag{5.1.14}$$

  with

$$\begin{aligned} s_M &= \min\{k_1 + k_2, k_3 + k_4\} - 2\,, \\ t_M &= \min\{k_1 + k_4, k_2 + k_3\} - 2\,, \\ u_M &= \min\{k_1 + k_3, k_2 + k_4\} - 2\,, \end{aligned}$$

  and a, b, c are non-negative integers. Moreover, it turns that the residue at each pole is a polynomial in the other Mandelstam variable.

One also can notice that the R-symmetry implies that the amplitudes $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ are the polynomials in σ and τ of degree L and $L-2$ respectively. Experimentation with the known results for the supergravity correlators allowed the authors to conjecture the following formula for $\widetilde{\mathcal{M}}$, satisfying all the properties mentioned above:

$$\widetilde{\mathcal{M}}(s, t, \tilde{u}, \sigma, \tau) = \sum_{\substack{i+j+k=L-2,\\ 0 \leqslant i,j,k \leqslant L-2}} \frac{a_{ijk}\sigma^i\tau^j}{(s - s_M + 2k)(t - t_M + 2j)(\tilde{u} - u_M + 2i)}\,, \tag{5.1.15}$$

with the coefficients a_{ijk} being

$$a_{ijk} = \frac{C_{k_1k_2k_3k_4}\binom{L-2}{i,j,k}}{(1 + \frac{|k_1-k_2+k_3-k_4|}{2})_i(1 + \frac{|k_1+k_4-k_2-k_3|}{2})_j(1 + \frac{|k_1+k_2-k_3-k_4|}{2})_k}\,, \tag{5.1.16}$$

where $\binom{L-2}{i,j,k}$ is the trinomial coefficient and the overall normalization term

$$C_{p_1p_2p_3p_4} = \frac{f(k_1, k_2, k_3, k_4)}{N^2} \tag{5.1.17}$$

remained unknown. However, this coefficient still can be determined.

Let us comment on the subtlety related to the validity of the procedure described above. The Mellin transform of the free part $\mathcal{G}_{\text{free}}$ is ill-defined and was ignored (set to zero). However, it is completely recovered as a regularization effect when transforming back to the coordinate space. The inverse Mellin transform requires to choose the appropriate contour of integration - one must integrate inside the regions where the integrals converge. The operator $\widehat{R}$ splits those regions into several domains which may not intersect. The way to proceed is to perform an infinitesimal shift such that they overlap. This regularization recovers the correct answer for $\mathcal{G}_{\text{free}}$ and allows to determine the unknown factor $f(k_1, k_2, k_3, k_4)$ by requiring the result to match with $\mathcal{G}_{\text{free}}$ computed by Wick's theorem. Later this factor was derived in [52] by analyzing the light-like limit.

5.2 Consistency with the Mellin formula

Having the correlator computed in the coordinate space, we can go one step further by checking whether the result matches the conjectured formula (5.1.15) from [20, 22]. Using the free part extracted in the verification process, see section 4.3, or computed via performing Wick contractions, as in section 4.4, we can easily find an expression for $R\mathcal{H}$. We can furthermore find $\mathcal{H}$ in terms of $\bar{D}$ functions by solving a set of linear equations obtained from the decomposition into different tensor components. The resulting expressions for $\mathcal{H}$ are expressed as a linear combination of $\bar{D}$ functions:

$$\sum a_{k_1k_2k_3k_4}(u,v)\bar{D}_{k_1k_2k_3k_4} + \sum b_{k_1k_2k_3k_4}(u,v), \tag{5.2.1}$$

where the sums are finite and run over the k_i, and a and b are rational functions of u and v. Here we suppress their polynomial dependence on σ and τ. It is straightforward to find the corresponding Mellin-space expression: as discussed we can consistently send the functions b to zero and use that

$$\begin{aligned}\bar{D}_{\Delta_1\ldots\Delta_4}(u,v) = 2\int \frac{ds}{2}\frac{dt}{2} u^{\frac{s}{2}-\frac{\Delta_1+\Delta_2}{2}} v^{\frac{t}{2}-\frac{\Delta_2+\Delta_3}{2}} \Gamma\left(\tfrac{-s+\Delta_1+\Delta_2}{2}\right)\Gamma\left(\tfrac{-s+\Delta_3+\Delta_4}{2}\right)\times \\ \Gamma\left(\tfrac{-t+\Delta_1+\Delta_4}{2}\right)\Gamma\left(\tfrac{-t+\Delta_2+\Delta_3}{2}\right)\Gamma\left(\tfrac{s+t-\Delta_2-\Delta_4}{2}\right)\Gamma\left(\tfrac{s+t-\Delta_1-\Delta_3}{2}\right),\end{aligned} \tag{5.2.2}$$

to find the Mellin transform (5.1.8) of $\mathcal{H}$. The resulting Mellin-space expressions are rational functions in the coordinates s and t, and further simplification of them yields an exact match with the conjecture for all checked correlation functions up to an expected normalization constant f (5.1.17): these are all the 91 (of which 61 new) correlation functions with weights, up to and including 8 except for the three with lowest weight $k_1 \geqslant 7$,[1] as well as $\langle 7\,10\,12\,17\rangle$. In particular, we find agreement with the derived normalization function from [52], which in our notation becomes

$$f(k_1,k_2,k_3,k_4) = \frac{2^4\sqrt{k_1k_2k_3k_4}}{\left(\frac{k_4-k_3+k_2-k_1}{2}\right)!\left(\frac{k_4+k_3-k_2-k_1}{2}\right)!\left(\frac{|k_4-k_3-k_2+k_1|}{2}\right)!\,(L-2)!}\,. \tag{5.2.3}$$

This, therefore, corroborates the conjecture.

[1] This exception is due to computational limits.

5.3 From Mellin to coordinate representation

Another interesting task is to recover the correlation function in the position space knowing its Mellin-space representation. To see how one can do this we only need to consider a single summand: so let us for simplicity assume

$$\mathcal{M} \sim \frac{1}{(s-s_0)(t-t_0)(\tilde{u}-\tilde{u}_0)}, \tag{5.3.1}$$

with $\tilde{u} = k_1 + k_2 + k_3 + k_4 - 4 - s - t$ and $s_0, t_0, \tilde{u}_0$ non-negative integers. The inverse Mellin transform is defined as

$$\mathcal{H} = \int \frac{ds}{2}\frac{dt}{2} u^{\frac{s}{2} - \frac{k_3+k_4}{2} + L} v^{\frac{t}{2} - \frac{\min(k_1+k_4, k_2+k_3)}{2}} \mathcal{M}(s,t) \Gamma_{k_1k_2k_3k_4} \,. \tag{5.3.2}$$

Let us consider part of the integrand in (5.3.2), namely

$$\frac{\Gamma_{k_1k_2k_3k_4}}{(s-s_0)(t-t_0)(\tilde{u}-\tilde{u}_0)}. \tag{5.3.3}$$

If we manage to rewrite this expression as a linear combination of the $\Gamma_{p_1p_2p_3p_4}$, each summand in that sum gives rise to an integral of the form (5.2.2) after an appropriate identification of the Δ_i with the p_i. Therefore, the problem of finding a coordinate-space expression is reduced to finding a linear combination of $\Gamma_{p_1p_2p_3p_4}$ such that

$$\frac{\Gamma_{k_1k_2k_3k_4}}{(s-s_0)(t-t_0)(\tilde{u}-\tilde{u}_0)} = \sum c_{p_1p_2p_3p_4} \Gamma_{p_1p_2p_3p_4}, \tag{5.3.4}$$

where the c are numbers and the sum is finite over the p_i. The representation on the right-hand side of (5.3.4) is usually not unique, which reflects the fact that the $\bar{D}$ functions are not independent.

The first step towards an expression as in the right-hand side of (5.3.4) is to rewrite its left-hand side as a pure product of gamma functions and linear factors. This can be done by applying the basic property $x\Gamma(x) = \Gamma(x+1)$ repeatedly to some of the gamma functions in the numerator, such that finally factors in the numerator cancel the denominator. For example

$$\frac{\Gamma(\frac{-s+5}{2})}{(s-1)} = \frac{\frac{-s+3}{2}\frac{-s+1}{2}\Gamma(\frac{-s+1}{2})}{(s-1)} = -\frac{1}{2}\frac{-s+3}{2}\Gamma\left(\frac{-s+1}{2}\right). \tag{5.3.5}$$

This yields the intermediate form

$$C\frac{-s+s_1}{2}\dots\frac{-s+s_n}{2}\frac{-t+t_1}{2}\dots\frac{-t+t_m}{2}\dots\frac{-\tilde{u}+\tilde{u}_1}{2}\dots\frac{-\tilde{u}+\tilde{u}_l}{2}\Gamma(x_1)\dots\Gamma(x_6), \tag{5.3.6}$$

with C some constant and $x_{1,2}$ a linear factor in s, $x_{3,4}$ a linear factor in t and $x_{5,6}$ a linear factor in $\tilde{u}$. Note that it could happen that each of the three sets of prefactors in s, t and $\tilde{u}$ might be empty. Suppose first for simplicity that there is only one factor $\frac{-s+s_1}{2}$. Let us consider the linear equation

$$\frac{-s+s_1}{2} = \sum_{i=1}^{6} \lambda_i x_i \tag{5.3.7}$$

for the six unknowns λ_i. Working out the arguments x_i one sees that there are four independent equations (one for s, t and $\tilde{u}$ and one for the constant part), such that we are guaranteed a solution. With this solution we can now rewrite

$$\frac{-s+s_1}{2}\Gamma(x_1)\ldots\Gamma(x_6) = \sum_{i=1}^{6}\lambda_i x_i \Gamma(x_1)\ldots\Gamma(x_6) = \sum_{i=1}^{6}\lambda_i \Gamma(x_1)\ldots\Gamma(x_i+1)\ldots\Gamma(x_6) \tag{5.3.8}$$

and see that we have succeeded in our goal: by repeating the procedure described above recursively for the list of factors in (5.3.6) we can find a linear combination of products of gamma functions that are equal to (5.3.3), such that we have found a representation as in (5.3.4). Exchanging sum and integral we find that each summand is of the form (5.2.2) such that after matching the coefficients we find an expression for the inverse Mellin-transform in terms of $\bar{D}$ functions.

We have applied this algorithm to all the correlation functions in our database. All cases have been checked explicitly with our coordinate-space results. In some cases, a more minimal representation may exist, but due to the automatized nature of our application, this is unavoidable. It is noteworthy that in exchanging the sum and integral, we do not run into any domain issues that exist for the full correlator, as described in [20], that give rise to the free part upon inverse Mellin-transforming. After all, all we are doing is rewriting the integrand using a global property of the gamma function.

Appendix A

D-functions

Here we collect the useful identities involving the D-functions. They are defined as integrals over AdS_5:

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \int \frac{d^5z}{z_0^5} K_{\Delta_1}(z,\vec{x}_1)K_{\Delta_2}(z,\vec{x}_2)K_{\Delta_3}(z,\vec{x}_3)K_{\Delta_4}(z,\vec{x}_4) \quad \text{(A.0.1)}$$

with

$$K_\Delta(z,\vec{x}) = \left(\frac{z_0}{z_0^2+(\vec{z}-\vec{x})^2}\right)^\Delta . \quad \text{(A.0.2)}$$

D-functions have also a representation in terms of integrals over Feynman parameters:

$$D_{\Delta_1\Delta_2\Delta_3\Delta_4}(\vec{x}_1,\vec{x}_2,\vec{x}_3,\vec{x}_4) = \frac{\pi^2\Gamma(\Sigma-2)\Gamma(\Sigma)}{2\prod_i \Gamma(\Delta_i)} \int \prod_j d\alpha_j \alpha_j^{\Delta_j-1} \frac{\delta(\sum_j \alpha_j - 1)}{(\sum_{k<l}\alpha_k\alpha_l x_{kl}^2)^\Sigma} , \quad \text{(A.0.3)}$$

where $\Sigma = \frac{1}{2}\sum_i \Delta_i$.
It is convenient to define the corresponding $\bar{D}$-functions, which are functions of conformal invariant ratios, $u = \frac{x_{12}^2x_{34}^2}{x_{13}^2x_{24}^2}$ and $v = \frac{x_{14}^2x_{23}^2}{x_{13}^2x_{24}^2}$, by:

$$\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) = \kappa \frac{|\vec{x}_{31}|^{2\Sigma-2\Delta_4}|\vec{x}_{24}|^{2\Delta_2}}{|\vec{x}_{41}|^{2\Sigma-2\Delta_1-2\Delta_4}|\vec{x}_{34}|^{2\Sigma-2\Delta_3-2\Delta_4}} D_{\Delta_1\Delta_2\Delta_3\Delta_4} , \quad \text{(A.0.4)}$$

where

$$\kappa = \frac{2}{\pi^2}\frac{\Gamma(\Delta_1)\Gamma(\Delta_2)\Gamma(\Delta_3)\Gamma(\Delta_4)}{\Gamma(\Sigma-2)} . \quad \text{(A.0.5)}$$

By applying differential operators, one can obtain identities relating different $\bar{D}$-functions

$$\begin{aligned}
\bar{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4} &= -\partial_u \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} , \\
\bar{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4} &= -\partial_v \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} , \\
\bar{D}_{\Delta_1\Delta_2\Delta_3+1\Delta_4+1} &= (\Delta_3+\Delta_4-\Sigma-u\partial_u)\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} , \\
\bar{D}_{\Delta_1+1\Delta_2\Delta_3\Delta_4+1} &= (\Delta_1+\Delta_4-\Sigma-v\partial_v)\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} , \\
\bar{D}_{\Delta_1\Delta_2+1\Delta_3\Delta_4+1} &= (\Delta_2+u\partial_u+v\partial_v)\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} , \\
\bar{D}_{\Delta_1+1\Delta_2\Delta_3+1\Delta_4} &= (\Sigma-\Delta_4+u\partial_u+v\partial_v)\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} . \quad \text{(A.0.6)}
\end{aligned}$$

This allows one to reduce any $\bar{D}$-function to a $\bar{D}_{1111} = \Phi(u,v)$ by appropriate differentiation with respect to u and v. The function $\Phi(u,v)$ is given in terms of a standard four-dimensional one-loop (box) integral and has an explicit representation in terms of the dilogarithm function Li_2 [53]:

$$\Phi(u,v) = \frac{1}{\lambda}\left\{ 2\Big(\mathrm{Li}_2(-\rho u) + \mathrm{Li}_2(-\rho v)\Big) + \ln\frac{v}{u}\ln\frac{1+\rho v}{1+\rho u} + \ln(\rho u)\ln(\rho v) + \frac{\pi^2}{3}\right\}, \quad \text{(A.0.7)}$$

with [1]

$$\lambda(u,v) = \sqrt{(1-u-v)^2 - 4uv}, \qquad \rho(u,v) = 2\,(1-u-v+\lambda)^{-1}. \qquad \text{(A.0.8)}$$

The action of the derivatives on Φ is given by [27]

$$\begin{aligned} \partial_u\Phi(u,v) &= \frac{1}{\lambda^2}\left(\Phi(u,v)(1-u+v) + 2\ln u - \frac{u+v-1}{u}\ln v\right), \\ \partial_v\Phi(u,v) &= \frac{1}{\lambda^2}\left(\Phi(u,v)(1-v+u) + 2\ln v - \frac{u+v-1}{v}\ln u\right). \end{aligned} \qquad \text{(A.0.9)}$$

By repeated use of $(A.0.6)$ one can obtain additional identities which relate $\bar{D}$-functions with different values of Σ:

$$\begin{aligned} (\Delta_2+\Delta_4-\Sigma)\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} &= \bar{D}_{\Delta_1\Delta_2+1\Delta_3\Delta_4+1} - \bar{D}_{\Delta_1+1\Delta_2\Delta_3+1\Delta_4} \\ (\Delta_1+\Delta_4-\Sigma)\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} &= \bar{D}_{\Delta_1+1\Delta_2\Delta_3\Delta_4+1} - v\bar{D}_{\Delta_1\Delta_2+1\Delta_3+1\Delta_4} \\ (\Delta_3+\Delta_4-\Sigma)\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} &= \bar{D}_{\Delta_1\Delta_2\Delta_3+1\Delta_4+1} - u\bar{D}_{\Delta_1+1\Delta_2+1\Delta_3\Delta_4} \end{aligned} \qquad \text{(A.0.10)}$$

Furthermore, there are identities relating $\bar{D}$-functions with the same Σ:

$$\Delta_4\bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4} = \bar{D}_{\Delta_1\Delta_2\Delta_3+1\Delta_4+1} + \bar{D}_{\Delta_1\Delta_2+1\Delta_3\Delta_4+1} + \bar{D}_{\Delta_1+1\Delta_2\Delta_3\Delta_4+1} \qquad \text{(A.0.11)}$$

Another useful identities involve the various symmetries that these functions exhibit. By means of conformal symmetry, one can see that:

$$\begin{aligned} \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) &= v^{-\Delta_2}\bar{D}_{\Delta_1\Delta_2\Delta_4\Delta_3}(u/v,1/v) \\ \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) &= v^{\Delta_4-\Sigma}\bar{D}_{\Delta_2\Delta_1\Delta_3\Delta_4}(u/v,1/v) \\ \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) &= v^{\Delta_1+\Delta_4-\Sigma}\bar{D}_{\Delta_2\Delta_1\Delta_4\Delta_3}(u,v) \\ \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) &= u^{\Delta_3+\Delta_4-\Sigma}\bar{D}_{\Delta_4\Delta_3\Delta_2\Delta_1}(u,v) \\ \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) &= \bar{D}_{\Delta_3\Delta_2\Delta_1\Delta_4}(v,u) \\ \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) &= \bar{D}_{\Sigma-\Delta_3\Sigma-\Delta_4\Sigma-\Delta_1\Sigma-\Delta_2}(u,v) \end{aligned} \qquad \text{(A.0.12)}$$

[1]The case $\lambda^2 > 0$ is assumed; the case $\lambda^2 < 0$ requires an appropriate analytic continuation.

Appendix B

Vanishing of $L^{(4)}$

As has been already discussed, the vanishing of the four-derivative quartic Lagrangian was observed in all the previously computed correlators and is believed to be true. Proving this statement, in general, became important: it is one of the assumptions under which the authors of (5.1.15) conjectured the Mellin-space formula for the four-point functions of arbitrary weights. In this appendix, we provide an explicit computation showing that these terms indeed vanish identically. The proof is based on the new reduction formulae relating sums of products of different tensor structures which the quartic couplings depend on.

B.1 Quartic couplings with four derivatives

Remind that the quartic Lagrangian for the fields s^I with four derivatives was found to be of the following form

$$\mathcal{L}_4^{(4)} = \sum_{1,2,3,4} \left(S_{1234}^{(4)} + A_{1234}^{(4)}\right) s^1 \nabla_a s^2 \nabla_b^2 (s^3 \nabla^a s^4) \,. \tag{B.1.1}$$

Here ∇_a is a covariant derivative along AdS space and each summation label $j = 1, \ldots, 4$ stands for a concise notation for the representation index I_j running over a basis of of an irreducible representation $[0, k_j, 0]$ of SU(4). The couplings A_{1234} and S_{1234} have the following symmetry properties

$$\begin{aligned} A_{1234}^{(4)} &= -A_{2134}^{(4)} = A_{3412}^{(4)} \,, \\ S_{1234}^{(4)} &= S_{2134}^{(4)} = S_{3412}^{(4)} \,. \end{aligned} \tag{B.1.2}$$

Explicitly, A_{1234} is given by the sum of the following individual terms

$$\begin{aligned} (A_3)_{1234}^{(4)} &= \frac{1}{4\delta} f_5^3 \left(a_{145}a_{235} - a_{135}a_{245}\right) . \\ (A_2)_{1234}^{(4)} &= -\frac{1}{4\delta} (3(f_1 + f_2 + f_3 + f_4) - 28) f_5^2 \left(a_{145}a_{235} - a_{135}a_{245}\right) . \\ (A_1)_{1234}^{(4)} &= -\frac{3}{4\delta} (f_1 - f_2)(f_3 - f_4) f_5 a_{125} a_{345} \\ &+ \frac{1}{2\delta} (f_1 + f_2 + f_3 + f_4 - 2)(f_1 + f_2 + f_3 + f_4 - 12) f_5 \left(a_{145}a_{235} - a_{135}a_{245}\right) . \end{aligned}$$

$$
\begin{aligned}
(A_0)^{(4)}_{1234} &= \frac{21}{4\delta}(f_1-f_2)(f_3-f_4)a_{125}a_{345}.\\
(A_{-1})^{(4)}_{1234} &= -\frac{12}{\delta}(f_1-f_2)(f_3-f_4)f_5^{-1}a_{125}a_{345}.\\
(A_{t2})^{(4)}_{1234} &= -\frac{3}{\delta}(f_5-1)^2 t_{125}t_{345}.
\end{aligned}
$$

The symmetric coupling is

$$
S^{(4)}_{1234} = \frac{7}{4\delta}\left(2f_1f_2+2f_3f_4-(f_1+f_2)(f_3+f_4)\right)a_{125}a_{345}.
$$

In the formulae above $f_i \equiv f(k_i) = k_i(k_i+4)$, $\delta = \prod_{i=1}^4(k_i+1)$ and summation over the index 5 is assumed. The couplings a_{123} and t_{123} are given as the following integrals over the five-sphere of the spherical harmonics

$$
a_{123} = \int Y^1Y^2Y^3\,, \qquad t_{123} = \int \nabla^\alpha Y^1 Y^2 Y^3_\alpha\,. \tag{B.1.3}
$$

Here Y^k are scalar spherical harmonics and Y^k_α are vector spherical harmonics satisfying the irreducibility condition $\nabla^\alpha Y^k_\alpha = 0$. Both Y^k and Y^k_α are eigenvalues of the sphere Laplacian ∇^2 with the following eigenvalues

$$
\nabla^2 Y^k = -f_k Y^k\,, \qquad \nabla^2 Y^k_\alpha = (1-f_k)Y^k_\alpha\,. \tag{B.1.4}
$$

In what follows we will also need the following product formulae which follow from the orthogonality relation for scalar harmonics

$$
Y^1Y^2 = a_{125}Y^5, \quad \nabla^\alpha Y^1\nabla_\alpha Y^2 = b_{125}Y^5, \quad \nabla^\alpha\nabla^\beta Y^1\nabla_\alpha\nabla_\beta Y^2 = c_{125}Y^5\,, \tag{B.1.5}
$$

where the coefficients are[1]

$$
\begin{aligned}
b_{123} &= \int \nabla^\alpha Y^1 \nabla_\alpha Y^2 Y^3 = \frac{1}{2}(f_1+f_2-f_3)a_{123},\\
c_{123} &= \int \nabla^\alpha\nabla^\beta Y^1\nabla_\alpha\nabla_\beta Y^2Y^3 = \frac{1}{2}(f_1+f_2-f_3-8)(f_1+f_2-f_3)a_{123}\,.
\end{aligned} \tag{B.1.6}
$$

This completes our discussion of the known results on the quartic Lagrangian with four-derivative vertices, for further information and derivation of the above formulae we refer the reader to [12].

To proceed with proving, we employ the same strategy as in [24], where the vanishing of quartic four-derivative-vertices were shown for the so-called sub-extremal and sub-sub-extremal cases. Recall that we are ultimately interested in the four-point function of BPS operators corresponding to arbitrary weights $k_1,\dots k_4$. We can therefore restrict the infinite sum in (B.1.1) to representations which correspond to these weights. The sum in (B.1.1) is not ordered and, therefore, there are 24 ordered sets of the indices $k_1,\dots,k_4$ which split into 3 equivalence classes due to the symmetries (B.1.2). Further, integrating

[1] The formula for c_{123} in terms of a_{123} is different from the one in [12], because there the combination $\nabla^\alpha\nabla^\beta$ stands for the traceless symmetric combination of derivatives $\nabla^\alpha\nabla^\beta \equiv \nabla^{(\alpha}\nabla^{\beta)}$.

by parts and using (B.1.2), we represent the part of the Lagrangian (B.1.1) contributing to the four-point function $\langle k_1k_2k_3k_4\rangle$ in the form similar to that in [24]

$$
\begin{aligned}
\mathcal{L}_4^{(4),k_1k_2k_3k_4} = & - 8\sum_{1,2,3,4}\Big(S^{(4)}_{1234}+A^{(4)}_{1324}+A^{(4)}_{1423}\Big)\nabla_a s^1\nabla^a s^2\nabla_b s^3\nabla^b s^4 \\
& - 8\sum_{1,2,3,4}\Big(S^{(4)}_{1324}+A^{(4)}_{1234}+A^{(4)}_{1432}\Big)\nabla_a s^1\nabla^a s^3\nabla_b s^2\nabla^b s^4 \qquad (B.1.7) \\
& - 8\sum_{1,2,3,4}\Big(S^{(4)}_{1432}+A^{(4)}_{1342}+A^{(4)}_{1243}\Big)\nabla_a s^1\nabla^a s^4\nabla_b s^2\nabla^b s^3 .
\end{aligned}
$$

Since we are interested here in the four-derivative vertices only, in the above formula we have omitted the contribution of two-derivative terms and terms without derivatives which arise upon integrating by parts and using equations of motion. There terms however should be taken into account in subsequent analysis of the remaining part of the quartic effective action. We also note that the meaning of the sums in (B.1.7) is different from that in (B.1.1) – in (B.1.7) the sums are ordered, *i.e.* summation over 1 means summation over index I_1 corresponding to the representation with a given weight k_1 and so on. It is now obvious that it is enough to analyse the coupling

$$\mathscr{C}_{1234} \equiv S^{(4)}_{1234}+A^{(4)}_{1324}+A^{(4)}_{1423} , \qquad (B.1.8)$$

because the other two couplings in (B.1.7) differ from it by permutation of indices only.

Obviously, among the couplings there is a distinguished one, namely, $(A_{t2})^{(4)}_{1234}$, as the latter involves vector spherical harmonics. Its contribution into (B.1.8) comes in the combination

$$W^{1234} \equiv (f_5-1)^2(t_{135}t_{245}+t_{145}t_{235}) . \qquad (B.1.9)$$

Our further strategy will be to reduce this combination to structures of the type $f_5^n a_{125}a_{345}$ and permutations thereof. After this is done, all the couplings become comparable and we can add them up according to (B.1.8).

B.2 Reduction formula

The reduction of (B.1.9) is based on the following formula [12]

$$\nabla_\alpha Y^1 Y^2 = t_{125}Y^5_\alpha + \frac{b_{152}}{f_5}\nabla_\alpha Y^5 . \qquad (B.2.1)$$

In what follows it appears advantageous to split (B.2.1) into anti-symmetric and symmetric part with respect to indices 1 and 2, namely,

$$
\begin{aligned}
\nabla_\alpha Y^{[1}Y^{2]} &\equiv \frac{1}{2}(\nabla_\alpha Y^1Y^2-\nabla_\alpha Y^2Y^1) = t_{125}Y^5_\alpha+\frac{f_1-f_2}{2f_5}a_{125}\nabla_\alpha Y^5 , \qquad (B.2.2) \\
\nabla_\alpha(Y^1Y^2) &= a_{125}\nabla_\alpha Y^5 . \qquad (B.2.3)
\end{aligned}
$$

Acting on (B.2.2) with the Laplacian and taking into account that

$$\nabla^2\nabla_\alpha Y^k = -(f_k-4)\nabla_\alpha Y^k , \qquad (B.2.4)$$

we obtain

$$(1-f_5)t_{125}Y^5_\alpha = \nabla^2(\nabla_\alpha Y^{[1}Y^{2]}) - \frac{f_1-f_2}{2f_5}(4-f_5)a_{125}\nabla_\alpha Y^5\,. \qquad \text{(B.2.5)}$$

Now we multiply this relation with a similar one where indices $1,2,5$ are replaced by $3,4,6$ and integrate over the five-sphere. The orthogonality relation for the vector spherical harmonics together with (B.2.4) used upon integrating the Laplacian by parts leads to the following formula

$$\begin{aligned}(f_5-1)^2 t_{125}t_{345} = \int \nabla^2(\nabla_\alpha Y^{[1}Y^{2]})\nabla^2(\nabla^\alpha Y^{[3}Y^{4]}) - \\ - \frac{(4-f_5)^2}{4f_5}(f_1-f_2)(f_3-f_4)a_{125}a_{345}\,,\end{aligned}$$

which for (B.1.9) implies the following relation

$$W^{1234} = K^{1234} - \frac{(4-f_5)^2}{4f_5}\Big((f_1-f_3)(f_2-f_4)a_{135}a_{245} + (f_1-f_4)(f_2-f_3)a_{145}a_{235}\Big)\,,$$

where K^{1234} is the following integral

$$K^{1234} = \int \big(\nabla^2(\nabla_\alpha Y^{[1}Y^{4]})\nabla^2(\nabla_\alpha Y^{[2}Y^{3]}) + \nabla^2(\nabla_\alpha Y^{[1}Y^{3]})\nabla^2(\nabla_\alpha Y^{[2}Y^{4]})\big) \qquad \text{(B.2.6)}$$

with the symmetry properties

$$K^{1234} = K^{2134} = K^{1243} = K^{3412}\,. \qquad \text{(B.2.7)}$$

Thus, we have reduced evaluation of our main quantity W^{1234} to the computation of the integral K^{1234}. In order not to overload our discussion with heavy formulae we perform the computation of K^{1234} in the next section and here present only the final reduction formula for W^{1234}:

$$\begin{aligned}
&W^{1234} = \\
&-\frac{a_{135}a_{245}}{24f_5}\Big(-2f_5^4 + 2(3f_1+3f_2+3f_3+3f_4-28)f_5^3 - 2(2f_1^2-(f_2-5f_3-8f_4+28)f_1 \\
&+2f_2^2+2f_4^2+2(f_3-12)(f_3-2)-(f_3+28)f_4+f_2(8f_3+5f_4-28))f_5^2 \\
&+((f_2+3f_3+4f_4-12)f_1^2+(f_2^2+4(f_3+f_4-20)f_2+3(f_3-4)^2+4f_4^2+4(f_3+4)f_4)f_1 \\
&+(3f_2+f_3-12)f_4^2+4((f_3-3)f_2^2+(f_3(f_3+4)+12)f_2-3(f_3-4)f_3)+(3(f_2-4)^2 \\
&+f_3^2+4(f_2-20)f_3)f_4)f_5+96(f_1-f_3)(f_2-f_4)\Big) \\
&-\frac{a_{145}a_{235}}{24f_5}\Big(-2f_5^4+2(3f_1+3f_2+3f_3+3f_4-28)f_5^3-2(2f_1^2-(f_2-8f_3-5f_4+28)f_1 \\
&+2f_2^2+2f_4^2+2(f_3-12)(f_3-2)-(f_3+28)f_4+f_2(5f_3+8f_4-28))f_5^2 \\
&+((f_2+4f_3+3(f_4-4))f_1^2+(f_2^2+4(f_3+f_4-20)f_2+4f_3^2+3(f_4-4)^2+4f_3(f_4+4))f_1 \\
&+(4f_2+f_3-12)f_4^2+3(f_2-4)(f_3-4)(f_2+f_3) \\
&+(4f_2^2+4(f_3+4)f_2+(f_3-80)f_3+48)f_4)f_5+96(f_2-f_3)(f_1-f_4)\Big) \\
&+\frac{a_{125}a_{345}}{24}\Big(-4f_5^3+4(3f_3+3f_4-28)f_5^2-4(2f_4^2+5f_3f_4-28f_4+2(f_3-14)f_3+48)f_5 \\
&+6(f_3-4)(f_4-4)(f_3+f_4)+f_2(5f_3^2+8f_4f_3-64f_3+5f_4^2+12f_5^2-64f_4 \\
&-14(f_3+f_4-8)f_5+96)+f_1(6f_2^2+4(2(f_3+f_4-6)-5f_5)f_2+5f_3^2+5f_4^2+12f_5^2 \\
&-64f_3+8f_3f_4-64f_4-14(f_3+f_4-8)f_5+96)+f_2^2(5f_3+5f_4-8(f_5+3)) \\
&+f_1^2(6f_2+5f_3+5f_4-8(f_5+3))\Big)\,.
\end{aligned} \qquad \text{(B.2.8)}$$

Finally, we sum $-\frac{3}{\delta}W^{1234}$ with the remaining couplings in (B.1.8) and observe that all the terms are neatly canceled delivering thereby $\mathscr{C}_{1234}=0$. In this way, we have shown that the quartic four-derivative couplings of the effective supergravity action vanish.

B.3 Evaluation of K^{1234}

In this part we compute the integral (B.2.6) in terms of structures $f_5^n a_{125}a_{345}$ and permutations thereof. In the computation process the following identity valid for any co-vector ξ_α

$$[\nabla_\alpha\,,\,\nabla_\beta]\,\xi_\gamma = g_{\alpha\gamma}\xi_\beta - g_{\beta\gamma}\xi_\alpha\,. \tag{B.3.1}$$

will be heavily used. Here $g_{\alpha\beta}$ is the metric of the unit five-sphere. Note also that on a scalar function two covariant derivatives commute.

We start with computing

$$\nabla^2(\nabla_\alpha Y^1 Y^4) \;=\; 2\nabla_\beta\nabla_\alpha Y^1\nabla^\beta Y^4 - (f_1+f_4-4)\nabla_\alpha Y^1 Y^4\,. \tag{B.3.2}$$

The latter formula gives rise to the following identity

$$\begin{aligned}
&\nabla^2(\nabla_\alpha Y^1 Y^4)\nabla^2(\nabla^\alpha Y^2 Y^3) = 4\nabla_\beta\nabla_\alpha Y^1\nabla^\beta Y^4\nabla_\gamma\nabla^\alpha Y^2\nabla^\gamma Y^3 \\
&\quad -2(f_2+f_3-4)\nabla_\beta\nabla_\alpha Y^1\nabla^\beta Y^4\nabla^\alpha Y^2 Y^3 - 2(f_1+f_4-4)\nabla_\alpha Y^1 Y^4\nabla_\gamma\nabla^\alpha Y^2\nabla^\gamma Y^3 \\
&\quad +(f_1+f_4-4)(f_2+f_3-4)\nabla_\alpha Y^1 Y^4\nabla^\alpha Y^2 Y^3
\end{aligned} \tag{B.3.3}$$

and a similar one with indices 3 and 4 interchanged. To simplify our presentation, in the sequel we will drop the integration sign and always identify expressions differing by a total derivative. Using (B.3.3) we then get

$$K^{1234} \;=\; U^{1234} + V^{1234}\,, \tag{B.3.4}$$

where

$$\begin{aligned}
&U^{1234} = \\
&-2(f_2+f_3-4)\nabla_\beta\nabla_\alpha Y^{[1}\nabla^\beta Y^{4]}\nabla^\alpha Y^{[2}Y^{3]} - 2(f_2+f_4-4)\nabla_\beta\nabla_\alpha Y^{[1}\nabla^\beta Y^{3]}\nabla^\alpha Y^{[2}Y^{4]} \\
&-2(f_1+f_4-4)\nabla_\alpha Y^{[1}Y^{4]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{3]} - 2(f_1+f_3-4)\nabla_\alpha Y^{[1}Y^{3]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{4]} \\
&+(f_1+f_4-4)(f_2+f_3-4)\nabla_\alpha Y^{[1}Y^{4]}\nabla^\alpha Y^{[2}Y^{3]} \\
&+(f_1+f_3-4)(f_2+f_4-4)\nabla_\alpha Y^{[1}Y^{3]}\nabla^\alpha Y^{[2}Y^{4]}\,.
\end{aligned} \tag{B.3.5}$$

and

$$V^{1234} = 4\big(\nabla_\beta\nabla_\alpha Y^{[1}\nabla^\beta Y^{4]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{3]} + \nabla_\beta\nabla_\alpha Y^{[1}\nabla^\beta Y^{3]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{4]}\big)\,. \tag{B.3.6}$$

We continue our further treatment with evaluating the quantity U^{1234}. To this end, we compute

$$\begin{aligned}
&\nabla_\beta\nabla_\alpha Y^{[i}\nabla^\beta Y^{j]}\nabla^\alpha Y^{[k}Y^{l]} \\
&= -\nabla_\alpha Y^{[i}\nabla^2 Y^{j]}\nabla^\alpha Y^{[k}Y^{l]} - \nabla_\alpha Y^{[i}\nabla_\beta Y^{j]}\nabla^\beta\nabla^\alpha Y^{[k}Y^{l]} - \nabla_\alpha Y^{[i}\nabla_\beta Y^{j]}\nabla^\alpha Y^{[k}\nabla^\beta Y^{l]} \\
&= \frac{1}{4}(f_j\nabla_\alpha Y^i Y^j - f_i\nabla_\alpha Y^j Y^i)(\nabla^\alpha Y^k Y^l - \nabla^\alpha Y^l Y^k) \\
&-\frac{1}{4}(\nabla_\alpha Y^i\nabla_\beta Y^j - \nabla_\alpha Y^j\nabla_\beta Y^i)(\nabla^\alpha Y^k\nabla^\beta Y^l - \nabla^\alpha Y^l\nabla^\beta Y^k) \\
&= \frac{1}{4}f_j(b_{ik5}a_{jl5} - b_{il5}a_{jk5}) - \frac{1}{4}f_i(b_{jk5}a_{il5} - b_{jl5}a_{ik5}) - \frac{1}{2}(b_{ik5}b_{jl5} - b_{il5}b_{jk5}) \\
&= \frac{1}{8}((f_5-f_k)(f_5-f_l) - f_i f_j)(a_{il5}a_{jk5} - a_{ik5}a_{jl5})\,,
\end{aligned} \tag{B.3.7}$$

and

$$
\begin{aligned}
&\nabla_\alpha Y^{[i}Y^{j]}\nabla^\alpha Y^{[k}Y^{l]} = \frac{1}{4}(\nabla_\alpha Y^i Y^j - \nabla_\alpha Y^j Y^i)(\nabla^\alpha Y^k Y^l - \nabla^\alpha Y^l Y^k) \\
&= \frac{1}{4}(b_{ik5}a_{jl5} - b_{il5}a_{jk5}) - \frac{1}{4}(b_{jk5}a_{il5} - b_{jl5}a_{ik5}) \\
&= \frac{1}{8}(-f_i - f_j - f_k - f_l + 2f_5)(a_{il5}a_{jk5} - a_{ik5}a_{jl5})\,.
\end{aligned}
\tag{B.3.8}
$$

As the result, for the quantity U^{1234} we get

$$
\begin{aligned}
U^{1234} = &\frac{1}{8}(-2(f_1+f_4-4)((f_5-f_1)(f_5-f_4)-f_2f_3) \\
&-(f_1+f_4-4)(f_2+f_3-4)(f_1+f_2+f_3+f_4-2f_5) \\
&-2(f_2+f_3-4)((f_5-f_2)(f_5-f_3)-f_1f_4))(a_{135}a_{245}-a_{125}a_{345}) \\
&+\frac{1}{8}(-2(f_2+f_4-4)((f_5-f_2)(f_5-f_4)-f_1f_3) \\
&-(f_1+f_3-4)(f_2+f_4-4)(f_1+f_2+f_3+f_4-2f_5) \\
&-2(f_1+f_3-4)((f_5-f_1)(f_5-f_3)-f_2f_4))(a_{145}a_{235}-a_{125}a_{345})\,.
\end{aligned}
\tag{B.3.9}
$$

In this way U^{1234} has been reduced to the desired structure.

Now we look for a similar reduction of the quantity V^{1234}. Here we perform a sequence of the following transformations. First, we have

$$
\begin{aligned}
V^{1234} = 4\big(&-\nabla_\alpha Y^{[1}\nabla^2 Y^{4]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{3]} - \nabla^\alpha Y^{[1}\nabla^\beta Y^{4]}\nabla_\beta\nabla_\alpha\nabla_\gamma Y^{[2}\nabla^\gamma Y^{3]} \\
&-\nabla_\alpha Y^{[1}\nabla_\beta Y^{4]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\beta\nabla^\gamma Y^{3]} \\
&-\nabla_\alpha Y^{[1}\nabla^2 Y^{3]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{4]} - \nabla^\alpha Y^{[1}\nabla^\beta Y^{3]}\nabla_\beta\nabla_\alpha\nabla_\gamma Y^{[2}\nabla^\gamma Y^{4]} \\
&-\nabla_\alpha Y^{[1}\nabla_\beta Y^{3]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\beta\nabla^\gamma Y^{4]}\big)\,.
\end{aligned}
\tag{B.3.10}
$$

Here the combinations $\nabla^\alpha Y^{[1}\nabla^\beta Y^{4]}$ and $\nabla^\alpha Y^{[1}\nabla^\beta Y^{3]}$ entering in the 2nd and 5th terms are anti-symmetric in α and β and, therefore, in these terms one can replace $\nabla_\beta\nabla_\alpha\nabla_\gamma$ with $\frac{1}{2}[\nabla_\beta,\nabla_\alpha]\nabla_\gamma$ and then apply identity (B.3.1). In this way we get

$$
\begin{aligned}
V^{1234} &= -4\big(\nabla_\alpha Y^{[1}\nabla_\beta Y^{4]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\beta\nabla^\gamma Y^{3]} + \nabla_\alpha Y^{[1}\nabla_\beta Y^{3]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\beta\nabla^\gamma Y^{4]} \\
&+ \nabla_\alpha Y^{[1}\nabla^2 Y^{4]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{3]} - \nabla_\alpha Y^{[1}\nabla_\beta Y^{4]}\nabla^\beta Y^{[2}\nabla^\alpha Y^{3]} \\
&+ \nabla_\alpha Y^{[1}\nabla^2 Y^{3]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{4]} - \nabla_\alpha Y^{[1}\nabla_\beta Y^{3]}\nabla^\beta Y^{[2}\nabla^\alpha Y^{4]}\big)\,.
\end{aligned}
\tag{B.3.11}
$$

As the next step, we consider the first line in the expression above and transform it in the following way

$$
\begin{aligned}
I^{1234} &\equiv -4\big(\nabla_\alpha Y^{[1}\nabla_\beta Y^{4]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\beta\nabla^\gamma Y^{3]} + \nabla_\alpha Y^{[1}\nabla_\beta Y^{3]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\beta\nabla^\gamma Y^{4]}\big) \\
&= -(\nabla^\gamma Y^1\nabla^\beta Y^4 - \nabla^\gamma Y^4\nabla^\beta Y^1)(\nabla_\gamma\nabla_\alpha Y^2\nabla_\beta\nabla^\alpha Y^3 - \nabla_\gamma\nabla_\alpha Y^3\nabla_\beta\nabla^\alpha Y^2) \\
&\quad -(\nabla^\gamma Y^1\nabla^\beta Y^3 - \nabla^\gamma Y^3\nabla^\beta Y^1)(\nabla_\gamma\nabla_\alpha Y^2\nabla_\beta\nabla^\alpha Y^4 - \nabla_\gamma\nabla_\alpha Y^4\nabla_\beta\nabla^\alpha Y^2) \\
&= -2\nabla_\beta\nabla_\alpha Y^3\nabla^\beta Y^4\nabla_\gamma\nabla^\alpha Y^2\nabla^\gamma Y^1 + 2\nabla_\beta\nabla_\alpha Y^3\nabla^\beta Y^1\nabla_\gamma\nabla^\alpha Y^2\nabla^\gamma Y^4 \\
&\quad -2\nabla_\beta\nabla_\alpha Y^4\nabla^\beta Y^3\nabla_\gamma\nabla^\alpha Y^2\nabla^\gamma Y^1 + 2\nabla_\beta\nabla_\alpha Y^4\nabla^\beta Y^1\nabla_\gamma\nabla^\alpha Y^2\nabla^\gamma Y^3\,.
\end{aligned}
\tag{B.3.12}
$$

The resulting expression undergoes further transformation

$$
\begin{aligned}
I^{1234} = &-2\nabla_\alpha(\nabla_\beta Y^3\nabla^\beta Y^4)\nabla_\gamma\nabla^\alpha Y^2\nabla^\gamma Y^1 \qquad \text{(B.3.13)} \\
&+2\big(\nabla_\beta\nabla_\alpha Y^{[3}\nabla^\beta Y^{1]} + \frac{1}{2}\nabla_\alpha(\nabla_\beta Y^3\nabla^\beta Y^1)\big)\big(\nabla_\gamma\nabla_\alpha Y^{[2}\nabla^\gamma Y^{4]} + \frac{1}{2}\nabla_\alpha(\nabla_\gamma Y^2\nabla^\gamma Y^4)\big) \\
&+2\big(\nabla_\beta\nabla_\alpha Y^{[4}\nabla_\beta Y^{1]} + \frac{1}{2}\nabla_\alpha(\nabla_\beta Y^4\nabla_\beta Y^1)\big)\big(\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{3]} + \frac{1}{2}\nabla^\alpha(\nabla_\gamma Y^2\nabla^\gamma Y^3)\big)\,,
\end{aligned}
$$

which finally results into

$$
\begin{aligned}
I^{1234} \;=-\; & 2\big(\nabla_\beta\nabla_\alpha Y^{[1}\nabla^\beta Y^{4]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{3]}+\nabla_\beta\nabla_\alpha Y^{[1}\nabla^\beta Y^{3]}\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{4]}\big)\\
&-\nabla_\beta\nabla_\alpha Y^{[1}\nabla^\beta Y^{3]}\nabla^\alpha(\nabla_\gamma Y^2\nabla^\gamma Y^4)+\nabla_\alpha(\nabla_\beta Y^3\nabla^\beta Y^1)\nabla_\gamma\nabla^\alpha Y^{[2}\nabla^\gamma Y^{4]}\\
&+\frac{1}{2}\nabla_\alpha(\nabla_\beta Y^3\nabla^\beta Y^1)\nabla^\alpha(\nabla_\gamma Y^2\nabla^\gamma Y^4) \qquad\qquad\qquad\qquad \text{(B.3.14)}\\
&-\nabla_\beta\nabla_\alpha Y^{[1}\nabla^\beta Y^{4]}\nabla^\alpha(\nabla_\gamma Y^2\nabla^\gamma Y^3)+\nabla_\alpha(\nabla_\beta Y^4\nabla^\beta Y^1)\nabla^\gamma\nabla^\alpha Y^{[2}\nabla_\gamma Y^{3]}\\
&+\frac{1}{2}\nabla_\alpha(\nabla_\beta Y^4\nabla^\beta Y^1)\nabla^\alpha(\nabla_\gamma Y^2\nabla^\gamma Y^3)-2\nabla_\alpha(\nabla_\beta Y^3\nabla^\beta Y^4)\nabla^\gamma\nabla^\alpha Y^2\nabla_\gamma Y^1\,.
\end{aligned}
$$

Comparing the first line in the above formula with the original expression (B.3.6) for V^{1234} we observe that it coincides with $-\frac{1}{2}V^{1234}$. This allows us to find the following answer for V^{1234}

$$
\begin{aligned}
V^{1234} \;=\; & \frac{2}{3}\Big(\nabla^2\nabla_\beta Y^{[1}\nabla^\beta Y^{3]}\nabla_\gamma Y^2\nabla^\gamma Y^4-\nabla_\beta Y^3\nabla^\beta Y^1\nabla^2\nabla_\gamma Y^{[2}\nabla^\gamma Y^{4]}\\
&-\frac{1}{2}\nabla_\beta Y^3\nabla^\beta Y^1\nabla^2(\nabla_\gamma Y^2\nabla^\gamma Y^4)-\frac{1}{2}\nabla_\beta Y^4\nabla^\beta Y^1\nabla^2(\nabla_\gamma Y^2\nabla^\gamma Y^3)\\
&+\nabla^2\nabla_\beta Y^{[1}\nabla^\beta Y^{4]}\nabla_\gamma Y^2\nabla^\gamma Y^3-\nabla_\beta Y^4\nabla^\beta Y^1\nabla^2\nabla_\gamma Y^{[2}\nabla^\gamma Y^{3]} \qquad \text{(B.3.15)}\\
&+2\nabla_\beta Y^3\nabla^\beta Y^4\nabla_\gamma\nabla_\alpha Y^2\nabla^\alpha\nabla^\gamma Y^1+2\nabla_\beta Y^3\nabla^\beta Y^4\nabla^2\nabla_\gamma Y^2\nabla^\gamma Y^1\\
&-4\nabla_\alpha Y^{[1}\nabla^2 Y^{4]}\nabla^\gamma\nabla^\alpha Y^{[2}\nabla_\gamma Y^{3]}+4\nabla_\alpha Y^{[1}\nabla_\beta Y^{4]}\nabla^\beta Y^{[2}\nabla^\alpha Y^{3]}\\
&-4\nabla_\alpha Y^{[1}\nabla^2 Y^{3]}\nabla^\gamma\nabla^\alpha Y^{[2}\nabla_\gamma Y^{4]}+4\nabla_\alpha Y^{[1}\nabla_\beta Y^{3]}\nabla^\beta Y^{[2}\nabla^\alpha Y^{4]}\Big)\,.
\end{aligned}
$$

All the terms in the right hand side of the last formula are reducible, *i.e.* by using eqs.(B.1.4), (B.1.5), (B.1.6), (B.2.4) they can be written via $f_5^n a_{125}a_{345}$ and permutations thereof. For instance,

$$
\begin{aligned}
\nabla^2\nabla_\beta Y^{[1}\nabla^\beta Y^{3]}\nabla_\gamma Y^2\nabla^\gamma Y^4 \;&=\; \frac{1}{2}(-f_1\nabla_\beta Y^1\nabla^\beta Y^3+f_3\nabla_\beta Y^3\nabla^\beta Y^1)\nabla_\gamma Y^2\nabla^\gamma Y^4\\
&=\; \frac{1}{2}(f_3-f_1)b_{135}b_{245}\,. \qquad \text{(B.3.16)}
\end{aligned}
$$

Proceeding in a similar manner, after tedious computation we find

$$
\begin{aligned}
V^{1234}=\;&\frac{1}{6}a_{125}a_{345}\Big(-(f_3+f_4-f_5)f_1^2+(-f_3^2+(2(f_2+f_4-2)+f_5)f_3\\
&-(-2f_2+f_4+4)(f_4-f_5))f_1-(f_2-f_5)(f_3^2+(f_2-2f_4+4)f_3\\
&+(f_4-f_5)(f_2+f_4+f_5+4))\Big)\\
&+\frac{1}{12}a_{145}a_{235}\Big((f_2+f_3-f_5)f_1^2+(f_2^2-(2(f_3+f_4-2)+f_5)f_2\\
&+(f_3-2f_4+4)(f_3-f_5))f_1+(f_4-f_5)(f_2^2+(-2f_3+f_4+4)f_2 \qquad \text{(B.3.17)}\\
&+(f_3-f_5)(f_3+f_4+f_5+4))\Big)\\
&+\frac{1}{12}a_{135}a_{245}\Big((f_2+f_4-f_5)f_1^2+(f_2^2-(2(f_3+f_4-2)+f_5)f_2\\
&+(-2f_3+f_4+4)(f_4-f_5))f_1+(f_3-f_5)(f_2^2+(f_3-2f_4+4)f_2\\
&+(f_4-f_5)(f_3+f_4+f_5+4))\Big)\,,
\end{aligned}
$$

which according to eq.(B.3.4) gives the final result for K^{1234}

$$
\begin{aligned}
&K^{1234} = \\
&\frac{1}{24}a_{125}a_{345}\Big(-4f_5^3+4(3f_3+3f_4-28)f_5^2-4(2f_4^2+5f_3f_4-28f_4+2(f_3-14)f_3+48)f_5 \\
&+6(f_3-4)(f_4-4)(f_3+f_4)+f_2(5f_3^2+8f_4f_3-64f_3+5f_4^2+12f_5^2-64f_4 \\
&-14(f_3+f_4-8)f_5+96)+f_1(6f_2^2+4(2(f_3+f_4-6)-5f_5)f_2+5f_3^2+5f_4^2+12f_5^2-64f_3 \\
&+8f_3f_4-64f_4-14(f_3+f_4-8)f_5+96)+f_2^2(5f_3+5f_4-8(f_5+3)) \\
&+f_1^2(6f_2+5f_3+5f_4-8(f_5+3))\Big) \\
&+\frac{1}{24}a_{145}a_{235}\Big(6(-f_2-f_4+4)((f_2-f_5)(f_4-f_5)-f_1f_3) \\
&-3(f_1+f_3-4)(f_2+f_4-4)(f_1+f_2+f_3+f_4-2f_5)-8f_1f_4(f_2+f_3-f_5) \\
&-8f_2f_3(f_1+f_4-f_5)+2(f_2-f_3)(f_2+f_3-f_5)(f_1+f_4-f_5) \\
&+4(f_1+f_3)(f_2+f_3-f_5)(f_1+f_4-f_5)+2(f_4-f_1)(f_2+f_3-f_5)(f_1+f_4-f_5) \\
&+8(f_2+f_3-f_5)(f_1+f_4-f_5)+2(f_2+f_3-f_5)(f_1+f_4-f_5)f_5 \\
&+6(-f_1-f_3+4)((f_5-f_1)(f_5-f_3)-f_2f_4)\Big) \\
&+\frac{1}{24}a_{135}a_{245}\Big(6(-f_1-f_4+4)((f_1-f_5)(f_4-f_5)-f_2f_3) \\
&-3(f_2+f_3-4)(f_1+f_4-4)(f_1+f_2+f_3+f_4-2f_5)-8f_2f_4(f_1+f_3-f_5) \\
&-8f_1f_3(f_2+f_4-f_5)+2(f_3-f_1)(f_1+f_3-f_5)(f_2+f_4-f_5) \\
&+2(f_2-f_4)(f_1+f_3-f_5)(f_2+f_4-f_5)+4(f_1+f_4)(f_1+f_3-f_5)(f_2+f_4-f_5) \\
&+8(f_1+f_3-f_5)(f_2+f_4-f_5)+2(f_1+f_3-f_5)(f_2+f_4-f_5)f_5 \\
&+6(-f_2-f_3+4)((f_5-f_2)(f_5-f_3)-f_1f_4)\Big)\,. \qquad \text{(B.3.18)}
\end{aligned}
$$

Appendix C

Quartic couplings

Here we collect the quartic couplings of the scalars s^I. The following notations are used

$$x \equiv k_1, \quad y \equiv k_2, \quad t \equiv k_3, \quad w \equiv k_4, \quad z \equiv k_5,$$
$$\delta = (x+1)(y+1)(t+1)(w+1).$$

Quartic couplings of 4-derivative vertices

$$
\begin{aligned}
(A_3)^{(4)}_{I_1I_2I_3I_4} &= \frac{1}{4\delta} f_5^3 \left(a_{145}a_{235} - a_{135}a_{245}\right). \\
(A_2)^{(4)}_{I_1I_2I_3I_4} &= -\frac{1}{4\delta}(3(f_1+f_2+f_3+f_4)-28) f_5^2 \left(a_{145}a_{235} - a_{135}a_{245}\right). \\
(A_1)^{(4)}_{I_1I_2I_3I_4} &= -\frac{3}{4\delta}(f_1-f_2)(f_3-f_4) f_5 a_{125}a_{345} \\
&- \frac{1}{\delta}(f_1+f_2+f_3+f_4-2)(f_1+f_2+f_3+f_4-12) f_5 \left(a_{145}a_{235} - a_{135}a_{245}\right). \\
(A_0)^{(4)}_{I_1I_2I_3I_4} &= \frac{21}{4\delta}(f_1-f_2)(f_3-f_4) a_{125}a_{345}. \\
(S_0)^{(4)}_{I_1I_2I_3I_4} &= \frac{7}{4\delta}\left(2f_1f_2 + 2f_3f_4 - (f_1+f_2)(f_3+f_4)\right) a_{125}a_{345}. \\
(A_{-1})^{(4)}_{I_1I_2I_3I_4} &= -\frac{12}{\delta}(f_1-f_2)(f_3-f_4) f_5^{-1} a_{125}a_{345}. \\
(A_{t2})^{(4)}_{I_1I_2I_3I_4} &= -\frac{3}{\delta}(f_5-1)^2 t_{125}t_{345}.
\end{aligned}
$$

Quartic couplings of 2-derivative vertices

$$
\begin{aligned}
(A_4)^{(2)}_{I_1I_2I_3I_4} &= \frac{5}{48\delta} f_5^4 \left(a_{145}a_{235} - a_{135}a_{245}\right). \\
(A_3)^{(2)}_{I_1I_2I_3I_4} &= -\frac{1}{2\delta}(k_1-k_2)(k_3-k_4) f_5^3 a_{125}a_{345}. \\
(S_3)^{(2)}_{I_1I_2I_3I_4} &= \frac{1}{16\delta}\Big(137 - 80(k_1+k_2+k_3+k_4) + 2(f_1+f_2+f_3+f_4) \\
&+ 32(k_1k_2+k_3k_4) + 24(k_1+k_2)(k_3+k_4)\Big) f_5^3 a_{125}a_{345}. \\
(A_2)^{(2)}_{I_1I_2I_3I_4} &= \frac{(k_1-k_2)(k_3-k_4)}{4\delta}\Big(40 - 12(k_1+k_2+k_3+k_4) + 2(f_1+f_2+f_3+f_4) \\
&+ 16(k_1k_2+k_3k_4) + (k_1+k_2)(k_3+k_4)\Big) f_5^2 a_{125}a_{345}. \\
(S_2)^{(2)}_{I_1I_2I_3I_4} &= -\frac{1}{16\delta}\Big(-3741 + 2984t - 342t^2 - 56t^3 + 31t^4 + 2984w - 2272tw + 376t^2w
\end{aligned}
$$

$$
\begin{aligned}
&+\ 128t^3w - 342w^2 + 376tw^2 + 42t^2w^2 - 56w^3 + 128tw^3 + 31w^4 + 2984x \\
&-\ 1760tx + 144t^2x + 88t^3x - 1760wx + 832twx + 88t^2wx + 144w^2x + 88tw^2x \\
&+\ 88w^3x - 342x^2 + 144tx^2 + 40t^2x^2 + 144wx^2 + 192twx^2 + 40w^2x^2 - 56x^3 \\
&+\ 88tx^3 + 88wx^3 + 31x^4 + 2984y - 1760ty + 144t^2y + 88t^3y - 1760wy \\
&+\ 832twy + 88t^2wy + 144w^2y + 88tw^2y + 88w^3y - 2272xy + 832txy + 192t^2xy \\
&+\ 832wxy - 128twxy + 192w^2xy + 376x^2y + 88tx^2y + 88wx^2y + 128x^3y - 342y^2 \\
&+\ 144ty^2 + 40t^2y^2 + 144wy^2 + 192twy^2 + 40w^2y^2 + 376xy^2 + 88txy^2 + 88wxy^2 \\
&+\ \left. 42x^2y^2 - 56y^3 + 88ty^3 + 88wy^3 + 128xy^3 + 31y^4 \right) f_5^2 a_{125}a_{345}.
\end{aligned}
$$

$$
\begin{aligned}
(A_1)^{(2)}_{I_1I_2I_3I_4} &= \frac{(t-w)(x-y)}{48\delta}\left(-1840 - 1964t + 160t^2 + 156t^3 + 16t^4 - 1964w \right. \\
&+\ 1312tw - 388t^2w - 128t^3w + 160w^2 - 388tw^2 - 120t^2w^2 + 156w^3 - 128tw^3 \\
&+\ 16w^4 - 1964x + 645tx - 48t^2x - 25t^3x + 645wx - 952twx - 73t^2wx \\
&-\ 48w^2x - 73tw^2x - 25w^3x + 160x^2 - 48tx^2 - 56t^2x^2 - 48wx^2 - 328twx^2 \\
&-\ 56w^2x^2 + 156x^3 - 25tx^3 - 25wx^3 + 16x^4 - 1964y + 645ty - 48t^2y \\
&-\ 25t^3y + 645wy - 952twy - 73t^2wy - 48w^2y - 73tw^2y - 25w^3y + 1312xy \\
&-\ 952txy - 328t^2xy - 952wxy - 656twxy - 328w^2xy - 388x^2y - 73tx^2y \\
&-\ 73wx^2y - 128x^3y + 160y^2 - 48ty^2 - 56t^2y^2 - 48wy^2 - 328twy^2 - 56w^2y^2 \\
&-\ 388xy^2 - 73txy^2 - 73wxy^2 - 120x^2y^2 + 156y^3 - 25ty^3 - 25wy^3 \\
&-\ \left. 128xy^3 + 16y^4 \right) f_5 a_{125}a_{345}
\end{aligned}
$$

$$
\begin{aligned}
(S_1)^{(2)}_{I_1I_2I_3I_4} &= \frac{1}{48\delta}\left(20979 - 53784t + 18666t^2 + 4056t^3 - 1197t^4 + 192t^5 + 72t^6 \right. \\
&-\ 53784w + 59648tw - 17792t^2w - 2816t^3w + 1896t^4w + 256t^5w + 18666w^2 \\
&-\ 17792tw^2 + 2736t^2w^2 + 1344t^3w^2 + 98t^4w^2 + 4056w^3 - 2816tw^3 + 1344t^2w^3 \\
&+\ 256t^3w^3 - 1197w^4 + 1896tw^4 + 98t^2w^4 + 192w^5 + 256tw^5 + 72w^6 \\
&-\ 53784x + 65168tx - 11900t^2x - 3296t^3x + 1428t^4x + 208t^5x + 65168wx \\
&-\ 53760twx + 7296t^2wx + 4000t^3wx + 144t^4wx - 11900w^2x + 7296tw^2x \\
&+\ 1760t^2w^2x + 104t^3w^2x - 3296w^3x + 4000tw^3x + 104t^2w^3x + 1428w^4x \\
&+\ 144tw^4x + 208w^5x + 18666x^2 - 11900tx^2 + 801t^2x^2 + 1488t^3x^2 \\
&+\ 173t^4x^2 - 11900wx^2 + 3840twx^2 + 4472t^2wx^2 + 704t^3wx^2 + 801w^2x^2 \\
&+\ 4472tw^2x^2 + 252t^2w^2x^2 + 1488w^3x^2 + 704tw^3x^2 + 173w^4x^2 + 4056x^3 \\
&-\ 3296tx^3 + 1488t^2x^3 + 424t^3x^3 - 3296wx^3 + 5632twx^3 + 464t^2wx^3 \\
&+\ 1488w^2x^3 + 464tw^2x^3 + 424w^3x^3 - 1197x^4 + 1428tx^4 + 173t^2x^4 \\
&+\ 1428wx^4 + 576twx^4 + 173w^2x^4 + 192x^5 + 208tx^5 + 208wx^5 + 72x^6 \\
&-\ 53784y + 65168ty - 11900t^2y - 3296t^3y + 1428t^4y + 208t^5y + 65168wy \\
&-\ 53760twy + 7296t^2wy + 4000t^3wy + 144t^4wy - 11900w^2y + 7296tw^2y \\
&+\ 1760t^2w^2y + 104t^3w^2y - 3296w^3y + 4000tw^3y + 104t^2w^3y + 1428w^4y \\
&+\ 144tw^4y + 208w^5y + 59648xy - 53760txy + 3840t^2xy + 5632t^3xy \\
&+\ 576t^4xy - 53760wxy + 23040twxy + 3264t^2wxy - 384t^3wxy + 3840w^2xy \\
&+\ 3264tw^2xy - 128t^2w^2xy + 5632w^3xy - 384tw^3xy + 576w^4xy - 17792x^2y \\
&+\ 7296tx^2y + 4472t^2x^2y + 464t^3x^2y + 7296wx^2y + 3264twx^2y + 40t^2wx^2y \\
&+\ 4472w^2x^2y + 40tw^2x^2y + 464w^3x^2y - 2816x^3y + 4000tx^3y + 704t^2x^3y \\
&+\ 4000wx^3y - 384twx^3y + 704w^2x^3y + 1896x^4y + 144tx^4y + 144wx^4y + 256x^5y \\
&+\ 18666y^2 - 11900ty^2 + 801t^2y^2 + 1488t^3y^2 + 173t^4y^2 - 11900wy^2 + 3840twy^2 \\
&+\ 4472t^2wy^2 + 704t^3wy^2 + 801w^2y^2 + 4472tw^2y^2 + 252t^2w^2y^2 + 1488w^3y^2 \\
&+\ 704tw^3y^2 + 173w^4y^2 - 17792xy^2 + 7296txy^2 + 4472t^2xy^2 + 464t^3xy^2 \\
&+\ 7296wxy^2 + 3264twxy^2 + 40t^2wxy^2 + 4472w^2xy^2 + 40tw^2xy^2 + 464w^3xy^2 \\
&+\ 2736x^2y^2 + 1760wx^2y^2 - 128twx^2y^2 + 252w^2x^2y^2 + 1344x^3y^2 + 104tx^3y^2
\end{aligned}
$$

$$
\begin{aligned}
&+ && 104wx^3y^2 + 98x^4y^2 + 4056y^3 - 3296ty^3 + 1488t^2y^3 + 424t^3y^3 \\
&- && 3296wy^3 + 5632twy^3 + 464t^2wy^3 + 1488w^2y^3 + 464tw^2y^3 + 424w^3y^3 \\
&- && 2816xy^3 + 4000txy^3 + 704t^2xy^3 + 4000wxy^3 - 384twxy^3 + 704w^2xy^3 \\
&+ && 1344x^2y^3 + 104tx^2y^3 + 104wx^2y^3 + 256x^3y^3 - 1197y^4 + 1428ty^4 + 173t^2y^4 \\
&+ && 1428wy^4 + 576twy^4 + 173w^2y^4 + 1896xy^4 + 144txy^4 + 144wxy^4 + 98x^2y^4 \\
&+ && \left. 192y^5 + 208ty^5 + 208wy^5 + 256xy^5 + 72y^6 \right) f_5 a_{125} a_{345}.
\end{aligned}
$$

$$
\begin{aligned}
(A_0)^{(2)}_{I_1I_2I_3I_4} &= && -\frac{(x-y)(t-w)}{192\delta}\Big(-144288 + 74776t + 10752t^2 - 5264t^3 \\
&+ && 992t^4 + 440t^5 + 32t^6 + 74776w + 37504tw - 11664t^2w \\
&+ && 2016t^3w + 1400t^4w + 128t^5w + 10752w^2 - 11664tw^2 + 4512t^2w^2 \\
&+ && 2088t^3w^2 + 176t^4w^2 - 5264w^3 + 2016tw^3 + 2088t^2w^3 + 256t^3w^3 \\
&+ && 992w^4 + 1400tw^4 + 176t^2w^4 + 440w^5 + 128tw^5 + 32w^6 \\
&+ && 74776x - 26042tx - 5888t^2x + 3948t^3x + 888t^4x + 46t^5x \\
&- && 26042wx - 7648twx + 6380t^2wx + 1784t^3wx + 142t^4wx - 5888w^2x \\
&+ && 6380tw^2x + 1880t^2w^2x + 170t^3w^2x + 3948w^3x + 1784tw^3x + 170t^2w^3x \\
&+ && 888w^4x + 142tw^4x + 46w^5x + 10752x^2 - 5888tx^2 + 832t^2x^2 \\
&+ && 1272t^3x^2 + 160t^4x^2 - 5888wx^2 + 768twx^2 + 1784t^2wx^2 + 144t^3wx^2 \\
&+ && 832w^2x^2 + 1784tw^2x^2 + 16t^2w^2x^2 + 1272w^3x^2 + 144tw^3x^2 + 160w^4x^2 \\
&- && 5264x^3 + 3948tx^3 + 1272t^2x^3 + 5t^3x^3 + 3948wx^3 + 1480twx^3 \\
&- && 91t^2wx^3 + 1272w^2x^3 - 91tw^2x^3 + 5w^3x^3 + 992x^4 + 888tx^4 \\
&+ && 160t^2x^4 + 888wx^4 + 208twx^4 + 160w^2x^4 + 440x^5 + 46tx^5 \\
&+ && 46wx^5 + 32x^6 + 74776y - 26042ty - 5888t^2y + 3948t^3y \\
&+ && 888t^4y + 46t^5y - 26042wy - 7648twy + 6380t^2wy + 1784t^3wy \\
&+ && 142t^4wy - 5888w^2y + 6380tw^2y + 1880t^2w^2y + 170t^3w^2y + 3948w^3y \\
&+ && 1784tw^3y + 170t^2w^3y + 888w^4y + 142tw^4y + 46w^5y + 37504xy \\
&- && 7648txy + 768t^2xy + 1480t^3xy + 208t^4xy - 7648wxy + 448twxy \\
&+ && 1608t^2wxy + 32t^3wxy + 768w^2xy + 1608tw^2xy - 352t^2w^2xy + 1480w^3xy \\
&+ && 32tw^3xy + 208w^4xy - 11664x^2y + 6380tx^2y + 1784t^2x^2y - 91t^3x^2y \\
&+ && 6380wx^2y + 1608twx^2y - 571t^2wx^2y + 1784w^2x^2y - 571tw^2x^2y - 91w^3x^2y \\
&+ && 2016x^3y + 1784tx^3y + 144t^2x^3y + 1784wx^3y + 32twx^3y + 144w^2x^3y \\
&+ && 1400x^4y + 142tx^4y + 142wx^4y + 128x^5y + 10752y^2 - 5888ty^2 \\
&+ && 832t^2y^2 + 1272t^3y^2 + 160t^4y^2 - 5888wy^2 + 768twy^2 + 1784t^2wy^2 \\
&+ && 144t^3wy^2 + 832w^2y^2 + 1784tw^2y^2 + 16t^2w^2y^2 + 1272w^3y^2 + 144tw^3y^2 \\
&+ && 160w^4y^2 - 11664xy^2 + 6380txy^2 + 1784t^2xy^2 - 91t^3xy^2 + 6380wxy^2 \\
&+ && 1608twxy^2 - 571t^2wxy^2 + 1784w^2xy^2 - 571tw^2xy^2 - 91w^3xy^2 + 4512x^2y^2 \\
&+ && 1880tx^2y^2 + 16t^2x^2y^2 + 1880wx^2y^2 - 352twx^2y^2 + 16w^2x^2y^2 + 2088x^3y^2 \\
&+ && 170tx^3y^2 + 170wx^3y^2 + 176x^4y^2 - 5264y^3 + 3948ty^3 + 1272t^2y^3 \\
&+ && 5t^3y^3 + 3948wy^3 + 1480twy^3 - 91t^2wy^3 + 1272w^2y^3 - 91tw^2y^3 \\
&+ && 5w^3y^3 + 2016xy^3 + 1784txy^3 + 144t^2xy^3 + 1784wxy^3 + 32twxy^3 \\
&+ && 144w^2xy^3 + 2088x^2y^3 + 170tx^2y^3 + 170wx^2y^3 + 256x^3y^3 + 992y^4 \\
&+ && 888ty^4 + 160t^2y^4 + 888wy^4 + 208twy^4 + 160w^2y^4 + 1400xy^4 \\
&+ && 142txy^4 + 142wxy^4 + 176x^2y^4 + 440y^5 + 46ty^5 + 46wy^5 \\
&+ && \left. 128xy^5 + 32y^6 \right) a_{125} a_{345}.
\end{aligned}
$$

$$
\begin{aligned}
(S_0)^{(2)}_{I_1I_2I_3I_4} &= && \frac{1}{576\delta}\Big(-288576tw + 149552t^2w + 21504t^3w - 10528t^4w \\
&+ && 1984t^5w + 880t^6w + 64t^7w + 149552tw^2 - 52084t^2w^2 - 11776t^3w^2 \\
&+ && 7896t^4w^2 + 1776t^5w^2 + 92t^6w^2 + 21504tw^3 - 11776t^2w^3 + 1664t^3w^3
\end{aligned}
$$

$$
\begin{aligned}
+ &\quad 2544t^4w^3 + 320t^5w^3 - 10528tw^4 + 7896t^2w^4 + 2544t^3w^4 + 10t^4w^4 \\
+ &\quad 1984tw^5 + 1776t^2w^5 + 320t^3w^5 + 880tw^6 + 92t^2w^6 + 64tw^7 \\
+ &\quad 144288tx - 74776t^2x - 10752t^3x + 5264t^4x - 992t^5x - 440t^6x \\
- &\quad 32t^7x + 144288wx + 26752t^2wx - 6400t^3wx + 1024t^4wx + 960t^5wx \\
+ &\quad 96t^6wx - 74776w^2x + 26752tw^2x - 3520t^2w^2x + 2368t^3w^2x + 1104t^4w^2x \\
+ &\quad 144t^5w^2x - 10752w^3x - 6400tw^3x + 2368t^2w^3x + 1024t^3w^3x - 112t^4w^3x \\
+ &\quad 5264w^4x + 1024tw^4x + 1104t^2w^4x - 112t^3w^4x - 992w^5x + 960tw^5x \\
+ &\quad 144t^2w^5x - 440w^6x + 96tw^6x - 32w^7x - 74776tx^2 + 26042t^2x^2 \\
+ &\quad 5888t^3x^2 - 3948t^4x^2 - 888t^5x^2 - 46t^6x^2 - 74776wx^2 - 53504twx^2 \\
+ &\quad 1760t^2wx^2 + 2560t^3wx^2 + 480t^4wx^2 + 26042w^2x^2 + 1760tw^2x^2 + 96t^3w^2x^2 \\
+ &\quad 28t^4w^2x^2 + 5888w^3x^2 + 2560tw^3x^2 + 96t^2w^3x^2 - 256t^3w^3x^2 - 3948w^4x^2 \\
+ &\quad 480tw^4x^2 + 28t^2w^4x^2 - 888w^5x^2 - 46w^6x^2 - 10752tx^3 + 5888t^2x^3 \\
- &\quad 832t^3x^3 - 1272t^4x^3 - 160t^5x^3 - 10752wx^3 + 12800twx^3 - 4928t^2wx^3 \\
- &\quad 512t^3wx^3 + 176t^4wx^3 + 5888w^2x^3 - 4928tw^2x^3 - 192t^2w^2x^3 + 128t^3w^2x^3 \\
- &\quad 832w^3x^3 - 512tw^3x^3 + 128t^2w^3x^3 - 1272w^4x^3 + 176tw^4x^3 - 160w^5x^3 \\
+ &\quad 5264tx^4 - 3948t^2x^4 - 1272t^3x^4 - 5t^4x^4 + 5264wx^4 - 2048twx^4 \\
- &\quad 1584t^2wx^4 - 64t^3wx^4 - 3948w^2x^4 - 1584tw^2x^4 - 56t^2w^2x^4 - 1272w^3x^4 \\
- &\quad 64tw^3x^4 - 5w^4x^4 - 992tx^5 - 888t^2x^5 - 160t^3x^5 - 992wx^5 \\
- &\quad 1920twx^5 - 144t^2wx^5 - 888w^2x^5 - 144tw^2x^5 - 160w^3x^5 - 440tx^6 \\
- &\quad 46t^2x^6 - 440wx^6 - 192twx^6 - 46w^2x^6 - 32tx^7 - 32wx^7 \\
+ &\quad 144288ty - 74776t^2y - 10752t^3y + 5264t^4y - 992t^5y - 440t^6y \\
- &\quad 32t^7y + 144288wy + 26752t^2wy - 6400t^3wy + 1024t^4wy + 960t^5wy \\
+ &\quad 96t^6wy - 74776w^2y + 26752tw^2y - 3520t^2w^2y + 2368t^3w^2y + 1104t^4w^2y \\
+ &\quad 144t^5w^2y - 10752w^3y - 6400tw^3y + 2368t^2w^3y + 1024t^3w^3y - 112t^4w^3y \\
+ &\quad 5264w^4y + 1024tw^4y + 1104t^2w^4y - 112t^3w^4y - 992w^5y + 960tw^5y \\
+ &\quad 144t^2w^5y - 440w^6y + 96tw^6y - 32w^7y - 288576xy - 53504t^2xy \\
+ &\quad 12800t^3xy - 2048t^4xy - 1920t^5xy - 192t^6xy - 53504w^2xy - 512t^2w^2xy \\
- &\quad 768t^3w^2xy - 320t^4w^2xy + 12800w^3xy - 768t^2w^3xy - 768t^3w^3xy - 2048w^4xy \\
- &\quad 320t^2w^4xy - 1920w^5xy - 192w^6xy + 149552x^2y + 26752tx^2y + 1760t^2x^2y \\
- &\quad 4928t^3x^2y - 1584t^4x^2y - 144t^5x^2y + 26752wx^2y + 256t^2wx^2y + 384t^3wx^2y \\
+ &\quad 160t^4wx^2y + 1760w^2x^2y + 256tw^2x^2y - 256t^3w^2x^2y - 4928w^3x^2y + 384tw^3x^2y \\
- &\quad 256t^2w^3x^2y - 1584w^4x^2y + 160tw^4x^2y - 144w^5x^2y + 21504x^3y - 6400tx^3y \\
+ &\quad 2560t^2x^3y - 512t^3x^3y - 64t^4x^3y - 6400wx^3y + 384t^2wx^3y + 384t^3wx^3y \\
+ &\quad 2560w^2x^3y + 384tw^2x^3y + 512t^2w^2x^3y - 512w^3x^3y + 384tw^3x^3y - 64w^4x^3y \\
- &\quad 10528x^4y + 1024tx^4y + 480t^2x^4y + 176t^3x^4y + 1024wx^4y + 160t^2wx^4y \\
+ &\quad 480w^2x^4y + 160tw^2x^4y + 176w^3x^4y + 1984x^5y + 960tx^5y + 960wx^5y \\
+ &\quad 880x^6y + 96tx^6y + 96wx^6y + 64x^7y - 74776ty^2 + 26042t^2y^2 \\
+ &\quad 5888t^3y^2 - 3948t^4y^2 - 888t^5y^2 - 46t^6y^2 - 74776wy^2 - 53504twy^2 \\
+ &\quad 1760t^2wy^2 + 2560t^3wy^2 + 480t^4wy^2 + 26042w^2y^2 + 1760tw^2y^2 + 96t^3w^2y^2 \\
+ &\quad 28t^4w^2y^2 + 5888w^3y^2 + 2560tw^3y^2 + 96t^2w^3y^2 - 256t^3w^3y^2 - 3948w^4y^2 \\
+ &\quad 480tw^4y^2 + 28t^2w^4y^2 - 888w^5y^2 - 46w^6y^2 + 149552xy^2 + 26752txy^2 \\
+ &\quad 1760t^2xy^2 - 4928t^3xy^2 - 1584t^4xy^2 - 144t^5xy^2 + 26752wxy^2 + 256t^2wxy^2 \\
+ &\quad 384t^3wxy^2 + 160t^4wxy^2 + 1760w^2xy^2 + 256tw^2xy^2 - 256t^3w^2xy^2 - 4928w^3xy^2 \\
+ &\quad 384tw^3xy^2 - 256t^2w^3xy^2 - 1584w^4xy^2 + 160tw^4xy^2 - 144w^5xy^2 - 52084x^2y^2 \\
- &\quad 3520tx^2y^2 - 192t^3x^2y^2 - 56t^4x^2y^2 - 3520wx^2y^2 - 512twx^2y^2 + 512t^3wx^2y^2 \\
- &\quad 192w^3x^2y^2 + 512tw^3x^2y^2 - 56w^4x^2y^2 - 11776x^3y^2 + 2368tx^3y^2 + 96t^2x^3y^2 \\
+ &\quad 128t^3x^3y^2 + 2368wx^3y^2 - 768twx^3y^2 - 256t^2wx^3y^2 + 96w^2x^3y^2 - 256tw^2x^3y^2 \\
+ &\quad 128w^3x^3y^2 + 7896x^4y^2 + 1104tx^4y^2 + 28t^2x^4y^2 + 1104wx^4y^2 - 320twx^4y^2 \\
+ &\quad 28w^2x^4y^2 + 1776x^5y^2 + 144tx^5y^2 + 144wx^5y^2 + 92x^6y^2
\end{aligned}
$$

$$
\begin{aligned}
&- && 10752ty^3 + 5888t^2y^3 - 832t^3y^3 - 1272t^4y^3 - 160t^5y^3 - 10752wy^3 \\
&+ && 12800twy^3 - 4928t^2wy^3 - 512t^3wy^3 + 176t^4wy^3 + 5888w^2y^3 - 4928tw^2y^3 \\
&- && 192t^2w^2y^3 + 128t^3w^2y^3 - 832w^3y^3 - 512tw^3y^3 + 128t^2w^3y^3 - 1272w^4y^3 \\
&+ && 176tw^4y^3 - 160w^5y^3 + 21504xy^3 - 6400txy^3 + 2560t^2xy^3 - 512t^3xy^3 \\
&- && 64t^4xy^3 - 6400wxy^3 + 384t^2wxy^3 + 384t^3wxy^3 + 2560w^2xy^3 + 384tw^2xy^3 \\
&+ && 512t^2w^2xy^3 - 512w^3xy^3 + 384tw^3xy^3 - 64w^4xy^3 - 11776x^2y^3 + 2368tx^2y^3 \\
&+ && 96t^2x^2y^3 + 128t^3x^2y^3 + 2368wx^2y^3 - 768twx^2y^3 - 256t^2wx^2y^3 + 96w^2x^2y^3 \\
&- && 256tw^2x^2y^3 + 128w^3x^2y^3 + 1664x^3y^3 + 1024tx^3y^3 - 256t^2x^3y^3 + 1024wx^3y^3 \\
&- && 768twx^3y^3 - 256w^2x^3y^3 + 2544x^4y^3 - 112tx^4y^3 - 112wx^4y^3 + 320x^5y^3 \\
&+ && 5264ty^4 - 3948t^2y^4 - 1272t^3y^4 - 5t^4y^4 + 5264wy^4 - 2048twy^4 \\
&- && 1584t^2wy^4 - 64t^3wy^4 - 3948w^2y^4 - 1584tw^2y^4 - 56t^2w^2y^4 - 1272w^3y^4 \\
&- && 64tw^3y^4 - 5w^4y^4 - 10528xy^4 + 1024txy^4 + 480t^2xy^4 + 176t^3xy^4 \\
&+ && 1024wxy^4 + 160t^2wxy^4 + 480w^2xy^4 + 160tw^2xy^4 + 176w^3xy^4 + 7896x^2y^4 \\
&+ && 1104tx^2y^4 + 28t^2x^2y^4 + 1104wx^2y^4 - 320twx^2y^4 + 28w^2x^2y^4 + 2544x^3y^4 \\
&- && 112tx^3y^4 - 112wx^3y^4 + 10x^4y^4 - 992ty^5 - 888t^2y^5 - 160t^3y^5 \\
&- && 992wy^5 - 1920twy^5 - 144t^2wy^5 - 888w^2y^5 - 144tw^2y^5 - 160w^3y^5 \\
&+ && 1984xy^5 + 960txy^5 + 960wxy^5 + 1776x^2y^5 + 144tx^2y^5 + 144wx^2y^5 \\
&+ && 320x^3y^5 - 440ty^6 - 46t^2y^6 - 440wy^6 - 192twy^6 - 46w^2y^6 \\
&+ && 880xy^6 + 96txy^6 + 96wxy^6 + 92x^2y^6 - 32ty^7 - 32wy^7 \\
&+ && 64xy^7 \Big) a_{125}a_{345}.
\end{aligned}
$$

$$
\begin{aligned}
(A_{-1})^{(2)}_{I_1I_2I_3I_4} &= && -\frac{4}{\delta}(f_1-f_2)(f_3-f_4)\Big(36 - 20(k_1+k_2+k_3+k_4) + f_1+f_2+f_3+f_4 \\
&+ && 10(k_1+k_2)(k_3+k_4)\Big) f_5^{-1} a_{125}a_{345}. \\
(S_{-1})^{(2)}_{I_1I_2I_3I_4} &= && -\frac{8}{\delta}(f_1-f_2)(f_3-f_4)(k_1-k_2)(k_3-k_4) f_5^{-1} a_{125}a_{345}. \\
(A_{t3})^{(2)}_{I_1I_2I_3I_4} &= && -\frac{2}{\delta}(f_5-1)^3 t_{125}t_{345}. \\
(A_{t2})^{(2)}_{I_1I_2I_3I_4} &= && -\frac{1}{\delta}(f_5-1)^2 t_{125}t_{345}(k_1^2+k_2^2+k_3^2+k_4^2 \\
&- && 16(k_1+k_2+k_3+k_4) + 10(k_1+k_2)(k_3+k_4) + 44). \\
(S_{t2})^{(2)}_{I_1I_2I_3I_4} &= && -\frac{2}{\delta}(k_1-k_2)(k_3-k_4)(f_5-1)^2 t_{125}t_{345}. \\
(S_{p2})^{(2)}_{I_1I_2I_3I_4} &= && \frac{2}{\delta} f_5^2 p_{125}p_{345}\left(k_1^2+k_2^2+k_3^2+k_4^2 - 2(k_1+k_2+k_3+k_4) + 2(k_1k_2+k_3k_4) - 4\right). \\
(S_d)^{(2)}_{I_1I_2I_3I_4} &= && \frac{9a_{125}a_{345}}{16\delta(f_5-5)}(-1+k_1-k_2)(1+k_1-k_2)(3+k_1+k_2)(5+k_1+k_2) \\
&\times && (-1+k_3-k_4)(1+k_3-k_4)(3+k_3+k_4)(5+k_3+k_4).
\end{aligned}
$$

Quartic couplings of non-derivative vertices

$$
\begin{aligned}
(S_5)^{(0)}_{I_1I_2I_3I_4} &= && \frac{3}{64\delta} f_5^5 a_{125}a_{345}. \\
(S_4)^{(0)}_{I_1I_2I_3I_4} &= && -\frac{1}{192\delta} f_5^4 a_{125}a_{345}\Big(747 - 368(k_1+k_2+k_3+k_4) + 65(k_1^2+k_2^2+k_3^2+k_4^2) \\
&+ && 132(k_1k_2+k_3k_4) - 96(k_1+k_2)(k_3+k_4)\Big). \\
(S_3)^{(0)}_{I_1I_2I_3I_4} &= && \frac{1}{64\delta} f_5^3 a_{125}a_{345}\Big(-3293 + 4036t - 1012t^2 - 96t^3
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 35t^4+4036w-2428tw+48t^2w+88t^3w-1012w^2+48tw^2\\
&+ \quad 122t^2w^2-96w^3+88tw^3+35w^4+4036x-2976tx+360t^2x\\
&+ \quad 40t^3x-2976wx+1056twx-8t^2wx+360w^2x-8tw^2x+40w^3x\\
&- \quad 1012x^2+360tx^2+44t^2x^2+360wx^2-8twx^2+44w^2x^2-96x^3\\
&+ \quad 40tx^3+40wx^3+35x^4+4036y-2976ty+360t^2y+40t^3y\\
&- \quad 2976wy+1056twy-8t^2wy+360w^2y-8tw^2y+40w^3y-2428xy\\
&+ \quad 1056txy-8t^2xy+1056wxy-8w^2xy+48x^2y-8tx^2y-8wx^2y\\
&+ \quad 88x^3y-1012y^2+360ty^2+44t^2y^2+360wy^2-8twy^2+44w^2y^2\\
&+ \quad 48xy^2-8txy^2-8wxy^2+122x^2y^2-96y^3+40ty^3\\
&+ \quad 40wy^3+88xy^3+35y^4\Big).
\end{aligned}
$$

$$
\begin{aligned}
(S_2)^{(0)}_{I_1I_2I_3I_4} \quad = \quad & -\frac{1}{64\delta}f_5^2a_{125}a_{345}\Big(8273-20116t+9396t^2+1008t^3\\
&- \quad 1227t^4+36t^5+26t^6-20116w+25644tw-2688t^2w\\
&- \quad 3544t^3w+356t^4w+76t^5w+9396w^2-2688tw^2-2778t^2w^2\\
&+ \quad 664t^3w^2+46t^4w^2+1008w^3-3544tw^3+664t^2w^3+104t^3w^3\\
&- \quad 1227w^4+356tw^4+46t^2w^4+36w^5+76tw^5+26w^6-20116x\\
&+ \quad 32384tx-9032t^2x-1696t^3x+492t^4x+8t^5x+32384wx\\
&- \quad 23776twx+224t^2wx+1152t^3wx-104t^4wx-9032w^2x+224tw^2x\\
&+ \quad 1096t^2w^2x-224t^3w^2x-1696w^3x+1152tw^3x-224t^2w^3x+492w^4x\\
&- \quad 104tw^4x+8w^5x+9396x^2-9032tx^2+332t^2x^2+288t^3x^2\\
&+ \quad 60t^4x^2-9032wx^2+2152twx^2-96t^2wx^2+144t^3wx^2+332w^2x^2\\
&- \quad 96tw^2x^2+96t^2w^2x^2+288w^3x^2+144tw^3x^2+60w^4x^2+1008x^3\\
&- \quad 1696tx^3+288t^2x^3+80t^3x^3-1696wx^3+608twx^3+32t^2wx^3\\
&+ \quad 288w^2x^3+32tw^2x^3+80w^3x^3-1227x^4+492tx^4+60t^2x^4\\
&+ \quad 492wx^4-20twx^4+60w^2x^4+36x^5+8tx^5+8wx^5+26x^6\\
&- \quad 20116y+32384ty-9032t^2y-1696t^3y+492t^4y+8t^5y\\
&+ \quad 32384wy-23776twy+224t^2wy+1152t^3wy-104t^4wy-9032w^2y\\
&+ \quad 224tw^2y+1096t^2w^2y-224t^3w^2y-1696w^3y+1152tw^3y-224t^2w^3y\\
&+ \quad 492w^4y-104tw^4y+8w^5y+25644xy-23776txy+2152t^2xy\\
&+ \quad 608t^3xy-20t^4xy-23776wxy+10496twxy+32t^2wxy-128t^3wxy\\
&+ \quad 2152w^2xy+32tw^2xy+168t^2w^2xy+608w^3xy-128tw^3xy-20w^4xy\\
&- \quad 2688x^2y+224tx^2y-96t^2x^2y+32t^3x^2y+224wx^2y+32twx^2y-16t^2wx^2y\\
&- \quad 96w^2x^2y-16tw^2x^2y+32w^3x^2y-3544x^3y+1152tx^3y+144t^2x^3y\\
&+ \quad 1152wx^3y-128twx^3y+144w^2x^3y+356x^4y-104tx^4y-104wx^4y\\
&+ \quad 76x^5y+9396y^2-9032ty^2+332t^2y^2+288t^3y^2+60t^4y^2\\
&- \quad 9032wy^2+2152twy^2-96t^2wy^2+144t^3wy^2+332w^2y^2-96tw^2y^2\\
&+ \quad 96t^2w^2y^2+288w^3y^2+144tw^3y^2+60w^4y^2-2688xy^2+224txy^2\\
&- \quad 96t^2xy^2+32t^3xy^2+224wxy^2+32twxy^2-16t^2wxy^2-96w^2xy^2-16tw^2xy^2\\
&+ \quad 32w^3xy^2-2778x^2y^2+1096tx^2y^2+96t^2x^2y^2+1096wx^2y^2+168twx^2y^2\\
&+ \quad 96w^2x^2y^2+664x^3y^2-224tx^3y^2-224wx^3y^2+46x^4y^2+1008y^3\\
&- \quad 1696ty^3+288t^2y^3+80t^3y^3-1696wy^3+608twy^3+32t^2wy^3\\
&+ \quad 288w^2y^3+32tw^2y^3+80w^3y^3-3544xy^3+1152txy^3+144t^2xy^3\\
&+ \quad 1152wxy^3-128twxy^3+144w^2xy^3+664x^2y^3-224tx^2y^3-224wx^2y^3\\
&+ \quad 104x^3y^3-1227y^4+492ty^4+60t^2y^4+492wy^4-20twy^4\\
&+ \quad 60w^2y^4+356xy^4-104txy^4-104wxy^4+46x^2y^4+36y^5
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 8ty^5+8wy^5+76xy^5+26y^6\Big).\\
(S_1)^{(0)}_{I_1I_2I_3I_4} &= \frac{(w-x)(t-y)}{288\delta}f_5a_{125}a_{345}\Big(163692-128440t+28616t^2+2052t^3\\
&- \quad 3460t^4+484t^5+80t^6-128440w+72314tw-3096t^2w\\
&+ \quad 393t^3w+468t^4w-119t^5w+28616w^2-3096tw^2-1208t^2w^2\\
&- \quad 864t^3w^2+184t^4w^2+2052w^3+393tw^3-864t^2w^3+41t^3w^3\\
&- \quad 3460w^4+468tw^4+184t^2w^4+484w^5-119tw^5+80w^6-128440x\\
&+ \quad 72314tx-3096t^2x+393t^3x+468t^4x-119t^5x+88352wx\\
&- \quad 31064twx+13912t^2wx-2360t^3wx-392t^4wx-18716w^2x+15049tw^2x\\
&- \quad 5472t^2w^2x+641t^3w^2x-416w^3x-1416tw^3x+312t^2w^3x+1380w^4x\\
&- \quad 335tw^4x-256w^5x+28616x^2-3096tx^2-1208t^2x^2-864t^3x^2\\
&+ \quad 184t^4x^2-18716wx^2+15049twx^2-5472t^2wx^2+641t^3wx^2-880w^2x^2\\
&- \quad 4496tw^2x^2+2392t^2w^2x^2+992w^3x^2+524tw^3x^2+104w^4x^2+2052x^3+393tx^3\\
&- \quad 864t^2x^3+41t^3x^3-416wx^3-1416twx^3+312t^2wx^3+992w^2x^3\\
&+ \quad 524tw^2x^3-368w^3x^3-3460x^4+468tx^4+184t^2x^4+1380wx^4\\
&- \quad 335twx^4+104w^2x^4+484x^5-119tx^5-256wx^5+80x^6\\
&- \quad 128440y+88352ty-18716t^2y-416t^3y+1380t^4y-256t^5y\\
&+ \quad 72314wy-31064twy+15049t^2wy-1416t^3wy-335t^4wy-3096w^2y\\
&+ \quad 13912tw^2y-5472t^2w^2y+312t^3w^2y+393w^3y-2360tw^3y+641t^2w^3y\\
&+ \quad 468w^4y-392tw^4y-119w^5y+72314xy-31064txy+15049t^2xy\\
&- \quad 1416t^3xy-335t^4xy-31064wxy+34928twxy-15928t^2wxy+1664t^3wxy\\
&+ \quad 15049w^2xy-15928tw^2xy+4313t^2w^2xy-1416w^3xy+1664tw^3xy-335w^4xy\\
&- \quad 3096x^2y+13912tx^2y-5472t^2x^2y+312t^3x^2y+15049wx^2y-15928twx^2y\\
&+ \quad 4313t^2wx^2y-4496w^2x^2y+3488tw^2x^2y+524w^3x^2y+393x^3y-2360tx^3y\\
&+ \quad 641t^2x^3y-1416wx^3y+1664twx^3y+524w^2x^3y+468x^4y-392tx^4y\\
&- \quad 335wx^4y-119x^5y+28616y^2-18716ty^2-880t^2y^2+992t^3y^2\\
&+ \quad 104t^4y^2-3096wy^2+15049twy^2-4496t^2wy^2+524t^3wy^2-1208w^2y^2\\
&- \quad 5472tw^2y^2+2392t^2w^2y^2-864w^3y^2+641tw^3y^2+184w^4y^2-3096xy^2\\
&+ \quad 15049txy^2-4496t^2xy^2+524t^3xy^2+13912wxy^2-15928twxy^2+3488t^2wxy^2\\
&- \quad 5472w^2xy^2+4313tw^2xy^2+312w^3xy^2-1208x^2y^2-5472tx^2y^2+2392t^2x^2y^2\\
&- \quad 5472wx^2y^2+4313twx^2y^2+2392w^2x^2y^2-864x^3y^2+641tx^3y^2+312wx^3y^2\\
&+ \quad 184x^4y^2+2052y^3-416ty^3+992t^2y^3-368t^3y^3+393wy^3\\
&- \quad 1416twy^3+524t^2wy^3-864w^2y^3+312tw^2y^3+41w^3y^3+393xy^3\\
&- \quad 1416txy^3+524t^2xy^3-2360wxy^3+1664twxy^3+641w^2xy^3-864x^2y^3\\
&+ \quad 312tx^2y^3+641wx^2y^3+41x^3y^3-3460y^4+1380ty^4+104t^2y^4\\
&+ \quad 468wy^4-335twy^4+184w^2y^4+468xy^4-335txy^4-392wxy^4\\
&+ \quad 184x^2y^4+484y^5-256ty^5-119wy^5-119xy^5+80y^6\Big).\\
(S_0)^{(0)}_{I_1I_2I_3I_4} &= \frac{1}{576\delta}a_{125}a_{345}\Big(-18225-24300t+292830t^2-71028t^3\\
&- \quad 111795t^4+33444t^5+8640t^6-5124t^7-618t^8+192t^9\\
&+ \quad 24t^{10}-24300w-256068tw-66756t^2w-120628t^3w+177012t^4w\\
&- \quad 17836t^5w-26900t^6w+132t^7w+800t^8w+32t^9w+292830w^2\\
&- \quad 66756tw^2+177178t^2w^2+160520t^3w^2-94964t^4w^2-31788t^5w^2+1416t^6w^2\\
&+ \quad 1136t^7w^2+112t^8w^2-71028w^3-120628tw^3+160520t^2w^3-11080t^3w^3\\
&- \quad 46764t^4w^3-3076t^5w^3+2640t^6w^3+336t^7w^3-111795w^4+177012tw^4\\
&- \quad 94964t^2w^4-46764t^3w^4+6638t^4w^4+2920t^5w^4+224t^6w^4+33444w^5\\
&- \quad 17836tw^5-31788t^2w^5-3076t^3w^5+2920t^4w^5+416t^5w^5+8640w^6
\end{aligned}
$$

$$
\begin{aligned}
- &\quad 26900tw^6 + 1416t^2w^6 + 2640t^3w^6 + 224t^4w^6 - 5124w^7 + 132tw^7 \\
+ &\quad 1136t^2w^7 + 336t^3w^7 - 618w^8 + 800tw^8 + 112t^2w^8 + 192w^9 \\
+ &\quad 32tw^9 + 24w^{10} - 24300x - 256068tx - 66756t^2x - 120628t^3x \\
+ &\quad 177012t^4x - 17836t^5x - 26900t^6x + 132t^7x + 800t^8x + 32t^9x \\
- &\quad 256068wx + 12816twx + 319756t^2wx + 370976t^3wx \\
- &\quad 220300t^4wx - 57456t^5wx + 5828t^6wx + 1024t^7wx - 96t^8wx \\
- &\quad 66756w^2x + 319756tw^2x + 184632t^2w^2x - 247264t^3w^2x \\
- &\quad 68020t^4w^2x - 2724t^5w^2x + 1792t^6w^2x + 272t^7w^2x - 120628w^3x \\
+ &\quad 370976tw^3x - 247264t^2w^3x - 124960t^3w^3x + 6940t^4w^3x + 4000t^5w^3x \\
+ &\quad 352t^6w^3x + 177012w^4x - 220300tw^4x - 68020t^2w^4x + 6940t^3w^4x \\
+ &\quad 1672t^4w^4x + 268t^5w^4x - 17836w^5x - 57456tw^5x - 2724t^2w^5x + 4000t^3w^5x \\
+ &\quad 268t^4w^5x - 26900w^6x + 5828tw^6x + 1792t^2w^6x + 352t^3w^6x + 132w^7x \\
+ &\quad 1024tw^7x + 272t^2w^7x + 800w^8x - 96tw^8x + 32w^9x + 292830x^2 \\
- &\quad 66756tx^2 + 177178t^2x^2 + 160520t^3x^2 - 94964t^4x^2 - 31788t^5x^2 \\
+ &\quad 1416t^6x^2 + 1136t^7x^2 + 112t^8x^2 - 66756wx^2 + 319756twx^2 + 184632t^2wx^2 \\
- &\quad 247264t^3wx^2 - 68020t^4wx^2 - 2724t^5wx^2 + 1792t^6wx^2 \\
+ &\quad 272t^7wx^2 + 177178w^2x^2 + 184632tw^2x^2 - 335076t^2w^2x^2 \\
- &\quad 134456t^3w^2x^2 - 1458t^4w^2x^2 + 4296t^5w^2x^2 \\
+ &\quad 848t^6w^2x^2 + 160520w^3x^2 - 247264tw^3x^2 - 134456t^2w^3x^2 + 7688t^3w^3x^2 \\
+ &\quad 4056t^4w^3x^2 + 592t^5w^3x^2 - 94964w^4x^2 - 68020tw^4x^2 - 1458t^2w^4x^2 \\
+ &\quad 4056t^3w^4x^2 - 56t^4w^4x^2 - 31788w^5x^2 - 2724tw^5x^2 + 4296t^2w^5x^2 + 592t^3w^5x^2 \\
+ &\quad 1416w^6x^2 + 1792tw^6x^2 + 848t^2w^6x^2 + 1136w^7x^2 + 272tw^7x^2 + 112w^8x^2 \\
- &\quad 71028x^3 - 120628tx^3 + 160520t^2x^3 - 11080t^3x^3 - 46764t^4x^3 \\
- &\quad 3076t^5x^3 + 2640t^6x^3 + 336t^7x^3 - 120628wx^3 + 370976twx^3 \\
- &\quad 247264t^2wx^3 - 124960t^3wx^3 + 6940t^4wx^3 + 4000t^5wx^3 + 352t^6wx^3 \\
+ &\quad 160520w^2x^3 - 247264tw^2x^3 - 134456t^2w^2x^3 + 7688t^3w^2x^3 + 4056t^4w^2x^3 \\
+ &\quad 592t^5w^2x^3 - 11080w^3x^3 - 124960tw^3x^3 + 7688t^2w^3x^3 + 6720t^3w^3x^3 \\
- &\quad 1096t^4w^3x^3 - 46764w^4x^3 + 6940tw^4x^3 + 4056t^2w^4x^3 \\
- &\quad 1096t^3w^4x^3 - 3076w^5x^3 + 4000tw^5x^3 + 592t^2w^5x^3 \\
+ &\quad 2640w^6x^3 + 352tw^6x^3 + 336w^7x^3 - 111795x^4 + 177012tx^4 - 94964t^2x^4 \\
- &\quad 46764t^3x^4 + 6638t^4x^4 + 2920t^5x^4 + 224t^6x^4 + 177012wx^4 - 220300twx^4 \\
- &\quad 68020t^2wx^4 + 6940t^3wx^4 + 1672t^4wx^4 + 268t^5wx^4 - 94964w^2x^4 - 68020tw^2x^4 \\
- &\quad 1458t^2w^2x^4 + 4056t^3w^2x^4 - 56t^4w^2x^4 - 46764w^3x^4 + 6940tw^3x^4 + 4056t^2w^3x^4 \\
- &\quad 1096t^3w^3x^4 + 6638w^4x^4 + 1672tw^4x^4 - 56t^2w^4x^4 + 2920w^5x^4 + 268tw^5x^4 \\
+ &\quad 224w^6x^4 + 33444x^5 - 17836tx^5 - 31788t^2x^5 - 3076t^3x^5 + 2920t^4x^5 \\
+ &\quad 416t^5x^5 - 17836wx^5 - 57456twx^5 - 2724t^2wx^5 + 4000t^3wx^5 + 268t^4wx^5 \\
- &\quad 31788w^2x^5 - 2724tw^2x^5 + 4296t^2w^2x^5 + 592t^3w^2x^5 - 3076w^3x^5 + 4000tw^3x^5 \\
+ &\quad 592t^2w^3x^5 + 2920w^4x^5 + 268tw^4x^5 + 416w^5x^5 + 8640x^6 - 26900tx^6 \\
+ &\quad 1416t^2x^6 + 2640t^3x^6 + 224t^4x^6 - 26900wx^6 + 5828twx^6 + 1792t^2wx^6 \\
+ &\quad 352t^3wx^6 + 1416w^2x^6 + 1792tw^2x^6 + 848t^2w^2x^6 + 2640w^3x^6 + 352tw^3x^6 \\
+ &\quad 224w^4x^6 - 5124x^7 + 132tx^7 + 1136t^2x^7 + 336t^3x^7 + 132wx^7 \\
+ &\quad 1024twx^7 + 272t^2wx^7 + 1136w^2x^7 + 272tw^2x^7 + 336w^3x^7 - 618x^8 \\
+ &\quad 800tx^8 + 112t^2x^8 + 800wx^8 - 96twx^8 + 112w^2x^8 + 192x^9 + 32tx^9 \\
+ &\quad 32wx^9 + 24x^{10} - 24300y - 256068ty - 66756t^2y - 120628t^3y \\
+ &\quad 177012t^4y - 17836t^5y - 26900t^6y + 132t^7y + 800t^8y + 32t^9y \\
- &\quad 256068wy + 12816twy + 319756t^2wy + 370976t^3wy - 220300t^4wy \\
- &\quad 57456t^5wy + 5828t^6wy + 1024t^7wy - 96t^8wy - 66756w^2y \\
+ &\quad 319756tw^2y + 184632t^2w^2y - 247264t^3w^2y - 68020t^4w^2y - 2724t^5w^2y \\
+ &\quad 1792t^6w^2y + 272t^7w^2y - 120628w^3y + 370976tw^3y - 247264t^2w^3y
\end{aligned}
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\begin{aligned}
- &\ 124960t^3w^3y + 6940t^4w^3y + 4000t^5w^3y + 352t^6w^3y + 177012w^4y - 220300tw^4y \\
- &\ 68020t^2w^4y + 6940t^3w^4y + 1672t^4w^4y + 268t^5w^4y - 17836w^5y \\
- &\ 57456tw^5y - 2724t^2w^5y + 4000t^3w^5y + 268t^4w^5y - 26900w^6y \\
+ &\ 5828tw^6y + 1792t^2w^6y + 352t^3w^6y + 132w^7y \\
+ &\ 1024tw^7y + 272t^2w^7y + 800w^8y - 96tw^8y \\
+ &\ 32w^9y - 256068xy + 12816txy + 319756t^2xy + 370976t^3xy - 220300t^4xy \\
- &\ 57456t^5xy + 5828t^6xy + 1024t^7xy - 96t^8xy + 12816wxy + 2404608twxy \\
- &\ 48096t^2wxy - 749568t^3wxy - 41136t^4wxy \\
+ &\ 4992t^5wxy - 2112t^6wxy - 384t^7wxy \\
+ &\ 319756w^2xy - 48096tw^2xy - 403336t^2w^2xy - 128736t^3w^2xy - 11588t^4w^2xy \\
+ &\ 2432t^5w^2xy + 688t^6w^2xy + 370976w^3xy - 749568tw^3xy - 128736t^2w^3xy \\
+ &\ 39680t^3w^3xy + 896t^4w^3xy + 384t^5w^3xy - 220300w^4xy - 41136tw^4xy \\
- &\ 11588t^2w^4xy + 896t^3w^4xy + 608t^4w^4xy \\
- &\ 57456w^5xy + 4992tw^5xy + 2432t^2w^5xy \\
+ &\ 384t^3w^5xy + 5828w^6xy - 2112tw^6xy + 688t^2w^6xy + 1024w^7xy - 384tw^7xy \\
- &\ 96w^8xy - 66756x^2y + 319756tx^2y + 184632t^2x^2y - 247264t^3x^2y - 68020t^4x^2y \\
- &\ 2724t^5x^2y + 1792t^6x^2y + 272t^7x^2y + 319756wx^2y - 48096twx^2y \\
- &\ 403336t^2wx^2y - 128736t^3wx^2y - 11588t^4wx^2y + 2432t^5wx^2y \\
+ &\ 688t^6wx^2y + 184632w^2x^2y - 403336tw^2x^2y - 169560t^2w^2x^2y \\
+ &\ 17288t^3w^2x^2y + 4136t^4w^2x^2y + 1112t^5w^2x^2y - 247264w^3x^2y \\
- &\ 128736tw^3x^2y + 17288t^2w^3x^2y + 2496t^3w^3x^2y \\
- &\ 1856t^4w^3x^2y - 68020w^4x^2y - 11588tw^4x^2y + 4136t^2w^4x^2y \\
- &\ 1856t^3w^4x^2y - 2724w^5x^2y + 2432tw^5x^2y \\
+ &\ 1112t^2w^5x^2y + 1792w^6x^2y + 688tw^6x^2y + 272w^7x^2y - 120628x^3y \\
+ &\ 370976tx^3y - 247264t^2x^3y - 124960t^3x^3y + 6940t^4x^3y + 4000t^5x^3y \\
+ &\ 352t^6x^3y + 370976wx^3y - 749568twx^3y - 128736t^2wx^3y + 39680t^3wx^3y \\
+ &\ 896t^4wx^3y + 384t^5wx^3y - 247264w^2x^3y - 128736tw^2x^3y + 17288t^2w^2x^3y \\
+ &\ 2496t^3w^2x^3y - 1856t^4w^2x^3y - 124960w^3x^3y + 39680tw^3x^3y + 2496t^2w^3x^3y \\
- &\ 6912t^3w^3x^3y + 6940w^4x^3y + 896tw^4x^3y \\
- &\ 1856t^2w^4x^3y + 4000w^5x^3y + 384tw^5x^3y \\
+ &\ 352w^6x^3y + 177012x^4y - 220300tx^4y - 68020t^2x^4y + 6940t^3x^4y + 1672t^4x^4y \\
+ &\ 268t^5x^4y - 220300wx^4y - 41136twx^4y \\
- &\ 11588t^2wx^4y + 896t^3wx^4y + 608t^4wx^4y \\
- &\ 68020w^2x^4y - 11588tw^2x^4y + 4136t^2w^2x^4y - 1856t^3w^2x^4y + 6940w^3x^4y \\
+ &\ 896tw^3x^4y - 1856t^2w^3x^4y + 1672w^4x^4y + 608tw^4x^4y + 268w^5x^4y - 17836x^5y \\
- &\ 57456tx^5y - 2724t^2x^5y + 4000t^3x^5y + 268t^4x^5y - 57456wx^5y + 4992twx^5y \\
+ &\ 2432t^2wx^5y + 384t^3wx^5y - 2724w^2x^5y \\
+ &\ 2432tw^2x^5y + 1112t^2w^2x^5y + 4000w^3x^5y \\
+ &\ 384tw^3x^5y + 268w^4x^5y - 26900x^6y + 5828tx^6y + 1792t^2x^6y + 352t^3x^6y \\
+ &\ 5828wx^6y - 2112twx^6y + 688t^2wx^6y + 1792w^2x^6y + 688tw^2x^6y + 352w^3x^6y \\
+ &\ 132x^7y + 1024tx^7y + 272t^2x^7y + 1024wx^7y - 384twx^7y + 272w^2x^7y \\
+ &\ 800x^8y - 96tx^8y - 96wx^8y + 32x^9y + 292830y^2 - 66756ty^2 \\
+ &\ 177178t^2y^2 + 160520t^3y^2 - 94964t^4y^2 - 31788t^5y^2 + 1416t^6y^2 \\
+ &\ 1136t^7y^2 + 112t^8y^2 - 66756wy^2 + 319756twy^2 + 184632t^2wy^2 \\
- &\ 247264t^3wy^2 - 68020t^4wy^2 - 2724t^5wy^2 + 1792t^6wy^2 + 272t^7wy^2 \\
+ &\ 177178w^2y^2 + 184632tw^2y^2 - 335076t^2w^2y^2 - 134456t^3w^2y^2 - 1458t^4w^2y^2 \\
+ &\ 4296t^5w^2y^2 + 848t^6w^2y^2 + 160520w^3y^2 - 247264tw^3y^2 - 134456t^2w^3y^2 \\
+ &\ 7688t^3w^3y^2 + 4056t^4w^3y^2 + 592t^5w^3y^2 - 94964w^4y^2 \\
- &\ 68020tw^4y^2 - 1458t^2w^4y^2 + 4056t^3w^4y^2 - 56t^4w^4y^2
\end{aligned}
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\begin{aligned}
- & \ 31788w^5y^2 - 2724tw^5y^2 + 4296t^2w^5y^2 + 592t^3w^5y^2 \\
+ & \ 1416w^6y^2 + 1792tw^6y^2 + 848t^2w^6y^2 + 1136w^7y^2 + 272tw^7y^2 + 112w^8y^2 \\
- & \ 66756xy^2 + 319756txy^2 + 184632t^2xy^2 - 247264t^3xy^2 - 68020t^4xy^2 \\
- & \ 2724t^5xy^2 + 1792t^6xy^2 + 272t^7xy^2 + 319756wxy^2 - 48096twxy^2 \\
- & \ 403336t^2wxy^2 - 128736t^3wxy^2 - 11588t^4wxy^2 + 2432t^5wxy^2 \\
+ & \ 688t^6wxy^2 + 184632w^2xy^2 - 403336tw^2xy^2 - 169560t^2w^2xy^2 \\
+ & \ 17288t^3w^2xy^2 + 4136t^4w^2xy^2 + 1112t^5w^2xy^2 - 247264w^3xy^2 \\
- & \ 128736tw^3xy^2 + 17288t^2w^3xy^2 + 2496t^3w^3xy^2 \\
- & \ 1856t^4w^3xy^2 - 68020w^4xy^2 - 11588tw^4xy^2 + 4136t^2w^4xy^2 - 1856t^3w^4xy^2 \\
- & \ 2724w^5xy^2 + 2432tw^5xy^2 + 1112t^2w^5xy^2 + 1792w^6xy^2 + 688tw^6xy^2 \\
+ & \ 272w^7xy^2 + 177178x^2y^2 + 184632tx^2y^2 - 335076t^2x^2y^2 - 134456t^3x^2y^2 \\
- & \ 1458t^4x^2y^2 + 4296t^5x^2y^2 + 848t^6x^2y^2 + 184632wx^2y^2 - 403336twx^2y^2 \\
- & \ 169560t^2wx^2y^2 + 17288t^3wx^2y^2 + 4136t^4wx^2y^2 + 1112t^5wx^2y^2 \\
- & \ 335076w^2x^2y^2 - 169560tw^2x^2y^2 + 62988t^2w^2x^2y^2 + 10224t^3w^2x^2y^2 \\
- & \ 3408t^4w^2x^2y^2 - 134456w^3x^2y^2 + 17288tw^3x^2y^2 + 10224t^2w^3x^2y^2 \\
- & \ 9680t^3w^3x^2y^2 - 1458w^4x^2y^2 + 4136tw^4x^2y^2 \\
- & \ 3408t^2w^4x^2y^2 + 4296w^5x^2y^2 + 1112tw^5x^2y^2 + 848w^6x^2y^2 + 160520x^3y^2 \\
- & \ 247264tx^3y^2 - 134456t^2x^3y^2 + 7688t^3x^3y^2 + 4056t^4x^3y^2 + 592t^5x^3y^2 \\
- & \ 247264wx^3y^2 - 128736twx^3y^2 + 17288t^2wx^3y^2 + 2496t^3wx^3y^2 \\
- & \ 1856t^4wx^3y^2 - 134456w^2x^3y^2 + 17288tw^2x^3y^2 + 10224t^2w^2x^3y^2 \\
- & \ 9680t^3w^2x^3y^2 + 7688w^3x^3y^2 + 2496tw^3x^3y^2 - 9680t^2w^3x^3y^2 \\
+ & \ 4056w^4x^3y^2 - 1856tw^4x^3y^2 + 592w^5x^3y^2 \\
- & \ 94964x^4y^2 - 68020tx^4y^2 - 1458t^2x^4y^2 + 4056t^3x^4y^2 - 56t^4x^4y^2 \\
- & \ 68020wx^4y^2 - 11588twx^4y^2 + 4136t^2wx^4y^2 - 1856t^3wx^4y^2 - 1458w^2x^4y^2 \\
+ & \ 4136tw^2x^4y^2 - 3408t^2w^2x^4y^2 + 4056w^3x^4y^2 - 1856tw^3x^4y^2 - 56w^4x^4y^2 \\
- & \ 31788x^5y^2 - 2724tx^5y^2 + 4296t^2x^5y^2 + 592t^3x^5y^2 - 2724wx^5y^2 \\
+ & \ 2432twx^5y^2 + 1112t^2wx^5y^2 + 4296w^2x^5y^2 + 1112tw^2x^5y^2 + 592w^3x^5y^2 \\
+ & \ 1416x^6y^2 + 1792tx^6y^2 + 848t^2x^6y^2 + 1792wx^6y^2 + 688twx^6y^2 \\
+ & \ 848w^2x^6y^2 + 1136x^7y^2 + 272tx^7y^2 + 272wx^7y^2 + 112x^8y^2 \\
- & \ 71028y^3 - 120628ty^3 + 160520t^2y^3 - 11080t^3y^3 - 46764t^4y^3 \\
- & \ 3076t^5y^3 + 2640t^6y^3 + 336t^7y^3 - 120628wy^3 + 370976twy^3 \\
- & \ 247264t^2wy^3 - 124960t^3wy^3 + 6940t^4wy^3 + 4000t^5wy^3 + 352t^6wy^3 \\
+ & \ 160520w^2y^3 - 247264tw^2y^3 - 134456t^2w^2y^3 + 7688t^3w^2y^3 + 4056t^4w^2y^3 \\
+ & \ 592t^5w^2y^3 - 11080w^3y^3 - 124960tw^3y^3 + 7688t^2w^3y^3 + 6720t^3w^3y^3 \\
- & \ 1096t^4w^3y^3 - 46764w^4y^3 + 6940tw^4y^3 + 4056t^2w^4y^3 - 1096t^3w^4y^3 \\
- & \ 3076w^5y^3 + 4000tw^5y^3 + 592t^2w^5y^3 + 2640w^6y^3 + 352tw^6y^3 \\
+ & \ 336w^7y^3 - 120628xy^3 + 370976txy^3 - 247264t^2xy^3 - 124960t^3xy^3 \\
+ & \ 6940t^4xy^3 + 4000t^5xy^3 + 352t^6xy^3 + 370976wxy^3 - 749568twxy^3 \\
- & \ 128736t^2wxy^3 + 39680t^3wxy^3 + 896t^4wxy^3 + 384t^5wxy^3 - 247264w^2xy^3 \\
- & \ 128736tw^2xy^3 + 17288t^2w^2xy^3 + 2496t^3w^2xy^3 - 1856t^4w^2xy^3 - 124960w^3xy^3 \\
+ & \ 39680tw^3xy^3 + 2496t^2w^3xy^3 - 6912t^3w^3xy^3 + 6940w^4xy^3 + 896tw^4xy^3 \\
- & \ 1856t^2w^4xy^3 + 4000w^5xy^3 + 384tw^5xy^3 + 352w^6xy^3 + 160520x^2y^3 \\
- & \ 247264tx^2y^3 - 134456t^2x^2y^3 + 7688t^3x^2y^3 + 4056t^4x^2y^3 + 592t^5x^2y^3 \\
- & \ 247264wx^2y^3 - 128736twx^2y^3 + 17288t^2wx^2y^3 + 2496t^3wx^2y^3 \\
- & \ 1856t^4wx^2y^3 - 134456w^2x^2y^3 + 17288tw^2x^2y^3 + 10224t^2w^2x^2y^3 \\
- & \ 9680t^3w^2x^2y^3 + 7688w^3x^2y^3 + 2496tw^3x^2y^3 - 9680t^2w^3x^2y^3 \\
+ & \ 4056w^4x^2y^3 - 1856tw^4x^2y^3 + 592w^5x^2y^3 \\
- & \ 11080x^3y^3 - 124960tx^3y^3 + 7688t^2x^3y^3 + 6720t^3x^3y^3 - 1096t^4x^3y^3 \\
- & \ 124960wx^3y^3 + 39680twx^3y^3 + 2496t^2wx^3y^3 - 6912t^3wx^3y^3 + 7688w^2x^3y^3
\end{aligned}
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\begin{aligned}
& + \quad 2496tw^2x^3y^3 - 9680t^2w^2x^3y^3 + 6720w^3x^3y^3 - 6912tw^3x^3y^3 - 1096w^4x^3y^3 \\
& - \quad 46764x^4y^3 + 6940tx^4y^3 + 4056t^2x^4y^3 - 1096t^3x^4y^3 + 6940wx^4y^3 \\
& + \quad 896twx^4y^3 - 1856t^2wx^4y^3 + 4056w^2x^4y^3 - 1856tw^2x^4y^3 - 1096w^3x^4y^3 \\
& - \quad 3076x^5y^3 + 4000tx^5y^3 + 592t^2x^5y^3 + 4000wx^5y^3 + 384twx^5y^3 \\
& + \quad 592w^2x^5y^3 + 2640x^6y^3 + 352tx^6y^3 + 352wx^6y^3 + 336x^7y^3 \\
& - \quad 111795y^4 + 177012ty^4 - 94964t^2y^4 - 46764t^3y^4 + 6638t^4y^4 \\
& + \quad 2920t^5y^4 + 224t^6y^4 + 177012wy^4 - 220300twy^4 - 68020t^2wy^4 \\
& + \quad 6940t^3wy^4 + 1672t^4wy^4 + 268t^5wy^4 - 94964w^2y^4 - 68020tw^2y^4 \\
& - \quad 1458t^2w^2y^4 + 4056t^3w^2y^4 - 56t^4w^2y^4 - 46764w^3y^4 + 6940tw^3y^4 \\
& + \quad 4056t^2w^3y^4 - 1096t^3w^3y^4 + 6638w^4y^4 + 1672tw^4y^4 - 56t^2w^4y^4 + 2920w^5y^4 \\
& + \quad 268tw^5y^4 + 224w^6y^4 + 177012xy^4 - 220300txy^4 - 68020t^2xy^4 \\
& + \quad 6940t^3xy^4 + 1672t^4xy^4 + 268t^5xy^4 - 220300wxy^4 - 41136twxy^4 \\
& - \quad 11588t^2wxy^4 + 896t^3wxy^4 + 608t^4wxy^4 - 68020w^2xy^4 - 11588tw^2xy^4 \\
& + \quad 4136t^2w^2xy^4 - 1856t^3w^2xy^4 + 6940w^3xy^4 + 896tw^3xy^4 - 1856t^2w^3xy^4 \\
& + \quad 1672w^4xy^4 + 608tw^4xy^4 + 268w^5xy^4 - 94964x^2y^4 - 68020tx^2y^4 \\
& - \quad 1458t^2x^2y^4 + 4056t^3x^2y^4 - 56t^4x^2y^4 - 68020wx^2y^4 - 11588twx^2y^4 \\
& + \quad 4136t^2wx^2y^4 - 1856t^3wx^2y^4 - 1458w^2x^2y^4 + 4136tw^2x^2y^4 - 3408t^2w^2x^2y^4 \\
& + \quad 4056w^3x^2y^4 - 1856tw^3x^2y^4 - 56w^4x^2y^4 - 46764x^3y^4 + 6940tx^3y^4 \\
& + \quad 4056t^2x^3y^4 - 1096t^3x^3y^4 + 6940wx^3y^4 + 896twx^3y^4 - 1856t^2wx^3y^4 \\
& + \quad 4056w^2x^3y^4 - 1856tw^2x^3y^4 - 1096w^3x^3y^4 + 6638x^4y^4 + 1672tx^4y^4 \\
& - \quad 56t^2x^4y^4 + 1672wx^4y^4 + 608twx^4y^4 - 56w^2x^4y^4 + 2920x^5y^4 \\
& + \quad 268tx^5y^4 + 268wx^5y^4 + 224x^6y^4 + 33444y^5 - 17836ty^5 \\
& - \quad 31788t^2y^5 - 3076t^3y^5 + 2920t^4y^5 + 416t^5y^5 - 17836wy^5 \\
& - \quad 57456twy^5 - 2724t^2wy^5 + 4000t^3wy^5 + 268t^4wy^5 - 31788w^2y^5 \\
& - \quad 2724tw^2y^5 + 4296t^2w^2y^5 + 592t^3w^2y^5 - 3076w^3y^5 + 4000tw^3y^5 \\
& + \quad 592t^2w^3y^5 + 2920w^4y^5 + 268tw^4y^5 + 416w^5y^5 - 17836xy^5 \\
& - \quad 57456txy^5 - 2724t^2xy^5 + 4000t^3xy^5 + 268t^4xy^5 - 57456wxy^5 \\
& + \quad 4992twxy^5 + 2432t^2wxy^5 + 384t^3wxy^5 - 2724w^2xy^5 + 2432tw^2xy^5 \\
& + \quad 1112t^2w^2xy^5 + 4000w^3xy^5 + 384tw^3xy^5 + 268w^4xy^5 - 31788x^2y^5 \\
& - \quad 2724tx^2y^5 + 4296t^2x^2y^5 + 592t^3x^2y^5 - 2724wx^2y^5 + 2432twx^2y^5 \\
& + \quad 1112t^2wx^2y^5 + 4296w^2x^2y^5 + 1112tw^2x^2y^5 + 592w^3x^2y^5 - 3076x^3y^5 \\
& + \quad 4000tx^3y^5 + 592t^2x^3y^5 + 4000wx^3y^5 + 384twx^3y^5 + 592w^2x^3y^5 \\
& + \quad 2920x^4y^5 + 268tx^4y^5 + 268wx^4y^5 + 416x^5y^5 + 8640y^6 \\
& - \quad 26900ty^6 + 1416t^2y^6 + 2640t^3y^6 + 224t^4y^6 - 26900wy^6 \\
& + \quad 5828twy^6 + 1792t^2wy^6 + 352t^3wy^6 + 1416w^2y^6 + 1792tw^2y^6 \\
& + \quad 848t^2w^2y^6 + 2640w^3y^6 + 352tw^3y^6 + 224w^4y^6 - 26900xy^6 \\
& + \quad 5828txy^6 + 1792t^2xy^6 + 352t^3xy^6 + 5828wxy^6 - 2112twxy^6 \\
& + \quad 688t^2wxy^6 + 1792w^2xy^6 + 688tw^2xy^6 + 352w^3xy^6 + 1416x^2y^6 \\
& + \quad 1792tx^2y^6 + 848t^2x^2y^6 + 1792wx^2y^6 + 688twx^2y^6 + 848w^2x^2y^6 \\
& + \quad 2640x^3y^6 + 352tx^3y^6 + 352wx^3y^6 + 224x^4y^6 - 5124y^7 + 132ty^7 \\
& + \quad 1136t^2y^7 + 336t^3y^7 + 132wy^7 + 1024twy^7 + 272t^2wy^7 \\
& + \quad 1136w^2y^7 + 272tw^2y^7 + 336w^3y^7 + 132xy^7 + 1024txy^7 + 272t^2xy^7 \\
& + \quad 1024wxy^7 - 384twxy^7 + 272w^2xy^7 + 1136x^2y^7 + 272tx^2y^7 \\
& + \quad 272wx^2y^7 + 336x^3y^7 - 618y^8 + 800ty^8 + 112t^2y^8 + 800wy^8 \\
& - \quad 96twy^8 + 112w^2y^8 + 800xy^8 - 96txy^8 - 96wxy^8 + 112x^2y^8 \\
& + \quad 192y^9 + 32ty^9 + 32wy^9 + 32xy^9 + 24y^{10}\Big).
\end{aligned}
$$

$$
(S_{-1})^{(0)}_{I_1I_2I_3I_4} = \frac{2}{\delta} f_5^{-1} a_{125}a_{345}(k_1-k_2)(k_3-k_4)(f_1-f_2)(f_3-f_4)
$$

$$
\begin{aligned}
&\times \quad \left(-36+2(k_1+k_2+k_3+k_4)+f_1+f_2+f_3+f_4-2k_1k_2-2k_3k_4\right).\\
(S_{t2})^{(0)}_{I_1I_2I_3I_4} &= \frac{(f_5-1)^2 t_{125}t_{345}}{2\delta}(k_1-k_2)(k_3-k_4)(f_1+f_2+f_3+f_4\\
&+ \quad 2(k_1+k_2+k_3+k_4)-2(k_1k_2+k_3k_4)-36).\\
(S_{p3})^{(0)}_{I_1I_2I_3I_4} &= -\frac{1}{\delta}f_5^3 p_{125}p_{345}.\\
(S_{p2})^{(0)}_{I_1I_2I_3I_4} &= \frac{2}{\delta}f_5^2 p_{125}p_{345}(k_1^2+k_2^2+k_3^2+k_4^2-2(k_1+k_2+k_3+k_4)+2(k_1k_2+k_3k_4)-4).\\
(S_d)^{(0)}_{I_1I_2I_3I_4} &= \frac{9a_{125}a_{345}}{64\delta(f_5-5)}(-1+k_1-k_2)(1+k_1-k_2)(3+k_1+k_2)(5+k_1+k_2)\\
&\times \quad (-1+k_3-k_4)(1+k_3-k_4)(3+k_3+k_4)(5+k_3+k_4)\\
&\times \quad (2k_1^2+2k_2^2+2k_3^2+2k_4^2+4k_1k_2+4k_3k_4-4(k_1+k_2+k_3+k_4)-5).
\end{aligned}
$$

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 1. November 2018 Unterschrift