Existence of a versal deformation for compact complex analytic spaces endowed with logarithmic structure

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Eidesstattliche Versicherung

Declaration on oath

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare, on oath, that I have written the present dissertation on my own and have not used other than the acknowledged resources and aids.

Hamburg, 15 October 2019

Raffaele Caputo

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Abstract

The first main result proved in the present work concerns the existence of a versal deformation for a compact complex space endowed with a fine logarithmic structure $(X_0, \mathcal{M}_{X_0})$. This is a deformation of the underlying compact complex space X_0 together with a deformation of the log structure $\mathcal{M}_{X_0}$. The *versality* property implies that such deformation contains all possible deformations of the pair $(X_0, \mathcal{M}_{X_0})$. In other words, any other deformation is obtained via a base change from a versal one. This result generalizes a classical theorem independently proved by A. Douady and H. Grauert, in the '70s, concerning the existence of a versal deformation for compact complex spaces, which, in turn, generalizes the same result about compact complex manifolds obtained by M. Kuranishi. Our work relies on the *theory of Banach analytic spaces*, which is an infinite dimensional kind of geometry developed by Douady. This theory is thoroughly reviewed in the first part of this thesis. Douady's solution to the versal deformation problem consists of an infinite dimensional construction of the deformation parameter space, followed by a finite dimensional reduction. To achieve our result, we carry out Douady's construction enhancing it with the log data. From Douady's results, one can naturally get the existence of a versal deformation for holomorphic maps between compact complex spaces. Analogously, as second main result of this dissertation, we prove the existence of a versal deformation for log morphisms between compact fine log complex spaces $f_0 : (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$. Moreover, we show that if f_0 is log flat or log smooth, then its versal deformation is log flat or log smooth respectively.

Earlier publications derived from this dissertation: –

Zusammenfassung

Im ersten Ergebnis dieser Arbeit geht es um die Existenz sogenannter versellen Deformationen eines mit einer logarithmischen Struktur versehenen kompakten Raums $(X_0, \mathcal{M}_{X_0})$. Eine solche Deformation ist eine Deformation des zugrunde liegenden Raums X_0 zusammen mit einer Deformation der logarithmischen Struktur $\mathcal{M}_{X_0}$. Im Wesentlichen, eine Deformation heißt *versal* (versell), wenn sie alle möglichen Deformationen enthält. Mit anderen Worten: alle anderen Deformationen gehen mithilfe einer Basistransformation aus einer versalen Deformation hervor. Unser Resultat verallgemeinert ein klassisches Theorem von A. Douady und H. Grauert aus den 1970er Jahren. Ihr Theorem ist selbst eine Verallgemeinerung eines Satzes von M. Kuranashi. Unsere Arbeit beruht auf Douadys *Theorie der Banach-analytischen Räumen*, eine Art unendlichdimensionaler Geometrie. Diese Theorie wird im ersten Teil der Dissertation sorgfältig entwickelt. Douadys Lösung des Problems der versalen Deformationen besteht in der Konstruktion des unendlichdimensionalen Parameterraums und einer anschließenden Reduktion auf einen endlichdimensionalen Raum. In dieser Arbeit erweitern wir diesen Ansatz so, dass er auch logarithmische Daten behandeln kann. Aus Douadys Ergebnissen lässt sich die Existenz einer versellen Deformation holomorpher Abbildungen zwischen kompakten komplexen Räumen auf eine natürliche Weise herleiten. Das zweite Resultat dieser Dissertation ist ein dazu analoger Satz über die Existenz verseller Deformationen von Logmorphismen $f_0 \colon (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$ zwischen kompakten, feinen log-komplexen Räumen. Ferner wird gezeigt, dass die so erhaltene verselle Deformation log-flach, bzw. log-glatt ist, wenn der Logmorphismus f_0 log-flach, bzw. log-glatt ist.

Aus dieser Dissertation hervorgegangene Vorveröffentlichungen: –

Contents

Introduction

Log geometry (see [Gro11] and [Ogu18]) is a branch of algebraic geometry designed to work with pairs (X, D), where X is a complex analytic space (or a scheme) and $D \subset X$ is a divisor, that is a closed analytic subspace (closed subscheme) of codimension 1. Log geometry associates to X a morphism of sheaves of monoids $\alpha_X : \mathcal{M}_{(X,D)} \to \mathcal{O}_X$, where $\mathcal{M}_{(X,D)}$ is the sheaf of holomorphic functions on X invertible on $X \setminus D$ and α_X is the inclusion. Since only multiplication is defined in $\mathcal{M}_{(X,D)}$ and the existence of the inverse of each element is not guaranteed, $\mathcal{M}_{(X,D)}$ comes endowed with the structure of a sheaf of monoids, hence a structure which is combinatorial in nature. The key feature is that the pair $(\mathcal{M}_{(X,D)}, \alpha_X)$ "remembers" some information about how D is embedded into X. Such a log structure on a complex space is called *divisorial*. A particular case is given by the *trivial log structure* obtained for $D = \emptyset$, hence $\mathcal{M}_{(X,D)} = \mathcal{O}_X^\times$. A definition of *log structure* on a complex analytic space can be given in much more generality (see Definition 2.12). Log geometry is naturally linked to toric geometry. Indeed, a classical example (see Example 2.19) is obtained taking as X an affine toric variety and as divisor D the complement in X of the big torus $(\mathbb{C}^*)^n$. Complex analytic spaces (see Chapter 1) endowed with a log structure which locally "looks like" the divisorial log structure of an affine toric variety are called *coherent log complex spaces* (see Definition 2.27). A coherent log structure consisting of a sheaf of *integral monoids*, that is monoids that can be naturally embedded into groups (see Definition 2.4), is called a *fine log structure*. In the present work, we prove the existence of a deformation of a compact fine log complex space $(X_0, \mathcal{M}_{X_0})$, that is a deformation of the underlying complex space together with a deformation of the associated log structure. We require this deformation to be *versal* (see Definition 4.5), which implies the notion of *completeness* (see Definition 4.1). This means that the deformation contains, not in a unique way, all possible deformations of the pair $(X_0, \mathcal{M}_{X_0})$. In other words, any other deformation of $(X_0, \mathcal{M}_{X_0})$ is obtained via a base change from a complete one. More precisely,

Definition 0.1. A *deformation of a compact fine log complex space* $(X_0, \mathcal{M}_{X_0})$ is triple $(((S, \mathcal{O}_S^\times), s_0), (\mathfrak{X}, \mathcal{M}_\mathfrak{X}), i)$, where $((S, \mathcal{O}_S^\times), s_0)$ is a germ of complex spaces (endowed with the trivial log structure), $p : (\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \to ((S, \mathcal{O}_S^\times), s_0)$ is a proper and flat log morphism between compact fine log complex spaces and $i : (X_0, \mathcal{M}_{X_0}) \to (\mathfrak{X}, \mathcal{M}_\mathfrak{X})(s_0) := p^{-1}(s_0)$ is a log isomorphism. The deformation is called *complete* if for any other deformation $(((T, \mathcal{O}_T^\times), t_0), (X, \mathcal{M}_X), i')$ of $(X_0, \mathcal{M}_{X_0})$, there exists a morphism $\psi : ((T, \mathcal{O}_T^\times), t_0) \to ((S, \mathcal{O}_S^\times), s_0)$ and a log T-isomorphism

$$\alpha : (X, \mathcal{M}_X) \to (\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \times_{(S, \mathcal{O}_S^\times)} (T, \mathcal{O}_T^\times),$$

such that $\alpha \circ i' = i$.

The main result of this work is the following

Theorem 0.2. *(Theorem 5.32) Let $(X_0, \mathcal{M}_{X_0})$ be a compact fine log complex space, then it admits a versal deformation $(p : (\mathfrak{X}, \mathcal{M}_\mathfrak{X}) \to ((S, \mathcal{O}_S^\times), s_0), i)$.*

In the '70s, A. Douady ([Dou74]) and H. Grauert ([Gra74]) independently prove the existence of a versal deformation for compact complex spaces (see [Pal90], p. 134, for a precise definition). In [Sti88], H. Stieber shows that Douady's construction can be axiomatized and carried out at a more abstract categorical level. We present a detailed account of Douady's construction in Chapter 4. His work is based on the notions and techniques of an infinite dimensional kind of geometry (Banach analytic) developed by himself in his doctoral thesis ([Dou66]). In more detail, let X be a (not necessarily compact) complex analytic space. A. Douady considers the problem of parametrizing the set of compact complex analytic subspaces of X by a complex analytic space H, positively answering to a conjecture by A. Grothendieck, posed a few years before in the H. Cartan Seminar, concerning the existence of a space similar to the Hilbert scheme, but valid in the more general framework of analytic geometry ([Car66]). The author solves a more general problem parametrizing the set of coherent quotient sheaves, with compact support, of a given coherent analytic sheaf $\mathcal{E}$ on X (see Section 3.4, Theorem 3.64). For the solution of the problem, Douady develops a theory of *Banach analytic spaces* (see Section 3), which are an infinite dimensional generalization of complex analytic spaces and a theory of *privileged subspaces of a polycylinder* (see Section 3.3). The latter are special closed subsets of a compact polycylinder K in $\mathbb{C}^n$ that can be used to decompose a compact complex space. The key feature is that to the collection of all privileged subspaces of a given polycylinder one can give the structure of a Banach analytic space, thus giving precise meaning to a notion of analytic deformation of such subspaces. Then, the solution of the problem is obtained via a two steps process consisting in an infinite dimensional construction of the parameter space, followed by a finite dimensional reduction (see Section 3.4). As an application of the main result, Douady shows that the space of morphisms $\mathrm{Mor}(X, Y)$, where X, Y are complex spaces (with X compact) carries a universal structure of complex analytic space (see Theorem 3.81). In [Pou69], Douady's results are extended by G. Pourcin to the relative setting, that is he considers the case of complex analytic spaces X, Y defined over another complex analytic space S (see Theorem 3.82). Furthermore, given a morphism $Z \to S$ of Banach analytic spaces, a polycylinder $K \subset \mathbb{C}^n$ and a "relative" privileged subspace of $S \times K$ (see Definition 3.62), in [Pou75], G. Pourcin endowes the set of relative morphisms $\mathrm{Mor}_S(Y, Z)$ with a universal Banach analytic structure (see Theorem 3.83). Banach analytic spaces of the form $\mathrm{Mor}_S(Y, Z)$ are the building blocks of Douady's versal deformation construction (see Definition 4.22). In Chapter 5, we achieve our goal by carrying out Douady's construction enhanced with the log data (see Definition 5.5). The notion of versal deformation of compact (fine log) complex spaces (Definition 4.5) extends naturally to the case of (log) morphisms between them (see, for instance, [Pal90], p. 163, for the non-log case). In particular, in the log setting the notion of *completeness* can be stated as follows

Definition 0.3. A *deformation over a base* $((S, \mathcal{O}_S^\times), s_0)$ *of a morphism of compact fine log complex spaces* $f_0 : (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$ is a triple consisting of a deformation (p, i) of $(X_0, \mathcal{M}_{X_0})$, a deformation (q, i') of $(Y_0, \mathcal{M}_{Y_0})$ and a log morphism $f : (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \to (\mathcal{Y}, \mathcal{M}_{\mathcal{Y}})$ over $(S, \mathcal{O}_S^\times)$, such that $q \circ f = p$ and $f \circ i = i' \circ f_0$. That is, we have the following commutative diagram:

$$\begin{tikzcd}
(X_0, \mathcal{M}_{X_0}) \arrow[r, "i"] \arrow[dd, "f_0"'] & (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \arrow[dd, "f"'] \arrow[dr, "p"] & \\
& & ((S, \mathcal{O}_S^\times), s_0) \\
(Y_0, \mathcal{M}_{Y_0}) \arrow[r, "i'"] & (\mathcal{Y}, \mathcal{M}_{\mathcal{Y}}) \arrow[ur, "q"'] &
\end{tikzcd} \quad . \tag{1}$$

The deformation is called *complete* if given any other deformation $g : (\mathfrak{X}', \mathcal{M}_{\mathfrak{X}'}) \to (\mathcal{Y}', \mathcal{M}_{\mathcal{Y}'})$ of f_0 over a base $((T, \mathcal{O}_T^\times), t_0)$, there exists a morphism $\psi : ((T, \mathcal{O}_T^\times), t_0) \to ((S, \mathcal{O}_S^\times), s_0)$ such that $g = \psi^* f$.

The existence of versal deformations of morphisms between compact complex spaces follows naturally from Douady and Pourcin's results (see [Fle79], p. 130). Analogously, we take a further step in our work studying versal deformations of log morphisms. Given a morphism of log complex spaces, we have the notion of *log smoothness* and *log flatness* (see Definition 2.35). These notions generalize and extend the classical notions of smoothness and flatness (see Section 1.3), which are retrieved if we consider complex spaces endowed with trivial log structures. In [Kat], K. Kato writes that a log structure is "magic by which a degenerate scheme begins to behave as being non-degenerate". For example, the affine toric variety $\mathrm{Spec}\, \mathbb{C}[P]$, with its canonical divisorial log structure, is log smooth over $\mathrm{Spec}\, \mathbb{C}$ (equipped with the trivial log structure), despite almost always not being smooth in the usual sense. We prove the following

Theorem 0.4. *(Theorem* 5.35 *and Proposition* 5.40*) Let* $f_0 : (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$ *be a morphism of compact fine log complex spaces. Then* f_0 *admits a versal deformation* f *over a germ of complex spaces* $((S, \mathcal{O}_S^\times), s_0)$*. Moreover, if* f_0 *is log flat (log smooth), then* f *is log flat (log smooth) in an open neighborhood of* s_0*.*

A successful application of log geometry can be found in the *toric degeneration program* carried out by M. Gross and B. Siebert (see [GS03], [GS06] and [GS07]) in the context of Mirror Symmetry (see, for instance, [Gro12]). At the center of this program is the notion of *toric degeneration*, which is a degeneration $\mathfrak{X}$ of algebraic varieties (for instance, Calabi-Yau), whose central fibre is a union of toric varieties glued pairwise along toric divisors. This program provides canonical such families out of purely discrete data. Mirror symmetry then works as a perfect duality on the discrete data. The families come endowed with a canonical log structure, namely the divisorial log structure, where as a divisor is taken the central fibre X_0. The key role of log geometry is that it gives a way to reproduce the whole family starting from X_0 together with the log structure on it, which is the restriction to the central fibre of the divisorial log structure on the family. Hence, instead of working with a degenerating family $f : (\mathfrak{X}, X_0) \to (\mathcal{S}, 0)$, one can work with the induced morphism of log schemes $f_0 : (X_0, \mathcal{M}_{X_0}) \to 0^\dagger$, where $0^\dagger$ denotes the *standard log point* (see Example 2.17). In nice cases, a versal deformation of f_0 can be used to obtain an analytic local model of the degenerating family f (see [RS19]). Future steps can be taken to extend the results of this thesis to more general types of log structures. For instance, to the case of *relatively coherent* log structures. These are log structures locally isomorphic to a sheaf of faces of a fine log structure (see [NO10]). A simple example is obtained taking X in $\mathbb{C}^4$ defined by $xy - tw = 0$ and the divisor D in X defined by $t = 0$. The divisorial log structure on X only fails to be fine at the singular point $0 \in X$. This is a relatively coherent log structure on X contained in the fine log structure defined by the toric boundary ∂X of X. As first and useful step in generalizing the results of this thesis, one can try to deform X defined by $xy - tw^k = 0$, for $k \in \mathbb{N}_{>0}$, in $\mathbb{C}^4$, with the log structure given by the divisor D defined for $t = 0$, into $xy = f(w)t$, for any holomorphic function f, necessarily with the sum of orders of zeroes close to the origin equal to k.

Chapter 1

Complex analytic geometry

We recall the basic notions of complex analytic geometry. As main reference we use [Fis76], where the reader can find a more detailed treatment of the subject. In what follows, we assume that all rings occuring are commutative and have unit element.

1.1 Complex spaces

In a nutshell, a complex analytic space is a generalization of a complex manifold which allows the presence of singularities.

Definition 1.1. A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space and a sheaf of rings on it respectively. It is called a *locally ringed space*, if for every $p \in X$ the stalk $\mathcal{O}_{X,p}$ is a local ring. Its maximal ideal is denoted by $m_{X,p}$. A local ring is called $\mathbb{C}$-*ringed space*, if moreover $\mathcal{O}_X$ is a sheaf of $\mathbb{C}$-algebras and for every $p \in X$, there exists an isomorphism of $\mathbb{C}$-algebras

$$\mathcal{O}_{X,p}/m_{X,p} \to \mathbb{C}.$$

By abuse of notation we shall mostly write X instead of $(X, \mathcal{O}_X)$.

Definition 1.2. A morphism $(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of ringed spaces is a pair consisting of a continuous map $\varphi : X \to Y$ and a morphism of sheaves of rings $\tilde{\varphi} : \mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ respectively.

If X and Y are locally ringed spaces, then we moreover require that the stalkwise morphism

$$\tilde{\varphi}_p : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$$

is *local*, that is the condition:

$$\tilde{\varphi}_p(m_{Y,\varphi(p)}) \subset m_{X,p}$$

is satisfied, for each $p \in X$.

Definition 1.3. A *morphism of* $\mathbb{C}$*-ringed spaces* X and Y is a morphism of ringed spaces where $\tilde{\varphi}$ is a morphism of sheaves of $\mathbb{C}$-algebras.

Remark 1.4. In this case $\tilde{\varphi}_p$ is automatically local (see e.g. [Gro62]).

Again, by abuse of notation, we mostly denote a morphism of $\mathbb{C}$-ringed spaces with φ instead of $(\varphi, \tilde{\varphi})$. We can construct in an obvious way the composition of morphisms of $\mathbb{C}$-ringed spaces. We obtain the category of $\mathbb{C}$-ringed spaces. We define the notion of *coherent sheaf.* For a more detailed account of the subject we refer to [GR84] and [Fis76].

Definition 1.5. Let X be a ringed space and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules on X. $\mathcal{F}$ is called of *finite type* (resp. *locally free*) if for each $p \in X$ there exists an open neighborhood U of p and a surjective morphism (resp. isomorphism)

$$\alpha : \mathcal{O}_U^k \to \mathcal{F}|_U,$$

for some positive integer k. The sheaf $\mathcal{F}$ is called *coherent* if it is of finite type and for every open $U \subset X$ and every homomorphism

$$\mathcal{O}_U^k \to \mathcal{F}|_U$$

the kernel is of finite type.

We remark (see e.g. [GR84], p. 234) that if X is a ringed space and $\mathcal{F}$ an $\mathcal{O}_X$-module of finite type, then the *support*

$$\text{supp } \mathcal{F} := \{x \in X : \mathcal{F}_X \neq 0\}$$

is a closed subset of X.

Definition 1.6. Let $U \subset \mathbb{C}^n$ be an open subset and $\mathcal{I} \subset \mathcal{O}_U$ a coherent ideal sheaf. Then

$$V := \text{supp } \mathcal{O}_U/\mathcal{I}$$

is an analytic subset of U and $(V, (\mathcal{O}_U/\mathcal{I})|_V)$ is a $\mathbb{C}$-ringed space that we call a *local model.*

Definition 1.7. A *complex (analytic) space* is a Hausdorff $\mathbb{C}$-ringed space locally isomorphic to a local model. A $\mathbb{C}$-ringed space morphism between complex spaces is called a *holomorphic map.* A bijective holomorphic map, such that its inverse is again holomorphic, is called a *biholomorphic* map.

Theorem 1.8. *(Oka-Cartan, [GR84], p. 84) The structure sheaf $\mathcal{O}_X$ of every complex space X is coherent.*

Definition 1.9. A complex space is called *smooth* at a point $x \in X$, if there exists an open neighborhood U of x such that $(U, \mathcal{O}_U)$ is isomorphic to some local model of the form $(W, \mathcal{O}_W)$, where W is an open subset of $\mathbb{C}^n$. If X is smooth at every point, we call it a *complex manifold*.

Example 1.10. Let z be the coordinate function in $\mathbb{C}$. Let n be a positive integer and consider the ideal sheaf $\mathcal{I}_n \subset \mathcal{O}_\mathbb{C}$ generated by z^n. Then

$$\operatorname{supp} \mathcal{O}_\mathbb{C}/\mathcal{I}_n = \{0\}$$

and

$$(\mathcal{O}_\mathbb{C}/\mathcal{I}_n)|_{\{0\}} = \mathbb{C} + \mathbb{C}\epsilon + \ldots + \mathbb{C}\epsilon^{n-1},$$

with $\epsilon^n = 0$. The complex space $(\{0\}, (\mathcal{O}_\mathbb{C}/\mathcal{I}_n)|_{\{0\}})$ is called an n-*fold point*. For $n > 1$ it is not a complex manifold.

Definition 1.11. Let X be a complex space. A complex space Y is an *analytic closed subset* of X if there exists a coherent ideal sheaf of $\mathcal{I} \subset \mathcal{O}_X$ such that $Y = \operatorname{supp} \mathcal{O}_X/\mathcal{I}$ and $\mathcal{O}_Y = (\mathcal{O}_X/\mathcal{I})|_Y$

The category of complex spaces admits fiber products. Let $\varphi : X \to Z$ and $\psi : Y \to Z$ be morphisms of complex spaces, then $X \times_Z Y$ is the closed analytic subset of $X \times Y$ defined by

$$X \times_Z Y := \{(x, y) \in X \times Y | \varphi(x) = \psi(y)\}.$$

Let X be a complex space and $U \subset X$ an open subset. Any $f \in \mathcal{O}_X(U)$ is called an *holomorphic function* on U and it can be nilpotent, as in Example 1.10. In order to control this situation, we define the *nilradical sheaf* as the sheaf of ideals $\mathcal{N}_X \subset \mathcal{O}_X$ associated to the presheaf

$$U \mapsto \{f \in \mathcal{O}_X(U) : f^k = 0 \text{ for some } k \in \mathbb{N}\}.$$

Definition 1.12. Let X be a complex space. It is called *reduced* if

$$\mathcal{N}_X = 0$$

1.2 Proper and finite holomorphic maps

We recall some well known definitions from topology. Let $\varphi : X \to Y$ be a continuous map between locally compact topological spaces. The map φ is called *proper* if for any compact set $K \subset Y$, the set $\varphi^{-1}(K)$ is a compact subset of X. The map φ is called *finite* if every point $p \in X$ is an isolated point in the fibre $\varphi^{-1}(\varphi(p))$, or equivalently, if it is closed and has finite fibres. Using topological arguments one can prove the following useful results about proper and finite maps.

Proposition 1.13. *Let $\varphi : X \to Y$ be a continuous map between locally compact topological spaces and assume that there exists $q \in Y$ such that $\varphi^{-1}(q) \subset X$ is compact. Then there exist open neighborhoods U of $\varphi^{-1}(q)$ in X and V of q in Y such that $\varphi|_U : U \to V$ is proper.*

Similarly,

Proposition 1.14. *Let $\varphi : X \to Y$ be a holomorphic map which is finite at $p \in X$. Then there are neighborhoods $U \subset X$ of p and $V \subset Y$ of $\varphi(p)$, such that $\varphi|_U : U \to V$ is finite.*

We recall some important results about finite holomorphic maps.

Proposition 1.15. *([Fis76], p. 57) Let $\varphi : X \to Y$ be a finite holomorphic map, $\mathcal{F}$ an $\mathcal{O}_X$-module and $q \in Y$. Then*

$$(\varphi_*\mathcal{F})_q = \prod_{p\in\varphi^{-1}(q)} \mathcal{F}_p.$$

Let $\varphi : X \to Y$ be a finite holomorphic map and $p \in X$. Set $q := \varphi(p)$. Via the canonical homomorphism

$$\tilde{\varphi}_p : \mathcal{O}_{Y,q} \to \mathcal{O}_{X,p},$$

$\mathcal{O}_{X,p}$ is an $\mathcal{O}_{Y,q}$-module.

Definition 1.16. We say that $\mathcal{O}_{X,p}$ is *finite over* $\mathcal{O}_{Y,q}$, if it is a finitely generated $\mathcal{O}_{Y,q}$-module.

Theorem 1.17. *([Fis76], p. 57) If $\varphi : X \to Y$ is a holomorphic map, $p \in X$ and $q := \varphi(p)$, then the following are quivalent*

1. *$\mathcal{O}_{X,p}$ is finite over $\mathcal{O}_{Y,q}$;*

2. *p is an isolated point in the fibre X_q.*

The following result is called *Finite Coherence Theorem* and was first proved by Grauert and Remmert (see [GR84], p. 64).

Theorem 1.18. *If $\varphi : X \to Y$ is a finite holomorphic map and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module, then $\varphi_*\mathcal{F}$ is a coherent $\mathcal{O}_Y$-module.*

1.3 Flat and smooth holomorphic maps

Let us first recall the algebraic notion of flatness (see e.g. [AM16]).

Definition 1.19. Let R be a ring (commutative and with unit). An R-module M is called *R-flat* if and only if one of the following equivalent conditions holds:

1. For every exact sequence of R-modules

$$... \to N_{i-1} \to N_i \to N_{i+1} \to ...$$

the induced sequence

$$... \to N_{i-1} \otimes_R M \to N_i \otimes_R M \to N_{i+1} \otimes_R M \to ...$$

is again exact.

2. For every short exact sequence of R-modules

$$0 \to N_{i-1} \to N_i \to N_{i+1} \to 0$$

the induced sequence

$$0 \to N_{i-1} \otimes_R M \to N_i \otimes_R M \to N_{i+1} \otimes_R M \to 0$$

is again exact.

3. For every injective homomorphism of R-modules

$$N' \to N$$

the induced homomorphism

$$N' \otimes_R M \to N \otimes_R M$$

is again injective.

In general this concept is difficult to interpret in a geometric way. In [Mum76], D. Mumford writes: "The concept of flatness is a riddle that comes out of algebra, but which technically is the answer to many prayers". In analytic geometry the notion of flatness is used in the following way (see [Fis76]).

Definition 1.20. Let $\varphi : X \to Y$ be a holomorphic map between complex spaces and $\mathcal{F}$ be an $\mathcal{O}_X$-module. $\mathcal{F}$ is called *φ-flat* in $p \in X$, if F_p is $\mathcal{O}_{Y,\varphi(p)}$-flat. The map φ is called *flat* in $p \in X$, if $\mathcal{O}_{X,p}$ is $\mathcal{O}_{Y,\varphi(p)}$-flat.

We remark that $\mathcal{O}_{X,p}$-modules are considered as $\mathcal{O}_{Y,\varphi(p)}$-modules via the canonical homorphism

$$\tilde{\varphi}_p : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}.$$

Definition 1.21. The $\mathcal{O}_X$-module $\mathcal{F}$ and the map φ are called *φ-flat* respectively *flat* if Definition 1.20 holds at each point $p \in X$.

The following example illustrates a non-flat holomorphic map.

Example 1.22. Let $X := (\{\cdot\}, \mathbb{C})$ be a point and take $Y = \mathbb{C}$, where $\mathcal{O}_Y$ is the ordinary sheaf of holomorphic functions. Define:

$$\varphi : X \to Y$$

by

$$\varphi(\cdot) = 0.$$

If y is the coordinate function in $\mathbb{C}$, we denote by α the multiplication by y. This yields and exact sequence

$$0 \to \mathcal{O}_Y \stackrel{\alpha}{\to} \mathcal{O}_Y.$$

But the sequence obtained applying $\cdot \otimes_{\mathcal{O}_{Y,\varphi(p)}} O_{X,p}$ is

$$0 \to \mathbb{C} \stackrel{\cdot 0}{\to} \mathbb{C}.$$

The concept of flatness can be well illustrated in the case of a finite map. If $\varphi : X \to Y$ is a finite holomorphic map, then for any $q \in Y$ and $x \in X_q$, with $X_q := \varphi^{-1}(q)$ the fibre over q, the local ring $\mathcal{O}_{X_q,x}$ is a finite dimensional $\mathbb{C}$-vector space. We define

$$\nu_x(\varphi) := \dim_{\mathbb{C}} \mathcal{O}_{X_q,x}$$

and

$$\nu_q(\varphi) := \sum_{x \in X_q} \nu_x(\varphi).$$

Proposition 1.23. *([Fis76], p. 150) For a finite holomorphic map $\varphi : X \to Y$, with Y reduced, the following are equivalent*

1. *φ is flat;*
2. *$q \mapsto \nu_q(\varphi)$ is a locally constant function on Y.*

Example 1.24. (due to A. Douady) Let

$$Y := \{(p, q) \in \mathbb{C}^2 : 4p^3 + 27q^2 = 0\}$$

and

$$X := \{(p, q, x) \in \mathbb{C}^3 : 4p^3 + 27q^2 = 0, x^3 + x + q = 0\}.$$

Then Y is reduced, but X has a simple and a double branch. The projection

$$\mathbb{C}^3 \to \mathbb{C}^2,$$

$$(p, q, x) \to (p, q)$$

restricts to a finite holomorphic map

$$\varphi : X \to Y.$$

One can easily check that $\nu_y(\varphi) = 3$, for every $y \in Y$. Hence φ is flat. On the other hand, consider the restriction $\varphi_{red} : X_{red} \to Y$. The complex space X_{red} has two simple branches. One finds that $\nu_y(\varphi_{red}) = 2$ for $y \in Y \setminus \{0\}$ and $\nu_y(\varphi_{red}) = 3$ for $y = 0$. Hence, φ_{red} is not flat.

Definition 1.25. ([Fis76], p. 19) A holomorphic map $(\varphi, \tilde{\varphi}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ between complex spaces is called an *embedding* if φ is closed and injective and $\tilde{\varphi}_p : \mathcal{O}_{Y,\varphi(p)} \to \mathcal{O}_{X,p}$ is surjective for every $p \in X$.

Definition 1.26. ([Fis76], p. 100) A holomorphic map $\varphi : X \to Y$ between arbitrary complex spaces is called *smooth* (*submersion*) at $p \in X$ if there exists an open neighborhood $U \subset X$ of p, an open subset $V \subset Y$ with $\varphi(U) \subset V$, an open subset $Z \subset \mathbb{C}^k$ and a biholomorphic map $\psi : U \to Z \times V$ such that the diagram

$$\begin{tikzcd} U \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & Z \times V \arrow[dl, "\pi_2"] \\ & V & \end{tikzcd},$$

commutes.

We list a few useful results about flat holomorphic maps. For proofs we refer to [Fis76].

Proposition 1.27. *([Fis76], pp. 152-156)*

1. *Let $\varphi : X \to Y$ be a flat holomorphic map between complex spaces. Let $\psi : Z \to Y$ be a holomorphic map. Then the base change map $\psi^*\varphi : Z \times_Y X \to Z$ is flat.*

2. *Let X, Y be complex spaces. The projection $\pi : X \times Y \to X$ is flat.*

3. *Let $\varphi : X \to Y$ be a holomorphic map. The non-flat locus*

$$\{p \in X : \varphi \textit{ is not flat at } p\}$$

is an analytic subset of X.

4. *Every flat holomorphic map is open.*

The next result is due to A. Grothendieck in the algebraic geometry setting (see [Gro66], p. 138 and [Sta, $Tag00MP$]). The result can be extended to the analytic case as for any complex space $(X, \mathcal{O}_X)$ and $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a Noetherian local ring (see [KKBB83], p. 80).

Proposition 1.28. *(Critère de platitude par fibres)([Gro66], p. 138 and [Sta, Tag00MP]) Let S be a complex space. Let $f : X \to Y$ be a morphism of complex spaces over S. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Let $x \in X$. Set $y = f(x)$ and $s \in S$ the image of x in S. Let X_s and Y_s be the fibres of X and Y over s. Set:*

$$\mathcal{F}_s = (X_s \hookrightarrow X)^* \mathcal{F}.$$

Assume $\mathcal{F}_x \neq 0$. Then the following are equivalent:

1. *$\mathcal{F}$ is flat over S at x and $\mathcal{F}_s$ is flat over Y_s at x;*
2. *Y is flat over S at y and $\mathcal{F}$ is flat over Y at x.*

Proposition 1.29. *([Fis76], p. 159) Let $f : X \to Y$ be a morphism of complex spaces. Let $p \in X$. Then the following are equivalent*

1. *f is smooth (submersion) at $p \in X$ (see Definition 1.26);*
2. *f is flat at p and the fibre $X_{f(p)}$ is a manifold.*

Chapter 2

Logarithmic complex spaces

Our main references for this chapter are [Gro11], [Ogu18] and [KN99]. The latter, in particular, explicitly deals with log structures on complex analytic spaces.

2.1 Monoids

Definition 2.1. A *monoid* is a triple $(P, +, 0)$ consisting of a set P, a commutative associative binary operation $+$ and a unit $0 \in P$. A *homomorphism of monoids* is a map between two monoids $\alpha : P \to Q$ such that $\alpha(0) = 0$ and $\alpha(p+p') = \alpha(p)+\alpha(p')$.

Definition 2.2. Let P be a monoid and denote with $P^\times$ the subset of units of P. We set $P^+ := P \setminus P^\times$, the subset of nonunits of P. Let $\theta : P \to Q$ be a monoid homomorphism. We say that θ is *local* if $\theta^{-1}(Q^+) = P^+$. Furthermore, we denote with $\overline{P}$ the quotient $P/P^\times$.

Definition 2.3. The *Grothendieck group of a monoid* P is the abelian group defined as follows:

$$P^{gp} := P \times P/\sim$$

where $(x, y) \sim (x', y')$ if and only if there exists an element $p \in P$ such that $p+x+y' = p + y + x'$.

There is a natural map $P \to P^{gp}$, sending $p \mapsto (p, 0)$. For any abelian group G, we have $\mathrm{Hom}_{Mon}(P, G) = \mathrm{Hom}_{Ab}(P^{gp}, G)$.

Definition 2.4. A monoid is called *integral* if the map $P \to P^{gp}$ is injective. A monoid is called *fine* if it is integral and finitely generated. It is called *saturated* if it is integral and whenever $p \in P^{gp}$ such that $mp \in P$, for some $m \in \mathbb{Z}^+$, then $p \in P$. The monoid P is called *toric* if it is fine, saturated and P^{gp} is torsion free. A monoid P is called *sharp* if $M^\times = \{0\}$.

Complex spaces arising from toric monoids are the building blocks of toric geometry (see, for instance, [Ful93]).

Example 2.5. A ring R is not an integral monoid unless $R = \{0\}$, because $0 \cdot a = 0 \cdot 0$, for any $a \in R$. However, $R^\times$ is an integral monoid.

Let P be an integral monoid. The set

$$P^{\mathrm{sat}} := \{x \in P^{gp} : \exists n \in \mathbb{Z}^+ \text{ such that } nx \in P\}$$

is, by construction, a saturated submonoid of P^{gp}.

Example 2.6. The monoid of all integers greater than or equal to some natural number d, together with zero, is not saturated if $d > 1$.

Example 2.7. Let Q be the submonoid of $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ generated by $x := (0, 1)$ and $y := (1, e)$, where e is the nonzero element of $\mathbb{Z}/2\mathbb{Z}$. Then $2x = 2y$, so $z := (0, x-y) \in Q^{\mathrm{sat}} \setminus Q$. Thus Q is not saturated. In fact z is a nonzero unit of Q^{sat}, but Q is sharp.

Definition 2.8. ([Gro11], p. 109) A homomorphism of integral monoids $h : Q \to P$ is *integral* if whenever $q_1, q_2 \in Q$, $p_1, p_2 \in P$ and $h(q_1) + p_1 = h(q_2) + p_2$, there exists $q_3, q_4 \in Q$ and $p \in P$ such that $p_1 = h(q_3) + p$, $p_2 = h(q_4) + p$ and $q_1 + q_3 = q_2 + q_4$.

Definition 2.9. Let P be a monoid. We define the *monoid ring*

$$\mathbb{C}[P] := \bigoplus_{p \in P} \mathbb{C} z^p,$$

where z^p is a symbol. The multiplication is $\mathbb{C}$-bilinear and it is determined by

$$z^p \cdot z^q := z^{p+q}.$$

Example 2.10. Let r be a positive integer. Then, $\mathbb{C}[\mathbb{N}^r] \simeq \mathbb{C}[x_1, ..., x_r]$.

Let P be a finitely generated monoid. Then, there exists a positive integer r, such that P is isomorphic to $\mathbb{N}^r$ divided out by k binomial relations among its generators $\{e_1, ..., e_r\}$, for $k \in \mathbb{N}$. Hence, we get an iduced isomorphism

$$\mathbb{C}[P] \simeq \mathbb{C}[x_1, ..., x_r]/(f_1, ..., f_k),$$

where $f_i \in \mathbb{C}[x_1, ..., x_r]$. Let $V := \operatorname{supp}(\mathcal{O}_{\mathbb{C}^r}/(f_1, ..., f_k))$.

Definition 2.11. We define the *analytic spectrum* of $\mathbb{C}[P]$ to be the complex analytic space defined by

$$\operatorname{Spec}_{\mathrm{an}} \mathbb{C}[P] := (V, (\mathcal{O}_{\mathbb{C}^r}/(f_1, ..., f_k))|_V).$$

For the sake of simplicity, from now on we shall denote the analytic spectrum of a monoid ring $\mathbb{C}[P]$, for a finitely generated monoid P, simply by $\operatorname{Spec} \mathbb{C}[P]$.

2.2 Log and pre-log structures

Definition 2.12. A *pre-log structure* on a complex space X is a sheaf of monoids $\mathcal{M}_X$ on X together with a homomorphism of sheaves of monoids:

$$\alpha_X : \mathcal{M}_X \to \mathcal{O}_X,$$

where the monoid structure on $\mathcal{O}_X$ is given by multiplication. A pre-log structure is a *log structure* if:

$$\alpha_X : \alpha_X^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$$

is an isomorphism. Here $\mathcal{O}_X^\times$ is the sheaf of invertible elements of $\mathcal{O}_X$.

It is often useful to think of log structures by considering the exact sequence

$$1 \to \mathcal{O}_X^\times \xrightarrow{\alpha_X^{-1}} \mathcal{M}_X^{\mathrm{gp}} \to \mathcal{M}_X^{\mathrm{gp}} / \operatorname{Im} \alpha_X^{-1} \to 0.$$

Definition 2.13. The sheaf of monoids

$$\overline{\mathcal{M}}_X := \mathcal{M}_X / \operatorname{Im} \alpha_X^{-1} = \mathcal{M}_X / \mathcal{O}_X^\times,$$

written additively, is called the *ghost sheaf* of $\mathcal{M}_X$.

Definition 2.14. A morphism of (pre-)log complex spaces $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a holomorphic map $\underline{f} : X \to Y$ together with a homomorphism of sheaves of monoids $f^\flat : f^{-1}\mathcal{M}_Y \to \mathcal{M}_X$ such that the diagram

$$\begin{tikzcd}
\underline{f}^{-1}\mathcal{M}_Y \arrow[r, "f^\flat"] \arrow[d, "\alpha_Y"'] & \mathcal{M}_X \arrow[d, "\alpha_X"] \\
\underline{f}^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"] & \mathcal{O}_X
\end{tikzcd}$$

commutes. Here $f^\sharp$ is the morphism of the structure sheaves.

We denote the category of log complex spaces with *LAn*.

Example 2.15. *The trivial log structure.* Let X be a complex space. Set

$$\mathcal{M}_X := \mathcal{O}_X^\times$$

and let

$$\alpha_X : \mathcal{M}_X \hookrightarrow \mathcal{O}_X$$

be the canonical inclusion. This defines a log structure on X.

Remark 2.16. We clearly have $\mathrm{Mor}_{LAn}((X, \mathcal{O}_X^\times), (Y, \mathcal{O}_Y^\times)) = \mathrm{Mor}_{An}(X, Y)$.

From now on we shall mostly omit to write the log structure when a complex space is endowed with the trivial log structure. That is, we write S instead of $(S, \mathcal{O}_S^\times)$.

Example 2.17. *The log point.* Let $X := \operatorname{Spec} \mathbb{C}$. Let $\mathcal{M}_X := \mathbb{C}^\times \oplus Q$, where Q is a monoid whose only invertible element is 0. Then $\alpha_X : \mathcal{M}_X \to \mathbb{C}$ given by

$$\alpha_X(x, q) := \begin{cases} x, & \text{if } q = 0 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

defines a log structure on X. We have two special cases. If $Q = \{0\}$, then we get the trivial log structure. If $Q = \mathbb{N}$, we get what is called *the standard log point* and it is usually denoted by $\operatorname{Spec} \mathbb{C}^\dagger$.

Example 2.18. *The divisorial log structure.* Let X be a complex space and $D \subset X$ a divisor, that is a closed analytic subset of codimension 1. Let $j : X \setminus D \hookrightarrow X$ be the inclusion and consider

$$\mathcal{M}_{(X,D)} := (j_* \mathcal{O}_{X \setminus D}^\times) \cap \mathcal{O}_X.$$

This is the sheaf of holomorphic functions on X invertible on $X \setminus D$. Let $\alpha_X : \mathcal{M}_{(X,D)} \hookrightarrow \mathcal{O}_X$ be the canonical inclusion. This is clearly a log structure on X.

Example 2.19. Let P be a finitely generated monoid and consider the affine toric variety $X := \operatorname{Spec} \mathbb{C}[P]$. Let ∂X be the toric boundary divisor, that is the complement in X of the big torus. Then the *canonical log structure* on X is the divisorial log structure induced by the toric boundary divisor ∂X. The pair $(X, \mathcal{M}_{(X, \partial X)})$ is usually denoted by $\operatorname{Spec}(P \to \mathbb{C}[P])$.

Let $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ be a pre-log structure on X and consider the morphism of sheaves of monoids:

$$\alpha_X^a : \mathcal{M}_X \oplus \mathcal{O}_X^\times \to \mathcal{O}_X$$

defined by

$$\alpha_X^a(p, f) := \alpha_X(p) \cdot f.$$

We can define the kernel of α_X^a

$$\ker \alpha_X^a := \{(p, f) \in \mathcal{M}_X \oplus \mathcal{O}_X^\times | \alpha_X(p) \cdot f = 1\}.$$

Definition 2.20. The *log structure associated to the pre-log structure* $(\mathcal{M}_X, \alpha_X)$ is given by the sheaf of monoid:

$$\mathcal{M}_X^a := (\mathcal{M}_X \oplus \mathcal{O}_X^\times) / \ker \alpha_X^a$$

together with the induced map $\alpha_X^a : \mathcal{M}_X^a \to \mathcal{O}_X$.

Let $\beta_1 : P \to \mathcal{O}_X$ and $\beta_2 : P \to \mathcal{O}_X$ be pre-log structures on a complex space X. If there exists a map $f : P \to \mathcal{O}_X^\times$ such that $\beta_1 = f \cdot \beta_2$, then the corresponding associated log structures on X are isomorphic. The isomorphism between the corresponding associated log structures is clearly given by:

$$\phi_{\beta_1\beta_2} : \mathcal{M}^a_{\beta_1} \to \mathcal{M}^a_{\beta_2}, [(p, g)] \mapsto [(p, f(p) \cdot g)].$$

The pre-log structures β_1 and β_2 are said to be *equivalent* (see [Gro11], p. 106).

Definition 2.21. If $f : X \to Y$ is a morphism of complex spaces (see Definition 1.3) and Y has a pre-log structure $\alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y$, then the *pull-back pre-log structure* on X is given by the composition:

$$\alpha_X : f^{-1}(\mathcal{M}_Y) \xrightarrow{\alpha_Y} f^{-1}(\mathcal{O}_Y) \xrightarrow{\tilde{f}} \mathcal{O}_X. \tag{2.2}$$

We denote the pull-back pre-log structure with $(f^*\mathcal{M}_Y, f^*\alpha_Y)$. If $\alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y$ is a log structure, then the *pull-back log structure* on X is the associated log structure to the pull-back pre-log structure given in (2.2).

Definition 2.22. A morphism of log complex spaces $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is called *strict* if the pull-back of $\mathcal{M}_Y$ via f is isomorphic to $\mathcal{M}_X$.

2.3 Log charts

The notion of *log chart* was first introduced by K. Kato in [Kat89]. In a nutshell, a log complex space $(X, \mathcal{M}_X)$ admits a log chart at a point $x \in X$ if locally, around x, the log structure $\mathcal{M}_X$ "looks like" the divisorial log structure of an affine toric variety (see Example 2.19).

Definition 2.23. Let P be a monoid, $(X, \mathcal{M}_X)$ a log complex space. Let $\underline{P}$ be the constant sheaf on X with stalk P. A *chart* for $\mathcal{M}_X$ subordinate to P is a morphism of pre-log structures:

$$\begin{tikzcd} \underline{P} \arrow[rr, "\theta"] \arrow[dr, "\beta"'] & & \mathcal{M}_X \arrow[dl, "\alpha_X"] \\ & \mathcal{O}_X & \end{tikzcd},$$

such that $\theta^a : \underline{P}^a \to \mathcal{M}_X$ is an isomorphism.

A chart is determined by the morphism θ. Thus, we usually identify the chart with the morphism θ. For the sake of simplicity, we shall mostly write P instead of $\underline{P}$ for the constant sheaf with stalk the monoid P.

Remark 2.24. Let X be a complex space and $\alpha_X : P \to \mathcal{O}_X$ a pre-log structure on X. Let $(\mathcal{M}_X^a, \alpha_X^a)$ be the associated log structure. Then, $\theta : P \to \mathcal{M}_X^a$, given by $\theta(p) = (p, 1)$ is a chart for $\mathcal{M}_X^a$ subordinate to P.

Example 2.25. Let $\mathrm{Spec}(P \to \mathbb{C}[P])$ be the log affine toric variety considered in Example 2.19. Set $X := \mathrm{Spec}\,\mathbb{C}[P]$, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}[P]$. Let $\theta : P \to \Gamma(X, \mathcal{M}_{(X,\partial X)})$ be the map sending p to the holomorphic function z^p on X, which is invertible on the big torus orbit, that is on the complement of the toric boundary divisor ∂X. The map θ defines a log chart on $\mathrm{Spec}(P \to \mathbb{C}[P])$.

Proposition 2.26. *([Ogu06], p. 114) Let $(X, \mathcal{M}_X)$ be a log complex space. Let P be a finitely generated monoid. The following data are equivalent:*

1. *a log chart $P \to \mathcal{M}_X$;*
2. *a strict morphism of log complex spaces $(X, \mathcal{M}_X) \to \mathrm{Spec}(P \to \mathbb{C}[P])$.*

Definition 2.27. A log structure $\mathcal{M}_X$ on a complex space X is called *coherent* if locally on X it admits charts subordinate to finitely generated monoids. A log structure $\mathcal{M}_X$ on X is called *integral* if $\mathcal{M}_X$ is a sheaf of integral monoids. An integral and coherent log structure is called *fine*. A log structure is called *fine and saturated (fs)* if it is fine and for every $x \in X$ the stalk at x of the ghost sheaf $\overline{M}_{X,x}$ is a saturated monoid.

Remark 2.28. ([Ogu06], p. 19) If $\overline{M}_{X,x}$ is a saturated monoid, then $\overline{M}_{X,x}^{gp}$ is torsion free.

The category of fine and saturated log complex spaces is a full subcategory of the category of fine log complex spaces, which in turn is a full subcategory of coherent log complex spaces. The category of log complex spaces admits fiber products (see [Ogu06], p. 137). More precisely,

Definition 2.29. Let $(X, \mathcal{M}_X), (Y, \mathcal{M}_Y), (Z, \mathcal{M}_Z)$ be log complex spaces and let $\psi : (Y, \mathcal{M}_Y) \to (Z, \mathcal{M}_Z)$ be a log morphism. The fiber product $(X, \mathcal{M}_X) \times_{(Z, \mathcal{M}_Z)} (Y, \mathcal{M}_Y)$ is given by the pair $(X \times_Z Y, \mathcal{M}_{X \times_Z Y})$, where the underlying complex space is the fiber product of the corresponding underlying complex spaces and the log structure is given by the log structure associated to

$$\pi_X^{-1}\mathcal{M}_X \oplus_{\pi_Z^{-1}\mathcal{M}_Z} \pi_Y^{-1}\mathcal{M}_Y \to \mathcal{O}_{X_Z Y},$$

which is the pushout of the following diagram:

$$\begin{tikzcd} & \pi_Z^{-1}\mathcal{M}_Z \arrow[dl] \arrow[dr] & \\ \pi_X^{-1}\mathcal{M}_X & & \pi_Y^{-1}\mathcal{M}_Y \end{tikzcd}.$$

Proposition 2.30. *([Ogu18], p. 13, Proposition 1.3.4) Let $u_i : P \to Q_i$ be monoid homomorphisms, with $i \in \{1, 2\}$, and consider the pushout $Q := Q_1 \oplus_P Q_2$. If P, Q_1, Q_2 are integral and any one of them is a group, then Q is integral.*

We immediately get the following

Corollary 2.31. *Let $(X, \mathcal{M}_X) \to S$ and $\psi : T \to S$ be morphisms of fine log complex spaces (where S and T are endowed with the trivial log structure, see Example 2.15). Then $\psi^*(X, \mathcal{M}_X) := (X, \mathcal{M}_X) \times_T S$ is a fine log complex space.*

2.4 Gluing log charts

We want to find a universal setup for constructing log structures from gluing of charts. This is quite analogous to the case of sheaves, presented for example in [Har77], Example II.1.22. Let $(X_0, \mathcal{M}_{X_0})$ be a compact fine log complex space. Denote by $\alpha : \mathcal{M}_{X_0} \to \mathcal{O}_{X_0}$ the structure map.

Assume we have a covering of X_0 by open sets U_i for an ordered index set J_0, and for each U_i a chart (see Definition 2.23)

$$\theta_i : P_i \longrightarrow \Gamma(U_i, \mathcal{M}_{X_0}).$$

We identify θ_i with the corresponding map of monoid sheaves $\underline{P}_i \to \mathcal{M}_{X_0}|_{U_i}$. For $l = 1, 2$, let

$$J_l := \{(i_0, ..., i_l) \in J_0^{l+1} : U_{i_0} \cap ... \cap U_{i_l} \neq \emptyset\}.$$

We get maps $d_m : J_l \to J_{l-1}$, for $0 \le m \le l$ and $1 \le l \le 2$, sending $(i_0, .., i_m, .., i_l)$ to $(i_0, .., i_{m-1}, i_{m+1}, .., i_l)$. We set

$$\begin{aligned} J &:= \bigcup_{i=0}^{2} J_l, \\ \partial i &:= \{d_0 i, ..., d_l i\}, \text{ if } i \in J_l. \end{aligned}$$

The set J, together with the maps (d_m), is called a *simplicial set of order 2* (see Definition 4.7). For each $j := (i_0, i_1) \in J_1$, we assume that there is a chart

$$\theta_j : P_j \longrightarrow \Gamma(U_j, \mathcal{M}_{X_0})$$

and comparison maps

$$\varphi_j^i : P_i \quad \longrightarrow \quad P_j \oplus \Gamma(U_j, \mathcal{O}_{X_0}^{\times}),$$

for $i \in \partial j$, with the property

$$\left(\theta_j \cdot \mathrm{id}_{\mathcal{O}_{X_0}^{\times}|_{U_j}}\right) \circ \varphi_j^i = \theta_i|_{U_j}. \tag{2.3}$$

By the local existence of charts, φ_j^i exists locally, so at the expense of introducing another index, we could always cover (a compactum in) U_j by finitely many V_j^k supporting such charts. For simplicity, let's rather assume our original charts θ_i have been chosen intelligently enough to have a comparison chart θ_j on all of U_j.

From a slightly different point of view, each θ_i defines an isomorphism of $\mathcal{M}_{U_i}$ with the log structure $\mathcal{M}_i$ associated to the pre-log structure $\beta_i := \alpha \circ \theta_i$. Similarly, the pre-log structure $\beta_j := \alpha \circ \theta_j$ defines a log structure $\mathcal{M}_j$ and θ_j defines an isomorphism of log structures $\mathcal{M}_{U_j} \simeq \mathcal{M}_j$. From this point of view, equation 2.3 means that φ_j^i provides an isomorphism between $\mathcal{M}_i|_{U_j}$ and $\mathcal{M}_j$, and this isomorphism is compatible with the isomorphisms $\mathcal{M}_i|_{U_j} \simeq \mathcal{M}_{U_j}$ and $\mathcal{M}_j \simeq \mathcal{M}_{U_j}$.

Now, if we have θ_i, θ_j, φ_j^i, fulfilling (2.3), we need compatibility on triple intersections for the patching of the $\mathcal{M}_i$ to be consistent. To formulate this cocycle condition in terms of charts we assume, for each $k := (i_0, i_1, i_2) \in J_2$, a third system of charts

$$\theta_k : P_k \longrightarrow \Gamma(U_k, \mathcal{M}_{X_0})$$

and comparison maps

$$\varphi_k^j : P_j \quad \longrightarrow \quad P_k \oplus \Gamma(U_k, \mathcal{O}_{X_0}^\times),$$

for $j \in \partial k$. The analogue of the compatibility condition (2.3) is

$$\left(\theta_k \cdot \mathrm{id}_{\mathcal{O}_{X_0}^\times|_{U_k}}\right) \circ \varphi_k^j = \theta_j|_{U_k}. \tag{2.4}$$

Again, the φ_k^j define an isomorphism between the log structure $\mathcal{M}_j|_{U_k}$ on U_k and the log structure $\mathcal{M}_k$ associated to the pre-log structure $\beta_k := \alpha \circ \theta_k$. In particular, all the isomorphisms of log structures are compatible and the $(\mathcal{M}_i)_{i \in J_0}$ glue in a well-defined fashion, as do their structure maps, to a log structure on X_0 isomorphic to $\mathcal{M}_{X_0}$. This is just standard sheaf theory, for sheaves of monoids. We set $\mathscr{P}_l := \{P_i : i \in J_l\}$, for $l = 0, 1, 2$, and $\mathscr{P} := \bigcup_{l=0}^2 \mathscr{P}_l$.

Definition 2.32. A *directed set of log charts* is given by a set of log charts $(\theta_i : P_i \to \mathcal{M}_{U_i})_{j \in J}$ on $(X_0, \mathcal{M}_{X_0})$ together with a morphism

$$\varphi_j^i : P_i \to P_j \oplus \mathcal{O}_{U_j}^\times,$$

for each $j \in J_1 \cup J_2$ and $i \in \partial j$, such that

$$(\theta_j \cdot Id_{\mathcal{O}_{U_j}^\times}) \circ \varphi_j^i = \theta_i|_{U_j}.$$

Now, let us forget that the $(\theta_i)_{i \in J_0}$, $(\theta_j)_{j \in J_1}$ and $(\theta_k)_{k \in J_2}$ are charts for the given log structure. Let $(U_i)_{i \in J_0}$ be an open cover of X_0, J and $\mathcal{P}$ as above. Assume we have pre-log structures $(\beta_i)_{i \in J}$ and comparison maps $(\varphi_j^i)_{j \in J_1 \cup J_2, i \in \partial j}$ satisfying Equations 2.3 and 2.4. Then the log structures $(\mathcal{M}_i)_{i \in J_0}$ glue to a log structure $\mathcal{M}$ on X_0 in such a way that the gluing data $(\beta_j)_{j \in J_1}$ and compatibility $(\beta_k)_{k \in J_2}$ arise from identifying $\mathcal{M}_j$ and $\mathcal{M}_k$ with restrictions of $\mathcal{M}$ to U_j and U_k, respectively. We can collect these data in the notion of *pre-log atlas*.

Definition 2.33. Let X_0 be a compact complex space. With the above notation, a collection of data

$$\{(\beta_i : P_i \to \mathcal{O}_{U_i})_{i\in J}, (\varphi_j^i : P_i \to P_j \oplus \mathcal{O}_{U_j}^\times)_{j\in J_1\cup J_2, i\in\partial j}\}$$

satisfying

$$\left(\beta_j \cdot \mathrm{id}_{\mathcal{O}_{U_j}^\times}\right) \circ \varphi_j^i = \beta_i|_{U_j} \tag{2.5}$$

is called a *pre-log atlas* on X_0.

2.5 Log smooth and log flat morphisms

Definition 2.34. ([Ogu18], p. 259) Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a log morphism. A *chart for* f is a triple $(P \to \mathcal{M}_X, Q \to \mathcal{M}_Y, Q \to P)$ given respectively by a log chart for $(X, \mathcal{M}_X)$, a log chart for $(Y, \mathcal{M}_Y)$ and a monoid homomorphism $Q \to P$ such that the following diagram

$$\begin{array}{ccc} Q & \longrightarrow & f^{-1}\mathcal{N}_Y \\ \downarrow & & \downarrow \\ P & \longrightarrow & \mathcal{M}_X \end{array}$$

or equivalently (by Proposition 2.26), the diagram

$$\begin{array}{ccc} (X, \mathcal{M}_X) & \longrightarrow & \mathrm{Spec}(P \to \mathbb{C}[P]) \\ \downarrow & & \downarrow \\ (Y, \mathcal{M}_Y) & \longrightarrow & \mathrm{Spec}(Q \to \mathbb{C}[Q]) \end{array},$$

where the horizontal arrows are strict log morphisms, commutes.

Definition 2.35. ([Gro11], p. 107 and [INT13]) A morphism of fine log complex spaces $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is called *log smooth* (*log flat*), if locally there exists a commutative diagram:

$$\begin{array}{ccc} (U, \mathcal{M}_X|_U) & \longrightarrow & \mathrm{Spec}(P \to \mathbb{C}[P]) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ (V, \mathcal{M}_Y|_V) & \longrightarrow & \mathrm{Spec}(Q \to \mathbb{C}[Q]) \end{array}$$

such that the horizontal maps induce log charts $(\beta_U : P \to \mathcal{M}_U)$ and $(\beta_V : Q \to \mathcal{M}_V)$ for $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ respectively, the induced morphism of complex spaces

$$u : U \to V \times_{\operatorname{Spec} \mathbb{C}[Q]} \operatorname{Spec} \mathbb{C}[P]$$

is smooth (flat) and g is induced by an injective homomorphism of fine monoids $Q \to P$.

These notions are stable under compositions and base changes.

Remark 2.36. If $Q \to P$ is an integral and injective homomorphism of fine monoids (see Definition 2.8), then the induced morphism of affine toric varieties $\operatorname{Spec} \mathbb{C}[P] \to \operatorname{Spec} \mathbb{C}[Q]$ is flat (see [Ogu06], p. 91).

Proposition 2.37. *([Ogu18], p. 424) A log smooth morphism of fine log complex spaces is log flat.*

Chapter 3

The theory of Banach analytic spaces

In this chapter we present an account of the theory and results developed by A. Douady in [Dou66]. We use [Car66], [Dou68] and [Sti88] as auxiliary references.

3.1 Banach manifolds

Banach manifolds are a natural generalization, in the infinite dimensional setting, of the classical notion of complex manifolds. We first recall the definition (*á la Weierstrass*) of *holomorphic or analytic* maps between open subsets of Banach spaces.

Definition 3.1. Let E, F be Banach spaces over $\mathbb{C}$. A map $f_n : E \to F$ is called a *homogeneous polynomial* of degree n if there exists a continuous, n-multilinear and symmetric map:

$$\tilde{f}_n : \underbrace{E \times ... \times E}_{n} \to F$$

such that

$$f_n(x) = \tilde{f}_n(x, ..., x)$$

for each $x \in E$.

Definition 3.2. Let E, F be Banach spaces and $U \subset E$ open. A map $f : U \to F$ is called *holomorphic or analytic* if for each $a \in U$ there exists $r > 0$ and an expansion:

$$f(a + x) = \sum_{n \geq 0} f_n(x)$$

holding for $\|x\| \leq r$, where $f_n : E \to F$ are homogenous polynomial maps of degree n such that:

$$\sum_{n \geq 0} \sup_{\|x\| \leq r} \|f_n(x)\| < \infty$$

The composition of two holomorphic maps is again a holomorphic map.

Definition 3.3. We can define a category $\mathcal{B}$, whose objects are open subsets of Banach spaces and morphisms are holomorphic maps.

Definition 3.4. Let X be a topological space. A *Banach structure* on X is an atlas $\mathcal{A} := \{(U_i, \varphi_i)\}$, where U_i is open in X, $\varphi_i : U_i \to V_i$ is a homeomorphism of U_i onto an open subset V_i of a Banach space E_i, such that all transition maps are holomorphic. The pair $(X, \mathcal{A})$ is called a *Banach manifold*.

When all V_i are open subsets of finite dimensional complex vector spaces, we get the classical notion of complex manifold.

Definition 3.5. Let X, Y be Banach manifolds. A map $f : X \to Y$ is called *holomorphic* if, for any $x \in X$, any chart $\varphi_i : U_i \to V_i$ around x and any chart $\varphi_i' : U_i' \to V_i'$ around $f(x)$, the map $\varphi_i' \circ f \circ \varphi_i^{-1}$, which is defined on a neighborhood of $\varphi_i(x)$ in V_i, is a holomorphic map.

We get a category, whose objects are Banach manifolds and whose morphisms are holomorphic maps.

Definition 3.6. Let E be a Banach space. A subspace L of E is called *direct* if it is closed and if there exists another closed subspace $G \subset E$ such that $L \oplus G = E$. That is, there exists a linear homomorphism $p \in \mathcal{L}(E, E)$, such that $p \circ p = p$ and $\ker p = G$. The subspace G is called a *topological complement* of L. A continuous linear map $f : E \to F$ between Banach spaces is called *direct* if $\ker f$ and $\operatorname{Im} f$ are direct subspaces of E and F respectively.

Example 3.7. ([Dou66], p. 16) *The Grassmannian of a Banach space.* The set

$$\mathcal{G}(E) := \{\text{direct subspaces of E}\}.$$

can be endowed with a Banach structure. For any $G \in \mathcal{G}(E)$, let $i_G : G \hookrightarrow E$ be the canonical injection. Let

$$U_G := \{F \in \mathcal{G}(E) : F \oplus G = E\}.$$

Fix $F \in U_G$ and for any $F' \in U_G$, let $p_{F',G} : F' \oplus G \to G$ be the projection. We define

$$\phi_{F,G} : U_G \to \mathcal{L}(F, G)$$

via

$$F' \mapsto \lambda := p_{F',G} \circ i_F.$$

The map $i_F - i_G \circ \lambda : F \to E$ is an isomorphism of F onto F' and conversely, given any $\lambda \in \mathcal{L}(F, G)$, the map $i_F - i_G \circ \lambda$ is the restriction to F of some automorphism of E, hence it is an isomorphism of F onto a complementary subspace of G. Thus, the morphism:

$$\phi'_{F;G} : \mathcal{L}(F, G) \to U_G$$

defined by $\phi'_{F;G}(\lambda) := (i_F - i_G \circ \lambda)(F)$ gives a two-sided inverse of $\phi_{F,G}$. Now, let us consider the set

$$\hat{\mathcal{G}}(E) := \{\text{closed subvectorspaces of } E\}.$$

For each $F, F' \in \hat{\mathcal{G}}(E)$ let

$$\theta(F, F') := \sup_{x \in F, \|x\| \leq 1} \inf_{y \in F', \|y\| \leq 1} \|x - y\|.$$

We can define a metric on $\hat{\mathcal{G}}(E)$ setting $d(F, F') := \max(\theta(F, F'), \theta(F', F))$. This metric induces a topology on $\hat{\mathcal{G}}(E)$ for which $\mathcal{G}(E)$ is an open subset. The induced topology on $\mathcal{G}(E)$ is the unique topology for which all the U_G are open subsets and the maps $\phi_{F,G}$ are homeomorphisms. The transition maps are analytic (see [Dou66], pp. 16-17) and the charts $\phi_{F,G} : U_G \to \mathcal{L}(F, G)$ endow $\mathcal{G}(E)$ with the structure of a Banach manifold.

Definition 3.8. Let X be a Banach manifold and $Y \subset X$ a subset. Then Y is called a *(direct) submanifold* of X, if for any $y \in Y$ there exists a chart $\varphi : U \to V \subset E$ for X such that $\varphi(y) = 0$ and there exists a closed (direct) subspace F of the Banach space E such that $\varphi(U \cap Y) = F \cap V$.

Definition 3.9. Let X be a Banach manifold and $x \in X$. The *tangent space* of X at x is defined as

$$T_xX := \{(\varphi, t) | \varphi : U \to V \subset E \text{ is a chart on a neigh. of } x \text{ s.t. } \varphi(x) = 0, t \in E\}/\sim,$$

with $(\varphi, t) \sim (\varphi', t')$, if $t' = \alpha(t)$, where α is the linear part of the series expansion of the transition function $\varphi' \circ \varphi^{-1}$ in 0.

Definition 3.10. Let $f : X \to Y$ be a holomorphic map between Banach manifolds. Let $x \in X$ and set $y = f(x)$. The *linear tangent map* to f at x

$$T_xf : T_xX \to T_yY$$

is defined via $T_xf((\varphi, t)) := (\psi, \alpha(t))$, where α is the linear part of the series expansion of the expression of f in the charts φ and ψ.

Theorem 3.11. *(Implicit function Theorem)([Dou66], p. 13) Let $f : X \to Y$ be a morphism of Banach manifolds. Let $x \in X$ and set $y := f(x)$. If $T_xf : T_xX \to T_yY$ is an isomorphism, then f is a local isomorphism at x.*

Definition 3.12. ([Dou66], p. 14) A continuous linear map $f : E \to F$ between Banach spaces is called *strict*, if it induces a topological isomorphism of $E/\ker f$ onto $\operatorname{Im} f$. Moreover, f is called a *monomorphism* if it is strict and injective. It is called an *epimorphism* if it is surjective (it is automatically strict). The homomorphism f is called *direct* if $\ker f$ and $\operatorname{Im} f$ are direct subspaces of E and F respectively. Every direct homomorphism is strict.

Corollary 3.13. *([Dou66], p. 14) Let $f : X \to Y$ be a morphism of Banach manifolds. Let $x \in X$ and set $y := f(x)$.*

1. *If $T_x f$ is a direct monomorphism, then f is a direct immersion at x, that is X is isomorphic to a direct submanifold of Y.*
2. *If $T_x f$ is a direct epimorphism, then f is a direct submersion at x, that is $f^{-1}(f(x))$ is a direct submanifold of X in a neighborhood of x.*

3.2 Banach analytic spaces

We recall from Definition 1.7, that a complex space $(X, \mathcal{O}_X)$ is a locally ringed space locally isomorphic to a model. This model is defined by holomorphic functions $f_1, ..., f_k$ defined on an open subset $U \subset \mathbb{C}^n$. Indeed, we get a holomorphic map $f := (f_1, ..., f_k) : U \subset \mathbb{C}^n \to \mathbb{C}^k$ and then we take $X := f^{-1}(0)$ as model. To define a complex analytic structure on X, for each W open in X, for each $l \in \mathbb{N}$ and open subset V in $\mathbb{C}^l$, we need to specify the set $\operatorname{Mor}(W, V)$ of holomorphic maps from W into V. Since,

$$\operatorname{Mor}(W, \mathbb{C}^l) = \operatorname{Mor}(W, \mathbb{C}) \otimes_{\mathbb{C}} \mathbb{C}^l$$

see [GR84], p. 22, it is enough to consider the case $l = 1$. Hence the complex analytic structure on X is determined by the sheaf of rings $\mathcal{O}_U$ of holomorphic functions on $U \subset \mathbb{C}^n$ modulo the ideal sheaf generated by the functions $f_1, .., f_k$. Now, if we consider a holomorphic map $f : U \subset E \to F$, where E and F are infinite dimensional Banach spaces, for any Banach space G and open subset W in $X := f^{-1}(0)$, we have

$$\operatorname{Mor}(W, G) \neq \operatorname{Mor}(W, \mathbb{C}) \otimes_{\mathbb{C}} G.$$

Hence, to define a Banach analytic structure on X, we need to specify the set of holomorphic maps from each open subset W of X into any open subset V of any Banach space G. That is, we need to define a functor Φ from the category of open subsets of Banach spaces $\mathcal{B}$ (see Definition 3.3) into the category of sheaves of sets on X (if $V = G$, then we get a sheaf of $\mathbb{C}$-vector spaces on X). This leads to the following notion of *$\mathcal{K}$-functored spaces* and, in particular, to the notion of *Banach analytic spaces* (see Definition 3.16).

Definition 3.14. Let $\mathcal{K}$ be a category. A *$\mathcal{K}$-functored space* is a topological space X together with a covariant functor Φ from $\mathcal{K}$ into the category of sheaves of sets on X. Moreover, let (X, Φ) and (X', Φ') be two $\mathcal{K}$-functored spaces. A morphism of (X, Φ) into (X', Φ') is a pair (f_0, f_1), where $f_0 : X \to X'$ is a continuous map and $f_1 : f_0^{-1}\Phi' \to \Phi$ is a natural transformation.

The collection of $\mathcal{K}$-functored spaces and morphisms between them form a category.

Example 3.15. A Banach manifold (see Definition 3.4) can be defined as a $\mathcal{B}$-functored space such that each point admits a neighborhood isomorphic (as $\mathcal{B}$-functored space) to a *local model* defined as follows. For any open subset V of a Banach space G, that is, for any object in $\mathcal{B}$, and for any open subset W of X, the set of morphisms

$$\mathcal{H}(W, V) := \{f : W \to V | f \text{ is holomorphic}\}.$$

For any open subset W' in W, we naturally get restriction maps

$$\mathcal{H}(W, V) \to \mathcal{H}(W', V).$$

Thus, for any object V in $\mathcal{B}$, we get a sheaf of sets $\mathcal{H}_X(V)$ on X. If $\phi : V \to V'$ is a morphism of open subsets of Banach spaces, we can naturally define a morphism of sheaves of sets

$$\mathcal{H}_X(\phi) : \mathcal{H}_X(V) \to \mathcal{H}_X(V'),$$

which, for any $W \subset X$ open and $f \in \mathcal{H}_X(V)(W)$, sends f to $\phi \circ f$. The category of Banach manifolds is a full subcategory of $\mathcal{B}$-functored spaces.

Banach analytic spaces are $\mathcal{B}$-functored spaces locally isomorphic to local models defined as follows. Let E, F be Banach spaces, U open in E and $f : U \to F$ a holomorphic map. Set $X := f^{-1}(0)$. For any Banach space G, set

$$\mathcal{H}(U, G) := \{g : U \to G | g \text{ is holomorphic}\}$$

and

$$\mathcal{N}(f, G) := \{\lambda.f | \lambda \in \mathcal{H}(U, \mathcal{L}(F, G))\},$$

where $\mathcal{L}(F, G)$ is the space of linear maps from F into G, and $\lambda.f(x) := \lambda(x)(f(x))$, for each $x \in U$. The space $\mathcal{N}(f, G)$ is a subspace of the vector space $\mathcal{H}(U, G)$, so we can perform the quotient $\mathcal{H}(U, G)/\mathcal{N}(f, G)$. For any open subset U' in U, the map:

$$\mathcal{H}_U(G) : U' \subset U \mapsto \mathcal{H}(U', G)/\mathcal{N}(f_{|_{U'}}, G),$$

defines a presheaf of vector spaces on U. Let $\mathcal{H}_U^\dagger(G)$ be its associated sheaf. Since for each $g \in \mathcal{N}(f, G)$ we have $g(X) = 0$, then Supp $\mathcal{H}_U^\dagger(G) = X$. Now, for each open subset $V \subset G$, we get a sheaf of sets on X by restricting $\mathcal{H}_U^\dagger(G)$ to the subsheaf $\Phi(V)$ defined for each W open in X, by all sections $s \in \Gamma(W, \mathcal{H}_U^\dagger(G))$ such that $s : W \to V$. Lastly, if $h : V \to V'$ is a morphism between open subspaces of Banach spaces, then we define $\Phi(h) : \Phi(V) \to \Phi(V')$, sending $[g]$ to $[h \circ g]$. This map is well defined (see [Dou66], p. 23). We write $\mu(U, F, f)$ for the local model (X, Φ) defined by the analytic map $f : U \subset E \to F$.

Definition 3.16. ([Dou66], pp. 22-25) A *Banach analytic space* is a $\mathcal{B}$-functored space (X, Φ) locally isomorphic to a model $\mu(U, F, f)$. The topological space X is called the *support* of (X, Φ).

If (X, Φ) is a Banach analytic space, then $\Phi(\mathbb{C})$ is a sheaf of $\mathbb{C}$-algebras on X. If G is a Banach space, then $\Phi(G)$ is a sheaf of vector spaces on X. By abuse of notation we shall mostly write X instead of (X, Φ).

Definition 3.17. A Banach analytic space is called of *finite dimension* if every point admits a neighborhood isomorphic to a local model defined by a holomorphic map $f : W \subset E \to F$, where E, F are finite dimensional Banach spaces.

Let $F := \mathbb{C}^k$, $f := (f_1, .., f_k)$ and $G := \mathbb{C}$. Since $\mathcal{L}(F, G) \simeq \mathbb{C}^k$, we have that $\mathcal{N}(f_{|_{U'}}, G)$ is the ideal in $\mathcal{H}(U', G)$ generated by $f_1|_{U'}, ..., f_k|_{U'}$. Thus, we see that the notion of finite dimensional Banach analytic spaces is equivalent to the notion of complex analytic spaces (see Definition 1.7). The category of complex analytic spaces is a full subcategory of the category of Banach analytic spaces.

Definition 3.18. A model $\mu(U, F, f)$ is called *smooth* if $f = 0$. A Banach analytic space is called *smooth* if every point admits a neighborhood isomorphic to a smooth local model.

From Example 3.15, we see that the notion of smooth Banach analytic spaces is equivalent to the notion of Banach manifolds. The category of Banach manifolds is a full subcategory of the category of Banach analytic spaces. We can draw the following diagram

$$\begin{array}{ccc} \text{Banach analytic spaces} & \xrightarrow{\text{smooth}} & \text{Banach manifolds} \\ {\scriptstyle \text{finite dimensional}}\Big\downarrow & & \Big\downarrow{\scriptstyle \text{finite dimensional}} \\ \text{complex analytic spaces} & \xrightarrow{\text{smooth}} & \text{complex manifolds} \end{array}.$$

Definition 3.19. Let $h = (h_0, h_1) : (X, \Phi) \to (X', \Phi')$ be a morphism of Banach analytic spaces. It is called an *immersion* if h_0 is injective and for any U open in a Banach space G, the sheaf morphism

$$h_1(U) : h_0^{-1}(\Phi'(U)) \to \Phi(U)$$

is surjective. In this case (X, Φ) is called *a Banach analytic subspace* of (X', Φ').

Example 3.20. Let X be a Banach manifold and F a Banach space. Let $f : X \to F$ be a holomorphic map. Then, f defines a Banach analytic subspace of X, whose support is the zero set of f.

Example 3.21. ([Dou66], p. 29) *The Grassmannian of a Banach module.* Let A be a Banach algebra and E a Banach space. Assume that E is endowed with the structure of A-module, that is there exists a continuous bilinear map

$$A \times E \to E$$

$$(a, x) \mapsto a.x,$$

satisfying the A-module axioms. We have already defined the Grassmannian $\mathcal{G}(E)$ (see Example 3.7). The set

$$\mathcal{G}_A(E) := \{F \in \mathcal{G}(E) | \text{F is a } A\text{-submodule of E}\}$$

can be endowed with the structure of a Banach analytic subspace of the Banach manifold $\mathcal{G}(E)$. For $G \in \mathcal{G}(E)$, let U_G be as in Example 3.7. For any $F \in U_G$, let $i_F : F \hookrightarrow E$ be the canonical inclusion. We recall from Example 3.7 that for any $\lambda \in \mathcal{L}(F, G)$, the map $i_F - i_G \circ \lambda$ is an isomorphism of F onto a complement subspace F' of G. Let $p_{F',G} : F' \oplus G \to G$ be the projection and set

$$\rho_\lambda : A \times F \to G$$

$$(a, x) \mapsto p_{F',G}(a.(i_F(x) - i_G \circ \lambda(x))).$$

We can define a map

$$f : \mathcal{L}(F, G) \to \mathcal{L}(A \times F, G)$$

$$\lambda \mapsto \rho_\lambda.$$

This map is holomorphic being quadratic in λ. Now, let $\phi_{F,G} : U_G \to \mathcal{L}(F, G)$ be as in Example 3.7 and set $f' := f \circ \phi_{F,G} : U_G \to \mathcal{L}(A \times F, G)$. Let us consider the local model $\mu(U_G, \mathcal{L}(A \times F, G), f')$. We want to determine $X := f'^{-1}(0)$. For $\lambda \in \mathcal{L}(F, G)$, let as before $F' := (i_F - i_G \circ \lambda)(F)$, thus $f'(F') = f(\lambda)$. We have that $f'(F') = 0$ is equivalent to the condition $F' \in U_G \cap \mathcal{G}_A(E)$. Indeed, $f(\lambda) = 0$ is equivalent to $p_{F',G}(A.(i_F - i_G \circ \lambda)(F)) = 0$, that is $p_{F',G}(A.F') = 0$, which means $A.F' \subset F'$. Thus, $X = U_G \cap \mathcal{G}_A(E)$ and this does not depend on the choice of F. The collection $(U_G \cap \mathcal{G}_A(E))_{G \in \mathcal{G}(E)}$ covers $\mathcal{G}_A(E)$, hence $(\mu(U_G, \mathcal{L}(A \times F, G), f'))_{G \in \mathcal{G}(E)}$ defines a Banach analytic structure on $\mathcal{G}_A(E)$.

We describe some important operations that can be performed with Banach analytic spaces.

Proposition 3.22. *([Dou66], p. 26) The category of Banach analytic spaces admits finite products.*

Proof. Let $\mu(U', F', f')$ and $\mu(U'', F'', f'')$ be local models of Banach analytic spaces. We can define their *cartesian product* as the model $\mu(U' \times U'', F' \times F'', f' \times f'')$. Let (X, Φ) and (X', Φ') be Banach analytic spaces. We define their *cartesian product* in the following way. As underlying topological space we take the cartesian product $X \times X'$. Now, to each open subset W of a Banach space G we need to associated a sheaf of sets $\Phi_{X\times X'}(W)$ on $X \times X'$. Let $(U_i)_{i\in I}$ be an open cover of $X \times X'$, where each U_i is of the form $U_i' \times U_i''$ such that $U_i' \subset X$ and $U_i'' \subset X'$ give local models as above. We get local models $(\mu(U_i' \times U_i'', F_i' \times F_i'', f_i' \times f_i''))_{i\in I}$, which give sheaves of sets $(\Phi_{U_i'\times U_i''}(W))_{i\in I}$, which, in turn, give $\Phi_{X\times X'}(W)$ via the usual gluing procedure of sheaves. □

Proposition 3.23. *([Dou66], p. 25) Let E, F be Banach spaces, U an open subset of E and $f : U \to F$ a holomorphic map. Let (X, Φ) be a Banach analytic space and $i : \mu(U, F, f) \hookrightarrow \mu(U, 0)$ the canonical injection. Then*

$$\mathrm{Mor}((X, \Phi), \mu(U, F, f)) \to \{u : (X, \Phi) \to \mu(U, 0) : f \circ u = 0\}$$

$$h \mapsto u := i \circ h$$

is a bijection.

Proposition 3.24. *([Dou66], p. 25) In the category of Banach analytic spaces there exists the kernel of a double arrow. In other words, let X, X' be Banach analytic spaces and $u, v : X \to X'$ morphisms. Then, there exists a unique analytic subspace Z of X satisfying the following property: for any Banach analytic space Y*

$$\mathrm{Mor}(Y, Z) = \{h : Y \to X : u \circ h = v \circ h\}.$$

Proof. The uniqueness is clear. For the existence, we assume that X and X' are local models respectively given by $f : U \subset E \to F$ and $f' : U' \subset E' \to F'$. Let us assume that u, v, defined on $X = f^{-1}(0) \subset U$, can be extended to analytic maps on U. We claim that

$$Z = \mu(U, F \times E', (f, v - u)) \subset X.$$

Let Y be any Banach analytic space. By Proposition 3.23 we have

$$\mathrm{Mor}(Y, Z) = \{h' : Y \to \mu(U, 0) : (f, u - v) \circ h' = 0\}$$

that is

$$\mathrm{Mor}(Y, Z) = \{h' : Y \to \mu(U, 0) : f \circ h' = 0, u \circ h' = v \circ h'\}.$$

Again by Proposition 3.23, we have

$$\mathrm{Mor}(Y, \mu(U, F, f)) = \{h' : Y \to \mu(U, F, f) : f \circ h' = 0\}.$$

Thus,

$$\mathrm{Mor}(Y, Z) = \{h : Y \to \mu(U, F, f) : u \circ h = v \circ h\}.$$

□

Definition 3.25. The Banach analytic space Z in Proposition 3.24 is called *the kernel of the double arrow* (u, v). As set, we have:

$$Z = \{x \in X : u(x) = v(x)\}.$$

Corollary 3.26. *The category of Banach analytic spaces admits fibre products.*

Proof. Let X, Y, S be Banach analytic spaces and $u : X \to S$, $v : Y \to S$ morphisms. The *fibre product* $X \times_S Y$ of X and Y over S is the kernel of the double arrow $(u \circ \pi_1, v \circ \pi_2) : X \times Y \rightrightarrows S$, where π_1 and π_2 are the canonical projections of $X \times Y$ respectively onto X and Y. □

Definition 3.27. Let $X \to Y$ be a morphism of Banach analytic spaces and let $y \in Y$. We define the *fibre* of h over y to be

$$X(y) := h^{-1}(y) := (X \times_Y \{y\}).$$

Definition 3.28. ([Dou74], p. 573) A morphism $\varphi : X \to Y$ between Banach analytic spaces is called *submersion* at $p \in X$ if there exists an open neighborhood $U \subset X$ of p, an open subset $V \subset Y$ with $\varphi(U) \subset V$, an open subset Z of a Banach space E and an isomorphism $\psi : U \to Z \times V$ such that the diagram

$$\begin{tikzcd} U \arrow[rr, "\psi"] \arrow[dr, "\varphi"'] & & Z \times V \arrow[dl, "\pi_2"] \\ & V & \end{tikzcd},$$

commutes. The map φ is called *smooth* at $p \in X$, if one can choose $V = Y$.

Definition 3.29. ([Dou74], p. 572) Let X be a Banach analytic space and F a Banach space. The *trivial bundle* F_X is the product $X \times F$ together with the projection onto X. Let E be a Banach analytic space over X. A *vector atlas* on E over an open cover (U_i) of X is a family of isomorphisms $f_i : E|_{U_i} \to U_i \times F_i$, where (F_i) are Banach spaces, such that

$$\gamma_{ij} = f_i \circ f_j^{-1} \in Hom((F_j)_{U_{ij}}; (F_i)_{U_{ij}})$$

for each (i, j). One can show that the condition $\gamma_{ik} = \gamma_{ij} \circ \gamma_{jk}$ is automatically satisfied. A structure of *vector bundle* on E is an equivalence class of vector atlas.

We can define the *vertical tangent bundle* $T_S X$ of a Banach analytic space X smooth over S.

Definition 3.30. ([Dou74], p. 573) Let S be a Banach analytic space, S_1 an analytic subspace of S and W an open subset of $S_1 \times E$, where E is a Banach space. We denote by $T_S W$ the trivial bundle E_W and we call it the *vertical tangent bundle* of W over S.

If W' is an open subset of E', where E' is another Banach space, and $f : W \to W'$ is a S_1-morphism, we can define a f-morphism $T_S f : T_S W \to T_S W'$ in the following way. If S_1 is a Banach manifold, $T_S f(s, x)$ is the linear tangent map at x to the map $f(s) : W(s) \to W'(s)$ (see Definition 3.10). In general, one can locally embeds S into a Banach manifold U and f into an U-morphism $\bar{f}$ from an open subset $\tilde{W}$ of $U \times E$ into $U \times E'$. It can be shown (see ([Dou74], p. 573)) that the morphism $T_S f : E_W \to E_{W'}$ induced by $T_S \bar{f}$ does not depend on the choice of $\bar{f}$. The vertical tangent bundle has functoriality properties. In particular, if f is an isomorphism, then the same holds for $T_S f$. Therefore, by a gluing procedure, we can define *the vertical tangent bundle* $T_S X$ of a space X smooth over S. If $f : X \to X'$ is a S-morphism of two Banach analytic space smooth over S, one can define a f-morphism $T_S f : T_S X \to T_S X'$ as before.

Theorem 3.31. *(Implicit function theorem)([Dou74], p 574 and [Sti88], p. 141) Let $p : X \to S$ and $q : Y \to S$ be Banach analytic spaces smooth over a Banach analytic space S. Let $f : X \to Y$ be an S-morphism. Let $x \in X$ and $y \in Y$ with $y = f(x)$. Then*

1. *If $T_S f(x) : T_S X(x) \to T_S Y(y)$ is an isomorphism, then f is an isomorphism in a neighborhood of x.*

2. *Let $\sigma : S \to Y$ be a section with $\sigma(p(x)) = y$ and assume that $T_S f(x)$ is a direct epimorphism. Then $\ker(f, \sigma \circ p)$ is smooth over S.*

Let $h = (h_0, h_1) : (X, \Phi) \to (X', \Phi')$ be a morphism of Banach analytic spaces. Let $x \in X$ and assume that $h_0(x) = x' \in X'$. Since we can restrict a morphism to any open subset of X, we have the concept of a *germ of morphisms* $h = (h_0, h_1) : (X, \Phi) \to (X', \Phi')$ at the point $x \in X$, mapping x to x'.

Definition 3.32. Let X, X' be two Banach analytic spaces, $h : X \to X'$ a morphism and $x \in X$. Set $x' := h(x)$. We say that h is *compact* in x, if there exists an isomorphism φ of a neighborhood U of x onto a model $\mu(U, F, f)$, an isomorphism φ' of a neighborhood U' of x' onto a model $\mu(U', F', f')$ and a germ $\bar{h}$ of holomorphic maps at the point $\varphi(x)$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\ U \arrow[r, "\bar{h}"] & U' \end{tikzcd}$$

and such that the linear tangent map to $\bar{h}$ in $\varphi(x)$ is compact in the usual functional analytic sense, that is the image of any bounded subset of E is a relatively compact subset (has compact closure) in E'.

Proposition 3.33. *([Dou66], p. 29) Let X be a Banach analytic space and $x \in X$. The space X is of finite dimension at x if and only if the identity morphism is compact in x.*

Proof. The necessity is clear as any linear map of finite-dimensional vector spaces is compact. Let us assume $X = \mu(U, F, f)$, with U open in a Banach space E. Let us consider the chart diagram

$$\begin{tikzcd} X \arrow[r, "Id_x"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ U \arrow[r, "\bar{h}"] & U \end{tikzcd}.$$

Let $\lambda : E \to E$ be the linear tangent map at $\bar{h}$ in $\varphi(x)$. Since λ is compact, $\mathrm{Id} - \lambda$ is a Fredholm operator (see [Lan83], p. 221), hence $\ker(\mathrm{Id} - \lambda)$ and $\mathrm{coker}(\mathrm{Id} - \lambda)$ are finite dimensional and $\mathrm{Id} - \lambda$ is direct. Set $E' := \mathrm{Im}(\mathrm{Id} - \lambda)$ and let $p : E \to E'$ be a linear projection. Since $\bar{h} \circ \varphi = \varphi$, we have that $p \circ \bar{h} \circ \varphi = p \circ \varphi$. Hence, $\varphi(X)$ is an analytic subspace of the model defined by $p \circ (\mathrm{Id} - \bar{h}) : U \subset E \to E'$. Since $p \circ (\mathrm{Id} - \lambda)$ is a direct epimorphism, by Corollary 3.13 we get that $p \circ (\mathrm{Id} - \bar{h})$ is a direct submersion at $\varphi(x)$. That is, $(p \circ (\mathrm{Id} - \bar{h}))^{-1}(0)$ is a direct submanifold of U in a neighborhood of $\varphi(x)$. Hence, since E' is of finite codimension in E, $(p \circ (\mathrm{Id} - \bar{h}))^{-1}(0)$ is (smooth) of finite dimension in a neighborhood of $\varphi(x)$. $\square$

Proposition 3.33 is a Banach analytic adaptation of an analogous result in Functional analysis.

Lemma 3.34. *(Riesz's Lemma)([Kre13], p. 27, Lemma 2.24) Let X be a normed space, $Y \subset X$ a proper and closed subspace and $\alpha \in (0, 1)$. Then, there exists $x \in X$, with $\|x\| = 1$, such that $\|x - y\| \geq \alpha$, for each $y \in Y$.*

Proposition 3.35. *([Kre13], p. 27, Theorem 2.25) Let X be a normed vector space. Then, the identity operator $\mathrm{Id} : X \to X$ is compact if and only if X is finite dimensional.*

Proof. Assume that Id is compact and X is not finite dimensional. Choose $x_1 \in X$ such that $\|x_1\| = 1$. Then, $X_1 := \mathrm{span}\{x_1\}$ is a closed subspace of X and by the Riesz's Lemma 3.34, there exists $x_2 \in X$ such that $\|x_2\| = 1$ and $\|x_2 - x_1\| \geq 1/2$. Now, let $X_2 := \mathrm{span}\{x_1, x_2\}$. Again by the Riesz's Lemma 3.34, there exists $x_3 \in X$ with $\|x_3\| = 1$, $\|x_3 - x_1\| \geq 1/2$ and $\|x_3 - x_2\| \geq 1/2$. Continuing in this way, we get a sequence $\{x_n\}$ in X such that $\|x_n\| = 1$ and $\|x_n - x_m\| \geq 1/2$, for $n \neq m$. Hence, $\{x_n\}$ does not contain a convergent subsequence, thus Id is not compact. This is a contradiction to our assumption. Conversely, if X has finite dimension, for any $B \subset X$ bounded, $\mathrm{Id}(B)$ is finite dimensional and by the Bolzano-Weierstrass Theorem (see ([Kre13], p. 10, Theorem 1.17) it is relatively compact in X. Hence, Id is compact. $\square$

Definition 3.36. ([Sti88], p. 140) Let $\pi : X \to S$ be a morphism of Banach analytic spaces. The space X is called of *relative finite dimension* at $x \in X$, if there exists an open neighborhood V of x in X, an open subset $S' \subset S$ with $\pi(V) \subset S'$, an open subset $U \subset \mathbb{C}^n$ and an embedding $i : V \hookrightarrow S' \times U$ such that the following diagram commutes

$$\begin{tikzcd} V \arrow[rr, "i"] \arrow[dr, "\pi"'] & & S' \times U \arrow[dl] \\ & S' & \end{tikzcd},$$

Proposition 3.37. *([Dou74], p. 596 and [LP75], p. 271) Let $\pi : X \to S$ be a morphism of Banach analytic spaces. Let $s \in S$ and $x \in X(s)$. If $X(s)$ is of finite dimension at x, then X is of relative finite dimension at x.*

3.3 Privileged subspaces of a polycylinder and anaflat sheaves

Definition 3.38. A *polycylinder* in $\mathbb{C}^n$ is a compact set K of the form $K_1 \times .... \times K_n$, where each K_i is a compact, convex subset of $\mathbb{C}$ with nonempty interior.

We recall Theorem A and B by H. Cartan:

Theorem 3.39. *([GR04], p. 95 and [Dou66], p. 51) Let K be a polycylinder contained in some open subset U of $\mathbb{C}^n$. Let $\mathcal{F}$ be a coherent sheaf on U.*

A. There exists an open neighborhood of K over which $\mathcal{F}$ admits a finite free resolution

$$0 \to \mathcal{L}_n \to ... \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0.$$

B. $H^q(K, \mathcal{F}) = 0$ for $q > 0$.

Let $\mathcal{F}$ be a coherent sheaf on $U \subset \mathbb{C}^n$ and $K \subset U$ a polycylinder. If V is an open neighborhood of K, then $\mathcal{F}(V)$ can be equipped with a Fréchet-space structure (see [Mal68], p. 23). Hence, we can give $\mathcal{F}(K) := H^0(K; \mathcal{F})$ the structure of inductive limit of Fréchet-spaces. However, we can get a Banach structure choosing a space slightly different from $\mathcal{F}(K)$ and choosing K in a *privileged* way. Let F be a Banach space and set

$$B(K; F) := \{f : K \to F | f \text{ continuous on } K \text{ and holomorphic on } \mathring{K}\}.$$

It is a Banach algebra with the uniform convergence norm

$$\|f\| := \sup_{x \in K} \|f(x)\|.$$

Let

$$C(K;F) := \{f : K \to F | f \text{ continuous on } K\}.$$

We have $B(K;F) \subset C(K;F)$ and $B(K;F)$ is in fact the uniform closure in $C(K;F)$ of the space of holomorphic maps $f : U \to F$, for an open neighborhood U of K in $\mathbb{C}^n$. Let E, F be Banach spaces. The *norm* ϵ on the tensor product $E \otimes F$ is the norm induced by the norm on the space of linear maps $\mathcal{L}(F^*, E)$, where $F^* := \mathcal{L}(F;\mathbb{C})$, by means of the *Kronecker injection* $E \otimes F \to \mathcal{L}(F^*, E)$ (see [Dou66], p. 40).

Proposition 3.40. *([Dou66], p. 40) We have*

$$B(K;F) \simeq B(K;\mathbb{C})\hat{\otimes}_\epsilon F.$$

From now on we set $B(K) := B(K;\mathbb{C})$. We define now a Banach space of sections $B(K;\mathcal{F})$, where $\mathcal{F}$ is a coherent sheaf on a neighborhood of K.

Definition 3.41. ([Dou66], p. 54) Let $\mathcal{F}$ be a coherent sheaf on $U \subset \mathbb{C}^n$ and $K \subset U$ a polycylinder. We define a $\mathbb{C}$-vector space

$$B(K;\mathcal{F}) := B(K) \otimes_{\mathcal{O}_U(K)} \mathcal{F}(K).$$

Definition 3.42. ([Dou66], p. 54) Let K be a polycylinder in $U \subset \mathbb{C}^n$. Let $\mathcal{F}$ be a coherent sheaf on U. We say that K is *$\mathcal{F}$-privileged*, if there exists a free resolution

$$0 \to \mathcal{L}_n \to ... \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0 \tag{3.1}$$

on an open neighborhood of K, such that the sequence of $\mathbb{C}$-vector spaces

$$0 \to \mathcal{B}(K;\mathcal{L}_n) \to ... \to \mathcal{B}(K;\mathcal{L}_1) \to \mathcal{B}(K;\mathcal{L}_0) \to \mathcal{B}(K;\mathcal{F}) \to 0 \tag{3.2}$$

is direct (as $\mathbb{C}$-vector space sequence) and exact.

Remark 3.43.

1. The property of being $\mathcal{F}$-privileged is independent of the choice of the resolution. The exactness of the sequence (3.2) can be expressed by

$$\mathrm{Tor}_i^{\mathcal{O}_K}(B(K), \mathcal{F}(K)) = 0,$$

for every $i > 0$.

2. By Theorem 3.39 A, any polycylinder K admits a finite free resolution and the resolution can be taken of length n. By Theorem 3.39 B, it is enough to require exactness of the complex $B(K;\mathcal{L}_\bullet)$.

Since (3.1) is a free resolution of $\mathcal{F}$, we have

$$B(K;\mathcal{L}_n) \simeq B(K;\mathcal{O}_U^{\oplus r_n}) \simeq B(K)^{\oplus r_n}.$$

Hence, $B(K;\mathcal{L}_n)$ is a Banach $B(K)$-module. Therefore, if K is $\mathcal{F}$-privileged, then

$$B(K;\mathcal{F}) = \text{ coker } (\mathcal{B}(K;\mathcal{L}_1) \to \mathcal{B}(K;\mathcal{L}_0))$$

is a Banach $B(K)$-module.

Let $U \subset \mathbb{C}^2$ be an open connected neighborhood of the origin and let $h : U \to \mathbb{C}$ be holomorphic, with $h \neq 0$. Let X be the curve given by h, that is $(X, \mathcal{O}_X) = (h^{-1}(0), \mathcal{O}_U/(h))$. We illustrate the notion of O_X-privileged polycylinders in $\mathbb{C}^2$. We have an exact sequence $0 \to \mathcal{O}_U \xrightarrow{h\cdot} \mathcal{O}_U \to \mathcal{O}_X \to 0$. Let $K = K_1 \times K_2 \subset U$ be a polycylinder. By definition K is $\mathcal{O}_X$-privileged if and only if the induced morphism $h : B(K) \to B(K)$ is a direct monomorphism (see Definition 3.12). Let ∂K_i be the boundary of the polycylinder K_i, for $i = 1, 2$. Set $\partial K := \partial K_1 \times \partial K_2$.

Proposition 3.44. *([Dou68], p. 68) The following are equivalent*

1. $h : B(K) \to B(K)$ *is a monomorphism.*
2. $\exists a > 0$ *such that* $\|hf\| \geq a\|f\|$, $\forall f \in B(K)$.
3. $X \cap \partial K = \emptyset$.

Now, let us assume $U = U_1 \times U_2$, with $U_i \subset \mathbb{C}$, $i = 1, 2$. Let $K_1 \subset U_1$ be a polycylinder and choose $x_2 \in U_2$.

Proposition 3.45. *([Dou68], p. 68 and [Dou66], p. 61) If* $X \cap (\partial K_1 \times \{x_2\}) = \emptyset$, *then there exists a polycylinder* $K_2 \subset U_2$, *with* $x_2 \in K_2$, *such that* $K := K_1 \times K_2$ *is a* $\mathcal{O}_X$*-privileged polycylinder.*

Proof. Let us define the function $h(x_2) : B(K_1) \to B(K_1)$, sending f to $h(x_2) \cdot f$. By assumption, no zeros of $h(x_2)$ is in the boundary of K_1. Let l be the number of zeros of $h(x_2)$, counted with multiplicity, contained in the interior of K_1. Let P_l be the $\mathbb{C}$-vector space of polynomials of degree less than l. By the Weierstrass Division Formula (see, for instance, [KKBB83], p. 72), we get $B(K_1) \simeq h(x_2)(B(K_1)) \oplus P_l$. Now, by [Dou66], p. 60, Proposition 6, there exists $V \subset U_2$, with $x_2 \in V$, such that for each polycylinder $K_2 \subset V$, with $x_2 \in K_2$, we have $B(K) \simeq h(B(K)) \oplus B(K_2; P_l)$, where $B(K_2; P_l)$ is the Banach space of continuous functions $g : K_2 \to P_l$, which are analytic on the interior of K_2 (see Proposition 3.40). □

Proposition 3.46. *([Dou66], p. 56) Let* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *be an exact sequence of coherent sheaves on* $U \subset \mathbb{C}^n$. *If* $K \subset U$ *is a* $\mathcal{F}', \mathcal{F}''$*-privileged polycylinder, then* K *is* $\mathcal{F}$*-privileged and*

$$0 \to B(K;\mathcal{F}') \to B(K;\mathcal{F}) \to B(K;\mathcal{F}'') \to 0$$

is a direct exact sequence of $\mathbb{C}$*-vector spaces.*

Let $K \subset U$ be a polycylinder and $\mathcal{E}$ a coherent sheaf on U. From Proposition 3.46 it follows that if $\mathcal{F}$ is a coherent quotient sheaf of $\mathcal{E}$ such that K is $\mathcal{F}$-privileged, then $B(K;\mathcal{F})$ is a direct Banach $B(K)$-submodule of $B(K;\mathcal{E})$.

Theorem 3.47. *([Dou66], p. 62) Let $\mathcal{F}$ be a coherent sheaf on an open subset $U \subset \mathbb{C}^n$. For any $x \in U$ there exists a fundamental system of neighborhoods of x in U, which are $\mathcal{F}$-privileged polycylinders.*

Given a polycylinder K in $\mathbb{C}^n$, we can define a sheaf of algebras on K. For any $U \subset K$ open, we set

$$\mathcal{B}_K : U \mapsto B_K(U) := \{f : U \to \mathbb{C} : f \text{ is continuous and holomorphic on } U \cap K^\circ\}.$$

Therefore, we get a ringed space $(K, \mathcal{B}_K)$.

Definition 3.48. ([Dou74], p. 577 and [LP75], p. 256) Let $\mathcal{F}$ be a sheaf of $\mathcal{B}_K$-modules. We say that $\mathcal{F}$ is *K-privileged* if there exists a finite free resolution $\mathcal{L}_\bullet$ of $\mathcal{F}$ on K such that the complex $B(K;\mathcal{L}_\bullet) := \mathcal{L}_\bullet(K)$ is direct and acyclic in degree $d > 0$. This complex is a resolution of $B(K;\mathcal{F}) := \mathcal{F}(K)$. We call a *privileged subspace* of K a ringed space $(Y, \mathcal{B}_Y)$, where Y is a subspace of K and the sheaf $\mathcal{B}_Y$, extended by 0 over K, is a quotient sheaf of $\mathcal{B}_K$ by a privileged ideal subsheaf $\mathcal{I}$ with support Y.

We notice that the sheaf $\mathcal{B}_K$ restricted to K° coincides with $\mathcal{O}_{K^\circ}$. If Y is a privileged subspace of K, then the space $Y \cap K^\circ$ is a complex subspace of K°. We set $Y^\circ := Y \cap K^\circ$.

Definition 3.49. ([Dou74], p. 577 and [LP75], p. 256) Let $Y \subset K$ be a privileged subspace. Let $L \subset K$ be another polycylinder. We say that L is *Y-privileged* if there exists a finite free resolution of $\mathcal{B}_Y$ on K

$$0 \to \mathcal{B}_K^{r_n} \to \mathcal{B}_K^{r_{n-1}} \to ... \to \mathcal{B}_K^{r_1} \to \mathcal{B}_K \to \mathcal{B}_Y \to 0$$

such that the sequence

$$0 \to \mathcal{B}_L^{r_n} \to \mathcal{B}_L^{r_{n-1}} \to ... \to \mathcal{B}_L^{r_1} \to \mathcal{B}_L,$$

obtained applying $- \otimes_{\mathcal{B}_K} \mathcal{B}_L$, is direct and exact.

In this case $(Y \cap L, \mathcal{B}_Y \otimes_{\mathcal{B}_K} \mathcal{B}_L)$ is a privileged subspace of L.

Theorem 3.50. *([Pou75], p. 159) Let $K \subset \mathbb{C}^n$ be a polycylinder and $Y \subset K$ a privileged subspace. Then, every x in K admits a fundamental system of neighborhoods which are Y-privileged polycylinders.*

If $x \in K^\circ$, this result is nothing but Theorem 3.47. If $x \notin K^\circ$, the proof is due to G. Pourcin ([Pou75]).

Let A be a Banach algebra and E a Banach A-module. Let $\mathcal{G}_A(E)$ be the Grassmannian of direct A-submodules of E (see Example 3.21). Let $\mathcal{G}_A(A^{r_n}, ..., A^{r_0}; E)$ be the Banach analytic space of direct finite free resolutions of E

$$0 \to A^{r_n} \overset{d_n}{\to} .... \overset{d_1}{\to} A^{r_0} \overset{d_0}{\to} E$$

see [Dou66], p. 33.

Proposition 3.51. *([Dou66], p. 33) The canonical morphism*

$$\begin{aligned} \operatorname{Im} : \mathcal{G}_A(A^{r_n}, ..., A^{r_0}; E) &\to \mathcal{G}_A(E) \\ (d_n, ..., d_0) &\mapsto \operatorname{Im} d_0 \end{aligned} \tag{3.3}$$

is smooth.

From Proposition 3.51, it follows that the set of all privileged ideals I in $B(K)$ is an open subset of the Grassmannian $\mathcal{G}_{B(K)}(B(K))$ of direct ideals of $B(K)$ (see [Dou66], p. 34).

Definition 3.52. Let K be a polycylinder. The set of all privileged subspaces of K is a Banach analytic space denoted by $\mathcal{G}(K)$.

Let (X, Φ) be a Banach analytic space. We naturally get a ringed space $(X, \mathcal{O}_X)$ setting $\mathcal{O}_X := \Phi(\mathbb{C})$. We want to give an exact definition of what is a coherent sheaf depending analytically on a parameter s belonging to a Banach analytic space.

Definition 3.53. ([Dou66], p. 64) Let $f : X \to X'$ be a morphism of Banach analytic spaces and $\mathcal{F}$ a sheaf of $\mathcal{O}_{X'}$-modules on X'. We set:

$$f^*\mathcal{F} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_{X'}} f^{-1}\mathcal{F}.$$

If $\mathcal{F}$ is a sheaf of $\mathcal{O}_{S\times X}$-modules on $S \times X$, for each $s \in S$, we set:

$$\mathcal{F}(s) := i_s^*\mathcal{F},$$

where $i_s : X \hookrightarrow S \times X$ is the natural embedding defined by s.

Remark 3.54. If S is of finite dimension, then $\mathcal{F}(s) \simeq \mathcal{F}/\mathfrak{m}_s\mathcal{F} \simeq \mathbb{C} \otimes_{\mathcal{O}_{S,s}} \mathcal{F}$.

Definition 3.55. ([Dou66], p. 66) Let $U \subset \mathbb{C}^n$ an open subset and S a Banach analytic space. A sheaf $\mathcal{F}$ of $S \times U$- modules on $S \times U$ is called *S-anaflat* if for each $(s, x) \in S \times U$ there exists a finite free resolution $0 \to \mathcal{L}_\bullet \to \mathcal{F} \to 0$ of $\mathcal{F}$ in a neighborhood of (s, x) in $S \times U$ such that the complex:

$$0 \to \mathcal{L}_\bullet(s) \to \mathcal{F}(s) \to 0$$

is an exact sequence in an neighborhood of x in U.

Theorem 3.56. *([Dou66], p. 66) Let S be a complex space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{S\times U}$-modules on $S \times U$. Then $\mathcal{F}$ is S-anaflat if and only if it is coherent and flat.*

Let S be a Banach analytic space and U an open subset of $\mathbb{C}^n$. Let $\mathcal{F}$ be a S-anaflat sheaf on $S \times U$, $s \in S$ and $K \subset U$ a $\mathcal{F}(s)$-privileged polycylinder. From Theorem 3.39 A, we get the existence of a finite free resolution on a neighborhood $V_0 \times V_1$ of $\{s_0\} \times K$ in $S \times U$

$$0 \to \mathcal{L}_n \to ... \to \mathcal{L}_1 \to \mathcal{L}_0 \to \mathcal{F} \to 0.$$

Since $\mathcal{F}$ is S-anaflat

$$0 \to \mathcal{L}_n(s') \to ... \to \mathcal{L}_1(s') \to \mathcal{L}_0(s') \to \mathcal{F}(s') \to 0$$

is exact for each $s' \in V_0$. Let us consider the complex

$$0 \to \mathcal{L}_n(s') \to ... \to \mathcal{L}_1(s') \to \mathcal{L}_0(s').$$

It is acyclic in degree $d > 0$ and it defines a complex of Banach spaces

$$0 \to B(K; \mathcal{L}_n(s')) \to ... \to B(K; \mathcal{L}_1(s')) \to B(K; \mathcal{L}_0(s'))$$

for each $s' \in V_0$, which, in turn, defines a complex of Banach vector bundles over V_0

$$0 \to B(K; \mathcal{L}_n) \to ... \to B(K; \mathcal{L}_1) \to B(K; \mathcal{L}_0).$$

Since K is $\mathcal{F}(s)$-privileged, this complex is direct and acyclic in degree $d > 0$ over a neighborhood V_s of s in V_0. Hence, let

$$B(K; \mathcal{F})_{V_s} := H^0(B(K; \mathcal{L}_\bullet)) = \operatorname{coker}(B(K; \mathcal{L}_1) \to B(K; \mathcal{L}_0)).$$

This is a trivial Banach vector bundle over V_s, with fibres

$$B(K; \mathcal{F}(s'))_{V_s} := \ H^0(B(K; \mathcal{L}_\bullet(s'))) = \operatorname{coker}(B(K; \mathcal{L}_1(s')) \to B(K; \mathcal{L}_0(s'))),$$

for $s' \in V_s$. The Banach vector bundle $B(K; \mathcal{F})_{V_s}$ does not depend on the choice of the resolution $\mathcal{L}_\bullet$ of $\mathcal{F}$ (see [Dou66], p. 68).

Theorem 3.57. *([Dou68], p. 68)(Flatness and privilege) Let S be a Banach analytic space, $U \subset \mathbb{C}^n$ open, $\mathcal{F}$ a S-anaflat sheaf on $S \times U$ and $K \subset U$ a polycylinder. The set S' of all points $s \in S$ such that K is $\mathcal{F}(s)$-privileged is open in S and the Banach spaces $B(K; \mathcal{F}(s))$ are the fibres of a Banach vector bundle $B(K; \mathcal{F})$ over S'*

This result expresses the idea that if $\mathcal{F}$ is S-anaflat, the sheaf $\mathcal{F}(s)$ depends continuously on the point $s \in S$.

Corollary 3.58. *([Dou68], p. 73) Let $\pi : X \to S$ be a morphism of complex spaces and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module, which is flat as $\mathcal{O}_S$-module. Then $\pi|_{Supp\, \mathcal{E}}$ is an open map.*

Proof. Assume that X can be embedded in $S \times U$, with $U \subset \mathbb{C}^n$ open, and that $\mathcal{E}$ is extended by zero to $S \times U$. Let $x_0 \in \mathrm{Supp}\, \mathcal{E}$ and V a neighborhood of x_0 in $S \times U$. Let $s_0 := \pi(x_0)$ and choose an $\mathcal{E}(s_0)$-privileged polycylinder K in U, such that $\{s_0\} \times K \subset \pi^{-1}(W) \subset V$, where W is a neighborhood of s_0 in S. We have the Banach vector bundle $B(K; \mathcal{E}|_{\pi^{-1}(W)})$, whose fiber over s is $B(K; \mathcal{E}(s))$. Since $x_0 \in \mathrm{Supp}\, \mathcal{E}(s_0)$ and K is a neighborhood of x_0, $B(K; \mathcal{E}(s_0)) \neq 0$. Since all the fibres are isomorphic, then for any $s \in U$, $B(K; \mathcal{E}(s)) \neq 0$ and therefore $\{s\} \times K \cap \mathrm{Supp}\, \mathcal{E} \neq 0$, and $s \in \pi(\mathrm{Supp}\, \mathcal{E})$. Thus, π is open. $\square$

Proposition 3.59. *([Dou66], p. 69) (Change of base) Let $\mathcal{F}$ be a S-anaflat sheaf on $S \times U$. Let S' be a Banach analytic space and $f : S' \to S$ a morphism. The sheaf*

$$\mathcal{F}' := (f \times Id_U)^* \mathcal{F}$$

is a S'-anaflat sheaf on $S' \times U$.

Definition 3.60. ([Dou66], p. 76) Let S be a Banach analytic space and X a complex space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_{S\times X}$-modules on $S \times X$. We say that $\mathcal{F}$ is *S-anaflat* if for any $x \in X$ there exists an open subset U of $\mathbb{C}^n$ and a local embedding $i : X \hookrightarrow S \times U$, such that $i_* \mathcal{F}$ is a S-anaflat sheaf on $S \times U$.

Definition 3.61. Let X, S be Banach anlytic spaces and $f : X \to S$ a morphism. We say that X *is anaflat over* S if for each $x \in X$ there exists a neighborhood V of x in X, an open subset $U \subset \mathbb{C}^n$ and an embedding $i : V \hookrightarrow S \times U$ such that f is induced by the projection of $S \times U$ onto S and $i_* \mathcal{O}_V$ is a S-anaflat sheaf of $S \times U$-modules.

Definition 3.62. ([LP75], p. 258 and [Pou75], p. 183, Theorem 4.13) Let S be a Banach analytic space and $K \subset \mathbb{C}^n$ a polycylinder. Let $\mathcal{G}(K)$ given by Definition 3.52. A morphism $S \to \mathcal{G}(K)$ is called a *S-anaflat subspace* of $S \times K$.

Definition 3.63. We denote with $\underline{Y}$ the universal $\mathcal{G}(K)$-anaflat subspace of $\mathcal{G}(K) \times K$ defined by the identity morphism $\mathrm{Id} : \mathcal{G}(K) \to \mathcal{G}(K)$.

3.4 The Douady space

Let $f : X \to X'$ be a morphism of complex spaces and $\mathcal{F}$ a sheaf of $\mathcal{O}_{X'}$-modules on X'. We recall that

$$f^* \mathcal{F} := \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_{X'}} f^{-1} \mathcal{F}.$$

Let X be a complex space and $\mathcal{E}$ a coherent sheaf on X. If T is another complex space, we set $\mathcal{E}_T := \pi_2^* \mathcal{E}$, where $\pi_2 : T \times X \to X$ is the projection. The main result in [Dou66] is the following

Theorem 3.64. *([Dou66], p. 83)Let X be a complex space and $\mathcal{E}$ a coherent sheaf on X. Then there exists a complex space H and a coherent quotient sheaf $\mathcal{R}$ of $\mathcal{E}_H$ such that $\mathcal{R}$ is H-proper and H-flat, that is the map* $\mathrm{supp}\, \mathcal{R} \to H$ *is proper and flat, and* *such that the following universal property is satisfied: for any complex space S and coherent quotient sheaf $\mathcal{F}$ of $\mathcal{E}_S$, which is S-proper and S-flat, there exists a unique morphism $f : S \to H$ such that*

$$\mathcal{F} = (f \times \mathrm{Id}_X)^* \mathcal{R}.$$

If we take S to be a single point $S = \{pt\}$, we see that set-theoretically H coincides with the set of coherent quotient sheaves $\mathcal{F}$ of $\mathcal{E}$, with compact support. Moreover, if we take $\mathcal{E} = \mathcal{O}_X$, then

$$H = \{\text{compact complex subspaces of } X\}.$$

The complex space H is called the *Douady space* of X. If $X = \mathbb{CP}^n$, the complex projective space, then, in the algebraic geometry setting, H is called the *Hilbert scheme* of X (see [Gro61]). The main idea in the construction of the space H is the following. For the sake of clarity, assume $S = \{pt\}$. Let $\mathcal{E}$ be a coherent sheaf on X and $\mathcal{F}$ a coherent quotient sheaf of $\mathcal{E}$ as in Theorem 3.64. Since $\operatorname{supp} \mathcal{F}$ is compact, we can cover it with finitely many polycylinders $M := (K_i)_{i \in I}$, satisfying gluing relations on the intersections (a *cuirasse*, see Definition 3.72). We assume $K_i \subset U_i$, with $U_i \subset \mathbb{C}^n$ open. If we choose M such that each K_i is a $\mathcal{E}$- and $\mathcal{F}$-privileged polycylinder (see Definition 3.42), then, for each $i \in I$, $B(K_i; \mathcal{F}) := B(K_i) \otimes_{\mathcal{O}_{U}(K_i)} \mathcal{F}(K_i)$ is a direct quotient of $B(K_i; \mathcal{E})$ admitting a direct finite free resolution (see Section 3.3). The set of all direct quotients of $B(K_i; \mathcal{E})$ admitting a direct finite free resolution, which we denote with $\mathcal{G}_{K_i}(\mathcal{E})$ (see Definition 3.65), is an open subset of the Grassmannian of direct $B(K_i)$-submodules of $B(K_i; \mathcal{E})$ (see Example 3.21). By Theorem 3.39, the coherent sheaf $\mathcal{F}$ is uniquely determined over K_i by $B(K_i; \mathcal{F})$. Hence, $\mathcal{F}$ is uniquely determined by a point in a subspace H_M of $\prod_{i \in I} \mathcal{G}_{K_i}(\mathcal{E})$ constructed by using the gluing relations of the cuirasse M. The union of all spaces H_M for all cuirasses $M := (K_i)_{i \in I}$, such that each K_i is a $\mathcal{E}$-privileged polycylinder, is the desired space H (see Proposition 3.78).

Let $K \subset U \subset \mathbb{C}^n$ be a polycylinder. Let $\mathcal{E}$ be a coherent sheaf on U such that K is $\mathcal{E}$-privileged (see Definition 3.42). Let $B(K; \mathcal{E})$ be the Banach $B(K)$-module as in Definition 3.41. Consider the Banach manifold (see Example 3.7)

$$\mathcal{G}(B(K; \mathcal{E})) := \{\text{direct } \mathbb{C}\text{-vector subpaces of } B(K; \mathcal{E})\}$$

and its Banach analytic subspace (see Example 3.21)

$$\mathcal{G}_{B(K)}(B(K; \mathcal{E})) := \{F \in \mathcal{G}(B(K; \mathcal{E})) : F \text{ is a } B(K)\text{-submodule of } B(K; \mathcal{E})\}.$$

Definition 3.65. We denote with $\mathcal{G}_K(\mathcal{E})$ the set of $B(K)$-submodules of $B(K; \mathcal{E})$ admitting a direct finite free resolution.

By Proposition 3.51, we get that $\mathcal{G}_K(\mathcal{E})$ is open in $\mathcal{G}_{B(K)}(B(K; \mathcal{E}))$. Moreover, $\mathcal{G}_K(\mathcal{E})$ can be identified with the set of *direct quotients* of $B(K; \mathcal{E})$ admitting a direct finite free resolution (see [Dou66], p. 34). Let S be a Banach analytic space and $\pi : S \times U \to U$ the canonical projection. Consider $\mathcal{E}_S := \pi^*\mathcal{E}$ and take a S-anaflat quotient sheaf $\mathcal{F} = \mathcal{E}_S/\mathcal{F}'$ of $\mathcal{E}_S$. Let S_1 be the set of all $s \in S$ such that K is $\mathcal{F}(s)$-privileged. By Theorem 3.57, S_1 is open in S and there exists a Banach vector bundle $B(K;\mathcal{F})$ over S_1, whose fibres are the Banach spaces $B(K;\mathcal{F}(s))$ (see [Dou66], p.68). On S_1, we have a direct exact sequence of Banach vector bundles

$$0 \to B(K;\mathcal{F}') \to B(K;\mathcal{E})_S \to B(K;\mathcal{F}) \to 0,$$

where $B(K;\mathcal{E})_S$ denotes the *trivial* vector bundle over S.

Definition 3.66. ([Dou66], p. 72) We can define a morphism

$$\beta_K(\mathcal{F}) : S_1 \to \mathcal{G}_K(\mathcal{E}), s \mapsto B(K;\mathcal{F}(s)). \tag{3.4}$$

This morphism has the following functorial property. If $f : S' \to S$ is a morphism of Banach analytic spaces and we set $\mathcal{F}' := (f \times \mathrm{Id}_U)^*\mathcal{F}$, then

$$S_1' := f^{-1}(S_1) \text{ and } \beta_K(\mathcal{F}') = \beta_K(\mathcal{F}) \circ f. \tag{3.5}$$

On the other hand, let $f : S \to \mathcal{G}_K(\mathcal{E})$ be a morphism, we can associate to it a S-anaflat quotient sheaf $\gamma_K(f)$ of $\mathcal{E}_S$ defined on $S \times K^\circ$. Indeed, for each $s \in S$ we get an element $f(s) = F = B(K;\mathcal{E})/F'$ of $\mathcal{G}_K(\mathcal{E})$. The $B(K)$-module F admits a direct finite free resolution

$$0 \to B(K)^{r_n} \to .... \to B(K)^{r_0} \xrightarrow{\epsilon} B(K;\mathcal{E}) \tag{3.6}$$

such that $\operatorname{coker} \epsilon = F$. Let

$$\mathcal{G} := \mathcal{G}_{B(K)}(B(K)^{r_n}, ..., B(K)^{r_0}; B(K;\mathcal{E}))$$

be the set of all direct exact sequences of the form (3.6). It is a Banach analytic space (see [Dou66], p. 33). By Proposition 3.51, the canonical morphism $\mathrm{Im} : \mathcal{G} \to \mathcal{G}_K(\mathcal{E})$ is smooth, hence it admits local sections. Thus, there exists an open neighborhood V of F in $\mathcal{G}_K(\mathcal{E})$ and a section $\sigma : V \to \mathcal{G}$ of Im. Set $S' := f^{-1}(V) \subset S$, we get a map $g := \sigma \circ f_{|S'} : S' \to \mathcal{G}$. For each point in $f(S') \subset \mathcal{G}$ we get a direct exact sequence

$$0 \to \mathcal{O}_{K^\circ}^{r_n} \to ... \to \mathcal{O}_{K^\circ}^{r_n} \to \mathcal{E}_{|K^\circ},$$

from which, taking the inverse image by g, we get a direct exact sequence

$$0 \to \mathcal{O}_{S'\times K^\circ}^{r_n} \to ... \to \mathcal{O}_{S'\times K^\circ}^{r_n} \xrightarrow{u} \mathcal{E}_{|S'\times K^\circ}.$$

Set $\gamma_K(f)_{|S'\times K^\circ} := \operatorname{coker} u$. The sheaves $(\gamma_K(f)_{|S'\times K^\circ})_{s\in S}$ glue to a sheaf $\gamma_K(f)$ on $S \times K^\circ$, which, by construction, is S-anaflat (see [Dou66], p. 74).

Definition 3.67. Let $f : S \to \mathcal{G}_K(\mathcal{E})$ be a morphism. We denote with $\gamma_K(f)$ the S-anaflat quotient sheaf of $\mathcal{E}_S$, defined on $S \times K^\circ$, associated to f.

The construction of $\gamma_K(f)$ is independent of the choice of the local section σ. Moreover, we have the following functorial property. If $h : S' \to S$ is a morphism of Banach analytic spaces, then $\gamma_K(f \circ h) = (h \times \mathrm{Id}_{K^\circ})^* \gamma_K(f)$.

Proposition 3.68. *([Dou66], p. 74)(Aller-retours) Let S be a Banach analytic space and $\pi : S \times U \to U$ the canonical projection. Consider $\mathcal{E}_S := \pi^*\mathcal{E}$ and take a S-anaflat quotient sheaf $\mathcal{F} = \mathcal{E}_S/\mathcal{F}'$ of $\mathcal{E}_S$. Let $\beta_K(\mathcal{F})$ be the morphism associated to $\mathcal{F}$ (Definition* 3.66*) and $\gamma_K(\beta_K(\mathcal{F}))$ the S-anaflat quotient sheaf of $\mathcal{E}_S$ associated to $\beta_K(\mathcal{F})$ (Definition* 3.67*). Then*

$$\gamma_K(\beta_K(\mathcal{F})) = \mathcal{F}_{|S_1 \times K^\circ}.$$

Definition 3.69. In Definition 3.67, take $S := \mathcal{G}_K(\mathcal{E})$. Then the $\mathcal{G}_K(\mathcal{E})$-anaflat sheaf

$$\mathcal{R} := \gamma_K(\mathrm{Id}_{\mathcal{G}_K(\mathcal{E})})$$

is called *the universal sheaf* on $\mathcal{G}_K(\mathcal{E}) \times K^\circ$.

We have the following universal property.

Proposition 3.70. *([Dou66], p. 74) Let S be a Banach analytic space and $f : S \to \mathcal{G}_K(\mathcal{E})$ a morphism. Then*

$$\gamma_K(f) = (f \times \mathrm{Id}_{|K^\circ})^* \mathcal{R}.$$

Using Propositions 3.68 and 3.70, we immediately get the following.

Theorem 3.71. *Let $\mathcal{E}$ be a coherent sheaf on U such that K is $\mathcal{E}$-privileged. Let S be a Banach analytic space and $\pi : S \times U \to U$ the canonical projection, set $\mathcal{E}_S := \pi^*\mathcal{E}$. For any S-anaflat quotient sheaf $\mathcal{F}$ of $\mathcal{E}_S$ on $S \times K^\circ$, let S_1 be the set of all $s \in S$ such that K is $\mathcal{F}(s)$-privileged. Then there exists a morphism $\beta_K(\mathcal{F}) : S_1 \to \mathcal{G}_K(\mathcal{E})$ such that*

$$\mathcal{F}_{|_{S_1 \times K^\circ}} = (\beta_K(\mathcal{F}) \times \mathrm{Id}_{K^\circ})^* \mathcal{R}.$$

This local result is very close to Theorem 3.64. In order to pass from this local version to a global version of this result, we cover X with finitely many $\mathcal{E}$-privileged polycylinders satisfying gluing relations.

Definition 3.72. ([Dou66], p. 78) Let X be a complex space. A *cuirasse* M on X is given by

1. a finite family of charts $(\varphi_i)_{i \in I}$ on X, that is for each i we have an isomorphism φ_i of an open subset X_i of X onto an open subset U_i of $\mathbb{C}^{n_i}$;

2. a polycylinder $K_i \subset U_i$, for each $i \in I$;

3. a compact subset $T_i \subset \varphi_i^{-1}(K_i^\circ) \subset X_i$;

4. a chart φ_{ij} defined on $X_{ij} := X_i \cap X_j$, for each $(i, j) \in I^2$;

5. a family of polycylinders $(K_{ij\alpha})_{\alpha\in A_{ij}}$, for each $(i,j)\in I^2$, contained in U_{ij}, such that
$$T_i\cap T_j\subset\bigcup_{\alpha\in A_{ij}}\varphi_{ij}^{-1}(K_{ij\alpha}^{\circ})$$
and
$$\bigcup_{\alpha\in A_{ij}}\varphi_{ij}^{-1}(K_{ij\alpha})\subset\varphi_i^{-1}(K_i^{\circ})\cap\varphi_j^{-1}(K_j^{\circ});$$
6. a closed subset L of X such that $L\cup\bigcup\mathring{T}_i=X$.

Definition 3.73. Let X be a complex space, $\mathcal{E}$ a coherent sheaf on X. A cuirasse M on X is called $\mathcal{E}$*-semiprivileged* if, for any $i\in I$, the polycylinder K_i is $\mathcal{E}$-privileged, for any $j\in I$ and $\alpha\in A_{ij}$, the polycylinder $K_{ij\alpha}$ is $\mathcal{E}$-privileged. Moreover, let $\mathcal{F}$ be a quotient sheaf of $\mathcal{E}$. The cuirasse M is called $(\mathcal{E},\mathcal{F})$*-privileged* if it is $\mathcal{E}$-privileged and $\mathcal{F}$-privileged and, moreover, $\operatorname{supp}\mathcal{F}\cap L=\emptyset$. Thus, $\mathcal{F}$ has compact support.

Let X be a complex space and $\mathcal{E}$ a coherent sheaf on it. Let M be a $\mathcal{E}$-semiprivileged cuirasse on X. For each polycylinder K_i in M, let $\mathcal{R}_i$ be the $\mathcal{G}_{K_i}(\mathcal{E})$-anaflat sheaf on $\mathcal{G}_{K_i}(\mathcal{E})\times K_i^{\circ}$ given by Definition 3.69. We want to glue the sheaves $(\mathcal{R}_i)_{i\in I}$. For each $i\in I$, let $(\mathcal{G}_{K_i}(\mathcal{E}))_1$ be the set of all $s\in\mathcal{G}_{K_i}(\mathcal{E})$ such that K_i is $\mathcal{R}_i(s)$-privileged. Let $\beta_{K_i}(\mathcal{R}_i):(\mathcal{G}_{K_i}(\mathcal{E}))_1\to\mathcal{G}_{K_i}(\mathcal{E})$ given by Definition 3.66. For each $j\in I$ and $\alpha\in A_{ij\alpha}$, we get restriction maps $\rho'_{ij\alpha}:B(K_i)\to B(K_{ij\alpha})$ and $\rho''_{ij\alpha}:B(K_i;\mathcal{E})\to B(K_{ij\alpha};\mathcal{E})$. Set
$$G_i:=\{s\in\mathcal{G}_{K_i}(\mathcal{E}):K_{ij\alpha}\text{ is }\mathcal{R}_i(s)\text{-privileged and }\operatorname{supp}R_i(s)\cap L\cap T_i=\emptyset\}.\qquad(3.7)$$
We get a map
$$\beta_{K_{ij\alpha}}(\mathcal{R}_i):=\beta_{K_i}(\mathcal{R}_i)_{|G_i}:G_i\to\prod_{j\in I,\alpha\in A_{ij\alpha}}\mathcal{G}_{K_{ij\alpha}}(\mathcal{E})$$
and hence, taking the products, a map
$$\rho':=\prod_{i\in I}\beta_{K_{ij\alpha}}(\mathcal{R}_i):\prod_{i\in I}G_i\to\prod_{i,j\in I,\alpha\in A_{ij\alpha}}\mathcal{G}_{K_{ij\alpha}}(\mathcal{E}).$$
Moreover, since $K_{ij\alpha}=K_{ji\alpha}$, we can define another morphism
$$\rho'':=\prod_{i\in I}G_i\to\prod_{i,j\in I,\alpha\in A_{ij\alpha}}\mathcal{G}_{K_{ij\alpha}}(\mathcal{E}),$$
via the assignment $\rho''(s_i):=\beta_{K_{ji\alpha}}(\mathcal{R}_j)(s_j)$. Therefore, we get a double arrow:
$$(\rho',\rho''):\prod_{i\in I}G_i\rightrightarrows\prod_{i,j\in I,\alpha\in A_{ij\alpha}}\mathcal{G}_{K_{ij\alpha}}(\mathcal{E}).$$
Let Θ be its kernel. This procedure has the final effect of allowing the sheaves $(\mathcal{R}_i)_{i\in I}$ to glue to a sheaf on $\Theta\times X$, by matching them on the intersections $K_i\cap K_j$, for $i,j\in I$. Indeed, let $p_i:\Theta\to G_i$ be the projection. The sheaf
$$\tilde{\mathcal{R}}_i:=(p_i\times Id_{\varphi_i^{-1}(K_i^{\circ})})^*\mathcal{R}_i$$
is a quotient sheaf of $\mathcal{E}_\Theta$ defined on $\Theta\times\varphi_i^{-1}(K_i^{\circ})$.

Lemma 3.74. *([Dou66], p. 81) We have*

$$\tilde{\mathcal{R}}_{i|T_i\cap T_j} = \tilde{\mathcal{R}}_{j|T_i\cap T_j} \text{ and } \tilde{\mathcal{R}}_{i|T_i\cap L} = \emptyset.$$

Proof. The first equality follows from the definition of Θ. The second equality follows from the condition $\mathcal{R}_i(s) \cap L \cap T_i = \emptyset$ in the definition of G_i (3.7). □

Lemma 3.74 shows that there exists a unique quotient sheaf $\tilde{\mathcal{R}}$ of $\mathcal{E}_\Theta$ on $\Theta \times X$ such that for each $i \in I$, $\tilde{\mathcal{R}}_{|T_i} = \mathcal{R}_{i|T_i}$ and $\tilde{\mathcal{R}}_{|L} = \emptyset$. Moreover, $\tilde{\mathcal{R}}$ is Θ-anaflat by construction and $\operatorname{supp}\tilde{\mathcal{R}}$ is contained in $\Theta \times (X \setminus L)^-$. Since $(X \setminus L)^-$ is compact, $\tilde{\mathcal{R}}$ is Θ-proper. Furthermore, let S be a Banach analytic space and $\mathcal{F}$ a S-anaflat and S-proper quotient sheaf of $\mathcal{E}_S$. Let S_1 be the open subset of S given by all $s \in S$ such that the cuirasse M is $(\mathcal{E}, \mathcal{F}(s))$-privileged. The morphisms $(\beta_{K_i}(\mathcal{F}))_{i\in I}$ (see Definition 3.66) induce morphisms $(\beta_{K_i}(\mathcal{F}) : S_1 \to \mathcal{G}_i)_{i\in I}$ and hence a morphism $\beta_M(\mathcal{F}) : S_1 \to \prod_{i\in I} G_i$. It can be shown (see [Dou66], p. 80) that $\rho' \circ \beta_M(\mathcal{F}) = \rho'' \circ \beta_M(\mathcal{F})$, thus $\beta_M(\mathcal{F}) : S_1 \to \Theta$. Using Theorem 3.71, we get the following

Proposition 3.75. *([Dou66], p. 81) Let S be a Banach analytic space and $\mathcal{F}$ a S-anaflat and S-proper quotient sheaf of $\mathcal{E}_S$. Then there exists a morphism $f := \beta_M(\mathcal{F}) : S_1 \to \Theta$ such that*

$$\mathcal{F}_{S_1\times X} = (f \times \mathrm{Id}_X)^*\tilde{\mathcal{R}}.$$

Proposition 3.75 is a global version of Theorem 3.71. We have achieved it using the notion of cuirasse (Definition 3.72). The space Θ is still too big as the morphism f is not unique. We want to reduce it. Let Θ_1 be the open subset of Θ given by all $s \in \Theta$ such that the cuirasse M is $(\mathcal{E}, \tilde{\mathcal{R}}(s))$-privileged. Let us consider the morphism $\beta_M(\tilde{\mathcal{R}}) : \Theta_1 \to \Theta$ given by Proposition 3.75. It can be shown ([Dou66], p. 82) that $\beta_M(\tilde{\mathcal{R}})(\Theta_1) \subset \Theta_1$ and $\beta_M(\tilde{\mathcal{R}}) \circ \beta_M(\tilde{\mathcal{R}}) = \beta_M(\tilde{\mathcal{R}})$. Thus, we can consider the double arrow

$$(\beta_M(\tilde{\mathcal{R}}), \mathrm{Id}) : \Theta_1 \rightrightarrows \Theta_1.$$

Definition 3.76. Denote with H_M the kernel of the double arrow $(\beta_M(\tilde{\mathcal{R}}), \mathrm{Id})$. Moreover, let $i : H_M \hookrightarrow \Theta$ be the canonical inclusion. Set

$$\mathcal{R}_M := (i \times \mathrm{Id}_X)^*\tilde{\mathcal{R}}.$$

The sheaf $\mathcal{R}_M$ deserves to be called *universal*. Indeed, it is H_M-anaflat and the cuirasse M is $(\mathcal{E}, \mathcal{R}_M(h))$-privileged, for each $h \in H_M$. Moreover,

Proposition 3.77. *([Dou66], p. 83) Let S be a Banach analytic space, $\mathcal{F}$ a S-anaflat quotient sheaf of $\mathcal{E}_S$ such that M is $(\mathcal{E}, \mathcal{F}(s))$-privileged for each $s \in S$, hence $S_1 = S$. Then there exists a unique morphism $f := \beta_M(\mathcal{F}) : S \to H_M$ such that*

$$\mathcal{F} = (f \times \mathrm{Id}_X)^*\mathcal{R}_M.$$

Proof. Proposition 3.75 tells us that $\beta_M(\mathcal{F})$ satisfies the equation defining the universal property. For the uniqueness, let $g : S \to H_M$ be another morphism such that

$$\mathcal{F} = (g \times \mathrm{Id}_X)^*\mathcal{R}_M.$$

Then

$$\beta_M(\mathcal{F}) = \beta_M((g \times \mathrm{Id}_X)^*\mathcal{R}_M) = \beta_M(\mathcal{R}_M) \circ g,$$

where the last equality follows from the functoriality property of the morphism $\beta_M(\mathcal{F})$, see (3.5). Since $H_M = \ker(\beta_M(\tilde{\mathcal{R}}), \mathrm{Id})$, we have that $\beta_M(\mathcal{R}_M) \circ g = g$, thus $\beta_M(\mathcal{F}) = g$. $\square$

Now, if we take $S = \{pt\}$, then $X \times S$ is of finite dimension. Thus, the notion of anaflatness (see Definition 3.55) is equivalent to coherence and S-flatness (see Theorem 3.56). Then, from Proposition 3.77, we see that H_M coincides with the set of coherent quotient sheaves $\mathcal{F}$ of $\mathcal{E}_S$ such that M is $(\mathcal{E}, \mathcal{F})$-privileged. We remark that if M is $(\mathcal{E}, \mathcal{F})$-privileged, then $\mathcal{F}$ has compact support (see Definition 3.73). Using Theorem 3.47, one can show

Proposition 3.78. *([Dou66], p. 79) Let $\mathcal{E}$ be a coherent sheaf on X. Let $\mathcal{F}$ be a coherent quotient sheaf of $\mathcal{E}$ with compact support. Then there exists a cuirasse M on X such that M is $(\mathcal{E}, \mathcal{F})$-privileged.*

Let X be a complex space and $\mathcal{E}$ a coherent sheaf on X. Eventually (see [Dou66], p. 83), Douady shows that the Banach analytic spaces (H_M) (Definition 3.76), for any $\mathcal{E}$-semiprivileged cuirasse M on X, glue to a Banach analytic space H. Set-theoretically

$$|H| = \bigcup_{M \text{ is } \mathcal{E}\text{-semiprivileged}} |H_M|,$$

(see [Dou66], p. 84). By construction, if M is $\mathcal{E}$-semiprivileged, then M is $(\mathcal{E}, \mathcal{R}_M)$-privileged. Thus, from Proposition 3.77 and Proposition 3.78, it follows that as sets

$$|H| = \{\text{coherent quotient sheaves } \mathcal{F} \text{ of } \mathcal{E} \text{ with compact support}\}.$$

Moreover, Douady shows (see [Dou66], p. 84) that the universal sheaves $(\mathcal{R}_M)$ on (H_M) (see Definition 3.76) glue to a universal sheaf $\mathcal{R}$ on H. Using Proposition 3.77, it immediately follows that the pair $(H, \mathcal{R})$ is the solution to the universal problem stated in Theorem 3.64. The last step is

Proposition 3.79. *([Dou66], p. 84) The Banach analytic space H is finite dimensional.*

Proof. Let $s \in H$, we can construct two cuirasses

$$M = ((\varphi_i), (K_i), ...) \text{ and } M' = ((\varphi'_i), (K'_i), ...)$$

such that $K_i' \subset K_i^\circ$, for each $i \in I$ and such that the two cuirasses are both $(\mathcal{E}, \mathcal{R}(s))$-privileged, hence $s \in H_M \cap H_{M'}$. From the construction of the space $H_{M'}$, we see that the morphism $\beta_{M'}(\mathcal{R})$ coincides with the identity on $H_M \cap H_{M'}$. Using the restriction maps $\rho' : B(K_i) \to B(K_i')$ and $\rho'' : B(K_i; \mathcal{E}) \to B(K_i'; \mathcal{E})$, one can define a morphism (see [Dou66], p. 35)

$$\rho_* : W \subset \mathcal{G}_{B(K_i)}(B(K_i; \mathcal{E})) \to \mathcal{G}_{B(K_i')}(B(K_i'; \mathcal{E})),$$

with W open, such that for $F \in \mathcal{G}_{B(K_i)}(B(K_i; \mathcal{E}))$, $F' := \rho_*(F)$ is the $B(K_i')$-submodule of $B(K_i'; \mathcal{E})$ generated by $\rho''(F)$. Since the morphisms ρ' and ρ'' are compact in the functional analytic sense, it can be shown that ρ_* is compact as well ([Dou66], p. 39). The morphism $\beta_M(\mathcal{R})$ is induced by the morphism $\prod_i \beta_{K_i}(\mathcal{R}_i)$ from an open subset G_1 of $\prod_i \mathcal{G}_{K_i}(\mathcal{E})$ into $\prod_i \mathcal{G}_{K_i'}(\mathcal{E})$. Now, the morphism $\beta_{M'}(\mathcal{R})$ can be seen as the restriction of ρ_* to G_1. Since ρ_* is compact, $\beta_{M'}(\mathcal{R})$ is compact and hence $\mathrm{Id}_{H_M \cap H_{M'}}$ is compact. Using Proposition 3.33, we get that H is finite dimensional in s. □

As an application of Theorem 3.64, Douady proves that if X and Y are complex spaces, with X compact, then the set $\mathrm{Mor}(X, Y)$ of morphisms from X to Y is a complex space.

Proposition 3.80. *([Dou66], p. 86) Let X and Y be complex spaces proper and flat over S and $f : X \to Y$ a S-morphism. The set S' of $s \in S$ such that $f(s) : X(s) \to Y(s)$ is an isomorphism is open in S and f induces an isomorphism of $X_{S'}$ onto $Y_{S'}$.*

Theorem 3.81. *([Dou66], p. 87) Let X and Y be complex spaces, with X compact. Let H be the Douady space of $X \times Y$ and R the universal subspace of $H \times X \times Y$.*

1. *The set $\mathrm{Mor}(X, Y)$ of the points $s \in H$ such that $R(s)$ is the graph of a morphism $f : X \to Y$ is open in H;*
2. *$R_{|M}$ is the graph of a morphism $m : \mathrm{Mor}(X, Y) \times X \to Y$;*
3. *The morphism m satisfies the following universal property. For any complex space S and any morphism $u : S \times X \to Y$, there exists a unique morphism $f : S \to M$ such that $u = m \circ (f \times Id_X)$.*

Proof. The spaces R and $H \times X$ are proper and flat over H. Let $p : R \to H \times X$ be the projection. A point $s \in H$ is a point in $\mathrm{Mor}(X, Y)$ if and only if $p(s)$ is an isomorphism of $R(s)$ onto X. From Proposition 3.80, $\mathrm{Mor}(X, Y)$ is open in H and $p_{|\mathrm{Mor}(X,Y)}$ is an isomorphism of $R_{|\mathrm{Mor}(X,Y)}$ onto $\mathrm{Mor}(X, Y) \times X$, hence $R_{|\mathrm{Mor}(X,Y)}$ is the graph of a morphism $m : \mathrm{Mor}(X, Y) \times X \to Y$. Now, if S is another complex space, the morphisms $S \times X \to Y$ correspond bijectively to the subspaces Z of $S \times X \times Y$ such that the projection $Z \to S \times X$ is an isomorphism, that is the subspaces $Z \subset S \times X \times Y$ proper and flat over S, such that, for any $s \in S$, $Z(s)$ is the graph of a morphism $X \to Y$, hence to a morphism $f : S \to H$ such that, for any $s \in S$, $f(s) \in \mathrm{Mor}(X, Y)$. This is a functorial correspondence. □

In [Pou69], p. 452, Theorem 3.64 is extended by G. Pourcin to the relative setting. That is, he considers a complex space X over another complex space S.

Theorem 3.82. *([Fle79], p. 130) Let S be a complex space, X a complex space proper and flat over S and Y another complex space over S. The contravariant functor that to every complex space T over S associates the set* $\mathrm{Mor}_T(X \times_S T, Y \times_S T)$ *is representable by a complex space* $\mathfrak{Mor}_S(X,Y)$.

In other words, there exists a complex space $\mathfrak{Mor}_S(X,Y)$ together with a morphism p to S and a morphism $m : p^*X \to p^*Y$ over $\mathfrak{Mor}_S(X,Y)$ with the following universal property: for each complex space T over S, $\psi : T \to S$, and for each T-morphism $g : \psi^*X \to \psi^*Y$, there exists a unique morphism $h : T \to \mathfrak{Mor}_S(X,Y)$, with $p \circ h = \psi$, such that $h^*m = g$. As set

$$\mathfrak{Mor}_S(X,Y) := \{(s,f) | s \in S, f : X(s) \to Y(s) \text{ is a holomorphic map}\}.$$

Set-theoretically, the map $h : T \to \mathfrak{Mor}_S(X,Y)$ is defined by

$$h : t \mapsto (\psi(t), g(t)).$$

Theorem 3.83. *([Pou75], p. 192, Theorem 5.13) Let Y be a S-anaflat subspace of $S \times K$ and X a Banach analytic space of finite presentation over S. The contravariant functor that to every Banach analytic space T over S associates the set* $\mathrm{Mor}_T(Y \times_S T, X \times_S T)$ *is representable by a Banach analytic space* $\mathfrak{Mor}_S(Y,X)$.

Chapter 4

Versal deformation of compact complex spaces

In this chapter we present an account of the proof by A. Douady of the existence of a versal deformation for compact complex spaces. As in the construction of the Douady space (see Chapter 3, section 3.4), the proof of the main Theorem (see Theorem 4.43) is a two steps process consisting of an infinite dimensional construction of the family (see section 4.1) followed by a finite dimensional reduction (see section 4.2). Our main references are [Dou74], [LP75] and [Sti88]. We first give a brief recall of the main historical facts and definitions in deformation theory. The notion of deformation of a complex manifold is due to Kodaira and Spencer (see [KS58]).

Definition 4.1. Let X_0 be a compact complex space. A *deformation* of X_0 is a triple $((S,0),X,i)$, where $(S,0)$ is a germ of complex spaces and $X \to S$ is a proper and flat family of compact complex spaces such that $i : X_0 \to X(0)$ is an isomorphism. The deformation is called *complete* if for any other deformation $((T,0),Y,j)$ of X_0 there exists a morphism $f : (T,0) \to (S,0)$ such that f^*X is isomorphic to Y via an isomorphism α and $\alpha \circ f^*i = j$.

In deformation theory a special role is played by the *double point* D. This is the analytic subspace of $\mathbb{C}$ defined by the function $f := z^2$, so that its structure sheaf is the algebra of dual numbers $\mathbb{C}[\epsilon]/\epsilon^2$. The tangent space TD to the double point is one-dimensional. If we choose a generator ϵ in the algebra of dual numbers, then a basis element $v \in TD$ can be selected by the condition $v(\epsilon) = 1$. Let $(S,0)$ be a germ of complex spaces and denote with $\mathrm{Hom}(D,(S,0))$ the set of morphisms of germs $D \to (S,0)$. We have a bijection

$$\mathrm{Hom}(D,(S,0)) \to T_0S,$$

where T_0S is the Zariski tangent space of S at 0, sending $u : D \to (S,0)$ to $du(v) \in T_0S$. Let $\pi : X \to (S,0)$ be a complete deformation of a compact complex space X_0. If we denote with $\mathrm{Ex}^1(0)$ the set of classes of isomorphic deformations of X_0 over D, which has a natural structure of $\mathbb{C}$-vector space, we get a natural epimorphism

$$\mathrm{ks} : T_0Y \to \mathrm{Ex}^1(0)$$

via $u \mapsto u^*\pi$. This morphism is called the *Kodaira-Spencer map* (see [Pal90], p. 130).

Definition 4.2. ([Pal90], p. 134) A deformation is called *effective* if ks is an isomorphism.

In 1958, Kodaira, Nirenberg and Spencer ([KNS58]) proved that if X_0 is a compact complex manifold with $H^2(X_0; \mathcal{T}_{X_0}) = 0$, then X_0 admits a complete and effective deformation with smooth base space. In 1962, Kuranishi ([Kur62]) proved the existence of a complete and effective deformation without the condition $H^2(X_0; \mathcal{T}_{X_0}) = 0$. In this case, the base space is a germ of complex spaces, in general singular.

Example 4.3. ([Sti88], p. XI) *Hopf Surfaces.* Let $W := \mathbb{C}^2 \setminus \{0\}$, $\alpha \in \mathbb{C}$, $0 < |\alpha| < 1$ and $g : W \times \mathbb{C} \to W \times \mathbb{C}$ defined by

$$(z_1, z_2, t) \mapsto (\alpha z_1 + tz_2, \alpha z_2, t).$$

Let G be the cyclic group generated by g. Then G acts properly discontinuously and freely on $W \times \mathbb{C}$. The space $M := W \times \mathbb{C}/G$ is a compact complex manifold. Let $\pi : M \to \mathbb{C}$ be the map induced by the projection onto the second factor. The fibres M_t, for $t \in \mathbb{C}$, are all diffeomorphic to $S^1 \times S^3$ (Hopf surfaces). One can show that $M_1 \simeq M_t \not\simeq M_0$, for any $t \in \mathbb{C} \setminus \{0\}$. This is a deformation of M_0. Now, let

$$S := \{s \in GL(2, \mathbb{C}) : \text{all eigenvalues of s have absolute value} < 1\}$$

and define $F : W \times S \to W \times S$ via $F(x, s) := (sx, x)$. Let G' be the cyclic group of automorphisms generated by F. This group acts properly discontinuously and freely on $W \times S$. Let $\mathfrak{X} := W \times S/G'$ be. One can show (see [Weh81]) that the family $p : \mathfrak{X} \to S$ is a complete and effective deformation of the Hopf surface M_0.

Definition 4.4. Let X_0 be a compact complex space. If given any deformation $\pi : X \to (S, 0)$ of X_0, there exists an isomorphism $f : X \to X_0 \times S$, such that $\pi = f \circ pr_2$, then X_0 is said to be *rigid*.

Examples of rigid spaces are $\mathbb{P}^n$ and $\mathbb{P}^1 \times \mathbb{P}^1$. In 1964, A. Douady (see [Dou64]), using his theory of Banach analytic spaces , succeeded in giving a very elegant exposition of the results of Kuranishi. Afterwards, he succeeded in generalizing Kuranishi's result to the category of complex analytic spaces.

Definition 4.5. ([Dou74], p. 601, Proposition 1) Let X_0 be a compact complex space. A complete and effective deformation $((S, 0), X, i)$ is called *versal* if given any deformation $((T, 0), Y, j)$ of X_0, a subgerm $(T', 0)$ of $(T, 0)$ and a morphism $h' : (T', 0) \to (S, 0)$ such that $Y|_{T'} \simeq h'^*X$, there exists a morphism $h : (T, 0) \to (S, 0)$, such that $Y \simeq h^*X$ and $h|_{T'} = h'$.

Theorem 4.6. *(Grauert [Gra74], Douady [Dou74], Foster-Knorr [FK79], Palamodov [Pal78]) Every compact complex space admits a versal deformation.*

4.1 Infinite dimensional construction

Let X be a compact complex space. Then one can find a finite set I_0 and, for each $i \in I_0$, an open subset $U_i \subset \mathbb{C}^{n_i}$, a closed subset $Z_i \subset U_i$, an open subset $X_i^\circ \subset X$ and an isomorphism $f_i : Z_i \to X_i^\circ$, such that

$$X = \bigcup_{i \in I_0} X_i^\circ.$$

For $l = 1, 2$, let $I_l := \{(i_0, ..., i_l) \in I_0^{l+1} : X_{i_0}^\circ \cap ... \cap X_{i_l}^\circ \neq \emptyset\}$. For each $j = (i_0, ..., i_l) \in I_l$, one can find an open subset $U_j \subset \mathbb{C}^{n_j}$, a closed subset $Z_j \subset U_j$ and an isomorphism $f_j : Z_j \to X_{i_0}^\circ \cap ... \cap X_{i_l}^\circ$. Set $I := I_0 \cup I_1 \cup I_2$. Let $\eta := (Z_i, f_i)_{i \in I}$. For $i \in I_0$ and $j = (i, -) \in I_1$, let $g_i^j := f_i^{-1} \circ f_j$ be the transition maps. From the family $(Z_i, g_i^j)_{(i \in I_0, j=(i,-) \in I_1)}$ one can reconstruct X, more precisely a complex space isomorphic to X, gluing $(Z_i)_{i \in I_0}$ via $(g_i^j \circ (g_{i'}^j)^{-1})_{(j=(i,i') \in I_1)}$. This process of decomposition and recomposition is at the core of Douady's techniques. What changes, in the deformation context, is that the charts are taken of the form $f_i : Y_i \to X$, where Y_i is a privileged subspace of a polycylinder $K_i \subset \mathbb{C}^{n_i}$ (see Definition 3.48). Since intersections of privileged polycylinders are not in general privileged, one needs to cover the intersections too. In order to have the transition maps well-defined, one needs to work with two polycylinders $K_i' \subset K_i^\circ$ for double intersections and three polycylinders $K_i' \subset \tilde{K}_i^\circ$, $\tilde{K}_i \subset K_i^\circ$ for triple intersections. This leads to the notion of *cuirasse of order 2* (see Definition 4.10).

Definition 4.7. A *simplicial set $I_\bullet$ of order 2* is given by three disjoint sets I_0, I_1, I_2 and maps $d_i : I_m \to I_{m-1}$, for $0 \leq i \leq m$ and $1 \leq m \leq 2$ such that for each $i < j$

$$d_i \circ d_j = d_{j-1} \circ d_i.$$

We set

$$\begin{aligned} I &:= \bigcup_{j=0}^{2} I_j, \\ \partial i &:= \{d_0 i, ..., d_m i\} \qquad\qquad, \text{ if } i \in I_m, \\ di &:= (d_0 i, ..., d_m i) \in I_{m-1}^{m+1}, \text{ if } i \in I_m. \end{aligned}$$

Definition 4.8. ([Dou74], p. 587) The *type of a 2-cuirasse* is a collection

$$\mathfrak{I} := (I_\bullet, (K_i)_{i \in I}, (\tilde{K}_i)_{i \in I}, (K_i')_{i \in I_0 \cup I_1}),$$

where $I_\bullet$ is a finite simplicial set of dimension 2, for each $i \in I_0 \cup I_1$, we have three compact polycylinders in $\mathbb{C}^{n_i}$

$$K_i' \subset \mathring{\tilde{K}}_i, \tilde{K}_i \subset \mathring{K}_i$$

and, furthermore, for each $i \in I_2$, we have two compact polycylinders in $\mathbb{C}^{n_i}$

$$\tilde{K}_i \subset \mathring{K}_i.$$

Let Y_i be a priviliged subspace of a polycylinder K_i (see Definition 3.52) and $(I_\bullet, (K_i)_{i\in I}, (\tilde{K}_i)_{i\in I}, (K_i')_{i\in I_0\cup I_1})$ a type of a 2-cuirasse, from now on we set:

$$Y_i^\circ := Y_i \cap K_i^\circ,\ Y_i'^\circ := Y_i \cap K_i'^\circ \text{ and } \tilde{Y}_i^\circ := Y_i \cap \tilde{K}_i^\circ.$$

Definition 4.9. Let X be a complex space. A *chart* for X is a triple (X', φ, U), where X' is open in X, U is open in $\mathbb{C}^n$ and $\varphi : X' \to Z \subset U$ is an isomorphism onto a local model in U(see Definition 1.6).

Definition 4.10. ([Dou74], p. 587 and [Sti88], p. 11)
Let X_0 be a compact complex space. Let $\mathfrak{I} := (I_\bullet, (K_i)_{i\in I}, (\tilde{K}_i)_{i\in I}, (K_i')_{i\in I_0\cup I_1})$ be a type of a 2-cuirasse . A *2-cuirasse* of type $\mathfrak{I}$ on X_0 is a collection $\{(Y_i, f_i)\}_{i\in I}$, where Y_i is a privileged subspace of K_i and $f_i : Y_i \to X$ is a morphism such that the following conditions are satisfied:

(C0) There exist charts $(V_i, \varphi_i, U_i)_{i\in I_0}$ for X_0, such that $K_i \subset U_i$ is privileged for the structure sheaf of $Z_i := \varphi(V_i)$ (see Definition 3.42), $Y_i = K_i \cap Z_i$ and $f_i = \varphi^{-1}|_{Y_i}$;

(C1)
$$X_0 = \bigcup_{i\in I_0} f_i(Y_i'^\circ);$$

(C2) If $(i_0, i_1) \in I_0^2$, then
$$f_{i_0}(Y_{i_0}') \cap f_{i_1}(Y_{i_1}') \subset \bigcup_{dj=(i_0,i_1)} f_j(Y_j'^\circ)$$
and
$$\bigcup_{dj=(i_0,i_1)} f_j(Y_j) \subset f_{i_0}(\tilde{Y}_{i_0}^\circ) \cap f_{i_1}(\tilde{Y}_{i_1}^\circ);$$

(C3) If $(j_0, j_1, j_2) \in I_1^3$, then
$$f_{j_0}(Y_{j_0}') \cap f_{j_1}(Y_{j_1}') \cap f_{j_2}(Y_{j_2}') \subset \bigcup_{dk=(j_0,j_1,j_2)} f_k(\tilde{Y}_k^\circ)$$
and
$$\bigcup_{dk=(j_0,j_1,j_2)} f_k(Y_k) \subset f_{j_0}(\tilde{Y}_{j_0}^\circ) \cap f_{j_1}(\tilde{Y}_{j_1}^\circ) \cap f_{j_2}(\tilde{Y}_{j_2}^\circ).$$

We remark that f_i restricted to $Y_i'^\circ$ is an isomorphism onto the image. Using the Theorem of privileged neighborhoods (Theorem 3.50), one can prove the following

Proposition 4.11. *([Dou74], p. 588) Every compact complex space X_0 admits a cuirasse of order 2.*

From now on, for the sake of simplicity, we shall mostly call *2-cuirasse* simply *cuirasse*. Let $\mathfrak{I} := (I_\bullet, (K_i)_{i\in I}, (\tilde{K}_i)_{i\in I}, (K_i')_{i\in I_0\cup I_1})$ be a type of cuirasse. For each $i \in I$, let $\mathcal{G}(K_i)$ be the Banach analytic space of privileged subspaces of K_i (see Definition 3.52) and let $\underline{Y}_i$ be the universal $\mathcal{G}(K_i)$-anaflat subspace of $\mathcal{G}(K_i) \times K_i$ (see Definition 3.63). Then the set of cuirasses of type $\mathfrak{I}$ on X_0 is an open subset of the Banach analytic space (see Theorem 3.83)

$$\prod_{i\in I} \mathfrak{Mor}_{\mathcal{G}(K_i)}(\underline{Y}_i, \mathcal{G}(K_i) \times X_0),$$

see [Dou74], p. 583 and 588.

Definition 4.12. We denote the space of cuirasses of type $\mathfrak{I}$ on X_0 by $\mathcal{Q}(\mathfrak{I}; X_0)$.

Definition 4.13. ([Dou74], p. 588) Let S be a Banach analytic space and X a Banach analytic space proper and anaflat over S. We set:

$$\mathcal{Q}_S(\mathfrak{I}; X) := \{(s, (Y_i, f_i)_{i\in I}) | s \in S, (Y_i, f_i)_{i\in I} \in \mathcal{Q}(\mathfrak{I}; X(s))\}.$$

That is,

$$\mathcal{Q}_S(\mathfrak{I}; X) = \bigsqcup_{s\in S} \mathcal{Q}(\mathfrak{I}; X(s)).$$

Analogously, the set $\mathcal{Q}_S(\mathfrak{I}; X)$ is an open subset of the Banach analytic space

$$\prod_{i\in I} \mathfrak{Mor}_{S\times\mathcal{G}(K_i)}(S \times \underline{Y}_i, \mathcal{G}(K_i) \times X),$$

see [Dou74], p. 583 and 588.

Definition 4.14. A section $\sigma : S \to \mathcal{Q}_S(\mathfrak{I}; X)$ of the canonical projection is called a *relative cuirasse of type* $\mathfrak{I}$ on X.

Remark 4.15. In what follows, for the sake of readability, we shall sometimes write Banach analytic morphisms just set-theoretically.

Clearly, relative cuirasses satisfy a relative version of the axioms in Definition 4.10. We introduce a special class of cuirasses.

Definition 4.16. ([Dou74], p. 585 and [Sti88], p. 100)

1. A *triangular set* in $\mathbb{R}_{>0}$ is any interval of the form $]0, a[$, for $a \in \mathbb{R}_{>0}$. We define by induction a triangular set for any $n \in \mathbb{N}$. A *triangular set* in $\mathbb{R}^n_{>0}$ is a set of the form
$$\triangle := \{(x', x_n) : x' \in \triangle', 0 < x_n < h(x')\},$$
where $\triangle'$ is a triangular set in $\mathbb{R}^{n-1}_{>0}$ and $h : \triangle' \to \mathbb{R}_{>0}$ is a lower semi-continuous function.

2. Let $K \subset \mathbb{C}^n$ be a polycylinder and $Y \subset K$ a privileged subspace. The space Y is called *triangularly privileged* if there exists a point $c \in Y^\circ$ and a triangular set $\triangle \subset \mathbb{R}^n_{>0}$, with $(1, ..., 1) \in \triangle$, such that for any $t \in \triangle \cap]0, 1]^n$ the polycylinder

$$K_t := (1 - t) \cdot c + t \cdot K,$$

with component-wise addition and multiplication, is Y-privileged (see Definition 3.49).

We say that Y is *continuously privileged* in K.

Definition 4.17. A cuirasse $q = (Y_i, f_i) \in \mathcal{Q}(\mathfrak{I}; X_0)$ on X_0 is called *triangularly privileged*, if Y_i is triangularly privileged for each $i \in I$.

Using Theorem 3.50, one can construct a triangularly privileged cuirasse on every compact complex space (see [Dou74], p. 588). Let X be a Banach analytic space proper and anaflat over a Banach analytic space S. Let Y be a triangularly privileged subspace of a polycylinder $K \subset \mathbb{C}^n$, $s \in S$ and $f : Y \to X$ a morphism. Let

$$\pi : \mathfrak{Mor}_{S \times \mathcal{G}(K_i)}(S \times \underline{Y}_i, \mathcal{G}(K_i) \times X) \to S$$

be the canonical projection. We are going to show that π is smooth at (s, Y, f). Let $\mathfrak{M}$ be the germ of $\mathfrak{Mor}_{S \times \mathcal{G}(K_i)}(S \times \underline{Y}_i, \mathcal{G}(K_i) \times X)$ at (s, Y, f). Let $T \subset \mathfrak{M}$ be the subset of points in $\mathfrak{M}$ where π is a submersion (see Definition 3.28) and G the subset of points in $\mathfrak{M}$ where π is smooth (see Definition 3.28). Let $\triangle \subset \mathbb{R}^n_{>0}$, $c \in Y^\circ$ and

$$\begin{aligned} h_t : K &\to \mathbb{C}^n \\ z &\mapsto (1 - t) \cdot c + t \cdot z \end{aligned}$$

be as in Definition 4.17. Let $T(s)$ be the fibre of $\pi|_T$ over s.

Lemma 4.18. *([Dou74], p. 585 and [Sti88], p. 101) The association $t \mapsto (s, h_t^{-1}(Y), f \circ h_t)$ defines a map $\chi : \triangle \cap]0, 1]^n \to T(s)$. Moreover, for $t \in \triangle \cap]0, 1]^n$ close to 0, we have $(s, h_t^{-1}(Y), f \circ h_t) \in G(s)$.*

Lemma 4.19. *([Sti88], p. 101) The fibre $G(s)$ is open and closed in $T(s)$.*

Proof. It is clearly open. Moreover, let $(Y, f) \in T(s) \setminus G(s)$. By definition of submersion there exists a germ $(S', s) \subset (S, s)$, with $S' \neq S$, and a Banach space U such that, on a neighborhood of (s, Y, f), $\mathfrak{M}$ is isomorphic to $S' \times U$. Now, if (s, Y', f') is a point in a neighborhood of (s, Y, f), then $(s, Y', f') \notin G(s)$. □

Proposition 4.20. *([Dou74], p. 585) The projection*

$$\pi : \mathfrak{Mor}_{S \times \mathcal{G}(K_i)}(S \times \underline{Y}_i, \mathcal{G}(K_i) \times X) \to S$$

is smooth at (s, Y, f).

Proof. Since $\triangle \cap]0,1]^n$ is connected, by Lemma 4.18, the connected component of (s, Y, f) in $T(s)$ intersects $G(s)$. Since, by Lemma 4.19, $G(s)$, is open and closed, the connected component of (s, Y, f) is entirely contained in $G(s)$.

□

Corollary 4.21. *([Dou74], p. 589) Let X be a Banach analytic space proper and anaflat over a Banach analytic space S and $q_0 \in \mathcal{Q}(\mathfrak{I}; X(s_0))$ a triangularly privileged cuirasse. Then the projection $\pi : \mathcal{Q}_S(\mathfrak{I}; X) \to S$ is smooth in a neighborhood of q_0.*

In particular, given any triangularly privileged cuirasse on $X(s_0)$, since the projection $\pi : \mathcal{Q}_S(\mathfrak{I}; X) \to S$ is smooth (see Definition 3.28), there exists a relative cuirasse σ (that is a local section of π) defined on a neighborhood S' of s_0 in S such that $\sigma(s_0) = q_0$.

Definition 4.22. ([Dou74], p. 589 and [Sti88], pp. 53-54)
Let $\mathfrak{I} := (I_\bullet, (K_i)_{i\in I}, (\tilde{K}_i)_{i\in I}, (K'_i)_{i\in I_0\cup I_1})$ be a type of cuirasse. A *puzzle* of type $\mathfrak{I}$ is given by a privileged subspace Y_i of K_i, for each $i \in I$, and, for each $j \in I$ with $i \in \partial j$, a map $g_i^j : Y_j \to Y_i$, such that the following conditions are satisfied:

(P0) $g_i^j(Y_j)$ is open in Y_i and $g_i^j|_{Y_j'^\circ}$ is an isomorphism onto the image.

(P1) (Simplicial functoriality) If $k \in I_2$, $j \in \partial k$, $j' \in \partial k$ and $i \in \partial j \cap \partial j'$, then

$$g_i^j \circ g_j^k = g_i^{j'} \circ g_{j'}^k.$$

(P2) (Kan condition) Let $i \in I_0$ and $j, j' \in I_1$ such that $dj = (i', i)$ and $dj' = (i, i'')$. If $y \in Y'_j$ and $y' \in Y'_{j'}$ are such that

$$g_i^j(y) = g_i^{j'}(y'),\ g_{i'}^j(y) \in Y'_{i'} \text{ and } g_{i''}^{j'}(y') \in Y'_{i''}$$

then there exists $k \in I_2$ with $dk = (i', i, i'')$ and $z \in \tilde{Y}_k^\circ$ such that $g_j^k(z) = y$ and $g_{j'}^k(z) = y'$.

(P3) (Compactness) If $i \in I_0$ and $x \in Y_i^{'}$, then there exists $i' \in I_0$, $x' \in Y_{i'}^{'\circ}$, $j \in I_1$ such that $dj = (i, i')$ and $y \in Y_j^{'\circ}$ such that $g_i^j(y) = x$ and $g_{i'}^j(y) = x'$.

(P4) (Separation) If $j \in I_1$, with $dj = (i, i')$ and $y \in Y_j$ such that $g_i^j(y) \in Y'_{i'}$ and $g_{i'}^j(y) \in Y'_{i'}$, then there exists $j' \in I_1$ and $y' \in Y_{j'}^{'\circ}$ such that $dj' = (i, i')$ and

$$g_i^{j'}(y') = g_i^j(y) \text{ and } g_{i'}^{j'}(y') = g_{i'}^j(y).$$

Conditions $(P1)$ and $(P2)$ give the following commutative diagram:

$$\begin{tikzcd}
 & & Y_k \arrow[dl, "g_j^k"'] \arrow[dr, "g_{j'}^k"] & & \\
 & Y_j \arrow[dl, "g_{i'}^j"'] \arrow[dr, "g_i^j"] & & Y_{j'} \arrow[dl, "g_i^{j'}"'] \arrow[dr, "g_{i''}^{j'}"] & \\
Y_{i'} & & Y_i & & Y_{i''}
\end{tikzcd}.$$

The family $(Y_i')_{i\in I_0}$ will be glued via the maps $(g_{i'}^j \circ (g_i^j)^{-1})_{dj=(i,i')}$ to get a complex space (see Lemma 4.25). In this case, condition (P_3) guarantees that the double intersections are covered by $(Y_j')_{j\in I_1}$ and condition (P_2) guarantees that the triple intersections are covered by $(Y_k)_{k\in I_2}$. Moreover, condition (P_1) yields the cocycle condition.

Proposition 4.23. *([Dou74], p. 590) Let $\mathfrak{I}$ be a type of cuirasse. The set of puzzles of type $\mathfrak{I}$ is a Banach analytic space.*

Definition 4.24. Let $\mathfrak{I}$ be a type of cuirasse. We denote the space of puzzles of type $\mathfrak{I}$ with $\mathfrak{Z}^{\mathfrak{I}}$ or simply with $\mathfrak{Z}$, when no risk of confusion is possible.

Let us fix a type of cuirasse $\mathfrak{I}$, thus a type of puzzle. Given a puzzle, one can get a (compact) complex space out of it. Moreover, one can collect the complex spaces obtained from puzzles into a Banach analytic space. More precisely, let S be a Banach analytic space and $\zeta : S \to \mathfrak{Z}$ a morphism. Identifying ζ with its graph $Y := \mathrm{graph}(\zeta) \subset S \times \mathfrak{Z}$, we see that Y is given by a S-anaflat privileged subspace (see Definition 3.62) $Y_i \subset S \times \mathcal{G}(K_i) \subset S \times K_i$, for each $i \in I$, and a S-morphism $g_i^j : Y_j \to Y_i$, for each $j \in I$ with $i = \partial j$. The collection $\{(Y_i), (g_i^j)\}$ satisfies a relative version $(P0)_S$–$(P4)_S$ of conditions $(P0)$–$(P4)$. That is, the relative conditions are conditions $(P0)$–$(P4)$ holding for each $s \in S$. Now, for each $i, i' \in I_0$ and $j \in I_1$ such that $dj = (i, i')$, we get morphisms $g_i^j : Y_j \to Y_i$ and $g_{i'}^j : Y_j \to Y_{i'}$. Let $Y_{ii'}^j$ be the subset of all $x \in Y_i'^{\circ}$ such that there exists $y \in Y_j'^{\circ}$ with $g_i^j(y) = x$ and $g_{i'}^j(y) \in Y_{i'}'^{\circ}$, that is $Y_{ii'}^j = g_i^j(Y_j'^{\circ} \cap (g_{i'}^j)^{-1}(Y_{i'}'^{\circ}))$. Let J be the set of all $j \in I_1$, with $dj = (i, i')$, such that $Y_{ii'}^j$ is not empty. For each $j \in J$, we have that $Y_{ii'}^j$ is open in $Y_i'^{\circ}$ and we get a well defined morphism $g_i^j \circ (g_{i'}^j)^{-1} : Y_{ii'}^j \to Y_{i'}'^{\circ}$, which is an isomorphism onto the image. Let $Y_{ii'}$ be the open subset of $Y_i'^{\circ}$ given by the union of all $Y_{ii'}^j$, for $j \in J$. We want to glue the family of morphisms $(g_i^j \circ (g_{i'}^j)^{-1} : Y_{ii'}^j \to Y_{i'}'^{\circ})_{j\in J}$ to a morphism $\gamma_{ii'} : Y_{ii'} \to Y_{i'}'^{\circ}$. To do that, we have to check that for each $j, j' \in J$ such that $Y_{ii'}^j \cap Y_{ii'}^{j'} \neq \emptyset$, we have, on this intersection, that

$$g_{i'}^j \circ (g_i^j)^{-1} = g_{i'}^{j'} \circ (g_i^{j'})^{-1}. \tag{4.1}$$

To check this, we first apply the Kan condition $(P2)_S$, in order to apply then the simplicial functoriality condition $(P1)_S$. Indeed, using the Kan condition $(P2)_S$, we have that if $j, j' \in J$, $y \in Y_j^{'\circ}$ and $y' \in Y_{j'}^{'\circ}$ are both mapped to the pair $(x, x') \in Y_i^{'\circ} \times Y_{i'}^{'\circ}$, then there exists some $k \in I_2$ with $j, j' \in \partial k$ and some $z \in \tilde{Y}_k^\circ$, which is mapped to y and y' via g_j^k and $g_{j'}^k$ respectively. This means that, if $Y_{ii'}^j \cap Y_{ii'}^{j'} \neq \emptyset$, then there exists some open subset $V \subset \tilde{Y}_k^\circ$, with $g_j^k(V) = (g_i^j)^{-1}(Y_{ii'}^j \cap Y_{ii'}^{j'})$ and $g_{j'}^k(V) = (g_i^{j'})^{-1}(Y_{ii'}^j \cap Y_{ii'}^{j'})$. Therefore, (4.1) becomes:

$$g_i^j \circ g_j^k = g_i^{j'} \circ g_{j'}^k. \tag{4.2}$$

Now, from the simplicial functoriality condition $(P1)_S$, we get that this equation in fact holds. Thus, we get a morphism $\gamma_{ii'} : Y_{ii'} \to Y_{i'}^{'\circ}$, which is an isomorphism onto the image. At this point, we get a family of morphisms $(\gamma_{ii'} : Y_{ii'} \to Y_{i'}^{'\circ})_{(i,i') \in I_0^2}$. This family satisfies the following Lemma.

Lemma 4.25. *([Dou74], p. 591)*

(a) $Y_{ii} = Y_i^{'\circ}$ *and* $\gamma_{ii} = \mathrm{Id}$*;*

(b) $Y_{i'i} = \gamma_{ii'}(Y_{ii'})$ *and* $\gamma_{i'i} = \gamma_{ii'}^{-1}$*;*

(c) if $(i, i', i'') \in I_0^3$*, set* $Y_{ii'i''} := \gamma_{ii'}^{-1}(Y_{i'i''})$*, then* $Y_{ii'i''} \subset Y_{ii''}$ *and*

$$\gamma_{i'i''} \circ \gamma_{ii'|Y_{ii'i''}} = \gamma_{ii''|Y_{ii'i''}}.$$

Using Lemma 4.25, we get

Proposition 4.26. *([Dou74], p. 591) Let S be a Banach analytic space and $\zeta : S \to \mathfrak{Z}$ a morphism. The family of S-anaflat spaces $(Y_i^{'\circ})_{i \in I_0}$ glues to a Banach analytic space X_ζ over S.*

For each $i \in I_0$, we get an isomorphism χ_i of $Y_i^{'\circ}$ onto an open subset of X_ζ and the images of these isomorphisms cover X_ζ. Therefore, X_ζ is a Banach analytic space anaflat over S.

Proposition 4.27. *([Dou74], p. 592) The Banach analytic space X_ζ is Hausdorff and proper over S.*

Proof. Let us consider the topological space

$$\coprod_{i \in I_0} Y_i^{'},$$

and write $\mathcal{R}(x, x^{'})$ if $x \in Y_i^{'}$ and $x \in Y_{i'}^{'}$ are such that there exists $j \in I_1$ and $y \in Y_i^{'}$ with $dj = (i, i')$, $g_i^j(y) = x$ and $g_{i'}^j(y) = x'$. From the axiom $(P2)$ it follows that $\mathcal{R}$ is an equivalence relation. The quotient:

$$\hat{X} := \coprod_{i \in I_0} Y_i^{'}/\mathcal{R},$$

is proper over S since the $(Y_i^{'})$ are proper over S and they are finitely many. Since the graph of $\mathcal{R}$ is proper over S, the space $\hat{X}$ is Hausdorff. From axiom $(P4)$ it follows that $\mathcal{R}$ coincides with the equivalence relation defined for the $(Y_j^{\circ'})_{j\in I_1}$. Therefore, X_ζ identifies with an open subset of $\hat{X}$. From axiom $(P3)$ it follows that in fact this open subset coincides with the whole space $\hat{X}$. □

Definition 4.28. If we take $S = \mathfrak{Z}$ and $\zeta = \mathrm{Id}_{\mathfrak{Z}}$ in Proposition 4.26, we get a universal Banach analytic space proper and anaflat over the space of puzzles $\mathfrak{Z}$. We denote such a space with $\mathfrak{X}$.

Remark 4.29. The space $\mathfrak{X}$ is given by

$$\coprod_{i\in I_0} \underline{Y}_i^{'}/\mathcal{R},$$

where $\underline{Y}_i^{'}$ is the universal $\mathcal{G}(K_i')$-anaflat subspace of $\mathcal{G}(K_i') \times K_i'$ (see Definition 3.63) and $\mathcal{R}$ is the equivalence relation defined in Proposition 4.27. It is easy to show that $X_\zeta = \zeta^*\mathfrak{X}$, for any morphism $\zeta : S \to \mathfrak{Z}$ (see [Dou74], p. 592).

Now, from a cuirasse $q := (Y_i, f_i)_{i\in I} \in \mathcal{Q}(\mathfrak{I}; X_0)$ on X_0, one can get a puzzle $z_q \in \mathfrak{Z}$. Indeed, if $i \in \partial j$, using the second part of conditions $(C2)$ and $(C3)$ in Definition 4.10, we have that $f_j(Y_j) \subset f_i(\tilde{Y}_i^{\circ})$. Moreover, f_i induces an isomorphism of $\tilde{Y}_i^{\circ}$ onto its image. Thus, we can define a morphism $g_i^j := f_i^{-1} \circ f_j : Y_j \to \tilde{Y}_i^{\circ}$. Hence, we get a collection of data $z_q := (Y_i, (g_i^j))$

Proposition 4.30. *([Dou74], p. 592) The collection of data $z_q = (Y_i, (g_i^j))$ is a puzzle of type $\mathfrak{I}$.*

Proof. By construction, conditions $(P0)$ and $(P1)$ immediately holds. To verify $(P2)$, let j, j', i, y and y' satisfying the hypothesis of $(P2)$. Then,

$$x = f_j(y) = f_{j'}(y') \in f_{i'}(Y_{i'}') \cap f_{i''}(Y_{i''}'),$$

using $(C2)$ we get that there exists $j'' \in I_1$ with $dj'' = (i', i'')$ such that $x \in f_{j''}(Y_{j''}'^{\circ})$ and by $(C3)$ there exists $k \in I_2$ such that $x \in f_k(\tilde{Y}_k^{\circ})$, then set $z := f_k^{-1}(x)$. Conditions $(P3)$ and $(P4)$ follow from $(C1)$ and $(C2)$ respectively. □

If S is a Banach analytic space, X another Banach analytic space proper and anaflat over S via a morphism π and $q : S \to \mathcal{Q}_S(\mathfrak{I}; X)$ a relative cuirasse on X, then one can naturally construct a morphism $\varphi_q : S \to \mathfrak{Z}$ with the property that for each $s \in S$, the image $\varphi_q(s)$ is the puzzle associated to the cuirasse $q(s) \in \mathcal{Q}(\mathfrak{I}; X(s))$ (see Proposition 4.30). Indeed, the morphism (the relative cuirasse) q is given by S-anaflat subspaces $Y_i \subset S \times K_i$ (see Definition 3.62) and S-morphisms $f_i : Y_i \to X$. As in Proposition 4.30, for $i \in \partial j$, we get a S-morphism $g_i^j := f_i^{-1} \circ f_j : Y_j \to \tilde{Y}_i$.

Definition 4.31. We can define a map $\varphi_q : S \to \mathfrak{Z}$ via $s \mapsto z_{q(s)} = (Y_i(s), (g_i^j(s)))$.

By Proposition 4.30, this morphism is well defined. Moreover, this morphism is compatible with the change of basis. In fact, if $h : S' \to S$ is a morphism, setting $X' := h^*X$ and $q' := h^*q$, we have $\varphi_{q'} = \varphi_q \circ h$. The relative cuirasse $q : S \to \mathcal{Q}(\mathfrak{I}; X)$ describes the space X over S by describing the fibres $(\pi^{-1}(s))_{s\in S}$ of $\pi : X \to S$ in terms of a family of cuirasses $(q(s))_{s\in S}$. Using Proposition 4.30, we get a puzzle $z_{q(s)}$ out of each cuirasse $q(s)$. In light of the definition of cuirasse (see Definition 4.10) and puzzle associated to it (see Proposition 4.30), it is natural to think that $z_{q(s)}$ is isomorphic to $\pi^{-1}(s)$, for each $s \in S$, and hence, that the space X_{φ_q} (see Proposition 4.26), obtained by gluing together the puzzles $(z_{q(s)})_{s\in S}$ using Lemma 4.25, is S-isomorphic to X.

Proposition 4.32. *([Dou74], p. 592) There exists a S-isomorphism $\alpha_q : X_{\varphi_q} \to X$.*

Proof. From Lemma 4.25 we get isomorphisms χ_i from $Y_i^{'\circ}$ onto open subsets U_{i,φ_q} of X_{φ_q}. On the other hand, the cuirasse q gives us isomorphisms f_i from $Y_i^{'\circ}$ onto open subsets U_i of X. Therefore, the collection of maps $(f_i \circ \chi_i^{-1})_{i\in I}$ gives us S-isomorphims of an open cover of X_{φ_q} onto an open cover of X. This family of maps glue to a S-isomorphism $\alpha_q : X_{\varphi_q} \to X$. $\square$

Now, let X_0 be a compact complex space and q_0 a triangularly privileged cuirasse of type $\mathfrak{I}$ (see Definition 4.17) on it. Let z_0 be the puzzle associated to q_0. Let $\mathfrak{X}$ be the Banach analytic space proper and anaflat over the space of puzzles $\mathfrak{Z}$ given by Proposition 4.26 and Definition 4.28. Let $\alpha_{q_0} : \mathfrak{X}(z_0) \to X_0$ be the isomorphism given by Proposition 4.32.

Theorem 4.33. *([Dou74], p. 592) The pair $(\mathfrak{X} \to \mathfrak{Z}, \alpha_{q_0})$ is a complete deformation of X_0. In other words, let X be a Banach analytic space proper and anaflat over a Banach analytic space S, let $s_0 \in S$ and let i be an isomorphism of $X(s_0)$ onto X_0. Then there exists a neighborhood S' of s_0 in S, a morphism $h : S' \to \mathfrak{Z}$ such that $h(s_0) = z_0$ and an isomorphism $u : h^*\mathfrak{X} \to X_{|S'}$ such that $i \circ u(s_0) = \alpha_{q_0}$.*

Proof. Since i^*q_0 is a triangularly privileged cuirasse (see Definition 4.17) on $X(s_0)$, we have that $\mathcal{Q}_S(\mathfrak{I}; X)$ is smooth over S (Corollary 4.21). Therefore there exists a local relative cuirasse q on X defined on a neighborhood S' of s_0 in S. Then set $h := \varphi_q$ (Definition 4.31) and $u := \alpha_q : X_{\varphi_q} = \varphi_q^*\mathfrak{X} \to X_{|S'}$ (Proposition 4.32). $\square$

4.2 Finite dimensional reduction

In this section we present an account of the finite dimensional reduction procedure developed by A. Douady in [Dou74], pp. 593-599. We largely follow [Sti88], pp. 20-46. We recall from Section 4.1 that the space $\mathfrak{Z}$ (see Definition 4.24) parametrizes all puzzles $z := (Y_i, g_i^j)_{i,j\in I}$ of type $\mathfrak{I}$ (see Definition 4.8). Each fibre $\mathfrak{X}(z)$ of $\mathfrak{X} \to \mathfrak{Z}$ (Proposition 4.26 and Remark 4.29), for $z \in \mathfrak{Z}$, is obtained by gluing the "pieces" $(Y_i)_{i\in I_0}$ of the puzzle z. Let us consider the space $\mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X})$ of relative cuirasses on $\mathfrak{X}$

over $\mathfrak{Z}$ (see Definition 4.14). Each point in $\mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X})$ is a pair (z, q), where $z \in \mathfrak{Z}$ is a puzzle and $q := (Y_i, f_i) \in \mathcal{Q}(\mathfrak{X}(z))$ is a cuirasse on the fibre $\mathfrak{X}(z)$. To each cuirasse $q := (Y_i, f_i)$ on $\mathfrak{X}(z)$, we can associate another puzzle $z_q \in \mathfrak{Z}$ (see Proposition 4.30). In principle, given a pair $(z, q) \in \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X})$, we have $z \neq z_q$, although $\mathfrak{X}(z) \simeq \mathfrak{X}(z_q)$. However, we can consider the subspace $Z \subset \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X})$ set-theoretically defined by the condition

$$Z := \{(z, q) \in \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) : z_q = z\}. \tag{4.3}$$

This condition selects, in each fibre $\mathcal{Q}(\mathfrak{X}(z))$ of the canonical projection $\pi : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$, all cuirasses q on $\mathfrak{X}(z)$ whose associated puzzle z_q coincides exactly with z. This means that, given a cuirasse $q := (Y_i, f_i)_{i \in I}$ on $\mathfrak{X}(z)$ such that $z_q = z$, we fix the privileged subspaces Y_i but we allow the embeddings $f_i : Y_i \to \mathfrak{X}(z)$ to change as long as the resulting gluing maps $g_i^j := f_i \circ f_j^{-1}$ among the pieces $(Y_i)_{i \in I}$ of the obtained puzzle z_q stay the same. In other words, we deform each Y_i inside $\mathfrak{X}(z)$ but we keep fixed the gluing relations. The space Z parametrizes all cuirasses of all compact complex spaces "close" to X_0. More precisely, let $\pi : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$ be the canonical projection. Let $\mathfrak{q}$ be the canonical cuirasse on $\pi^*\mathfrak{X}$ defined by

$$\begin{aligned} \mathfrak{q} : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) &\to \mathcal{Q}_{\mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X})}(\pi^*\mathfrak{X}) \\ (z, q) &\mapsto (z, q, q). \end{aligned} \tag{4.4}$$

By Definition 4.31, the cuirasse $\mathfrak{q}$ induces a morphism

$$\begin{aligned} \varphi_{\mathfrak{q}} : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) &\to \mathfrak{Z} \\ (z, q) &\mapsto z_q. \end{aligned} \tag{4.5}$$

Then, we get that the space Z (4.3) is given by

$$Z = \ker(\varphi_{\mathfrak{q}}, \pi). \tag{4.6}$$

In general, given any Banach analytic space X proper and anaflat over a Banach analytic space S, we get a map from the space of relative cuirasses $\mathcal{Q}_S(X)$ on X over S into the space of puzzles $\mathfrak{Z}$ by

$$\begin{aligned} \varphi_{X/S} : \mathcal{Q}_S(X) &\to \mathfrak{Z} \\ (s, q) &\mapsto z_q. \end{aligned} \tag{4.7}$$

(see Proposition 4.30). If $\sigma : S \to \mathcal{Q}_S(X)$ is a relative cuirasse on X over S, that is a section of the projection $\pi : \mathcal{Q}_S(X) \to S$, the composition $\varphi_\sigma := \varphi_{X/S} \circ \sigma : S \to \mathfrak{Z}$ is a morphism satisfying the versality property (see Definition 4.31 and Theorem 4.33). By the versality property, for each $s \in S$, the fibre $\mathfrak{X}(\varphi_\sigma(s))$ is isomorphic to the fibre $X(s)$ via an isomorphism $\alpha_\sigma(s) : \mathfrak{X}(\varphi_\sigma(s)) \to X(s)$ (see Proposition 4.32). Therefore, a cuirasse q on $X(s)$ gives a cuirasse $\alpha_\sigma(s)^* q$ on $\mathfrak{X}(\varphi_\sigma(s))$. Hence, we get a map

$$\begin{aligned} \psi_\sigma : S &\to Q_{\mathfrak{Z}}(\mathfrak{X}) \\ s &\mapsto (z_{\alpha_\sigma(s)^*\sigma(s)}, \alpha_\sigma(s)^*\sigma(s)) \end{aligned}. \tag{4.8}$$

For the sake of simplicity, if we identify $X(s)$ with $\mathfrak{X}(\varphi_\sigma(s))$ and hence $\sigma(s)$ with $\alpha_\sigma(s)^*\sigma(s)$, then we can rewrite (4.8) as

$$\begin{aligned} \psi_\sigma : S &\to Q_{\mathfrak{Z}}(\mathfrak{X}) \\ s &\mapsto (z_{\sigma(s)}, \sigma(s)) \end{aligned}. \tag{4.9}$$

We have the following commutative diagram

$$\begin{tikzcd}
 & & Q_{\mathfrak{Z}}(\mathfrak{X}) \arrow[d, "\pi"] \\
S \arrow[r, "\sigma"] \arrow[urr, "\psi_\sigma"] & \mathcal{Q}_S(X) \arrow[r, "\varphi_{X/S}"] & \mathfrak{Z}
\end{tikzcd}.$$

In fact, ψ_σ is the unique morphism from S to $Q_{\mathfrak{Z}}(\mathfrak{X})$ making the above diagram commutative and such that $\psi_\sigma^*\mathfrak{q} = (\alpha_\sigma^{-1})_*\sigma$ (see [Dou74], p. 593). By construction, ψ_σ factors through $Z \subset Q_{\mathfrak{Z}}(\mathfrak{X})$. Set $\mathfrak{X}_Z := \pi^*\mathfrak{X}|_Z$.

Proposition 4.34. *([Dou74], p. 593) The family $\mathfrak{X}_Z \to Z$ gives a complete deformation of a compact complex space X_0, in a neighborhood of a triangularly privileged cuirasse q_0 on X_0.*

Now, the restriction of $\mathfrak{q}$ (4.4) to Z gives a canonical relative cuirasse on $\mathfrak{X}_Z$ over Z, namely

$$\begin{aligned} \sigma_Z : Z &\to \mathcal{Q}_Z(\mathfrak{X}_Z) \\ (z_q, q) &\mapsto (z_q, q, q) \end{aligned} \tag{4.10}$$

Using the projection $\pi : \mathcal{Q}_Z(\mathfrak{X}_Z) \to Z$, we get a canonical relative cuirasse $\pi^*\sigma_Z$ on $\pi^*\mathfrak{X}_Z$ over $\mathcal{Q}_Z(\mathfrak{X}_Z)$, namely

$$\begin{aligned} \pi^*\sigma_Z : \mathcal{Q}_Z(\mathfrak{X}_Z) &\to \mathcal{Q}_{\mathcal{Q}_Z(\mathfrak{X}_Z)}(\pi^*\mathfrak{X}_Z) \\ (z_q, q, p) &\mapsto (z_q, q, p, p) \end{aligned}. \tag{4.11}$$

By Proposition 4.32, we get a $\mathcal{Q}_Z(\mathfrak{X}_Z)$-isomorphism

$$\alpha_{\pi^*\sigma_Z} : \varphi_{\pi^*\sigma_Z}^*\mathfrak{X} \to \pi^*\mathfrak{X}_Z. \tag{4.12}$$

Let

$$\begin{aligned} \psi_Z : \mathcal{Q}_Z(\mathfrak{X}_Z) &\to Z \\ s := (z_q, q, p) &\mapsto (z_{(\alpha_{\pi^*\sigma_Z}(s))^*p}, (\alpha_{\pi^*\sigma_Z}(s))^*p) \end{aligned} \tag{4.13}$$

be the morphism associated to the relative cuirasse $\pi^*\sigma_Z$ on $\pi^*\mathfrak{X}_Z$ over $\mathcal{Q}_Z(\mathfrak{X}_Z)$ (see (4.8)). In simpler terms, by (4.9), we can write

$$\begin{aligned} \psi_Z : \mathcal{Q}_Z(\mathfrak{X}_Z) &\to Z \\ (z_q, q, p) &\mapsto (z_p, p) \end{aligned}. \tag{4.14}$$

Proposition 4.35. *([Sti88], p. 33) Let S be a Banach analytic space and $f_1, f_2 : S \to Z$ morphisms. If $f_1^*\mathfrak{X}_Z \simeq f_2^*\mathfrak{X}_Z$, then there exists a morphism $k : S \to \mathcal{Q}_Z(\mathfrak{X}_Z)$ such that the following diagram commutes*

$$\begin{tikzcd}
 & Z \\
S \arrow[ur, "f_1"] \arrow[r, "k"] \arrow[dr, "f_2"'] & Q_Z(\mathfrak{X}_Z) \arrow[u, "\psi_Z"'] \arrow[d, "\pi"] \\
 & Z
\end{tikzcd}. \tag{4.15}$$

Proof. Let

$$\begin{aligned} f_1 : S &\to Z \\ s &\mapsto (z_{q_1(s)}, q_1(s)) \end{aligned} \tag{4.16}$$

and

$$\begin{aligned} f_2 : S &\to Z \\ s &\mapsto (z_{q_2(s)}, q_2(s)) \end{aligned}. \tag{4.17}$$

Let $\alpha : f_1^*\mathfrak{X}_Z \to f_2^*\mathfrak{X}_Z$ be a S-isomorphism. For each $s \in S$, it induces an isomorphism of spaces of absolute cuirasses

$$\begin{aligned} \alpha_{Q,s} : \mathcal{Q}(\mathfrak{X}_Z(f_1(s))) &\to \mathcal{Q}(\mathfrak{X}_Z(f_2(s))) \\ q_1 := (Y_i, f_i) &\mapsto \alpha(s)_* q_1 = (Y_i, \alpha(s) \circ f_i) \end{aligned}. \tag{4.18}$$

Hence, we have $(f_2(s), \alpha(s)_* q_1(s)) \in \mathcal{Q}_Z(\mathfrak{X}_Z)$. Let

$$\alpha_{\pi^*\sigma_Z} : \varphi^*_{\pi^*\sigma_Z}\mathfrak{X} \to \pi^*\mathfrak{X}_Z$$

be the $\mathcal{Q}_Z(\mathfrak{X}_Z)$-isomorphism given by (4.12). By the definition of the fibre product (Corollary 3.26), we can canonically write

$$(\pi^*\mathfrak{X}_Z)(f_2(s), \alpha(s)_* q_1(s)) = \mathfrak{X}_Z(f_2(s)).$$

On the other hand, we can canonically write

$$(\varphi^*_{\pi^*\sigma_Z}\mathfrak{X})(f_2(s), \alpha(s)_* q_1(s)) = \mathfrak{X}(z_{\alpha(s)_* q_1(s)})$$

see Definition 4.31. Since $\alpha(s)$ is an isomorphism, by the construction of puzzle associated to a cuirasse (Proposition 4.30), we have $z_{\alpha(s)_* q_1(s)} = z_{q_1(s)}$. Hence, $\mathfrak{X}(z_{\alpha(s)_* q_1(s)}) =$

$\mathfrak{X}(z_{q_1(s)})$. Moreover, again by the definition of the fibre product, we can canonically write $\mathfrak{X}(z_{q_1(s)}) = \mathfrak{X}_Z(z_{q_1(s)}, q_1(s))$. Therefore,

$$(\varphi^*_{\pi^*\sigma_Z}\mathfrak{X})(f_2(s), \alpha(s)_* q_1(s)) = \mathfrak{X}_Z(f_1(s)).$$

Thus, for each $s \in S$, $\alpha_{\pi^*\sigma_Z}$ naturally induces an isomorphism

$$\alpha'(s) : \mathfrak{X}_Z(f_1(s)) \to \mathfrak{X}_Z(f_2(s)).$$

Therefore, the map

$$\begin{aligned} k : S &\to \mathcal{Q}_Z(\mathfrak{X}_Z) \\ s &\mapsto (f_2(s), \alpha'(s)_* q_1(s)) \end{aligned} \tag{4.19}$$

is well defined. Then, by (4.13) and by the construction of the isomorphism $\alpha'(s)$, the statement follows. □

Let us fix a compact complex space X_0. For the sake of simplicity, we set $Q_0 := \mathcal{Q}(X_0)$, the space of cuirasses on X_0 (see Definition 4.10). Let $q_0 \in \mathcal{Q}(X_0)$ be a triangularly privileged cuirasse on X_0 (see Definition 4.17) and $z_{q_0} \in \mathfrak{Z}$ the associated puzzle. We get a point $(z_{q_0}, q_0) \in Z$. Since q_0 is triangularly privileged, the projection $\pi : \mathcal{Q}_Z(\mathfrak{X}_Z) \to Z$ is smooth in a neighborhood of (z_{q_0}, q_0, q_0) (see Corollary 4.21). Hence, we can find a local trivialization

$$\gamma : Z \times Q_0 \to \mathcal{Q}_Z(\mathfrak{X}_Z). \tag{4.20}$$

Let $\alpha_{q_0} : \mathfrak{X}_Z(z_{q_0}, q_0) \to X_0$ be the isomorphism given by Proposition 4.32, we naturally get an induced isomorphism $\alpha_{Q_0} : Q_0 \to \mathcal{Q}(\mathfrak{X}_Z(z_{q_0}, q_0))$. Moreover, let σ_Z be as in (4.10). We choose γ such that

$$\gamma|_{Z\times q_0} = \sigma_Z \text{ and } \gamma|_{(z_{q_0}, q_0)\times Q_0} = \alpha_{Q_0}.$$

Now, let ψ_Z be as in (4.13). We define

$$\omega : Z \times Q_0 \xrightarrow{\gamma} \mathcal{Q}_Z(\mathfrak{X}_Z) \xrightarrow{\psi_Z} Z \tag{4.21}$$

via $\omega := \psi_Z \circ \gamma$. Since $\gamma|_{Z\times q_0} = \sigma_Z$, we get that

$$\omega|_{Z\times q_0} = \psi_Z \circ \sigma_Z = Id_Z. \tag{4.22}$$

Proposition 4.36. *([Sti88], p. 36, [Sti90], p. 271) Let S be a Banach analytic space and $f, g : S \to Z$ morphisms. Then $f^*\mathfrak{X}_Z \simeq g^*\mathfrak{X}_Z$, if and only if there exists $h : S \to$*

Q_0 such that the following diagram commutes

$$\begin{tikzcd} & Z \times Q_0 \arrow[dd, "\omega"] \\ S \arrow[ur, "(f,h)"] \arrow[dr, "g"'] & \\ & Z \end{tikzcd} \quad . \tag{4.23}$$

Proof. Let $f, g : S \to Z$ be morphisms, such that $f^*\mathfrak{X}_Z \simeq g^*\mathfrak{X}_Z$. Using Proposition 4.35, with $f_1 := g$ and $f_2 := f$, we get a morphism $k : S \to \mathcal{Q}_Z(\mathfrak{X}_Z)$, such that $\psi_Z \circ k = g$ and $\pi \circ k = f$. Let $\gamma : Z \times Q_0 \to \mathcal{Q}_Z(\mathfrak{X}_Z)$ be the local trivialization chosen in (4.20). Then, setting $h := \pi_2 \circ \gamma^{-1} \circ k$ and using (4.21), we get

$$\omega \circ (f, h) = \psi_Z \circ \gamma \circ (f, \pi_2 \circ \gamma^{-1} \circ k),$$

that is, using $\pi \circ k = f$

$$\omega \circ (f, h) = \psi_Z \circ \gamma \circ (\pi \circ k, \pi_2 \circ \gamma^{-1} \circ k),$$

and equivalently

$$\omega \circ (f, h) = \psi_Z \circ k.$$

Using $\psi_Z \circ k = g$, the statement follows. Conversely, let $h : S \to Q_0$ be such that diagram (4.23) commutes. Set $\rho := \gamma \circ (f, h)$. Then, we immediately have $f = \pi \circ \rho$ and, since $\omega = \psi_Z \circ \gamma$ (by (4.21)), we get $g = \psi_Z \circ \rho$. But, by definition (4.13) and Proposition 4.34, we have that $\psi_Z^*\mathfrak{X}_Z \simeq \pi^*\mathfrak{X}_Z$. $\square$

In other words, $f^*\mathfrak{X}_Z \simeq g^*\mathfrak{X}_Z$ if and only if for each $s \in S$, $g(s)$ is obtained by "changing" $f(s)$ by a cuirasse q on the central fibre X_0. Now, if we set

$$\delta := \omega|_{0 \times Q_0} \tag{4.24}$$

and we take for S the double point $D := (\{\cdot\}, \mathbb{C}[\epsilon]/\epsilon^2)$, we get that the set $\mathrm{Ex}^1(X_0)$ of equivalence classes of infinitesimal deformations of X_0 over D is given by

$$\mathrm{Ex}^1(X_0) = T_{(z_{q_0}, q_0)} Z / \operatorname{Im} T_{q_0}\delta.$$

In Proposition 4.40, we will show that $T_0\delta$ is a Fredholm operator of index 0. Then, it follows that $\operatorname{Im} T_0\delta$ admits a finite dimensional topological complement T_0R. Hence, $\mathrm{Ex}^1(X_0) = T_0R < \infty$. In this way, we get a complete and effective deformation of X_0 parametrized by a finite dimensional complex space R.

Definition 4.37. Let $\mathfrak{I} = (I_\bullet, (K_i), (\tilde{K}_i), (K_i'))$ and $\hat{\mathfrak{I}} = (I_\bullet, (\hat{K}_i), (\hat{\tilde{K}}_i), (\hat{K'}_i))$ be two types of cuirasses having the same underlying simplicial set. We write $\mathfrak{I} \Subset \hat{\mathfrak{I}}$, if $K_i \subset \mathring{\hat{K}}_i$, $\tilde{K}_i \subset \hat{\tilde{K}}_i$ and $K_i' \subset \hat{K'}_i$.

Definition 4.38. ([Dou74], p. 594) Let X be a Banach analytic space proper and anaflat over S. Let $q : S \to \mathcal{Q}_S(\mathfrak{I}; X)$ be a relative cuirasse. We say that q is *extendable* if there exists a type of cuirasse $\hat{\mathfrak{I}}$ and a relative cuirasse $\hat{q} : S \to \mathcal{Q}_S(\hat{\mathfrak{I}}; X)$ such that

1. $\mathfrak{I} \Subset \hat{\mathfrak{I}}$;
2. $Y_i = K_i \cap \hat{Y}_i$, for any $i \in I$;
3. $f_i = \hat{f}_i|_{Y_i}$, for any $i \in I$.

If $\mathfrak{I} \Subset \hat{\mathfrak{I}}$ are two types of cuirasses, then we can construct the spaces of puzzles $\mathfrak{Z}$ and $\hat{\mathfrak{Z}}$ of type $\mathfrak{I}$ and $\hat{\mathfrak{I}}$ respectively (see Definition 4.24). We get the restriction morphism

$$\begin{aligned} j : \hat{\mathfrak{Z}} &\to \mathfrak{Z} \\ ((\hat{Y}_i), (\hat{g}_i^j)) &\mapsto ((\hat{Y}_i \cap K_i), (\hat{g}_i^j|_{\hat{Y}_j \cap K_j})) \end{aligned}. \tag{4.25}$$

One can show (see [Dou74], p. 595) that j is compact in the sense of Definition 3.32. Every compact complex space X_0 admits an extendable triangularly privileged cuirasse $q_0 \in \mathcal{Q}(\mathfrak{I}; X_0)$, whose extension $\hat{q}_0 \in \mathcal{Q}(\mathfrak{I}; X_0)$ is triangularly privileged. Hence, let q_0 be an extendable triangularly privileged cuirasse on X_0 admitting a triangularly privileged extension $\hat{q}_0$ (see [Dou74], p. 595). Since the projection $\pi : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$ is smooth in a neighborhood of q_0 (see Corollary 4.21), we can find a local relative cuirasse

$$\sigma : \mathfrak{Z} \to \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \tag{4.26}$$

on $\mathfrak{X}$ over $\mathfrak{Z}$ such that $\sigma(z_{q_0}) = (z_{q_0}, q_0)$. Moreover, since $\hat{q}_0$ is also triangularly privileged, we can find a local relative $\hat{\sigma} : \hat{\mathfrak{Z}} \to \mathcal{Q}_{\hat{\mathfrak{Z}}}(\mathfrak{I}, \mathfrak{X})$ on $\mathfrak{X}$ over $\hat{\mathfrak{Z}}$, which extends σ (see Definition 4.38). Let us choose a local trivialization

$$\rho : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z} \times Q_0 \tag{4.27}$$

of the form $\rho = (\pi, p)$, where $\pi : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$ is the projection and

$$p : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to Q_0 \tag{4.28}$$

is a morphism such that

$$p \circ \sigma = \{q_0\}. \tag{4.29}$$

Moreover, let $\alpha_{q_0} : \mathfrak{X}(z_{q_0}) \to X_0$ be the isomorphism given by Proposition 4.32. We naturally get an induced isomorphism $\alpha_{Q_0} : Q_0 \to \mathcal{Q}(\mathfrak{X}(z_{q_0}))$. We choose p so that

$$p|_{\mathcal{Q}(\mathfrak{X}(z_{q_0}))} = \alpha_{q_0}^{-1}. \tag{4.30}$$

Proposition 4.39. *([Dou74], p. 596 and [LP75], p. 269) The morphism $p|_Z : Z \to Q_0$ is of relative finite dimension in a neighborhood of (z_{q_0}, q_0) (see Definition 3.36).*

Proof. Let $\varphi_{\mathfrak{q}}$ be as in (4.5), $\pi : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$ the canonical projection, σ the extendable relative cuirasse on $\mathfrak{X}$ over $\mathfrak{Z}$ chosen in (4.26) and $p : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to Q_0$ the morphism chosen in (4.28). We can construct two morphisms

$$\begin{aligned}(p,\pi)\circ\sigma : \mathfrak{Z} &\to \mathfrak{Z}\times\{q_0\}\\ z &\mapsto (z, q_0)\end{aligned} \tag{4.31}$$

and

$$\begin{aligned}(p,\varphi_{\mathfrak{q}})\circ\sigma : \mathfrak{Z} &\to \mathfrak{Z}\times\{q_0\}\\ z &\mapsto (z_{\sigma(z)}, q_0).\end{aligned} \tag{4.32}$$

On one hand we have that $(p,\pi)\circ\sigma$ is nothing but the identity map $Id_{\mathfrak{Z}}$ on $\mathfrak{Z}$. On the other hand, since σ is extendable (see Definition 4.38), the morphism $\varphi_{\mathfrak{q}}\circ\sigma$ factors through the restriction morphism $j : \mathfrak{Z}^{\hat{\mathfrak{I}}} \to \mathfrak{Z}$ (4.25), with $\mathfrak{I} \Subset \hat{\mathfrak{I}}$, which is a compact morphism. Hence, $\varphi_{\mathfrak{q}}\circ\sigma$ is compact. By Proposition 3.33, we get that

$$T := \ker((p,\varphi_{\mathfrak{q}})\circ\sigma, Id_{\mathfrak{Z}}) \subset \mathfrak{Z}$$

is of finite dimension. From (4.29), we get $\sigma(T) = (p,\pi)^{-1}(\{q_0\}\times\mathfrak{Z}) = p^{-1}(q_0)$. Since $Z = \ker((p\circ\varphi_{\mathfrak{q}}),(p\circ\pi))$, we have $\sigma(T) = p|_Z^{-1}(q_0)$. Hence, we have that the fibre $p|_Z^{-1}(q_0)$ of $p|_Z : Z \to Q_0$ over q_0 is of finite dimension. Using Proposition 3.37, the statement follows. $\square$

Thus, we get the existence of an embedding $\iota : Z \hookrightarrow Q_0\times\mathbb{C}^m$ making the following diagram commutative

$$\begin{tikzcd} Z \arrow[rr, "\iota"] \arrow[dr, "p"'] & & Q_0\times\mathbb{C}^m \arrow[dl, "\pi_1"] \\ & Q_0 & \end{tikzcd} \quad . \tag{4.33}$$

Let $\delta : Q_0 \to \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X})$ given by (4.24). We can draw the following diagram

$$\begin{tikzcd} Q_0 \arrow[r, "\delta"] & Z \arrow[rr, "\iota"] \arrow[dr, "p"'] & & Q_0\times\mathbb{C}^m \arrow[dl, "\pi_1"] \\ & & Q_0 & \end{tikzcd} \quad . \tag{4.34}$$

Proposition 4.40. *([Dou74], p. 595 and [Sti88], p. 40) The linear tangent map:*

$$T_{q_0}(p \circ \delta) : T_{q_0}Q_0 \to T_{q_0}Q_0$$

is of the form $1 - v$*, where* v *is compact.*

Proof. Let $Q_0 \times X_0 \to Q_0$ be the trivial deformation of X_0 over Q_0. The triangularly privileged cuirasse q_0 on X_0 induces a relative cuirasse $\mathfrak{q}_0 : Q_0 \to \mathcal{Q}_{Q_0}(Q_0 \times X_0) = Q_0 \times Q_0$ on $Q_0 \times X_0$ over Q_0 via $q \mapsto (q, q_0)$. Let $\underline{\beta} : Q_0 \times X_0 \to \mathfrak{X}$ be the morphism induced by the morphism $\varphi_{\mathfrak{q}_0} : Q_0 \to \mathfrak{Z}$ associated to $\mathfrak{q}_0$ (see Definition 4.31). We have the following commutative diagram

$$\begin{tikzcd} Q_0 \times X_0 \arrow[r, "\underline{\beta}"] \arrow[d, "\pi_1"'] & \mathfrak{X} \arrow[d] \\ Q_0 \arrow[r, "\varphi_{\mathfrak{q}_0}"] & \mathfrak{Z} \end{tikzcd}. \tag{4.35}$$

Let $\rho := (\pi, p) : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z} \times Q_0$ be the local trivialization chosen in (4.27). From (4.35), we get the following induced commutative diagram

$$\begin{tikzcd} & \mathfrak{Z} \times Q_0 & \\ Q_0 \times Q_0 \arrow[ur, "(\varphi_{\mathfrak{q}_0}, \tilde{p})"] \arrow[rr, "\beta"] \arrow[d, "\pi_1"'] & & \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \arrow[ul, "(\pi, p)"'] \arrow[d, "\pi"] \\ Q_0 \arrow[rr, "\varphi_{\mathfrak{q}_0}"] & & \mathfrak{Z} \end{tikzcd}. \tag{4.36}$$

where $\tilde{p} := p \circ \beta$. Let $\sigma : \mathfrak{Z} \to \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X})$ be the extendable relative cuirasse chosen in (4.26). Set $\tau := \varphi_{\mathfrak{q}_0}^* \sigma : Q_0 \to Q_0 \times Q_0$. Since σ is a section of the projection $\pi : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}) \to \mathfrak{Z}$, we get that τ is of the form (Id_{Q_0}, θ). Moreover, since σ is extendable, $T_{q_0}\theta$ is a compact morphism (see Definition 3.32). Indeed, θ is of the form $j \circ \hat{\theta}$, where $\hat{\theta} : Q_0 \to \mathcal{Q}(\hat{\mathfrak{I}}; X_0)$, with $\mathfrak{I} \Subset \hat{\mathfrak{I}}$, and $j : \mathcal{Q}(\hat{\mathfrak{I}}; X_0) \to Q_0$ is the restriction morphism, which is a compact morphism (see [Dou74], p. 595). Because $\tau = \varphi_{\mathfrak{q}_0}^* \sigma$, we get that

$$p \circ \sigma \circ \varphi_{\mathfrak{q}_0} = \tilde{p} \circ \tau.$$

Hence, using (4.29), we get that

$$T_{q_0}(\tilde{p} \circ \tau) = 0. \tag{4.37}$$

Let $i_1 : Q_0 \to Q_0 \times Q_0$ and $i_2 : Q_0 \to Q_0 \times Q_0$ be the injections defined by $q \mapsto (q, q_0)$ and $q \mapsto (q_0, q)$ respectively. Then, since $\tau = (Id_{Q_0}, \theta)$, we get

$$T_{q_0}(\tilde{p} \circ \tau) = T_{q_0}(\tilde{p} \circ (Id_{Q_0}, \theta)) = T_{q_0}(\tilde{p} \circ i_1) + T_{q_0}(\tilde{p} \circ i_2) \circ T_{q_0}\theta. \tag{4.38}$$

From (4.30), we obtain

$$\tilde{p} \circ i_2 = p \circ \beta \circ i_2 = Id_{Q_0}. \tag{4.39}$$

Thus, since $T_{q_0}(\tilde{p} \circ \tau) = 0$ by (4.37), from (4.38), using (4.39), we get

$$T_{q_0}(\tilde{p} \circ i_1) = -T_{q_0}\theta. \tag{4.40}$$

Let $\psi_{\mathfrak{q}_0} : Q_0 \to Z$ be the morphism associated to $\mathfrak{q}_0$ (see (4.8)). From the definition of $\delta : Q_0 \to \mathcal{Q}_3(\mathfrak{X})$ (see (4.24)), we get that $\delta = \psi_{\mathfrak{q}_0}$. Let $\triangle : Q_0 \to Q_0 \times Q_0$, $q \mapsto (q, q)$, be the diagonal morphism. From the commutative diagram (4.36) and from $\pi \circ \psi_{\mathfrak{q}_0} = \varphi_{\mathfrak{q}_0}$ (4.8), we get $\beta \circ \triangle = \delta$. Hence,

$$p \circ \delta = p \circ \beta \circ \triangle = \tilde{p} \circ \triangle. \tag{4.41}$$

Thus, using (4.41), (4.40) and (4.39), we get

$$T_{q_0}(p \circ \delta) = T_{q_0}(\tilde{p} \circ \triangle) = T_{q_0}(\tilde{p} \circ i_1) + T_{q_0}(\tilde{p} \circ i_2) = -T_{q_0}\theta + Id_{Q_0}.$$

□

From Proposition 4.40, we get that $\ker T_{q_0}(p \circ \delta)$ is a finite dimensional direct subspace of $T_{q_0}Q_0$. By Proposition 4.39

$$\ker T_{q_0}(p \circ \delta) \supset \ker T_{q_0}\delta = \ker T_{q_0}(\iota \circ \delta),$$

hence we have that $\ker T_{q_0}(\iota \circ \delta)$ is of finite dimension. Moreover, from Proposition 4.40 we have that $\operatorname{Im} T_{q_0}(p \circ \delta)$ has finite codimension in $T_{(z_{q_0}, q_0)}Z$. Therefore, using again Proposition 4.39, we get that $\operatorname{Im} T_{q_0}(\iota \circ \delta)$ is of finite codimension. We identify Z with its image in $Q_0 \times \mathbb{C}^m$ under ι. Denote with Σ the subspace of Q_0 such that

$$T_{q_0}\Sigma \oplus \ker T_{q_0}\delta = T_{q_0}Q_0.$$

We choose a retraction $r : Q_0 \times \mathbb{C}^m \to \delta(\Sigma)$ and we set $R := r^{-1}(q_0) \cap Z$.

Lemma 4.41. *([Dou74], p. 598) Let Σ_1 be an open subset of a Banach space, H an open subset of $\mathbb{C}^n$, R an analytic subspace of H and Z an analytic subspace of $\Sigma_1 \times H$. Furthermore, assume $0 \in H$, $0 \in \Sigma_1$, $\Sigma_1 \times R \subset Z$ and $Z \cap (\{0\} \times H) = R$. Then, $Z = \Sigma_1 \times R$ on a neighborhood of 0.*

Proposition 4.42. *([Dou74], p. 597 and [Sti88], p. 38) The morphism $\omega|_{R\times\Sigma} : R \times \Sigma \to Z$ (see Definition 4.21) is an isomorphism.*

Proof. Let $\tilde{\omega} : R \times \delta(\Sigma) \to Z$ defined by $\tilde{\omega} := \omega \circ (id \times \delta^{-1})$. From (4.22) and (4.24) it follows that $\tilde{\omega}$ restricted to each factor is the identity map. Therefore, using Lemma 4.41, the statement follows. $\square$

Let $i : R \hookrightarrow Z$ be the inclusion and set $\mathfrak{X}_R := i^*\mathfrak{X}_Z$. Let $\alpha_{q_0} : \mathfrak{X}_R(r_0) \to X_0$ be the isomorphism given by Proposition 4.32.

Theorem 4.43. *([Dou74], p. 598 and p. 601; [Sti88], p. 39) The morphism $\mathfrak{X}_R \to (R, r_0)$, endowed with the isomorphism $\alpha_{q_0} : \mathfrak{X}_R(r_0) \to X_0$, is a versal deformation of X_0.*

Proof. Let $((S, s_0), X, i)$ be a deformation of X_0. Since $(\mathfrak{X}_Z \to (Z, (z_{q_0}, q_0)))$ is complete (Proposition 4.34), there exists a morphism $\psi : S \to Z$ such that X is isomorphic to $\psi^*\mathfrak{X}_Z$. Let $\pi_R : R \times \Sigma \to R$ and $\pi_\Sigma : R \times \Sigma \to \Sigma$ be the projections. Since $\omega|_{R\times\Sigma}$ is an isomorphism (see Proposition 4.42), we can set $g := \pi_R \circ \omega|_{R\times\Sigma}^{-1} \circ \psi$ and $h := \pi_\Sigma \circ \omega|_{R\times\Sigma}^{-1} \circ \psi$. Then we have $\omega \circ (g, h) = \psi$ and by Proposition 4.36, we get $g^*\mathfrak{X}_R \simeq \psi^*\mathfrak{X}_Z \simeq X$. Hence, we get completeness. Moreover, by construction, $T_0R = \mathrm{Ex}^1(X_0)$. Hence, our complete deformation is also effective. Now, let $((S, s_0), X, i)$ be a deformation of X_0 and (S', s_0) a subgerm of (S, s_0). Let $h' : (S', s_0) \to (R, r_0)$ such that $X|_{S'} \simeq h'^*\mathfrak{X}_R$. Let $\mathfrak{q}$ be the canonical relative cuirasse on $\mathfrak{X}_Z$ over Z (see (4.4)). Then, $h'^*\mathfrak{q}$ is a relative cuirasse on $X|_{S'}$ over S', whose associated morphism (see (4.8)) coincides with h'. Since, by Corollary 4.21, $\mathcal{Q}_S(X)$ is smooth over S in a neighborhood of $q_0 \in \mathcal{Q}(X(s_0))$, there exists a relative cuirasse q on X over S, such that $q|_{S'} = h'^*\mathfrak{q}$. Let $\tilde{h} : S \to Z$ be the morphism associated to q (see (4.8)) and $\pi_R : Z \to R$ the projection. Then, $h := \pi_R \circ \tilde{h}$ satisfies $X \simeq h^*\mathfrak{X}_R$ and $h|_{S'} = h'$. $\square$

The existence of a versal deformation of a morphism of compact complex spaces (see [Pal90], p. 163) follows naturally from Douady's results. We outline the proof of such existence following [Fle79], p. 130. Let $f_0 : X_0 \to Y_0$ be a holomorphic map between compact complex spaces. Let $(p : \mathfrak{X} \to (S, s_0), i)$ be a versal deformation of X_0 and $(q : \mathcal{Y} \to (T, t_0), i')$ a versal deformation of Y_0 given by Theorem 4.43. We can make the two deformations over the same base $(S \times T, (s_0, t_0))$ taking the products $\mathfrak{X}_T := \mathfrak{X} \times T$ and $\mathcal{Y}_S := \mathcal{Y} \times S$. From Theorem 3.82, we get a universal complex space

$$\mathfrak{M} := \mathfrak{Mor}_{S\times T}(\mathfrak{X}_T, \mathcal{Y}_S)$$

and a $\mathfrak{M}$-morphism $m : \mathfrak{X}_\mathfrak{M} \to \mathcal{Y}_\mathfrak{M}$, where $\mathfrak{X}_\mathfrak{M} := \mathfrak{X}_T \times_{(S\times T)} \mathfrak{M}$ and $\mathcal{Y}_\mathfrak{M} := \mathcal{Y}_S \times_{(S\times T)} \mathfrak{M}$. Using the universal property in Theorem 3.82, we immediatly get the following result.

Corollary 4.44. *([Fle79], p. 130) The $\mathfrak{M}$-morphism $m : \mathfrak{X}_\mathfrak{M} \to \mathcal{Y}_\mathfrak{M}$ is a versal deformation of $f_0 : X_0 \to Y_0$.*

Chapter 5

Versal deformation of compact log complex spaces

We extend Douady's result (Theorem 4.43) to the case of compact complex analytic spaces endowed with a logarithmic structure (see Section 2). We first show that a compact complex space endowed with a divisorial log structure (see Example 2.18) admits a versal deformation. Then we consider the case of a compact complex space endowed with a fine log structure (see Definition 2.27). Before starting, we notice that the notion of log structure (see Section 2) can be naturally extended to the category of Banach analytic spaces. Indeed, let (X, Φ) be a Banach analytic space (see Definition 3.16). Setting $\mathcal{O}_X := \Phi(\mathbb{C})$, we get a ringed space $(X, \mathcal{O}_X)$.

Definition 5.1. A *pre-log structure* on a Banach analytic space (X, Φ) is a sheaf of monoids $\mathcal{M}_X$ on X together with a homomorphism of sheaves of monoids:

$$\alpha_X : \mathcal{M}_X \to \mathcal{O}_X,$$

where the monoid structure on $\mathcal{O}_X$ is given by multiplication. A pre-log structure is a called a *log structure* if

$$\alpha_X : \alpha_X^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$$

is an isomorphism.

The notion of fine log structure (see Chapter 2) extends naturally to the Banach analytic setting. Let $(S, \mathcal{O}_S^\times)$ be a log Banach analytic space endowed with the trivial log structure. In what follows, for the sake of readability, we shall mostly write just S instead of $(S, \mathcal{O}_S^\times)$.

5.1 Versal deformation of divisorial log structures

Let X_0 be a compact complex space and $D \subset X_0$ a divisor. We recall that a divisor D in X_0 is a closed complex analytic subspace of codimension 1. Let $i_0 : D \hookrightarrow X_0$ be the canonical embedding. From Corollary 4.44, we obtain a versal deformation of i_0:

$$\begin{tikzcd} \mathfrak{D}_{\mathfrak{M}} \arrow[rr, "i"] \arrow[dr] & & \mathfrak{X}_{\mathfrak{M}} \arrow[dl] \\ & (\mathfrak{M}, m_0) & \end{tikzcd},$$

where $i(m_0) = i_0$. We only need to show that i, restricted to a neighborhood of m_0, is an embedding. This immediately follows from:

Proposition 5.2. *(see [Dou66], p. 86) Let X and Y be complex spaces proper and flat over S and $f : X \to Y$ a S-morphism. The set S' of $s \in S$ such that $f(s) : X(s) \to Y(s)$ is an embedding is open in S and f induces an embedding of $X_{S'}$ into $Y_{S'}$.*

Proof. Let $s \in S$ such that $f(s)$ is an embedding. Then $f(s)$ is a finite holomorphic map and hence f is finite in a neighborhood of $X(s)$, which we can assume to be of the form $X|_{S_1}$, with S_1 a neighborhood of s in S. The morphism f induces a finite and proper morphism of $X|_{S_1}$ in $Y|_{S_1}$ and it defines on $Y|_{S_1}$ a morphism $\tilde{f}$ of $\mathcal{O}_Y$ in the coherent sheaf $f_*\mathcal{O}_X$. The morphism $\tilde{f}(s)$ is surjective. Let $\mathcal{F} :=$ coker $\tilde{f}$, we have $\mathcal{F}(s) = 0$. Then, by Nakayama's Lemma (see for instance [Mat89]), $\mathcal{F} = 0$ in a neighborhood of $Y(s)$, which we can assume of the form $Y|_{S_2}$, where S_2 is a neighborhood of s in S. Thus, f induces an embedding of $X|_{S_2}$ into $Y|_{S_2}$. □

Example 5.3. Let $\mathbb{P}^1$ be endowed with the divisorial log structure induced by a point $\{pt\} \in \mathbb{P}^1$. Since a point and $\mathbb{P}^1$ are rigid, we have that the base of the versal deformation of $\{pt\} \hookrightarrow \mathbb{P}^1$ is $\mathfrak{Mor}(\{pt\}, \mathbb{P}^1) = \mathbb{P}^1$. Hence, we get the following commutative diagram

$$\begin{tikzcd} \{pt\} \times \mathbb{P}^1 \arrow[rr, "m"] \arrow[dr, "Id"'] & & \mathbb{P}^1 \times \mathbb{P}^1 \arrow[dl, "\pi_2"] \\ & (\mathbb{P}^1, \{pt\}) & \end{tikzcd},$$

where $m : \{pt\} \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ is the closed embedding given by $m(pt, x) = (x, x)$.

5.2 Versal deformation of compact fine log complex spaces

In what follows, we construct a versal deformation in the general case of a compact complex space X_0 endowed with a fine log structure $\mathcal{M}_{X_0}$ (see Definition 0.1). From now on, we always assume that $\overline{\mathcal{M}}^{gp}_{X_0}$ (see Definition 2.13) is torsion free.

Definition 5.4. Let us assume $(X_0, \mathcal{M}_{X_0})$ admits a directed collection of log charts $(\theta_{i,0} : P_i \to \mathcal{M}_{U_i})_{i\in J}$, with comparison morphisms $(\varphi^i_{j,0} := (\phi^i_{j,0}, \eta^i_{j,0}) : P_i \to P_j \oplus \mathcal{O}^\times_{U_j})_{j\in J_1\cup J_2, i\in\partial j}$ (see Definition 2.32). We set

$$l_0 := ((\theta_{i,0})_{i\in J}, (\eta^i_{j,0})_{j\in J_1\cup J_2, i\in\partial j}).$$

5.2.1 Infinite dimensional construction

Let

$$\mathfrak{I} := (I_\bullet, (K_i)_{i\in I}, (\tilde{K}_i)_{i\in I}, (K'_i)_{i\in I_0\cup I_1})$$

be a type of cuirasse (see Definition 4.8). We assume that the index sets I and J (Definition 5.4) coincide. Let $\mathfrak{Z}$ be the space of puzzles of type $\mathfrak{I}$ (see Definition 4.22).

Definition 5.5. A *log puzzle* is a pair (z, l), where $z := (Y_i, g^i_j) \in \mathfrak{Z}$ is a puzzle and l is a collection of data

$$((\beta_i : P_i \to \mathcal{O}_{Y^\circ_i})_{i\in I}, (\eta^i_j : P_i \to \mathcal{O}^\times_{Y^\circ_j})_{j\in I_1\cup I_2, i\in\partial j}),$$

satisfying

$$\Big((\beta_j \cdot \mathrm{id}_{\mathcal{O}^\times_{Y^\circ_j}}) \circ \varphi^i_j = \beta_i|_{Y^\circ_j}\Big)_{j\in I_1\cup I_2, i\in\partial j}, \tag{5.1}$$

where $\varphi^i_j := (\phi^i_{j,0}, \eta^i_j)$, with $\phi^i_{j,0} : P_i \to P_j$ given by Definition 5.4.

Definition 5.6. We denote the set of log puzzles by $\mathfrak{Z}^{\log}$.

The set of log puzzles $\mathfrak{Z}^{\log}$ can be endowed with a Banach analytic structure. Indeed, let $\underline{Y}_i$ be the universal $\mathcal{G}(K_i)$-anaflat subspace of $\mathcal{G}(K_i) \times K_i$, for $i \in I$ (see Definition 3.52). Let us consider the Banach analytic space

$$\mathfrak{M} := \prod_{i\in I}{}_{\mathfrak{Z}} \mathfrak{Mor}_{\mathfrak{Z}}(\underline{Y}_i, \operatorname{Spec}\mathbb{C}[P_i] \times \mathfrak{Z}) \times_{\mathfrak{Z}} \prod_{j\in I_1\cup I_2}{}_{\mathfrak{Z}} \prod_{i\in\partial j}{}_{\mathfrak{Z}} \mathfrak{Mor}_{\mathfrak{Z}}(\underline{Y}_j, \operatorname{Spec}\mathbb{C}[P^{gp}_i] \times \mathfrak{Z}).$$

Each morphism

$$(\eta^i_j(z))^{gp} : P^{gp}_i \to \mathcal{O}^\times_{\underline{Y}^\circ_j(z)}$$

induces a morphism $\eta^i_j(z) : P_i \to \mathcal{O}^\times_{\underline{Y}^\circ_j(z)}$. Hence, a point in $\mathfrak{M}$ can be written as

$$(z, (\beta_i(z) : P_i \to \mathcal{O}_{\underline{Y}^\circ_i(z)})_{i\in I}, (\eta^i_j(z) : P_i \to \mathcal{O}^\times_{\underline{Y}^\circ_j(z)})_{j\in I_1\cup I_2, i\in\partial j}).$$

We naturally get an injective map

$$\rho : \mathfrak{Z}^{\log} \to \prod_{i\in I}{}_{\mathfrak{Z}} \mathfrak{Mor}_{\mathfrak{Z}}(\underline{Y}_i, \operatorname{Spec}\mathbb{C}[P_i] \times \mathfrak{Z}) \times_{\mathfrak{Z}} \prod_{j\in I_1\cup I_2}{}_{\mathfrak{Z}} \prod_{i\in\partial j}{}_{\mathfrak{Z}} \mathfrak{Mor}_{\mathfrak{Z}}(\underline{Y}_j, \operatorname{Spec}\mathbb{C}[P^{gp}_i] \times \mathfrak{Z}). \tag{5.2}$$

Proposition 5.7. *The universal space of log puzzles $\mathfrak{Z}^{\log}$ is Banach analytic.*

Proof. Let $(\phi^i_{j,0} : P_i \to P_j)_{j\in I_1\cup I_2, i\in\partial j}$ given by Definition 5.4, we set

$$\varphi^i_j(z) := (\phi^i_{j,0}, \eta^i_j(z)) : P_i \to P_j \oplus \mathcal{O}^\times_{\underline{Y}^\circ_j(z)}.$$

The subset $\rho(\mathfrak{Z}^{\log})$ of $\mathfrak{M}$ (5.2) is defined by the equations

$$\Big((\beta_j(z)\cdot \mathrm{id}_{\mathcal{O}^\times_{\underline{Y}^\circ_j(z)}}) \circ \varphi^i_j(z) = \beta_i(z)|_{\underline{Y}^\circ_j(z)}\Big)_{j\in I_1\cup I_2, i\in\partial j}. \tag{5.3}$$

Thus, we define a double arrow

$$(\rho_1,\rho_2) : \mathfrak{M} \rightrightarrows \prod_{j\in I_1\cup I_2}{}_{\mathfrak{Z}} \mathfrak{Mor}_{\mathfrak{Z}}(\underline{Y}_j, \operatorname{Spec}\mathbb{C}[P_j]\times\mathfrak{Z})$$

by

$$\rho_1 : (z, (\beta_i(z)), (\eta^i_j(z))) \mapsto (z, (\beta_i(z)|_{\underline{Y}^\circ_j(z)}))$$

and

$$\rho_2 : (z, (\beta_i(z)), (\eta^i_j(z))) \mapsto (z, ((\beta_j(z)\cdot \mathrm{id}_{\mathcal{O}^\times_{\underline{Y}^\circ_j(z)}}) \circ \varphi^i_j(z))).$$

Then $\mathfrak{Z}^{\log}$ is given by the kernel of the double arrow defined by ρ_1 and ρ_2:

$$\mathfrak{Z}^{\log} = \ker(\rho_1,\rho_2).$$

□

Let $p : \mathfrak{Z}^{\log} \to \mathfrak{Z}$ be the canonical projection. Consider the Banach analytic space $\mathfrak{X}_{\log} := p^*\mathfrak{X}$ over $\mathfrak{Z}^{\log}$.

Proposition 5.8. *The Banach analytic space $\mathfrak{X}_{\log}$ comes naturally endowed with a fine log structure $\mathcal{M}_{\mathfrak{X}_{\log}}$.*

Proof. By Theorem 3.83, we have universal morphisms

$$\begin{array}{c}(\underline{\beta}_i : P_i \to \mathcal{O}_{p^*\underline{Y}^\circ_i})_{i\in I}\\ (\underline{\varphi}^i_j := (\phi^i_{j,0}, \underline{\eta}^i_j) : P_i \to P_j \oplus \mathcal{O}^\times_{p^*\underline{Y}^\circ_j})_{j\in I_1\cup I_2, i\in\partial j}\end{array}. \tag{5.4}$$

By construction, they satisfy

$$\Big((\underline{\beta}_j\cdot \mathrm{id}_{\mathcal{O}^\times_{\underline{Y}^\circ_j}}) \circ \underline{\varphi}^i_j = \underline{\beta}_i|_{\underline{Y}^\circ_j}\Big)_{j\in I_1\cup I_2, i\in\partial j}.$$

On the other hand, we have that the space $\mathfrak{X}$ is canonically isomorphic to

$$\coprod_{i\in I_0} \underline{Y}'_i/\mathcal{R},$$

where $\mathcal{R}(x,x^{'})$ if $x \in \underline{Y}'_i$ and $x' \in \underline{Y}'_{i'}$ are such that there exists $j \in I_1$ and $y \in \underline{Y}'_j$ with $dj = (i,i')$, $\underline{g}^j_i(y) = x$ and $\underline{g}^j_{i'}(y) = x'$ (see Remark 4.29). Therefore, $\mathfrak{X}_{\log}$ is canonically isomorphic to

$$\coprod_{i\in I_0} p^*\underline{Y}'_i/\mathcal{R}.$$

Hence, the collection of universal morphisms $((\underline{\beta}_i),(\underline{\varphi}^i_j))$ defines a pre-log atlas (see Definition 2.33) on $\mathfrak{X}_{\log}$, which glues to a fine log structure $\mathcal{M}_{X_{\log}}$ on $\mathfrak{X}_{\log}$ (see Section 2.4). □

We show that the universal family of log puzzles $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$ gives a *complete* deformation of $(X_0, \mathcal{M}_{X_0})$. To do that, we introduce the notion of *log cuirasse.* We recall (see Chapter 4, Section 4.1) that if S is a Banach analytic space, X a Banach analytic space proper and anaflat over S and q a relative cuirasse on X, then we get a morphism $\varphi_q : S \to \mathfrak{Z}$ (see Definition 4.31), a Banach analytic space X_{φ_q} over S obtained by gluing the pieces of the puzzle z_q associated to q (see Propositions see Proposition 4.26 and 4.30), and a S-isomporphism $\alpha_q : X_{\varphi_q} \to X$ (Proposition 4.32). Now, let $(X_0, \mathcal{M}_{X_0})$ be a compact fine log complex space admitting a collection of directed log charts $((\theta_i : P_i \to \mathcal{M}_{U_i})_{i\in I}, (\varphi^i_j := (\phi^i_j, \eta^i_j) : P_i \to P_j \oplus \mathcal{O}^{\times}_{U_j})_{j\in I_1\cup I_2, i\in\partial j})$ (see Definition 2.32). We assume that ϕ^i_j coincide with the $\phi^i_{j,0}$ given by Definition 5.4. Let $q_0 \in \mathcal{Q}(\mathfrak{I}; X_0)$ be a cuirasse on X_0. By Proposition 4.32, we get an isomorphism

$$\alpha_{q_0} : X_{\varphi_{q_0}} \to X_0.$$

Definition 5.9. We naturally get a fine log structure on $X_{\varphi_{q_0}}$ via $\mathcal{M}_{X_{\varphi_{q_0}}} := \alpha^*_{q_0}\mathcal{M}_{X_0}$.

Definition 5.10. A *log cuirasse* $q^{\dagger}_0$ on $(X_0, \mathcal{M}_{X_0})$ is a pair given by a cuirasse $q_0 = (Y_i, f_i)_{i\in I}$ on X_0 and a directed collection of log charts $((\theta_i : P_i \to \mathcal{M}_{X_{\varphi_{q_0}}}|_{Y^{\circ}_i}), (\eta^i_j : P_i \to O^{\times}_{Y^{\circ}_j}))$ on $(X_{\varphi_{q_0}}, \mathcal{M}_{X_{\varphi_{q_0}}})$ (see Definition 2.32). We denote the set of log cuirasses on $(X_0, \mathcal{M}_{X_0})$ by $\mathcal{Q}(X_0, \mathcal{M}_{X_0})$.

Remark 5.11. In Definition 5.10 we need to give the set of comparison morphisms (η^i_j) in order to define, in Definition 5.20, the *log puzzle associated to a log cuirasse.*

Analogously to the classical case (see Definitin 4.13), we can define the notion of relative log cuirasse. Let S be a Banach analytic space and $(X, \mathcal{M}_X)$ a fine log Banach analytic space proper and anaflat over S. Given the local nature of the problem, we can assume that $(X, \mathcal{M}_X)$ can be covered by finitely many log charts $(\theta_i : P_i \to \mathcal{M}_{U_i})_{i\in I}$ such that $U_i \cap X(s) \neq \emptyset$, for each $i \in I$ and $s \in S$.

Definition 5.12. Let S be a Banach analytic space and $(X, \mathcal{M}_X)$ a fine log Banach analytic space proper and anaflat over S. We define the set of relative log cuirasses on $(X, \mathcal{M}_X)$ over S by

$$\mathcal{Q}_S(X, \mathcal{M}_X) := \{(s,q) | s \in S, q \in \mathcal{Q}(X(s), \mathcal{M}_{X|_{X(s)}})\} = \coprod_{s\in S} \mathcal{Q}(X(s), \mathcal{M}_{X|_{X(s)}}).$$

Definition 5.13. A section $q^\dagger : S \to \mathcal{Q}_S(X, \mathcal{M}_X)$ is called a *relative log cuirasse* on $(X, \mathcal{M}_X)$.

Definition 5.14. Let $(X_0, \mathcal{M}_{X_0})$ be a compact fine log complex space. A log cuirasse $q_0^\dagger$ on $(X_0, \mathcal{M}_{X_0})$ is called *triangularly privileged* if the underlying cuirasse $q_0 \in \mathcal{Q}(\mathfrak{I}; X_0)$ on X_0 is triangularly privileged (see Definition 4.17).

Since every compact complex space X_0 admits a triangularly privileged cuirasse, we have that every compact fine log complex space $(X_0, \mathcal{M}_{X_0})$ admits a triangularly privileged log cuirasse. The set of log cuirasses can be endowed with the structure of a Banach analytic space in a neighborhood of a triangularly privileged log cuirasse. To prove it, we need the following three Lemmas.

Lemma 5.15. *Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space over a Banach analytic space S. Let $q_0 = (Y_{i,0}, f_{i,0})$ be a triangularly privileged cuirasse on the central fibre $(X_0, \mathcal{M}_{X_0})$ over $s_0 \in S$. Assume that $\Gamma(Y_{i,0}, f_{i,0}^{-1}\overline{\mathcal{M}}_{X_0})$ is globally generated, for each $i \in I$. Then there exists a local relative cuirasse $q = (Y_i, f_i)$ on X defined on a neighborhood S' of s_0 in S, such that $\Gamma(Y_i, f_i^{-1}\overline{\mathcal{M}}_X)$ is globally generated, for each $i \in I$.*

Proof. Since q_0 is triangularly privileged, there exists a local relative cuirasse $q = (Y_i, f_i)$ on X defined in a neighborhood S' of s_0 in S (see Proposition 4.20). Now, up to shrinking $Y_{i,0}$, for each $i \in I$ and for each $y \in Y_{i,0}$ there exists an open set $U_y = V_y \times W_y \subset S \times K_i$, $y \in U_y$, such that the canonical map $\Gamma(U_y, f_i^{-1}\overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_{X, f_i(y)}$ is an isomorphism. Since each Y_i is compact, we can find a finite set J and finitely many subsets U_j such that $\Gamma(\bigcup_{j \in J} U_j, f_i^{-1}\overline{\mathcal{M}}_X)$ is isomorphic to $\Gamma(Y_i, f_i^{-1}\overline{\mathcal{M}}_X)$. Hence, possibly after shrinking S', we can assume that for each $j \in J$, $V_j = S'$ and $f_i^{-1}(X) \subset \bigcup_{j \in J} U_j$. Thus, we get that for each $i \in I$, $\Gamma(Y_i, f_i^{-1}\overline{\mathcal{M}}_X)$ is isomorphic to $\Gamma(Y_{i,0}, f_{i,0}^{-1}\overline{\mathcal{M}}_X)$. $\square$

Lemma 5.16. *(see [Gra58]) Let X be a Stein space and $\mathcal{L}$ a $\mathcal{O}_X^\times$-torsor. If $c_1(\mathcal{L}) = 0$, then $\mathcal{L}$ is trivial.*

Lemma 5.17. *Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space. Assume that $P := \Gamma(X, \overline{\mathcal{M}}_X)$ is globally generated and torsion free. Assume that for each $\overline{m} \in \Gamma(X, \overline{\mathcal{M}}_X)$, the torsor $\mathcal{L}_{\overline{m}} = \kappa^{-1}(\overline{m})$, with $\kappa : \mathcal{M}_X \to \overline{\mathcal{M}_X}$ the canonical map, is trivial. Then there exists a chart $P \to \Gamma(X, \mathcal{M}_X)$.*

Proof. Let $p_1, ..., p_r \in P$ be generators, that is we have a surjective map $\mathbb{N}^r \to \Gamma(X, \overline{\mathcal{M}}_X)$ sending e_i to p_i. For each $i \in \{1, ..., r\}$, choose a section $m_i \in \mathcal{L}_{\overline{m}}$. We obtain a chart $\phi : \mathbb{N}^r \to \Gamma(X, \mathcal{M}_X)$. Now, we want to modify ϕ so that it factors through P. Let $K := \ker(\mathbb{Z}^r \to P^{gp})$, we have $P = \mathbb{N}^r/K$. We get the following exact sequence

$$0 \to K \to \mathbb{Z}^r \to P^{gp} \to 0.$$

Since, by assumption, P is torsion free, we can find a section $\pi : \mathbb{Z}^r \to K$. Set $h_i := \phi^{gp}(\pi(p_i))$. Clearly, if $\sum a_i p_i = \sum b_j p_j$, for $a_i, b_j \geq 0$, then it holds $\prod h_i^{a_i} = \prod h_j^{b_j}$ in $\Gamma(X, \mathcal{M}_X^{gp})$. We get a chart by

$$\begin{aligned} \tilde{\psi} : \mathbb{N}^r &\to \Gamma(X, \mathcal{M}_X) \\ e_i &\mapsto h_i^{-1} m_i \end{aligned} . \tag{5.5}$$

Now, let $\tilde{\psi}^{gp} : \mathbb{Z}^r \to \Gamma(X, \mathcal{M}_X^{gp})$ and $\sum a_i e_i \in K$. If $\tilde{\psi}^{gp}(\sum a_i e_i) = 1$, we get that $\tilde{\psi}^{gp}$ induces a chart $\psi : P \to \Gamma(X, \mathcal{M}_X)$. Hence, assume $\sum a_i e_i \in K$, $a_i \in \mathbb{Z}$. Then $\tilde{\psi}^{gp}(\sum a_i e_i) = \prod \tilde{\psi}^{gp}(e_i)^{a_i} = \prod h_i^{-a_i} m_i^{a_i} = \prod \phi^{gp}(\pi(p_i))^{-a_i} \phi^{gp}(e_i)^{a_i} = \phi^{gp}(\pi(-\sum a_i p_i) + \sum a_i e_i)) = 1$.

Proposition 5.18. *Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space over a Banach analytic space S. Let $s_0 \in S$ and $q_0^\dagger$ a triangularly privileged log cuirasse on $(X(s_0), \mathcal{M}_{X(s_0)})$. Then the set of log cuirasses $\mathcal{Q}_S(X, \mathcal{M}_X)$ on $(X, \mathcal{M}_X)$ over S can be endowed with the structure of a Banach analytic space in a neighborhood of $(s_0, q_0^\dagger)$.*

Proof. Let us consider the projection $\pi : \mathcal{Q}_S(X, \mathcal{M}_X) \to Q_S(X)$. By Lemma 5.15 and Lemma 5.16, we can use Lemma 5.17 and get the existence around (s_0, q_0) of a local section $\rho : Q_S(X) \to \mathcal{Q}_S(X, \mathcal{M}_X)$, such that $\rho(s_0, q_0) = q_0^\dagger$. Now, let $(s, q) \in Q_S(X)$, in a small neighborhood of (s_0, q_0), and consider $\rho(s, q) \in \mathcal{Q}_S(X, \mathcal{M}_X)$. We have that $\rho(s, q) = (s, q = (Y_i, f_i), (\theta_i), (\eta_j^i))$, where $(\theta_i), (\eta_j^i)$ is a directed collection of log charts on $(X_{\varphi_q}, \mathcal{M}_{X_{\varphi_q}})$ (see Definition 5.10). Any other directed set of log charts $((\theta_i'), (\eta_j'^i))$ on $(X_{\varphi_q}, \mathcal{M}_{X_{\varphi_q}})$ is obtained by $\theta_i' = \chi_i \cdot \theta_i$ and $\eta_j'^i = \chi_i^{-1} \cdot \chi_j \cdot \eta_j^i$, for morphisms $\chi_i : P_i \to \mathcal{O}_{Y_i^\circ}^\times$, for $i \in I$ (see Chapter 2, Section 2.4). Therefore, let $\underline{Y}_i$ be the universal $\mathcal{G}(K_i)$-anaflat subspace of $\mathcal{G}(K_i) \times K_i$, for $i \in I$ (see Definition 3.52). We can define a map

$$\gamma : \mathcal{Q}_S(X) \times_{\prod \mathcal{G}(K_i) \times S} \prod_{i \in I} {}_S \, \mathfrak{Mor}_{\mathcal{G}(K_i) \times S}(\underline{Y}_i \times S, \operatorname{Spec} \mathbb{C}[P_i^{gp}] \times \mathcal{G}(K_i) \times S) \to \mathcal{Q}_S(X, \mathcal{M}_X)$$

via

$$(s, q, (\chi_i)) \mapsto (s, q, (\chi_i \cdot \theta_i), (\chi_i^{-1} \cdot \chi_j \cdot \eta_j^i)),$$

which defines a structure of Banach analytic space on $\mathcal{Q}_S(X, \mathcal{M}_X)$ in a neighborhood of $(s_0, q_0^\dagger)$.

□

Proposition 5.19. *Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space proper and anaflat over a Banach analytic space S. Let $s \in S$. Let $q^\dagger(s)$ be a triangularly privileged log cuirasse on $(X(s), \mathcal{M}_{X(s)})$. Then*

$$\pi : \mathcal{Q}_S(X, \mathcal{M}_X) \to S$$

is smooth in a neighborhood of $q^\dagger(s)$.

Proof. Let $q^\dagger(s)$ be a triangularly privileged log cuirasse on $(X(s), \mathcal{M}_{X(s)})$. By Proposition 5.18, we have that in a neighborhood of $(s, q^\dagger(s))$, the space $\mathcal{Q}_S(X, \mathcal{M}_X)$ is isomorphic to

$$\mathcal{Q}_S(X) \times_{\prod \mathcal{G}(K_i)\times S} \prod_{i\in I}{}_S \mathfrak{Mor}_{\mathcal{G}(K_i)\times S}(\underline{Y}_i \times S, \operatorname{Spec}\mathbb{C}[P_i^{gp}] \times \mathcal{G}(K_i) \times S).$$

Let $q(s) = (Y_i, f_i)$ be the triangularly privileged cuirasse on $X(s)$ underlying $q^\dagger(s)$. By Corollary 4.21, $\pi : \mathcal{Q}_S(X) \to S$ is smooth in a neighborhood of $(s, q(s))$. Furthermore, by Proposition 4.20, we have that

$$\mathfrak{Mor}_{\mathcal{G}(K_i)\times S}(\underline{Y}_i \times S, \operatorname{Spec}\mathbb{C}[P_i^{gp}] \times \mathcal{G}(K_i) \times S) \to S$$

is smooth in a neighborhood of (s, Y_i, χ_i). Hence, the statement follows. □

Analogously to the classical case (see Proposition 4.30), we are going to define the notion of log puzzle *associated* to a log cuirasse. Let $(X_0, \mathcal{M}_{X_0})$ be a compact fine log complex space. Let q_0 be a cuirasse on X_0. By Definition 5.9, we get a compact fine log complex space $(X_{\varphi_{q_0}}, \mathcal{M}_{X_{\varphi_{q_0}}})$, which is isomorphic to $(X_0, \mathcal{M}_{X_0})$. Let $q_0^\dagger = (q_0, (\theta_i), (\eta_j^i))$ be a log cuirasse on $(X_0, \mathcal{M}_{X_0})$ (see Definition 5.10). Let $\alpha_{X_{\varphi_{q_0}}} : \mathcal{M}_{X_{\varphi_{q_0}}} \to \mathcal{O}_{X_{\varphi_{q_0}}}$ be the structure log morphism (see Definition 2.12) and $z_{q_0} \in \mathfrak{Z}$ the puzzle associated to q_0 (see Proposition 4.30).

Definition 5.20. We call

$$z_{q_0^\dagger} := (z_{q_0}, (\alpha_{X_{\varphi_{q_0}}} \circ \theta_i), (\eta_j^i))$$

the *log puzzle associated* to $q_0^\dagger$.

Clearly, $z_{q_0^\dagger}$ is a log puzzle in $\mathfrak{Z}^{\log}$ (Definition 5.5). Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space proper and anaflat over a Banach analytic space S. Let $q^\dagger$ be a relative log cuirasse on $(X, \mathcal{M}_X)$ over S.

Definition 5.21. We can define a morphism

$$\begin{aligned} \varphi_{q^\dagger} : S &\to \mathfrak{Z}^{\log} \\ s &\mapsto z_{q^\dagger(s)}. \end{aligned} \tag{5.6}$$

Let $q^\dagger = (q, (\theta_i), (\eta_j^i))$ be a log cuirasse on $(X, \mathcal{M}_X)$ over S. Let $(X_{\varphi_q}, \mathcal{M}_{X_{\varphi_q}})$ given by Definition 5.9 and $\alpha_{X_{\varphi_q}} : \mathcal{M}_{X_{\varphi_q}} \to \mathcal{O}_{X_{\varphi_q}}$ the structure log morphism. For each $i \in I$, let $\mathcal{M}^a_{X_{\varphi_q},i}$ be the log structure associated to the pre-log structure $\alpha_{X_{\varphi_q}} \circ \theta_i$ (see Definition 2.20). The collection of log structures $(\mathcal{M}^a_{X_{\varphi_q},i})$ glues to a log structure $\mathcal{M}^a_{X_{\varphi_q}}$ on X_{φ_q} (see Section 2.4).

Definition 5.22. We set

$$(X_{\varphi_{q^\dagger}}, \mathcal{M}_{X_{\varphi_{q^\dagger}}}) := (X_{\varphi_q}, \mathcal{M}^a_{X_{\varphi_q}}).$$

The fine log Banach analytic space $(X_{\varphi_{q^\dagger}}, \mathcal{M}_{X_{\varphi_{q^\dagger}}})$ is obtained by gluing the pieces of the log puzzle $z_{q^\dagger}$ associated to the cuirasse $q^\dagger$.

Proposition 5.23. *Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space proer and anaflat over a Banach analytic space S. Let $q^\dagger$ be a relative log cuirasse on $(X, \mathcal{M}_X)$ over S. Then, there exists a S-log isomorphism*

$$\begin{tikzcd}
(X_{\varphi_{q^\dagger}}, \mathcal{M}_{X_{\varphi_{q^\dagger}}}) \arrow[rr, "\alpha_{q^\dagger}"] \arrow[dr] & & (X, \mathcal{M}_X) \arrow[dl] \\
& S &
\end{tikzcd}$$

.

Proof. By Proposition 4.32, we have a S-isomorphism $\alpha_q : X_{\varphi_q} \to X$. Moreover, we have $\mathcal{M}_{X_{\varphi_q}} := \alpha_q^* \mathcal{M}_X$ (see Definition 5.10). Hence, α_q induces a S-log isomorphism $\alpha_q : (X_{\varphi_q}, \mathcal{M}_{X_{\varphi_q}}) \to (X, \mathcal{M}_X)$. Now, the log cuirasse $q^\dagger$ gives us a collection of directed log charts $((\theta_i), (\eta^i_j))$ for $\mathcal{M}_{X_{\varphi_q}}$. Let $\alpha_{X_{\varphi_q}} : \mathcal{M}_{X_{\varphi_q}} \to \mathcal{O}_{X_{\varphi_q}}$ be the structure log morphism and $\mathcal{M}^a_{X_{\varphi_q},i}$ the log structure associated to the pre-log structure $\alpha_{X_{\varphi_q}} \circ \theta_i$ (see Definition 2.20), for each $i \in I$. By the definition of log chart (Definition 2.23), we have an isomorphism $\alpha_i^\flat : \mathcal{M}^a_{X_{\varphi_q},i} \to \mathcal{M}_{X_{\varphi_q}}$. Then the collection of log structures $(\mathcal{M}^a_{X_{\varphi_q},i})$, together with the isomorphisms $(\alpha_i^\flat)$, glues to a log structure $\mathcal{M}^a_{X_{\varphi_q}}$ on X_{φ_q}, together with an isomorphism $\alpha^\flat : \mathcal{M}^a_{X_{\varphi_q}} \to \mathcal{M}_{X_{\varphi_q}}$ (see Section 2.4). Hence, we get an isomorphism $\alpha := (\mathrm{Id}, \alpha^\flat) : (X_{\varphi_{q^\dagger}}, \mathcal{M}_{X_{\varphi_{q^\dagger}}}) := (X_{\varphi_q}, \mathcal{M}^a_{X_{\varphi_q}}) \to (X_{\varphi_q}, \mathcal{M}_{X_{\varphi_q}})$. Set $\alpha_{q^\dagger} := \alpha_q \circ \alpha$. $\square$

Remark 5.24. We clearly have $(X_{\varphi_{q^\dagger}}, \mathcal{M}_{X_{\varphi_{q^\dagger}}}) = \varphi_{q^\dagger}^* (\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$.

We are ready to prove the existence of an infinite dimensional complete deformation of a fine compact log complex space $(X_0, \mathcal{M}_{X_0})$. With the due modifications, the proof of Theorem 5.25 is identical to the one of Theorem 4.33. Let $q_0 = (Y_{i,0}, f_{i,0})$ be a triangularly privileged cuirasse on X_0 and $l_0 := ((\theta_{i,0}), (\eta^i_{j,0}))$ the directed collection of log charts on $(X_0, \mathcal{M}_{X_0})$ as in Definition 5.4. Then, $q_0^\dagger := (q_0, (f_{i,0}^* \theta_i), (f_{j,0}^* \eta^i_j))$ is a triangularly privileged log cuirasse on $(X_0, \mathcal{M}_{X_0})$ (see Definition 5.14). Let $z_{q_0^\dagger}$ be the log puzzle associated to $q_0^\dagger$ (see Definition 5.20). Let $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$ be the universal space of log puzzles (see Proposition 5.8) and

$$\alpha_{q_0^\dagger} : (\mathfrak{X}_{\log}(z_{q_0^\dagger}), \mathcal{M}_{\mathfrak{X}_{\log}(z_{q_0^\dagger})}) \to (X_0, \mathcal{M}_{X_0})$$

the log isomorphism given by Proposition 5.23.

Theorem 5.25. *The pair $((\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to (\mathfrak{Z}^{\log}, z_{q_0^\dagger}), \alpha_{q_0^\dagger})$ is a complete deformation of $(X_0, \mathcal{M}_{X_0})$.*

Proof. Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space proper and anaflat over a Banach analytic space S. Let $s_0 \in S$ and $i : (X(s_0), \mathcal{M}_{X(s_0)}) \to (X_0, \mathcal{M}_{X_0})$ a log isomorphism. Since $i^* q_0^\dagger$ is a triangularly privileged log cuirasse (see Definition 5.14) on $(X(s_0), \mathcal{M}_{X(s_0)})$, we have that $\mathcal{Q}_S(X, \mathcal{M}_X)$ is smooth over S in a neighborhood of $i^* q_0^\dagger$ (Proposition 5.19). Therefore there exists a local relative log cuirasse $q^\dagger$ on $(X, \mathcal{M}_X)$ defined in a neighborhood S' of s_0 in S. Hence, taking $\varphi_{q^\dagger} : S \to \mathfrak{Z}^{\log}$ (Definition 5.21) and the S'-isomorphism $\alpha_{q^\dagger}|_{S'} : (X_{\varphi_{q^\dagger}}, \mathcal{M}_{X_{\varphi_{q^\dagger}}}) \to (X, \mathcal{M}_X)$ (Proposition 5.23), the statement follows. □

5.2.2 Finite dimensional reduction

Let $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$ be given by Theorem 5.25 and $\mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ the space of relative log cuirasses on $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ over $\mathfrak{Z}^{\log}$ (see Definition 5.10). Since the finite dimensional reduction is performed on the Banach analytic space $\mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$, which does not come endowed with a non-trivial log structure, the finite dimensional reduction in the log setting is identical to the one in the classical setting (see Section 4.2). In what follows, we check all the details. Let us consider the canonical projection $\pi : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$. As in the classical case (see Section 4.2), we have a canonical relative cuirasse on $\pi^*(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ over $\mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ defined by

$$\begin{aligned} \mathfrak{q}^\dagger : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) &\to \mathcal{Q}_{\mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})}(\pi^*(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})) \\ (z^\dagger, q^\dagger) &\mapsto (z^\dagger, q^\dagger, q^\dagger) \end{aligned}. \tag{5.7}$$

By Definition 5.21, we get an associated morphism

$$\begin{aligned} \varphi_{\mathfrak{q}^\dagger} : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) &\to \mathfrak{Z}^{\log} \\ (z^\dagger, q^\dagger) &\mapsto z_{q^\dagger} \end{aligned}. \tag{5.8}$$

Analogously to the classical case (4.6), we get a Banach analytic space

$$Z^{\log} := \ker(\pi, \varphi_{\mathfrak{q}^\dagger}) \subset \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}). \tag{5.9}$$

The space $Z^{\log}$ parametrizes all log cuirasses on compact fine log complex spaces close to $(X_0, \mathcal{M}_{X_0})$. Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space proper and anaflat over a Banach analytic space S and $\sigma^\dagger : S \to \mathcal{Q}_S(X, \mathcal{M}_X)$ a relative log cuirasse on $(X, \mathcal{M}_X)$ over S. By Proposition 5.23, for each $s \in S$, the fibre $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(\varphi_{\sigma^\dagger}(s))$ is isomorphic to the fibre $(X, \mathcal{M}_X)(s)$ via an isomorphism $\alpha_{\sigma^\dagger}(s)$. Therefore, a log cuirasse $q^\dagger$ on $(X, \mathcal{M}_X)(s)$ gives a cuirasse $\alpha_{\sigma^\dagger}(s)^* q^\dagger$ on $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(\varphi_{\sigma^\dagger}(s))$. Hence, we get a map

$$\begin{aligned} \psi_{\sigma^\dagger} : S &\to Q_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \\ s &\mapsto (z_{\alpha_{\sigma^\dagger}(s)^*\sigma^\dagger(s)}, \alpha_{\sigma^\dagger}(s)^*\sigma^\dagger(s)) \end{aligned}. \tag{5.10}$$

For the sake of simplicity, if we identify $(X, \mathcal{M}_X)(s)$ with $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(\varphi_{\sigma^\dagger}(s))$ and hence $\sigma^\dagger(s)$ with $\alpha_{\sigma^\dagger}(s)^*\sigma^\dagger(s)$, then we can rewrite (5.10) as

$$\begin{aligned} \psi_{\sigma^\dagger} : S &\to Q_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \\ s &\mapsto (z_{\sigma^\dagger(s)}, \sigma^\dagger(s)) \end{aligned}. \tag{5.11}$$

By construction, $\psi_{\sigma^\dagger}$ factors through $Z^{\log}$ (5.9). Let $\mathfrak{q}$ and $\psi_{\mathfrak{q}} : S \to Z$ given by (4.4) and (4.8) respectively. If we denote with $\pi_Z : Z^{\log} \to Z$ the forgetful morphism, then, clearly, $\pi_Z \circ \psi_{\mathfrak{q}^\dagger} = \psi_{\mathfrak{q}}$. Let $q_0^\dagger$ be a triangularly privileged cuirasse on $(X_0, \mathcal{M}_{X_0})$ (see Definition 5.14) and $z_{q_0^\dagger} \in \mathfrak{Z}^{\log}$ the associated log puzzle (see Definition 5.20), hence $(z_{q_0^\dagger}, q_0^\dagger) \in Z^{\log}$. Let

$$\alpha_{q_0^\dagger} : (\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(z_{q_0^\dagger}) \to (X_0, \mathcal{M}_{X_0})$$

be the log isomorphism given by Proposition 5.23. Let $\pi : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$ be the projection and $i : Z^{\log} \hookrightarrow \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ the canonical injection. Set $(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) := i^*\pi^*(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ and consider the log isomorphism

$$i^*\pi^*\alpha_{q_0^\dagger} : (\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(z_{q_0^\dagger}, q_0^\dagger) \to (X_0, \mathcal{M}_{X_0}).$$

Proposition 5.26. *The morphism* $((\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \to (Z^{\log}, (z_{q_0^\dagger}, q_0^\dagger)), i^*\pi^*\alpha_{q_0^\dagger})$ *is a complete deformation of* $(X_0, \mathcal{M}_{X_0})$.

Proof. Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space proper and anaflat over a Banach analytic space S. Let $s_0 \in S$ and $i : (X(s_0), \mathcal{M}_{X(s_0)}) \to (X_0, \mathcal{M}_{X_0})$ a log isomorphism. Since $i^*q_0^\dagger$ is a triangularly privileged log cuirasse (see Definition 5.14) on $(X(s_0), \mathcal{M}_{X(s_0)})$, we have that $\mathcal{Q}_S(X, \mathcal{M}_X)$ is smooth over S in a neighborhood of $i^*q_0^\dagger$ (Proposition 5.19). Therefore, there exists a local relative log cuirasse $q^\dagger$ on $(X, \mathcal{M}_X)$ defined in a neighborhood S' of s_0 in S. Let $\varphi_{q^\dagger} : S \to \mathfrak{Z}^{\log}$ be the morphism given by Definition 5.21 and $\psi_{q^\dagger} : S \to Z^{\log}$ given by Definition 5.10. Let $\pi \circ i : Z^{\log} \to \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$ be the composition of the canonical projection π with the canonical injection i. Since $\pi \circ i \circ \psi_{q^\dagger} = \varphi_{q^\dagger}$, we have $\psi_{q^\dagger}^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) = \varphi_{q^\dagger}^*(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$. □

In what follows, in the same way as done in Section 4.2, we are going to decompose $Z^{\log}$ into a product $\Sigma^{\log} \times R^{\log}$, where $\Sigma^{\log}$ is a Banach manifold (see Definition 3.4) and $R^{\log}$ is a finite dimensional complex analytic space satisfying the versality property in a neighborhood of $(z_{q_0^\dagger}, q_0^\dagger)$. To do that, we start by noticing that the restriction of $\mathfrak{q}^\dagger$ (5.7) to $Z^{\log}$ gives us a canonical relative log cuirasse on $(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$ over $Z^{\log}$

$$\sigma_{Z^{\log}} : Z^{\log} \to \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}). \tag{5.12}$$

Via the projection $\pi : \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \to Z^{\log}$, we get a canonical relative log cuirasse $\pi^*\sigma_{Z^{\log}}$ on $\pi^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$ over $\mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$, namely

$$\begin{aligned} \pi^*\sigma_{Z^{\log}} : \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) &\to \mathcal{Q}_{\mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}},\mathcal{M}_{\mathfrak{X}_{Z^{\log}}})}(\pi^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})) \\ (z_{q^\dagger}, q^\dagger, p^\dagger) &\mapsto (z_{q^\dagger}, q^\dagger, p^\dagger, p^\dagger) \end{aligned}. \tag{5.13}$$

By Proposition 5.23, we get a log $\mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$-isomorphism

$$\alpha_{\pi^*\sigma_{Z^{\log}}} : \varphi^*_{\pi^*\sigma_{Z^{\log}}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \pi^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}). \tag{5.14}$$

Let

$$\begin{aligned} \psi_{Z^{\log}} : \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) &\to Z^{\log} \\ s^\dagger := (z_{q^\dagger}, q^\dagger, p^\dagger) &\mapsto (z_{(\alpha_{\pi^*\sigma_{Z^{\log}}}(s^\dagger))^* p^\dagger}, (\alpha_{\pi^*\sigma_{Z^{\log}}}(s^\dagger))^* p^\dagger) \end{aligned} \tag{5.15}$$

be the versal morphism associated to the relative log cuirasse $\pi^*\sigma_{Z^{\log}}$ on $\pi^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$ over $\mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$ (see (5.10)). In simpler terms, by (5.11), we can write

$$\begin{aligned} \psi_{Z^{\log}} : \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) &\to Z^{\log} \\ (z_{q^\dagger}, q^\dagger, p^\dagger) &\mapsto (z_{p^\dagger}, p^\dagger) \end{aligned}. \tag{5.16}$$

The proof of Proposition 5.27 is identical to the one of Proposition 4.35.

Proposition 5.27. *Let S be a Banach analytic space and $f_1, f_2 : S \to Z^{log}$ morphisms. If $f_1^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \simeq f_2^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$, then there exists a morphism $k^\dagger : S \to \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$, such that the following diagram commutes*

$$\begin{tikzcd} & Z^{\log} \\ S \arrow[ur, "f_1"] \arrow[r, "k^\dagger"] \arrow[dr, "f_2"'] & Q_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \arrow[u, "\psi_{Z^{log}}"'] \arrow[d, "\pi"] \\ & Z^{\log} \end{tikzcd}. \tag{5.17}$$

Proof. Let

$$\begin{aligned} f_1 : S &\to Z^{\log} \\ s &\mapsto (z_{q_1^\dagger(s)}, q_1^\dagger(s)) \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} f_2 : S &\to Z^{\log} \\ s &\mapsto (z_{q_2^\dagger(s)}, q_2^\dagger(s)) \end{aligned}. \tag{5.19}$$

Let $\alpha : f_1^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \to f_2^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$ be a log S-isomorphism. For each $s \in S$, it induces an isomorphism of spaces of absolute log cuirasses

$$\begin{aligned}\alpha_{Q,s} : \mathcal{Q}((\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(f_1(s))) &\to \mathcal{Q}((\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(f_2(s))) \\ q_1^\dagger &\mapsto \alpha(s)_* q_1^\dagger\end{aligned}. \tag{5.20}$$

Hence, we have $(f_2(s), \alpha(s)_* q_1^\dagger(s)) \in \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$. Let

$$\alpha_{\pi^*\sigma_{Z^{\log}}} : \varphi^*_{\pi^*\sigma_{Z^{\log}}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \pi^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$$

be the log $\mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$-isomorphism given by (5.14). By the definition of the fibre product (Corollary 3.26 and Corollary 2.31), we can canonically write

$$\pi^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(f_2(s), \alpha(s)_* q_1^\dagger(s)) = (\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(f_2(s)).$$

On the other hand, we can canonically write

$$(\varphi^*_{\pi^*\sigma_{Z^{\log}}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}))(f_2(s), \alpha(s)_* q_1^\dagger(s)) = (\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(z_{\alpha(s)_* q_1^\dagger(s)})$$

see Definition 5.21. Since $\alpha(s)$ is a log isomorphism, by the construction of log puzzle associated to a log cuirasse (Definition 5.20), we have $z_{\alpha(s)_* q_1^\dagger(s)} = z_{q_1^\dagger(s)}$. Hence, $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(z_{\alpha(s)_* q_1^\dagger(s)}) = (\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(z_{q_1^\dagger(s)})$. Moreover, again by the definition of the fibre product, we can canonically write

$$(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(z_{q_1^\dagger(s)}) = (\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(z_{q_1^\dagger(s)}, q_1^\dagger(s)).$$

Therefore,

$$(\varphi^*_{\pi^*\sigma_{Z^{\log}}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}))(f_2(s), \alpha(s)_* q_1^\dagger(s)) = (\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(f_1(s)).$$

Thus, for each $s \in S$, $\alpha_{\pi^*\sigma_{Z^{\log}}}$ naturally induces a log isomorphism

$$\alpha'(s) : (\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(f_1(s)) \to (\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(f_2(s)).$$

Therefore, the map

$$\begin{aligned}k^\dagger : S &\to \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \\ s &\mapsto (f_2(s), \alpha'(s)_* q_1^\dagger(s))\end{aligned} \tag{5.21}$$

is well defined. Then, by (5.15) and by the construction of the isomorphism $\alpha'(s)$, the statement follows. $\square$

Now, by Proposition 5.19, we have that the canonical projection

$$\pi : \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \to Z^{\log}$$

is smooth in a neighborhood of $(z_{q_0^\dagger}, q_0^\dagger, q_0^\dagger)$. Therefore, setting $Q_0^{\log} := \mathcal{Q}(X_0, \mathcal{M}_{X_0})$, the space of log cuirasses on $(X_0, \mathcal{M}_{X_0})$ (see Definition 5.10), we can choose a local trivialization

$$\gamma^\dagger : Z^{\log} \times Q_0^{\log} \to \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}). \tag{5.22}$$

Let $\sigma_{Z^{\log}}$ given by (5.12) and

$$\alpha_{q_0^\dagger} : (\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(z_{q_0^\dagger}, q_0^\dagger) \to (X_0, \mathcal{M}_{X_0})$$

the log isomorphism given by Proposition 5.23[1]. We naturally get an induced isomorphism $\alpha_{Q_0^{\log}} : Q_0^{\log} \to \mathcal{Q}((\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})(z_{q_0^\dagger}, q_0^\dagger))$. We choose $\gamma^\dagger$ such that

$$\gamma^\dagger|_{Z^{\log} \times \{q_0^\dagger\}} = \sigma_{Z^{\log}} \text{ and } \gamma^\dagger|_{\{(z_{q_0^\dagger}, q_0^\dagger)\} \times Q_0^{\log}} = \alpha_{Q_0^{\log}}. \tag{5.23}$$

Let $\psi_{Z^{\log}}$ given by (5.15). We can construct a morphism

$$\omega^\dagger : Z^{\log} \times Q_0^{\log} \xrightarrow{\gamma^\dagger} \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \xrightarrow{\psi_{Z^{\log}}} Z^{\log} \tag{5.24}$$

via $\omega^\dagger := \psi_{Z^{\log}} \circ \gamma^\dagger$. Since $\gamma^\dagger|_{Z^{\log} \times \{q_0^\dagger\}} = \sigma_{Z^{\log}}$ (5.23), we get that

$$\omega^\dagger|_{Z^{\log} \times \{q_0^\dagger\}} = \psi_{Z^{\log}} \circ \sigma_{Z^{\log}} = \mathrm{Id}_{Z^{\log}}. \tag{5.25}$$

The proof of Proposition 5.28 is identical to the proof of Proposition 4.36.

Proposition 5.28. *Let S be a Banach analytic space and $f, g : S \to Z^{\log}$ morphisms. Then $f^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \simeq g^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$, if and only if there exists $h^\dagger : S \to Q_0^{\log}$ such that the following diagram commutes*

$$\begin{tikzcd}
 & Z^{\log} \times Q_0^{\log} \arrow[dd, "\omega^\dagger"] \\
S \arrow[ur, "{(f, h^\dagger)}"] \arrow[dr, "g"'] & \\
 & Z^{\log}
\end{tikzcd} \tag{5.26}$$

Proof. Let $f, g : S \to Z^{\log}$ be morphisms, with $f^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \simeq g^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$. Using Proposition 5.27, with $f_1 := g$ and $f_2 := f$, we get a morphism $k^\dagger : S \to \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$, such that $\psi_{Z^{\log}} \circ k^\dagger = g$ and $\pi \circ k^\dagger = f$. Let $\gamma^\dagger : Z^{\log} \times Q_0^{\log} \to$

[1] Let $\pi \circ i : Z^{\log} \to \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$ be the canonical projection, we identify $\alpha_{q_0^\dagger}$ with $i^*\pi^*\alpha_{q_0^\dagger}$.

$\mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$ be the local trivialization chosen in (5.22). Then, setting $h^\dagger := \pi_2 \circ (\gamma^\dagger)^{-1} \circ k^\dagger$ and using (5.24), we get

$$\omega^\dagger \circ (f, h^\dagger) = \psi_{Z^{\log}} \circ \gamma^\dagger \circ (f, \pi_2 \circ (\gamma^\dagger)^{-1} \circ k^\dagger),$$

that is, using $\pi \circ k^\dagger = f$

$$\omega^\dagger \circ (f, h^\dagger) = \psi_{Z^{\log}} \circ \gamma^\dagger \circ (\pi \circ k^\dagger, \pi_2 \circ (\gamma^\dagger)^{-1} \circ k^\dagger),$$

and equivalently

$$\omega^\dagger \circ (f, h^\dagger) = \psi_{Z^{\log}} \circ k^\dagger.$$

Using $\psi_{Z^{\log}} \circ k^\dagger = g$, the statement follows. Conversely, let us assume the existence of a morphism $h^\dagger : S \to Q_0^{\log}$ such that $\omega^\dagger \circ (f, h^\dagger) = g$. Let $\pi : \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \to Z^{\log}$ be the projection and $\psi_{Z^{\log}} : \mathcal{Q}_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \to Z^{\log}$ given by (5.15). By Definition 5.15 and Proposition 5.26,

$$\pi^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \simeq \psi^*_{Z^{\log}}(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}).$$

Set $\rho := \gamma^\dagger \circ (f, h^\dagger)$. Since $\pi \circ \rho = f$ and $\psi_{Z^{\log}} \circ \rho = g$ (by (5.24)), the statement follows.

$\square$

Let us denote with $\operatorname{Ex}^1(X_0, \mathcal{M}_{X_0})$ the set of equivalence classes of infinitesimal deformations of $(X_0, \mathcal{M}_{X_0})$. Using (5.24), we define

$$\delta^\dagger := \omega^\dagger|_{\{(z_{q_0^\dagger}, q_0^\dagger)\} \times Q_0^{\log}}. \tag{5.27}$$

Let $D = (\{\cdot\}, \mathbb{C}[\epsilon]/\epsilon^2)$ be the double point. Using Proposition 5.28 for $S = D$, we get that

$$\operatorname{Ex}^1(X_0, \mathcal{M}_{X_0}) = T_{(z_{q_0^\dagger}, q_0^\dagger)} Z^{\log} / \operatorname{Im} T_{q_0^\dagger} \delta^\dagger.$$

As in the classical case (see Section 4.2), we show in Proposition 5.31 that $T_{q_0^\dagger}\delta^\dagger$ is a Fredholm operator of index 0. This fact, together with Proposition 5.28, allows us to get a finite dimensional versal deformation space (see Theorem 5.32).

Definition 5.29. Let $(X, \mathcal{M}_X)$ be a fine log Banach analytic space proper and anaflat over a Banach analytic space S. Let $q^\dagger : S \to \mathcal{Q}_S(\mathfrak{I}; (X, \mathcal{M}_X))$ be a relative log cuirasse on $(X, \mathcal{M}_X)$ over S. We recall from Definition 5.10 that $q^\dagger(s) = (q(s), (\theta_i(s) : P_i \to \mathcal{M}_{X_{\varphi_{q(s)}}}|_{Y_i^\circ(s)}), (\eta_j^i(s) : P_i \to O^\times_{Y_j^\circ(s)}))$, $s \in S$. We say that $q^\dagger$ is *extendable* if there exists a type of cuirasse $\hat{\mathfrak{I}}$ (see Definition 4.8) and a relative log cuirasse $\hat{q}^\dagger : S \to \mathcal{Q}_S(\hat{\mathfrak{I}}; (X, \mathcal{M}_X))$ of type $\hat{\mathfrak{I}}$ on $(X, \mathcal{M}_X)$ over S such that the relative cuirasse $\hat{q}$ underlying $\hat{q}^\dagger$ extends the relative cuirasse q underlying $q^\dagger$ (see Definition 4.38) and, for each $i \in I$,

1. $\theta_i = \hat{\theta}_i|_{Y_i}$;
2. $\eta^i_j = \hat{\eta}^i_j|_{Y_j}$.

Exactly as in the classical case (see Definition 4.38), extendable relative log cuirasses exist on a fine log Banach analytic spaces $(X, \mathcal{M}_X)$ proper and anaflat over a Banach analytic space S. Indeed, let $\hat{\mathfrak{I}} := (\hat{I}_\bullet, (\hat{K}_i)_{i\in\hat{I}}, (\hat{\tilde{K}}_i)_{i\in\hat{I}}, (\hat{K'}_i)_{i\in\hat{I}_0\cup\hat{I}_1})$ be a type of cuirasse and $\hat{q}$ a relative log cuirasse of type $\hat{\mathfrak{I}}$ on $(X, \mathcal{M}_X)$ over S. Then, by slightly shrinking each polycylinder $\hat{K}_i$, $\hat{\tilde{K}}_i$ and $\hat{K}'_i$ in $\hat{\mathfrak{I}}$, we can get polycylinders K_i, $\tilde{K}_i$ and K'_i respectively and hence a type of cuirasse $\mathfrak{I} := (\hat{I}_\bullet, (K_i)_{i\in\hat{I}}, (\tilde{K}_i)_{i\in\hat{I}}, (K'_i)_{i\in\hat{I}_0\cup\hat{I}_1})$, such that $\mathfrak{I} \Subset \hat{\mathfrak{I}}$. Then, $q := \hat{q}|_{\mathfrak{I}}$ is an extendable relative log cuirasse on $(X, \mathcal{M}_X)$ over S. By Proposition 5.19, the projection $\pi : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$ is smooth in a neighborhood of $(z_{q_0^\dagger}, q_0^\dagger)$. Therefore, we can find an extendable local relative log cuirasse

$$\sigma^\dagger : \mathfrak{Z}^{\log} \to \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \tag{5.28}$$

on $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ over $\mathfrak{Z}^{\log}$, such that $\sigma^\dagger(z_{q_0^\dagger}) = (z_{q_0^\dagger}, q_0^\dagger)$. Moreover, we can find a local trivialization

$$\rho^\dagger : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log} \times Q_0^{\log} \tag{5.29}$$

of the form $(\pi, p^\dagger)$, where

$$p^\dagger : \mathcal{Q}_{\mathfrak{Z}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to Q_0^{\log} \tag{5.30}$$

is a morphism such that

$$p^\dagger \circ \sigma^\dagger = \{q_0^\dagger\}. \tag{5.31}$$

Furthermore, let $\alpha_{q_0^\dagger} : (\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(z_{q_0^\dagger}) \to (X_0, \mathcal{M}_{X_0})$ be the log isomorphism given by Proposition 5.23. We naturally get an induced isomorphism

$$\alpha_{Q_0^{\log}} : \mathcal{Q}((\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(z_{q_0^\dagger})) \to Q_0^{\log}.$$

We choose $p^\dagger$ so that

$$p^\dagger|_{\mathcal{Q}((\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})(z_{q_0^\dagger}))} = \alpha_{q_0^\dagger}. \tag{5.32}$$

The proof of Proposition 5.30 is identical to the proof of Proposition 4.39.

Proposition 5.30. *The morphism $p^\dagger : Z^{\log} \to Q_0^{\log}$ is of relative finite dimension in a neighborhood of $(z_{q_0^\dagger}, q_0^\dagger)$.*

Proof. Let $\varphi_{\mathfrak{q}^\dagger}$ be as in (5.8), $\pi : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$ the canonical projection, $\sigma^\dagger$ the extendable relative log cuirasse on $(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ over $\mathfrak{Z}^{\log}$ chosen in (5.28) and $p^\dagger : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{X_{\log}}) \to Q_0^{\log}$ the morphism chosen in (5.30). We can construct two morphisms

$$\begin{aligned} (p^\dagger, \pi) \circ \sigma^\dagger : \mathfrak{Z}^{\log} &\to \mathfrak{Z}^{\log} \times \{q_0^\dagger\} \\ z^\dagger &\mapsto (z^\dagger, q_0^\dagger) \end{aligned} \tag{5.33}$$

and

$$\begin{aligned}(p^\dagger, \varphi_{\mathfrak{q}^\dagger}) \circ \sigma^\dagger : \mathfrak{Z}^{\log} &\to \mathfrak{Z}^{\log} \times \{q_0^\dagger\} \\ z^\dagger &\mapsto (z_{\sigma^\dagger(z^\dagger)}, q_0^\dagger).\end{aligned} \tag{5.34}$$

On one hand we have that $(p^\dagger, \pi) \circ \sigma^\dagger$ is nothing but the identity map $\mathrm{Id}_{\mathfrak{Z}^{\log}}$ on $\mathfrak{Z}^{\log}$. On the other hand, since $\sigma^\dagger$ is extendable (see Definition 5.29), the morphism $\varphi_{\mathfrak{q}^\dagger} \circ \sigma^\dagger$ factors through the restriction morphism

$$\begin{aligned}j^\dagger : \mathfrak{Z}^{\hat{\mathfrak{I}},\log} &\to \mathfrak{Z}^{\log} \\ ((\hat{Y}_i), (\hat{g}_i^j), (\hat{\beta}_i), (\hat{\eta}_j^i)) &\mapsto ((\hat{Y}_i \cap K_i), (\hat{g}_i^j|_{\hat{Y}_j \cap K_j}), (\hat{\beta}_i|_{\hat{Y}_i \cap K_i}), (\hat{\eta}_j^i|_{\hat{Y}_j \cap K_j}))\end{aligned}, \tag{5.35}$$

with $\mathfrak{I} \Subset \hat{\mathfrak{I}}$. Since the morphism $j^\dagger$ is a compact morphism (see [Dou74], p. 595 and [Sti88], p. 17), we get that $\varphi_{\mathfrak{q}^\dagger} \circ \sigma^\dagger$ is compact. By Proposition 3.33, we get that

$$T^{\log} := \ker((p^\dagger, \varphi_{\mathfrak{q}^\dagger}) \circ \sigma^\dagger, \mathrm{Id}_{\mathfrak{Z}^{\log}}) \subset \mathfrak{Z}^{\log}$$

is of finite dimension. From (5.31), we get $\sigma^\dagger(T^{\log}) = (p^\dagger, \pi)^{-1}(\{q_0^\dagger\} \times \mathfrak{Z}^{\log}) = (p^\dagger)^{-1}(q_0^\dagger)$. Since $Z^{\log} = \ker((p^\dagger \circ \varphi_{\mathfrak{q}^\dagger}), (p^\dagger \circ \pi))$, we have that $\sigma^\dagger(T^{\log}) = (p^\dagger|_{Z^{\log}})^{-1}(q_0^\dagger)$. Hence, we have that the fibre $(p^\dagger|_{Z^{\log}})^{-1}(q_0^\dagger)$ of $p^\dagger|_{Z^{\log}} : Z^{\log} \to Q_0^{\log}$ over $q_0^\dagger$ is of finite dimension. Using Proposition 3.37, the statement follows. □

We can draw the following diagram

$$\begin{tikzcd} Q_0^{\log} \arrow[r, "\delta^\dagger"] & Z^{\log} \arrow[rr, "\iota^\dagger"] \arrow[dr, "p^\dagger"'] & & Q_0^{\log} \times \mathbb{C}^m \arrow[dl, "\pi_1"] \\ & & Q_0^{\log} & \end{tikzcd} . \tag{5.36}$$

Analogously, the proof of the Proposition 5.31 is identical to the proof of Proposition 4.40.

Proposition 5.31. *The linear tangent map:*

$$T_{q_0^\dagger}(p^\dagger \circ \delta^\dagger) : T_{q_0^\dagger} Q_0^{\log} \to T_{q_0^\dagger} Q_0^{\log}$$

is of the form $1 - v^\dagger$*, with* $v^\dagger$ *compact.*

Proof. Let $Q_0^{\log} \times (X_0, \mathcal{M}_{X_0}) \to Q_0^{\log}$ be the trivial deformation of $(X_0, \mathcal{M}_{X_0})$ over $Q_0^{\log}$. The triangularly privileged log cuirasse $q_0^\dagger$ on $(X_0, \mathcal{M}_{X_0})$ induces a relative log cuirasse $\mathfrak{q}_0^\dagger : Q_0^{\log} \to \mathcal{Q}_{Q_0^{\log}}(Q_0^{\log} \times (X_0, \mathcal{M}_{X_0})) = Q_0^{\log} \times Q_0^{\log}$ on $Q_0^{\log} \times (X_0, \mathcal{M}_{X_0})$ over $Q_0^{\log}$ via $q^\dagger \mapsto (q^\dagger, q_0^\dagger)$. Let $\underline{\beta}^\dagger : Q_0^{\log} \times (X_0, \mathcal{M}_{X_0}) \to (\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ be the morphism induced by the morphism $\varphi_{\mathfrak{q}_0^\dagger} : Q_0^{\log} \to \mathfrak{Z}^{\log}$ associated to $\mathfrak{q}_0^\dagger$ (see Definition 5.21). We have the following commutative diagram

$$\begin{tikzcd}
Q_0^{\log} \times (X_0, \mathcal{M}_{X_0}) \arrow[r, "\beta^\dagger"] \arrow[d, "\pi_1"'] & (\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \arrow[d] \\
Q_0^{\log} \arrow[r, "\varphi_{\mathfrak{q}_0^\dagger}"] & \mathfrak{Z}^{\log}
\end{tikzcd} . \tag{5.37}$$

Let $\rho^\dagger := (\pi, p^\dagger) : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log} \times Q_0^{\log}$ be the local trivialization chosen in (5.29). From (5.37), we get the following induced commutative diagram

$$\begin{tikzcd}
 & \mathfrak{Z}^{\log} \times Q_0^{\log} & \\
Q_0^{\log} \times Q_0^{\log} \arrow[ur, "(\varphi_{\mathfrak{q}_0^\dagger}, \tilde{p}^\dagger)"] \arrow[rr, "\beta^\dagger"] \arrow[d, "\pi_1"'] & & \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \arrow[ul, "(\pi, p^\dagger)"'] \arrow[d, "\pi"] \\
Q_0^{\log} \arrow[rr, "\varphi_{\mathfrak{q}_0^\dagger}"] & & \mathfrak{Z}^{\log}
\end{tikzcd} , \tag{5.38}$$

where $\tilde{p}^\dagger := p^\dagger \circ \beta^\dagger$. Let $\sigma^\dagger : \mathfrak{Z}^{\log} \to \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{,,log}})$ be the extendable relative log cuirasse chosen in (5.28). Set $\tau^\dagger := \varphi_{\mathfrak{q}_0^\dagger}^* \sigma^\dagger : Q_0^{\log} \to Q_0^{\log} \times Q_0^{\log}$. Since $\sigma^\dagger$ is a section of the projection $\pi : \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}}) \to \mathfrak{Z}^{\log}$, we get that $\tau^\dagger$ is of the form $(\mathrm{Id}_{Q_0^{\log}}, \theta^\dagger)$. Moreover, since $\sigma^\dagger$ is extendable (see Definition 5.29), $T_{q_0^\dagger}\theta^\dagger$ is a compact morphism (see Definition 3.32). Indeed, $\theta^\dagger$ is of the form $j^\dagger \circ \hat{\theta}^\dagger$, where $\hat{\theta}^\dagger : Q_0^{\log} \to \mathcal{Q}(\hat{\mathfrak{I}}; (X_0, \mathcal{M}_{X_0}))$, with $\mathfrak{I} \Subset \hat{\mathfrak{I}}$, and $j^\dagger : \mathcal{Q}(\hat{\mathfrak{I}}; (X_0, \mathcal{M}_{X_0})) \to Q_0^{\log}$ is the restriction morphism, which is a compact morphism (see [Dou74], p. 595 and [Sti88], p. 17). Because $\tau^\dagger = \varphi_{\mathfrak{q}_0^\dagger}^* \sigma^\dagger$, we get that

$$p^\dagger \circ \sigma^\dagger \circ \varphi_{\mathfrak{q}_0^\dagger} = \tilde{p}^\dagger \circ \tau^\dagger.$$

Hence, using (5.31), we get that

$$T_{q_0^\dagger}(\tilde{p}^\dagger \circ \tau^\dagger) = 0. \tag{5.39}$$

Let $i_1 : Q_0^{\log} \to Q_0^{\log} \times Q_0^{\log}$ and $i_2 : Q_0^{\log} \to Q_0^{\log} \times Q_0^{\log}$ be the injections defined by $q^\dagger \mapsto (q^\dagger, q_0^\dagger)$ and $q^\dagger \mapsto (q_0^\dagger, q^\dagger)$ respectively. Then, since $\tau^\dagger = (\mathrm{Id}_{Q_0^{\log}}, \theta^\dagger)$, we get

$$T_{q_0^\dagger}(\tilde{p}^\dagger \circ \tau^\dagger) = T_{q_0^\dagger}(\tilde{p}^\dagger \circ (\mathrm{Id}_{Q_0^{\log}}, \theta^\dagger)) = T_{q_0^\dagger}(\tilde{p}^\dagger \circ i_1) + T_{q_0^\dagger}(\tilde{p}^\dagger \circ i_2) \circ T_{q_0^\dagger}\theta^\dagger. \tag{5.40}$$

From (5.32), we obtain

$$\tilde{p}^\dagger \circ i_2 = p^\dagger \circ \beta^\dagger \circ i_2 = Id_{Q_0^{\log}}. \tag{5.41}$$

Thus, since $T_{q_0^\dagger}(\tilde{p}^\dagger \circ \tau^\dagger) = 0$ by (5.39), from (5.40), using (5.41), we get

$$T_{q_0^\dagger}(\tilde{p}^\dagger \circ i_1) = -T_{q_0^\dagger}\theta^\dagger. \tag{5.42}$$

Let $\psi_{\mathfrak{q}_0^\dagger} : Q_0^{\log} \to Z^{\log}$ be the morphism associated to $\mathfrak{q}_0^\dagger$ (see (5.10)). From the definition of $\delta^\dagger : Q_0^{\log} \to \mathcal{Q}_{\mathfrak{Z}^{\log}}(\mathfrak{X}_{\log}, \mathcal{M}_{\mathfrak{X}_{\log}})$ (see (5.27)), we get that $\delta^\dagger = \psi_{\mathfrak{q}_0^\dagger}$. Let $\triangle : Q_0^{\log} \to Q_0^{\log} \times Q_0^{\log}$, $q^\dagger \mapsto (q^\dagger, q^\dagger)$, be the diagonal morphism. From the commutative diagram (5.38) and from $\pi \circ \psi_{\mathfrak{q}_0^\dagger} = \varphi_{\mathfrak{q}_0^\dagger}$ (5.10), we get $\beta^\dagger \circ \triangle = \delta^\dagger$. Hence,

$$p^\dagger \circ \delta^\dagger = p^\dagger \circ \beta^\dagger \circ \triangle = \tilde{p}^\dagger \circ \triangle. \tag{5.43}$$

Thus, using (5.43), (5.42) and (5.41), we get

$$T_{q_0^\dagger}(p^\dagger \circ \delta^\dagger) = T_{q_0^\dagger}(\tilde{p}^\dagger \circ \triangle) = T_{q_0^\dagger}(\tilde{p}^\dagger \circ i_1) + T_{q_0^\dagger}(\tilde{p}^\dagger \circ i_2) = -T_{q_0^\dagger}\theta^\dagger + \mathrm{Id}_{Q_0^{\log}}.$$

□

From Proposition 5.31, we get that $\ker T_{q_0^\dagger}(p^\dagger \circ \delta^\dagger)$ is finite dimensional and a direct subspace of $T_{q_0^\dagger}Q_0^{\log}$. By Proposition 5.30

$$\ker T_{q_0^\dagger}(p^\dagger \circ \delta^\dagger) \supset \ker T_{q_0^\dagger}\delta^\dagger = \ker T_{q_0^\dagger}(\iota^\dagger \circ \delta^\dagger),$$

we get that $\ker T_{q_0^\dagger}(\iota^\dagger \circ \delta^\dagger)$ is of finite dimension. Moreover, from Proposition 5.31 we have that $\operatorname{Im} T_{q_0^\dagger}(p^\dagger \circ \delta^\dagger)$ has finite codimension in $T_{(z_{q_0^\dagger}, q_0^\dagger)}Z^{\log}$. Therefore, using again Proposition 5.30, we get that $\operatorname{Im} T_{q_0^\dagger}(\iota^\dagger \circ \delta^\dagger)$ is of finite codimension. Thus, let us identify $Z^{\log}$ with its image in $Q_0^{\log} \times \mathbb{C}^m$ under $\iota^\dagger$ (5.36). Let $\Sigma^{\log}$ be the subspace of $Q_0^{\log}$ such that

$$T_{q_0^\dagger}\Sigma^{\log} \oplus \ker T_{q_0^\dagger}\delta^\dagger = T_{q_0^\dagger}Q_0^{\log}.$$

We choose a retraction $r^\dagger : Q_0^{\log} \times \mathbb{C}^m \to \delta^\dagger(\Sigma^{\log})$ and we set $R^{\log} := (r^\dagger)^{-1}(q_0^\dagger) \cap Z^{\log}$. Let $i^\dagger : R^{\log} \hookrightarrow Z^{\log}$ be the inclusion and set

$$(\mathfrak{X}_{R^{\log}}, \mathcal{M}_{\mathfrak{X}_{R^{\log}}}) := (i^\dagger)^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}).$$

Moreover, we set $r_0^\dagger := (z_{q_0^\dagger}, q_0^\dagger)$. Let $\alpha_{q_0^\dagger} : (\mathfrak{X}_{R^{\log}}, \mathcal{M}_{\mathfrak{X}_{R^{\log}}})(r_0^\dagger) \to (X_0, \mathcal{M}_{X_0})$ be the log isomorphism given by Proposition 5.23. The proof of the next Theorem is identical to the one of Theorem 4.43.

Theorem 5.32. *The morphism $(\mathfrak{X}_{R^{\log}}, \mathcal{M}_{\mathfrak{X}_{R^{\log}}}) \to (R^{\log}, r_0^\dagger)$, endowed with the isomorphism $\alpha_{q_0^\dagger}$ is a versal deformation of $(X_0, \mathcal{M}_{X_0})$.*

Proof. Let $((X, \mathcal{M}_X) \to (S, s_0), i)$ be a deformation of $(X_0, \mathcal{M}_{X_0})$. Since, by Proposition 5.26, we have that $((\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \to Z^{\log}, (z_{q_0^\dagger}, q_0^\dagger))$ is versal, there exists a morphism $\psi_\dagger : S \to Z^{\log}$ such that $(X, \mathcal{M}_X)$ is isomorphic to $\psi_\dagger^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$. Let $\pi_{R^{\log}} : R^{\log} \times \Sigma^{\log} \to R^{\log}$ and $\pi_{\Sigma^{\log}} : R^{\log} \times \Sigma^{\log} \to \Sigma^{\log}$ be the projections. Exactly as in Proposition 4.42, using Lemma 4.41, we get that the morphism $\omega^\dagger|_{R^{\log}\times\Sigma^{\log}} : R^{\log} \times \Sigma^{\log} \to Z^{\log}$ (5.24) is an isomorphism. Thus, setting $g := \pi_{R^{\log}} \circ (\omega^\dagger|_{R^{\log}\times\Sigma^{\log}})^{-1} \circ \psi_\dagger$ and $h^\dagger := \pi_{\Sigma^{\log}} \circ (\omega^\dagger|_{R^{\log}\times\Sigma^{\log}})^{-1} \circ \psi_\dagger$, we obtain

$$\omega^\dagger \circ (g, h^\dagger) = \psi_\dagger.$$

Hence, by Proposition 5.28, we get

$$g^*(\mathfrak{X}_{R^{\log}}, \mathcal{M}_{\mathfrak{X}_{R^{\log}}}) \simeq \psi_\dagger^*(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}}) \simeq (X, \mathcal{M}_X).$$

Hence, we get completeness. Moreover, by construction, $T_{q_0^\dagger} R^{\log} = \mathrm{Ex}^1(X_0, \mathcal{M}_{X_0})$. Hence, our deformation is effective. Now, let $((S, s_0), (X, \mathcal{M}_X), i)$ be a log deformation of $(X_0, \mathcal{M}_{X_0})$ and (S', s_0) a subgerm of (S, s_0). Let $h' : (S', s_0) \to (R^{\log}, r_0)$ such that $(X, \mathcal{M}_X)|_{S'} \simeq h'^*(\mathfrak{X}_{R^{\log}}, \mathcal{M}_{\mathfrak{X}_{R^{\log}}})$. Let $\mathfrak{q}^\dagger$ be the canonical relative log cuirasse on $(\mathfrak{X}_{Z^{\log}}, \mathcal{M}_{\mathfrak{X}_{Z^{\log}}})$ over $Z^{\log}$ (see (5.7)). Then, $h'^*\mathfrak{q}^\dagger$ is a relative log cuirasse on $(X, \mathcal{M}_X)|_{S'}$ over S', whose associated morphism (see (5.10)) coincides with h'. Since, by Proposition 5.19, $\mathcal{Q}_S(X, \mathcal{M}_X)$ is smooth over S in a neighborhood of $q_0^\dagger \in \mathcal{Q}((X(s_0), \mathcal{M}_{X(s_0)})$, there exists a relative cuirasse $q^\dagger$ on $(X, \mathcal{M}_X)$ over S, such that $q^\dagger|_{S'} = h'^*\mathfrak{q}^\dagger$. Let $\tilde{h} : S \to Z^{\log}$ be the morphism associated to $q^\dagger$ (see (5.10)) and $\pi_{R^{\log}} : Z^{\log} \to R^{\log}$ the projection. Then, $h := \pi_{R^{\log}} \circ \tilde{h}$ satisfies $(X, \mathcal{M}_X) \simeq h^*(\mathfrak{X}_{R^{\log}}, \mathcal{M}_{\mathfrak{X}_{R^{\log}}})$ and $h|_{S'} = h'$.

□

5.3 Versal deformation of log morphisms

In what follows, we construct a versal deformation of a morphism $f_0 : (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$ of compact fine log complex spaces (see Definition 0.3). Let X be a complex space and $\alpha_i : \mathcal{M}_i \to \mathcal{O}_X$, $i = 1, 2$, two fine log structures on X. Let $\gamma : \overline{\mathcal{M}}_1 \to \overline{\mathcal{M}}_2$ be a morphism of the ghost sheaves (see Definition 2.13). Let $f : T \to X$ be a morphism of complex spaces and set $\gamma_T : (\overline{\mathcal{M}}_1)_T \to (\overline{\mathcal{M}}_2)_T$, the pull-back of γ via f.

Lemma 5.33. *([GS13], p. 474) The functor*

$$\mathrm{Mor}_X^{\log} : An_X \to Sets$$

defined on the objects by

$$(f : T \to X) \mapsto \{\varphi : (T, f^*\mathcal{M}_1) \to (T, f^*\mathcal{M}_2) | \overline{\varphi}^\flat = \gamma_T\}$$

is represented by a complex space $\mathfrak{Mor}_X^{\log}(\mathcal{M}_1, \mathcal{M}_2)$ over X.

Proof. By the universal property, the statement is local in X. Hence, let $\beta_i : P_i \to \Gamma(X, \mathcal{M}_i)$, $i = 1, 2$, be two log charts for $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Let $p_1, ..., p_n \in P_1$ be a generating set for P_1 as monoid. Consider the sheaf of finitely generated $\mathcal{O}_X$-algebras

$$\mathcal{F}_X := \mathcal{O}_X[P_1^{gp}]/\langle \alpha_1(\beta_1(p_i)) - z^{p_i}\alpha_2(\beta_2(\gamma(p_i)))|1 \leq i \leq n\rangle.$$

Set $\mathfrak{Mor}_X^{\log}(\mathcal{M}_1, \mathcal{M}_2) := \mathbf{Spec_{an}}\mathcal{F}_X$, the relative analytic spectrum of $\mathcal{F}_X$ over X. Now, we check the universal property. Let $f : T \to X$ be given. We want to show that giving a commutative diagram of complex spaces

$$\begin{tikzcd} T \arrow[r, "g"] \arrow[dr, "f"'] & \mathbf{Spec_{an}}\mathcal{F}_X \arrow[d] \\ & X \end{tikzcd}$$

is equivalent to giving a log morphism $\varphi : (T, f^*\mathcal{M}_1) \to (T, f^*\mathcal{M}_2)$, which is the identity on X and such that $\overline{\varphi}^\flat = \gamma_T$. Giving a morphism g is equivalent to giving a section of $(\mathbf{Spec_{an}}\mathcal{F}_X) \times_X T$ over T. But

$$(\mathbf{Spec_{an}}\mathcal{F}_X) \times_X T = \mathbf{Spec_{an}}\mathcal{O}_T[P_1^{gp}]/\langle f^*(\alpha_1(\beta_1(p_i))) - z^{p_i} f^*(\alpha_2(\beta_2(\gamma(p_i))))|1 \leq i \leq n\rangle,$$

and the latter complex space is $\mathbf{Spec_{an}}\mathcal{F}_T$ associated to the data $(T, f^*\mathcal{M}_1)$,$(T, f^*\mathcal{M}_2)$ with charts $f^*(\beta_i) = f^\flat \circ \beta_i : P_i \to \Gamma(Y, f^*\mathcal{M}_i)$. Thus, without loss of generality, we can assume $T = X$ and f is the identity. Now, giving $\varphi : (X, \mathcal{M}_1) \to (X, \mathcal{M}_2)$, with $\overline{\varphi}^\flat = \gamma$, is equivalent to specifying $\varphi^\flat$. From $\varphi^\flat$ we obtain a map $\eta : P_1 \to \Gamma(X, \mathcal{O}_X^\times)$ with the property that for all $p \in P_1$,

$$\varphi^\flat(\beta_1(p)) = \eta(p) \cdot \beta_2(\gamma(p)).$$

Conversely, η completely determines $\varphi^\flat$. In addition, $\varphi^\flat$ is a homomorphism of monoids if and only if η is a homomorphism, and since η takes values in the group $\mathcal{O}_X^\times$, specifying $\varphi^\flat$ is equivalent to specifying a section of $\mathbf{Spec_{an}}\mathcal{O}_X[P_1^{gp}]$. Indeed, a section of $\mathbf{Spec_{an}}\mathcal{O}_X[P_1^{gp}]$ over X is the same as a morphism $X \to \operatorname{Spec}\mathbb{C}[P_1^{gp}]$, which in turn is the same as an element of $\operatorname{Hom}(P_1, \Gamma(X, \mathcal{O}_X^\times))$. Second, since $\varphi^* = id$, we must have $\alpha_1 = \alpha_2 \circ \varphi^\flat$, so for each $p \in P_1$, we must have

$$\alpha_1(\beta_1(p)) = \alpha_2(\varphi^\flat(\beta_1(p))) = \eta(p) \cdot \alpha_2(\beta_2(\gamma(p))).$$

If this holds for each p_i, it holds for all p. Thus a section of $\mathbf{Spec_{an}}\mathcal{O}_X[P_1^{gp}]$ over X determines a morphism of log structures if and only if it lies in the subspace determined by the equations

$$\alpha_1(\beta_1(p_i)) - z^{p_i}\alpha_2(\beta_2(\gamma(p_i))),$$

demonstrating the result. □

Now, assume the complex space X is proper over a germ of complex spaces (S, s_0).

Proposition 5.34. *([GS13], p. 475) The functor*

$$\mathrm{Mor}^{\log}_{X/S} : (f : (T, t_0) \to (S, s_0)) \mapsto \{\varphi : (X_T, (\mathcal{M}_1)_T) \to (X_T, (\mathcal{M}_2)_T) | \overline{\varphi}^\flat = \gamma_T\}$$

is represented by a germ $\mathfrak{Mor}^{\log}_{X/S}(\mathcal{M}_1, \mathcal{M}_2)$ *of complex spaces over* (S, s_0).

Proof. Let $Z = \mathfrak{Mor}^{\log}_{X}(\mathcal{M}_1, \mathcal{M}_2)$. By Lemma 5.33, $\mathfrak{Mor}^{\log}_{X/S}(\mathcal{M}_1, \mathcal{M}_2)$ is isomorphic to the functor

$$(\psi : T \to S) \mapsto \{\text{sections of } \psi^* Z \to \psi^* X\}.$$

This is exactly the functor of sections $\prod_{X/S}(Z/X)$ discussed, in the algebraic-geometric setting, in [Gro95], p. 267 and here it is represented by an open subspace of the relative Douady space of Z over S (see Section 3.4 and [Pou69]). □

Theorem 5.35. *Every morphism* $f_0 : (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$ *of compact fine log complex spaces admits a versal deformation parametrized by a germ of complex spaces* (S, s_0).

Proof. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \to (R, r_0)$ and $(\mathcal{Y}, \mathcal{M}_{\mathcal{Y}}) \to (R, r_0)$ be the versal deformations of $(X_0, \mathcal{M}_{X_0})$ and $(Y_0, \mathcal{M}_{Y_0})$ respectively given by Theorem 5.32. By pulling-back to the product of the base spaces, we can assume that the two deformations are defined over the same base space. Let us consider the finite dimensional complex analytic space $\mathfrak{Mor}_R(\mathfrak{X}, \mathcal{Y})$ given by Theorem 3.82. Let $p : \mathfrak{Mor}_R(\mathfrak{X}, \mathcal{Y}) \to R$ be the projection and set $\underline{m}_0 := (r_0, \underline{f}_0)$. By Theorem 3.82, we get a universal morphism $\underline{f} : p^*\mathfrak{X} \to p^*\mathcal{Y}$, such that the restriction of $\underline{f}$ to the central fibre $p^*\mathfrak{X}(\underline{m}_0)$ equals $\underline{f}_0$. We can consider two fine log structures on $p^*\mathfrak{X}$, namely $\mathcal{M}_1 := p^*\mathcal{M}_{\mathfrak{X}}$ and $\mathcal{M}_2 := \underline{f}^* p^*\mathcal{M}_{\mathcal{Y}}$. Set $\gamma := \overline{f}_0^\flat$ and $m_0 := (\underline{m}_0, f_0^\flat)$. For the sake of clarity, denote $\mathfrak{M} := \mathfrak{Mor}_R(\mathfrak{X}, \mathcal{Y})$. Now, consider the germ of complex spaces $(\mathfrak{Mor}^{\log}_{p^*\mathfrak{X}/\mathfrak{M}}(\mathcal{M}_1, \mathcal{M}_2), m_0)$, together with the projection

$$\pi : (\mathfrak{Mor}^{\log}_{p^*\mathfrak{X}/\mathfrak{M}}(\mathcal{M}_1, \mathcal{M}_2), m_0) \to (\mathfrak{Mor}_R(\mathfrak{X}, \mathcal{Y}), \underline{m}_0),$$

provided by Proposition 5.34. Moreover, by Proposition 5.34, we get a morphism $f^\flat : \pi^*(p^*\mathfrak{X}, \mathcal{M}_1) \to \pi^*(p^*\mathfrak{X}, \mathcal{M}_2)$. Hence, we get a log morphism

$$f := (\pi^*\underline{f}, f^\flat) : \pi^* p^*(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \to \pi^* p^*(\mathcal{Y}, \mathcal{M}_{\mathcal{Y}})$$

over $(\mathfrak{Mor}^{\log}_{p^*\mathfrak{X}/\mathfrak{M}}(\mathcal{M}_1, \mathcal{M}_2), m_0)$. Set $(S, s_0) := (\mathfrak{Mor}^{\log}_{p^*\mathfrak{X}/\mathfrak{M}}(\mathcal{M}_1, \mathcal{M}_2), m_0)$. Using the universal property of (S, s_0) (see Proposition 5.34) and the versality property of (R, r_0) (see Theorem 5.32), the statement follows. □

Moreover, we can deform $(X_0, \mathcal{M}_{X_0})$ as relative log space over $(Y_0, \mathcal{M}_{Y_0})$. That is, we can deform $(X_0, \mathcal{M}_{X_0})$ together with the morphism f_0 into $(Y_0, \mathcal{M}_{Y_0})$. In this case, Y_0 needs not to be compact. More precisely,

Definition 5.36. Let $f_0 : (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$ be a log morphism of fine log complex spaces, with X_0 compact. A versal deformation of $(X_0, \mathcal{M}_{X_0})$ *over* $(Y_0, \mathcal{M}_{Y_0})$, with base a germ of complex spaces (S, s_0), is a commutative diagram

$$\begin{tikzcd}
(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \arrow[rr, "f"] \arrow[dr, "p"'] & & (Y_0, \mathcal{M}_{Y_0}) \times S \arrow[dl, "\pi_2"] \\
& (S, s_0) &
\end{tikzcd},$$

where p is a versal deformation of $(X_0, \mathcal{M}_{X_0})$, together with an isomorphism $i : (X_0, \mathcal{M}_{X_0}) \to (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})(s_0)$, such that $f \circ i = f_0$.

The same proof of Theorem 5.35, with $(\mathcal{Y}, \mathcal{M}_{\mathcal{Y}}) := (Y_0, \mathcal{M}_{Y_0}) \times R$, gives us the following

Corollary 5.37. *Let $f_0 : (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$ be a log morphism of fine log complex spaces, with X_0 compact. Then $(X_0, \mathcal{M}_{X_0})$ admits a versal deformation over $(Y_0, \mathcal{M}_{Y_0})$.*

Remark 5.38. If f_0 is a log embedding, then Corollary 5.37 gives us a versal deformation of a log subspace $(X_0, \mathcal{M}_{X_0})$ in a fixed ambient log space $(Y_0, \mathcal{M}_{Y_0})$.

Now, assume f_0 to be a log flat (log smooth) morphism (see Definition 2.35). We show that, in this case, we get a log flat (log smooth) versal deformation of f_0.

Lemma 5.39. *Let $f : X \to Y$ be a continuous map between topological spaces. If f is closed, then for all $y \in Y$ and open subset $U \subset X$ satisfying $f^{-1}(y) \subset U$, there exists an open neighborhood V of y satisfying $f^{-1}(V) \subset U$.*

Proof. Let us consider the closed subset $X \backslash U$. Since f is closed, $f(X\backslash U)$ is closed in Y. Therefore, $Y \backslash f(X\backslash U)$ is open in Y and it contains y as $f^{-1}(y) \subset U$. Take $V := f^{-1}(Y \backslash f(X\backslash U))$. □

Let $f : (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \to (\mathcal{Y}, \mathcal{M}_{\mathcal{Y}})$ be the versal deformation of $f_0 : (X_0, \mathcal{M}_{X_0}) \to (Y_0, \mathcal{M}_{Y_0})$, over a germ of complex spaces (S, s_0), given by Theorem 5.35 or Corollary 5.37. Denote with π_1 and π_2 the morphisms of $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and $(\mathcal{Y}, \mathcal{M}_{\mathcal{Y}})$ into (S, s_0) respectively.

Proposition 5.40. *If f_0 is log flat (log smooth), then f is log flat (log smooth) in an open neighborhood of s_0.*

Proof. Let us assume that there exists an open neighborhood U' of X_0 in $\mathfrak{X}$ such that $f|_{(U', \mathcal{M}_{U'})}$ is log flat (log smooth). Then, since $\pi_1 : \mathfrak{X} \to S$ is a proper map between locally compact Hausdorff spaces, it is closed. Hence, by Lemma 5.39, we can find an open neighborhood W of s_0 such that $\pi_1^{-1}(W)$ is contained in U'. This ensures us that f is log flat (log smooth) as relative morphism over $(W, s_0) \subset (S, s_0)$. Since log flatness (log smoothness) is a local property, we choose a log chart for f (see Definition 2.34). We have the following commutative diagram

$$\begin{tikzcd} U \arrow[r, "\beta"] \arrow[d, "\underline{f}"'] & \operatorname{Spec} \mathbb{C}[P] \arrow[d, "g"] \\ V \arrow[r, "\gamma"] & \operatorname{Spec} \mathbb{C}[Q] \end{tikzcd}.$$

Let us consider the universal morphism $\underline{u} : U \to V \times_{\operatorname{Spec} \mathbb{C}[Q]} \operatorname{Spec} \mathbb{C}[P]$. Let $p : V \times_{\operatorname{Spec} \mathbb{C}[Q]} \operatorname{Spec} \mathbb{C}[P] \to V$ be the projection. We get the following commutative diagram

$$\begin{tikzcd} U \arrow[rr, "\underline{u}"] \arrow[dr, "\pi_1|_U"'] & & V \times_{\operatorname{Spec} \mathbb{C}[Q]} \operatorname{Spec} \mathbb{C}[P] \arrow[dl, "\pi_2 \circ p"] \\ & (S, s_0) & \end{tikzcd}.$$

Assume f_0 log flat. By Definition 2.35, we have that $\underline{u}$ is flat at s_0. Moreover, by Theorem 5.32, $\pi_1|_U$ is flat too. For the sake of clarity, set $A := V \times_{\operatorname{Spec} \mathbb{C}[Q]} \operatorname{Spec} \mathbb{C}[P]$. We use Proposition 1.28 for $\mathcal{F} = \mathcal{O}_U$. Since *condition* 1 holds, by *condition* 2, we get that $\mathcal{O}_{U,x}$ is a flat $\mathcal{O}_{A,\underline{u}(x)}$-module, for each $x \in \pi_1^{-1}|_U(s_0)$. By Proposition 1.27, *statement 4.*, we get the existence of an open subset U' of U, containing $\pi_1^{-1}|_U(s_0)$, such that $\underline{u}_{|U'}$ is flat. This proves the first part of the statement. Now, assume f_0 log smooth. By Proposition 2.37, f_0 is log flat. Hence, by the first part of this proof, we get the existence of an open subset U' in U such that $\underline{u}_{|U'}$ is flat. Let $x \in \pi_1^{-1}|_{U'}(s_0)$ and set $y := \underline{u}(x) \in V \times_{\operatorname{Spec} \mathbb{C}[Q]} \operatorname{Spec} \mathbb{C}[P]$. By Definition 2.35, $\underline{u}$ is smooth at s_0. Hence, by Definition 1.26, we get that the fibre U'_y of $\underline{u}_{|U'}$ over y is a manifold. Therefore, using Proposition 1.29, we get the second part of the statement.

□

Example 5.41. Let $(\operatorname{Spec} \mathbb{C}, Q)$ be a log point (see Example 2.17). Let $(X_0, \mathcal{M}_{X_0})$ be a compact fine log complex space and $f_0 : (X_0, \mathcal{M}_{X_0}) \to (\operatorname{Spec} \mathbb{C}, Q)$ a log smooth morphism. Since $\operatorname{Hom}((Q, +), (\mathbb{C}, \cdot)) = \operatorname{Spec} \mathbb{C}[Q]$, a versal deformation of the log point $(\operatorname{Spec} \mathbb{C}, Q)$ is given by the affine toric variety $\operatorname{Spec} \mathbb{C}[Q]$ endowed with the canonical log structure (see Example 2.25). Let $p_0 \in \operatorname{Spec} \mathbb{C}[Q]$ be the base point. Let $(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \to (R, r_0)$ be the versal deformation of $(X_0, \mathcal{M}_{X_0})$ given by Theorem 5.32. Let $R \times \operatorname{Spec} \mathbb{C}[Q]$ and consider the projections π_1, π_2 onto the first and second factor respectively. Then, $\pi_1^*(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$ and $\pi_2^* \operatorname{Spec}(Q \to \mathbb{C}[Q])$ are versal deformations of $(X_0, \mathcal{M}_{X_0})$ and $(\operatorname{Spec} \mathbb{C}, Q)$ over $R \times \operatorname{Spec} \mathbb{C}[Q]$ respectively . Let $(r_0, p_0) \in$

$R \times \operatorname{Spec} \mathbb{C}[Q]$ be the base point. By Theorem 5.35, we get a germ of complex spaces (S, s_0), together with a morphism of germs $p : (S, s_0) \to (R \times \operatorname{Spec} \mathbb{C}[Q], (r_0, p_0))$, and a log S-morphism $f : p^*\pi_1^*(\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) \to p^*\pi_2^* \operatorname{Spec}(Q \to \mathbb{C}[Q])$, which is a versal deformation of f_0 . By Proposition 5.40, f is log smooth in a neighborhood of $(X_0, \mathcal{M}_{X_0})$.

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