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Adaptive control for boundary control systems

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Abstract

In the present thesis we consider systems which are modeled by partial differential equations taking the form of evolution equations. Observation and control of the state x typically occur at the boundary of the spatial domain where the state evolves in. From technical and practical considerations, the inputs u and outputs y of the system are assumed to be finite-dimensional, which means that the system has a finite amount of actuators.

The aim of this thesis is to design adaptive controllers for such systems in order to track a prescribed reference signal y_{ref}. In fact, one could attempt to use the high-gain controller $u = -k(y - y_{\mathrm{ref}})$ for $k > 0$. However, the performance of this controller strongly depends on the gain k. To solve this problem, we will make use of the funnel controller, where one defines a time-varying gain $k(\cdot)$, so that only large values of $k(t)$ are used when required. Moreover, the funnel controller takes into account the transient behavior of the error $e \coloneqq y - y_{\mathrm{ref}}$: for a specified performance funnel ψ, it can be guaranteed that $\|e(t)\| < \psi(t)$.

As a motivating example, we consider a linearized model of a water tank, for which we control the force of the motor and observe the absolute distance from the origin to the tank. Inspired by this example, we also study a large class of systems with infinite-dimensional internal dynamics. By using the existing theory on the funnel controller, we are able to show that the controller is feasible for both the model and this class.

However, it is known that the results about funnel control are not always applicable, in particular when dealing with systems modeled by partial differential equations. The evolution equations that one often encounters resemble $\dot{x} = \mathfrak{A}x$, where $\mathfrak{A}$ is a differential operator acting on a spatial domain Ω. Moreover, the observation and control interactions are often modeled by two additional operators which include evaluations of the state x at the boundary of Ω. The inputs and outputs of the system can be represented as $u = \mathfrak{B}x$ and $y = \mathfrak{C}x$, respectively. We want to study the applicability of the funnel controller to boundary control systems of the form described above. These systems enclose both parabolic and hyperbolic equations. In order to capture this feature arising from the infinite-dimensionality of the problem, the controller needs to be slightly modified. These systems usually come from physical models and have a rich underlying structure that can be exploited. We make use of nonlinear, m-dissipative operator theory to show that the funnel controller is feasible for the system class. Moreover, we are able to take advantage of the parabolic structure of the problem to show more regularity of the solution. On top of that, we demonstrate the application of these results with some examples. By using the funnel controller one could for instance control heat transfer, diffusion processes, voltage of electric circuits or the bending of beams.

Finally, we consider a nonlinear parabolic system which is often used to model the electric current in heart cells. We study the applicability of the funnel controller not

only in the boundary control scenario, but also in the distributed one, that is, the control takes place in the spatial domain Ω. This controller could be used to develop heart pacemakers, where the reference signal y_{ref} represents the natural heart rhythm. We prove the feasibility of the funnel controller for this system and use again the parabolicity of the problem to show more regularity of the solution.

Zusammenfassung

In der vorliegenden Dissertation werden Systeme betrachtet, die durch partielle Differentialgleichungen modelliert werden. Beobachtung und Steuerung des Zustandes x geschehen typischerweise am Rand des Gebiets, in dem sich der Zustand entwickelt. Aus technischen und praktischen Erwägungen wird angenommen, dass die Systemeingänge u und Systemausgänge y endlich dimensional sind, was bedeutet, dass das System eine endliche Menge Aktuatoren besitzt.

Das Ziel dieser Dissertation ist es, adaptive Regler für solche Systeme zu entwerfen, um ein vorgeschriebenes Referenzsignal y_{ref} zu verfolgen. Natürlich könnte man versuchen, den Hochverstärkungsregler $u = -k(y - y_{\text{ref}})$ für $k > 0$ zu benutzen. Die Leistung dieses Reglers hängt jedoch stark von der Hochverstärkungskonstante k ab. Um dieses Problem zu lösen, werden wir den Trichter-Regler (Funnel controller) anwenden. Bei Deisem benutzt man einen zeitabhängigen Hochverstärkungsfaktor $k(\cdot)$, sodass nur große Werte $k(t)$ angenommen werden, wenn sie nötig sind. Darüber hinaus fließt das transiente Verhalten des Fehlers $e := y - y_{\text{ref}}$ durch den Trichter-Regler in das Reglermodell ein: für einen gewählten Performanz-Trichter ψ kann gewährleistet werden, dass $\|e(t)\| < \psi(t)$.

Als motivierendes Beispiel betrachten wir ein linearisiertes Modell eines Wassertanks, dessen Motorkraft wir regulieren und dessen absolute Distanz vom Ursprung zum Tank wir beobachten. Durch dieses Beispiel inspiriert, studieren wir zusätztlich eine breite Klasse von Systemen mit endlich dimensionaler interner Dynamik. Die Anwendung existenter Theorie über den Trichter-Regler ermöglicht uns zu zeigen, dass der Regler für sowohl das Modell als auch die Systemklasse im gewünschten Sinne funktioniert.

Die bisherigen Resultate über Trichter-Regelung schließen interessante Anwendungsbeispiele nicht mit ein, insbesondere durch partielle Differentialgleichungen modellierte Systeme. Die Evolutionsgleichungen, welchen man oft begegnet, können durch $\dot{x} = \mathfrak{A}x$ beschrieben werden, wobei $\mathfrak{A}$ ein Differentialoperator ist, der im Gebiet Ω agiert. Zusätztlich werden die Interaktionen von Beobachtung und Steuerung oft durch zwei Operatoren modelliert. Klassischerweise sind Diese Auswertungen des Zustandes x am Rand von Ω. Die Ein- und Ausgänge des Systems können jeweils als $u = \mathfrak{B}x$ und $y = \mathfrak{C}x$ dargestellt werden. Wir wollen den Trichter-Regler auf Randsteuerungssysteme anwenden, die die beschriebene Form besitzen. Diese Systeme enthalten sowohl parabolische als auch hyperbolische Gleichungen. Um diese Besonderheit einzufangen, die aus der unendlichen Dimensionalität des Problems entsteht, muss der Regler leicht modifiziert werden. Normalerweise ergeben sich diese Systeme aus physikalischen Modellen und besitzen eine spezielle Struktur, welche man ausnutzen kann. Wir werden Theorie über nichtlineare, m-dissipative Operatoren benutzen, um zu beweisen, dass der Trichter-Regler für diese Systemklasse funktioniert. Außerdem sind wir imstande, die parabolische Struktur des Problems auszunutzen, um eine höhere Regularität der Lösung zu beweisen. Zusätzlich dazu veranschaulichen wir die Anwendung dieser

Ergebnisse mit einigen Beispielen. Durch die Trichter-Regelung könnten beispielsweise Wärmeleitung, Diffusionsprozesse, elektrische Spannungen von Schaltkreisen oder Krümmungen von Balken geregelt werden.
Zum Scluss betrachten wir ein nichtlineares parabolisches System, welches oft benutzt wird, um die elektrische Spannung in Herzzellen zu modellieren. Wir studieren die Anwendbarkeit des Trichter-Reglers nicht nur im Randsteuerungsszenario, sondern auch mit verteilter Steuerung, das heißt, die Reglung geschieht innerhalb des Gebiets Ω. Diese Regelung könnte benutzt werden, um Herzschrittmacher zu entwickeln, bei denen das Referenzsignal y_{ref} der natürlichen Herzfrequenz entspricht. Wir beweisen, dass der Trichter-Regler auf dieses System anwendbar ist und wir nutzen die Parabolizität des Problems aus, um eine höhere Regularität der Lösung zu zeigen.

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Contents

Introduction

Since the late XVIII$^{\text{th}}$ century with the contribution of Joseph-Louise Lagrange and the *Lagrangian formalism* of mechanics, the modeling of time-continuous dependent systems has become quite straightforward. With the reformulation of the classical mechanics due to William R. Hamilton in the early XIX$^{\text{th}}$ century, an energy-based approach arouse by simply applying a Legendre transformation of the Lagrangian with the generalized coordinates q and momenta p, which gave rise to the *Hamiltonian formalism*. This lead to the well-known equations of motion

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}, \\ \dot{p} &= -\frac{\partial H}{\partial q}, \end{aligned}$$

where H is the Hamiltonian, which in many cases coincides with the total energy of the system [38]. By using the *Hamilton's Principle* or the *Principle of Minimal Action* and this abstract formalism, one can additionally incorporate the dependence of q and p with respect to the position. One of the most well-known examples of such a system is the wave equation, which describes the vertical displacement of a string on a segment of length L over the time. For $t \in (0, \infty)$ and $\zeta \in (0, L)$, this is commonly given as

$$\begin{aligned} \rho w_{tt}(t,\zeta) &= T w_{\zeta\zeta}(t,\zeta), \\ w(0,\zeta) &= w_0(\zeta), \\ w_t(0,\zeta) &= v_0(\zeta), \end{aligned} \tag{0.1}$$

where T is the Young modulus, ρ is the linear mass density, w_0 is the initial elongation of the string and v_0 its initial speed. Since the function w is implicitly given by its partial derivatives, the wave equation is a *Partial Differential Equation (PDE)*. This is in fact one of the so-called equations of the mathematical physics, which also comprehend the heat and the Laplace equations [120]. All of them are linear PDEs of second order which are representative for one of the most common form of classification: hyperbolic, parabolic and elliptic, respectively.

Hyperbolic and parabolic PDEs play an important role in the modeling of physical systems which have time and position dependence. In the hyperbolic scenario one has most of equations of theoretical physics such as the Schrödinger, Klein-Gordon or the Dirac equations [86, 104]. Parabolic equations are typically used to model diffusive

systems like fluids or thermal propagation. This kind of PDEs can be viewed in such a way that for each time, one needs to find a function, which represents the current state, in contrast to the usual Cauchy problem where at each time one has a point in $\mathbb{R}^n$. This is why we will often use the terminology *infinite-dimensional* to refer to problems like (0.1), while we will use *finite-dimensional* for *Ordinary Differential Equation (ODE).*
In several applications, these models include *inputs* and *outputs*, that is, one can somehow influence the system and measure something from it. For instance, one could control the force per unit length applied to (0.1) at the boundary, namely,

$$Tw_\zeta(t, L) = u_1(t), \quad -Tw_\zeta(t, 0) = u_2(t),$$

and $u = (u_1, u_2)^\top$ are the controls, which are prescribed. One could also measure the speed at the boundary, that is,

$$y_1(t) = w_t(t, L), \quad y_2(t) = w_t(t, 0),$$

and $y = (y_1, y_2)^\top$ is the output of the system, which is observed. The energy of the system is the sum of the kinetic and potential energy and is given by the expression

$$E(t) = \frac{1}{2}\int_0^L \rho w_t(t,\zeta)^2 + Tw_\zeta(t,\zeta)^2 \,\mathrm{d}\zeta\,.$$

Note that by using integration by parts and (0.1), the power is then given by

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = u_2(t)y_2(t) + u_1(t)y_1(t), \tag{0.2}$$

that is, the power equals the product of the force and velocity.
The variable which is described by the PDE is usually called *state* of the system or *state variable*, and it is important to bear in mind how it relates to the inputs and outputs of the system. Often one wants to achieve that the output y of a system behaves in a specific, desired way. The several approaches to this issue are known as control theory, and among them, we will use adaptive control. Roughly speaking, one couples u and y in such a way that the resulting *closed-loop system* has a solution for which the output behaves as a prescribed reference trajectory y_{ref}. The simplest case is the stabilization of the system, where $y_{\mathrm{ref}} \equiv 0$. In the example of the wave equation, this can be achieved by using the high-gain controller $u = -ky$, with $k > 0$.
The aim of this work is to develop an adaptive controller for a large class of systems whose state will be mostly described by a linear partial differential equation where the control and observation of the system occur at the boundary of the spatial domain of the state variable.
Adaptive control of infinite-dimensional systems has been addressed over the past years by several authors, [10, 19, 55, 68, 71–74, 77–79]. Needless to say, the fact of dealing with PDEs makes it of course challenging and the approaches used strongly lay on

the particular cases which are investigated. In these works, the problems investigated often distinguish between parabolic and hyperbolic PDEs and many of them only deal with systems with one spatial dimension. Moreover, their controller design specifically uses parameters or estimations of the parameters of the model, which could jeopardize the robustness of the controller. Importantly, none of them deals with the problem of tracking a prescribed reference signal which is a measurement of the system.

The funnel controller

Ideally, one would like to only make structural assumptions on the system such as the energy dissipation (0.2) in order to design a controller so that for a given reference signal y_{ref}, the error $e := y - y_{\text{ref}}$ can be controlled.
By comparing to the finite-dimensional case, one realizes that two major problems quickly arise. For instance, for the linear prototype

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t)$$

the input affects the state directly, which does not seem to be a realistic assumption when dealing with PDEs, since it is unlikely that one can influence the whole spatial domain. Hence, one needs to think more carefully about the meaning of the operator B. Moreover, A will be a differential operator, so the space on which it acts and where it is defined plays a crucial role. Secondly, the design of the controller strongly depends on the structure of the *transfer function*, which is a rational function describing the input-output behavior of the system in the frequency domain. In the infinite-dimensional case, this function is no longer rational, so the usual techniques involving concepts such as the *relative degree* may not be applicable.
However, the finite-dimensional setting already provides a large amount of possible controllers one could attempt to use or extend to the infinite-dimensional scenario. The simplest one might possibly be the high-gain controller, that is, a feed-back relation of the form

$$u(t) = -k(y(t) - y_{\text{ref}}(t)),$$

where $k > 0$ is the gain. A more sophisticated alternative can be realized by designing an adaptive gain k. More precisely, k should grow when required and remain small when the output is close to the reference signal. To do so without increasing the mathematical complexity excessively, this k should not add any extra dynamics — differential equations— to the system. Furthermore, the controller should be easy to implement. Taking all of this into consideration, we choose the *funnel controller (FC)*, first introduced by [60] and developed in [61], which satisfies all the requirements with the additional property, that one can control the error $e(t)$ during the transient behavior.

The FC is a model-free output-error feedback of high-gain type. Therefore, it is inherently robust and of striking simplicity. The main idea of the controller is to make the gain k adaptive in such a way that if the error e comes closer to a time depending boundary, the gain grows so that the error decreases due to the *internal dynamics (ID)* of the state. In its simplest form the controller looks like

$$u(t) = -\frac{k_0}{1-\varphi(t)^2 e(t)^2} e(t),$$

and φ^{-1} serves as the aforementioned boundary. Note that if φ tends asymptotically to a value λ, then the error will be asymptotically bounded by λ^{-1}. By choosing a specific function φ, one obtains that the error is enclosed in a concrete region usually denoted by $\mathcal{F}_\varphi$, which is defined by

$$\mathcal{F}_\varphi := \{(t,e) \in [0,\infty) \times \mathbb{R} \mid \varphi(t)|e| < 1\}.$$

In fact, this region resembles a *funnel* when depicted, see Fig. 0.1, and $\mathcal{F}_\varphi$ is called *performance funnel*.

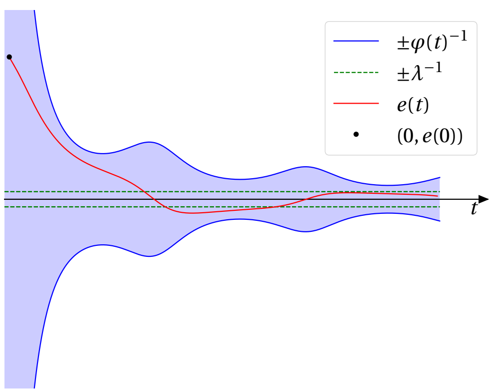

Figure 0.1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$.

In [60] the feasibility of the funnel controller for a class of functional differential equations has been shown. These encompass infinite-dimensional systems with very restrictive assumptions on the operators involved, a special class of nonlinear finite-dimensional systems and nonlinear delay systems. In fact, finite-dimensional linear prototype systems with relative degree one are treated therein. The relative degree is a well-known magnitude for finite-dimensional systems and can roughly be understood as the number of times one needs to differentiate y so that u appears in the equation. This quantity turned out to be relevant when considering the funnel controller and has been used to generalize the results of [60]. For instance, in [57], the funnel controller was proved to

be applicable for systems with known but arbitrary relative degree. The problem is that the ansatz used therein requires very large powers of the gain factor $k(t)$. This problem has been overcome in [11] by introducing a funnel controller which involves derivatives of the output and reference signal, and feasibility of this controller in the case of nonlinear finite-dimensional systems with *strict relative degree* having *stable internal dynamics* has been proven. For a survey regarding the first steps of the FC and high-gain adaptive control we refer to [58].
Moreover, the FC has been successfully applied e.g. in temperature control of chemical reactor models [64], control of industrial servo-systems [49] and underactuated multibody systems [12], speed control of wind turbine systems [46, 48, 49], current control for synchronous machines [47, 49], DC-link power flow control [105], voltage and current control of electrical circuits [16], oxygenation control during artificial ventilation therapy [96], control of peak inspiratory pressure [97] and adaptive cruise control [15].
For infinite-dimensional systems the FC has so far only attracted attention in special configurations [13, 63, 101]. The recent article [13] deals with a linearized model of a moving water tank by showing that this system belongs to the class being treated in [11]. On the other hand, not even every linear, infinite-dimensional system has a well-defined and integer-valued relative degree, in which case the results as in [11, 61] cannot be applied. Instead, the feasibility of funnel control has to be investigated directly for the (nonlinear) closed-loop system, see [101] for a boundary controlled heat equation and [99] for a general class of boundary control systems. In [63], a class of infinite-dimensional systems has been considered that allows to prove feasibility of the funnel controller in a similar way as for finite-dimensional systems. More precisely, this class consists of systems which possess a so-called *Byrnes-Isidori form* via bounded and boundedly invertible state space transformation. The existence of such a form however requires that the control and observation operators fulfill very strong boundedness conditions, which in particular exclude boundary control and observation. Funnel control of a heat equation with Neumann boundary control and co-located Dirichlet output has been treated in [101]. The proof of feasibility of funnel control uses the spectral properties of the Laplacian, whence this technique is hardly transferable to further classes of boundary control systems.

Content overview

The dissertation contains a mathematical introduction, four major projects and conclusions. The chapters corresponding to the respective projects are organized in increasing complexity. In Chapter 1 we provide the basic notation and develop the mathematical essentials for the subsequent parts. In Chapter 2 we deal with a practical example to which we aim to apply the FC, which already motivates to develop a theory for the FC when the dynamics of the system are given by a PDE. There we consider a

linerized model of a linear water tank, where we control the force acting on the system (for instance the one provided by the motor of a truck) and observe the position. In order to show feasibility of the FC we make use of the existing theory for the FC. The content presented in this chapter corresponds to the results given in [13].
In Chapter 3 we work with an abstract setting which resembles the one seen in Chapter 2 and it has been presented in [14]. Nevertheless, we make no particular assumptions in how the operators look like, or what they model. With the appropriate structure it is quite straightforward to show that the FC is feasible for this system class by using the existing theory regarding the applicability of the FC.
Noting that in Chapter 2 we have treated systems with boundary control and observation and that the internal dynamics of Chapter 2 could describe boundary control systems, we move into Chapter 4 bearing in mind what it has been discussed exemplary with Equation (0.1). This chapter could be seen as the core of this work. There we introduce a class of *boundary control systems (BCS)* and consider from the very beginning the closed-loop system induced by the feedback law of the FC, which requires some minor modifications in order to deal with the system class in the most general way. The results presented in this chapter will make use of nonlinear analysis, in particular nonlinear, m-dissipative operators. We will provide a set of assumptions that the operators describing the BCS need to satisfy in order to apply the FC successfully. We illustrate the main result by applying it to three different classes of systems which fit in the framework. This is an extended version of [99]. It is worth mentioning, that the results presented in Chapter 4 can not be proved in general with the existing funnel theory as we did in Chapter 2 & 3.
With some of the techniques and knowledge acquired in Chapter 4, we approach in Chapter 5 a fully nonlinear parabolic PDE, which represents a reaction diffusion equation that models defibrillation processes of the human heart. From the application, it is possible to use both distributed and boundary control, that is, the control can be performed in some cells of the spatial domain or at its boundary. In order to make the whole setting as complete and compact as possible, we consider a co-located control and observation scenario with arbitrary operators that will cover these two options, which is achieved by introducing abstract Sobolev spaces.
Finally we conclude with Chapter 6, where we gather some thoughts and conclusions from the previous chapters.

1 Mathematical background

In the present chapter we give the notation that will be used in this thesis, which happens to be quite standard. Later on we move to defining and giving known results in functional analysis that will be used afterwards. $\mathbb{N}$ is the set of the natural numbers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, whereas $\mathbb{Z}$ is the set of integers. $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are the fields of rations, real and complex numbers, respectively. We will use $\mathbb{K}$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ indistinguishably. The imaginary unit will be denoted by i. For a complex number $z \in \mathbb{C}$, we denote by $\operatorname{Re} z$ the real part, by $\operatorname{Im} z$ the imaginary part, by $\overline{z}$ the complex conjugate and the absolute value by $|z|$. For $\alpha \in \mathbb{R}$, we define $\mathbb{C}_\alpha := \{z \in \mathbb{C} \mid \operatorname{Re} z > \alpha\}$. For $n, m \in \mathbb{N}$, the sets $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the vector spaces of n-tuples of real and complex numbers, respectively. In the same way, $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ denote the sets of real and complex $n \times m$-matrices. The set of real and complex invertible $n \times n$-matrices is abbreviated by $\mathbf{Gl}_n(\mathbb{R})$ and $\mathbf{Gl}_n(\mathbb{C})$, respectively. $A^\top$ denotes the transpose and A^* the Hermitian of $A \in \mathbb{C}^{n\times m}$. The identity matrix in $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ is I_n.

1.1 Normed vector spaces

For a normed space X, the norm is denoted by $\|\cdot\|_X$, and for an inner product space X, the scalar product by $\langle \cdot, \cdot \rangle_X$. For normed spaces $X_1, \dots, X_n$ the *product space*

$$X := \bigotimes_{i=1}^{n} X_i = X_1 \times \cdots \times X_n$$

is made a normed space via $\|(x_1, \dots, x_n)\|_X^2 := \sum_{i=1}^n \|x_i\|_{X_i}^2$. Note that one may define equivalent norms in the product space, see [2, Theorem 1.22]. In the case that the normed or inner product spaces are complete, we shall call them *Banach* or *Hilbert spaces* as it is common. From the *parallelogram law* —see [103, pp. 307]—, speaking of an inner product space X and saying that it has norm $\|\cdot\|_X$, implicitly defines the scalar product $\langle \cdot, \cdot \rangle_X$. For a subset $S \subset X$ we denote its topological closure by $\overline{S}$ and interior by $\operatorname{int} S$. We say that S is *dense* in X if $\overline{S} = X$. It is known that every normed space X can be densely embedded into a Banach space which is called the *completion* of X. Similarly, it holds that the completion of an inner product space is a Hilbert space. Banach spaces which posses a countable and dense subset are called *separable.* For X and Y arbitrary normed spaces, $\mathscr{L}(X, Y)$ denotes the vector space of bounded linear operators from X to Y. It is known that, if at least Y is a Banach space,

$\mathscr{L}(X,Y)$ is also a Banach space with the operator norm

$$\|T\|_{\mathscr{L}(X,Y)} := \sup_{\substack{x\in X\\ \|x\|_X\leq 1}} \|Tx\|_Y.$$

We simply write $\mathscr{L}(X) := \mathscr{L}(X,X)$. As in the matrix case, the identity operator will be denoted by $I_X \in \mathscr{L}(X)$ or simply I.
We briefly introduce the dual space. For a normed vector space X over $\mathbb{K}$, the *topological dual* X' consists of all $x' \in \mathscr{L}(X,\mathbb{K})$. The elements of X' are called *linear functionals* and with the usual operator norm, X' is known to be a Banach space. Further, for $x' \in X'$ we denote $\langle x', x\rangle := x'(x)$. If X is a Hilbert space, one can bijectively embedded X into X' with a mapping $J_X : X' \to X$ by using the [37, 1.7.18 Riesz Representation Theorem], so that $\langle x', x\rangle = \langle J_X x', x\rangle_X$ for all $x \in X$.

1.2 Weak and weak* convergence

We now briefly introduce the *weak* and *weak* convergence*. A sequence $(x_n)_{n\in\mathbb{N}}$ in a normed linear space X is said to be *weakly convergent* if

$$\lim_{n\to\infty} \langle x', x_n\rangle$$

exists and it is finite for each $x' \in X'$. $(x_n)_{n\in\mathbb{N}}$ is said to be *weakly convergent to* $x_\infty \in X$ if

$$\lim_{n\to\infty} \langle x', x_n\rangle = \langle x', x_\infty\rangle$$

for all $x' \in X$. In the latter case, x_∞ is uniquely determined in virtue of the *Hahn-Banach Theorem*, see [127, Chapter IV.6, Corollary 2 of Theorem 1]. We shall write

$$w - \lim_{n\to\infty} x_n = x_\infty$$

or, in short $x_n \to x_\infty$ weakly.
X is said to be *sequentially weakly complete* if every weakly convergent sequence of X converges weakly to an element of X. It is well-known that Banach spaces are sequentially weakly complete — [127, Chapter V.1, Theorem 7]— and that a weakly convergent sequence $(x_n)_{n\in\mathbb{N}}$ is strongly bounded and, in particular, if

$$w - \lim_{n\to\infty} x_n = x_\infty,$$

then $(\|x_n\|_X)_{n\in\mathbb{N}}$ is bounded and

$$\|x_\infty\|_X \leq \liminf_{n\to\infty} \|x_n\|_X,$$

see [127, Chapter V.1 Theorem 1]. Moreover, if X is a reflexive Banach space and $(x_n)_{n\in\mathbb{N}}$ is an arbitrary sequence in X which is norm bounded, then we can choose

a subsequence $(x_{n_k})_{n_k \in \mathcal{I}}$, with $\mathcal{I} \subset \mathbb{N}$, which converges weakly to an element of X — [127, Chapter V.2, Theorem 1].
A sequence $(x'_n)_{n\in\mathbb{N}}$ in the dual space X' of a normed vector space X is said to be *weakly⋆ convergent* if

$$\lim_{n\to\infty} \langle x'_n, x\rangle$$

exists and it is finite for each $x \in X$. $(x'_n)_{n\in\mathbb{N}}$ is said to *converge weakly⋆ to* $x'_\infty \in X'$ if

$$\lim_{n\to\infty} \langle x'_n, x\rangle = \langle x'_\infty, x\rangle$$

for all $x \in X$. In the latter case, we write

$$w^\star - \lim_{n\to\infty} x'_n = x'_\infty$$

or, in short $x'_n \to x'_\infty$ weakly⋆ or weak⋆. If X is a Banach space, then a weakly⋆ convergent sequence $(x'_n)_{n\in\mathbb{N}}$ converges weakly⋆ to an element $x'_\infty \in X'$ and

$$\|x'_\infty\|_{X'} \leq \liminf_{n\to\infty} \|x'_n\|_{X'},$$

see [127, Chapter V.1, Theorem 9].

1.3 Unbounded operators

Let X, Y be normed spaces and T be a linear operator defined in a subset of X, $\mathcal{D}(T)$, and with range in a subspace of Y, $\mathcal{R}(T)$. The kernel of T, that is, the elements $x \in \mathcal{D}(T)$ such that $Tx = 0$ is denoted by $\ker T$. The set $\mathcal{G}(T) := \{(x, Tx) \mid x \in \mathcal{D}(T)\}$ is called the *graph of* T and since T is linear, $\mathcal{G}(T)$ is a subspace of $X \times Y$. If $\mathcal{G}(T)$ is closed in $X \times Y$, then T is said to be *closed* in X. Note that T is closed if, and only if, for all $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(T)$ with $x_n \to x$ in X, $Tx_n \to y$ in Y for some $y \in Y$, imply $x \in \mathcal{D}(T)$ and $Tx = y$, see [37, Chapter 2]. If $\mathcal{D}(T)$ is dense in X, then T is called densely defined.
If X is complete, then T is closed if, and only if, $\mathcal{D}(T)$ associated with the *graph norm* $\|\cdot\|_{\mathcal{D}(T)}$, where $\|x\|^2_{\mathcal{D}(T)} := \|x\|^2_X + \|Tx\|^2_Y$ for $x \in \mathcal{D}(T)$, is a Banach space. If X is a Hilbert space and Y an inner product space, $\mathcal{D}(T)$ is even a Hilbert space with the graph norm. Moreover, the kernel of closed operators is closed in X.
For vector spaces X, Y and a linear operator $T : \mathcal{D}(T) \subset X \to Y$, we denote the restriction of T in $U \subset X$ by $T|_U$. Another operator $S : \mathcal{D}(S) \subset X \to Y$ is called a *restriction of* T *to* $\mathcal{D}(S)$, denoted by $T|_{\mathcal{D}(S)}$, if $\mathcal{D}(S) \subset \mathcal{D}(T)$ and $Tx = Sx$ for all $x \in \mathcal{D}(S)$. Moreover, S is called an *extension of* T if T is a restriction of S.
For two Banach spaces X, Y and $T : \mathcal{D}(T) \subset X \to Y$ densely defined, the *dual operator* $T' : \mathcal{D}(T') \subset Y' \to X'$ is defined on the domain

$$\mathcal{D}(T') := \{y' \in Y' \mid \exists x' \in X' : \langle y', Tx\rangle = \langle x', x\rangle \;\forall\; x \in \mathcal{D}(T)\}.$$

For x', y' according to the definition of $\mathcal{D}(T')$, we define $T'y' = x'$. In the case where X and Y are Hilbert spaces, the *adjoint operator* $T^* : \mathcal{D}(T^*) \subset Y \to X$ is defined on

$$\mathcal{D}(T^*) := \{y^* \in Y \mid \exists x^* \in X : \langle y^*, Tx \rangle_Y = \langle x^*, x \rangle_X \ \forall\, x \in \mathcal{D}(T)\}.$$

Similarly we define $T^*y^* = x^*$. Note that by using the Riesz isomorphisms one has $T^* = J_X T' J_Y^{-1}$. An operator $T \in \mathscr{L}(X)$ is called *left-invertible* if there exists $S \in \mathscr{L}(X)$ such that $ST = I$. It can be seen that an operator $T \in \mathscr{L}(X)$ is left-invertible if there exists $m > 0$ for which

$$\|Tz\|_X \geq m\|z\|_X, \quad \forall z \in X,$$

i.e., the kernel of T is trivial, $\ker T = \{0\}$. It is called *right-invertible* if there exists an operator $R \in \mathscr{L}(X)$ such that $TR = I$. It can be also easily seen that this is equivalent to $\mathcal{R}(T) = X$, that is, T is *onto*.
For the sake of simplicity, we only introduce the following concepts when X is an infinite-dimensional Hilbert space instead of a Banach space, but the theory can be also done in the Banach context. If $T : \mathcal{D}(T) \subset X \to X$ densely defined, then the *resolvent set* of T, denoted by $\rho(T)$, is the set of those points $s \in \mathbb{C}$ for which the operator $sI - T : \mathcal{D}(T) \to X$ is invertible and $(sI - T)^{-1} \in \mathscr{L}(X)$ is called *resolvent operator* of T, or simply *resolvent* of T. The *spectrum* of T, denoted by $\sigma(T)$, is the complement of $\rho(T)$ in $\mathbb{C}$. Note that if $\rho(T)$ is not empty, then T is closed. Further for $\alpha, \beta \in \rho(T)$ we have the *resolvent identity*

$$(\alpha I - T)^{-1} - (\beta I - T)^{-1} = (\beta - \alpha)(\alpha I - T)^{-1}(\beta I - T)^{-1}.$$

A value $\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if there exists a $z_\lambda \in \mathcal{D}(T)$, $z_\lambda \neq 0$, such that $Tz_\lambda = \lambda z_\lambda$. In this case, z_λ is called an *eigenvector* of T corresponding to λ. The set of all the eigenvalues of T is called the *point spectrum* of T and it is denoted by $\sigma_p(T)$. For $n \in \mathbb{N}$, we define the space $\mathcal{D}(T^n)$ recursively:

$$\mathcal{D}(T^n) := \{z \in \mathcal{D}(T) \mid Tz \in \mathcal{D}(T^{n-1})\}.$$

The powers of T, $T^n : \mathcal{D}(T^n) \to X$ are defined in the obvious way. Further,

$$\mathcal{D}(T^\infty) := \bigcap_{n\in\mathbb{N}} \mathcal{D}(T^n).$$

For every $\beta \in \rho(T)$, the space $\mathcal{D}(T)$ with the norm

$$\|z\|_{X_1} = \|(\beta I - T)z\|_X, \quad z \in \mathcal{D}(T)$$

is a Hilbert space, denoted by X_1. The norms generated as above for different $\beta \in \rho(T)$ are equivalent to the graph norm. The embedding $X_1 \subset X$ is continuous. If

$L \in \mathscr{L}(X)$ is such that $L\mathcal{D}(T) \subseteq \mathcal{D}(T)$, then $L \in \mathscr{L}(X_1)$. One can define the spaces X_n recursively, that is, $X_n = \mathcal{D}(T^n)$ and

$$\|x\|_{X_n} := \|(\beta I - T)^n x\|_X,$$

so that it becomes a Banach space [110, Section 3.6].
We denote by X_{-1} the completion of X with respect to the norm

$$\|z\|_{X_{-1}} := \|(\beta I - T)^{-1} z\|_X, \quad z \in X.$$

Then the norms generated as before for different $\beta \in \rho(T)$ are equivalent (in particular, X_{-1} is independent of the choice of β). Moreover, X_{-1} is the dual of X_1^d with respect to the pivot space, where X_1^d the corresponding X_1 space for the operator T^*. If $L \in \mathscr{L}(X)$ is such that $L^*\mathcal{D}(T^*) \subseteq \mathcal{D}(T^*)$, then L has a unique extension to an operator $\tilde{L} \in \mathscr{L}(X_{-1})$. $T \in \mathscr{L}(X_1, X)$ and has a unique extension $\tilde{T} \in \mathscr{L}(X, X_{-1})$. Moreover,

$$(\beta I - T)^{-1} \in \mathscr{L}(X, X_1), \quad (\beta I - \tilde{T})^{-1} \in \mathscr{L}(X_{-1}, X),$$

and these two operators are unitary. We often denote $\tilde{T}$ by T. Using a similar construction we can define the spaces X_{-n} for $n \in \mathbb{N}$, see [110, Section 3.6].
The operator $T : \mathcal{D}(T) \to X$ is called *dissipative* if

$$\operatorname{Re} \langle Tz, z \rangle_X \leq 0, \quad \forall z \in \mathcal{D}(T).$$

From [115, Proposition 3.1.2], the operator $T : \mathcal{D}(T) \to X$ is dissipative if, and only if,

$$\|(\lambda I - T)z\|_X \geq \lambda \|z\|_X, \quad \forall z \in \mathcal{D}(T), \lambda \in (0, \infty)$$

which is further equivalent to

$$\|(sI - T)z\|_X \geq \operatorname{Re} s \|z\|_X, \quad \forall z \in \mathcal{D}(T), s \in \mathbb{C}_0.$$

In [115, Theorem 3.1.7] one finds the usual characterization of the so-called m-dissipative operators in Hilbert space. Let $T : \mathcal{D}(T) \to X$ be dissipative. Then the following statements are equivalent:

(i) $\mathcal{R}(sI - T) = X$ for some $s \in \mathbb{C}_0$;

(ii) $\mathcal{R}(sI - T) = X$ for all $s \in \mathbb{C}_0$;

(iii) $\mathcal{D}(T)$ is dense and if $\tilde{T}$ is a dissipative extension of T, then $\tilde{T} = T$.

Such an operator is called *maximal dissipative* or *m-dissipative*.
Let $T : \mathcal{D}(T) \subset X \to X$ with $\mathcal{D}(T)$ dense in X. Then T is called *symmetric* if

$$\langle Tw, v \rangle_X = \langle w, Tv \rangle_X, \quad \forall v, w \in \mathcal{D}(T).$$

It is called *self-adjoint* if $T = T^*$, meaning that $\mathcal{D}(T) = \mathcal{D}(T^*)$ and $Tx = T^*x$ for all $x \in \mathcal{D}(T)$. It is well-known that if T is self-adjoint, then $\sigma(T) \subset \mathbb{R}$ —see [115, Proposition 3.2.6]. T is called *skew-symmetric* if

$$\langle Tw, v \rangle_X = -\langle w, Tv \rangle_X, \quad \forall v, w \in \mathcal{D}(T).$$

In this context, one can see that iT is a symmetric operator. It is also clear that T is dissipative. T is called *skew-adjoint* if $T = -T^*$. Clearly, the latter is equivalent to iT being self-adjoint and with $\sigma(T) \subset i\mathbb{R}$.

1.4 Function spaces

In this section we present the *Sobolev spaces* $W^{s,p}(\Omega)$, gather some embedding results and define the traces of functions. Thereafter we introduce the concept of *Bochner integral* for functions f which are defined on a measurable set of $\mathbb{R}$ and take values in a Banach space B and in particular we introduce the *Bochner spaces*. The (k-th) derivative of a function f of one variable t will be denoted by $\frac{\mathrm{d}f}{\mathrm{d}t}$ $\left(\frac{\mathrm{d}^k f}{\mathrm{d}t^k}\right)$ or $\dot{f}$ $\left(f^{(k)}\right)$. The (k-th) partial derivative of a function f of several variables with respect to the variable ζ will be denoted by $\frac{\partial f}{\partial \zeta}$ $\left(\frac{\partial^k f}{\partial \zeta^k}\right)$, $\partial_\zeta f$ $\left(\partial_\zeta^k f \text{ or } \partial_{\overset{(k)}{\zeta \cdots \zeta}} f\right)$ or f_ζ $\left(f_{\overset{(k)}{\zeta \cdots \zeta}}\right)$.
We will follow the lines of [2], but most of the books in PDEs include excellent introductory chapters about Sobolev and Bochner spaces, see for instance [21, 31, 44, 51, 87, 94].

1.4.1 Sobolev spaces

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers, we call α a multi-index with degree $|\alpha| := \sum_{k=1}^n \alpha_k$. If $D_k = \partial/\partial x_k$ for $k = 1, \ldots, n$, then

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$$

denotes the differential operator of order $|\alpha|$. $D^{(0,\ldots,0)}u = u$. We will also use the operator ∇ to indicate the gradient of a scalar function f, ∇f, or the divergence of a vector field F, $\nabla \cdot F$.
Let $\Omega \subset \mathbb{R}^n$ be an open and connected domain. For $k \in \mathbb{N}_0$ let $C^k(\Omega)$ be the vector space consisting of all functions ϕ which, together with their partial derivatives $D^\alpha \phi$ of order $|\alpha| \leq k$, are continuous on Ω. We abbreviate $C^0(\Omega) \equiv C(\Omega)$. Let $C^\infty(\Omega) := \bigcap_{k=0}^\infty C^k(\Omega)$. The space $C_0^\infty(\Omega)$ consists of all functions in $C^\infty(\Omega)$ with compact support. If $\phi \in C^k(\Omega)$ is bounded and uniformly continuous on Ω, then it possesses a unique, bounded, continuous extension to the closure of Ω, $\overline{\Omega}$. Accordingly, we define the vector space $C^k(\overline{\Omega})$ to consist of all those functions $\phi \in C^k(\Omega)$ for which

$D^\alpha \phi$ is bounded and uniformly continuous on Ω for $|\alpha| \le k$. It is well-known that $C^k(\Omega)$ is a Banach space with norm given by

$$\|\phi\|_{C^k} := \max_{|\alpha|\le k} \sup_{\zeta\in\Omega} |D^\alpha \phi(\zeta)|.$$

For $0 < \lambda \le 1$, we define $C^{k,\lambda}(\overline{\Omega})$ to be the subspace of $C^k(\overline{\Omega})$ consisting of those functions ϕ for which, for $|\alpha| < k$, $D^\alpha\phi$ satisfies in Ω a Hölder condition of exponent λ. This means that there exists a constant $K \ge 0$ such that

$$|D^\alpha \phi(\zeta_2) - D^\alpha \phi(\zeta_1)| \le K|\zeta_2 - \zeta_1|^\lambda, \quad \zeta_1, \zeta_2 \in \Omega.$$

$C^{k,\lambda}(\overline{\Omega})$ is a Banach space with norm given by

$$\|\phi\|_{C^{k,\lambda}} := \|\phi\|_{C^k} + \max_{|\alpha|\le k} \sup_{\substack{\zeta_1,\zeta_2\in\Omega \\ \zeta_1\ne\zeta_2}} \frac{|D^\alpha \phi(\zeta_2) - D^\alpha \phi(\zeta_1)|}{|\zeta_2 - \zeta_1|^\lambda}.$$

It should be noted that for $0 < \nu < \lambda \le 1$,

$$C^{k,\lambda}(\overline{\Omega}) \subset C^{k,\nu}(\overline{\Omega}) \subset C^k(\overline{\Omega})$$

and $C^{k,1}(\overline{\Omega}) \nsubseteq C^{k+1}(\overline{\Omega})$. We formulate the result that connects $C^{k,\lambda}(\overline{\Omega})$, $C^{k,\nu}(\overline{\Omega})$ and $C^{k+1}(\overline{\Omega})$. We briefly recall the concept of a *compact operator*. Let X and Y be Banach spaces, U the open unit ball in X and $T \in \mathscr{L}(X,Y)$. T is said to be compact if $\overline{T(U)}$ is compact in Y. It is well-known —see for instance [103, p. 103]— that an operator $T \in \mathscr{L}(X,Y)$ is compact if, and only if, every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in X contains a subsequence (x_{n_k}) such that Tx_{n_k} converges to a point in Y.

Theorem 1.4.1. *Let $k \in \mathbb{N}_0$ and $0 < \nu < \lambda \le 1$. Then the following embeddings exist:*

$$C^{k+1}(\overline{\Omega}) \hookrightarrow C^k(\overline{\Omega}), \tag{1.1}$$
$$C^{k,\lambda}(\overline{\Omega}) \hookrightarrow C^k(\overline{\Omega}), \tag{1.2}$$
$$C^{k,\lambda}(\overline{\Omega}) \hookrightarrow C^{k,\nu}(\overline{\Omega}). \tag{1.3}$$

If Ω is bounded, then embeddings (1.2) *and* (1.3) *are compact. If Ω is convex we further have the embeddings*

$$C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k,1}(\overline{\Omega}), \tag{1.4}$$
$$C^{k+1}(\overline{\Omega}) \hookrightarrow C^{k,\nu}(\overline{\Omega}). \tag{1.5}$$

If Ω is convex and bounded, then embeddings (1.1) *and* (1.5) *are compact.*

Proof. This is [2, Theorem 1.31]. □

In the following we will recover some results concerning *Sobolev spaces*. We assume the reader to be familiar with the concepts of Lebesgue measure and integration over domains of $\mathbb{R}^n$. By a *measure* μ on a σ-algebra Σ, we mean a function on Σ taking values in either $\mathbb{R} \cup \{\infty\}$ or $\mathbb{C}$ which is countably additive. For a more detailed discussion of the Lebesgue theory we refer to [91].
For a measurable set $\Omega \subset \mathbb{R}^n$, we say that the set has measure zero if $|\Omega| = \mu(\Omega) = 0$. We say that a property happens *almost everywhere* if it holds except, possibly, in a set of measure zero. We abbreviate it by *a.e.*. We denote by $L^1(\Omega)$ the class of integrable functions on Ω, that is, the set of integrable functions on Ω which coincide a.e. in Ω. It is well-known that $L^1(\Omega)$ is a Banach space with the norm

$$\|f\|_{L^1} \coloneqq \int_\Omega |f| \,\mathrm{d}\mu \,.$$

For $p \in [1, \infty)$ one can hence define the *L^p-spaces*

$$L^p(\Omega) \coloneqq \left\{ f : \Omega \to \mathbb{K} \,\middle|\, f \text{ is measurable and } \int_\Omega |f|^p \,\mathrm{d}\lambda < \infty \right\}$$

with the norm

$$\|f\|_{L^p} \coloneqq \left(\int_\Omega |f|^p \,\mathrm{d}\lambda \right)^{1/p},$$

which are also Banach spaces. Moreover, one can define $L^\infty(\Omega)$ as the set of measurable functions on Ω which are essentially bounded with the norm

$$\|f\|_{L^\infty} \coloneqq \operatorname{ess\,sup}_{\zeta \in \Omega} |f(\zeta)|,$$

which is a Banach space as well, see [2, Theorem 2.10]. For $p \in [1, \infty]$ we define the *Hölder conjugate* q to be such that

$$\frac{1}{p} + \frac{1}{q} = 1,$$

where for $p = 1$ we set $q = \infty$ and for $p = \infty$ we set $q = 1$. For an arbitrary $p \in [1, \infty]$ and its Hölder conjugate q, $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ we have that $fg \in L^1(\Omega)$ and *Hölder's inequality* — [21, Theorem 4.6]— holds

$$\int_\Omega |fg| \,\mathrm{d}\lambda \le \|f\|_{L^p} \|g\|_{L^q}.$$

Another useful relation is *Young's inequality for products*, namely,

$$\int_\Omega |fg| \,\mathrm{d}\lambda \le \frac{1}{p} \|f\|_{L^p}^p + \frac{1}{q} \|g\|_{L^q}^q,$$

for $p \in (1, \infty)$ and its Hölder conjugate q, [127, Lemma I.3.1].

For the particular case $p = 2$, one has that $L^2(\Omega)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_{L^2} = \int_\Omega \overline{f} g \,\mathrm{d}\lambda\,.$$

Hölder's inequality becomes the well-known *Cauchy-Schwarz inequality*

$$|\langle f, g\rangle_{L^2}| \leq \|f\|_{L^2}\|g\|_{L^2}.$$

It is also well-known that $C_0^\infty(\Omega)$ —set of C^∞ functions whose support is a compact subset of Ω for all derivatives— is dense in L^p for all $p \in [1,\infty)$ and that L^p is separable also for $p \in [1,\infty)$ but not for L^∞.

By the Riesz-Representation [2, Theorem 2.33 and Theorem 2.34], for $p \in [1,\infty)$, the topological dual space of $L^p(\Omega)$, $(L^p(\Omega))'$, can be identified with $L^q(\Omega)$, where q is the Hölder conjugate of p. Moreover, for $p \in (1,\infty)$, $L^p(\Omega)$ is reflexive but L^∞ is not. We will always make use of these facts.

Before we move on to the Sobolev spaces, we briefly define the ℓ^p spaces of sequences for $p \in [1,\infty]$. For $p \geq 1$ and a sequence $x := (x_n)_{n\in\mathbb{N}} \subset \mathbb{K}$ we define

$$\|x\|_{\ell^p} := \left(\sum_{i=1}^\infty |x_i|^p\right)^{1/p}$$

and

$$\ell^p := \{x \in \mathbb{K}^\mathbb{N} \mid \|x\|_{\ell^p} < \infty\}.$$

In the case $p = \infty$ we set

$$\|x\|_{\ell^\infty} := \sup_{i\in\mathbb{N}} |x_i|$$

and

$$\ell^\infty := \{x \in \mathbb{K}^\mathbb{N} \mid \|x\|_{\ell^\infty} < \infty\}.$$

As the discrete analogue of the L^p spaces, the ℓ^p spaces are Banach spaces and ℓ^2 is a Hilbert space, as it can be checked in [76, Chapters 1 & 2].

Now we present the Sobolev spaces, which in the literature appear with different nomenclature. We shall use $W^{k,p}(\Omega)$. There are two possible ways of defining these spaces. First, the one we will use, which is the set of functions in L^p whose derivatives of order $|\alpha| \leq k$ are again in L^p —in the sense of distributions— or, alternatively, as the completion of $C^k(\Omega)$ with the $W^{k,p}$-norm that will be defined next. Both of them happen to be equivalent due to [88]. For $p \in [1,\infty]$ and $k \in \mathbb{N}_0$ we define the Sobolev spaces to be

$$W^{k,p}(\Omega) := \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega) \quad \forall |\alpha| \leq k\}.$$

With the norm

$$\|f\|_{W^{k,p}} := \left(\sum_{|\alpha|\leq k} \|D^\alpha f\|_{L^p}^p\right)^{1/p}$$

if $p < \infty$ and

$$\|f\|_{W^{k,\infty}} := \max_{|\alpha|\leq k} \|D^\alpha f\|_{L^\infty}.$$

The Sobolev spaces are Banach spaces [2, Theorem 3.2]. Moreover, for $p \in [1,\infty)$ they are separable and for $p = 2$ are Hilbert spaces [2, Theorem 3.5] with the inner product

$$\langle f, g\rangle_{W^{k,p}} := \sum_{|\alpha|\leq k} \langle D^\alpha f, D^\alpha g\rangle_{L^2},$$

where

$$\langle f, g\rangle_{L^2} = \int_\Omega f\overline{g}\,\mathrm{d}\lambda.$$

For the geometrical properties of domains Ω and boundaries $\Gamma := \partial\Omega$ we refer to [2, Chapter IV] and use the notation therein. We note that for $k \geq 1$, the following holds

$$\begin{aligned}\text{uniform } C^k\text{-regularity property} &\Rightarrow \text{strong local Lipschitz property}\\ &\Rightarrow \text{uniform cone property}\\ &\Rightarrow \text{segment property.}\end{aligned}$$

Recall that if Ω is convex, then Γ is Lipschitz, see for instance [44, Corollary 1.2.2.3]. Sometimes we will say that the boundary Γ is $C^{k,1}$ to refer to the uniform C^k-regularity property, as in [44, Section 1.3.3].
We define the following spaces, which will appear when considering Sobolev embedding theorems. Let

$$BC^k(\Omega) := \{f \in C^k(\Omega) \mid D^\alpha f \text{ is bounded on } \Omega \text{ for } |\alpha| \leq k\}$$

with norm

$$\|f\|_{CB^k} := \max_{|\alpha|\leq k} \sup_{\zeta\in\Omega} |D^\alpha f(\zeta)|.$$

$BC^k(\Omega)$ is a Banach space which is larger than $C^k(\overline{\Omega})$.

Theorem 1.4.2 (The Sobolev Embedding Theorem)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded. Let $j, k \in \mathbb{N}_0$ and $p \in [1,\infty)$. If Ω has the cone property, then there exist the following embeddings:*

(a) Suppose $kp < n$. Then

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \quad p \leq q \leq \frac{np}{n-kp}.$$

(b) Suppose $kp = n$. Then

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), \quad p \leq q \leq \infty.$$

Moreover, if $p = 1$ so that $k = n$,

$$W^{j+n,p}(\Omega) \hookrightarrow BC^j(\Omega), \quad p \leq q \leq \infty.$$

(c) *Suppose $kp > n$. Then*

$$W^{j+k,p}(\Omega) \hookrightarrow BC^j(\Omega).$$

Proof. This is contained in [2, Theorem 5.4, Part I]. □

Another important result regarding the Sobolev embeddings is the case in which the embeddings are compacts. This plays an important role in the spectral decomposition of self-adjoint operators.

Theorem 1.4.3 (The Rellich-Kondrachov Theorem)**.** *Let Ω be a domain in $\mathbb{R}^n$ and Ω_0 a bounded subdomain of Ω. Let $k \in \mathbb{N}$, $j \in \mathbb{N}_0$ and $p \in [1, \infty)$.*

(a) *If Ω has the cone property and $kp \leq n$, then the following embeddings are compact*

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega_0), \quad kp < n, 1 \leq q < np/(n-kp),$$

and

$$W^{j+k,p}(\Omega) \hookrightarrow W^{j,q}(\Omega_0), \quad kp = n, 1 \leq q < \infty.$$

(b) *If Ω has the cone property and $kp > n$, then the following embeddings are compact*

$$W^{j+k,p}(\Omega) \hookrightarrow BC^j(\Omega).$$

Proof. This is contained in [2, Theorem 6.2]. □

The Sobolev spaces can also be defined in the case that k is not an nonnegative integer. For the fractional Sobolev spaces, we consider domains Ω which are $\mathbb{R}^n$, a half-space of $\mathbb{R}^n$ or a domain in $\mathbb{R}^n$ which is uniformly C^1-regular and has a bounded boundary. In this case, if $s = k + \sigma$, where $\sigma \in (0, 1)$ and $k \in \mathbb{N}_0$, the Sobolev spaces are defined as interpolation spaces [2, Chapter VII] in an abstract way for $p \in [1, \infty]$. However, there are natural norms for them, namely,

$$\|f\|_{W^{s,p}} := \left(\|f\|^p_{W^{k,p}} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|D^\alpha f(\zeta_1) - D^\alpha g(\zeta_2)|^p}{|\zeta_1 - \zeta_2|^{n+\sigma p}} \, \mathrm{d}\zeta_1 \, \mathrm{d}\zeta_2 \right)^{1/p}$$

for $p \in [1, \infty)$ and

$$\|f\|_{W^{s,\infty}} := \max \left(\|f\|_{W^{k,\infty}}, \max_{|\alpha|=k} \sup_{\substack{\zeta_1,\zeta_2 \in \Omega \\ \zeta_1 \neq \zeta_2}} \frac{|D^\alpha f(\zeta_1) - D^\alpha g(\zeta_2)|}{|\zeta_1 - \zeta_2|^\sigma} \right).$$

Under some regularity conditions, $C_0^\infty(\mathbb{R}^n)$ restricted to a domain $\Omega \subset \mathbb{R}^n$ is dense in $W^{s,p}(\Omega)$.

Using a similar definition as for $W^{k,p}(\Omega)$, we note that the Sobolev spaces can also be defined on the boundary of Ω, $\Gamma := \partial\Omega$, when this is regular enough. In fact, if Ω is a domain in $\mathbb{R}^n$ having the C^k-regularity property, $kp < n$ and $p \leq q \leq (n-1)p/(n-kp)$, then the *trace operator* $\gamma_0 f = f|_\Gamma$ is well-defined and bounded from $W^{k,p}(\Omega)$ to $L^q(\Gamma)$. If $kp = n$ then also for $p \leq q < \infty$, see [2, Theorem 5.22]. In fact, if Ω is a domain in $\mathbb{R}^n$ having the C^k-regularity property, one can also define the spaces $W^{s,p}(\Gamma)$ for $s \geq 0$ and $p \in (1, \infty)$, see for instance [83, pp. 58]. It can also be checked that $C^\infty(\Gamma)$ is dense in $W^{s,p}(\Gamma)$. With the fractional Sobolev spaces defined, we now generalize the concept of trace operator. Let $f \in C^\infty(\mathbb{R}^n)$ —recall that the restriction to such functions in Ω is dense in $W^{s,p}(\Omega)$— and let γ denote the linear mapping

$$f \mapsto \gamma f := (\gamma_0 f, \ldots, \gamma_{k-1} f); \quad \gamma_j f = \left.\frac{\partial^j f}{\partial n^j}\right|_\Gamma, \tag{1.6}$$

where $\partial^j/\partial n^j$ denotes the j-th directional derivative in the direction of the inward normal vector to Γ. By [2, Theorem 7.53], for $p \in (1, \infty)$ the mapping given by (1.6) extends by continuity to an isomorphism and homeomorphism of $W^{k,p}(\Omega)/\ker\gamma$ onto

$$\bigotimes_{j=0}^{k-1} W^{k-j-1/p,p}(\Gamma).$$

We now define $W_0^{k,p}(\Omega)$ as the functions $f \in W^{k,p}(\Omega)$ such that $\gamma f = 0$, and this coincides with the alternative definition of $W_0^{k,p}(\Omega)$, namely, the closure of $C_0^\infty(\Omega)$ in the space $W^{k,p}(\Omega)$. For the case $\Omega = \mathbb{R}^n$ we have $W_0^{k,p}(\mathbb{R}^n) = W^{k,p}(\mathbb{R}^n)$, that is, the functions in $W^{k,p}(\mathbb{R}^n)$ vanish at infinity, see [2, Corollary 3.19]. In fact, with enough regularity at the boundary, the same result holds for $\gamma : W^{s,p}(\Omega) \to \bigotimes_{j=0}^{k-1} W^{s-j-1/p,p}(\Gamma)$, see [44, Theorem 1.5.1.2 & Corollary 1.5.1.6].

We now define the Sobolev spaces with negative exponents s as $W^{s,p}(\Omega) := (W_0^{-s,q}(\Omega))'$ and $W^{s,p}(\Gamma) := (W^{-s,q}(\Gamma))'$, for $s < 0$ —see [83, Proposition 2.10]. Moreover, as in the non-negative integer case, for $p \in [1, \infty]$ and $s \geq 0$, $W^{s,p}(\Omega)$ and $W^{s,p}(\Gamma)$ are Banach spaces, for $p \neq \infty$ they are separable and in the case $p = 2$ they are Hilbert spaces. For an accurate description see [51, Section 4.5].

We identify spaces of $\mathbb{K}^n$-valued functions with the Cartesian product of spaces of scalar-valued functions, such as, for instance $(W^{k,p}(\Omega))^n \cong W^{k,p}(\Omega; \mathbb{K}^n)$. Similarly with the other spaces defined in the section.

The statement regarding γ can be generalized when considering a more general version of differential operators than the powers of $\partial/\partial\nu$. We introduce here a result that we will need later on.

Lemma 1.4.4. *Let Ω be a bounded subset of $\mathbb{R}^n$ having the uniform C^1-regularity property. Let $a \in C^\infty(\overline{\Omega}; \mathbb{K}^{m\times m})$ be noncharacteristic on Γ, that is, $a\nu$ is nontangential or in other words*

$$\langle \nu(\zeta), a(\zeta)\nu(\zeta)\rangle_{\mathbb{K}^m} \neq 0$$

for a.e. $\zeta \in \Gamma$. *Then* $\gamma_a : W^{2,2}(\Omega) \to W^{3/2,2}(\Gamma) \times W^{1/2,2}(\Gamma)$ *defined by* $\gamma_a f = (\gamma_0 f, \gamma_0(\nu \cdot a\nabla f))$ *is onto.*

Proof. This follows from [44, Theorem 1.6.1.3]. □

We conclude this part with a generalized Poincaré inequality, which can be shown by using [128, Corollary 4.4.7]. However, we provide a proof of our own since the techniques used therein are not presented in this work.

Lemma 1.4.5. *Let* Ω *be a bounded domain in* $\mathbb{R}^n$ *having the uniform* C^1*-regularity property. Let* $f \in W^{1,2}(\Omega)$ *then there exists a constant* $K > 0$ *such that*

$$\|f\|_{L^2}^2 \le K \left(\|\nabla f\|_{L^2}^2 + \left(\int_\Gamma \gamma_0 f \,\mathrm{d}\sigma \right)^2 \right).$$

Proof. We prove it by contradiction. Assume that there exists $f_n \in W^{1,2}(\Omega)$ such that

$$\|f_n\|_{L^2}^2 > n \left(\|\nabla f_n\|_{L^2}^2 + \left(\int_\Gamma \gamma_0 f_n \,\mathrm{d}\sigma \right)^2 \right).$$

Let $T_\Gamma : W^{1,2}(\Omega) \to \mathbb{R}$ be the bounded, linear operator

$$T_\Gamma g := \int_\Gamma \gamma_0 g \,\mathrm{d}\sigma\,.$$

The latter implies that $\|f_n\|_{L^2} > 0$ for all $n \in \mathbb{N}$ and hence, $g_n := f_n/\|f_n\|_{L^2}$ satisfies

$$\|\nabla g_n\|_{L^2}^2 + (T_\Gamma g_n)^2 < \frac{1}{n}.$$

Thus, $\nabla g_n \to 0$ in $L^2(\Omega)$. Since v_n is bounded in $W^{1,2}(\Omega)$, by [127, Theorem V.2.1] we have that there exists a subsequnce for which $g_{n_k} \to g$ weakly in $W^{1,2}(\Omega)$, which implies that $\nabla g_{n_k} \to \nabla g$ with $\nabla g = 0$, so that $g = c \in \mathbb{R}$. By the 1.4.3 we have that there exists a subsequence g_{n_l} that converges strongly to $g = c$ in $L^2(\Omega)$. This yields $1 = \|g_n\|_{L^2} \to \|c\|_{L^2} = 1$, so that $c \neq 0$. Moreover, since T_Γ is continuous, $(T_\Gamma g_n)^2 \to (T_\Gamma c)^2 = c^2|\Gamma| \neq 0$, but

$$(T_\Gamma g_n)^2 < \frac{1}{n},$$

which is a contradiction. □

1.4.2 Bochner spaces

Here we follow [2, Chapter VII]. For a more complete and systematic discussion on the topic we refer to [127, Section V.5].

Let B be a Banach space. Let $\{A_1, \ldots, A_m\}$ be a finite collection of mutually disjoint, (Lebesgue) measurable subsets of $\mathbb{R}$, each having finite measure. Let $\{b_1, \ldots, b_m\}$ be a corresponding collection of points in B. The function $f : \mathbb{R} \to B$ defined by

$$f(t) = \sum_{j=1}^{m} \chi_{A_j}(t) b_j,$$

χ_A being the characteristic function of A, is called a *simple function.* For simple functions, as in the Lebesgue measure scenario, we define

$$\int_{\mathbb{R}} f(t)\, \mathrm{d}t := \sum_{j=1}^{m} \mu(A_j) b_j,$$

where $\mu(A)$ denotes the Lebesgue measure of A. Let $A \subset \mathbb{R}$ be a measurable set and $f : A \to B$ defined a.e. on A. The function f is called *(strongly) measurable* on A if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions with supports in A such that

$$\lim_{n\to\infty} \|f_n(t) - f(t)\|_B = 0, \quad \text{a.e. in } A.$$

Suppose that a sequence of simple functions satisfying the former limit can be chosen in such a way that

$$\lim_{n\to\infty} \int_{\mathbb{R}} \|f_n(t) - f(t)\|_B\, \mathrm{d}t = 0.$$

Then f is called *Bochner integrable* on A and we define

$$\int_A f(t)\, \mathrm{d}t = \lim_{n\to\infty} \int_{\mathbb{R}} f_n(t)\, \mathrm{d}t\,.$$

A measurable function f is Bochner integrable on A if, and only if, $\|f(\cdot)\|_B$ is Lebesgue integrable on A. In fact,

$$\left\| \int_A f(t)\, \mathrm{d}t \right\|_B \leq \int_A \|f(t)\|_B\, \mathrm{d}t\,.$$

For an interval $J \subset \mathbb{R}$, a separable Banach space B and $p \in [1, \infty]$. We denote by $L^p(J; B)$ the vector space of equivalence classes of functions f strongly measurable on J into B such that $\|f(\cdot)\|_B \in L^p(J)$. The space $L^p(J; B)$ is a Banach space with respect to the norm

$$\|f\|_{L^p(J;B)} := \begin{cases} \left(\int_J \|f(t)\|_B^p\right)^{1/p} & \text{if } p \in [1, \infty), \\ \operatorname{ess\,sup}_{t\in J} \|f(t)\|_B & \text{if } p = \infty. \end{cases}$$

If J has finite measure, for $p \in [1, \infty)$, $(L^p(J; B))'$ can be isometrically identified with $L^q(J; B')$. In fact, if B is a Hilbert space, $L^2(J; B)$ is a Hilbert space with the natural inner product, see [54, Theorem 1.31]. For $k \in \mathbb{N}_0$, $C^k(J;B)$ is defined as the space of k-times continuously differentiable functions on J. The space of k-times continuously differentiable functions with bounded derivatives is denoted by $BC^k(J;B)$ and it is endowed with the usual norm to be a Banach space. The space of bounded and uniformly continuous functions will be denoted by $BUC(J;B)$. The Banach space of Hölder continuous functions $C^{0,\alpha}(J;B)$ with $\alpha \in (0,1)$ is given by

$$C^{0,\alpha}(J;B) := \left\{ f \in BC(J;X) \,\middle|\, [f]_\alpha := \sup_{t,s\in J, s<t} \frac{\|f(t)-f(s)\|}{(t-s)^\alpha} < \infty \right\},$$
$$\|f\|_\alpha := \|f\|_\infty + [f]_\alpha,$$

see [84, Chapter 0]. One could also define $C^{k,\alpha}(J;B)$ for $\alpha \in (0,1)$ and $k \in \mathbb{N}$, but we will only need $C^{0,\alpha}(J;B)$.
In the following we refer to [54, Section 1.3.2.2] and [31, Section 5.9.2]. For $p \in [1,\infty]$ we set

$$W^{1,p}_{\text{loc}}(J;B) := \{f \in L^p(J;B) \mid \dot{f} \in L^p(J;B), j = 0,\dots,k\},$$

which is to be understood in the Bochner sense. By [31, Theorem 5.9.2.2], for $f \in W^{1,p}(J;B)$ it holds that $f \in C(J;B)$ and for $s,t \in J$, $s \le t$, we have

$$f(t) = f(s) + \int_s^t \dot{f}(r)\,\mathrm{d}r\,.$$

The spaces $L^p_{\text{loc}}(J;B)$ and $W^{1,p}_{\text{loc}}(J;B)$ consist of all those functions f whose restriction to any compact interval $K \subset J$ are in $L^p(K;B)$ and $W^{1,p}(K;B)$, respectively. We defined one particular class of weighted spaces. For $\omega \in \mathbb{R}$ we set $L^2_\omega(J;B) := \left\{ \mathrm{e}^{\omega\cdot} f(\cdot) \;\middle|\; f \in L^2(J;B) \right\}$ with norm $\|\mathrm{e}^{\omega\cdot} f\|_{L^2_\omega} := \|f\|_{L^2}$.
The next result guarantees the convergence of subsequences in Bochner spaces. This is mainly a consequence of the characterization of sequentially weak and weak* compactness provided by the Banach-Alaoglu Theorem [103, Theorem 3.15].

Lemma 1.4.6. *Let $T > 0$ and Z be a reflexive and separable Banach space. Then*

(i) every bounded sequence $(w_n)_{n\in\mathbb{N}}$ in $L^\infty([0,T];Z)$ has a weak* convergent subsequence and the limit is again in $L^\infty([0,T];Z)$;

(ii) every bounded sequence $(w_n)_{n\in\mathbb{N}}$ in $L^2([0,T];Z)$ has a weakly convergent subsequence and the limit is again in $L^2([0,T];Z)$.

Proof. Let $p \in [1,\infty)$. Then $W := L^p([0,T];Z')$ is a separable Banach space, see [28, Chapter IV]. Since Z is reflexive, by [28, Corollary III.4] it has the Radon-Nikodým property. Then by [28, Theorem IV.1], $W' = L^q([0,T];Z)$ is the dual of W, where q is such that $p^{-1} + q^{-1} = 1$ and $q = \infty$, if $p = 1$. For $p = 1$ so that $q = \infty$, assertion (i) follows from [103, Theorem 3.17]. For $p = 2$ so that $q = 2$, W is reflexive and assertion (ii) is a direct consequence of [127, Theorem V.2.1]. □

1.5 Diagonalizable operators

Now that we have presented the unbounded operators and the Sobolev spaces, we introduce in the following the concepts of diagonal operators and Riesz basis. Consider the Hilbert space ℓ^2 and let the sequence (e_k) be the standard orthonormal basis in ℓ^2. Thus, e_k has a 1 in the kth position and zero everywhere else. Clearly $\langle e_i, e_j \rangle = \delta_{ij}$. A sequence (ϕ_k) in a Hilbert space X is called a *Riesz basis* in X if there exists an invertible operator $Q \in \mathscr{L}(X, l^2)$ such that $Q\phi_k = e_k$ for all $k \in \mathbb{N}$. In this case, the sequence $(\tilde{\phi}_k)$ defined by

$$\tilde{\phi}_k = Q^* Q \phi_k,$$

is called the *biorthogonal sequence* to (ϕ_k). Then every $z \in X$ can be expressed as

$$z = \sum_{k\in\mathbb{N}} \left\langle \tilde{\phi}_k, z \right\rangle_X \phi_k.$$

Moreover, denoting $m = \|Q^{-1}\|^{-1}$ and $M = \|Q\|$, we have

$$m^2 \|z\|_X^2 \leq \sum_{k\in\mathbb{N}} | \left\langle \tilde{\phi}_k, z \right\rangle_X |^n \leq M^2 \|z\|_X^2, \quad \forall z \in X.$$

Note that if ϕ is orthonormal, then Q is unitary and $m = M = 1$. Further, the converse of the statement also holds true —see [115, Proposition 2.5.2 and Proposition 2.5.3]. Let $T : \mathcal{D}(T) \subset X \to X$. T is called *diagonalizable* if $\rho(T) \neq \emptyset$ and there exists a Riesz basis (ϕ_k) in X consisting of eigenvectors of T. In virtue of [115, Proposition 2.6.2], one can construct diagonalizable operators as follows. Let (ϕ_k) be a Riesz basis in X and let $(\tilde{\phi}_k)$ be the biorthogonal sequence to (ϕ_k). Let (λ_k) be a sequence in $\mathbb{C}$ which is not dense in $\mathbb{C}$. Define an operator $\tilde{T} : \mathcal{D}(\tilde{T}) \to X$ by

$$\mathcal{D}(\tilde{T}) = \left\{ z \in X \;\middle|\; \sum_{k\in\mathbb{N}} (1 + |\lambda_k|^2) | \left\langle \tilde{\phi}_k, z \right\rangle_X |^2 < \infty \right\},$$
$$\tilde{T} z = \sum_{k\in\mathbb{N}} \lambda_k \left\langle \tilde{\phi}_k, z \right\rangle_X \phi_k, \quad z \in \mathcal{D}(\tilde{T}).$$

Then $\tilde{T}$ is diagonalizable, we have $\sigma_p(\tilde{T}) = \{\lambda_k \mid k \in \mathbb{N}\}$, $\sigma(\tilde{T})$ is the closure of $\sigma_p(\tilde{T})$ and for every $s \in \rho(\tilde{A})$ we have

$$(sI - \tilde{T})^{-1} = \sum_{k\in\mathbb{N}} \frac{1}{s - \lambda_k} \left\langle \tilde{\phi}_k, z \right\rangle_X \phi_k, \quad \forall z \in X.$$

By [115, Proposition 2.6.3] the converse holds also true, that is, let $T : \mathcal{D}(T) \subset X \to X$ be diagonalizable. Let (ϕ_k) be a Riesz basis consisting of eigenvectors of T. Let $(\tilde{\phi}_k)$ be the biorthogonal sequence to (ϕ_k) and denote the eigenvalue corresponding to the eigenvector ϕ_k by λ_k. Then

$$\mathcal{D}(T) = \left\{ z \in X \;\middle|\; \sum_{k\in\mathbb{N}} (1+|\lambda_k|^2) |\langle \tilde{\phi}_k, z\rangle_X|^2 < \infty \right\},$$
$$Tz = \sum_{k\in\mathbb{N}} \lambda_k \langle \tilde{\phi}_k, z\rangle_X \phi_k, \quad z \in \mathcal{D}(T).$$

If $T : \mathcal{D}(T) \subset X \to X$ is self-adjoint and diagonalizable, then there exists in X an orthonormal basis $(\varphi_k)_{k\in\mathcal{I}}$ of eigenvectors of T (here, $\mathcal{I} \subseteq \mathbb{Z}$). Denoting the eigenvalue corresponding to ϕ_k by λ_k, we have for $\lambda_k \in \mathbb{R}$,

$$\mathcal{D}(T) = \left\{ z \in X \;\middle|\; \sum_{k\in\mathcal{I}} (1+|\lambda_k|^2) |\langle \phi_k, z\rangle_X|^2 < \infty \right\},$$
$$Tz = \sum_{k\in\mathcal{I}} \lambda_k \langle \phi_k, z\rangle_X \phi_k.$$

The following result gives a sufficient condition for a self-adjoint operator to be diagonalizable.

Proposition 1.5.1. *Let X be an infinite-dimensional Hilbert space and let $T : \mathcal{D}(T) \subset X \to X$ be a self-adjoint operator with compact resolvents. Then T is diagonalizable with an orthonormal basis $(\phi_k)_{k\in\mathcal{I}}$ of eigenvectors, where $\mathcal{I} \subseteq \mathbb{Z}$, and the corresponding family of real eigenvalues $(\lambda_k)_{k\in\mathcal{I}}$ satisfies $\lim_{k\to\infty} |\lambda_k| = \infty$.*

Proof. This is [115, Proposition 3.2.12]. □

1.6 Strongly continuous semigroups

In this section we will follow the lines of [115]. However, the literature regarding strongly continuous semigroups is vast, and one can find appropriate references in [30, 40, 53, 94, 127].

Even though the construction can be done in Banach spaces, we will consider only the Hilbert space scenario, so that X will be an infinite-dimensional Hilbert space. A family $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ of operators in $\mathscr{L}(X)$ is a *strongly continuous semigroup* or in short *C_0-semigroup* on X if

(i) $\mathbb{T}_0 = I$;

(ii) $\mathbb{T}_{t+s} = \mathbb{T}_t \mathbb{T}_s$ for every $t, s \geq 0$;

(iii) $\lim_{t\to 0, t>0} \mathbb{T}_t z = z$ for all $z \in X$.

The *growth bound* of a strongly continuous semigroup $\mathbb{T}$ is the number $\omega_0(\mathbb{T})$ defined by

$$\omega_0(\mathbb{T}) = \inf_{t\in(0,\infty)} \frac{1}{t}\log\|\mathbb{T}_t\|.$$

Clearly $\omega_0(\mathbb{T}) \in [-\infty,\infty)$. The name is justified by the fact that

(i) $\omega_0(\mathbb{T}) = \lim_{t\to\infty} t^{-1}\log\|\mathbb{T}_t\|$;

(ii) for any $\omega > \omega_0(\mathbb{T})$ there exists an $M_\omega \in [1,\infty)$ such that

$$\|\mathbb{T}_t\| \le M_\omega \mathrm{e}^{\omega t}, \quad \forall t\in[0,\infty);$$

(iii) the function $\varphi : \mathbb{R}_{\ge 0}\times X \to X$ defined by $\varphi(t,z) = \mathbb{T}_t z$ is continuous;

as it can be checked in [115, Proposition 2.1.2]. We call a C_0-semigroup $(\mathbb{T}_t)_{t\ge 0}$ *exponentially stable* if $\omega_0(\mathbb{T}) < 0$.
The linear operator $A : \mathcal{D}(A) \subset X \to X$ defined by

$$\mathcal{D}(A) = \left\{ z\in X \;\middle|\; \lim_{t\to 0, t>0} \frac{\mathbb{T}_t z - z}{t} \text{ exists} \right\},$$
$$Az = \lim_{t\to 0,t>0} \frac{\mathbb{T}_t z - z}{t}, \quad \forall z\in\mathcal{D}(A),$$

is called the *infinitesimal generator* or simply *generator* of the semigroup $\mathbb{T}$. It is well-known that $\mathcal{D}(A)$ is dense in X —see [115, Corollary 2.1.8].
We begin now to hatch what we have presented about closed operators with the semigroup theory. For a strongly continuous semigroup $\mathbb{T}$ on X with generator A and for every $s\in\mathbb{C}$ with $\operatorname{Re} s > \omega_0(\mathbb{T})$ we have $s\in\rho(A)$, and hence A is closed. Moreover

$$(sI - A)^{-1} z = \int_0^\infty \mathrm{e}^{-st}\mathbb{T}_t z \,\mathrm{d}t\,, \quad \forall z\in X$$

and $\mathcal{D}(A^\infty)$ is dense in X —check [115, Proposition 2.3.1 and Proposition 2.3.6]. In fact, the resolvent of the semigroup generator is the Laplace transform of the semigroup. Further, for $z_0 \in \mathcal{D}(A)$, the function $z : [0,\infty) \to \mathcal{D}(A)$ defined by $z(t) := \mathbb{T}_t z_0$ is continuous if we consider in $\mathcal{D}(A)$ the graph norm and $C^1([0,\infty); X)$. Actually, z is the unique solution of the *abstract Cauchy problem (ACP)*

$$\dot{z} = Az, \quad z(0) = z_0.$$

From [115, Proposition 2.6.5] it follows that a diagonalizable operator A is the generator of a C_0-semigroup $\mathbb{T}$ on X if, and only if,

$$\sup_{k\in\mathbb{N}} \operatorname{Re}\lambda_k < \infty.$$

If this is the case, then

$$\sup_{k\in\mathbb{N}} \operatorname{Re} \lambda_k = \omega_0(\mathbb{T})$$

and for every $t \geq 0$,

$$\mathbb{T}_t z = \sum_{k\in\mathbb{N}} \mathrm{e}^{\lambda_k t} \left\langle \tilde{\phi}_k, z \right\rangle_X \phi_k, \quad \forall z \in X.$$

Such a semigroup is called *diagonalizable.*

By having a closer look at a diagonal semigroup, it is clear that if eigenvalues of A are, for instance, purely imaginary, one could extend the semigroup for $t < 0$, so that we end up having a group. In the following we define such a concept.

Let $\mathbb{T}$ be a C_0-semigroup on X. $\mathbb{T}$ is called *left-invertible* (respectively, *right-invertible*) if for some $t > 0$, $\mathbb{T}_t$ is left-invertible (respectively, right-invertible). The semigroup is called *invertible* if it is both left-invertible and right-invertible.

A family $\mathbb{T} = (\mathbb{T}_t)_{t\in\mathbb{R}}$ of operators in $\mathscr{L}(X)$ is a *strongly continuous group* in short *C_0-group* on X if

(i) $\mathbb{T}_0 = I$;

(ii) $\mathbb{T}_{t+s} = \mathbb{T}_t\mathbb{T}_s$ for every $t, s \in \mathbb{R}$;

(iii) $\lim_{t\to 0} \mathbb{T}_t z = z$ for all $z \in X$.

The generator of such a group is defined in the same way as for semigroups. Note that given a C_0-semigroup $\mathbb{T}$, if for some $\tau > 0$ the operator $\mathbb{T}_\tau$ is invertible, then $\mathbb{T}_t$ is invertible for all $t > 0$ and $\mathbb{T}$ can be extended to a group by setting $\mathbb{T}_{-t} = \mathbb{T}_t^{-1}$ —see [115, Proposition 2.7.4]. Moreover, from [115, Proposition 2.7.8], if A generates a C_0-semigroup $\mathbb{T}$ and $-A$ a C_0-semigroup $\mathbb{S}$, then we can extend the family of $\mathbb{T}$ to all of $\mathbb{R}$ by putting $\mathbb{T}_{-t} = \mathbb{S}_t$ and $\mathbb{T}$ is a C_0-group.

From [115, Proposition 2.8.5] we have that for a C_0-semigroup $\mathbb{T}$ the family of operators $\mathbb{T}^* = (\mathbb{T}_t^*)_{t\geq 0}$ is also a C_0-semigroup and its generator is A^*. It is an immediate consequence that if $A : \mathcal{D}(A) \subset X \to X$ is a diagonalizable operator, (ϕ_k) is a Riesz basis consisting of eigenvectors of A, $(\tilde{\phi}_k)$ the biorthogonal sequence to (ϕ_k) and we denote the eigenvalue corresponding to the eigenvector ϕ_k by λ_k. Then A^* is diagonalizable operator with eigenvectors $\tilde{\phi}_k$ and eigenvalues $\overline{\lambda_k}$, see [115, Proposition 2.8.6].

Assuming that A generates a C_0-semigroup $\mathbb{T}$ on X, consider the spaces X_1 and X_{-1}. Then the restriction of $\mathbb{T}_t$ to X_1 is the image of $\mathbb{T}_t \in \mathscr{L}(X)$ through the unitary operator $(\beta I - A)^{-1} \in \mathscr{L}(X, X_1)$. Therefore, these operators form a C_0-semigroup on X_1, whose generator is the restriction of A to $\mathcal{D}(A^2)$. The operator $\tilde{\mathbb{T}}_t \in \mathscr{L}(X_{-1})$ is the image of $\mathbb{T}_t \in \mathscr{L}(X)$ though the unitary operator $(\beta I - \tilde{A}) \in \mathscr{L}(X, X_{-1})$. Therefore, these operators form a C_0-semigroup on $\tilde{\mathbb{T}} = (\tilde{\mathbb{T}}_t)_{t\geq 0}$ on X_{-1}, whose generator is $\tilde{A}$, see [115, Proposition 2.10.4]. We will often refer to this extensions as $((\mathbb{T}|_{-1})_t)_{t\geq 0}$.

Before we conclude this section, we define an important class of semigroups. A C_0-semigroup $\mathbb{T}$ on X is called *contractive*, *semigroup of contractions* or *contraction semigroup* if for all $t > 0$ it holds $\|\mathbb{T}_t\| \leq 1$. Further, these semigroups have a well-known characterisation, namely, for any $A : \mathcal{D}(A) \subset X \to X$ the following statements are equivalent:

(i) A is the generator of a contraction semigroup on X.

(ii) A is m-dissipative.

This is in fact [115, Theorem 3.8.4]. The particular case in which the norm of the semigroup is exactly one, goes via the following definition. An operator $U \in \mathscr{L}(X)$ is called *unitary* if $UU^* = U^*U = I$. A strongly continuous semigroup $\mathbb{T}$ on X is called *unitary* if $\mathbb{T}_t$ is unitary for every $t > 0$. It is clear that a unitary semigroup can be extended to a group, which is then called a *unitary group*. From [115, Theorem 3.8.6], for any $A : \mathcal{D}(A) \subset X \to X$ the following statements are equivalent:

(i) A is the generator of a unitary group on X.

(ii) A is skew-adjoint.

1.6.1 Analytic semigroups and fractional powers

Here we briefly present the concept of analytic C_0-semigroup, which plays a role when solving PDEs of parabolic type, whose solutions enjoy of some extra smoothness properties. For this part we refer to [110, Section 3.10] and present only the case where X is a Hilbert space, even though it can be done in the case of a Banach space.
Let $0 < \delta \leq \pi/2$, and let Δ_δ be the open sector

$$\Delta_\delta := \{t \in \mathbb{C} \mid t \neq 0,\ |\arg t| < \delta\}.$$

The family of operators $\mathbb{T}_t \in \mathscr{L}(X)$, $t \in \Delta_\delta$, is an *analytic C_0-semigroup* (with uniformly bounded growth bound ω) in Δ_δ if the following conditions hold:

(i) $t \mapsto \mathbb{T}_t$ is analytic in Δ_δ;

(ii) $\mathbb{T}_0 = I$ and $\mathbb{T}_s\mathbb{T}_t = \mathbb{T}_{s+t}$ for all $s, t \in \Delta_\delta$;

(iii) there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|\mathbb{T}_t\| \leq M\mathrm{e}^{\omega t}, \quad t \in \Delta_\delta;$$

(iv) for all $x \in X$, $\lim_{\substack{t \to 0 \\ t \in \Delta_\delta}} \mathbb{T}_t x = x$.

This leads us to the concept of *sectorial operator*. For each $\gamma \in \mathbb{R}$ and $\pi/2 < \theta < \pi$, let $\Sigma_{\theta,\gamma}$ be the open sector

$$\Sigma_{\theta,\gamma} = \{\lambda \in \mathbb{C} \mid \lambda \neq \gamma, |\arg(\lambda - \gamma)| < \theta\}.$$

A closed, densely defined linear operator $A : \mathcal{D}(A) \subset X \to X$ is *sectorial* on $\Sigma_{\theta,\gamma}$ if the resolvent set of A contains $\Sigma_{\theta,\gamma}$ and if

$$\|(\lambda I - A)^{-1}\| \leq \frac{C}{|\lambda - \gamma|} \quad \lambda \in \Sigma_{\theta,\gamma},$$

for some $C \geq 1$. The operator A is *sectorial* if it is sectorial on some sector $\Sigma_{\theta,\gamma}$.
Let $A : \mathcal{D}(A) \subset X \to X$ be a closed, linear operator and $\gamma \in \mathbb{R}$. Then the following are equivalent:

(i) A is the generator of an analytic semigroup $(\mathbb{T}_t)_{t\geq 0}$ with uniformly bounded growth bound γ on a sector Δ_δ, $\delta > 0$;

(ii) every $\lambda \in \mathbb{C}_\gamma$ belongs to the resolvent set of A and there exists a constant C such that
$$\|(\lambda I - A)^{-1}\| \leq \frac{C}{|\lambda - \gamma|} \quad \operatorname{Re}\lambda > \gamma;$$

(iii) A is sectorial on some sector $\Sigma_{\theta,\gamma}$ with $\pi/2 < \theta < \pi$;

(iv) A is the generator of a semigroup $(\mathbb{T}_t)_{t\geq 0}$ which is differentiable on $(0, \infty)$, and there exist non-negative constants M_0, M_1 such that
$$\|\mathbb{T}_t\| \leq M_0 \mathrm{e}^{\gamma t}, \quad \|(\gamma I - A)\mathbb{T}_t\| \leq M_1 t^{-1} \mathrm{e}^{\gamma t}, \quad t > 0,$$

see [110, Theorem 3.10.6].
This last part of the section is devoted to introduce an recall some basic facts about the fractional powers X_α, for $\alpha \in \mathbb{R}$ induced by the semigroup generator A and the space X. This construction can be done in the general Banach setting, but we will consider only the Hilbert space scenario. For a more detailed discussion on the topic we refer to [110, Section 3.9].
Let A be the generator of a C_0-semigroup $\mathbb{T} \coloneqq (\mathbb{T}_t)_{t\geq 0}$ on the Hilbert space X with growth bound $\omega_0 \in \mathbb{R}$. For each $\gamma \in \mathbb{C}_{\omega_0}$ and $\alpha \geq 0$ we define $(\gamma I - A)^\alpha$ as follows

$$(\gamma I - A)^0 = I,$$
$$(\gamma I - A)^{-\alpha} x = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} \mathrm{e}^{-\gamma t} \mathbb{T}_t x \, \mathrm{d}t\,, \quad \alpha > 0, \quad x \in X,$$

where Γ is the Euler Gamma function. These operators are bounded and linear on X, injective and $\alpha \to (\gamma I - A)^{-\alpha}$ defines a semigroup, see [110, Lemma 3.9.5]. Now we

define $(\gamma I-A)^\alpha$ for $\alpha>0$ to be the inverse of $(\gamma I-A)^{-\alpha}$, with domain $\mathcal{D}((\gamma I-A)^\alpha) = \mathcal{R}((\gamma I-A)^{-\alpha})$.
Having the powers of $\gamma I - A$ at our disposal, we can construct a scale of spaces X_α, for $\alpha\in\mathbb{R}$, as in the same fashion we constructed the spaces X_n, $n\in\mathbb{Z}$. For $\alpha>0$, we set

$$X_\alpha := \mathcal{R}((\gamma I-A)^{-\alpha}) = (\gamma I-A)^{-\alpha}X.$$

with norm

$$\|x\|_{X_\alpha} := \|(\gamma I-A)^\alpha x\|_X.$$

For $\alpha<0$, we let X_α be the completion of X with the weaker norm

$$\|x\|_{X_{-\alpha}} := \|(\gamma I-A)^{-\alpha}x\|_X, \quad \alpha>0.$$

Different choices of $\gamma\in\mathbb{C}_{\omega_0}$ yield identical spaces with equivalent norms. The spaces X_α are interpreted as *interpolation spaces* of X_n for $n\in\mathbb{Z}$, see [84, Chapters 1 & 2].
We conclude this section with an auxiliary result that will be used later on. This delivers an estimate for the norm of $\mathbb{T}_t$ in $\mathscr{L}(X, X_\alpha)$ when the semigroup is analytic.

Lemma 1.6.1. *Let A be the generator of an exponentially stable analytic semigroup $(\mathbb{T}_t)_{t\geq 0}$. Let X_α be the interpolation spaces defined as above associated to A. Then for all $k\in\mathbb{N}$ and $\alpha\in[0,1)$ there exist constants $M := M(k,\alpha)$ and $\omega>0$ such that*

$$\|\mathbb{T}_t\|_{\mathscr{L}(X,X_{k+\alpha})} \leq M(1+t^{-k-\alpha})\mathrm{e}^{-\omega t}, \quad t>0.$$

Hence, there exists $K := K(k,\alpha)$ such that

$$\sup_{t\in[0,\infty)} t^{k+\alpha}\|\mathbb{T}_t\|_{\mathscr{L}(X,X_{k+\alpha})} \leq K.$$

Proof. For the cases in which $\alpha=0$ and $k\in\mathbb{N}$, this is [110, Corollary 3.10.8]. For $k=0$ and $\alpha\in(0,1)$, this follows from [110, Lemma 3.10.9] and using the exponential stability of $(\mathbb{T}_t)_{t\geq 0}$. For $k>1$ and $\alpha\in(0,1)$ the result follows by induction using the former an interpolating between $k, k+1$ with [110, Lemma 3.9.8]. □

1.6.2 The abstract Cauchy problem

Throughout this section $\mathbb{T}$ is a C_0-semigroup on X with generator A and growth bound $\omega_0(\mathbb{T})$. As we have already seen, the semigroup fully characterizes the solutions to the abstract Cauchy problem. This part of the section is dedicated to present a *nonhomogeneous abstract Cauchy problem*, in order to motivate the theory of abstract linear systems, where one also has inputs and outputs.
Consider the differential equation

$$\dot{z}(t) = Az(t) + f(t),$$

where $f \in L^1_{\text{loc}}([0,\infty); X_{-1})$. A *solution of the problem in* X_{-1} is a function

$$z \in L^1_{\text{loc}}([0,\infty); X) \cap C([0,\infty); X_{-1})$$

which satisfies the following equation in X_{-1}:

$$z(t) - z(0) = \int_0^t [Az(s) + f(s)]\, \mathrm{d}s\,, \quad \forall t \geq 0.$$

Suppose that z is a solution of the differential equation in X_{-1} and denote $z_0 = z(0)$. Then z is given by

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-s} f(s)\, \mathrm{d}s\,.$$

In particular, for every $z_0 \in X$ there exists at most one solution in X_{-1} which satisfies the initial condition $z(0) = z_0$. We call this a *mild solution* —see [115, Propostion 4.1.4]. Further, from [115, Theorem 4.1.6], if $z_0 \in X$ and $f \in W^{1,2}_{\text{loc}}([0,\infty); X_{-1})$, then the differential equation has a unique solution in X_{-1}, denoted by z, that satisfies $z(0) = z_0$. Moreover,

$$z \in C([0,\infty); X) \cap C^1([0,\infty); X_{-1})$$

1.7 Well-posed linear systems

In this section we introduce the concept of *abstract linear systems* or, more precisely, *well-posed linear systems*, which has been intensively studied in [110, 115] and [111, 112, 122]. Let $p \in [1,\infty)$. For any Hilbert space W and any $\tau \geq 0$, $\mathbf{S}_\tau$ will denote the operator of the *right shift* by τ on $L^p([0,\infty); W)$. $\mathbf{P}_\tau$ will denote the *projection* of $L^p([0,\infty); W)$ onto $L^p([0,\tau]; W)$ by truncation, the latter space being regarded as a subspace of the former. For $u, v \in L^p([0,\infty); W)$, the *$\tau$-concatenation* of u and v, denoted by $u \underset{\tau}{\diamond} v$, is defined by

$$u \underset{\tau}{\diamond} v = \mathbf{P}_\tau u + \mathbf{S}_\tau v.$$

In other words, $\big(u \underset{\tau}{\diamond} v\big)(t) = u(t)$ for $t \in [0,\tau)$, while $\big(u \underset{\tau}{\diamond} v\big)(t) = v(t-\tau)$ for $t \geq \tau$. The shift and projection operators and the operation of concatenation have natural extensions to $L^p_{\text{loc}}([0,\infty); W)$, and we denote these extensions using the same symbols. We regard $L^p_{\text{loc}}([0,\infty); W)$ as a Fréchet space, with the topology given by the family of seminorms $p_n(u) = \|\mathbf{P}_n u\|_{L^p}$, so that $L^p([0,\infty); W)$ is dense in $L^p_{\text{loc}}([0,\infty); W)$.
Let U, X and Y be Hilbert spaces, $\mathcal{U} := L^p([0,\infty); U)$ and $\mathcal{Y} := L^p([0,\infty); Y)$. A *well-posed linear system* on $\mathcal{U}$, X and $\mathcal{Y}$ is a quadruple $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$, where

(i) $\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$ is a C_0-semigroup on X with generator A;

(ii) $\Phi = (\Phi_t)_{t\geq 0}$ is a family of bounded linear operators from $\mathcal{U}$ to X such that

$$\Phi_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v,$$

for any $u, v \in \mathcal{U}$ and any $t, \tau \geq 0$.

(iii) $\Psi = (\Psi_t)_{t\geq 0}$ is a family of bounded linear operators from X to $\mathcal{Y}$ such that

$$\Psi_{\tau+t} x = \Psi_\tau x \underset{\tau}{\diamond} \Psi_t \mathbb{T}_\tau x,$$

for any $x \in X$ and any $t, \tau \geq 0$, and $\Psi_0 = 0$.

(iv) $\mathbb{F} = (\mathbb{F}_t)_{t\geq 0}$ is a family of bounded linear operators from $\mathcal{U}$ to $\mathcal{Y}$ such that

$$\mathbb{F}_{\tau+t}(u \underset{\tau}{\diamond} v) = \mathbb{F}_\tau u \underset{\tau}{\diamond} (\Psi_t \Phi_\tau u + \mathbb{F}_t v),$$

for any $u, v \in \mathcal{U}$ and any $t, \tau \geq 0$, and $\mathbb{F}_0 = 0$.

U is the *input space*, X the *state space* and Y the *output space* (all of them referring to Σ). The operators Φ_t are called *input maps*. The operators Ψ_t are called *output maps*. The operators $\mathbb{F}_t$ are called *input-output maps*.
Recall that $\mathbb{T}|_{-1}$ is the semigroup extended to X_{-1}. There is a unique $B \in \mathscr{L}(U, X_{-1})$, called the *control operator* of Σ, such that for any $t \geq 0$

$$\Phi_t u = \int_0^t (\mathbb{T}|_{-1})_{t-s} B u(s)\,\mathrm{d}s\,.$$

B is called *bounded* if it belongs to $\mathscr{L}(U, X)$, and *unbounded* otherwise.
For any $x_0 \in X$ and $u \in L^p_{\mathrm{loc}}([0,\infty); U)$ the function $x : [0,\infty) \to X$ defined by $x(t) = \mathbb{T}_t x_0 + \Phi_t u$ is called *state trajectory*. It follows that x is a *strong solution* in X_{-1} of

$$\dot{x}(t) = Ax(t) + Bu(t).$$

We shall often need the Fréchet spaces

$$\tilde{\mathcal{U}} = L^p_{\mathrm{loc}}([0,\infty); U), \quad \tilde{\mathcal{Y}} = L^p_{\mathrm{loc}}([0,\infty); Y).$$

The operators $\mathbb{F}_t$ have natural continuous extensions to $\tilde{\mathcal{U}}$. If we regard the operators Ψ_t as elements of $\mathscr{L}(X, \tilde{\mathcal{Y}})$ and the operators $\mathbb{F}_t$ as elements of $\mathscr{L}(\tilde{\mathcal{U}}, \tilde{\mathcal{Y}})$, then these operator families have strong limits as $t \to \infty$, denoted Ψ_∞ and $\mathbb{F}_\infty$. Hence,

$$\Psi_t = \mathbf{P}_t \Psi_\infty, \quad \mathbb{F}_t = \mathbf{P}_t \mathbb{F}_\infty.$$

One usually defines the output of the system to be

$$y = \Psi_\infty x_0 + \mathbb{F}_\infty u.$$

There is a unique $C \in \mathscr{L}(X_1, Y)$, called the *observation operator* of Σ, such that for $x_0 \in X_1$ and $t \geq 0$

$$(\Psi_\infty x_0)(t) = C\mathbb{T}_t x_0.$$

C is called *bounded* if it can be extended continuously to X and *unbounded* otherwise. A useful consequence is that for any $x_0 \in X$

$$\int_0^t (\Psi_\infty x_0)(s)\,\mathrm{d}s = C\int_0^t \mathbb{T}_s x_0\,\mathrm{d}s\,.$$

The *Lebesgue extension* of C is defined by

$$C_L x_0 = \lim_{t\to 0} C\frac{1}{t}\int_0^t \mathbb{T}_s x_0\,\mathrm{d}s\,,$$

which a domain $\mathcal{D}(C_L)$ defined where the former expression exists, and it satisfies $X_1 \subset \mathcal{D}(C_L) \subset X$. For any $x_0 \in X$ we have that $\mathbb{T}_t x_0 \in \mathcal{D}(C_L)$ for almost every $t \geq 0$ and

$$(\Psi_\infty x_0)(t) = C_L\mathbb{T}_t x_0$$

almost everywhere.
For any $v \in U$, the function

$$y_v = \mathbb{F}_\infty(\chi u)$$

is the *step response* of Σ corresponding to v, where χ is the constant function on $[0,\infty)$ equal to 1 everywhere.
Σ is called *regular* if for any $v \in U$ the corresponding step response y_v has a Lebesgue point at 0, i.e., the following limit exists in Y:

$$Dv = \lim_{t\to 0}\frac{1}{t}\int_0^t y_v(s)\,\mathrm{d}s\,.$$

In that case, the operator $D \in \mathscr{L}(U, Y)$ is called *feed-through operator* of Σ.
For operators $A \in \mathscr{L}(X_1, X)$, $B \in \mathscr{L}(U, X_{-1})$, $C \in \mathscr{L}(X_1, Y)$ and $D \in \mathscr{L}(U, Y)$ we call

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

an *abstract linear system*.
Let U, X, Y be Hilbert spaces and $A : \mathcal{D}(A) \subseteq X \to X$ be the generator of a C_0-semigroup $(\mathbb{T}_t)_{t\geq 0}$. The notion of *admissible* operators is well-known in infinite-dimensional linear systems theory with unbounded control and observation operators, as present in boundary control, see e.g. [116], and is motivated by interpreting a PDE on a larger space in order to define solutions. Let $p \in [1, \infty]$. We recall that

$B \in \mathscr{L}(U, X_{-1})$ is an L^p*-admissible control operator* (for $(\mathbb{T}_t)_{t\geq 0}$), if for all $t \geq 0$ and all $u \in L^p([0,t];U)$ we have

$$\Phi_t u := \int_0^t (\mathbb{T}|_{-1})_{t-s} Bu(s)\, \mathrm{d}s \in X.$$

By a closed graph theorem argument this property implies that, for any $t \geq 0$, the operator Φ_t is bounded from $L^p([0,t];U)$ to X. We call B *infinite-time* L^p*-admissible*, if

$$\sup_{t>0} \|\Phi_t\| < \infty.$$

An operator $C \in \mathscr{L}(\mathcal{D}(A), Y)$ is called L^p*-admissible observation operator* for the semigroup $(\mathbb{T}_t)_{t\geq 0}$, if for some (and hence all) $t \geq 0$ the mapping

$$\Psi_t : \mathcal{D}(A) \to L^p([0,t];Y),\ x \mapsto C\mathbb{T}_t x$$

can be extended to a bounded operator from X to $L^p([0,t];Y)$ — this extension will again be denoted by Ψ_t. We call C *infinite-time* L^p*-admissible*, if

$$\sup_{t>0} \|\Psi_t\| < \infty.$$

Both admissibility notions are combined in the stronger concept of well-posedness. Let (A, B, C) represent an abstract linear system where A generates a C_0-semigroup $\mathbb{T}$, B is an L^2-admissible control operator and C is an L^2-admissible observation operator in the sense described above. If there exists a function $G : \mathbb{C}_\omega \to \mathscr{L}(U, Y)$, $\omega > \omega_0(\mathbb{T})$, which satisfies

$$G(\beta) - G(\gamma) = C((\beta I - A)^{-1} - (\gamma I - A)^{-1})B \tag{1.7}$$

for all $\beta, \gamma \in \mathbb{C}_\omega$ and G is proper, i.e., $\sup_{s\in\mathbb{C}_\omega} \|G(s)\| < \infty$, then we say that (A, B, C) —implicitly meaning the corresponding tuple $(\mathbb{T}, \Psi, \Phi, \mathbb{F})$, where $\mathbb{F}$ is the inverse Laplace transform of G— is well-posed. Note that G is uniquely determined up to a constant. This definition of well-posedness for the tuple (A, B, C) is equivalent to the one presented at the beginning of this section, see [115, Proposition 4.9] and [110]. If $\lim_{\operatorname{Re} s\to\infty} G(s)v$ exists for any $v \in U$, then the system (A, B, C) is called *regular*. We call G the *transfer function* of the system.

1.8 Nonlinear functional analysis

This section is devoted to present a specific class of nonlinear operators, which extend the notion of semigroup generators to the nonlinear case. The tools developed at the end of the section will be used to prove one of our main results. In this setting, the notion of contractive nonlinear operator appears naturally, and so does the fixed point theorem.

1.8.1 Fixed point theorem

We begin by briefly recalling a fixed point result.

Proposition 1.8.1. *Let K be a closed nonempty subset of a Banach space V and let $T : K \to K$ be a strict contraction, that is,*

$$\|T(x) - T(x)\| \leq \rho\|x - y\|, \quad x, y \in K,$$

where $0 \leq \rho < 1$. Then T has a unique fixed point*, an $z \in K$ such that $T(z) = z$.*

Proof. This is [106, Proposition I.2.3]. □

Corollary 1.8.2. *Let K and V as above and $T : K \to K$. Assume that T^n is a strict contraction for some integer $n \geq 1$. Then T has a unique fixed point in K.*

Proof. This is [106, Corollary I.2.5]. □

1.8.2 Nonlinear m-dissipative operators on Hilbert spaces

In this section we present the basic theory of nonlinear analysis to deal in the Hilbert space case with m-dissipative operators and we refer to [69, 90, 106]. For another construction of evolution operators one could also check [93]. We consider a real or complex valued Hilbert space X and an operator A with domain and range in X and for simplicity single-valued. As usual, we denote the domain by $\mathcal{D}(A)$, the range by $\mathcal{R}(A)$ and the graph by $\mathcal{G}(A)$. We denote by Ax or $A(x)$ the image of a point $x \in \mathcal{D}(A)$ under the operator A.

The concept of dissipativity can be extended in the nonlinear case. An operator $A : \mathcal{D}(A) \subset X \to X$ is called *dissipative* if for each $x, y \in \mathcal{D}(A)$, we have

$$\mathrm{Re}\,\langle A(x) - A(y), x - y\rangle \leq 0.$$

If further for one $\lambda > 0$ —and hence for all $\lambda > 0$, see [90, Lemma 2.13]— it holds that $\mathcal{R}(I - \lambda A) = X$, then we call A *m-dissipative*. The concept of *maximal dissipative* appears often in the literature in contrast to the one of m-dissipative. In fact, a maximal dissipative operator is a dissipative operator for which all extensions in X coincide with it. In the Hilbert space scenario, the concepts are equivalent, see [90, Lemma 2.12 (iii) and Corollary 2.27]. A counterexample in the more general Banach space case is given in [90, Example 2.6].

For an m-dissipative operator A one can define the nonlinear version of the *resolvent operator*, namely, for $\lambda > 0$

$$J_\lambda := (I - \lambda A)^{-1},$$

which is a single-valued operator such that $\mathcal{D}(J_\lambda) = \mathcal{R}(I - \lambda A) = X$ and $\mathcal{R}(J_\lambda) = \mathcal{D}(A)$. Further, J_λ satisfies

$$\|J_\lambda x - J_\lambda y\| \leq \|x - y\|$$

for all $x, y \in X$, that is, J_λ is Lipschitz with Lipschitz constant $L = 1$, see [90, Corollary 2.10]. As in the linear case, this makes possible to define the *Yosida approximation* of A, that is,

$$A_\lambda := \lambda^{-1}(J_\lambda - I).$$

Moreover,

(i) $A_\lambda x = AJ_\lambda x$, for $x \in X$;

(ii) $\|A_\lambda x\| \leq \|Ax\|$ for $x \in \mathcal{D}(A)$;

(iii) if $0 < \mu \leq \lambda$ and $x \in X$, then $\|A_\lambda x\| \leq \|A_\mu x\|$;

which is contained in [90, Lemma 2.11].
As in the linear case, A is called a *closed operator* if $x_n \in \mathcal{D}(A)$, $x_n' = Ax_n$ with $\lim_{n\to\infty} x_n = x$, and $\lim_{n\to\infty} x_n' = x'$ imply that $x \in \mathcal{D}(A)$ and $x' = Ax$, that is, the graph $\mathcal{G}(A)$ of A is closed in $X \times X$. A is called *demiclosed* if $x_n \in \mathcal{D}(A)$, $x_n' = Ax_n$, $\lim_{n\to\infty} x_n = x$, and $w - \lim_{n\to\infty} x_n' = x'$ imply that $x \in \mathcal{D}(A)$ and $x' = Ax$. From [90, Lemma 2.16 and Lemma 2.17 (ii)] it follows that an m-dissipative operator is closed and demiclosed.

Lemma 1.8.3. *Let $A : \mathcal{D}(A) \subset X \to X$ be m-dissipative and $\lambda > 0$. Then*

(i) for all $x, y \in X$ it holds that $\operatorname{Re}\langle A_\lambda x - A_\lambda y, x - y\rangle \leq 0$*;*

(ii) for all $x, y \in X$ it holds that $\operatorname{Re}\langle A_\lambda x - A_\lambda y, J_\lambda x - J_\lambda y\rangle \leq 0$*;*

(iii) A_λ is Lipschitz with constant λ^{-1}.

Proof.

(i) Let $u = J_\lambda x, v = J_\lambda y \in \mathcal{D}(A)$. Then $x = u - \lambda Au$ and $y = v - \lambda Av$. Further, using the definition of A_λ we have that

$$\begin{aligned}
\operatorname{Re}\langle A_\lambda x - A_\lambda y, x - y\rangle &= \lambda^{-1} \operatorname{Re}\langle J_\lambda x - x - J_\lambda y + y, x - y\rangle \\
&= \lambda^{-1} \operatorname{Re}\langle u - (u - \lambda Au) - v + (v - \lambda Av), u - \lambda Au - v + \lambda Av\rangle \\
&= \operatorname{Re}\langle Au - Av, u - \lambda Au - v + \lambda Av\rangle \\
&= -\lambda\|Au - Av\|^2 + \operatorname{Re}\langle Au - Av, u - v\rangle \\
&\leq 0,
\end{aligned}$$

since A is dissipative.

(ii) As in (i), let $u = J_\lambda x, v = J_\lambda y \in \mathcal{D}(A)$. Then $x = u - \lambda Au$ and $y = v - \lambda Av$. Further, using the definition of A_λ we have that

$$\operatorname{Re}\langle A_\lambda x - A_\lambda y, J_\lambda x - J_\lambda y\rangle = \lambda^{-1} \operatorname{Re}\langle J_\lambda x - x - J_\lambda y + y, J_\lambda x - J_\lambda y\rangle$$

$$
\begin{aligned}
&= \lambda^{-1} \operatorname{Re} \langle u - (u - \lambda A u) - v + (v - \lambda A v), u - v \rangle \\
&= \operatorname{Re} \langle Au - Av, u - v \rangle \\
&\leq 0.
\end{aligned}
$$

(iii) It suffices to use that $x = J_\lambda x - \lambda A_\lambda x$ for all $x \in X$ to obtain

$$
\begin{aligned}
\operatorname{Re} \langle A_\lambda x - A_\lambda y, y - x \rangle &= -\operatorname{Re} \langle A_\lambda x - A_\lambda y, J_\lambda x - J_\lambda y \rangle + \lambda \|A_\lambda x - A_\lambda y\|^2 \\
&\geq \lambda \|A_\lambda x - A_\lambda y\|^2.
\end{aligned}
$$

Using now the Cauchy-Schwarz inequality delivers the result.

□

It follows from Lemma 1.8.3 (i) that the Yosida approximation of an m-dissipative operator is also dissipative.

Lemma 1.8.4. *Let $x, y \in X$. Then $\operatorname{Re} \langle x, y \rangle \leq 0$ if, and only if, $\|x\| \leq \|x - \alpha y\|$ for all $\alpha > 0$.*

Proof. Follows from [127, Lemma XIV.6.1]. □

Lemma 1.8.5. *Let $A : \mathcal{D}(A) \subset X \to X$ be an m-dissipative operator.*

(i) If $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$, $x_n \to x \in X$, and if $(\|Ax_n\|)_{n\in\mathbb{N}}$ is bounded, then $x \in \mathcal{D}(A)$ and $Ax_n \to Ax$ weakly;

(ii) If $(x_n)_{n\in\mathbb{N}} \subset \mathcal{D}(A)$, $x_n \to x \in X$ and if $(\|A_{1/n} x_n\|)_{n\in\mathbb{N}}$ is bounded, then $x \in \mathcal{D}(A)$ and $A_{1/n} x_n \to Ax$ weakly;

(iii) For all $x \in \mathcal{D}(A)$ it holds that $A_{1/n} x \to Ax$.

Proof.

(i) [90, Lemma 2.18] already gives that $x \in \mathcal{D}(A)$. Moreover, for any $y \in \mathcal{D}(A)$ we have

$$\operatorname{Re} \langle Ay - Ax_n, y - x_n \rangle \leq 0.$$

Since X is a Hilbert space, a bounded sequence has a subsequence that converges weakly, that is, $(\|Ax_n\|)_{n\in\mathbb{N}}$ is bounded, then $Ax_{n_k} \to w \in X$ weakly. By $x_n \to x$ and continuity we obtain that $\operatorname{Re} \langle Ay - w, y - x \rangle \leq 0$. Now using Lemma 1.8.4 with $\alpha = \lambda$ we have

$$\|y - x\| \leq \|y - x - \lambda(Ay - w)\|.$$

By setting $y = J_\lambda(x - \lambda w)$ so that $y \in \mathcal{D}(A)$ and $y - \lambda Ay = x - \lambda w$, we see that $\|y - x\| \leq 0$, hence $x = y$ and $Ax = w$. Thus $Ax_{n_k} \to w = Ax$ weakly. Since we could have started with any subsequence of (x_n) instead of (x_{n_k}), the result obtained shows that $Ax_n \to Ax$ weakly.

(ii) Set $y_n = J_{1/n}x_n \in \mathcal{D}(A)$. Then $Ay_n = A_{1/n}x_n$ and the $\|Ay_n\|$ are bounded. Also $y_n - x_n = J_{1/n}x_n - x_n = n^{-1}A_{1/n}x_n \to 0$ so that $y_n \to x$. Thus (i) is applicable, with the result that $x \in \mathcal{D}(A)$ and $A_{1/n}x_n = Ay_n \to Ax$ weakly.

(iii) This is a direct consequence of [90, Lemma 2.22].

□

The following result clearly differs from the linear case, since the domain of a linear m-dissipative operator in a Hilbert space is dense. In the nonlinear case one can only show in general that the closure of the domain is convex.

Proposition 1.8.6. *If $A : \mathcal{D}(A) \subset X \to X$ is m-dissipative, then $\overline{\mathcal{D}(A)}$ is a convex set.*

Proof. This is [90, Theorem 2.20 (ii)]. □

Lemma 1.8.7. *Let X be a Hilbert space, $A : \mathcal{D}(A) \subset X \to X$ a single-valued, m-dissipative, possibly non-linear operator and $B : \overline{\mathcal{D}(A)} \subseteq \mathcal{D}(B) \subseteq X \to X$ continuous. Then if $A + B$ is a dissipative operator, then $A + B$ is m-dissipative.*

Proof. This is [90, Corollary 6.19 (a)]. □

Remark 1.8.8. If $A : \mathcal{D}(A) \subset X \to X$ is m-dissipative, then for all $f \in X$ and $\lambda > 0$ there exists some $z \in \mathcal{D}(A)$ with $\lambda z - A(z) = f$. The element z is indeed unique, since for any $x \in \mathcal{D}(A)$ with $\lambda x - A(x) = f$, we obtain by taking the difference that

$$\lambda(x - z) - (A(x) - A(z)) = 0$$

and taking the inner product with $x - z$ gives

$$\lambda\|x - z\|^2 = \mathrm{Re}\,\langle A(x) - A(z), x - z\rangle\,.$$

Dissipativity of A leads to non-positivity of the latter expression, whence $x = z$.

In the following we recall some concepts about differentiability of functionals and introduce the concept of subdifferential. It will be very convenient to define $\mathbb{R}_\infty := (-\infty, +\infty]$.

Definition 1.8.9. Let H be a Hilbert space and let $\Psi : H \to \mathbb{R}_\infty$ be a single-valued functional with $\Psi \neq \infty$, that is, $\{x \in H \mid \Psi(x) < \infty\} \neq \emptyset$. We define the *subdifferential* $\partial\Psi$ of Ψ as follows

$$\partial\Psi(x) := \{y \in H \mid \Psi(u) - \Psi(x) \geq \mathrm{Re}\,\langle y, u - x\rangle\,, \forall u \in H\},$$

for $x \in \mathcal{D}(\partial\Psi)$, which means that the set on the right-hand side of the above equation is not empty.

If Ψ is such that $\{x \in H \mid \Psi(x) < \infty\} \neq \emptyset$, we say that Ψ is *proper* and its effective domain is $\mathcal{D}(\Psi) := \{x \in H \mid \Psi(x) < \infty\}$. Moreover, we say that Ψ is *convex* if

$$\Psi(tu + (1-t)v) \leq t\Psi(u) + (1-t)\Psi(v), \quad u, v \in \mathcal{D}(\Psi), t \in [0,1].$$

Lemma 1.8.10. *With the notation of the former definition, the operator $-\partial\Psi$ is dissipative. Assume further that the functional Ψ is convex and lower-semi-continuous. Then $-\partial\Psi$ is m-dissipative.*

Proof. This is [90, Example 2.2]. ☐

Lemma 1.8.11. *Let H be a real Hilbert space, $\Psi : H \to \mathbb{R}_\infty$ be proper, convex and lower-semi-continuous on the real Hilbert space H. Denote the subgradient by $\partial\Psi$. If $u, \dot{u} \in L^2([0,T]; H)$ and if there exists a $g \in L^2([0,T]; H)$, with $g \in \partial\Psi(u)$ a.e. on $[0,T]$, then the functional $\Psi \circ u$ is absolutely continuous on $[0,T]$ and*

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(u(t)) = \left\langle h(t), \frac{\mathrm{d}}{\mathrm{d}t}u(t) \right\rangle, \qquad a.e.\ t \in [0,T],$$

for any function h with $h \in \partial\Psi(u)$ a.e. on $[0,T]$.

Proof. This is [106, Lemma IV.4.3]. ☐

The notion of subdifferential is closely related with the concept of *G-differential* or *Gateaux differential*. Given a real Hilbert space H and $\Psi : H \to \mathbb{R}_\infty$, the *directional derivative* of Ψ at $u \in \mathcal{D}(\Psi)$ in the direction v is the one-sided limit

$$\Psi'(u,v) := \lim_{t \to 0^+} \frac{\Psi(u+tv) - \Psi(u)}{t}$$

where it exists. If Ψ is convex, then the limit exists in $[-\infty, \infty]$. The G-differential of $\Psi : H \to \mathbb{R}_\infty$ at $u \in \mathcal{D}(\Psi)$ is an $f \in H'$ for which $f(v) = \Psi'(u,v)$ for all $v \in H$. Such an f is unique, it is denoted by $\Psi'(u)$ and we say that Ψ is G-differentiable at u, see [106, pp. 80]. The following result relates the notion of G-differentiability and subgradient.

Proposition 1.8.12. *Let H be a real Hilbert space and $\Psi : H \to \mathbb{R}_\infty$ be convex and proper. If Ψ is G-differentiable at $u \in \operatorname{int}(\mathcal{D}(\Psi))$, then $\partial\Psi(u) = \{\Psi'(u,\cdot)\}$. If Ψ is somewhere continuous and $\partial\Psi(u)$ is a singleton, then Ψ is G-differentiable at u.*

Proof. This is [106, Proposition II.7.6]. ☐

1.8.3 The nonlinear abstract Cauchy problem

The following results can be performed when starting at $t_0 \in \mathbb{R}$ with $z(t_0) = z_0$, but for the sake simplicity we will set $t_0 = 0$. One of the cornerstones of our results resides in the fact of being able to show that the equation

$$\begin{aligned} \dot{z}(t) &= A(z(t)) + \omega(t)z(t) + f(t), \\ z(0) &= z_0 \end{aligned}$$

has a solution in some sense, where A is m-dissipative, nonlinear operator in a Hilbert space and ω and f are regular enough. In [90, Theorem 6.20] it has been shown that there exists a unique solution when the domain of A is closed, which in general will not be the case in this thesis, since we will deal with differential operators. For the real valued case, one could attempt to use the notion of *nonlinear evolution operator* in [75] to obtain existence and uniqueness. Also in the real valued scenario, the result has been shown when $\omega(t)$ is independent of t in [106, Theorem IV.4.1], which already follows from [39]. The original result in [106] is due to T. Kato. Following this thread, with the results given in [69, Section 3], one can indeed show that the *abstract nonlinear Cauchy problem (ANCP)* has a unique solution. To this end, one defines $\mathcal{A}(t)z := A(z) + \omega(t)z + f(t)$. Clearly, $\mathcal{D}(\mathcal{A}(t)) = \mathcal{D}(A)$ is independent of t. Further, if ω and f are Lipschitz, then $\mathcal{A}(t)$ it is uniformly Lipschitz in t. In order to apply the main theorems in [69], it is also required $\mathcal{A}(t)$ to be m-dissipative, which is not the case. However, in the remarks subsequent to Theorem 3, in particular Remark 5., this condition can be weakened to the fact that for some $\lambda > 0$, $\mathcal{A}(t) - \lambda I$ is m-dissipative, and then we can apply the results there for our particular case. However, in Remark 5. one can read the following:

> *It should be noted that this is not a trivial generalization. If $\mathcal{A}(t)$ were linear, the transformation $x(t) = \mathrm{e}^{-\lambda t} z(t)$ would change*
>
> $$\dot{z}(t) = \mathcal{A}(t)z(t)$$
>
> *into*
>
> $$\dot{x}(t) = (\mathcal{A}(t) - \lambda I)x(t).$$
>
> *But the same transformation does not always work in the nonlinear case, for the transformed equation involves the operator $\mathrm{e}^{-\lambda t}(\mathcal{A}(t) - \lambda I)\mathrm{e}^{\lambda t}$, the domain of which may depend on t when $\mathcal{D}(\mathcal{A}(t))$ does not.*

No further comments are given about the reason why this holds true or the way to prove it, even though it is clear when having a closer look at the proof, namely, by modifying the fixed point argument. For the sake of self-containment, we provide here a detailed version of the proof following the lines of [106, Theorem IV.4.1] and [69]. Nevertheless,

the proof of Theorem IV.4.1 is not well-structured and some steps are incomplete, so this requires more effort than only adapting the steps to the time dependence of ω.
The next result will be often used in the proof, so we introduce it here. This is a Grönwall-type inequality.

Lemma 1.8.13. *Let $a, b \in L^1([0,T];\mathbb{R})$ with $b \geq 0$ a.e. and let the absolutely continuous function $v : [0,T] \to (0,\infty)$ satisfy*

$$(1-\rho)v'(t) \leq a(t)v(t) + b(t)v(t)^\rho, \qquad a.e.\ t \in [0,T],$$

where $0 \leq \rho < 1$. Then

$$v(t)^{1-\rho} \leq v(0)^{1-\rho} \exp\left(\int_0^t a(s)\,\mathrm{d}s\right) + \int_0^t \exp\left(\int_s^t a(r)\,\mathrm{d}r\right) b(s)\,\mathrm{d}s\,, \quad t \in [0,T].$$

Proof. This is [106, Lemma IV.4.1]. □

Theorem 1.8.14. *Let $T > 0$, X be a Hilbert space and $A : \mathcal{D}(A) \subset X \to X$ be m-dissipative in X with $0 \in \mathcal{D}(A)$ and $A(0) = 0$.*
Then for each $z_0 \in \mathcal{D}(A)$, $\omega \in W^{1,\infty}([0,T];\mathbb{R})$ and $f \in W^{1,\infty}([0,T];X)$ there exists a unique Lipschitz continuous $z : [0,T] \to X$, such that

1. *$z(t) \in \mathcal{D}(A)$ at every $t \in [0,T]$;*
2. *for a.e. $t \in [0,T]$ it holds that*

$$\begin{aligned} \dot{z}(t) &= A(z(t)) + \omega(t)z(t) + f(t), \\ z(0) &= z_0 \end{aligned} \tag{1.8}$$

3. *$\dot{z}$ and $A(z)$ are continuous except at a countable number of values in $[0,T]$.*

Proof. The proof is divided in three parts: *uniqueness*, *existence* and *smoothness of z*.
Step 1. Uniqueness
For any two solutions z_1, z_2 of (1.8) we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_1(t) - z_2(t)\|^2 \leq \omega(t)\|z_1(t) - z_2(t)\|^2, \quad t \geq 0,$$

since A is dissipative. From Lemma 1.8.13 with $\rho = 1/2$ it follows that

$$\|z_1(t) - z_2(t)\| \leq \|z_1(0) - z_2(0)\|\mathrm{e}^{\Omega(t,0)}, \quad t \geq 0,$$

where

$$\Omega(t,s) := \int_s^t \omega(r)\,\mathrm{d}r\,.$$

Uniqueness is now immediate from the initial condition.

Step 2. Existence
To obtain existence, for $\alpha > 0$ we consider the Yosida approximation of the operator A in (1.8), which is given by $A_\alpha := \alpha^{-1}(J_\alpha - I)$, $J_\alpha := (I - \alpha A)^{-1}$. This leads to

$$\dot{z}_\alpha(t) = \alpha^{-1}(J_\alpha - I)(z_\alpha(t)) + \omega(t) z_\alpha(t) + f(t), \quad t \in [0,T], \tag{1.9}$$

with $z_\alpha(0) = z_0$. Multiplying the former equation by $\mathrm{e}^{-t/\alpha}$ and integrating we obtain

$$z_\alpha(t) = \mathrm{e}^{-t/\alpha} z_0 + \int_0^t \mathrm{e}^{-(t-s)/\alpha}(\alpha^{-1} J_\alpha(z_\alpha(s)) + \omega(s) z_\alpha(s) + f(s))\,\mathrm{d}s\,, \quad 0 \le t \le T. \tag{1.10}$$

Let us solve (1.10) in the Banach space $C([0,T];X)$ by a fixed point theorem. Consider the closed, convex set $\mathcal{K} = \{z \in C([0,T];X) \mid z(t) \in X \ \forall t \in [0,T]\}$, and define

$$\mathbb{T}(z)(t) := \mathrm{e}^{-t/\alpha} z_0 + \int_0^t \mathrm{e}^{-(t-s)/\alpha}(\alpha^{-1} J_\alpha(z(s)) + \omega(s) z(s) + f(s))\,\mathrm{d}s\,, \quad 0 \le t \le T,$$

for $z \in \mathcal{K}$. Then $\mathbb{T}(z(\cdot)) \in C([0,T];X)$ and the indicated convex combination satisfies

$$\mathbb{T}(z)(t) = \mathrm{e}^{-t/\alpha} z_0 + (1 - \mathrm{e}^{-t/\alpha}) z_1(t) \in X, t > 0,$$

where

$$z_1(t) = \alpha^{-1} \int_0^t \mathrm{e}^{-(t-s)/\alpha}(\alpha^{-1} J_\alpha(z(s)) + \omega(s) z(s) + f(s))\,\mathrm{d}s \left(\int_0^t \mathrm{e}^{-(t-s)/\alpha}\,\mathrm{d}s \right)^{-1},$$

for $t > 0$.
Thus $\mathbb{T}$ maps $\mathcal{K}$ into itself. For any pair $z_1, z_2 \in \mathcal{K}$ we have

$$\|\mathbb{T}(z_1)(t) - \mathbb{T}(z_2)(t)\| \le (\alpha^{-1} + \|\omega\|_{L^\infty}) \int_0^t \mathrm{e}^{-(t-s)/\alpha} \|z_1(s) - z_2(s)\|\,\mathrm{d}s\,,$$

since J_α is Lipschitz with constant 1. This shows that

$$\|\mathbb{T}(z_1)(t) - \mathbb{T}(z_2)(t)\| \le (\alpha^{-1} + \|\omega\|_{L^\infty}) t \|z_1 - z_2\|_{C([0,T];X)}, \quad 0 \le t \le T.$$

Assume that for an integer $k \ge 1$ arbitrary but fixed

$$\|\mathbb{T}^k(z_1)(t) - \mathbb{T}^k(z_2)(t)\| \le \frac{(\alpha^{-1} + \|\omega\|_{L^\infty})^k t^k}{k!} \|z_1 - z_2\|_{C([0,T];X)}, \quad 0 \le t \le T.$$

Then, for $k+1$ we have

$$\begin{aligned} \|\mathbb{T}^{k+1}(z_1)(t) - \mathbb{T}^{k+1}(z_2)(t)\| &= \|\mathbb{T}(\mathbb{T}^k(z_1))(t) - \mathbb{T}(\mathbb{T}^k(z_2))(t)\| \\ &\le (\alpha^{-1} + \|\omega\|_{L^\infty}) \int_0^t \mathrm{e}^{-(t-s)/\alpha} \|\mathbb{T}^k(z_1) - \mathbb{T}^k(z_2)\|_{C([0,T];X)}\,\mathrm{d}s \end{aligned}$$

$$\begin{aligned}
&\leq (\alpha^{-1} + \|\omega\|_{L^\infty}) \int_0^t \frac{(\alpha^{-1} + \|\omega\|_{L^\infty})^k t^k}{k!} \|z_1 - z_2\|_{C([0,T];X)} \, \mathrm{d}s \\
&= \frac{(\alpha^{-1} + \|\omega\|_{L^\infty})^{k+1} t^{k+1}}{(k+1)!} \|z_1 - z_2\|_{C([0,T];X)}, \quad 0 \leq t \leq T.
\end{aligned}$$

Thus, using induction,

$$\|\mathbb{T}^k(z_1)(t) - \mathbb{T}^k(z_2)(t)\| \leq \frac{(\alpha^{-1} + \|\omega\|_{L^\infty})^k t^k}{k!} \|z_1 - z_2\|_{C([0,T];X)}, \quad 0 \leq t \leq T,$$

which implies that

$$\|\mathbb{T}^k(z_1) - \mathbb{T}^k(z_2)\|_{C([0,T];X)} \leq \frac{(\alpha^{-1} + \|\omega\|_{L^\infty})^k T^k}{k!} \|z_1 - z_2\|_{C([0,T];X)},$$

and hence $\mathbb{T}^k$ is a strict contraction for k sufficiently large, so (1.10) has a unique solution in $\mathcal{K}$.

Now let z_α be the unique solution for each $\alpha > 0$ of (1.10) with $z_\alpha(0) = z_0$. Since by assumption $A(0) = 0$, it follows that $A_\alpha(0) = 0$. Hence, we obtain the inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_\alpha(t)\|^2 \leq \omega(t)\|z_\alpha(t)\|^2 + \|f(t)\|\|z_\alpha(t)\|.$$

Applying Lemma 1.8.13 with $\rho = 1/2$ yields

$$\|z_\alpha(t)\| \leq \|z_0\|\mathrm{e}^{\Omega(t,0)} + \int_0^t \mathrm{e}^{\Omega(t,s)}\|f(s)\| \, \mathrm{d}s \,. \tag{1.11}$$

We now estimate $\dot{z}_\alpha$ using a similar technique. If $h > 0$, it follows that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_\alpha(t+h) - z_\alpha(t)\|^2 &\leq \mathrm{Re}\,\langle \omega(t+h)z_\alpha(t+h) - \omega(t)z_\alpha(t), z_\alpha(t+h) - z_\alpha(t)\rangle \\
&\quad + \mathrm{Re}\,\langle f(t+h) - f(t), z_\alpha(t+h) - z_\alpha(t)\rangle \\
&= \omega(t+h)\|z_\alpha(t+h) - z_\alpha(t)\|^2 \\
&\quad + (\omega(t+h) - \omega(t))\,\mathrm{Re}\,\langle z_\alpha(t), z_\alpha(t+h) - z_\alpha(t)\rangle \\
&\quad + \mathrm{Re}\,\langle f(t+h) - f(t), z_\alpha(t+h) - z_\alpha(t)\rangle \\
&\leq \omega(t+h)\|z_\alpha(t+h) - z_\alpha(t)\|^2 \\
&\quad + |\omega(t+h) - \omega(t)|\|z_\alpha(t)\|\|z_\alpha(t+h) - z_\alpha(t)\| \\
&\quad + \|f(t+h) - f(t)\|\|z_\alpha(t+h) - z_\alpha(t)\|.
\end{aligned}$$

Hence, applying Lemma 1.8.13 with $\rho = 1/2$, dividing by h and letting $h \to 0$ gives

$$\begin{aligned}
\|\dot{z}_\alpha(t)\| &\leq \|\dot{z}_\alpha(0)\|\mathrm{e}^{\Omega(t,0)} + \int_0^t \mathrm{e}^{\Omega(t,s)} b(s) \, \mathrm{d}s \\
&= \|A_\alpha(z_0) + \omega(0)z_0 + f(0)\|\mathrm{e}^{\Omega(t,0)} + \int_0^t \mathrm{e}^{\Omega(t,s)} b(s) \, \mathrm{d}s \,,
\end{aligned} \tag{1.12}$$

where $b(t) := \|\dot{f}(s)\| + |\dot{\omega}(s)|\|z_\alpha(s)\|$. Since $\|A_\alpha(z_0)\| \le \|A(z_0)\|$, it follows that $\dot{z}_\alpha, z_\alpha$ and $A_\alpha(z_\alpha)$ are bounded in $C([0,T];X)$.
Let $\alpha, \beta > 0$ be given, then

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_\alpha(t) - z_\beta(t)\|^2 = \omega(t)\|z_\alpha(t) - z_\beta(t)\|^2 \\ + \operatorname{Re}\left\langle A_\alpha(z_\alpha(t)) - A_\beta(z_\beta(t)), z_\alpha(t) - z_\beta(t)\right\rangle .\end{aligned}$$

Write $z_\alpha = J_\alpha z_\alpha - \alpha A_\alpha z_\alpha$ and likewise for z_β to get

$$\begin{aligned}\operatorname{Re}\langle A_\alpha(z_\alpha) - A_\beta(z_\beta), z_\alpha - z_\beta\rangle = &- \operatorname{Re}\langle A_\alpha(z_\alpha) - A_\beta(z_\beta), \alpha A_\alpha(z_\alpha) - \beta A_\beta(z_\beta)\rangle \\ &+ \operatorname{Re}\langle A_\alpha(z_\alpha) - A_\beta(z_\beta), J_\alpha(z_\alpha) - J_\beta(z_\beta)\rangle\end{aligned}$$

For the first term in the right-hand side we have

$$\begin{aligned}&- \operatorname{Re}\langle A_\alpha(z_\alpha) - A_\beta(z_\beta), \alpha A_\alpha(z_\alpha) - \beta A_\beta(z_\beta)\rangle \\ &= -\alpha\|A_\alpha(z_\alpha)\|^2 - \beta\|A_\beta(z_\beta)\|^2 + (\alpha+\beta)\operatorname{Re}\langle A_\alpha(z_\alpha), A_\beta(z_\beta)\rangle \\ &\le -\alpha\|A_\alpha(z_\alpha)\|^2 - \beta\|A_\beta(z_\beta)\|^2 \\ &\quad + \alpha\left(\|A_\alpha(z_\alpha)\|^2 + \frac{1}{4}\|A_\beta(z_\beta)\|^2\right) + \beta\left(\|A_\beta(z_\beta)\|^2 + \frac{1}{4}\|A_\alpha(z_\alpha)\|^2\right) \\ &\le \frac{\alpha+\beta}{4}K^2.\end{aligned}$$

where $K := \sup_{t\in[0,T]}\{\|A_\alpha(z_\alpha(t))\| \mid \alpha > 0\}$. For the second one, using that $A_\alpha = \alpha^{-1}(J_\alpha - I)$ and the same with β we obtain

$$\begin{aligned}&\operatorname{Re}\langle A_\alpha(z_\alpha) - A_\beta(z_\beta), J_\alpha(z_\alpha) - J_\beta(z_\beta)\rangle \\ &\quad = \operatorname{Re}\left\langle \alpha^{-1}(J_\alpha(z_\alpha) - z_\alpha) - \beta^{-1}(J_\beta(z_\beta) - z_\beta), J_\alpha(z_\alpha) - J_\beta(z_\beta)\right\rangle .\end{aligned}$$

Setting $J_\alpha(z_\alpha) = u_\alpha$ and $J_\beta(z_\beta) = u_\beta$ together with $z_\alpha = u_\alpha - \alpha A(u_\alpha)$ and likewise for z_β yield

$$\begin{aligned}&\operatorname{Re}\left\langle \alpha^{-1}(u_\alpha - (u_\alpha - \alpha A(u_\alpha))) - \beta^{-1}(u_\beta - (u_\beta - \beta A(u_\beta))), u_\alpha - u_\beta\right\rangle \\ &\quad = \operatorname{Re}\langle A(u_\alpha) - A(u_\beta), u_\alpha - u_\beta\rangle \le 0,\end{aligned}$$

since A is dissipative. Hence

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_\alpha(t) - z_\beta(t)\|^2 \le \omega(t)\|z_\alpha(t) - z_\beta(t)\|^2 + \frac{\alpha+\beta}{4}K^2.$$

Thus we have

$$\|z_\alpha(t) - z_\beta(t)\|^2 \le \frac{\alpha+\beta}{2}K^2\int_0^t \mathrm{e}^{2\Omega(t,s)}\,\mathrm{d}s\,, \quad 0 \le t \le T.$$

In particular $(z_{1/n})_{n\in\mathbb{N}}$ is Cauchy in $C([0,T];X)$ and define $z \in C([0,T];X)$ as its limit.

Step 3. Smoothness of z

Since $z_{1/n}$ is Lipschitz continuous uniformly in t and n by (1.12), it follows that z is also Lipschitz continuous uniformly in t with $z(0) = z_0$ and $\dot{z} \in L^\infty([0,T];X)$. Moreover, for each t we have $z_{1/n}(t) \to z(t)$ and there exists $K > 0$ independent of t and n such that $\|A_{1/n} z_{1/n}(t)\| \leq K$. It follows from Lemma 1.8.5 ((ii)) that $z(t) \in \mathcal{D}(A)$ for all t and $A_{1/n} z_{1/n}(t) \to Az(t)$ weakly. Thus, $\|Az(t)\| \leq K$ as well. Moreover, for a sequence $t_k \to t$, we have by Lemma 1.8.5 (i) that $Az(t_k) \to Az(t)$ weakly, which shows that $t \mapsto Az(t)$ is weakly continuous.

Since $z_{1/n}$ satisfies (1.9), for every $\theta \in X$ we have that

$$\left\langle z_{1/n_k}(t), \theta \right\rangle = \langle z_0, \theta \rangle + \int_0^t \left\langle A_{1/n_k}(z_{1/n_k}(s)) + \omega(s) z_{1/n_k}(s) + f(s), \theta \right\rangle \mathrm{d}s \,.$$

Moreover, $z_{1/n}(t) \to z(t)$, $A_{1/n} z_{1/n}(t) \to Az(t)$ weakly and $\left| \left\langle A_{1/n} z_{1/n}(t), \theta \right\rangle \right| \leq K \|\theta\|$, we obtain

$$\langle z(t), \theta \rangle = \int_0^t \langle A(z(s)) + \omega(s) z(s) + f(s), \theta \rangle \,\mathrm{d}s$$

by bounded convergence [28, Theorem II.4.1]. Given that the integrand is continuous in s, $\langle z(t), \theta \rangle$ is continuously differentiable on $[0,T]$ with

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle z(t), \theta \rangle = \langle A(z(t)) + \omega(t) z(t) + f(t), \theta \rangle \,.$$

Further since $A(z(t))$ is weakly continuous it follows that it is strongly measurable. Together with the fact of being bounded, it follows that it is Bochner integrable. Then the former shows that

$$z(t) = z_0 + \int_0^t A(z(s)) + \omega(s) z(s) + f(s) \,\mathrm{d}s \,,$$

and thus $\dot{z}$ is continuous except possibly in a set of measure zero. If we consider (1.8) in $[s,T]$ with the initial value $z(s)$, the solution must coincide with our $z(t)$ on $[s,T]$ owing this to the uniqueness of the solution. As we have obtained (1.11), for $r \in [s,t]$ with $t \leq T$, we have

$$\mathrm{e}^{-\Omega(r,0)} \|z(r)\| \leq \|z_0\| + \|f\|_\infty r.$$

In a similar way as we have computed (1.12), using the former we obtain

$$\begin{aligned} \|\dot{z}(t)\| \mathrm{e}^{-\Omega(t,0)} - \|\dot{z}(s)\| \mathrm{e}^{-\Omega(s,0)} &\leq \|\dot{\omega}\|_\infty \int_s^t \mathrm{e}^{-\Omega(r,0)} \|z(r)\| \,\mathrm{d}r + \|\dot{f}\|_\infty \int_s^t \mathrm{e}^{-\Omega(r,0)} \,\mathrm{d}r \\ &\leq \|\dot{\omega}\|_\infty \|z_0\| (t-s) + \frac{\|\dot{\omega}\|_\infty \|f\|_\infty}{2} (t^2 - s^2) \end{aligned}$$

$$+ \frac{\|\dot{f}\|_\infty}{\|\omega\|_\infty}(\mathrm{e}^{\|\omega\|_\infty t} - \mathrm{e}^{\|\omega\|_\infty s}).$$

Thus, defining $c_1, c_2, c_3, c_4 > 0$ according to the former inequality, the function $t \mapsto \|\dot{z}(t)\|\mathrm{e}^{-\Omega(t,0)} - c_1 t - c_2 t^2 - c_3 \mathrm{e}^{c_4 t}$ is monotonically decreasing and it is therefore continuous except possibly at a countable number of points. Since $t \mapsto \|\dot{z}(t)\|$ is weakly continuous, it follows that it is strongly continuous at each point t where $t \mapsto \|\dot{z}(t)\|$ is continuous. Hence, since $t \mapsto \dot{z}(t)$ is strongly continuous except possibly at a countable number of points and since $t \mapsto \omega(t)z(t) + f(t)$ is continuous, $t \mapsto A(z(t))$ is also continuous except possibly at a countable number of points. □

1.9 Funnel control and solution concepts

The last section of this chapter is devoted to clarify the relation between the uncontrolled and funnel controlled systems by using the appropriate solution concepts. As we will see in the following chapters, when dealing with PDEs, there is something that plays an important role that does not come up in the finite-dimensional scenario, which is the initial value of the system. This is then strictly related with the kind of funnel controller that we use.

Assume that we have reflexive Banach spaces U, X, Y and a system Σ in the triple (U, X, Y) with input $u(t) \in U$, state $x(t) \in X$ and output $y(t) \in Y$ for a.e. $t \in [0, T]$ for $T > 0$ and $x(0) = x_0 \in X_0 \subset X$ given, where X_0 is a dense subspace of X. Let us assume that on the system Σ one defines a *solution concept (SC)*, that is, (u, x, y) satisfy certain properties —for instance, in the case of the linear infinite-dimensional prototype one could think about well-posedness. Of course, if we introduce a feed-back law as the FC,

$$u(t) = -\frac{k_0}{1 - \varphi(t)^2 \|y(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}}(y(t) - y_{\mathrm{ref}}(t)),$$

one wishes that the resulting solution of the closed-loop system, if it exists, satisfies that (u, x, y) is also a solution as defined in SC. Note that we have chosen finite-dimensional input and output spaces and in particular, we will work in the case in which they have the same dimension, that is $U = Y = \mathbb{R}^m$ for some $m \in \mathbb{N}$, which will be a general assumption.

As an example, assume that the system Σ is L^2-well-posed on $(\mathbb{R}^m, X, \mathbb{R}^m)$ for some $m \in \mathbb{N}$, so that $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$. Then we investigate the problem

$$\begin{aligned} x(t) &= \mathbb{T}_t x_0 + \Phi_t u, \\ y(t) &= \Psi_t x_0 + \mathbb{F}_t u, \\ u(t) &= -\frac{k_0}{1 - \varphi(t)^2 \|y(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}}(y(t) - y_{\mathrm{ref}}(t)), \end{aligned} \qquad (\Sigma_{CL})$$

for some appropriate and regular $\varphi, y_{\mathrm{ref}}$. Assuming that the closed-loop system (Σ_{CL}) has a solution according to our SC, this would imply that $u, y \in L^2([0,T];\mathbb{R}^m)$ and further we have that the state $x \in C([0,T];X)$. However, this does not provide enough information about the regularity of (u,x,y) or whether the funnel controller is performing as desired, that is, the error $y - y_{\mathrm{ref}}$ is bounded away from the funnel boundary φ^{-1}. In general, these two things are the ones that take most effort to prove, together with showing existence of a solution of (Σ_{CL}).

In the following chapters we will study different cases to which we will apply the FC. In Chapters 2 & 3 we consider systems that fit within the existing theory of the FC, so even though we deal with infinite-dimensional systems, we reduce the problem to a functional differential equation, for which the solution concept is clear. On the other hand, in Chapter 4 we start from the very beginning with a system class for which one can not always guarantee that there is a solution —we do not assume the system to be well-posed. In fact, we consider the fully nonlinear problem induced by the closed-loop system and on top of that we define a solution concept for the resulting system. Finally, in Chapter 5 we deal with a nonlinear, parabolic PDE for which one can define a solution concept by means of the weak formulation using sesquilinear forms. Since all the cases are different and the FC needs to be slightly modified in the various scenarios, we will recall and introduce in each chapter the basics related to funnel control and explicitly define what the control objective is.

2 Adaptive control for a moving water tank

When a liquid-filled containment is subject to movement, the motion of the fluid may have a significant effect on the dynamics of the overall system and is known as *sloshing*. The latter phenomenon can be understood as internal dynamics of the system and it is of great importance in a range of applications such as aeronautics and control of containers and vehicles, and has been studied in engineering for a long time, see e.g. [23, 32, 42, 45, 118, 126].

The standard model for the one-dimensional movement of a fluid is given by the Saint-Venant equations, which is a system of nonlinear hyperbolic partial differential equations (PDEs). Models of a moving water tank involving these equations without friction have been studied in various articles. The first approach appears in [29] where a *flat output* for the linearized model is constructed (see e.g. [34] for the flatness approach). Several additional control problems related to this model are studied in [95] and it is proved that the linearization is steady-state controllable. Even more so, the seminal work [26] shows that the (nonlinear) model is locally controllable around any steady state. However, as an interesting addition, in [25] it is shown that the two-dimensional version is not locally controllable under some generic condition, where the control acts on the boundary and only depends on time. Different stabilization approaches by state and output feedback using Lyapunov functions are studied in [98]. In [8] observers are designed to estimate the horizontal currents by exploiting the symmetries in the Saint-Venant equations. Convergence of the estimates to the actual states is studied for the linearized model. In [23] a port-Hamiltonian formulation of the system is provided as a mixed finite-infinite-dimensional port-Hamiltonian system. For a recent numerical treatment of a truck with a fluid basin see [35].

In the present chapter we consider output trajectory tracking for moving water tank systems by funnel control. The moving water tank system that we consider in the present chapter contains a non-vanishing friction term as modeled in the Saint-Venant equations e.g. in [9]. It is our aim to show that the funnel controller introduced in [11] is feasible for these systems. While a very large class of functional differential equations with higher relative degree is considered in [11] and funnel control is shown to work for those systems (cf. also Section 2.2), it is not clear exactly which systems containing PDEs are encompassed by this class. It is our main result that the linearized model of the moving water tank, where the above mentioned effect of sloshing appears, belongs to the aforementioned system class.

2.1 Mathematical model

In the present chapter we investigate the (frictionless) horizontal movement of a water tank as depicted in Fig. 2.1.

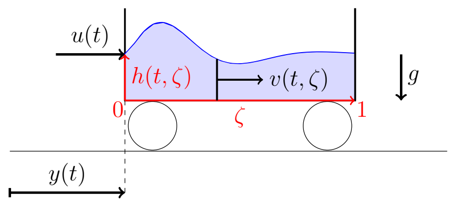

Figure 2.1: Horizontal movement of a water tank.

We assume that there is an external force acting on the water tank, which we denote by $u(t)$ as this will be the control input of the resulting system. The measurement output is the horizontal position $y(t)$ of the water tank, and the mass of the empty tank is denoted by m_T. The dynamics of the water under gravity g are described by the *Saint-Venant equations* (also called one-dimensional *shallow water equations*)

$$\begin{aligned} \partial_t h + \partial_\zeta (hv) &= 0, \\ \partial_t v + \partial_\zeta \left(\frac{v^2}{2} + gh \right) + hS\left(\frac{v}{h}\right) &= -\ddot{y} \end{aligned} \tag{2.1}$$

with boundary conditions $v(t,0) = v(t,1) = 0$. Here $h : [0,\infty) \times [0,1] \to \mathbb{R}$ denotes the height profile and $v : [0,\infty) \times [0,1] \to \mathbb{R}$ the (relative) horizontal velocity profile. The first equation in (2.1) is the so called *continuity equation* and describes the conservation of water volume in the tank. The second equation in (2.1) is the so called *momentum equation* and describes the balance between forces and momentum change rate. The boundary conditions require a zero velocity profile at the boundaries so that the movement is restricted to the container, the length of which is normalized to 1.
The friction term $S : \mathbb{R} \to \mathbb{R}$ is typically modeled by a high velocity coefficient of the form $C_S v^2/h^2$ and another one which plays the role of a viscous drag of the form $C_D v/h$ for some positive constants C_S, C_D. In the present chapter, we do not specify S, but we do assume that $S(0) = 0$ and $S'(0) > 0$. The condition $S(0) = 0$ means that, whenever the velocity is zero, then there is no friction. The condition $S'(0) > 0$ indicates the viscous drag does not vanish and hence the friction term is not conservative; this is the case in most real-world non-ideal situations. However, we stress that in the literature the friction term is usually assumed to be conservative, see e.g. [9, Sec. 1.4].

For a derivation of the Saint-Venant equations (2.1) of a moving water tank we refer to [23, 95], see also the references therein. The friction term in the model is the general version of that used in [9, Sec. 1.4]. Let us emphasize that in our framework the input is the force acting on the water tank, which can be manipulated using an engine for instance. In contrast to this, in [26, 95] the acceleration of the tank is used as input, but this can usually not be influenced directly. Note that — in the presence of sloshing — the applied force does not equal the product of the tanks's mass and acceleration. We also stress that, if the acceleration is used as input, then the input-output relation is given by the simple double integrator $\ddot{y} = u$, and the Saint-Venant equations (2.1) do not affect this relation. Of course, in [26, 95] controllability of the complete state including the Saint-Venant equations is considered, but here we study output tracking, which does not require to influence the complete state.
As shown in [29, 95], the linearization of the Saint-Venant equations is relevant in the context of control since it provides a model which is much simpler to solve (both analytically and numerically) and it can be an insightful approximation for motion planning purposes. Therefore, we restrict ourselves to the linearization of (2.1). In order to derive the linearization we first consider the general operator differential equation

$$\partial_t x(t) - F\big(x(t)\big) = f(t), \tag{2.2}$$

where $F : \mathcal{D}(F) \subseteq X \to Y$, $f : [0, \infty) \to Y$, X, Y are suitable Hilbert spaces and $\mathcal{D}(F)$ is the domain of the operator F. Different notions of a solution of (2.2) may be used, such as classical, mild or weak solution, see e.g. [67]. We call a point $x^* \in \mathcal{D}(F)$ an *equilibrium* or *steady-state* of (2.2), if $F(x^*) = 0$. In this case, $t \mapsto x^*$ is a solution (in any sense) of the homogeneous part $\partial_t x(t) - F\big(x(t)\big) = 0$. If F is Fréchet differentiable in x^* with Fréchet derivative $\mathfrak{A} := D_{x^*}F : X \to Y$ ($\mathfrak{A}$ is linear and bounded), then the *linearization* of (2.2) around the steady-state x^* is given by

$$\partial_t x(t) - \mathfrak{A}x(t) = f(t). \tag{2.3}$$

In the setting of the Saint-Venant equations (2.1) we have $X = W^{1,2}([0, 1]; \mathbb{R}^2)$, $Y = L^2([0, 1]; \mathbb{R}^2)$, $f(t) = \left(\begin{smallmatrix} 0 \\ -\ddot{y}(t) \end{smallmatrix}\right)$ and the operator

$$F(x_1, x_2) = -\begin{pmatrix} \partial_\zeta(x_1 x_2) \\ \partial_\zeta\left(\frac{1}{2}x_2^2 + g x_1\right) + x_1 S\left(\frac{x_2}{x_1}\right) \end{pmatrix} \tag{2.4}$$

with

$$\mathcal{D}(F) = \left\{ (x_1, x_2) \in X \;\middle|\; \begin{array}{l} x_2(0) = x_2(1) = 0, \\ \forall\, \zeta \in [0, 1] : \; x_1(\zeta) > 0 \end{array} \right\}. \tag{2.5}$$

A steady-state $x^* = (H, V) \in \mathcal{D}(F)$ is a solution of the boundary-value problem

$$\partial_\zeta(HV) = 0,$$

$$V\partial_\zeta V + g\partial_\zeta H + HS\left(\frac{V}{H}\right) = 0, \quad V(0) = V(1) = 0.$$

In order to be able to deal with the aforementioned linearization we give the following result.

Lemma 2.1.1. *Consider the operator* $F : \mathcal{D}(F) \subseteq X \to Y$ *given by* (2.4)*, where* $X = W^{1,2}([0,1];\mathbb{R}^2)$*,* $Y = L^2([0,1];\mathbb{R}^2)$*, and* $\mathcal{D}(F)$ *is as in* (2.5)*. Then* F *is Fréchet differentiable in any point* $x = (x_1, x_2) \in \mathcal{D}(F)$ *with Fréchet derivative*

$$D_xFh = \begin{pmatrix} h_1\partial_\zeta x_2 + x_1\partial_\zeta h_2 + x_2\partial_\zeta h_1 + h_2\partial_\zeta x_1 \\ h_2\partial_\zeta x_2 + x_2\partial_\zeta h_2 + g\partial_\zeta h_1 + h_1 S\left(\frac{x_2}{x_1}\right) + h_2 S'\left(\frac{x_2}{x_1}\right) - h_1\frac{x_2}{x_1}S'\left(\frac{x_2}{x_1}\right)\end{pmatrix}$$

for $h = (h_1, h_2) \in X$*. In particular,* $D_xF : X \to Y$ *is bounded.*

Proof. It is straightforward to check that D_xF as defined above is the Gateaux derivative of F in x. It remains to show that F is Fréchet differentiable in x. Define the auxiliary function

$$\bar{S} : D \to \mathbb{R},\ (x_1, x_2) \mapsto x_1 S\left(\frac{x_2}{x_1}\right),$$

where

$$D = \left\{\ (x_1, x_2) \in \mathbb{R}^2\ \middle|\ x_1 \neq 0\ \right\}.$$

Then $\bar{S}$ is differentiable in D with

$$\bar{S}'(x_1, x_2) = \left[S\left(\frac{x_2}{x_1}\right) - \frac{x_2}{x_1}S'\left(\frac{x_2}{x_1}\right), S'\left(\frac{x_2}{x_1}\right)\right].$$

Now we compute that

$$\begin{aligned}
&\|F(x+h) - F(x) - D_xFh\|_Y^2 \\
&\qquad = \|h_1\partial_\zeta h_2 + h_2\partial_\zeta h_1\|_{L^2}^2 + \|h_2\partial_\zeta h_2\|_{L^2}^2 \\
&\qquad\quad + \left\|\bar{S}(x_1 + h_1, x_2 + h_2) - \bar{S}(x_1, x_2) - \bar{S}'(x_1, x_2)\left(\begin{smallmatrix} h_1 \\ h_2 \end{smallmatrix}\right)\right\|_{L^2}^2 \\
&\qquad \leq 2\|h_1\|_{L^2}^2\|\partial_\zeta h_2\|_{L^2}^2 + 2\|h_2\|_{L^2}^2\|\partial_\zeta h_1\|_{L^2}^2 + \|h_2\|_{L^2}^2\|\partial_\zeta h_2\|_{L^2}^2 \\
&\qquad\quad + \|R(x+h)\|_{L^2}^2\ \|h\|_Y^2 \\
&\qquad \leq 5\|h\|_X^4 + \|R(x+h)\|_{L^2}^2\|h\|_X^2,
\end{aligned}$$

where $R : \mathbb{R}^2 \to \mathbb{R}$, which depends on $x(\zeta)$, is such that $\lim_{z\to x(\zeta)} R(z) = 0$ for all $\zeta \in [0,1]$. Therefore, $\lim_{h\to 0, h\in X} \|R(x+h)\|_{L^2} = 0$ and hence we find that

$$\lim_{h\to 0}\frac{\|F(x+h) - F(x) - D_xFh\|_Y}{\|h\|_X} = 0.$$

Finally, boundedness of $D_xF : X \to Y$ follows from

$$D_xFh = \begin{bmatrix} \partial_\zeta x_2 & \partial_\zeta x_1 \\ S\left(\frac{x_2}{x_1}\right) - \frac{x_2}{x_1} S'\left(\frac{x_2}{x_1}\right) & \partial_\zeta x_2 + S'\left(\frac{x_2}{x_1}\right) \end{bmatrix} h + \begin{bmatrix} x_2 & x_1 \\ g & x_2 \end{bmatrix} \partial_\zeta h.$$

□

Since $S(0) = 0$ and $H(\zeta) > 0$ for all $\zeta \in [0,1]$, we may infer that $V \equiv 0$ and $H \equiv h_0 > 0$. It follows from Lemma 2.1.1 that F is Fréchet differentiable in $x^* = (h_0, 0) \in \mathcal{D}(F)$ with Fréchet derivative $\mathfrak{A} := D_{x^*}F : X \to Y$ given by

$$\mathfrak{A}z = -\begin{pmatrix} h_0 \partial_\zeta z_2 \\ g\partial_\zeta z_1 + S'(0) z_2 \end{pmatrix}, \quad z \in X.$$

Note that $\mathfrak{A} : X \to Y$ is bounded, but $\mathfrak{A} : (X, \|\cdot\|_Y) \subseteq Y \to Y$ will be unbounded since a weaker norm is used.
Define $\mu := \frac{1}{2} S'(0)$ and

$$P_1 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad \mathcal{H} := \begin{bmatrix} g & 0 \\ 0 & h_0 \end{bmatrix}, \quad P_0 := \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}, \quad b := \begin{pmatrix} 0 \\ -1 \end{pmatrix}.$$

Then the linearization of the Saint-Venant equations (2.1) is given by the system

$$\partial_t z = \mathfrak{A}z + b\ddot{y} = P_1 \partial_\zeta (\mathcal{H}z) - \mu P_0 z + b\ddot{y} \tag{2.6}$$

with boundary conditions

$$z_2(t,0) = z_2(t,1) = 0. \tag{2.7}$$

For the convenience of the reader let us also restate (2.6) line by line:

$$\begin{aligned} \partial_t z_1 + h_0 \partial_\zeta z_2 &= 0, \\ \partial_t z_2 + g\partial_\zeta z_1 + 2\mu z_2 &= -\ddot{y}. \end{aligned}$$

Note that by the first equation (conservation of mass) we have

$$\partial_t \int_0^1 z_1(t,\zeta)\,\mathrm{d}\zeta = -h_0 \int_0^1 \partial_\zeta z_2(t,\zeta)\,\mathrm{d}\zeta = -h_0\big(z_2(t,1) - z_2(t,0)\big) \overset{(2.7)}{=} 0,$$

hence $\int_0^1 z_1(t,\zeta)\,\mathrm{d}\zeta = \text{const}$ for all t. Furthermore, if (z_1, z_2) is a solution of (2.6) (in any sense), then also $(z_1, z_2) + (c, 0)$ is a solution of (2.6) for all $c \in \mathbb{R}$. Hence, without loss of generality we may restrict ourselves to solutions which satisfy $\int_0^1 z_1(t,\zeta)\,\mathrm{d}\zeta = 0$ for all $t \geq 0$. This justifies to choose

$$\hat{X} = \left\{ (f_1, f_2) \in L^2([0,1]; \mathbb{R}^2) \;\middle|\; \int_0^1 f_1(\zeta)\,\mathrm{d}\zeta = 0 \right\} \tag{2.8}$$

as new state space and to consider the operator $A : \mathcal{D}(A) \subseteq \hat{X} \to Y$, where $Az = \mathfrak{A}z$, for $z \in \mathcal{D}(A)$ and

$$\mathcal{D}(A) = \left\{ \, (z_1, z_2) \in \hat{X} \;\middle|\; \begin{array}{l} z_1, z_2 \in W^{1,2}([0,1];\mathbb{R}), \\ z_2(0) = z_2(1) = 0 \end{array} \right\}. \tag{2.9}$$

Note that for any $z \in \mathcal{D}(A)$ we have $\int_0^1 \partial_\zeta z_2(\zeta)\,\mathrm{d}\zeta = 0$, hence $Az \in \hat{X}$. Therefore, $A : \mathcal{D}(A) \subseteq \hat{X} \to \hat{X}$ and we like to stress that A may be unbounded. From now on, with some abuse of notation, we write X instead of $\hat{X}$.
In order to complete the model, we introduce the *momentum*

$$p(t) := m_T \dot{y}(t) + \int_0^1 \big(z_1(t,\zeta) + h_0\big)\big(z_2(t,\zeta) + \dot{y}(t)\big)\,\mathrm{d}\zeta\,,$$

and consider the balance law $\dot{p}(t) = u(t)$. Using (2.6) we calculate

$$\begin{aligned}
\dot{p}(t) &= m_T \ddot{y}(t) + \int_0^1 \partial_t z_1(t,\zeta)(z_2(t,\zeta) + \dot{y}(t))\,\mathrm{d}\zeta \\
&\quad + \int_0^1 \big(z_1(t,\zeta) + h_0\big)(\partial_t z_2(t,\zeta) + \ddot{y}(t))\,\mathrm{d}\zeta \\
&= m_T \ddot{y}(t) - \int_0^1 h_0 \partial_\zeta z_2(t,\zeta)(z_2(t,\zeta) + \dot{y}(t))\,\mathrm{d}\zeta \\
&\quad - \int_0^1 \big(z_1(t,\zeta) + h_0\big)\left(g\partial_\zeta z_1(t,\zeta) + 2\mu z_2(t,\zeta)\right)\,\mathrm{d}\zeta \\
&= m_T \ddot{y}(t) - \frac{g}{2}\left(z_1(t,1)^2 - z_1(t,0)^2\right) - h_0 g\big(z_1(t,1) - z_1(t,0)\big) \\
&\quad - 2\mu \int_0^1 \big(z_1(t,\zeta) + h_0\big) z_2(t,\zeta)\,\mathrm{d}\zeta\,.
\end{aligned}$$

Altogether the model that we consider in the present chapter is described by the following nonlinear equations,

$$\begin{aligned}
\partial_t z &= P_1 \partial_\zeta(\mathcal{H}z) - \mu P_0 z + b\ddot{y}, \\
\ddot{y}(t) &= \frac{g}{2m_T}\left(z_1(t,1) - z_1(t,0)\right)\big(2h_0 + z_1(t,1) + z_1(t,0)\big) \\
&\quad + \frac{2\mu h_0}{m_T}\int_0^1 z_2(t,\zeta)\,\mathrm{d}\zeta + \frac{2\mu}{m_T}\int_0^1 z_1(t,\zeta) z_2(t,\zeta)\,\mathrm{d}\zeta \\
&\quad + \frac{u(t)}{m_T}, \\
z_2(t,0) &= z_2(t,1) = 0
\end{aligned} \tag{2.10}$$

on the state space X, with input u, state z and output y.

2.2 Funnel control

The objective is to design an output error feedback $u(t) = F\big(t, e(t), \dot{e}(t)\big)$, where $y_{\text{ref}} \in W^{2,\infty}(0,\infty;\mathbb{R})$ is a reference signal, which applied to (2.10) results in a closed-loop system where the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \{\ (t,e) \in [0,\infty) \times \mathbb{R} \ \mid \ \varphi(t)|e| < 1\ \}, \tag{2.11}$$

which is determined by a function φ belonging to

$$\Phi := \left\{ \varphi \in C^1([0,\infty);\mathbb{R}) \ \middle| \ \begin{array}{l} \varphi, \dot{\varphi} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, all signals $u, e, \dot{e}$ should remain bounded.
The funnel boundary is given by the reciprocal of φ, see Fig. 2.2. The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)|e(0)| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.

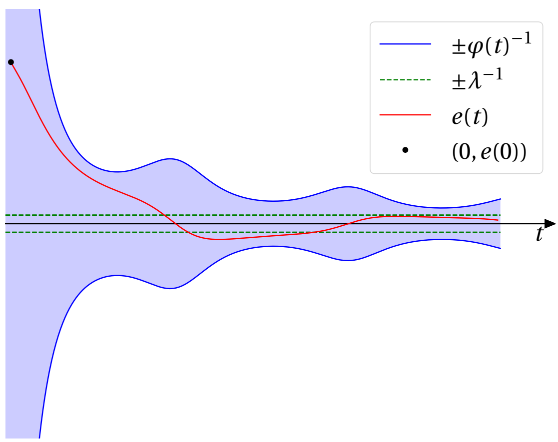

Figure 2.2: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$.

An important property is that each performance funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi$ is bounded away from zero, i.e., boundedness of φ implies that there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, for instance in the presence of periodic disturbances or strongly varying reference signals. For typical choices of funnel boundaries see e.g. [56, Section 3.2].

It was shown in [11] that the funnel controller

$$\begin{aligned} u(t) &= -k_1(t)\big(\dot{e}(t) + k_0(t)e(t)\big), \\ k_0(t) &= \frac{1}{1 - \varphi_0(t)^2\|e(t)\|^2}, \\ k_1(t) &= \frac{1}{1 - \varphi_1(t)^2\|\dot{e}(t) + k_0(t)e(t)\|^2}, \end{aligned} \tag{2.12}$$

where $\varphi_0, \varphi_1 \in \Phi$, achieves the above described control objective for a large class of nonlinear systems with relative degree two. In the present chapter we aim to extend this result and show feasibility of (2.12) for the (linearized) moving water tank described by (2.10).
In Section 2.2 we recall a recent result in funnel control from [11]. We show that in order to achieve the control objective it suffices to show that a certain operator is causal, locally Lipschitz continuous and maps bounded functions to bounded functions. To this end, we consider the linearized Saint-Venant equations in an abstract framework in Section 2.3 and show that the homogeneous part is an operator which generates a contraction semigroup. This then allows to study admissibility of certain control and observation operators for the system and, finally, to show that the inverse Laplace transform of the transfer function corresponding to these systems defines a measure with bounded total variation. In Section 2.4 we exploit this result to show that the operator associated with the internal dynamics of (2.10) is well-defined and has the properties mentioned above. Some conclusions are given in Section 2.5.
In this section we formulate how the funnel controller (2.12) described above achieves the control objective for system (2.10) — this is the main result of this chapter. The initial conditions for (2.10) are

$$\begin{aligned} \big(z_1(0,\cdot), z_2(0,\cdot)\big) &= \big(\tilde{h}_0(\cdot), v_0(\cdot)\big) \in \mathcal{D}(A), \\ \big(y(0), \dot{y}(0)\big) &= \big(y^0, y^1\big) \qquad \in \mathbb{R}^2, \end{aligned} \tag{2.13}$$

since the initial value for z needs to belong to the domain of the operator A in (2.9). In [11] the controller (2.12) is shown to be feasible for a large class of nonlinear systems of the form

$$\begin{aligned} \ddot{y}(t) &= f\big(d(t), \mathcal{S}(y,\dot{y})(t)\big) + \Gamma\, u(t) \\ \big(y(0), \dot{y}(0)\big) &= \big(y^0, y^1\big) \in \mathbb{R}^2 \end{aligned} \tag{2.14}$$

where

(N1) the disturbance satisfies $d \in L^\infty([0,\infty); \mathbb{R}^p)$, $p \in \mathbb{N}$;

(N2) $f \in C(\mathbb{R}^p \times \mathbb{R}^q; \mathbb{R})$, $q \in \mathbb{N}$,

(N3) the high-frequency gain satisfies $\Gamma > 0$, (N4) $\mathcal{S} : C([0,\infty);\mathbb{R}^2) \to L^\infty_{\mathrm{loc}}(0,\infty;\mathbb{R}^q)$ is an operator with the following properties:

a) $\mathcal{S}$ maps bounded trajectories to bounded trajectories, i.e, for all $c_1 > 0$, there exists $c_2 > 0$ such that for all $\zeta \in BC([0,\infty);\mathbb{R}^2)$ we have $\mathcal{S}(\zeta) \in L^\infty([0,\infty);\mathbb{R}^q)$ and

$$\|\zeta\|_\infty \le c_1 \quad \Rightarrow \quad \|\mathcal{S}(\zeta)\|_\infty \le c_2,$$

b) $\mathcal{S}$ is causal, i.e, for all $t \ge 0$ and all $\zeta, \xi \in C([0,\infty);\mathbb{R}^2)$,

$$\zeta|_{[0,t)} = \xi|_{[0,t)} \quad \Rightarrow \quad \mathcal{S}(\zeta)|_{[0,t)} \overset{\text{a.e.}}{=} \mathcal{S}(\xi)|_{[0,t)}.$$

c) $\mathcal{S}$ is locally Lipschitz continuous in the following sense: for all $t \ge 0$ and all $\xi \in C([0,t];\mathbb{R}^2)$ there exist $\tau, \delta, c > 0$ such that, for all $\zeta_1, \zeta_2 \in C([0,\infty);\mathbb{R}^2)$ with $\zeta_i|_{[0,t]} = \xi$ and $\|\zeta_i(s) - \xi(t)\| < \delta$ for all $s \in [t, t+\tau]$ and $i = 1, 2$, we have

$$\left\| \left(\mathcal{S}(\zeta_1) - \mathcal{S}(\zeta_2)\right)|_{[t,t+\tau]} \right\|_\infty \le c \left\| (\zeta_1 - \zeta_2)|_{[t,t+\tau]} \right\|_\infty .$$

In [50, 59, 61, 62] it is shown that the class of systems (2.14) encompasses linear and nonlinear finite-dimensional systems with strict relative degree two and input-to-state stable internal dynamics. The operator $\mathcal{S}$ allows for infinite-dimensional (linear) systems, systems with hysteretic effects or (when a slightly more general version of (2.14) with a memory component is considered) nonlinear delay elements, and combinations thereof. The linear infinite-dimensional systems that are considered in [61, 62] are in a special Byrnes-Isidori form that is discussed in detail in [63]. While the internal dynamics in these systems is allowed to correspond to a strongly continuous semigroup, all other operators are assumed to be bounded. In contrast to this, the equation (2.10) that we consider here is nonlinear and involves unbounded operators.

In [11], the existence of solutions of the initial value problem resulting from the application of the funnel controller (2.12) to a system (2.14) is investigated. By a *solution* of (2.12), (2.14) on $[0,\omega)$ we mean a function $y \in C^1([0,\omega);\mathbb{R})$, $\omega \in (0,\infty]$, such that $\dot{y}$ is weakly differentiable and satisfies (2.14) with u defined in (2.12) for almost all $t \in [0,\omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of solutions of functional differential equations has been investigated in [61] for instance.

The following result is from [11]. Note that in [11] a slightly stronger version of condition (N4) c) is used. However, the existence part of the proof there relies on a result from [59] where the version from the present chapter is used.

Theorem 2.2.1. *Consider a system* (2.14) *with properties (N1)–(N4). Let* $y_{\mathrm{ref}} \in W^{2,\infty}([0,\infty);\mathbb{R})$*,* $\varphi_0, \varphi_1 \in \Phi$ *and* $(y^0, y^1) \in \mathbb{R}^2$ *be initial conditions such that*

$$\varphi_0(0)|y^0 - y_{\mathrm{ref}}(0)| < 1$$

$$\text{and} \quad \varphi_1(0)|y^1 - \dot{y}_{\text{ref}}(0) + k_0(0)\big(y^0 - y_{\text{ref}}(0)\big)| < 1.$$

Then the funnel controller (2.12) applied to (2.14) yields an initial-value problem which has a solution, and every solution can be extended to a maximal solution $y : [0, \omega) \to \mathbb{R}$, $\omega \in (0, \infty]$, which has the following properties:

1. *The solution is global (i.e., $\omega = \infty$).*
2. *The input $u : [0, \infty) \to \mathbb{R}$, the gain functions $k_0, k_1 : [0, \infty) \to \mathbb{R}$ and $y, \dot{y} : [0, \infty) \to \mathbb{R}$ are bounded.*
3. *The tracking error $e = y - y_{\text{ref}}$ is uniformly bounded away from the funnel boundary in the following sense:*

$$\exists\, \varepsilon > 0\ \forall\, t > 0 : \ |e(t)| \le \varphi_0(t)^{-1} - \varepsilon. \tag{2.15}$$

In order to show that the funnel controller (2.12) is feasible for (2.10), (2.13), we will show that (2.10), (2.13) belongs to the class of systems (2.14). Then feasibility is a consequence of the above Theorem 2.2.1.
Using the change of variables $x(t) = z(t) - b\eta(t)$ where we use the notation $\eta(t) := \dot{y}(t) - \dot{y}(0)$, system (2.10) can be rewritten as

$$\ddot{y}(t) = \mathcal{S}(y, \dot{y})(t) + \frac{u(t)}{m_T}, \tag{2.16}$$

where $\mathcal{S} : C([0, \infty); \mathbb{R}^2) \to L^\infty_{\text{loc}}([0, \infty); \mathbb{R})$ is given by

$$\mathcal{S}(y_1, y_2) := \mathcal{T}(y_2 - y_2(0)) \tag{2.17}$$

for the operator $\mathcal{T} : C_0([0, \infty); \mathbb{R}) \to L^\infty_{\text{loc}}([0, \infty); \mathbb{R})$, where

$$C_0([0, \infty); \mathbb{R}) := \{f \in C([0, \infty); \mathbb{R}) \mid f(0) = 0\},$$

defined by

$$\begin{aligned}
\mathcal{T}(\eta)(t) :=\ & \frac{g}{2m_T}\left(x_1(t,1) - x_1(t,0)\right)\big(2h_0 + x_1(t,1) + x_1(t,0)\big) \\
& + \frac{2\mu h_0}{m_T}\int_0^1 x_2(t,\zeta)\,\mathrm{d}\zeta\ + \frac{2\mu}{m_T}\int_0^1 x_1(t,\zeta)x_2(t,\zeta)\,\mathrm{d}\zeta \\
& - \frac{2\mu h_0}{m_T}\eta(t),
\end{aligned} \tag{2.18}$$

$$\dot{x}(t) = Ax(t) + Ab\eta(t), \quad x(0) = x_0 = (\tilde{h}_0, v_0). \tag{2.19}$$

Note that $\mathcal{T}$ depends on $x = x(t, \zeta)$ which in turn is given through η and x_0 as the solution of the linear PDE (2.19) that is a one-dimensional wave equation. We like to point out that the operator $\mathcal{S}$ essentially models the internal dynamics of system (2.10).

Theorem 2.2.2. *For* $\mu > 0$ *the system consisting of* (2.10), (2.13) *belongs to the class of systems* (2.14). *More precisely, the operator* $\mathcal{S}$ *from (N4) is given by* (2.17). *Therefore, the assertions of Theorem 2.2.1 hold for the considered system.*

Proof. First observe that for equation (2.16) conditions (N1)–(N3) are obviously satisfied, so it remains to show the properties of the operator $\mathcal{S}$ as required in (N4). By Proposition 2.4.1 the operator $\mathcal{T}$ given by (2.18), (2.19) is well-defined, locally Lipschitz continuous and maps bounded functions to bounded functions. As it can be seen that $\mathcal{S}$ is causal it thus follows that it satisfies (N4). □

Remark 2.2.3. In the case $\mu = 0$ the statement of Theorem 2.2.2 is false in general, because the operator $\mathcal{S}$ does not satisfy condition a) in (N4). To be more precise we need to consider the later results derived in Sections 2.3 and 2.4. If $\mu = 0$, then $\mathfrak{h} = \mathcal{L}^{-1}(G)$ derived in Lemma 2.3.4 does not have bounded total variation and thus an inspection of the proof of Proposition 2.4.1 reveals that $\mathcal{T}$ does not map bounded functions to bounded functions. For instance, $\mathcal{T}(\sin)(\cdot)$ is unbounded.

Here we illustrate the application of the funnel controller (2.12) to the linearized moving water tank system (2.10). In the following we present the numerical method used to simulate the corresponding closed-loop system. Using the change of variables

$$z(\zeta,t) = Q\eta(\zeta,t), \quad \text{where} \quad Q := \begin{bmatrix} 1 & 1 \\ \frac{g}{c} & -\frac{g}{c} \end{bmatrix},$$

we may rewrite (2.10) as

$$\begin{aligned}
\partial_t \eta_1(t,\zeta) + c\partial_\zeta \eta_1(t,\zeta) &= \mu(\eta_2(t,\zeta) - \eta_1(t,\zeta)) - \frac{c}{2g}\ddot{y}(t), \\
\partial_t \eta_2(t,\zeta) - c\partial_\zeta \eta_2(t,\zeta) &= \mu(\eta_1(t,\zeta) - \eta_2(t,\zeta)) + \frac{c}{2g}\ddot{y}(t), \\
\ddot{y}(t) - \frac{u(t)}{m_T} &= \frac{2\mu c}{m_T}\int_0^1 \eta_1(t,\zeta) - \eta_2(t,\zeta)\,\mathrm{d}\zeta \\
&\quad + \frac{2c^2}{m_T}(\eta_1(t,1) - \eta_1(t,0)) \\
&\quad + \frac{2g}{m_T}(\eta_1(t,1)^2 - \eta_1(t,0)^2) \\
&\quad + \frac{2\mu g}{m_T c}\int_0^1 \eta_1(t,\zeta)^2 - \eta_2(t,\zeta)^2\,\mathrm{d}\zeta\,, \\
\eta_1(t,0) &= \eta_2(t,0), \\
\eta_1(t,1) &= \eta_2(t,1).
\end{aligned}$$

Using an implicit method for the PDE corresponding to η_1 and an explicit method for the PDE corresponding to η_2 we can easily solve the closed-loop system numerically using finite differences. For the simulation we have used the parameters

$$m_T = 1\text{kg},\ h_0 = 0.5\text{m},\ g = 9.8\text{ms}^{-2},\ \mu = 0.1\text{s}^{-1}$$

and the reference signal

$$y_{\text{ref}}(t) = A\tanh^2(\omega t),$$

where $A = 1\text{m}$ and $\omega = 2\pi f$ with $f = \sqrt{h_0/g}$. The initial values (2.13) are chosen as

$$(\tilde{h}_0(\zeta), v_0(\zeta)) = (0\text{m}, 0.1\sin^2(4\pi\zeta)\text{ms}^{-1}),$$

and

$$(y^0, y^1) = (0\text{m}, 0\text{ms}^{-1}).$$

For the controller (2.12) we chose the funnel functions

$$\varphi_0(t) = \varphi_1(t) = 100\tanh(\omega t).$$

Clearly, the initial errors lie within the funnel boundaries as required in Theorem 2.2.1. For the finite differences we have used a grid in t with $M = 4000$ points for the interval $[0, 2\tau]$ with $\tau = f^{-1}$, and a grid in ζ with $N = \lfloor ML/(4c\tau)\rfloor$ points, where for $r \in [0,\infty)$, $\lfloor r\rfloor := \max\{n \in \mathbb{N} \mid n \le r\}$. Furthermore, we have used a tolerance of 10^{-6}. The method has been implemented in Python and the simulation results are shown in Figs. 2.3 & 2.4.

It can be seen that even in the presence of sloshing effects a prescribed performance of the tracking error can be achieved with the funnel controller (2.12), while at the same time the generated input is bounded and shows an acceptable performance.

The remainder of the chapter is concerned with the proof of Proposition 2.4.1, for which the crucial preliminaries are developed in the following section.

2.3 Linearized model – abstract framework

In this section we derive preliminary results concerning the operator associated with the linearized Saint-Venant equations (2.6). Furthermore, for later use we consider admissibility with respect to a certain control operator and compute the transfer functions with respect to certain observation operators. Finally, we show that the inverse Laplace transform of these transfer functions defines measures with bounded total variation.

We consider the complexification of the state space from (2.8) given by

$$\begin{aligned} X &= \left\{ (f_1, f_2) \in L^2([0,1];\mathbb{C}^2) \;\middle|\; \int_0^1 f_1(\zeta)\,\mathrm{d}\zeta = 0 \right\} \\ &= L^2([0,1];\mathbb{C}^2) \ominus [\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)] \end{aligned}$$

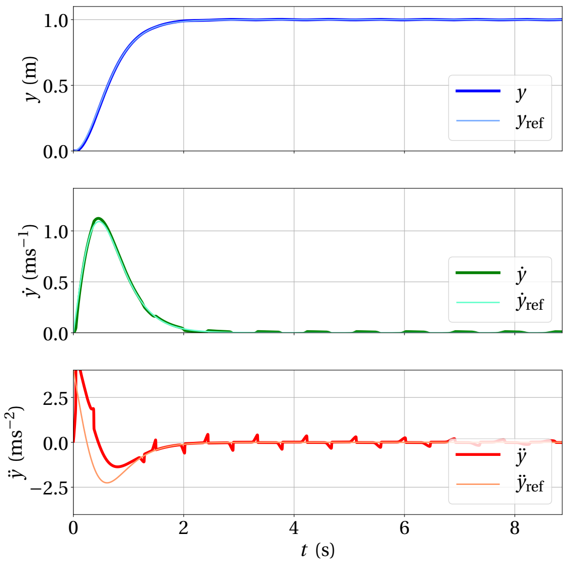

Figure 2.3: Output y and reference signal y_{ref} and corresponding first and second derivatives.

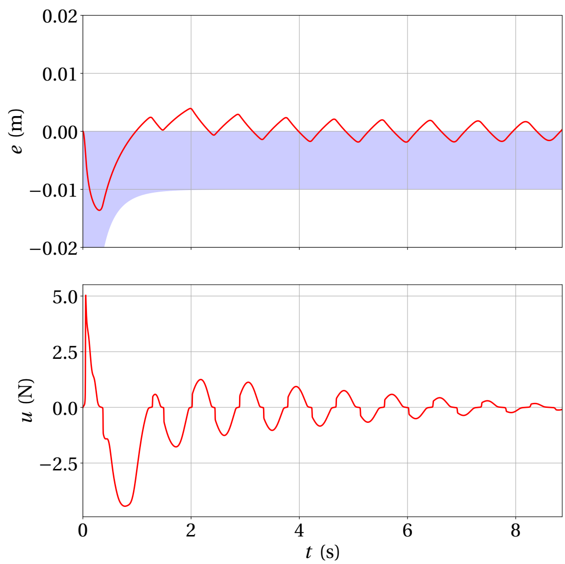

Figure 2.4: Performance funnel with tracking error e and generated input function u.

and the linear operators $A_\mu : \mathcal{D}(A_\mu) \subseteq X \to X$ given by

$$A_\mu z := P_1 \partial_\zeta (\mathcal{H} z) - \mu P_0 z$$

with domain $\mathcal{D}(A_\mu) = \mathcal{D}(A)$ as the complexification of (2.9). Here

$$L^2([0,1];\mathbb{C}^2) \ominus [(\begin{smallmatrix}1\\0\end{smallmatrix})]$$

refers to the orthogonal complement of the span of the function $(\begin{smallmatrix}1\\0\end{smallmatrix})$ in $L^2([0,1];\mathbb{C}^2)$. The equation (2.6) motivates to consider energy-based norms given through the Hamiltonian $\mathcal{H}$, i.e., for $z_1, z_2 \in X$ let

$$\langle z_1, z_2 \rangle_X = \frac{1}{2} \int_0^1 \overline{z_1(\zeta)}\, \mathcal{H}\, z_2(\zeta)\, \mathrm{d}\zeta\,.$$

Clearly, the solution of the linear damped wave equation $\dot{z} = A_\mu z$ with $z(0) = z_0$ can be derived by a Fourier ansatz. More general, the solution theory for linear PDEs can be derived in the framework of semigroup theory which corresponds to well-posedness in the sense of Hadamard.
The proof of the following result is a standard argument; we include a proof in the semigroup context.

Proposition 2.3.1. *Let $c := \sqrt{gh_0}$ and assume that $\mu \in [0, \pi c)$. The operator A_μ generates a contraction semigroup $(\mathbb{T}_t^\mu)_{t\geq 0}$ in X. The spectrum of A_μ consists of the eigenvalues*

$$\theta_n^\pm = -\mu \pm i\phi_n,$$

where

$$\phi_n = \sqrt{\sigma_n^2 - \mu^2}, \quad \sigma_n = n\pi c, \quad n \in \mathbb{N}. \tag{2.20}$$

If $\mu \in (0, \pi c)$, then $(\mathbb{T}_t^\mu)_{t\geq 0}$ is exponentially stable and $\omega_0(\mathbb{T}^\mu) = -\mu$. Furthermore, for $\mu = 0$ we have that for all $z \in X$ and $t \geq 0$,

$$\mathbb{T}_t z := \mathbb{T}_t^0 z = \sum_{n \in \mathbb{Z}\setminus\{0\}} z^n \psi_n \mathrm{e}^{i\sigma_n t},$$

where

$$\psi_n(\zeta) := \sqrt{\frac{2}{g}} \begin{pmatrix} \cos(\pi n \zeta) \\ -ich_0^{-1} \sin(\pi n \zeta) \end{pmatrix} \tag{2.21}$$

and $z^n = \langle \psi_n, z \rangle_X$ for $n \in \mathbb{Z}$.

Proof. Let us denote by $A_\mu^{X_0}$ the operator $A := A_0$ considered on the larger space $X_0 = L^2([0,1];\mathbb{C}^2)$, where

$$\mathcal{D}(A_\mu^{X_0}) = \left\{ z \in X_0 \;\middle|\; z \in W^{1,2}([0,1];\mathbb{C}^2),\ z_2(0) = z_2(1) = 0 \right\}.$$

It is well-known that A^{X_0} generates a unitary group $\mathbb{T}^{X_0}$. This can e.g. be argued by general results on port-Hamiltonian systems; in particular it suffices to show that A^{X_0} and $-A^{X_0}$ are dissipative, see [67, Ch. 7], which easily follows from the fact that $P_1 = P_1^*$ and integration by parts and the boundary conditions incorporated in the domain. Hence, $A^{X_0} = -(A^{X_0})^*$ by Stone's theorem and since A^{X_0} has compact resolvent (due to a Sobolev embedding argument, see Theorem 1.4.3) the spectrum of A^{X_0} consists only of countably many eigenvalues tending to ∞ with corresponding eigenvectors $(\psi_n)_{n\in\mathbb{Z}}$ forming an orthonormal basis, see Proposition 1.5.1. By a standard calculation one can compute both the eigenvalues $\lambda_n = i\sigma_n$ and the eigenfunctions ψ_n, $n \in \mathbb{Z}$, as defined in (2.21).
Since $A_\mu^{X_0}$ is a bounded, dissipative perturbation[1] of A^{X_0}, $A_\mu^{X_0}$ generates a contraction semigroup as well and has compact resolvent. Computing the eigenvalues of $A_\mu^{X_0}$ in a similar fashion yields the above values for $\theta_n^\pm$, $n \in \mathbb{N}$.
The part[2] of A^{X_0} in $X = X_0 \ominus [\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)]$ equals A and since the eigenfunction corresponding to the eigenvalue 0 of $A_\mu^{X_0}$ is $\psi_0 = \sqrt{\frac{2}{g}}\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$, it follows that A_μ generates a contraction semigroup. This also yields the representation of $\mathbb{T}_t^0 z$. If $\mu \in (0, \pi c)$, then it is obvious from the representation of the eigenvalues that $(\mathbb{T}_t^\mu)_{t\geq 0}$ is exponentially stable with $\omega_0(\mathbb{T}^\mu) = -\mu$. □

In order to complete the proof of Theorem 2.2.1 we study the PDE (2.19) in combination with two observation operators which appear in the definition of the operator $\mathcal{T}$ in (2.18), that is we investigate the input-output behaviour of the linear systems

$$\begin{aligned} \dot{x} &= A_\mu x + A_\mu b\eta, \\ v_i &= C_i x := \frac{1}{2}(x_1(1) + (-1)^i x_1(0)) \end{aligned} \tag{Σ_i}$$

for $i = 1, 2$, where $C_i : \mathcal{D}(A) \to \mathbb{C}$. This kind of systems is sometimes called distributed/boundary control system, see [110, Definition 5.2.14 & Theorem 5.2.16] for the respective representations. Whereas it is essential to show that the associated input-output map $u \mapsto v_i$ is bounded with respect to L^∞-norms, we first restrict ourselves to the classical case of boundedness with respect to L^2-norms. In Lemma 2.3.3 below we prove that (Σ_i) is regular and well-posed. This then implies by definition, cf. [110,115], that the input-output map

$$\begin{aligned} F_i : W_0^{1,\infty}([0,\infty);\mathbb{C}) \cap L^2_\omega([0,\infty);\mathbb{C}) &\to L^2_\omega(0,\infty;\mathbb{C}), \\ \eta &\mapsto \left(t \mapsto C_i \int_0^t (T_\mu)_{-1}(t-s) B\eta(s)\,\mathrm{d}s \right), \end{aligned} \tag{2.22}$$

[1] We say that A is a bounded perturbation of B, if $A = B + C$ for a bounded operator C.
[2] The *part* of an operator $B : D(B) \subseteq Y \to Y$ in $Z \subseteq Y$ is $B|_Z : D(B|_Z) \subseteq Z \to Z$ with $D(B|_Z) = \{\, z \in D(B) \cap Z \mid Bz \in Z \,\}$.

where

$$W_0^{1,\infty}([0,\infty);\mathbb{K}) = \left\{ \ f \in W^{1,\infty}([0,\infty);\mathbb{K}) \ \middle| \ f(0) = 0 \ \right\},$$

is well-defined for all $\omega > \omega_0(\mathbb{T}^\mu) = -\mu$ and can be continuously extended to the space $L^2_\omega([0,\infty);\mathbb{C})$ (here, we identify C_i with a suitable extension, see [115, Section 5] for details). Therefore, the *transfer function* of (Σ_i) can be defined by representing F_i in terms of the Laplace transform $\mathcal{L}(\cdot)$, that is

$$\mathcal{L}(v_i)(s) = \mathcal{L}(F_i\eta)(s) = G^i(s)\mathcal{L}(\eta)(s), \tag{2.23}$$

where $G^i : \mathbb{C}_\omega \to \mathbb{C}$, $i = 1, 2$.
In the following two lemmas, we prove admissibility and well-posedness of system (Σ_i) for $i = 1, 2$ as well as a representation of the transfer functions. The subsequent result can be shown in several standard ways; for the convenience of the reader we include the proof.

Lemma 2.3.2. *Let $\mu \in [0, \pi c)$. Consider A_μ and $(\mathbb{T}^\mu_t)_{t\geq 0}$ from Proposition 2.3.1, and let $b = \left(\begin{smallmatrix} 0 \\ -1 \end{smallmatrix}\right)$. Then we have that*

1. *$B = A_\mu b \in \mathscr{L}(\mathbb{C}, X_{-1})$ is an L^p-admissible control operator for all $p \in [2, \infty]$;*
2. *$C_i \in \mathscr{L}(\mathcal{D}(A), \mathbb{C})$ defined in (Σ_i) are L^2-admissible observation operators for $i = 1, 2$.*

For $\mu \in (0, \pi c)$, the operators B, C_1 and C_2 are even infinite-time admissible.

Proof. First note that $\mathbb{T}^\mu_t$ is boundedly invertible for any $t \geq 0$. Therefore, to show L^2-admissibility of B, by [116, Theorem 5.2.2] it suffices to show that

$$\sup_{\operatorname{Re}\lambda = \alpha} \|(\lambda I - A_\mu)^{-1} B\|_X < \infty$$

for some $\alpha > \omega_0(\mathbb{T}^\mu) = -\mu$. As A_μ and $B = A_\mu b$ are bounded perturbations of A_0 and $A_0 b$, resp., it moreover suffices to consider the case $\mu = 0$; cf. e.g. [116, Rem. 2.11.3.] and note that any bounded operator is L^2-admissible. By the resolvent identity

$$(\lambda I - A_0)^{-1} A_0 b = -b + \lambda(\lambda I - A_0)^{-1} b,$$

and as $\omega_0(\mathbb{T}^0) = 0$ we may restrict ourselves to showing that $\|\lambda(\lambda I - A_0)^{-1} b\|$ is uniformly bounded for $\operatorname{Re}\lambda = 1$. This is equivalent to show that the solution $z = z_\lambda$ of the ordinary differential equation $(\lambda I - A_\mu)z = b$ satisfies that $\sup_{\operatorname{Re}\lambda=1} \|\lambda z_\lambda\|_X < \infty$, which can be shown by an elementary calculation. Thus, B is L^2-admissible for $(\mathbb{T}^\mu_t)_{t\geq 0}$ and hence L^p-admissible for all $p \in [2, \infty]$ by the nesting property of L^p spaces. For $\mu > 0$, the semigroup is exponentially stable by Proposition 2.3.1, and in this case admissibility and infinite-time admissibility coincide, see e.g. [66, Lemma 2.9].

To show that C_k is L^2-admissible for $k = 1, 2$, it suffices to consider $\mu = 0$ and show L^2-admissibility of $\tilde{C}_k : \mathcal{D}(A) \to \mathbb{C}^2$ defined by $\tilde{C}_k x = x(k-1)$ for $(\mathbb{T}_t^0)_{t\geq 0}$ — in fact this is well-known for the one-dimensional wave equation. For completeness we provide a short argument for $\tilde{C}_2$; the assertion for $\tilde{C}_1$ follows analogously. Let $x \in X_1$ and write, in virtue of Proposition 2.3.1, $x = \sum_{n\in\mathbb{Z}} x^n \psi_n$. Then, using $\tilde{C}_2\psi_n = \sqrt{\frac{2}{g}} \left(\begin{smallmatrix} (-1)^n \\ 0 \end{smallmatrix}\right)$ and again Proposition 2.3.1, we obtain that

$$\int_0^t |\tilde{C}_2 \mathbb{T}_t^0 x|^2 \, dt \leq 2\sqrt{\frac{2}{g}} \int_0^t \left| \sum_{n\in\mathbb{N}} \mathrm{e}^{i\sigma_{2n} t} x^{2n} \right|^2 + \left| \sum_{n\in\mathbb{N}} \mathrm{e}^{i\sigma_{2n+1} t} x^{2n+1} \right|^2 \, dt.$$

Choosing $t = 2/c$ and recalling that $\sigma_n = n\pi c$ we infer, using Parseval's identity, that

$$\int_0^t |\tilde{C}_2 \mathbb{T}_t^0 x|^2 \, dt \leq K \|x\|_X^2$$

for some $K > 0$. Thus $\tilde{C}_k$ is admissible for $(\mathbb{T}_t^0)_{t\geq 0}$ and since C_1 and C_2 are projection of the sum of two admissible operators, they are admissible as well. Since admissibility is preserved under bounded perturbations of the generator, it follows that C_k is also L^2-admissible for $(\mathbb{T}_t^\mu)_{\geq 0}$. □

Lemma 2.3.3. *Let $\mu \in [0, \pi c)$ and $\omega > -\mu$. Consider (A_μ, B, C_i) with $A_\mu, B = A_\mu b, C_i$, $i = 1, 2$, as in Lemma 2.3.2. Then the following assertions hold.*

1. *(A_μ, B, C_i) is well-posed and regular for $i = 1, 2$.*
2. *The transfer functions $G^i : \mathbb{C}_\omega \to \mathbb{C}$ of (Σ_i), $i = 1, 2$, are given by, for $\lambda \in \mathbb{C}_\omega$,*

$$G(\lambda) \coloneqq G^1(\lambda) = -2\sqrt{\frac{h_0}{g}} \sqrt{\frac{\lambda}{\lambda + 2\mu}} \tanh\left(\frac{\sqrt{\lambda(\lambda+2\mu)}}{2c}\right) \tag{2.24}$$

and

$$G^2(\lambda) = 0.$$

Proof. To show that the system is well-posed we construct functions $\tilde{G}_i : \mathbb{C}_\omega \to \mathbb{C}$ which satisfy

$$\tilde{G}_i(\lambda_1) - \tilde{G}_i(\lambda_2) = C_i((\lambda_1 I - A_\mu)^{-1} - (\lambda_2 I - A_\mu)^{-1})B$$

for all $\lambda_1, \lambda_2 \in \mathbb{C}_\omega$. To this end, using $B = A_\mu b$ we compute

$$x \coloneqq (\lambda I - A_\mu)^{-1} B\eta = -b + \lambda(\lambda I - A_\mu)^{-1} b\eta.$$

Thus it remains to solve the linear ordinary differential equation $\lambda z = A_\mu z + b$, the solution $z(\zeta)$ of which is given by

$$z(\zeta) = \frac{h_0}{\theta} \left(\frac{\cosh\left(\frac{\theta}{c^2}\right) - 1}{\sinh\left(\frac{\theta}{c^2}\right)} \begin{bmatrix} \cosh\left(\frac{\theta\zeta}{c^2}\right) \\ -\frac{\lambda g}{\theta} \sinh\left(\frac{\theta\zeta}{c^2}\right) \end{bmatrix} + \begin{bmatrix} -\sinh\left(\frac{\theta\zeta}{c^2}\right) \\ \frac{\lambda g}{\theta}\left(\cosh\left(\frac{\theta\zeta}{c^2}\right) - 1\right) \end{bmatrix} \right),$$

where $\theta = c\sqrt{\lambda(\lambda+2\mu)}$. Therefore, $x = -b + \lambda z$ and computing $x_1(1) + (-1)^i x_1(0)$ gives that $\tilde{G}_i$ can be chosen as G^i defined in the statement of the Lemma. Since G^i are proper and the limits $\lim_{\mathrm{Re}\,\lambda\to\infty} G^i(\lambda)$ exist, the systems (A_μ, B, C_i) are well-posed and regular. This also implies that (2.23) holds, which shows that G^i is the transfer function of the system. □

In the next step we obtain a series representation for $G(\lambda)$ and its inverse Laplace transform, which is a sum of an integrable function and a measure of bounded total variation. The latter set is denoted by $\mathrm{M}([0,\infty))$ and the total variation by $\|f\|_{\mathrm{M}([0,\infty))}$ for $f \in \mathrm{M}([0,\infty))$; we refer to the textbook [41] for more details.

Lemma 2.3.4. *Let $\mu \in (0, \pi c)$, $\omega > -\mu$ and $\sigma_n = n\pi c$ as in* (2.20)*. The transfer function $G : \mathbb{C}_\omega \to \mathbb{C}$ defined in* (2.24) *can be represented as*

$$G(\lambda) = -8h_0 \sum_{n\in\mathbb{N}} G_n(\lambda) = -8h_0 \sum_{n\in 2\mathbb{N}_0+1} \frac{\lambda}{\lambda^2 + 2\mu\lambda + \sigma_n^2},$$

is bounded and analytic with inverse Laplace transform $\mathfrak{h} = \mathcal{L}^{-1}(G)$ given by a measure of bounded total variation $\|\mathfrak{h}\|_{\mathrm{M}([0,\infty))}$. Moreover,

$$\mathfrak{h} = \mathfrak{h}_{L^1} + \frac{1}{4c}\mathfrak{h}_\delta,$$

where

$$\mathfrak{h}_{L^1}(t) := \mathrm{e}^{-\mu t}(t^2\mathfrak{f}_2(t) + t\mathfrak{f}_1(t) + \mathfrak{f}_0(t)), \quad t \geq 0,$$
$$\mathfrak{h}_\delta := \delta_0 - 2\mathrm{e}^{-\mu/c}\delta_{1/c} + 2\sum_{k\in\mathbb{N}} \left(\mathrm{e}^{-2k\mu/c}\delta_{2k/c} - \mathrm{e}^{-(2k+1)\mu/c}\delta_{(2k+1)/c}\right),$$

for some $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{f}_2 \in L^\infty([0,\infty);\mathbb{R})$, and δ_t denotes the Dirac delta distribution at $t \in \mathbb{R}$.

Proof. By Lemma 2.3.3, G is bounded and analytic on $\mathbb{C}_\omega$. Let us first show the series representation of G. Recall that

$$\tanh(z) = 8z\sum_{k=1}^{\infty} \frac{1}{\pi^2(2k-1)^2 + 4z^2}, \quad z \notin i\pi(1+2\mathbb{Z}),$$

(which can be obtained from the representation of cosh as an infinite product and differentiation of the composition $\log\circ\cosh$). Using this in (2.24) gives the desired formula for G.

We now study the inverse Laplace transform of G; in particular, $G_n(\lambda) = 0$ for $n \in 2\mathbb{N}_0$. It is not difficult to see that G is also continuous on $\overline{\mathbb{C}_0}$ and that the series converges locally uniformly along the imaginary axis. This implies that the partial sums converge to $\alpha \mapsto G(i\alpha)$ in the distributional sense when considered as tempered distributions on $i\mathbb{R}$. By continuity of the Fourier transform $\mathcal{F}(\cdot)$, this gives that the series

$$-8h_0 \sum_{n\in\mathbb{N}} \mathcal{F}^{-1}(G_n(i\cdot)) = -8h_0 \sum_{n\in\mathbb{N}} \mathcal{L}^{-1}(G_n)$$

converges to $\mathfrak{h} = \mathcal{F}^{-1}(G(i\cdot)) = \mathcal{L}^{-1}(G)$ in the distributional sense[3]. It remains to study $\mathcal{L}^{-1}(G_n)$ and the limit of the corresponding sum. By well-known rules for the Laplace transform we have

$$\mathcal{L}^{-1}(G_n)(t) = \mathrm{e}^{-\mu t}\mathfrak{g}_n(t), \quad t \geq 0,$$

where

$$\mathfrak{g}_n(t) = \cos(\phi_n t) - \mu\phi_n^{-1}\sin(\phi_n t), \quad n \in 2\mathbb{N}_0 + 1.$$

The idea of the proof is to use well-known Fourier series that are related to the frequencies σ_n in contrast to the 'perturbed' harmonics $\sin\phi_n$ and $\cos\phi_n$. We write

$$\mathfrak{g}_n(t) = [\cos(\phi_n t) - \cos(\sigma_n t)] + \frac{\mu}{\phi_n}[\sin(\sigma_n t) - \sin(\phi_n t)] + \cos(\sigma_n t) + \frac{\mu}{\phi_n}\sin(\sigma_n t)$$

In the following we will use the identity $\sigma_n^2 - \phi_n^2 = \mu^2$ from (2.20) several times. By the mean value theorem there exist $\alpha_n, \beta_n \in [\phi_n, \sigma_n]$ and $\omega_n \in [\alpha_n, \sigma_n]$ such that

$$\begin{aligned}
\cos(\phi_n t) - \cos(\sigma_n t) = t(\sigma_n - \phi_n)\sin(\alpha_n t) &= \frac{\mu^2 t\sin(\alpha_n t)}{\sigma_n + \phi_n}, \\
\sin(\alpha_n t) = t(\alpha_n - \sigma_n)\cos(\omega_n t) &+ \sin(\sigma_n t), \\
\sin(\sigma_n t) - \sin(\phi_n t) = t(\sigma_n - \phi_n)\cos(\beta_n t) &= \frac{\mu^2 t\cos(\beta_n t)}{\sigma_n + \phi_n}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathfrak{g}_n(t) = {} & t^2\frac{\mu^2(\alpha_n - \sigma_n)}{\sigma_n + \phi_n}\cos(\omega_n t) + \frac{\mu^3 t}{\phi_n(\sigma_n + \phi_n)}\cos(\beta_n t) + \cos(\sigma_n t) \\
& + \left[t(\sigma_n - \phi_n) + \frac{\mu}{\phi_n}\right]\sin(\sigma_n t)
\end{aligned}$$

The coefficient sequences of the first two terms in the sum,

$$a_n := \mu^2\frac{\alpha_n - \sigma_n}{\sigma_n + \phi_n}, \quad b_n := \frac{\mu^3}{\phi_n(\sigma_n + \phi_n)},$$

are absolutely summable sequences since

$$0 > a_n > \mu^2\frac{\phi_n - \sigma_n}{\sigma_n + \phi_n} = \frac{-\mu^4}{(\sigma_n + \phi_n)^2}.$$

[3]Here we identify functions on $[0,\infty)$ with their trivial extension to $\mathbb{R}$ and use the relation between Fourier and Laplace transform.

Let us further rewrite the coefficient of the last term, recalling that $\sigma_n^2 - \phi_n^2 = \mu^2$ implies that $\frac{1}{\sigma_n+\phi_n} - \frac{1}{2\sigma_n} = \frac{\mu^2}{2\sigma_n(\sigma_n+\phi_n)^2}$, and hence

$$t(\sigma_n - \phi_n) = \frac{\mu^2 t}{\sigma_n + \phi_n} = \frac{\mu^4 t}{2\sigma_n(\sigma_n+\phi_n)^2} + \frac{\mu^2 t}{2\sigma_n},$$
$$\frac{\mu}{\phi_n} = \frac{\mu}{\phi_n} + \frac{\mu}{\sigma_n} - \frac{\mu}{\sigma_n} = \frac{\mu}{\sigma_n} + \frac{\mu^3}{\sigma_n\phi_n(\sigma_n+\phi_n)}.$$

Thus, with $c_n = \frac{\mu^4}{2\sigma_n(\sigma_n+\phi_n)^2}$ and $d_n = \frac{\mu^3}{\sigma_n\phi_n(\sigma_n+\phi_n)}$, which define absolutely summable sequences, we have

$$\begin{aligned}\mathfrak{g}_n(t) &= t^2 a_n \cos(\omega_n t) + t b_n \cos(\beta_n t) + [t c_n + d_n]\sin(\sigma_n t) + \cos(\sigma_n t) \\ &\quad + (\mu t + 2)\frac{\mu}{2\sigma_n}\sin(\sigma_n t).\end{aligned}$$

Let us study the last two terms of the sum $\sum_{n\in 2\mathbb{N}_0+1} \mathfrak{g}_n(t)$ in more detail: Since $\sigma_n = n\pi c$, we have by basic facts on Fourier series that $4c\sum_{n\in 2\mathbb{N}_0+1}\sigma_n^{-1}\sin(\sigma_n t)$ converges to

$$H_0(t) = \begin{cases} 1, & t\in[2k/c,(2k+1)/c),\ k\in\mathbb{N}_0 \\ -1, & t\in[(2k+1)/c,(2k+2)/c),\ k\in\mathbb{N}_0 \end{cases}$$

for almost all $t \geq 0$. Therefore, for almost all $t \geq 0$ we have

$$\sum_{n\in 2\mathbb{N}_0+1} \frac{\mu}{2\sigma_n}\sin(\sigma_n t) = \frac{\mu}{8c}H_0(t).$$

Since the coefficients $\frac{\mu}{\sigma_n}$ are square summable, the series even converges in L^2 on any bounded interval and thus particularly in the distributional sense on $[0,\infty)$.
Finally, note — by well-known facts on the Fourier series of Dirac delta distributions — that $4c\sum_{n\in 2\mathbb{N}_0+1}\cos(\sigma_n\cdot)$ converges to the $2c^{-1}$-periodic extension of $(\delta_0 - 2\delta_{1/c} + \delta_{2/c})$ in the distributional sense as we have

$$\begin{aligned}\lim_{N\to\infty}\left\langle 4c\sum_{n=1,n\text{ odd}}^{N}\cos(\sigma_n\cdot),\psi\right\rangle &= \lim_{N\to\infty}\int_0^{\frac{2}{c}} 4c\sum_{n=1,n\text{ odd}}^{N}\cos(\sigma_n s)\psi(s)\,\mathrm{d}s \\ &= \langle\delta_0 - 2\delta_{1/c} + \delta_{2/c},\psi\rangle\end{aligned}$$

for any function $\psi \in C^\infty([0,\frac{2}{c}];\mathbb{R})$. Altogether, since multiplying with $\mathrm{e}^{-\mu t}$ preserves the distributional convergence, this yields that

$$\sum_{n\in 2\mathbb{N}_0+1}\mathcal{L}^{-1}(G_n)(\cdot) = \sum_{n\in 2\mathbb{N}_0+1}\mathrm{e}^{-\mu\cdot}\mathfrak{g}_n(\cdot) = \mathfrak{h}_{L^1}(\cdot) + \frac{1}{4c}\mathfrak{h}_\delta$$

with $\mathfrak{h}_{L^1}$, $\mathfrak{h}_\delta$ as in the assertion and where the functions

$$\mathfrak{f}_2(t) := \sum_{n\in 2\mathbb{N}_0+1} a_n\cos(\omega_n t)$$

$$\mathfrak{f}_1(t) := \frac{\mu^2}{8c} H_0(t) + \sum_{n \in 2\mathbb{N}_0+1} b_n \cos(\beta_n t) + c_n \sin(\sigma_n t),$$
$$\mathfrak{f}_0(t) := \frac{\mu}{4c} H_0(t) + \sum_{n \in \mathbb{N}} d_n \sin(\sigma_n t), \qquad t \geq 0,$$

are bounded since a_n, b_n, c_n, d_n are absolutely summable sequences. By this representation, $\mathfrak{h}_{L^1} \in L^1([0,\infty);\mathbb{R})$ and can thus be identified with an element in $\mathrm{M}([0,\infty))$, while obviously $\mathfrak{h}_\delta \in \mathrm{M}([0,\infty))$ as the total variation $\|h_\delta\|_{\mathrm{M}([0,\infty))} = 1 + 2\sum_{k\in\mathbb{N}} \mathrm{e}^{-\mu k/c}$ is finite. □

Remark 2.3.5. The assumption $\mu \in (0, \pi c)$ is not a loss of generality, but it simplifies the computations. For arbitrary $\mu > 0$, there exists $N \in \mathbb{N}$ such that $\sigma_n < \mu$ for all $n \leq N$ and the spectrum of A_μ consists of the eigenvalues

$$\theta_n^\pm := -\mu \pm \sqrt{\mu^2 - \sigma_n^2}, \quad n \leq N$$

and

$$\theta_n^\pm := -\mu \pm i\phi_n, \quad n > N.$$

Note that $\operatorname{Re}\theta_n^\pm < 0$ for all $n \in \mathbb{N}$, and hence the semigroup is still exponentially stable. However, the calculations in the previous results become more involved.

In order to provide some intuition about the former calculations, we show in Fig 2.5 how the impulse-response looks like.

2.4 The operator $\mathcal{T}$

We show next some properties of the operator $\mathcal{T}$.

Proposition 2.4.1. *Let $x_0 \in \mathcal{D}(A)$ as defined in* (2.9). *Then the operator $\mathcal{T}$ given by* (2.18), (2.19) *is well-defined from $W_0^{1,\infty}([0,\infty);\mathbb{R})$ to $L^\infty([0,\infty);\mathbb{R})$ and there exist $k_1, k_2, k_3, k_4 > 0$ such that for every $\eta \in W_0^{1,\infty}([0,\infty);\mathbb{R})$ we have*

$$\begin{aligned}\|\mathcal{T}(\eta)\|_\infty \leq\ & k_1(\|x_0\|_X + \|A_\mu x_0\|_X + \|\eta\|_\infty) \\ & + k_2(\|x_0\|_X + \|\eta\|_\infty)^2 + k_3(\|x_0\|_X^2 + \|A_\mu x_0\|_X^2) \\ & + k_4\|A_\mu x_0\|_X\|\eta\|_\infty.\end{aligned}$$

Moreover, $\mathcal{T}$ can be extended to an operator defined from the space $C_0([0,\infty);\mathbb{R})$ to $L^\infty_{\mathrm{loc}}([0,\infty);\mathbb{R})$, which is locally Lipschitz continuous in the sense of condition (N4) c) and the above estimate extends to $\eta \in C_0([0,\infty);\mathbb{R}) \cap L^\infty([0,\infty);\mathbb{R})$.

Proof. Recall that the (mild) solution to the PDE (2.19) is given by

$$x(t) = \mathbb{T}_t^\mu x_0 + \int_0^t (\mathbb{T}^\mu|_{-1})_{t-s} A_\mu b\eta(s)\,\mathrm{d}s\,, \quad t \geq 0. \tag{2.25}$$

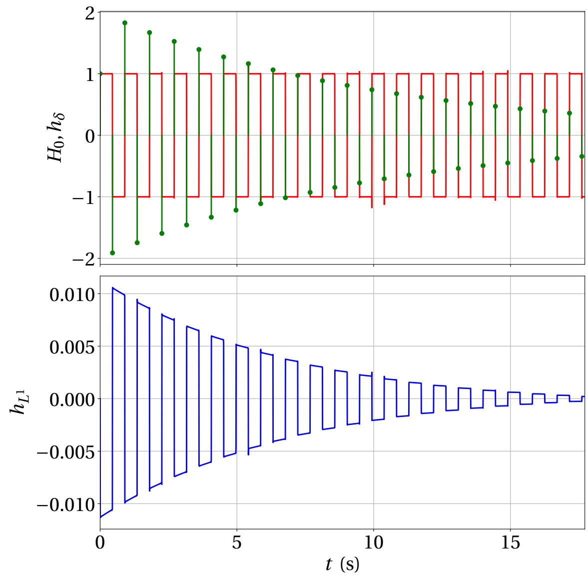

Figure 2.5: Approximation of the distributional part of the impulse-response and the remaining non-distributional part.

By Lemma 2.3.2, $B = A_\mu b \in \mathscr{L}(\mathbb{C}, X_{-1})$ is infinite-time L^∞-admissibile, hence $x \in C([0,\infty); X)$ and there exists $\tilde{k} > 0$ such that

$$\|x(t)\|_X \leq \tilde{k}(\|x_0\|_X + \|\eta\|_{L^\infty(0,t;\mathbb{R})})$$

for all $t \geq 0$, any $x_0 \in X$ and $\eta \in C_0([0,\infty);\mathbb{R})$. Furthermore, since $x_0(\cdot)$ and $\eta(\cdot)$ are real-valued we have that x, as a function in time and space, is real-valued as well.
Let $C_i \in \mathscr{L}(\mathcal{D}(A), \mathbb{C})$ denote the operators from (Σ_i) and define the operators

$$\mathcal{M} : X \to \mathbb{R},\ x \mapsto \frac{2\mu h_0}{m_T} \int_0^1 x_2(\zeta)\,\mathrm{d}\zeta\,,$$
$$\mathcal{N} : X \to \mathbb{R},\ x \mapsto \frac{2\mu}{m_T} \int_0^1 x_1(\zeta) x_2(\zeta)\,\mathrm{d}\zeta\,.$$

Then $\mathcal{T}$ defined in (2.18) can be written as

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2$$

where, for $\eta \in C_0([0,\infty);\mathbb{R})$,

$$\mathcal{T}_1(\eta)(t) = \frac{g}{2m_T}\big(C_1(x(t))\big)\big(2h_0 + C_2(x(t))\big)$$

$$\mathcal{T}_2(\eta)(t) = \mathcal{M}(x(t)) + \mathcal{N}(x(t)) - \frac{2\mu h_0}{m_T}\eta(t), \quad t \geq 0,$$

and x is given by (2.25). While it is obvious that $\mathcal{T}_2$ is well-defined on $C_0([0,\infty);\mathbb{R})$, this is not yet clear for $\mathcal{T}_1$.
In order to estimate $\|\mathcal{T}(\eta)\|_\infty$, we first study the operator $\mathcal{T}_2$. From the definition of $\mathcal{M}$ and $\mathcal{N}$ we readily get for $x \in X$ that

$$|\mathcal{M}(x)| \leq \frac{2\mu h_0}{m_T}\|x\|_X \quad \text{and} \quad |\mathcal{N}(x)| \leq \frac{\mu}{m_T}\|x\|_X^2.$$

Hence, for $\eta \in C_0([0,\infty);\mathbb{R}) \cap L^\infty([0,\infty);\mathbb{R})$ we obtain

$$\|\mathcal{T}_2(\eta)\|_\infty \leq \frac{2\mu h_0}{m_T}\tilde{k}(\|x_0\|_X + \|\eta\|_\infty) + \frac{\mu}{m_T}\tilde{k}^2(\|x_0\|_X + \|\eta\|_\infty)^2.$$

In the remainder of the proof we consider $\mathcal{T}_1$. Let $\eta \in W_0^{1,\infty}([0,\infty);\mathbb{R}) \cap L^2([0,\infty);\mathbb{R})$ in the following. First note that $C_2(x(\cdot))$ only depends on x_0 and is hence constant as a function of η. In fact, by Lemma 2.3.3 we have that $g_2(\lambda) = 0$ which implies that

$$C_2(x(\cdot)) = C_2\mathbb{T}^\mu_{(\cdot)}x_0,$$

which is well-defined since $x_0 \in \mathcal{D}(A_\mu)$ and moreover bounded, i.e.,

$$|C_2(x(t))| \leq \|C_2\|_{\mathscr{L}(\mathcal{D}(A),\mathbb{R})}\|\mathbb{T}^\mu_t A_\mu x_0\|_X \leq \|C_2\|_{\mathscr{L}(\mathcal{D}(A),\mathbb{R})}M\|A_\mu x_0\|_X$$

with $M = \sup_{t\geq 0}\|\mathbb{T}^\mu_t\|$. Analogously, $C_1\mathbb{T}^\mu_{(\cdot)}x_0$ is bounded by

$$\|C_1\|_{\mathscr{L}(\mathcal{D}(A),\mathbb{R})}M\|A_\mu x_0\|_X.$$

Using the input-output map F_1 defined in (2.22) we may infer from the variation of constants formula that

$$C_1(x(\cdot)) = C_1\mathbb{T}^\mu_{(\cdot)}x_0 + F_1(\eta)(\cdot).$$

It remains to investigate whether the real-valued extension of F_1 to L^2, which we again denote by F_1, that is the map

$$\begin{aligned} F_1 : W_0^{1,\infty}([0,\infty);\mathbb{R}) \cap L^2([0,\infty);\mathbb{R}) \to L^2([0,\infty);\mathbb{R}), \\ \eta \mapsto \left(t \mapsto C_1\int_0^t (\mathbb{T}^\mu|_{-1})_{t-s}B\eta(s)\,\mathrm{d}s\right), \end{aligned}$$

is bounded in the L^∞-norms. By Lemma 2.3.3, the transfer function G is an element of $H^\infty(\mathbb{C}_+;\mathbb{C})$ and thus

$$\mathcal{L}(F_1(\eta))(\lambda) = \mathbf{H}(\lambda)\cdot\mathcal{L}(\eta)(\lambda), \quad \lambda \in \mathbb{C}_+.$$

Therefore, there exists a tempered distribution $\mathfrak{h} = \mathcal{L}^{-1}(G)$ such that

$$F_1(\eta) = \mathfrak{h} * \eta \tag{2.26}$$

for Schwartz-class functions η with support in $[0,\infty)$ — here and in the following we extend functions defined on $[0,\infty)$ to $\mathbb{R}$ by zero. By Lemma 2.3.4, $\mathfrak{h}$ can be identified with a Radon measure on $[0,\infty)$ with bounded total variation $\|\mathfrak{h}\|_{\mathrm{M}([0,\infty))}$. Hence, by a variant of Young's integral inequality, $F_1(\eta) \in L^\infty([0,\infty);\mathbb{R})$ and

$$\|F_1(\eta)\|_\infty \le \|\mathfrak{h}\|_{\mathrm{M}([0,\infty))}\, \|\eta\|_\infty \tag{2.27}$$

for all Schwartz functions η supported in $[0,\infty)$; we refer to [41, Section 2.5.4] for details on convolution operators with $\mathfrak{h} \in \mathrm{M}([0,\infty))$. Thus, F_1 (and hence also $\mathcal{T}_1$ and $\mathcal{T}$) can, in the form (2.26), be extended to $C_0([0,\infty);\mathbb{R})$ and we find that for $\eta \in C_0([0,\infty);\mathbb{R}) \cap L^\infty([0,\infty);\mathbb{R})$

$$\begin{aligned}\|\mathcal{T}_1(\eta)\|_\infty &\le \frac{g}{2m_T}(\|C_1\mathbb{T}^\mu_{(\cdot)}x_0\|_\infty + \|\mathfrak{h}\|_{\mathrm{M}(\mathbb{R}_{\ge 0})}\, \|\eta\|_\infty)(2h_0 + \|C_2\mathbb{T}^\mu_{(\cdot)}x_0\|_\infty)\\ &\le k_3\|A_\mu x_0\|_X^2 + k_4(\|A_\mu x_0\|_X + 1)\|\eta\|_\infty + k_5\|A_\mu x_0\|_X\end{aligned}$$

for some $k_3, k_4, k_5 > 0$. Finally, it remains to show that $\mathcal{T}$ satisfies condition (N4) c). To this end, first observe that $\mathcal{T}(\eta) - \mathcal{N}(x)$, where x is as in (2.25), is linear in η and hence trivially locally Lipschitz. To show (N4) c) for $\mathcal{N}(x)$ fix $t \ge 0$ and $\xi \in C([0,t];\mathbb{R})$ as well as $\eta_i \in C_0([0,\infty);\mathbb{R})$ with $\eta_i|_{[0,t]} = \xi$ and $|\eta_i(s) - \xi(t)| < 1$ for all $s \in [t, t+1]$ and $i = 1, 2$. Let x^i denote the mild solution as in (2.25) corresponding to $\eta = \eta_i$ for $i = 1, 2$. Then, for $s \in [t, t+1]$, we have

$$x_1^1(s)x_2^1(s) - x_1^2(s)x_2^2(s) = \big(x_1^1(s) - x_1^2(s)\big)x_2^2(s) + x_1^1(s)\big(x_2^1(s) - x_2^2(s)\big)$$

and hence

$$\begin{aligned}|\mathcal{N}(x^1)(s) - \mathcal{N}(x^2)(s)| &\le \frac{\mu}{m_T}\|x^1(s) - x^2(s)\|_X \left(\|x^1(s)\|_X + \|x^2(s)\|_X\right)\\ &\le \frac{\mu}{m_T}\tilde{k}^2\|\eta_1 - \eta_2\|_\infty \left(2\|x_0\|_X + \|\eta_1|_{[0,t+1]}\|_\infty + \|\eta_2|_{[0,t+1]}\|_\infty\right).\end{aligned}$$

Clearly, $\|\eta_i|_{[0,t+1]}\|_\infty \le \|\xi\|_\infty + 1$ and thus the assertion is true for $\tau = \delta = 1$ and

$$c = \frac{2\mu}{m_T}\tilde{k}^2\left(\|x_0\|_X + \|\xi\|_\infty + 1\right). \qquad \square$$

Remark 2.4.2. Although BIBO stability for linear systems is a well-known topic, it may be involved to check this property, see e.g. [1]. An inspection of the proof of Proposition 2.4.1 reveals that it required a lot of effort to show that the linear system (Σ_i) is *BIBO stable*, that is, (essentially) bounded inputs are taken to (essentially) bounded outputs, and moreover, that the bound is uniform in time. The reason is that it is difficult to determine whether a function which is bounded and analytic in the open right half-plane is the Laplace transform of a measure with bounded total variation. However, we like to remark that the (closure of the) space of measures consisting of the L^1-induced measures and the Dirac measures is also well-known in the literature, see [27].

2.5 Outlook

In the present chapter we have shown that the funnel controller (2.12) is feasible for the moving water tank system (2.10) which includes the linearized Saint-Venant equations. We stress that the system (2.10) is nonlinear and the operators involved in it are unbounded. Even in the linearized case the motion of the fluid affects the dynamics of the overall system which leads to the effect of sloshing. That such impulses at discrete time points indeed appear can be seen by the part $\mathfrak{h}_\delta$ of the inverse Laplace transform of the transfer function derived in Lemma 2.3.4, which is an exponentially decaying infinite sum of Dirac delta distributions. A careful inspection of the proof of Proposition 2.4.1 then reveals that the convolution of this sum with η, i.e., $\mathfrak{h}_\delta * \eta$, explicitly appears in $\ddot{y}$; the decaying impulses can be seen in Fig. 2.3. Overall, the funnel controller is able to handle sloshing as shown in Theorems 2.2.1 and 2.2.2.

We also like to point out that the controller (2.12) requires that the derivative of the output is available for control. This may not be true in practice, and it may even be hard to obtain suitable estimates of the output derivative. This drawback may be resolved by combining the controller (2.12) with a funnel pre-compensator as developed in [17, 18], which results in a pure output feedback.

Several extensions of the moving water tank system (2.10) may be considered in future research, such as a slope at the bottom of the tank, the interconnection of the tank with a truck as in [35] and, of course, the general nonlinear Saint-Venant equations (2.1) as well as the two-dimensional case.

3 Adaptive control in the presence of infinite-dimensional internal dynamics

We study output trajectory tracking for uncertain nonlinear systems by funnel control. As a crucial assumption, we require that the internal dynamics of the system, typically arising from a PDE in our framework, are *bounded-input, bounded-output (BIBO)* stable.

The present chapter is devoted to systems which have a relative degree, but in the presence of internal dynamics that are modeled by a PDE system. We generalize the findings from [13] and develop a general system class containing PDE models for which funnel control is feasible; this result is presented in Section 3.3. As an example, we consider a system internally driven by a transport equation, and illustrate the funnel controller by a simulation. Some conclusions are given in Section 3.4.

3.1 System class

In the remainder of the present chapter we consider abstract differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), T(y, \dot y, \ldots, y^{(r-1)})(t)\big) \\ &\qquad + \Gamma\big(d(t), T(y, \dot y, \ldots, y^{(r-1)})(t)\big)\, u(t) \\ y|_{[-h,0]} &= y^0 \in W^{r-1,\infty}(-h, 0; \mathbb{R}^m), \end{aligned} \tag{3.1}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the relative degree. This differential equation typically comes by differentiating the output of a system until the input u does not vanish from the equation. In fact, the number of times that one needs to differentiate the output so that this happens, goes by the name aforementioned, that is, relative degree. Moreover, we assume

(N1) the disturbance satisfies $d \in L^\infty([0, \infty); \mathbb{R}^p)$, $p \in \mathbb{N}$;

(N2) $f \in C(\mathbb{R}^p \times \mathbb{R}^q; \mathbb{R}^m)$, $q \in \mathbb{N}$;

(N3) the high-frequency gain matrix function $\Gamma \in C(\mathbb{R}^p \times \mathbb{R}^q; \mathbb{R}^{m\times m})$ satisfies $\Gamma(d, \eta) + \Gamma(d, \eta)^\top > 0$ for all $(d, \eta) \in \mathbb{R}^p \times \mathbb{R}^q$;

(N4) $T : C([-h,\infty);\mathbb{R}^{rm}) \to L^\infty_{\text{loc}}([0,\infty);\mathbb{R}^q)$ is an operator with the following properties:

a) T maps bounded trajectories to bounded trajectories, i.e, for all $c_1 > 0$, there exists $c_2 > 0$ such that for all $\zeta \in C([-h,\infty);\mathbb{R}^{rm})$,

$$\sup_{t\in[-h,\infty)} \|\zeta(t)\| \le c_1 \Rightarrow \sup_{t\ge 0} \|T(\zeta)(t)\| \le c_2,$$

b) T is causal, i.e, for all $t \ge 0$ and all $\zeta, \xi \in C([-h,\infty);\mathbb{R}^{rm})$,

$$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \Rightarrow T(\zeta)|_{[0,t)} \overset{\text{a.e.}}{=} T(\xi)|_{[0,t)}.$$

c) T is locally Lipschitz continuous in the following sense: for all $t \ge 0$ and all $\xi \in C([-h,t];\mathbb{R}^{rm})$ there exist $\tau, \delta, c > 0$ such that, for all $\zeta_1, \zeta_2 \in C([-h,\infty);\mathbb{R}^{rm})$ with $\zeta_i|_{[-h,t]} = \xi$ and $\|\zeta_i(s) - \xi(t)\| < \delta$ for all $s \in [t, t+\tau]$ and $i = 1, 2$, we have

$$\left\| (T(\zeta_1) - T(\zeta_2))\,|_{[t,t+\tau]} \right\|_\infty \le c \left\| (\zeta_1 - \zeta_2)|_{[t,t+\tau]} \right\|_\infty.$$

In [11, 50, 59, 61, 62] it is shown that the class of systems (3.1) encompasses linear and nonlinear systems with strict relative degree r and BIBO stable internal dynamics. The operator T allows for infinite-dimensional (linear) systems, systems with hysteretic effects or nonlinear delay elements, and combinations thereof. Note that T is typically the solution operator corresponding to a (partial) differential equation which describes the internal dynamics of the system. The linear infinite-dimensional systems that are considered in [61, 62] are in a special Byrnes-Isidori form that is discussed in detail in [63]. While the internal dynamics in these systems is allowed to correspond to a strongly continuous semigroup, all other operators are assumed to be bounded and to satisfy additional restrictive conditions. In contrast to this, in the present chapter we consider nonlinear equations which, in particular, involve unbounded operators. This complements and generalizes the findings in [13].

3.2 Funnel control

The objective is to design an output error feedback

$$u(t) = F\big(t, e(t), \dot e(t), \dots, e^{(r-1)}(t)\big),$$

where $y_{\text{ref}} \in W^{r,\infty}([0,\infty);\mathbb{R}^m)$ is a reference signal, which applied to (3.1) results in a closed-loop system where the tracking error $e(t) := y(t) - y_{\text{ref}}(t)$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \{\ (t,e) \in [0,\infty) \times \mathbb{R}^m \ \mid \varphi(t)\|e\| < 1\ \}, \tag{3.2}$$

which is determined by a function φ belonging to

$$\Phi_r := \left\{ \varphi \in C^r([0,\infty);\mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau\to\infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, all signals $u, e, \dot{e}, \ldots, e^{(r-1)}$ should remain bounded.
The funnel boundary is given by $1/\varphi$. The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)\|e(0)\| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.
An important property is that each performance funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi_r$ is bounded away from zero, because boundedness of φ implies existence of $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, for instance in the presence of periodic disturbances or strongly varying reference signals. For typical choices of funnel boundaries see also [56, Section 3.2].
It was shown in [11] that the funnel controller

$$\begin{aligned} u(t) &= -k_1(t)\big(\dot{e}(t) + k_0(t)e(t)\big), \\ e_0(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t), \\ e_1(t) &= \dot{e}_0(t) + k_0(t)\, e_0(t), \\ e_2(t) &= \dot{e}_1(t) + k_1(t)\, e_1(t), \\ &\;\;\vdots \\ e_{r-1}(t) &= \dot{e}_{r-2}(t) + k_{r-2}(t)\, e_{r-2}(t), \\ k_i(t) &= \frac{1}{1 - \varphi_i(t)^2 \|e_i(t)\|^2}, \quad i = 0, \ldots, r-1, \end{aligned} \tag{3.3}$$

where

$$\varphi_0 \in \Phi_r, \varphi_1 \in \Phi_{r-1}, \ldots, \varphi_{r-1} \in \Phi_1, \tag{3.4}$$

achieves the control objective described above for any system which belongs to the class (3.1). We stress that while the derivatives $\dot{e}_0, \ldots, \dot{e}_{r-2}$ appear in (3.3), they only serve as short-hand notations and may be resolved in terms of the tracking error, the funnel functions and their derivatives, cf. [11, Remark 2.1].
The existence of solutions of the initial value problem resulting from the application of the funnel controller (3.3) to a system (3.1) must be treated carefully. By a *solution* of (3.3), (3.1) on $[-h, \omega)$ we mean a function $y \in C^{r-1}([-h,\omega);\mathbb{R}^m)$, $\omega \in (0,\infty]$, with $y|_{[-h,0]} = y^0$ such that $y^{(r-1)}|_{[0,\omega)}$ is weakly differentiable and satisfies the differential equation in (3.1) with u defined in (3.3) for almost all $t \in [0,\omega)$; y is called *maximal*,

if it has no right extension that is also a solution. Existence of solutions of functional differential equations has been investigated in [61] for instance.
The following result is from [11]. Note that in [11] a slightly stronger version of conditions (N3) and (N4) c) is used. However, the proof does not change; in particular, regarding (N4) c), the existence part of the proof in [11] relies on a result from [59] where the version from the present chapter is used.

Theorem 3.2.1. *Consider a system* (3.1) *with properties (N1)–(N4) for some* $r \in \mathbb{N}$ *and* $h > 0$. *Let* $y_{\text{ref}} \in W^{r,\infty}([0,\infty);\mathbb{R}^m)$, $\varphi_0, \ldots, \varphi_{r-1}$ *as in* (3.4) *and* $y^0 \in W^{r-1,\infty}([-h,0];\mathbb{R}^m)$ *be an initial condition such that* $e_0, \ldots, e_{r-1}$ *defined in* (3.3) *satisfy*

$$\varphi_i(0)\|e_i(0)\| < 1 \quad \text{for } i = 0, \ldots, r-1.$$

Then the funnel controller (3.3) *applied to* (3.1) *yields an initial-value problem which has a solution, and every solution can be extended to a maximal solution* $y : [-h,\omega) \to \mathbb{R}^m$, $\omega \in (0,\infty]$, *which has the following properties:*

1. *The solution is global, i.e.,* $\omega = \infty$.
2. *The input* $u : [0,\infty) \to \mathbb{R}^m$, *the gain functions* $k_0, \ldots, k_{r-1} : [0,\infty) \to \mathbb{R}$ *and* $y, \dot{y}, \ldots, y^{(r-1)} : [0,\infty) \to \mathbb{R}^m$ *are bounded.*
3. *The functions* $e_0, \ldots, e_{r-1} : [0,\infty) \to \mathbb{R}^m$ *evolve in their respective performance funnels and are uniformly bounded away from the funnel boundaries in the sense*

$$\forall\, i = 0, \ldots, r-1 \;\exists\, \varepsilon_i > 0 \;\forall\, t > 0 : \|e_i(t)\| \le \varphi_i(t)^{-1} - \varepsilon_i.$$

3.3 A class of operators for funnel control

While the class of functional differential equations (3.1) appears to be rather general and funnel control is feasible for these systems by Theorem 3.2.1, it is not clear exactly which kind of systems that contain PDEs are encompassed by the class (3.1). In this section we develop a description for a class of operators T which include certain BIBO stable linear PDEs and satisfy condition (N4). The aforementioned PDEs may either be coupled with a nonlinear observation operator which is polynomially bounded, or with a linear observation operator which is possibly unbounded, but with respect to which the system is regular well-posed and the inverse Laplace transform of the corresponding transfer function defines a measure of bounded total variation. This structure is illustrated in Fig. 3.1.
We give a precise definition of the operator class in the following.

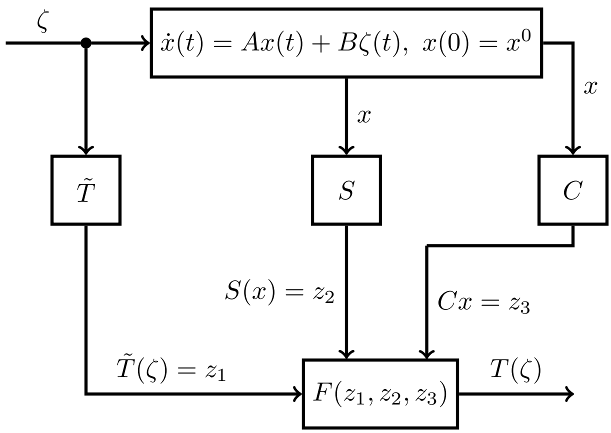

Figure 3.1: Structure of an operator $T \in \mathcal{T}_h^{\ell,q}$.

Definition 3.3.1. Let $h \geq 0$ and $\ell, q \in \mathbb{N}$. Then $\mathcal{T}_h^{\ell,q}$ is defined as the set of all operators

$$T : C([-h,\infty);\mathbb{R}^\ell) \to L^\infty_{\rm loc}([0,\infty);\mathbb{R}^q)$$

which, for any $\zeta \in C([-h,\infty);\mathbb{R}^\ell)$, are given by

$$T(\zeta)(t) = F\big(\tilde{T}(\zeta)(t), S(x)(t), (Cx)(t)\big), \quad t \geq 0,$$

where x, for some $x^0 \in \mathcal{D}(A)$, is the mild solution of the PDE

$$\dot{x}(t) = Ax(t) + B\zeta(t), \quad x(0) = x^0, \tag{3.5}$$

and

(P1) $A : \mathcal{D}(A) \subseteq X \to X$ is the generator of a bounded C_0-semigroup in X, X a real Hilbert space, and $B \in \mathscr{L}(\mathbb{R}^\ell, X_{-1})$ is an L^2-admissible control operator such that $\dot{x}(t) = Ax(t) + B\zeta(t)$ is BIBO stable, i.e., there exists $\gamma \in C^1([0,\infty);\mathbb{R})$ such that for all $\zeta \in C([-h,\infty);\mathbb{R}^\ell)$ the mild solution of (3.5) satisfies

$$\forall\, t \geq 0 : \quad \|x(t)\|_X \leq \gamma(\|\zeta|_{[-h,t]}\|_\infty);$$

(P2) $F \in C^1(\mathbb{R}^{q_1} \times \mathbb{R}^{q_2} \times \mathbb{R}^{q_3};\mathbb{R}^q)$;

(P3) $\tilde{T} : C([-h,\infty);\mathbb{R}^\ell) \to L^\infty_{\rm loc}([0,\infty);\mathbb{R}^{q_1})$ satisfies condition (N4) in Section 3.1 with $\ell = rm$;

(P4) $S : X \to \mathbb{R}^{q_2}$ is a Fréchet differentiable operator with continuous Fréchet derivative and satisfies

$$\forall\, x \in X : \ \|S(x)\| \leq p(\|x\|_X)$$

for some polynomial $p(s)$;

(P5) $C \in \mathscr{L}(\mathcal{D}(A), \mathbb{R}^{q_3})$ is an L^2-admissible observation operator such that the system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B\zeta(t),\\ \nu(t) &= Cx(t)\end{aligned}$$

is well-posed, i.e., for some $\omega \in \mathbb{R}$ the transfer function $G : \mathbb{C}_\omega \to \mathbb{C}^{q_3\times\ell}$, which is uniquely determined (up to a constant) by

$$\frac{1}{t-s}(G(s) - G(t)) = C\big((sI - A)^{-1}(tI - A)^{-1}\big)B$$

for all $s, t \in \mathbb{C}_\omega$, $s \neq t$, exists and is proper, that is $\sup_{s\in\mathbb{C}_\omega} \|G(s)\| < \infty$. Furthermore, we require that the system is regular, i.e., $\lim_{\operatorname{Re} s\to\infty} G(s)v$ exists for all $v \in \mathbb{C}^\ell$, and we require that G satisfies that the inverse Laplace transform of its components $h_{ij} = \mathcal{L}^{-1}(G_{ij})$ is a real-valued measure with bounded total variation for all $i = 1, \dots, q_3$ and $j = 1, \dots, \ell$.

Remark 3.3.2.

1. We note that the notion of *admissible* operators is well-known in infinite-dimensional linear systems theory with unbounded control and observation operators, see e.g. [116], and is motivated by interpreting a PDE on a larger space in order to define solutions. Further, note that any operator T as given in Definition 3.3.1 with the properties (P1)–(P5) is indeed well-defined from $C([-h,\infty); \mathbb{R}^\ell)$ to $L^\infty_{\text{loc}}([0,\infty); \mathbb{R}^q)$.

2. We emphasize that the assumption of BIBO stability of (3.5) as in (P1) is quite weak. A sufficient condition for this is *input-to-state stability*, which has been introduced by Sontag [108]. This concept was studied extensively for nonlinear systems, see [109], and for systems containing PDEs it is investigated in [66, 89]. However, the state of an input-to-state stable system converges to zero whenever the input is zero, which is not required for BIBO stable systems considered here.

In the following main result we show that any operator which belongs to the class $\mathcal{T}_h^{\ell,q}$ satisfies the condition (N4) in Section 3.1.

Theorem 3.3.3. *Any $T \in \mathcal{T}_h^{\ell,q}$ satisfies condition (N4) in Section 3.1.*

Proof. Step 1: We show property (N4) a). To this end, observe that by continuity of F it suffices to show this for the maps $\zeta \mapsto \tilde{T}(\zeta)$, $\zeta \mapsto S(x)$ and $\zeta \mapsto Cx$. By (P3), $\tilde{T}$ satisfies (N4) a) and by (P2) together with (P1) we have

$$\|S(x)(t)\| \le p\big(\|x(t)\|_X\big) \le p\left(\gamma(\|\zeta\|_\infty)\right)$$

for all bounded $\zeta \in C([-h,\infty);\mathbb{R}^\ell)$. It remains to show that Cx is bounded. From (P5) the system (A,B,C) is regular well-posed, from which it follows by the variation of constants formula that, see [115] for instance,

$$Cx(\cdot) = C\mathbb{T}^A_{(\cdot)}x_0 + (h*\zeta)(\cdot),$$

where $(\mathbb{T}^A_t)_{t\geq 0}$ is the C_0-semigroup generated by A and $h=(h_{ij})_{i=1,\dots,q_3;j=1,\dots,\ell}$ is the inverse Laplace transform of the transfer function G. By assumption we have that $h \in \mathrm{M}([0,\infty);\mathbb{R}^{q_3\times\ell})$. Thus, for all $t\geq 0$,

$$\begin{aligned}\|Cx(t)\| &\leq \|C\mathbb{T}^A_t x_0\| + \|(h*\zeta)(t)\| \\ &\leq \|C\|_{\mathscr{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|A\mathbb{T}^A_t x_0\| + \|h\|_{\mathrm{M}([0,\infty))}\|\zeta\|_\infty \\ &= \|C\|_{\mathscr{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|\mathbb{T}^A_t Ax_0\| + \|h\|_{\mathrm{M}([0,\infty))}\|\zeta\|_\infty \\ &\leq \|C\|_{\mathscr{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|\mathbb{T}^A_t\|_{\mathscr{L}(X)}\|Ax_0\|_X + \|h\|_{\mathrm{M}([0,\infty))}\|\zeta\|_\infty \\ &\leq M\|C\|_{\mathscr{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|Ax_0\|_X + \|h\|_{\mathrm{M}([0,\infty))}\|\zeta\|_\infty,\end{aligned}$$

where we have used that $x_0 \in \mathcal{D}(A)$ and $(\mathbb{T}^A_t)_{t\geq 0}$ is bounded, that is, $\|\mathbb{T}^A_t\|_{\mathscr{L}(X)} \leq M$ for some $M\geq 1$ and all $t\geq 0$. Thus,

$$\|Cx(\cdot)\|_\infty \leq M\|C\|_{\mathscr{L}(\mathcal{D}(A),\mathbb{R}^{q_3})}\|Ax_0\|_X + \|h\|_{\mathrm{M}([0,\infty))}\|\zeta\|_\infty.$$

Step 2: We show property (N4) b). This is a straightforward consequence of the definition of $\tilde{T}$.

Step 3: We show property (N4) c). Fix $t\geq 0$ and $\xi \in C([-h,t];\mathbb{R}^\ell)$. Let $\tilde\tau,\tilde\delta,\tilde c$ be the constants given by property (N4) c) of $\tilde{T}$. Set $\tau := \tilde\tau$ and $\delta := \tilde\delta$. Further let $\zeta_i \in C([-h,\infty);\mathbb{R}^\ell)$ with $\zeta_i|_{[-h,t]} = \xi$ and $\|\zeta_i(s)-\xi(t)\| < \delta$ for all $s\in[t,t+\tau]$ and $i=1,2$. Let x_i denote the mild solution of (3.5) corresponding to ζ_i for $i=1,2$. Then, by linearity, x_1-x_2 is the mild solution corresponding to $\zeta_1-\zeta_2$. Since S is Fréchet differentiable with continuous Fréchet derivative $DS : X\to\mathscr{L}(X,\mathbb{R}^{q_2})$ by (P4), the mean value theorem implies that it is locally Lipschitz continuous. Therefore, we find that for all $s\in[t,t+\tau]$

$$\begin{aligned}\|S(x_1(s)) - S(x_2(s))\| &\leq L_1\|\big(x_1-x_2\big)(s)\| \\ &\leq L_1\gamma(\|\big(\zeta_1-\zeta_2\big)|_{[-h,s]}\|_\infty) \\ &\leq L_1L_2\|\big(\zeta_1-\zeta_2\big)|_{[t,t+\tau]}\|_\infty,\end{aligned}$$

where, with $\tilde x$ denoting the mild solution of (3.5) corresponding to $\tilde\xi$ for $\tilde\xi|_{[-h,t]}=\xi$ and $\tilde\xi|_{[t,\infty)} \equiv \xi(t)$, we have $\|x_i(s)-\tilde x(t)\|_X \leq \gamma(\|\zeta_i|_{[t,s]}-\xi(t)\|_\infty) < \gamma(\delta)$, which justifies to set

$$L_1 := \sup_{\|x-\tilde x(t)\|_X\leq\gamma(\delta)} \|DS(x)\|_{\mathscr{L}(X,\mathbb{R}^{q_2})},$$

$$L_2 := \sup_{s\in[0,2\delta]} |\gamma'(s)|.$$

Furthermore, by linearity and (P5) we have

$$\begin{aligned}\|Cx_1(s) - Cx_2(s)\| &= \|(h * (\zeta_1 - \zeta_2))(s)\| \\ &\le L_3 \|(\zeta_1 - \zeta_2)|_{[t,t+\tau]}\|_\infty\end{aligned}$$

for all $s \in [t, t+\tau]$ and $L_3 := \|h\|_{\mathrm{M}([0,\infty))}$. Now define $\hat{c} := \tilde{c} + L_1 L_2 + L_3$ and

$$L_4 := \sup \left\{ \|F'(z)\| \;\middle|\; \left\| z - \begin{pmatrix} \tilde{T}(\tilde{\xi})(t) \\ S(\tilde{x})(t) \\ C\tilde{x}(t) \end{pmatrix} \right\| \le \hat{c}\delta \right\}$$

and set

$$c := \hat{c} L_4.$$

Then we have

$$\|\tilde{T}(\zeta_1)(s) - \tilde{T}(\zeta_2)(s)\| \le c \|(\zeta_1 - \zeta_2)|_{[t,t+\tau]}\|_\infty$$

for all $s \in [t, t+\tau]$ and this finishes the proof of the theorem. □

It is shown in [13] that the operator associated with the internal dynamics of a linearized model of a moving water tank system belongs to the class $\mathcal{T}_h^{\ell,q}$. In the subsequent section we consider another example which contains a transport equation.

Example: The transport equation

We illustrate our results by considering the following system whose internal dynamics are described by a transport equation, that is

$$\begin{aligned} \dot{y}(t) &= T(y)(t) + \gamma u(t) \\ T(y)(t) &= z(t,0) \\ \frac{\partial z}{\partial t}(t,\xi) &= c\frac{\partial z}{\partial \xi}(t,\xi) + h(\xi)y(t), \\ z(0,\xi) &= 0, \end{aligned} \tag{3.6}$$

for $(t,\xi) \in (0,\infty) \times [0,\infty)$, where $c, \gamma > 0$ and $h \in \mathrm{M}([0,\infty))$ is a Borel measure of bounded total variation. It is well-known that the third and fourth equations in (3.6) constitute a regular well-posed linear system (A, B, C) on $X = L^2([0,\infty); \mathbb{R})$, the so-called *shift-realization* of the Laplace transform $\mathcal{L}(h)$, see e.g. [52, 124, 125]. More precisely, the PDE is then understood on an abstract Sobolev space X_{-1} to make sense of the term $h(\xi)y(t)$ and the solutions are *mild solutions* in general. Also note that the generated (left-)shift-semigroup is not exponentially stable. In particular, the Laplace transform $\mathcal{L}(h)$ of the measure h is defined on the closed right half-plane and bounded analytic on this domain. Moreover, the impulse response of the PDE equals h. More precisely, for sufficiently smooth y we have the representation

$$T(y)(t) = z(t,0) = (h * y)(t) = \int_0^t y(t-s)\,\mathrm{d}h(s).$$

As h is of bounded total variation, it follows that T is a bounded operator from $BC([0,\infty);\mathbb{R})$ to $L^\infty([0,\infty);\mathbb{R})$ and hence $T \in \mathcal{T}_0^{1,1}$. Therefore, the first equation in (3.6) formally reads

$$\dot{y}(t) = (h * y)(t) + \gamma u,$$

which is a differential-integral Volterra equation. Also note that for the following simple cases

- $h = \delta_0$, we obtain a finite-dimensional linear system:

$$\dot{y}(t) = y(t) + \gamma u(t);$$

- $h = \delta_{t_0}$, $t_0 > 0$, we obtain a delay differential equation:

$$\dot{y}(t) = \begin{cases} y(t-t_0) + \gamma u(t), & t \geq t_0, \\ \gamma u(t), & 0 \leq t < t_0. \end{cases}$$

- $h(t) = f(t)\mathrm{d}t$ with $f \in L^1([0,\infty);\mathbb{R})$, i.e. h is represented by its L^1-density with respect to the Lebesgue measure. If additionally $f \in L^2([0,\infty);\mathbb{R})$, then the input operator B is bounded.

For the simulation we have chosen $h(\xi) = \mathrm{e}^{-\xi}/\sqrt{\xi}$, which is integrable but not square integrable on $[0,\infty)$. Furthermore, we use the parameters $c = \gamma = 1$ and the reference signal

$$y_{\mathrm{ref}}(t) = \cos t, \quad t \geq 0.$$

The initial value is chosen as $y(0) = 0$ and for the controller (3.3) we chose the funnel function

$$\varphi(t) = \left(2\mathrm{e}^{-2t} + 0.1\right)^{-1}, \quad t \geq 0.$$

Clearly, the initial error lies within the funnel boundaries as required in Theorem 3.2.1. Furthermore, by Theorem 3.3.3 the operator T satisfies (N4) and hence funnel control is feasible.

The PDE is solved using explicit finite differences with a grid in t with $M = 1000$ points for the interval $[0,T]$, where $T = 15$, and a grid in ξ with $N = \lfloor M(b-a)/(\alpha T) \rfloor$ points for $\alpha = 0.4$ and $a = 0$, $b = 10$. The method has been implemented in Python and the simulation results are shown in Fig. 3.2.

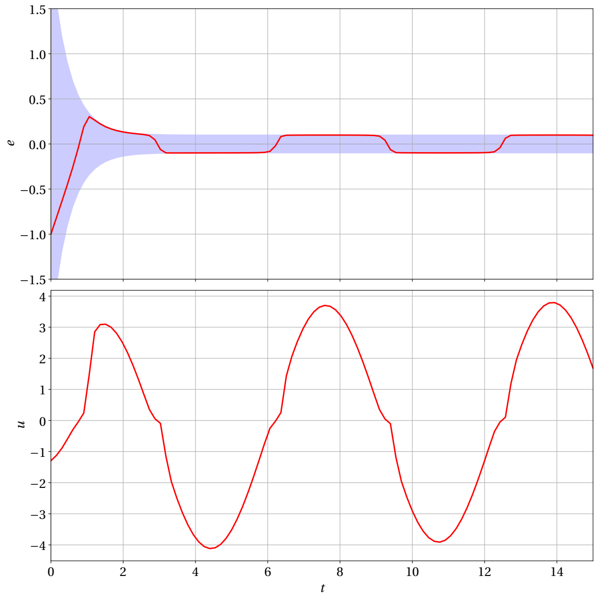

Fig. 3.2a: Performance funnel with tracking error e and generated input function u.

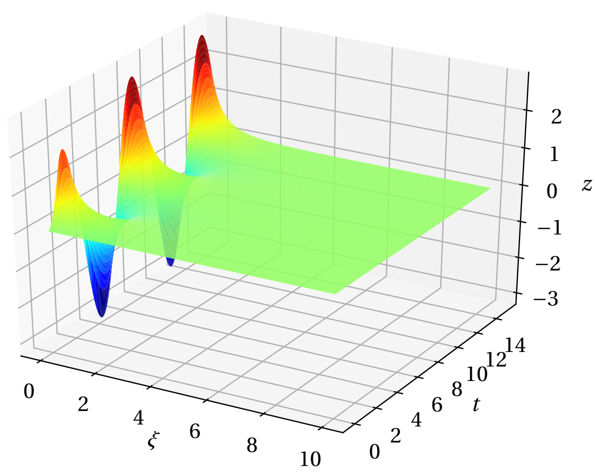

Fig. 3.2b: State z of the PDE.

Figure 3.2: Simulation of the funnel controller (3.3) for the system (3.6).

It can be seen that even in the presence of infinite-dimensional internal dynamics which are not exponentially stable a prescribed performance of the tracking error can be achieved with the funnel controller (3.3). At the same time the input generated by the controller is bounded with a satisfactory performance.

3.4 Outlook

In the present chapter we considered the question which classes of systems with infinite-dimensional internal dynamics are encompassed by the abstract system class (3.1) for which funnel control is feasible by Theorem 3.2.1. We have defined a class of operators $\mathcal{T}_h^{\ell,q}$, which model the internal dynamics of the system, that encompass BIBO stable linear PDEs. These PDEs may either be coupled with a nonlinear, but polynomially bounded observation operator, or with a linear observation operator which may be unbounded. For the latter we additionally assumed that the resulting system is regular well-posed such that the inverse Laplace transform of its transfer function defines a measure with bounded total variation. In Theorem 3.3.3 we have proved that any operator belonging to $\mathcal{T}_h^{\ell,q}$ satisfies the conditions of the system class (3.1).
Several extensions of the operator class $\mathcal{T}_h^{\ell,q}$ and Theorem 3.2.1 may be investigated in future research. In particular, extensions to nonlinear PDE systems with unbounded observation operators are of interest as well as systems with infinite-dimensional input and output spaces which do not have an integer-valued relative degree.

4 Adaptive control for boundary control systems

Here we consider a class of *boundary control systems (BCS)* of the form

$$\begin{aligned}\dot{x}(t) &= \mathfrak{A}x(t), \quad t > 0, \quad x(0) = x_0,\\ u(t) &= \mathfrak{B}x(t),\\ y(t) &= \mathfrak{C}x(t),\end{aligned}$$

where $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are linear operators. The function u is interpreted as the input, y as the measured output and x is called the state of the system. Typically, $\mathfrak{A}$ is a differential operator on the state space X and $\mathfrak{B}, \mathfrak{C}$ are evaluation operators of the state at the boundary of the spatial domain, that is, the domain of the functions lying in X.

The aim of this chapter is to develop an adaptive controller for boundary control systems which, roughly speaking, achieves the following goal:

> *For any prescribed reference signal $y_{\text{ref}} \in W^{2,\infty}(0,\infty)$, the output y of the system tracks y_{ref} in the sense that the transient behavior of the error $e(t) \coloneqq y(t) - y_{\text{ref}}(t)$ is controlled.*

Shortly, we will elaborate on the class of possible reference signals and the meaning of "controlling the transient behavior" in more detail. The goal will be achieved by using a *funnel controller*, which, in the simplest case, has the form

$$u(t) = -\frac{1}{1 - \varphi(t)^2 \|e(t)\|^2} e(t)$$

for some positive function φ. Under this feedback, the error is supposed to evolve in the *performance funnel*

$$\mathcal{F}_\varphi \coloneqq \{(t,e) \in [0,\infty) \times \mathbb{C}^m \mid \varphi(t)\|e\| < 1\}$$

and would hence satisfy

$$\|e(t)\| \le \varphi(t)^{-1}, \quad \text{for all } t \ge 0.$$

In fact, if φ tends asymptotically to a large value λ, then the error remains at some point bounded by λ^{-1}, see Fig. 4.1.

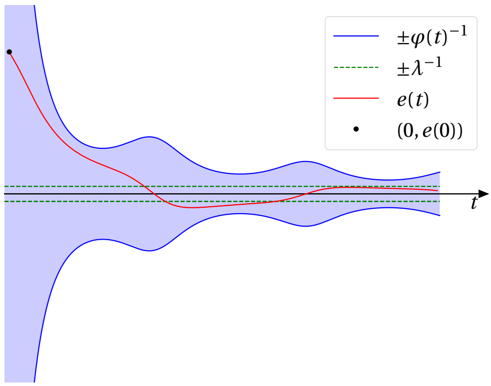

Figure 4.1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$.

We consider a class of *boundary control systems* which satisfy a certain energy balance [9, 67]. The feedback law of the funnel controller naturally induces a nonlinear closed-loop system. For the corresponding solution theory, the concept of (nonlinear) m-dissipative operators in a Hilbert space will play an important role. For an appropriate introduction to this classical topic we refer to [69, 90, 106]. For the sake of completeness we have provided the basics on the topic in Chapter 1 Sections 1.8.2 & 1.8.3.

The chapter is organized as follows. In Section 4.1 we introduce the system class that is subject of our results. In Section 4.2 we present the details about the controller and as well as the main results which refer to the applicability of the funnel controller to the considered system class. In Section 4.3 we present some examples of partial differential equations for which the funnel controller is applicable and accompany them by showing some numerical simulations. Section 4.4 contains the proof of the main results together with some preliminary auxiliary results. We conclude the chapter with an overlook in Section 4.5

4.1 System class

In the following we introduce our system class, define our controller and discuss the solution concept to the resulting nonlinear feedback system.

Definition 4.1.1 (System class)**.** Let X be a complex Hilbert space and let $m \in \mathbb{N}$ be given. Let $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \subset X \to X$ be a closed linear operator, $\mathfrak{B}, \mathfrak{C} : \mathcal{D}(\mathfrak{A}) \subset X \to \mathbb{C}^m$ be linear operators to which we associate the system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t), \\ y(t) &= \mathfrak{C}x(t). \end{aligned} \tag{4.1}$$

We will refer to (4.1) by $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and call it a *boundary control system (BCS)*.

In the sequel we specify the system class.

Assumption 4.1.2. Let a BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given.

(i) The system is *(generalized) impedance passive*, i.e., there exists $\alpha \in \mathbb{R}$ such that

$$\operatorname{Re} \langle \mathfrak{A}x, x \rangle_X \leq \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x \rangle_{\mathbb{C}^m} + \alpha \|x\|_X^2 \quad \text{for all } x \in \mathcal{D}(\mathfrak{A}). \tag{4.2}$$

(ii) $\mathfrak{A}|_{\ker \mathfrak{B}}$ (the restriction of $\mathfrak{A}$ to $\ker \mathfrak{B}$) generates a strongly continuous semigroup on X.

(iii) The operator

$$\begin{bmatrix} \mathfrak{B} \\ \mathfrak{C} \end{bmatrix} : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}^{2m} \tag{4.3}$$

is onto, $\ker \mathfrak{B} \cap \ker \mathfrak{C} \subset X$ is dense and $\mathfrak{C} : \mathcal{D}(\mathfrak{A}|_{\ker \mathfrak{B}}) \to \mathbb{C}^m$ is continuous with respect to the graph norm $\|x\|_{\mathcal{D}(\mathfrak{A})} = (\|x\|_X^2 + \|\mathfrak{A}x\|_X^2)^{1/2}$.

Remark 4.1.3.

a) By setting $u = 0$, the above assumptions imply that the semigroup $\mathbb{T}_{(\cdot)} : [0, \infty) \to \mathscr{L}(X)$ generated by $\mathfrak{A}|_{\ker \mathfrak{B}}$ fulfills $\|\mathbb{T}_t\| \leq e^{\alpha t}$. In particular, the semigroup is contractive, if $\alpha \leq 0$.

b) The Lumer–Phillips theorem [30, Theorem 3.15] implies that $\mathfrak{A}|_{\ker \mathfrak{B}}$ generates a strongly continuous semigroup $(\mathbb{T}_t)_{t \geq 0}$ on X with $\|\mathbb{T}_t\| \leq e^{\alpha t}$ for all $t > 0$ if, and only if, $\mathcal{R}(\mathfrak{A}|_{\ker \mathfrak{B}} - \lambda I) = X$ for some (and hence any) $\lambda \geq \alpha$, together with $\operatorname{Re} \langle \mathfrak{A}x, x \rangle_X \leq \alpha \|x\|_X^2$ for all $x \in D(\mathfrak{A})$. As a consequence, Assumption 4.1.1(ii) can be replaced by the condition that $\mathcal{R}(\mathfrak{A}|_{\ker \mathfrak{B}} - \lambda I) = X$ for some (and hence any) $\lambda \geq \alpha$.

c) The operator (4.3) is onto if, and only if, there exist $P, Q : \mathscr{L}(\mathbb{C}^m, \mathcal{D}(\mathfrak{A}))$ with

$$\begin{bmatrix} \mathfrak{B} \\ \mathfrak{C} \end{bmatrix} \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_m \end{bmatrix} \tag{4.4}$$

d) We are dealing with complex spaces in this chapter for sake of simplicity. A comment on real systems can be found in Remark 4.2.5e).

e) An oftentimes considered class in infinite-dimensional linear systems theory is that of *well-posed linear systems*, see e.g. [110]. That is, the controllability map, observability map and input-output map are bounded operators. Note that we do not impose such a well-posedness assumption throughout this chapter. The well-posed case has been for instance studied in [24] and [116, Section 10].

Example 4.1.4. There are several systems which fit in our description. A class of examples of hyperbolic type is given by so-called *port-Hamiltonian systems* such as the *lossy transmission line*

$$\begin{aligned} V_\zeta(\zeta,t) &= -LI_t(\zeta,t) - RI(\zeta,t), \\ I_\zeta(\zeta,t) &= -CV_t(\zeta,t) - GV(\zeta,t), \\ u(t) &= \begin{pmatrix} V(a,t) \\ V(b,t) \end{pmatrix}, \\ y(t) &= \begin{pmatrix} I(a,t) \\ -I(b,t) \end{pmatrix}, \end{aligned}$$

where V and I are the voltage and the electric current at a point ζ of a segment (a,b) over the time t. A precise definition of port-Hamiltonian systems will be given in Section 4.3.1.

In Section 4.3.3 we will also apply the theoretical results to parabolic systems given through a general second-order elliptic operator on a regular domain Ω. A particular case is the heat equation,

$$\begin{aligned} \partial_t x(t,\zeta) &= \Delta x(t,\zeta), \\ \nu \cdot \nabla x(t,\zeta)|_{\partial\Omega} &= u(t), \\ \int_{\partial\Omega} x(t,\zeta)\,\mathrm{d}\zeta &= y(t), \end{aligned}$$

where the control variable is the heat flux at the boundary and the observation is the total temperature along the boundary.

4.2 Funnel control

The following definition presents the cornerstone of our controller, the class of admissible funnel boundaries.

Definition 4.2.1. Let

$$\Phi := \left\{ \varphi \in W^{2,\infty}([0,\infty);\mathbb{R}) \;\middle|\; \forall\delta > 0,\ \inf_{t\geq\delta} \varphi(t) > 0 \right\}.$$

With $\varphi \in \Phi$ we associate the *performance funnel*

$$\mathcal{F}_\varphi := \{(t,e) \in [0,\infty) \times \mathbb{C}^m \mid \varphi(t)\|e\| < 1\}.$$

In this context we refer to $1/\varphi(\cdot)$ as *funnel boundary*, see also Fig. 4.1.

Now we define our controller, which is a slight modification of the original controller introduced in [60]. For $x_0 \in \mathcal{D}(\mathfrak{A})$, we define the funnel controller as

$$u(t) = \left(u_0 + \frac{1}{1-\varphi_0^2\|e_0\|^2} e_0\right) p(t) - \frac{1}{1-\varphi(t)^2\|e(t)\|^2} e(t), \tag{4.5}$$

where $\varphi_0 = \varphi(0)$, $e_0 := \mathfrak{C}x_0 - y_{\mathrm{ref}}(0)$, $u_0 := \mathfrak{B}x_0$, and p is a function with compact support and $p(0) = 1$. In the following we collect assumptions on the functions involved in the funnel controller and the initial value of the BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$. Particularly, this includes that the expressions $\mathfrak{C}x_0$, $y_{\mathrm{ref}}(0)$, $\mathfrak{B}x_0$ and $p(0)$ are well-defined.

Assumption 4.2.2 (Reference signal, performance funnel, initial value)**.** The initial value x_0 of the BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ and the functions in the controller (4.5) fulfill

(i) $y_{\mathrm{ref}} \in W^{2,\infty}([0,\infty); \mathbb{C}^m)$;

(ii) $p \in W^{2,\infty}([0,\infty); \mathbb{R})$ with compact support and $p(0) = 1$;

(iii) $x_0 \in \mathcal{D}(\mathfrak{A})$ and $\varphi \in \Phi$ with $\varphi(0)\|\mathfrak{C}x_0 - y_{\mathrm{ref}}(0)\| < 1$ and $\varphi(0) > 0$.

Remark 4.2.3. Apart from regularity of the reference signal and performance funnel, the assumptions on the controller basically include two points:

a) The initial value is "smooth", i.e., $x_0 \in \mathcal{D}(\mathfrak{A})$. The reason is that - especially for hyperbolic systems - the initialization with $x_0 \in X \setminus \mathcal{D}(\mathfrak{A})$ might result in a discontinuous output. This effect typically occurs when the semigroup generated by $\mathfrak{A}|_{\ker \mathfrak{B}}$ is not analytical, such as, for instance, when a wave equation is considered.

b) The output of the system at $t = 0$ is already in the performance funnel and $\varphi^{-1} \in L^\infty([0,\infty); \mathbb{R})$ since $\varphi(0) > 0$, so that $\inf_{t\geq 0} \varphi > 0$.

The funnel controller (4.5) differs from the classical one in [60] by the addition of the term

$$\left(u_0 + \frac{1}{1-\varphi_0^2\|e_0\|^2} e_0\right) p(t)$$

for some (arbitrary) smooth function with $p(0) = 1$ and compact support. This ensures that the controller is consistent with the initial value, that is, u in (4.5) satisfies

$$u(0) = \left(u_0 + \frac{1}{1-\varphi_0^2\|e_0\|^2} e_0\right) p(0) - \frac{1}{1-\varphi(0)^2\|e(0)\|^2} e(0) = u_0 = \mathfrak{B}x_0 = \mathfrak{B}x(0).$$

The funnel controller therefore requires the knowledge of the "initial value of the input" $u_0 = \mathfrak{B}x_0$. This means that, loosely speaking, the "actuator position" has to be known at the initial time, which is —by the opinion of the authors— no restriction from a practical point of view.
We would like to emphasize that the application of the funnel controller does not need any further "internal information" on the system, such as system parameters or the full knowledge of the initial state.

The funnel controller (4.5) applied to a BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ results in the closed-loop system

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), & x(0) &= x_0, \\ \mathfrak{B}x(t) &= u(t), \\ \mathfrak{C}x(t) &= y(t), \\ e(t) &= y(t) - y_{\mathrm{ref}}(t), \quad e_0 = \mathfrak{C}x_0 - y_{\mathrm{ref}}(0), \quad \varphi_0 = \varphi(0), \\ u(t) &= (\mathfrak{B}x_0 + \psi(\varphi_0, e_0))p(t) - \psi(\varphi(t), e(t)), \end{aligned} \tag{4.6a}$$

where

$$\begin{aligned} \psi(\varphi, e) &:= \frac{1}{1 - \varphi^2 \|e\|^2} e, \\ \mathcal{D}(\psi) &:= \{(\varphi, e) \in (0, \infty) \times \mathbb{C}^m \mid \varphi \|e\| < 1\}. \end{aligned} \tag{4.6b}$$

We see immediately that the closed-loop system is nonlinear and time-variant. In the sequel we present our main results which state that the funnel controller is functioning in a certain sense. Note that this result includes the specification of the solution concept with which we are working. First we show that the funnel controller applied to any system fulfilling Assumption 4.1.2 has a solution. Such a solution however might not be bounded on the infinite time horizon. Thereafter, we show that boundedness on $[0, \infty)$ is guaranteed, if the constant α in the energy balance (4.2) is negative. The proofs of these results can be found in Section 4.4.

Theorem 4.2.4 (Feasibility of funnel controller, arbitrary α)**.** *Let a BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given which satisfies Assumption 4.1.2 and assume that the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 4.2.2. Then, for all $T > 0$ the closed-loop system* (4.6) *has a unique solution $x \in W^{1,\infty}([0, T]; X)$ in the following sense:*

(i) $\dot{x}$ is continuous except possibly at a countable number of points in $[0, T]$, and

(ii) for all $t \in [0, T]$ holds $x(t) \in D(\mathfrak{A})$ and (4.6)*.*

Remark 4.2.5.

a) The solution concept which is subject of Theorem 4.2.4 is strong in the sense that the weak derivative of x is evolving in the space X and not in some larger space as used e.g. in [116].

b) The property $x \in W^{1,\infty}([0,T];X)$ of a solution implies $\mathfrak{A}x = \dot{x} \in L^\infty([0,T];X)$, whence $x \in L^\infty([0,T];D(\mathfrak{A}))$. As a consequence, for $u = \mathfrak{B}x$ and $y = \mathfrak{C}x$ holds that $u, y \in L^\infty([0,T];\mathbb{C}^m)$. By the same argumentation, we see that the continuity of $\dot{x}$ except possibly in a countable set in $[0,T]$ implies that u and y are continuous except possibly in a countable set of $[0,T]$.

c) For $T_1 < T_2$ consider solutions x_1 and x_2 of the closed-loop system (4.6) on $[0,T_1]$ and $[0,T_2]$, respectively. Uniqueness of the solution implies that $x_1 = x_2|_{[0,T_1]}$. As a consequence, there exists a unique $x \in W^{1,\infty}_{\mathrm{loc}}([0,\infty);X)$ with the property that for all $T > 0$ holds that $x|_{[0,T]}$ is a solution of (4.6). Accordingly, the input satisfies $u \in L^\infty_{\mathrm{loc}}([0,\infty);\mathbb{C}^m)$. Note that, by the fact that the output evolves in the funnel, we have that y is essentially bounded, that is $y \in L^\infty([0,\infty);\mathbb{C}^m)$.

d) The properties $u, y, y_{\mathrm{ref}} \in L^\infty([0,T];\mathbb{C}^m)$ imply that the error $e = y - y_{\mathrm{ref}}$ is uniformly bounded away from the funnel boundary. That is, there exists some $\varepsilon > 0$ such that

$$\varphi(t)\|e(t)\| < 1 - \varepsilon \quad \text{for almost all } t \in [0,T].$$

e) The typical situation is that the system is real in the sense that the input, output and state evolve in the real spaces $\mathbb{R}^m$ and X. By using a complexification $X + iX$, the results presented in this chapter can be applied to such systems yielding that a (not yet necessarily real) solution $x \in W^{1,\infty}([0,T];X+iX)$ the closed-loop system (4.6) exists which is moreover unique. A closer look yields that the pointwise complex conjugate $\bar{x}$ is as well a solution of (4.6), and uniqueness gives $x = \bar{x}$, whence x has to be real in this case.

Though bounded on each bounded interval, the solution x of the closed-loop system (4.6) might satisfy

$$\limsup_{t\to\infty} \|x(t)\| = \infty, \qquad \limsup_{t\to\infty} \|u(t)\| = \infty$$

In the following we show that this unboundedness does not occur when the constant α in (4.2) in Assumption 4.1.2 is negative.

Theorem 4.2.6. *Let a BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given which satisfies Assumption 4.1.2 such that Assumption 4.1.2(i) holds with $\alpha < 0$. Assume that the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 4.2.2. Then the solution $x : [0,\infty) \to X$ of the closed-loop system* (4.6) *(which exists by Theorem 4.2.4) fulfills*

$$x \in W^{1,\infty}([0,\infty);X) \quad \textit{and} \quad u = \mathfrak{B}x \in L^\infty([0,\infty);\mathbb{C}^m).$$

Remark 4.2.7. In particular when the input and output of a system have different physical dimensions, it might be essential that the funnel controller is dilated by some constant $k_0 > 0$. More precisely, one might consider the controller

$$u(t) = \left(u_0 + \frac{k_0}{1-\varphi_0^2\|e_0\|^2} e_0\right) p(t) - \frac{k_0}{1-\varphi(t)^2\|e(t)\|^2} e(t). \tag{4.7}$$

The feasibility of this controller is indeed covered by Theorems 4.2.4 & 4.2.6, which can be seen by the following argumentation: Consider the BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ with transformed input $\tilde{u} = k_0^{-1} u$. That is, a system $\mathfrak{S} = (\mathfrak{A}, k_0^{-1}\mathfrak{B}, \mathfrak{C})$. Providing X with the equivalent inner product $\langle \cdot, \cdot \rangle_{\text{new}} := k_0^{-1} \langle \cdot, \cdot \rangle_X$, we obtain

$$\operatorname{Re} \langle \mathfrak{A}x, x \rangle_{\text{new}} \leq \operatorname{Re} \left\langle k_0^{-1}\mathfrak{B}x, \mathfrak{C}x \right\rangle_{\mathbb{C}^m} + \alpha \|x\|_{\text{new}}^2 \quad \text{for all } x \in \mathcal{D}(\mathfrak{A}).$$

Consequently, by Theorem 4.2.4, the funnel controller

$$\tilde{u}(t) = \Big(\underbrace{\tilde{u}_0}_{=k_0^{-1}u_0} + \frac{1}{1-\varphi_0^2\|e_0\|^2} e_0\Big) p(t) - \frac{1}{1-\varphi(t)^2\|e(t)\|^2} e(t)$$

results in feasibility of the closed-loop. Now resolving $\tilde{u} = k_0^{-1}u$ in the previous formula, we obtain exactly the controller (4.7). Further note that, by the same argumentation together with Theorem 4.2.6, we obtain that all the trajectories are bounded in the case where $\alpha < 0$.

4.2.1 Unbounded funnel boundary at the origin

The assumption that for $\varphi \in \Phi$ we have $\inf_{t \geq 0} \varphi(t) > 0$ is quite technical. It has to do with the fact that we need $\varphi^{-1} \in L^\infty([0,\infty); \mathbb{R})$ in order to show existence of solutions to (4.6). However, under suitable assumptions this can be generalized. In the following we discuss the case in which we allow an unbounded funnel boundary at the origin under the assumption that the BCS is well-posed. In the following assumption, we replace Assumption 4.2.2(iii) so that the case $\varphi(0) = 0$ is not excluded.

Assumption 4.2.8. Let a BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given. In addition to Assumption 4.1.2 and Assumption 4.2.2(i) & (ii) assume that the initial state x_0 and the function φ in the controller (4.5) fulfill that $x_0 \in \mathcal{D}(\mathfrak{A})$ and $\varphi \in \Phi$ with $\varphi(0)\|\mathfrak{C}x_0 - y_{\text{ref}}(0)\|_{\mathbb{C}^m} < 1$. Assume further that

(i) the BCS system is L^2-well-posed and given by $\Sigma = (\mathbb{T}, \Phi, \Psi, \mathbb{F})$;

(ii) for all $f \in W^{1,2}_{\text{loc}}([0,\infty); \mathbb{C}^m)$ there exists $\delta > 0$ and $e \in W^{1,2}_{\text{loc}}([0,\delta]; \mathbb{C}^m)$ such that

$$e(t) = f(t) + \left(\mathbb{F}_\infty \left(\psi(\varphi(0), e(0))p - \psi(\varphi, e)\right)\right)(t), \tag{4.8}$$

and $\varphi(t)\|e(t)\|_{\mathbb{C}^m} < 1$ for all $t \in [0,\delta]$.

Remark 4.2.9.

a) As we will see in the proof of Theorem 4.2.12, Assumption 4.2.8(ii) delivers the existence of a solution to the closed-loop system. In fact, this assumption is not so restrictive. In the case where $\mathfrak{A}|_{\ker \mathfrak{B}}$ generates an analytic semigroup, one typically has that there exists $h \in L^1_{\mathrm{loc}}([0,\infty);\mathbb{R}^{m\times m})$ such that

$$(\mathbb{F}v)(t) = (h * v)(t), \quad v \in L^2_{\mathrm{loc}}([0,\infty);\mathbb{R}^m),$$

see for instance [110, Theorem 5.7.3]. If that is the case, by means of a standard fixed point argument and slightly modifying [43, Theorem XIII.3.3 & XIII.3.3], one obtains that the nonlinear Volterra equation (4.8) has a solution with the desired properties on $[0, T_{max})$ for some $T_{max} > 0$. It suffices then to choose $\delta := T_{max}/2$ for instance.

b) Since the system is well-posed $v := \psi(\varphi(0), e(0))p - \psi(\varphi, e)$ is in $W^{1,2}([0,\delta];\mathbb{C}^m)$ with $v(0) = 0$, [116, Proposition 4.2.10] implies that $x \in C([0,\delta];\mathcal{D}(\mathfrak{A}))$ so that $(\mathbb{F}_\infty v)(0) = 0$ and $e(0) = f(0)$.

c) BCS $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ being well-posed and making use of operators $P, Q \in \mathscr{L}(\mathbb{C}^m, \mathcal{D}(\mathfrak{A}))$ from Remark 4.1.3c) we can easily compute that

$$\dot{x}(t) = \mathfrak{A}|_{\ker \mathfrak{B}} x(t) + Bu(t),$$

where $B \in \mathscr{L}(\mathbb{C}^m, X_{-1})$ is given by $B = \mathfrak{A}P - \mathfrak{A}|_{\ker \mathfrak{B}}P$, see for instance [116, Theorem 5.2.13(iii)].

Of course, the main problem is to guarantee the existence and uniqueness of a solution in a small interval $[0,\delta]$, since for $t \geq \delta$, we can apply Theorems 4.2.4 & 4.2.6 with a time shift so that we start with $x_0 = x(\delta)$. Before we prove the main theorem of this section, we need an auxiliary result.

Lemma 4.2.10. *Let a BCS* $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ *be* L^2*-well-posed and given by* $\Sigma = (\mathbb{T},\Phi,\Psi,\mathbb{F})$*. Let the initial value* $x_0 \in \mathcal{D}(\mathfrak{A})$ *and* $p \in W^{2,\infty}([0,\infty);\mathbb{R})$ *with* $p(0) = 1$ *be given. Let* $u_0 := \mathfrak{B}x_0$ *and set* $u = u_0 p + v$*, for* $v \in L^2_{\mathrm{loc}}([0,\infty);\mathbb{C}^m)$*. Then* $g := \Psi_\infty x_0 + \mathbb{F}_\infty(u_0 p) \in W^{1,2}_{\mathrm{loc}}([0,\infty);\mathbb{C}^m)$ *and it holds that*

$$y = g + \mathbb{F}_\infty v$$

with $g(0) = \mathfrak{C}x_0$*.*

Proof. Since by BCS $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is well-posed, we can write it as

$$\begin{aligned} x(t) &= \mathbb{T}_t x_0 + \Phi_t u, \\ y &= \Psi_\infty x_0 + \mathbb{F}_\infty u. \end{aligned}$$

Thus,

$$\Phi_t u = \Phi_t(u_0 p) + \Phi_t v.$$

Making use of Remark 4.2.9c) it follows that $B = \mathfrak{A}P - \mathfrak{A}|_{\ker\mathfrak{B}}P$. Hence,

$$\Phi_t(u_0 p) = \int_0^t \mathbb{T}_{t-s}(\mathfrak{A}P - \mathfrak{A}|_{\ker\mathfrak{B}}P)u_0 p(s)\,\mathrm{d}s\,.$$

Since p is at least once weakly differentiable, integration by parts leads to

$$\Phi_t(u_0 p) = -\mathbb{T}_t P u_0 + P u_0 p(t) + \int_0^t \mathbb{T}_{t-s}\mathfrak{A}P u_0 p(s)\,\mathrm{d}s\, - \int_0^t \mathbb{T}_{t-s} P u_0 \dot{p}(s)\,\mathrm{d}s\,,$$

since $p(0) = 1$, so that we can define $f(t) := \Phi_t(u_0 p)$ which is continuous in X. If the semigroup $(\mathbb{T}_t)_{t\geq 0}$ is exponentially stable, then by direct calculation one can show that f is bounded in $[0,\infty)$ with values in X. Using $y = \mathfrak{C}x$ leads to

$$y = g + \mathbb{F}_\infty v,$$

where

$$g(t) := \mathfrak{C}\mathbb{T}_t(x_0 - P u_0) + \mathfrak{C}\int_0^t \mathbb{T}_{t-s}\mathfrak{A}P u_0 p(s)\,\mathrm{d}s\, - \mathfrak{C}\int_0^t \mathbb{T}_{t-s} P u_0 \dot{p}(s)\,\mathrm{d}s\,,$$

since $\mathfrak{C}P = 0$ so that $\mathfrak{C}P u_0 p(t) = 0$.
Note that $x_0 - P u_0 \in \mathcal{D}(\mathfrak{A}|_{\ker\mathfrak{B}})$, since $\mathfrak{B}(x_0 - P u_0) = \mathfrak{B}x_0 - u_0 = 0$ by construction. By considering the auxiliary problems

$$\dot{w}_1(t) = \mathfrak{A}|_{\ker\mathfrak{B}} w_1(t) + \mathfrak{A}P u_0 p(t), \quad w_1(0) = 0,$$

and

$$\dot{w}_2(t) = \mathfrak{A}|_{\ker\mathfrak{B}} w_2(t) + P u_0 \dot{p}(t), \quad w_2(0) = 0,$$

it is not difficult to see that since $\mathfrak{A}P, P \in \mathscr{L}(\mathbb{C}^m, X)$, we get $w_1, w_2 \in C([0,\infty); \mathcal{D}(\mathfrak{A}))$, see for instance [116, Proposition 4.2.10]. This implies that g is continuous and we have that $g(0) = \mathfrak{C}(x_0 - P u_0) = \mathfrak{C}x_0$, which holds by $\mathfrak{C}P = 0$.
Since the system is well-posed and $\mathfrak{A}|_{\ker\mathfrak{B}}(x_0 - P u_0) \in X$ we further have that

$$t \mapsto \mathfrak{C}\mathbb{T}_t\mathfrak{A}|_{\ker\mathfrak{B}}(x_0 - P u_0) \in L^2_{\mathrm{loc}}([0,\infty); \mathbb{C}^m).$$

By considering the control operators $b_1 := \mathfrak{A}P u_0$ and $b_2 := P u_0$ and the observation operator $\mathfrak{C}$, we have that the systems $(\mathfrak{A}|_{\ker\mathfrak{B}}, b_i, \mathfrak{C})$, $i = 1, 2$, are thus well-posed in $(\mathbb{R}, X, \mathbb{C}^m)$. If we consider the respective inputs $(u_1, u_2) = (p, \dot{p}) \in W^{1,\infty}([0,\infty); \mathbb{R}) \times W^{1,\infty}([0,\infty); \mathbb{R})$, by [110, Theorem 4.6.5] it follows that

$$t \mapsto \mathfrak{C}\int_0^t \mathbb{T}_{t-s}\mathfrak{A}P u_0 p(s)\,\mathrm{d}s\, - \mathfrak{C}\int_0^t \mathbb{T}_{t-s} P u_0 \dot{p}(s)\,\mathrm{d}s\, \in W^{1,2}_{\mathrm{loc}}([0,\infty); \mathbb{R}^m).$$

Hence $g \in W^{1,2}_{\mathrm{loc}}([0,\infty); \mathbb{C}^m)$. $\square$

Remark 4.2.11. Note that Lemma 4.2.10 does not make use of the Assumption 4.2.8.

We now present the main result of this section, which shows existence and uniqueness of a solution in a small interval $[0, \delta]$. In fact, the following result extends naturally the solution concept which is subject to Theorem 4.2.4.

Theorem 4.2.12. *Let a BCS* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$, *the initial value* x_0 *and the functions* y_{ref}, p, φ *be given which satisfy Assumption 4.2.8. Then, there exists a* $\delta > 0$ *such that the closed-loop system* (4.6) *has a unique solution* $x \in W^{1,\infty}([0, \delta]; X)$ *in the following sense:*

(i) $\dot{x} \in C([0, \delta]; X)$, *and*

(ii) for all $t \in [0, \delta]$ *holds* $x(t) \in D(\mathfrak{A})$ *and* (4.6).

Proof. We construct a solution on an interval $[0, T_{max})$ for some $T_{max} > 0$ and then choose $\delta := T_{max}/2$. We need to solve (4.6). Note that we can rewrite u as

$$u(t) = u_0 p(t) + \psi(\varphi_0, e_0)p(t) - \psi(\varphi(t), e(t)),$$

where $e_0 = e(0)$ and $\varphi_0 = \varphi(0)$. We thus consider the equation

$$y(t) = g(t) + (\mathbb{F}_\infty v)(t)$$

with

$$v(t) := \psi(\varphi_0, e_0)p(t) - \psi(\varphi(t), e(t))$$

and g from Lemma 4.2.10 such that $y(0) = \mathfrak{C}x_0 = g(0)$. Subtracting y_{ref} on both sides and setting $e := y - y_{\mathrm{ref}}$ leads to

$$e(t) = f(t) + (\mathbb{F}_\infty v)(t)$$

with $f := g - y_{\mathrm{ref}}$. If we find a solution to the former, then by Lemma 4.2.10, u is well-defined and satisfies

$$y = \Psi_\infty x_0 + \mathbb{F}_\infty u.$$

However, this is guaranteed by Assumption 4.2.8(ii), since $f \in W^{1,2}_{\mathrm{loc}}([0, \infty); \mathbb{C}^m)$ and thus, there exists $\delta > 0$ and $e \in W^{1,2}_{\mathrm{loc}}([0, \delta]; \mathbb{C}^m)$ with $\varphi(t)\|e(t)\|_{\mathbb{C}^m} < 1$ for all $t \in [0, \delta]$ such that

$$e(t) = f(t) + \left(\mathbb{F}_\infty \left(\psi(\varphi_0, e_0)p(t) - \psi(\varphi(t), e(t))\right)\right)(t)$$

Since in particular e is continuous and $\varphi(t)\|e(t)\|_{\mathbb{R}^m} < 1$ for $t \in [0, \delta]$, so that

$$\frac{1}{1 - \varphi^2\|e\|^2_{\mathbb{C}^m}} \in BC([0, \delta]; \mathbb{R}^m)$$

and thus $u \in W^{1,2}_{\mathrm{loc}}([0,\delta];\mathbb{C}^m)$. With this choice of u, combining [110, Theorem 4.6.5 & Theorem 5.2.13] it follows that there exists a unique $x \in C^1([0,\delta];X) \cap C([0,\delta];\mathcal{D}(\mathfrak{A}))$ such that

$$\begin{aligned}\dot{x}(t) &= x(t), \quad x(0) = x_0,\\ \mathfrak{B}x(t) &= u(t),\end{aligned}$$

so that x is a solution on $[0,\delta]$.
Now we show the uniqueness. Assuming that there exist two solutions of the Volterra equation e_1, e_2 with respective u_1, u_2, we obtain x_1, x_2. Let $z := x_1 - x_2$ and note that $z(0) = 0$. Hence, using Lemma 4.4.4 it follows that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_X^2 &= \mathrm{Re}\,\langle z(t), \mathfrak{A}z(t)\rangle_X \le \mathrm{Re}\,\langle \mathfrak{B}z(t), \mathfrak{C}z(t)\rangle_{\mathbb{R}^m} + \alpha\|z(t)\|_X^2\\ &= \mathrm{Re}\,\langle u_1 - u_2, y_1 - y_2\rangle_{\mathbb{R}^m} + \alpha\|z(t)\|_X^2\\ &= -\,\mathrm{Re}\,\langle \psi(\varphi(t), e_1(t)) - \psi(\varphi(t), e_2(t)), y_1 - y_2\rangle_{\mathbb{R}^m} + \alpha\|z(t)\|_X^2\\ &= -\,\mathrm{Re}\,\langle \psi(\varphi(t), e_1(t)) - \psi(\varphi(t), e_2(t)), e_1 - e_2\rangle_{\mathbb{R}^m} + \alpha\|z(t)\|_X^2\\ &\le \alpha\|z(t)\|_X^2.\end{aligned}$$

Thus, by Lemma 1.8.13 we have

$$\|z(t)\|_X \le \|z(0)\|_X \mathrm{e}^{\alpha t} = 0$$

which implies $x_1 = x_2$ in $[0,\delta]$ and this concludes the proof. □

4.2.2 Analytic semigroups and regularity

Throughout this section we will assume that $\mathfrak{A}|_{\ker\mathfrak{B}}$ generates an analytic semigroup on X. As we have already seen in Remark 4.2.9c), the BCS $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ can be brought to the form

$$\begin{aligned}\dot{x}(t) &= \mathfrak{A}|_{\ker\mathfrak{B}}x(t) + Bu(t), \quad x(0) = x_0,\\ y(t) &= \mathfrak{C}x(t).\end{aligned} \tag{4.9}$$

We call B the *abstract control operator* associated to $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$. We will next show, that if the operator $B \in \mathscr{L}(\mathbb{C}^m, X_{-1})$ is more regular, then we obtain more time regularity of the solution described in Theorem 4.2.4. For $r \in \mathbb{R}$ we denote by X_r the interpolation spaces associated to $\mathfrak{A}|_{\ker\mathfrak{B}}$ as introduced in Section 1.6.1.
In the next result we show that the solution of Theorem 4.2.4 enjoys of a certain regularity if the semigroup generated by $\mathfrak{A}|_{\ker\mathfrak{B}}$ is analytic and the abstract control operator associated to $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ is regular enough.

Theorem 4.2.13. *Let a BCS $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ be given which satisfies Assumption 4.1.2 and assume that the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 4.2.2.*

Assume that there exists $r \in (0,1/2)$ such that the abstract control operator satisfies $B \in \mathscr{L}(\mathbb{C}^m, X_{-r})$ and further $\mathfrak{C} \in \mathscr{L}(X_s, \mathbb{C}^m)$ for some $s \in (0,1/2)$. For $T > 0$ let $x \in W^{1,\infty}([0,T]; X)$ be the unique solution of closed-loop system (4.6) from Theorem 4.2.4 with input $u := \mathfrak{B}x$ and output $y := \mathfrak{C}x$. Then

$$u \in C^{0,1-r-s}([0,T]; \mathbb{C}^m), \quad y \in C^{0,1-r-s}([0,T]; \mathbb{C}^m).$$

Moreover, if Assumption 4.1.2(i) holds with $\alpha < 0$, we have that

$$u \in C^{0,1-r-s}([0,\infty); \mathbb{C}^m), \quad y \in C^{0,1-r-s}([0,\infty); \mathbb{C}^m).$$

Proof. With this choice of B and $\mathfrak{C}$, we have that $r+s < 1$. Hence, [110, Theorem 5.7.3] implies that the system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is L^2-well-posed on $(\mathbb{C}^m, X, \mathbb{C}^m)$ with operators $\Sigma = (\mathbb{T}, \Psi, \Phi, \mathbb{F})$ and that there exists $h \in L^1_{\mathrm{loc}}([0,\infty); \mathbb{C}^{m\times m}) \cap C([0,\infty); \mathbb{C}^{m\times m})$ such that

$$(\mathbb{F}_\infty u)(t) = (h * u)(t)$$

for all $u \in L^p_{\mathrm{loc}}([0,\infty); \mathbb{C}^m)$. In particular, $(\mathbb{T}_t)_{t\geq 0}$ is the semigroup generated by $\mathfrak{A}|_{\ker \mathfrak{B}}$.

Theorem 4.2.4 implies that there exists a unique solution $x \in W^{1,\infty}([0,T]; X)$ such that $u := \mathfrak{B}x \in L^\infty([0,T]; \mathbb{C}^m)$ and $y \in L^\infty([0,T]; \mathbb{C}^m)$ are continuous except, possibly, in a countable number of point.

Let $z := x - Pu_0p$, where $P \in \mathscr{L}(\mathbb{C}^m, \mathcal{D}(\mathfrak{A}))$ is from Remark 4.1.3c), that is $\mathfrak{C}P = 0$, $\mathfrak{B}P = I$. Clearly, z has the same properties as x by construction and $z_0 := z(0) = x_0 - Pu_0$. Further, $y = \mathfrak{C}x = \mathfrak{C}(z + Pu_0p) = \mathfrak{C}z$. Moreover, since

$$\dot{x}(t) = \mathfrak{A}|_{\ker \mathfrak{B}} x(t) + Bu(t)$$

it follows that

$$\dot{z}(t) = \mathfrak{A}|_{\ker \mathfrak{B}} z(t) + \mathfrak{A}|_{\ker \mathfrak{B}} Pu_0p(t) - Pu_0\dot{p}(t) + Bu(t).$$

Using the definition of u we have that

$$u = u_0p + v,$$

where

$$v := \psi(\varphi_0, e_0)p - \psi(\varphi, e)$$

and $e := y - y_{\mathrm{ref}}$ as usual. Note that $v \in L^\infty([0,T]; \mathbb{C}^m)$ and has the same properties as u. With that and using that $B = \mathfrak{A}P - \mathfrak{A}|_{\ker \mathfrak{B}}P$ we have that

$$\dot{z}(t) = \mathfrak{A}|_{\ker \mathfrak{B}} z(t) + \mathfrak{A}Pu_0p(t) - Pu_0\dot{p}(t) + Bv(t).$$

Set $f := \mathfrak{A}Pu_0p - Pu_0\dot{p} \in W^{1,\infty}([0,\infty); X)$. Thus,

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-\tau} f(\tau)\,\mathrm{d}\tau + \int_0^t (\mathbb{T}|_{-r})_{t-\tau} Bv(\tau)\,\mathrm{d}\tau\,.$$

By [84, Proposition 4.2.1] it follows that

$$t \mapsto \int_0^t \mathbb{T}_{t-\tau} f(\tau)\,\mathrm{d}\tau \in C^{0,1-r}([0,T];X) \cap C^{0,1-r-s}([0,T];X_s).$$

Let

$$z_v(t) := \int_0^t (\mathbb{T}|_{-r})_{t-\tau} Bv(\tau)\,\mathrm{d}\tau\,.$$

By using Lemma 1.6.1 it follows that there exists $K > 0$ such that for all $t_1, t_2 \in [0,T]$, $t_1 < t_2$, it holds that

$$\|z_v(t_2) - z_v(t_1)\|_{X_s} \le K \|Bv\|_{L^\infty([0,T];X_{-r})} (t_2 - t_1)^{1-r-s}.$$

Further, since by construction $z_0 \in \mathcal{D}(\mathfrak{A}|_{\ker \mathfrak{B}})$, we have $t \mapsto \mathbb{T}_t z_0 \in BC^1([0,T];X)$, and thus, it follows that it has (at least) the same regularity as z_v. Hence, $z \in C^{0,1-r-s}([0,T];X_s)$. Since $\mathfrak{C} \in \mathscr{L}(X_s, \mathbb{C}^m)$ and $y = \mathfrak{C}z$, the former yields $y \in C^{0,1-r-s}([0,T];\mathbb{C}^m)$. By the closed-loop it follows that u has the same regularity as y.
If $\alpha < 0$, then it follows that $(\mathbb{T}_t)_{t\ge 0}$ is exponentially stable. Theorem 4.2.6 implies that $u, y \in L^\infty([0,\infty);\mathbb{C}^m)$, so that $z_v \in C^{1-r-s}([0,\infty);X_s)$. Moreover, [84, Proposition 4.4.1 (i)] implies that

$$t \mapsto \int_0^t \mathbb{T}_{t-\tau} f(\tau)\,\mathrm{d}\tau \in C^{0,1-r}([0,\infty);X) \cap C^{0,1-r-s}([0,\infty);X_s),$$

so that $z \in C^{1-r-s}([0,\infty);X_s)$ and the result follows by continuity of $\mathfrak{C} \in \mathscr{L}(X_s, \mathbb{C}^m)$ and the closed-loop. □

4.3 Some PDE examples

We now present three different system classes for which we can apply the previously presented results. The first two have state variables which are described by hyperbolic PDEs and the third one by a parabolic PDE.

4.3.1 Port-Hamiltonian systems in one spatial variable

The systems considered in this chapter enclose a class of *port-Hamiltonian* hyperbolic system in one spatial dimension with boundary control and observation, which has been treated in [5–7, 9, 67] and is subject of the subsequent definition. Typically they are considered in a bounded interval $[a,b] \subset \mathbb{R}$. We may consider $\mathbb{I} := [a,b] = [0,1]$ without loss of generality.

Definition 4.3.1 (Port-Hamiltonian hyperbolic BCS in one spatial variable)**.** Let $N, d \in \mathbb{N}$ and for $k = 0, \dots, N$ consider $P_k \in \mathbb{C}^{d\times d}$. We assume that $P_k = (-1)^{k+1} P_k^*$ for $k \neq 0$ with P_N invertible and $P_0 + P_0^* \leq 0$. Further let $W_B, W_C \in \mathbb{C}^{Nd\times 2Nd}$ such that the matrix

$$W := \begin{bmatrix} W_B \\ W_C \end{bmatrix} \in \mathbb{C}^{2Nd\times 2Nd}$$

is invertible.

a) Let $\mathcal{H} \in L^\infty\left([0,1]; \mathbb{C}^{d\times d}\right)$ with $\mathcal{H}(\zeta) = \mathcal{H}(\zeta)^*$ for almost every $\zeta \in [0,1]$ and assume that there exist $m, M > 0$ such that $mI_d \leq \mathcal{H}(\zeta) \leq MI_d$ for almost every $\zeta \in [0,1]$. We consider $X := L^2([0,1]; \mathbb{C}^d)$ equipped with the scalar product induced by $\mathcal{H}$,

$$\langle y, x\rangle_X := \langle y, \mathcal{H}x\rangle_{L^2} = \int_0^1 y(\zeta)^* \mathcal{H}(\zeta) x(\zeta)\, \mathrm{d}\zeta\,, \quad x, y \in L^2([0,1]; \mathbb{C}^d). \tag{4.10}$$

The *port-Hamiltonian operator* $\mathfrak{A} : \mathcal{D}(\mathfrak{A}) \subset X \to X$ is given by

$$\mathfrak{A}x = \sum_{k=0}^{N} P_k \frac{\partial^k}{\partial \zeta^k}(\mathcal{H}x), \quad x \in \mathcal{D}(\mathfrak{A}), \tag{4.11a}$$

with domain

$$\mathcal{D}(\mathfrak{A}) = \left\{x \in X \mid \mathcal{H}x \in W^{N,2}([0,1]; \mathbb{C}^d)\right\} \tag{4.11b}$$

b) Denote the spatial derivative of f by f'. For a port-Hamiltonian operator $\mathfrak{A}$ and $x \in \mathcal{D}(\mathfrak{A})$ we define the *boundary flow* $f_{\partial,\mathcal{H}x} \in \mathbb{C}^{Nd}$ and *boundary effort* $e_{\partial,\mathcal{H}x} \in \mathbb{C}^{Nd}$ by

$$\begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix} := R_0 \begin{pmatrix} (\mathcal{H}x)(1) \\ (\mathcal{H}x)'(1) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(1) \\ (\mathcal{H}x)(0) \\ (\mathcal{H}x)'(0) \\ \vdots \\ (\mathcal{H}x)^{(N-1)}(0) \end{pmatrix}, \tag{4.12}$$

where the matrix $R_0 \in \mathbb{C}^{2Nd\times 2Nd}$ is defined by

$$R_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} \Lambda & -\Lambda \\ I_{Nd} & I_{Nd} \end{bmatrix}, \tag{4.13}$$

with

$$\Lambda := \begin{bmatrix} P_1 & P_2 & \cdots & \cdots & P_N \\ -P_2 & -P_3 & \cdots & -P_N & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{N-1}P_N & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

c) For a port-Hamiltonian operator $\mathfrak{A}$ we define the *input map* $\mathfrak{B} : \mathcal{D}(\mathfrak{A}) \subset X \to \mathbb{C}^{Nd}$ and the *output map* $\mathfrak{C} : \mathcal{D}(\mathfrak{A}) \subset X \to \mathbb{C}^{Nd}$ as

$$\mathfrak{B}x := W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}, \tag{4.14}$$

$$\mathfrak{C}x := W_C \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix}. \tag{4.15}$$

We call $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ a *port-Hamiltonian hyperbolic BCS in one spatial variable* to which we associate the boundary control and observation problem

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \quad x(0) = x_0, \\ u(t) &= \mathfrak{B}x(t), \\ y(t) &= \mathfrak{C}x(t) \end{aligned} \tag{4.16}$$

with a state $x(t) := x(t, \cdot) \in X$ and $t \geq 0$.

From the former definition we have the following result.

Lemma 4.3.2. *With operators $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ as in Definition 4.3.1, there exist $P, Q \in \mathscr{L}(\mathbb{C}^{2Nd}, \mathcal{D}(\mathfrak{A}))$ with*

$$\begin{array}{rclcrcl} \mathfrak{B}P &=& I_{Nd}, && \mathfrak{B}Q &=& 0, \\ \mathfrak{C}P &=& 0, && \mathfrak{C}Q &=& I_{Nd}. \end{array}$$

Consequently, $\mathfrak{A}P, \mathfrak{A}Q \in \mathscr{L}(\mathbb{C}^{Nd}, X)$.

Proof. Consider the trace operator $\mathcal{T} : W^{N,2}([0,1]; \mathbb{C}^d) \to \mathbb{C}^{2Nd}$ as the linear map

$$\mathcal{T}z = \begin{pmatrix} z(1) \\ z'(1) \\ \vdots \\ z^{(N-1)}(1) \\ z(0) \\ z'(0) \\ \vdots \\ z^{(N-1)}(0) \end{pmatrix},$$

so that

$$\begin{bmatrix}\mathfrak{B}x\\ \mathfrak{C}x\end{bmatrix} = WR_0\mathcal{T}\mathcal{H}x, \qquad \text{where } W = \begin{bmatrix}W_B\\ W_C\end{bmatrix}.$$

Consider the standard orthogonal basis $\{e_j\}_{j=1}^{2Nd}$ in $\mathbb{C}^{2Nd}$ and for $j = 1, \dots, 2Nd$ choose some $f_j \in W^{N,2}([0,1];\mathbb{C}^d)$ with $\mathcal{T}(f_j) = e_j$. Since W, R_0 are invertible, we can define $M_p, M_q \in \mathbb{C}^{2Nd\times Nd}$ by

$$M_p = R_0^{-1}W^{-1}\begin{bmatrix}I_{Nd}\\ 0\end{bmatrix}, \quad M_q = R_0^{-1}W^{-1}\begin{bmatrix}0\\ I_{Nd}\end{bmatrix}.$$

Let M_p, M_q be decomposed as

$$M_p = \begin{bmatrix}M_{p,1}\\ \vdots\\ M_{p,2Nd}\end{bmatrix}, \quad M_q = \begin{bmatrix}M_{q,1}\\ \vdots\\ M_{q,2Nd}\end{bmatrix},$$

with $M_{p,j}, M_{q,j} \in \mathbb{C}^{1\times Nd}$ for $j = 1, \dots, 2Nd$. Now set for almost every $\zeta \in [0,1]$,

$$(Pu)(\zeta) \coloneqq \mathcal{H}^{-1}(\zeta)\sum_{j=1}^{2Nd} M_{p,j}uf_j(\zeta), \quad \forall u \in \mathbb{C}^{Nd},$$

$$(Qy)(\zeta) \coloneqq \mathcal{H}^{-1}(\zeta)\sum_{j=1}^{2Nd} M_{q,j}yf_j(\zeta), \quad \forall y \in \mathbb{C}^{Nd}.$$

By construction P, Q have the desired properties. □

Remark 4.3.3. Note that for a port-Hamiltonian hyperbolic BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ in one spatial variable holds that $C_0^\infty([0,1];\mathbb{C}^d) \subset \ker\mathfrak{B} \cap \ker\mathfrak{C}$ is a dense subspace of X. With an appropriate choice of W_B and W_C, integration by parts gives

$$\operatorname{Re}\langle\mathfrak{A}x, x\rangle_X \le \operatorname{Re}\langle\mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^{Nd}} + \operatorname{Re}\langle P_0\mathcal{H}x, \mathcal{H}x\rangle \quad \text{for all } x \in \mathcal{D}(\mathfrak{A}). \tag{4.17}$$

Since $P_0 + P_0^* \le 0$, it follows that the BCS fulfills Assumption 4.1.2(i) with $\alpha = 0$.

The class of impedance passive port-Hamiltonian systems meets the requirements of Assumption 4.1.2. We summarize it in the following statement.

Theorem 4.3.4. *Any port-Hamiltonian hyperbolic BCS* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *with one spatial variable satisfies Assumption 4.1.2. If, moreover, there exists some* $\mu > 0$ *such that* $P_0 + P_0^* + \mu I$ *is pointwise negative definite, then Assumption 4.1.2(i) holds for some* $\alpha < 0$.

Proof. It is stated in Remark 4.3.3 that $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ satisfies Assumption 4.1.2(i) with $\alpha \le 0$. Further, $\mathfrak{A}|_{\ker\mathfrak{B}}$ generates a (contractive) semigroup by [7, Theorem 2.3], whence $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ satisfies Assumption 4.1.2(ii). We can further infer from Remark 4.3.3 that $\ker \mathfrak{B} \cap \ker \mathfrak{C}$ is dense in X, and Lemma 4.3.2 guarantees the existence of P, Q such that (4.4) holds. This implies that the condition in Assumption 4.1.2(4.3) is fulfilled by $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$.

If, moreover, $P_0 + P_0^* + \mu I$ is negative definite for some $\mu > 0$, then we can conclude from (4.17) that Assumption 4.1.2(i) holds with $\alpha := -\mu m/(2M)$, where $m, M > 0$ are given in Definition 4.3.1. □

Theorem 4.3.4 allows to directly apply Theorems 4.2.4 & 4.2.6. Namely, if the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 4.2.2, the application of the funnel controller (4.5) results in a unique global solution $x \in W^{1,\infty}_{\mathrm{loc}}([0,\infty); X)$ in the sense of Theorem 4.2.4. If, moreover, $P_0 + P_0^* + \mu I$ is negative definite for some $\mu > 0$, then $x, \dot{x}$ and u are moreover essentially bounded by Theorem 4.2.6.

Lossy transmission line

Here we consider the dissipative version of the Telegrapher's Equation with constant coefficients given by

$$\begin{aligned} V_\zeta(\zeta,t) &= -LI_t(\zeta,t) - RI(\zeta,t), \\ I_\zeta(\zeta,t) &= -CV_t(\zeta,t) - GV(\zeta,t), \\ u(t) &= \begin{pmatrix} V(a,t) \\ V(b,t) \end{pmatrix}, \\ y(t) &= \begin{pmatrix} I(a,t) \\ -I(b,t) \end{pmatrix}. \end{aligned}$$

R is the resistance, C the capacitance, L the inductance and G the conductance —all of them per unit length.

The system can be written in port-Hamiltonian form as

$$\begin{aligned} \partial_t x(\zeta,t) &= P_1 \partial_\zeta(\mathcal{H}(\zeta)x(\zeta,t)) + P_0\mathcal{H}(\zeta)x(\zeta,t), \\ u(t) &= W_B R_0 \begin{pmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(a,t) \end{pmatrix}, \\ y(t) &= W_C R_0 \begin{pmatrix} (\mathcal{H}x)(b,t) \\ (\mathcal{H}x)(a,t) \end{pmatrix}, \end{aligned} \tag{4.18a}$$

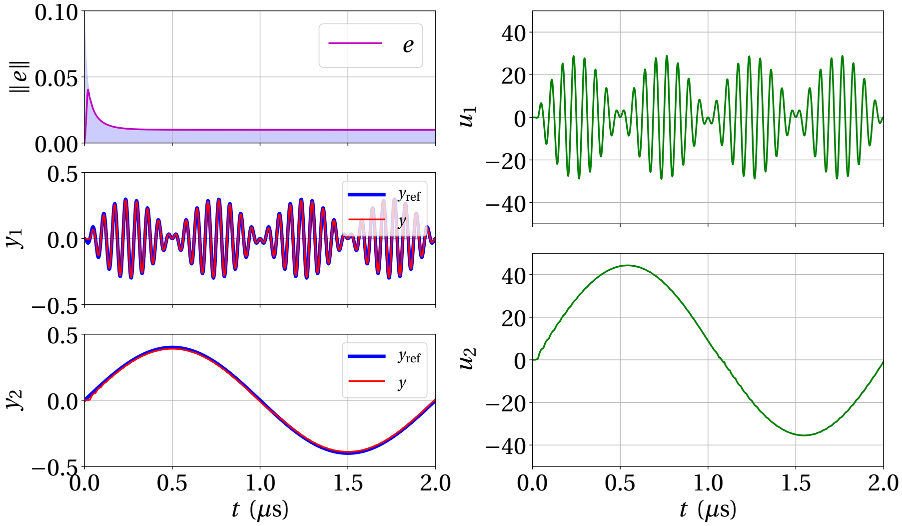

Figure 4.2: Left: Norm of the error within the funnel boundary followed by the two reference signals and the respective outputs. Right: Inputs obtained from the feedback law.

where

$$x(\zeta,t) := \begin{pmatrix} LI(\zeta,t) \\ CV(\zeta,t) \end{pmatrix}, P_1 := \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, P_0 := \begin{bmatrix} -R & 0 \\ 0 & -G \end{bmatrix}, \mathcal{H}(\zeta) := \begin{bmatrix} L^{-1} & 0 \\ 0 & C^{-1} \end{bmatrix},$$

$$W_B := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}, W_C := \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix}. \tag{4.18b}$$

We have chosen the reference signals and funnel boundary of the following form

$$y_{\text{ref}}(t) = \begin{pmatrix} A_1 \sin(\omega_1 t)\sin(\omega_2 t) \\ A_2 \sin(\omega_3 t) \end{pmatrix},$$
$$\varphi(t) = \varphi_0 \varepsilon^{-2} \tanh(\omega t + \varepsilon).$$

In this case the system is impedance passive and $P_0 + P_0^* \leq -2\min\{R,G\}I_2$ and Theorem 4.3.4 implies that $u, y \in L^\infty([0,\infty);\mathbb{R}^2)$. The simulated system is shown in Fig. 4.2.

The parameter values are $\zeta \in (a,b)$ with $a = 0$ m, $b = 1$ m,

$$\begin{aligned} R &= 463.59\Omega\text{m}^{-1}, & L &= 0.5062 \text{ mH m}^{-1}, \\ G &= 29.111\ \mu\text{S m}^{-1}, & C &= 51.57 \text{ nF m}^{-1}. \end{aligned}$$

Further, set $c_0 = (LC)^{-1/2}$, $f = 1$ MHz, $\omega = 2\pi f$, $\varphi_0 = 1\ \text{A}^{-1}$, $\varepsilon = 0.1$ and amplitudes $A_1 = -0.3$ A, $A_2 = 0.4$ A. The other angular frequencies are $\omega_1 = \omega$, $\omega_2 = 16\omega$ and $\omega_3 = \omega/2$. For the time interval we have defined $T_0 = f^{-1}$ and $t \in [0, T]$, where $T = 2T_0$. We have used semi-explicit finite differences with a tolerance of 10^{-3}. The mesh in ζ has $M = 1000$ points and the mesh in t has

$$N = \left\lfloor \frac{b-a}{2c_0 T} M \right\rfloor$$

points. We further assume that the initial state is zero, i.e., $x_0 = 0$ and we apply the controller (4.7) from Remark 4.2.7 with $k_0 = 1\ \Omega$.

4.3.2 Hyperbolic systems in several spatial variables

The following setting is presented in [119, Section 8.2]. We give a summary of the main results. For the particular case of the higher dimensional wave equation we refer to [116].

Definition 4.3.5. Let $d \in \mathbb{N}$ and matrices $P_j \in \mathbb{R}^{n\times n}$ for $j = 0, \ldots, d$ such that $P_j^\top = P_j$ for all $j \neq 0$ and $P_0^\top = -P_0$. Let Ω be a bounded open subset of $\mathbb{R}^d$ with smooth boundary Γ and outward unit normal vector field η. We define the first order differential operator

$$\begin{aligned} \mathfrak{A}x &:= P_0 x + \sum_{j=1}^{d} P_j \frac{\partial x}{\partial \zeta_j}, \quad x \in \mathcal{D}(\mathfrak{A}), \\ \mathcal{D}(\mathfrak{A}) &:= \{x \in L^2(\Omega;\mathbb{R}^n) \mid \mathfrak{A}x \in L^2(\Omega;\mathbb{R}^n)\}. \end{aligned} \tag{4.19}$$

We also define the symmetric operator $Q_\eta := \sum_{j=1}^{d} \eta_j P_j : \Gamma \to \mathbb{R}^{n\times n}$.

Remark 4.3.6. Note that $\mathcal{D}(\mathfrak{A})$ in (4.19) is the maximal domain of definition of the operator $\mathfrak{A}$. This is further a Hilbert space when endowed with the graph norm, see [100].

Assumption 4.3.7.

(i) Γ is characteristic with constant multiplicity, that is, for all $\zeta \in \Gamma$ we have that

$$\dim\ker Q_\eta(\zeta) = n - 2r \Leftrightarrow \operatorname{rank} Q_\eta(\zeta) = 2r$$

where $n > 2r \in \mathbb{N}$ is constant.

(ii) The spectrum of $Q_\eta(\zeta)$, $\zeta \in \Gamma$, is symmetric with respect to the imaginary axis and the sign of its eigenvalues is independent of $\zeta \in \Gamma$, that is, there exist r positive eigenvalues.

Under Assumption 4.3.7, there exists a unitary operator $U \in \mathscr{L}(L^2(\Gamma;\mathbb{R}^n))$ and a diagonal matrix Λ such that $Q_\eta = U\Lambda U^*$ with $U^*U = I_{L^2(\Gamma;\mathbb{R}^n)}$ and

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & -\Lambda_1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

see [119]. Here $\Lambda_1 \in \mathscr{L}(L^2(\Gamma;\mathbb{R}^r))$ contains the positive eigenvalues of Q_η. Further we have the following decomposition

$$\Lambda = \begin{bmatrix} R_0^*\Sigma R_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.20}$$

where

$$R_0 := \frac{1}{\sqrt{2}} \begin{bmatrix} \Lambda_1 & -\Lambda_1 \\ I & I \end{bmatrix} \in \mathscr{L}(L^2(\Gamma;\mathbb{R}^{2r})), \quad \Sigma := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \in \mathscr{L}(L^2(\Gamma;\mathbb{R}^{2r})).$$

According to (4.20) we partition the unitary operator $U \in \mathscr{L}(L^2(\Gamma)^n)$ as follows

$$U^* = \begin{bmatrix} R^* \\ S^* \end{bmatrix} : L^2(\Gamma;\mathbb{R}^n) \to L^2(\Gamma;\mathbb{R}^{2r}) \times L^2(\Gamma;\mathbb{R}^{n-2r}).$$

Definition 4.3.8. Let $r \in \mathbb{N}$ be given as in Assumption 4.3.7 and $\mathcal{T}_0 : W^{1,2}(\Omega;\mathbb{R}^n) \to L^2(\Gamma;\mathbb{R}^n)$ be the trace operator of order zero, i.e., $\mathcal{T}_0 x = x|_\Gamma$ for $x \in W^{1,2}(\Omega;\mathbb{R}^n)$. Then the *boundary port-variables* associated with the differential operator $\mathfrak{A}$ are the operators $e_\partial, f_\partial \in \mathscr{L}(W^{1,2}(\Omega;\mathbb{R}^n), L^2(\Gamma;\mathbb{R}^r))$ defined by

$$\begin{bmatrix} f_\partial x \\ e_\partial x \end{bmatrix} := R_0 R^* \mathcal{T}_0 x, \quad x \in W^{1,2}(\Omega;\mathbb{R}^n).$$

We make the following assumption as in [119], which is a natural extension of the integration by parts formula for this systems. Recall that $W^{1/2,2}(\Gamma;\mathbb{R}^r)$ equals the range of trace operator on $W^{1,2}(\Omega;\mathbb{R}^r)$.

Assumption 4.3.9. Assume that the mapping

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} : W^{1,2}(\Omega;\mathbb{R}^n) \to L^2(\Gamma;\mathbb{R}^r) \times L^2(\Gamma;\mathbb{R}^r)$$

can be continuously extended to a linear mapping

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} : \mathcal{D}(\mathfrak{A}) \to W^{1/2,2}(\Gamma;\mathbb{R}^r) \times W^{-1/2,2}(\Gamma;\mathbb{R}^r).$$

Furthermore assume that Green's identity holds for all $x, z \in \mathcal{D}(\mathfrak{A})$, that is

$$\langle \mathfrak{A}x, z\rangle_{L^2} + \langle x, \mathfrak{A}z\rangle_{L^2} = \langle e_\partial x, f_\partial z\rangle_{W^{-1/2,2},W^{1/2,2}} + \langle e_\partial z, f_\partial x\rangle_{W^{-1/2,2},W^{1/2,2}}.$$

Definition 4.3.10. Let $\mathfrak{A}_0 := \mathfrak{A}$ with $\mathcal{D}(\mathfrak{A}_0) := \{x \in \mathcal{D}(\mathfrak{A}) \mid \exists b \in \mathbb{R}^r : e_\partial x = b\}$. To the operator $\mathfrak{A}_0$ we associate the (BCS) $\mathfrak{S} = (\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})$ with

$$\mathfrak{B}x = e_\partial x, \quad x \in \mathcal{D}(\mathfrak{A}_0)$$

and

$$\mathfrak{C}x = \int_\Gamma f_\partial x \,\mathrm{d}\sigma\,, \quad x \in \mathcal{D}(\mathfrak{A}_0).$$

For our purposes, we make the following assumption, which is for instance satisfied by the wave equation.

Remark 4.3.11. Note that if we restrict $\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}$ to $W^{1,2}(\Omega; \mathbb{R}^n)$, we obtain that

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} (W^{1,2}(\Omega; \mathbb{R}^n)) = W^{1/2,2}(\Gamma; \mathbb{R}^{2r}),$$

see [119, pp. 212]. Since $\mathbb{R}^{2r} \subset W^{1/2,2}(\Gamma; \mathbb{R}^{2r})$, the former implies that there exist $p, q : W^{1,2}(\Omega) \subset \mathcal{D}(\mathfrak{A}_0) \to \mathbb{R}^r$ such that

$$\begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix} \begin{bmatrix} q & p \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_r \end{bmatrix}.$$

Thus, by setting $P := p$ and $Q := |\Gamma|^{-1} q$ we have that

$$\begin{bmatrix} \mathfrak{B} \\ \mathfrak{C} \end{bmatrix} \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} I_r & 0 \\ 0 & I_r \end{bmatrix}.$$

Theorem 4.3.12. *Under Assumptions 4.3.7 & 4.3.9 and the notation of Definitions 4.3.8 & 4.3.10, it follows that*

$$\operatorname{Re} \langle \mathfrak{A}_0 x, x \rangle_{L^2} = \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x \rangle_{\mathbb{R}^r} \qquad \forall x \in \mathcal{D}(\mathfrak{A}_0)$$

and that the operator $\mathfrak{A}_0|_{\ker \mathfrak{B}}$ is skew-adjoint and generates a unitary C_0-semigroup.

Proof. This is [119, Theorem 8.18]. □

We show that the class belongs to that which is subject of Section 4.1, which consequences that the funnel controller is applicable.

Theorem 4.3.13. *Let $\mathfrak{S} = (\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})$ be as in Definition 4.3.10 and let Assumptions 4.3.7 & 4.3.9 be satisfied. Then $\mathfrak{S} = (\mathfrak{A}_0, \mathfrak{B}, \mathfrak{C})$ satisfies Assumption 4.1.2.*

Proof. The result follows immediately from Theorem 4.3.12 and Remark 4.3.11 together with the fact that $C_0^\infty(\Omega; \mathbb{R}^n) \subset \ker \begin{bmatrix} f_\partial \\ e_\partial \end{bmatrix}$ is a dense subspace. □

Example 4.3.14. Consider the 2-dimensional wave equation with boundary control in an open bounded domain Ω with smooth boundary Γ, namely,

$$\begin{aligned} \partial_{tt} w(t,\zeta) &= \Delta w(t,\zeta), \\ u(t) &= \left.\frac{\partial w(t,\zeta)}{\partial \eta}\right|_{\Gamma}, \\ y(t) &= \int_{\Gamma} \partial_t w(t,\zeta)|_{\Gamma} \,\mathrm{d}\sigma\,, \end{aligned} \tag{4.21}$$

and $w(0,\cdot) = a(\cdot) \in W^{2,2}(\Omega)$ with $\partial_\eta a(\cdot)|_\Gamma = 0$, $w_t(0,\cdot) = v(\cdot) \in W^{1,2}(\Omega)$. Then the funnel controller is applicable for (4.21) for every finite time-horizon.

Proof. The wave equation can be transformed into a port-Hamiltonian system of the form (4.19), c.f. [119, Example 8.12] with $P_0 = 0$ and

$$P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

and state variable

$$x = \begin{bmatrix} p \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} \partial_t w \\ \partial_{\zeta_1} w \\ \partial_{\zeta_2} w \end{bmatrix}.$$

Further

$$\begin{bmatrix} e_\partial x \\ f_\partial x \end{bmatrix} = \begin{bmatrix} \eta \cdot q|_\Gamma \\ p|_\Gamma \end{bmatrix},$$

where η is the normal unit vector. The domain of the operator $\mathfrak{A}_0$ is given by

$$\mathcal{D}(\mathfrak{A}_0) := \left\{ \begin{bmatrix} p \\ q_1 \\ q_2 \end{bmatrix} \in L^2(\Omega;\mathbb{R}^3) \mid p \in W^{1,2}(\Omega), q \in H_{\mathrm{div}}(\Omega), \exists b \in \mathbb{R} : \eta \cdot q|_\Gamma = b \right\},$$

where

$$H_{\mathrm{div}}(\Omega) := \{ x \in L^2(\Omega) \mid \nabla \cdot x \in L^2(\Omega) \}.$$

It is clear that $x_0 \in \mathcal{D}(\mathfrak{A}_0)$. From [36, Theorem 1.3] the range of f_∂ is precisely $W^{1/2,2}(\Gamma)$ and e_∂ from $H_{\mathrm{div}}(\Omega)$ is surjective onto $W^{-1/2,2}(\Gamma)$, see [36, Theorem 2.2] and [36, Corollary 2.4].
In this case P, Q are explicitly given by

$$(Pu)(\zeta) = \begin{bmatrix} 0 \\ \eta_1(\zeta) \\ \eta_2(\zeta) \end{bmatrix} u, \quad (Qy)(\zeta) = \frac{1}{|\Gamma|} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} y, \qquad u, y \in \mathbb{R}.$$

Further $C_0^\infty(\Omega) \subset \ker \mathfrak{B} \cap \ker \mathfrak{C}$ is dense. Hence, Theorem 4.3.13 gives the result. □

Wave equation in two spatial dimensions

Here we consider the situation described in Example 4.3.14, given by the system in polar coordinates on the unit disc

$$\begin{aligned}\partial_{tt}w(t,r,\theta) &= \partial_{rr}w(t,r,\theta) + r^{-1}\partial_r w(t,r,\theta) + r^{-2}\partial_{\theta\theta}w(t,r,\theta),\\ u(t) &= (\partial_r w(t,r,\theta))|_{r=1},\\ y(t) &= \int_0^{2\pi} \partial_t w(t,1,\theta)\,\mathrm{d}\theta\,,\end{aligned}$$

and use again a funnel boundary of the form $\varphi(t) = \varphi_0\varepsilon^{-2}\tanh(\omega t+\varepsilon)$ and a reference signal of the form $y_{\mathrm{ref}}(t) = A\tanh(\omega t) + B\sin(\omega t)$. The results are given in Fig. 4.3. Note that by setting the speed of propagation to 1, the units of t coincide with the ones of r.

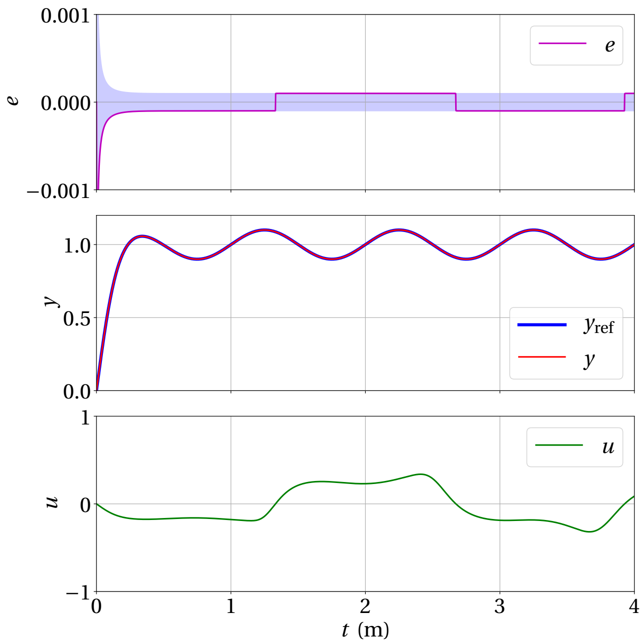

Figure 4.3: Performance funnel with the error, reference signal with the output of the closed-loop system and input of the closed-loop.

The parameter values are $r \in (a,b)$ with $a = 0$ m$, b = 1$ m, $\theta \in (0, 2\pi)$, $f = 1$ m^{-1}, $\omega = 2\pi f$, $\varepsilon = 10^{-2}$, $\varphi_0 = 1$. The amplitudes are $A = 1$ and $B = 0.1$. We define $T_0 = f^{-1}$ and $T = 4T_0$. The initial state of the system is

$$w(0,r,\theta) = 0 \text{ m}, \quad w_t(0,r,\theta) = 0,$$

which leads to a problem with radial symmetry, so the partial derivatives with respect to θ vanish and we use explicit finite differences in r with $M = 2000$ points and in $t \in [0, T]$ with N points, where

$$N = \left\lfloor \frac{b-a}{2T} \right\rfloor M.$$

4.3.3 A parabolic system

A particular case of the boundary controlled heat equation was already discussed in [101], with a slightly different funnel controller. Here we present a parabolic problem and refer to [44] for more details on second order elliptic operators.

Definition 4.3.15. Let $n \in \mathbb{N}$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary Γ and outward normal unit vector ν. Assume that $a \in C^\infty(\Omega; \mathbb{C}^{n\times n})$ is self-adjoint and satisfies the ellipticity condition

$$\exists \alpha > 0 : \forall v \in \mathbb{C}^n \quad \operatorname{Re} \sum_{i,j=1}^{n} a_{ij}(\zeta) v_i v_j^* \geq \alpha \|v\|^2_{\mathbb{C}^n}.$$

Let $\kappa \geq 0$ and consider the BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ defined by

$$\begin{aligned} \mathfrak{A}x &:= \nabla \cdot (a\nabla x) - \kappa x, \quad x \in \mathcal{D}(\mathfrak{A}), \\ \mathcal{D}(\mathfrak{A}) &:= \left\{ x \in W^{1,2}(\Omega) \mid \nabla \cdot a\nabla x \in L^2(\Omega) \text{ and } \exists b \in \mathbb{C} : \gamma_0(\nu \cdot a\nabla x) = b \right\} \\ \mathfrak{B}x &:= \gamma_0(\nu \cdot a\nabla x), \\ \mathfrak{C}x &:= \int_\Gamma \gamma_0 x \, \mathrm{d}\sigma \,, \end{aligned} \tag{4.22}$$

where $\gamma_0 : W^{1,2}(\Omega) \to W^{1/2,2}(\Gamma)$ denotes trace operator, $\gamma_0 x = x|_\Gamma$.

Remark 4.3.16. We have the following comments on the former definition.

1. The operator $\gamma_0 : W^{1,2}(\Omega) \to W^{1/2,2}(\Gamma)$ is onto;
2. it is well-known that the realization of $\mathfrak{A}$ in $\ker \mathfrak{B}$ with $\kappa = 0$ corresponds to the Neumann elliptic problem, e.g. [44, Theorem 2.2.2.5], and $\mathfrak{A}|_{\ker \mathfrak{B}}$ generates a contractive semigroup for $\kappa \geq 0$.
3. for $x \in \mathcal{D}(\mathfrak{A})$
$$\operatorname{Re} \langle \mathfrak{A}x, x \rangle_{L^2} \leq \operatorname{Re} \langle \mathfrak{B}x, \mathfrak{C}x \rangle_{\mathbb{C}} - \kappa \|x\|^2_{L^2}.$$

Lemma 4.3.17. *There are operators $P, Q : \mathcal{D}(\mathfrak{A}) \to \mathbb{C}$ such that*

$$\begin{bmatrix} \mathfrak{B} \\ \mathfrak{C} \end{bmatrix} \begin{bmatrix} P & Q \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In fact, $Q = |\Gamma|^{-1}$ is constant.

Proof. Let $\gamma_0 x := x|_\Gamma$ and $\gamma_\nu x := \gamma_0(\nu \cdot a\nabla x)$. From Lemma 1.4.4 the combined trace operator

$$\begin{bmatrix}\gamma_\nu x\\ \gamma_0 x\end{bmatrix} : W^{2,2}(\Omega) \to W^{1/2,2}(\Gamma) \times W^{3/2,2}(\Gamma)$$

is onto. Hence, there exist p_ν, q_0 such that

$$\begin{array}{rclrcl} \nu \cdot a\nabla p_\nu|_\Gamma &=& 1, & \nu \cdot a\nabla q_0|_\Gamma &=& 0,\\ p_\nu|_\Gamma &=& 0, & q_0|_\Gamma &=& 1. \end{array}$$

Note that $q_0 = 1$ is a solution. Considering p_ν, q_0 as operators from $\mathbb{C}$ to $W^{2,2}(\Omega) \subset \mathcal{D}(\mathfrak{A})$ yields that $P := p_\nu$ and $Q := |\Gamma|^{-1} q_0$ have the desired properties. □

Next we show that this class satisfies the preliminaries of Theorem 4.2.6.

Theorem 4.3.18. *For any BCS* $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ *as introduced in Definition 4.3.15 with, additionally,* $\kappa > 0$*, satisfies Assumption 4.1.2 with* $\alpha < 0$*.*

Proof. It follows immediately from the conditions and previous considerations, together with $C_0^\infty(\Omega) \subset \ker \mathfrak{B} \cap \ker \mathfrak{C}$ being a dense subspace and Theorem 4.2.6. □

Surprisingly, for the parabolic case the requirement $\kappa > 0$ can be relaxed to $\kappa \geq 0$. This has to do with the fact that the operator $\mathfrak{A}$ in this case satisfies the extra inequality

$$\langle \mathfrak{A}x, x\rangle_{L^2(\Omega)} \leq -\theta \|x\|^2_{L^2(\Omega)} + \nu(\mathfrak{C}x)^2 + \mathfrak{B}x \cdot \mathfrak{C}x,$$

so that the system is high-gain stabilizable and the state is in $L^\infty([0,\infty); L^2(\Omega))$. The essential difference is that here one can show that $u \in L^\infty([0,\infty); \mathbb{R})$ without showing that $\dot{x} \in L^\infty([0,\infty); X)$.
Before proving the general result, we need the following lemma. Similar results can be found in [106, Example IV.2.E, Example IV.2.F, Example V.5.A] and the case in which ϕ is defined in the whole space can be found in [44, Section 3.2.2].

Lemma 4.3.19. *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded domain with* C^2 *boundary* Γ *and normal outward unit vector* ν*. Let* $a \in C^\infty(\overline{\Omega}; \mathbb{R}^{n\times n})$ *and* $\mathfrak{C}$ *as in Definition 4.3.15 and* $\mathfrak{a} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ *be the symmetric, positive form*

$$\mathfrak{a}(x_1, x_2) := \int_\Omega \langle \nabla x_1, a\nabla x_2\rangle_{\mathbb{R}^n} \,\mathrm{d}\lambda\,.$$

Let $\Psi : X \to \mathbb{R}_\infty$ *be given by*

$$\Psi(x) := \frac{1}{2}\begin{cases}\mathfrak{a}(x,x) + \Psi_0(\mathfrak{C}x), & x \in \mathcal{D}(\Psi)\\ \infty, & \textit{otherwise,}\end{cases}$$

where

$$\Psi_0(e) := \log\left(\frac{1}{1-e^2}\right), \quad e \in (-1,1),$$

and

$$\mathcal{D}(\Psi) := \{x \in W^{1,2}(\Omega) \mid |\mathfrak{C}x| < 1\}$$

Then Ψ *defines a proper, convex and lower-semi-continuous functional with* $\partial\Psi = -\mathcal{A}$*, where*

$$\begin{aligned} \mathcal{A}(x) &:= \nabla \cdot a\nabla x, \quad x \in \mathcal{D}(\mathcal{A}) \\ \mathcal{D}(\mathcal{A}) &:= \{x \in W^{1,2}(\Omega) \mid \nabla \cdot a\nabla x \in L^2(\Omega), |\mathfrak{C}x| < 1, \mathfrak{B}x + \phi(\mathfrak{C}x) = 0\}. \end{aligned}$$

Further, for all $x_1 \in \mathcal{D}(\Psi)$*,* $x_2 \in W^{1,2}(\Omega)$ *it holds that*

$$\langle \partial\Psi(x_1), x_2\rangle = \mathfrak{a}(x_1,x_2) + k_0\frac{\mathfrak{C}x_1\mathfrak{C}x_2}{1-(\mathfrak{C}x_1)^2}.$$

Proof. Observe that $C_0^\infty(\Omega) \subset \mathcal{D}(\Psi)$, so that Ψ is proper. The convexity follows from the convexity of the respective functions as well as their lower-semi-continuity together with the supper-additivity of $\liminf$.

To compute the subgradient, note that the functional Ψ is G-differentiable, and from Proposition 1.8.12 the subdifferential is a singelton $\partial\Psi(x) = \{\Psi'(x)\}$ for $x \in \mathcal{D}(\Psi)$, where $\Psi'(x)$ is the G-derivative of Ψ at $x \in \mathcal{D}(\Psi)$. It is a straightforward exercise to compute it and for $x_1 \in \mathcal{D}(\Psi)$ and $x_2 \in W^{1,2}(\Omega)$ it holds

$$\begin{aligned} \langle \partial\Psi(x_1), x_2\rangle_{L^2} &= \int_\Omega \langle \nabla x_1, a\nabla x_2\rangle_{\mathbb{R}^n}\,\mathrm{d}\lambda + \frac{\mathfrak{C}x_1\mathfrak{C}x_2}{1-(\mathfrak{C}x_1)^2} \\ &= \int_\Omega (\nabla x_1)^\top a\nabla x_2\,\mathrm{d}\lambda + \int_\Gamma \frac{\mathfrak{C}x_1}{1-(\mathfrak{C}x_1)^2}x_2\,\mathrm{d}\sigma\,. \end{aligned}$$

For $x_2 \in C_0^\infty(\Omega)$ we have that

$$\partial\Psi(x_1) = -\nabla \cdot a\nabla x_1$$

in the distributional sense.

If $\partial\Psi(x_1) \in L^2(\Omega)$ and noting that

$$\frac{\mathfrak{C}x_1}{1-(\mathfrak{C}x_1)^2} \in \mathbb{R}$$

we have the abstract Green's Theorem — [106, Proposition II.5.3]—

$$\int_\Omega \langle \nabla x_1, a\nabla x_2\rangle_{\mathbb{R}^n}\,\mathrm{d}\lambda + \int_\Omega x_2\nabla \cdot a\nabla x_1\,\mathrm{d}\lambda = \langle \mathfrak{B}x_1, \mathfrak{C}x_2\rangle_{\mathbb{R}}\,,$$

where the right hand side generalizes the normal derivative at the boundary. Putting both expressions together we obtain the boundary condition

$$\mathfrak{B}x = -(\partial\Psi_0)(x) = -\frac{\mathfrak{C}x}{1-(\mathfrak{C}x)^2}.$$

Now we have that $\mathcal{D}(\partial\Psi) \subseteq \mathcal{D}(-\mathcal{A})$ and the converse is clear. □

Remark 4.3.20. If we use the controller described in (4.7), the functional Ψ_0 needs to be modified as

$$\Psi_0(e) := k_0 \log\left(\frac{1}{1-e^2}\right), \quad e \in (-1,1),$$

Theorem 4.3.21. *For any BCS* $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$ *as introduced in Definition 4.3.15 over the real numbers, with* $\kappa = 0$*, we have that the solution of the closed-loop system* (4.6a) *from Theorem 4.2.4 satisfies* $x \in L^\infty([0,\infty);L^2(\Omega))$ *and* $y, u \in L^\infty([0,\infty);\mathbb{R})$*.*

Proof. Theorem 4.3.18 guarantees the existence of a unique solution x. If we now perform the usual change of variables

$$x(t) = \varphi(t)^{-1}z(t) + Qy_{\text{ref}}(t) + P(u_0 + \psi(\varphi_0, e_0))p(t),$$

we obtain

$$\begin{aligned} \dot{z} &= \mathcal{A}(z) + \omega z + f, \\ z(0) &= z_0, \end{aligned} \tag{4.23}$$

where $\mathcal{A}$ is given in Lemma 4.3.19, ω, f, z_0 are given in (4.32) and $z_0 \in \mathcal{D}(\mathcal{A})$, see Lemma 4.4.7. Note that $\mathcal{A}$ coincides with (4.30). From Kato's Theorem 1.8.14 there exists a unique solution z of (4.23) with $z(0) = z_0$ such that $z(t) \in \mathcal{D}(\mathcal{A})$ for all $t \geq 0$ and $\dot{z}$ is continuous except possibly in a countable number of points.
By showing that $|\mathfrak{C}z(t)|$ remains uniformly bounded away from 1 in t, when we undo the change of variables, we will have that the error e remains uniformly bounded away from the funnel boundary. For that we will need that $z \in L^\infty([0,\infty);L^2(\Omega))$. Define

$$\hat{e}_n := \mathfrak{C}z_n = \int_\Gamma z_n \,\mathrm{d}\sigma\,.$$

Taking the scalar product of (4.23) with $z(t)$ we obtain the following energy balance

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 = -\mathfrak{a}(z,z) - \frac{\hat{e}^2}{1-\hat{e}^2} + \omega(t)\|z(t)\|^2 + \langle z(t), f(t)\rangle \tag{4.24}$$

where $\mathfrak{a}$ is given in Lemma 4.3.19.
Using the positivity of $\mathfrak{a}$ and the Poincaré-type inequality in Lemma 1.4.5 we have that there exist positive constants α, ν, θ such that

$$-\mathfrak{a}(z,z) \leq -\alpha\|\nabla z\|^2 \leq \nu\left(\int_\Gamma z\,\mathrm{d}\sigma\right)^2 - \theta\|z\|^2 = \nu\hat{e}^2 - \theta\|z\|^2 \leq \nu - \theta\|z\|^2,$$

since $\hat{e}^2 < 1$. Using that

$$\langle z(t), f(t)\rangle \leq \frac{\|f(t)\|^2}{2\theta} + \frac{\theta}{2}\|z(t)\|^2,$$

we can estimate the energy balance by

$$\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|^2 \leq -\theta\|z(t)\|^2 + 2\omega(t)\|z(t)\|^2 + 2\nu + \frac{\|f(t)\|^2}{\theta}. \tag{4.25}$$

We integrate (4.25) using Lemma 1.8.13 and obtain

$$\|z(t)\|^2 \leq \varphi_0^{-2}\|z_0\|^2 \mathrm{e}^{-\theta t}\varphi(t)^2 + \varphi(t)^2 C$$

where

$$C := 2\nu\|\varphi^{-2}\|_\infty + \frac{\|\varphi^{-1}f\|_\infty^2}{\theta}.$$

This implies that $z \in L^\infty([0,\infty); L^2(\Omega))$.
Next we derive a weak formulation type equation to show that $|\hat{e}|$ is uniformly bounded away from 1. Taking the scalar product of the differential equation with $\dot{z}(t)$ leads to

$$\|\dot{z}(t)\|^2 - \langle \mathcal{A}(z(t)), \dot{z}(t)\rangle = \omega(t)\langle z(t), \dot{z}(t)\rangle + \langle f(t), \dot{z}(t)\rangle.$$

By Lemma 4.3.19 we have that $\partial\Psi = -\mathcal{A}$, so that the latter becomes

$$\|\dot{z}(t)\|^2 - \langle \partial\Psi(z(t)), \dot{z}(t)\rangle = \omega(t)\langle z(t), \dot{z}(t)\rangle + \langle f(t), \dot{z}(t)\rangle.$$

Using now Lemma 1.8.11 leads to

$$\|\dot{z}(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\Psi(z(t)) = \omega(t)\langle z(t), \dot{z}(t)\rangle + \langle f(t), \dot{z}(t)\rangle.$$

We can now make use of

$$\langle f(t), z(t)\rangle \leq \|f(t)\|^2 + \frac{1}{4}\|\dot{z}(t)\|^2$$

and

$$\omega(t)\langle z(t), \dot{z}(t)\rangle \leq \|\omega\|_\infty^2\|z(t)\|^2 + \frac{1}{4}\|\dot{z}(t)\|^2$$

to obtain

$$\|\dot{z}(t)\|^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\Psi(z(t)) = K, \tag{4.26}$$

where

$$K = 2(\|f\|_\infty^2 + \|\omega\|_\infty^2\|z\|_\infty^2).$$

Adding and subtracting

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2$$

in (4.26) we obtain

$$\|\dot{z}\|^2 + 2\frac{\mathrm{d}}{\mathrm{d}t}\Psi(z) \leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + K.$$

Since $\|\dot{z}\|^2 \geq 0$ the former implies

$$2\frac{\mathrm{d}}{\mathrm{d}t}\Psi(z) \leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + K - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2.$$

Using now (4.24) and the definition of Ψ yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\mathfrak{a}(z,z) - \log(1-\hat{e}^2)) &\leq \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + K - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 \\ &\leq -\mathfrak{a}(z,z) - \frac{\hat{e}^2}{1-\hat{e}^2} + \omega\|z\|^2 + \langle f, z\rangle + K - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 \\ &\leq -\mathfrak{a}(z,z) - \frac{1}{1-\hat{e}^2} - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + D,\end{aligned}$$

where we have used that

$$\frac{\hat{e}^2}{1-\hat{e}^2} = \frac{1}{1-\hat{e}^2} - 1$$

and

$$D \coloneqq 1 + K + \|\omega\|_\infty\|z\|_\infty^2 + \|f\|_\infty\|z\|_\infty.$$

If we define

$$\rho \coloneqq \mathfrak{a}(z,z) - \log(1-\hat{e}^2)$$

then we have

$$\dot{\rho} \leq -\rho - \left(\frac{1}{1-\hat{e}^2} - \log\left(\frac{1}{1-\hat{e}^2}\right)\right) - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + D \leq -\rho - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z\|^2 + D,$$

since

$$\frac{1}{1-\zeta^2} \geq \log\left(\frac{1}{1-\zeta^2}\right)$$

for all $\zeta \in (-1,1)$. Using the integrating factor e^t we have

$$\rho(t) \leq \rho(0)\mathrm{e}^{-t} + E \leq \rho(0) + E$$

where $E > 0$ is independent of t and $\rho(0) = \mathfrak{a}(z_0, z_0) - \log(1-\hat{e}_0^2)$. Hence, ρ is uniformly bounded in t, which implies that there exists $F > 0$ such that

$$\frac{1}{1-\hat{e}(t)^2} \leq F, \quad \forall t \geq 0.$$

This shows that $\hat{e}$ is uniformly bounded away from 1. Inverting the change concludes the proof. □

Heat equation

Here we consider the following boundary controlled 2D heat equation on the unit disc given by

$$\begin{aligned}\partial_t x(t,r,\theta) &= \alpha(\partial_{rr}x(t,r,\theta) + r^{-1}\partial_r x(t,r,\theta) + r^{-2}\partial_{\theta\theta}x(t,r,\theta)),\\ u(t) &= \alpha(\partial_r x(t,r,\theta))|_{r=1},\\ y(t) &= \int_0^{2\pi} x(t,1,\theta)\,\mathrm{d}\theta\,,\end{aligned}$$

where $\alpha > 0$ is the thermal diffusivity.
In this case, making use of Theorem 4.3.18, we choose a funnel boundary of the form $\varphi(t) = \varphi_0\varepsilon^{-2}\tanh(\omega t + \varepsilon)$. The reference signal is given by $y_{\mathrm{ref}}(t) = A\sin(\omega t)$ and the simulated system is shown in Fig. 4.4. In Fig. 4.5 we show the evolution of the plate at four different times.

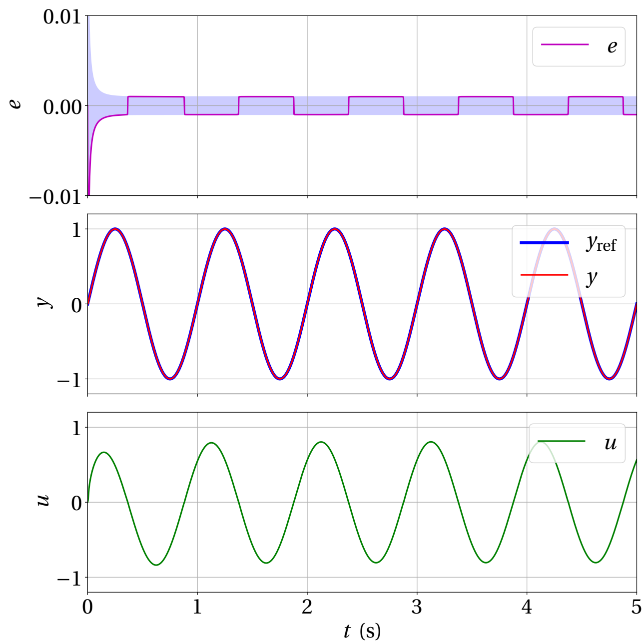

Figure 4.4: Performance funnel with the error, reference signal with the output of the closed-loop system and input of the closed-loop.

The parameter values are $\alpha = 1\ \mathrm{m}^2\mathrm{s}^{-1}$, $r \in (r_0, r_1)$, with $r_0 = 0$ m and $r_1 = 1$ m, and $\theta \in (0, 2\pi)$. The amplitude values are $A = 1$ J, $\varphi_0 = 0.1$ J and $\varepsilon = 10^{-1}$. We have set $T_0 = 1$ s, $\omega = 2\pi T_0^{-1}$ and $T = 5T_0$. We have used explicit finite differences with a

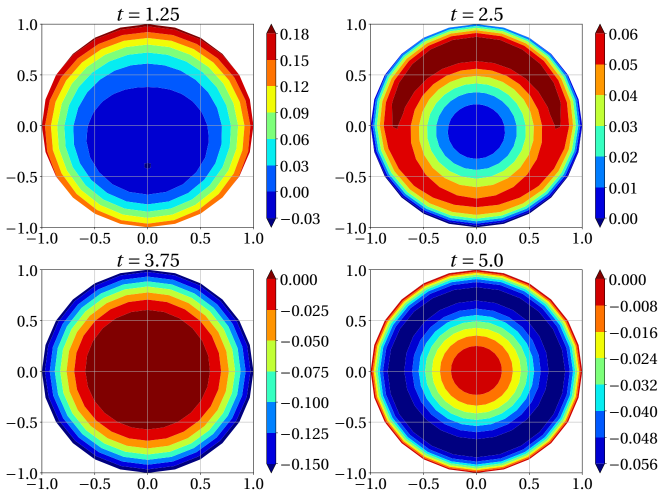

Figure 4.5: From left to right, top to bottom, the temperature of the plate for different increasing times.

partition in r and θ of $N = 25$ points for each variable and in $t \in [0, T]$ of

$$M = \left\lfloor 10T \left(\frac{N^2}{(r_1 - r_0)^2} + \frac{N}{r_1 - r_0} + \frac{N^2}{4\pi^2} \right) \right\rfloor$$

points. The initial state of the system is

$$x(0, r, \theta) = x_0 (r_1 - r)^2 \sin(\theta),$$

where $x_0 = 0.5 \text{ Jm}^{-2}$. We apply the controller (4.7) from Remark 4.2.7 with $k_0 = 1 \text{ s}^{-2}$. Note that Theorem 4.3.21 implies that the output $u \in L^\infty([0, \infty); \mathbb{R})$.

4.4 Proof of the Main Theorems

We develop some auxiliary results to conclude with the proof of the main results. A part of following Lemma has been shown in [24] under the additional assumption of well-posedness, cf. Remark 4.1.3e).

Lemma 4.4.1. *Assume that $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ satisfies Assumption 4.1.2 with $\alpha \in \mathbb{R}$. For all $\beta > \alpha$, $u \in \mathbb{C}^m$ and $f \in X$ there exist unique $x \in \mathcal{D}(\mathfrak{A})$ and $y \in \mathbb{C}^m$ with*

$$\begin{aligned} (\beta I - \mathfrak{A}) x &= f, \\ u &= \mathfrak{B} x, \\ y &= \mathfrak{C} x. \end{aligned} \tag{4.27}$$

Furthermore, there exist bounded operators $H(\beta) \in \mathscr{L}(X)$, $J(\beta) \in \mathscr{L}(\mathbb{C}^m, X)$, $F(\beta) \in \mathscr{L}(X, \mathbb{C}^m)$ and $G(\beta) \in \mathscr{L}(\mathbb{C}^m) = \mathbb{C}^{m\times m}$ which connect the solution of (4.27) *via*

$$\begin{aligned} x &= H(\beta)f + J(\beta)u, \\ y &= F(\beta)f + G(\beta)u. \end{aligned} \tag{4.28}$$

Thereby, the matrix $G(\beta) + G(\beta)^*$ is positive definite, and $G(\beta)$ is invertible with positive definite $G(\beta)^{-1} + (G(\beta)^*)^{-1}$.

Proof. Step 1: We show uniqueness of the solution of (4.27). To this end, we have to show that the choice $f = 0$ and $u = 0$ leads to $x = 0$ and $y = 0$. Assuming that $x \in D(\mathfrak{A})$, $y \in \mathbb{C}^m$ fulfills (4.27) with $f = 0$ and $u = 0$, we obtain from (4.2) in Assumption 4.1.2(i) that

$$\beta\|x\|_X^2 = \operatorname{Re}\langle \mathfrak{A}x, x\rangle_X \le \operatorname{Re}\langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^m} + \alpha\|x\|_X^2,$$

and thus $(\beta - \alpha)\|x\|^2 \le 0$. Invoking $\beta > \alpha$, we obtain $x = 0$ and, consequently, $y = \mathfrak{C}x = 0$.
Step 2: We show the existence of bounded operators $H(\beta)$, $J(\beta)$, $F(\beta)$ and $G(\beta)$ such that the solutions of (4.27) fulfill (4.28): By Remark 4.1.3b), Assumption 4.1.2(i)&(ii) imply that $\beta I - \mathfrak{A}|_{\ker\mathfrak{B}}$ is bijective. Further, invoking Remark 4.1.3c), Assumption 4.1.2 (iii) leads to the existence of $P, Q \in \mathscr{L}(\mathbb{C}^m, \mathcal{D}(\mathfrak{A}))$, such that (4.4) holds. Considering

$$\begin{aligned} x &= \underbrace{(\beta I - \mathfrak{A}|_{\ker\mathfrak{B}})^{-1}}_{=:H(\beta)} f + \underbrace{((\beta I - \mathfrak{A}|_{\ker\mathfrak{B}})^{-1}(\mathfrak{A}P - \beta P) + P)}_{=:J(\beta)} u, \\ y &= \underbrace{\mathfrak{C}(\beta I - \mathfrak{A}|_{\ker\mathfrak{B}})^{-1}}_{=:F(\beta)} f + \underbrace{\mathfrak{C}(\beta I - \mathfrak{A}|_{\ker\mathfrak{B}})^{-1}(\mathfrak{A}P - \beta P)}_{=:G(\beta)} u, \end{aligned}$$

a straightforward calculation shows that (4.27) holds. Further, the operators $H(\beta)$, $J(\beta)$, $F(\beta)$ and $G(\beta)$ are bounded as they are compositions of bounded operators.
Step 3: We show that $G(\beta) + G(\beta)^*$ is positive definite, and $G(\beta)$ is invertible with positive definite $G(\beta)^{-1} + (G(\beta)^*)^{-1}$: Considering (4.27) with $f = 0$ and taking the real part of inner product in X, we obtain

$$\operatorname{Re}\beta\|x\|^2 = \operatorname{Re}\langle \mathfrak{A}x, x\rangle \le \operatorname{Re}\langle u, y\rangle_{\mathbb{C}^m} + \alpha\|x\|^2 = \left\langle u, \frac{1}{2}(G(\beta) + G(\beta)^*)u\right\rangle_{\mathbb{C}^m} + \alpha\|x\|^2,$$

whence

$$(\beta - \alpha)\|x\|^2 \le \left\langle u, \frac{1}{2}(G(\beta) + G(\beta)^*)u\right\rangle_{\mathbb{C}^m},$$

so that $G(\beta) + G(\beta)^*$ is positive semidefinite. If for $u \in \mathbb{C}^m$ holds

$$\left\langle u, \frac{1}{2}(G(\beta) + G(\beta)^*)u\right\rangle_{\mathbb{C}^m} = 0,$$

then $(\beta - \alpha)\|x\|_X^2 \leq 0$ which implies $x = 0$ and thus $u = \mathfrak{B}x = 0$. This implies the positive definiteness of $G(\beta) + G(\beta)^*$, and we can immediately conclude that $G(\beta)$ is invertible with positive definite $G(\beta)^{-1} + (G(\beta)^*)^{-1}$. □

Example 4.4.2. Here we show how to compute the transfer function of the two systems presented in Example 4.1.4. We could bring the lossy transmition line in port-Hamiltonian form as in (4.18), but the original form of the system is more convenient for our purposes. Let $\beta \geq 0$. The firs step is to solve

$$\begin{aligned} V_\zeta(\zeta) &= -L\beta I(\zeta) - RI(\zeta), \\ I_\zeta(\zeta) &= -C\beta V(\zeta) - GV(\zeta), \\ V(a) &= u_1, \\ V(b) &= u_2. \end{aligned}$$

One can easily compute the exponential matrix of the system and obtain

$$\begin{pmatrix} V(\zeta) \\ I(\zeta) \end{pmatrix} = \begin{bmatrix} \cosh(\gamma(\zeta - a)) & -\tau^{-1}\sinh(\gamma(\zeta - a)) \\ -\tau\sinh(\gamma(\zeta - a)) & \cosh(\gamma(\zeta - a)) \end{bmatrix} \begin{pmatrix} V(a) \\ I(a) \end{pmatrix},$$

where $\gamma := \sqrt{(L\beta + R)(C\beta + G)}$ and $\tau := \sqrt{(C\beta + G)/(L\beta + R)}$. The condition $V(a) = u_1$ gives

$$\begin{aligned} V(\zeta) &= \cosh(\gamma(\zeta - a))u_1 - \tau^{-1}\sinh(\gamma(\zeta - a))I(a), \\ I(\zeta) &= \cosh(\gamma(\zeta - a))I(a) - \tau\sinh(\gamma(\zeta - a))u_1. \end{aligned}$$

If we use $V(b) = u_2$, we can find $I(a)$ so that

$$I(a) = \frac{\tau\cosh(\gamma(b - a))}{\sinh(\gamma(b - a))}u_1 - \frac{\tau}{\sinh(\gamma(b - a))}u_2.$$

Now,

$$\begin{aligned} y_1 &= I(a), \\ y_2 &= -I(b), \end{aligned}$$

which leads to

$$y_1 = \tau\coth(\gamma(b - a))u_1 - \frac{\tau}{\sinh(\gamma(b - a))}u_2,$$

and

$$y_2 = \tau\coth(\gamma(b - a))u_2 - \frac{\tau}{\sinh(\gamma(b - a))}u_1.$$

Hence

$$G(\beta) = \tau(\beta)\begin{bmatrix} \coth(\gamma(\beta)L) & -\dfrac{1}{\sinh(\gamma(\beta)L)} \\ -\dfrac{1}{\sinh(\gamma(\beta)L)} & \coth(\gamma(\beta)L) \end{bmatrix},$$

where $L := b - a$.
For the boundary controlled heat equation, we consider the problem in $1D$ and $2D$. Let $k^2 := \beta > 0$ be given. For the $1D$ case we need to solve

$$\begin{aligned} k^2 x(\zeta) &= x_{\zeta\zeta}(\zeta), \\ x_\zeta(1) &= u, \\ x_\zeta(0) &= -u, \\ y &= x(1) + x(0). \end{aligned}$$

By simply integrating the equation we have

$$x(\zeta) = A\cosh(k\zeta) + B\sinh(k\zeta)$$

Using $x_\zeta(1) = u$ and $x_\zeta(0) = -u$ leads to

$$\begin{aligned} A &= \frac{1+\cosh(k)}{\sinh(k)}\frac{u}{k}, \\ B &= -\frac{u}{k}. \end{aligned}$$

Computing now y yields

$$y = A(\cosh(k)+1) + B\sinh(k) = \frac{2\coth(k/2)}{k}u,$$

so that

$$G(\beta) = \frac{2}{\sqrt{\beta}}\coth\left(\frac{\sqrt{\beta}}{2}\right).$$

For the $2D$ case, we need to solve

$$\begin{aligned} k^2 x(r,\theta) &= x_{rr}(r,\theta) + r^{-1}x_r(r,\theta) + r^{-2}x_{\theta\theta}(r,\theta), \\ u &= x_r(1,\theta), \\ y &= \int_0^{2\pi} x(1,\theta)\,\mathrm{d}\theta\,. \end{aligned}$$

It seems that things are more involved as in the $1D$ case. However, since we are prescribing a constant u for all $\theta \in [0, 2\pi)$ by using a separation of variables ansatz, namely $x(r,\theta) = R(r)\Theta(\theta)$, we obtain that $\Theta(\theta) = c \in \mathbb{R} \setminus \{0\}$. Without loss of generality we can set $c \equiv 1$. Thus, we have to solve

$$\begin{aligned} k^2 R(r) &= R_{rr}(r) + r^{-1}R_r(r), \\ u &= R_r(1), \\ y &= 2\pi R(1). \end{aligned}$$

We complete the system by assuming that $R_r(0) = 0$. With that, the solution can be given by using Bessel functions of first kind

$$R(r) = \frac{iJ_0(ikr)}{J_1(ik)}u.$$

Hence,

$$y = 2\pi\frac{iJ_0(ik)}{J_1(ik)}u.$$

By using the definition of the Bessel functions J_0, J_1 we obtain that

$$G(\beta) = 2\pi\frac{2}{\sqrt{\beta}}\frac{\sum_{l=0}^{\infty}\frac{1}{(l!)^2}\left(\frac{\sqrt{\beta}}{2}\right)^{2l}}{\sum_{l=0}^{\infty}\frac{1}{(l!)^2(l+1)}\left(\frac{\sqrt{\beta}}{2}\right)^{2l}}.$$

Note that both transfer functions for the parabolic PDE resemble one another in spite of the spatial dimension.

Proposition 4.4.3. *Let $\phi : \mathcal{D}(\phi) \subset \mathbb{C}^m \to \mathbb{C}^m$ be defined by*

$$\begin{aligned} \phi(y) &:= \frac{1}{1-\|y\|^2}y, \\ \mathcal{D}(\phi) &:= \{y \in \mathbb{C}^m \mid \|y\| < 1\}. \end{aligned} \tag{4.29}$$

Then $-\phi$ is m-dissipative.

Proof. Step 1: We prove that $-\phi$ is dissipative. We first like to note that the function $g : [0,1) \to \mathbb{R}$ with $r \mapsto \frac{r}{1-r^2}$ is monotonically increasing on $[0,1)$, which follows by nonnegativity of its derivative. As a consequence $(g(a) - g(b))(a - b) \geq 0$ for all $a, b \in [0,1)$. Using this, we obtain that for $w, y \in \mathcal{D}(\phi)$ holds

$$\begin{aligned} \operatorname{Re}\langle\phi(w) - \phi(y), w - y\rangle &= \operatorname{Re}\langle\phi(w), w\rangle + \operatorname{Re}\langle\phi(y), y\rangle - \operatorname{Re}\langle\phi(y), w\rangle - \operatorname{Re}\langle\phi(w), y\rangle \\ &= \left(\frac{\|w\|^2}{1-\|w\|^2} + \frac{\|y\|^2}{1-\|y\|^2} - \frac{\operatorname{Re}\langle w, y\rangle}{1-\|y\|^2} - \frac{\operatorname{Re}\langle y, w\rangle}{1-\|w\|^2}\right) \\ &\geq \left(\frac{\|w\|^2}{1-\|w\|^2} + \frac{\|y\|^2}{1-\|y\|^2} - \frac{\|w\|\|y\|}{1-\|y\|^2} - \frac{\|y\|\|w\|}{1-\|w\|^2}\right) \\ &= \left(\frac{\|w\|}{1-\|w\|^2} - \frac{\|y\|}{1-\|y\|^2}\right)(\|w\| - \|y\|) \\ &= (g(\|w\|) - g(\|y\|)) \cdot (\|w\| - \|y\|) \geq 0. \end{aligned}$$

Step 2: We show that $\lambda I + \phi(\cdot)$ is surjective for all $\lambda > 0$. Consider $f \in \mathbb{C}^m$ and $\lambda > 0$. Since $\lambda I + \phi(\cdot)$ maps zero to zero, it suffices to prove that any $f \neq 0$ is in the range of

$\lambda I + \phi(\cdot)$. To this end, consider the real polynomial p with

$$p(\rho) = \lambda\rho^3 - \|f\|\rho^2 - (\lambda+1)\rho + \|f\|.$$

We observe that $p(0) = \|f\| > 0$ and $p(1) = -1 < 0$, whence there exists some $\tilde{\rho} \in (0,1)$ with $p(\tilde{\rho}) = 0$. Now choosing $y = \frac{\tilde{\rho}}{\|f\|} f$, we obtain by simple arithmetics that

$$\begin{aligned} \lambda y + \phi(y) &= \frac{f}{(1-\tilde{\rho}^2)\|f\|} \cdot \left(-\lambda\tilde{\rho}^3 + (\lambda+1)\tilde{\rho}\right) \\ &\overset{p(\tilde{\rho})=0}{=} \frac{f}{(1-\tilde{\rho}^2)\|f\|} \cdot \left(-\|f\| \cdot \tilde{\rho}^2 + \|f\|\right) = f, \end{aligned}$$

which shows that $\lambda I + \phi(\cdot)$ is surjective. □

We also record a similar result but only regarding the dissipativity of the function. Since the result is used for both the real and complex valued cases, we use $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

Lemma 4.4.4. *Let $\varphi \in [0,\infty)$ and $a \in \mathbb{K}^m$ be given. Define*

$$\psi(y) = -\frac{1}{1-\varphi^2\|y-a\|^2}(y-a), \quad y \in \mathcal{D}(\psi),$$

where

$$\mathcal{D}(\psi) := \left\{ \ y \in \mathbb{K}^m \ \middle| \ \varphi^2\|y-a\|^2 < 1 \ \right\}.$$

Then, for all $y_1, y_2 \in \mathcal{D}(\psi)$ we have that

$$\operatorname{Re}\langle \psi(y_1) - \psi(y_2), y_1 - y_2\rangle \geq 0.$$

Proof. The case $\varphi = 0$ is clear, so let $\varphi > 0$. We first like to note that the function $g : [0,\varphi^{-1}) \to \mathbb{R}$, $r \mapsto \frac{r}{1-\varphi^2 r^2}$ is strictly monotonically increasing, which follows from positivity of its derivative. As a consequence $(g(a)-g(b))(a-b) \geq 0$ for all $a, b \in [0,\varphi^{-1})$. Using this, define $e_i := y_i - a$ for $i = 1, 2$ so that it holds

$$\begin{aligned} &\operatorname{Re}\langle \psi(y_1) - \psi(y_2), y_1 - y_2\rangle \\ &\quad = \operatorname{Re}\langle \psi(y_1), e_1\rangle + \operatorname{Re}\langle \psi(y_2), e_2\rangle - \operatorname{Re}\langle \psi(y_1), e_2\rangle - \operatorname{Re}\langle \psi(y_2), e_1\rangle \\ &\quad = \left(\frac{\|e_1\|^2}{1-\varphi^2\|e_1\|^2} + \frac{\|e_2\|^2}{1-\varphi^2\|e_2\|^2} - \frac{\operatorname{Re}\langle e_1, e_2\rangle}{1-\varphi^2\|e_1\|^2} - \frac{\operatorname{Re}\langle e_2, e_1\rangle}{1-\varphi^2\|e_2\|^2}\right) \\ &\quad \geq \left(\frac{\|e_1\|^2}{1-\varphi^2\|e_1\|^2} + \frac{\|e_2\|^2}{1-\varphi^2\|e_2\|^2} - \frac{\|e_1\|\|e_2\|}{1-\varphi^2\|e_1\|^2} - \frac{\|e_1\|\|e_2\|}{1-\varphi^2\|e_2\|^2}\right) \\ &\quad = \left(\frac{\|e_1\|}{1-\varphi^2\|e_1\|^2} - \frac{\|e_2\|}{1-\varphi^2\|e_2\|^2}\right)(\|e_1\| - \|e_2\|) \\ &\quad = (g(\|e_1\|) - g(\|e_2\|)) \cdot (\|e_1\| - \|e_2\|) \geq 0. \end{aligned}$$

□

The next result is a modification of [5, Theorem 4.3] in which the function ϕ is defined on the whole space $\mathbb{C}^m$ instead of a domain $\mathcal{D}(\phi)$ as in our situation.

Lemma 4.4.5. *Let $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be a BCS and let Assumption 4.1.2 be satisfied with $\alpha \in \mathbb{R}$. Let $\phi : \mathcal{D}(\phi) \subset \mathbb{C}^m \to \mathbb{C}^m$ be given by (4.29). Then the nonlinear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ with*

$$\begin{aligned} \mathcal{A}(z) &:= (\mathfrak{A} - \alpha I)|_{\mathcal{D}(\mathcal{A})} z, \\ \mathcal{D}(\mathcal{A}) &:= \{z \in \mathcal{D}(\mathfrak{A}) \mid \|\mathfrak{C}z\| < 1, \mathfrak{B}z + \phi(\mathfrak{C}z) = 0\} \end{aligned} \tag{4.30}$$

is m-dissipative and $\overline{\mathcal{D}(\mathcal{A})} = X$.

Proof. Step 1: Note that $\mathcal{D}(\mathcal{A})$ is not empty, since $0 \in \mathcal{D}(\mathcal{A})$ with $\mathcal{A}(0) = 0$. We show that $\mathcal{A}$ is a densely defined: For given $(v, e) \in \mathbb{C}^m \times \mathcal{D}(\phi)$ with $v = -\phi(e)$ we can find $z_0 \in D(\mathcal{A})$ such that

$$\begin{pmatrix} \mathfrak{B}z_0 \\ \mathfrak{C}z_0 \end{pmatrix} = \begin{pmatrix} v \\ e \end{pmatrix},$$

e.g., by setting $z_0 = Pv + Qe$, where P, Q are chosen as in Remark 4.1.3c). It follows that $z_0 + \ker \mathfrak{B} \cap \ker \mathfrak{C} \subset \mathcal{D}(\mathcal{A})$ is a dense subset of X by Assumption 4.1.2.
Step 2: For given $\lambda > 0$, we show that $\lambda I - \mathcal{A}$ is surjective:
Let $f \in X$. Our aim is to find some $z \in \mathcal{D}(\mathcal{A})$ with $(\lambda I - \mathcal{A})(z) = f$, that is,

$$\begin{aligned} ((\lambda + \alpha)I - \mathfrak{A})z &= f \\ \mathfrak{B}z &= -\phi(\mathfrak{C}z). \end{aligned} \tag{4.31}$$

Set $\beta := \lambda + \alpha > \alpha$ and consider the operators $H(\lambda) \in \mathscr{L}(X)$, $J(\lambda) \in \mathscr{L}(\mathbb{C}^m, X)$, $F(\lambda) \in \mathscr{L}(X, \mathbb{C}^m)$ and $G(\lambda) \in \mathscr{L}(\mathbb{C}^m) = \mathbb{C}^{m \times m}$ from Lemma 4.4.1. Since the matrix $G(\beta)^{-1} + (G(\beta)^*)^{-1}$ is positive definite by Lemma 4.4.1, there exists some $\delta > 0$ such that $G(\beta)^{-1} + (G(\beta)^*)^{-1} - 2\delta I$ is positive definite. The function $-\phi$ is m-dissipative by Proposition 4.4.3, whence

$$\Psi(\cdot) := -\phi(\cdot) - G(\beta)^{-1} + \delta I$$

is dissipative. Then Lemma 1.8.7 gives rise to m-dissipativity of Ψ. In particular, $\Psi(\cdot) - \delta I = -\phi(\cdot) - G(\beta)^{-1} : D(\phi) \to \mathbb{C}^m$ is bijective, whence there exists some $e \in D(\phi)$ with

$$\Psi(e) - \delta e = G(\beta)^{-1} F(\beta) f,$$

which is equivalent to

$$-\phi(e) = G(\beta)^{-1} e - G(\beta)^{-1} F(\beta) f,$$

and thus

$$e = F(\beta) f + G(\beta)(-\phi(e)).$$

Then Lemma 4.4.1 implies that $z = H(\beta)f + J(\beta)(-\phi(e))$ indeed fulfills (4.31).
Step 3: We show that $\mathcal{A}$ is dissipative: Let $z_1, z_2 \in \mathcal{D}(\mathcal{A})$, then

$$\begin{aligned}\operatorname{Re}\langle \mathcal{A}(z_1) - \mathcal{A}(z_2), \quad & z_1 - z_2\rangle_X \\ &\overset{\text{Assumption}}{\underset{4.1.2(i)}{=}} \operatorname{Re}\left\langle (\mathfrak{A} - \alpha I)|_{\mathcal{D}(\mathcal{A})} z_1 - (\mathfrak{A} - \alpha I)|_{\mathcal{D}(\mathcal{A})} z_2, z_1 - z_2 \right\rangle_X \\ &\leq - \operatorname{Re}\langle \phi(\mathfrak{C}z_1) - \phi(\mathfrak{C}z_2), \mathfrak{C}z_1 - \mathfrak{C}z_2\rangle_{\mathbb{C}^n} \\ &\overset{\text{Proposition}}{\underset{4.4.3}{\leq}} 0.\end{aligned}$$

□

Remark 4.4.6. Note that the statement concerning the density of $\mathcal{D}(\mathcal{A})$ is not trivial. In the linear case, it is always so, but in the nonlinear setting it does not need to hold, see Proposition 1.8.6.

An intrinsic technical problem when investigating solvability of (4.6) is that the feedback is varying in time, i.e. it depends on t explicitly. To circumvent this problem, we perform a change of variables leading to an evolution equation with a constant operator. This is subject of the subsequent auxiliary result.

Lemma 4.4.7. *Let a BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ be given which satisfies Assumption 4.1.2 and assume that the initial value x_0 and the functions y_{ref}, p, φ fulfill Assumption 4.2.2. Then for $\varphi_0 = \varphi(0)$, $e_0 = \mathfrak{C}x_0 - y_{\text{ref}}(0)$, $u_0 = \mathfrak{B}x_0$, operators $P, Q \in \mathscr{L}(\mathbb{C}^m, \mathcal{D}(\mathfrak{A}))$ with (4.4), and the nonlinear m-dissipative operator $\mathcal{A}$ given in (4.30) and*

$$\begin{aligned}\omega &= \frac{\dot\varphi}{\varphi}, \\ f &= \varphi \cdot \Big(\mathfrak{A}Qy_{\text{ref}} - Q\dot y_{\text{ref}} + \mathfrak{A}P(u_0 + \psi(\varphi_0, e_0))p - P(u_0 + \psi(\varphi_0, e_0))\dot p(t)\Big) \\ z_0 &= \varphi_0 \cdot \Big(x_0 - Qy_{\text{ref}}(0) - P(u_0 + \psi(\varphi_0, e_0))\Big).\end{aligned} \tag{4.32}$$

holds $\omega \in W^{1,\infty}([0,\infty); \mathbb{R})$, $f \in W^{1,\infty}([0,\infty); X)$ and $z_0 \in D(\mathcal{A})$. Furthermore, the following holds for $T > 0$:

a) If $x \in W^{1,\infty}([0,T]; X)$ and for almost every $t \in [0,T]$ holds that $\dot x$ is continuous at t, $x(t) \in D(\mathfrak{A})$ for all $t \in [0,T]$ and (4.6), then for

$$z(t) = \varphi(t)\Big(x(t) - Qy_{\text{ref}}(t) - P(\mathfrak{B}x_0 + \psi(\varphi_0, e_0))p(t)\Big), \tag{4.33}$$

holds $z \in W^{1,\infty}([0,T]; X)$ and for almost every $t \in [0,T]$ holds that $\dot z$ is continuous at t, $z(t) \in D(\mathcal{A})$ for all $t \in [0,T]$ and

$$\begin{aligned}\dot z(t) &= \mathcal{A}(z(t)) + (\omega(t) + \alpha)z(t) + f(t), \\ z(0) &= z_0\end{aligned} \tag{4.34}$$

b) *Conversely, if* $z \in W^{1,\infty}([0,T];X)$ *and for almost every* $t \in [0,T]$ *holds that* $\dot{z}$ *is continuous at* t*,* $z(t) \in D(\mathcal{A})$ *for all* $t \in [0,T]$ *and* (4.34)*, then for*

$$x(t) = \varphi(t)^{-1}z(t) + Qy_{\mathrm{ref}}(t) + P(\mathfrak{B}x_0 + \psi(\varphi_0, e_0))p(t), \tag{4.35}$$

$x \in W^{1,\infty}([0,T];X)$ *and for almost every* $t \in [0,T]$ *holds that* $\dot{x}$ *is continuous at* t*,* $x(t) \in D(\mathfrak{A})$ *for all* $t \in [0,T]$ *and* (4.6)*.*

c) *If* $z \in W^{1,\infty}([0,\infty);X)$*, then* x *as in* (4.35) *fulfills* $x \in W^{1,\infty}([0,\infty);X)$*.*

Proof. The statements $\omega \in W^{1,\infty}([0,\infty);\mathbb{R})$, $f \in W^{1,\infty}([0,\infty);X)$ follow from the product rule for weak derivatives [3, p. 124]. Since P maps to $D(\mathfrak{A})$, we have $z_0 \in D(\mathfrak{A})$. Further, by using $\mathfrak{B}P = I$, $\mathfrak{B}Q = 0$, $\mathfrak{C}P = 0$ and $\mathfrak{C}Q = I$, we obtain

$$\phi(\mathfrak{C}z_0) = \frac{\varphi_0 \cdot e_0}{1 - \varphi_0^2\|e_0\|^2} = -\mathfrak{B}z_0,$$

whence $z_0 \in D(\mathcal{A})$.
To prove statement a), assume that $x \in W^{1,\infty}([0,T];X)$ has a derivative which is continuous almost everywhere and in the domain of $\mathfrak{A}$ for all $t \in [0,T]$. First note that the twice weak differentiability of p and φ together with the fact that P and Q map to $D(\mathfrak{A})$ implies that $z \in W^{1,\infty}([0,T];X)$ with $z(t)$ being in $D(\mathfrak{A})$ for all $t \in [0,T]$. By further using that (4.6) holds for all $t \in [0,T]$, we obtain —analogously to the above computations for z_0— that

$$\phi(\mathfrak{C}z(t)) = \frac{\varphi(t)e(t)}{1 - \varphi(t)^2\|e(t)\|^2} = -\mathfrak{B}z(t),$$

which implies that $z(t) \in D(\mathcal{A})$ for all $t \in [0,T]$. Further, a straightforward calculation shows that (4.6) implies that $z(t)$ fulfills (4.34).
Statement b) follows by an argumentation straightforward to that in the proof of a). Statement c) is a simple consequence of $\inf_{t\geq 0}\varphi(t) > 0$, $\varphi, p \in W^{2,\infty}([0,\infty);\mathbb{R})$, $y_{\mathrm{ref}} \in W^{2,\infty}([0,\infty);\mathbb{C}^m)$ and the product rule for weak derivatives. □

The previous lemma is indeed the key step to prove Theorems 4.2.4 & 4.2.6 on the feasibility of the funnel controller. By using the state transformation (4.33) with inversion (4.35), the analysis of feasibility of the funnel controller reduces to the proof of existence of a solution to the nonlinear evolution equation (4.34) in which the time-dependence is now extracted to the inhomogeneity.

Proof of Theorem 4.2.4. Let $T > 0$, and consider the nonlinear operator $\mathcal{A}$ as in (4.30) and $\omega \in W^{1,\infty}([0,\infty);\mathbb{R})$, $f \in W^{1,\infty}([0,\infty);X)$ and $z_0 \in D(\mathcal{A})$ as in (4.32). Then Theorem 1.8.14 implies that the nonlinear evolution equation (4.34) has a unique solution $z \in W^{1,\infty}([0,T];X)$ in the sense that for all $t \in [0,T]$ holds $z(t) \in D(\mathcal{A})$, $\dot{z}$ is continuous at except at a countable number of points in $[0,T]$, and (4.34). Then

Lemma 4.4.7b) yields that $x \in W^{1,\infty}([0,T];X)$ as in (4.35) has the desired properties. It remains to show uniqueness: Assume that $x_i \in W^{1,\infty}([0,T];X)$ are solutions of the closed-loop system (4.6) for $i = 1,2$. Then

$$z_i(t) = \varphi(t)\Big(x_i(t) - Qy_{\mathrm{ref}}(t) - P(\mathfrak{B}x_0 + \psi(\varphi_0, e_0))p(t)\Big), \tag{4.36}$$

fulfills $\dot{z}_i(t) = \mathcal{A}(z_i(t)) + (\omega(t)+\alpha)z(t) + f(t)$ with $z_i(0) = z_0$, and the uniqueness statement in Theorem 1.8.14 gives $z_1 = z_2$. Now resolving (4.36) for x_i and invoking $z_1 = z_2$ gives $x_1 = x_2$. □

It remains to prove Theorem 4.2.6 which states that the global solution and its derivative are bounded in case of negativity of the constant α in Assumption 4.1.2 (i).

Proof of Theorem 4.2.6. Let $\alpha < 0$, let $\mathcal{A}$ be the nonlinear operator in (4.30) and $\omega \in W^{1,\infty}([0,\infty);\mathbb{R})$, $f \in W^{1,\infty}([0,\infty);X)$ and $z_0 \in D(\mathcal{A})$ as in (4.32).
Step 1: We show that the solution $z \in W^{1,\infty}_{\mathrm{loc}}([0,\infty);X)$ of (4.34) (which exists by Theorem 1.8.14) is bounded:
Then we obtain that for almost all $t \geq 0$ holds

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t)\|_X &= \mathrm{Re}\,\langle z(t), \dot{z}(t)\rangle_X \\
&= \mathrm{Re}\,\langle z(t), \mathcal{A}z(t) + (\omega(t)+\alpha)z(t) + f(t)\rangle_X \\
&\leq \mathrm{Re}\,\langle z(t), \mathcal{A}z(t)\rangle_X + (\omega(t)+\alpha)\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X \\
&\overset{(4.30)}{=} \mathrm{Re}\,\langle z(t), \mathfrak{A}z(t)\rangle_X + \omega(t)\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X \\
&\overset{(4.2)}{\leq} \mathrm{Re}\,\langle \mathfrak{B}z(t), \mathfrak{C}z(t)\rangle_{\mathbb{C}^n} + \alpha\|z(t)\|_X^2 + \omega(t)\|z(t)\|_X^2\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X \\
&\overset{(4.30)}{=} -\frac{\|\mathfrak{C}z(t)\|^2}{1-\|\mathfrak{C}z(t)\|^2} + \alpha\|z(t)\|_X^2 + \omega(t)\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X \\
&\leq \alpha\|z(t)\|_X^2 + \omega(t)\|z(t)\|_X^2 + \|z(t)\|_X\|f(t)\|_X.
\end{aligned}$$

Now applying Lemma 1.8.13 with $\rho = 1/2$, using that the definition of ω in (4.32) leads to $\omega = \frac{\mathrm{d}}{\mathrm{d}t}\log(\varphi)$, we obtain that for almost all $t \geq 0$ holds

$$\|z(t)\|_X \leq \varphi_0^{-1}\|z_0\|_X\varphi(t)\mathrm{e}^{\alpha t} + \varphi(t)\mathrm{e}^{\alpha t}\int_0^t \varphi(s)^{-1}\mathrm{e}^{-\alpha s}\|f(s)\|_X\,\mathrm{d}s\,.$$

The definition of f in (4.32) leads to the existence of $c_0, c_1 > 0$ such that for almost all $t \geq 0$ holds $\|f(t)\|_X \leq \varphi(t)(c_0 + c_1\|y_{\mathrm{ref}}\|_{W^{1,\infty}})$. Thus,

$$\|z(t)\|_X \leq \varphi_0^{-1}\|z_0\|_X\varphi(t)\mathrm{e}^{\alpha t} - \alpha^{-1}\varphi(t)(c_0 + c_1\|y_{\mathrm{ref}}\|_{W^{1,\infty}})(1-\mathrm{e}^{\alpha t}),$$

whence $z \in L^\infty([0,\infty);X)$.
Step 2: We show that $\dot{z} \in L^\infty([0,\infty);X)$:

To this end, let $h > 0$ and, by using the dissipativity of $\mathcal{A}$, consider

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z(t+h)-z(t)\|_X^2 \le \alpha\|z(t+h)-z(t)\|_X^2 + \omega(t+h)\|z(t+h)-z(t)\|_X^2 \\ + |\omega(t+h)-\omega(t)|\|z(t)\|_X\|z(t+h)-z(t)\|_X \\ + \|f(t+h)-f(t)\|_X\|z(t+h)-z(t)\|_X,\end{aligned}$$

Again applying the Grönwall type inequality from Lemma 1.8.13 with $\rho = 1/2$, dividing by h and letting $h \to 0$ yields

$$\begin{aligned}\|\dot{z}(t)\|_X &\le \varphi_0^{-1}\|\dot{z}(0)\|_X\varphi(t)\mathrm{e}^{\alpha t} + \varphi(t)\mathrm{e}^{\alpha t}\int_0^t \mathrm{e}^{-\alpha s}\varphi(s)^{-1}(\|z\|_{L^\infty}|\dot{\omega}(s)| + \|\dot{f}(s)\|_X)\,\mathrm{d}s \\ &\le \varphi_0^{-1}\|\mathcal{A}(z_0)+\omega(0)z_0+f(0)\|_X\|\varphi\|_{L^\infty} \\ &\quad + \|\varphi\|_{L^\infty}\left\|\varphi^{-1}\right\|_{L^\infty}(\|z\|_{L^\infty}\|\dot{\omega}\|_{L^\infty} + d_0 + d_1\|y_{\mathrm{ref}}\|_{W^{2,\infty}})\end{aligned}$$

for some $d_0, d_1 > 0$. Hence, $\dot{z}(t) \in L^\infty([0,\infty);X)$.
Step 3: We conclude that the solution x in (4.6) (which exists by Theorem 4.2.4) fulfills $x \in W^{1,\infty}([0,\infty);X)$:
We know from the first two steps that $z \in W^{1,\infty}([0,\infty);X)$. Then Lemma 4.4.7c) leads to $x \in W^{1,\infty}([0,\infty);X)$.
Step 4: We finally show that $u = \mathfrak{B}x$ fulfills $u \in L^\infty([0,\infty);\mathbb{C}^m)$:
We know from the third step, we know that $x \in W^{1,\infty}([0,\infty);X)$. Since we have $x(t) \in D(\mathfrak{A})$ with $\dot{x}(t) = \mathfrak{A}x(t)$ for almost all $t \ge 0$, we can conclude that $\mathfrak{A}x \in L^\infty([0,\infty);X)$, and thus $x \in L^\infty([0,\infty);D(\mathfrak{A}))$. Then $\mathfrak{B} \in \mathscr{L}(D(\mathfrak{A}),\mathbb{C}^m)$ gives $u = \mathfrak{B}x \in L^\infty([0,\infty);\mathbb{C}^m)$. □

4.5 Outlook

In this chapter we have shown that the funnel controller (4.5) is feasible for BCS of the form $(\mathfrak{A},\mathfrak{B},\mathfrak{C})$. Even though the mathematical tools used happened to be non-standard, the resulting theory is very reach as it has been shown by applying our main results, Theorems 4.2.4 & 4.2.6, to different system classes, whose state variable is described by both hyperbolic and parabolic partial differential equations. Moreover, Assumption 4.1.2 provides a recipe to check whether one can apply the funnel controller to a particular system.
It is worth mentioning that the methodology used here differs from the other results regarding the funnel controller, since from the beginning on we have considered the closed-loop system, while in the literature, the open-loop problem is first considered.
The results of this chapter have been mostly gathered in [99]. Nevertheless, some things have been added. For didactic purposes we have included Example 4.4.2. We have also given enough conditions for which one can apply the funnel controller with an

infinitely open funnel boundary at $t = 0$ in Section 4.2.1. Moreover, we have included Section 4.2.2 to show that if the semigroup generated by $\mathfrak{A}|_{\ker \mathfrak{B}}$ is analytic, the input and output are Hölder continuous. Following the intuition from the finite-dimensional case, in particular the notion of *high-gain stabilizability*, and the fact that in [101] it has been shown that a funnel controller can be feasible for the boundary controller heat equation, we have proved Theorem 4.3.21. This requires of special techniques which tie the notion of nonlinear m-dissipative operators and subdifferentials together. In order to build bridges between the finite-dimensional scenario and the infinite-dimensional case at hand, we have also included the following part, which attempts to explain the intuition behind the applicability of the funnel controller.

4.5.1 The zero dynamics

In the existing literature about the funnel controller, both finite and infinite-dimensional, the concept of *zero dynamics* plays an important role. Roughly speaking, the zero dynamics consists of all pairs of state and input trajectories (x, u) for which the output vanishes, that is, $y = 0$.

Here we elucidate by means of two examples what the zero dynamics in this infinite-dimensional setting is. As we will see in the examples, the systems we have considered are *flat*, this means that there is an equation which relates the input and output without having any derivatives involved. This corresponds in the finite-dimensional literature to the case of *relative degree* 0. This concept is defined by using the transfer function, which in the finite-dimensional case is a rational function or a matrix whose entries are rational functions. As seen in Example 4.4.2, the transfer function of infinite-dimensional systems is no longer a rational function. In fact, for the port-Hamiltonian example of the lossy transmission line, the relative degree is 0 and for the heat equation, the relative degree is 1/2.

First, consider $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ to be a port-Hamiltonian system with $N = 1$, that is,

$$\mathfrak{A}x = P_1 \frac{\partial}{\partial \zeta} \mathcal{H}x + P_0 \mathcal{H}x, \quad x \in \mathcal{D}(\mathfrak{A}),$$

and

$$\mathfrak{B}x := W_B \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix},$$

$$\mathfrak{C}x := W_C \begin{pmatrix} f_{\partial,\mathcal{H}x} \\ e_{\partial,\mathcal{H}x} \end{pmatrix},$$

as in Definition 4.3.1, so that W_B, W_C have full row rank and $\left[\begin{smallmatrix} W_B \\ W_C \end{smallmatrix} \right]$ is invertible. We further assume that $P_1 \mathcal{H}(\zeta)$ can be written as

$$P_1 \mathcal{H}(\zeta) = S^{-1}(\zeta) \Delta(\zeta) S(\zeta),$$

where S, Δ are continuously differentiable matrix-valued functions and Δ is diagonal. It follows from [67, Theorem 13.2.2] that $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is an L^2 well-posed linear system, that is,

$$\begin{aligned} x(t) &= \mathbb{T}_t^B x_0 + \Phi_t^B u, \\ y &= \Psi_\infty^B x_0 + \mathbb{F}_\infty^B u, \end{aligned} \tag{Σ_B}$$

where $\mathbb{T}^B := (\mathbb{T}_t^B)_{t\geq 0}$ is the C_0-semigroup generated by $\mathfrak{A}|_{\ker \mathfrak{B}}$. The properties of the other operators are given in Section 1.7. Note that this system satisfies

$$\operatorname{Re} \langle \mathfrak{A}x, x\rangle_X \leq \langle \mathfrak{B}x, \mathfrak{C}x\rangle_{\mathbb{C}^m}, \quad x \in \mathcal{D}(\mathfrak{A}),$$

so that u and y are interchangeable, because $\mathfrak{A}|_{\ker \mathfrak{C}}$ generates also a C_0-semigroup —combine [5, Proposition 3.2.15] and [67, Theorem 7.2.4]— and [67, Theorem 13.2.2] holds as well. This means that the system $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ is *invertible* as a well-posed linear system and

$$\begin{aligned} x(t) &= \mathbb{T}_t^C x_0 + \Phi_t^C y, \\ u &= \Psi_\infty^C x_0 + \mathbb{F}_\infty^C y, \end{aligned} \tag{Σ_C}$$

where $\mathbb{T}^C := (\mathbb{T}_t^C)_{t\geq 0}$ is the C_0-semigroup generated by $\mathfrak{A}|_{\ker \mathfrak{C}}$. The operators from (Σ_B) and (Σ_C) are related by certain transformations, see [112, Section 5],

$$\begin{aligned} \mathbb{T}_t^C x_0 &= [\mathbb{T}_t^B - \Phi_t^B (\mathbb{F}_\infty^B)^{-1} \Psi_\infty^B] x_0, \\ \Phi_t^C y &= \Psi_t^B (\mathbb{F}_\infty^B)^{-1} y, \\ \Psi_\infty^C x_0 &= -(\mathbb{F}_\infty^B)^{-1} \Psi_\infty^B x_0, \\ \mathbb{F}_\infty^C y &= (\mathbb{F}_\infty^B)^{-1} y. \end{aligned}$$

Moreover, the transfer functions of (Σ_B) and (Σ_C) are inverse of one another, $G_B(\beta) = G_C(\beta)^{-1}$, [112, Corollary 5.2]. Hence, the zero dynamics which is given by

$$\begin{aligned} \dot{x}(t) &= \mathfrak{A}x(t), \\ u &= \mathfrak{B}x(t), \\ 0 &= \mathfrak{C}x(t), \end{aligned}$$

can be obtained by simply setting $y = 0$ in (Σ_C),

$$\begin{aligned} x(t) &= \mathbb{T}_t^C x_0, \\ u &= \Psi_\infty^C x_0. \end{aligned}$$

From an heuristic viewpoint, if we want the zero dynamics to be asymptotically stable, that is, (x, u) go to zero (in some sense) as $t \to \infty$, then the norm semigroup $\mathbb{T}^C$ needs to decay as t grows after some transient period. Since the port-Hamiltonian systems are described by hyperbolic PDEs, the spectrum of $\mathfrak{A}|_{\ker \mathfrak{C}}$, $\sigma(\mathfrak{A}|_{\ker \mathfrak{C}})$, is contained

in an imaginary line with real part α, which depends on P_0. For the zero dynamics to be asymptotically stable it is necessary that $\alpha < 0$, so that the semigroup $\mathbb{T}^C$ is exponentially stable. From the dissipativity equation, this implies that $\mathbb{T}^B$ is also exponentially stable.
For the second example, let us consider the BCS $(\mathfrak{A}, \mathfrak{B}, \mathfrak{C})$ described in Definition 4.3.15 with $a = I_n$ and $\kappa = 0$. This system is also well-posed, see for instance [22, Corollary 1]. Nevertheless, it is not invertible, so that the characterization of (x, u) such that the following holds

$$\begin{aligned} \dot{x}(t) &= \Delta x(t), \\ u(t) &= \gamma_0(\nu \cdot a \nabla x(t)), \\ 0 &= \int_\Gamma \gamma_0 x(t) \, \mathrm{d}\sigma \, , \end{aligned}$$

requires some extra work. However, by using the form a, it can be inferred that

$$\langle \Delta x, x \rangle \leq -\theta \|x\|^2 + \nu y^2 + uy. \tag{4.37}$$

Thus, by considering $\Delta|_{\ker \mathfrak{B} \cap \ker \mathfrak{C}}$ as in [102, Section 5] it can be shown that the zero dynamics are asymptotically stable.
As we have shown in Theorems 4.3.4 & 4.3.21, the funnel controller is feasible in the sense that $u \in L^\infty(0, \infty; \mathbb{R}^m)$, and in both cases the zero dynamics were stable. Hence, this notion plays also an underlying role in the feasibility of the funnel controller in infinite-dimension.

4.5.2 Open questions

In the lines of the former comments regarding the zero dynamics, even though it seems to provide a good intuition regarding the class of systems for which the funnel controller may be feasible, it is still unclear how to exploit this stability in general with the methodology presented in this chapter. In fact, the characterization of operators $\mathfrak{A}$ satisfying an inequality of the form (4.37) together with the generalized impedance passivity inequality (4.2) seems to be not known. Perhaps it is not possible by using only the inequalities, since the amount of information lost may be very relevant, as in the example with the Laplace operator, where one uses the form a. Moreover, conditions resembling the ones given in Theorem 4.3.21 so that we can show $u \in L^\infty([0, \infty); \mathbb{R}^m)$ without needing to prove that $\dot{x} \in L^\infty([0, \infty); X)$ are also unknown at the present time. However, it does seem to be a strong connection between this fact and (4.37), as seen by using the subdifferential presented in Lemma 4.3.19.
One of the drawbacks of working with BCS is that we do not distinguish between hyperbolic and parabolic PDEs. In particular, parabolic problems enjoy of smoothing properties of the solution which we have not fully exploited. For instance, following

the lines of [101], one could modify the FC 4.5 with a funnel boundary which infinitely open for a small period of time so that the solution becomes smooth. This would allow for arbitrary initialization of the system, that is, $x_0 \in X$ instead of $x_0 \in \mathcal{D}(\mathfrak{A})$. Hence, the factor associated to the function p could be dropped. This will be in fact illustrated in the next chapter.

It seems to be also possible to consider infinite-dimensional inputs and outputs, by slightly modifying the proofs of the results given in Section 4.4. This can be for instance achieved by assuming that $\mathfrak{B} : \mathcal{D}(\mathfrak{A}) \subset X \to U$ and $\mathfrak{C} : \mathcal{D}(\mathfrak{A}) \subset X \to Y$ with U, Y infinite-dimensional reflexive spaces and $U' = Y$, so that one can form the duality $\langle u, y \rangle_{U \times Y}$. However, for some of the systems considered as examples, the joint operator $\left[\begin{smallmatrix} \mathfrak{B} \\ \mathfrak{C} \end{smallmatrix}\right]$ may not be surjective when considering the infinite-dimensional versions of $\mathfrak{B}$ and $\mathfrak{C}$, for instance when considering the Maxwell's Equations in the form of Section 4.3.2 and $\mathfrak{B}x = e_\partial x$ and $\mathfrak{C}x = f_\partial x$, see for instance [36, 114]. Hence, the natural question arises, whether this can be somehow circumvented.

5 Adaptive control for a nonlinear parabolic problem

We study output trajectory tracking for a class of nonlinear reaction diffusion equations such that a prescribed performance of the tracking error is achieved. To this end, we utilize the method of funnel control.
The reaction diffusion equation that we consider in the present chapter is known as the monodomain model and represents defibrillation processes of the human heart [117]. The monodomain equations are a reasonable simplification of the well accepted bidomain equations, which arise in cardiac electrophysiology [113]. In the monodomain model the dynamics are governed by a parabolic reaction diffusion equation which is coupled with a linear ordinary differential equation that models the ionic current.
The present chapter is organized as follows. In Section 5.2 we introduce the mathematical model that will be considered and the general framework where we will work, in Section 5.3 we define the controller present the funnel controller that will be used to achieve the control objective and give our main result, Theorem 5.3.3. After we move on to Section 5.4 where we provide the necessary tools for the proof of the main theorem and in Section 5.6 we give the respective proof.

5.1 Neumann elliptic operators

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Denote the scalar product in $L^2(\Omega;\mathbb{R}^d)$ by $\langle\cdot,\cdot\rangle$ and the norm in $L^2(\Omega)$ by $\|\cdot\|$. Further, let $D \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be symmetric-valued and satisfying the *ellipticity condition*

$$\text{for a.e.}\, \zeta \in \Omega \ \forall\, \xi \in \mathbb{R}^d : \ \xi^\top D(\zeta)\xi = \sum_{i,j=1}^{d} D_{ij}(\zeta)\xi_i\xi_j \geq \delta\|\xi\|^2_{\mathbb{R}^d} \tag{5.1}$$

for some $\delta > 0$. Consider the sesquilinear form $\mathfrak{a} : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R}$ with

$$\mathfrak{a}(z_1, z_2) = \langle \nabla z_1, D\nabla z_2\rangle\,. \tag{5.2}$$

We can associate a linear operator to the above sequilinear form.

Proposition 5.1.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary and let $D \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be symmetric-valued and satisfying the ellipticity condition* (5.1)*. Then there exists exactly one operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega)$ with*

$$\mathcal{D}(\mathcal{A}) = \left\{ \ z_2 \in W^{1,2}(\Omega) \ \middle| \ \exists y_2 \in L^2(\Omega) \ : \ \mathfrak{a}(z_1, z_2) = -\langle z_1, y_2 \rangle \ \forall x \in W^{1,2}(\Omega) \ \right\},$$

and

$$\mathfrak{a}(z_1, z_2) = -\langle z_1, \mathcal{A}z_2 \rangle \quad \forall z_1 \in W^{1,2}(\Omega), z_2 \in \mathcal{D}(\mathcal{A}).$$

We call $\mathcal{A}$ the Neumann elliptic operator on Ω associated to D*. The operator $\mathcal{A}$ is closed, self-adjoint, and $\mathcal{D}(\mathcal{A})$ is dense in $W^{1,2}(\Omega)$.*

Proof. Existence, uniqueness and closedness of $\mathcal{A}$ as well as the density of $\mathcal{D}(\mathcal{A})$ in $W^{1,2}(\Omega)$ follow from Kato's First Representation Theorem [70, Section VI.2, Theorem 2.1], whereas self-adjointness is an immediate consequence of the property

$$\mathfrak{a}(z_1, z_2) = \mathfrak{a}(z_2, z_1)$$

for all $z_1, z_2 \in W^{1,2}(\Omega)$. □

Note that above operator does not require any further smoothness of $\partial\Omega$. In particular, the classical definition of the Neumann boundary trace, i.e., the derivative of a function in the direction of the outward normal unit vector $\nu : \partial\Omega \to \mathbb{R}^d$ does not need to exist. If however $\partial\Omega$ and the coefficient matrix D are sufficiently smooth, we have

$$\mathcal{A}z = \mathrm{div} D\nabla z, \quad z \in \mathcal{D}(\mathcal{A}) := \left\{ \ z \in W^{2,2}(\Omega) \ \middle| \ (\nu^\top \cdot D\nabla z)|_\Gamma = 0 \ \right\},$$

see [44, Theorem 2.2.2.5]. This justifies that we call $\mathcal{A}$ a *Neumann elliptic operator.*

5.2 The FitzHugh-Nagumo model

Let $d \leq 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary. We consider a model for the interaction of the electric current in a cell, namely

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} v(t) &= \mathcal{A}v(t) + p_3(v(t)) - u(t) + I_{s,i}(t) + \mathcal{B}I_{s,e}(t), && v(0) = v_0 \\ \frac{\mathrm{d}}{\mathrm{d}t} u(t) &= c_5 v(t) - c_4 u(t), && u(0) = u_0 \\ y(t) &= \mathcal{B}' v(t), \end{aligned} \tag{5.3}$$

where

$$p_3(v) := -c_1 v + c_2 v^2 - c_3 v^3,$$

with constants $c_i > 0$ for $i = 1, \ldots, 5$, initial values $v_0, u_0 \in L^2(\Omega)$, the Neumann elliptic operator $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq L^2(\Omega) \to L^2(\Omega)$ associated to $D \in L^\infty(\Omega)$ and control

operator $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, (W^{1,2}(\Omega))')$. The output operator $\mathcal{B}'$, i.e., the adjoint of $\mathcal{B}$, is consequently in $\mathscr{L}(W^{1,2}(\Omega), \mathbb{R}^m)$.
System (5.3) is known as the FitzHugh-Nagumo model for the ionic current [33], where

$$I_{ion}(u, v) = p_3(v) - u.$$

The functions $I_{s,i} \in L^2_{\text{loc}}([0,T); L^2(\Omega))$, $I_{s,i} \in L^2_{\text{loc}}([0,T); \mathbb{R}^m)$ are the intracellular and extracellular stimulation currents, respectively. In particular, $I_{s,i}$ is the control of the system, whereas y is the output.
Next we introduce the solution concept.

Definition 5.2.1. Assume that $d \leq 3$ and $\Omega \subset \mathbb{R}^d$ is a bounded Lipschitz domain, let $D \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be symmetric-valued and satisfying the ellipticity condition (5.1). Let $\mathcal{A}$ be a Neumann elliptic operator on Ω associated to D (see Proposition 5.1.1), let $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, (W^{1,2}(\Omega))')$, and let $u_0, v_0 \in L^2(\Omega)$ be given. Further, let $T \in (0, \infty]$ and $I_{s,i} \in L^2_{\text{loc}}([0,T); L^2(\Omega))$, $I_{s,e} \in L^2_{\text{loc}}([0,T); \mathbb{R}^m)$. A triple of functions (u, v, y) is called *solution* of (5.3) in $[0,T)$, if

(i) $v \in L^2([0,T); W^{1,2}(\Omega)) \cap C([0,T); L^2(\Omega)))$ with $v(0) = v_0$;

(ii) $u \in C([0,T); L^2(\Omega))$ with $u(0) = u_0$;

(iii) for all $\chi \in L^2(\Omega)$, $\theta \in W^{1,2}(\Omega)$, the scalar functions $t \mapsto \langle u(t), \chi\rangle$, $t \mapsto \langle v(t), \theta\rangle$ are weakly differentiable on $[0,T]$, and it holds that, for almost all $t \in (0,T)$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle v(t), \theta\rangle &= -\mathfrak{a}(v(t), \theta) + \langle p_3(v(t)) - u(t) + I_{s,i}(t), \theta\rangle + \langle I_{s,e}(t), \mathcal{B}'\theta\rangle_{\mathbb{R}^m},\\ \frac{\mathrm{d}}{\mathrm{d}t}\langle u(t), \chi\rangle &= \langle c_5 v(t) - c_4 u(t), \chi\rangle,\\ y(t) &= \mathcal{B}' v(t),\end{aligned} \tag{5.4}$$

where $\mathfrak{a} : W^{1,2}(\Omega) \times W^{1,2}(\Omega) \to \mathbb{R}$ is the sesquilinear form (5.2).

Remark 5.2.2.

a) Weak differentiability of $t \mapsto \langle u(t), \chi\rangle$, $t \mapsto \langle v(t), \theta\rangle$ for all $\chi \in L^2(\Omega)$, $\theta \in W^{1,2}(\Omega)$ on $(0,T)$ further leads to

$$v \in W^{1,2}([0,T]; (W^{1,2}(\Omega))') \text{ and } u \in W^{1,2}([0,T]; L^2(\Omega)).$$

b) The Sobolev Embedding Theorem [2, Theorem 5.4] implies that the inclusion map $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ is bounded. This guarantees that $p_3(v) \in L^2([0,T]; L^2(\Omega))$, whence the first equation in (5.4) is well-defined.

c) Let $w \in L^2(\Omega)$. An input operator of the form $\mathcal{B}u = u \cdot w$ corresponds to distributed input, and we have $\mathcal{B} \in \mathscr{L}(\mathbb{R}, L^2(\Omega))$. In this case, the output is given by

$$y(t) = \int_\Omega w(\xi) \cdot (v(t))(\xi) d\xi.$$

A typical situation is that z is an indicator function on a subset of Ω; such choices have been considered in [81] for instance.

d) Let $w \in L^2(\partial\Omega)$. An input operator with

$$\mathcal{B}' z = \int_\Gamma w(\xi) \cdot z(\xi) d\sigma \tag{5.5}$$

corresponds to a Neumann boundary control

$$\nu(\xi)^\top \cdot (\nabla v(t))(\xi) = w(\xi) \cdot I_{s,e}(t).$$

In this case, the output is given by a weighted integral of the Dirichlet boundary values. More precisely

$$y(t) = \int_\Gamma w(\xi) \cdot (v(t))(\xi) d\sigma.$$

Note that $\mathcal{B}'$ is the composition of the trace operator

$$\mathrm{tr}: \quad z \mapsto z|_{\partial\Omega}$$

with taking the inner product in $L^2(\partial\Omega)$ with w. The trace operator fulfills for all $\varepsilon > 0$ that $\mathrm{tr} \in \mathscr{L}(W^{1/2+\varepsilon,2}(\Omega), L^2(\partial\Omega))$ by the Trace Theorem [123, Theorem 1.39]. In particular, $\mathrm{tr} \in \mathscr{L}(W^{1,2}(\Omega), L^2(\partial\Omega))$, thus $\mathcal{B}' \in \mathscr{L}(W^{1,2}(\Omega), \mathbb{R})$ and $\mathcal{B} \in \mathscr{L}(\mathbb{R}, (W^{1,2}(\Omega))')$.

e) The case $T = \infty$ is to be understood in the following sense. The triple (u, v, y) is a solution of (5.3) in $[0, \infty)$ if for all $T > 0$ the restrictions $(u|_{[0,T)}, v|_{[0,T)}, y|_{[0,T)})$ are solutions of (5.3).

5.3 Funnel control

The objective is that the output y of the system (5.3) tracks a given reference signal which is $y_{\mathrm{ref}} \in W^{1,\infty}([0, \infty); \mathbb{R}^m)$ with a prescribed performance of the tracking error $e := y - y_{\mathrm{ref}}$, that is e evolves within the performance funnel

$$\mathcal{F}_\varphi := \{ \ (t, e) \in [0, \infty) \times \mathbb{R}^m \ \mid \varphi(t) \|e\|_{\mathbb{R}^m} < 1 \ \}$$

defined by a function φ belonging to

$$\Phi_\gamma := \left\{ \varphi \in W^{1,\infty}([0,\infty);\mathbb{R}) \;\middle|\; \varphi|_{[0,\gamma]} \equiv 0,\ \forall \delta > 0 : \inf_{t>\gamma+\delta} \varphi(t) > 0 \right\},$$

for some $\gamma > 0$. The situation is illustrated in Fig. 5.1. The funnel boundary given by $1/\varphi$ is unbounded in a small interval $[0,\gamma]$ to allow for an arbitrary initial tracking error. Since φ is bounded there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. Thus, we seek practical tracking with arbitrary small accuracy $\lambda > 0$, but asymptotic tracking is not required in general.

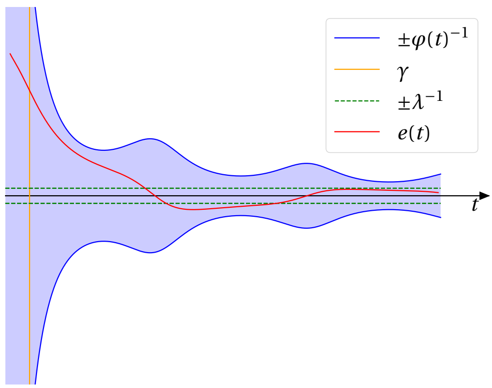

Figure 5.1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$.

The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. Sometimes, widening the funnel over some later time interval might be beneficial, for instance in the presence of periodic disturbances or strongly varying reference signals. For typical choices of funnel boundaries see e.g. [56, Section 3.2].
A controller which achieves the above described control objective is the funnel controller. In the present chapter it suffices to use the simple version developed in [61], which is the feedback law

$$I_{s,e}(t) = -\frac{k_0}{1-\varphi(t)^2\|\mathcal{B}'v(t)-y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}}(\mathcal{B}'v(t)-y_{\mathrm{ref}}(t)), \tag{5.6}$$

where $k_0 > 0$ is some constant used for scaling and agreement of physical units. Note that for $t \leq \gamma$, the controller is merely

$$I_{s,e}(t) = -k_0(\mathcal{B}'v(t)-y_{\mathrm{ref}}(t)),$$

since $\varphi|_{[0,\gamma]} \equiv 0$.

The application of the controller (5.6) results in the nonlinear and time-varying PDE system
We are interested in considering solutions of (5.7), which leads to the following weak solution framework.

Definition 5.3.1. Assume that $d \leq 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, let $D \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be symmetric-valued and satisfying the ellipticity condition (5.1). Let $T \in (0,\infty]$ and $\mathcal{A}$ be the Neumann elliptic operator on Ω associated to D (see Proposition 5.1.1), let $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m,(W^{1,2}(\Omega))')$, and let $u_0, v_0 \in L^2(\Omega)$ as well as $I_{s,i} \in L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega))$ be given. Further, let $k_0 > 0$, $y_{\mathrm{ref}} \in W^{1,\infty}([0,\infty);\mathbb{R}^m)$, $\gamma > 0$ and $\varphi \in \Phi_\gamma$. A triple of functions (u,v,y) is called *solution* of the system (5.3) with feedback (5.7) in $[0,T)$, if (u,v,y) is a solution of (5.3) in $[0,T)$ with (5.6) for all $t \in [0,T)$.

Remark 5.3.2.

a) Plugging the feedback law (5.6) into the system (5.3), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}v(t) &= \mathcal{A}v(t) + p_3(v)(t) - u(t) + I_{s,i}(t) - \frac{k_0\mathcal{B}(\mathcal{B}'v(t)-y_{\mathrm{ref}}(t))}{1-\varphi(t)^2\|\mathcal{B}'v(t)-y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}},\\ \frac{\mathrm{d}}{\mathrm{d}t}u(t) &= c_5v(t) - c_4u(t),\end{aligned} \tag{5.7}$$

Consequently, (u,v,y) is a solution of system (5.3) with feedback (5.7), if, and only if

(i) $v \in L^2([0,T);W^{1,2}(\Omega)) \cap C([0,T);L^2(\Omega)))$ with $v(0) = v_0$;

(ii) $u \in C([0,T);L^2(\Omega))$ with $u(0) = u_0$;

(iii) for all $\chi \in L^2(\Omega)$, $\theta \in W^{1,2}(\Omega)$, the scalar functions $t \mapsto \langle u(t),\chi\rangle$, $t \mapsto \langle v(t),\theta\rangle$ are weakly differentiable on $[0,T]$, and it holds that, for almost all $t \in (0,T)$,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\langle v(t),\theta\rangle &= -\mathfrak{a}(v(t),\theta) + \langle p_3(v(t)) - u(t) + I_{s,i}(t),\theta\rangle\\ &\quad - \frac{k_0\langle \mathcal{B}'v(t)-y_{\mathrm{ref}}(t),\mathcal{B}'\theta\rangle_{\mathbb{R}^m}}{1-\varphi(t)^2\|\mathcal{B}'v(t)-y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}},\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\langle u(t),\chi\rangle &= \langle c_5v(t) - c_4u(t),\chi\rangle,\\ y(t) &= \mathcal{B}'v(t),\end{aligned} \tag{5.8}$$

The system is a nonlinear and non-autonomous PDE and any solution needs to satisfy that the tracking error evolves in the prescribed performance funnel $\mathcal{F}_\varphi$. Therefore, existence and uniqueness of solutions is a nontrivial problem and even if a solution exists on a finite time interval $[0,T)$, it is not clear that it can be extended to a global solution.

b) For global solutions it is desirable that $I_{s,e} \in L^\infty([\delta,\infty);\mathbb{R}^m)$ for all $\delta > 0$. Note that this is equivalent to

$$\limsup_{t\to\infty} \varphi(t)^2 \|\mathcal{B}' v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m} < 1.$$

It is as well desirable that y and $I_{s,e}$ have a certain smoothness.

In the following we state the main theorem of this chapter. We will show that the closed-loop system (5.7) has a unique global solution so that all signals remain bounded. Furthermore, the tracking error stays uniformly away from the funnel boundary. We further show that we gain more regularity, if $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, (W^{r,2}(\Omega))')$ for some $r \in [0,1)$.

Theorem 5.3.3. *Assume that $d \le 3$ and $\Omega \subset \mathbb{R}^d$ is a bounded domain with Lipschitz boundary, let $D \in L^\infty(\Omega;\mathbb{R}^{d\times d})$ be symmetric-valued and satisfying the ellipticity condition* (5.1)*. Let $\mathcal{A}$ be the Neumann elliptic operator on Ω associated to D (see Proposition 5.1.1), let $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, (W^{1,2}(\Omega))')$ with $\ker \mathcal{B} = \{0\}$, and let $u_0, v_0 \in L^2(\Omega)$ as well as $I_{s,i} \in L^\infty([0,\infty); L^2(\Omega))$ be given. Further, let $k_0 > 0$, $y_{\mathrm{ref}} \in W^{1,\infty}([0,\infty);\mathbb{R}^m)$, $\gamma > 0$ and $\varphi \in \Phi_\gamma$. Then, for all $T > 0$, there exists a unique solution of* (5.7) *in* $[0,T)$*. The solution is thus global and*

(i) $u, \dot{u}, v \in BC([0,\infty); L^2(\Omega))$;

(ii) for all $\delta > 0$ holds

$$v \in C^{0,1/2}([\delta,\infty); L^2(\Omega)),$$
$$y, I_{s,e} \in BUC([\delta,\infty);\mathbb{R}^m),$$

and

(iii) there exists some $\varepsilon_0 > 0$ such that for all $\delta > 0$

$$\forall\, t \ge \delta : \varphi(t)^2 \|\mathcal{B}' v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m} \le 1 - \varepsilon_0.$$

a) If, further, $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, (W^{r,2}(\Omega))')$ for some $r \in (0,1)$, then for all $\delta > 0$

$$v \in C^{0,1-r/2}([\delta,\infty); L^2(\Omega)),$$
$$y, I_{s,e} \in C^{0,1-r}([\delta,\infty);\mathbb{R}^m),$$

b) If $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, L^2(\Omega))$, then for all $\delta > 0$ and for all $\lambda \in (0,1)$

$$v \in C^{0,\lambda}([\delta,\infty); L^2(\Omega)),$$
$$y, I_{s,e} \in C^{0,\lambda}([\delta,\infty);\mathbb{R}^m),$$

c) If $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, W^{1,2}(\Omega))$, then for all $\delta > 0$ holds $y, I_{s,e} \in C^{0,1}([\delta,\infty);\mathbb{R}^m)$.

If in particular, $v_0 \in \mathcal{D}(\mathcal{A})$, then the former hold with $\delta = 0$.

Remark 5.3.4.

a) The condition $\ker \mathcal{B} = \{0\}$ is equivalent to $\operatorname{im} \mathcal{B}'$ being dense in $\mathbb{R}^m$. The latter is equivalent to $\operatorname{im} \mathcal{B}' = \mathbb{R}^m$ by the finite-dimensionality of $\mathbb{R}^m$.
Note that surjectivity of $\mathcal{B}'$ is evident for tracking control, since any reference signal $y_{\text{ref}} \in W^{1,\infty}([0,\infty);\mathbb{R}^m)$ has to be tracked by the output $y = \mathcal{B}'v$.

b) If the input operator corresponds to Neumann boundary control, i.e. $\mathcal{B}$ is as in (5.5) for some $w \in L^2(\partial\Omega)$, then $r \in (1/2, 1)$, see Remark 5.2.2d). Consequently, for all $\varepsilon > 0$, we have $v \in C^{0,3/4-\varepsilon}([\delta,\infty);L^2(\Omega))$ and $y, I_{s,e} \in C^{0,1/2-\varepsilon}([\delta,\infty);\mathbb{R}^m)$ in this case.

c) If the input operator corresponds to distributed control, i.e. $\mathcal{B}u = uw$ for some $w \in L^2(\Omega)$, see Remark 5.2.2c), then for all $\varepsilon > 0$ holds $v \in C^{0,1-\varepsilon}([\delta,\infty);L^2(\Omega))$ and $y, I_{s,e} \in C^{0,1-\varepsilon}([\delta,\infty);\mathbb{R}^m)$.

Before we begin to develop the necessary results to prove Theorem 5.3.3, we show the simulated system (5.7).

A numerical example

In this section, we illustrate the practical applicability of the funnel controller for a numerical example. The setup chosen here is a standard test example for termination of reentry waves and has been considered similarly in, e.g., [20, 80]. All simulations are generated on an AMD Ryzen 7 1800X @ 3.68 GHz x 16, 64 GB RAM, MATLAB® Version 9.2.0.538062 (R2017a). The solutions of the ODE systems are obtained by the MATLAB® routine `ode23`. The parameters for the FitzHugh-Nagumo model (5.3) used here are as follows:

$$\Omega = (0,1)^2, \quad D = \begin{pmatrix} 0.015 & 0 \\ 0 & 0.015 \end{pmatrix}, \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{pmatrix} \approx \begin{pmatrix} 1.614 \\ 0.1403 \\ 0.012 \\ 0.00015 \\ 0.015 \end{pmatrix}.$$

The spatially discrete system of ODEs corresponds to a finite element discretization with piecewise linear finite elements on a uniform 64×64 mesh. The uncontrolled system is stimulated such that reentry phenomena occur, see Figure 5.2. Let us emphasize that the associated dynamics are obtained from (5.3) with $I_{s,i} = 0 = I_{s,e}$.
For the control interaction, we assume that $\mathcal{B} \in \mathscr{L}(\mathbb{R}^4, (W^{1,2}(\Omega))')$ where the Neumann control operator is defined such that

$$\mathcal{B}^* z = \left(\int_{\Gamma_1} z(\xi)\,\mathrm{d}\sigma, \int_{\Gamma_2} z(\xi)\,\mathrm{d}\sigma, \int_{\Gamma_3} z(\xi)\,\mathrm{d}\sigma, \int_{\Gamma_4} z(\xi)\,\mathrm{d}\sigma \right)^\top,$$

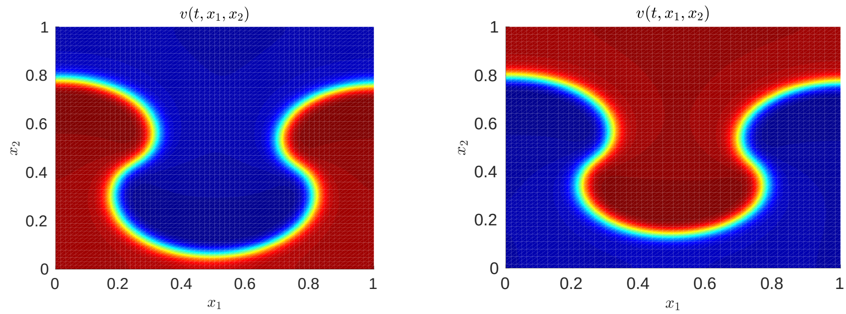

Figure 5.2: Snapshots of reentry waves for $t = 100$ (left) and $t = 200$ (right).

$$\Gamma_1 = \{1\} \times [0,1], \quad \Gamma_2 = [0,1] \times \{1\}, \quad \Gamma_3 = \{0\} \times [0,1], \quad \Gamma = [0,1] \times \{0\}.$$

Note that this scenario is the one described in Remark 5.2.2 d), so Theorem 5.3.3 and Remark 5.3.4 b) imply that the solution $y, I_{s,e} \in C^{0,1/2-\varepsilon}([\delta,\infty);\mathbb{R}^m)$ for all $\varepsilon > 0$. The function φ characterizing the performance funnel (see Figure 5.3) is chosen as

$$\varphi(t) = \begin{cases} 0, & t \in [0, 0.05], \\ \tanh(\frac{t}{100}), & t > 0.05. \end{cases}$$

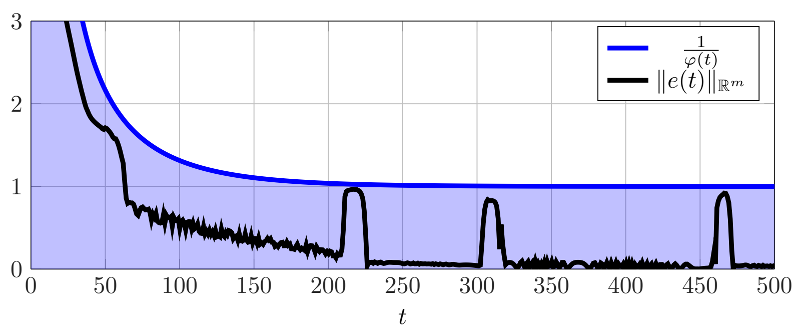

Figure 5.3: Error dynamics and funnel boundary.

The simulations correspond to a finite element discretization of (5.3) with piecewise linear finite elements.
For replication of a natural (desired) heart rhythm, the funnel reference signal is obtained by a periodic excitation wave spreading out from the stimulation point in the center of the domain. The smoothness of the signal is guaranteed by convoluting the

original signal with a triangular function. Figure 5.4 show the results of the closed-loop system obtained with the control law

$$I_{s,e}(t) = -\frac{0.75}{1-\varphi(t)^2\|\mathcal{B}'v(t)-y_{\text{ref}}(t)\|^2_{\mathbb{R}^m}}(\mathcal{B}'v(t)-y_{\text{ref}}(t)),$$

which is visualized in Figure 5.5. The initial condition is taken as a snapshot of the reentry wave which without control will not terminate. Let us note that the sudden changes in the feedback law are due to the jump discontinuities of the intracellular stimulation current $I_{s,i}$ used for simulating a regular heart beat.

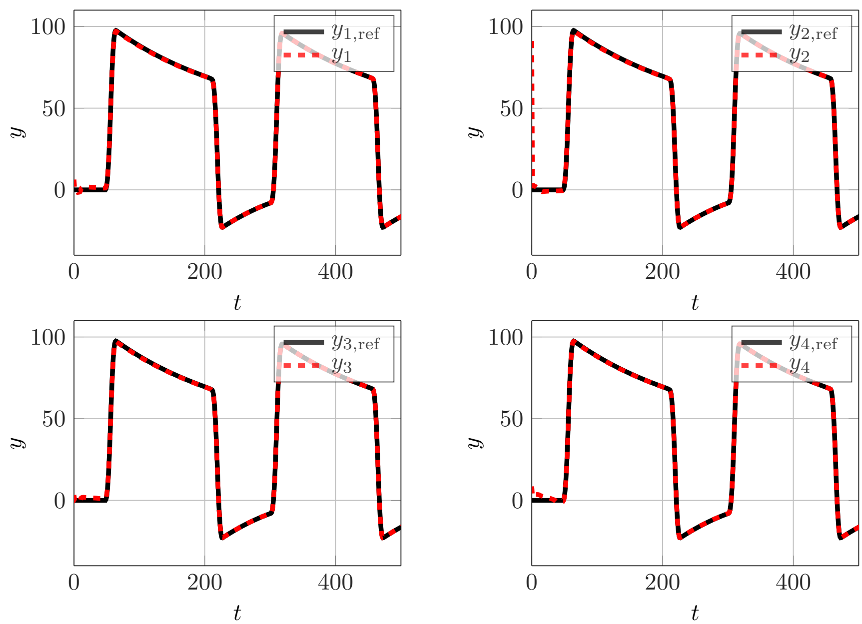

Figure 5.4: Reference signals and outputs of the funnel controlled system.

It is seen from Figure 5.4 that the controlled system faithfully reproduces the desired reference signal except for the very beginning where the control is not active yet. Figure 5.5 further shows that this is achieved with a comparably small control effort.

5.4 Preparations for the proof of the Main Theorem - Part I

We now collect some results on Neumann elliptic operators and interpolation spaces which are necessary for the proof of Theorem 5.3.3.

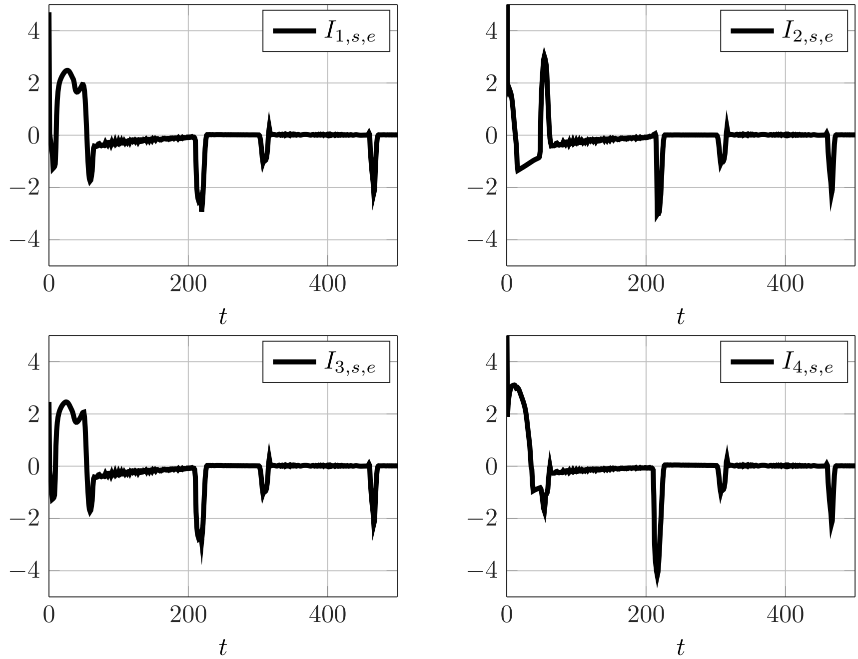

Figure 5.5: Funnel control laws.

Proposition 5.4.1. *Let $d \leq 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, and let $D \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be symmetric-valued and satisfying the ellipticity condition* (5.1)*. Then the Neumann elliptic operator $\mathcal{A}$ on Ω associated to D has the following properties:*

a) there exists some $\nu > 0$ such that $\mathcal{D}(\mathcal{A}) \subset C^{0,\nu}(\Omega)$;

b) $\mathcal{A}$ has compact resolvent;

c) there exists a real-valued and monotonically increasing sequence $(\alpha_j)_{j\in\mathbb{N}_0}$ such that

(i) $\alpha_0 = 0$, $\alpha_1 > 0$ and $\lim_{j\to\infty} \alpha_j = \infty$, and

(ii) the spectrum of $\mathcal{A}$ reads $\sigma(\mathcal{A}) = \{-\alpha_j \mid j \in \mathbb{N}_0\}$

and an orthonormal basis $(\theta_j)_{j\in\mathbb{N}_0}$ of $L^2(\Omega)$, such that

$$\forall\, x \in \mathcal{D}(\mathcal{A}) :\ \mathcal{A}x = -\sum_{j=0}^{\infty} \alpha_j \langle x, \theta_j\rangle\, \theta_j, \tag{5.9}$$

and the domain of $\mathcal{A}$ reads

$$\mathcal{D}(\mathcal{A}) = \left\{ \sum_{j=0}^{\infty} \lambda_j \cdot \theta_j \;\middle|\; (\lambda_j)_{j\in\mathbb{N}_0} \text{ with } \sum_{j=1}^{\infty} \alpha_j^2 |\lambda_j|^2 < \infty \right\}. \tag{5.10}$$

Proof. Statement a) follows from [92, Proposition 3.6].
To prove b), we first use that the ellipticity condition (5.1) implies

$$\delta\|z\| + \|\mathcal{A}z\| \geq \|z\|_{W^{1,2}}. \tag{5.11}$$

Since $\partial\Omega$ is Lipschitz, Ω has the cone property [2, p. 66], and we can apply the Rellich-Kondrachov Theorem [2, Theorem 6.3], which states that $W^{1,2}(\Omega)$ is compactly embedded in $L^2(\Omega)$. Combining this with (5.11), we obtain that $\mathcal{A}$ has compact resolvent. We finally prove c) about the spectral properties of $\mathcal{A}$: Since $\mathcal{A}$ has compact resolvent and is self-adjoint by Proposition 5.1.1, we obtain from [116, Proposition 3.2.9 & 3.2.12] that there exists a real valued sequence $(\alpha_j)_{j\in\mathbb{N}_0}$ with $\lim_{j\to\infty} |\alpha_j| = \infty$ and (5.9), and the domain of $\mathcal{A}$ has the representation (5.10). Further taking into account that

$$\langle z, \mathcal{A}z\rangle = -\mathfrak{a}(z,z) \leq 0 \;\; \forall z \in \mathcal{D}(\mathcal{A}),$$

we obtain that $\alpha_j \geq 0$ for all $j \in \mathbb{N}_0$. Consequently, it is no loss of generality to assume that $(\alpha_j)_{j\in\mathbb{N}_0}$ is monotonically increasing. It remains to prove that $\alpha_0 = 0$, $\alpha_1 > 0$: On the one hand, we have that the constant function $1_\Omega \in L^2(\Omega)$ fulfills $\mathcal{A}1_\Omega = 0$, since

$$\langle z, \mathcal{A}1_\Omega\rangle = \mathfrak{a}(z, 1_\Omega) = \langle \nabla z_1, D\nabla 1_\Omega\rangle = 0 \;\; \forall z \in W^{1,2}(\Omega).$$

On the other hand, if $z \in \ker\mathcal{A}$, we have

$$0 = \langle z, \mathcal{A}z\rangle = \mathfrak{a}(z,z) = \langle \nabla z, D\nabla z\rangle\,,$$

and the pointwise positive definiteness of D implies $\nabla z = 0$, whence z is a constant function. This altogether gives $\dim\ker\mathcal{A} = 1$, which leads to $\alpha_0 = 0$ and $\alpha_1 > 0$. □

We further need interpolation spaces to introduce our considerations. For general interpolation theory, we refer to [85]. This complements and connects the results given in Section 1.6.1 and yields a characterization of the fractional power spaces in terms of Sobolev spaces of fractional order.

Definition 5.4.2. Let X, Y be Hilbert spaces and let $\alpha \in [0,1]$. Consider the function $K : (0,\infty) \times (X+Y) \to \mathbb{R}$ with

$$K(t,x) = \inf_{x=a+b,\, a\in X, b\in Y} \|a\|_X + t\|b\|_Y.$$

The *interpolation space* $(X,Y)_\alpha$ is defined by

$$(X,Y)_\alpha := \left\{ \, x \in X+Y \;\middle|\; t \mapsto t^{-\alpha} K(t,x) \in L^2(0,\infty) \, \right\},$$

and it is a Hilbert space provided with the norm

$$\|x\|_{(X,Y)_\alpha} = \|t \mapsto t^{-\alpha} K(t,x)\|_{L^2}.$$

Note that interpolation can be performed in a more general fashion for Banach spaces X, Y: Namely, an L^p-norm with general $p \in [1,\infty)$ can be taken from $t \mapsto t^{-\theta} K(t,x)$ instead of the L^2-norm in the above definition. Note that this however does not lead to Hilbert spaces even when X and Y are Hilbert spaces.
For a self-adjoint operator $A : \mathcal{D}(A) \subset X \to X$ and $n \in \mathbb{N}$, we can define the space $X_n := \mathcal{D}(A^n)$, i.e., by $X_0 = X$ and $X_{n+1} := \{ \, x \in X_n \mid Ax \in X_n \, \}$. This becomes a Hilbert space with norm $\|z\|_{X_{n+1}} = \|-\lambda z + Az\|_{X_n}$, where $\lambda \in \mathbb{C}$ is in the resolvent set of A. Likewise, we introduce X_{-n} by the completion of X with the norm $\|z\|_{X_{-n}} = \|(-\lambda I + A)^{-n} z\|$. Note that X_{-n} is the dual of X_n with respect to the pivot space X [116, Section 2.10]. By using interpolation theory, we can further introduce X_α for non-integer $\alpha \in \mathbb{R}$:

Definition 5.4.3. Let $A : \mathcal{D}(A) \subset X \to X$ be self-adjoint and let $\alpha \in \mathbb{R}$. Further, let $n \in \mathbb{Z}$ with $\alpha \in [n, n+1)$. The space X_α is given by the interpolation space

$$X_\alpha = (X_n, X_{n+1})_{\alpha - n}.$$

The Reiteration Theorem [85, Corollary 1.24] together with [85, Proposition 3.8] yields

$$(X_{\alpha_1}, X_{\alpha_2})_\alpha = X_{\alpha_1 + \alpha(\alpha_2 - \alpha_1)}.$$

Our special interest is in the interpolation spaces of the Neumann elliptic operator $\mathcal{A}$. Next we characterize these spaces.

Proposition 5.4.4. *Let $d \leq 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, let $D \in L^\infty(\Omega; \mathbb{R}^{d\times d})$ be symmetric-valued and satisfying the ellipticity condition* (5.1)*, let $\mathcal{A}$ be the corresponding Neumann elliptic operator on Ω associated to D, and let X_α be the corresponding interpolation spaces with, in particular, $X = X_0 = L^2(\Omega)$. Then*

$$X_{r/2} = W^{r,2}(\Omega) \quad \textit{for all } r \in [0, 1/2];$$

Proof. The equation $X_{1/2} = W^{1,2}(\Omega)$ in an immediate consequence of Kato's Second Representation Theorem [70, Section VI.2, Theorem 2.23]. For general $r \in [0,1/2]$, we can use the Reiteration Theorem [85, Corollary 1.24], which implies

$$X_{r/2} = (X_1, X_0)_{r/2} = (X_{1/2}, X_0)_r.$$

Now using that $X_0 = L^2(\Omega)$ by definition, and, as already stated, $X_{1/2} = W^{1,2}(\Omega)$, we can use [123, Theorem 1.35] to obtain

$$(W^{1,2}(\Omega), L^2(\Omega))_r = W^{r,2}(\Omega),$$

and thus $X_{r/2} = W^{r,2}(\Omega)$. □

Remark 5.4.5. In terms of the spectral decomposition (5.9), the space X_α can be represented by

$$X_\alpha = \left\{ \sum_{j=0}^{\infty} \lambda_j \cdot \theta_j \;\middle|\; (\lambda_j)_{j\in\mathbb{N}_0} \text{ with } \sum_{j=1}^{\infty} \alpha_j^{2\alpha} |\lambda_j|^2 < \infty \right\}. \tag{5.12}$$

This follows by a combination of [85, Theorem 4.33] with [85, Theorem 4.36].

5.5 Preparations for the proof of the Main Theorem - Part II

We consider mild solutions of certain abstract Cauchy problems and the concept of admissible control operators. This notion is well-known in infinite-dimensional linear systems theory with unbounded control and observation operators and we refer to [116] for further details.

If X is a Hilbert space and $A : \mathcal{D}(A) \subset X \to X$ is self-adjoint with $\langle x, Ax\rangle \le 0$ for all $x \in \mathcal{D}(A)$, then it generates a contractive, analytic semigroup $(\mathbb{T}_t)_{t\ge0}$ on X [4, Theorem 4.2]. It can be further concluded that, if some $\omega_0 > 0$ exists such that $\langle x, Ax\rangle \le -\omega_0\|x\|^2$ for all $x \in \mathcal{D}(A)$, then the semigroup $(\mathbb{T}_t)_{t\ge0}$ generated by A is exponentially stable with $\|\mathbb{T}_t\| \le \mathrm{e}^{-\omega_0 t}$ for all $t \ge 0$. We can further conclude from [94, Theorem 6.13(b)] that, for all $\alpha \in \mathbb{R}$, $(\mathbb{T}_t)_{t\ge0}$ restricts (resp. extends) to an analytic semigroup $((\mathbb{T}|_\alpha)_t)_{t\ge0}$ on X_α with same growth bound as $(\mathbb{T}_t)_{t\ge0}$. Furthermore, we have $\operatorname{im} \mathbb{T}_t \subset X_r$ for all $t > 0$ and $r \in \mathbb{R}$ [94, Theorem 6.13(a)].

Next we consider the abstract Cauchy problem with source term. Note that the following results also hold when considering some $t_0 \in \mathbb{R}$, $T \in (t_0, \infty]$ and

$$\dot{x}(t) = Ax(t) + f(t) + Bu(t), \quad x(t_0) = x_0$$

by doing the necessary modifications, c.f. [110, Section 3.8]. For the sake of simplicity we set $t_0 = 0$.

Definition 5.5.1. Let X be a Hilbert space, $A : \mathcal{D}(A) \subset X \to X$ be self-adjoint with $\langle x, Ax\rangle \le 0$ for all $x \in \mathcal{D}(A)$, $T \in (0, \infty]$, and $\alpha \in [0, 1]$. Let $(\mathbb{T}_t)_{t\ge0}$ be the semigroup on X generated by A, and let $B \in \mathscr{L}(\mathbb{R}^m, X_{-\alpha})$. For $x_0 \in X$, $p \in [1, \infty]$,

$f \in L^p_{\mathrm{loc}}([0,T);X)$ and $u \in L^p_{\mathrm{loc}}([0,T);\mathbb{R}^m)$, we call $x:[0,T) \to X$ a *mild solution* of

$$\dot{x}(t) = Ax(t) + f(t) + Bu(t), \quad x(0) = x_0 \tag{5.13}$$

on $[0,T)$ the function obtain by using the variation of constants formula

$$\forall\, t \in [0,T): \; x(t) = \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s + \int_0^t (\mathbb{T}|_{-\alpha})_{t-s} Bu(s)\,\mathrm{d}s\,, \tag{5.14}$$

where $((\mathbb{T}|_{-\alpha})_t)_{t\geq 0}$ is the extension of $(\mathbb{T}_t)_{t\geq 0}$ to $X_{-\alpha}$.
We further call $x:[0,T) \to X$ a *strong solution* of (5.13) on $[0,T)$, if x in (5.14) further satisfies $x \in C([0,T);X) \cap W^{1,p}_{\mathrm{loc}}([0,T);X_{-1})$.

The above definition includes that the integral $\int_0^t (\mathbb{T}|_{-\alpha})_{t-s} Bu(s)\,\mathrm{d}s$ is in X, whilst die integrand is not necessarily in X.
In the following we show that the mild solution of the abstract Cauchy problem is indeed a strong solution.

Lemma 5.5.2. *Let X be a Hilbert space, $A : \mathcal{D}(A) \subset X \to X$ be self-adjoint with $\langle x, Ax\rangle \leq 0$ for all $x \in \mathcal{D}(A)$, $T \in (0,\infty]$, $\alpha \in [0,1/2]$ and let $B \in \mathscr{L}(\mathbb{R}^m, X_{-\alpha})$. Let $(\mathbb{T}_t)_{t\geq 0}$ be the analytic semigroup generated by A. Then for all $p \in [2,\infty]$ B is L^p-admissible for $(\mathbb{T}_t)_{t\geq 0}$.*
Further, for all $x_0 \in X$, $f \in L^p_{\mathrm{loc}}([0,T);X)$ and $u \in L^p_{\mathrm{loc}}([0,T);\mathbb{R}^m)$, the function x in (5.14) *is a strong solution of* (5.13).

Proof. For $p > 2$, set $\tilde{f} := f + Bu$ and apply [110, Theorem 3.10.10] with $\tilde{f} \in L^\infty_{\mathrm{loc}}([0,T);X_{-\alpha})$.
For the case $p = 2$, there exists a unique strong solution in X_{-1} —that is, replacing X by X_{-1} and X_{-1} by X_{-2}— given by (5.14) and at most one strong solution in X, see for instance [110, Theorem 3.8.2 (i) & (ii)], so we only need to check that all the elements are in the correct spaces. Since A is self-adjoint, the semigroup that generates is also self-adjoint, that is, $\mathbb{T}_t = \mathbb{T}_t^*$ for all $t \geq 0$. Further, by [116, Proposition 5.1.3], B^* is an L^2-admissible observation operator for the semigroup $(\mathbb{T}_t)_{t\geq 0}$ for $\alpha = 1/2$, so it follows that it is an L^2-admissible control operator for all $\alpha \in (0,1/2]$. By the duality of observation and control admissibility (see for instance [116, Theorem 4.4.3]) we have that B is an L^2-admissible control operator for $(\mathbb{T}_t)_{t\geq 0}$. By the nesting property of L^p, B is an L^p-admissible control operator for $(\mathbb{T}_t)_{t\geq 0}$ all $p \in [2,\infty]$. Moreover, by [116, Proposition 4.2.5] we have that

$$t \mapsto \mathbb{T}_t x_0 + \int_0^t \mathbb{T}_{t-s} Bu(s)\,\mathrm{d}s \in C([0,T];X) \cap W^{1,2}_{\mathrm{loc}}(0,T;X_{-1}).$$

From [110, Theorem 3.8.2 (iv)],

$$t \mapsto \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s \in C([0,T);X) \cap W^{1,2}_{\mathrm{loc}}([0,T);X_{-1}),$$

so $x \in C([0,T);X) \cap W^{1,2}_{\text{loc}}([0,T);X_{-1})$. □

Observe that if x is the unique solution of (5.13) under the conditions of Lemma 5.5.2 and A is the operator associated to a form $a : X_{1/2} \times X_{1/2} \to \mathbb{R}$, then this is also the unique solution of the weak formulation

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle x(t), \theta \rangle = -a(x(t), \theta) + \langle f(t), \theta \rangle + \langle u(t), B^* \theta \rangle, \quad \forall \theta \in X_{1/2},$$

which can be easily seen by adapting [54, Theorem 1.33] and [27, Theorem 3.1.7] to our setting.
In the proof of the next result, we will implicitly use for $a, b \geq 0$ and $p \in (0,1)$ the inequality $(a+b)^p \leq a^p + b^p$, see for instance [103, §1.47].

Proposition 5.5.3. *Let $d \leq 3$ and $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, let $D \in L^\infty(\Omega; \mathbb{R}^{d \times d})$ be symmetric-valued and satisfying the ellipticity condition (5.1). Let $\mathcal{A}$ be the corresponding Neumann elliptic operator on Ω associated to D, and let $T \in (0, \infty]$ and $c > 0$. Let $X = X_0 = L^2(\Omega)$ and let X_r be the interpolation space constructed by $\mathcal{A}$. Define $\mathcal{A}_0 := \mathcal{A} - cI$ with $\mathcal{D}(\mathcal{A}_0) = \mathcal{D}(\mathcal{A})$ and consider $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, X_{-\alpha})$ for $\alpha \in [0, 1/2]$, $u \in L^2_{\text{loc}}([0,T); \mathbb{R}^m) \cap L^\infty([\delta, T); \mathbb{R}^m)$ and $f \in L^2_{\text{loc}}([0,T); X) \cap L^\infty([\delta, T); X)$ for all $\delta > 0$. Then for all $x_0 \in X$ and all $\delta > 0$ the mild solution (5.14) of (5.13) on $[0,T)$ satisfies*

(i) if $\alpha = 0$, then for all $\lambda \in (0,1)$

$$x \in BC([0,T);X) \cap C^{0,\lambda}([\delta,T);X);$$

(ii) if $\alpha \in (0, 1/2)$, then

$$x \in BC([0,T);X) \cap C^{0,1-\alpha}([\delta,T);X) \cap C^{0,1-2\alpha}([\delta,T);X_\alpha);$$

(iii) if $\alpha = 1/2$, then

$$x \in BC([0,T);X) \cap C^{0,1/2}([\delta,T);X) \cap BUC([\delta,T);X_{1/2}).$$

Moreover, if $u \in L^\infty([0,\infty); \mathbb{R}^m)$, $f \in L^\infty([0,\infty); X)$ and $x_0 \in X_1$, the former hold with $\delta = 0$.

Proof. Note that for all $T > 0$ the continuity of x from Lemma 5.5.2 guarantees its boundedness, so we only need to explicitly check it for the case $T = \infty$. Moreover, the Hölder regularity respectively bounded and uniform continuity will be clear for all $T > 0$, so we will focus on the case $T = \infty$.
Consider the form $\tilde{\mathfrak{a}}(\cdot,\cdot) = \mathfrak{a}(\cdot,\cdot) + c \langle \cdot, \cdot \rangle$, which is coercive, hence the operator associated to $\tilde{\mathfrak{a}}$ is $\mathcal{A}_0$ and generates an analytic, contractive semigroup denoted by $(\mathbb{T}_t)_{t \geq 0}$.

Moreover, since $\mathcal{A}_0$ is strictly dissipative, the semigroup $(\mathbb{T}_t)_{t\geq 0}$ is exponentially stable and, in particular, satisfies $\|\mathbb{T}_t x\| \leq \mathrm{e}^{-ct}\|x\|$ for all $t \geq 0$ and all $x \in X$.
Define, for $t \geq 0$, the functions

$$x_h(t) := \mathbb{T}_t x_0, \quad x_f(t) := \int_0^t \mathbb{T}_{t-s} f(s)\,\mathrm{d}s\,, \quad x_u(t) := \int_0^t (\mathbb{T}|_{-\alpha})_{t-s}\mathcal{B}u(s)\,\mathrm{d}s\,,$$

so that $x = x_h + x_f + x_u$. By Lemma 5.5.2, $x \in C([0,\infty);X) \cap W^{1,2}_{\mathrm{loc}}(0,\infty;X_{-1})$. Since $(\mathbb{T}_t)_{t\geq 0}$ is exponentially stable, it follows that $\mathcal{B}$ is also infinite-time L^p-admissible for $p \in [2,\infty]$, see [66, Lemma 2.9 (i)], which implies that $x_u \in BC([0,\infty);X)$. A direct calculation using the exponential stability of $(\mathbb{T}_t)_{t\geq 0}$ shows that $x_h, x_f \in BC([0,\infty);X)$.
We study the three different cases: $\alpha = 0$, $\alpha \in (0,1/2)$ and $\alpha = 1/2$. Let $\alpha = 0$. Then, we can define $\tilde{f} := f + Bu \in L^2_{\mathrm{loc}}(0,\infty;X) \cap L^\infty(\delta,\infty;X)$ for all $\delta > 0$. We thus have from [84, Proposition 4.2.3 & Proposition 4.4.1 (i)] that for all $\lambda \in (0,1)$ it holds that

$$x \in C^{0,\lambda}([\delta,\infty);X) \cap BC([0,\infty);X).$$

Consider now $\alpha \in (0,1/2)$. Since $x_0 \in X$ and $f \in L^\infty([\delta,\infty);X)$, we may infer from [84, Proposition 4.2.3 & Proposition 4.4.1 (i)] that

$$x_h + x_f \in C^{0,1-2\alpha}([\delta,\infty);X_\alpha) \cap C^{0,1-\alpha}([\delta,\infty);X) \cap BC([0,\infty);X).$$

By using Lemma 1.6.1 it can be easily computed that there exists $K_\alpha > 0$ such that for all $t \geq s \geq \delta$ we have that

$$\|x_u(t) - x_u(s)\|_{X_\alpha} \leq K_\alpha \|\mathcal{B}u\|_{L^\infty([\delta,\infty);X_{-\alpha})}(t-s)^{1-2\alpha},$$

so that $x_u \in C^{0,1-2\alpha}([\delta,\infty);X_\alpha)$. Similarly,

$$\|x_u(t) - x_u(s)\|_{X} \leq G_\alpha \|\mathcal{B}u\|_{L^\infty([\delta,\infty);X_{-\alpha})}(t-s)^{1-\alpha},$$

for some $G_\alpha > 0$, so that $x_u \in C^{0,1-\alpha}([\delta,\infty);X)$.
The case $\alpha = 1/2$ needs a special treatment, because the previous calculations are not feasible in this case. We obtain from Proposition 5.4.1c) that $\mathcal{A}_0$ has an eigendecomposition of type (5.9) with eigenvalues $(-\beta_j)_{j\in\mathbb{N}_0}$, $\beta_j := \alpha_j + c$, and eigenfunctions $(\theta_j)_{j\in\mathbb{N}_0}$. Moreover, there exist $b_i \in X_{-1/2}$ for $i = 1,\ldots,m$ such that $\mathcal{B}\xi = \sum_{i=1}^m b_i \cdot \xi_i$ for all $\xi \in \mathbb{R}^m$. Therefore,

$$\begin{aligned} x_u(t) &= \int_0^t \sum_{j=0}^\infty \mathrm{e}^{-\beta_j(t-\tau)}\theta_j \sum_{i=1}^m \langle b_i \cdot u_i(\tau), \theta_j\rangle \,\mathrm{d}\tau \\ &= \int_0^t \sum_{j=0}^\infty \mathrm{e}^{-\beta_j(t-\tau)}\theta_j \sum_{i=1}^m u_i(\tau)\,\langle b_i, \theta_j\rangle \,\mathrm{d}\tau\,, \end{aligned}$$

where the last equality holds since $u_i(\tau) \in \mathbb{R}$ and can be treated as a constant in X. By considering each of the factors in the sum over $i = 1, \dots, m$, we can assume without loss of generality that $m = 1$ and $b \coloneqq b_1$, so that

$$x_u(t) = \int_0^t \sum_{j=0}^{\infty} \mathrm{e}^{-\beta_j (t-\tau)} u(\tau) \langle b, \theta_j \rangle \, \theta_j \, \mathrm{d}\tau \, .$$

Define $b^j \coloneqq \langle b, \theta_j \rangle$ for $j \in \mathbb{N}_0$. We show that $x_u \in BUC([\delta, \infty); X_{1/2})$. For that we use the diagonal representation as in Remark 5.4.5. In particular, since $b \in X_{-1/2}$ we have that

$$S \coloneqq \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j} < \infty. \tag{5.15}$$

Now let $t > s > \delta$ and $\sigma > 0$ such that $t - s < \sigma$. By dominated convergence [28, Theorem II.2.3], summation and integration can be interchanged, so that

$$\begin{aligned} &\|x_u(t) - x_u(s)\|^2_{X_{1/2}} \\ &\leq \|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sum_{j=0}^{\infty} \beta_j (b^j)^2 \left(\int_\delta^s \mathrm{e}^{-\beta_j (s-\tau)} - \mathrm{e}^{-\beta_j (t-\tau)} \, \mathrm{d}\tau + \int_s^t \mathrm{e}^{-\beta_j (t-\tau)} \, \mathrm{d}\tau \right)^2 \\ &\leq 4 \|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j} \left(1 - \mathrm{e}^{-\beta_j (t-s)}\right)^2 \\ &\leq 4 \|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j} \left(1 - \mathrm{e}^{-\beta_j \sigma}\right)^2 . \end{aligned}$$

We can conclude from (5.15) that the series $F : (0, \infty) \to (0, S)$ with

$$F(\sigma) \coloneqq \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j} (1 - \mathrm{e}^{-\beta_j \sigma})^2$$

converges uniformly to a strictly monotone, continuous and surjective function. Therefore, F has an inverse. The function x_u is thus uniformly continuous on $[\delta, \infty)$ and by exponential stability of $(\mathbb{T}_t)_{t \geq 0}$ we obtain boundedness, i.e., $x_u \in BUC([\delta, \infty); X_{1/2})$. By using similar estimates and the exponential stability of $(\mathbb{T}_t)_{t \geq 0}$, it is straightforward to see that $x_h, x_f \in C^{0,1/2}([\delta, \infty); X_{1/2}) \cap C^{0,1/2}([\delta, \infty); X)$ by applying [84, Proposition 4.2.3 & Proposition 4.4.1 (i)], which further implies $x_h, x_f \in BUC([\delta, \infty); X_{1/2})$. To see that also $x_u \in C^{0,1/2}([\delta, \infty); X)$, we proceed as we did before. Let $t > s > \delta$ and observe that

$$\begin{aligned} &\|x_u(t) - x_u(s)\|^2_X \\ &\leq \|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sum_{j=0}^{\infty} (b^j)^2 \left(\int_0^s \mathrm{e}^{-\beta_j (s-\tau)} - \mathrm{e}^{-\beta_j (t-\tau)} \, \mathrm{d}\tau + \int_s^t \mathrm{e}^{-\beta_j (t-\tau)} \, \mathrm{d}\tau \right)^2 \end{aligned}$$

$$\leq 4\|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j^2} \left(1 - \mathrm{e}^{-\beta_j(t-s)}\right)^2.$$

Hence,

$$\frac{\|x_u(t) - x_u(s)\|^2_X}{t-s} \leq 4\|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j^2} \frac{\left(1 - \mathrm{e}^{-\beta_j(t-s)}\right)^2}{t-s}.$$

By the mean value theorem, there exists $\xi \in (0, t-s)$ such that

$$\begin{aligned}
\frac{\|x_u(t) - x_u(s)\|^2_X}{t-s} &\leq 8\|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j} \left(1 - \mathrm{e}^{-\beta_j \xi}\right) \\
&\leq 8\|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sum_{j=0}^{\infty} \frac{(b^j)^2}{\beta_j} \\
&= 8\|u\|^2_{L^\infty([\delta,\infty);\mathbb{R}^m)} S.
\end{aligned}$$

Hence,

$$\|x_u(t) - x_u(s)\|_X \leq \sqrt{8S}\|u\|_{L^\infty([\delta,\infty);\mathbb{R}^m)} \sqrt{t-s},$$

and $x_u \in C^{0,1/2}([\delta,\infty); X)$.
Since $\delta > 0$ is arbitrary, the result follows. In order to show that the case $\delta \to 0$ is consistent, note that in the estimates involving x_u, we can set $\delta = 0$, and everything holds. For x_f, this follows from [84, Proposition 4.2.1], so that we only need to check x_h. Since $x_0 \in X_1$, it follows that $x_0 \in X_\lambda$ for all $\lambda \in (0,1)$. Thus, for $\lambda \in (0,1)$, $\alpha \in [0, 1/2)$ and $t, s \in [0,\infty)$ with $s < t$ we have from Lemma 1.6.1 that there exist $K_1, K_2 > 0$ such that

$$\|x_h(t) - x_h(s)\|_X \leq K_1 \|x_0\|_{X_\lambda} (t-s)^\lambda$$

and

$$\|x_h(t) - x_h(s)\|_{X_\alpha} \leq K_2 \|x_0\|_{X_{1-\alpha}} (t-s)^{1-2\alpha}$$

Thus the case $\alpha \in [0, 1/2)$ is clear. For the case $\alpha = 1/2$ using Lemma 1.6.1 leads to

$$\|x_h(t) - x_h(s)\|_{X_{1/2}} \leq K_3 \|x_0\|_{X_1} (t-s)^{1/2},$$

for some $K_3 > 0$, so that $x_h \in BUC([0,\infty); X_{1/2})$, which concludes the proof. □

Remark 5.5.4. Note that in the latter results, when $T \in (0,\infty]$ is finite, the solutions are defined in the compact $[0,T]$ instead of $[0,T)$ or $[\delta,T]$ instead of $[\delta,T)$.

5.6 Proof of the Main Theorem

The proof is inspired by [65], which uses arguments from [82] for the convergence of certain subsequences. We divide the proof in two major parts. First, we show that there exists a unique solution for $t \in [0,\gamma]$. Secondly we show that the solution also exists for $t \in (\gamma,\infty)$ and is continuous at $t=\gamma$ and has the desired properties.

5.6.1 Solution for $t \in [0,\gamma]$

Assuming that $t \in [0,\gamma]$, we have that $\varphi(t) \equiv 0$ so that we need to show existence of a solution of

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle v_n(t),\theta\rangle &= -\mathfrak{a}(v_n(t),\theta) + \langle p_3(v_n(t)) - u_n(t) + I_{s,i}(t),\theta\rangle + \left\langle I_{s,e}^n(t),\mathcal{B}'\theta\right\rangle_{\mathbb{R}^m},\\
\frac{\mathrm{d}}{\mathrm{d}t}\langle u_n(t),\chi\rangle &= \langle c_5 v_n(t) - c_4 u_n(t),\chi\rangle,\\
I_{s,e}^n(t) &= -k_0(\mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t)).
\end{aligned} \tag{5.16}$$

Step 1. Existence and uniqueness
Let $(\theta_i)_{i\in\mathbb{N}_0}$ be the eigenfunctions of $-\mathcal{A}$ and let α_i be the corresponding eigenvalues, with $\alpha_i \geq 0$ for all $i \in \mathbb{N}_0$. Recall that $(\theta_i)_{i\in\mathbb{N}_0}$ form an orthonormal basis of $L^2(\Omega)$ by Proposition 5.4.1c). Hence, with $a_i := \langle v_0,\theta_i\rangle$ and $b_i := \langle u_0,\theta_i\rangle$ for $i \in \mathbb{N}_0$ and

$$v_0^n := \sum_{i=0}^{n} a_i\theta_i, \quad u_0^n := \sum_{i=0}^{n} b_i\theta_i, \quad n \in \mathbb{N},$$

we have that $v_0^n \to v_0$ and $u_0^n \to u_0$ strongly in $L^2(\Omega)$.
Let $\gamma_i := \mathcal{B}'\theta_i$ for $i = 0,\dots,n$. Consider

$$\begin{aligned}
\dot{\mu}_j(t) &= -\alpha_j\mu_j(t) - \nu_j(t) - \left\langle k_0\left(\sum_{i=0}^{n}\gamma_i\mu_i(t) - y_{\mathrm{ref}}(t)\right),\gamma_j\right\rangle_{\mathbb{R}^m} + \langle I_{s,i}(t),\theta_j\rangle\\
&\quad + \left\langle p_3\left(\sum_{i=0}^{n}\mu_i(t)\theta_i\right),\theta_j\right\rangle,\\
\dot{\nu}_j(t) &= -c_4\nu_j(t) + c_5\mu_j(t),
\end{aligned}$$

whit $\mu_j(0) = a_j$ and $\nu_j(0) = b_j$ defined on $\mathbb{D} := [0,\infty)\times\mathbb{R}^{2(n+1)}$. Since the functions on the right hand side of the differential equations are continuous, the set $\mathbb{D}$ is relatively open in $[0,\infty)\times\mathbb{R}^{2(n+1)}$ and the initial condition is in $\mathbb{D}$ it follows from ODE theory, see e.g. [121, § 10, Theorem XX], that there exists a weakly differentiable solution $(\mu,\nu) = (\mu_0,\dots,\mu_n,\nu_0,\dots,\nu_n) : [0,T_n) \to \mathbb{R}^{2(n+1)}$ such that $T_n \in (0,\gamma]$ is maximal. Furthermore, the closure of the graph of (μ,ν) is not a compact subset of $\mathbb{D}$.

Consequently, let $v_n(t) = \sum_{i=0}^{n} \mu_n(t)\theta_i$ and $u_n(t) = \sum_{i=0}^{n} \nu_n(t)\theta_i$. By using the functions θ_j we have that for $j = 1, \dots, n$ the former ODE system is equivalent to the truncated weak formulation

$$\begin{aligned} \langle \dot{v}_n(t), \theta_j \rangle = -\mathfrak{a}(v_n(t), \theta_j) - \langle u_n(t), \theta_j \rangle + \langle p_3(v_n(t)), \theta_j \rangle + \langle I_{s,i}(t), \theta_j \rangle \\ - \langle k_0(\mathcal{B}' v_n(t) - y_{\mathrm{ref}}(t)), \mathcal{B}'\theta_j \rangle_{\mathbb{R}^m}, \end{aligned} \tag{5.17a}$$

$$\langle \dot{u}_n(t), \theta_j \rangle = -c_4 \langle u_n(t), \theta_j \rangle + c_5 \langle v_n(t), \theta_j \rangle . \tag{5.17b}$$

Consider the Lyapunov function candidate

$$\begin{aligned} V: \quad L^2(\Omega) \times L^2(\Omega) &\to \mathbb{R}, \\ (v, u) &\mapsto \frac{1}{2}(c_5\|v\|^2 + \|u\|^2). \end{aligned} \tag{5.18}$$

Observing that, since $(\theta_i)_{i\in\mathbb{N}_0}$ are orthonormal, we have $\|v_n\|^2 = \sum_{j=0}^{n} \mu_j^2$ and $\|u_n\|^2 = \sum_{j=0}^{n} \nu_j^2$ and hence we find that, for all $t \in [0, T_n)$,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(v_n(t), u_n(t)) =& c_5 \sum_{j=0}^{n} \mu_j(t)\dot{\mu}_j(t) + \sum_{j=0}^{n} \nu_j(t)\dot{\nu}_j(t) \\ =& - c_5 \sum_{j=0}^{n} \alpha_j \mu_j(t)^2 - c_4 \sum_{j=0}^{n} \nu_j(t)^2 \\ & - c_5 \left\langle k_0 \left(\sum_{i=0}^{n} \gamma_i \mu_i(t) - y_{\mathrm{ref}}(t) \right), \sum_{i=0}^{n} \gamma_i \mu_i(t) \right\rangle_{\mathbb{R}^m} \\ & + c_5 \left\langle p_3 \left(\sum_{i=0}^{n} \mu_i(t)\theta_i \right), \sum_{i=0}^{n} \mu_i(t)\theta_i \right\rangle + c_5 \langle I_{s,i}(t), v_n \rangle \end{aligned}$$

hence, omitting the argument t for brevity in the following,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(v_n, u_n) =& - c_5 \mathfrak{a}(v_n, v_n) - c_4 \|u_n\|^2 + c_5 \langle I_{s,i}(t), v_n \rangle \\ & - c_5 k_0 \|\overline{e}_n\|^2_{\mathbb{R}^m} + c_5 k_0 \langle \overline{e}_n, y_{\mathrm{ref}} \rangle_{\mathbb{R}^m} + c_5 \langle p_3(v_n), v_n \rangle , \end{aligned} \tag{5.19}$$

where

$$\overline{e}_n(t) \coloneqq \sum_{i=0}^{n} \gamma_i \mu_i(t) - y_{\mathrm{ref}}(t)$$

First recall Young's inequality for products, i.e., for $a, b \geq 0$ and $p, q \geq 1$ such that $1/p + 1/q = 1$ we have that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q},$$

which we will frequently use in the following. Note that

$$\langle p_3(v_n), v_n \rangle = -c_1 \|v_n\|^2 + c_2 \left\langle v_n^2, v_n \right\rangle - c_3 \|v_n^4\|_{L^4}$$

and

$$c_2|\left\langle v_n^2, v_n\right\rangle| = |\left\langle \epsilon v_n^3, \epsilon^{-1}c_2\right\rangle| \leq \frac{3\epsilon^{4/3}}{4}\|v\|_{L^4}^4 + \frac{c_2^4}{4\epsilon^4}|\Omega|.$$

By choosing ϵ such that $3\epsilon^{4/3} = 2c_3$ we have that

$$\langle p_3(v_n), v_n\rangle \leq \frac{c_2^4}{4\epsilon^4}|\Omega| - c_1\|v\|^2 - \frac{c_3}{2}\|v_n\|_{L^4}^4.$$

Moreover,

$$\langle \overline{e}_n, y_{\text{ref}}\rangle_{\mathbb{R}^m} \leq \frac{1}{2}\|\overline{e}\|_{\mathbb{R}^m}^2 + \frac{1}{2}\|y_{\text{ref}}(t)\|_{\mathbb{R}^m}^2$$

and

$$\langle I_{s,i}(t), v_n\rangle \leq \frac{c_1}{2}\|v_n\|^2 + \frac{1}{2c_1}\|I_{s,i}(t)\|^2$$

so that (5.19) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(v_n, u_n) \leq & - c_5\mathfrak{a}(v_n, v_n) - \frac{c_1c_5}{2}\|v_n\|^2 - \frac{c_5k_0}{2}\|\overline{e}\|_{\mathbb{R}^m}^2 - \frac{c_3}{2}\|v_n\|_{L^4}^4 \\ & + \frac{k_0c_5}{2}\|y_{\text{ref}}(t)\|_{\mathbb{R}^m}^2 + \frac{1}{2c_1}\|I_{s,i}(t)\|^2 + \frac{c_2^4}{4\epsilon^4}|\Omega| \\ & \leq c_5\mathfrak{a}(v_n, v_n) - \frac{c_3}{2}\|v_n\|_{L^4}^4 + \frac{1}{2c_1}\|I_{s,i}(t)\|^2 + \frac{k_0c_5}{2}\|y_{\text{ref}}(t)\|_{\mathbb{R}^m}^2 + \frac{c_2^4}{4\epsilon^4}|\Omega|\end{aligned}$$

Set

$$C_\infty := \frac{k_0c_5}{2}\|y_{\text{ref}}\|_\infty^2 + \frac{1}{2c_1}\|I_{s,i}\|_\infty^2 + \frac{c_2^4}{4\epsilon^4}|\Omega|$$

to obtain that

$$V(v_n(t), u_n(t)) - V(v_0^n, u_0^n) + c_5\int_0^t \mathfrak{a}(v_n(s), v_n(s))\,\mathrm{d}s + \frac{c_3}{2}\int_0^t \|v_n(s)\|_{L^4}^4\,\mathrm{d}s \leq C_\infty\gamma.$$

Since $(u_n^0, v_n^0) \to (u_0, v_0)$ strongly in $L^2(\Omega)$ and for any $p \in L^2(\Omega)$ we have

$$\left\|\sum_{i=0}^n \langle p, \theta_i\rangle\,\theta_i\right\|^2 = \sum_{i=0}^n \langle p, \theta_i\rangle^2 \leq \sum_{i=0}^\infty \langle p, \theta_i\rangle^2 = \left\|\sum_{i=1}^\infty \langle p, \theta_i\rangle\,\theta_i\right\|^2 = \|p\|^2,$$

we have that

$$\begin{aligned}c_5\|v_n(t)\|^2 + \|u_n(t)\|^2 + c_5\int_0^t \mathfrak{a}(v_n(s), v_n(s))\,\mathrm{d}s + \frac{c_5k_0}{2}\int_0^t \|\overline{e}_n(s)\|_{\mathbb{R}^m}^2\,\mathrm{d}s \\ + \frac{c_3}{2}\int_0^t \|v_n(s)\|_{L^4}^4\,\mathrm{d}s \leq C_\infty\gamma + c_5\|u_0\|^2 + \|v_0\|^2.\end{aligned} \tag{5.20}$$

In particular, $T_n = \gamma$, since otherwise we would have a contradiction and hence the solution is defined for all $t \in [0, \gamma]$. Moreover

$$\|v_n(t)\|^2 \leq c_5^{-1} C_\infty \gamma, \quad \|u_n(t)\|^2 \leq C_\infty \gamma \tag{5.21}$$

and there exist constants $C_1, C_2 > 0$ such that

$$\|v_n\|^4_{L^4(Q_\gamma)} \leq C_1, \quad \|v_n\|_{L^2(0,\gamma;W^{1,2}(\Omega))} \leq C_2, \tag{5.22}$$

where $Q_\gamma := (0, \gamma) \times \Omega$. Note that (5.22) directly implies that

$$\|v_n^2\|^2_{L^2(Q_\gamma)} \leq C_1, \tag{5.23}$$

and using Hölder's inequality, there exists a $\tilde{C}_1$ such that

$$\|v_n^3\|_{L^{4/3}(Q_\gamma)} \leq \tilde{C}_1 \tag{5.24}$$

Equations (5.17b) and (5.21) imply that there exists $C_3 > 0$ such that

$$\|u\|_{W^{1,2}([0,\gamma];L^2(\Omega))} \leq C_3.$$

Let P_n be the projection of $L^2(\Omega)$ onto the subspace generated by θ_i, $i = 1, \ldots, n$ with the norm

$$\|v\|_{W^{1,2}} = \left(\sum_{i=0}^{n} |\langle v, \theta_i \rangle|^2 + \sum_{i=1}^{n} \alpha_i |\langle v, \theta_i \rangle|^2 \right)^{1/2},$$

on $W^{1,2}(\Omega)$ as by Proposition 5.4.4 and Remark 5.4.5. By the duality, we have that

$$\|v\|_{(W^{1,2})'} = \left(|\langle v, \theta_0 \rangle|^2 + \sum_{i=1}^{n} (1 + \alpha_i)^{-1} |\langle v, \theta_i \rangle|^2 \right)^{1/2},$$

is a norm on $(W^{1,2}(\Omega))'$, c.f. [116, Proposition 3.4.8]. Each P_n is a bounded linear functional from $(W^{1,2}(\Omega))'$ to $(W^{1,2}(\Omega))'$ with norm 1 independent of n. From the weak formulation (5.17) we have that

$$\dot{v}_n = P_n \mathcal{A} v_n + P_n p_3(v_n) - P_n u_n + P_n I_{s,i} - P_n \mathcal{B} k_0 (\mathcal{B}' v_n(t) - y_{\text{ref}}(t)).$$

Since $v_n \in L^2([0, \gamma]; W^{1,2}(\Omega))$,be the Sobolev embedding, $v_n \in L^2([0, \gamma]; L^p(\Omega))$ for all $2 \leq p \leq 6$ and thus $p_3(v_n) \in L^2([0, \gamma]; L^2(\Omega))$. Moreover, $\mathcal{A}v \in L^2([0, \gamma]; (W^{1,2}(\Omega))')$ so that there exists a constant $C_4 > 0$ such that

$$\|\dot{v}_n\|_{L^2([0,\gamma];(W^{1,2}(\Omega))')} \leq C_4.$$

Now, by Lemma 1.4.6 we have that there exists a subsequence for which

$$\begin{aligned} u_n &\to u \in W^{1,2}([0,\gamma]; L^2(\Omega)) \text{ weakly}, \\ u_n &\to u \in W^{1,\infty}([0,\gamma]; L^2(\Omega)) \text{ weak}^\star, \\ v_n &\to v \in L^2([0,\gamma]; W^{1,2}(\Omega)) \text{ weakly}, \\ v_n &\to v \in L^\infty([0,\gamma]; L^2(\Omega)) \text{ weak}^\star, \\ v_n &\to v \in L^4(Q_\gamma) \text{ weakly}, \\ \dot{v}_n &\to \dot{v} \in L^2([0,\gamma]; W^{1,2}(\Omega)') \text{ weakly}. \end{aligned} \tag{5.25}$$

Moreover, let $p_0 = p_1 = 2$ and $X = W^{1,2}(\Omega)$, $Y = L^2(\Omega)$, $Z = W^{1,2}(\Omega)'$. Then, [82, Chapitre 1, Théorème 5.1] implies that

$$W := \{\ u \in L^{p_0}([0,\gamma]; X)\ \mid \dot{u} \in L^{p_1}([0,\gamma]; Z)\ \}$$

with norm $\|u\|_{L^{p_0}([0,\gamma];X)} + \|\dot{u}\|_{L^{p_1}([0,\gamma];Y)}$ has a compact injection into $L^{p_0}([0,\gamma]; Y)$, so the weakly convergent sequence $v_n \to v \in W$ converges strongly in $L^2([0,\gamma]; L^2(\Omega))$ by [54, Lem. 1.6].

Further, $(u(0), v(0)) = (u_0, v_0)$ and by $v \in W^{1,2}([0,\gamma]; L^2(\Omega))$, $v \in L^2([0,\gamma]; W^{1,2}(\Omega))$ and $\dot{v} \in L^2([0,\gamma]; (W^{1,2}(\Omega))')$ it follows that $u, v \in C([0,\gamma]; L^2(\Omega))$, see for instance [54, Theorem 1.32]. Note that $\mathcal{B}'v - y_{\text{ref}} \in L^2([0,\gamma]; \mathbb{R}^m)$ as well. Hence, (u, v) is a solution of (5.7) in $[0,\gamma]$ and

$$\dot{v}(t) = \mathcal{A}v(t) + p_3(v(t)) - u(t) + I_{s,i}(t) - \mathcal{B}k_0(\mathcal{B}'v(t) - y_{\text{ref}}(t)) \tag{5.26}$$

holds in $(W^{1,2}(\Omega))'$. Moreover, from (5.23) & (5.24), by [82, Chapitre 1, Lemme 1.3] and $v_n \to v$ in $L^4(Q_\gamma)$ we have that $v_n^3 \to v^3$ weakly in $L^{4/3}(Q_\gamma)$ and $v_n^2 \to v^2$ weakly in $L^2(Q_\gamma)$.

To show uniqueness, assume now that there exist two solutions (u_1, v_1) and (u_2, v_2) of the former. Define

$$\Sigma(t,\zeta) := \frac{c_2}{c_3} + |v_1(t,\zeta)| + |v_2(t,\zeta)|,$$

and let

$$Q_\Lambda := \{(t,\zeta) \in Q_\gamma \mid \Sigma(t,\zeta) \geq \Lambda\}.$$

Since $v_1, v_2 \in L^4(Q_\gamma)$, we can choose Λ such that

$$\int_{Q_\gamma \setminus Q_\Lambda} \Sigma^4 < \epsilon.$$

Let $V_i := v_i - \Lambda$ for $i = 1, 2$, $V := V_2 - V_2 = v_2 - v_1$ and $U := u_2 - u_1$. By [54, Theorem 1.32], we have that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|^2 = \left\langle \dot{V}(t), V(t) \right\rangle, \quad \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|^2 = \left\langle \dot{U}(t), U(t) \right\rangle$$

holds for all $t \in (0,\gamma)$. Using Lemma 4.4.4, it can be computed that

$$\frac{c_5}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|V(t)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|U(t)\|^2 \le \langle p_3(v_2) - p_3(v_1), V_2 - V_1\rangle ,$$

so that

$$\begin{aligned}\frac{c_5}{2}\|V(t)\|^2 + \frac{1}{2}\|U(t)\|^2 \le &\int_{Q_\Lambda} (p_3(v_2) - p_3(v_1))(V_2 - V_1)\\ &+ \int_{Q_\gamma \setminus Q_\Lambda} (p_3(v_2) - p_3(v_1))(V_2 - V_1).\end{aligned}$$

However,

$$\begin{aligned}(p_3(v_2) - p_3(v_1))(V_2 - V_1) &= (p_3(V_2+\Lambda) - p_3(V_1+\Lambda))(V_2 - V_1)\\ &= - c_1V^2 - (3c_3\Lambda - c_2)V^2(V_1+V_2) - (3c_3\Lambda - 2c_2)\Lambda V^2\\ &\quad - c_3V^2(V_2^2 + V_1V_2 + V_1^2)\end{aligned}$$

which is non-positive on Q_Λ. Hence,

$$\frac{c_5}{2}\|V(t)\|^2 + \frac{1}{2}\|U(t)\|^2 < 2\epsilon,$$

which yields $u_2 = u_1$ and $v_2 = v_1$.

Step 2. For all $\epsilon \in (0,\gamma)$, $v(t) \in W^{1,2}(\Omega)$ for $t \in [\epsilon,\gamma]$.

We will next show that for all $\epsilon > 0$, $v \in BUC([\epsilon,\gamma]; W^{1,2}(\Omega))$. Consider again the approximation of the weak formulation (5.17). Multiplying (5.17a) by $\dot{\mu}_j$ and adding over $j = 0,\dots,n$ we have

$$\begin{aligned}\|\dot v_n\|^2 &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(v_n,v_n) - \langle u_n, \dot v_n\rangle + \langle p_3(v_n(t)), \dot v_n\rangle + \langle I_{s,i}(t), \dot v_n\rangle\\ &\quad - k_0\left\langle \mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t), \mathcal{B}'\dot v_n\right\rangle_{\mathbb{R}^m}\\ &= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(v_n,v_n) - \langle u_n, \dot v_n\rangle + \langle p_3(v_n(t)), \dot v_n\rangle + \langle I_{s,i}(t), \dot v_n\rangle\\ &\quad - k_0\left\langle \mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t), \mathcal{B}'\dot v_n - \dot y_{\mathrm{ref}}(t)\right\rangle_{\mathbb{R}^m} + k_0\left\langle \mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t), \dot y_{\mathrm{ref}}(t)\right\rangle_{\mathbb{R}^m}\end{aligned}$$

By using (5.21) and Young's inequality, it can be computed that there exist constants $Q_1, Q_2 > 0$ independent of n such that

$$\begin{aligned}\|\dot v_n\|^2 = &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(v_n,v_n) - \frac{c_3}{4}\frac{\mathrm{d}}{\mathrm{d}t}\|v_n\|_{L^4}^4 - \frac{k_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathcal{B}'v_n - y_{\mathrm{ref}}\|_{\mathbb{R}^m}^2 + \frac{1}{2}\|\dot v_n\|^2\\ &+ Q_1\|v_n\|_{L^4}^4 + Q_2 + \frac{k_0}{2}\|\mathcal{B}'v_n - y_{\mathrm{ref}}\|_{\mathbb{R}^m}^2.\end{aligned}$$

Thus,

$$\begin{aligned}&\|\dot v_n\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathfrak{a}(v_n,v_n) + \frac{c_3}{2}\|v_n\|_{L^4}^4 + k_0\|\mathcal{B}'v_n - y_{\mathrm{ref}}\|_{\mathbb{R}^m}^2\right)\\ &\qquad = 2Q_1\|v_n\|_{L^4}^4 + 2Q_2 + k_0\|\mathcal{B}'v_n - y_{\mathrm{ref}}\|_{\mathbb{R}^m}^2.\end{aligned} \tag{5.27}$$

Multiplying the former by t, using the chain rule and the fact that $t \leq \gamma$ for $t \in [0,\gamma]$ leads to

$$\begin{aligned} t\|\dot{v}_n(t)\|^2 &+ \frac{\mathrm{d}}{\mathrm{d}t}\left(t\mathfrak{a}(v_n(t),v_n(t)) + \frac{c_3 t}{2}\|v_n(t)\|_{L^4}^4 + k_0 t\|\mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m}^2\right) \\ &= \left(2Q_1\gamma + \frac{c_3}{2}\right)\|v_n(t)\|_{L^4}^4 + \mathfrak{a}(v_n(t),v_n(t)) \\ &\quad + 2Q_2\gamma + k_0(\gamma+1)\|\mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m}^2. \end{aligned}$$

Since $t\|\dot{v}_n(t)\|^2 \geq 0$ for all $t \in [0,\gamma]$, we further have that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(t\mathfrak{a}(v_n(t),v_n(t)) + \frac{c_3 t}{2}\|v_n(t)\|_{L^4}^4 + k_0 t\|\mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m}^2\right) \\ &= \left(2Q_1\gamma + \frac{c_3}{2}\right)\|v_n(t)\|_{L^4}^4 + \mathfrak{a}(v_n(t),v_n(t)) \\ &\quad + 2Q_2\gamma + k_0(\gamma+1)\|\mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m}^2. \end{aligned}$$

Integrating the former and using (5.20), there exist $P_1, P_2 > 0$ independent of n such that for $t \in [0,\gamma]$ we have

$$t\mathfrak{a}(v_n(t),v_n(t)) + \frac{c_3 t}{2}\|v_n(t)\|_{L^4}^4 + k_0 t\|\mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m}^2 = P_1 + P_2 t.$$

Thus, there exist constants $C_5, C_6 > 0$ independent of n such that for all $t \in [0,\gamma]$

$$t\mathfrak{a}(v_n(t),v_n(t)) \leq C_5, \quad t\|\mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m} \leq C_6.$$

Hence, for all $\epsilon \in (0,\gamma)$, $v_n \in L^\infty([\epsilon,\gamma];W^{1,2}(\Omega))$ and $\mathcal{B}'v_n - y_{\mathrm{ref}} \in L^\infty([\epsilon,\gamma];\mathbb{R}^m)$, so that in addition to (5.25), by using Lemma 1.4.6 we further have that there exists a subsequence such that

$$v_n \to v \in L^\infty([\epsilon,\gamma];W^{1,2}(\Omega))$$

and $\mathcal{B}'v \in L^\infty([\epsilon,\gamma];\mathbb{R}^m)$ for all $\epsilon \in (0,\gamma)$, so $I_{s,e} \in L^2([0,\gamma];\mathbb{R}^m) \cap L^\infty([\epsilon,\gamma];\mathbb{R}^m)$. By the Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^p(\Omega)$ for $2 \leq p \leq 6$ we have that $p_3(v) \in L^\infty([\epsilon,\gamma];L^2(\Omega))$. Moreover, since (5.26) holds, we can rewrite it as

$$\dot{v}(t) = (\mathcal{A} - c_1 I)v(t) + I_r(t) + \mathcal{B}I_{s,e}(t),$$

where $I_r := c_2 v^2 - c_3 v^3 - u + I_{s,i} \in L^2([0,\gamma];L^2(\Omega)) \cap L^\infty([\epsilon,\gamma];L^2(\Omega))$ and Proposition 5.5.3 with $c = c_1$ implies that the solution $v \in BUC([\epsilon,\gamma];W^{1,2}(\Omega))$. Hence, for all $\epsilon \in (0,\gamma)$, $v(t) \in W^{1,2}(\Omega)$ for $t \in [\epsilon,\gamma]$, so that in particular $v(\gamma) \in W^{1,2}(\Omega)$.

Note that if $v_0 \in \mathcal{D}(\mathcal{A})$, then it follows that $v_0 \in W^{1,2}(\Omega)$ and the former can be carried out for $\epsilon = 0$. Since $v_0 \in W^{1,2}(\Omega)$ so that by the Sobolev embedding it holds $v_0 \in L^4(\Omega)$. We can integrate (5.27) and use (5.20) so that there exist $\tilde{P}_1, \tilde{P}_2$ independent of n such that

$$\mathfrak{a}(v_n(t),v_n(t)) + \frac{c_3}{2}\|v_n(t)\|_{L^4}^4 + k_0\|\mathcal{B}'v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m}^2 = \tilde{P}_1 + \tilde{P}_2 t.$$

Thus, there exist constants $\tilde{C}_5, \tilde{C}_6 > 0$ independent of n such that for all $t \in [0, \gamma]$

$$\mathfrak{a}(v_n(t), v_n(t)) \le \tilde{C}_5, \quad \|\mathcal{B}' v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m} \le \tilde{C}_6.$$

Analogously, by using Lemma 1.4.6 we further have that there exists a subsequence such that

$$v_n \to v \in L^\infty([0, \gamma]; W^{1,2}(\Omega)).$$

In fact,

$$\begin{aligned} v &\in L^\infty([0, \gamma]; W^{1,2}(\Omega)), \\ I_{s,e} &\in L^\infty([0, \gamma]; \mathbb{R}^m). \end{aligned} \tag{5.28}$$

5.6.2 Solution for $t \in (\gamma, \infty)$

The crucial step in the proof is to show that the error remains uniformly bounded away from the funnel boundary while $v \in L^\infty([\gamma, \infty); W^{1,2}(\Omega))$. The proof is divided into several steps.

Step 1. We show existence of an approximate solution by means of a time-varying state-space transformation.

Once more, let $(\theta_i)_{i \in \mathbb{N}_0}$ be the eigenfunctions of $-\mathcal{A}$ and let α_i be the corresponding eigenvalues, with $\alpha_i \ge 0$ for all $i \in \mathbb{N}_0$. Recall that $(\theta_i)_{i \in \mathbb{N}_0}$ form an orthonormal basis of $L^2(\Omega)$ by Proposition 5.4.1c). Let $(u_\gamma, v_\gamma) := (u(\gamma), v(\gamma))$, $a_i := \langle v_\gamma, \theta_i \rangle$ and $b_i := \langle u_\gamma, \theta_i \rangle$ for $i \in \mathbb{N}_0$ and

$$v_\gamma^n := \sum_{i=0}^{n} a_i \theta_i, \quad u_\gamma^n := \sum_{i=0}^{n} b_i \theta_i, \quad n \in \mathbb{N},$$

we have that $v_\gamma^n \to v_\gamma$ strongly in $W^{1,2}(\Omega)$ and $u_\gamma^n \to u_\gamma$ strongly in $L^2(\Omega)$.

We have stated in Remark 5.3.4a) that $\ker \mathcal{B} = \{0\}$ implies $\mathcal{B}'\mathcal{D}(\mathcal{A}) = \mathbb{R}^m$. As a consequence, there exist $q_1, \dots, q_m \in \mathcal{D}(\mathcal{A})$ such that $\mathcal{B}' q_k = \mathrm{e}_k$ for $k = 1, \dots, m$. By Proposition 5.4.1a), we further have $q_k \in C^{0,\nu}(\Omega)$ for some $\nu > 0$.

Note that $U := \bigcup_{n \in \mathbb{N}} \operatorname{span}\{\theta_i\}_{i=0}^n$ satisfies $\overline{U} = W^{1,2}(\Omega)$ with the respective norm. Moreover, $\overline{\mathcal{B}' U} = \mathbb{R}^m$. Since $\mathbb{R}^m$ is complete and finite dimensional and $\mathcal{B}'$ is linear and continuous it follows that $\mathcal{B}' U = \mathbb{R}^m$. By the surjectivity of $\mathcal{B}'$ we have that for all $k \in \{1, \dots, m\}$ there exist $n_k \in \mathbb{N}$ and $q_k \in U_{n_k}$ such that $\mathcal{B}' q_k = \mathrm{e}_k$, where $U_N = \bigcup_{n=1}^N \operatorname{span}\{\theta_i\}_{i=0}^n = \operatorname{span}\{\theta_i\}_{i=0}^N$ for $N \in \mathbb{N}$. Thus, there exists $n_0 \in \mathbb{N}$ such that $q_k \in U_{n_0}$ for all $k = \{1, \dots, m\}$, so the q_k are a (finite) linear combination of θ_i.

Define $q \in W^{1,2}(\Omega; \mathbb{R}^m) \cap C^{0,\nu}(\Omega; \mathbb{R}^m)$ by $q(\zeta) = \big(q_1(\zeta), \dots, q_m(\zeta)\big)^\top$ and $q \cdot y_{\mathrm{ref}}$ by

$$(q \cdot y_{\mathrm{ref}})(t, \zeta) := \sum_{k=1}^{m} q_k(\zeta) y_{\mathrm{ref},k}(t), \quad \zeta \in \Omega,\ t \ge 0.$$

Analogously for $q \cdot \dot{y}_{\text{ref}}$. Note that we have $(q \cdot y_{\text{ref}}) \in BC([0,\infty) \times \Omega)$, because

$$|\left\langle q, y_{\text{ref}}\right\rangle_{\mathbb{R}^m}(t,\zeta)| \le \sum_{k=1}^{m} \|q_k\|_\infty \|y_{\text{ref},k}\|_\infty$$

for all $\zeta \in \Omega$ and $t \ge 0$, where we write $\|\cdot\|_\infty$ for the supremum norm. We denote by $q_{k,j} := \langle q_k, \theta_j\rangle$ for $k = 1,\dots,m$, $j \in \mathbb{N}_0$ and $q_k^n := \sum_{j=0}^n q_{k,j}$ for $n \in \mathbb{N}$. Similarly, $q^n := (q_1^n,\dots,q_m^n)^\top$ for $n \in \mathbb{N}$, so that $q^n \to q$ strongly in $W^{1,2}(\Omega)$.
Since $\mathcal{B}' : W^{r,2}(\Omega) \to \mathbb{R}^m$ is continuous with $r \in [0,1]$, it follows that for all $\theta \in W^{r,2}(\Omega)$ there exists a unique $\Gamma_r > 0$ such that

$$\|\mathcal{B}'\theta\|_{\mathbb{R}^m} \le \Gamma_r \|\theta\|_{W^{r,2}}.$$

For $n \in \mathbb{N}_0$, let

$$\kappa_n := \frac{1}{n+1}\frac{1}{\Gamma_r}\frac{1}{1 + \|v_\gamma^n - q^n \cdot y_{\text{ref}}(\gamma)\|_{W^{r,2}}^2}.$$

Note that for $v_\gamma \in W^{1,2}(\Omega)$ it holds that $\kappa_n > 0$ for all $n \in \mathbb{N}_0$, $(\kappa_n)_{n\in\mathbb{N}_0}$ is bounded by Γ_r^{-1} (and monotonically decreasing) and $\kappa_n \to 0$ as $n \to \infty$ and by construction

$$\kappa_n \|\mathcal{B}'(v_\gamma^n - q^n \cdot y_{\text{ref}}(\gamma))\|_{\mathbb{R}^m} < 1, \quad \forall n \in \mathbb{N}.$$

Consider a modification of φ induced by κ_n, namely, for $n \in \mathbb{N}_0$

$$\varphi_n := \varphi + \kappa_n.$$

It is clear that for each $n \in \mathbb{N}_0$ it holds $\varphi_n \in W^{1,\infty}([\gamma,\infty);\mathbb{R})$, $\|\varphi_n\|_\infty \le \|\varphi\|_\infty + \Gamma_r^{-1}$ and $\|\dot{\varphi}_n\|_\infty = \|\dot{\varphi}\|_\infty$ are independent of n and $\varphi_n \to \varphi \in \Phi_\gamma$ uniformly in $BC([\gamma,\infty);\mathbb{R})$. Moreover,$\inf_{t>\gamma} \varphi_n(t) > 0$.
Fix $n \in \mathbb{N}$. For $t \ge \gamma$, define

$$\begin{aligned}
\phi(e) &:= \frac{k_0}{1 - \|e\|_{\mathbb{R}^m}^2} e, \quad e \in \mathbb{R}^m,\ \|e\|_{\mathbb{R}^m} < 1,\\
\omega_0(t) &:= \dot{\varphi}_n(t)\varphi_n(t)^{-1},\\
F(t,z) &:= \varphi_n(t) f_{-1}(t) + \varphi_n(t) f_0(t) + f_1(t) z + \varphi_n(t)^{-1} f_2(t) z^2 - c_3 \varphi_n(t)^{-2} z^3,\\
f_{-1}(t) &:= I_{s,i}(t) + \sum_{k=1}^{m} y_{\text{ref},k}(t) \mathcal{A} q_k,\\
f_0(t) &:= -q \cdot (\dot{y}_{\text{ref}}(t) + c_1 y_{\text{ref}}(t)) + c_2 (q \cdot y_{\text{ref}}(t))^2 - c_3 (q \cdot y_{\text{ref}}(t))^3,\\
f_1(t) &:= (q \cdot y_{\text{ref}}(t))(2c_2 - 3c_3(q \cdot y_{\text{ref}}(t))),\\
f_2(t) &:= c_2 - 3c_3(q \cdot y_{\text{ref}}(t)),\\
g(t) &:= c_5(q \cdot y_{\text{ref}}(t)).
\end{aligned}$$

We have that $f_{-1} \in L^\infty([0,\infty); L^2(\Omega))$, since

$$\begin{aligned}\|f_{-1}\|_{2,\infty} &:= \operatorname{ess\,sup}_{t\geq 0} \left(\int_\Omega f_{-1}(\zeta,t)^2 \,\mathrm{d}\lambda \right)^{1/2} \\ &\leq \|I_{s,i}\|_{L^2} + \sum_{k=1}^m \|y_{\mathrm{ref},k}\|_\infty \|\mathcal{A}q_k\|_{L^2} < \infty.\end{aligned}$$

Furthermore, we have that $f_0 \in L^\infty([0,\infty)\times\Omega)$, because

$$\begin{aligned}|f_0(\zeta,t)| \leq\ & (\|\dot{y}_{\mathrm{ref}}\|_\infty + c_1\|y_{\mathrm{ref}}\|_\infty) \sum_{k=1}^m \|q_k\|_\infty + c_2 \|y_{\mathrm{ref}}\|_\infty^2 \left(\sum_{k=1}^m \|q_k\|_\infty \right)^2 \\ &+ c_3 \|y_{\mathrm{ref}}\|_\infty^3 \left(\sum_{k=1}^m \|q_k\|_\infty \right)^3\end{aligned}$$

for all $\zeta \in \Omega$ and $t \geq 0$. Hence

$$\|f_0\|_{\infty,\infty} := \operatorname{ess\,sup}_{(t,\zeta)\in[t,\infty)\times\Omega} |f_0(\zeta,t)| < \infty.$$

Moreover, $f_1, f_2, g \in BC([0,\infty)\times\Omega)$, so that $\|f_1\|_{\infty,\infty} < \infty$, $\|f_2\|_{\infty,\infty} < \infty$ and $\|g\|_{\infty,\infty} < \infty$.

Consider the system of $2(n+1)$ ODEs

$$\begin{aligned}\dot{\mu}_j(t) &= -\alpha_j \mu_j(t) - (c_1 - \omega_0(t))\mu_j(t) - \nu_j(t) - \left\langle \phi\left(\sum_{i=0}^n \gamma_i \mu_i(t) \right), \gamma_j \right\rangle_{\mathbb{R}^m} \\ &\quad + \left\langle F\left(t, \sum_{i=0}^n \mu_i(t)\theta_i \right), \theta_j \right\rangle, \\ \dot{\nu}_j(t) &= -(c_4 - \omega_0(t))\nu_j(t) + c_5 \mu_j(t) + \varphi_n(t) \langle g(t), \theta_j \rangle,\end{aligned} \tag{5.29}$$

defined on

$$\mathbb{D} := \left\{ (t, \mu_0, \ldots, \mu_n, \nu_0, \ldots, \nu_n) \in [\gamma,\infty) \times \mathbb{R}^{2(n+1)} \;\middle|\; \left\| \sum_{i=0}^n \gamma_i \mu_i \right\|_{\mathbb{R}^m} < 1 \right\},$$

with initial value

$$\mu_j(\gamma) = \kappa_n \left(a_j - \sum_{k=1}^m q_{k,j} y_{\mathrm{ref},k}(\gamma) \right), \quad \nu_j(\gamma) = \kappa_n b_j, \quad j \in \mathbb{N}_0.$$

Since the functions on the right hand side of the differential equations are continuous, the set $\mathbb{D}$ is relatively open in $[\gamma,\infty)\times\mathbb{R}^{2(n+1)}$ and by construction

$$(\gamma, \mu_0(\gamma), \ldots, \mu_n(\gamma), \nu_0(\gamma), \ldots, \nu_n(\gamma)) \in \mathbb{D}$$

it follows from ODE theory, see e.g. [121, § 10, Theorem XX], that there exists a weakly differentiable solution

$$(\mu, \nu) = (\mu_0, \ldots, \mu_n, \nu_0, \ldots, \nu_n) : [\gamma, T_n) \to \mathbb{R}^{2(n+1)}$$

such that $T_n \in (\gamma, \infty]$ is maximal. Furthermore, the closure of the graph of (μ, ν) is not a compact subset of $\mathbb{D}$.
With that, we can define

$$z_n(t) \coloneqq \sum_{i=0}^{n} \mu_i(t)\theta_i, \quad w_n(t) \coloneqq \sum_{i=0}^{n} \nu_i(t)\theta_i, \quad e_n(t) \coloneqq \sum_{i=0}^{n} \gamma_i \mu_i(t)$$

and note that

$$z_\gamma^n \coloneqq z_n(\gamma) = \kappa_n(v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma)), \quad w_\gamma^n \coloneqq w_n(\gamma) = \kappa_n u_\gamma^n.$$

From the orthonormality of the θ_i we have that

$$\begin{aligned} \langle \dot{z}_n(t), \theta_j \rangle = &-\mathfrak{a}(z_n(t), \theta_j) - (c_1 - \omega_0(t)) \langle z_n(t), \theta_j \rangle - \langle w_n(t), \theta_j \rangle \\ &- \langle \phi(\mathcal{B}' z_n(t)), \mathcal{B}'\theta_j \rangle_{\mathbb{R}^m} + \langle F(t, z_n(t)), \theta_j \rangle, \end{aligned} \tag{5.30a}$$

$$\langle \dot{w}_n(t), \theta_j \rangle = -(c_4 - \omega_0(t)) \langle w_n(t), \theta_j \rangle + c_5 \langle z_n(t), \theta_j \rangle + \varphi_n \langle g(t), \theta_j \rangle. \tag{5.30b}$$

Define now

$$\begin{aligned} v_n(t) &\coloneqq \varphi_n(t)^{-1} z_n(t) + q^n \cdot y_{\mathrm{ref}}(t), \\ u_n(t) &\coloneqq \varphi_n(t)^{-1} w_n(t). \end{aligned} \tag{5.31}$$

and let $v_n(t) = \sum_{i=0}^n \tilde{\mu}_n(t)\theta_i$ and $u_n(t) = \sum_{i=0}^n \tilde{\nu}_n(t)\theta_i$. The transformation (5.31) is bijective for all $n \in \mathbb{N}$. In fact, there exists a relation between μ_i, ν_i and $\tilde{\mu}_i, \tilde{\nu}_i$, namely,

$$\begin{aligned} \mu_i(t) &= \varphi_n(t) \left(\tilde{\mu}_i(t) - \sum_{k=1}^{m} q_{k,i} y_{\mathrm{ref},k}(t) \right), \\ \nu_i(t) &= \varphi_n(t) \tilde{\nu}_i(t). \end{aligned}$$

With this transformation we obtain the truncated equation

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \langle v_n(t), \theta \rangle = &- \mathfrak{a}(v_n(t), \theta) + \langle p_3(v_n(t) + (q - q^n) \cdot y_{\mathrm{ref}}(t)) - u_n(t), \theta \rangle \\ &+ \left\langle I_{s,i}(t) - (q - q^n) \cdot \dot{y}_{\mathrm{ref}}(t) + \sum_{k=1}^{m} y_{\mathrm{ref},k}(t) \mathcal{A}(q_k - q_k^n), \theta \right\rangle \\ &+ \left\langle I_{s,e}^n(t), \mathcal{B}'\theta \right\rangle_{\mathbb{R}^m}, \\ \frac{\mathrm{d}}{\mathrm{d}t} \langle u_n(t), \chi \rangle = &\langle c_5(v_n(t) + (q - q^n) \cdot y_{\mathrm{ref}}(t)) - c_4 u_n(t), \chi \rangle, \\ I_{s,e}^n(t) = &- \frac{k_0}{1 - \varphi_n(t)^2 \|\mathcal{B}'(v_n(t) - q^n \cdot y_{\mathrm{ref}}(t))\|_{\mathbb{R}^m}^2} (\mathcal{B}'(v_n(t) - q^n \cdot y_{\mathrm{ref}}(t))), \end{aligned} \tag{5.32}$$

with $(u_n(\gamma), v_n(\gamma)) = (u_\gamma, v_\gamma)$. Since there exists $n_0 \in \mathbb{N}$ for which $q^{n_0} = q$, we have that for $n \geq n_0$ the following holds

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle v_n(t), \theta \rangle &= -\mathfrak{a}(v_n(t), \theta) + \langle p_3(v_n(t)) - u_n(t), \theta \rangle + \langle I_{s,i}(t), \theta \rangle + \left\langle I_{s,e}^n(t), \mathcal{B}'\theta \right\rangle_{\mathbb{R}^m}, \\
\frac{\mathrm{d}}{\mathrm{d}t} \langle u_n(t), \chi \rangle &= \langle c_5 v_n(t) - c_4 u_n(t), \chi \rangle, \\
I_{s,e}^n(t) &= -\frac{k_0}{1 - \varphi_n(t)^2 \|\mathcal{B}' v_n(t) - y_{\mathrm{ref}}(t)\|_{\mathbb{R}^m}^2} (\mathcal{B}' v_n(t) - y_{\mathrm{ref}}(t)),
\end{aligned}$$

Step 2. We show boundedness of (z_n, w_n) in terms of φ_n.
Consider again the Lyapunov function (5.18). We find that, for all $t \in [0, T_n)$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(z_n(t), w_n(t)) &= c_5 \sum_{j=0}^n \mu_j(t) \dot{\mu}_j(t) + \sum_{j=0}^n \nu_j(t) \dot{\nu}_j(t) \\
&= -c_5 \sum_{j=0}^n \alpha_j \mu_j(t)^2 - c_5(c_1 - \omega_0(t)) \sum_{j=0}^n \mu_j(t)^2 \\
&\quad - (c_4 - \omega_0(t)) \sum_{j=0}^n \nu_j(t)^2 - c_5 \langle \phi(e_n(t)), e_n(t) \rangle_{\mathbb{R}^m} \\
&\quad + \varphi_n(t) \left\langle g(t), \sum_{i=0}^n \nu_i(t) \theta_i \right\rangle \\
&\quad + c_5 \left\langle F\left(t, \sum_{i=0}^n \mu_i(t) \theta_i \right), \sum_{i=0}^n \mu_i(t) \theta_i \right\rangle,
\end{aligned}$$

hence, omitting the argument t for brevity in the following,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} V(z_n, w_n) = &- c_5 \mathfrak{a}(z_n, z_n) - c_5(c_1 - \omega_0) \|z_n\|^2 - (c_4 - \omega_0) \|w_n\|^2 \\
&- c_5 \frac{k_0 \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} + c_5 \langle F(t, z_n), z_n \rangle + \varphi_n \langle g, w_n \rangle.
\end{aligned} \tag{5.33}$$

Next we use some Young and Hölder inequalities to estimate the term involving $F(t, z_n)$, that is

$$\begin{aligned}
\langle F(t, z_n), z_n \rangle = &\underbrace{\varphi_n(t) \langle f_{-1}(t), z_n \rangle}_{I_{-1}} + \underbrace{\varphi_n(t) \langle f_0(t), z_n \rangle}_{I_0} + \underbrace{\langle f_1(t) z_n, z_n \rangle}_{I_1} \\
&+ \underbrace{\varphi_n(t)^{-1} \left\langle f_2(t) z_n^2, z_n \right\rangle}_{I_2} - c_3 \varphi_n(t)^{-2} \underbrace{\left\langle z_n^3, z_n \right\rangle}_{= \|z_n\|_{L^4}^4}.
\end{aligned}$$

For the first term we derive using Young's inequality with $p = 4/3$ and $q = 4$ that

$$I_{-1} \leq \left\langle \frac{2^{1/2} \varphi_n^{3/2} |I_{s,i}|}{c_3^{1/4}}, \frac{c_3^{1/4} |z_n|}{2^{1/2} \varphi_n^{1/2}} \right\rangle + \sum_{k=1}^m \left\langle \frac{(4m)^{1/4} \varphi_n^{3/2} \|y_{\mathrm{ref}}\|_\infty |A q_k|}{c_3^{1/4}}, \frac{c_3^{1/4} |z_n|}{(4m)^{1/4} \varphi_n^{1/2}} \right\rangle$$

$$\leq \frac{2^{2/3}3\varphi_n^2 \|I_{s,i}\|_{2,\infty}^{4/3} |\Omega|^{1/3}}{4c_3^{1/3}} + \sum_{k=1}^{m} \frac{3(4m)^{1/3}\varphi_n^2 \|y_{\mathrm{ref}}\|_\infty^{4/3} \|Aq_k\|^{4/3} |\Omega|^{1/3}}{4c_3^{1/3}} + \frac{c_3\|z_n\|_{L^4}^4}{8\varphi_n^2}.$$

and with the same choice we obtain for the second term

$$I_0 \leq \left\langle \frac{2^{1/4}\varphi_n^{3/2}\|f_0\|_{\infty,\infty}}{c_3^{1/4}}, \frac{c_3^{1/4}|z_n|}{2^{1/4}\varphi_n^{1/2}} \right\rangle \leq \frac{2^{1/3}3\varphi_n^2\|f_0\|_{\infty,\infty}^{4/3}|\Omega|}{4c_3^{1/3}} + \frac{c_3\|z_n\|_{L^4}^4}{8\varphi_n^2}.$$

Using $p = q = 2$ we find that the third term satisfies

$$I_1 \leq \left\langle \frac{2\varphi_n\|f_1\|_{\infty,\infty}}{\sqrt{c_3}}, \frac{\sqrt{c_3}|z_n|^2}{2\varphi_n} \right\rangle \leq \frac{2\varphi_n^2\|f_1\|_{\infty,\infty}^2|\Omega|}{c_3} + \frac{c_3\|z_n\|_{L^4}^4}{8\varphi_n^2},$$

and finally, with $p = 4$ and $q = 4/3$,

$$\begin{aligned} I_2 &\leq \left\langle \varphi_n^{-1}\|f_2\|_{\infty,\infty}, |z_n|^3 \right\rangle = \left\langle \frac{3^{3/2}\varphi_n^{1/2}\|f_2\|_{\infty,\infty}}{c_3^{3/4}}, \left| \frac{c_3^{1/4} z_n}{\varphi_n^{1/2}\sqrt{3}} \right|^3 \right\rangle \\ &\leq \frac{9^3\varphi_n^2\|f_2\|_{\infty,\infty}^4|\Omega|}{4c_3^3} + \frac{c_3}{12\varphi_n^2}\|z_n\|_{L^4}^4. \end{aligned}$$

Summarizing, we have shown that

$$\langle F(t,z_n), z_n\rangle \leq K_0\varphi_n^2 - \frac{13c_3}{24\varphi_n^2}\|z_n\|_{L^4}^4 \leq K_0\varphi_n^2 - \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4,$$

where

$$\begin{aligned} K_0 :=& \frac{2^{2/3}3\|I_{s,i}\|_{2,\infty}^{4/3}|\Omega|^{1/3}}{4c_3^{1/3}} + \sum_{k=1}^{m} \frac{3(4m)^{1/3}\|y_{\mathrm{ref}}\|_\infty^{4/3}\|Aq_k\|^{4/3}|\Omega|^{1/3}}{4c_3^{1/3}} \\ &+ \frac{2^{1/3}3\|f_0\|_{\infty,\infty}^{4/3}|\Omega|}{4c_3^{1/3}} + \frac{2\|f_1\|_{\infty,\infty}^2|\Omega|}{c_3} + \frac{9^3\|f_2\|_{\infty,\infty}^4|\Omega|}{4c_3^3}. \end{aligned}$$

Finally, using Young's inequality with $p = q = 2$, we estimate the last term in (5.33) as follows

$$\varphi_n \langle g, w_n\rangle \leq \frac{\varphi_n^2\|g\|_{\infty,\infty}^2|\Omega|}{2c_4} + \frac{c_4}{2}\|w_n\|^2.$$

We have thus obtained the estimate

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}V(z_n,w_n) \leq& -(\sigma - 2\omega_0)V(z_n,w_n) \\ & - c_5\mathfrak{a}(z_n,z_n) - c_5\frac{k_0\|e_n\|_{\mathbb{R}^m}^2}{1-\|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3c_5}{2\varphi_n^2}\|z_n\|_{L^4}^4 + \varphi_n^2K_1, \end{aligned} \tag{5.34}$$

where

$$\sigma := 2\min\{c_1, c_4\}, \quad K_1 := c_5K_0 + \frac{\|g\|_{\infty,\infty}^2|\Omega|}{2c_4}.$$

In particular, we have the conservative estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}V(z_n,w_n) \le -(\sigma-2\omega_0)V(z_n,w_n)+\varphi_n^2 K_1$$

on $[\gamma, T_n)$, which implies that

$$V(z_n(t),w_n(t)) \le \mathrm{e}^{-K(t,\gamma)}V(z_n(\gamma),w_n(\gamma)) + \int_\gamma^t \mathrm{e}^{-K(t,s)}\varphi_n(s)^2 K_1 \,\mathrm{d}s\,,$$

where

$$K(t,s)=\int_s^t \sigma-2\omega_0(\tau)\,\mathrm{d}\tau \;=\sigma(t-s)-2\ln\varphi_n(t)+2\ln\varphi_n(s),\quad \gamma\le s\le t<T_n.$$

Therefore, invoking $\varphi_n(\gamma)=\kappa_n$, for all $t\in[\gamma,T_n)$ we have

$$\begin{aligned}V(z_n(t),w_n(t)) &\le \mathrm{e}^{-\sigma(t-\gamma)}\frac{\varphi_n(t)^2}{\kappa_n^2}V(z_n(\gamma),w_n(\gamma))+\frac{K_1}{\sigma}\varphi_n(t)^2\\ &= \frac{\varphi_n(t)^2}{2}\Big((c_5\|v_\gamma^n-q^n\cdot y_{\mathrm{ref}}(\gamma)\|^2+\|u_\gamma^n\|^2)\mathrm{e}^{-\sigma(t-\gamma)}+2K_1\sigma^{-1}\Big)\\ &\le \frac{\varphi_n(t)^2}{2}\Big((c_5\|v_\gamma-q\cdot y_{\mathrm{ref}}(\gamma)\|^2+\|u_\gamma\|^2)\mathrm{e}^{-\sigma(t-\gamma)}+2K_1\sigma^{-1}\Big).\end{aligned}$$

Hence

$$c_5\|z_n(t)\|^2+\|w_n(t)\|^2\le\varphi_n(t)^2(c_5\|v_\gamma-q\cdot y_{\mathrm{ref}}(\gamma)\|^2+\|u_\gamma\|^2+2K_1\sigma^{-1})$$

for all $t\in[\gamma,T_n)$. Thus there exist $M,N>0$ which are independent of n and t such that

$$\forall\, t\in[\gamma,T_n):\ \|z_n(t)\|^2\le M\varphi_n(t)^2 \ \text{ and }\ \|w_n(t)\|^2\le N\varphi_n(t)^2. \tag{5.35}$$

Hence

$$\forall\, t\in[\gamma,T_n):\ \|v_n(t)-q^n\cdot y_{\mathrm{ref}}(t)\|^2\le M \ \text{ and }\ \|u_n(t)\|^2\le N. \tag{5.36}$$

Step 3. We show $T_n=\infty$ and that e_n is uniformly bounded away from 1 on $[\gamma,\infty)$.
Step 3a. We derive some estimates for $\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2$ and for an integral involving $\|z_n(s)\|_{L^4}^4$.
In a similar way in which we have derived (5.34) we can obtain the estimate

$$\begin{aligned}\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2\le &-\mathfrak{a}(z_n,z_n)-(c_1-\omega_0)\|z_n\|^2+\|z_n\|\|w_n\|\\ &-\frac{k_0\|e_n\|_{\mathbb{R}^m}^2}{1-\|e_n\|_{\mathbb{R}^m}^2}-\frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4+K_0\varphi_n^2.\end{aligned} \tag{5.37}$$

Using (5.35) and $-c_1\|z_n\|^2\le 0$ leads to

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2\le -\mathfrak{a}(z_n,z_n)-\frac{k_0\|e_n\|_{\mathbb{R}^m}^2}{1-\|e_n\|_{\mathbb{R}^m}^2}-\frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4$$

$$+\|\dot{\varphi}\|_\infty M\varphi_n + (K_0+\sqrt{MN})\varphi_n^2.$$

Hence,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 \leq -\mathfrak{a}(z_n,z_n) - \frac{k_0\|e_n\|^2_{\mathbb{R}^m}}{1-\|e_n\|^2_{\mathbb{R}^m}} - \frac{c_3}{2\varphi_n^2}\|z_n\|^4_{L^4} + K_1\varphi_n + K_2\varphi_n^2 \tag{5.38}$$

on $[\gamma, T_n)$, where $K_1 := M\|\dot{\varphi}\|_\infty$ and $K_2 := K_0+\sqrt{MN}$. Observe that

$$\frac{c_3}{2}\varphi_n^{-3}\|z_n\|^4_{L^4} \leq -\frac{\varphi_n^{-1}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + K_3,$$

where $K_3 := K_1 + K_2\|\varphi\|_\infty$. Therefore,

$$\begin{aligned}
&\frac{c_3}{2}\int_\gamma^t \mathrm{e}^s\varphi_n(s)^{-3}\|z_n(s)\|^4_{L^4}\,\mathrm{d}s\\
&\leq K_3(\mathrm{e}^t-\mathrm{e}^\gamma) - \frac{1}{2}\int_\gamma^t \mathrm{e}^s\varphi_n(s)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n(s)\|^2\,\mathrm{d}s\\
&= K_3(\mathrm{e}^t-\mathrm{e}^\gamma) - \frac{1}{2}\left(\mathrm{e}^t\varphi_n(t)^{-1}\|z_n(t)\|^2 - \frac{\|z^n_\gamma\|^2}{\kappa_n}\mathrm{e}^\gamma\right)\\
&\quad + \frac{1}{2}\int_\gamma^t \mathrm{e}^s\varphi_n(s)^{-2}(\varphi_n(s)-\dot{\varphi}_n(s))\|z_n(s)\|^2\,\mathrm{d}s\\
&\leq \frac{\mathrm{e}^t}{2}(2K_3 + (\|\varphi\|_\infty + \Gamma_r^{-1} + \|\dot{\varphi}\|_\infty)M) + \kappa_n\mathrm{e}^\gamma(\|v_\gamma\|^2 + \|q\cdot y_{\mathrm{ref}}(\gamma)\|^2),
\end{aligned}$$

and hence there exist $D_0, D_1 > 0$ independent of n and t such that

$$\forall\, t\in[\gamma,T_n):\ \int_\gamma^t \mathrm{e}^s\varphi_n(s)^{-3}\|z_n(s)\|^4_{L^4}\,\mathrm{d}s \leq D_1\mathrm{e}^t + \kappa_n D_0. \tag{5.39}$$

Step 3b. We derive an estimate for $\|\dot{z}_n\|^2$. Multiplying (5.30a) by $\dot{\mu}_j$ and summing up over $j\in\{0,\dots,n\}$ we obtain

$$\begin{aligned}
\|\dot{z}_n\|^2 = &-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(z_n,z_n) - \frac{c_1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + \frac{k_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\ln(1-\|e_n\|^2_{\mathbb{R}^m})\\
&+ \langle \omega_0 z_n + F(t,z_n) - w_n, \dot{z}_n\rangle\,.
\end{aligned}$$

We can estimate the last term above by

$$\begin{aligned}
\langle \omega_0 z_n, \dot{z}_n\rangle &\leq \frac{7}{2}\|\dot{\varphi}\|^2_\infty\varphi_n^{-2}\|z_n\|^2 + \frac{1}{14}\|\dot{z}_n\|^2 \overset{(5.35)}{\leq} \frac{7}{2}\|\dot{\varphi}\|^2_\infty M + \frac{1}{14}\|\dot{z}_n\|^2,\\
\langle -w_n, \dot{z}_n\rangle &\leq \frac{7}{2}\|w_n\|^2 + \frac{1}{14}\|\dot{z}_n\|^2,\\
\langle F(t,z_n), \dot{z}_n\rangle &\leq \frac{7}{2}\varphi_n^2\left(m\sum_{k=1}^m \|y_{\mathrm{ref},k}\|^2_\infty\|\mathcal{A}q_k\|^2 + \|I_{s,i}\|^2_{2,\infty} + \|f_0\|^2_{\infty,\infty}|\Omega|\right)
\end{aligned}$$

$$+ \frac{7}{2}\|f_1\|^2_{\infty,\infty}\|z_n\|^2 + \frac{7}{2}\varphi_n^{-2}\|f_2\|^2_{\infty,\infty}\|z_n\|^4_{L^4} + \frac{5}{14}\|\dot z_n\|^2 - \frac{c_3}{4\varphi_n^2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^4_{L^4}.$$

Inserting these inequalities, substracting $\frac{1}{2}\|\dot z_n\|^2$ and then multiplying by 2 gives

$$\begin{aligned}\|\dot z_n\|^2 = &- \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(z_n, z_n) - c_1\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + k_0\frac{\mathrm{d}}{\mathrm{d}t}\ln(1 - \|e_n\|^2_{\mathbb{R}^m}) - \frac{c_3}{2\varphi_n^2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^4_{L^4}\\ &+ 7\varphi_n^2\left(m\sum_{k=1}^m \|y_{\mathrm{ref},k}\|^2_\infty\|\mathcal{A}q_k\|^2 + \|I_{s,i}\|^2_{2,\infty} + \|f_0\|^2_{\infty,\infty}|\Omega| + \|f_1\|^2_{\infty,\infty}M + N\right)\\ &+ 7\|\dot\varphi\|^2_\infty M + 7\varphi_n^{-2}\|f_2\|^2_{\infty,\infty}\|z_n\|^4_{L^4}.\end{aligned}$$

Now we add and subtract $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2$, thus we obtain

$$\begin{aligned}\|\dot z_n\|^2 \le &-\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(z_n, z_n) - \left(c_1 + \frac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + k_0\frac{\mathrm{d}}{\mathrm{d}t}\ln(1 - \|e_n\|^2_{\mathbb{R}^m}) - \frac{c_3}{2\varphi_n^2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^4_{L^4}\\ &+ 7(\|\varphi\|_\infty + \Gamma_r^{-1})^2\\ &\times\left(m\sum_{k=1}^m \|y_{\mathrm{ref},k}\|^2_\infty\|\mathcal{A}q_k\|^2 + \|I_{s,i}\|^2_{2,\infty} + \|f_0\|^2_{\infty,\infty}|\Omega| + \|f_1\|^2_{\infty,\infty}M\right)\\ &+ 7(N(\|\varphi\|_\infty + \Gamma_r^{-1})^2 + \|\dot\varphi\|^2_\infty M) + 7\varphi_n^{-2}\|f_2\|^2_{\infty,\infty}\|z_n\|^4_{L^4} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2\end{aligned}$$

By the product rule

$$-\frac{c_3}{2\varphi_n^2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^4_{L^4} = -\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c_3}{2\varphi_n^2}\|z_n\|^4_{L^4}\right) - c_3\varphi_n^{-3}\dot\varphi_n\|z_n\|^4_{L^4}$$

we find that

$$\begin{aligned}\|\dot z_n\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}(z_n, z_n) - k_0\frac{\mathrm{d}}{\mathrm{d}t}\ln(1 - \|e_n\|^2_{\mathbb{R}^m}) + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c_3}{2\varphi_n^2}\|z_n\|^4_{L^4}\right)\\ \le -\left(c_1 + \frac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_1 + E_2\varphi_n^{-3}\|z_n\|^4_{L^4} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2,\end{aligned} \tag{5.40}$$

where

$$\begin{aligned}E_1 :=\ &7(\|\varphi\|_\infty + \Gamma_r^{-1})^2\\ &\times\left(m\sum_{k=1}^m \|y_{\mathrm{ref},k}\|^2_\infty\|Aq_k\|^2 + \|I_{s,i}\|^2_{2,\infty} + \|f_0\|^2_{\infty,\infty}|\Omega| + \|f_1\|^2_{\infty,\infty}M\right)\\ &+ 7(N(\|\varphi\|_\infty + \Gamma_r^{-1})^2 + \|\dot\varphi\|^2_\infty M),\\ E_2 :=\ &7\|f_2\|^2_{\infty,\infty}(\|\varphi\|_\infty + \Gamma_r^{-1}) + c_3\|\dot\varphi\|_\infty\end{aligned}$$

are independent of n and t.

Step 3c. We show uniform boundedness of e_n. Using (5.38) in (5.40) we obtain

$$\begin{aligned}\|\dot{z}_n\|^2 + \dot{\rho}_n \le &- \left(c_1 + \frac{1}{2}\right) \frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_1 + E_2\varphi_n^{-3}\|z_n\|_{L^4}^4 \\ &- \mathfrak{a}(z_n, z_n) - \frac{k_0\|e_n\|_{\mathbb{R}^m}^2}{1-\|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4 + K_1\varphi_n + K_2\varphi_n^2 \\ = &- \left(c_1 + \frac{1}{2}\right) \frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_2\varphi_n^{-3}\|z_n\|_{L^4}^4 \\ &- \mathfrak{a}(z_n, z_n) - \frac{k_0}{1-\|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4 + \Lambda\end{aligned}$$

where

$$\begin{aligned}\rho_n &:= \mathfrak{a}(z_n, z_n) - k_0 \ln(1 - \|e_n\|_{\mathbb{R}^m}^2) + \frac{c_3}{2\varphi_n^2}\|z_n\|_{L^4}^4, \\ \Lambda &:= E_1 + K_1(\|\varphi\|_\infty + \Gamma_r^{-1}) + K_2(\|\varphi\|_\infty + \Gamma_r^{-1})^2 + k_0,\end{aligned}$$

and we have used the equality

$$\frac{\|e_n\|_{\mathbb{R}^m}^2}{1-\|e_n\|_{\mathbb{R}^m}^2} = -1 + \frac{1}{1-\|e_n\|_{\mathbb{R}^m}^2}.$$

Adding and subtracting $k_0 \ln(1 - \|e_n\|_{\mathbb{R}^m}^2)$ leads to

$$\begin{aligned}\|\dot{z}_n\|^2 + \dot{\rho}_n \le &- \rho_n - \left(c_1 + \frac{1}{2}\right) \frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_2\varphi_n^{-3}\|z_n\|_{L^4}^4 \\ &- k_0\left(\frac{1}{1-\|e_n\|_{\mathbb{R}^m}^2} + \ln(1-\|e_n\|_{\mathbb{R}^m}^2)\right) + \Lambda \\ \le &- \rho_n - \left(c_1 + \frac{1}{2}\right) \frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_2\varphi_n^{-3}\|z_n\|_{L^4}^4 + \Lambda,\end{aligned} \tag{5.41}$$

where for the last inequality we have used that

$$\forall\, p \in (-1,1): \ \frac{1}{1-p^2} \ge \ln\left(\frac{1}{1-p^2}\right) = -\ln(1-p^2).$$

We may now use the integrating factor e^t to obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t\rho_n\right) = \mathrm{e}^t(\rho_n + \dot{\rho}_n) \le -\mathrm{e}^t\left(c_1 + \frac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\|z_n\|^2 + E_2\mathrm{e}^t\varphi_n^{-3}\|z_n\|_{L^4}^4 + \Lambda\mathrm{e}^t \underbrace{-\mathrm{e}^t\|\dot{z}_n\|^2}_{\le 0}$$

Integrating and using (5.39) yields that for all $t \in [\gamma, T_n)$ we have

$$\mathrm{e}^t\rho_n(t) - \rho_n(\gamma)\mathrm{e}^\gamma \le (E_2D_1 + \Lambda)\mathrm{e}^t + \kappa_n E_2 D_0 - \int_\gamma^t \mathrm{e}^s\left(c_1 + \frac{1}{2}\right)\frac{\mathrm{d}}{\mathrm{d}t}\|z_n(s)\|^2\,\mathrm{d}s$$

$$\le (E_2 D_1 + \Lambda)\mathrm{e}^t + \kappa_n E_2 D_0 + \left(c_1 + \frac{1}{2}\right) \|z_\gamma^n\|^2 \mathrm{e}^\gamma$$
$$+ \left(c_1 + \frac{1}{2}\right) \int_\gamma^t \mathrm{e}^s \|z_n(s)\|^2 \,\mathrm{d}s$$
$$\overset{(5.35)}{\le} (E_2 D_1 + \Lambda)\mathrm{e}^t + \kappa_n E_2 D_0 + \left(c_1 + \frac{1}{2}\right) \kappa_n^2 \mathrm{e}^\gamma (\|v_\gamma - q \cdot y_{\mathrm{ref}}(\gamma)\|^2)$$
$$+ \left(c_1 + \frac{1}{2}\right) (\|\varphi\|_\infty + \Gamma_r^{-1})^2 M \mathrm{e}^t.$$

Thus, there exist $\Xi_1, \Xi_2, \Xi_3 > 0$ which are independent of n and t such that

$$\rho_n(t) \le \rho_n(\gamma)\mathrm{e}^{-(t-\gamma)} + \Xi_1 + \kappa_n(\Xi_2 + \kappa_n \Xi_3)\mathrm{e}^{-(t-\gamma)}.$$

Invoking the definition of ρ_n and that $\mathrm{e}^{-(t-\gamma)} \le 1$ for $t \ge \gamma$ we find that

$$\forall\, t \in [\gamma, T_n): \;\; \rho_n(t) \le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3, \tag{5.42}$$

where

$$\rho_n^0 := \kappa_n^2 \mathfrak{a}(v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma), v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma)) - k_0 \ln(1 - \kappa_n^2 \|\mathcal{B}'(v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma))\|_{\mathbb{R}^m}^2)$$
$$+ \kappa_n^2 \|v_\gamma^n - q^n \cdot y_{\mathrm{ref}}(\gamma)\|_{L^4}^4 = \rho_n(\gamma).$$

Note that by construction of κ_n and the Sobolev embedding, $(\rho_n^0)_{n\in\mathbb{N}}$ is bounded, $\rho_n^0 \to 0$ as $n \to \infty$, so that ρ_n^0 can be bounded independently of n.
Again using the definition of ρ_n and (5.42) we find that

$$k_0 \ln\left(\frac{1}{1 - \|e_n\|_{\mathbb{R}^m}^2}\right) = \rho_n - \mathfrak{a}(z_n, z_n) - \frac{c_3}{2\varphi_n^2} \|z_n\|_{L^4}^4 \le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3,$$

and hence

$$\frac{1}{1 - \|e_n\|_{\mathbb{R}^m}^2} \le \exp\left(\frac{1}{k_0}\left(\rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3\right)\right) =: \varepsilon(n).$$

We may thus conclude that

$$\forall\, t \in [\gamma, T_n): \;\; \|e_n(t)\|_{\mathbb{R}^m}^2 \le 1 - \varepsilon(n). \tag{5.43}$$

This means that

$$\forall\, t \in [\gamma, T_n): \;\; \varphi_n(t)^2 \|\mathcal{B}'(v_n(t) - q^n \cdot y_{\mathrm{ref}}(t))\|_{\mathbb{R}^m}^2 \le 1 - \varepsilon(n). \tag{5.44}$$

Moreover,

$$\forall\, t \in [\gamma, T_n): \;\; \delta\varphi_n(t)^2 \|\nabla(v_n(t) - q^n \cdot y_{\mathrm{ref}}(t))\|^2 + \varphi_n(t)^2 \|v_n(t) - q^n \cdot y_{\mathrm{ref}}(t)\|_{L^4}^4$$
$$\le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3, \tag{5.45}$$

which implies that for all $t > 0$ it holds $v_n(t) \in W^{1,2}(\Omega)$.
Step 3d. We show that $T_n = \infty$. Assuming $T_n < \infty$ it follows from (5.43) that the graph of the solution (μ, ν) from Step 2 would be a compact subset of $\mathbb{D}$, a contradiction. Therefore, we have $T_n = \infty$.
Step 4. We show convergence of the approximate solution, uniqueness and regularity of the solution in $[\gamma, \infty) \times \Omega$.
Step 4a. Some inequalities. From (5.42) we have that

$$\varphi_n^{-2} \|z_n\|_{L^4}^4 \le \rho_n^0 + \Xi_1 + \kappa_n \Xi_2 + \kappa_n^2 \Xi_3.$$

Using a similar procedure as in (5.39) we can derive the following estimate

$$\int_\gamma^t \varphi_n(s)^{-3} \|z_n(s)\|_{L^4}^4 \, \mathrm{d}s \le \kappa_n d_0 + d_1 t, \tag{5.46}$$

for $d_0, d_1 > 0$ independent of n and t. Further, we can integrate (5.41) on the interval $[0, t]$ to obtain, invoking $\rho_n(t) \ge 0$ and (5.46),

$$\int_\gamma^t \|\dot{z}_n(s)\|^2 \, \mathrm{d}s \le \rho_n^0 + \left(c_1 + \frac{1}{2}\right) \kappa_n^2 (\|v_\gamma - q \cdot y_{\mathrm{ref}}(\gamma)\|^2) + E_2(\kappa_n d_0 + d_1 t) + \Lambda t$$

for all $t \ge \gamma$. Hence, there exist $S_0, S_1, S_2 > 0$ which are independent of n and t such that

$$\forall\, t \ge \gamma : \int_\gamma^t \|\dot{z}_n(s)\|^2 \, \mathrm{d}s \le \rho_n^0 + S_0 \kappa_n + S_1 \kappa_n^2 + S_2 t. \tag{5.47}$$

Hence, there exist $S_3, S_4 > 0$ such that

$$\forall\, t \ge \gamma : \int_\gamma^t \left\| \frac{\mathrm{d}}{\mathrm{d}t} (\varphi_n v_n) \right\|^2 \mathrm{d}s \le \rho_n^0 + S_0 \kappa_n + S_1 \kappa_n^2 + S_3 t + S_4. \tag{5.48}$$

Using (5.37) we can slightly improve (5.46), since

$$\begin{aligned}
\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|z_n\|^2 \le & - \mathfrak{a}(z_n, z_n) - (c_1 - \omega_0) \|z_n\|^2 + \|z_n\| \|w_n\| \\
& - \frac{k_0 \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} - \frac{c_3}{2\varphi_n^2} \|z_n\|_{L^4}^4 + K_0 \varphi_n^2 \\
\le & \, \omega_0 \|z_n\|^2 - \frac{c_3}{2\varphi_n^2} \|z_n\|_{L^4}^4 + (K_0 + \sqrt{NM}) \varphi_n^2 \\
& - \mathfrak{a}(z_n, z_n) - \frac{k_0 \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2} \\
= & \, \omega_0 \|z_n\|^2 - \frac{c_3}{2\varphi_n^2} \|z_n\|_{L^4}^4 + K_2 \varphi_n^2 \\
& - \mathfrak{a}(z_n, z_n) - \frac{k_0 \|e_n\|_{\mathbb{R}^m}^2}{1 - \|e_n\|_{\mathbb{R}^m}^2}
\end{aligned}$$

The former is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_n^{-2}\|z_n\|^2 \leq 2K_2 - c_3\varphi_n^{-4}\|z_n\|_{L^4}^4 - 2\varphi_n^{-2}\mathfrak{a}(z_n, z_n) - \frac{2k_0\varphi_n^{-2}\|e_n\|_{\mathbb{R}^m}^2}{1-\|e_n\|_{\mathbb{R}^m}^2}.$$

This implies that $\forall\, t \geq \gamma$

$$\begin{aligned} c_3\int_\gamma^t \varphi_n(s)^{-4}\|z_n(s)\|_{L^4}^4 + 2\varphi_n(s)^{-2}\mathfrak{a}(z_n(s), z_n(s)) - \frac{2k_0\varphi_n(s)^{-2}\|e_n(s)\|_{\mathbb{R}^m}^2}{1-\|e_n(s)\|_{\mathbb{R}^m}^2}\,\mathrm{d}s \\ \leq 2K_2 t + \|v_\gamma - q\cdot y_{\mathrm{ref}}(\gamma)\|^2, \end{aligned} \tag{5.49}$$

which is bounded independently of n. This shows that $\forall\, t \geq \gamma$ it holds

$$\begin{aligned} c_3\int_\gamma^t \|v_n(s) - q^n\cdot y_{\mathrm{ref}}(s)\|_{L^4}^4\,\mathrm{d}s + \int_\gamma^t 2\mathfrak{a}(v_n(s) - q^n\cdot y_{\mathrm{ref}}(s), v_n(s) - q^n\cdot y_{\mathrm{ref}}(s))\,\mathrm{d}s \\ + \int_\gamma^t \frac{2k_0\|\mathcal{B}'(v_n(s) - q^n\cdot y_{\mathrm{ref}}(s))\|_{\mathbb{R}^m}^2}{1-\varphi_n(s)^2\|\mathcal{B}'(v_n(s) - q^n\cdot y_{\mathrm{ref}}(s))\|_{\mathbb{R}^m}^2}\,\mathrm{d}s \leq 2K_2 t + \|v_\gamma - q\cdot y_{\mathrm{ref}}(\gamma)\|^2. \end{aligned} \tag{5.50}$$

We require a last calculation to prove that $\|\dot{w}_n\|^2$ is bounded independently of n and t. To this end multiply (5.30b) by $\dot{\nu}_j$ and sum over j to obtain

$$\|\dot{w}_n\|^2 = -(c_4 - \omega_0)\langle w_n, \dot{w}_n\rangle + c_5\langle z_n, \dot{w}_n\rangle + \varphi_n\langle g, \dot{w}_n\rangle.$$

Using $(\omega_0 - c_4)w_n = (\dot{\varphi}_n - c_4\varphi_n)\varphi_n^{-1}w_n$ and the inequalities

$$\begin{aligned} -(c_4 - \omega_0)\langle w_n, \dot{w}_n\rangle &\leq \frac{3}{2}\|\dot{\varphi} - c_4\varphi\|_\infty^2\varphi_n^{-2}\|w_n\|^2 + \frac{\|\dot{w}_n\|^2}{6} \\ &\leq \frac{3}{2}(\|\dot{\varphi}\|_\infty + c_4(\|\varphi\|_\infty + \Gamma_r^{-1}))^2 N + \frac{\|\dot{w}_n\|^2}{6}, \\ c_5\langle z_n, \dot{w}_n\rangle &\leq \frac{3c_5^2}{2}\|z_n\|^2 + \frac{1}{6}\|\dot{w}_n\|^2 \\ &\leq \frac{3c_5^2 M}{2}(\|\varphi\|_\infty + \Gamma_r^{-1})^2 + \frac{1}{6}\|\dot{w}_n\|^2, \\ \varphi_n\langle g, \dot{w}_n\rangle &\leq \frac{3}{2}(\|\varphi\|_\infty + \Gamma_r^{-1})^2\|g\|_{\infty,\infty}^2|\Omega| + \frac{1}{6}\|\dot{w}_n\|^2. \end{aligned}$$

It follows now that for all $t \geq \gamma$ it holds

$$\begin{aligned} \|\dot{w}_n(t)\|^2 \leq 3\|(\|\dot{\varphi}\|_\infty + c_4(\|\varphi\|_\infty + \Gamma_r^{-1}))^2 N + 3c_5^2 M(\|\varphi\|_\infty + \Gamma_r^{-1})^2 \\ + 3(\|\varphi\|_\infty + \Gamma_r^{-1})^2\|g\|_{\infty,\infty}^2|\Omega|, \end{aligned} \tag{5.51}$$

which is bounded independently of n and t. Multiplying (5.30b) by φ_n^{-1} and θ_i and adding over $i \in \{0, \dots, n\}$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_n^{-1}w_n) = -\varphi^{-2}\dot{\varphi}_n w_n + \varphi_n^{-1}\dot{w}_n = -c_4\varphi_n^{-1}w_n + c_5\varphi_n^{-1}z_n + g_n,$$

where

$$g_n := \sum_{i=0}^{n} \langle g, \theta_i \rangle \theta_i.$$

If we now take the norm of the latter we have that

$$\begin{aligned} \left\| \frac{\mathrm{d}}{\mathrm{d}t}(\varphi_n^{-1} w_n) \right\| &\leq c_4 \varphi_n^{-1} \|w_n\| + c_5 \varphi_n^{-1} \|z_n\| + \|g_n\| \\ &\leq c_4 N + c_5 M + \|g\|_{\infty,\infty}. \end{aligned}$$

Thus,

$$\forall\, t \geq \gamma : \ \|\dot{u}_n(t)\| \leq c_4 N + c_5 M + \|g\|_{\infty,\infty}. \tag{5.52}$$

Step 4b. Weak convergence of the solution.
Let $T > \gamma$ be given. As in Section 5.6.1, we have $v_n \in L^4((\gamma, T) \times \Omega)$, $v_n \in L^2([\gamma, T]; W^{1,2}(\Omega))$, $\dot{v}_n \in L^2([\gamma, T]; (W^{1,2}(\Omega))')$, since (5.50) together with (5.44) implies that $I_{s,e}^n \in L^2([\gamma, T]; \mathbb{R}^m)$ and $v_n \in L^2([\gamma, T]; W^{1,2}(\Omega))$.
Analogously to Section 5.6.1 we have that there exist subsequences such that

$$\begin{aligned} u_n &\to u \in W^{1,2}([\gamma, T]; L^2(\Omega)) \text{ weakly}, \\ v_n &\to v \in L^2([\gamma, T]; W^{1,2}(\Omega)) \text{ weakly}, \\ \dot{v}_n &\to \dot{v} \in L^2([\gamma, T]; (W^{1,2}(\Omega))') \text{ weakly}, \end{aligned}$$

so that $u, v \in C([\gamma, T]; L^2(\Omega))$. Also $v_n^2 \to v^2$ weakly in $L^2((\gamma, T) \times \Omega)$ and $v_n^3 \to v^3$ weakly in $L^{4/3}((\gamma, T) \times \Omega)$.
We have further properties of u and v. By (5.36), (5.45), (5.48) & (5.52) we have that $u_n, \dot{u}_n$ are in a bounded set of $L^\infty([\gamma, T]; L^2(\Omega))$ and that v_n is in a bounded set of $L^\infty([\gamma, T]; L^2(\Omega))$ and the bound is independent of T. Moreover, $\frac{\mathrm{d}}{\mathrm{d}t}(\varphi_n v_n)$ is in $L^2([\gamma, T]; L^2(\Omega))$. Using Lemma 1.4.6, we have subsequences such that

$$\begin{aligned} u_n &\to u \in L^\infty([\gamma, T]; L^2(\Omega)) \text{ weak}^\star, \\ \dot{u}_n &\to \dot{u} \in L^\infty([\gamma, T]; L^2(\Omega)) \text{ weak}^\star, \\ v_n &\to v \in L^\infty([\gamma, T]; L^2(\Omega)) \text{ weak}^\star, \\ \varphi_n v_n &\to \varphi v \in L^\infty([\gamma, T]; W^{1,2}(\Omega)) \text{ weak}^\star, \\ \dot{v}_n &\to \dot{v} \in L^2([\gamma, T]; (W^{1,2}(\Omega))') \text{ weakly}, \\ \varphi_n \dot{v}_n &\to \varphi \dot{v} \in L^2([\gamma, T]; L^2(\Omega)) \text{ weakly}, \end{aligned}$$

since $\varphi_n \to \varphi$ in $BC([\gamma, T]; \mathbb{R})$. Moreover, by $\inf_{t > \gamma} \varphi(t) > 0$, we also have that $v \in L^\infty([\gamma + \delta, T]; W^{1,2}(\Omega))$ and $\dot{v} \in L^2([\gamma + \delta, T]; L^2(\Omega))$ for all $\delta > 0$.
Further, $\kappa_n, \rho_n^0 \to 0$ and

$$\varepsilon(n) \to \varepsilon_0 := \exp\left(-k_0^{-1} \Xi_1\right)$$

as $n \to \infty$.

Thus, by (5.36), (5.44), (5.45) & (5.50) we have $v \in L^4((\gamma,T)\times\Omega)$ and for a.e. $t \in [\gamma,T)$ the following estimates hold:

$$\begin{aligned} &\|v(t) - q\cdot y_{\mathrm{ref}}(t)\| \le \sqrt{M},\\ &\|u(t)\| \le \sqrt{N},\\ &\varphi(t)^2\|\mathcal{B}'v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m} \le 1-\varepsilon_0,\\ &\delta\varphi(t)^2\|\nabla(v(t) - q\cdot y_{\mathrm{ref}}(t))\|^2 + \varphi(t)^2\|v(t) - q\cdot y_{\mathrm{ref}}(t)\|^4_{L^4} \le \Xi_1,\\ &\int_\gamma^t \|v(s) - q\cdot y_{\mathrm{ref}}(s)\|^4_{L^4}\,\mathrm{d}s \le 2K_2 t + \|v_\gamma - q\cdot y_{\mathrm{ref}}(\gamma)\|^2. \end{aligned} \tag{5.53}$$

Moreover, as in Section 5.6.1, $v_n \to v$ strongly in $L^2(\gamma,T;L^2(\Omega))$. Further, $u,v \in C([\gamma,T);L^2(\Omega))$ and $(u(\gamma),v(\gamma)) = (u_\gamma,v_\gamma)$.
Hence, for $\chi \in L^2(\Omega)$ and $\theta \in W^{1,2}(\Omega)$ we have that (u_n,v_n) fulfill the integrated version of (5.32), so we obtain that for $t \in (\gamma,T)$ it holds

$$\begin{aligned} \langle v(t),\theta\rangle &= \langle v_\gamma,\theta\rangle + \int_\gamma^t -\mathfrak{a}(v(s),\theta) + \langle p_3(v(s)) - u(s) + I_{s,i}(s),\theta\rangle\,\mathrm{d}s\,,\\ &\quad + \int_\gamma^T \langle I_{s,e}(s),\mathcal{B}'\theta\rangle_{\mathbb{R}^m}\,\mathrm{d}s\,,\\ \langle u(t),\chi\rangle &= \langle u_\gamma,\chi\rangle + \int_\gamma^t \langle c_5 v(s) - c_4 u(s),\chi\rangle\,\mathrm{d}s\,,\\ I_{s,e}(t) &= -\frac{k_0}{1-\varphi(t)^2\|\mathcal{B}'v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m}}(\mathcal{B}'v(t) - y_{\mathrm{ref}}(t)) \end{aligned}$$

by bounded convergence [28, Theorem II.4.1]. Thus, (u,v) is a solution of (5.7) in (γ,T). Moreover, (5.26) also holds in $(W^{1,2}(\Omega))'$ for $t \ge \gamma$, that is

$$\dot v(t) = \mathcal{A}v(t) + p_3(v(t)) + \mathcal{B}I_{s,e}(t) - u(t) + I_{s,i}(t). \tag{5.54}$$

Step 4c. Uniqueness of the solution and regularity.
By using a similar argumentation as in Section 5.6.1 it can be seen that the solution (u,v) is unique on (γ,T) and this holds for all $T > \gamma$ by invoking Lemma 4.4.4 and using a similar construction of a region Q_Λ. Hence, we can choose $T = \infty$ and define the solution for all $t > \gamma$ as the restriction on subintervals (γ,t). By (5.53) we further have that $u,v \in BC([\gamma,\infty);L^2(\Omega))$ with $I_{s,e} \in L^\infty([\gamma,\infty);\mathbb{R}^m)$, $v \in L^\infty([\gamma+\delta,\infty);W^{1,2}(\Omega))$ for all $\delta > 0$. Hence, by the continuity, (u,v) is the unique solution of 5.7 in $[0,\infty)$.
In order to show the regularity of the solution, note that for all $\delta > 0$ we have that

$$v \in L^2_{\mathrm{loc}}([\gamma,\infty);W^{1,2}(\Omega)) \cap L^\infty([\gamma+\delta,\infty);W^{1,2}(\Omega)),$$

so that $I_r := I_{s,i} + c_2 v^2 - c_3 v^3 - u \in L^2_{\mathrm{loc}}([\gamma,\infty);L^2(\Omega)) \cap L^\infty([\gamma+\delta,\infty);L^2(\Omega))$, and the application of Proposition 5.5.3 yields $v \in BC([\gamma,\infty);L^2(\Omega)) \cap BUC((\gamma,\infty);W^{1,2}(\Omega))$.

By the uniform continuity of v and the completeness of $W^{1,2}(\Omega)$, v has a limit at $t = \gamma$, see for instance [107, Theorem II.13.D]. Thus, $v \in L^\infty([\gamma, \infty); W^{1,2}(\Omega))$. From Section 5.6.1 and the latter we have that $v \in L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)) \cap L^\infty([\delta, \infty); W^{1,2}(\Omega))$ for all $\delta > 0$, so we have

$$\begin{aligned} I_{s,e} &\in L^2_{\text{loc}}([0, \infty); \mathbb{R}^m) \cap L^\infty([\delta, \infty); \mathbb{R}^m), \\ v &\in L^2_{\text{loc}}([0, \infty); W^{1,2}(\Omega)) \cap L^\infty([\delta, \infty); W^{1,2}(\Omega)), \end{aligned}$$

so that $I_r := I_{s,i} + c_2 v^2 - c_3 v^3 - u \in L^2_{\text{loc}}([0, \infty); L^2(\Omega)) \cap L^\infty([\delta, \infty); L^2(\Omega))$.
Iterating Proposition 5.5.3, for all $\delta > 0$, we have that the unique solution of (5.54) satisfies

$$v \in BC([\delta, \infty); L^2(\Omega)) \cap C^{0,\lambda}([\delta, \infty); L^2(\Omega)) \tag{5.55a}$$

for $r = 0$ and all $\lambda \in (0, 1)$;

$$v \in BC([\delta, \infty); L^2(\Omega)) \cap C^{0,1-r/2}([\delta, \infty); L^2(\Omega)) \cap C^{0,1-r}([\delta, \infty); W^{r,2}(\Omega)) \tag{5.55b}$$

for $r \in (0, 1)$; and

$$v \in BC([\delta, \infty); L^2(\Omega)) \cap C^{0,1/2}([\delta, \infty); L^2(\Omega)) \cap BUC([\delta, \infty); W^{1,2}(\Omega)) \tag{5.55c}$$

for $r = 1/2$. Since $u, v \in BC([0, \infty); L^2(\Omega))$ and $\dot{u} = c_4 v - c_5 u$, we have as well $\dot{u} \in BC([0, \infty), L^2(\Omega))$.
From (5.55) and $\mathcal{B}' \in \mathscr{L}(W^{r,2}(\Omega), \mathbb{R}^m)$ for $r \in [0, 1]$ it holds that

- for $r = 0$ and $\lambda \in (0, 1)$
$$y = \mathcal{B}' v \in C^{0,\lambda}([\delta, \infty); \mathbb{R}^m);$$
- for $r \in (0, 1)$
$$y = \mathcal{B}' v \in C^{0,1-r}([\delta, \infty); \mathbb{R}^m);$$
- for $r = 1$
$$y = \mathcal{B}' v \in BUC([\delta, \infty); \mathbb{R}^m).$$

If we further have that $\mathcal{B} \in \mathscr{L}(\mathbb{R}^m, W^{1,2}(\Omega))$, there exist $b_1, \ldots, b_m \in W^{1,2}(\Omega)$ such that $(\mathcal{B}' x)_i = \langle x, b_i \rangle$ for all $i = 1, \ldots, m$ and $x \in L^2(\Omega)$. By using the b_i in the weak formulation for $i = 1, \ldots, m$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle v(t), b_i \rangle = -\mathfrak{a}(v(t), b_i) + \langle p_3(v(t)) - u(t) + I_{s,i}(t), b_i \rangle + \langle I_{s,e}(t), \mathcal{B}' b_i \rangle_{\mathbb{R}^m}.$$

Since $(\mathcal{B}' v(t))_i = \langle v(t), b_i \rangle$, this leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} (\mathcal{B}' v(t)_i) = -\mathfrak{a}(v(t), b_i) + \langle p_3(v(t)) - u(t) + I_{s,i}(t), b_i \rangle + \langle I_{s,e}(t), \mathcal{B}' b_i \rangle_{\mathbb{R}^m}.$$

Taking the absolute value and using the Cauchy-Schwarz inequality yields

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}'v(t))_i\right| \leq \|D\|_{L^\infty}\|v(t)\|_{W^{1,2}}\|b_i\|_{W^{1,2}} + \|p_3(v(t)) - u(t) + I_{s,i}(t)\|_{L^2}\|b_i\|_{L^2} + \|I_{s,e}(t)\|_{\mathbb{R}^m}\|\mathcal{B}'b_i\|_{\mathbb{R}^m},$$

and therefore

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{B}'v)_i\right\|_{L^\infty(\delta,\infty;\mathbb{R}^m)} < \infty. \quad \forall\,\delta > 0. \tag{5.56}$$

Further, from (5.53) we have

$$\varphi(t)^2\|\mathcal{B}'v(t) - y_{\mathrm{ref}}(t)\|^2_{\mathbb{R}^m} \leq 1 - \varepsilon_0 \quad \forall\,t \geq \delta.$$

Hence, $I_{s,e} \in L^\infty([\delta,\infty);\mathbb{R}^m)$ has the same regularity as y, since we have that $\varphi \in \Phi_\gamma$ and $y_{\mathrm{ref}} \in W^{1,\infty}([0,\infty);\mathbb{R}^m)$.
Last but not least, if $v_0 \in \mathcal{D}(\mathcal{A})$, it follows from (5.28) that $v \in L^\infty([0,\infty);W^{1,2}(\Omega))$ and $I_{s,e} \in L^\infty([0,\infty);\mathbb{R}^m)$, so that $I_r := I_{s,i} + c_2v^2 - c_3v^3 - u \in L^\infty([0,\infty);L^2(\Omega))$. Proposition 5.5.3 implies that (5.55) holds as well with $\delta = 0$. The continuity of $\mathcal{B}'$ together with the closed loop yield now the regularity of $y, I_{s,e}$ at $\delta = 0$. In particular, (5.56) also holds with $\delta = 0$. □

5.7 Outlook

Under minimal assumptions, we have been able to successfully show the feasibility of the system described by 5.3, which includes distributed and boundary control in a multiple-input-multiple-output setting. Moreover, we did not restrict the operator $\mathcal{A}$ to be the Neumann Laplacian, but considered a general symmetric, Neumann elliptic operator, which under enough regularity of the diffusion matrix D and the boundary Γ, corresponds to the diffusion operator

$$\mathcal{A}z = \mathrm{div}D\nabla z, \quad (\nu \cdot D\nabla z)|_\Gamma = 0.$$

Moreover, we have been able to show Hölder regularity of the solution, input and outputs depending on the observation operator.
Having a look at Lemma 5.5.2, one could think of approaching the problem in a different fashion by trying to solve the equation by means of a fixed point argument. Of course two problems quickly arise. On the one hand, because of the feedback law, the domain in which the solution lays becomes time-dependent. Secondly, it is not clear how to exploit the positivity encountered when considering the weak formulation of the system, which guarantees the uniform boundedness of the error. However, this procedure could hasten the process of obtaining a solution of the equation so that the proof of Theorem 5.3.3 becomes shorter or more elegant. From both a mathematical

and practical viewpoint, it would be also interesting to see if the funnel controller used in [101] can be also applied in this context, since it has been proved to be feasible for the heat equation in a similar setting.

6 Conclusions

For the drawing of the conclusions we first wish to highlight the problems encountered in the different chapters. A common denominator in all of them is that the funnel controller is a nonlinear, non-autonomous feedback relation, so that we end up dealing with a nonlinear, non-autonomous PDE, independently of the fact whether the initial open-loop system was linear or not. Since there is no general theory of PDEs, this already represents a major difficulty in dealing with this setting. Moreover, most of the existing theory on nonlinear PDEs consider the *initial value problem (IVP)* on a compact time interval $[0, T]$, that is, finite time horizon, which somehow seems inappropriate given the nature of the problem, since one wants to show that the funnel controller is feasible for all $t \in [0, \infty)$ and the error remains uniformly bounded away from the funnel boundary.

Whereas in Chapters 2 & 3 it has been sufficient to show that the dynamics of the system fit within the framework of the existing theory on the funnel controller given in [11], Chapters 4 & 5 have required much more work, but at the same time, they deliver general results for a broad class of systems.

Even though Chapter 2 serves as an example to motivate the FC with internal dynamics involving PDEs, it becomes clear that the important relation to study is the input-output behavior, so that the PDE goes into a second plane. Trying to distill the key arguments, we show in Chapter 3 when the FC is feasible for a specific class of systems whose internal dynamics is described by an infinite-dimensional system. Since these systems have unbounded observation and control operators, which usually come from a boundary-control-like system, the natural step is to study such structures. Since BCS can be both hyperbolic and parabolic systems, the well-posedness approach does not seem to be the best one, since these systems exhibit complete different properties, and hence, we consider from the very beginning the closed-loop system.

In fact, focusing on Chapter 4, the literature does not specifically tackle the problem at hand. The key idea resides in the transformation carried out to remove the time-dependence from the funnel controller, which leads to an abstract nonlinear Cauchy problem of the form (1.8). The nonlinearity is hidden in the domain of an operator, which happens to be nonlinear and m-dissipative. The price of obtaining an autonomous operator is a time-varying linear perturbation and an inhomogeneity. Dealing with the closed-loop system from the beginning has several advantages. First, there are no assumptions on the open-loop system, in particular no well-posedness assumptions

—in any sense— are made. Secondly, the construction of a solution on a compact interval already shows feasibility of the FC in all intervals with finite horizon. Under extra dissipativity assumptions it is then easy to show that the error remains uniformly bounded away from the funnel boundary. Intrinsically, the methods used to prove the main results lay on the theory of nonlinear, m-dissipative operators, which enjoy of a certain mathematical elegance. Moreover, in the parabolic scenario, one can exploit the structure of the problem to obtain a better result than in the general case. This brings to light that the FC seems to be feasible for systems having stable internal dynamics also in the infinite-dimensional scenario.

Following this train of thought and the techniques learned in Chapter 4, in Chapter 5, we have been able to show that the FC is feasible for a particular nonlinear parabolic system, which involves the so-called FitzHugh-Nagumo potential. The length of the proof does not completely represent its complexity, since most of it is devoted to obtain the necessary inequalities to argue convergence of subsequences in weak and weak* topologies. The main difficulty is indeed the step in which one guarantees that the error remains uniformly bounded away from the funnel boundary while showing that the state satisfies some regularity properties. Hence, nonlinearities or non-autonomous systems do not seem to be an inconvenient, as long as the structure of the problem is reach enough.

As of today, the content presented in this dissertation represents the major work involving the funnel controller for systems whose state is described by a PDE with unbounded control and observation operators. It brings the applicability of the controller to a completely new world and pushes it way beyond by utterly leaving the finite-dimensional constellation. It lies the fundamental pillars to further explore the feasibility of the FC in the infinite-dimensional scenario and also to consider infinite-dimensional observation and control, even though from a practical viewpoint this may not be of interest. Moreover, it provides the main tools for extending the topic into nonlinear or non-autonomous systems. If one considers that the principal aim of the funnel controller is to track a reference signal with prescribed transient behavior, this project generalizes in some sense the one and only similar result in this direction given in [101].

6.1 Open questions

While the current problems in Chapter 2 have to do with the modeling of the system, that is, whether we work with the fully nonlinear equations or we use higher spatial dimensions, the other chapters present more challenging issues. The natural step in Chapter 3 would be to consider nonlinear internal dynamics and bounded control and observation operators. A reasonable assumption to generalize the linear case could be *input-to-state stability (ISS)* [109], and from there start studying the input-output

behavior.
As already mentioned in Chapter 4, it is not clear yet how to exploit the structure of the problem when the internal dynamics of the system are stable. Ideally, one should aim to find a characterization between the stability of the internal dynamics and the uniform boundedness of u as well as finding out when it is possible to show boundedness of u without needing boundedness of $\dot{x}$.
A second problem has to do with the nature of the system treated in Chapter 5. It is well-known —see for instance [94, Section 8.2]— that one can find a solution to the nonlinear parabolic problem related to the FitzHugh-Nagumo model by means of the mild solution and a fixed point argument. However, there are two aspects of this method which are not directly transferable to our setting. The first one has of course to do with the non-autonomous feedback, since the set on which one would apply the fixed point argument becomes time-dependent. Secondly, in order to obtain a global solution of the system together with having a uniformly bounded input, one requires the weak formulation of the system and the positivity of some operators, like the form associated to the elliptic operator. For that, it is not sufficient to have an integrated equation which corresponds to the mild solution, since this positivity cannot be exploited.

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List of notations

Acronyms

ACP	abstract Cauchy problem – p. 24
ANCP	abstract nonlinear Cauchy problem – p. 38
BCS	boundary control system – p. 6
BIBO	bounded-input, bounded-output – p. 73
FC	funnel controller – p. 3
ID	internal dynamics – p. 4
ISS	input-to-state stability – p. 176
IVP	initial value problem – p. 175
ODE	Ordinary Differential Equation – p. 2
PDE	Partial Differential Equation – p. 1
SC	solution concept – p. 44

General numbers, sets and spaces

$\bigotimes_{i=1}^{n} X_i$	product space, often denoted by $X_1 \times \cdots \times X_n$ – p. 7
X'	topological dual space of X, that is, $X' = \mathscr{L}(X, \mathbb{K})$ – p. 8
$\langle x', x \rangle$	duality pairing, that is, $x'(x)$ for $x \in X$ and $x' \in X'$ – p. 8
$\overline{S}$	closure of a set S – p. 7
$\operatorname{int} S$	interior of a set S – p. 7
$\lfloor r \rfloor$	truncation of $r \in [0, \infty)$ – p. 58

$\mathbb{C}$	set of complex numbers – p. 7
$\mathbb{C}_\alpha$	set of complex numbers whose real part is larger than $\alpha \in \mathbb{R}$ – p. 7
$\mathbb{C}^n$	space of comples n-dimensional vectors – p. 7
$\mathbb{C}^{n\times m}$	space of comples $n \times m$-matrices – p. 7
$\mathcal{D}(T)$	domain of the operator $T : \mathcal{D}(T) \subset X \to Y$ – p. 9
$\mathbf{Gl}_n(\mathbb{C})$	set of invertible $n \times n$-matrices – p. 7
$\mathbf{Gl}_n(\mathbb{R})$	set of invertible $n \times n$-matrices – p. 7
$\mathcal{G}(T)$	graph of the operator $T : \mathcal{D}(T) \subset X \to Y$ – p. 9
i	imaginary unit – p. 7
$\mathbb{K}$	either $\mathbb{R}$ or $\mathbb{C}$ – p. 7
$\ker T$	kernel of the operator $T : \mathcal{D}(T) \subset X \to Y$ – p. 9
$\mathscr{L}(X,Y)$	space of bounded linear operators mapping from X to Y – p. 7
$\mathscr{L}(X)$	space of bounded linear operators mapping from X to X – p. 8
$\mathbb{N}$	set of natural numbers – p. 7
$\mathbb{N}_0$	set of natural numbers including 0 – p. 7
$\mathbb{Q}$	set of rational numbers – p. 7
$\mathbb{R}$	set of real numbers – p. 7
$\mathbb{R}_\infty$	set of real numbers including $+\infty$ – p. 36
$\mathbb{R}^n$	space of real n-dimensional vectors – p. 7
$\mathbb{R}^{n\times m}$	space of real $n \times m$-matrices – p. 7
$\mathcal{R}(T)$	range of the operator $T : \mathcal{D}(T) \subset X \to Y$ – p. 9
$\rho(T)$	resolvent set of the operator T – p. 10
$\sigma(T)$	spectrum of the operator T – p. 10
$\sigma_p(T)$	point spectrum of the operator T – p. 10

$\mathbb{Z}$	set of integers – p. 7

General operators and functions

$\Vert\cdot\Vert_X$	norm in the space X – p. 7
$\Vert\cdot\Vert_{\mathcal{D}(T)}$	graph norm – p. 9
$\langle\cdot,\cdot\rangle_X$	scalar or inner product in the space X – p. 7
$\overline{z}$	complex conjugate of $z \in \mathbb{C}$ – p. 7
$\lvert z\rvert$	absolute value of $z \in \mathbb{C}$ – p. 7
$\operatorname{Im} z$	imaginary part of $z \in \mathbb{C}$ – p. 7
$\operatorname{Re} z$	real part of $z \in \mathbb{C}$ – p. 7
A^*	Hermitian of a matrix – p. 7
$A^\top$	transposed of a matrix – p. 7
I_n	identity matrix, sometimes just I – p. 7
I_X	identity operator from X to X, sometimes just I – p. 8
T^*	adjoint of an operator T – p. 10
T'	dual of an operator T – p. 9
$T\vert_U$	restriction of the operator T to the space U – p. 9

Function spaces

$BC^k(\Omega)$	space of k-times continuously differentiable functions with bounded derivatives up to order k – p. 16
$BC^k(J;B)$	space of k-times continuously differentiable Banach-valued functions with bounded derivatives up to order k – p. 21
$BUC(J;B)$	space of bounded and uniformly continuous Banach-valued functions – p. 21
$C(\Omega)$	set of continuous functions $\phi : \Omega \to \mathbb{K}$ – p. 12
$C^k(\Omega)$	set of k-times continuously differentiable functions $\phi : \Omega \to \mathbb{K}$. $C^0(\Omega) \coloneqq C(\Omega)$ – p. 12
$C^k(J;B)$	set of k-times continuously differentiable Banach-valued functions $f : J \to B$ – p. 21

$C^\infty(\Omega)$	set of infinite times continuously differentiable functions $\phi : \Omega \to \mathbb{K}$ – p. 12
$C_0^\infty(\Omega)$	set of infinite times continuously differentiable functions $\phi : \Omega \to \mathbb{K}$ with compact support – p. 12
$C^{k,\lambda}(\overline{\Omega})$	set of Hölder k-times continuously differentiable functions $\phi : \overline{\Omega} \to \mathbb{K}$ of order λ – p. 13
$C^{0,\alpha}(J;B)$	set of Hölder continuous Banach-valued functions of order α – p. 21
ℓ^∞	space of sequences which are bounded – p. 15
ℓ^p	space of sequences which the p-th power of the absolute value is sumable – p. 15
$L^\infty(\Omega)$	space of equivalence classes of measurable functions on Ω which are essentially bounded – p. 14
$L^p(\Omega)$	space of equivalence classes of functions for which the p-th power of the absolute value is Lebesgue integrable – p. 14
$L^p(J;B)$	space of equivalence classes of functions f strongly measurable on J into B such that $\|f(\cdot)\|_B \in L^p(J)$ – p. 20
$L^2_\omega(J;B)$	exponentially weighted $L^2(J;B)$ space – p. 21
$L^p_{\mathrm{loc}}(J;B)$	space of functions which are locally in $L^p(J;B)$ – p. 21
$W^{k,p}(\Omega)$	Sobolev space of integer order k – p. 15
$W^{1,p}_{\mathrm{loc}}(J;B)$	space of functions which are locally in $W^{1,p}(J;B)$ – p. 21
$W^{1,p}_{\mathrm{loc}}(J;B)$	space of functions which are locally in $W^{1,p}(J;B)$ – p. 21
$W^{s,p}(\Omega)$	Sobolev space of real order s – p. 17

Special notations

$u \underset{\tau}{\diamond} v$	τ-concatenation of u and v – p. 29
$\partial\Psi$	subdifferential of the functional Ψ – p. 36
$\Psi'(u,v)$	directional derivative of Ψ at u in the direction v – p. 37
$\Psi'(u)$	G-derivative of Ψ at u – p. 37

$e_{\partial,\mathcal{H}x}$	usually denotes the boundary effort of a port-Hamiltonian system. For higher spatial dimensional port-Hamiltonian systems, $\mathcal{H}x$ is omitted – p. 99
$f_{\partial,\mathcal{H}x}$	usually denotes the boundary flow of a port-Hamiltonian system. For higher spatial dimensional port-Hamiltonian systems, $\mathcal{H}x$ is omitted – p. 99
$C_0([0,\infty);\mathbb{R})$	set of continuous functions $f:[0,\infty)\to\mathbb{R}$ with $f(0)=0$ – p. 56
δ_t	Dirac delta distribution at $t\in\mathbb{R}$ – p. 64
$\frac{\mathrm{d}f}{\mathrm{d}t}$ $\left(\frac{\mathrm{d}^k f}{\mathrm{d}t^k}\right)$	derivative of the funtion of one variable f – p. 12
$\frac{\partial f}{\partial \zeta}$ $\left(\frac{\partial^k f}{\partial \zeta^k}\right)$	derivative of the function of several variables f with respect to the variable ζ – p. 12
D_k	differential operator $\frac{\partial}{\partial x_k}$ – p. 12
D^α	differential operator of order $\lvert\alpha\rvert$ – p. 12
e	usually denotes the error between output and reference signal, $y-y_{\text{ref}}$ – p. 3
$\mathcal{F}(\cdot)$	Fourier transform – p. 65
$\mathfrak{A}$	usually the linear differential operator of a BCS – p. 85
$\mathfrak{B}$	usually the linear control operator a BCS – p. 85
$\mathfrak{C}$	usually the linear observation operator a BCS – p. 85
$(\mathfrak{A},\mathfrak{B},\mathfrak{C})$	the tuple denotes a BCS – p. 87
$\mathcal{F}_\varphi$	performance funnel with funnel boundary $1/\varphi$ – p. 4
G	usually denotes the transfer function of a linear system – p. 32
$\mathcal{H}$	usually denotes Hamiltonian density matrix of a port-Hamiltonian system – p. 99
$\mathbb{F}=(\mathbb{F}_t)_{t\geq 0}$	input to output map – p. 30
$\Phi=(\Phi_t)_{t\geq 0}$	input to state map – p. 30

$\mathcal{L}(\cdot)$	Laplace transform – p. 62
$\mathbf{P}_\tau$	right shift operator – p. 29
$\mathrm{M}([0,\infty))$	set of measures of bounded total variation – p. 64
∇	differential operator $(\partial_{x_1} \cdots \partial_{x_n})^\top$ – p. 12
$\nabla \cdot F$	divergence of a vector field F – p. 12
∇f	gradient of a scalar function f – p. 12
φ	usually denotes the reciprocal function of the funnel boundary in the funnel controller – p. 4
Φ	usually denotes the class of admissible funnel boundaries – p. 53
Φ_r	usually denotes the class of admissible funnel boundaries which are r times continuously differentiable – p. 75
P_k	usually denotes the matrices of the port-Hamiltonian differential operator – p. 98
$\mathbf{S}_\tau$	right shift operator – p. 29
$\mathbb{T} = (\mathbb{T}_t)_{t\geq 0}$	C_0-semigroup – p. 23
$\mathbb{T} = (\mathbb{T}_t)_{t\in\mathbb{R}}$	C_0-group – p. 25
$\omega_0(\mathbb{T})$	growth bound of the C_0-semigroup $\mathbb{T}$ – p. 24
Σ	usually a well-posed linear system – p. 29
$\Psi = (\Psi_t)_{t\geq 0}$	state to output map – p. 30
U	usually the input space – p. 29
u	usually the input of a system – p. 30
$W_0^{1,\infty}([0,\infty);\mathbb{K})$	functions f in $W^{1,\infty}(0,\infty;\mathbb{K})$ with $f(0)=0$ – p. 61
W_B, W_C	usually denotes the input and output matrices of a port-Hamiltonian system – p. 99
X	usually the state space – p. 29

x	usually the state of a system – p. 30
x_0	usually the initial state of a system, sometimes also x^0 – p. 30
X_1	completion of a space X with the graph norm of $\beta I - T$, T a semigroup generator, $\beta \in \rho(T)$ – p. 10
X_{-1}	completion of a space X with the graph norm of $(\beta I - T)^{-1}$, T a semigroup generator, $\beta \in \rho(T)$ – p. 11
X_α	abstract Sobolev space of order $\alpha \in \mathbb{R}$. – p. 27
Y	usually the output space – p. 29
y	usually the output of a system – p. 30
y_{ref}	reference signal aimed to track – p. 2

Index

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Erklärung zum Eigenanteil bei der Zusammenarbeit

Kapitel 2 basiert auf [13]. Bei [13] habe ich die Struktur und Ideen für die Beweise konzipiert. Ich habe auch den Haupttext geschrieben und die numerischen Beispiele durchgeführt.
Kapitel 3 basiert auf [14]. Das Projekt extrapoliert die notwendigen Voraussetzungen aus [14], um eine allgemeine Theorie für eine neue Klasse Systeme zu entwickeln. Da die Grundideen und Methoden direkt aus [14] folgen, habe ich mich bei der Konzeption entsprechend beteiligt. Ich habe verschiedene Teile des ursprünglichen Artikels geschrieben, die numerische Beispiele durchgeführt und das Ganze ins Kapitel 3 eingebettet, sodass sich gewisse Sachen nicht wiederholen bezüglich Kapitel 3.
Kapitel 4 basiert auf [99]. Dabei habe ich die Ideen für die Beweise, Beispiele und Struktur konzipiert, eine erste Fassung geschrieben und anschließend lediglich die Kommentare meiner Koautoren umgesetzt. Das Kapitel wurde deutlich durch neue Beispiele und Sektionen erweitert, welche bei [99] nicht zu finden sind.
Das im Kapitel 5 studierte Problem wurde mir von Jr. Prof T. Berger und Dr. T. Breiten vorgeschlagen. Ich habe die Beweise und Struktur konzipiert und das Kapitel entsprechend verfasst.

Ort, Datum Marc Puche Niubó