TOPICS IN THREE-DIMENSIONAL DESCENT THEORY

Dissertation zur Erlangung des Doktorgrades

an der
Fakultät für Mathematik, Informatik und Naturwissenschaften
Fachbereich Mathematik
der
Universität Hamburg

vorgelegt von

Lukas Buhné

Hamburg, 2015

Gutachter waren:
Prof. Dr. Christoph Schweigert
Dr. Nick Gurski
Prof. Dr. Ross Street

Tag der Disputation:
29. April 2015

Contents

Introduction

TOPICS IN THREE-DIMENSIONAL DESCENT THEORY

Structures satisfying a property only up to a distinguished natural coherence isomorphism appear frequently in mathematics and physics. Category theory provides a formal framework, in which to speak and reason about these structures. In particular, since many constructions are much easier carried out in the strict situation where the coherence isomorphisms are trivial, one wishes to know if any such structure is equivalent to a strict one. This is the concept of strictification. On the other hand, the definition of weak notions of higher categories and their homomorphisms is a rich source of examples for these situations. This is because the definition of weak higher categories involves categorification, the process where axioms and more generally equations are replaced by new coherence isomorphisms subject to new axioms. It is a highly nontrivial task to identify which equations have to be replaced, and what kind of and how many axioms have to be enforced in order to allow for strictification. In this thesis we are concerned with the concept of strictification in three-dimensional category theory. Specifically, our interest concerns strictification for the higher cells of weak 3-categories and is driven by fundamental questions in three-dimensional category theory arising from the construction of bundle gerbes in terms of descent [61], which was motivated by Mathematical Physics.

A strict 3-category can be defined as a category enriched in the symmetric monoidal closed category of small 2-categories equipped with the cartesian product and a suitable internal hom. The weak concept of a 3-category is the concept of a tricategory introduced by Gordon, Power, and Street [21]. In contrast to monoidal categories, and more generally bicategories, not every tricategory is triequivalent to a strict 3-category. For example, just as a monoidal category gives rise to a bicategory with one object, a braided monoidal category gives rise to a tricategory with one object and one 1-cell. If this tricategory were equivalent to a strict 3-category, every braided monoidal category would have to be strictly symmetric cf. [21, Rem. 8.8]. Rather, any tricategory is equivalent to a semi-strict 3-category: a $\mathcal{G}ray$-category [21]. A $\mathcal{G}ray$-category is a category enriched in the symmetric monoidal closed category $\mathcal{G}ray$ of small 2-categories and strict functors equipped with the strong Gray product [22] of 2-categories and a suitable internal hom cf. Section 1.1.3. The semi-strictness of $\mathcal{G}ray$-categories has its roots in the definition of the monoidal structure of $\mathcal{G}ray$: The Gray product encodes the non-trivial interchange corresponding to the braiding of a monoidal category. The other side of the story is that

the closed structure of $\mathcal{G}ray$ is given by pseudonatural transformations and modifications of strict functors of 2-categories, but there are two possibilities to compose pseudonatural transformations, which coincide only up to an invertible interchange modification. While the question of strictification for a tricategory has basically been answered by the coherence theorem of Gordon, Power, and Street, and later by Gurski [24] using free constructions with coherence theorems in the form of commuting diagrams and the explicit construction of a strictification $\mathrm{Gr}T$, a $\mathcal{G}ray$-category, for a tricategory T, much less is known about the higher cells of tricategories.

In enriched category theory the basic object of interest is a category enriched in a monoidal category. For enriched categories $\mathcal{A}$ and $\mathcal{B}$, there are two basic notions of cells between them: that of a functor $F\colon \mathcal{A} \to \mathcal{B}$, and that of an enriched natural transformation between two such functors. Under suitable assumptions, these organize into an enriched category themselves: the enriched functor category $[\mathcal{A}, \mathcal{B}]$. Its underlying category has objects the enriched functors between $\mathcal{A}$ and $\mathcal{B}$ and morphisms the enriched natural transformations. For example, for enrichment in the category $\mathcal{S}et$ of small sets with the cartesian product, an enriched category is a locally small ordinary category, and the functor category consists exactly of the ordinary functors between such categories and natural transformations. Hence, in three-dimensional category theory, there is the well-understood concept of the functor $\mathcal{G}ray$-category $[\mathcal{A}, \mathcal{B}]$ between $\mathcal{G}ray$-categories $\mathcal{A}$ and $\mathcal{B}$. By the definition of an enriched category, for two objects F, G in $[\mathcal{A}, \mathcal{B}]$, that is, two $\mathcal{G}ray$-functors $F, G\colon \mathcal{A} \to \mathcal{B}$, we have a hom object $[\mathcal{A}, \mathcal{B}](F, G)$ in the category of enrichment $\mathcal{G}ray$. Thus, $[\mathcal{A}, \mathcal{B}](F, G)$ is a 2-category with objects the $\mathcal{G}ray$-natural transformations between F and G, but there are also 1- and 2-cells in this 2-category, and these constitute the higher cells between $\mathcal{G}ray$-categories. By the definition of the hom object of the enriched functor $\mathcal{G}ray$-category in terms of an end—a basic form of a limit in enriched category theory—each of these higher cells will consist of component cells in the target $\mathcal{G}ray$-category $\mathcal{B}$ and each of them will be subject to a naturality condition in the form of an equation of cells in the target $\mathcal{B}$ cf. Corollary 1.5.

The weak notions of cells between $\mathcal{G}ray$-categories arise if axioms are only enforced on cells of the highest dimension, that is, on 2-cells in the hom 2-categories of the target, which we call the 3-cells of the target $\mathcal{G}ray$-category. We are in particular interested in the case, where an equation of each kind of cell is replaced by the appropriate notion of 'equivalence' between such cells and where axioms are only allowed for 3-cells. Since 3-cells between two fixed 2-cells form a set, the only notion of 'equivalence' is equality. Hence, 3-cells will be subject to axioms. The higher cells between two fixed 1-cells form a category, in which the appropriate notion of 'equivalence' is an isomorphism. Finally, the higher cells between two fixed objects form a 2-category—the hom object of the category enriched in $\mathcal{G}ray$. The notion of equivalence in a 2-category that we use is that of an adjoint equivalence. Hence, an equation of 2-cells is replaced by an invertible 3-cell subject to new axioms that have to be specified. An equation of 1-cells is replaced by an adjoint equivalence 2-cell, but since equations of 2-cells are not allowed, further data in the form of invertible 3-cells subject to yet new axioms has to be specified. If the axioms of a $\mathcal{G}ray$-category are themselves replaced in this fashion, one recovers the notion of a cubical tricategory.

On the other hand, we have the definitions of weak higher cells between two tricategories T and S. These are trihomomorphisms, tritransformations, trimodifications and perturbations, which are defined in a similar fashion involving bicategories, pseudofunctors, and the cartesian product instead of 2-categories, strict functors, and the Gray product. One expects that these organize into a functor tricategory $\mathcal{T}ricat(T,S)$, but in fact, the precise structure of this tricategory has yet only been described in special cases. For example, Gurski [27, Th. 9.4] has shown that $\mathcal{T}ricat(T,S)$ forms a $\mathcal{G}ray$-category if the target S is a $\mathcal{G}ray$-category. The question how this $\mathcal{G}ray$-category compares to the functor $\mathcal{G}ray$-category if also T is a $\mathcal{G}ray$-category is the content of the first chapter of this thesis. In fact, even if T and S are $\mathcal{G}ray$-categories, $\mathcal{T}ricat(T,S)$ is not the $\mathcal{G}ray$-category that one obtains by weakening the concept of the functor $\mathcal{G}ray$-category as described above. Rather, just as a weak $\mathcal{G}ray$-category corresponds to a cubical tricategory, a weak functor of $\mathcal{G}ray$-categories or Gray homomorphism as we call it in Chapter 1, corresponds to a cubical trihomomorphism. The only difference between these two notions of trihomomorphisms is that a Gray homomorphism $A\colon T \to S$ has as its basic data, just as a $\mathcal{G}ray$-functor, a function on objects $t \to At$, and strict hom functors $A_{t,t'}\colon T(t,t') \to S(At,At')$, where $t,t' \in \mathrm{ob}T$; while for a trihomomorphism the hom functors are allowed to be non-strict i.e. honest pseudofunctors. Apart from this difference in objects, the weak notions of higher cells between $\mathcal{G}ray$-categories coincide with the tricategorical ones and thus form a full sub-$\mathcal{G}ray$-category of $\mathcal{T}ricat(T,S)$ as we will show in Chapter 1.

However, it is well-known [46] that not every pseudofunctor is equivalent to a strict functor as we also show in the Example 0.1 in this introduction. This basic example can in fact be extended to show that not every trihomomorphism is biequivalent to a locally strict trihomomorphism cf. Ex. 2.2. Hence, even up to triequivalence, the $\mathcal{G}ray$-categorical notions fail to capture all of the functor tricategory $\mathcal{T}ricat(T,S)$. Since the replacement of a tricategory by a triequivalent $\mathcal{G}ray$-category is a basic form of reasoning in three-dimensional category theory, the question arose whether this failure might severely limit the applicability of this strategy. On the other hand, whenever there is a comparison with an enriched context in higher-dimensional category theory, one can often transport the enriched Yoneda lemma through the comparison to prove a higher-categorical Yoneda lemma. Since the Yoneda lemma is a basic tool in category theory, one wonders whether the comparison of the functor tricategory with the $\mathcal{G}ray$-categorical notions is, in spite of its shortcomings, good enough to allow for a proof of a Yoneda lemma for tricategories. This is the ultimate goal of Chapter 2 of this thesis.

The final third Chapter is concerned with an application of the theory of Chapters 1 and 2 and was motivated by the desire to find a reformulation of the descent construction of bundle gerbes from Nikolaus and Schweigert [61] in three-dimensional category theory. To explain this desire a word on bundle gerbes and their physical motivation seems in order.

PHYSICAL MOTIVATION

Hermitian line bundles with connection are a well-established mathematical formalization for ordinary abelian gauge theories, in which the gauge field is described locally by a 1-form. String theory and, more generally, two-dimensional sigma models lead to gauge theories which involve locally defined 2-forms. The Kalb-Ramond field in string theory is an example of such a 2-form. An interesting class of two-dimensional quantum field theories or more specifically two-dimensional conformal field theories is given by non-linear sigma models with Wess-Zumino term. The topological term in the action has the mathematical interpretation of a surface holonomy assigned to a field $\Sigma \to M$ of the sigma model on a target manifold M. The appropriate and intrinsically local geometric framework for this is given by bundle gerbes. These were introduced by Murray [58] as an alternative geometric realization of classes in third integral cohomology $H^3(M, \mathbf{Z})$.

Bundles over a manifold M naturally form a category – morphisms are local gauge transformations. Bundle gerbes turn out to be objects of a higher category, a bicategory. In physical terms, 2-cells have been called gauge transformations of gauge transformations. In fact, whether bundle gerbes over a manifold form a 2-category or a bicategory depends very much on their precise construction. This is where strictification enters the picture. As we will see below, this has to do with the fact that transition functions provide a strictification of the higher categorical structures borne by abelian principal bundles.

Bundle gerbes, as objects in a higher category, are not in one-to-one correspondence to classes in $H^3(M, \mathbf{Z})$; rather their bicategorical equivalence classes are classified by such cohomology classes. The fact that $H^3(M, \mathbf{Z})$ is an abelian group, suggests that bundle gerbes have even more structure than the one of a bicategory. Indeed, there is a tensor product of bundle gerbes, and there is a dual bundle gerbe with respect to this tensor product. One can rephrase this by saying that bundle gerbes over a manifold M form a symmetric monoidal bicategory [29] with duality.

The theory of the final Chapter 3 of this thesis is concerned with the question how the bicategorical structure of bundle gerbes arises locally. Locally means, in mathematical terms, by descent. A proper understanding of locality by descent techniques requires much more than the knowledge of the equivalence classes of objects. One needs a firm control over the whole of the bicategorical structure of bundle gerbes and the way in which this structure varies over different base manifolds. This is illustrated by examples from physics, where 1- and 2-cells of bundle gerbes explicitly turn up.

- 1-cells play an explicit role for D-branes i.e. for sigma models on surfaces with boundary. Here, one needs the structure of a bundle gerbe $\mathcal{G}$-module. This is an equivalence 1-cell with a trivial bundle gerbe on the submanifold of the D-brane, see [15].

- In fact, bundle gerbe modules were introduced to define a notion of bundle gerbe K-theory, which is intimately linked to twisted K-theory with its various applications in string theory (for example for D-brane charges), see [9].

- 2-cells play an explicit role for unoriented surface holonomy. On the geometric side, a Jandl structure $\mathcal{J}$ on the bundle gerbe is needed. The definition of such a Jandl structure includes a 2-cell isomorphism, see [76].

Initially, it took some efforts to unravel the right notion of equivalence of bundle gerbes. The notion in [58] was too restrictive; it did not lead to a classification of equivalence classes of bundle gerbes by $H^3(X, \mathbf{Z})$. This was achieved with the notion of stable isomorphism in [59], but only with the drawback that composition of such isomorphisms was very cumbersome [71, p. 31-33].

The notions introduced in [77] and finally in [61], although giving rise to more 1-cells, achieve the desired classification of equivalence classes, while horizontal composition of 1-cells is transparent. In particular, the construction in [61] also shows that bundle gerbes naturally form a higher categorical stack in the way in which they arise: Informally, bundle gerbes may be glued together just as principal bundles may be glued together from suitable local data. This ensures the local character of bundle gerbes as geometric objects and, in applications to physics, locality of target space geometries.

To show that bundle gerbes are closed under descent, one needs control over the bicategorical structure and in particular over the way in which this structure varies with respect to maps of manifolds: Just as bundles may be pulled back, so there is a pullback operation for bundle gerbes. More precisely, the construction of bundle gerbes is a contravariant assignment of a bicategory to a manifold M. This is where three-dimensional category theory enters the picture because such a structure corresponds to a trihomomorphism into the tricategory of bicategories. The stack property of bundle gerbes is formulated as a property of this trihomomorphism with respect to some class of coverings of manifolds—such as for example open coverings. The construction of bundle gerbes in terms of descent resembles the construction of a bundle from transition functions in a higher categorical sense. Transition functions involve a cocyle condition cf. (0.22). For bundle gerbes, which are of a higher categorical nature, the transition functions are replaced by (adjoint) equivalences in a bicategory and the cocycle condition is replaced by an invertible 2-cell subject to a new axiom cf. Lem. 3.5. At this point, it becomes more complicate as coherence isomorphisms turn up. In fact, these make it much harder to show that the resulting structures form a bicategory again and even more so that they form a trihomomorphism into the tricategory of bicategories. Coherence isomorphisms in particular have to be considered if bundle gerbes are defined with respect to more general coverings than open coverings, namely surjective submersions. These appear in important examples such as lifting bundle gerbes [58], [60] and the bundle gerbes over compact connected simple Lie groups [20], [56]. However, pullback and fiber product of surjective submersions as limits are necessarily non-strict. The fiber product, for example, is just as non-associative and non-unital as the cartesian product is.

This was the starting point for the idea to tackle the problem from another direction by searching for a genuinely three-dimensional construction, which builds in coherence isomorphisms automatically, and which manifestly gives rise to a trihomomorphism again. The basic idea for this is to replace the bicategories of explicit descent data by bicategories of tritransformations as we do in Chapter 3. In hindsight, it was noticed that this also makes contact to Street's approach to descent [75]. The central question ad-

dressed in Chapter 3 is how these bicategories of tritransformations behave under the strictification from Chapters 1 and 3. We now take a step back to illustrate the concept of strictification in a simpler setting.

THE CONCEPT OF STRICTIFICATION

A particularly rich source of examples for coherence isomorphisms is given by structures that arise from a universal property. For example, the cartesian product $X \times Y$ of two sets X and Y satisfies the universal property that for any set Z and any two functions $f\colon Z \to X$ and $g\colon Z \to Y$, there is a unique function $h\colon Z \to X \times Y$ such that $\pi_1 h = f$ and $\pi_2 h = g$, where $\pi_1\colon X \times Y \to X$ and $\pi_2\colon Y \times Y \to Y$ are the cartesian projections and where composition is denoted by juxtaposition. From the uniqueness in the universal property, one deduces that given three sets X, Y, Z, there must be an isomorphism

$$(X \times Y) \times Z \cong X \times (Y \times Z)\ , \tag{0.1}$$

and that this isomorphism must be natural in X, Y, and Z. In this sense, the cartesian product is associative up to the natural coherence isomorphism (0.1).

Many mathematical objects have an underlying set, and it is frequently the case that the cartesian product of the underlying sets gives rise to a mathematical object of the same kind and which fulfills a similar universal property in the given context. Sets and functions organize into a category $\mathcal{S}et$, and this phenomenon can often be described by the existence of a forgetful functor

$$U : \mathcal{C} \to \mathcal{S}et \tag{0.2}$$

sending an object in the category $\mathcal{C}$ to its underlying set and which has the property that it preserves products. A product in a category $\mathcal{C}$ is an example of the concept of a limit, which is characterized by a universal property as above.

Limits are preserved by right adjoints, and it is indeed often the case that the forgetful functor (0.2) has a left adjoint, which associates the 'free' object in $\mathcal{C}$ to a set X. For example, this is the case when the category $\mathcal{C}$ is monadic over $\mathcal{S}et$. In particular, the monadicity ensures the existence of finite limits and thus for example of finite products. Now in a category $\mathcal{C}$ with finite products the binary product is associative up to natural isomorphism as in (0.1) as follows by an analogous argument. Thus the cartesian product of sets underlies many mathematical contexts, in which natural coherence isomorphisms as in (0.1) arise. Further well-known examples are given by the associativity and unitality of the tensor product of vector spaces up to natural isomorphism and the tensor product of principal bundles with abelian structure group, which takes a special, guiding position for this thesis.

The idea that one could get rid of coherence isomorphisms by passing over to isomorphism classes turns out to be a tempting misconception. The relevant categorical notion to pass to isomorphism classes is that of a skeleton of a category, which is a

full subcategory containing exactly one object in each isomorphism class[1]. Now by an argument attributed to Isbell, the skeleton $\mathcal{S}et_0$ of the category of small sets does not have a strictly associative product, as we will now explain in more detail cf. MacLane [55, VII.1., p. 160]. In fact, to get slightly ahead of our narrative, $\mathcal{S}et_0$ cannot be given the structure of a strict monoidal category such that the inclusion into $\mathcal{S}et$ is a monoidal equivalence. First, $\mathcal{S}et_0$ has a unique product[2] induced from the one in $\mathcal{S}et$. Now the denumerable set D in $\mathcal{S}et_0$ i.e. the set in the isomorphism class of the set of natural numbers $\mathbf{N}$ gives an example for which the natural coherence isomorphism

$$a_{D,D,D}\colon\ (D \times D) \times D \cong D \times (D \times D) \tag{0.3}$$

cannot be an identity in $\mathcal{S}et_0$. To see this, recall that there is an isomorphism $\mathbf{N} \times \mathbf{N} \cong \mathbf{N}$ of sets. In terms of the cartesian product, such an isomorphism is provided by the Cantor pairing function $\pi\colon\ \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ defined by

$$\pi(n, m) = \frac{1}{2}(n + m)(n + m + 1) + n\ . \tag{0.4}$$

By the definition of a skeleton, we must have

$$D \times D = D \tag{0.5}$$

in $\mathcal{S}et_0$. By the universal property of a product, there is a unique morphism $g\colon\ D \to D$ such that

$$\pi_1 g = 1_D \quad \text{and} \quad \pi_2 g = 1_D\ , \tag{0.6}$$

where $\pi_1\colon\ D \to D$ and $\pi_2\colon\ D \to D$ are the two projections of the product (0.5) in the category $\mathcal{S}et_0$. The naturality of the coherence isomorphisms (0.3) in particular requires that for three arbitrary morphisms $f, g, h\colon\ D \to D$ the following naturality condition holds.

$$a_{D,D,D}((f \times g) \times h) = (f \times (g \times h))a_{D,D,D}\ . \tag{0.7}$$

Assuming that $a_{D,D,D}$ is the identity thus implies that

$$(f \times g) \times h = f \times (g \times h)\ . \tag{0.8}$$

Now there clearly exist two functions $j, k\colon\ \mathbf{N} \to \mathbf{N}$ such that $j \neq k$. Since $\mathcal{S}et_0$ is a full subcategory of $\mathcal{S}et$, this means that there are two morphisms $j', k'\colon\ D \to D$ in $\mathcal{S}et_0$ such that $j' \neq k'$. On the other hand, by definition of the product of morphisms and the equality (0.8),

$$j'\pi_1 = \pi_1(j' \times (k' \times 1_D)) = \pi_1((j' \times k') \times 1_D) = (j' \times k')p_1 \tag{0.9}$$

[1]Being a full subcategory, a skeleton still contains all automorphisms of its objects.

[2]Here we mean a binary product in the sense of category theory. If $\mathcal{S}et_0$ is to be monoidally equivalent to $\mathcal{S}et$, we have to define the tensor product in $\mathcal{S}et_0$ to be this product induced from $\mathcal{S}et$.

and

$$(j' \times k')\pi_2 = \pi_2(1_D \times (j' \times k')) = \pi_2((1_D \times j') \times k') = k'\pi_1 \ . \tag{0.10}$$

Yet, together with (0.6) these equalities imply that $j' = j' \times k' = k'$, which is a contradiction.

MONOIDAL CATEGORIES

It follows that the notion of a skeleton gives no justification for ignoring the natural isomorphisms (0.1) arising from binary cartesian products. Rather, the practice of ignoring such natural isomorphisms is justified by suitable coherence theorems. One way of stating such a result is the assertion that the structure satisfying a property up to a distinguished natural isomorphism is 'equivalent' to a structure where this property holds strictly. To make this more precise, one has to give a framework formalizing the appearance of such structures and giving a means to define an equivalence of these. One such framework is provided by the concept of a monoidal category.

A monoidal category is a category M together with a functor

$$\otimes\colon\ M \times M \to M \tag{0.11}$$

on the cartesian product of M with itself—called the tensor product—an object $I \in M$—called the unit—and three natural transformations $a_{XYZ}\colon\ (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, $l_X\colon\ I \otimes X \to X$, $r_X\colon\ X \otimes I \to X$—called the associators and unitors—which are subject to the pentagon identity

$$\begin{array}{ccccc} ((W \otimes X) \otimes Y) \otimes Z & \xrightarrow{a} & (W \otimes X) \otimes (Y \otimes Z) & \xrightarrow{a} & W \otimes (X \otimes (Y \otimes Z)) \\ \downarrow{\scriptstyle a\otimes 1} & & & & \uparrow{\scriptstyle 1\otimes a} \\ (W \otimes (X \otimes Y)) \otimes Z & & \xrightarrow[a]{} & & W \otimes ((X \otimes Y) \otimes Z) \end{array} \tag{0.12}$$

and the triangle identity

$$\begin{array}{ccc} (X \otimes I) \otimes Y & \xrightarrow{\ a\ } & X \otimes (I \otimes Y) \\ & {\scriptstyle r\otimes 1}\searrow \quad \swarrow{\scriptstyle 1\otimes l} & \\ & X \otimes Y\ , & \end{array} \tag{0.13}$$

which we have here expressed as commutative diagrams. A monoidal category is said to be strict if the natural transformations a, l, and r are identity natural transformations.

To give a notion of equivalence of monoidal categories, one first needs a means to compare monoidal categories: A (strong) monoidal functor $F\colon\ A \to B$ between monoidal

categories A and B is given by a functor of the underlying categories and natural isomorphisms with components

$$F_{a,a'}\colon\ Fa \otimes_B Fa' \to F(a \otimes_A a') \tag{0.14}$$

and

$$F_*\colon\ I_B \to F(I_A)\ , \tag{0.15}$$

which have to be appropriately compatible with associators and unitors. The isomorphisms in (0.14) and (0.15) are called the composition and unit constraints or coherence isomorphisms of F respectively. The monoidal functor is said to be strict if the composition and unit constraints are given by identities.

Finally, there is the notion of a monoidal transformation $\sigma\colon\ F \Rightarrow G\colon\ A \to B$ between monoidal functors F and G. This consists of a natural transformation of the underlying functors such that the components are compatible with the constraints (0.14) and (0.15).

A monoidal equivalence between monoidal categories A and B is given by monoidal functors $F\colon\ A \to B$, $G\colon\ B \to A$ and invertible monoidal transformations $FG \Rightarrow 1_B$ and $1_A \Rightarrow GF$.

The definition of a weak 2-category as introduced by Bénabou [2] with the notion of a bicategory is a generalization of the concept of a monoidal category. A bicategory is given by a set of objects and hom categories between pairs of such objects with composition functors which resemble the tensor product (0.11) of a monoidal category. In particular, a bicategory with one object is exactly a monoidal category. A monoidal functor precisely corresponds to the notion of a pseudofunctor between bicategories, but a monoidal transformation is a special kind of pseudonatural transformation: an icon [48].

A strict bicategory, where all the coherence isomorphisms are identities, is called a 2-category and corresponds to a category enriched the category $\mathcal{C}at$ of small categories equipped with the cartesian product and the functor category as its internal hom. Monoidal categories themselves form a 2-category $\mathcal{M}on\mathcal{C}at$ with 1-cells functors and 2-cells monoidal transformations.

STRICTIFICATION OF MONOIDAL CATEGORIES

A choice of binary and nullary cartesian products in a category $\mathcal{C}$ with finite limits gives rise to the structure of a monoidal category on $\mathcal{C}$. Since the composite products are again finite products, these satisfy associativity and unitality up to the unique isomorphisms determined by the universal property. In fact, by uniqueness, these isomorphisms must be natural—they form the components of natural transformations, and again by uniqueness, they satisfy the pentagon and triangle identity, and in fact any identity built from them. This is a general phenomenon for monoidal categories: the pentagon and

triangle identity (0.12) and (0.13) are enough[3] to ensure that any diagram of coherence isomorphisms i.e. a diagram built solely from associators and unitors commutes. This is one form of MacLane's famous coherence theorem for monoidal categories [54].

The other, closely related form of the coherence theorem is that any monoidal category is monoidally equivalent to a strict monoidal category, cf. MacLane [55]. This general strictification can be explicitly constructed, and as a concept it is opposite to the concept of a skeleton. Instead of having possibly less objects, new objects are added in the process of strictification, and in contrast to the skeleton, which is a full subcategory, M is 'embedded' in its strictification $\mathrm{st}M$.

The strictification [32] $\mathrm{st}M$ of a monoidal category M is a strict monoidal category with two monoidal functors $e\colon \mathrm{st}M \to M$ and $f\colon M \to \mathrm{st}M$ giving rise to a monoidal equivalence such that

$$ef = 1_M \ . \tag{0.16}$$

An object of the monoidal category $\mathrm{st}M$ is a string of objects in M. The monoidal functor e sends a string $g_m \cdot g_{m-1} \cdot \ldots \cdot g_1$ to the tensor product $(\ldots((g_m * g_{m-1}) * g_{m-2}) * \ldots) * g_1$ and the empty string $\emptyset$ to the unit I. The set of morphisms between strings g and h is defined by the set of morphisms $M(e(g), e(h))$ and e sends such a morphism to the morphism of the same name between $e(g)$ and $e(h)$ in M. Composition of morphisms is defined via composition in M. The tensor product is defined on objects by string concatenation. On morphisms $\alpha\colon h \to h'$ and $\beta\colon g \to g'$ in $\mathrm{st}M$, the tensor product is defined by use of the unique coherence isomorphisms arising from the coherence theorem:

$$e(hg) \cong e(h) * e(g) \xrightarrow{\alpha * \beta} e(h') * e(g') \cong e(h'g') \ . \tag{0.17}$$

Uniqueness of the coherence isomorphisms implies that this prescription is suitably functorial to give a monoidal structure, and the coherence isomorphisms give e the structure of a monoidal functor as required. The monoidal functor $f\colon M \to \mathrm{st}M$ is defined by including objects as length one strings and by sending a morphism to the morphism of the same name in $\mathrm{st}M$. The constraints of this functor are represented by identity morphisms in M, but they are not identity cells in $\mathrm{st}M$ since source and target do not coincide. The strictification of a monoidal category has an obvious extension to bicategories [27, 2.3].

STRICTIFICATION OF FUNCTORS

MacLane's coherence theorem alone is not justification enough to just forget about coherence isomorphisms at all. This becomes more evident when considering the strictification of a monoidal functor. Notice that a monoidal functor can be strict even if the monoidal

[3] In fact, MacLane [54] first came up with a redundant amount of axioms for a monoidal category as Kelly [33] later showed when he determined the minimal amount of coherent axioms i.e. such that MacLane's coherence theorem still holds. A similar thing happens for trihomomorphisms, which are subject to only two axioms corresponding to the pentagon and the triangle identity for a monoidal category. This will be of importance in Chapter 1.

categories in the domain and codomain are not strict. Hence, the question arises whether a monoidal functor between monoidal categories A and B is isomorphic by an invertible monoidal transformation to a strict monoidal functor between A and B.

However, this is not true even if source and target are strict monoidal categories as we show in the following elementary example.

Example 0.1. Let I be the 2-category with one object x, a single nontrivial idempotent endo-1-cell g of x, and only identity 2-cells: 1_g and 1_{1_x}. Equivalently, this can be considered as a strict monoidal category with two objects: the unit object 1_x and g such that $g * g = g$. The monoidal functor $f\colon I \to \mathrm{st}I$ is clearly not isomorphic to a strict monoidal functor: Indeed, the only idempotent in $\mathrm{st}I$ is the empty string, which thus has to be the image of g under a strict monoidal functor $f'\colon I \to \mathrm{st}I$. However, there is no 2-cell in $\mathrm{st}I$ between the empty string $\emptyset$ and the one-letter string $f(g)$ as there is no 2-cell between $1_x = e(\emptyset)$ and $g = e(f(g))$ in I. Hence, there can be no natural transformation between f and f' and thus certainly no monoidal isomorphism. Consequently, $f\colon I \to \mathrm{st}I$ is an example of a monoidal functor between strict monoidal categories that is not (monoidally) isomorphic to a strict monoidal functor.

Moreover, f is in fact also an example of a pseudofunctor of 2-categories that is not equivalent to a strict functor i.e. a 2-functor $f'\colon I \to \mathrm{st}I$. Again, f' has to send g to the empty string $\emptyset$. In this context, however, there are pseudonatural transformations $\sigma\colon f \Rightarrow f'$ with component g (or any string $g \cdot g \cdot \ldots \cdot g$ of length greater or equal 1) and a nontrivial naturality 2-cell $g \cdot g \Rightarrow g$ represented by the identity $e(g \cdot g) = g = e(g)$ in I. It is indeed not hard to see that this defines a pseudonatural transformation: The naturality 2-cell at 1_x is determined by respect for units[4] to be the identity. Since there are only identity 2-cells in I, naturality is trivial. Respect for composition is easily seen to hold since I is a 2-category—thus has trivial constraints—and since the condition involves only identity 2-cells in I. Similarly, there are pseudonatural transformations $\sigma'\colon f' \Rightarrow f$, but for any two such transformations, there can be no modifications $\sigma' * \sigma \Rrightarrow 1_f$, $1_f \Rrightarrow \sigma' * \sigma$, $\sigma * \sigma' \Rrightarrow 1_{f'}$, and $1_{f'} \Rrightarrow \sigma * \sigma'$ since these would all have to have components 2-cells $g \cdot g \cdot \ldots \cdot g \Rightarrow \emptyset$ or $\emptyset \Rightarrow g \cdot g \cdot \ldots \cdot g$ for strings of length greater or equal 2, but there are no 2-cells $e(g \cdot g \cdot \ldots \cdot g) = g \Rightarrow 1_x = e(\emptyset)$ in I and thus no such 2-cells in $\mathrm{st}I$. Consequently, f cannot be equivalent to a 2-functor.

The 2-category I considered here is also given as an example of a 2-category that is not strictly biequivalent to any 2-category of the form $\mathrm{st}B$ in Gurski [27, Rem. 8.22].

On the other hand, it is true that any diagram in the target built from constraints of the source, the target, and the monoidal functor commutes [32]. This corresponds to the result that a monoidal functor can be meaningfully replaced by a strict monoidal functor between the strictifications of the source and target monoidal categories cf. Joyal and Street [32]. Namely, for a functor $F\colon A \to B$ as above, there is a strict monoidal functor

[4] We refer to the axioms of a pseudonatural transformation as naturality, respect for composition, and respect for units.

$\mathrm{st}F\colon \mathrm{st}A \to \mathrm{st}B$ between the strictifications of A and B and there are two diagrams

$$\begin{array}{ccc} \mathrm{st}A & \xrightarrow{\mathrm{st}F} & \mathrm{st}B \\ {\scriptstyle f_A}\uparrow & & \uparrow{\scriptstyle f_B} \\ A & \xrightarrow{F} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathrm{st}A & \xrightarrow{\mathrm{st}F} & \mathrm{st}B \\ {\scriptstyle e_A}\downarrow & \Downarrow \omega & \downarrow{\scriptstyle e_B} \\ A & \xrightarrow{F} & B \end{array} \tag{0.18}$$

such that the left hand diagram commutes, and the right hand diagram commutes up to an invertible monoidal transformation ω.

The strictification $\mathrm{st}F\colon \mathrm{st}A \to \mathrm{st}$ of the monoidal functor F is defined on a length 1-string in an object g of A by $\mathrm{st}F(f(g)) = f(F(g))$. This definition is extended to arbitrary strings by strictness. On a morphism $\alpha\colon g \to h$ in $\mathrm{st}A$, $\mathrm{st}F$ is defined by use of the unique coherence isomorphisms arising from the coherence theorem for monoidal functors:

$$e(F(g)) \cong F(e(g)) \xrightarrow{F(e(\alpha))} F(e(h)) \cong e(F(h)) \ . \tag{0.19}$$

It is easy to see that st is functorial on monoidal functors.

Finally, given a monoidal transformation $\sigma\colon F \Rightarrow G\colon A \to B$, one constructs a monoidal transformation $\mathrm{st}\sigma\colon \mathrm{st}F \Rightarrow \mathrm{st}G\colon \mathrm{st}A \to \mathrm{st}B$ as follows. The component of $\mathrm{st}\sigma$ at a string of objects g in $\mathrm{st}A$ is defined by use of the unique coherence isomorphisms arising from coherence for F and G:

$$(\mathrm{st}\sigma)_g\colon e(\mathrm{st}F(g)) \cong F(e(g)) \xrightarrow{\alpha_{e(g)}} G(e(g)) \cong e(\mathrm{st}G(g)) \ . \tag{0.20}$$

The strictification $\mathrm{st}A$ of a monoidal category also has another strictification property. Any monoidal functor $F\colon \mathrm{st}A \to B$ with B strict monoidal category is isomorphic by an invertible monoidal transformation to a strict functor $F'\colon \mathrm{st}A \to B$. One can again easily give an explicit definition of F', but its existence also follows from (0.18), the obvious fact that e_B is a strict monoidal functor if B is strict, and (0.16). In fact one can show that strictification provides a left 2-adjoint to the inclusion of the 2-category of strict monoidal categories into the 2-category of monoidal categories and (0.16) corresponds exactly to a triangle identity [28, 4.]. On the other hand, there is the notion of a free strict monoidal category $F_s\mathcal{C}$ on a category $\mathcal{C}$. The strict monoidal category $F_s\mathcal{C}$ has again objects given by strings of objects in $\mathcal{C}$ and the morphisms are given by the corresponding strings of morphisms in $\mathcal{C}$. It is easy to see that the strict monoidal functor $e\colon \mathrm{st}F_s\mathcal{C} \to F_s\mathcal{C}$ has strict monoidal equivalence-inverse. Note that this implies by the above that any monoidal functor on a free strict monoidal category and with target a strict monoidal category is isomorphic to a strict monoidal functor.

This is much like the strictification that we study in Chapter 2, where we show that the codescent object of a free algebra—which corresponds to the strictification—is strictly biequivalent to the free algebra itself.

We now want to give a short account of how one is naturally led to consider the question of coherence for trihomomorphisms starting from the problem of coherence for monoidal categories.

THE MONOIDAL CATEGORY OF ABELIAN PRINCIPAL BUNDLES

The monoidal category $\mathcal{B}un_A(M)$ of principal bundles over some fixed manifold M for an abelian structure group A is central for the definition of bundle gerbes. With respect to strictification, it has in fact quite unusual properties. The morphisms of this category are bundle maps, and the tensor product is given by the tensor product of principal A-bundles. We shortly recall the definition of the contracted product and tensor product of principal A-bundles:

Definition 0.1. Let P and P' be two principal smooth A-bundles over a smooth manifold M. The contracted product $P \times_A P'$ of P and P' is defined to be the cartesian product $P \times P'$ of P and P' modulo the equivalence relation

$$(a.p, p') \sim (p, a.p') \tag{0.21}$$

where $a \in A$ and $p, p' \in P, P'$. It is straightforward to prove that the contracted product $P \times_A P'$ is locally trivial and gives rise to principal A-bundle again.

The tensor product $P \otimes P'$ of P and P' is defined as the pullback of $P \times_A P'$ with respect to the diagonal $d\colon\ M \to M \times M$.

Given choices of local trivializations, there is a monoidal equivalence from $\mathcal{B}un_A(M)$ to the *strict* monoidal category $\mathcal{B}un'_A(M)$ of principal A-bundles in terms of open coverings and transition functions, which we now describe in more detail. An object of the category $\mathcal{B}un'_A(M)$ is a *principal A-bundle over M in terms of open coverings and transition functions*, which consists of an open covering (U_i) of the manifold M and smooth functions $\tau_{ij}\colon\ U_{ij} \to A$ on two-fold intersections $U_{ij} = U_i \cap U_j$ that satisfy the cocycle condition on three fold intersections U_{ijk},

$$\tau_{ik} = \tau_{jk}\tau_{ij}\ , \tag{0.22}$$

and where τ_{ii} is the constant function at the identity.

A morphism in $\mathcal{B}un'_A(M)$ is given by a common refinement (V_k) and smooth functions $f_k\colon\ V_K \to A$ into the structure group that are compatible with the transition functions on intersections. The composition of two such morphisms is given by the intersection of the refinements and point-wise multiplication in the structure group. It is to be observed that this composition is strictly associative and unital because so is intersection of open coverings and multiplication in the structure group. Hence, principal bundles in terms of open coverings and transition functions form a category.

In terms of open coverings and transition functions, the tensor product of two such abelian principal bundles is given by the intersection of open coverings and the pointwise product of transition functions in the structure group. For the definition of the tensor product on morphisms note that the intersection $(W_j \cap W_m)$ of refinements (W_j) and (W'_m) of open coverings $(U_i), (V_k)$ and $(U'_l), (V'_n)$ is a refinement of the intersections $(U_i \cap U'_l)$ and $(V_k \cap V'_n)$. For the same reasons as for the composition of morphisms, the tensor product is strictly associative and unital.

It is well-known how to construct an ordinary principal A-bundle from one in terms of transition functions as a quotient of the disjoint union

$$\Sigma_i U_i \times A \tag{0.23}$$

with an equivalence relation determined by the transition functions. Moreover, a morphism in $\mathcal{B}un'_A(M)$ gives rise to a bundle map between the principal bundles arising from (0.23) and in fact gives rise to a strong monoidal functor

$$\mathcal{B}un'_A(M) \to \mathcal{B}un_A(M) \tag{0.24}$$

as is easily verified from the construction (0.23) and the definition of the tensor products of $\mathcal{B}un'_A(M)$ and $\mathcal{B}un_A(M)$.

On the other hand, by choosing a local trivialization, an ordinary principal A-bundle gives rise to one in terms of an open covering and transition functions. Moreover, a bundle map between principal A-bundles with chosen local trivializations gives rise to a morphism in $\mathcal{B}un'_A(M)$. Under the assumption that a local trivialization for any principal A-bundle has been chosen, this process has the structure of a monoidal functor

$$\mathcal{B}un_A(M) \to \mathcal{B}un'_A(M)\ . \tag{0.25}$$

The two functors (0.24) and (0.25) constitute a monoidal equivalence between $\mathcal{B}un_A(M)$ and $\mathcal{B}un'_A(M)$. Thus, transition function give rise to an explicit strictification $\mathcal{B}un'_A(M)$ of the monoidal category of principal A-bundles, which is in fact quite close to the original category. It shares some similarity with the general strictification of a monoidal category insofar as there are again more objects than in the original monoidal category of principal bundles. In fact, principal bundles with abelian structure group are a symmetric monoidal category and the above gives a symmetric monoidal equivalence with a strictly symmetric strict monoidal category. This is an unusual property since a symmetric monoidal category is in general not equivalent to a strictly symmetric strict monoidal category [64, Ex. 7.4].

THE PRESHEAF OF ABELIAN PRINCIPAL BUNDLES

Up to now the base of the bundles has been fixed. In fact, by pullback, principal bundles give rise to a contravariant assignment of a monoidal category to a manifold. However, since the pullback is only determined up to isomorphism, this does not give rise to an ordinary functor $\mathcal{M}an^{\mathrm{op}} \to \mathcal{M}on\mathcal{C}at$. Instead, monoidal transformations have to be taken into account, which makes principal bundles into a pseudofunctor

$$\mathcal{B}un_A\colon\ \mathcal{M}an^{\mathrm{op}} \to \mathcal{M}on\mathcal{C}at \tag{0.26}$$

of 2-categories, where the category $\mathcal{M}an$ is considered as a locally discrete 2-category[5]. On the other hand, open coverings and transition functions provide a pseudonatural

[5]Here and in the following, a 'local' property means a property of all hom objects. Hence, locally discrete means that all hom categories of the 2-category are discrete categories. In other words, $\mathcal{M}an$ has only trivial 2-cells.

transformation to a strict functor

$$\mathcal{B}un'_A\colon\ \mathcal{M}an^{\mathrm{op}} \to \mathcal{M}on\mathcal{C}at\ . \tag{0.27}$$

Since the components of this pseudonatural transformation are equivalences, this is in fact an equivalence with a strict functor in the 2-category

$$\mathcal{B}icat(\mathcal{M}an^{\mathrm{op}}, \mathcal{M}on\mathcal{C}at) \tag{0.28}$$

of pseudofunctors from the 2-category $\mathcal{M}an^{\mathrm{op}}$ to the 2-category $\mathcal{M}on\mathcal{C}at$, pseudonatural transformations of such functors, and modifications of such transformations. Recall from Ex. 0.1 that such a strictification does in general not exist.

TWO-DIMENSIONAL MONAD THEORY

However, if we restrict to a small subcategory of $\mathcal{M}an$, this fits into a coherence result from two-dimensional monad theory proving that an arbitrary non-strict functor between a small 2-category P and a cocomplete 2-category L is equivalent to a 2-functor cf. Power [63]. In particular, higher dimensional monad theory provides a well-developed general framework, in which to address the question of strictification.

A 2-monad on a 2-category M, just as an ordinary monad, is given by a 2-functor $T\colon\ M \to M$ and two 2-natural transformations $\mu\colon\ T^2 \Rightarrow T$ and $\eta\colon\ 1_M \Rightarrow T$ subject to the following axioms in terms of commutative diagrams:

$$\begin{array}{ccc} T^3 & \xrightarrow{T\mu} & T^2 \\ {\scriptstyle \mu T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow{\mu} & T \end{array} \qquad\qquad \begin{array}{ccccc} T & \xrightarrow{\eta T} & T^2 & \xleftarrow{T\eta} & T \\ & \searrow & \downarrow{\scriptstyle \mu} & \swarrow & \\ & & T\ . & & \end{array} \tag{0.29}$$

An algebra for a 2-monad is given by an object A of M and a 1-cell $a\colon\ TA \to A$ such that the following equations of 1-cells hold:

$$aTa = a\mu_A\colon\ T^2A \to A \quad \text{and} \quad a\eta_A = 1_A\colon\ A \to A\ . \tag{0.30}$$

Similarly, there are the notions of 1- and 2-cells between such algebras involving 1- and 2-cells in M subject to axioms on 1- and 2-cells respectively. Indeed, algebras for T organize into a 2-category $T\text{-Alg}_{\mathrm{s}}$ with a forgetful functor

$$U\colon\ \text{T-Alg}_{\mathrm{s}} \to M\ . \tag{0.31}$$

However, in a 2-category we can replace the axioms (0.30) for algebras and 1-cells by invertible 2-cells subject to new coherence axioms cf. [44]. The resulting notions of pseudo algebras and pseudo algebra 1-cells give rise to a 2-category $T\text{-Alg}$ and an inclusion

$$i\colon\ T\text{-Alg}_s \to T\text{-Alg}\ . \tag{0.32}$$

The idea to study coherence and strictification in terms of the 2-categorical properties of this inclusion has been championed in [4]. In particular, one asks under which conditions there exists a left 2-adjoint

$$L\colon\ T\text{-Alg} \to T\text{-Alg}_s \tag{0.33}$$

to the inclusion (0.32) and if so whether the unit of the adjunction is an equivalence.

Lack [44] has given an elegant treatment of this situation using the concept of a codescent object, which is a certain 2-categorical limit. Under suitable assumptions on the existence of these colimits and their preservation, the codescent object gives rise to a left adjoint as in (0.32).

For example, this applies to monoidal categories as we have illustrated above. On the other hand, for a small 2-category P and a cocomplete 2-category L, the functor 2-category $[P, L]$ is the 2-category of algebras for a 2-monad, and T-Alg corresponds to the 2-category $\mathcal{B}icat(P, L)$ of pseudofunctors, pseudonatural transformations, and modifications. In this case the left adjoint L exists and the unit of the adjunction is an equivalence. In Chapter 1, we analyze the three-dimensional counterpart of this situation.

THREE-DIMENSIONAL MONAD THEORY

Three-dimensional monad theory is the study of $\mathcal{G}ray$-monads and their different kinds of algebras. The general coherence theory of $\mathcal{G}ray$-monads has been developed by Power [65]. Pseudo algebras and codescent objects in the three-dimensional context have later been introduced by Gurski [27, III] leading to a similar coherence theorem as the one proved by Lack [44] in the 2-dimensional context.

The concept of a codescent object and the resulting coherence theorem underlie the first Chapter of this thesis. As a consequence of the general theory of enriched monad theory, the $\mathcal{G}ray$-functor category $[\mathcal{P}, \mathcal{L}]$ is the category of algebras for a $\mathcal{G}ray$-monad over $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$. In the first Chapter, we identify the $\mathcal{G}ray$-category of pseudo algebras for this $\mathcal{G}ray$-monad with the $\mathcal{G}ray$-category of locally strict trihomomorphisms $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$. The concept of a codescent object turns up again in Chapter 2, where we concentrate on the special case of the codescent object of a strict codescent diagram to analyze the codescent object of a free algebra for a $\mathcal{G}ray$-monad. Finally, in Chapter 3, we analyze the codescent object of the weight of the three-dimensional descent construction introduced ibidem.

We end this introduction with a word on the general strategy that we adopt with regards to the application of three-dimensional monad theory to the theory of tricategories. The identification of the $\mathcal{G}ray$-category Ps-T-Alg that we present in Chapter 1 is an explicit identification of pseudo algebras, their cells, and the $\mathcal{G}ray$-category structure rather than the application of a recognition theorem such as Beck's monadicity theorem. On the one hand, the reason for this is that we do not have at hand a recognition theorem in three-dimensional monad theory, which would allow to identify the $\mathcal{G}ray$-category of pseudo algebras. In two-dimensional monad theory a recognition theorem going into

this direction has been provided by Bourke [8, Th. 25], though it only applies to strict algebras. However, the extension to *Gray*-monads and pseudo algebras seems nontrivial.

On the other hand, to apply a recognition theorem, one certainly will have to verify properties of the forgetful *Gray*-functor

$$\mathcal{T}\mathit{ricat}_{\mathrm{ls}}(\mathcal{P}, \mathcal{L}) \to [\mathrm{ob}\mathcal{P}, \mathcal{L}] \tag{0.34}$$

such as the question if it reflects internal biequivalences. However, the verification of such properties is usually nontrivial in the three-dimensional context. For example, the reflection of internal biequivalences in (0.34) means that a tritransformation of locally strict trihomomorphisms $\mathcal{P} \to \mathcal{L}$ is a biequivalence in $\mathcal{T}\mathit{ricat}_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$ if its components are biequivalences in $\mathcal{L}$. Since there is no coherence theorem for tritransformations, the verification of such properties quickly becomes cumbersome.

Our explicit identification of the *Gray*-category Ps-T-Alg with $\mathcal{T}\mathit{ricat}_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$ allows us to deduce such properties from three-dimensional monad theory. Namely, for a *Gray*-monad T on $\mathcal{K}$, the forgetful functor

$$U\colon \text{Ps-}T\text{-Alg} \to \mathcal{K}\,, \tag{0.35}$$

reflects internal biequivalences cf. Gurski [27, Lem. 15.11], and hence so must (0.34). We use this property in Proposition A.16 to extend this result to the full functor *Gray*-category $\mathcal{T}\mathit{ricat}(\mathcal{P}, \mathcal{L})$.

Similarly, in 2.2, we prove a Yoneda lemma for *Gray*-categories and locally strict trihomomorphisms via the identification of pseudo algebras from Chapter 1 and the *Gray*-enriched Yoneda lemma.

We now give a summary of the results, and a detailed outline of each of the three chapters and the appendix afterwards.

Summary of results

In Chapter 1, we introduce new notions of homomorphisms between $\mathcal{G}ray$-categories cf. (1.5.1). Conceptually, they can be described as functors between categories enriched in the monoidal tricategory $\mathcal{G}ray$ i.e. between cubical tricategories restricted to the case that the domain $\mathcal{P}$ and the codomain $\mathcal{L}$ are $\mathcal{G}ray$-categories. For small $\mathcal{P}$ and cocomplete $\mathcal{L}$, we show that the resulting $\mathcal{G}ray$-category $\mathsf{Gray}(\mathcal{P},\mathcal{L})$ can be identified with the $\mathcal{G}ray$-category Ps-T-Alg of pseudo algebras for the $\mathcal{G}ray$-monad T on $[\mathrm{ob}\mathcal{P},\mathcal{L}]$ given by left Kan extension on the one hand, and with the $\mathcal{G}ray$-category $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P},\mathcal{L})$ of locally-strict trihomomorphisms on the other hand. This comparison depends on two purely enriched identities for tensor products that we prove in Lemmata 1.11 and 1.12 in 1.3.3; and the observation that two axioms in the definition of a pseudo algebra for a $\mathcal{G}ray$-monad are redundant cf. Proposition 1.3, which is reminiscent of Kelly's proof that two axioms in the definition of a monoidal category [33] are redundant. We then employ the coherence theory of three-dimensional monad theory, to prove the existence of a left $\mathcal{G}ray$-adjoint to the inclusion $[\mathcal{P},\mathcal{L}] \to \mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P},\mathcal{L})$ into the $\mathcal{G}ray$-category of locally strict trihomomorphisms, tritransformations, trimodifications, and perturbations such that the components of the unit of this adjunction are internal biequivalences. As an application, we prove that a locally strict trihomomorphism from a small $\mathcal{G}ray$-category to the $\mathcal{G}ray$-category $\mathcal{G}ray$ of 2-categories, strict functors, pseudonatural transformations, and modifications is biequivalent to a $\mathcal{G}ray$-functor. We also provide a counterexample in Example 1.1, which shows that the condition of local strictness cannot be dropped.

In Chapter 2, we collect various intermediate results, which culminate in a Yoneda lemma for tricategories cf. Theorem 2.12 in 2.6. The Yoneda lemma for tricategories is proved via the $\mathcal{G}ray$-enriched Yoneda lemma, and further depends on the coherence theory from Chapter 1 and on the results presented in Appendix A. The intermediate results are of independent interest and include Theorem 2.1 about the preservation of free algebras up to strict biequivalence under the left adjoint L given on pseudo algebras by taking the codescent object of the associated codescent diagram; a Yoneda lemma for $\mathcal{G}ray$-categories and locally strict trihomomorphisms proved in Theorem 2.3 in 2.3; Theorem 2.5 in 2.3 about invariance of trihomomorphisms between $\mathcal{G}ray$-categories under change of local functors; and Theorems 2.7-2.11 in 2.4 and 2.5, where we prove a comparison of representables under pseudo-icon biequivalences of tricategories and under 'change of base' with respect to the strictification functor $\mathrm{st}\colon \mathcal{B}icat \to \mathcal{G}ray$.

In Chapter 3, we introduce a new three-dimensional descent construction as a simple tricategorical limit. We show that this construction is equivalent under strictification to the descent construction given in [61]. To do so, we have to identify the latter with a $\mathcal{G}ray$-enriched limit given by 2-categories of $\mathcal{G}ray$-natural transformations. We then identify the weight of this simple limit as the codescent object of the weight of the tricategorical limit.

Finally, new results on functor tricategories are collected in Appendix A. In particular, we give an account of whiskering on the right in case that the target is a $\mathcal{G}ray$-category and some partial results about whiskering on the left, which allow to give the bicategory enriched graph $\mathcal{T}ricat(T,\mathcal{B}icat)$ the structure of a tricategory via transport of structure

cf. Theorem A.4. Specifically, we show in Theorem A.9 that whiskering on the right with a trihomomorphism that is part of a triequivalence gives rise to a triequivalence of functor tricategories.

Outline

Chapter 1: **Trihomomorphisms as pseudo algebras.**

In the first chapter we show how under suitable conditions on domain and codomain the locally strict trihomomorphisms between $\mathcal{G}ray$-categories $\mathcal{P}$ and $\mathcal{L}$ correspond to pseudo algebras for a $\mathcal{G}ray$-monad T on the functor $\mathcal{G}ray$-category $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$, where $\mathrm{ob}\mathcal{P}$ is considered as a discrete $\mathcal{G}ray$-category. The conditions are that the domain $\mathcal{P}$ is a small and that the codomain $\mathcal{L}$ is a cocomplete $\mathcal{G}ray$-category. In fact, we prove that the $\mathcal{G}ray$-categories Ps-T-Alg and $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$ are isomorphic as $\mathcal{G}ray$-categories, which extends the local result mentioned in [65, Ex. 3.5]. On the other hand, the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ for this monad is given by the functor $\mathcal{G}ray$-category $[\mathcal{P}, \mathcal{L}]$, and there is an obvious inclusion of $[\mathcal{P}, \mathcal{L}]$ into Ps-T-Alg. The relation of these two $\mathcal{G}ray$-categories was studied locally by Power [65] and by Gurski using codescent objects [27]. We readily show that a corollary of Gurski's central coherence theorem [27, Th. 15.13] applies to the $\mathcal{G}ray$-monad on $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$: The inclusion of the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ into the $\mathcal{G}ray$-category Ps-T-Alg of pseudo T-algebras has a left adjoint such that the components of the unit of this adjunction are internal biequivalences.

The unique $\mathcal{G}ray$-functor $H\colon \mathrm{ob}\mathcal{P} \to \mathcal{P}$ which is the identity on objects, induces a $\mathcal{G}ray$-functor $[H, 1]\colon [\mathcal{P}, \mathcal{L}] \to [\mathrm{ob}\mathcal{P}, \mathcal{L}]$ of functor $\mathcal{G}ray$-categories, which is given on objects by precomposition with H. This functor sends any cell of $[\mathcal{P}, \mathcal{L}]$ such as a $\mathcal{G}ray$-functor or a $\mathcal{G}ray$-natural transformation to its family of values and components in $\mathcal{L}$ respectively. By the theorem of Kan adjoints, left Kan extension Lan_H along H provides a left adjoint to $[H, 1]$, and T is the $\mathcal{G}ray$-monad corresponding to this adjunction. This is all as in the 2-dimensional context, and the story is then usually told as follows: The enriched Beck's monadicity theorem shows that $[H, 1]\colon [\mathcal{P}, \mathcal{L}] \to [\mathrm{ob}\mathcal{P}, \mathcal{L}]$ is strictly monadic. That is, the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ is isomorphic to the functor category $[\mathcal{P}, \mathcal{L}]$ such that the forgetful functor factorizes through this isomorphism and $[H, 1]$. On the other hand, the monad has an obvious explicit description. Namely, by the description of the left Kan extension in terms of tensor products and coends we must have:

$$(TA)Q = \int^{P \in \mathrm{ob}\mathcal{P}} \mathcal{P}(P, Q) \otimes AP \tag{0.36}$$

where A is a $\mathcal{G}ray$-functor $\mathrm{ob}\mathcal{P} \to \mathcal{L}$ and where Q is an object of $\mathcal{P}$. The tensor product is a simple weighted colimit. For enrichment in a general symmetric monoidal closed category $\mathcal{V}$, it is characterized by an appropriately natural isomorphism in $\mathcal{V}$:

$$\mathcal{L}(\mathcal{P}(P, Q) \otimes AP, AQ) \cong [\mathcal{P}(P, Q), \mathcal{L}(AP, AQ)] \ , \tag{0.37}$$

where $[-,-]$ denotes the internal hom of $\mathcal{V}$. Thus, in the case of enrichment in $\mathcal{G}ray$, (0.37) is an isomorphism of 2-categories. In fact, the tensor product gives rise to a $\mathcal{G}ray$-adjunction, and its hom $\mathcal{G}ray$-adjunction is given by (0.37).

To achieve the promised identification of Ps-T-Alg with $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P},\mathcal{L})$, we have to determine how the data and $\mathcal{G}ray$-category structure of Ps-T-Alg transforms under the adjunction of the tensor product. Doing so, one notices that the identification of the Eilenberg-Moore object with the functor $\mathcal{G}ray$-category is a mere byproduct. For example, an object of $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ is an algebra for the monad T. The definition of such an algebra is just as for an ordinary monad. Thus, it consists of a 1-cell $a\colon TA \to A$ subject to two algebra axioms. According to the expression (0.36), the $\mathcal{G}ray$-natural transformation a is determined by components $a_{PQ}\colon \mathcal{P}(P,Q)\otimes AP \to AQ$. These are objects in the 2-category $\mathcal{L}(\mathcal{P}(P,Q)\otimes AP, AQ)$. The internal hom of $\mathcal{G}ray$ is given by the 2-category of strict functors, pseudonatural transformations, and modifications. Thus a_{PQ} corresponds under the hom adjunction (0.37) to a strict functor $A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP,AQ)$, and the axioms of an algebra imply that this gives A the structure of a $\mathcal{G}ray$-functor $\mathcal{P} \to \mathcal{L}$.

Given a $\mathcal{G}ray$-monad on a $\mathcal{G}ray$-category $\mathcal{K}$, the notions of pseudo algebras, pseudo functors, transformations, and modifications are all given by cell data of the $\mathcal{G}ray$-category $\mathcal{K}$. In the case that $\mathcal{K} = [\mathrm{ob}\mathcal{P},\mathcal{L}]$, this means that the data consists of families of cells in the target $\mathcal{L}$. Parts of these data transform under the adjunction of the tensor product into families of cells in the internal hom, that is, families of strict functors of 2-categories, pseudonatural transformations of those, and modifications of those. This already shows that we only have a chance to recover locally strict trihomomorphisms from pseudo algebras because a general trihomomorphism might consist of nonstrict functors of 2-categories, which is in contrast to the two-dimensional context where pseudo algebras correspond precisely to pseudofunctors of 2-categories.

We now give a short overview of how Chapter 1 is organized. In Section 1.1, we describe the symmetric monoidal closed category $\mathcal{G}ray$ and extend some elementary results on the correspondence of cubical functors and strict functors on Gray products.

In Section 1.2, we reproduce Gurski's definition of Ps-T-Alg and prove that two lax algebra axioms are redundant for a pseudo algebra.

In Section 1.3, we introduce the monad T on $[\mathrm{ob}\mathcal{P},\mathcal{L}]$ in 1.3.1 and describe it explicitly in 1.3.2. In 1.3.3 we expand on tensor products and derive the two technical Lemmata 1.11 and 1.12, which play a critical role in the bulk of our calculations.

In Section 1.4, we explicitly identify the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ in the general situation where $\mathcal{V}$ is a complete and cocomplete locally small symmetric monoidal closed category. Instead of appealing to Beck's enriched monadicity theorem, we provide an explicit identification in accordance with the general strategy adopted in Chapter 1.

In Section 1.5, we establish the identification of Ps-T-Alg and $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P},\mathcal{L})$, on which we now comment in more detail. To characterize how Ps-T-Alg transforms under the adjunction of the tensor product, in 1.5.1 we introduce the notion of homomorphisms of $\mathcal{G}ray$-categories, Gray transformations, Gray modifications, and Gray perturbations. With the help of Lemmata 1.11 and 1.12 from 1.3.3, these are seen to be exactly the transforms of pseudo algebras, pseudo functors, transformations, and modifications re-

spectively. This also equips the Gray data with the structure of a $\mathcal{G}ray$-category.

Elementary observations in 1.5.3 then give that a homomorphism of $\mathcal{G}ray$-categories is the same thing as a locally strict trihomomorphism, much like a $\mathcal{G}ray$-category is the same thing as a strict, cubical tricategory. Similarly, the notion of a Gray transformation corresponds exactly to a tritransformation between locally strict functors, and Gray modifications and perturbations correspond exactly to trimodifications and perturbations of those. This follows from the general correspondence, mediated by Theorem 1.2 from 1.1.4, of data for the cubical composition functor and the cartesian product on the one hand and data for the composition law of the $\mathcal{G}ray$-category and the Gray product on the other hand. The only thing left to check is that the axioms correspond to each other. Namely, the Gray notions being the transforms of the pseudo notions of three-dimensional monad theory, the axioms are equations of modifications, while the axioms for the tricategorical constructions are equations involving the components of modifications. That these coincide is mostly straightforward, less transparent is only the comparison of interchange cells. Gurski's coherence theorem then gives that the inclusion of the full sub-$\mathcal{G}ray$-category of $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$ determined by the locally strict functors, denoted by $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$, into the functor $\mathcal{G}ray$-category $[\mathcal{P}, \mathcal{L}]$ has a left adjoint and the components of the unit of this adjunction are internal biequivalences. As a corollary, we show that in this situation any locally strict trihomomorphism $\mathcal{P} \to \mathcal{L}$ is biequivalent to a $\mathcal{G}ray$-functor in $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$. Finally, in Example 1.1, we show that the latter assertion is not true if one drops the assumption of local strictness: In the case of $\mathcal{L} = \mathcal{G}ray$, we provide a counterexample with a small $\mathcal{G}ray$-category $\mathcal{D}$ and a trihomomorphism $\mathcal{D} \to \mathcal{G}ray$ that is not locally strict and that is not biequivalent to a $\mathcal{G}ray$-functor $\mathcal{D} \to \mathcal{G}ray$.

Chapter 2: **A Yoneda lemma for tricategories.**

In the second chapter we provide a proof of a Yoneda lemma for tricategories via the $\mathcal{G}ray$-enriched Yoneda lemma and strictification of the $\mathcal{G}ray$-category of locally strict trihomomorphisms as developed in Chapter 1. Given a trihomomorphism $\mathcal{P}^{\mathrm{op}} \to \mathcal{B}icat$ on a tricategory $\mathcal{P}$, the Yoneda lemma for tricategories states that there are natural biequivalences:

$$\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)(\mathcal{P}(-, P), A-) \cong AP \ . \tag{0.38}$$

It is well known that we can replace a tricategory $\mathcal{P}$ by a triequivalent $\mathcal{G}ray$-category, and $\mathcal{B}icat$ is triequivalent to a full sub-$\mathcal{G}ray$-category $\mathcal{G}ray'$ of $\mathcal{G}ray$ via strictification $\mathrm{st}\colon \mathcal{B}icat \to \mathcal{G}ray$[27], [46, Rem. 3]. These replacements give rise to triequivalences of functor tricategories as we show in Appendix A. To apply the $\mathcal{G}ray$-enriched Yoneda lemma, we first have to show that the transform of the representable $\mathcal{P}(-, P)$ under the replacements is suitably biequivalent to a representable of $\mathcal{P}'$. Second, we have to show that the transform of A is suitably biequivalent to a $\mathcal{G}ray$-functor.

The latter leads us to the first obstacle on the path of a proof of the Yoneda lemma: our coherence result from three-dimensional monad theory only applies to locally strict trihomomorphisms. We show that the transform of A is biequivalent to a locally strict

trihomomorphism if the correct replacement $\mathcal{P}'$ of $\mathcal{P}$ is chosen: this is the $\mathcal{G}ray$-category obtained by first replacing $\mathcal{P}$ by the cubical tricategory $\mathrm{st}P$ which is locally given by the strictification of bicategories [27], and then by the image of $\mathrm{st}P$ under the cubical Yoneda embedding [27], which is locally strict with locally strict triequivalence-inverse. Since the strictification of bicategories is left adjoint to the inclusion, the resulting local functors on $\mathrm{st}\mathcal{P}$ are equivalent to strict functors. Since the cubical Yoneda embedding and its inverse on its image are locally strict, we are given a trihomomorphism between $\mathcal{G}ray$-categories for which each local functor is equivalent to a strict functor. By general principles, the definition of a trihomomorphism should be invariant under change of local functors. We present a proof of this in the given situation that domain and codomain are $\mathcal{G}ray$-categories. Thus the given trihomomorphism can be replaced by a biequivalent locally strict trihomomorphism, which since the unit of the adjunction $L \dashv i$ from Chapter 1 is a biequivalence, can in turn be replaced by a biequivalent $\mathcal{G}ray$-functor.

The comparison of the representables of $\mathcal{P}$ and $\mathcal{P}'$ is of a different nature. Conceptually, this involves two results about enrichment in monoidal tricategories. First a comparison of representables under change of base with respect to the monoidal strictification functor $\mathrm{st}\colon \mathcal{B}icat \to \mathcal{G}ray$, and a comparison of representables under the triequivalences corresponding to the enrichment. We give proofs of these results in the specific situation. The comparison under change of base with respect to the strictification st can be proved by reference to coherence for the tricategory $\mathcal{P}$ although this necessarily involves some deeper analysis of functor tricategories cf. Appendix A. The comparison of representables under the triequivalence induced from the cubical Yoneda embedding y, proceeds mainly by reference to coherence for the weak inverse w of y. However, to make this comparison sufficiently natural by a coherence argument, we would have to refer to coherence for the triequivalence. Conceptually, this would follow from coherence for a tetrahomomorphism. Yet, since the cubical Yoneda embedding is bijective-on-objects, the resulting triequivalence is actually a biequivalence in the tricategory $\mathcal{T}ricat_3^c$ of cubical tricategories [18], and one can prove naturality in this situation by substantial calculation. Finally, we are in the situation to apply the hom $\mathcal{G}ray$-adjunction from Chapter 1, but before the $\mathcal{G}ray$-enriched Yoneda lemma can be applied, we have to compare $L\mathcal{P}'(-,P)$ with a representable. Since the representable $\mathcal{P}'(-,P)$ is in fact a free algebra for the monad on $[\mathrm{ob}\mathcal{P}'^{\mathrm{op}}, \mathcal{G}ray]$, we appeal to the following result established in Chapter 2: Given a $\mathcal{G}ray$-monad T on a $\mathcal{G}ray$-category $\mathcal{K}$ and $A \in \mathrm{ob}\mathcal{K}$, the codescent object of the codescent diagram associated to the free algebra TA is biequivalent to TA itself.

Chapter 2 is organized as follows.

In Section 2.1 we introduce the codescent object of a strict codescent diagram. In Section 2.1.1 we prove that the codescent object preserves free algebras up to strict biequivalence.

In Section 2.2 we provide a Yoneda lemma for $\mathcal{G}ray$-categories and locally strict trihomomorphisms.

In Section 2.3 we prove the change of local functors theorem.

In Sections 2.4-2.5 we provide the comparisons of representables.

Finally, in Section 2.6 we state and prove the Yoneda lemma for tricategories.

Chapter 3: **A three-dimensional descent construction.**

In the third chapter we give a new definition of the descent construction for a trihomomorphism

$$\mathfrak{X}\colon C^{\mathrm{op}} \to \mathcal{B}icat\ , \tag{0.39}$$

where C is a small category considered as a locally discrete tricategory, and where $\mathcal{B}icat$ is the tricategory of bicategories. Such a trihomomorphism has been called a presheaf in bicategories in Nikolaus and Schweigert [61]. The descent construction for $\mathfrak{X}$ is another such trihomomorphism

$$\mathrm{Desc}(\mathfrak{X})\colon [\Delta_{\delta}^{\mathrm{op}}, C]^{\mathrm{op}} \to \mathcal{G}ray\ . \tag{0.40}$$

Rather than by an explicit definition in terms of descent data, we define (0.40) by a tricategorical limit given by bicategories of tritransformations. The main question that we are then concerned with is how this new definition relates to the descent construction from Nikolaus and Schweigert [61]. This requires an analysis of the codescent object of the weight of the tricategorical limit. In the end, we are able to identify the strictification of the three-dimensional descent construction with the one from Nikolaus and Schweigert [61]. Chapter 3 is organized as follows.

In Section 3, we introduce the three-dimensional descent construction in Definition 3.1.

In Section 3.1 we analyze how the descent construction behaves under strictification and thus propose a descent construction for $\mathcal{G}ray$-functors.

Section 3.3 contains the central result of Chapter 3: in Theorem 3.1 we explicitly identify the weight of the descent construction for $\mathcal{G}ray$-functors with the codescent object of the weight for the general descent construction

In Section 3.4, we can then easily identify the descent construction for $\mathcal{G}ray$-functors with the descent construction from Nikolaus and Schweigert [61].

Finally, in Section 3.5, we give an outlook on how one might use these results to construct the stackification of a trihomomorphism with values in bicategories.

Acknowledgments

First, I want to thank my supervisor Christoph Schweigert for his constant help at each stage of this project over the last years. I am especially grateful for his encouragement since I first became interested in the subject. Second, I want to thank Nick Gurski for many decisive ideas, discussions, and his hospitality during a research stay in Sheffield. I also want to thank Richard Garner, Ross Street, and Daniel Schäppi for interesting discussions that had an impact on this thesis. Further, I want to thank Benjamín Alarcón Heredia for many interesting discussions and taking a look at the draft. Financial support by the Research Training Group 1670 "Mathematics Inspired by String Theory and Quantum Field Theory" and a "DAAD-Kurzstipendium für Doktoranden" funding a six-month research stay at the University of Sheffield is gratefully acknowledged. Finally, I want to thank my incomparable family and Sophia, who also helped drawing diagrams. Without you, this would have not been possible.

The thesis, in particular the first chapter, is based on the following preprint:

Lukas Buhné. Homomorphisms of Gray-categories as pseudo algebras. arXiv preprint. arXiv:1408.3481 [math.CT], 2014.

Chapter 1

Homomorphisms of Gray-categories as pseudo algebras

1.1 Preliminaries

We assume familiarity with enriched category theory. Regarding enriched category theory, we stay notationally close to Kelly's book [38], from which we shall cite freely. We also assume a fair amount of bicategory theory, see for example [2] or the short [51]. The appropriate references for tricategories are the memoir by Gordon, Power, and Street [21], Gurski's thesis [24], and his later book [27]. Since this is also our primary reference for three-dimensional monad theory, we will usually cite from [27]. In particular, we use the same letters as in [27] to denote the adjoint equivalences and invertible modifications of the tricategorical definitions. The tricategories considered in the first chapter are all *Gray*-categories, and we will describe them in terms of enriched notions to the extent possible. We do supply definitions in terms of enriched notions that correspond precisely to locally strict trihomomorphisms, tritransformations, trimodifications, and perturbations, but in describing this correspondence we assume knowledge of the tricategorical definitions.

Only basic knowledge of the general theory of monads in a 2-category is required cf. [73]. For monads in enriched category theory see also [14].

1.1.1 Conventions

Horizontal composition in a bicategory is generally denoted by the symbol $*$, while vertical composition is denoted by the symbol $\diamond$. We use the term functor or pseudofunctor for what is elsewhere called weak functor or homomorphism of bicategories and shall indicate whether the functor is strict where it is not clear from context. By an isomorphism we always mean an honest isomorphism, e.g. an isomorphism on objects and hom objects in enriched category theory. The symbol $\otimes$ is reserved both for a monoidal

structure and tensor products in the sense of enriched category theory. If not otherwise stated, $\mathcal{V}$ denotes a locally small symmetric monoidal closed category with monoidal structure $\otimes$; associators and unitors a, l, and r; internal hom $[-,-]$; unit d and counit or evaluation e. We shall usually use the prefix $\mathcal{V}$- to emphasize when the $\mathcal{V}$-enriched notions are meant, although this is occasionally dropped where it would otherwise seem overly redundant. The composition law of a $\mathcal{V}$-category $\mathcal{K}$ is denoted by $M_{\mathcal{K}}$. The unit at the object $K \in \mathcal{K}$ is denoted by j_K or occasionally 1_K, for example when it shall be emphasized that it is also the unit at K in the underlying category $\mathcal{K}_0$. The identification of $\mathcal{V}_0(I, [X, Y])$ and $\mathcal{V}_0(X, Y)$ induced from the closed structure for objects $X, Y \in \mathcal{V}$ of $\mathcal{V}$ is to be understood and usually implicit. We use the terms weighted limit and weighted colimit for what is elsewhere also called indexed limit and indexed colimit[1]. The concepts of ordinary and extraordinary $\mathcal{V}$-naturality cf. [38, Ch. 1] and the corresponding composition calculus are to be understood, and we freely use the underlying ordinary and extraordinary naturality too.

Composition in a monoidal category is generally denoted by juxtaposition. Composition of $\mathcal{V}$-functors is in general also denoted by juxtaposition. For cells of a *Gray*-category, juxtaposition is used as shorthand for the application of its composition law.

1.1.2 The category *2Cat*

Let *Cat* denote the category of small categories and functors. It is well-known that *Cat* is complete and cocomplete: it clearly has products and equalizers, thus is complete. Coproducts are given by disjoint union, and there is a construction for coequalizers in [22, I,1.3, p. 25]. In fact, the same strategy applies to the category of $\mathcal{V}$-enriched categories and $\mathcal{V}$-functors in general, where $\mathcal{V}$ is a complete and cocomplete symmetric monoidal closed category, see [78] and [3, Th. 7] for a full proof. Products are given by the cartesian product of the object sets and the cartesian product of the hom objects. Equalizers of $\mathcal{V}$-functors are given by the equalizer of the maps on objects and the equalizer of the hom morphisms. Coproducts are given by the coproduct of the object sets i.e. the disjoint union and by the original hom object or the initial object in $\mathcal{V}$. The construction of coequalizers in [22, I,1.3, p. 25] can be transferred to this context, see also [3, Prop. 5]. In particular, the category *2Cat* of small 2-categories and strict functors is complete and cocomplete.

1.1.3 The symmetric monoidal closed category *Gray*

We now describe the symmetric monoidal closed category in which we will usually enrich. Its underlying category is *2Cat*, which has a symmetric monoidal closed structure given by the Gray product. We only provide a brief description of the Gray product here. For details, the reader is referred to [27, 3.1, p. 36ff.] and to [22, I,4.9, p. 73ff.] for a lax variant. We always use the former strong variant of the Gray product in the following.

[1] In particular in Kelly [38].

The Gray product of 2-categories X and Y is a 2-category denoted by $X \otimes Y$, which can be characterized as follows cf. [27, 3.4]. Considering the sets of objects $\mathrm{ob}X$ and $\mathrm{ob}Y$ as discrete 2-categories, we denote by $X \square Y$ the pushout in $2\mathcal{C}at$ of the diagram below, where $\times$ denotes the cartesian product of 2-categories, and where the morphisms are given by products of the inclusions $\mathrm{ob}X \to X$ and $\mathrm{ob}Y \to Y$ and identity functors respectively.

$$\begin{array}{ccc} \mathrm{ob}X \times \mathrm{ob}Y & \longrightarrow & X \times \mathrm{ob}Y \\ \downarrow & & \\ \mathrm{ob}X \times Y & & \end{array} \tag{1.1}$$

By the universal property of the pushout, the products of the inclusions and identity functors, $\mathrm{ob}X \times Y \to X \times Y$ and $X \times \mathrm{ob}Y \to X \times Y$, induce a strict functor $j\colon X \square Y \to X \times Y$.

It is well-known that there is an orthogonal factorization system on $2\mathcal{C}at$ with left class the strict functors which are bijective on objects and 1-cells and right class the strict functors which are locally fully faithful, see for example[27, Corr. 3.20, p. 51]. The Gray product $X \otimes Y$ may be characterized by factorizing j with respect to this factorization system. More precisely, $X \otimes Y$ is uniquely characterized (up to unique isomorphism in $2\mathcal{C}at$) by the fact that there is a strict functor $m\colon X \square Y \to X \otimes Y$ which is an isomorphism on the underlying categories i.e. bijective on objects and 1-cells and a strict functor $i\colon X \otimes Y \to X \times Y$ which is locally fully faithful such that $j = im$.

There is an obvious explicit description of $X \otimes Y$ in terms of generators and relations, which can be used to construct a functor $\otimes\colon 2\mathcal{C}at \times 2\mathcal{C}at \to 2\mathcal{C}at$. Clearly, $X \otimes Y$ has the same objects as $X \times Y$, and we have the images of the 1-cells and 2-cells from $X \times \mathrm{ob}Y$ and $\mathrm{ob}X \times Y$, for which we use the same name in $X \otimes Y$. That is, there are 1-cells $(f, 1)\colon (A, B) \to (A', B)$ for 1-cells $f\colon A \to A'$ in X and objects B in Y, and there are 1-cells $(1, g)\colon (A, B) \to (A, B')$ for objects A in X and 1-cells $g\colon B \to B'$ in Y. All 1-cells in $X \otimes Y$ are up to the obvious relations generated by horizontal strings of those 1-cells, the identity 1-cell being $(1, 1)$. Apart from the obvious 2-cells $(\alpha, 1)\colon (f, 1) \Rightarrow (f', 1)\colon (A, B) \to (A', B)$ and $(1, \beta)\colon (1, g) \Rightarrow (1, g')\colon (A, B) \to (A, B')$, there must be unique invertible interchange 2-cells $\Sigma_{f,g}\colon (f, 1) * (1, g) \Rightarrow (1, g) * (f, 1)$ mapping to the identity of (f, g) under j because the latter is fully faithful—domain and codomain cleary both map to (f, g) under j. In particular, by uniqueness i.e. because j is locally fully faithful, these must be the identity if either f or g is the identity. There are various relations on horizontal and vertical composites of those cells, all rather obvious from the characterization above. We omit those as well as the details how equivalence classes and horizontal and vertical composition are defined.

For functors $F\colon X \to X'$ and $G\colon Y \to Y'$, it is not hard to give a functorial definition of the functor $F \otimes G\colon X \otimes Y \to X' \otimes Y'$. We confine ourselves with the observation that on interchange 2-cells,

$$(F \otimes G)_{(A,B),(A'B')}(\Sigma_{f,g}) = \Sigma_{F_{A,A'}(f),G_{B,B'}(g)} \ . \tag{1.2}$$

From the characterization above it is then clear how to define associators and unitors for $\otimes$. We only mention here that

$$a(\Sigma_{f,g}, 1) = \Sigma_{f,(g,1)} \tag{1.3}$$

and

$$a(\Sigma_{(f,1),h}) = \Sigma_{f,(1,h)} \tag{1.4}$$

and

$$a(\Sigma_{(1,g),h}) = (1, \Sigma_{g,h}) \ . \tag{1.5}$$

We omit the details that this gives a monoidal structure on $\mathit{2Cat}$ (pentagon and triangle identity follow from pentagon and triangle identity for the cartesian product and $\square$).

There is an obvious symmetry c for the Gray product, which on interchange cells is given by

$$c(\Sigma_{f,g}) = \Sigma^{-1}_{g,f} \ . \tag{1.6}$$

As for any two bicategories (by all means for small domain), there is a functor bicategory $\mathit{Bicat}(X, Y)$ given by functors of bicategories, pseudonatural transformations, and modifications. As the codomain Y is a 2-category, this is in fact again a 2-category. We denote by $[X, Y]$ the full sub-2-category of $\mathit{Bicat}(X, Y)$ given by the strict functors. One can show that this gives $\mathit{2Cat}$ the structure of a symmetric monoidal closed category with internal hom $[X, Y]$:

Theorem 1.1. *[27, Th. 3.16] The category $\mathit{2Cat}$ of small 2-categories and strict functors has the structure of a symmetric monoidal closed category. As such, it is referred to as Gray. The monoidal structure is given by the Gray product and the terminal 2-category as the unit object, the internal hom is given by the functor 2-category of strict functors, pseudonatural transformations, and modifications.*

Remark 1.1. In fact, we will not have to specify the closed structure of Gray apart from the fact that its evaluation is (partly) given by taking components. This is because our ultimate goal is to compare definitions from three-dimensional i.e. Gray-enriched monad theory to definitions from the theory of tricategories, and we do so in the case where all tricategories are in fact Gray-categories, that is, equivalently, strict, cubical tricategories. These definitions will only formally involve the cubical composition functor, which relates to the composition law of the Gray-category – we will usually not have to specify the composition. Of course, one can explicitly identify the enriched notions, and then there are alternative explicit arguments. However, we think that the formal argumentation is more adequate. The closed structure is worked out in [27, 3.3], and the enriched notions usually turn out to be just as one would expect. We spell out a few explicit prescriptions below the following lemma, but in fact we just need a few consequences of these, for example equation (1.8) below.

Next recall that a locally small symmetric monoidal closed category $\mathcal{V}$ can be considered as a category enriched in itself i.e. as a $\mathcal{V}$-category. Also recall that if the underlying category $\mathcal{V}_0$ of $\mathcal{V}$ is complete and cocomplete, $\mathcal{V}$ is complete and cocomplete considered as a $\mathcal{V}$-category. This means it has any small weighted limit and any small weighted colimit. For the concept of a weighted limit see [38, Ch. 3][2]. In fact, completeness follows from the fact that a limit is given by an end, and if the limit is small, this end exists and is given by an equalizer in $\mathcal{V}_0$, see [38, (2.2)]. It is cocomplete because, $\mathcal{V}$ being complete, $\mathcal{V}^{\mathrm{op}}$ is tensored and thus also admits small conical limits because $\mathcal{V}_0$ is cocomplete, hence $\mathcal{V}$ admits small coends because it is also tensored, but then since by [38, (3.70)] any small colimit is given by a small coend over tensor products, it is cocomplete.

Recall that the underlying category $2\mathcal{C}at$ of $\mathcal{G}ray$ is complete and cocomplete cf. 1.1.2. Thus in particular, we have the following :

Lemma 1.1. *The $\mathcal{G}ray$-category $\mathcal{G}ray$ is complete and cocomplete.* □

The composition law of the $\mathcal{G}ray$-category $\mathcal{G}ray$ is given by strict functors $[Y,Z] \otimes [X,Y] \to [X,Y]$, where X, Y and Z are 2-categories . It is given on objects by composition of strict functors. On 1-cells of the form $(\theta, 1)\colon (F,G) \to (F',G)$ it is given by the pseudonatural transformation denoted $G^*\theta$ with components θ_{Gx} and naturality 2-cells θ_{Gf}. On 1-cells of the form $(1,\sigma)\colon (F,G) \to (F,G')$ it is given by the pseudonatural transformation denoted $F_*\sigma$ with components $F_{Gx,G'x}(\sigma_x)$ and naturality 2-cells $F_{Gx,G'x'}(\sigma_f)$. Similarly, on 2-cells of the form $(\Gamma, 1)\colon (\theta,1) \Rightarrow (\theta',1)\colon (F,G) \to (F',G)$ it is given by the modification denoted $G^*\Gamma$ with components Γ_{Gx}, and on 2-cells of the form $(1,\Delta)\colon (1,\sigma) \Rightarrow (1,\sigma')\colon (F,G) \to (F,G')$ it is given by the modification denoted $F^*\Delta$ with components $F_{Gx,G'x}(\Delta_X)$. Finally, on interchange cells of the form $\Sigma_{\theta,\sigma}$, it is given by the naturality 2-cell θ_{σ_x} of θ at σ_x, hence,

$$(M_{\mathcal{G}ray}(\Sigma_{\theta,\sigma}))_x = \theta_{\sigma_x}\colon\ \theta_{G'x} * F_{Gx,G'x}(\sigma_x) \Rightarrow F'_{Gx,G'x}(\sigma_x) * \theta_{Gx}\ . \tag{1.7}$$

This follows from the general form of $M_{\mathcal{V}}$ in enriched category theory by inspection of the closed structure of $\mathcal{G}ray$ cf. [27, Prop. 3.10].

Also recall that there is a functor $\mathsf{Ten}\colon\ \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ which is given on objects by the monoidal structure. For $\mathcal{V} = \mathcal{G}ray$, its strict hom functor

$$\mathsf{Ten}_{(X,X'),(Y,Y')}\colon\ [X,X'] \otimes [Y,Y'] \to [X \otimes Y, X' \otimes Y']$$

sends an object (F,G) to the functor $F \otimes G$. It sends a transformation $(\theta, 1_G)\colon (F,G) \Rightarrow (F',G)$ to the transformation with component the 1-cell $(\theta_x, 1_{Gy})$ in $X' \otimes Y'$ at the object (x,y) in $X \otimes Y$; and naturality 2-cells $(\sigma_f, 1_{Gg})$ and $\Sigma_{\theta_x,Gg}$ at 1-cells $(f,1_y)$ and $(1_x,g)$ respectively. Its effect on a transformation $(1_F,\iota)\colon (F,G) \Rightarrow (F,G')$ is analogous. It sends a modification $(\Gamma, 1_{1_G})\colon (\theta,1_G) \Rrightarrow (\theta',1_G)$ to the modification with component the 2-cell $(\Gamma_X, 1_{1_{Gy}})$ in $X' \otimes Y'$ at (x,y) in $X \otimes Y$. Its effect on a modification $(1_{1_F},\Delta)\colon (1_F,\iota) \Rrightarrow (1_F,\iota')$ is

[2] where a weighted limit is called and indexed limit

analogous. Finally, it sends the interchange 2-cell $\Sigma_{\theta,\iota}$ to the modification with component the interchange 2-cell $\Sigma_{\theta_x,\iota_y}$, hence,

$$(\mathrm{Ten}_{(X,X'),(Y,Y')}(\Sigma_{\theta,\iota}))_{x,y} = \Sigma_{\theta_x,\iota_y} \ . \tag{1.8}$$

All of this again follows from inspection of the closed structure of $\mathcal{G}ray$, cf. [27, Prop. 3.10]. See also equation (1.57) below.

1.1.4 Cubical functors

Given 2-categories X, Y, Z, recall that a cubical functor in two variables is a functor $\hat{F}\colon X \times Y \to Z$ such that for all 1-cells (f, g) in $X \times Y$, the composition constraint

$$\hat{F}_{(1,g),(f,1)}\colon \hat{F}(1,g) * \hat{F}(f,1) \Rightarrow \hat{F}(f,g) \ ,$$

is the identity 2-cell, and such that for all composable 1-cells $(f', 1)$, $(f, 1)$ in $X \times Y$,

$$\hat{F}_{(f',1),(f,1)}\colon \hat{F}(f',1) * \hat{F}(f,1) \Rightarrow \hat{F}(f' * f,1) \ ,$$

is the identity 2-cell, and such that for all composable 1-cells $(1, g')$, $(1, g)$ in $X \times Y$,

$$\hat{F}_{(1,g'),(1,g)}\colon \hat{F}(1,g') * \hat{F}(1,g) \Rightarrow \hat{F}(1,g' * g) \ ,$$

is the identity 2-cell. For composable $(1, g'), (f, g)$ and $(f', g'), (f, 1)$, the constraint cells are then automatically identities by compatibility of $\hat{F}$ with associators i.e. a functor axiom for $\hat{F}$; it also automatically preserves identity 1-cells.

We start with the following elementary result, which extends the natural $\mathcal{S}et$-isomorphism in [27, Th. 3.7] to a $\mathcal{C}at$-isomorphism.

Proposition 1.1. *Given 2-categories X, Y, Z, there is a universal cubical functor $C\colon X \times Y \to X \otimes Y$ natural in X and Y such that precomposition with C induces a natural isomorphism of 2-categories (i.e. a $\mathcal{C}at$-isomorphism)*

$$[X \otimes Y, Z] \cong \mathcal{B}icat_c(X, Y; Z) \ ,$$

where $\mathcal{B}icat_c(X, Y; Z)$ denotes the full sub-2-category of $\mathcal{B}icat(X \times Y, Z)$ determined by the cubical functors.

Proof. The functor C is determined by the requirements that it be the identity on objects, that $C(f, 1) = (f, 1)$, $C(\alpha, 1) = (\alpha, 1)$, $C(1, g) = (1, g)$, $C(1, \beta) = (1, \beta)$, and that it be a cubical functor. In particular, observe that this means that $C(f, g) = (1, g) * (f, 1)$ and that the constraint $C_{(f,1),(1,g)}$ is given by the interchange cell $\Sigma_{f,g}$.

As for an arbitrary functor of bicategories, precomposition with C induces a strict functor

$$C^*\colon \mathcal{B}icat(X \otimes Y, Z) \to \mathcal{B}icat(X \times Y, Z) \ .$$

It sends a functor $G\colon X \otimes Y \to Z$ to the composite functor $GC\colon X \times Y \to C$. In fact, if F is a strict functor $X \otimes Y \to Z$, recalling the definition of the composite of two functors of bicategories, a moment's reflection affirms that $\hat{F} \coloneqq FC$ is a cubical functor with constraint $\hat{F}_{(f,1),(1,g)} = F(\Sigma_{f,g})$. Thus by restriction, C^* gives rise to a functor $[X \otimes Y, Z] \to \mathcal{B}icat_c(X, Y; Z)$ which we also denote by C^*.

If $\sigma\colon F \Rightarrow G\colon X \otimes Y \to Z$ is a pseudonatural transformation, $C^*\sigma\colon FC \Rightarrow GC$ is the pseudonatural transformation with component

$$(C^*\sigma)_{(A,B)} = \sigma_{C(A,B)} = \sigma_{(A,B)}$$

at an object $(A, B) \in X \times Y$, and naturality 2-cell

$$(C^*\sigma)_{(f,g)} = \sigma_{C(f,g)} = \sigma_{(f,1)*(1,g)} = (\sigma_{(f,1)} * 1) \diamond (1 * \sigma_{(1,g)})$$

at a 1-cell $(f, g) \in X \times Y$, where the last equality is by respect for composition of σ. If σ is the identity pseudonatural transformation, it is immediate that the same applies to $C^*\sigma$. Given another pseudonatural transformation of strict functors $\tau\colon G \Rightarrow H$, we maintain that $(C^*\tau) * (C^*\sigma) = C^*(\tau * \sigma)$. It is manifest that the components coincide: both are given by $\tau_{(A,B)} * \sigma_{(A,B)}$ at the object $(A, B) \in X \times Y$. That the naturality 2-cells at a 1-cell $(f, g) \in X \times Y$ coincide,

$$(\tau_{C(f,g)} * 1) \diamond (1 * \sigma_{C(f,g)}) = (\tau * \sigma)_{C(f,g)}$$

is simply the defining equation for the naturality 2-cell of the horizontal composite $\tau * \sigma$.

If $\Delta\colon \sigma \Rrightarrow \pi$ is a modification of pseudonatural transformations $F \Rightarrow G$ of strict functors $X \otimes Y \to Z$, then $C^*\Delta$ is the modification $C^*\sigma \Rrightarrow C^*\pi$ with component

$$(C^*\Delta)_{(A,B)} = \Delta_{C(A,B)} = \Delta_{(A,B)}$$

at an object $(A, B) \in X \times Y$, and this prescription clearly strictly preserves identities and vertical composition of modifications. Given another modification $\Lambda\colon \tau \Rrightarrow \rho\colon G \Rightarrow H$ where H is strict, one readily checks that $(C^*\Lambda) * (C^*\Delta) = C^*(\Lambda * \Delta)$ both having component $\Lambda_{(A,B)} * \Delta_{(A,B)}$ at an object $(A, B) \in X \times Y$. Thus, C^* is indeed a strict functor.

As a side note, we remark that because we only consider 2-categories, C^* is the same as the functor $\mathcal{B}icat(C, Z)$ induced by the composition of the tricategory $\mathcal{T}ricat$ of bicategories, functors, pseudonatural transformations, and modifications.

Let $\hat{F}\colon X \times Y \to Z$ be an arbitrary cubical functor, then the prescriptions $F(f, 1) = \hat{F}(f, 1)$, $F(\alpha, 1) = \hat{F}(\alpha, 1)$, $F(1, g) = \hat{F}(1, g)$, $F(1, \beta) = \hat{F}(1, \beta)$, and $F(\Sigma_{f,g}) = \hat{F}_{(f,1),(1,g)}$, provide a strict functor $F\colon X \otimes Y \to Z$ such that $FC = \hat{F}$. The latter equation and the requirement that F be strict, clearly determine F uniquely. That this is well-defined e.g. that it respects the various relations for the interchange cells is by compatibility of $\hat{F}$ with associators and naturality of

$$\hat{F}_{(A,B),(A',B'),(A'',B'')}\colon \ {*_Z}\,(\hat{F}_{(A',B'),(A'',B'')} \times \hat{F}_{(A,B),(A',B')}) \Rightarrow \hat{F}_{(A,B),(A'',B'')} * {*_{X \times Y}}\ , \qquad (1.9)$$

where $*$ denotes the corresponding horizontal composition functors. For example, for the relation

$$\Sigma_{f' * f,g} \sim (\Sigma_{f',g} * (1_f, 1)) \diamond ((1_{f'}, 1) * \Sigma_{f,g}) \qquad (1.10)$$

one has to use that axiom twice giving

$$\hat{F}_{(f'*f,1),(1,g)} = \hat{F}_{(f',1)*(f,1),(1,g)} = \hat{F}_{(f',1),(f,g)} \diamond (\hat{F}(1_{f'}, 1) * \hat{F}_{(f,1),(1,g)})$$

and $\hat{F}_{(f',1),(f,g)} = \hat{F}_{(f',1),(1,g)*(f,1)} = \hat{F}_{(f',1),(g,1)} * \hat{F}(1_f, 1)$. Alternatively, one uses coherence for the functor $\hat{F}$—then any relation in the Gray product must clearly be mapped to an identity in Z because the constraints in $\mathcal{F}_{\hat{F}}Z$ are mapped to identities in $\mathcal{F}_{2C}Z$, where these are the corresponding free constructions on the underlying category-enriched graphs cf. [27, 2.].

Now let $\hat{\sigma}\colon \hat{F} \Rightarrow \hat{G}$ be an arbitrary pseudonatural transformation of cubical functors. We have already shown that $\hat{F}$ and $\hat{G}$ have the form FC and GC respectively, where F and G were determined above. We maintain that there is a unique pseudonatural transformation $\sigma\colon F \Rightarrow G$ such that $\hat{\sigma} = C^*\sigma$. By the above, the latter equation uniquely determines both the components, $\sigma_{(A,B)} = \hat{\sigma}_{(A,B)}$, and the naturality 2-cells of σ, namely $\sigma_{(f,1)} = \hat{\sigma}_{(f,1)}$ and $\sigma_{(1,g)} = \hat{\sigma}_{(1,g)}$, and thus σ is uniquely determined by respect for composition. That this is compatible with the relations $(f', 1) * (f, 1) \sim (f' * f, 1)$ and $(1, g') * (1, g) \sim (1, g' * g)$ in the Gray product follows from the fact that respect for composition is in this case tantamount to respect for composition of $\hat{\sigma}$ because the constraints are identities here due to the axioms for cubical functors. Hence, what is left to prove is that this is indeed a pseudonatural transformation. First observe that the prescriptions for σ have been determined by the requirement that it respects composition, and respect for units is tantamount to respect for units of $\hat{\sigma}$. Naturality with respect to 2-cells of the form $(\alpha, 1)$ and $(1,\beta)$ is tantamount to the corresponding naturality condition for $\hat{\sigma}$. Naturality with respect to an interchange cell $\Sigma_{f,g}$, i.e.

$$(G(\Sigma_{f,g}) * 1_{\sigma_{(A,B)}}) \diamond \sigma_{(f,1)*(1,g)} = \sigma_{(1,g)*(f,1)} \diamond (1_{\sigma_{(A',B')}} * F(\Sigma_{f,g}))$$

is—by the requirement that σ respects composition:

$$\sigma_{(f,1)*(1,g)} = (1_{G(f,1)} * \sigma_{(1,g)}) \diamond (\sigma_{(f,1)} * 1_{F(1,g)}) = (1_{\hat{G}(f,1)} * \hat{\sigma}_{(1,g)}) \diamond (\hat{\sigma}_{(f,1)} * 1_{\hat{F}(1,g)})$$

and by respect for composition of $\hat{\sigma}$:

$$\sigma_{(1,g)*(f,1)} = (1_{G(1,g)} * \sigma_{(f,1)}) \diamond (\sigma_{(1,g)} * 1_{F(f,1)}) = (1_{\hat{G}(1,g)} * \hat{\sigma}_{(f,1)}) \diamond (\hat{\sigma}_{(1,g)} * 1_{\hat{F}(f,1)}) = \hat{\sigma}_{(f,g)}$$

(the constraints are trivial here)—tantamount to respect for composition of $\hat{\sigma}$:

$$(\hat{G}_{(f,1),(1,g)} * 1_{\hat{\sigma}_{(A,B)}}) \diamond ((1_{\hat{G}(f,1)} * \hat{\sigma}_{(1,g)}) \diamond (\hat{\sigma}_{(f,1)} * 1_{\hat{F}(1,g)})) = \hat{\sigma}_{(f,g)} \diamond (1_{\hat{\sigma}_{(A',B')}} * \hat{F}_{(f,1),(1,g)}) \ .$$

Notice that in general, naturality with respect to a vertical composite is implied by naturality with respect to the individual factors. Similarly, naturality with respect to a horizontal composite is implied by functoriality of $F_{,,}$ and $G_{,,}$ (cf. (1.9)), respect for composition, and naturality with respect to the individual factors.

Finally, let $\hat{\Delta}\colon C^*\sigma \Rrightarrow C^*\pi\colon FC \Rightarrow GC$ be an arbitrary modification. Then we maintain that there is a unique modification $\Delta\colon \sigma \Rightarrow \pi$ such that $\hat{\Delta} = C^*\Delta$. By the above, the latter equation uniquely determines Δ's components, $\Delta_{(A,B)} = \hat{\Delta}_{(A,B)}$ and thus Δ itself, but we have to show that Δ exists i.e. that this gives Δ the structure of a modification. The

modification axiom for 1-cells of the form $(f, 1)$ is tantamount to the modification axiom for $\hat{\Delta}$ and the corresponding 1-cell in $X \times Y$ of the same name. The same applies to the modification axiom for 1-cells of the form $(1, g)$. This proves that σ is a modification because the modification axiom for a horizontal composite is implied by respect for composition of σ and π, and the modification axiom for the individual factors.

□

Given 2-categories X_1, X_2, X_3, it is an easy observation that

$$a(C(C \times 1)) = C(1 \times C)a_{\times} \colon\ X_1 \times X_2 \times X_3 \to X_1 \otimes (X_2 \otimes X_3)\ , \tag{1.11}$$

where $a_{\times}$ is the associator of the cartesian product.

It is well-known that a strict, cubical tricategory is the same thing as a $\mathcal{G}ray$-category. To prove this, one has to replace the cubical composition functor by the composition law of a $\mathcal{G}ray$-category. This uses the underlying $\mathcal{S}et$-isomorphism of Proposition 1.1. In the same fashion, in order to compare locally strict trihomomorphisms between $\mathcal{G}ray$-categories with Gray homomorphisms as it is done in Theorem 1.6 in 1.5.3 below, we need the following many-variable version of Proposition 1.1 to replace adjoint equivalences and modifications of cubical functors by adjoint equivalences and modifications of the corresponding strict functors on Gray products.

Theorem 1.2. *Given a natural number n and 2-categories $Z, X_1, X_2, ..., X_n$, composition with*

$$C(C(C(...) \times 1_{X_{n-1}}) \times 1_{X_n}) \colon\ X_1 \times X_2 \times ... \times X_n \to (...((X_1 \otimes X_2) \otimes X_3) \otimes ...) \otimes X_n\ ,$$

where C is the universal cubical functor, induces a natural isomorphism of 2-categories (i.e. a $\mathcal{C}at$-isomorphism)

$$[(...((X_1 \otimes X_2) \otimes X_3) \otimes ...) \otimes X_n, Z] \cong \mathcal{B}icat_c(X_1, X_2, ..., X_n; Z)\ ,$$

where $\mathcal{B}icat_c(X_1, X_2, ..., X_n; Z)$ denotes the full sub-2-category of $\mathcal{B}icat(X_1 \times X_2 \times ... \times X_n, Z)$ determined by the cubical functors in n variables. The same is of course true for any other combination of universal cubical functors (mediated by the unique isomorphism in terms of associators for the Gray product).

Proof. Recall that the composition $(F \circ (F_1 \times ... \times F_k))$ of cubical functors is again a cubical functor. This shows that the restriction of $(C(C(C(...) \times 1_{X_{n-1}}) \times 1_{X_n}))^*$ to $[(...((X_1 \otimes X_2) \otimes X_3) \otimes ...) \otimes X_n, Z]$ does indeed factorize through $\mathcal{B}icat_c(X_1, X_2, ..., X_n; Z)$. The proof that this gives an isomorphism as wanted is then a straightforward extension of the two-variable case. There are Gray product relations on combinations of interchange cells, which correspond to relations for the constraints holding by coherence. Indeed any diagram of interchange cells commutes because these map to identities in the cartesian product.

For example, a cubical functor in three variables is determined by compatible partial cubical functors in two variables and a relation on their constraints cf. the diagram in [27, Prop. 3.3, p. 42]. This corresponds to a combination of the Gray product relation

$$\Sigma_{f,g'*g} \sim ((1_{g'}, 1) * \Sigma_{f,g}) \diamond (\Sigma_{f,g'} * (1, 1_g)) \tag{1.12}$$

for $f = f_1$ and $g' = (f_2, 1)$ and $g = (1, f_3)$ and $g' = (1, f_3)$ and $g = (f_2, 1)$ respectively, and the Gray product relation

$$((1,\beta) * (\alpha, 1)) \diamond \Sigma_{f,g} \sim \Sigma_{f',g'} \diamond ((\alpha, 1) * (1,\beta)) \tag{1.13}$$

for $f = f_1 = f'$, $\alpha = 1_{f_1}$, $g = (f_2, 1) * (1, f_3)$, $g' = (1, f_3) * (f_2, 1)$, and $\beta = \Sigma_{f_2,f_3}$, which reads

$$\begin{aligned}&((1, \Sigma_{f_2,f_3}) * (1_{f_1}, 1)) \diamond ((1_{(f_2,1)}, 1) * \Sigma_{f_1,(1,f_3)}) \diamond (\Sigma_{f_1,(f_2,1)} * (1, 1_{(1,f_3)}))\\ \sim&((1_{(1,f_3)}, 1) * \Sigma_{f_1,(f_2,1)}) \diamond (\Sigma_{f_1,(1,f_3)} * (1, 1_{(f_2,1)})) \diamond ((1_{f_1}, 1) * (1, \Sigma_{f_2,f_3}))\ .\end{aligned}$$

For pseudonatural transformations and modifications, the arguments are entirely analogous to the two-variable case. □

1.1.5 $\mathcal{V}$-enriched monad theory

Recall from enriched category theory that there is a $\mathcal{V}$-functor $\mathbf{Hom}_{\mathcal{L}}\colon \mathcal{L}^{\mathrm{op}} \otimes \mathcal{L} \to \mathcal{V}$, which sends and object (M, N) in the tensor product of $\mathcal{V}$-categories $\mathcal{L}^{\mathrm{op}} \otimes \mathcal{L}$ to the hom object $\mathcal{L}(M, N)$ in $\mathcal{V}$. As is common, the corresponding partial $\mathcal{V}$-functors are denoted $\mathcal{L}(M, -)$ and $\mathcal{L}(-, N)$ with hom morphisms determined by the equations

$$e^{\mathcal{L}(M,N)}_{\mathcal{L}(M,N')}(\mathcal{L}(M,-)_{N,N'} \otimes 1) = M_{\mathcal{L}} \tag{1.14}$$

and

$$e^{\mathcal{L}(M',N)}_{\mathcal{L}(M,N)}(\mathcal{L}(-,N)_{M,M'} \otimes 1) = M_{\mathcal{L}}c \tag{1.15}$$

where e denotes evaluation, i.e. counits of the adjunctions of the closed structure with a superscript indicating which adjunction we mean and a subscript indicating a component of the counit of this adjunction.

For an element $f\colon N \to N'$ in $\mathcal{L}(N, N')$ i.e. a morphism in the underlying category of $\mathcal{L}$, we usually denote by $\mathcal{L}(M, f)\colon \mathcal{L}(M, N) \to \mathcal{L}(M, N')$ the morphism in $\mathcal{V}$ corresponding to the image of f under the underlying functor of $\mathcal{L}(M, -)$ with respect to the identification $\mathcal{V}_0(I, [X, Y]) \cong \mathcal{V}_0(X, Y)$ induced from the closed structure of $\mathcal{V}$ for objects $X, Y \in \mathcal{V}$. Also note that we will occasionally write $\mathcal{L}(1, f)$ instead e.g. if the functoriality of

$$\mathbf{hom}_{\mathcal{L}}\colon \mathcal{L}_0^{\mathrm{op}} \times \mathcal{L}_0 \longrightarrow (\mathcal{L}^{\mathrm{op}} \otimes \mathcal{L})_0 \xrightarrow{(\mathbf{Hom}_{\mathcal{L}})_0} \mathcal{V}_0 \tag{1.16}$$

is to be emphasized, where the first arrow is the canonical comparison functor [38]. This notation is obviously extended in the case that $\mathcal{V} = \mathcal{G}ray$, e.g. given a 1-cell $\alpha\colon f \to g$ in the 2-category $\mathcal{L}(N, N') \cong [I, \mathcal{L}(N, N')]$, then $\mathcal{L}(M, \alpha)$ denotes the pseudonatural transformation $\mathcal{L}(M, f) \Rightarrow \mathcal{L}(M, g)$ given by $\mathcal{L}(M, -)_{N,N'}(\alpha)$ i.e. the 1-cell in $[\mathcal{L}(M, N), \mathcal{L}(M, N')]$.

Recall that the 2-category $\mathcal{V}$-$\mathcal{CAT}$ of $\mathcal{V}$-categories, $\mathcal{V}$-functors, and $\mathcal{V}$-natural transformations is a symmetric monoidal 2-category with monoidal structure the tensor product of $\mathcal{V}$-categories and unit object the unit $\mathcal{V}$-category $\mathcal{I}$ with a single object 0 and hom object I. Recall that the 2-functor $(-)_0 = \mathcal{V}\text{-}\mathcal{CAT}(\mathcal{I}, -)\colon \mathcal{V}\text{-}\mathcal{CAT} \to \mathcal{CAT}$ sends

a $\mathcal{V}$-category to its underlying category, a $\mathcal{V}$-functor to its underlying functor, and a $\mathcal{V}$-natural transformation to its underlying natural transformation.

Let T be a $\mathcal{V}$-monad on a $\mathcal{V}$-category $\mathcal{M}$. Recall that this means that T is a monad in the 2-category $\mathcal{V}$-$\mathcal{CAT}$. Thus T is a $\mathcal{V}$-functor $\mathcal{M} \to \mathcal{M}$, and its multiplication and unit are $\mathcal{V}$-natural transformations $\mu\colon TT \Rightarrow T$ and $\eta\colon 1_{\mathcal{M}} \Rightarrow T$ respectively such that

$$\mu(\mu T) = \mu(T\mu) \qquad \text{and} \qquad \mu(\eta T) = 1_T = \mu(T\eta)\ , \tag{1.17}$$

where μT and $T\mu$ are as usual the $\mathcal{V}$-natural transformations with component

$$\mu_{TM}\colon I \to \mathcal{M}(TTTM, TM) \quad \text{and} \quad T_{TTM,M}\mu_M\colon I \to \mathcal{M}(TTTM, TM)$$

respectively at the object $M \in \mathcal{M}$, and similarly for ηT and $T\eta$.

Under the assumption that $\mathcal{V}$ has equalizers e.g. if $\mathcal{V}$ is complete, the Eilenberg-Moore object $\mathcal{M}^T$ exists and has an explicit description, on which we expand below. For now, recall that the Eilenberg-Moore object is formally characterized by the existence of an isomorphism

$$\mathcal{V}\text{-}\mathcal{C}at(\mathcal{K}, \mathcal{M}^T) \cong \mathcal{V}\text{-}\mathcal{C}at(\mathcal{K}, \mathcal{M})^{T_*} \tag{1.18}$$

of categories which is $\mathcal{C}at$-natural in $\mathcal{K}$ and where T_* is the ordinary monad induced by composition with T.

In particular, setting $\mathcal{K} = \mathcal{I}$ shows that the underlying category $\mathcal{M}^T$ of the Eilenberg-Moore object in $\mathcal{V}$-$\mathcal{CAT}$ is isomorphic to the Eilenberg-Moore object for the underlying monad T_0 on the underlying category $\mathcal{M}_0$ of $\mathcal{M}$.

Thus an object of $\mathcal{M}^T$, i.e. a T-algebra, is the same thing as a T_0 algebra. This means, that it is given by a pair (A, a) where A is an object of $\mathcal{M}$ and a is an element $I \to \mathcal{M}(TA, A)$ such that the two algebra axioms hold true:

$$M_{\mathcal{M}}(a, T_{TA,A}a) = M_{\mathcal{M}}(a, \mu_A) \qquad \text{and} \qquad 1_A = M_{\mathcal{M}}(a, \eta_A)\ . \tag{1.19}$$

Here, the notation is already suggestive for the situation for $\mathcal{V} = \mathcal{G}ray$. Namely, $(a, T_{TA,A}a)$ is considered as an element of the underlying set $V(\mathcal{M}(TA, A) \otimes \mathcal{M}(TTA, TA))$, and we apply the underlying function $VM_{\mathcal{M}}$ to this, where V is usually dropped because for $\mathcal{V} = \mathcal{G}ray$ the equations in (1.19) make sense as equations of the values of strict functors on objects in the Gray product.

Given T-algebras (A, a) and (B, b), the hom object of $\mathcal{M}^T$ is given by the following equalizer

$$\mathcal{M}^T((A, a), (B, b)) \xrightarrow{(U^T)_{(A,a),(B,b)}} \mathcal{M}(A, B) \underset{\mathcal{M}(1,b)T_{A,B}}{\overset{\mathcal{M}(a,1)}{\rightrightarrows}} \mathcal{M}(TA, B)\ . \tag{1.20}$$

In fact, it is not hard to show that the composition law $M_{\mathcal{M}}$ and the units j_A of $\mathcal{M}$ induce a $\mathcal{V}$-category structure on $\mathcal{M}^T$ such that U^T is a faithful $\mathcal{V}$-functor $\mathcal{M}^T \to \mathcal{M}$, which we call the forgetful functor. The explicit arguments may be found in [53].

In the case that $\mathcal{V} = \mathcal{G}ray$ and $\mathcal{K}$ is a $\mathcal{G}ray$-category with a $\mathcal{V}$-monad T on it, Gurski identifies $\mathcal{K}^T$ explicitly in [27, 13.1]. This is also what the equalizer description gives when it is spelled out:

Proposition 1.2. *The $\mathcal{G}ray$-category of algebras for a $\mathcal{G}ray$-monad T on a $\mathcal{G}ray$-category $\mathcal{K}$, i.e. the Eilenberg-Moore object $\mathcal{K}^T$, can be described in the following way. Objects are T-algebras: they are given by an object X in $\mathcal{K}$ and a 1-cell $x\colon TX \to X$, i.e. an object in $\mathcal{K}(TX, X)$, satisfying $M_{\mathcal{K}}(x, Tx) = M_{\mathcal{K}}(x, \mu_X)$ and $1_X = M_{\mathcal{K}}(x, \eta_X)$. These algebra axioms are abbreviated by $xTx = x\mu_X$ and $1_X = x\eta_X$ respectively.*

An algebra 1-cell $f\colon (X, x) \to (Y, y)$ is given by a 1-cell $f\colon X \to Y$, i.e. an object in $\mathcal{K}(X, Y)$, such that $M_{\mathcal{K}}(f, x) = M_{\mathcal{K}}(y, Tf)$, which is abbreviated by $fx = yTf$. An algebra 2-cell $\alpha\colon f \Rightarrow g\colon (X, x) \to (Y, y)$ is given by a 2-cell $\alpha\colon f \to g$, i.e. a 1-cell in $\mathcal{K}(X, Y)$, such that $M_{\mathcal{K}}(1_y, T\alpha) = M_{\mathcal{K}}(\alpha, 1_x)$, which is abbreviated by $1_yT\alpha = \alpha 1_x$. An algebra 3-cell $\Gamma\colon \alpha \Rrightarrow \beta\colon f \Rightarrow g\colon (X, x) \to (Y, y)$ is given by a 3-cell $\Gamma\colon \alpha \Rrightarrow \beta$ i.e. a 2-cell in $\mathcal{K}(X, Y)$ such that $M_{\mathcal{K}}(1_{1_y}, T\Gamma) = M_{\mathcal{K}}(\Gamma, 1_{1_x})$, which is abbreviated $1_{1_y}T\Gamma = \Gamma 1_{1_x}$. The compositions are induced from the $\mathcal{G}ray$-category structure of $\mathcal{K}$. □

Observe here that the common notation $xTx = x\mu_X$ for equations of (composites of) morphisms in the underlying categories has been obviously extended for $\mathcal{V} = \mathcal{G}ray$ to 2-cells and 3-cells i.e. 1- and 2-cells in the hom 2-categories, where juxtaposition now denotes application of the composition law of $\mathcal{K}$, and the axioms for algebra 2- and 3-cells are whiskered equations with respect to this composition on 2-cells and 3-cells in $\mathcal{K}$.

1.2 The Gray-category of pseudo algebras

Let again T be a $\mathcal{G}ray$-monad on a $\mathcal{G}ray$-category $\mathcal{K}$. Since the underlying category $2\mathcal{C}at$ of the $\mathcal{G}ray$-category $\mathcal{G}ray$ is complete, it has equalizers in particular, so we have a convenient description of the $\mathcal{G}ray$-category $\mathcal{K}^T$ of T-algebras in terms of equalizers as in 1.1.5.

Recall that for enrichment in $\mathcal{C}at$, there is a pseudo and a lax version of the 2-category of algebras with obvious inclusions of the stricter into the laxer ones respectively. Under suitable conditions on the monad and its (co)domain, there are two coherence results relating those different kinds of algebras. First, each of the inclusions has a left adjoint. Second, each component of the unit of the adjunction is an internal equivalence. The primary references for these results are [4] and [44]. In particular, in the second, Lack provides an analysis of the coherence problem by use of codescent objects. In the case of enrichment in $\mathcal{G}ray$, there are partial results along these lines by Power [65], and a local version of the identification of pseudo notions for the monad (0.36) from the Introduction with tricategorical structures is mentioned in [65, Ex. 3.5, p. 319]. A perspective similar to Lack's treatment is given by Gurski in [27, Part III] for $\mathcal{G}ray$-monads.

For a $\mathcal{G}ray$-monad T on a $\mathcal{G}ray$-category $\mathcal{K}$, Gurski gives a definition of lax algebras, lax functors of lax algebras, transformations of lax functors, and modifications of those, and shows that these assemble into a $\mathcal{G}ray$-category Lax-T-Alg. Further, he defines pseudo algebras, pseudo functors of pseudo algebras, and shows that these, together with transformations of pseudo functors and modifications of those, form a $\mathcal{G}ray$-category Ps-T-Alg, which embeds as a locally full sub-$\mathcal{G}ray$-category in the $\mathcal{G}ray$-category

Lax-T-Alg of lax algebras. Finally, there is an obvious 2-locally full inclusion of the $\mathcal{G}ray$-category $\mathcal{K}^T$ of algebras into Ps-T-Alg and Lax-T-Alg.

1.2.1 Definitions and two identities

We here reproduce Gurski's definition of Ps-T-Alg in equational form. In Section 1.5 we will identify this $\mathcal{G}ray$-category for a particular monad on the functor $\mathcal{G}ray$-category $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ where $\mathcal{P}$ is a small and $\mathcal{L}$ is a cocomplete $\mathcal{G}ray$-category. Namely, we show that it is isomorphic as a $\mathcal{G}ray$-category to the full sub-$\mathcal{G}ray$-category $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$ determined by the locally strict trihomomorphisms.

Definition 1.1. [27, Def. 13.4, Def. 13.8] A pseudo T-algebra consists of

- an object X of $\mathcal{K}$;
- a 1-cell $x\colon TX \to X$ i.e. an object in $\mathcal{K}(TX, X)$;
- 2-cell adjoint equivalences[3] $(m, m^\bullet)\colon M_{\mathcal{K}}(x, Tx) \to M_{\mathcal{K}}(x, \mu_X)$ or abbreviated $(m, m^\bullet)\colon xTx \to x\mu_X$ and $(i, i^\bullet)\colon 1_X \to M_{\mathcal{K}}(x, \eta_X)$ or abbreviated $(i, i^\bullet)\colon 1 \to x\eta_X$ i.e. 1-cells in $\mathcal{K}(T^2X, X)$ and $\mathcal{K}(X, X)$ respectively which are adjoint equivalences;
- and three invertible 3-cells π, λ, ρ as in **(PSA1)**-**(PSA3)** subject to the four axioms **(LAA1)**-**(LAA4)** of a lax T-algebra:

(PSA1) An invertible 3-cell π given by an invertible 2-cell in $\mathcal{K}(T^3X, X)$:

$$\pi\colon (m1_{\mu_{TX}}) * (m1_{T^2x}) \Rightarrow (m1_{T\mu_X}) * (1_x Tm) \ ,$$

which is shorthand for

$$\pi\colon M_{\mathcal{K}}(m, 1_{\mu_{TX}}) * M_{\mathcal{K}}(m, 1_{T^2x}) \Rightarrow M_{\mathcal{K}}(m, 1_{T\mu_X}) * M_{\mathcal{K}}(1_x, Tm) \ ;$$

where the horizontal factors on the left compose due to $\mathcal{G}ray$-naturality of μ and the codomains match by the monad axiom $\mu(\mu T) = \mu(T\mu)$.
(PSA2) An invertible 3-cell λ given by an invertible 2-cell in $\mathcal{K}(TX, X)$:

$$\lambda\colon (m1_{\eta_{TX}}) * (i1_x) \Rightarrow 1_x \ ,$$

where the horizontal factors compose due to $\mathcal{G}ray$-naturality of η and the codomains match by the monad axiom $\mu(\eta T) = 1_T$.
(PSA3) An invertible 3-cell ρ given by an invertible 2-cell in $\mathcal{K}(TX, X)$:

$$\rho\colon (m1_{T\eta_X}) * (1_x Ti) \Rightarrow 1_x \ ,$$

where the codomains match by the monad axiom $\mu(T\eta) = 1_T$.

The four lax algebra axioms are:

[3] For adjunctions in a 2-category see [41, §2.].

(LAA1) The following equation in $\mathcal{K}(T^4X, X)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&((\pi 1) * 1_{11T^2m}) \diamond (1_{m11} * \Sigma^{-1}_{m,T^2m}) \diamond ((\pi 1) * 1_{m11}) \\ &= (1_{m11} * (1T\pi)) \diamond ((\pi 1) * 1_{1Tm1}) \diamond (1_{m11} * (\pi 1)) \ ,\end{aligned}$$

where Σ^{-1}_{m,T^2m} is shorthand for $M_{\mathcal{K}}(\Sigma^{-1}_{m,T^2m})$. A careful inspection shows that the horizontal and vertical factors do indeed compose. Note that any mention of the object X has been omitted, e.g. $T\mu_T$ stands for $T\mu_{TX}$. We refer to this as the *pentagon-like axiom* for π.
(LAA2) The following equation in $\mathcal{K}(T^2X, X)$ of vertical composites of whiskered 3-cells is required:

$$((\rho 1) * 1_{m1}) \diamond (1_{m11} * \Sigma_{m,T^2i}) = (1_{m11} * (1_x T\rho)) \diamond ((\pi 1) * 1_{11T^2i}) \ .$$

(LAA3) The following equation in $\mathcal{K}(T^2X, X)$ of vertical composites of whiskered 3-cells is required:

$$1_{m11} * (\lambda 1) = ((\lambda 1) * 1_{1m}) \diamond (1_{m11} * \Sigma^{-1}_{i,m}) \diamond ((\pi 1) * 1_{i11}) \ .$$

(LAA4) The following equation in $\mathcal{K}(T^4X, X)$ of vertical composites of whiskered 3-cells is required:

$$(1_{m11} * (1T\lambda)) \diamond ((\pi 1) * 1_{1Ti1}) = 1_{m11} * (\rho 1) \ .$$

We refer to this as the *triangle-like axiom* for λ, ρ, and π. Diagrams for these axioms may be found in Gurski's definition.

Remark 1.2. In the shorthand notation juxtaposition stands for application of $M_{\mathcal{K}}$, an instance of a power of T in an index refers to its effect on the object X, any other instance of a power of T is shorthand for a hom 2-functor and only applies to the cell directly following it. This notation is possible due to the functor axiom for T.

Remark 1.3. The definition of pseudo algebras and their higher cells may be illuminated by the following viewpoint, which the author learned from Ross Street. Let T be a *Gray*-monad on $\mathcal{K}$ and X be an object of $\mathcal{K}$. The objects of the 2-category $\mathcal{K}(TX, X)$ should form the objects of a multi-2-category with 1- and 2-cells from $\mathcal{K}(T^nX, X)$ for $n > 0$. Then a pseudo algebra for T should correspond to a pseudomonoid in this multi-2-category with multiplication m and unit i, and the higher cells defined in Definitions 1.2-1.4 below should correspond to the higher cells between pseudomonoids.

The definition of a pseudo algebra is derived from the definition of a lax algebra by requiring the 2-cells m and i to be adjoint equivalences and the 3-cells π, λ, ρ to be invertible. In fact, under these circumstances we do not need all of the axioms[4]. This is

[4]In fact, while used in the proof below, the same applies if m and i are not required not be adjoint equivalences, that is, for a lax algebra with π, λ, ρ invertible. To show the redundancy of **(LAA3)** under these circumstances, the equation has to be whiskered on the left with the domain $(m1) * (i1)$ of λ, which gives an equivalent equation since λ is an isomorphism with the identity 1_x. The argument then parallels Kelly's proof [33] even more closely than the one shown here.

proved in the following proposition, which is central for the comparison of trihomomorphisms of $\mathcal{G}ray$-categories with pseudo algebras. Namely, there are only two axioms for a trihomomorphism, while there are four in the definition of a lax algebra. Proposition 1.3 shows that, in general, two of the axioms suffice for a pseudo algebra. In light of Remark 1.3, it is natural to expect such a redundancy.

Proposition 1.3. *Given a pseudo T-algebra, the pentagon-like axiom* **(LAA1)** *and the triangle-like axiom* **(LAA4)** *imply the other two axioms* **(LAA2)**-**(LAA3)**, *i.e. these are redundant.*

Proof. We proceed analogous to Kelly's classical proof that the two corresponding axioms in MacLane's original definition of a monoidal category are redundant [33]. The associators and unitors in Kelly's proof here correspond to π, λ, and ρ. Commuting naturality squares for associators have to be replaced by instances of the middle four interchange law, and there is an additional complication due to the appearance of interchange cells – these have no counterpart in Kelly's proof, so that it is gratifying that the strategy of the proof can still be applied. We only show the proof for the axiom **(LAA3)** involving π and λ here. The one for the axiom **(LAA2)** involving π and ρ is entirely analogous.

The general idea of the proof is to transform the equation of the axiom **(LAA3)** into an equivalent form, namely (1.24) below, which we can manipulate by use of the pentagon-like and triangle-like axiom. Diagrammatically, this has the form of adjoining

$$\begin{array}{ccc}
x1Tx T^2x & \xrightarrow{1Ti11} & xTxT\eta TxT^2x \\
{\scriptstyle 11Tm}\downarrow & \Downarrow \Sigma^{-1}_{1Ti,Tm} & {\scriptstyle 111Tm}\downarrow \\
x1TxT\mu & \xrightarrow{1Ti11} & xTxT\eta TxT\mu \\
\| & \Downarrow 1T\lambda 1 & {\scriptstyle 1Tm11}\downarrow \\
xTxT\mu T\eta_T T\mu & = & xTxT\mu T^2\mu T\eta_{T^2}
\end{array}$$

to the right hand side of some image of the pentagon-like axiom **(LAA1)**, and then a diagram equivalent to the right hand side of **(LAA3)** can be identified as a subdiagram of this.

Since $i\colon\ 1_X \to x\eta_X$ is an equivalence in $\mathcal{L}(X,X)$, and since

$$\mathcal{L}(X', 1_X)\colon\ \mathcal{L}(X',X) \to \mathcal{L}(X',X)$$

is the identity for arbitrary $X' \in \mathrm{ob}\mathcal{L}$, we have that $\mathcal{L}(X', x\eta_X)$ is equivalent to the identity functor. In particular, it is 2-locally fully faithful i.e. a bijection on the sets of 2-cells. On the other hand, by naturality of η we have:

$$\mathcal{L}(X', x\eta_X) = \mathcal{L}(X',x)\mathcal{L}(X',\eta_X) = \mathcal{L}(X',x)\mathcal{L}(\eta_{X'}, TX)T_{,}\ ,$$

where the subscript of T on the right indicates a hom morphism of T. This means that the equation of **(LAA3)** is equivalent to its image under

$$\mathcal{L}(T^2X, x)\mathcal{L}(\eta_{T^2X}, TX)T_{,} = \mathcal{L}(\eta_{T^2X}, X)\mathcal{L}(T^3X, x)T_{,}\ ,$$

where we have used the underlying functoriality of $\mathbf{hom}_{\mathcal{L}}$. We will actually show that the image of the equation under $\mathcal{L}(T^3X, x)T_{,}$ holds, which of course implies that the image of the equation under $\mathcal{L}(\eta_{T^2X}, X)\mathcal{L}(T^3X, x)T_{,}$ holds.

Applying $\mathcal{L}(T^3X, x)T_{,}$ to the lax algebra axiom **(LAA3)** gives

$$1_{1Tm11} * (1T\lambda 1) = ((1T\lambda 1) * 1_{11Tm}) \diamond (1_{1Tm11} * (1T\Sigma^{-1}_{i,m})) \diamond ((1T\pi 1) * 1_{1Ti11}) \ . \tag{1.21}$$

Observe that since $\Sigma^{-1}_{i,m}$ is shorthand for $M_{\mathcal{L}}(\Sigma^{-1}_{i,m})$, we have $1T\Sigma^{-1}_{i,m} = 1\Sigma^{-1}_{Ti,Tm}$ by the functor axiom for T and the equality (1.2) from 1.1.3, which is in fact shorthand for

$$\begin{aligned} M_{\mathcal{L}}((1, M_{\mathcal{L}}(\Sigma^{-1}_{Ti,Tm}))) &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)(a^{-1}(1, \Sigma^{-1}_{Ti,Tm})) && \text{(by a } \mathcal{Gray}\text{-category axiom)} \\ &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)(\Sigma^{-1}_{(1,Ti),Tm}) && \text{(by eq. (1.5) from 1.1.3)} \\ &= M_{\mathcal{L}}(\Sigma^{-1}_{1Ti,Tm}) \ , && \text{(by eq. (1.2) from 1.1.3)} \end{aligned}$$

for which the corresponding shorthand is just $\Sigma^{-1}_{1Ti,Tm}$.

Next, equation (1.21) is clearly equivalent to the one whiskered with $m111$ on the left because $m111$ is an (adjoint) equivalence by the definition of m, i.e. to

$$\begin{aligned} &1_{m111} * 1_{1Tm11} * (1T\lambda 1) \\ &= (1_{m111} * (1T\lambda 1) * 1_{11Tm}) \diamond (1_{m111} * 1_{1Tm11} * (\Sigma^{-1}_{1Ti,Tm})) \diamond (1_{m111} * (1T\pi 1) * 1_{1Ti11}) \ . \end{aligned} \tag{1.22}$$

Here we have used functoriality of $*$ i.e. the middle four interchange law, and it is understood that because horizontal composition is associative, we can drop parentheses.

Now observe that $1_{m111} * (1T\pi 1)$ is the image under $\mathcal{L}(T\eta_{T^2X}, X)$ of the leftmost vertical factor in the right hand side of the pentagon-like axiom **(LAA1)** for π. Namely, the image of **(LAA1)** under $\mathcal{L}(T\eta_{T^2X}, X)$ is

$$\begin{aligned} &((\pi 11) * 1_{11T^2m1}) \diamond (1_{m111} * (\Sigma^{-1}_{m,T^2m} 1)) \diamond ((\pi 11) * 1_{m111}) \\ &= (1_{m111} * (1T\pi 1)) \diamond ((\pi 11) * 1_{1Tm11}) \diamond (1_{m111} * (\pi 11)) \ , \end{aligned}$$

where $\Sigma^{-1}_{m,T^2m} 1$ is shorthand for

$$\begin{aligned} M_{\mathcal{L}}(M_{\mathcal{L}}(\Sigma^{-1}_{m,T^2m}), 1) &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)(\Sigma^{-1}_{m,T^2m}, 1) && \text{(by definition of } (M_{\mathcal{L}} \otimes 1)) \\ &= M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})a(\Sigma^{-1}_{m,T^2m}, 1) && \text{(by a } \mathcal{Gray}\text{-category axiom)} \\ &= M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})(\Sigma^{-1}_{m,(T^2m,1)}) && \text{(by eq. (1.3) from 1.1.3)} \\ &= M_{\mathcal{L}}(\Sigma^{-1}_{m,T^2m1}) \ , && \text{(by eq. (1.2) from 1.1.3)} \end{aligned}$$

for which the corresponding shorthand is just Σ^{-1}_{m,T^2m1}.

In fact, we have that $T^2m1 = 1Tm$ where on the left the identity is $1_{T\eta_{T^2X}}$ and on the right it is $1_{T\eta_X}$, thus this is simply naturality of $T\eta$. Hence, $\Sigma^{-1}_{m,T^2m1} = \Sigma^{-1}_{m,1Tm}$. In turn, this is shorthand for

$$\begin{aligned} M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})(\Sigma^{-1}_{m,(1,Tm)}) &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)a^{-1}(\Sigma^{-1}_{(m,(1,Tm))}) && \text{(by a } \mathcal{Gray}\text{-category axiom)} \\ &= M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)(\Sigma^{-1}_{(m,1),Tm}) && \text{(by eq. (1.4) from 1.1.3)} \\ &= M_{\mathcal{L}}(\Sigma^{-1}_{m1,Tm}) \ . && \text{(by eq. (1.2) from 1.1.3)} \end{aligned}$$

Implementing these transformations, the image of **(LAA1)** has the form

$$\begin{aligned}&((\pi 11) * 1_{111Tm}) \diamond (1_{m111} * \Sigma^{-1}_{m1,Tm}) \diamond ((\pi 11) * 1_{m111}) \\ &= (1_{m111} * (1T\pi 1)) \diamond ((\pi 11) * 1_{1Tm11}) \diamond (1_{m111} * (\pi 11)) \ . \end{aligned} \tag{1.23}$$

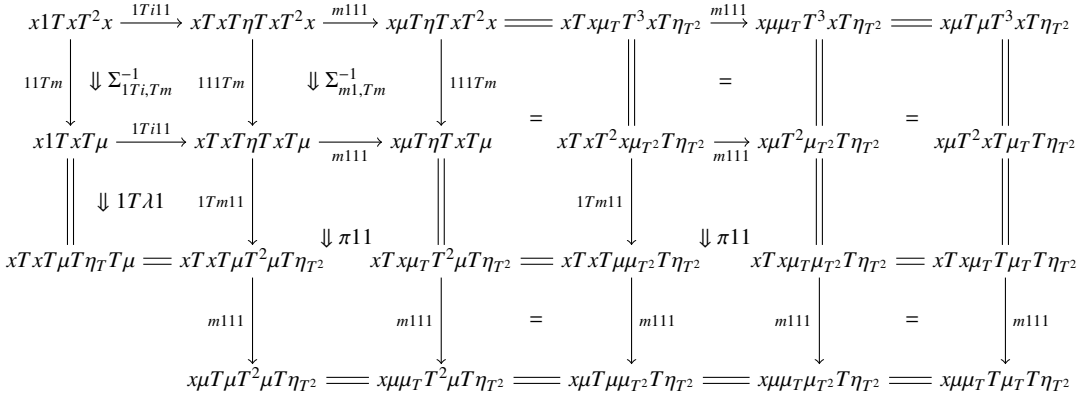
Since they are invertible, composing equation (1.22) with the other two factors from the pentagon-like axiom for π whiskered with the (adjoint) equivalence $1Ti11$ on the right, gives an equivalent equation. Thus, our goal is now to prove the following equation:

$$\begin{aligned}&(1_{m111} * 1_{1Tm11} * (1T\lambda 1)) \diamond ((\pi 11) * 1_{1Tm11} * 1_{1Ti11}) \diamond (1_{m111} * (\pi 11) * 1_{1Ti11}) \\ &= (1_{m111} * (1T\lambda 1) * 1_{11Tm}) \diamond (1_{m111} * 1_{1Tm11} * (\Sigma^{-1}_{1Ti,Tm})) \\ &\quad \diamond \Big(\big((1_{m111} * (1T\pi 1)) \diamond ((\pi 11) * 1_{1Tm11}) \diamond (1_{m111} * (\pi 11))\big) * 1_{1Ti11}\Big) \ . \end{aligned} \tag{1.24}$$

This is proved by transforming the right hand side by use of the pentagon-like and triangle-like axiom until we finally obtain the left hand side. Namely, using the image of the pentagon-like axiom for π in the form (1.23) above, the right hand side of (1.24) is equal to

$$\begin{aligned}&(1_{m111} * (1T\lambda 1) * 1_{11Tm}) \diamond (1_{m111} * 1_{1Tm11} * (\Sigma^{-1}_{1Ti,Tm})) \\ &\quad \diamond \Big(\big((\pi 11) * 1_{111Tm}) \diamond (1_{m111} * \Sigma^{-1}_{m1,Tm}) \diamond ((\pi 11) * 1_{m111})\big) * 1_{1Ti11}\Big) \ . \end{aligned}$$

The diagrammatic form of this is drawn below.



The rectangle composed of the two interchange cells is shorthand for

$$M_{\mathcal{L}}((1_{m1}, 1) * \Sigma^{-1}_{1Ti,Tm}) \diamond M_{\mathcal{L}}(\Sigma^{-1}_{m1,Tm} * (1_{1Ti}, 1)) = M_{\mathcal{L}}(\Sigma_{(m1)*(1Ti),Tm}) \ ,$$

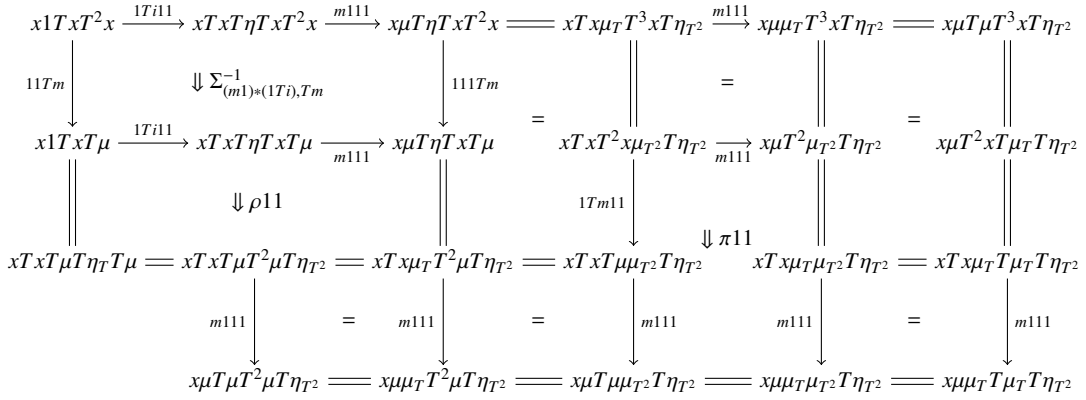
for which the corresponding shorthand is just $\Sigma_{(m1)*(1Ti),Tm}$. Notice that we made use here of the image under $M_{\mathcal{L}}$ of the Gray product relation

$$(\Sigma_{f',g} * (1_f, 1)) \diamond ((1_{f'}, 1) * \Sigma_{f,g}) \sim \Sigma_{f'*f,g} \ .$$

Next, the subdiagram formed by $1T\lambda 1$ and $\pi 11$ may be transformed by use of the image under $\mathcal{L}(T\mu, X)$ of the triangle-like axiom **(LAA3)**:

$$(1_{m111} * (1T\lambda 1)) \diamond ((\pi 11) * 1_{1Ti1}) = 1_{m111} * (\rho 11) \ .$$

Implementing these transformations in the diagram, gives the one drawn below.



The subdiagram formed by the interchange cell and $\rho 11$ is

$$((\rho 11) * 1_{11Tm}) \diamond \Sigma^{-1}_{(m1)*(1Ti),Tm} \ .$$

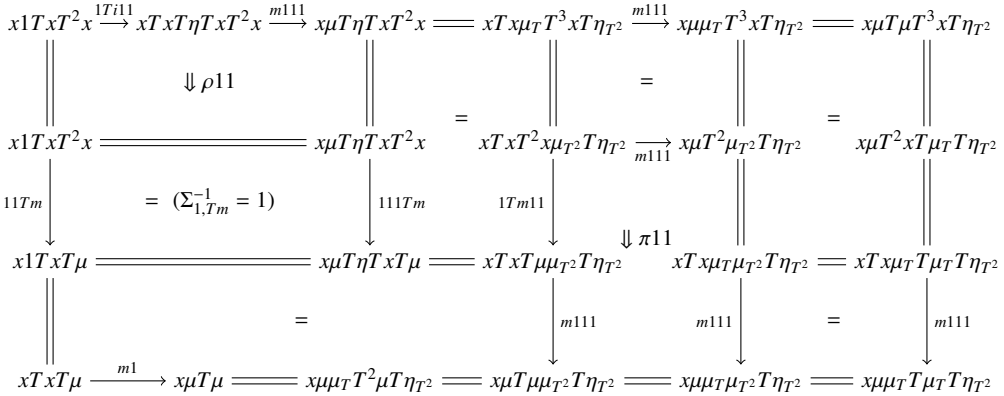
This is shorthand for

$$M_{\mathcal{L}}(((\rho, 1) * (1, 1_{Tm})) \diamond \Sigma^{-1}_{(m1)*(1Ti),Tm}) = M_{\mathcal{L}}(\Sigma^{-1}_{1,Tm} \diamond ((1, 1_{Tm}) * (\rho, 1))) = M_{\mathcal{L}}((1, 1_{Tm}) * (\rho, 1))$$

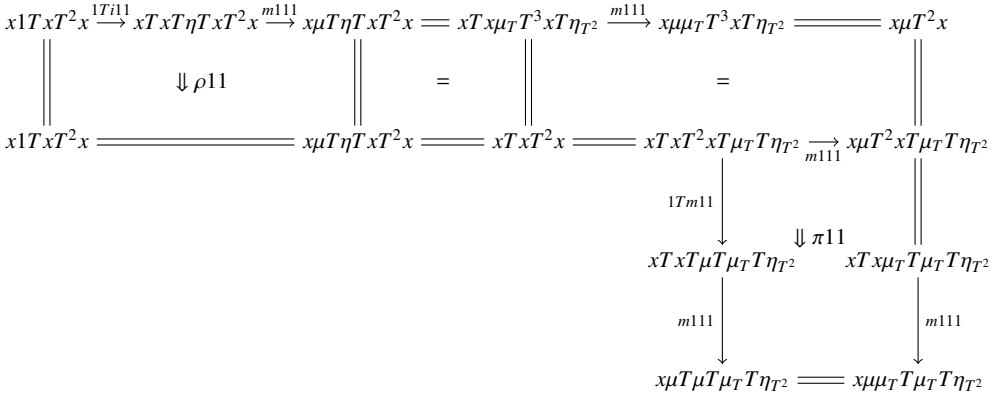
or $1_{111Tm} * (\rho 11)$, where we have used the relation

$$((1, \beta) * (\alpha, 1)) \diamond \Sigma_{f,g} \sim \Sigma_{f',g'} \diamond ((\alpha, 1) * (1, \beta)) \tag{1.25}$$

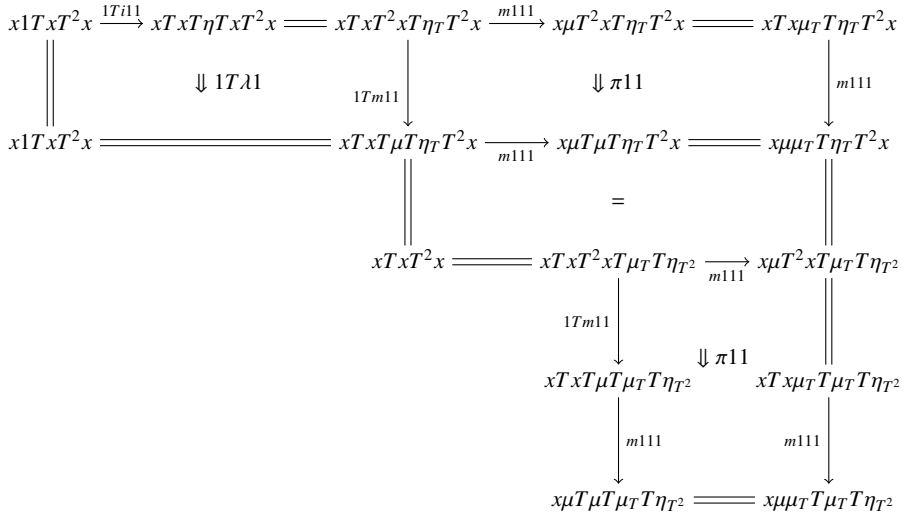
for interchange cells in the Gray product and the fact that $\Sigma^{-1}_{1,Tm}$ is the identity 2-cell cf. 1.1.3. This means we now have the following diagram.



Slightly rewritten, this is the same as the diagram below.



For the upper right entry we have used another identity to make commutativity obvious. Finally, by another instance of the triangle identity—in fact its image under $\mathcal{L}(T^2x, X)$—we end up with the diagram below. It is easily seen to be the diagrammatic form of the left hand side of (1.24), so this ends the proof.



□

Remark 1.4. One can impose a fifth axiom in the definition of a lax algebra, which corresponds to the fifth axiom in the definition of a skew-monoidal category. One can similarly prove that this fifth axiom is implied by the pentagon-like and triangle-like

axiom in the case of a pseudo T-algebra.

Definition 1.2. [27, Def. 13.6 and Def. 13.9] A pseudo T-functor

$$(f, F, \hbar, m)\colon (X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$$

consists of

- a 1-cell $f\colon X \to Y$ in $\mathcal{K}$ i.e. an object of $\mathcal{K}(X, Y)$;
- a 2-cell adjoint equivalence $(F, F^\bullet)\colon fx \to yTf$ i.e. a 1-cell adjoint equivalence internal to $\mathcal{K}(TX, Y)$;
- and two invertible 3-cells $\hbar, m$ as in **(PSF1)**-**(PSF2)** subject to the three axioms **(LFA1)**-**(LFA3)** of a lax T-functor:

(PSF1) An invertible 3-cell $\hbar$ given by an invertible 2-cell in $\mathcal{K}(X, Y)$:

$$\hbar\colon (F1_{\eta X}) * (1_f i^X) \Rightarrow (i^Y 1_f)$$

where the codomains match by *Gray*-naturality of η.
(PSF2) An invertible 3-cell m given by an invertible 2-cell in $\mathcal{K}(T^2X, Y)$:

$$m\colon (m^Y 1_{T^2 f}) * (1_y TF) * (F1_x) \Rightarrow (F1_\mu) * (1_f m)$$

where the codomains match by *Gray*-naturality of μ.

The three lax T-functor axioms are:
(LFA1) The following equation in $\mathcal{K}(T^3X, Y)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&(1_{F11} * (1\pi^X)) \diamond ((m1) * 1_{1m^X 1}) \diamond (1_{m^Y 11} * 1_{11T^2F} * (m1)) \diamond (1_{m^Y 11} * \Sigma_{m^Y, T^2F} * 1_{1TF1} * 1_{F11})\\ &= ((m1) * 1_{11Tm^X}) \diamond (1_{m^Y 11} * 1_{1TF1} * \Sigma^{-1}_{F, Tm^X}) \diamond (1Tm) \diamond ((\pi^Y 1) * 1_{11T^2F} * 1_{TF1} * 1_{F11})\ .\end{aligned}$$

(LFA2) The following equation in $\mathcal{K}(TX, Y)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&((\lambda^Y 1) * 1_{1F}) \diamond (1_{m^Y 11} * \Sigma^{-1}_{i^Y, F}) \diamond (1_{m^Y 11} * 1_{11F} * (\hbar 1))\\ &= (1_F * (1\lambda^Y)) \diamond ((m1) * 1_{1i^X 1})\ .\end{aligned}$$

(LFA3) The following equation in $\mathcal{K}(TX, Y)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&((\rho^Y 1) * 1_{F1}) \diamond (1_{m^Y 11} * (1T\hbar) * 1_{F1})\\ &= (1_F * (1\rho^X)) \diamond ((m1) * 1_{1Ti^X}) \diamond (1_{m^Y 1} * 1_{1TF1} * \Sigma^{-1}_{F, Ti^X})\ .\end{aligned}$$

A careful inspection shows that the horizontal and vertical factors do indeed compose in all of these axioms. Diagrams may be found in Gurski's definition.

Definition 1.3. [27, Def. 13.6 and Def. 13.10] A T-transformation

$$(\alpha, A)\colon\ (f, F, \hbar^f, \mathfrak{m}^f) \Rightarrow (g, G, \hbar^g, \mathfrak{m}^g)\colon\ (X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$$

consists of

- a 2-cell $\alpha\colon\ f \Rightarrow g$ i.e. an object of $\mathcal{K}(X, Y)$;
- an invertible 3-cell A as in **(T1)** subject to the two axioms **(LTA1)**-**(LTA2)** of a lax T-algebra:

(T1) An invertible 3-cell A given by an invertible 2-cell in $\mathcal{K}(TX, Y)$:

$$A\colon\ (1_y T\alpha) * F \Rightarrow G * (\alpha 1_x)\ .$$

The two lax T-transformation axioms are:
(LTA1) The following equation in $\mathcal{K}(X, Y)$ of vertical composites of whiskered 3-cells is required:

$$(\hbar^g * 1_{\alpha 1}) \diamond (1_{G1} * \Sigma_{\alpha, i^X}) \diamond ((A1) * 1_{1i^X}) = \Sigma^{-1}_{i^Y, \alpha} \diamond (1_{1T\alpha 1} * \hbar^f)\ .$$

(LTA2) The following equation in $\mathcal{K}(T^2X, X)$ of vertical composites of whiskered 3-cells is required:

$$\begin{aligned}&(\mathfrak{m}^g * 1_{\alpha 1}) \diamond (1_{m^Y 1} * 1_{1TG} * (A1)) \diamond (1_{m^Y 1} * (1TA) * 1_{F1}) \diamond (\Sigma^{-1}_{m^Y, T^2\alpha} * 1_{1TF} * 1_{F1})\\ &= (1_{G1} * \Sigma_{\alpha, m^X}) \diamond ((A1) * 1_{1m^X}) \diamond (1_{1T^2\alpha} * \mathfrak{m}^f)\ .\end{aligned}$$

A careful inspection shows that the horizontal and vertical factors do indeed compose in the two axioms. Diagrams may be found in Gurski's definition.

Definition 1.4. A T-modification $\Gamma\colon\ (\alpha, A) \Rrightarrow (\beta, B)$ of T-transformations

$$(f, F, \hbar^f, \mathfrak{m}^f) \Rightarrow (g, G, \hbar^g, \mathfrak{m}^g)\colon\ (X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$$

consists of a

- 3-cell $\Gamma\colon\ \alpha \Rrightarrow \beta$ i.e. a 2-cell in $\mathcal{K}(X, Y)$;
- subject to one axiom **(MA1)**:

(MA1) The following equation in $\mathcal{K}(TX, Y)$ of vertical composites of whiskered 3-cells is required:

$$B \diamond ((1T\Gamma) * 1_F) = (1_G * (\Gamma 1)) \diamond A\ .$$

Finally, we provide the *Gray*-category structure of Ps-T-Alg. We begin with its hom 2-categories.

Definition 1.5. Given T-algebras $(X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X)$ and $(Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$, the prescriptions below give the 2-globular set Ps-T-Alg(X, Y) whose objects are pseudo T-functors from X to Y, whose 1-cells are T-transformations between pseudo T-functors, and whose 2-cells are T-modifications between those, the structure of a 2-category [27, Prop. 13.11].

Given T-modifications $\Gamma\colon (\alpha, A) \Rrightarrow (\beta, B)$ and $\Delta\colon (\beta, B) \Rrightarrow (\epsilon, E)$ of T-transformations

$$(f, F, \hbar^f, m^f) \Rightarrow (g, G, \hbar^g, m^g)\colon (X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)\ ,$$

their vertical composite $\Delta \diamond \Gamma$ is defined by the vertical composite $\Delta \diamond \Gamma$ of 2-cells in $\mathcal{K}(X, Y)$. The identity T-modification of (α, A) as above is defined by the 2-cell 1_α in $\mathcal{K}(X, Y)$.

Given T-transformations $(\alpha, A)\colon (f, F, \hbar^f, m^f) \Rightarrow (g, G, \hbar^g, m^g)$ and $(\beta, B)\colon (g, G, \hbar^g, m^g) \Rightarrow (h, H, \hbar^h, m^h)$ of pseudo T-functors

$$(X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)\ ,$$

their horizontal composite $(\beta, B) * (\alpha, A)$ is defined by

$$(\beta * \alpha, (B * 1_{\alpha 1}) \diamond (1_{1T\beta} * A))\ .$$

The identity T-transformation of $(f, F, \hbar^f, m^f)$ is defined by $(1_f, 1_F)$.

Given T-modifications $\Gamma\colon (\alpha, A) \Rrightarrow (\alpha', A')\colon (f, F, \hbar^f, m^f) \Rightarrow (g, G, \hbar^g, m^g)$ and $\Delta\colon (\beta, B) \Rrightarrow (\beta', B')\colon (g, G, \hbar^g, m^g) \Rightarrow (h, H, \hbar^h, m^h$ of pseudo T-functors

$$(X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X) \to (Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)\ ,$$

their horizontal composite is defined by the horizontal composite $\Delta * \Gamma$ of 2-cells in $\mathcal{K}(X, Y)$.

We omit the proof that this is indeed a 2-category.

Definition 1.6. The prescriptions below give the set of pseudo T-algebras with the hom 2-categories from the proposition above, the structure of a *Gray*-category denoted Ps-T-Alg, see [27, Prop. 13.12 and Th. 13.13].

Given pseudo T-algebras $(X, x, m^X, i^X, \pi^X, \lambda^X, \rho^X)$, $(Y, y, m^Y, i^Y, \pi^Y, \lambda^Y, \rho^Y)$ and $(Z, z, m^Z, i^Z, \pi^Z, \lambda^Z, \rho^Z)$, the composition law is defined by the strict functor

$$\boxtimes\colon \text{Ps-}T\text{-Alg}(Y, Z) \otimes \text{Ps-}T\text{-Alg}(X, Y) \to \text{Ps-}T\text{-Alg}(X, Z)$$

specified as follows.

On an object (g, f) in Ps-T-Alg$(Y, Z) \otimes$ Ps-T-Alg(X, Y) i.e. on functors $(g, G, \hbar^g, m^g)$ and $(f, F, \hbar^f, m^f)$, $\boxtimes$ is defined by

$$\Big(gf, (G1_{Tf}) * (1_g F), (\hbar^g 1) \diamond (1_{G11} * (1\hbar^f)),$$
$$(1_{G11} * (1 m^f)) \diamond ((m^g 1) * 1_{11TF} * 1_{G11} * 1_{1F1}) \diamond (1_{m^Y 11} * 1_{1TG1} * \Sigma^{-1}_{G,TF} * 1_{1F1})\Big)\ ,$$

which we denote by $g \boxtimes f$.

On a generating 1-cell of the form $((\alpha, A), 1)\colon (g, f) \to (g', f)$ in the Gray product Ps-T-Alg$(Y, Z) \otimes$ Ps-T-Alg(X, Y), where (α, A) is a T-transformation

$$(g, G, h^g, m^g) \Rightarrow (g', G', h^{g'}, m^{g'}),$$

and f is as above, $\boxtimes$ is defined by

$$(\alpha 1_f, (1_{G'1} * \Sigma_{\alpha,F}) \diamond ((A1) * 1_{1F}))$$

and denoted $\alpha \boxtimes 1_f$.

Similarly, on a generating 1-cell of the form $(1, (\beta, B))\colon (g, f) \to (g, f')$ in the Gray product Ps-T-Alg$(Y, Z) \otimes$ Ps-T-Alg(X, Y), where (β, B) is a T-transformation

$$(f, F, h^f, m^f) \Rightarrow (f', F', h^{f'}, m^{f'}),$$

and g is as above, $\boxtimes$ is defined by

$$(1_g \beta, (1_{G1} * (1B)) \diamond (\Sigma^{-1}_{G,\beta} * 1_{1F}))$$

and denoted $\alpha \boxtimes 1_f$.

On a generating 2-cell of the form $(\Gamma, 1)\colon ((\alpha, A), 1) \Rrightarrow ((\alpha', A'), 1)$, $\boxtimes$ is defined by the underlying 2-cell $\Gamma 1_f$ in $\mathcal{K}(X, Y)$ and denoted by $\Gamma \boxtimes 1$, and similarly for 2-cells of the form $(1, \Delta)\colon (1, (\beta, B)) \Rrightarrow (1, (\beta', B'))$.

Finally, on an interchange cell $\Sigma_{(\alpha,A),(\beta,B)}$ in Ps-T-Alg$(Y, Z) \otimes$ Ps-T-Alg(X, Y), $\boxtimes$ is defined by the 2-cell $M_{\mathcal{K}}(\Sigma_{\alpha,\beta})$, the shorthand of which is $\Sigma_{\alpha,\beta}$.

The unit at an object $(X, x, m, i, \pi, \lambda, \rho)$, that is, the functor $j_X\colon I \to$ Ps-T-Alg(X, X) is determined by strictness and the requirement that it sends the unique object $*$ of I to the T-functor $(1_X, 1_x, 1_i, 1_m)$.

We omit the proof that this is well-defined and that Ps-T-Alg is indeed a $\mathcal{G}ray$-category.

1.2.2 Coherence via codescent

Recall from [38, 3.1] that given a complete and cocomplete locally small symmetric monoidal closed category $\mathcal{V}$, $\mathcal{V}$-categories $\mathcal{K}$ and $\mathcal{B}$, and $\mathcal{V}$-functors $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ and $\mathcal{G}\colon \mathcal{K} \to \mathcal{B}$, the colimit of G weighted by F is a representation $(F * G, \nu)$ of the $\mathcal{V}$-functor $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}](F-, \mathcal{B}(G-, ?))\colon \mathcal{B} \to \mathcal{V}$ (where this is assumed to exist) with representing object $F * G$ in $\mathcal{B}$ and unit $\nu\colon F \to \mathcal{B}(G-, F * G)$. For the concept of representable functors see [38, 1.10]. In particular, there is a $\mathcal{V}$-natural (in B) isomorphism

$$\mathcal{B}(F * G, B) \xrightarrow{\cong} [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](F, \mathcal{B}(G-, B)) \,, \tag{1.26}$$

and the unit is obtained by Yoneda when this is composed with the unit j_{F*G} of the $\mathcal{V}$-category $\mathcal{B}$ at the object $F * G$.

Definition 1.7. A $\mathcal{V}$-functor $T\colon \mathcal{B} \to \mathcal{C}$ preserves the colimit of $G\colon \mathcal{K} \to \mathcal{B}$ weighted by $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ if when $(F * G, \nu)$ exists, the composite

$$T_{G-,B}\nu\colon F \to \mathcal{B}(G-, F * G) \to \mathcal{L}(TG-, T\{F, G\})$$

exhibits $T(F * G)$ as the colimit of TG weighted by F i.e. the composite corresponds under Yoneda to an isomorphism as in (1.26) with TG instead of G.

Now let again $\mathcal{K}$ be a *Gray*-category and T be a *Gray*-monad on it. Below we will make use of the following corollary of the central coherence theorem from three-dimensional monad theory [27, Corr. 15.14].

Theorem 1.3 (Gurski's coherence theorem)**.** *Assume that $\mathcal{K}$ has codescent objects of codescent diagrams, and that T preserves them. Then the inclusion $i\colon \mathcal{K}^T \hookrightarrow$ Ps-T-Alg has a left adjoint $L\colon$ Ps-T-Alg $\to \mathcal{K}^T$ and each component $\eta_X\colon X \to iLX$ of the unit of this adjunction is a biequivalence in* Ps-T-Alg.

Remark 1.5. Codescent objects are certain weighted colimits, see [27, 12.3] and 2.1 for the codescent object of a strict codescent diagram. In fact, they are built from co-2-inserters, co-3-inserters and coequifiers. These are classes of weighted colimits where each of these classes is determined separately by considering all weighted colimits $F * G$ with a particular fixed *Gray*-functor $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$. Thus there is no other restriction on G apart from the fact that it must have the same domain as F. In particular, if T is a *Gray*-monad and T preserves co-2-inserters, co-3-inserters and coequifiers, then also TT preserves co-2-inserters, co-3-inserters and coequifiers because G and TG have the same domain. This is used in the proof of the theorem to show that the Eilenberg-Moore object $\mathcal{K}^T$ has codescent objects and that they are preserved by the forgetful *Gray*-functor $\mathcal{K}^T \to \mathcal{K}$. Namely, in enriched monad theory one can show that the forgetful functor $\mathcal{K}^T \to \mathcal{K}$ creates any colimit that is preserved by T and TT, just as in ordinary monad theory.

For any of the classes of weighted colimits above, the domain of F is small, so codescent objects are small weighted colimits. Hence, if $\mathcal{K}$ is cocomplete, it has codescent objects of codescent diagrams in particular. This observation gives the following corollary.

Corollary 1.1. *Let $\mathcal{K}$ be cocomplete and let T be a monad on $\mathcal{K}$ that preserves small weighted colimits. Then the inclusion $i\colon \mathcal{K}^T \hookrightarrow$ Ps-T-Alg has a left adjoint $L\colon$ Ps-T-Alg $\to \mathcal{K}^T$ and each component $\eta_X\colon X \to iLX$ of the unit of the adjunction is an internal biequivalence in* Ps-T-Alg. □

1.3 The monad of the Kan adjunction

1.3.1 A $\mathcal{V}$-monad on $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$

Let $\mathcal{V}$ be a complete and cocomplete symmetric monoidal closed category such that the underlying category $\mathcal{V}_0$ is locally small. By cocompleteness we have an initial object which we denote by $\emptyset$. Let $\mathcal{P}$ be a small $\mathcal{V}$-category and let $\mathcal{L}$ be a cocomplete

$\mathcal{V}$-category. In this general situation, we now describe in more detail a $\mathcal{V}$-monad corresponding to the Kan adjunction with left adjoint left Kan extension Lan_H along a particular $\mathcal{V}$-functor H and right adjoint the functor $[H,1]$ from enriched category theory. For $\mathcal{V} = \mathcal{G}ray$, this is the $\mathcal{G}ray$-monad mentioned in the introduction, for which the pseudo algebras shall be compared to locally strict trihomomorphisms.

First, observe that the set $\mathrm{ob}\mathcal{P}$ of the objects of $\mathcal{P}$ may be considered as a discrete $\mathcal{V}$-category. More precisely, there is a $\mathcal{V}$-category structure on $\mathrm{ob}\mathcal{P}$ such that for objects $P, Q \in \mathcal{P}$ the hom object $(\mathrm{ob}\mathcal{P})(P,Q)$ is given by I if $P = Q$ and by $\emptyset$ otherwise and such that the nontrivial hom morphisms are given by $l_I = r_I$. The $\mathcal{V}$-functor H is defined to be the unique $\mathcal{V}$-functor $\mathrm{ob}\mathcal{P} \to \mathcal{P}$ such that the underlying map on objects is the identity.

Since $\mathcal{P}$ is small and since $\mathcal{V}_0$ is complete, the functor category $[\mathcal{P},\mathcal{L}]$ exists. For two $\mathcal{V}$-functors $A, B\colon\ \mathcal{P} \to \mathcal{L}$ the hom object $[\mathcal{P},\mathcal{L}](A,B)$ is the end

$$\int_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP)\ ,$$

which is given by an equalizer

$$\int_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP,BP) \longrightarrow \prod_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP,BP) \underset{\sigma}{\overset{\rho}{\rightrightarrows}} \prod_{P,Q \in \mathrm{ob}\mathcal{P}} [\mathcal{P}(P,Q), \mathcal{L}(AP,BQ)] \tag{1.27}$$

in $\mathcal{V}_0$, see [38, (2.2), p. 27], where—if we denote by π the cartesian projections— ρ and σ are determined by requiring $\pi_{P,Q}\rho$ and $\pi_{P,Q}\sigma$ to be π_P composed with the transform of $\mathcal{L}(AP, B-)_{PQ}$ and π_Q composed with the transform of $\mathcal{L}(A-, BQ)_{QP}$ respectively.

Now let $\mathcal{M}$ be another small $\mathcal{V}$-category and $K\colon\ \mathcal{M} \to \mathcal{P}$ be a $\mathcal{V}$-functor , e.g. $K = H$. The $\mathcal{V}$-functor K induces a $\mathcal{V}$-functor

$$[K,1]\colon\ [\mathcal{P},\mathcal{L}] \to [\mathcal{M},\mathcal{L}]\ ,$$

which sends a $\mathcal{V}$-functor $A\colon\ \mathcal{P} \to \mathcal{L}$ to the composite $\mathcal{V}$-functor AK, cf. [38, (2.26)], and its hom morphisms are determined by the universal property of the end and commutativity of the following diagram

$$\begin{array}{ccc} [\mathcal{P},\mathcal{L}](A,B) & \xrightarrow{[K,1]_{A,B}} & [\mathcal{M},\mathcal{L}](AK,BK) \\ \downarrow{\scriptstyle E_{KM}} & & \downarrow{\scriptstyle E_M} \\ \mathcal{L}(AKM,BKM) & = & \mathcal{L}(AKM,BKM)\ . \end{array} \tag{1.28}$$

Left Kan extension $\mathrm{Lan}_K\colon\ [\mathcal{M},\mathcal{L}] \to [\mathcal{P},\mathcal{L}]$ along K provides a left adjoint to $[K,1]$: this is the usual Theorem of Kan adjoints as given in [38, Th. 4.50, p. 67], and it applies since $\mathcal{M}$ and $\mathcal{P}$ are small and since $\mathcal{L}$ is cocomplete. In particular, we have a hom $\mathcal{V}$-adjunction

$$[\mathcal{P},\mathcal{L}](\mathrm{Lan}_K A, S) \cong [\mathcal{M},\mathcal{L}](A, [K,1](S))\ , \tag{1.29}$$

cf. [38, (4.39)], which is $\mathcal{V}$-natural in $A \in [\mathcal{M}, \mathcal{L}]$ and $S \in [\mathcal{P}, \mathcal{L}]$. Thus we have a monad

$$T = [H,1]\mathrm{Lan}_K \colon [\mathcal{M}, \mathcal{L}] \to [\mathcal{M}, \mathcal{L}] \tag{1.30}$$

on $[\mathcal{M}, \mathcal{L}]$, which we call the monad of the Kan adjunction. The unit $\eta \colon 1 \Rightarrow T$ of T is given by the unit η of the adjunction (1.29), while the multiplication $\mu \colon TT \Rightarrow T$, is given by

$$[H,1]\epsilon\mathrm{Lan}_H \colon [H,1]\mathrm{Lan}_H[H,1]\mathrm{Lan}_\mathrm{H} \Rightarrow [H,1]\mathrm{Lan}_H$$

where ϵ is the counit of the adjunction (1.29).

We now come back to the special case that $\mathcal{M} = \mathrm{ob}\mathcal{P}$ and $K = H$. Since $\mathcal{P}$ is small, we may identify a functor $\mathrm{ob}\mathcal{P} \to \mathcal{L}$ with its family of values in $\mathcal{L}$ i.e. the set of functors is identified with the (small) limit in $\mathcal{S}et$ given by the product $\prod_{\mathrm{ob}\mathcal{P}} \mathrm{ob}\mathcal{L}$.

In fact, the equalizer (1.27) is trivial for $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$, so for two functors $A, B \colon \mathrm{ob}\mathcal{P} \to \mathcal{L}$, the hom object $[\mathrm{ob}\mathcal{P}, \mathcal{L}](A, B)$ is given by the (small) limit in $\mathcal{V}_0$ given by the product $\prod_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP)$.

Namely, ρ and σ are equal in (1.27): Denoting by π the projections of the cartesian products, $\pi_{P,Q}\rho = \rho_{P,Q}\pi_P$ equals $\pi_{P,Q}\sigma = \sigma_{P,Q}\pi_Q$ because for $P \neq Q$, the two morphisms

$$\prod_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP) \to [\emptyset, \mathcal{L}(AP, BQ)]$$

must both be the transform of the unique morphism

$$\emptyset \to [\prod_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP), \mathcal{L}(AP, BQ)] \ ;$$

and for $P = Q$, we have $\rho_{P,P} = \sigma_{P,P}$ because these are the transforms of $\mathcal{L}(AP, B-)_{PP}$ and $\mathcal{L}(A-, BP)_{PP}$, which are both equal to

$$j_{\mathcal{L}(AP,BP)} \colon I = (\mathrm{ob}\mathcal{P})(P, P) \to [\mathcal{L}(AP, BP), \mathcal{L}(AP, BP)]$$

by the unit axioms for the $\mathcal{V}$-functors A, B, $\mathcal{L}(AP, -)$, and $\mathcal{L}(-, BP)$. To see this, note that $j_P \colon I \to I$ is the identity functor, so $\mathcal{L}(AP, B-)_{PP} = \mathcal{L}(AP, B-)_{PP} j_P$ and $\mathcal{L}(A-, BP)_{PP} = \mathcal{L}(A-, BP)_{PP} j_p$.

From diagram (1.28), we see that

$$[H,1]_{A,B} \colon [\mathcal{P}, \mathcal{L}](A, B) \to [\mathrm{ob}\mathcal{P}, \mathcal{L}](AH, BH)$$

is given by the strict functor of the equalizer (1.27),

$$\int_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP) \to \prod_{P \in \mathrm{ob}\mathcal{P}} \mathcal{L}(AP, BP) \ ,$$

that is, the strict functor into the product induced by the family of evaluation functors E_P where P runs through the objects of $\mathcal{P}$.

Lemma 1.2. *Let $\{F, G\}$ be a pointwise limit, then any representation is pointwise.*

Proof. Let (B, μ) be a pointwise representation and let (B', μ') be any other representation. By Yoneda, μ' has the form $[\mathcal{P}, \mathcal{L}](\alpha, G-)\mu$ for a unique isomorphism $\alpha\colon B' \Rightarrow B$. It follows that

$$E_P\mu' = E_P[\mathcal{P}, \mathcal{L}](\alpha, G-)\mu = \mathcal{L}(\alpha_P, E_PG-)E_P\mu$$

and by extraordinary naturality this induces

$$\mathcal{L}(L, B'P) \xrightarrow{\mathcal{L}(L,\alpha_P)} \mathcal{L}(L, BP) \xrightarrow{\beta} [\mathcal{K}, \mathcal{V}](F, \mathcal{L}(L, (G-)P))$$

where β is the isomorphism induced by $E_P\mu$, and this is an isomorphism that is $\mathcal{V}$-natural in L and P because (B, μ) is pointwise and α_P is an isomorphism that is $\mathcal{V}$-natural in P. This proves that (B', μ') is a pointwise limit. □

The following is the usual non-invariant notion of limit creation as in MacLane's book [55, p. 108] adapted to the enriched context:

Definition 1.8. A $\mathcal{V}$-functor $T\colon \mathcal{B} \to \mathcal{C}$ creates $F * G$ or creates colimits of $G\colon \mathcal{K} \to \mathcal{B}$ weighted by $F\colon \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ if (i) for every (C, ν) where $\nu\colon F \to \mathcal{C}(TG-, C)$ exhibits the object $C \in \mathcal{C}$ as the colimit $F*(TG)$, there is a unique (B, ξ) consisting of an object $B \in \mathcal{B}$ with $TB = C$ and a $\mathcal{V}$-natural transformation $\xi\colon F \to \mathcal{B}(G-, B)$ with $T_{G-,B}\xi = \nu$, and if, moreover; (ii) ξ exhibits B as the colimit $F * G$. There is a dual notion for creation of limits.

In particular, a colimit $F * G$ created by the $\mathcal{V}$-functor T is also preserved by T.

Lemma 1.3. *The functor $[H, 1]$ creates arbitrary pointwise (co)limits.*

Proof. We only prove the colimit case, the proof for limits is analogous. If the colimit $(C, \nu) = (F * [H, 1]G, \nu)$ exists pointwise, we have

$$CP = (F * [H, 1]G)P = F * ([H, 1](G-)P) = F * (G-)HP = F * (G-)P\ ,$$

which means that the value of C at P is a colimit (CP, ν^P) of $(G-)P$ weighted by F. In fact, this determines the $\mathcal{V}$-functor C uniquely since the domain $\mathrm{ob}\mathcal{P}$ is discrete. Further, it implies that the colimit $(F * G, \xi)$ exists pointwise because $F * (G-)P$ exists as (CP, ν^P), and this means that the functoriality of $F * G$ is induced from the pointwise representation and that $E_P\xi = \nu^P$. Now since

$$([H, 1](F * G))P = (F * G)HP = (F * G)P = F * (G-)P = CP\ ,$$

the two functors $[H, 1](F * G)$ and $F * ([H, 1]G)$ coincide pointwise, and this means that they must also coincide as functors $\mathrm{ob}\mathcal{P} \to \mathcal{L}$, i.e. $[H, 1](F * G) = F * ([H, 1]G)$. Moreover, since the units coincide pointwise, $E_P\xi = \nu^P$, we must have $[H, 1]_{G-,A}\xi = \nu = \Pi_P\nu^P$.

This proves the existence of a (B, ξ) as in Definition 1.8. Suppose there would be another (B', ξ') with $[H, 1]B' = C$ and $[H, 1]_{G-,B'}\xi' = \nu$. Then B and B' would coincide

pointwise i.e. $BP = BP'$ for any object $P \in \mathcal{P}$, and via ξ and ξ' would both give rise to the same representation isomorphism—by the fact that $[H,1]_{G-,A}\xi' = \nu = [H,1]_{G-,A}\xi$ and thus $E_P\xi' = E_p[H,1]_{G-,A}\xi' = E_P\nu = E_P[H,1]_{G-,A}\xi = E_P\xi$— and this representation isomomorphism is $\mathcal{V}$-natural in P as well as in L:

$$\mathcal{L}(BP, L) \cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](F, \mathcal{L}((G-)P, L)) \; . \tag{1.31}$$

But for such a representation isomorphism there is a unique way of making B a $\mathcal{V}$-functor $\mathcal{P} \to \mathcal{L}$ such that the representation isomorphism is $\mathcal{V}$-natural in P as well as in L, see for example [38, 1.10], so B and B' have to coincide as $\mathcal{V}$-functors. Clearly, by Yoneda, also $\xi = \xi'$ then as the representations of (B, ξ) and (B, ξ') coincide because the pointwise representations (1.31) do, cf. [38, 3.3]. Note here that (B', ξ') must be a pointwise colimit too because by assumption, it is preserved by $[H,1]$ and (C, ν) is preserved by any E_P, so (B', ξ') is preserved by any E_P and thus it is a pointwise colimit. On the other hand, this is just the general fact that if a colimit exists pointwise, then any representation must in fact be pointwise, see Lemma 1.2 above.

□

Corollary 1.2. *The functor* $[H,1]$ *preserves any limit and any pointwise colimit that exists.*

Proof. This follows from the lemma above and the fact that $[H,1]$ is a right adjoint. □

Remark 1.6. In case that $[H,1]$ is also a left adjoint, it in fact preserves any colimit that exists. This is for example the case when the target $\mathcal{L}$ is complete, where the right adjoint is given by right Kan extension $\mathbf{Ran}_H$ along H, which exists because $\mathcal{L}$ and $\mathrm{ob}\mathcal{P}$ were assumed to be complete and small respectively. In particular, this applies in the situation that $\mathcal{L} = \mathcal{V}$.

Corollary 1.3. *Let* $\mathcal{P}$ *be a small and* $\mathcal{L}$ *be a cocomplete* Gray*-category, and let* T *be the monad* $[H,1]\mathbf{Lan}_H$ *on* $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ *given by the Kan adjunction. Then the inclusion* $i\colon [\mathcal{P}, \mathcal{L}] \hookrightarrow \text{Ps-}T\text{-Alg}$ *has a left adjoint* $L\colon \text{Ps-}T\text{-Alg} \to [\mathcal{P}, \mathcal{L}]$ *and each component* $\eta_A\colon A \to iLA$ *of the unit of the adjunction is an internal biequivalence in* Ps-T-Alg. □

Proof. We aim at applying Corollary 1.1 of Gurski's coherence theorem. Thus, we have to show that $T = [H,1]\mathbf{Lan}_H$ preserves small colimits. Since $\mathbf{Lan}_H$ is a left adjoint, it preserves any colimit that exists. Since this colimit is again a small colimit and since, $\mathcal{L}$ being cocomplete, small colimits are pointwise colimits (cf. Lemma 1.2), Lemma 1.3 implies that it is preserved by $[H,1]$. This proves that any small colimit is preserved by T. □

1.3.2 Explicit description of the monad

In this paragraph, we will give an explicit description of the monad from 1.3.1 in terms of a coend over tensor products. As a matter of fact, the explicit identification of the monad structure is involved, and an alternative economical strategy adequate for the

purpose of this paper, would be to take the description in terms of coends and tensor products as a definition. By functoriality of the colimit it is then readily shown that this gives a monad on $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ as required, but one has to show that it preserves pointwise (and thus small) colimits in order to apply Corollary 1.1 from 1.2.2. This follows from an appropriate form of the interchange of colimits theorem. For this reason, we will be short on proofs below.

First, we recall the notions of tensor products and coends to present the well-known Kan extension formula (1.37) below. Then we determine the monad structure $\mu\colon TT \Rightarrow T$ and $\eta\colon 1 \Rightarrow T$ for the monad from 1.3.1. Given an object $X \in \mathcal{V}$ and an object $L \in \mathcal{L}$, recall that the tensor product $X \otimes L$ is defined as the colimit $X * L$ where X and L are considered as objects i.e. as $\mathcal{V}$-functors in the underlying categories $\mathcal{V}_0 = \mathcal{V}\text{-}\mathcal{CAT}(\mathcal{I}^{\mathrm{op}}, \mathcal{V})$ and $\mathcal{L}_0 = \mathcal{V}\text{-}\mathcal{CAT}(\mathcal{I}, \mathcal{L})$ where $\mathcal{I}$ is the unit $\mathcal{V}$-category. With the identification $[\mathcal{I}, \mathcal{L}] \cong \mathcal{L}$, the corresponding contravariant representation (1.26) from 1.2.2 has the form

$$n\colon\ \mathcal{L}(X \otimes L, M) \cong [X, \mathcal{L}(L, M)]\ , \tag{1.32}$$

and this is $\mathcal{V}$-natural in all variables by functoriality of the colimit cf. [38, (3.11)]. This means that tensor products are in fact $\mathcal{V}$-adjunctions, and we will dwell on this in the next paragraph 1.3.3. Because $\mathcal{L}$ is assumed to be cocomplete, tensor products indeed exist.

Next, recall that for a $\mathcal{V}$-functor $G\colon\ \mathcal{A}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{L}$, the coend

$$\int^{A} G(A, A) \tag{1.33}$$

is defined as the colimit $\mathbf{Hom}^{\mathrm{op}}_{\mathcal{A}} * G$. The corresponding representation (1.26) from 1.2.2 transforms under the extra-variable enriched Yoneda lemma cf. [38, (2.38)] into the following characteristic isomorphism of the coend:

$$\beta\colon\ \mathcal{L}(\int^{A} G(A, A), L) \cong \int_{A} \mathcal{L}(G(A, A), L)\ , \tag{1.34}$$

which is $\mathcal{V}$-natural in L and where on the right we have an end in the ordinary sense cf. [38, 2.1]. The unit of $\mathbf{Hom}^{\mathrm{op}}_{\mathcal{A}} * G$ corresponds to a $\mathcal{V}$-natural family

$$\kappa_A = \lambda_A \beta j_{\int^A G(A,A)}\colon\ G(A, A) \to \int^{A} G(A, A)\ , \tag{1.35}$$

where λ_A is the counit of the end, and (1.34) induces the following universal property of κ_P:

$$\mathcal{L}_0(\int^{A} G(A, A), L) \cong \mathcal{V}\text{-nat}(G(A, A), L)\ . \tag{1.36}$$

This is a bijection of sets and it is given by precomposition with κ_A, which proves that κ_A is the universal $\mathcal{V}$-natural family with domain $G(A, A)$. Since $\mathcal{L}$ was assumed to be cocomplete, small coends in $\mathcal{L}$ do in fact exist.

We are now ready to present the explicit description of left Kan extension and thus of the monad from 1.3.1. Since $\mathcal{L}$ admits tensor products and since $\mathcal{P}$ was assumed to be small, left Kan extension along the functor $H\colon \mathrm{ob}\mathcal{P} \to \mathcal{P}$ from 1.3.1 is given by the following small coend:

$$\mathrm{Lan}_H A \cong \int^P \mathcal{P}(P,-) \otimes AP \tag{1.37}$$

cf. [38, (4.25)].

Example. In case that $\mathcal{L} = \mathcal{V}$ and $\mathcal{A} = \mathrm{ob}\mathcal{P}$, the coend in (1.33) is given by a coproduct in $\mathcal{V}_0$: Indeed it is easy to verify that $\mathbf{Hom}^{\mathrm{op}}_{\mathrm{ob}\mathcal{P}} * G(A,A)$ reduces to a conical colimit in $\mathcal{V}$, which, $\mathcal{V}$ being cotensored, coincides with the ordinary colimit, hence the coproduct.

Next one observes that tensor products in $\mathcal{V}$ are given by the monoidal structure as is easily seen from (1.32) cf. (1.44) in 1.3.3. Therefore, (1.37) reduces to the coproduct

$$\mathrm{Lan}_H A \cong \sum_{P \in \mathrm{ob}\mathcal{P}} \mathcal{P}(P,-) \otimes AP\ . \tag{1.38}$$

Now let $\mathcal{M}$ be another small $\mathcal{V}$-category and $K\colon \mathcal{M} \to \mathcal{P}$ be a $\mathcal{V}$-functor. Then left Kan extension along K exists in the form of

$$\mathrm{Lan}_K A \cong \int^M \mathcal{P}(KM,-) \otimes AM\ , \tag{1.39}$$

the relevant functor categories exist, and we again have the Kan adjunction $\mathrm{Lan}_K \dashv [K,1]$.

Lemma 1.4. *The component at $A \in [\mathcal{P},\mathcal{L}]$ of the counit $\epsilon\colon \mathrm{Lan}_K[K,1] \Rightarrow 1_{[\mathcal{P},\mathcal{L}]}$ of the adjunction $\mathrm{Lan}_K \dashv [K,1]$ has component*

$$\epsilon_{A,Q}\colon \int^M \mathcal{P}(KM,Q) \otimes AKM \to AQ$$

at $Q \in \mathcal{P}$ induced from the $\mathcal{V}$-natural transform

$$\mathcal{P}(KM,Q) \otimes AKM \to AQ$$

of the hom morphism

$$A_{KM,Q}\colon \mathcal{P}(KM,Q) \to \mathcal{L}(AKM,AQ)$$

under the adjunction (1.32) *of the tensor product.*

Proof. The component at A is obtained by composing the unit

$$j_{AK}\colon I \to [\mathrm{ob}\mathcal{P},\mathcal{L}](AK,AK)$$

with the inverse of the $\mathcal{V}$-natural isomorphism of the Kan adjunction (1.29). The lemma then follows from inspection of the proof of the theorem of Kan adjoints [38, Th. 4.38]. In particular, the transform of the $\mathcal{V}$-natural $\mathcal{L}(AKM,-)_{AKM,AQ}A_{KM,Q}$, which gives rise to the extra-variable Yoneda isomorphism [38, (2.33)], enters in the inverse of (1.29), and this is the point where the hom morphism $A_{KM,Q}$ shows up.

□

We will show in the next paragraph 1.3.3 that there are obvious left unitors λ and associators α for the tensor products. These already show up in the following two lemmata, but since we mostly omit the proofs, it seems more stringent to state the lemmata here in order to have the explicit description of T at one place.

Lemma 1.5. *The component at $A \in [\mathcal{M}, \mathcal{L}]$ of the unit of the adjunction $\mathrm{Lan}_K \dashv [K, 1]$ and the corresponding monad $T = \mathrm{Lan}_K \dashv [K, 1]$ on $[\mathcal{M}, \mathcal{L}]$, i.e. the $\mathcal{V}$-natural transformation $\eta\colon 1_{[\mathcal{M},\mathcal{L}]} \Rightarrow [K, 1]\mathrm{Lan}_K$ has component*

$$\eta_{A,M}\colon AM \xrightarrow{\lambda^{-1}_{AM}} I \otimes AM \xrightarrow{j_{KM}\otimes 1} \mathcal{P}(KM, KM) \otimes AM \xrightarrow{\kappa_{M,KM}} \int^{O} \mathcal{P}(KO, KM) \otimes AO$$

at M in $\mathcal{M}$ where λ^{-1}_{AM} is the unitor of the tensor product cf. 1.3.3.

Proof. Note that we have stressed in the statement that the unit of the monad T is exactly given by the unit of the adjunction $\mathrm{Lan}_K \dashv [K, 1]$. Hence, its component at $A \in [\mathcal{M}, \mathcal{L}]$ is given by composing the $\mathcal{V}$-natural isomorphism of the Kan adjunction (1.29) with the unit

$$j_{\mathrm{Lan}_K A}\colon I \to [\mathcal{P}, \mathcal{L}](\mathrm{Lan}_K A, \mathrm{Lan}_K A) \ .$$

This gives an element $I \to [\mathrm{ob}\mathcal{P}, \mathcal{L}](A, [K, 1](\mathrm{Lan}_K A))$, that is, a $\mathcal{V}$-natural transformation $A \Rightarrow [K, 1](\mathrm{Lan}_K A) = TA$ cf. (1.29). Since the inverse of the extra-variable enriched Yoneda isomorphism [38, (2.33)] takes part in (1.29), this is converse to the situation in Lemma 1.5. Correspondingly, one has to consider the transform of $\mathcal{L}(AO, (\mathrm{Lan}_K A)-)_{O,KM}$, although one does not have to determine $((\mathrm{Lan}_K A)-)_{O,KM}$ in the argument as one only uses the unit axiom for a $\mathcal{V}$-functor. □

Lemma 1.6. *The hom morphism of $\mathrm{Lan}_K A = \int^M \mathcal{P}(KM, -) \otimes AM$,*

$$(\mathrm{Lan}_K A)_{Q,R}\colon \mathcal{P}(Q, R) \to \mathcal{L}(\int^M \mathcal{P}(KM, Q) \otimes AM, \int^M \mathcal{P}(KM, R) \otimes AM)$$

corresponds to the $\mathcal{V}$-natural family (in $M \in \mathcal{M}$ and also in $Q \in \mathcal{P}$ but Q is held constant here)

$$\kappa_{M,R}(M_{\mathcal{P}} \otimes 1)\alpha^{-1} \tag{1.40}$$

under (1.32)*, exchange of the colimits $\mathcal{P}(Q, R) \otimes -$ and $\int^M$, and* (1.36)*.*

Proof. A neat way of proving this is by showing that the prescription in the statement of the lemma gives rise to the correct unit of the representation for left Kan extension along K via

$$\mathcal{P}(KM, R) \xrightarrow{(\mathrm{Lan}_K)_{KM,R}} \mathcal{L}((\mathrm{Lan}_K A)KM, (\mathrm{Lan}_K A)R) \xrightarrow{\mathcal{L}(\eta_{A,M},1)} \mathcal{L}(AM, (\mathrm{Lan}_K A)R) \tag{1.41}$$

cf. [38, dual of Th. 4.6 (ii)], where $\eta_{A,M}$ was determined in Lemma 1.5, and the unit of the representation of the left Kan extension as a colimit in the form of (1.39) is quickly

determined to be $\mathcal{L}(1,\kappa_{M,R})\eta^{AM}_{\mathcal{P}(KM,R)}$. Namely, the unit of (1.34) is $\kappa_{M,R}$, then n is applied to this, which by (1.56) in 1.3.3 below gives

$$n(\kappa_{M,R}) = [\eta^{AM}_{\mathcal{P}(KM,R)}, 1]\mathcal{L}(AM,-)_{\mathcal{P}(KM,R)\otimes AM, \int^M \mathcal{P}(KM,R)\otimes AM}(\kappa_{M,R})$$

or $[\eta^{AM}_{\mathcal{P}(KM,R)}, 1]\mathcal{L}(AM,\kappa_{M,R})$, where η is the counit of the adjunction of the tensor product cf. (1.32). Thus this is the counit in question and it can be identified with $\mathcal{L}(AM,\kappa_Q)\eta^{AM}_{\mathcal{P}(KM,R)}$. One then proves that (1.41) in fact has exactly this form:

Denoting by x exchange of the tensor product and the coend $\int^M$, the relevant calculation is displayed below.

$$\begin{aligned}
&\mathcal{L}(\eta_{A,M},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})x)\eta^{\int^M \mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}\\
=&\mathcal{L}(\kappa_{M,KM}(j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})x)\eta^{\int^M \mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}\\
&\text{(by Lemma 1.5)}\\
=&\mathcal{L}((j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})x)\mathcal{L}(\kappa_{M,KM},1)\eta^{\int^M \mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}\\
&\text{(functoriality of } \mathbf{hom}_{\mathcal{L}})\\
=&\mathcal{L}((j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})x)\mathcal{L}(1,1\otimes\kappa_M)\eta^{\mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)}\\
&\text{(naturality of } \eta)\\
=&\mathcal{L}((j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,(V\beta)^{-1}(\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})\kappa_M)\eta^{\mathcal{P}(KM,KM)\otimes aAM}_{\mathcal{P}(KM,R)}\\
&\text{(exchange of colimits is induced by an iso of represented functors: } x(1\otimes\kappa_M)=\kappa_M)\\
=&\mathcal{L}((j_{KM}\otimes 1)\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})\alpha^{-1})\eta^{\mathcal{P}(KM,KM)\otimes AM}_{\mathcal{P}(KM,R)} \text{ (since } (V\beta)=V[\kappa_M,1])\\
=&\mathcal{L}(\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}\otimes 1_{AM})a^{-1}(1\otimes(j_{KM}\otimes 1)))\eta^{I\otimes AM}_{\mathcal{P}(KM,R)} \text{ (functoriality of } \mathbf{hom}_{\mathcal{L}})\\
=&\mathcal{L}(\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(M^{\mathcal{P}}_{KM,KM,R}(1\otimes j_{KM})\otimes 1_{AM})\alpha^{-1})\eta^{I\otimes AM}_{\mathcal{P}(KM,R)} \quad \text{(by naturality of } \alpha)\\
=&\mathcal{L}(\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(r_{\mathcal{P}(KM,R)}\otimes 1_{AM})\alpha^{-1})\eta^{I\otimes AM}_{\mathcal{P}(KM,R)} \quad \text{(by a } \mathcal{V}\text{-category axiom)}\\
=&\mathcal{L}(\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R}(1\otimes\lambda_{AM}))\eta^{I\otimes AM}_{\mathcal{P}(KM,R)} \quad \text{(by the triangle identity (1.53))}\\
=&\mathcal{L}(\lambda_{AM}\lambda^{-1}_{AM},1)\mathcal{L}(1,\kappa_{M,R})\eta^{AM}_{\mathcal{P}(KM,R)} \quad \text{(by naturality of } \eta)\\
=&\mathcal{L}(1,\kappa_{M,R})\eta^{AM}_{\mathcal{P}(KM,R)} = [\eta^{AM}_{\mathcal{P}(KM,R)},1]\mathcal{L}(AM,\kappa_{M,R})
\end{aligned}$$

Thus the prescription (1.40) leads to the right counit, but this means that the hom morphism $(\mathrm{Lan}_K)_{Q,R}$ must have precisely the claimed form since $\mathrm{Lan}_K A$ is uniquely functorial such that the representation, which is induced from this counit, is appropriately natural cf. [38, 1.10] (and indeed this is how the functoriality of (1.39) is defined). □

Corollary 1.4. *Let $T = [K,1]\mathrm{Lan}_K$ be the monad of the Kan adjunction from 1.3.1. The component at $A \in [\mathcal{M},\mathcal{L}]$ of the $\mathcal{V}$-natural transformation $\mu\colon TT \Rightarrow T$ has component corresponding to the $\mathcal{V}$-natural (in $M,N \in \mathcal{M}$) family*

$$\kappa_{M,R}(M_{\mathcal{P}}\otimes 1)\alpha^{-1} \tag{1.42}$$

under exchange of colimits, Fubini, and (1.36)*, where α is the associator of the tensor product cf. 1.3.3.*

Proof. The $\mathcal{V}$-natural transformation μ of the monad is determined by the counit ϵ of the adjunction $\mathrm{Lan}_H \dashv [H,1]$. Namely, it is given by the $\mathcal{V}$-natural transformation denoted

$$[H,1]\epsilon\mathrm{Lan}_H\colon\ [H,1]\mathrm{Lan}_H[H,1]\mathrm{Lan}_{\mathrm{H}} \Rightarrow [H,1]\mathrm{Lan}_H$$

with component

$$[H,1]_{\mathrm{Lan}_H((\mathrm{Lan}_H A)H),\mathrm{Lan}_H A}\epsilon_{\mathrm{Lan}_H A}\colon\ I \to [\mathrm{ob}\mathcal{P},\mathcal{L}]((\mathrm{Lan}_H((\mathrm{Lan}_H A)H))H,(\mathrm{Lan}_H A)H)$$

at $A \in [\mathrm{ob}\mathcal{P},\mathcal{L}]$. Since E_P factorizes through $[H,1]$ and π_P, the component at $Q \in \mathcal{P}$ of μ_A is simply given by the component of $\epsilon_{\mathrm{Lan}_H A}$ at Q. According to Lemma 1.4, the component of $\epsilon_{\mathrm{Lan}_H A}$ at $Q \in \mathcal{P}$ is induced from the transform of $(\mathrm{Lan}_H A)_{P,Q}$, and by Lemma 1.6, this transform is precisely given by (1.42). □

1.3.3 Some properties of tensor products

It is clear from the defining representation isomorphism (1.32) from 1.3.2 of the tensor product and its naturality in X,L, and M that tensor products, for any object L in a tensored $\mathcal{V}$-category $\mathcal{L}$, give an adjunction of $\mathcal{V}$-categories as below

$$(-\otimes L\colon\ \mathcal{V}\to\mathcal{L}) \quad \dashv \quad (\mathcal{L}(L,-)\colon\ \mathcal{L}\to\mathcal{V})\ . \tag{1.43}$$

Because the representation isomorphism is also $\mathcal{V}$-natural in L, it is a consequence of the extra-variable Yoneda lemma [38, 1.9] that the unit and counit of this adjunction are also extraordinarily $\mathcal{V}$-natural in L:

Lemma 1.7. *The unit $\eta^L_X\colon\ X\to\mathcal{L}(L,X\otimes L)$ and counit $\epsilon^L_M\colon\ \mathcal{L}(L,M)\otimes L\to M$ of these adjunctions are extraordinarily $\mathcal{V}$-natural in L (and ordinarily $\mathcal{V}$-natural in X and M).* □

Recall that there is a natural (in $X,Y,Z\in\mathcal{V}$) isomorphism

$$p\colon\ [X\otimes Y,Z]\cong[X,[Y,Z]]\ , \tag{1.44}$$

which is induced from the closed structure of $\mathcal{V}$ via the ordinary Yoneda lemma cf. [38, 1.5]. From p and the hom $\mathcal{V}$-adjunction (1.32) of the tensor product, we construct a $\mathcal{V}$-natural isomorphism

$$n^{-1}_{X,Y\otimes L,M}[X,n^{-1}_{Y,L,M}]pn_{X\otimes Y,L,M}\colon\ \mathcal{L}((X\otimes Y)\otimes L,M)\to\mathcal{L}(X\otimes(Y\otimes L),M)\ , \tag{1.45}$$

which, by Yoneda, must be of the form $\mathcal{L}(\alpha^{-1}_{X,Y,L},1)$ for a unique $\mathcal{V}$-natural (in X,Y,L) isomorphism

$$\alpha^{-1}_{X,Y,L}\colon\ X\otimes(Y\otimes L)\cong(X\otimes Y)\otimes L\ . \tag{1.46}$$

The natural isomorphism (1.46) is called the associator for the tensor product. Since tensor products reduce to the monoidal structure if $\mathcal{L} = \mathcal{V}$, the natural isomorphism (1.46) is in this special case, by uniqueness, given by the associator a^{-1} for the monoidal structure of $\mathcal{V}$.

In fact, there is a pentagon identity in terms of associators α and associators a:

Lemma 1.8 (Cf. Janelidze and Kelly [30])**.** *Given objects W, X, and Y in $\mathcal{V}$, and an object L in a tensored $\mathcal{V}$-category $\mathcal{L}$, the associators α and a satisfy the pentagon identity*

$$\alpha_{W,X,Y\otimes L}\alpha_{W\otimes X,Y,L} = (1_W \otimes \alpha_{X,Y,L})\alpha_{W,X\otimes Y,Z}(a_{W,X,Z} \otimes 1_Z) \ , \tag{1.47}$$

which is an identity of isomorphisms

$$((W \otimes X) \otimes Y) \otimes L \to W \otimes (X \otimes (Y \otimes L)) \ .$$

Proof. The corresponding identity for the inverses is proved by showing that the corresponding $\mathcal{V}$-natural isomorphisms

$$\mathcal{L}(((W \otimes X) \otimes Y) \otimes L, M) \cong \mathcal{L}(W \otimes (X \otimes (Y \otimes L)), M)$$

coincide. The $\mathcal{V}$-natural isomorphism corresponding to the inverse of the left hand side of (1.47) is readily seen to be given by

$$n^{-1}_{W,X\otimes(Y\otimes L),M}[W, n^{-1}_{X,Y\otimes L,M}[X, n^{-1}_{Y,L,M}]]ppn_{(W\otimes X)\otimes Y,L,M}$$

cf. (1.45), where we have used naturality of p and cancelled out two factors. Similarly, the $\mathcal{V}$-natural isomorphism corresponding to the inverse of the right hand side of (1.47) is given by

$$n^{-1}_{W,X\otimes(Y\otimes L),M}[W, n^{-1}_{X,Y\otimes L,M}[X, [n^{-1}_{Y,L,M}]]][W, p]p[\alpha^{-1}, \mathcal{L}(L, M)]n_{(W\otimes X)\otimes Y,L,M} \ .$$

Thus, the identity (1.47) is proved as soon as we show that

$$pp = [W, p]p[a^{-1}, \mathcal{L}(L, M)] \ .$$

In fact, this last equation reduces to the pentagon identity for a since p is defined via Yoneda by

$$\mathcal{V}_0(W, p) = \pi\pi\mathcal{V}_0(a, 1)\pi^{-1} \ ,$$

where π is the hom $\mathcal{S}et$-adjunction of the closed structure. Namely, one observes that on the one hand,

$$\begin{aligned}\mathcal{V}_0(V, pp) = \mathcal{V}_0(V, p)\mathcal{V}_0(V, p) = \pi\pi\mathcal{V}_0(a, 1)\pi^{-1}\pi\pi\mathcal{V}_0(a, 1)\pi^{-1} &= \pi\pi\mathcal{V}_0(a, 1)\pi\mathcal{V}_0(a, 1)\pi^{-1} \\ &= \pi\pi\pi\mathcal{V}_0(a \otimes 1, 1)\mathcal{V}_0(a, 1)\pi^{-1} \\ &= \pi\pi\pi\mathcal{V}_0(a(a \otimes 1), 1)\pi^{-1} \ ,\end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathcal{V}_0(V,[W,p]p[a^{-1},\mathcal{L}(L,M)]) &= \mathcal{V}_0(V,[W,p])\pi\pi\mathcal{V}_0(a^{-1},1)\pi^{-1}\pi\mathcal{V}_0(1\otimes a^{-1},1)\pi^{-1}\\ &= \mathcal{V}_0(V,[W,p])\pi\pi\mathcal{V}_0(a,1)\mathcal{V}_0(1\otimes a^{-1},1)\pi^{-1}\\ &= \pi\mathcal{V}_0(V\otimes W,p)\pi^{-1}\pi\pi\mathcal{V}_0(a,1)\mathcal{V}_0(1\otimes a^{-1},1)\pi^{-1}\\ &= \pi\pi\pi\mathcal{V}_0(a,1)\mathcal{V}_0(a,1)\mathcal{V}_0(1\otimes a^{-1},1)\pi^{-1}\\ &= \pi\pi\pi\mathcal{V}_0((1\otimes a^{-1})aa,1)\pi^{-1}\ .\end{aligned}$$

□

Similarly, recall that there is a natural (in $Z\in\mathcal{V}$) isomorphism

$$i\colon\ Z\cong[I,Z] \tag{1.48}$$

which is defined via Yoneda by

$$[X,i^{-1}]=[r_X^{-1},1]p^{-1}\colon\ [X,[I,Z]]\cong[X\otimes I,Z]\cong[X,Z]\ . \tag{1.49}$$

Thus from i and the hom $\mathcal{V}$-adjunction n of the tensor product, we construct a $\mathcal{V}$-natural (in L,M) isomorphism

$$n^{-1}_{I,L,M}i\colon\ \mathcal{L}(L,M)\cong[I,\mathcal{L}(L,M)]\cong\mathcal{L}(I\otimes L,M)\ . \tag{1.50}$$

By Yoneda, (1.50) must be of the form $\mathcal{L}(\lambda_L,1)$ for a unique isomorphism

$$\lambda_L\colon\ I\otimes L\to L\ , \tag{1.51}$$

which must moreover be $\mathcal{V}$-natural in L.

For $M=I\otimes L$, composing the inverse of the isomorphism (1.50) above with $j_{I\otimes L}$,

$$I\to\mathcal{L}(I\otimes L,I\otimes L)\cong[I,\mathcal{L}(L,I\otimes L)]\cong\mathcal{L}(L,I\otimes L)\ ,$$

must give λ_L^{-1}—since this is how we get hold of the counit of such a natural transformation in general—and it is of course also the map corresponding to η_I^L under the isomorphism $[I,\mathcal{L}(L,I\otimes L)]\cong\mathcal{L}(L,I\otimes L)$ —because this is exactly the inverse of the first isomorphism in (1.50). Conversely, composing (1.50) for $M=L$ with j_M must give λ_L.

For $Y=I$, consider the composition of the representation isomorphism (1.45) i.e.

$$\mathcal{L}(\alpha^{-1},1)=n^{-1}_{X,I\otimes L,M}[X,n^{-1}_{I,L,M}]pn_{X\otimes I,L,M}\ , \tag{1.52}$$

with $\mathcal{L}(r_X\otimes 1,1)$. By naturality of n, functoriality of $[-,-]$, and the definition of i via Yoneda cf. (1.49), we have the following chain of equalities:

$$\begin{aligned}\mathcal{L}((r_X\otimes 1)\alpha^{-1},1) &= n^{-1}_{X,I\otimes L,M}[X,n^{-1}_{I,L,M}]pn_{X\otimes I,L,M}\mathcal{L}(r_X\otimes 1,1)\\ &= n^{-1}_{X,I\otimes L,M}[X,n^{-1}_{I,L,M}]p[r_X,1]n_{X,L,M}\\ &= n^{-1}_{X,I\otimes L,M}[X,n^{-1}_{I,L,M}][X,i]n_{X,L,M}\\ &= n^{-1}_{X,I\otimes L,M}[X,\mathcal{L}(\lambda_L,1)]n_{X,L,M}\\ &= \mathcal{L}(1\otimes\lambda_L,1)\ .\end{aligned}$$

Hence, by Yoneda, we have proved the following lemma.

Lemma 1.9. *Given an object X in $\mathcal{V}$ and an object L in a tensored $\mathcal{V}$-category $\mathcal{L}$, there is a triangle identity for r, λ, and α,*

$$(r_X \otimes 1_L)\alpha^{-1}_{X,I,L} = 1_X \otimes \lambda_L \ , \tag{1.53}$$

which is an identity of isomorphisms

$$X \otimes (I \otimes L) \to X \otimes L \ .$$

□

Lemma 1.10 (Cf. Janelidze and Kelly [30])**.** *Let L, M, N be objects in a tensored $\mathcal{V}$-category $\mathcal{L}$. Then*

$$M_{\mathcal{L}} \colon \ \mathcal{L}(M,N) \otimes \mathcal{L}(L,M) \to \mathcal{L}(L,N)$$

can be identified in terms of the associator α for tensor products and units η and counits ϵ of the tensor product adjunctions:

$$M_{\mathcal{L}} = \mathcal{L}(L, \epsilon^M_N(1 \otimes \epsilon^L_M)\alpha)\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \ .$$

Proof. First, recall that as for any $\mathcal{V}$-adjunction, Yoneda implies the identity $\mathcal{L}(L,-)_{MN} = n\mathcal{L}(\epsilon^L_M, 1)$ cf. [38, (1.53), p. 24], and thus $M_{\mathcal{L}} = e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)})$ cf. (1.14) in 1.1.5.

The lemma is now proved by the following chain of equalities

$$\begin{aligned}
M_{\mathcal{L}} &= e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)}) \\
&= \mathcal{L}(1_L, \epsilon^L_N)\eta^L_{\mathcal{L}(L,N)} e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)}) \\
&\quad \text{(by a triangle identity for unit and counit of the adjunction (1.43))} \\
&= \mathcal{L}(1_L, \epsilon^L_N)\mathcal{L}(1_L, (e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)})) \otimes 1_L)\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \\
&\quad \text{(by ordinary naturality of } \eta^L_X \text{ in } X) \\
&= \mathcal{L}(1_L, \epsilon^{\mathcal{L}(L,M)\otimes L}_N \alpha((\mathcal{L}(\epsilon^L_M, 1) \otimes 1_{\mathcal{L}(L,M)}) \otimes 1_L))\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \\
&\quad \text{(see below (*), by functoriality and the identity } \epsilon = \epsilon((e(n \otimes 1)) \otimes 1)\alpha^{-1}) \\
&= \mathcal{L}(1_L, \epsilon^{\mathcal{L}(L,M)\otimes L}_N(\mathcal{L}(\epsilon^L_M, 1) \otimes (1_{\mathcal{L}(L,M)} \otimes 1_L))\alpha)\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \\
&\quad \text{(by ordinary naturality of } \alpha) \\
&= \mathcal{L}(1_L, \epsilon^M_N(1 \otimes \epsilon^L_M)\alpha)\eta^L_{\mathcal{L}(M,N)\otimes\mathcal{L}(L,M)} \\
&\quad \text{(by extraordinary naturality of } \epsilon^L_M \text{ in } L)
\end{aligned}$$

To prove the identity used in (*) consider the morphism

$$[\mathcal{L}(\mathcal{L}(L,M) \otimes L, N), \mathcal{L}(\mathcal{L}(L,M) \otimes L, N)] \to \mathcal{L}(\mathcal{L}(\mathcal{L}(L,M) \otimes L, N) \otimes (\mathcal{L}(L,M) \otimes L), N)$$

given by the composite

$$\mathcal{L}(\alpha,N)\mathcal{L}((n_{\mathcal{L}(L,M),L,N}\otimes 1)\otimes 1,N)n^{-1}_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]\otimes\mathcal{L}(L,M),L,N}p^{-1}[n^{-1}_{\mathcal{L}(M,N),L,N},n_{\mathcal{L}(M,N),L,N}] \tag{1.54}$$

in $\mathcal{V}_0$, where $p\colon [X\otimes Y,Z]\cong[X,[Y,Z]]$ is again the natural isomorphism (1.44) induced from the closed structure of $\mathcal{V}$, and where we have added subscripts such that the hom $\mathcal{V}$-adjunction (1.32) from 1.3.2 is now denoted by $n_{X,L,M}$. If $\mathcal{L}(\alpha,N)$ is spelled out in terms of hom $\mathcal{V}$-adjunctions n and p according to (1.45), then it is seen by naturality that (1.54) is in fact the same as

$$n^{-1}_{\mathcal{L}(\mathcal{L}(L,M)\otimes L,N),\mathcal{L}(L,M)\otimes L,N}\ . \tag{1.55}$$

In particular, it is an isomorphism (although this is also clear because each factor is an isomorphism) and appropriately $\mathcal{V}$-natural (and this follows from the composition calculus respectively). We now want to show that the unit of this natural isomorphism is given by

$$\epsilon^{L}_{N}((e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}(n_{\mathcal{L}(M,N),L,N}\otimes 1))\otimes 1)\alpha^{-1}$$

because then, by Yoneda, this must be the same as $\epsilon^{\mathcal{L}(L,M)\otimes L}_{N}$ since this is by definition the unit of (1.55).

The unit is obtained by applying (1.54) (or rather V of it) to $1_{\mathcal{L}(\mathcal{L}(L,M)\otimes L,N)}$, and we do this factor-by-factor. First, note that

$$[n^{-1}_{\mathcal{L}(M,N),L,N},n_{\mathcal{L}(M,N),L,N}](1_{\mathcal{L}(\mathcal{L}(L,M)\otimes L,N)})=1_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]}\ ,$$

and

$$p(1_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]})=e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}$$

because p is the hom $\mathcal{V}$-adjunction with underlying adjunction $-\otimes Y\dashv[Y,-]$ given by the closed structure i.e. $Vp=\pi$.

Now note that by ordinary naturality of n, we have

$$[e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)},1]=n_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]\otimes\mathcal{L}(L,M),L,N}\mathcal{L}(e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}\otimes 1_L,1)n^{-1}_{\mathcal{L}(L,N),L,N}\ .$$

Thus because $[e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)},1](1_{\mathcal{L}(L,N)})=e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}$, we may compute $n_{[\mathcal{L}(L,M),\mathcal{L}(L,N)]\otimes\mathcal{L}(L,M),L,N}(e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)})$ by applying $\mathcal{L}(e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}\otimes 1_L,1)n^{-1}_{\mathcal{L}(L,N),L,N}$ to $1_{\mathcal{L}(L,N)}$, the result of which is $\epsilon^{L}_{N}(e^{\mathcal{L}(L,M)}_{\mathcal{L}(L,N)}\otimes 1)$. Finally, applying the remaining factors $\mathcal{L}((n_{\mathcal{L}(L,M),L,N}\otimes 1)\otimes 1,N)$ and $\mathcal{L}(\alpha,N)$ indeed gives (1.54). □

Remark 1.7. A different strategy for the proof of the lemma is to first observe that the right hand side just as $M_{\mathcal{L}}$ is ordinarily $\mathcal{V}$-natural in L and N and extraordinarily $\mathcal{V}$-natural in M by Lemma 1.7, naturality of α, and the composition calculus. Then the identity in the lemma can be proved variable-by-variable by use of the Yoneda lemma where one considers the transforms in the case of the variable M.

For an object $X \in \mathcal{V}$ and objects $L, M \in \mathcal{L}$, recall that the hom $\mathcal{V}$-adjunction (1.32) from 1.3.2 of the tensor product has the following description in terms of the unit and the strict hom functor of the right adjoint $\mathcal{L}(L, -)$,

$$n = [\eta^L_X, 1]\mathcal{L}(L, -)_{X\otimes L, M} \colon \mathcal{L}(X \otimes L, M) \to [X, \mathcal{L}(L, M)] \ . \tag{1.56}$$

With this description of n we are able to derive two important identities for n stated in the two lemmata below. These are in fact the main technical tools that we employ to achieve the promised identification of Ps-T-Alg. Recall that there is a $\mathcal{V}$-functor $\mathsf{Ten} \colon \mathcal{V}\otimes\mathcal{V} \to \mathcal{V}$ which is given on objects by sending $(X, Y) \in \mathrm{ob}\mathcal{V}\times\mathrm{ob}\mathcal{V}$ to their product $X \otimes Y \in \mathrm{ob}\mathcal{V}$ and whose hom morphism $\mathsf{Ten}_{(X,X'),(Y,Y')} \colon [X, X'] \otimes [Y, Y'] \to [X \otimes Y, X' \otimes Y']$ is such that

$$e^{X\otimes Y}_{X'\otimes Y'}(\mathsf{Ten}_{(X,X'),(Y,Y')} \otimes 1_{X\otimes Y}) = (e^X_{X'} \otimes e^Y_{Y'})m \tag{1.57}$$

where e denotes evaluation i.e. the counits of the adjunctions comprising the closed structure of $\mathcal{V}$ and where m denotes interchange in $\mathcal{V}$.

The two lemmata specify how n behaves with respect to $\mathsf{Ten} \colon \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$ and $M_{\mathcal{L}}$ in two specific situations that we will constantly face below.

Lemma 1.11 (First Transformation Lemma)**.** *Given objects X, Y in $\mathcal{V}$, and objects L, M, N in $\mathcal{L}$, the following equality of $\mathcal{V}$-morphisms $\mathcal{L}(X \otimes M, N) \otimes \mathcal{L}(Y \otimes L, M) \to [X \otimes Y, \mathcal{L}(L, N)]$ holds.*

$$\begin{aligned}&[\eta^L_{X\otimes Y}, 1]\mathcal{L}(L, -)_{(X\otimes Y)\otimes L, N}\mathcal{L}(\alpha, 1)M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)} \otimes (X \otimes -)_{Y\otimes L,M})\\ &= [1, M_{\mathcal{L}}]\mathsf{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(Y\otimes L,M))}(([\eta^M_X, 1]\mathcal{L}(M, -)_{X\otimes M,N}) \otimes ([\eta^L_Y, 1]\mathcal{L}(L, -)_{Y\otimes L,M}))\end{aligned}$$

In terms of the hom $\mathcal{V}$-adjunction (1.32) *from 1.3.2, this means that*

$$\begin{aligned}&n\mathcal{L}(\alpha, 1)M_{\mathcal{L}}\mathsf{Ten}(\mathcal{L}(X \otimes M, N), -)_{\mathcal{L}(Y\otimes L,M),\mathcal{L}(X\otimes(Y\otimes L),X\otimes M)}((X \otimes -)_{Y\otimes L,M})\\ &= [1, M_{\mathcal{L}}]\mathsf{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(L,M))}(n \otimes n) \ .\end{aligned}$$

Proof. This is proved by the following chain of equalities.

$$
\begin{aligned}
&[\eta^L_{X\otimes Y}, 1]\mathcal{L}(L,-)_{(X\otimes Y)\otimes L,N}\mathcal{L}(\alpha,1)M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)}\otimes(X\otimes -)_{Y\otimes L,M})\\
&= [\mathcal{L}(L,\alpha)\eta^L_{X\otimes Y}, 1]\mathcal{L}(L,-)_{X\otimes(Y\otimes L),N}M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)}\otimes(X\otimes -)_{Y\otimes L,M})\\
&\quad \text{(by the functor axiom for } \mathcal{L}(L,-) \text{ or ordinary } \mathcal{V}\text{-naturality of } \mathcal{L}(L,-)_{M,N} \text{ in } M)\\
&= [\mathcal{L}(L,\alpha)\eta^L_{X\otimes Y}, 1]M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes(\mathcal{L}(L,-)_{X\otimes(Y\otimes L),X\otimes M}(X\otimes -)_{Y\otimes L,M}))\\
&\quad \text{(by the functor axiom for } \mathcal{L}(L,-) \text{)}\\
&= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([\mathcal{L}(L,\alpha)\eta^L_{X\otimes Y}, 1]\mathcal{L}(L,-)_{X\otimes(Y\otimes L),X\otimes M}(X\otimes -)_{Y\otimes L,M}))\\
&\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}} \text{ or a } \mathcal{V}\text{-category axiom if spelled out)}\\
&= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([\mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,Y\otimes L)}(X\otimes\eta^L_Y), 1]\mathcal{L}(L,-)_{X\otimes(Y\otimes L),X\otimes M}(X\otimes -)_{Y\otimes L,M}))\\
&\quad \text{(see below (A), by a triangle identity and naturality)}\\
&= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([(X\otimes\eta^L_Y), \mathcal{L}(L,(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}]\mathrm{Ten}(X,-)_{\mathcal{L}(L,Y\otimes L),\mathcal{L}(L,M)}\mathcal{L}(L,-)_{Y\otimes L,M}))\\
&\quad \text{(see below (C), by ordinary } \mathcal{V}\text{-naturality of } \mathcal{L}(L,(1_X\otimes\epsilon^L_K)\alpha)\eta^L_{X\otimes\mathcal{L}(L,K)} \text{ in } K)\\
&= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes([1, \mathcal{L}(L,(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}](\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M})))\\
&\quad \text{(ordinary } \mathcal{V}\text{-naturality of } \mathrm{Ten}(X,-)_{V,Z} \text{ in } V)\\
&= M_{\mathcal{V}}(([\mathcal{L}(L,(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}, 1]\mathcal{L}(L,-)_{X\otimes M,N})\otimes(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))\\
&\quad \text{(extraordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}})\\
&= M_{\mathcal{V}}(([M_{\mathcal{L}}(\eta^M_X\otimes 1_{\mathcal{L}(L,M)}), 1]\mathcal{L}(L,-)_{X\otimes M,N})\otimes(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))\\
&\quad \text{(see below (B), by a triangle identity, naturality, and Lemma 1.10)}\\
&= M_{\mathcal{V}}(([(\eta^M_X\otimes 1_{\mathcal{L}(L,M)}), M_{\mathcal{L}}]\mathrm{Ten}(-,\mathcal{L}(L,M))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(M,N)}\mathcal{L}(M,-)_{X\otimes M,N})\otimes\\
&\qquad (\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))\\
&\quad \text{(see below (D), by ordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{L}})\\
&= M_{\mathcal{V}}(([1, M_{\mathcal{L}}]\mathrm{Ten}(-,\mathcal{L}(L,M))_{X,\mathcal{L}(M,N)}[\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))\\
&\quad \text{(ordinary } \mathcal{V}\text{-naturality of } \mathrm{Ten}(-,Z)_{U,W} \text{ in } U)\\
&= [1, M_{\mathcal{L}}]M_{\mathcal{V}}((\mathrm{Ten}(-,\mathcal{L}(L,M))_{X,\mathcal{L}(M,N)}[\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes(\mathrm{Ten}(X,-)_{Y,\mathcal{L}(L,M)}[\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))\\
&\quad \text{(ordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}})\\
&= [1, M_{\mathcal{L}}]\mathrm{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(L,M))}(M_{\mathcal{V}\otimes\mathcal{V}}\\
&\qquad ((((([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes j^{\mathcal{V}}_{\mathcal{L}(L,M)})r^{-1}_{\mathcal{L}(X\otimes M,N)})\otimes((j^{\mathcal{V}}_X\otimes([\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))l^{-1}_{\mathcal{L}(Y\otimes L,M)})))\\
&\quad \text{(functor axiom for Ten (partial functors spelled out))}\\
&= [1, M_{\mathcal{L}}]\mathrm{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(L,M))}\\
&\qquad ((M_{\mathcal{V}}(([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes j^{\mathcal{V}}_X)r^{-1}_{\mathcal{L}(X\otimes M,N)})\otimes(M_{\mathcal{V}}(j^{\mathcal{V}}_{\mathcal{L}(L,M)}\otimes([\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))l^{-1}_{\mathcal{L}(Y\otimes L,M)}))\\
&\quad \text{(by } M_{\mathcal{V}_{\otimes\mathcal{V}}} = (M_{\mathcal{V}}\otimes M_{\mathcal{V}})m \text{ where } m \text{ is interchange, naturality of } m, \text{ and } m(r^{-1}\otimes l^{-1}) = r^{-1}\otimes l^{-1})\\
&= [1, M_{\mathcal{L}}]\mathrm{Ten}_{(X,Y),(\mathcal{L}(M,N),\mathcal{L}(L,M))}(([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N})\otimes([\eta^L_Y, 1]\mathcal{L}(L,-)_{Y\otimes L,M}))\\
&\quad (\mathcal{V}\text{-category axioms for } \mathcal{V})
\end{aligned}
$$

In (A) and (B) we have used the following identities of $\mathcal{V}$-morphisms. For (A), observe

that

$$\begin{aligned}
\mathcal{L}(L,\alpha)\eta^L_{X\otimes Y} &= \mathcal{L}(L,(1_X\otimes(\epsilon^L_{Y\otimes L}(\eta^L_Y\otimes 1_L)))\alpha)\eta^L_{X\otimes Y}\\
&\quad \text{(by a triangle identity)}\\
&= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})(1_X\otimes(\eta^L_Y\otimes 1_L))\alpha)\eta^L_{X\otimes Y}\\
&\quad \text{(by functoriality of } X\otimes -)\\
&= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha((1_X\otimes\eta^L_Y)\otimes 1_L))\eta^L_{X\otimes Y}\\
&\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } \alpha)\\
&= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,Y\otimes L)}(1_X\otimes\eta^L_Y)\\
&\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } \eta)\\
&= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,Y\otimes L)}(X\otimes\eta^L_Y)\\
&= \mathcal{L}(L,(1_X\otimes\epsilon^L_{Y\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,Y\otimes L)}\mathrm{Ten}(X,\eta^L_Y)\ .
\end{aligned}$$

Similarly, for (B) observe that

$$\begin{aligned}
\mathcal{L}(L,(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)} &= \mathcal{L}(L,\epsilon^M_{X\otimes M}(\eta^M_X\otimes 1_M)(1_X\otimes\epsilon^L_M)\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}\\
&\quad \text{(by a triangle identity)}\\
&= \mathcal{L}(L,\epsilon^M_{X\otimes M}(1_{\mathcal{L}(M,X\otimes M)}\otimes\epsilon^L_M)(\eta^M_X\otimes 1_{\mathcal{L}(L,M)\otimes L})\alpha)\eta^L_{X\otimes\mathcal{L}(L,M)}\\
&\quad \text{(by underlying functoriality of the functor } \otimes\ )\\
&= \mathcal{L}(L,\epsilon^M_{X\otimes M}(1_{\mathcal{L}(M,X\otimes M)}\otimes\epsilon^L_M)\alpha((\eta^M_X\otimes 1_{\mathcal{L}(L,M)})\otimes 1_L))\eta^L_{X\otimes\mathcal{L}(L,M)}\\
&\quad \text{(by underlying functoriality of } \otimes,\ 1_{\mathcal{L}(L,M)\otimes L}=1_{\mathcal{L}(L,M)}\otimes 1_L,\\
&\qquad \text{and by ordinary } \mathcal{V}\text{-naturality of } \alpha\ )\\
&= \mathcal{L}(L,\epsilon^M_{X\otimes M}(1_{\mathcal{L}(M,X\otimes M)}\otimes\epsilon^L_M)\alpha)\eta^L_{\mathcal{L}(M,X\otimes M)\otimes\mathcal{L}(L,M)}(\eta^M_X\otimes 1_{\mathcal{L}(L,M)})\\
&\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } \eta)\\
&= M_{\mathcal{L}}(\eta^M_X\otimes 1_{\mathcal{L}(L,M)})\\
&\quad \text{(by the identification of } M_{\mathcal{L}} \text{ in Lemma 1.10 above)}\\
&= M_{\mathcal{L}}(\eta^M_X\otimes\mathcal{L}(L,M)) = M_{\mathcal{L}}\mathrm{Ten}(\eta^M_X,\mathcal{L}(L,M))\ .
\end{aligned}$$

Finally, we comment on the $\mathcal{V}$-naturality used in (C) and (D):

For (C) recall that η^L_J is $\mathcal{V}$-natural in J. Then so is $\eta^L_{X\otimes\mathcal{L}(L,K)}$ because this is η^L_{PK} for $P=\mathcal{L}(L,-)(X\otimes -)$. Next, recall that α is ordinarily $\mathcal{V}$-natural in all of its variables, and that ϵ^L_K is ordinarily $\mathcal{V}$-natural in K. Then so is $1_X\otimes\epsilon^L_K$ because this is $Q_0(\epsilon^L_K)$ for $Q=(X\otimes -)$, and thus the composite $(1_X\otimes\epsilon^L_K)\alpha$ is ordinarily $\mathcal{V}$-natural in K. From this it follows that $\mathcal{L}(L,(1_X\otimes\epsilon^L_K)\alpha)$ is ordinarily $\mathcal{V}$-natural in K because this is $Q_0((1_X\otimes\epsilon^L_K)\alpha)$ for $Q=\mathcal{L}(L,-)$. Hence, we conclude that the composite family $\mathcal{L}(L,(1_X\otimes\epsilon^L_K)\alpha)\eta^L_{X\otimes\mathcal{L}(L,K)}$ is ordinarily $\mathcal{V}$-natural in K. That is, $\mathcal{L}(L,(1_X\otimes\epsilon^L_K)\alpha)\eta^L_{X\otimes\mathcal{L}(L,K)}$ is the component at $K\in\mathcal{L}$ of a $\mathcal{V}$-natural transformation

$$\mathrm{Ten}(X,-)\mathcal{L}(L,-)\Rightarrow\mathcal{L}(L,-)(X\otimes -)$$

where X and L are held constant, and where $X \otimes -\colon \mathcal{L} \to \mathcal{L}$ is the partial functor of the tensor product in contrast to the partial functor $\mathrm{Ten}(X,-)\colon \mathcal{V} \to \mathcal{V}$ induced by the Gray product.

For (D) recall that $M_{\mathcal{L}}\colon \mathcal{L}(M,K)\otimes\mathcal{L}(L,M) \to \mathcal{L}(L,K)$ is $\mathcal{V}$-natural. Since $\mathcal{V}$-naturality may be verified variable-by-variable, $M_{\mathcal{L}}$ is in particular ordinarily $\mathcal{V}$-natural in K. That is, it is the component at $K \in \mathcal{L}$ of a $\mathcal{V}$-natural transformation

$$\mathrm{Ten}(-, \mathcal{L}(L,M))\mathcal{L}(M,-) \Rightarrow \mathcal{L}(L,-)$$

where L and M are held constant. □

Lemma 1.12 (Second Transformation Lemma)**.** *Given an object X in $\mathcal{V}$, and objects L, M, N in $\mathcal{L}$, the following equality of $\mathcal{V}$-morphisms $\mathcal{L}(X\otimes M,N)\otimes\mathcal{L}(L,M) \to [X, \mathcal{L}(L,N)]$ holds, where c is the symmetry of $\mathcal{V}$.*

$$\begin{aligned}&[\eta^L_X, 1]\mathcal{L}(L,-)_{X\otimes L,N} M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)} \otimes (X \otimes -)_{L,M})\\ &= M_{\mathcal{V}}(\mathcal{L}(-,N)_{L,M} \otimes ([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N}))c\end{aligned}$$

In terms of the hom $\mathcal{V}$-adjunction (1.32) *from 1.3.2, this means that*

$$nM_{\mathcal{L}}\mathrm{Ten}(\mathcal{L}(X\otimes M,N),-)_{\mathcal{L}(L,M),\mathcal{L}(X\otimes L,X\otimes M)}((X\otimes -)_{L,M}) = M_{\mathcal{V}}(\mathcal{L}(-,N)_{L,M} \otimes n)c$$

Proof. This is proved by the following chain of equalities.

$$\begin{aligned}&[\eta^L_X, 1]\mathcal{L}(L,-)_{X\otimes L,N} M_{\mathcal{L}}(1_{\mathcal{L}(X\otimes M,N)} \otimes (X \otimes -)_{L,M})\\ &= [\eta^L_X, 1]M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N} \otimes (\mathcal{L}(L,-)_{X\otimes L,X\otimes M}(X\otimes -)_{L,M}))\\ &\quad \text{(functor axiom for } \mathcal{L}(L,-))\\ &= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N} \otimes ([\eta^L_X, 1]\mathcal{L}(L,-)_{X\otimes L,X\otimes M}(X\otimes -)_{L,M}))\\ &\quad \text{(ordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}})\\ &= M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N} \otimes ([\eta^L_X, 1]\mathcal{L}(-,X\otimes M)_{L,M}))\\ &\quad \text{(extraordinary } \mathcal{V}\text{-naturality of } \eta^L_X \text{ in } L)\\ &= [\eta^L_X, 1]M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N} \otimes \mathcal{L}(-,X\otimes M)_{L,M})\\ &\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}})\\ &= [\eta^M_X, 1]M_{\mathcal{V}}((\mathcal{L}(-,N)_{L,M}) \otimes \mathcal{L}(M,-)_{X\otimes M,N})c\\ &\quad \text{(see below, by extraordinary } \mathcal{V}\text{-naturality of } \mathcal{L}(L,-)_{M,N} \text{ in } L\ )\\ &= M_{\mathcal{V}}((\mathcal{L}(-,N)_{L,M}) \otimes ([\eta^M_X, 1]\mathcal{L}(M,-)_{X\otimes M,N}))c\\ &\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}})\end{aligned}$$

For the last equality, recall that $\mathcal{L}(L,-)_{M,N}\colon \mathcal{L}(M,N) \to [\mathcal{L}(L,M), \mathcal{L}(L,N)]$ is extraordinarily $\mathcal{V}$-natural in L when M and N are held constant. That is, $\mathcal{L}(L,-)_{M,N}$ has the form $I \to [\mathcal{L}(M,N), T(A,A)]$ for the $\mathcal{V}$-functor $T = \mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(-,M)^{\mathrm{op}}\otimes\mathcal{L}(-,N))c\colon \mathcal{L}^{\mathrm{op}}\otimes\mathcal{L} \to \mathcal{V}$, where $c\colon \mathcal{L}^{\mathrm{op}}\otimes\mathcal{L} \cong \mathcal{L}\otimes\mathcal{L}^{\mathrm{op}}$ is the $\mathcal{V}$-functor mediating the symmetry of the 2-category

$\mathcal{V}$-$\mathcal{CAT}$ of $\mathcal{V}$-categories, which is locally given by c, and the corresponding naturality condition is

$$\begin{aligned}&[\mathcal{L}(L,-)_{X\otimes M,N},1]\mathrm{Hom}_{\mathcal{V}}(-,\mathcal{L}(L,N))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(L,X\otimes M)}\mathcal{L}(-,X\otimes M)_{L,M}\\&=[\mathcal{L}(M,-)_{X\otimes M,N},1]\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(M,X\otimes M),-)_{\mathcal{L}(M,N),\mathcal{L}(L,N)}\mathcal{L}(-,N)_{L,M}\ .\end{aligned}$$

This is an equality of $\mathcal{V}$-morphisms $\mathcal{L}(L,M)\to[\mathcal{L}(X\otimes M,N),[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]]$, and it corresponds to an equality of $\mathcal{V}$-morphisms $\mathcal{L}(L,M)\otimes\mathcal{L}(X\otimes M,N)\to[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]$ under the adjunction of the closed structure, which in turn corresponds to an equality of $\mathcal{V}$-morphisms $\mathcal{L}(X\otimes M,N)\otimes\mathcal{L}(L,M)\to[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]$ by composition with c.

Recall that $\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(M,X\otimes M),-)_{\mathcal{L}(M,N),\mathcal{L}(L,N)}$ corresponds to $M_{\mathcal{V}}$ under the adjunction, while $\mathrm{Hom}_{\mathcal{V}}(-,\mathcal{L}(L,N))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(L,X\otimes M)}$ corresponds to $M_{\mathcal{V}}c$. The correspondence is given by application of $-\otimes\mathcal{L}(X\otimes M,N)$ and composition with $e^{\mathcal{L}(X\otimes M,N)}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]}$ (and we also compose with c).

Then the transform of the left hand side of the naturality condition is:

$$\begin{aligned}&e^{\mathcal{L}(X\otimes M,N)}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]}\\&\quad((([\mathcal{L}(L,-)_{X\otimes M,N},1]\mathrm{Hom}_{\mathcal{V}}(-,\mathcal{L}(L,N))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(L,X\otimes M)}\mathcal{L}(-,X\otimes M)_{L,M})\otimes 1_{\mathcal{L}(X\otimes M,N)})c\\&=e^{[\mathcal{L}(L,X\otimes M),\mathcal{L}(L,N)]}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]}((\mathrm{Hom}_{\mathcal{V}}(-,\mathcal{L}(L,N))_{\mathcal{L}(M,X\otimes M),\mathcal{L}(L,X\otimes M)}\mathcal{L}(-,X\otimes M)_{L,M})\otimes\mathcal{L}(L,-)_{X\otimes M,N})c\\&\quad\text{(by extraordinary }\mathcal{V}\text{-naturality of }e)\\&=M_{\mathcal{V}}c(\mathcal{L}(-,X\otimes M)_{L,M}\otimes\mathcal{L}(L,-)_{X\otimes M,N})c\\&=M_{\mathcal{V}}(\mathcal{L}(L,-)_{X\otimes M,N}\otimes\mathcal{L}(-,X\otimes M)_{L,M})\\&\quad\text{(by naturality of }c\text{ and }c^2=1)\ .\end{aligned}$$

Similarly, the transform of the right hand side of the naturality condition is:

$$\begin{aligned}&e^{\mathcal{L}(X\otimes M,N)}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]}\\&\quad((([\mathcal{L}(M,-)_{X\otimes M,N},1]\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(M,X\otimes M),-)_{\mathcal{L}(M,N),\mathcal{L}(L,N)}\mathcal{L}(-,N)_{L,M})\otimes 1_{\mathcal{L}(X\otimes M,N)})c\\&=e^{[\mathcal{L}(M,X\otimes M),\mathcal{L}(M,N)]}_{[\mathcal{L}(M,X\otimes M),\mathcal{L}(L,N)]}((\mathrm{Hom}_{\mathcal{V}}(\mathcal{L}(M,X\otimes M),-)_{\mathcal{L}(M,N),\mathcal{L}(L,N)}\mathcal{L}(-,N)_{L,M})\otimes\mathcal{L}(M,-)_{X\otimes M,N})c\\&\quad\text{(by extraordinary }\mathcal{V}\text{-naturality of }e)\\&=M_{\mathcal{V}}(\mathcal{L}(-,N)_{L,M}\otimes\mathcal{L}(M,-)_{X\otimes M,N})c\ .\end{aligned}$$

This proves the missing equality used in the chain of equalities above, and thus ends the proof. □

1.4 Identification of the $\mathcal{V}$-category of algebras

Let $\mathcal{V}$ be complete and cocomplete. Let $\mathcal{P}$ be a small and $\mathcal{L}$ be a cocomplete $\mathcal{V}$-category, and denote again by T the $\mathcal{V}$-monad $\mathrm{Lan}_H[H,1]\colon[\mathrm{ob}\mathcal{P},\mathcal{L}]\to[\mathrm{ob}\mathcal{P},\mathcal{L}]$ from

1.3.1 given by the Kan adjunction $\mathbf{Lan}_H \dashv [H, 1]$. Recall that we denote by $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ the Eilenberg-Moore object i.e. the $\mathcal{V}$-category of T-algebras, which we described in Proposition 1.2 from 1.1.5 explicitly in the special case that $\mathcal{V} = \mathcal{G}ray$.

We are now going to show that $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ is isomorphic as a $\mathcal{V}$-category to the functor $\mathcal{V}$-category $[\mathcal{P}, \mathcal{L}]$ i.e. that $[H, 1]$ is strictly monadic, which is the content of Theorem 1.4 below.

Lemma 1.13. *Let (A, a) be a T-algebra cf. 1.1.5. Then $A\colon \mathrm{ob}\mathcal{P} \to \mathcal{L}$ and the transforms $A_{PQ} \coloneqq n(a_{PQ})\colon \mathcal{P}(P, Q) \to \mathcal{L}$ under the adjunction* (1.32) *from 1.3.2 of the components a_{PQ} of a at objects $P, Q \in \mathcal{P}$ have the structure of a $\mathcal{V}$-functor. Conversely, if $A\colon \mathcal{P} \to \mathcal{L}$ is a $\mathcal{V}$-functor, then the function on objects considered as a functor $A\colon \mathrm{ob}\mathcal{P} \to \mathcal{L}$ and the transformation a induced by the transforms $n^{-1}(A_{PQ})$ of the strict hom functors are the underlying data of a T-algebra.*

Proof. The 1-cell $a\colon TA \to A$ has component

$$a_Q\colon (TA)Q = \int^{P \in \mathrm{ob}\mathcal{P}} \mathcal{P}(P, Q) \otimes AP \to AQ$$

at the object $Q \in \mathcal{P}$, and this is in turn induced from the $\mathcal{V}$-natural family of components

$$a_{PQ}\colon \mathcal{P}(P, Q) \otimes AP \to AQ \ .$$

These are elements of $\mathcal{L}(\mathcal{P}(P, Q) \otimes AP, AQ)$. Under the hom $\mathcal{V}$-adjunction (1.32) from 1.3.2 of the tensor product, these correspond to elements of the internal hom $[\mathcal{P}(P, Q), \mathcal{L}(AP, AQ)]$, i.e.

$$A_{PQ}\colon \mathcal{P}(P, Q) \to \mathcal{L}(AP, AQ) \ .$$

We now have to examine how the algebra axiom cf. 1.1.5

$$M_{[\mathrm{ob}\mathcal{P}, \mathcal{L}]}(a, Ta) = M_{[\mathrm{ob}\mathcal{P}, \mathcal{L}]}(a, \mu_A)$$

transforms under the adjunction.

This is an equation of morphisms in $[\mathrm{ob}\mathcal{P}, \mathcal{L}]_0$, which is equivalent to the equations

$$M_{\mathcal{L}}(a_{QR}, \mathcal{P}(Q, R) \otimes a_{PQ}) = M_{\mathcal{L}}(a_{PR}, (M_{\mathcal{P}} \otimes 1_{AP})\alpha^{-1})$$

of elements in $\mathcal{L}(\mathcal{P}(Q, R) \otimes (\mathcal{P}(P, Q) \otimes AP), AR)$ where P, Q, R run through the objects in $\mathcal{P}$. To apply the hom $\mathcal{V}$-adjunction (1.32) from 1.3.2 for $X = \mathcal{P}(Q, R) \otimes \mathcal{P}(P, Q)$ and $L = AP$ and $M = AQ$, we consider the equivalent equations

$$M_{\mathcal{L}}(a_{QR}, (\mathcal{P}(Q, R) \otimes a_{PQ})\alpha) = M_{\mathcal{L}}(a_{PR}, M_{\mathcal{P}} \otimes 1_{AP}) \ . \tag{1.58}$$

Applying Lemma 1.11 from 1.3.3 to the left hand side shows that its transform is given by[5]

$$M_{\mathcal{L}}(A_{QR} \otimes A_{PQ}) \ .$$

[5] In fact, we do not need the full strength of Lemma 1.11 here, and one could do with more elementary considerations if one was merely concerned with the identification of algebras.

On the other hand, the image of the right hand side under (1.56) from 1.3.3 is determined by the following elementary transformations [6].

$$
\begin{aligned}
&([\eta^{AP}_{\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q)}, 1]\mathcal{L}(L,-)_{(\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q))\otimes AP,AR} M_{\mathcal{L}})(a_{PR}, M_{\mathcal{P}} \otimes 1_{AP}) \\
&= [\eta^{AP}_{\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q)}, 1](M_{\mathcal{V}}(\mathcal{L}(AP,-)_{\mathcal{P}(P,R)\otimes AP,AR}(a_{PR}), \\
&\qquad \mathcal{L}(AP,-)_{(\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q))\otimes AP,\mathcal{P}(P,R)\otimes AR}(M_{\mathcal{P}} \otimes 1_{AP}))) \\
&\quad \text{(by the functor axiom for } \mathcal{L}(AP,-)) \\
&= M_{\mathcal{V}}(\mathcal{L}(AP,-)_{\mathcal{P}(P,R)\otimes AP,AR}(a_{PR}), \\
&\qquad [\eta^{AP}_{\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q)}, 1](\mathcal{L}(AP,-)_{(\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q))\otimes AP,\mathcal{P}(P,R)\otimes AR}(M_{\mathcal{P}} \otimes 1_{AP}))) \\
&\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}}) \\
&= M_{\mathcal{V}}(\mathcal{L}(AP,-)_{\mathcal{P}(P,R)\otimes AP,AR}(a_{PR}), [1, \eta^{AP}_{\mathcal{P}(P,R)}](M_{\mathcal{P}})) \\
&\quad \text{(by ordinary } \mathcal{V}\text{-naturality of } \eta) \\
&= M_{\mathcal{V}}(([\eta^{AP}_{\mathcal{P}(P,R)}, 1]\mathcal{L}(AP,-)_{\mathcal{P}(P,R)\otimes AP,AR})(a_{PR}), M_{\mathcal{P}}) \\
&\quad \text{(by extraordinary } \mathcal{V}\text{-naturality of } M_{\mathcal{V}}) \\
&= M_{\mathcal{V}}(A_{PR}, M_{\mathcal{P}}) = A_{PR}M_{\mathcal{P}}
\end{aligned}
$$

Hence, the algebra axiom is equivalent to the equation

$$M_{\mathcal{L}}(A_{QR} \otimes A_{PQ}) = A_{PQ}M_{\mathcal{P}} \ , \tag{1.59}$$

and this is exactly one of the two axioms for a $\mathcal{V}$-functor.

Now we want to determine the transform of the other axiom of a T-algebra:

$$1_A = M_{[\mathrm{ob}\mathcal{P},\mathcal{L}]}(a, \eta_A) \ .$$

First note that this equation is equivalent to the equations

$$1_{AP} = M_{\mathcal{L}}(a_{PP}, (j_P \otimes 1)\lambda^{-1}_{AP})$$

on objects in $\mathcal{L}(AP, AP)$ where P runs through the objects of $\mathcal{P}$. In turn, these are equivalent to the equations

$$\lambda_{AP} = M_{\mathcal{L}}(a_{PP}, j_P \otimes 1) \ .$$

By definition of the unitor for the tensor product, the transform of the left hand side is given by the unit $j_{AP}\colon I \to \mathcal{L}(AP, AP)$ of the $\mathcal{V}$-category at AP (under the identification of elements of the internal hom and morphisms in $\mathcal{V}$). On the other hand, it is routine to identify the transform of the right hand side as $A_{PP}j_P$. Thus the second axiom of a T-algebra is equivalent to

$$j_{AP} = A_{PP}j_P \ , \tag{1.60}$$

and this is exactly the other axiom of a $\mathcal{V}$-functor. □

[6] We will be short on such routine transformations below. The computation here should serve as an example for basic naturality transformations of the same kind. One could subsume this into another more elementary lemma.

Recall that the hom object of $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ at algebras (A, a) and (B, b) is given by the equalizer

$$[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T(A, B) \longrightarrow [\mathrm{ob}\mathcal{P}, \mathcal{L}](A, B) \underset{[\mathrm{ob}\mathcal{P},\mathcal{L}](1,b)T_{A,B}}{\overset{[\mathrm{ob}\mathcal{P},\mathcal{L}](a,1)}{\rightrightarrows}} [\mathrm{ob}\mathcal{P}, \mathcal{L}](TA, B) \ .$$

Spelling out the hom objects of $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$ as in 1.3.1, this is the same as the following equalizer

$$[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T(A, B) \longrightarrow \Pi_{P\in\mathrm{ob}\mathcal{P}}\mathcal{L}(AP, BP) \underset{\Pi\mathcal{L}(1,b_P)(T_{A,B})_P}{\overset{\Pi\mathcal{L}(a_P,1)}{\rightrightarrows}} \Pi_{\mathcal{P}\in\mathrm{ob}\mathcal{P}}\mathcal{L}(\int^{R\in\mathrm{ob}\mathcal{P}} \mathcal{P}(R, P) \otimes AR, BP)$$

where $E_P T_{A,B} = (T_{A,B})_P E_P$ for a unique $\mathcal{V}$-morphism $(T_{A,B})_P \colon \mathcal{L}(AP, BP) \to \mathcal{L}(TAP, TBP)$. By (1.34) and the universal property of the end, this equalizer is the same as the equalizer of the compositions with $\Pi_{P\in\mathrm{ob}\mathcal{P}}\mathcal{L}(\kappa_{RP}, 1)$ (by definition (1.35) of κ and by Yoneda), and since

$$n \colon \mathcal{L}(\mathcal{P}(R, P) \otimes AR, BP) \cong [\mathcal{P}(R, P), \mathcal{L}(AR, BP)] \ ,$$

this is in turn the same as the following equalizer

$$[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T(A, B)) \longrightarrow \Pi_{P\in\mathrm{ob}\mathcal{P}}\mathcal{L}(AP, BP) \underset{\Pi n\mathcal{L}(1,b_{RP})(\mathcal{P}(R,P)\otimes-)_{AR,BR}}{\overset{\Pi n\mathcal{L}(a_{RP},1)}{\rightrightarrows}} \Pi_{R,P\in\mathrm{ob}\mathcal{P}}[\mathcal{P}(R, P), \mathcal{L}(AR, BP)] \ .$$

(where we have used that $a_P\kappa_{R,P} = a_{RP}$ for the first morphism of the equalizer, and where we have used that $\mathcal{L}(\kappa^A_{R,P}, 1)(E_P)_{TA,TB}T_{A,B} = \mathcal{L}(1, \kappa^B_{R,P})(\mathcal{P}(R, P) \otimes -)_{AR,BR}(E_R)_{A,B}$ by ordinary $\mathcal{V}$-naturality of $\kappa^A_{R,P}$ in A and that $b_P\kappa_{R,P} = b_{RP}$ for the second morphism of the equalizer).

One can now use the Yoneda lemma to show that this is exactly the equalizer (1.27) from 1.3.1 which defines the hom object of the functor $\mathcal{V}$-category: one checks that both the first morphism given here and the transform of $\mathcal{L}(A-, AR)_{RP}$ map the identity at AP to A_{RP}, and both the second here and the transform of $\mathcal{L}(AP, B-)_{PR}$ map the identity at BR to B_{RP}.

Finally, note that the composition law of $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ is induced from the composition law of the functor category $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$, which in turn, is induced from the composition law of $\mathcal{L}$.

Likewise, the composition law of the functor category $[\mathcal{P}, \mathcal{L}]$ is induced via the evaluation functors from the composition law in $\mathcal{L}$, and since the evaluation functors $E_P \colon [\mathcal{P}, \mathcal{L}] \to \mathcal{L}$ where $P \in \mathrm{ob}\mathcal{P}$ factorize through $[H, 1]$ and $E_P \colon [\mathrm{ob}\mathcal{P}, \mathcal{L}] \to \mathcal{L}$, this means that $[H, 1] \colon \mathrm{ob}[\mathcal{P}, \mathcal{L}] \to \mathrm{ob}[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ and the $\mathcal{V}$-isomorphisms on the hom objects induced from the comparison of equalizers above satisfy the $\mathcal{V}$-functor axiom.

Similarly, this data is shown to satisfy the unit axiom for a $\mathcal{V}$-functor. This proves the following theorem.

Theorem 1.4. *Given a small $\mathcal{V}$-category $\mathcal{P}$ and a cocomplete $\mathcal{V}$-category $\mathcal{L}$, then the functor $[H,1]\colon [\mathcal{P},\mathcal{L}] \to [\mathrm{ob}\mathcal{P},\mathcal{L}]$ induced by the inclusion $H\colon \mathrm{ob}\mathcal{P} \to \mathcal{L}$ is strictly monadic for the $\mathcal{V}$-monad $T = [H,1]\mathrm{Lan}_H$ given by the Kan adjunction $\mathrm{Lan}_H \dashv [H,1]$. In particular, the functor $\mathcal{V}$-category $[\mathcal{P},\mathcal{L}]$ is isomorphic as a $\mathcal{V}$-category to the Eilenberg-Moore object $[\mathrm{ob}\mathcal{P},\mathcal{L}]^T$ in $\mathcal{V}$-$\mathcal{CAT}$.* □

Remark 1.8. An alternative strategy to achieve this result is to use an enriched version of Beck's monadicity theorem (see for example [14, Th. II.2.1]). The Kan adjunction meets the conditions of such a theorem because $[H,1]$ creates pointwise colimits, cf. Lemma 1.3 from 1.3.1.

Yet another strategy for specific $\mathcal{V}$ where an identification of the functor $\mathcal{V}$-category is known, is to be completely explicit: For example, if $\mathcal{V} = \mathcal{G}ray$, the functor category can be explicitly identified (cf. [27, Prop. 12.2]). With the help of the two Transformation Lemmata 1.11 and 1.12 from 1.3.1 it is then straightforward to identify the algebra 1-cells, algebra 2-cells, and algebra 3-cells from Proposition 1.2 from 1.1.5 explicitly as we have done it for algebras above.

1.5 Identification of pseudo algebras

Given $\mathcal{G}ray$-categories $\mathcal{P}$ and $\mathcal{L}$, the $\mathcal{G}ray$-category $\mathcal{T}ricat(\mathcal{P},\mathcal{L})$ of trihomomorphisms $\mathcal{P} \to \mathcal{L}$, tritransformations, trimodifications, and perturbations has been described by Gurski [27, Th. 9.4], see also Appendix A. The basic definitions of the objects and the 2-globular data of the local hom 2-categories, i.e. trihomomorphisms, tritransformations, trimodifications, and perturbations may be found in [27, 4.]. These are of course definitions for the general case that domain and codomain are honest tricategories. In our case, they simplify considerably because domain and codomain are always $\mathcal{G}ray$-categories.

Let again $T = [H,1]\mathrm{Lan}_H$ be the $\mathcal{G}ray$-monad on $[\mathrm{ob}\mathcal{P},\mathcal{L}]$ from 1.3.1 corresponding to the Kan adjunction $\mathrm{Lan}_H \dashv [H,1]$, where $\mathcal{P}$ is small and $\mathcal{L}$ is cocomplete. The aim of this section is to prove Theorem 1.6 below, which states that Ps-T-Alg is isomorphic to the full sub-$\mathcal{G}ray$-category of $\mathcal{T}ricat(\mathcal{P},\mathcal{L})$ determined by the locally strict trihomomorphisms.

The general idea of the proof is to identify how the pseudo data and axioms transform under the adjunction (1.32) from 1.3.2 of the tensor product. The main technical tools employed are the two Transformation Lemmata 1.11 and 1.12 from 1.3.3, and elementary identities involving the associators and unitors a, l, r, α, λ, and ρ, which are implied by the pentagon and triangle identity as presented in 1.3.3.

1.5.1 Homomorphisms of $\mathcal{G}ray$-categories

To characterize how Ps-T-Alg transforms under the adjunction of the tensor product, we now introduce the notions of Gray homomorphisms between $\mathcal{G}ray$-categories, say $\mathcal{P}$ and $\mathcal{L}$, Gray transformations, Gray modifications, and Gray perturbations in Definitions 1.9-1.12. If the 0-cell-domain $\mathcal{P}$ and -codomain $\mathcal{L}$ are fixed, these notions organize into a $\mathcal{G}ray$-category $\mathrm{Gray}(\mathcal{P},\mathcal{L})$.

On the other hand, we maintain that this is in fact the natural notion of a (locally strict) trihomomorphism when domain and target are *Gray*-categories, and when the definitions are to be given on Gray products and in terms of their composition laws, say $M_{\mathcal{P}}$ and $M_{\mathcal{L}}$, rather than on cartesian products and in terms of the corresponding cubical composition functors. Thus, the definitions below are easily seen to be mild context-related modifications of the definitions of (locally strict) trihomomorphisms between *Gray*-categories, tritransformations, trimodifications, and perturbations cf. [27, 4.]. All of the definitions are given in equational form. In 1.5.2 we will identify them with the transforms of the definitions from 1.2.1. To achieve this identification, it is at any rate necessary to unwind the diagrammatic presentations for these notions. Thus, we chose an equational notation to make the comparison—which is the object of the present chapter— as comprehensible as possible. While this might make it harder to grasp the content of the definitions on the one hand, the reader will note that the nontrivial cells are easily identified on the other hand.

Definition 1.9. A Gray homomorphism $A\colon \mathcal{P} \to \mathcal{L}$ consists of

- a function on the objects $P \mapsto AP$ denoted by the same letter as the Gray homomorphism itself;
- for objects $P, Q \in \mathcal{P}$, a strict functor $A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP, AQ)$;
- for objects $P, Q, R \in \mathcal{P}$, an adjoint equivalence

$$(\chi, \chi^{\bullet})\colon M_{\mathcal{L}}(A_{QR} \otimes A_{PQ}) \Rightarrow A_{PR} M_{\mathcal{P}}\colon \mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q) \to \mathcal{L}(AP, AR)\ ;$$

- for each object $P \in \mathcal{P}$, an adjoint equivalence

$$(\iota, \iota^{\bullet})\colon j_{AP} \Rightarrow A_{PP} j_P\colon I \to \mathcal{L}(AP, AP)\ ;$$

- and three families of invertible modifications **(GHM1)**-**(GHM3)** which are subject to two axioms **(GHA1)**-**(GHA2)**:

(GHM1) For objects $P, Q, R, S \in \mathcal{P}$, an invertible modification

$$\begin{aligned} \omega_{PQRS}:\quad & [(M_{\mathcal{P}} \otimes 1)a^{-1}, 1](\chi_{PQS}) * [a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\chi_{QRS}, 1_{A_{PQ}})) \\ \Rrightarrow\quad & [1 \otimes M_{\mathcal{P}}, 1](\chi_{PRS}) * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{A_{RS}}, \chi_{PQR})) \end{aligned}$$

of pseudonatural transformations

$$M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})(A_{RS} \otimes (A_{QR} \otimes A_{PQ})) \Rightarrow A_{PS}(1 \otimes M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}}))$$

of strict functors

$$\mathcal{P}(R,S) \otimes (\mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q)) \to \mathcal{L}(AP, AS)\ .$$

(GHM2) For objects $P, Q \in \mathcal{P}$, an invertible modification

$$\gamma_{PQ}\colon [(j_Q \otimes 1) l^{-1}_{\mathcal{P}(P,Q)}, 1](\chi_{PQQ}) * [l^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(\iota_Q, 1_{A_{PQ}})) \Rrightarrow 1_{A_{PQ}}$$

of pseudonatural transformations $A_{PQ} \Rightarrow A_{PQ} \colon \mathcal{P}(P,Q) \to \mathcal{L}(AP, AQ)$.
(GHM3) For objects $P, Q \in \mathcal{P}$, an invertible modification

$$\delta_{PQ} \colon 1_{A_{PQ}} \Rrightarrow [(1 \otimes j_P) r^{-1}_{\mathcal{P}(P,Q)}, 1](\chi_{PPQ}) * [r^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(1_{A_{PQ}}, \iota_P))$$

of pseudonatural transformations $A_{PQ} \Rightarrow A_{PQ} \colon \mathcal{P}(P,Q) \to \mathcal{L}(AP, AQ)$.
(GHA1) For objects $P, Q, R, S, T \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned}
& [1 \otimes (1 \otimes M_{\mathcal{P}}), 1](\omega) * 1_{[M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})](\mathrm{Ten}(1, \mathrm{Ten}(1, \chi)))} \\
\diamond \quad & 1_{[(M_{\mathcal{P}} \otimes M_{\mathcal{P}}) a^{-1}, 1](\chi)} * [a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\Sigma_{\chi,\chi})) \\
\diamond \quad & [(M_{\mathcal{P}} \otimes 1) a^{-1}, 1](\omega) * 1_{[a^{-1} a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(\mathrm{Ten}(\chi, 1), 1))} \\
= \quad & 1_{[1 \otimes (M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}})), 1](\chi)} * [1, M_{\mathcal{L}}](\mathrm{Ten}(1, \omega)) \\
\diamond \quad & [1 \otimes ((M_{\mathcal{P}} \otimes 1) a^{-1}), 1](\omega) * 1_{[a^{-1}(1 \otimes a^{-1}), M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(\mathrm{Ten}(1, \chi), 1))} \\
\diamond \quad & 1_{[(M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}}) a) \otimes 1, 1](\chi)} * [(a \otimes 1) a^{-1} a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\omega, 1))
\end{aligned}$$

To save space, we here employed the notation that vertical composition binds less strictly than horizontal composition $*$, which is also indicated by a line break. Also Ten always denotes the corresponding strict hom functor of the $\mathcal{G}ray$-functor $\mathrm{Ten} \colon \mathcal{G}ray \otimes \mathcal{G}ray \to \mathcal{G}ray$ cf. eq. (1.57) from 1.3.3 above. It is to be noted that in each vertical factor there appears only one nontrivial horizontal factor, and this applies generally to the following definitions.

The axiom is an equation of 2-cells i.e. modifications

$$\begin{aligned}
& [((M_{\mathcal{P}}(M_{\mathcal{P}} \otimes 1)) \otimes 1) a^{-1} a^{-1}, 1](\chi_{PQT}) \\
* \quad & [((M_{\mathcal{P}} \otimes 1) \otimes 1) a^{-1} a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\chi_{QRT}, 1)) \\
* \quad & [a^{-1} a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(\mathrm{Ten}(\chi_{RST}, 1), 1)) \\
\Rrightarrow \quad & [1 \otimes (M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}})), 1](\chi_{PST}) \\
* \quad & [1 \otimes (1 \otimes M_{\mathcal{P}}), M_{\mathcal{L}}](\mathrm{Ten}(1, \chi_{PRS})) \\
* \quad & [1, M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})](\mathrm{Ten}(1, \mathrm{Ten}(1, \chi_{PQR})))
\end{aligned}$$

between 1-cells i.e. pseudonatural transformations

$$M_{\mathcal{L}}(1 \otimes (M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})))(A_{ST} \otimes (A_{RS} \otimes (A_{QR} \otimes A_{PQ}))) \Rightarrow A_{PT} M_{\mathcal{P}}(1 \otimes (M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}})))$$

of strict functors

$$\mathcal{P}(S,T) \otimes (\mathcal{P}(R,S) \otimes (\mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q))) \to \mathcal{L}(AP, AT) \ .$$

We remark that we chose another bracketing than in the definition of a trihomomorphism in the references [27] and [21]. The difference is of course not substantive.
(GHA2) For objects $P, Q, R \in \mathcal{P}$, the following equation of modifications is required:

$$\begin{aligned}
& 1_{\chi} * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{1_{A_{QR}}}, \gamma_{PQ})) \\
\diamond \quad & [1 \otimes ((j_Q \otimes 1) l^{-1}_{\mathcal{P}(P,Q)}), 1](\omega_{PQQR}) * 1_{[r^{-1}_{\mathcal{P}(Q,R)} \otimes 1, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(\mathrm{Ten}(1, \iota), 1))} \\
= \quad & 1_{[((1 \otimes j) r^{-1}) \otimes 1](\chi)} * [1, M_{\mathcal{L}}](\mathrm{Ten}(\delta^{-1}_{QR}, 1_{1_{A_{PQ}}}))
\end{aligned}$$

This is an equation of 2-cells i.e. modifications

$$\chi_{PQR} \Rrightarrow \chi_{PQR} \colon\ M_{\mathcal{L}}(A_{QR} \otimes A_{PQ}) \Rightarrow A_{PR} M_{\mathcal{L}} \colon\ \mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q) \to \mathcal{L}(AP, AR)\ .$$

Definition 1.10. Let $A, B \colon\ \mathcal{P} \to \mathcal{L}$ be homomorphisms of *Gray*-categories. A Gray transformation $f \colon\ A \Rightarrow B$ consists of

- a family $(f_p)_{P \in \mathrm{ob}\mathcal{P}}$ of objects $f_P \colon\ AP \to BP$ in $\mathcal{L}(AP, BP)$;
- for objects $P, Q \in \mathcal{P}$, an adjoint equivalence

$$(f_{PQ}, f^{\bullet}_{PQ}) \colon\ \mathcal{L}(AP, f_Q) A_{PQ} \Rightarrow \mathcal{L}(f_P, BQ) B_{PQ} \colon\ \mathcal{P}(P,Q) \to \mathcal{L}(AP, BQ)\ ;$$

- and two families of invertible modifications **(GTM1)**-**(GTM2)** which are subject to three axioms **(GTA1)**-**(GTA3)**:

(GTM1) For objects $P, Q, R \in \mathcal{P}$, an invertible modification

$$\begin{aligned} \Pi_{PQR} \colon\ & [1, \mathcal{L}(f_P, BR)](\chi^B_{PQR}) * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{QR}}, f_{PQ})) * [1, M_{\mathcal{L}}](\mathrm{Ten}(f_{QR}, 1_{A_{PQ}})) \\ \Rrightarrow\ & [M_{\mathcal{P}}, 1](f_{PR}) * [1, \mathcal{L}(AP, f_R)](\chi^A_{PQR}) \end{aligned}$$

of pseudonatural transformations $M_{\mathcal{L}}((\mathcal{L}(AQ, f_R) A_{QR}) \otimes A_{PQ}) \Rightarrow \mathcal{L}(f_P, BR) B_{PR} M_{\mathcal{P}}$ of strict functors $\mathcal{P}(Q,R) \otimes \mathcal{P}(P,Q) \to \mathcal{L}(AP, BR)$;
(GTM2) For each object $P \in \mathcal{P}$, an invertible modification

$$M_P \colon\ [j_P, 1](f_{PP}) * [1, \mathcal{L}(AP, f_P)](\iota^A_P) \Rrightarrow [1, \mathcal{L}(f_P, BP)](\iota^B_P)$$

of pseudonatural transformations $f_P \Rightarrow \mathcal{L}(f_p, BP) B_{PP} j_P \colon\ I \to \mathcal{L}(AP, BP)$.
(GTA1) For objects $P, Q, R, S \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned} & 1_{[M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}}), 1](f_{PS})} * [1, \mathcal{L}(AP, f_S)](\omega^A) \\ \diamond\ & [(M_{\mathcal{P}} \otimes 1)a^{-1}, 1](\Pi) * 1_{[1 \otimes a^{-1}, M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})](\mathrm{Ten}(\mathrm{Ten}(1, \chi^A), 1))} \\ \diamond\ & 1_{[(M_{\mathcal{P}} \otimes 1)a^{-1}, \mathcal{L}(f_P, BS)](\chi^B)} * 1_{[(M_{\mathcal{P}} \otimes 1)a^{-1}, M_{\mathcal{L}}]\mathrm{Ten}(1, f_{PQ})} * [a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\Pi, 1_{1_{A_{PQ}}})) \\ \diamond\ & 1_{[(M_{\mathcal{P}} \otimes 1)a^{-1}, \mathcal{L}(f_P, BS)](\chi^B)} * [a^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\Sigma_{\chi^B, f_{PQ}})) \\ & \quad * 1_{[a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(1, f_{QR}), 1)} * 1_{[a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(f_{RS}, 1), 1)} \\ = \quad & [1 \otimes M_{\mathcal{P}}, 1](\Pi) * 1_{[1, \mathcal{L}(AP, f_S) M_{\mathcal{L}}]\mathrm{Ten}(1, \chi^A)} \\ \diamond\ & 1_{[1 \otimes M_{\mathcal{P}}, \mathcal{L}(f_P, BS)](\chi^B)} * 1_{[1 \otimes M_P, M_{\mathcal{L}}]\mathrm{Ten}(1, f_{PR})} * [1, M_{\mathcal{L}}](\mathrm{Ten}(\Sigma^{-1}_{f_{RS}, \chi^A})) \\ \diamond\ & 1_{[1 \otimes M_{\mathcal{P}}, \mathcal{L}(f_P, BS)](\chi^B)} * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{1_{B_{RS}}}, \Pi)) * 1_{[a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(f_{RS}, 1), 1)} \\ \diamond\ & [1, \mathcal{L}(f_P, BS)](\omega^B) * 1_{[1, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(1, \mathrm{Ten}(1, f_{PQ})))} \\ & \quad * 1_{[a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(1, f_{QR}), 1)} * 1_{[a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(f_{RS}, 1), 1)} \end{aligned}$$

This is an equation of modifications

$$\begin{aligned}
& [(M_{\mathcal{P}} \otimes 1)a^{-1}, \mathcal{L}(f_P, BS)](\chi^B) \\
* \quad & [a^{-1}, \mathcal{L}(f_P, BS)M_{\mathcal{L}}](\mathrm{Ten}(\chi^B, 1)) \\
* \quad & [1, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)](\mathrm{Ten}(1, \mathrm{Ten}(1, f_{PQ}))) \\
* \quad & [a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(1, f_{QR}), 1) \\
* \quad & [a^{-1}, M_{\mathcal{L}}(M_{\mathcal{L}} \otimes 1)]\mathrm{Ten}(\mathrm{Ten}(f_{RS}, 1), 1) \\
\Rrightarrow \quad & [M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}}), 1]f_{PS} \\
* \quad & [1 \otimes M_{\mathcal{P}}, \mathcal{L}(1, f_S)](\chi^A) \\
* \quad & [1, \mathcal{L}(1, f_S)M_{\mathcal{L}}]\mathrm{Ten}(1, \chi^A)
\end{aligned}$$

of pseudonatural transformations

$$\mathcal{L}(AR, f_S)M_{\mathcal{L}}(1 \otimes M_{\mathcal{L}})(A_{RS} \otimes (A_{QR} \otimes A_{PQ})) \Rightarrow \mathcal{L}(f_P, BS)B_{PS}M_{\mathcal{P}}(1 \otimes M_{\mathcal{P}})$$

of strict functors

$$\mathcal{P}(R, S) \otimes (\mathcal{P}(Q, R) \otimes \mathcal{P}(P, Q)) \to \mathcal{L}(AP, BS) \ .$$

(GTA2) For objects $P, Q \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned}
& 1_{f_{PQ}} * [1, \mathcal{L}(AP, f_Q)](\gamma^A_{PQ}) \\
\diamond \quad & [(j_Q \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, 1](\Pi_{PQQ}) * 1_{[l^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}([1, \mathcal{L}(1, f_Q)](\iota^A_Q), 1_{A_{PQ}}))} \\
= \quad & [1, \mathcal{L}(f_P, BQ)](\gamma^B_{PQ}) * 1_{f_{PQ}} \\
\diamond \quad & 1_{[(j_Q \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, \mathcal{L}(f_P, 1)](\chi^B_{PQQ})} * [l^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(\Sigma^{-1}_{\iota^B_Q, f_{PQ}})) \\
\diamond \quad & 1_{[(j_Q \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, \mathcal{L}(f_P, 1)](\chi^B_{PQQ})} * 1_{[((B_{QQ} j_Q) \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(1, f_{PQ}))} * [l^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(M_Q, 1_{1_{A_{PQ}}}))
\end{aligned}$$

This is an equation of modifications

$$\begin{aligned}
& [(j_Q \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, \mathcal{L}(f_P, BQ)](\chi_{PQQ}) * [((B_{QQ} j_Q) \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(1, f_{PQ})) \\
* \quad & [((j_Q \otimes 1)l^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(f_{QQ}, 1)) * [l^{-1}_{\mathcal{P}(P,Q)}, \mathcal{L}(AP, f_Q)](\iota_Q) \\
\Rrightarrow \quad & f_{PQ} \ .
\end{aligned}$$

(GTA3) For objects $P, Q \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned}
& [1, \mathcal{L}(f_P, BQ)]((\delta^B)^{-1}_{PQ}) * 1_{f_{PQ}} \\
\diamond \quad & 1_{[(1 \otimes j_P)r^{-1}_{\mathcal{P}(P,Q)}, \mathcal{L}(f_P, BQ)](\chi^B_{PPQ})} * [r^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(1_{1_{B_{PQ}}}, M_P)) * 1_{f_{PQ}} \\
= \quad & 1_{f_{PQ}} * [1, \mathcal{L}(AP, f_Q)]((\delta^A)^{-1}_{PQ}) \\
\diamond \quad & [(1 \otimes j_P)r^{-1}_{\mathcal{P}(P,Q)}, 1](\Pi_{PPQ}) * 1_{[r^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(1_{\mathcal{L}(AP, f_Q)A_{PQ}}, \iota^A_P))} \\
\diamond \quad & 1_{[(1 \otimes j_P)r^{-1}_{\mathcal{P}(P,Q)}, \mathcal{L}(f_P, BQ)](\chi^B_{PPQ})} * 1_{[(1 \otimes j_P)r^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{PQ}}, f_{PP}))} * [r^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(\Sigma^{-1}_{f_{PQ}, \iota^A_P}))
\end{aligned}$$

This is an equation of modifications

$$\begin{array}{rl} & [(1 \otimes j_P) r^{-1}_{\mathcal{P}(P,Q)}, \mathcal{L}(f_P, BQ)](\chi^B_{PPQ}) * [(1 \otimes j_P) r^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{PQ}}, f_{PP})) \\ * & [r^{-1}_{\mathcal{P}(P,Q)}, M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{PQ}}, [1, \mathcal{L}(AP, f_P)](\iota_P))) * f_{PQ} \\ \Rrightarrow & f_{PQ} \ . \end{array}$$

Definition 1.11. Let $f, g \colon A \Rightarrow B \colon \mathcal{P} \to \mathcal{L}$ be Gray transformations. A Gray modification $\alpha \colon f \Rrightarrow g$ consists of

- a family $(\alpha_P)_{P \in \mathrm{ob}\mathcal{P}}$ of 1-cells $\alpha_P \colon f_P \to g_P$ in $\mathcal{L}(AP, BP)$;
- and one family of invertible modifications **(GMM1)** which is subject to two axioms **(GMA1)**-**(GMA2)**:

(GMM1) For objects $P, Q \in \mathcal{P}$, an invertible modification

$$\alpha_{PQ} \colon [B_{PQ}, 1](\mathcal{L}(\alpha_P, BQ)) * f_{PQ} \Rrightarrow g_{PQ} * [A_{PQ}, 1](\mathcal{L}(AP, \alpha_Q))$$

of pseudonatural transformations

$$\mathcal{L}(AP, f_Q) A_{PQ} \Rrightarrow \mathcal{L}(g_P, BQ) B_{PQ} \colon \mathcal{P}(P, Q) \to \mathcal{L}(AP, BQ) \ .$$

(GMA1) For objects $P, Q, R \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{array}{rl} & \Pi^g * 1_{[1, M_{\mathcal{L}}](\mathrm{Ten}([A_{QR}, 1](\mathcal{L}(AQ, \alpha_R)), 1))} \\ \diamond & 1_{M_{\mathcal{G}ray}(1, \chi^B_{PQR})} * 1_{[1, M_{\mathcal{L}}](\mathrm{Ten}(1, g_{PQ}))} * [1, M_{\mathcal{L}}](\mathrm{Ten}(\alpha_{QR}, 1_{1_{A_{PQ}}})) \\ \diamond & 1_{M_{\mathcal{G}ray}(1, \chi^B_{PQR})} * [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{1_{B_{QR}}}, \alpha_{PQ})) * 1_{[1, M_{\mathcal{L}}](\mathrm{Ten}(f_{QR}, 1_{A_{PQ}}))} \\ \diamond & M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(\alpha_P, BR), \chi^B_{PQR}}) * 1_{[1, M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{QR}}, f_{PQ}))} * 1_{[1, M_{\mathcal{L}}](\mathrm{Ten}(f_{QR}, 1_{A_{PQ}}))} \\ = & 1_{[M_{\mathcal{P}}, 1](\alpha_{PR})} * M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP, \alpha_P), \chi^A_{PQR}}) \\ \diamond & [M_{\mathcal{P}}, 1](\alpha_{PR}) * 1_{M_{\mathcal{G}ray}(1, \chi^A_{PQR})} \\ \diamond & 1_{[B_{PR} M_{\mathcal{P}}, 1](\mathcal{L}(\alpha_P, BR))} * \Pi^f \end{array}$$

This is an equation of modifications

$$\begin{array}{rl} & [B_{PR} M_{\mathcal{P}}, 1](\mathcal{L}(\alpha_P, BR)) \\ * & 1_{M_{\mathcal{G}ray}(1, \chi^B_{PQR})} \\ * & [1, M_{\mathcal{L}}](\mathrm{Ten}(1_{B_{QR}}, f_{PQ})) \\ * & [1, M_{\mathcal{L}}](\mathrm{Ten}(f_{QR}, 1_{A_{PQ}})) \\ \Rrightarrow & [M_{\mathcal{P}}, 1](g_{PR}) \\ * & M_{\mathcal{G}ray}(1, \chi^A_{PQR}) \\ * & [1, M_{\mathcal{L}}](\mathrm{Ten}([A_{QR}, 1](\mathcal{L}(AQ, \alpha_R)), 1)) \end{array}$$

of pseudonatural transformations

$$M_{\mathcal{L}}((\mathcal{L}(AQ, f_R)A_{QR}) \otimes A_{PQ}) \Rightarrow \mathcal{L}(g_P, BR)B_{PR}M_{\mathcal{P}} \ .$$

(GMA2) For an object $P \in \mathcal{P}$, the following equation of vertical composites of whiskered modifications is required:

$$\begin{aligned} & M^g * 1_{\alpha_P} \\ \diamond \quad & 1_{[j_P,1](g_{PP})} * M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P),\iota_P^A}) \\ \diamond \quad & [j_P, 1](\alpha_{PP}) * 1_{[1,\mathcal{L}(AP,f_P)](\iota_P^A)} \\ = \quad & M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(\alpha_P,BP),\iota_P^B}) \\ \diamond \quad & 1_{[l_I^{-1},M_{\mathcal{L}}](\mathrm{Ten}(1_{G_{PP}j_P},\alpha_P))} * M^f \end{aligned}$$

This is an equation of modifications

$$[l_I^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(1_{G_{PP}j_P}, \alpha_P)) * f_{PP} * [l_I^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(1_{f_P}, \iota_P^A)) \Rrightarrow [l_I^{-1}, M_{\mathcal{L}}](\mathrm{Ten}(\iota_P^G, 1_{g_P})) * \alpha_P$$

of pseudonatural transformations

$$f_P \Rightarrow M_{\mathcal{L}}((G_{PP}j_P) \otimes g_P)l_I^{-1} \colon \ I \to \mathcal{L}(AP, BP) \ .$$

Definition 1.12. Let $\alpha,\beta \colon f \Rrightarrow g \colon A \Rightarrow B \colon \mathcal{P} \to \mathcal{L}$ be Gray modifications. A Gray perturbation $\Gamma \colon \alpha \Rrightarrow \beta$ consists of

- a family of 2-cells $\Gamma_P \colon \alpha_P \Rightarrow \beta_P$ in $\mathcal{L}(AP, BP)$;
- subject to one axiom **(GPA1)**:

(GPA1) For objects $P, Q \in \mathcal{P}$ the following equation of vertical composites of whiskered modifications is required:

$$\alpha_{PQ} \diamond ([B_{PQ}, 1](\mathcal{L}(\Gamma_P, BP)) * 1_{f_{PQ}}) = (1_{g_{PQ}} * [A_{PQ}, 1](\mathcal{L}(AP, \Gamma_Q))) \diamond \beta_{PQ}$$

This is an equation of modifications

$$[B_{PQ}, 1](\mathcal{L}(\alpha_P, BP)) * f_{PQ} \Rrightarrow [A_{PQ}, 1](\mathcal{L}(AP, \beta_Q)) \ .$$

Remark 1.9. Notice that the definitions above are notationally distinguished from the $\mathcal{G}ray$-enriched notions, which usually involve the symbol for the symmetric monoidal closed category $\mathcal{G}ray$. In fact, the notions above correspond to functor categorical notions enriched in the monoidal $\mathcal{G}ray$-category $\mathcal{G}ray$. Thus a Gray homomorphism is a cubical trihomomorphism between $\mathcal{G}ray$-categories, a Gray transformation is a cubical tritransformation between such trihomomorphisms, a Gray modification is a cubical trimodification, and a Gray perturbation is a cubical perturbation.

The 3-globular set formed by Gray homomorphisms, Gray transformations, Gray modifications, and perturbations for fixed domain $\mathcal{P}$ and codomain $\mathcal{L}$ has the structure of a $\mathcal{Gray}$-category: We write down definitions on components for the different compositions and interchange cells in Theorems A.1 and A.2 in the Appendix, see also Corollary A.1. There they are formulated in the case of a general domain for trihomomorphisms, tritransformations, trimodifications, and perturbations, but the definitions adapt to the present context without fuss.

Definition 1.13. Given $\mathcal{Gray}$-categories $\mathcal{P}$ and $\mathcal{L}$, the $\mathcal{Gray}$-category formed by Gray homomorphisms, Gray transformations, Gray modifications, and Gray perturbations with fixed 0-cell-domain $\mathcal{P}$ and -codomain $\mathcal{L}$ is denoted by $\mathsf{Gray}(\mathcal{P}, \mathcal{L})$.

1.5.2 The correspondence of Gray homomorphisms and pseudo algebras

We now show that the correspondence of algebras for the monad of the Kan adjunction and $\mathcal{Gray}$-functors as presented in Section 1.4 generalizes to a correspondence of pseudo algebras and Gray homomorphisms induced by the adjunction of the tensor product. Just as in Theorem 1.4, this correspondence extends to the higher cells between pseudo algebras and Gray homomorphisms. The 1-cell, 2-cell, and 3-cell data of pseudo algebras for the monad of the Kan adjunction and their higher cells consists of families of 1-cells, 2-cells, and 3-cells in the target $\mathcal{Gray}$-category $\mathcal{L}$. Under the adjunction of the tensor product (1.32) from 1.3.2, this data corresponds to data in the $\mathcal{Gray}$-category $\mathcal{Gray}$, i.e. strict functors, pseudonatural transformations, and modifications. In this way we can consider *the transform* of a pseudo algebra, a pseudo functor, a pseudo transformation, and a modification. Thus we maintain that the adjunction of the tensor product induces maps of 3-globular sets between $\mathsf{Ps}\text{-}T\text{-}\mathsf{Alg}$ and $\mathsf{Gray}(\mathcal{P}, \mathcal{L})$.

The following theorem is one of the main results, and forms the first part of the promised correspondence of pseudo algebras and locally strict trihomomorphisms.

Theorem 1.5. *Let $\mathcal{P}$ be a small $\mathcal{Gray}$-category and $\mathcal{L}$ be a cocomplete $\mathcal{Gray}$-category, and let T be the monad corresponding to the Kan adjunction. Then the notions of Gray homomorphism, Gray transformation, Gray modification, and Gray perturbation are precisely the transforms of the notions of a pseudo algebra, a pseudo functor, a pseudo transformation, and a pseudo modification respectively for the monad $T = [H, 1]\mathsf{Lan}_H$ on $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$. In fact, the correspondence induces an isomorphism*

$$\mathsf{Ps}\text{-}T\text{-}\mathsf{Alg} \cong \mathsf{Gray}(\mathcal{P}, \mathcal{L})$$

of $\mathcal{Gray}$-categories. □

We only present parts of the proof explicitly. The proof involves the determination of transforms under the hom $\mathcal{Gray}$-adjunction (1.32) from 1.3.2. These determinations involve the pentagon identity (1.47) and triangle identity (1.53) from 1.3.3 for associators and unitors both of the tensor products and of the monoidal category $\mathcal{Gray}$. We also need naturality of these associators and unitors as presented in the same paragraph. On the other hand, there are elementary identities due to naturality, which are similar to

the one we displayed for the transform of the right hand side of the algebra axiom (1.58) above in 1.4., and then there is heavy use of the two technical Transformation Lemmata 1.11 and 1.12 from 1.3.3. In pursuing the proof, one quickly notices that many of the determinations of transforms are similar to each other. While we cannot display all of the computations, it is our aim to at least characterize the arguments needed for these different classes of transforms. Thus, in the lemmata below we provide examples that should serve as a complete guideline for the rest of the proof.

Lemma 1.14. *Taking transforms under the adjunction* (1.32) *of the tensor product from 1.3.2 induces a one-to-one correspondence of Gray homomorphisms* $\mathcal{P} \to \mathcal{L}$ *and pseudo algebras for the monad* $T = [H, 1]\mathrm{Lan}_H$ *on* $[\mathrm{ob}\mathcal{P}, \mathcal{L}]$.

Proof. Let $(A, a, m, i, \pi, \lambda, \rho)$ be a pseudo T-algebra. From the identification of algebras we know that the components a_{PQ} of the 1-cell $a\colon TA \to A$ transform into strict functors $A_{PQ}\colon \mathcal{P}(P,Q) \to \mathcal{L}(AP, AQ)$ under the hom $\mathcal{Gray}$-adjunction (1.32) from 1.3.2 of the tensor product.

Since the adjoint equivalences m and i replace the two algebra axioms, we define adjoint equivalences $(\chi_{PQR}, \chi^{\bullet}_{PQR}) \coloneqq (n\mathcal{L}(\alpha, 1))(m_{PQR}, m^{\bullet}_{PQR})$ for objects $P, Q, R \in \mathcal{P}$ and $(\iota_P, \iota^{\bullet}_P) \coloneqq (n\mathcal{L}(\lambda, 1))(i_P, i^{\bullet}_P)$ for an object $P \in \mathcal{P}$. We have already determined domain and codomain of these transforms in the identification of the algebra axiom, and the adjoint equivalences in Definition 1.9 do indeed replace the axioms of a $\mathcal{Gray}$-functor cf. (1.59) and (1.60) in 1.4. above.

Next we have to show that the transforms of the components of the invertible 3-cells π, λ, and ρ^{-1} correspond to the invertible modifications ω_{PQRS}, γ_{PQ}, and δ_{PQ} in the definition of a Gray homomorphism.

The components of π at objects $P, Q, R, S \in \mathcal{P}$ are invertible 3-cells in $\mathcal{L}$. We apply the invertible strict functor $\mathcal{L}((1 \otimes \alpha)\alpha, 1)$ to bring them into a form where we can apply the hom $\mathcal{Gray}$-adjunction (1.32) from 1.3.2. We then obtain invertible 3-cells $\mathcal{L}((1 \otimes \alpha)\alpha, 1)(\pi_{PQRS})$ of the form

$$\begin{aligned} & M_{\mathcal{L}}(m_{PQS}, 1_{\alpha(((M_{\mathcal{P}} \otimes 1)a^{-1}) \otimes 1)}) * M_{\mathcal{L}}(m_{QRS}, 1_{(1_{\mathcal{P}(R,S)} \otimes \mathcal{P}(Q,R)} \otimes a_{PQ})\alpha(a^{-1} \otimes 1)}) \\ \Rrightarrow \quad & M_{\mathcal{L}}(m_{PRS}, 1_{(\mathcal{P}(R,S) \otimes ((M_{\mathcal{P}} \otimes 1_{AP})\alpha^{-1}))}) * M_{\mathcal{L}}(1_{a_{RS}}, \mathcal{L}(\alpha, 1)(\mathcal{P}(R,S) \otimes m_{PQR}))\ , \end{aligned}$$

where we have already used the pentagon identity (1.47) from 1.3.3 for α and a.

The computations below then determine the transforms of the horizontal factors and show that these coincide precisely with the horizontal factors in the domain and codomain of the invertible modification ω_{PQRS} in Definition 1.9 from 1.5.1. For brevity we have suppressed many indices e.g. those of hom morphisms where we leave a comma as a subscript to indicate that they are hom morphisms.

Transform of the right hand factor of the domain:

$$
\begin{aligned}
&([\eta,1]\mathcal{L}(AP,-)_{,})(M_{\mathcal{L}}(m_{QRS},1_{(1_{\mathcal{P}(R,S)}\otimes\mathcal{P}(Q,R)}\otimes a_{PQ})\alpha(a^{-1}\otimes 1)}))\\
=&([a^{-1},1][\eta,1]\mathcal{L}(AP,-)_{,}\mathcal{L}(\alpha,1)M_{\mathcal{L}}(1\otimes((\mathcal{P}(R,S)\otimes\mathcal{P}(Q,R))\otimes -)_{,}))(m_{QRS},1_{a_{PQ}})\\
&\text{(by naturality)}\\
=&[a^{-1},M_{\mathcal{L}}]\mathrm{Ten}_{,}(\chi_{QRS},1_{A_{PQ}})\\
&\text{(by Lemma 1.11 from 1.3.3)}
\end{aligned}
$$

Transform of the left hand factor of the domain:

$$
\begin{aligned}
&([\eta,1]\mathcal{L}(AP,-)_{,})(M_{\mathcal{L}}(m_{PQS},1_{\alpha(((M_{\mathcal{P}}\otimes 1)a^{-1})\otimes 1)}))\\
=&([(M_{\mathcal{P}}\otimes 1)a^{-1},1][\eta,1]\mathcal{L}(AP,-)_{,}\mathcal{L}(\alpha,1))(m_{PQS})\\
&\text{(by naturality)}\\
=&[(M_{\mathcal{P}}\otimes 1)a^{-1},1](\chi_{PQS})
\end{aligned}
$$

Transform of the right hand factor of the codomain:

$$
\begin{aligned}
&([\eta,1]\mathcal{L}(AP,-)_{,})(M_{\mathcal{L}}(1_{a_{RS}},\mathcal{L}(a,1)(\mathcal{P}(R,S)\otimes m_{PQR})))\\
=&[1,M_{\mathcal{L}}]\mathrm{Ten}_{,}(1_{A_{RS}},\chi_{PQR})\\
&\text{(by Lemma 1.11 from 1.3.3)}
\end{aligned}
$$

Transform of the left hand factor of the codomain:

$$
\begin{aligned}
&([\eta,1]\mathcal{L}(AP,-)_{,})(M_{\mathcal{L}}(m_{PRS},1_{(\mathcal{P}(R,S)\otimes((M_{\mathcal{P}}\otimes 1_{AP})\alpha^{-1}))}))\\
=&[1\otimes M_{\mathcal{P}},1](\chi_{PRS})\\
&\text{(by naturality)}
\end{aligned}
$$

Thus we may define ω_{PQRS} as the transform $(n\mathcal{L}((1\otimes\alpha)\alpha,1)(\pi_{PQRS})$.

Similarly, it is shown that γ_{PQ} and δ_{PQ} may be defined as the transforms of λ_{PQ} and ρ_{PQ}^{-1} respectively (where one has to use the first Transformation Lemma and the triangle identity).

Finally we have to show that the axioms of a Gray homomorphism are precisely the transforms of the axioms of a pseudo algebra. Observe that because there are only two axioms in the definition of a pseudo algebra, it is crucial that by Proposition 1.3 from 1.2.1 two of the lax algebra axioms are redundant for a pseudo algebra[7].

Now consider the pentagon-like axiom. The corresponding Gray homomorphism axiom and the pseudo algebra axiom are both composed out of three vertical factors on each side of the axiom. Each of the vertical factors is the horizontal composition of a nontrivial

[7]We show below that Gray homomorphisms correspond to locally strict trihomomorphisms. The two redundant axioms correspond to two identities for a trihomomorphism that hold generally: This can be shown because they hold for strict trihomomorphisms by the left and right normalization axiom of a tricategory, and then they hold for a general trihomomorphism by coherence.

2-cell and an identity 2-cell. Since the hom *Gray*-adjunction (1.32) of the tensor product from 1.3.2 is given by strict functors, it preserves vertical and horizontal composition and it preserves identity 2-cells. It follows that we only have to show that the nontrivial 2-cells in each vertical factor match. In diagrammatic language this means that we only have to compare the nontrivial subdiagrams.

In fact, the determination of the transforms of the nontrivial 2-cells of the pseudo algebra axiom is perfectly straightforward and similar to the identification of the transforms of π_{PQRS}'s domain and codomain above. For example, the transform of the interchange cell is:

$$
\begin{aligned}
&([\eta,1]\mathcal{L}(AP,-)_{,}M_{\mathcal{L}})(\Sigma_{m_{RST},\mathcal{L}(\alpha(a^{-1}\otimes 1),1)(\mathcal{P}(R,S)\otimes -)_{,}(m_{PQR})})\\
=&([\eta,1]\mathcal{L}(AP,-)_{,}M_{\mathcal{L}}(1\otimes(\mathcal{L}(\alpha(a^{-1}\otimes 1),1)(\mathcal{P}(R,S)\otimes -)_{,})))(\Sigma_{m_{RST},m_{PQR}})\\
&\text{(by equation (1.2) from 1.1.3)}\\
=&([a^{-1},1][\eta,1]\mathcal{L}(AP,-)_{,}\mathcal{L}(\alpha,1)M_{\mathcal{L}}(1\otimes(\mathcal{P}(R,S)\otimes -)_{,}))(\Sigma_{m_{RST},m_{PQR}})\\
&\text{(by naturality of } M_{\mathcal{L}},\ \mathcal{L}(AP,-)_{,}\text{, and } \eta)\\
=&[a^{-1},M_{\mathcal{L}}]\mathrm{Ten}_{,}(\Sigma_{\chi_{RST},\chi_{PQR}})\\
&\text{(by Lemma 1.11 from 1.3.3)}
\end{aligned}
$$

This is exactly the interchange cell appearing in the pentagon-like axiom of a Gray homomorphism. □

Notice that the proof above only involved the first Transformation Lemma from 1.3.3, namely Lemma 1.11. To show how the second Transformation Lemma i.e. Lemma 1.12 enters in the proof of Theorem 1.5, we provide the following lemma regarding the first axiom of a T-transformation. In fact, one has to employ Lemma 1.12 already in the proof of the correspondence for pseudo T-functors, but only for the axioms of a T-transformation, there appears a new class of interchange cells. Thus we skip the proof for the correspondence of pseudo T-functors and Gray transformations, and for the correspondence of the data of a T-transformation and the data of a Gray modification.

Lemma 1.15. *The transform of the first axiom* **(LTA1)** *of a T-transformation $\alpha\colon f \to g\colon A \to B$ is precisely the second axiom* **(GMA2)** *of a Gray modification.*

Proof. First note that the T-transformation axiom **(LTA1)** cf. Definition 1.3 from 1.5.1 is equivalent to the equations

$$
\begin{aligned}
&(\hbar^{g}_{P} * 1) \diamond (1 * M_{\mathcal{L}}(\Sigma_{\alpha_P,i^{A}_{P}})) \diamond (M_{\mathcal{L}}(A_{PP}, 1_{(j_P\otimes 1)\lambda^{-1}_{AP}}) * 1)\\
&= M_{\mathcal{L}}(\Sigma^{-1}_{i^{B}_{P},\alpha_P}) \diamond (1 * \hbar^{f}_{P})\ .
\end{aligned}
$$

where P runs through the objects of $\mathcal{P}$. We apply the invertible strict functor $\mathcal{L}(\lambda_{AP},1)$ to these equations, which gives the following equivalent equations:

$$
\begin{aligned}
&(\mathcal{L}(\lambda_{AP},1)(\hbar^{g}_{P}) * 1) \diamond (1 * M_{\mathcal{L}}(\Sigma_{\alpha_P,\mathcal{L}(\lambda_{AP},1)(i^{A}_{P})})) \diamond (M_{\mathcal{L}}(A_{PP}, 1_{j_P\otimes 1}) * 1)\\
&= M_{\mathcal{L}}(\Sigma^{-1}_{i^{B}_{P},\mathcal{L}(\lambda_{AP},1)(\alpha_P)}) \diamond (1 * \mathcal{L}(\lambda_{AP},1)(\hbar^{f}_{P}))\ .
\end{aligned}
$$

Here we have used naturality of $M_{\mathcal{L}}$ and the identity (1.2) from 1.1.3 for the manipulation of the interchange cells. As above we only have to compare the transforms of the nontrivial 2-cells. The transforms of $\mathcal{L}(\lambda_{AP},1)(\hbar^g_P)$ and $\mathcal{L}(\lambda_{AP},1)(\hbar^f_p)$ are by definition the modifications M^g and M^f of the Gray transformation corresponding to the pseudo T-functors f and g. The transform of $M_{\mathcal{L}}(A_{PP}, 1_{j_P\otimes 1})$ is by naturality $[j_P,1](\alpha_{PP})$.
The transform of the interchange cell $M_{\mathcal{L}}(\Sigma_{\alpha_P,\mathcal{L}(\lambda_{AP},1)i^A_P})$ is determined as follows:

$$
\begin{aligned}
&([\eta^{AP}_I,1]\mathcal{L}(AP,-)_{I\otimes AP,BP})(M_{\mathcal{L}}(\Sigma_{\alpha_P,\mathcal{L}(\lambda_{AP},1)i^A_P}))\\
=&M_{\mathcal{G}ray}(\mathcal{L}(AP,-)_{AP,BP}\otimes([\eta^{AP}_I,1]\mathcal{L}(AP,-)_{I\otimes AP,AP}))(\Sigma_{\alpha_P,(\mathcal{L}(\lambda_{AP},1)(i^A_P))})\\
&\text{(by the functor axiom for } \mathcal{L}(AP,-) \text{ and naturality of } M_{\mathcal{G}ray})\\
=&M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P),[\eta^{AP}_I,1]\mathcal{L}(AP,-)_{I\otimes AP,AP}(\mathcal{L}(\lambda_{AP},1)(i^A_P))})\\
&\text{(by equation (1.2) from 1.1.3)}\\
=&M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(AP,\alpha_P),\iota^A_P})
\end{aligned}
$$

The transform of the interchange cell left is:

$$
\begin{aligned}
&([\eta^{AP}_I,1]\mathcal{L}(AP,-)_{I\otimes AP,BP})(M_{\mathcal{L}}(\Sigma^{-1}_{i^B_P,\mathcal{L}(\lambda_{AP},1)(\alpha_P)}))\\
=&([\eta^{AP}_I,1]\mathcal{L}(AP,-)_{I\otimes AP,BP})((M_{\mathcal{L}}(1\otimes(I\otimes -),))(\Sigma^{-1}_{\mathcal{L}(\lambda_{AP},1)(i^B_P),\alpha_P}))\\
&\text{(by naturality of } \lambda, \text{ equation (1.2) from 1.1.3, and extraordinary naturality of } M_{\mathcal{G}ray})\\
=&(M_{\mathcal{G}ray}((\mathcal{L}(-,BP)_{AP,BP})\otimes([\eta^{AP}_I,1]\mathcal{L}(AP,-)_{(I\otimes AP,AP)})c)(\Sigma^{-1}_{\mathcal{L}(\lambda_{AP},1)(i^B_P),\alpha_P}))\\
&\text{(by Lemma 1.12 from 1.3.3)}\\
=&(M_{\mathcal{G}ray}((\mathcal{L}(-,BP)_{AP,BP})\otimes([\eta^{AP}_I,1]\mathcal{L}(AP,-)_{(I\otimes AP,AP)})))(\Sigma_{\alpha_P,\mathcal{L}(\lambda_{AP},1)(i^B_P)}))\\
&\text{(by equation (1.6) from 1.1.3)}\\
=&M_{\mathcal{G}ray}(\Sigma_{\mathcal{L}(\alpha_P,BP),\iota^B_P})\\
&\text{(by equation (1.2) from 1.1.3)}
\end{aligned}
$$

□

In the same way one goes on to show that the adjunction of the tensor product induces an isomorphism of 3-globular sets as stated in Theorem 1.5. It requires some more work to show that composition and interchange coincide under the correspondence since these are defined on components for $\mathcal{G}ray(\mathcal{P},\mathcal{L})$ cf. Appendix A, but the arguments closely parallel similar arguments in the following section, and thus we omit them. This finishes our exhibition of the proof of Theorem 1.5, and we end this paragraph with the following corollary:

Our explicit description of the $\mathcal{G}ray$-category of pseudo algebras also gives rise to an explicit description of the $\mathcal{G}ray$-category of strict algebras by restriction. Thus we can read off a description of the functor $\mathcal{G}ray$-category cf. Gurski [27, Prop. 12.2, p. 197]:

Corollary 1.5. *Given* Gray*-functors $T, S\colon \mathcal{A} \to \mathcal{B}$ of* Gray*-categories $\mathcal{A}$ and $\mathcal{B}$, the hom* Gray*-object i.e. the* 2*-category*

$$[\mathcal{A}, \mathcal{B}](T, S)$$

is isomorphic to the following 2*-category provided that $\mathcal{A}$ is small.*

- *The objects are given by the* Gray*-natural transformations. That is, an object α is given by objects α_A in $\mathcal{B}(TA, SA)$ for each $A \in \mathrm{ob}\mathcal{A}$ such that the following equation of strict functors $\mathcal{A}(A, A') \to \mathcal{B}(TA, SA')$ holds,*

$$\mathcal{B}(\alpha_{A'}, 1)T_{AA'} = \mathcal{B}(1, \alpha_A)S_{AA'} \ . \tag{1.61}$$

- *A* 1*-cell $s\colon \alpha \to \beta$ is given by* 1*-cells $s_A\colon \alpha_A \to \beta_A$ in $\mathcal{B}(TA, SA)$ for each $A \in \mathrm{ob}\mathcal{A}$ such that for every object κ in $\mathcal{A}(A, A')$,*

$$M_{\mathcal{B}}(s_{A'}, 1_{T\kappa}) = M_{\mathcal{B}}(1_{S\kappa}, s_A) \ , \tag{1.62}$$

  and for every 1*-cell $t\colon \kappa \to \lambda$ in $\mathcal{A}(A, A')$, the corresponding interchange cells coincide*[8]

$$M_{\mathcal{B}}(\Sigma_{s_{A'}, (T_{AA'})_{\kappa\lambda}(t)}) = M_{\mathcal{B}}(\Sigma^{-1}_{(S_{AA'})_{\kappa\lambda}(t), s_A}) \ . \tag{1.63}$$

- *A* 2*-cell $\rho\colon s \Rightarrow t\colon \alpha \to \beta$ is given by* 2*-cells $\rho_A\colon s_A \Rightarrow t_A$ in $\mathcal{B}(TA, SA)$ for every $A \in \mathrm{ob}\mathcal{A}$ such that for every object κ in $\mathcal{A}(A, A')$,*

$$M_{\mathcal{B}}(\rho_{A'}, 1_{1_{T\kappa}}) = M_{\mathcal{B}}(1_{1_{S\kappa}}, \rho_A) \ . \tag{1.64}$$

  Vertical and horizontal composition are given by the corresponding compositions in the hom 2*-categories of $\mathcal{B}$.*

Proof. In case that the target is cocomplete, this follows from comparison with the Definitions in 1.5.1: A strict algebra corresponds to a Gray homomorphism cf. Def. 1.9 where the adjoint equivalences χ and ι are the identity adjoint equivalences, and where all the invertible modifications are the identity modifications: This is exactly a *Gray*-functor. An algebra 1-cell corresponds to a Gray transformation cf. Def. 1.10 where the adjoint equivalences f_{PQ} are the identity adjoint equivalences, and where the invertible modifications Π_{PQR} and M_P are the identity modifications: This is exactly a *Gray*-natural transformation. An algebra 2-cell corresponds to a Gray modification cf. Def. 1.11 where the invertible modifications are the identity modifications. Pseudonatural transformations coincide if and only if the components and the naturality 2-cells coincide. The equation of components leads to equation (1.62) and the equation of naturality 2-cells leads to (1.63). Finally an algebra 3-cell corresponds to a Gray perturbation cf. Def. 1.12, and the axiom of a Gray perturbation between algebra 2-cells is exactly equation (1.64).

An explicit identification of the end representing the hom object in the functor *Gray*-category shows that this description is also valid in the case of a general target. □

[8] This requirement is not stated in Gurski [27, Prop. 12.2, p. 197].

1.5.3 The correspondence with locally strict trihomomorphisms

The next theorem forms the second part of the promised correspondence of pseudo algebras and locally strict trihomomorphisms, which is then proved in Theorem 1.7 below.

Theorem 1.6. *Given* $\mathcal{Gray}$*-categories* $\mathcal{P}$ *and* $\mathcal{L}$*, the* $\mathcal{Gray}$*-category* $\mathrm{Gray}(\mathcal{P},\mathcal{L})$ *is isomorphic as a* $\mathcal{Gray}$*-category to the full sub-*$\mathcal{Gray}$*-category* $\mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P},\mathcal{L})$ *of* $\mathcal{Tricat}(\mathcal{P},\mathcal{L})$ *determined by the locally strict trihomomorphisms.* □

Again, we just indicate how the proof works, but we want to stress that the steps of the proof not displayed have been explicitly checked and they are indeed entirely analogous to the situations we discuss in the lemmata below.

Lemma 1.16. *Given* $\mathcal{Gray}$*-categories* $\mathcal{P}$ *and* $\mathcal{L}$*, there is a one-to-one correspondence between locally strict trihomomorphisms* $\mathcal{P}\to\mathcal{L}$ *and Gray homomorphisms* $\mathcal{P}\to\mathcal{L}$*.*

Proof. Comparing the definitions, the first thing to be noticed is that Definitions 1.9-1.12 from 1.5.1 involve considerably less cell data than the tricategorical definitions cf. [27, 4.3]. Since $\mathcal{P}$ and $\mathcal{L}$ are $\mathcal{Gray}$-categories, these supernumerary cells are all trivial.

Consider, for example, a locally strict trihomomorphism $A\colon\ \mathcal{P}\to\mathcal{L}$. Recall that this is given by (i) a function on the objects $P\mapsto AP$; (ii) for objects $P,Q\in\mathcal{P}$, a strict functor $A_{PQ}\colon\ \mathcal{P}(P,Q)\to\mathcal{L}(AP,AQ)$; (iii) for objects $P,Q,R\in\mathcal{P}$, an adjoint equivalence

$$(\chi_{PQR},\chi^{\bullet}_{PQR})\colon\ M_{\mathcal{L}}C(A_{QR}\times A_{PQ})\Rightarrow A_{PR}M_{\mathcal{P}}C\ ,$$

where C is again the universal cubical functor, and an adjoint equivalence

$$(\iota_P,\iota^{\bullet}_P)\colon\ j_{AP}\Rightarrow A_{PP}j_P$$

if $P=Q=R$; (iv) and three families ω,γ,δ of invertible modifications subject to two axioms. Up to this point, this looks very similar to Definition 1.9 from 1.5.1, the difference being in the form of domain and codomain of the adjoint equivalence $(\chi_{PQR},\chi^{\bullet}_{PQR})$. However, observing that $C(A_{QR}\times A_{PQ})=(A_{QR}\otimes A_{PQ})C$ by naturality of C, it is clear from Proposition 1.1 from 1.1.4 that this corresponds to an adjoint equivalence $(\hat{\chi}_{PQR},\hat{\chi}^{\bullet}_{PQR})$ as in the definition of a Gray homomorphism such that $C^*(\hat{\chi}_{PQR},\hat{\chi}^{\bullet}_{PQR})=(\chi_{PQR},\chi^{\bullet}_{PQR})$.

Next, given objects $P,Q,R,S\in\mathcal{P}$, the modification ω_{PQRS} has the form[9]

$$\begin{aligned}\omega_{PQRS}:\quad & (((M_{\mathcal{P}}C)\times 1)a_{\times}^{-1})^*(\chi_{PQS})*(a_{\times}^{-1})^*(M_{\mathcal{L}}C)_*(\chi_{QRS}\times 1_{A_{PQ}})\\ \Rrightarrow\quad & (1\times(M_{\mathcal{P}}C))^*(\chi_{PRS})*(M_{\mathcal{L}}C)_*(1_{A_{RS}}\times\chi_{PQR})\end{aligned}$$

where we used strictness of the local functors, and where we made the monoidal structure of the cartesian product explicit, e.g. $a_{\times}$ denotes the corresponding associator.

[9] We again remark that we here and in fact always use a different bracketing than the one employed in [27]. Thus, specifying a modification as the one displayed is equivalent to specifying a modification as in [27].

This is the same as

$$\begin{aligned}\omega_{PQRS}\colon \quad & (C(1\times C))^*([(M_{\mathcal{P}}\otimes 1)a^{-1},1](\hat{\chi}_{PQS})*[a^{-1},M_{\mathcal{L}}](\mathsf{Ten}(\hat{\chi}_{QRS},1_{A_{PQ}})))\\ \Rrightarrow\ & (C(1\times C))^*([1\otimes M_{\mathcal{P}},1](\hat{\chi}_{PRS})*[1,M_{\mathcal{L}}](\mathsf{Ten}(1_{A_{RS}},\hat{\chi}_{PQR})))\ .\end{aligned}$$

Here we have used that $aC(C\times 1)a_\times^{-1} = C(1\times C)$ on the left hand side, that C commutes with the hom functors of Ten and $\times$ i.e.

$$C_*\times_{(X,Y),(X',Y')} = C^*\mathsf{Ten}_{(X,Y),(X',Y')}C\colon\ [X,X']\times[Y,Y']\to[X\times Y,X'\otimes Y']$$

where $\times_{(X,Y),(X',Y')}\colon [X,X']\times[Y,Y']\to[X\times Y,X'\times Y']$ (on objects, this is naturality of C), that $(FG)_* = F_*G_*$ and $(FG)^* = G^*F^*$, that $(C(1\times C))^*$ is strict, that $(-)^*$ and $(-)_*$ coincide with the partial functors of $[-,-]$ for strict functors, and that apart from the cubical functor C, all functors are strict.

By Theorem 1.2 from 1.1.4, ω_{PQRS} corresponds to an invertible modification $\hat{\omega}_{PQRS}$ as in the definition of a Gray homomorphism such that $(C(1\times C))^*\hat{\omega}_{PQRS} = \omega_{PQRS}$.

Given objects $P,Q\in\mathcal{P}$, the modifications γ_{PQ} and δ_{PQ} are of the same form as in the definition of a Gray homomorphism: Using strictness of the local functors, γ_{PQ} is seen to be of the form

$$\gamma_{PQ}\colon\ ((j_Q\times 1)l_\times^{-1})^*(\chi_{PQQ})*(l_\times^{-1})^*(M_{\mathcal{L}}C)_*(\iota_Q\times 1_{A_{PQ}})\Rrightarrow 1_{A_{PQ}}\ ,$$

and this is clearly the same as

$$\gamma_{PQ}\colon\ [(j_Q\otimes 1)l^{-1}_{\mathcal{P}(P,Q)},1](\hat{\chi}_{PQQ})*[l^{-1}_{\mathcal{P}(P,Q)},M_{\mathcal{L}}](\mathsf{Ten}(\iota_Q,1_{A_{PQ}}))\Rrightarrow 1_{A_{PQ}}\ .$$

Similarly, it is shown that δ_{PQ} is of the form required in the definition of a Gray homomorphism.

Finally, we have to compare the axioms. By Theorem 1.2 from 1.1.4, the axioms of a trihomomorphism correspond to equations involving the components of the modifications $\hat{\omega}_{PQRS}$, γ_{PQ}, and δ_{PQ}. On the other hand, the axioms of a Gray homomorphism are equations for the modifications $\hat{\omega}_{PQRS}$, γ_{PQ}, and δ_{PQ} themselves (involving an interchange modification in the case of the pentagon-like axiom). In fact, apart from the interchange cell in the pentagon-like axiom, it is obvious that the components of the nontrivial modifications in the Gray homomorphism axioms are precisely the nontrivial 2-cells in the axioms of the corresponding trihomomorphism axioms. Note here that the correspondence of Theorem 1.2 from 1.1.4 is trivial on components.

Recall that the interchange cell in the pentagon-like axiom of a Gray-homomorphism is given by $[a^{-1},M_{\mathcal{L}}]\mathsf{Ten}_{,}(\Sigma_{\chi_{RST},\chi_{PQR}})$. We maintain that at the object

$$(g,(h,(i,j)))\in\mathcal{P}(S,T)\otimes(\mathcal{P}(R,S)\otimes(\mathcal{P}(Q,R)\otimes\mathcal{P}(P,Q)))\ ,$$

the component $([a^{-1},M_{\mathcal{L}}]\mathsf{Ten}_{,}(\Sigma_{\chi_{RST},\chi_{PQR}}))_{ghij}$ is given by $M_{\mathcal{L}}(\Sigma^{-1}_{\chi_{gh},\chi_{ij}})$. This is because evaluation in $\mathcal{G}ray$ is in this case given by taking components and now the identity (1.57) from 1.3.3 for the strict hom functor $\mathsf{Ten}_{,}$ implies that

$$([a^{-1},M_{\mathcal{L}}]\mathsf{Ten}_{,}(\Sigma_{\chi_{RST},\chi_{PQR}}))_{g(h(ij))} = M_{\mathcal{L}}(\mathsf{Ten}_{,}(\Sigma_{\chi_{RST},\chi_{PQR}}))_{(gh)(ij)} = M_{\mathcal{L}}(\Sigma_{\chi_{gh},\chi_{ij}})\ .$$

This is exactly the interchange cell on the right hand side of the corresponding axiom for a trihomomorphism, cf. [27, p. 68]. □

As noted above, in the proof of Theorem 1.6, there appear additional classes of interchange cells of which the components have to be compared to the interchange cells appearing in the definitions of the data and the *Gray*-category structure of $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$. To give examples for these classes, we skip the proof for the correspondence of Gray transformations and tritransformations and for the correspondence of the data of Gray modifications and trimodifications, and we come back to the second axiom of a Gray modification cf. Lemma 1.15 from 1.5.1:

Lemma 1.17. *The components of the interchange cells in the second axiom of a Gray modification correspond precisely to the interchange cells appearing in the second axiom of a trimodification.*

Proof. The interchange cell appearing on the left hand side of the Gray modification axiom **(GMA2)** is $M_{Gray}(\Sigma_{\mathcal{L}(AP,\alpha_P),\iota_P^A})$. Note that $\Sigma_{\mathcal{L}(AP,\alpha_P),\iota_P^A}$ is an interchange 2-cell in the Gray product

$$[\mathcal{L}(AP, AP), \mathcal{L}(AP, BP)] \otimes [I, \mathcal{L}(AP, AP)] \ ,$$

and now we have to determine how M_{Gray} acts on such an interchange cell. Recall that $M_{Gray}\colon\ [Y, Z] \otimes [X, Y] \to [X, Z]$ is defined by

$$e_Z^X(M_{Gray} \otimes 1_X) = e_Z^Y(1 \otimes e_Y^X)a^{-1} \ .$$

For the component of $M_{Gray}(\Sigma_{\mathcal{L}(AP,\alpha_P),\iota_P^A})$ at the single object $* \in I$ this implies that

$$\begin{aligned}(M_{Gray}\Sigma_{\mathcal{L}(AP,\alpha_P),\iota_P^A})_* &= (e^{\mathcal{L}(AP,AP)}_{\mathcal{L}(AP,BP)}(\mathcal{L}(AP, -)_{AP,BP} \otimes 1))(\Sigma_{\alpha_P,(\iota_P^A)_*}) \\ &\quad \text{(by equation (1.2) from 1.1.3)} \\ &= M_{\mathcal{L}}(\Sigma_{\alpha_P,(\iota_P^A)_*}) \\ &\quad \text{(by definition of } \mathcal{L}(AP, -) \text{ see (1.14) from 1.1.5)}\end{aligned}$$

In fact, this is exactly the interchange cell for the left hand side of the axiom in [27, p. 77].

Similarly, the component of the interchange 2-cell $M_{Gray}(\Sigma_{\mathcal{L}(\alpha_P,BP),\iota_P^B})$ at the single object $* \in I$, is given by

$$\begin{aligned}(M_{Gray}(\Sigma_{\mathcal{L}(\alpha_P,BP),\iota_P^B}))_* &= e^{\mathcal{L}(BP,BP)}_{\mathcal{L}(AP,BP)}(\mathcal{L}(-, BP)_{AP,BP} \otimes 1)(\Sigma_{\alpha_P,(\iota_P^B)_*}) \\ &= M_{\mathcal{L}}c(\Sigma_{\alpha_P,(\iota_P^B)_*}) \\ &\quad \text{(by definition of } \mathcal{L}(-, BP) \text{ see (1.15) from 1.1.5)} \\ &= M_{\mathcal{L}}(\Sigma^{-1}_{(\iota_P^B)_*,\alpha_P}) \\ &\quad \text{(by equation (1.6) from 1.1.3) .}\end{aligned}$$

In fact, this is exactly the interchange cell for the right hand side of the axiom in [27, p. 77]. □

Remark 1.10. It is entirely analogous to show that the composition laws as given in [27, Th. 9.1 and 9.3] cf. Theorems A.1 and A.2 and Definitions 1.5 and 1.6 from 1.2.1 of the two $\mathcal{G}ray$-categories coincide under the correspondence. This concludes our exhibition of the critical ingredients of the proof of Theorem 1.6.

Combining Theorem 1.5 from 1.5.2 and Theorem 1.6, we have proved the main theorem of this chapter:

Theorem 1.7. *Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category and $\mathcal{L}$ be a cocomplete $\mathcal{G}ray$-category, and let T be the monad corresponding to the Kan adjunction. Then the $\mathcal{G}ray$-category* Ps-T-Alg *is isomorphic to the full sub-$\mathcal{G}ray$-category of $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$ determined by the locally strict trihomomorphisms.* □

The identification of the functor category $[\mathcal{P}, \mathcal{L}]$ with $[\mathrm{ob}\mathcal{P}, \mathcal{L}]^T$ in Theorem 1.4 from 1.4., and the coherence result for Ps-T-Alg given in Corollary 1.3 from 1.3.1, then prove the following coherence theorem for $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$:

Theorem 1.8. *Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category and $\mathcal{L}$ be a cocomplete $\mathcal{G}ray$-category. Then the inclusion $i\colon [\mathcal{P}, \mathcal{L}] \to \mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$ of the functor $\mathcal{G}ray$-category $[\mathcal{P}, \mathcal{L}]$ into the $\mathcal{G}ray$-category $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$ of locally strict trihomomorphisms has a left adjoint such that the components $\eta_A\colon A \to iLA$ for objects $A \in \mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{L})$ of the unit of this adjunction are internal biequivalences.* □

Recall that $\mathcal{G}ray$ considered as a $\mathcal{G}ray$-category is complete and cocomplete cf. Lemma 1.1 from 1.1.3. Thus Theorem 1.8 applies for $\mathcal{L} = \mathcal{G}ray$. As a consequence we have the following:

Corollary 1.6. *Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category. Then any locally strict trihomomorphism $\mathcal{P} \to \mathcal{G}ray$ is biequivalent to a $\mathcal{G}ray$-functor $\mathcal{P} \to \mathcal{G}ray$.* □

In particular, let $\mathcal{P}$ be a small category C considered as a discrete $\mathcal{G}ray$-category. Then locally strict trihomomorphisms $C \to \mathcal{G}ray$ are the homomorphisms of interest, and we have proved that any such homomorphism is biequivalent to a $\mathcal{G}ray$-functor $C \to \mathcal{G}ray$.

On the other hand, the following example shows that the assumption of local strictness in Corollary 1.6 cannot be dropped. It may come as a surprise that it is not true that any trihomomorphism into $\mathcal{G}ray$ is biequivalent to a $\mathcal{G}ray$-functor:

Example 1.1. Let J be the one-object 2-category with a single nontrivial idempotent endo-1-cell described in Example 0.1 from the Introduction. Now consider the $\mathcal{G}ray$-category $\mathcal{D}$ with two objects a and b, $\mathcal{D}(a,b) = J$, $\mathcal{D}(b,a) = \emptyset$, $\mathcal{D}(a,a) = I$, and $\mathcal{D}(b,b) = I$.

We aim to define a trihomomorphism $F\colon \mathcal{D} \to \mathcal{G}ray$ such that $Da = I$ and $Db = \mathrm{st}J$. Locally, this is given by the apparent strict $I \to [\mathrm{st}J, \mathrm{st}J]$ mapping $*$ to the identity functor, and 1_* to the identity natural transformation. Similarly, we define $I \to [I, I]$, and finally $f\colon J \to \mathrm{st}J \cong [I, \mathrm{st}J]$. This can clearly be no $\mathcal{G}ray$-functor since f is non-strict. The adjoint equivalences χ and ι can be taken to be the identity adjoint equivalences. Suppose there would be a tritransformation $\theta\colon G \Rightarrow F$ for a $\mathcal{G}ray$-functor $G\colon \mathcal{D} \to \mathcal{G}ray$. This involves strict functors $\theta_a\colon Ga \to I$ and $\theta_b\colon Gb \to \mathrm{st}J$ and among other

things a strict functor $G_{ab}\colon\ J \to [Ga, Gb]$. The adjoint equivalence of the hypothetical tritransformation has the form

$$\Theta\colon\ [1, \theta_b]G_{ab} \Rightarrow [\theta_a, 1]f\colon\ J \to [Ga, \mathrm{st}J]\ .$$

Clearly, for $Ga = I$ (then θ_a must be 1_I and the above has the form $\Theta\colon\ [1, \theta_b]G_{ab} \Rightarrow f\colon\ J \to \mathrm{st}J$), there can be no such equivalence since f is not equivalent to a strict functor. But if θ were a biequivalence, then $\theta_a\colon\ Ga \to I$ would be a biequivalence in $\mathcal{G}ray$ with strict biequivalence-inverse $\eta_a\colon\ I \to Ga$, and we would get a biequivalence between G and another $\mathcal{G}ray$-functor G' such that $G'a = I$. This $\mathcal{G}ray$-functor is defined by $G'b = b$ and $G'_{ab} = [\eta_a, 1]G_{ab}$ and a $\mathcal{G}ray$-natural isomorphism $\gamma\colon\ G' \cong G$ is given by $\gamma_a = \eta_a$ and $\gamma_b = 1_{Gb}$. This would be a biequivalence in $\mathcal{T}ricat(\mathcal{D}, \mathcal{G}ray)$ giving rise to another biequivalence $\theta'\colon\ G' \Rightarrow F$ where $G'a = I$, which cannot exist by the argument above.

Chapter 2

A Yoneda lemma for tricategories

In this chapter we prove a Yoneda lemma for tricategories. This involves again the concept of a codescent object. In contrast to Chapter 1 where the codescent object only appeared implicitly underlying the coherence theorem from three-dimensional monad theory, this chapter starts with the notion of a codescent object of a strict codescent diagram. The codescent object is a $\mathcal{G}ray$-enriched colimit and the restriction to strict codescent diagrams is precisely the restriction of this colimit to certain $\mathcal{G}ray$-functors.

2.1 Codescent objects

In this section we introduce the notion of a codescent object for a strict codescent diagram. The codescent object of a strict codescent diagram is just a simpler special case of the general definition of a codescent object for a codescent diagram cf. Gurski [27, 11.3]. However, we use a string diagram notation, which proves especially useful in the application in Chapter 3. The results of this section also apply to codescent objects of general codescent diagrams, but since we will apply them only to strict codescent diagrams below, we chose to present the simpler case to make the argument more easy to follow.

We present the definition of a codescent object of a strict codescent diagram in a form where the simplicial identities are implicit in the diagrams. That is, one has to apply simplicial identities to see that cells in the diagrams do indeed compose. On the other hand, this results in diagrams of a very simple shape, which enable us to easily recognize the weight of the colimit of the codescent object in Proposition 2.1. Namely, the axioms have the same shape as those for a pseudofunctor, which explains why the strictification st turns up in the weight of the colimit[1].

Definition 2.1. We denote by $\mathbf{n}$ the free category on the graph $0 \to 1 \to 2 \to \ldots \to n-1$. That is, $\mathbf{n}$ has a single morphism $i \to j$ if $i \leq j$ and the hom sets are empty otherwise.

[1] recall that the strictification is a left adjoint to the inclusion

Definition 2.2. We denote by I the left adjoint of the inclusion $\mathbf{Gpd} \to \mathcal{C}at$ of groupoids into small categories—in fact a left 2-adjoint—which formally inverts all arrows in a small category C.

Remark 2.1. The groupoid $I(C)$ is constructed as follows. One takes the underlying graph, adds inverses to all morphisms, then takes the free category on this graph, and finally forms the quotient category with relations for the composition in C (this also makes the length-1 string in the identity from C the identity for the quotient category) and making the adjoined morphisms left and right inverses c.f. [55, II.8. Prop. 1]. For functors one takes the underlying graph map, then the obvious graph map of graphs with adjoined inverses (an adjoined opposite arrow to an arrow f in the underlying graph $U(C)$ is sent to the adjoined opposite arrow to $F(f)$), takes the free functor on this graph map, composes with the quotient functor and uses the universal property of the quotiening c.f. [55, II.8. Prop. 1] to get a functor between $I(C)$ and $I(C')$. It also follows from the universal property that this prescription respects composition. For a transformation one takes the same components and notes that naturality with respect to a morphism is tantamount to naturality with respect to its inverse. Therefore, we get again a natural transformation. Since vertical and horizontal composition of transformations are defined in terms of components, the prescription clearly preserves those.

Definition 2.3. As a shorthand, we write I_n for the groupoid $I(\mathbf{n})$. The category I_n has the same objects as $\mathbf{n}$ and a single isomorphism between any two objects. That is, I_n is the indiscrete (chaotic) category on $\{0, 1, ..., n-1\}$.

Definition 2.4. We denote by Δ_3 the full subcategory of the simplex category Δ with objects [0], [1], [2], and [3].

Definition 2.5. We denote by Δ_3^- the subcategory of Δ_3 which differs from the latter only in that $\Delta_3^-([3],[2]) = \emptyset$—i.e. in Δ_3^- we discard the codegeneracy maps in the highest layer[2].

Definition 2.6. A strict codescent diagram in a $\mathcal{G}ray$-category $\mathcal{K}$ is a $\mathcal{G}ray$-functor

$$X_\bullet \colon (\Delta_3^-)^{\mathrm{op}} \to \mathcal{K} .$$

As the notation presumably suggests, we denote by X_j the object of $\mathcal{K}$ given by the image under $X_\bullet$ of $[j]$. Similarly, we use indexed letters d_i and s_j for the images of the coface and codegeneracy maps which are objects in the respective hom 2-categories of $\mathcal{K}$.

Remark 2.2. If, in Definition 2.6, Δ_3^- is replaced by Δ_3, one obtains the notion of a reflexive strict codescent diagram in a $\mathcal{G}ray$-category $\mathcal{K}$. Reflexive codescent diagrams are of theoretical importance, but they do not play an explicit role in this thesis, although most codescent diagrams are in fact reflective (as the strict codescent diagrams associated to an algebra, which gives a functor $\Delta_+^{\mathrm{op}} \to \mathcal{K}$).

[2]We have chosen a similar notation as in [7].

A codescent object for $X_\bullet$ is a representation for a weighted colimit of the strict codescent diagram $X_\bullet$:

Definition 2.7. The codescent object of a strict codescent diagram $X_\bullet\colon (\Delta_3^-)^{\mathrm{op}} \to \mathcal{K}$ is the colimit of $X_\bullet$ weighted by the following $\mathcal{G}ray$-functor $F_{\mathrm{codsc}}\colon \Delta_3^- \to \mathcal{G}ray$:

- $F_{\mathrm{codsc}}([j]) = \mathrm{st}I_j$
- $F_{\mathrm{codsc}}(\delta_j) = \mathrm{st}I(\delta_j)$
- $F_{\mathrm{codsc}}(\sigma_k) = \mathrm{st}I(\sigma_k)$

More precisely, as an ordinary functor, F_{codsc} is defined to be the composition of the inclusion $\Delta_3^- \to \Delta_+$, the full inclusion $\Delta_+ \to \mathcal{C}at$, the groupoidification $I\colon \mathcal{C}at \to \mathcal{G}pd$, the inclusion $\mathcal{G}pd \to \mathcal{C}at$, the inclusion $\mathcal{C}at \to \mathcal{G}ray$ of categories as locally discrete 2-categories, and the restriction of the strictification $\mathrm{st}\colon \mathcal{B}icat \to \mathcal{G}ray$ to $\mathcal{G}ray$. Since Δ_3^- is a category, this ordinary functor gives rise to a $\mathcal{G}ray$-functor $F_{\mathrm{codsc}}\colon \Delta_3^- \to \mathcal{G}ray$, which we also denote by $\mathrm{st}I_\bullet$.

To actually work with the notion of a codescent object, it is often useful to have a more explicit definition than the one given in Definition 2.7. A $\mathcal{G}ray$-enriched colimit is defined by a representation isomorphism of 2-categories. Definitions 2.8-2.13 give rise to a $\mathcal{G}ray$-functor with values in 2-categories cf. Prop. 2.3 that makes this representation explicit. A representation isomorphism of 2-categories gives rise to a three-part universal property, which we make explicit in Definition 2.15. On the other hand, the identification of the weight of the codescent object gives a good explanation for the specific form of this universal property.

Definition 2.8. Let $X_\bullet\colon (\Delta_3^-)^{\mathrm{op}} \to \mathcal{K}$ be a strict codescent diagram. A *cocone* from $X_\bullet$ to an object X in the $\mathcal{G}ray$-category $\mathcal{K}$ consists of

- an object x in the hom 2-category $\mathcal{K}(X_0, X)$,
- an adjoint equivalence 1-cell $\epsilon\colon xd_1 \to xd_0$ in $\mathcal{K}(X_1, X)$,
- an invertible 2-cell

$$M\colon (\epsilon d_0) * (\epsilon d_2) \Rightarrow \epsilon d_1\colon xd_1d_2 \to xd_0d_0$$

  in $\mathcal{K}(X_2, X)$,

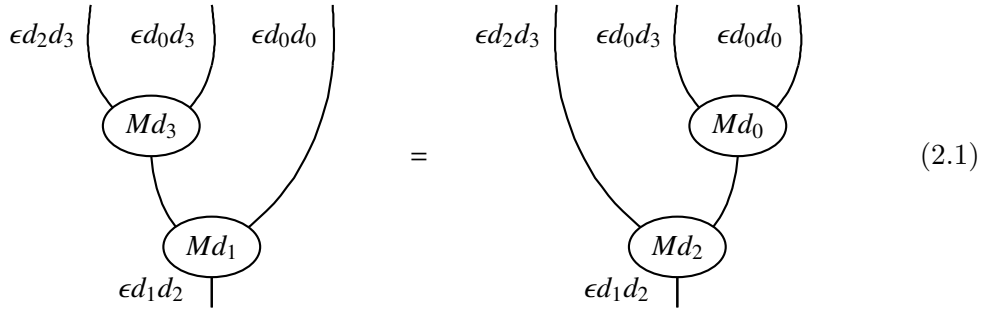
- an invertible 2-cell

$$U\colon \epsilon s_0 \Rightarrow 1\colon x \to x$$

  in $\mathcal{K}(X_0, X)$

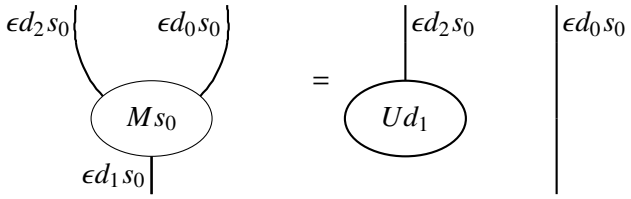
subject to the three axioms below, which are expressed as string diagrams[3] in hom-2-categories of the *Gray*-category $\mathcal{K}$.

First axiom.



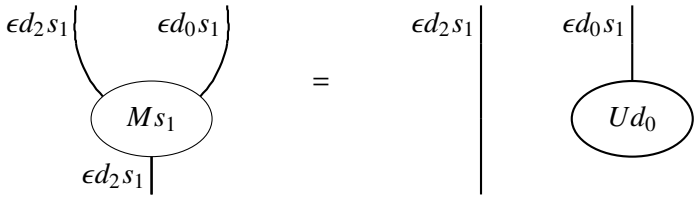
(2.1)

Second axiom.



(2.2)

Third axiom.



(2.3)

Proposition 2.1. *A cocone from $X_\bullet$ to X as in Def. 2.8 is precisely a cocone in the enriched sense from the weight F_{codsc} of the colimit of the codescent object to X.*

Proof. A cocone from F_{codsc} to X is given by a *Gray*-natural transformation

$$\beta\colon\ F_{\mathrm{codsc}} \Rightarrow \mathcal{K}(X_\bullet -, X)\ . \tag{2.4}$$

[3]Our string diagrams are read from left to right for horizontal composition and from top to bottom for vertical composition. Regions corresponding to objects in the 2-category are not labeled. As common, we also omit a label for the points corresponding to unit and counit 2-cells of an adjunction in the 2-category.

Since $F_{\mathrm{codsc}}([0]) = \mathrm{st}I_0 = \mathrm{st}[0]$, the component $\beta_{[0]}$ at $[0]$ of this transformation is determined by a pseudofunctor $\tilde{\beta}_{[0]}\colon [0] \to \mathcal{K}(X_0, X)$. Thus we exactly have to give an object $x \in \mathcal{K}(X_0, X)$ as in Definition 2.8 with $x = \tilde{\beta}_{[0]}(0)$ and the unit constraint of the functor.

Similarly, the component $\beta_{[1]}$ is determined by a pseudofunctor $\tilde{\beta}_{[1]}\colon I_1 \to \mathcal{K}(X_1, X)$. Forgetting about unit constraints for a moment, this is exactly given by an adjoint equivalence $\epsilon\colon xd_1 \to xd_0$ as in Definition 2.8 with $\epsilon = \tilde{\beta}_{[1]}(01)$, where we used that by $\mathcal{G}ray$-naturality of (2.4)

$$\beta_{[1]}(0) = \beta_{[1]}(\delta_1(0)) = \mathcal{K}(d_1, X)(\beta_{[0]}(0)) = xd_1$$

and similarly

$$\beta_{[1]}(1) = xd_0 \ .$$

On the other hand, the codegeneracy $I(\sigma_0)\colon I_1 \to I_0$ clearly sends the nontrivial isomorphism 01 in I_1 to the identity at 0 in I_0. Thus, writing 00 for the identity at 0 in I_0 , by naturality we have

$$\tilde{\beta}_{[0]}(00) = \tilde{\beta}_{[0]}(I(\sigma_0)(01)) = \mathcal{K}(s_0, X)(\tilde{\beta}_{[1]}(01)) = \epsilon s_0 \ ,$$

and so the unit constraint $(\tilde{\beta}_{[0]})_0$ for the pseudofunctor corresponding to $\beta_{[0]}$ is precisely an invertible 2-cell $U =: \epsilon s_0 \Rightarrow 1_x$ as in Definition 2.8.

Next note that the unit constraints for $\tilde{\beta}_{[1]}$ are determined by $\mathcal{G}ray$-naturality and the unit constraint for $\tilde{\beta}_{[0]}$. The composition constraints of $\tilde{\beta}_{[1]}$ are then determined by the pseudofunctor axioms.

The pseudofunctor $\tilde{\beta}_{[2]}\colon I_2 \to \mathcal{K}(X_2, X)$ corresponding to the component $\beta_{[2]}$ of (2.4) is already determined on cells by naturality and the only new constraint is the composition constraint

$$(\tilde{\beta}_{[2]})_{12,01}\colon (\epsilon d_0) * (\epsilon d_1) \Rightarrow (\epsilon d_2) \ ,$$

which is precisely an invertible 2-cell M as in Definition 2.8. Observe that, by naturality, Ms_0 and Ms_1 must be the composition constraints $(\beta_{[1]})_{01,00}$ and $(\beta_{[1]})_{11,01}$ respectively. The second and third axiom (2.2) and (2.3) in Definition 2.8 thus correspond literally to the two pseudofunctor axioms.

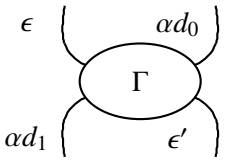
Finally, while the data of the pseudofunctor $\tilde{\beta}_{[3]}\colon I_3 \to \mathcal{K}(X_3, X)$ corresponding to the component $\beta_{[3]}$ of (2.4) is already completely determined by naturality, the pseudofunctor axiom for composition constraints imposes one new equation, which is exactly the first axiom (2.1) in Definition 2.8.

□

In light of Proposition 2.1, the set $\mathsf{Cocone}(X_\bullet, Y)$ of cocones from $X_\bullet$ to Y inherits the structure of a 2-category from the hom object $[\Delta_3^-, \mathcal{G}ray](F_{\mathrm{codsc}}, \mathcal{K}(X_\bullet -, X))$ of the functor $\mathcal{G}ray$-category cf. Corollary 1.5 from 1.5. Spelling this out leads to the following definitions.

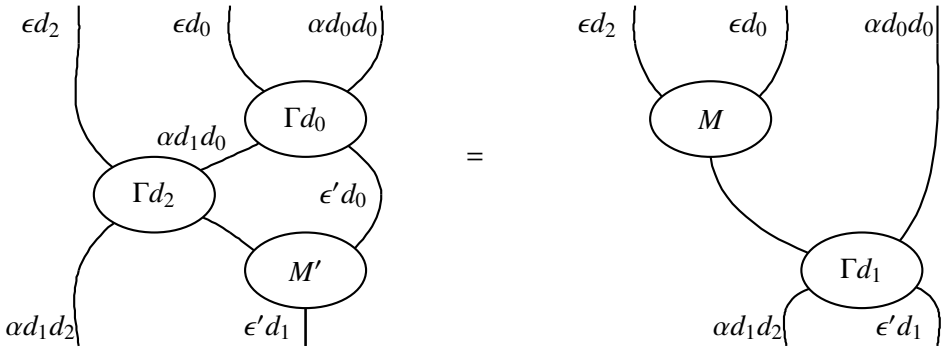
Definition 2.9. Let $X_\bullet\colon (\Delta_3^-)^{\mathrm{op}} \to \mathcal{K}$ be a strict codescent diagram, and let (Y, y, ϵ, M, U) and $(Y, y', \epsilon', M', U')$ be cocones from $X_\bullet$ to Y. Then a 1-cell from (Y, y, ϵ, M, U) to $(Y, y', \epsilon', M', U')$ consists of

- a 1-cell $\alpha\colon y \to y'$ in $\mathcal{K}(X_0, Y)$, and
- an invertible 2-cell

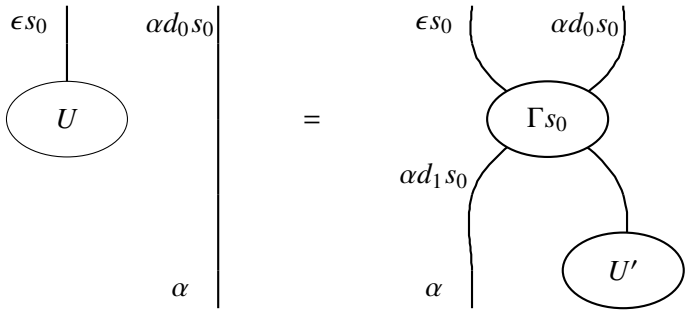


in $\mathcal{K}(X_1, Y)$ satisfying the two axioms below.

First axiom for Γ.



Second axiom for Γ.



Remark 2.3. A 1-cell as in Definition 2.9 corresponds to a 1-cell $s\colon \beta \to \beta'$ in $[\Delta_3^-, \mathcal{G}ray](F_{\mathrm{codsc}}, \mathcal{K}(X_\bullet -, X))$. The component $s_{[0]}$ essentially corresponds to the 1-cell α in Def. 2.9. The component $s_{[1]}$ corresponds to a pseudonatural transformation with components αd_1 and αd_0 determined by the naturality condition on s cf. (1.62) and a naturality 2-cell corresponding to the invertible 2-cell Γ in Def. 2.9. Notice that the condition on interchange cells (1.63) is vacuous because Δ_3^- has no nontrivial 2-cells. The two axioms for Γ in Def. 2.9 are then just respect for composition and units of a pseudonatural transformation. Also note that the naturality of the pseudonatural transformation corresponding to the component $s_{[j]}$ is automatic because the domain I_j possesses no nontrivial 2-cells.

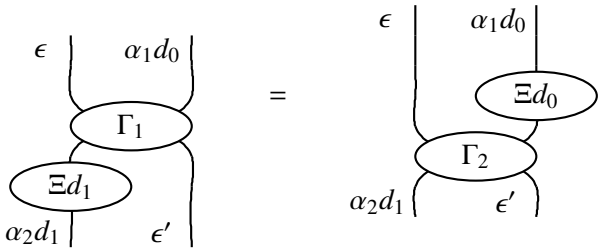
Definition 2.10. Let $X_\bullet\colon (\Delta_3^-)^{\mathrm{op}} \to \mathcal{K}$ be a strict codescent diagram, and let

$$(\alpha_1, \Gamma_1), (\alpha_2, \Gamma_2)\colon (Y, y, \epsilon, M, U) \to (Y, y', \epsilon', M', U')$$

be a pair of parallel 1-cells between cocones from $X_\bullet$ to Y. Then a 2-cell from (α_1, Γ_1) to (α_2, Γ_2) consists of

- a 2-cell $\Xi\colon \alpha_1 \Rightarrow \alpha_2$ in $\mathcal{K}(X_0, Y)$

satisfying the axiom below.



(2.5)

Remark 2.4. A 2-cell as in Definition 2.1 corresponds to a 2-cell $\rho\colon s \Rightarrow s'$ in $[\Delta_3^-, \mathcal{G}ray](F_{\mathrm{codsc}}, \mathcal{K}(X_\bullet -, X))$. Its component $\rho_{[0]}$ essentially corresponds to the 2-cell Ξ in Def. 2.1. The component $\rho_{[1]}$ corresponds to a modification with components determined by the naturality condition (1.64) and the axiom (2.5) in Def. 2.1 is exactly the modification axiom.

Lemma 2.1. *Vertical composition of* 2*-cells and unit* 2*-cells in* $\mathcal{K}(X_0, Y)$ *give*

$$\mathrm{Cocone}(X_\bullet, Y)((Y, y, \epsilon, M, U), (Y, y', \epsilon', M', U'))$$

the structure of a category.

Proof. Immediate since, for a vertical composite, the axiom in Definition 2.1 is implied by the axiom for the individual factors. Of course, this is also implied by the comparison with $[\Delta_3^-, \mathcal{G}ray](F_{\mathrm{codsc}}, \mathcal{K}(X_\bullet -, X))$. □

Definition 2.11. Let $X_\bullet\colon (\Delta_3^-)^{\mathrm{op}} \to \mathcal{K}$ be a strict codescent diagram. Then the identity 1-cell of a cocone (Y, y, ϵ, M, U) from $X_\bullet$ to Y is defined by

- the identity 1-cell of y in $\mathcal{K}(X_0, Y)$, and
- the identity 2-cell of ϵ in $\mathcal{K}(X_1, Y)$.

Definition 2.12. Let $X_\bullet\colon (\Delta_3^-)^{\mathrm{op}} \to \mathcal{K}$ be a strict codescent diagram, and let

$$(\alpha, \Gamma)\colon (Y, y, \epsilon, M, U) \to (Y, y', \epsilon', M', U')$$

and

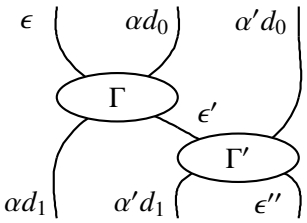
$$(\alpha', \Gamma')\colon (Y, y', \epsilon', M', U') \to (Y, y'', \epsilon'', M'', U'')$$

be 1-cells between cocones from $X_\bullet$ to Y. Then their horizontal composite

$$(\alpha', \Gamma') * (\alpha, \Gamma)\colon (Y, y, \epsilon, M, U) \to (Y, y', \epsilon', M', U')$$

is defined by

- the horizontal composite $\alpha' * \alpha\colon y \to y''$ of 1-cells in $\mathcal{K}(X_0, Y)$, and
- the vertical composite of whiskered 2-cells in $\mathcal{K}(X_1, Y)$ below.



(2.6)

Definition 2.13. Let $X_\bullet\colon (\Delta_3^-)^{\mathrm{op}} \to \mathcal{K}$ be a strict codescent diagram, and let

$$\Xi\colon (\alpha_1, \Gamma_1) \Rightarrow (\alpha_2, \Gamma_2)\colon (Y, y, \epsilon, M, U) \to (Y, y', \epsilon', M', U')$$

and

$$\Xi'\colon (\alpha_1', \Gamma_1') \Rightarrow (\alpha_2', \Gamma_2')\colon (Y, y', \epsilon', M', U') \to (Y, y'', \epsilon'', M'', U'')$$

be 2-cells between 1-cells of cocones from $X_\bullet$ to Y. Then their horizontal composite is defined by the horizontal composite $\Xi' * \Xi$ of 2-cells in $\mathcal{K}(X_0, Y)$.

Remark 2.5. Definitions 2.11-2.13 essentially correspond to horizontal composition of pseudonatural transformations and modifications. In particular, (2.6) corresponds to the naturality 2-cell for a horizontal composite of transformations.

Proposition 2.2. *Horizontal composition and* 1*-cell identities as introduced in Definitions 2.11-2.13 are well-defined and give* $\mathbf{Cocone}(X_\bullet, Y)$ *the structure of a* 2*-category.*

Proof. One either proves this by reference to the 2-category structure of $[\Delta_3^-, \mathcal{G}ray](F_{\mathrm{codsc}}, \mathcal{K}(X_\bullet -, X))$ or by a short direct argument: It is immediate that the identity 2-cell in Definition 2.11 satisfies the two axioms for a 1-cell of cocones. Further, for the 2-cell (2.6) in Definition 2.12, the first axiom is obvious from the string diagrams and reduces via the middle four interchange law to the corresponding axiom for Γ and Γ'. The second axiom is immediate from the ones for Γ and Γ'. Finally, for the 2-cell $\Xi' * \Xi$ in Definition 2.13, the axiom for a 2-cell of cocones follows from the axiom for Ξ and Ξ' and the form of the 2-cell (2.6) in Definition 2.12. It is obvious that horizontal composition is strictly associative and unital and suitably functorial—the middle-four interchange law reduces to the middle four interchange law in $\mathcal{K}(X_0, Y)$—to give $\mathbf{Cocone}(X_\bullet, Y)$ the structure of a 2-category. □

Proposition 2.3. *The covariant partial hom functors of the* $\mathcal{G}ray$*-category* $\mathcal{K}$ *induce a* $\mathcal{G}ray$*-functor* $\mathbf{Cocone}(X_\bullet, -)\colon \mathcal{K} \to \mathcal{G}ray$ *given by* $\mathbf{Cocone}(X_\bullet, Y)$ *on an object* Y *in* $\mathcal{K}$*.*

Proof. Again, this is proved by reference to the functor structure of the composite

$$[\Delta_3^-, \mathcal{G}ray](F_{\mathrm{codsc}}, -) \circ [X_\bullet^{\mathrm{op}}, 1] \circ y$$

where $y\colon \mathcal{K} \to [\mathcal{K}^{\mathrm{op}}, \mathcal{G}ray]$ denotes the Yoneda embedding and

$$[X_\bullet^{\mathrm{op}}, 1]\colon [\mathcal{K}^{\mathrm{op}}, \mathcal{G}ray] \to [\Delta_3^{\mathrm{op}}, \mathcal{G}ray]$$

is the $\mathcal{G}ray$-functor given by precomposition with $X_\bullet^{\mathrm{op}}$. An explicit proof is an easy exercise using the partial hom $\mathcal{G}ray$-functors $\mathcal{K}(X_i, -)$ for $i = 0, 1, 2, 3$. In particular, given a cocone (X, x, ϵ, M, U) from $X_\bullet$ to X and a 1-cell $f\colon X \to Y$, then $(Y, fx, f\epsilon, fM, fU)$ constitutes a cocone from $X_\bullet$ to Y. Given a 1-cell $\beta\colon f \to f'$ in the hom 2-category $\mathcal{K}(X, Y)$, notice that $(\beta x, \Sigma_{\beta,\epsilon})$ constitutes a 1-cell

$$(Y, fx, f\epsilon, fM, fU) \to (Y, f'x, f'\epsilon, f'M, f'U) \tag{2.7}$$

of cocones. This corresponds to the fact that the naturality 2-cell of the pseudonatural transformation

$$\mathcal{K}(X_1, \beta x)\colon \mathcal{K}(X_1, f) \Rightarrow \mathcal{K}(X_1, f')\colon \mathcal{K}(X_1, X) \to \mathcal{K}(X_1, Y) \tag{2.8}$$

at the 1-cell $\epsilon\colon xd_1 \to xd_0$ is given by the interchange cell $\Sigma_{\beta,\epsilon}$. Finally, a 2-cell $\Lambda\colon \beta \Rightarrow \beta'$ gives rise to a cocone-2-cell $\Lambda\colon (\beta x, \Sigma_{\beta,\epsilon}) \Rightarrow (\beta' x, \Sigma_{\beta',\epsilon})$ by naturality of interchange cells cf. (1.13) from 1.1.4. □

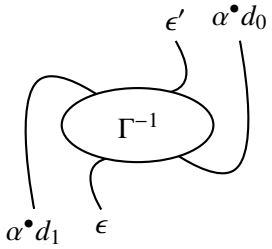
Lemma 2.2. *A* 2*-cell* $\Xi\colon (\alpha_1, \Gamma_1) \Rightarrow (\alpha_2, \Gamma_2)$ *in* $\mathbf{Cocone}(X_\bullet, Y)$ *is invertible if and only if it is invertible as a* 2*-cell in* $\mathcal{K}(X_0, Y)$*.*

Proof. Immediate since axiom (2.5) holds for Ξ if and only if it holds for Ξ^{-1} as is easily verified. □

Lemma 2.3. *A* 1*-cell*

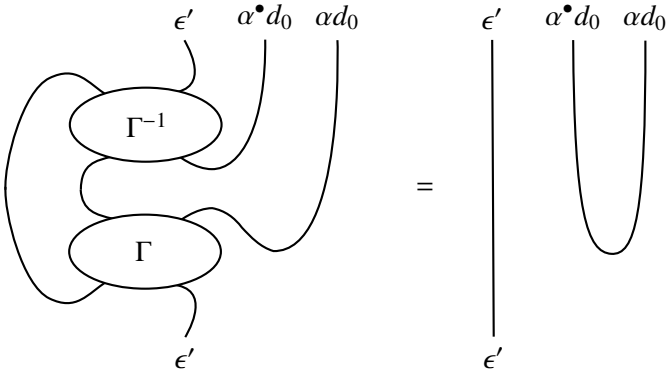
$$(\alpha, \Gamma)\colon\ (Y, y, \epsilon, M, U) \to (Y, y', \epsilon', M', U')$$

in $\mathbf{Cocone}(X_\bullet, Y)$ *is part of an adjoint equivalence* $((\alpha, \Gamma), (\alpha^\bullet, \Theta), \varepsilon, \eta)$ *if and only if* $(\alpha, \alpha^\bullet, \varepsilon, \eta)$ *is an adjoint equivalence in* $\mathcal{K}(X_0, Y)$ *and* Θ *is the mate of* Γ^{-1} *below.*



(2.9)

Proof. Suppose $(\alpha, \alpha^\bullet, \varepsilon, \eta)$ is an adjoint equivalence in $\mathcal{K}(X_0, Y)$ and Θ is the mate of Γ^{-1} in the statement. One easily checks the two axioms to show that $(\alpha^\bullet, \Theta)$ forms a 1-cell in $\mathbf{Cocone}(X_\bullet, Y)$ in the opposite direction: First, one takes the mate of the axioms, inverts the resulting equations, and pre- and postcomposes with the respective whiskerings of M' and M and U' and U respectively. This directly gives the second axiom for $(\alpha^\bullet, \Theta)$, while the first axiom is obtained after applying a triangle identity. Next, one checks that ε and η^{-1} form 2-cells in $\mathbf{Cocone}(X_\bullet, Y)$, which are then automatically invertible by Lemma 2.2: For the counit, equation (2.5) reduces to the following obvious identity of string diagrams.



For η^{-1}, recall that $(\alpha^\bullet, a, \eta^{-1}, \varepsilon^{-1})$ is again an adjoint equivalence. Therefore the above argument shows that the axiom holds for η^{-1}.

Conversely, if $((\alpha, \Gamma), (\alpha^\bullet, \Theta), \varepsilon, \eta)$ is an adjoint equivalence, so must be $(\alpha, \alpha^\bullet, \varepsilon, \eta)$, and one checks that Θ is the specified mate of Γ^{-1} from axiom (2.5) for either of the 2-cells of the adjoint equivalence in $\mathbf{Cocone}(X_\bullet, Y)$. □

Definition 2.14. Let $X_\bullet\colon \Delta \to \mathcal{K}$ be a strict codescent diagram. A codescent object for $X_\bullet$ is a representation of the $\mathcal{G}ray$-functor $\mathbf{Cocone}(X_\bullet, -)\colon \mathcal{K} \to \mathcal{G}ray$.

Remark 2.6. This definition is of course just a reformulation of Definition 2.7.

Thus a codescent object consists of an object X in $\mathcal{K}$ and a unit $I \to \mathbf{Cocone}(X_\bullet, X)$. The latter consists of a cocone from $X_\bullet$ to X such that the $\mathcal{G}ray$-natural transformation $\mathcal{K}(X, -) \Rightarrow \mathbf{Cocone}(X_\bullet, -)$ induced from the unit via Yoneda is an isomorphism. Since a $\mathcal{G}ray$-natural transformation is invertible if and only if its components are, the isomorphism condition gives rise to a three-part universal property. This leads to yet another reformulation of Definitions 2.7 and 2.14.

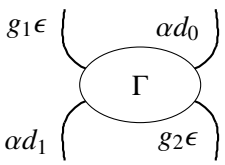
Definition 2.15. Let $X_\bullet \colon \Delta \to \mathcal{K}$ be a strict codescent diagram. A codescent object for $X_\bullet$ consists of a cocone (X, x, ϵ, M, U) from $X_\bullet$ to $X \in \mathrm{ob}\mathcal{K}$ that is universal in the sense of the three-part universal property below.

First part of the universal property. Given a cocone $(X', x', \epsilon', M', U')$ from $X_\bullet$ to X', there exists a unique 1-cell $f \colon X \to X'$ such that

- $fx = x'$;
- $f\epsilon = \epsilon'$;
- $fM = M'$; and
- $fU = U'$.

Second part of the universal property. Given

- an object Y in the $\mathcal{G}ray$-category $\mathcal{K}$;
- a pair of objects g_1, g_2 in the hom 2-category $\mathcal{K}(X, Y)$;
- a 1-cell $\alpha \colon g_1 x \to g_2 x$ in $\mathcal{K}(X_0, Y)$; and
- an invertible 2-cell

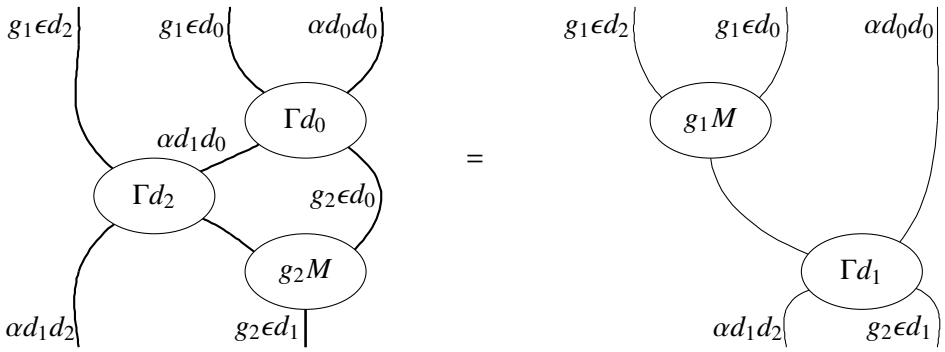


in $\mathcal{K}(X_1, Y)$ satisfying the two axioms below, there exists a unique 1-cell $\tilde{\alpha} \colon g_1 \to g_2$ in $\mathcal{K}(X, Y)$ such that

- $\tilde{\alpha}x = \alpha$, and
- $\Gamma = \Sigma_{\tilde{\alpha},\epsilon}$,

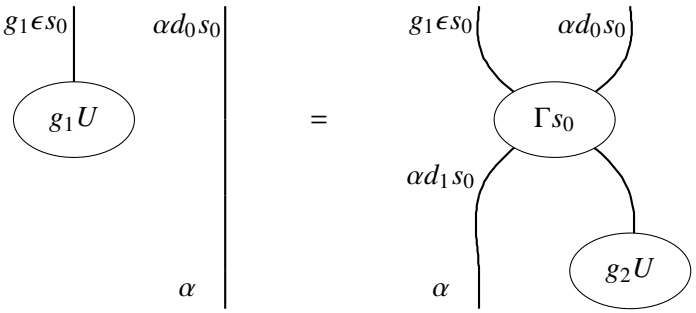
where the latter is the interchange cell in the *Gray*-category $\mathcal{K}$.

First axiom for Γ.



(2.10)

Second axiom for Γ.



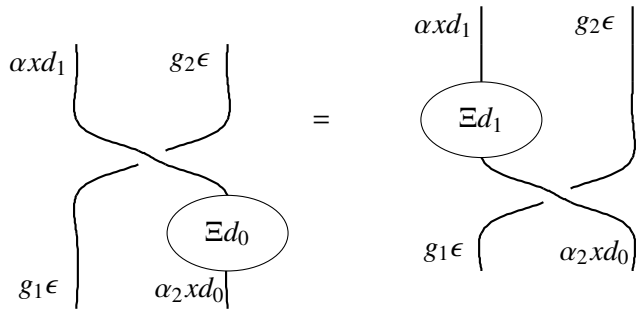
(2.11)

Third part of the universal property. Given

- an object Y of the *Gray*-category $\mathcal{K}$;
- a pair of objects g_1, g_2 in $\mathcal{K}(X, Y)$;
- a pair of 1-cells $\alpha_1, \alpha_2 \colon g_1 \to g_2$ in $\mathcal{K}(X, Y)$; and
- a 2-cell $\Xi \colon \alpha_1 x \Rightarrow \alpha_2 x$ in $\mathcal{K}(X_0, Y)$ satisfying the axiom below, where the crossings denote the corresponding interchange cells in the *Gray*-category $\mathcal{K}$,

there is a unique 2-cell $\tilde{\Xi} \colon \alpha_1 \Rightarrow \alpha_2$ such that

- $\tilde{\Xi}x = \Xi$.



Remark 2.7. The three parts of the universal property correspond to the bijections on objects, 1-cells, and 2-cells constituting the representation isomorphism

$$\mathcal{K}(X, Y) \xrightarrow{\cong} [\Delta_3^-, \mathcal{G}ray](F_{\mathrm{codsc}}-, \mathcal{K}(X_\bullet -, Y)) \cong \mathbf{Cocone}(X_\bullet, Y) \tag{2.12}$$

of 2-categories, which is induced from the universal cocone (X, x, ϵ, M, U) and the hom functors of the $\mathcal{G}ray$-functor $\mathbf{Cocone}(X_\bullet, -)$ cf. the proof of Proposition 2.3.

In particular, the condition that $\Gamma = \Sigma_{\tilde{\alpha},\epsilon}$ in the second part of the universal property is a consequence of the effect of $\mathbf{Cocone}(X_\bullet, -)$ on 2-cells described in the proof of Proposition 2.3.

Remark 2.8. The object X in the data of a codescent object is a representing object for a weighted colimit and thus determined up to isomorphism. This is also obvious from the first part of the universal property of the codescent object. Note also that we have the following trivial corollary from the description of the codescent object as a weighted colimit.

Corollary 2.1. *If the $\mathcal{G}ray$-category $\mathcal{K}$ is cocomplete, then $\mathcal{K}$ has codescent objects for all codescent diagrams.* □

Lemma 2.4. *The 2-cell Ξ in the third part of the universal property of the codescent object is invertible if and only if the 2-cell $\tilde{\Xi}$ ibidem is invertible.*

Proof. Immediate from the fact that Ξ is the image of $\tilde{\Xi}$ under the representation isomorphism $\mathcal{K}(X, Y) \cong \mathbf{Cocone}(X_\bullet, Y)$ and Lemma 2.2. □

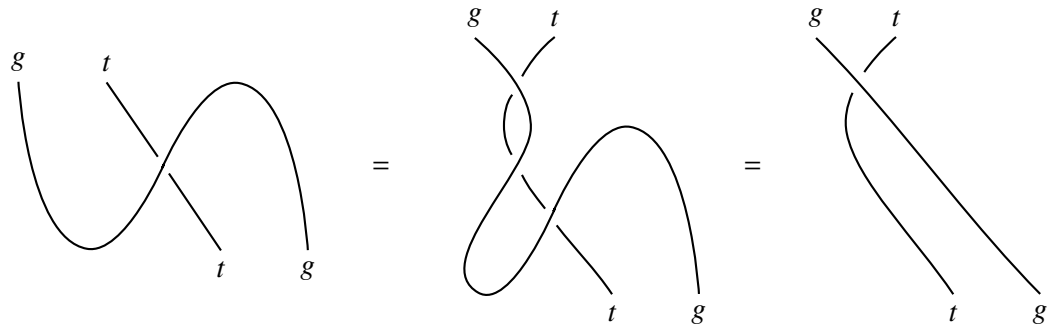
Lemma 2.5. *Given a 1-cell (α, Γ) in $\mathbf{Cocone}(X_\bullet, Y)$ as in the second part of the universal property of the codescent object such that α is part of an adjoint equivalence $(\alpha, \alpha^\bullet, \varepsilon, \eta)$ in $\mathcal{K}(X_0, Y)$, there is a unique adjoint equivalence $(\tilde{\alpha}, \tilde{\alpha}^\bullet, \tilde{\varepsilon}, \tilde{\eta})$ such that $\tilde{\alpha}x = \alpha$, $\tilde{\alpha}^\bullet x = \alpha^\bullet$, $\Gamma = \Sigma_{\tilde{\alpha},\epsilon}$, $\tilde{\varepsilon}x = \varepsilon$, and $\tilde{\eta}x = \eta$.*

Proof. By Lemma 2.3, $((\alpha, \Gamma), (\alpha^\bullet), \varepsilon, \eta)$, where Θ is the mate (2.9) of Γ^{-1}, is an adjoint equivalence in $\mathbf{Cocone}(X_\bullet, Y)$, which corresponds to a unique adjoint equivalence $(\tilde{\alpha}, \tilde{\alpha}^\bullet, \tilde{\varepsilon}, \tilde{\eta})$ under the representation isomorphism $\mathcal{K}(X, Y) \cong \mathbf{Cocone}(X_\bullet, Y)$. □

Remark 2.9. Note that $\Sigma_{\tilde{\alpha}^\bullet,\epsilon}$ is indeed the mate of $\Sigma^{-1}_{\tilde{\alpha},\epsilon}$ as we show in the following exemplifying Lemma. Technically, this fact is already implicit in the representation isomorphism $\mathcal{K}(X,Y) \to \mathbf{Cocone}(X_\bullet, Y)$.

Lemma 2.6. *Let $(f,g,\varepsilon,\eta)\colon A \to A'$ be an adjunction in a 2-category X, and let $t\colon B \to B'$ be a 1-cell in a 2-category Y. Then the mate of the interchange 2-cell $\Sigma_{f,t}\colon (f,1) * (1,t) \to (1,t)*(f,1)$ in the Gray product $X \otimes Y$ is given by the interchange 2-cell $\Sigma^{-1}_{g,t}\colon (1,t)*(g,1) \to (g,1)*(1,t)$.*

Proof. The Lemma is proved by the string diagram calculation below, where, for the first equation, one uses the invertibility of interchange cells, and for the second, one uses relation (1.10) for interchange cells, naturality of the interchange cell with respect to the counit ε, and a triangle identity.



□

Remark 2.10. Note that the mate $\Sigma^{-1}_{g,t}$ of $\Sigma_{f,t}$ is again invertible.

2.1.1 The codescent object of a free algebra

Let T be a *Gray*-monad on a *Gray*-category $\mathcal{K}$ with multiplication and unit *Gray*-natural transformations $\mu\colon TT \to T$ and $\eta\colon 1_{\mathcal{K}} \Rightarrow T$ respectively, and let $A \in \mathrm{ob}\mathcal{K}$ be an algebra for T with action a an object in the hom 2-category $\mathcal{K}(TA, A)$.

Definition 2.16. The strict codescent diagram in $\mathcal{K}^T$ associated with an algebra A, denoted by $X^A_\bullet$ or just $X_\bullet$, is defined by the following prescriptions:

- $X_i = T^{i+1}A$,
- $X(d_k) = T^k\mu_{T^{i-k-2}A}\colon T^iA \to T^{i-1}A$ if $0 \le k \le i-2$ and $X(d_{i-1}) = T^{i-1}x$,
- $X(s_k) = T^{k+1}\eta_{T^{i-k-1}A}\colon T^iA \to T^{i+1}A$ for $0 \le k \le i-1$.

Remark 2.11. The following argument shows that these prescriptions do indeed give rise to a strict codescent diagram in the *Gray*-category $\mathcal{K}^T$ of algebras i.e. a *Gray*-functor $(\Delta_3^-)^{\mathrm{op}} \to \mathcal{K}^T$. Recall [73, pp. 151-153] that the *Gray*-monad T is generated by the *Gray*-adjunction $U \dashv F$ where $U\colon \mathcal{K}^T \to \mathcal{K}$ is the forgetful functor and $F\colon \mathcal{K} \to \mathcal{K}^T$ is the

free T-algebra functor sending an object A to the free algebra TA with action μ_A, and we have $T = UF$ and $\mu = U\varepsilon F$, where ε is the counit of the adjunction, and the unit η of T is given by the unit of the adjunction, for which we use the same letter.

Now the adjunction $U \dashv F$ also generates a comonad FU on $\mathcal{K}^T$ with comultiplication $F\eta U$ and counit μ. As the augmented simplex category Δ_+ is the free monoidal category containing a monoid, such a comonad corresponds precisely to a strict monoidal *Gray*-functor

$$FU_\bullet \colon \Delta_+^{\mathrm{op}} \to [\mathcal{K}^T, \mathcal{K}^T] \tag{2.13}$$

where the latter is the functor *Gray*-category with monoidal structure given by composition—for the sake of the argument, we ignore possible issues with the existence of the functor *Gray*-category here. Since *Gray*-CAT is (partially) closed, the transpose of (2.13) gives rise to a *Gray*-functor

$$\overline{FU_\bullet} \colon \mathcal{K}^T \to [\Delta_+^{\mathrm{op}}, \mathcal{K}^T]\ . \tag{2.14}$$

Restriction now gives rise to a *Gray*-functor

$$(\overline{FU_\bullet})_3^- \colon \mathcal{K}^T \to [(\Delta_3^-)^{\mathrm{op}}, \mathcal{K}^T] \tag{2.15}$$

The argument is apparently purely enriched and thus the same as in two-dimensional monad theory, c.f. Bourke [7, Rem. 6.7].

Lemma 2.7. *Let $X_\bullet$ be the strict codescent diagram in $\mathcal{K}$ associated with a T-algebra A. Then the action $a \in \mathrm{ob}\,\mathcal{K}(TA, A)$ gives rise to a cocone from $X_\bullet$ to A.*

Proof. Explicitly, for $Y = A$ in Definition 2.8, we take $y \colon TA = X_0 \to Y = A$ to be the action a, ϵ is the identity adjoint equivalence, and M and U are the identity 2-cells. It is easy to see that this is well-defined. Namely,

$$yd_1 = aTa = a\mu_A = yd_0$$

by an axiom for the action of an algebra. □

Corollary 2.2. *The action a factorizes as fx where f is an object in $\mathcal{K}^T(X, A)$ such that $f\epsilon$ is an identity 1-cell and such that fF and fU are identity 2-cells.*

Proof. Immediate from Lemma 2.7 and the 1-dimensional part of the universal property of the codescent object. □

Theorem 2.1. *Let T be a *Gray*-monad on a *Gray*-category $\mathcal{K}$ and let A be an object in $\mathcal{K}$. Then the codescent object of the strict codescent diagram associated with the free T-algebra TA on the object A is biequivalent to TA in $\mathcal{K}^T$.*

Proof. Since the action of the free T-algebra TA is given by $\mu_A \in \mathcal{K}(T^2A, TA)$, by Corollary 2.2, μ_A factorizes as

$$\mu_A = fx \tag{2.16}$$

where f is an object in $\mathcal{K}^T(X, TA)$ such that

$$f\epsilon = 1\,,\ fM = 1\,,\ \text{and}\ fU = 1\ . \tag{2.17}$$

We will now construct a biequivalence-inverse g to f from the universal property of X. Consider the object

$$g \coloneqq xT\eta_A \in \mathcal{K}^T(TA, X)\ . \tag{2.18}$$

Notice that this is a section of f since x and f factorize μ_A c.f. eq. (2.16) and by a monad axiom:

$$fg = fxT\eta_A = \mu_A T\eta_A = 1_{TA}\ . \tag{2.19}$$

We now have to show that the other composition gf is equivalent to the identity 1_X in $\mathcal{K}(X, X)$. To do so, we construct an adjoint equivalence $gf \to 1_X$ with the help of Lemma 2.5. Thus what we have to show is that there is a 1-cell (α, Γ) of cocones from $X_\bullet$ to X as in the second part of the universal property of the codescent object with $g_1 = 1_X$ and $g_2 = gf$ such that α is part of an adjoint equivalence $(\alpha, \alpha^\bullet, \varepsilon, \eta)$.

First, notice that, by naturality of μ, $g_2 x$ may be transformed as follows[4].

$$gfx = xT\eta_A\mu_A = x\mu_{TA}T^2\eta_A = xd_0T^2\eta_A \tag{2.20}$$

On the other hand, by a monad axiom, $g_1 x$ has the form:

$$1_X x = x1_{T^2A} = xT\mu_A T^2\eta_A = xd_1T^2\eta_A\ . \tag{2.21}$$

Notice that $T^2\eta_A$ is not part of the data of the codescent diagram $X_\bullet$ for the free T-algebra TA, nor in fact of the data of the codescent diagram for T^2A. Rather, the 1-cell $T\eta_A\colon TA \to T^2A$ of (free) algebras in the $\mathcal{G}ray$-category $\mathcal{K}^T$ gives rise to a 1-cell in the functor $\mathcal{G}ray$-category $[(\Delta_3^-)^{\mathrm{op}}, \mathcal{K}^T]$, that is, a $\mathcal{G}ray$-natural transformation $(T\eta_A)_\bullet\colon X_\bullet^{TA} \Rightarrow X_\bullet^{T^2A}$ of codescent diagrams. Indeed this is implied by the functorial structure (2.15) of the prescription assigning the associated codescent diagram to an algebra as described in Remark 2.11 above.

On the other hand, the codescent diagram associated with the free algebra T^2A is obtained as the décalage [72, 2.] of the codescent diagram associated with TA—by forgetting about the highest face and degeneracy map and shifting everything up one dimension. Corresponding to the two décalage functors, the universal cocone (x, ϵ, M, U) from $X_\bullet^{TA}$ to X gives rise to two cocones from $X_\bullet^{T^2A}$ to X and an adjoint equivalence between them. The two cocones are given by $(xd_1, \epsilon d_2, Md_3, Ud_3)$ and $(xd_0, 1, 1, 1)$ respectively, and the adjoint equivalence is given by (ϵ, M). The adjoint equivalence that we describe explicitly below is exactly the adjoint equivalence of cocones obtained from (ϵ, M) under the functor

$$[(\Delta_3^-)^{\mathrm{op}}, \mathcal{G}ray](F_{\mathrm{codsc}}, \mathcal{K}^T(X_\bullet^{T^2A}, Y)) \to [(\Delta_3^-)^{\mathrm{op}}, \mathcal{G}ray](F_{\mathrm{codsc}}, \mathcal{K}^T(X_\bullet^{TA}, Y))\ ,$$

[4] In order to avoid complicating parentheses, we again employ the convention that the shorthand T for an instance of a hom functor only applies to the cell directly following it, cf. Remark 1.2 from 1.2.1.

which is induced by the transformation $(T\eta_A)_\bullet \colon X_\bullet^{TA} \Rightarrow X_\bullet^{T^2A}$ described above.

Namely, from the adjoint equivalence $\epsilon \colon xd_1 \to xd_0$ of the universal cocone, we can produce a 1-cell

$$\alpha = \epsilon T^2\eta_A \colon 1_X x \to gfx \tag{2.22}$$

as required in the second part of the universal property of the codescent object for $g_1 = 1_X$ and $g_2 = gf$. This 1-cell is evidently part of an adjoint equivalence—the image of the adjoint equivalence $(\epsilon, \epsilon^\bullet, \varepsilon, \eta)$ under the functor $\mathcal{K}(X, T^2\eta_A)$.

To apply Lemma 2.5, we only have to show that this adjoint equivalence underlies a 1-cell of cocones. That is, we have to provide an invertible 2-cell

$$\Gamma \colon (\alpha d_0) * (g_1\epsilon) \Rightarrow (g_2\epsilon) * (\alpha d_1) \tag{2.23}$$

as in the second part of the universal property and show that this satisfies the two axioms ibidem. To do so, we spell out domain and codomain of (2.23), which shows that $M(T^3\eta_A)$ provides an invertible 2-cell as in (2.23).

Namely, the domain is determined by the calculation below[5]:

$$\begin{aligned}
&(\alpha 1_{d_0}) * (1_{g_1}\epsilon) \\
=&(\epsilon 1_{T^2\eta_A d_0}) * \epsilon \\
&\text{(by def., functoriality of } \mathcal{M}_\mathcal{K}\text{, and } \mathcal{G}ray\text{-category axioms))} \\
=&(\epsilon 1_{d_0 T^3\eta_A}) * (\epsilon 1_{T^2\mu_A T^3\eta_A}) \\
&\text{(by naturality of } (T\eta_A)_\bullet \text{ and a monad axiom)} \\
=&(\epsilon 1_{d_0 T^3\eta_A}) * (\epsilon 1_{d_2 T^3\eta_A}) \\
&\text{(by def. of } X_\bullet \text{ for } TA\text{)} \\
=&((\epsilon 1_{d_0}) * (\epsilon 1_{d_2}))1_{T^3\eta_A} \\
&\text{(by functoriality of } \mathcal{M}_\mathcal{K} \text{ and } \mathcal{G}ray\text{-category structure)}
\end{aligned}$$

This calculation, in particular the step involving the monad axiom, can also be understood as a contraction of the codescent diagram associated with TA provided by the 1-cells $T^i\eta$ in the $\mathcal{G}ray$-category $\mathcal{K}$ as for usual augmented simplicial objects [72] .

Similarly the codomain is determined by the calculation below:

$$\begin{aligned}
&(1_{g_2}\epsilon) * (\alpha 1_{d_1}) \\
=&(1_g(1_f\epsilon)) * ((\epsilon 1_{T^2\eta_A})1_{d_1}) \\
&\text{(by definition and } \mathcal{G}ray\text{-category structure)} \\
=&(\epsilon 1_{T^2\eta_A})1_{d_1} \\
&\text{(by eq. (2.17))} \\
=&(\epsilon 1_{d_1})1_{T^3\eta_A} \\
&\text{(by naturality of } (T\eta_A)_\bullet \text{ and } \mathcal{G}ray\text{-category structure) .}
\end{aligned}$$

[5]where we have made identities explicit in order to improve readability

Thus an invertible 2-cell as required is given by

$$\Gamma = M1_{1_{T^3\eta_A}} : (\epsilon 1_{d_0} 1_{T^3\eta_A}) * (\epsilon 1_{d_2} 1_{T^3\eta A}) \Rightarrow \epsilon 1_{d_1} 1_{T^3\eta_A} .$$

We now have to show that this 2-cell satisfies the two axioms in the second part of the universal property of the codescent object.

Since by (2.17), we have $g_2 M = 1$, $g_2\epsilon = 1$, the first axiom (2.10) for Γ, in the present situation, has evidently the same shape as the first axiom (2.1) for a cocone (up to transposition of the left and right hand side of the axiom). Indeed it follows from (2.10) when whiskered with $T^4\eta_A$—i.e. when we consider its image under the functor $\mathcal{K}(X, T^4\eta_A)$. Explicitly, using the *Gray*-category structure of $\mathcal{K}$, the axiom for $\Gamma = M1_{1_{T^3\eta_A}}$ is given by

$$((M1_{1_{T^3\eta_A}})1_{1_{d_2}}) \diamond (((M1_{1_{T^3\eta_A}})1_{1_{d_0}}) * 1_{\epsilon 1_{d_2}}) = ((M1_{1_{T^3\eta_A}})1_{1_{d_1}}) \diamond (1_{\epsilon 1_{T^2\eta_A} 1_{d_0} 1_{d_0}} * M) . \quad (2.24)$$

On the other hand, the image of the axiom for M under whiskering i.e. under the functor $\mathcal{K}(X, T^4\eta_A)$ is similarly given by

$$(M1_{1_{d_2 T^4\eta_A}}) \diamond ((M1_{1_{d_0 T^4\eta_A}}) * 1_{\epsilon 1_{d_2} 1_{d_3} 1_{T^4\eta_A}}) = (M1_{1_{d_1 T^4\eta_A}}) \diamond (1_{\epsilon 1_{d_0} 1_{d_0} 1_{T^4\eta_A}} * (M1_{1_{d_3 T^4\eta_A}})) \quad (2.25)$$

where we exchanged the left and the right hand side. The axiom is thus evidently implied by naturality of $(T\eta_A)_\bullet$.

Similarly, since by (2.17), we have $g_2 U = 1$ and $g_2\epsilon = 1$, the second axiom (2.11) for Γ, in the present situation, has evidently the same shape as the second axiom (2.2) for a cocone (up to transposition of the left and right hand side of the axiom). Indeed it follows from (2.10) when we consider its image under the functor $\mathcal{K}(X, T^2\eta_A)$. Explicitly, using the *Gray*-category structure of $\mathcal{K}$, the axiom for $\Gamma = M1_{1_{T^3\eta_A}}$ is given by

$$1_{\epsilon 1_{T^2\eta}} * U = M1_{1_{T^3\eta_A s_0}} . \quad (2.26)$$

On the other hand, the image of the second axiom for a codescent object under the functor $\mathcal{K}(X, T^2\eta_A)$ is similarly given by

$$1_{\epsilon 1_{T^2\eta_A}} * (U1_{1_{d_1 T^2\eta_A}}) = M1_{1_{s_0 T^2\eta_A}} .$$

The axiom is thus evidently implied by naturality of $(T\eta_A)_\bullet$ and a monad axiom.

By the second part of the universal property of the codescent object, this means there exists a unique 1-cell $\tilde{\alpha}\colon 1_X \to gf$ such that $\tilde{\alpha} 1_x = \alpha$ and $\Sigma_{\tilde{\alpha},\epsilon} = \Gamma = M1_{T^3\eta_A}$. By Lemma 2.5 this 1-cell must be part of an adjoint equivalence $(\tilde{\alpha}, \tilde{\alpha}^\bullet, \tilde{\varepsilon}, \tilde{\eta})$.

□

Remark 2.12. To state Theorem 2.1 in other words, any free algebra is flexible [4, Rem. 4.5]. See [4, Cor. 5.6] for the corresponding result in two-dimensional monad theory.

Let T-Alg denote the full sub-*Gray*-category of Ps-T-Alg with objects (strict) algebras.

Lemma 2.8. *Suppose that the left adjoint L to the inclusion $i\colon \mathcal{K}^T \to T\text{-Alg}$ exists, for example if $\mathcal{K}$ has codescent objects of strict codescent diagram[6]. Let $F\colon \mathcal{K} \to \mathcal{K}^T$ be the free T-algebra functor.*

Given a $\mathcal{G}ray$-functor $G\colon \mathcal{M} \to \mathcal{K}$, the biequivalence in Theorem 2.1, for $A = GM$, forms the component at the object M in $\mathcal{M}$ of a $\mathcal{G}ray$-natural transformation

$$LiFG \cong FG \tag{2.27}$$

that is part of a biequivalence in $[\mathcal{M}, \mathcal{K}^T]$.

Proof. Since the 1-cell $f\colon X \to TA$ in the biequivalence $(f, g, 1_{fg}, (\tilde{\alpha}, \tilde{\alpha}^\bullet, \tilde{\varepsilon}, \tilde{\eta}))$ is in fact the component ε_{TA} of the counit of the adjunction $L \dashv i$, we indeed have a $\mathcal{G}ray$-natural transformation εFG as in (2.27). Since the forgetful functor

$$U\colon [\mathcal{M}, \mathcal{K}^T] \to \mathcal{K}^T \tag{2.28}$$

does in general not reflect internal biequivalences, we have to prove that this is indeed part of a biequivalence as required.

The 1-cell $g = xT\eta_A\colon TA \to X$ in the biequivalence is natural in A: Clearly $T\eta_A$ is natural in A, and $x\colon T^2A \to X$ is natural in A because this is a component of the unit of the colimit of the codescent object. The naturality of the representation isomorphism is equivalent to that of the unit i.e. the universal cocone

$$\nu\colon F_{\mathrm{codsc}} \Rightarrow \mathcal{K}(X^{TA}_\bullet -, X^{TA})$$

meaning that each $\nu_j\colon \mathrm{st}I_j \to \mathcal{K}(T^{j+2}A, X^{TA})$ is extraordinarily $\mathcal{G}ray$-natural in A (in fact in TA). By definition of the functor $\mathcal{G}ray$-category, this means each ν_j factorizes through the hom object of the functor $\mathcal{G}ray$-category. In particular, for $j = 0$, we see that x must be $\mathcal{G}ray$-natural in A (in fact in TA). For $j = 1$, we see that the adjoint equivalence $(\epsilon, \epsilon^\bullet, \varepsilon, \eta)$ underlies an adjoint equivalence in the hom object of the functor $\mathcal{G}ray$-category and that U underlies an invertible 2-cell in the hom object of the functor $\mathcal{G}ray$-category. For $j = 2$, we see that M underlies an invertible 2-cell in the hom object of the functor $\mathcal{G}ray$-category.

This implies that the cells involved in the construction of the adjoint equivalence $(\tilde{\alpha}, \tilde{\alpha}^\bullet, \tilde{\varepsilon}, \tilde{\eta})$ in $\mathcal{K}(X^{TA}, X^T)$ (in particular $\alpha = \epsilon 1_{T^2\eta_A}$ and $\Gamma = M1_{1_{T^4\eta_A}}$ where $T^2\eta$ and $T^4\eta$ are natural in A) via the universal property all underlie cells in the functor $\mathcal{G}ray$-category. It is then not hard to see that $(\tilde{\alpha}, \tilde{\alpha}^\bullet, \tilde{\varepsilon}, \tilde{\eta})$ in $\mathcal{K}(X^{TA}, X^{TA})$ underlies an adjoint equivalence in the functor $\mathcal{G}ray$-category $[\mathcal{K}, \mathcal{K}^T]$ (in fact in $[\mathcal{K}^T, \mathcal{K}^T]$).

For example, for $\tilde{\alpha}$, we have to prove that

$$\tilde{\alpha}_B 1_{(F_{\mathrm{codsc}}*(\overline{FU}_\bullet)_3^-)(Th)} = 1_{(F_{\mathrm{codsc}}*(\overline{FU}_\bullet)_3^-)(Th)} \tilde{\alpha}_A$$

for a 1-cell $h\colon A \to B$ in $\mathcal{K}$ cf. (1.62) from 1.5.2. By naturality of the representation isomorphism the left hand side is sent to $(\epsilon_B 1_{T^2\eta_B})1_{T^2h} = (\epsilon_B 1_{T^3h})1_{T^2\eta_A}$, where we used naturality of η and $\mathcal{G}ray$-category structure, while the right hand side is sent to $1_{(F_{\mathrm{codsc}}*(\overline{FU}_\bullet)_3^-)(Th)}(\epsilon_A 1_{T^2\eta_A})$. Hence, this is exactly implied by ϵ being a 2-cell in the functor $\mathcal{G}ray$-category. Similarly, one sees that the condition on interchange cells cf. (1.63) from 1.5.2 is satisfied. □

[6]The corresponding result in the present context cf. Theorem 1.3 in Chapter 1.

2.2 The Yoneda lemma for *Gray*-categories and locally strict trihomomorphisms

2.2.1 Representables as free algebras

The following elementary observation proves to be crucial in what follows.

Lemma 2.9. *Given a Gray-category $\mathcal{P}$ and an object P, the representable $\mathcal{P}(P,-)$ is isomorphic to the free algebra $T((\mathrm{ob}\mathcal{P})(P,-))$ in the functor Gray-category $[\mathcal{P}, \mathcal{G}ray]$, where T is the monad of the Kan adjunction described in Chapter 1.*

Proof. The representable $(\mathrm{ob}\mathcal{P}')(P,-)\colon \mathrm{ob}\mathcal{P}' \to \mathcal{G}ray$ sends Q to the initial i.e. empty 2-category $\emptyset$ if $Q \neq P$ and to the unit object i.e. the unit 2-category I with one object $*$, a single identity 1-cell 1_*, and a single identity 2-cell 1_{1_*} if $Q = P$.

Indeed then, we have the following isomorphism on objects

$$(T((\mathrm{ob}\mathcal{P})(P,-)))R = \sum_{Q} \mathcal{P}(Q,R) \otimes (\mathrm{ob}\mathcal{P})(P,Q) = \mathcal{P}(P,R) \otimes I \cong \mathcal{P}(P,R)\ , \tag{2.29}$$

and the hom morphism is induced from the monad multiplication i.e. the composition law of $\mathcal{P}'$ and so coincides with the one for $\mathcal{P}'(P,-)$ up to the isomorphism (2.29). Thus these functors are isomorphic in $[\mathcal{P}', \mathcal{G}ray]$. □

Recall that if $\mathcal{P}$ is small, there exists a left adjoint $L\colon \mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}, \mathcal{G}ray) \to [\mathcal{P}, \mathcal{G}ray]$ to the inclusion, given on objects by the codescent object of the associated codescent diagram cf. Theorem 1.8 from 1.5.3.

The representable $\mathcal{P}(P,-)$ is the value of the Yoneda embedding

$$Y\colon \mathcal{P}^{\mathrm{op}} \to [\mathcal{P}, \mathcal{G}ray] = [\mathrm{ob}\mathcal{P}, \mathcal{G}ray]^T \tag{2.30}$$

at the object P. However, it does not factorize through the free algebra functor $F\colon [\mathrm{ob}\mathcal{P}, \mathcal{G}ray] \to [\mathrm{ob}\mathcal{P}, \mathcal{G}ray]^T$. Thus, while $L\mathcal{P}(P,-)$ is strictly biequivalent to $\mathcal{P}(P,-)$ by Theorem 2.1, we cannot use Lemma 2.8 to show that this strict biequivalence underlies a biequivalence in the functor $\mathcal{G}ray$-category $[\mathcal{P}^{\mathrm{op}}, [\mathcal{P}, \mathcal{G}ray]]$.

The proof of Lemma 2.8 fails in this case because while $T\eta_A\colon TA \to T^2A$ is natural in A, it is in general not natural in TA. More precisely, given a functor $G\colon \mathcal{M} \to \mathcal{K}^T$ such that for every object $M \in \mathcal{M}$, GM is a free algebra TA for some object $A \in \mathcal{K}$, then $T\eta_A$ gives rise to a family $\beta_M\colon GM \to TGM$, which is not necessarily natural in M. However, this is the case for the Yoneda embedding:

Lemma 2.10. *Let Y be the Yoneda embedding* (2.30)*, then the family*

$$T\eta_{(\mathrm{ob}\mathcal{P})(P,-)}\colon YP \to TYP \tag{2.31}$$

is $\mathcal{G}ray$-natural in P.

Proof. First note that (2.31) is $\mathcal{G}ray$-natural in P if and only if

$$E_{P'}T\eta_{\mathrm{ob}(F,-)}\colon\ \mathcal{P}(P,P') \to (T\mathcal{P}(P,-))(P') = \Sigma_R \mathcal{P}(R,P') \otimes \mathcal{P}(P,R) \tag{2.32}$$

is $\mathcal{G}ray$-natural in P. Yet (2.32) is given by $\kappa_P(1 \otimes j_P)r^{-1}$, where κ_P is the inclusion into the coproduct. Clearly, the unitor r^{-1} is $\mathcal{G}ray$-natural in P, and it follows from the extraordinary naturality of κ_P and $1 \otimes j_P$ and the composition calculus that $\kappa_P(1 \otimes j_P)$ is ordinarily $\mathcal{G}ray$-natural in P. □

Corollary 2.3. *Let Y be the Yoneda embedding* (2.30)*, then the families*

$$T^j\eta_{(\mathrm{ob}\mathcal{P})(P,-)}\colon\ T^{j-1}YP \to T^jYP$$

are $\mathcal{G}ray$-natural in P for all $j > 1$.

Proof. Immediate from Lemma 2.10 and the 2-category structure of $\mathcal{G}ray$-CAT. □

Using this Corollary, one easily modifies the proof of Lemma 2.8 to prove the following result.

Proposition 2.4. *Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category and Y be the Yoneda embedding* (2.30)*, then the biequivalence from Theorem 2.1, for $A = (\mathrm{ob}\mathcal{P})(P,-)$, underlies a biequivalence*

$$LiY \Rightarrow Y$$

in the functor $\mathcal{G}ray$-category $[\mathcal{P}^{\mathrm{op}}, [\mathcal{P}, \mathcal{G}ray]]$. □

2.2.2 The $\mathcal{G}ray$-enriched Yoneda lemma.

Let $\mathcal{P}$ be a small $\mathcal{G}ray$-category. Recall that taking components at an object P has the form of an evaluation $\mathcal{G}ray$-functor $E_P\colon [\mathcal{P}, \mathcal{G}ray] \to \mathcal{G}ray$. In particular, given a $\mathcal{G}ray$-functor $A\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$ and an object P in $\mathcal{P}$, we have a 2-functor

$$(E_P)_{\mathcal{P}(-,P),A}\colon\ [\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](\mathcal{P}(-,P), A-) \to [\mathcal{P}(P,P), AP]$$

and precomposition $[j_p, 1]$ with the unit $j_P\colon I \to \mathcal{P}(P,P)$ of $\mathcal{P}$ at P and the isomorphism

$$[I, AP] \cong AP$$

from the closed structure of $\mathcal{G}ray$, give rise to a 2-functor

$$[\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](\mathcal{P}(-,P), A-) \to AP\ . \tag{2.33}$$

Theorem 2.2. *Given a small $\mathcal{G}ray$-category $\mathcal{P}$ and a $\mathcal{G}ray$-functor $A\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$ evaluation of the component at P at the identity* (2.33) *induces an isomorphism*

$$[\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](\mathcal{P}(-,P), A-) \cong AP\ ,$$

which is $\mathcal{G}ray$-natural in P and A.

Proof. See Kelly [38, 2.4, p. 33f.]. □

Remark 2.13. The functor $\mathcal{Gray}$-category is strictly monadic over $[\mathrm{ob}\mathcal{P}, \mathcal{Gray}]$ cf. Theorem 1.4 from 1.4. In particular, the forgetful functor $[\mathcal{P}, \mathcal{Gray}] \to [\mathrm{ob}\mathcal{P}, \mathcal{Gray}]$ given by precomposition with the inclusion is a $\mathcal{Gray}$-functor. Precomposition with the $\mathcal{Gray}$-functor $P\colon \mathcal{I} \to \mathrm{ob}\mathcal{P}^{\mathrm{op}}$ (sending the unique object in $\mathcal{I}$ to P)

$$[P, 1]\colon [\mathrm{ob}\mathcal{P}^{\mathrm{op}}\, \mathcal{Gray}] \to [\mathcal{I}, \mathcal{Gray}] \ ,$$

and the isomorphism

$$[\mathcal{I}, \mathcal{Gray}] \cong \mathcal{Gray}$$

give exactly rise to the evaluation $\mathcal{Gray}$-functor E_P.

2.2.3 The Yoneda lemma for $\mathcal{Gray}$-categories and locally strict trihomomorphisms

Theorem 2.3 (The locally strict Yoneda lemma)**.** *Given a small $\mathcal{Gray}$-category $\mathcal{P}$ and a locally strict trihomomorphism $A\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{Gray}$, evaluation of the component at P at the identity induces a biequivalence*

$$\mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{Gray})(i\mathcal{P}(-, P), A-) \cong AP \ , \tag{2.34}$$

which is natural in P and A meaning that these biequivalences form the components of a biequivalence

$$\mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{Gray})(-, A)i^{\mathrm{op}}Y^{\mathrm{op}} \Rightarrow A \tag{2.35}$$

in $\mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{Gray})$ where $Y\colon \mathcal{P} \to [\mathcal{P}^{\mathrm{op}}, \mathcal{Gray}]$ is the Yoneda embedding.

Similarly, the biequivalences (2.35) *form the components of a biequivalence*

$$\mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{Gray})(i\mathcal{P}(-, P), ?) \Rightarrow E_P \tag{2.36}$$

of $\mathcal{Gray}$-functors in $[\mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{Gray}), \mathcal{Gray}]$ where E_P is evaluation at P.

Remark 2.14. Notice that just as for the functor $\mathcal{Gray}$-category in Remark 2.13

$$E_P\colon \mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{Gray}) \to \mathcal{Gray}$$

is the composition of the forgetful functor

$$U\colon \mathcal{Tricat}_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{Gray}) \to [\mathrm{ob}\mathcal{P}^{\mathrm{op}}, \mathcal{Gray}] \ ,$$

precomposition with the $\mathcal{Gray}$-functor $P\colon \mathcal{I} \to \mathrm{ob}\mathcal{P}^{\mathrm{op}}$ (sending the unique object in $\mathcal{I}$ to P)

$$[P, 1]\colon [\mathrm{ob}\mathcal{P}^{\mathrm{op}}\, \mathcal{Gray}] \to [\mathcal{I}, \mathcal{Gray}] \ ,$$

and the isomorphism

$$[\mathcal{I}, \mathcal{Gray}] \cong \mathcal{Gray}$$

given by the partial closedness of $\mathcal{Gray}$-$\mathbf{Cat}$ c.f. [38, (2.29)]. It is thus clearly a $\mathcal{Gray}$-functor.

Proof. The existence of a biequivalence as in (2.35) is a direct consequence of the two assertions of Theorem 1.8 from 1.5.3; the fact that the representable $\mathcal{P}(-,P)$ is a free algebra c.f. Lemma 2.9 and that the left adjoint L preserves free algebras up to strict biequivalence cf. Theorem 2.1; and the $\mathcal{G}ray$-enriched Yoneda lemma:

$$\begin{aligned}
&\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(i\mathcal{P}(-,P), A-)\\
\cong&\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(i\mathcal{P}(-,P), iLA-)\\
\cong&[\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](Li\mathcal{P}(-,P), LA-)\\
\cong&[\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](\mathcal{P}(-,P), LA-)\\
\cong&LAP\\
\cong&AP\ .
\end{aligned}$$

The first biequivalence is trivially natural in P and $\mathcal{G}ray$-natural in A because so is the unit η of the adjunction $L \dashv i$—indeed the unit transformation η_A gives rise to a transformation $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(i\mathcal{P}(-,P), ?)\eta$. The second isomorphism is $\mathcal{G}ray$-natural in A and P because the hom adjunction for $L \dashv i$ is natural in both variables (and then this is just precomposition with the $\mathcal{G}ray$-functors given by the Yoneda embedding $Y\colon \mathcal{P} \to [\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray]$ and L). The third biequivalence is trivially $\mathcal{G}ray$-natural in A and natural in P because of Proposition 2.4. The fourth isomorphism is the $\mathcal{G}ray$-enriched Yoneda Lemma and thus $\mathcal{G}ray$-natural in P and A. Finally, the last biequivalence is given by the component at P oft the component at A of the biequivalence-inverse of the unit η and thus is natural in P and $\mathcal{G}ray$-natural in A.

That a biequivalence as in (2.35) is induced from evaluation as claimed follows from the lemma below. □

Lemma 2.11. *The chain of biequivalences appearing in the proof of Theorem 2.3 is equivalent to the component of the tritransformation*

$$\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(-,A)i^{\mathrm{op}}Y^{\mathrm{op}} \Rightarrow A$$

induced by taking the component at P (of a tritransformation, a trimodification, or perturbation) and

- *evaluating the functor $\mathcal{P}(P,P) \to AP$ at 1_P in the case of a tritransformation;*
- *taking the component at 1_P of the pseudonatural transformation of functors $\mathcal{P}(P,P) \to AP$ in the case of a trimodification;*
- *taking the component at 1_P of the modification of pseudonatural transformations of functors $\mathcal{P}(P,P) \to AP$ at 1_P in the case of a perturbation.*

Proof. We know that the $\mathcal{G}ray$-enriched Yoneda lemma is induced by evaluation of the component at P at the identity cf. (2.33), so the question is how this is modified by the remaining biequivalences. The hom adjunction of $L \dashv i$ is given by applying the hom functor of L and postcomposing with the counit of the adjunction i.e. the 1-cell $f\colon LiLA \to LA$) or $f\colon X^{LA} \to LA$ constructed above. The next biequivalence is given

by precomposition with the unit of the adjunction i.e. the 1-cell $g = xT\eta\colon TA \to X$ constructed above, but x is natural with respect to (pseudo) algebra cells in the sense that

$$[\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](x, LA)L_{i\mathcal{P}(-,P),iLA} = [\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](T\mathcal{P}(-, P), x)FU_{i\mathcal{P}(-,P),iLA}$$

(by the same argument as in the proof of Lemma 2.8, namely extraordinary naturality of the unit of the codescent colimit) where U is the forgetful functor and F is the free algebra functor.

Hence, since $f^{LA}x^{LA} = a^{LA}$, we have

$$\begin{aligned}
&[\mathrm{ob}\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](T\eta_{(\mathrm{ob}\mathcal{P})(-,P)}\eta_{(\mathrm{ob}\mathcal{P})(-,P)}, a^{LA})TU_{i\mathcal{P}(-,P),iLA}\\
=&[\mathrm{ob}\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](\eta_{\mathcal{P}(-,P)}\eta_{(\mathrm{ob}\mathcal{P})(-,P)}, a^{LA})TU_{i\mathcal{P}(-,P),iLA}\\
=&[\mathrm{ob}\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](\eta_{(\mathrm{ob}\mathcal{P})(-,P)}, a^{LA}\eta_{LA})U_{i\mathcal{P}(-,P),iLA}\\
=&[\mathrm{ob}\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray](\eta_{(\mathrm{ob}\mathcal{P})(-,P)}, 1)U_{i\mathcal{P}(-,P),iLA}
\end{aligned}$$

by a monad axiom and naturality of the unit of the monad, and where we have used that LA is a strict algebra in the last equation. We have injected an additional $\eta_{(\mathrm{ob}\mathcal{P})(-,P)}$, which we may do because evaluation of the component at 1_P factorizes through precomposition with $\eta_{(\mathrm{ob}\mathcal{P})(-,P)}$.

Since the last biequivalence is given by composition with f^A and since the first biequivalence is given by postcomposition with the unit $x^A\eta_A$ of the adjunction, in total, the chain of biequivalences appearing in the proof of Theorem 2.3 is induced by taking the component at P, whiskering with

$$f^A x^A \eta_A = a^A \eta_A \tag{2.37}$$

and evaluating at 1_P.

Yet whiskering with the adjoint equivalence $\iota^A\colon 1 \to a\eta_A$ of the pseudo algebra A cf. Def. 1.1 from 1.2.1 gives rise to an adjoint equivalence of the functors in question, and thus the functor induced by evaluation as required must be a biequivalence too. □

Corollary 2.4 (Strictification via Yoneda)**.** *Given a small $\mathcal{G}ray$-category $\mathcal{P}$ and a locally strict trihomomorphism $A\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$, the trihomomorphism appearing in the Yoneda lemma,*

$$\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(\mathcal{P}(-, ?), A-)\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray\ , \tag{2.38}$$

is a strictification of A i.e. a $\mathcal{G}ray$-functor biequivalent to A.

Proof. The trihomomorphism (2.38) is defined by the composition

$$\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(-, A)i^{\mathrm{op}}Y^{\mathrm{op}} \tag{2.39}$$

of the Yoneda embedding Y, the inclusion $i\colon [\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray] \to \mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)$ and a contravariant representable of the $\mathcal{G}ray$-category $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)$. All of these functors are $\mathcal{G}ray$-functors and thus so is their composition. That it is biequivalent to A is literally Theorem 2.3. □

Remark 2.15. According to Theorem 2.3, the $\mathcal{G}ray$-functor (2.39) is biequivalent to A in $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)$. At the same time, this $\mathcal{G}ray$-functor can be worked out explicitly for a given A. In contrast to Theorem 1.8 from 1.5.3, the Yoneda lemma hence provides an explicit strictification of a locally strict trihomomorphism $\mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$. One might wonder, how these two strictifications relate to each other. The following example shows that the strictification provided by the Yoneda lemma is in general very different from a codescent object.

Example 2.1. Let $*\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$ be the constant $\mathcal{G}ray$-functor at the unit 2-category I on a small $\mathcal{G}ray$-category $\mathcal{P}$. Then it is not hard to see that $*$ coincides with its strictification (2.38): First, on an object $\mathcal{P}$, all the components $\theta_Q\colon \mathcal{P}(Q,P) \to I$ of a tritransformation $\theta\colon \mathcal{P}(-,P) \Rightarrow *$, which have to be 1-cells in $\mathcal{G}ray$ i.e. strict functors, are determined to be the terminal morphisms. Since the internal hom

$$[X, I] \tag{2.40}$$

is isomorphic to I for any 2-category X, all the adjoint equivalences of the tritransformation θ must be the single pseudonatural transformation in such an internal hom—the identity transformation: First, the 0-cell target of these adjoint equivalences is given by such an internal hom (2.40), and second, this implies that these are adjoint equivalences in such an internal hom (2.40). Finally, the 0-cell target of the modifications Π and M for the tritransformation θ is given by an internal hom as in (2.40), and thus these are themselves modifications in such an internal hom (2.40), yet there is again only one such modification—the identity modification. It follows that the $\mathcal{G}ray$-functor (2.38) is given by I on objects for $A = *$, but since $[I, I] \cong I$, such a $\mathcal{G}ray$-functor must be the constant $\mathcal{G}ray$-functor $*$ itself.

On the other hand, $*$ does not have the universal property of the codescent object $L(*)$: Given a $\mathcal{G}ray$-functor $B\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$, a $\mathcal{G}ray$-natural transformation $\theta\colon * \Rightarrow B$ is given by an object $\theta_P \in BP$ for every $P \in \mathcal{P}$ such that

$$B_{QP}(f)(\theta_Q) = \theta_P \tag{2.41}$$

for every object $f \in \mathcal{P}(P,Q)$ with similar conditions for 1- and 2-cells in $\mathcal{P}(P,Q)$.

Now let $\mathcal{P}$ be a category $\mathcal{C}$ and B be a $\mathcal{G}ray$-functor such that BP is a one-object 2-category for every $P \in \mathrm{ob}\mathcal{C}$. The equation (2.41) is then automatically satisfied, and so there is exactly one $\mathcal{G}ray$-natural transformation $* \Rightarrow B$. This means the 2-category

$$[\mathcal{C}^{\mathrm{op}}, \mathcal{G}ray](*, B) \tag{2.42}$$

has exactly one object. However, the 2-category

$$\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{C}^{\mathrm{op}}, \mathcal{G}ray)(*, B) \tag{2.43}$$

is in general not isomorphic to a one-object 2-category. An example for this situation is given by bundle gerbes, which can be realized as tritransformations as in (2.43) where $\mathcal{C} = \Delta_\delta$ is the subcategory of the simplex category Δ spanned by the coface maps cf. Corollary

3.1 in Chapter 3. For simplicity, we restrict here to bundle gerbes without connection constructed by descent from principal $U(1)$-bundles in terms of transition functions and open coverings. In this case, B is a $\mathcal{G}ray$-functor assigning strict monoidal categories i.e. one-object 2-categories to any object in Δ_δ: Given a manifold M and a surjective submersion $Y \to M$, it assigns to $[n]$ the one-object 2-category of principal $U(1)$-bundles over the $(n+1)$-fold fiber product of Y over M with itself. While (2.42) has only one object, there are in general many bundle gerbes over a manifold M: The equivalence classes of these are classified by the third integral cohomology group $H^3(M;\mathbf{Z})$. For any connected, compact, simple Lie group $M = G$ this cohomology group is isomorphic to the integers $\mathbf{Z}$, which gives rise to a counterexample.

In the next three sections 2.3-2.5 we collect the remaining ingredients to prove a Yoneda lemma for tricategories. In Section 2.3, we prove an invariance result about trihomomorphisms called change of local functors. Change of local functors is much like the Transport of Structure, Change of Composition, and Change of Units Theorems mediating the invariance of tricategories cf. [27, Th. 7.22-7.24] and [21, 3.6-3.8]. Change of local functors enters in the proof of the Yoneda lemma in order to pass to locally strict trihomomorphisms. We can then appeal to the Yoneda lemma for $\mathcal{G}ray$-categories and locally strict trihomomorphisms from 2.2. This depends on replacing the domain tricategory $\mathcal{P}$ by the $\mathcal{G}ray$-category $\mathcal{P}'$ obtained from the local strictification $\mathrm{st}\mathcal{P}$, a cubical tricategory, and its image under the cubical Yoneda embedding—this is the original form of the coherence theorem for tricategories Gordon et al. [21, Theorem 8.1] cf. Gurski [27, Cor. 9.15]. It is crucial for our argument that the cubical strictification $\mathrm{st}\mathcal{P}$ is locally given by the strictification $\mathrm{st}\mathcal{P}(P,Q)$ of the hom bicategories $\mathcal{P}(P,Q)$ of the tricategory $\mathcal{P}$. Recalling that the strictification st is left adjoint to the inclusion in particular enables us to strictify the local functors of any trihomomorphism with a cubical target. We can then apply change of local functors to obtain a locally strict trihomomorphism. It is also important for the argument that the local functors of the cubical Yoneda embedding y_{PQ} and their triequivalence-inverses w_{PQ} are strict cf. Gurski [27, Prop. 9.14]. From the point of view that these correspond to the Yoneda embedding for enrichment in the monoidal tricategory $\mathcal{G}ray$ this might come as no surprise. Finally, we need to compare representables under the various replacements that mediate the transition to the $\mathcal{G}ray$-enriched context. This is the content of Sections 2.4 and 2.5.

2.3 Invariance of trihomomorphisms under change of local functors

In the case that domain $\mathcal{P}$ and target $\mathcal{L}$ are $\mathcal{G}ray$-categories, many cells in the definition of a trihomomorphism, tritransformation, trimodification, and perturbation cf. [27, 4.3] are identity cells. However, if the trihomomorphisms considered are not locally strict, there appear images of 2-cell identities with respect to the local nonstrict functors. These are isomorphic to identities via the corresponding unit constraints. Further, the naturality cells of the adjoint equivalences involved are not identities, but are rather given by unit constraints for the corresponding local functors. The images of the invert-

ible modifications of the domain tricategory—a *Gray*-category—are identities, but there remain composition constraints for the local functors which are implicit in the diagrams.

In fact, if these 2-cells are replaced by identities and if the 3-cells are simply discarded, a careful analysis shows that this gives valid 'reduced' notions of the tricategorical constructions, which form a *Gray*-category $\mathcal{T}ricat_r(\mathcal{P},\mathcal{L})$ that is isomorphic as a *Gray*-category to $\mathcal{T}ricat(\mathcal{P},\mathcal{L})$:

Theorem 2.4. *Given Gray-categories $\mathcal{P}$ and $\mathcal{L}$, the Gray-category $\mathcal{T}ricat(\mathcal{P},\mathcal{L})$ is isomorphic as a Gray-category to the Gray-category $\mathcal{T}ricat_r(\mathcal{P},\mathcal{L})$ of reduced trihomomorphisms.* □

In particular, the diagrams for the reduced versions contain no constraints of the local functors.

For the following theorem, recall that adjunctions in a 2-category form a category, and that a strict functor of 2-categories induces a functor on the categories of adjunctions.

Theorem 2.5 (Change of local functors)**.** *Let $A\colon \mathcal{P}\to\mathcal{L}$ be a reduced trihomomorphism of Gray-categories, let $(\tilde{A}_{PQ})_{P,Q\in\mathrm{ob}\mathcal{P}}$ be a family of functors*

$$\tilde{A}_{PQ}\colon\ \mathcal{P}(P,Q)\to\mathcal{L}(AP,AQ)\ ,$$

and let $(\sigma_{PQ},\sigma^{\bullet}_{PQ},E_{PQ},H_{PQ})_{P,Q\in\mathrm{ob}\mathcal{P}}$ be a family of adjoint equivalences

$$(\sigma_{PQ},\sigma^{\bullet}_{PQ},E_{PQ},H_{PQ})\colon\ \tilde{A}_{PQ}\to A_{PQ}\ .$$

Then $(\tilde{A}_{PQ})_{P,Q\in\mathrm{ob}\mathcal{P}}$ has the structure of a a reduced trihomomorphism $\tilde{A}\colon\ \mathcal{P}\to\mathcal{L}$ that agrees with A on objects such that $\tilde{A}$ is biequivalent to A in $\mathcal{T}ricat_r(\mathcal{P},\mathcal{L})$.

Proof. The remaining data of the trihomomorphism $\tilde{A}$ is defined as follows:

- for objects $P,Q,R\in\mathcal{P}$, the adjoint equivalence

  $$(\tilde{\chi}_{PQR},\tilde{\chi}^{\bullet}_{PQR},\tilde{E}_{PQR},\tilde{H}_{PQR})\colon\ M_{\mathcal{L}}C(\tilde{A}_{QR}\times\tilde{A}_{PQ})\Rightarrow\tilde{A}_{PR}M_{\mathcal{P}}C$$

  (where C is the universal cubical functor cf. 1.1.4) is defined to be the composite adjoint equivalence

  $$\begin{aligned}&(M_{\mathcal{P}}C)^{*}(\sigma^{\bullet}_{PR},\sigma_{PR},H^{-1}_{PR},,E^{-1}_{PR})\\ *\ \ &(\chi_{PQR},\chi^{\bullet}_{PQR},E_{PQR},H_{PQR})\\ *\ \ &(M_{\mathcal{L}}C)_{*}\times\Big((\sigma_{QR},\sigma^{\bullet}_{QR},E_{QR},H_{QR}),(\sigma_{PQ},\sigma^{\bullet}_{PQ},E_{PQ},H_{PQ})\Big)\ .\end{aligned}$$

  In particular, the component at composable 1-cells f,g in $\mathcal{P}$ of the left adjoint is given by

  $$\sigma^{\bullet}_{fg}*\chi_{f,g}*(Af\sigma_{g})*(\sigma_{f}\tilde{A}g)\ .$$

- for an object $P \in \mathcal{P}$, the adjoint equivalence

$$(\tilde{\iota}_P, \tilde{\iota}^\bullet_P, \tilde{E}_P, \tilde{H}_P) \colon\ j_{AP} \Rightarrow \tilde{A}_{PP} j_P$$

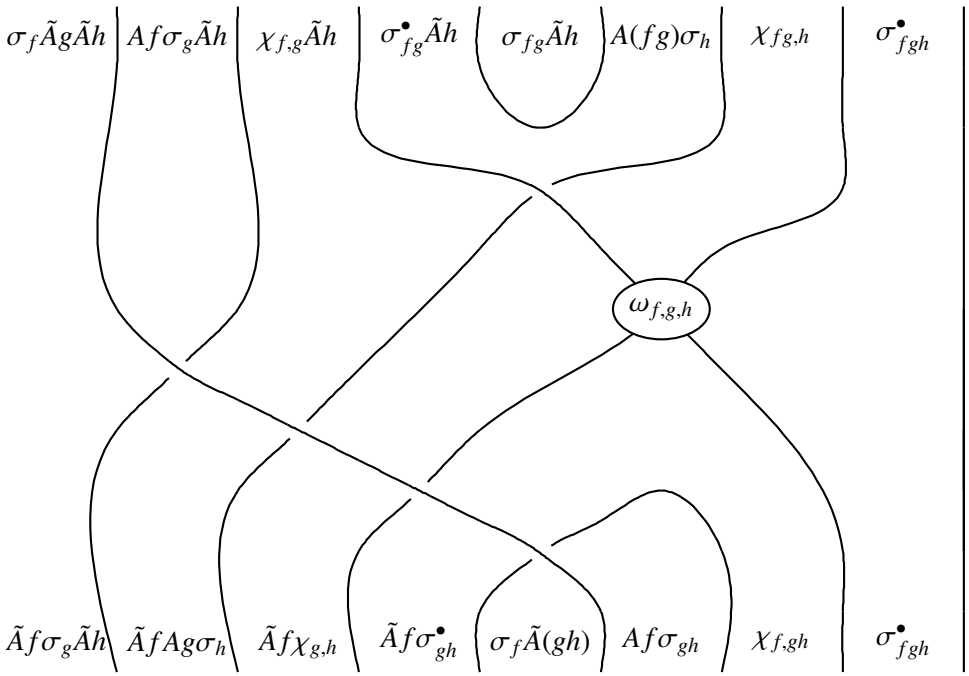
  is defined to be the composite adjoint equivalence

$$[j_P, 1](\sigma^\bullet_{PP}, \sigma_{PP}, H^{-1}_{PP}, E^{-1}_{PP}) * (\iota_P, \iota^\bullet_P, E_P, H_P) \ .$$

  In particular, the single component of the left adjoint is given by

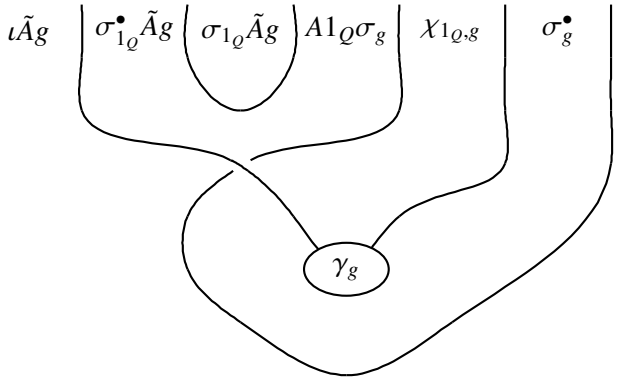
$$\sigma^\bullet_{1_P} * \iota_P \ .$$

- The modification $\tilde{\omega}$ is defined to be the modification with component at composable 1-cells f, g, h in $\mathcal{P}$ given by the following string diagram.

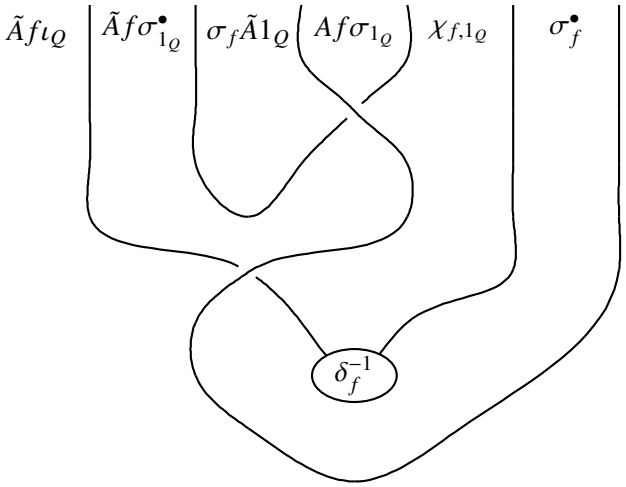


- The modification $\tilde{\gamma}$ is defined to be the modification with component at a 1-cell

$g\colon P \to Q$ in $\mathcal{P}$ given by the following string diagram.



- The inverse of the modification $\tilde{\delta}$ is defined to be the modification with component at a 1-cell $f\colon Q \to R$ in $\mathcal{P}$ given by the following string diagram.



A calculation shows that this data satisfies the two trihomomorphism axioms. The calculation is considerably simplified if performed on components with the help of string diagrams cf. Figures 2.1 and 2.2. The basic idea is to bring the diagrams into an equivalent form where one can apply the corresponding axiom for the trihomomorphism A.

Next tritransformations $\sigma\colon \tilde{A} \Rightarrow A$ and $\sigma^{\bullet}\colon A \Rightarrow \tilde{A}$ with identity components are defined by use of the adjoint equivalences $(\sigma_{PQ}, \sigma^{\bullet}_{PQ}, E_{PQ}, H_{PQ})$ and $(\sigma^{\bullet}_{PQ}, \sigma_{PQ}, H^{-1}_{PQ}, , E^{-1}_{PQ})$. That is, the components of σ and $\sigma_{\bullet}$ at an object P in $\mathcal{P}$ are both given by the identity 1-cell 1_{AP} at AP, and the components at a 1-cell $f\colon Q \to R$ in $\mathcal{P}$ of the left adjoints of the adjoint equivalences appearing in the definition of a tritransformation are given by σ_f and $\sigma^{\bullet}_f$ respectively. The invertible modifications of these tritransformations are defined as follows:

Figure 2.1.: First trihomomorphism axiom

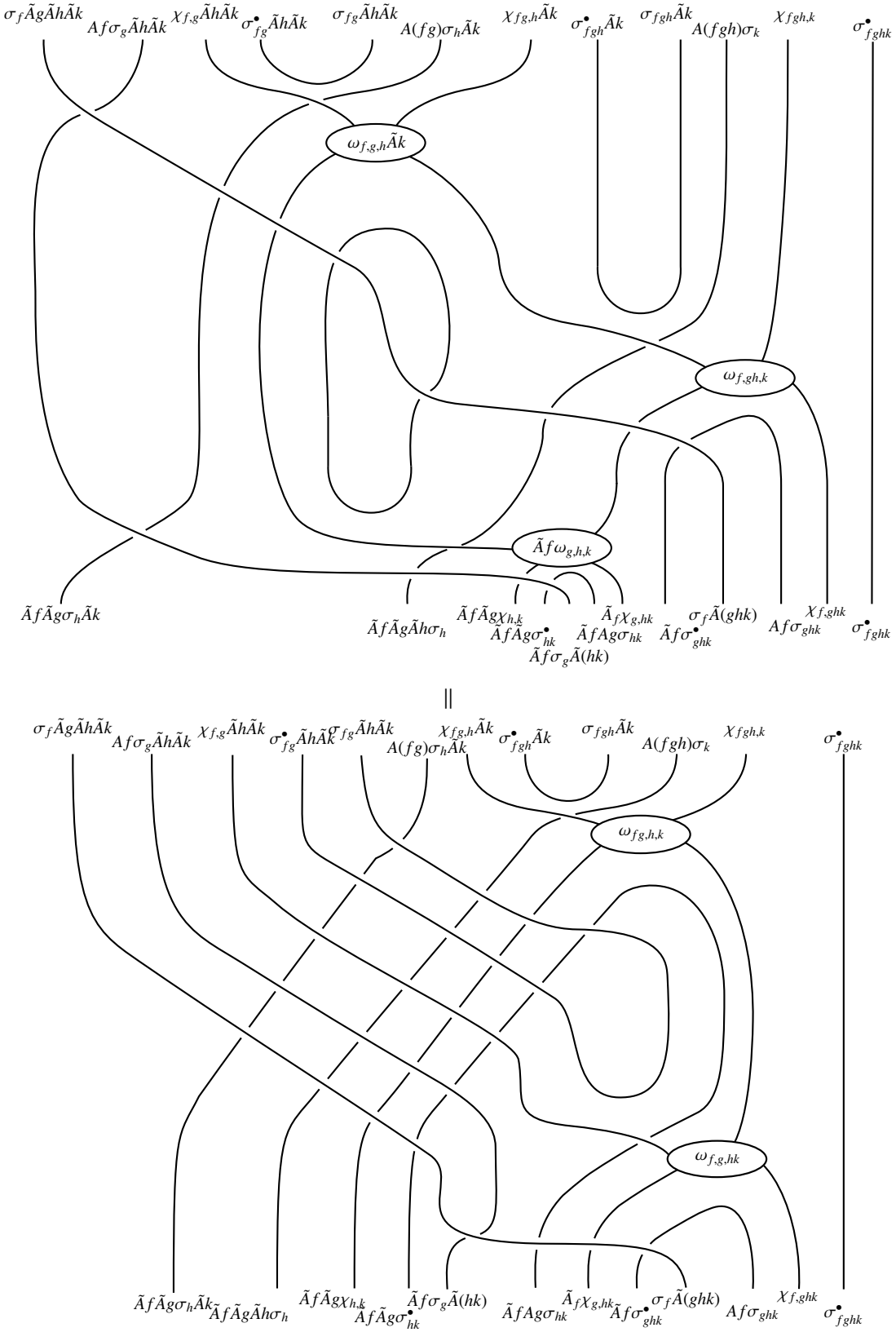
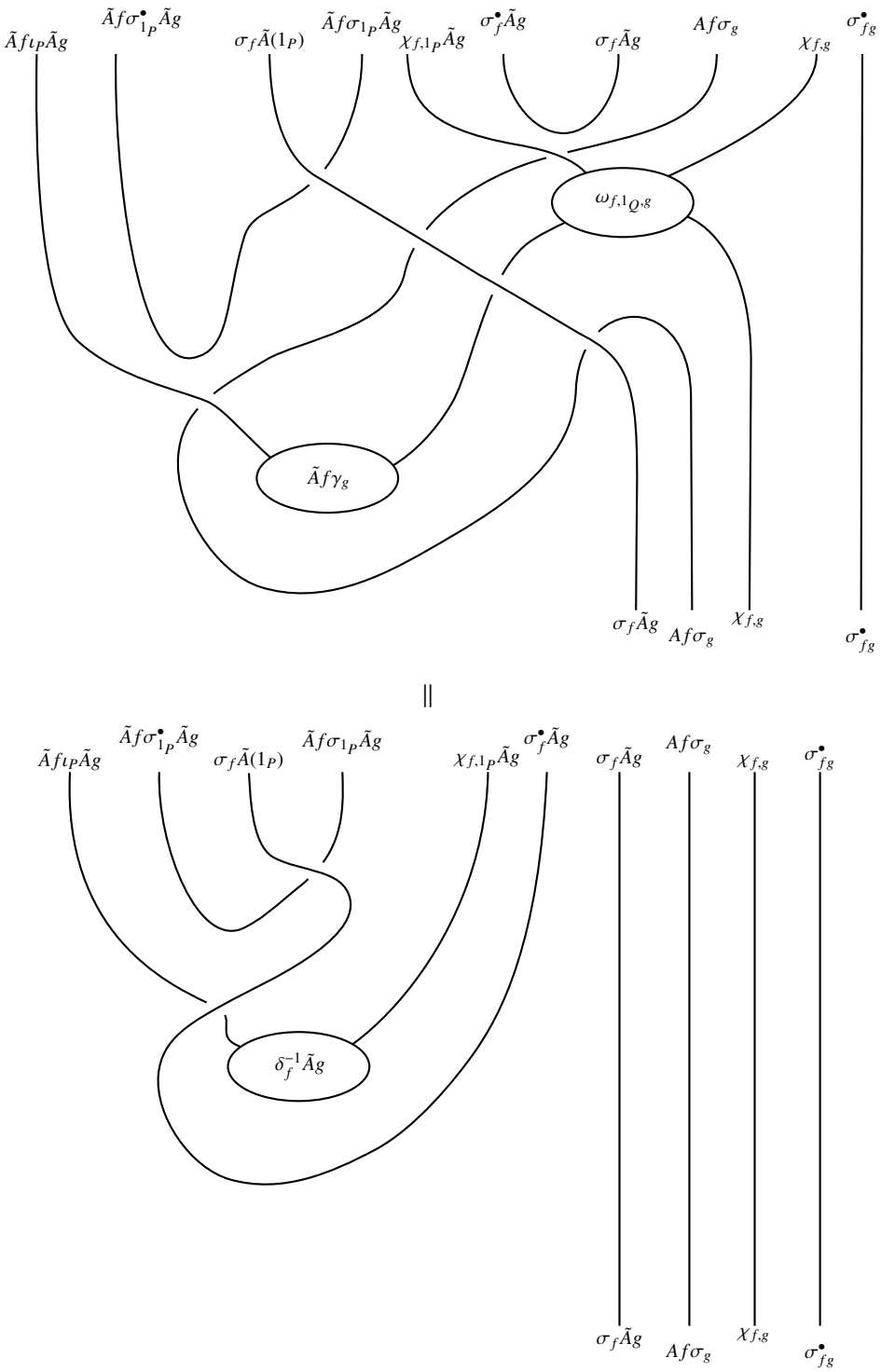
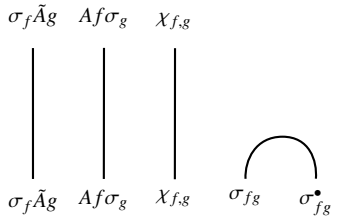


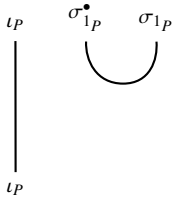
Figure 2.2.: Second trihomomorphism axiom



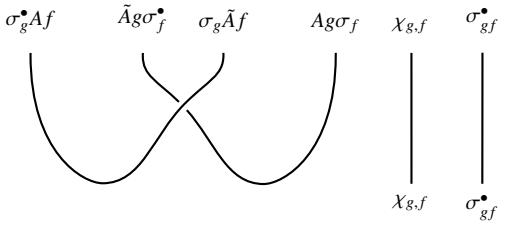
- The invertible modification Π for σ is defined to be the modification with component at composable 1-cells f and g in $\mathcal{P}$ given by the string diagram below.



- The invertible modification M for σ is defined to be the modification with single component given by the string diagram below.



- The invertible modification $\Pi^\bullet$ for $\sigma^\bullet$ is defined to be the modification with component at composable 1-cells f and g in $\mathcal{P}$ given by the string diagram below.

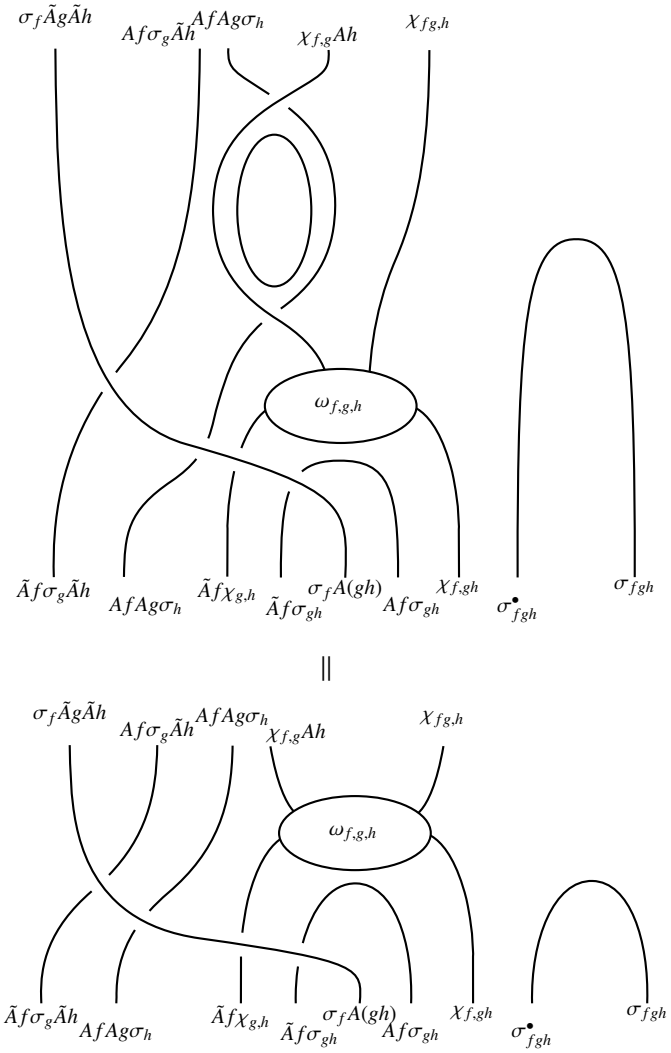


- The modification $M^\bullet$ for $\sigma^\bullet$ can be taken to be the identity:

$$M^\bullet_P\colon \quad j^*_P(\sigma^\bullet_{PP}) * \iota_P \Rrightarrow j^*_P(\sigma^\bullet_{PP}) * \iota_P$$

One then has to check the three tritransformation axioms for σ and $\sigma^\bullet$:

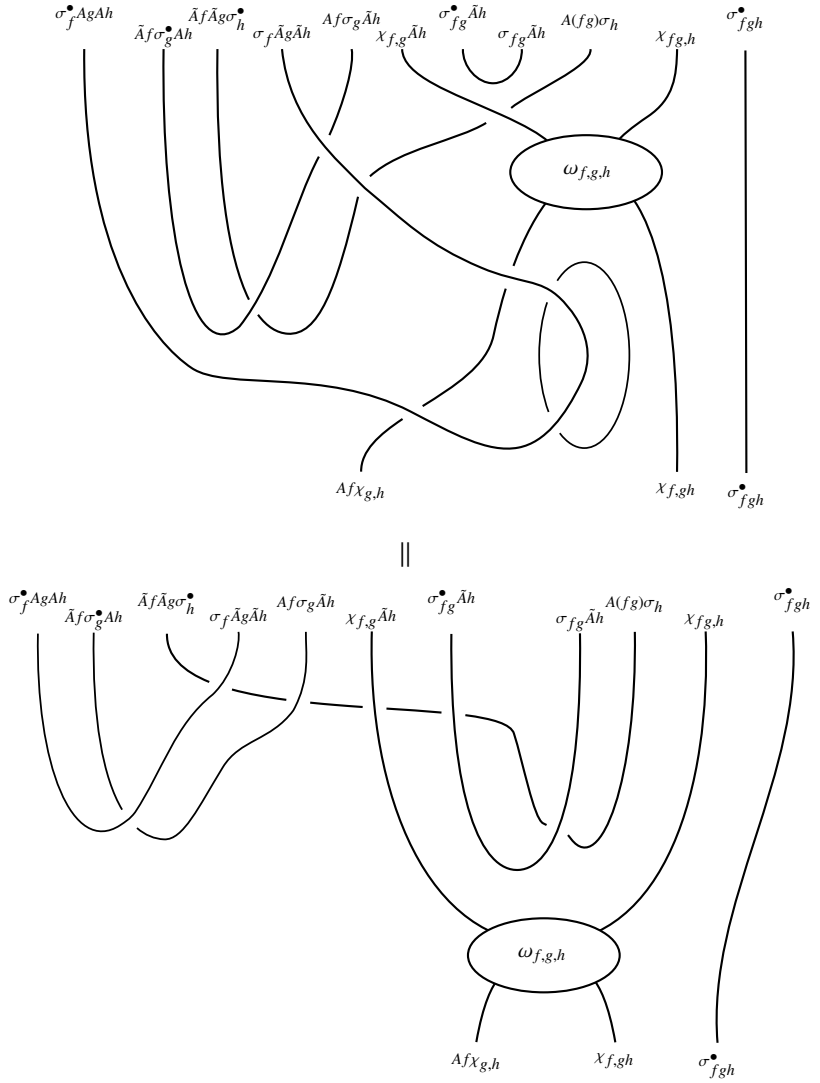
- First tritransformation axiom for $\sigma\colon \tilde{A} \Rightarrow A$: For objects $P, Q, R, S \in \mathcal{P}$, the following equation is required:



The axiom is easily proved by invertibility of the counit $E_{fg}Ah\colon\ (\sigma^\bullet_{fg}Ah)*(\sigma_{fg}Ah) \Rightarrow 1_{A(fg)Ah}$ and invertibility of interchange cells, which allows to eliminate the loop and the two adjacent interchange cells on the left hand side.

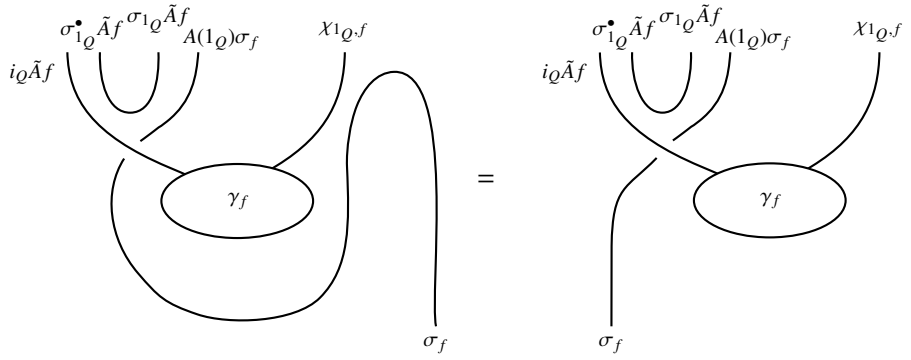
- First tritransformation axiom for $\sigma^\bullet\colon\ A \Rightarrow \tilde{A}$: For objects $P, Q, R, S \in \mathcal{P}$, the

following equation is required:



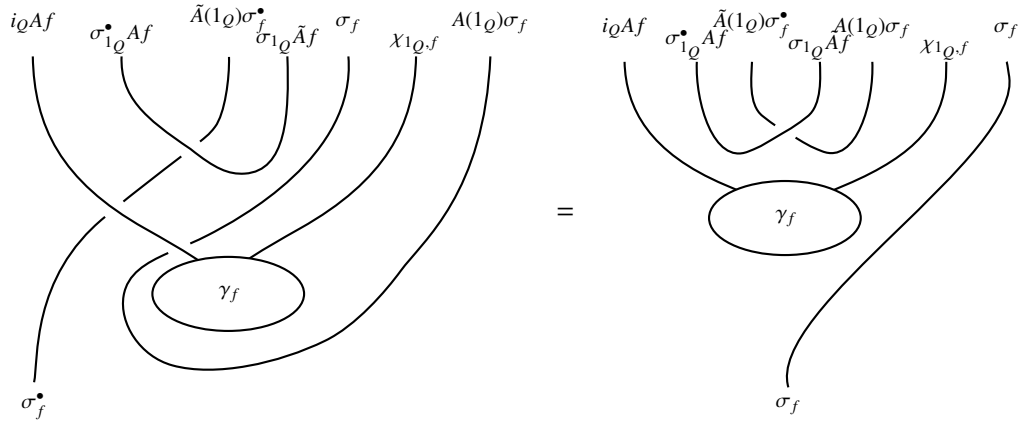
The axiom is proved by a short calculation involving similar arguments as above.

- Second tritransformation axiom for $\sigma\colon\ \tilde{A} \Rightarrow A$: For objects $P, Q \in \mathcal{P}$, the following equation is required:



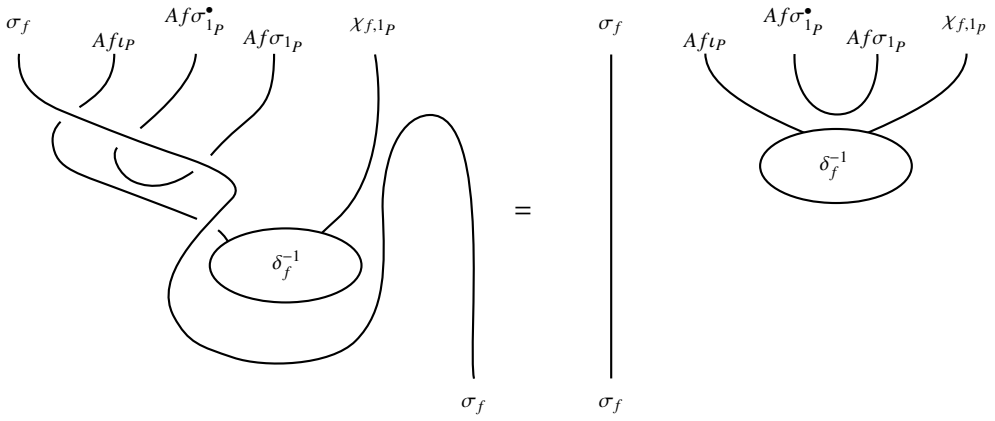
This equation is clearly satisfied by a triangle identity.

- Second tritransformation for $\sigma^{\bullet}\colon A \Rightarrow \tilde{A}$: For objects $P, Q \in \mathcal{P}$, the following equation is required:



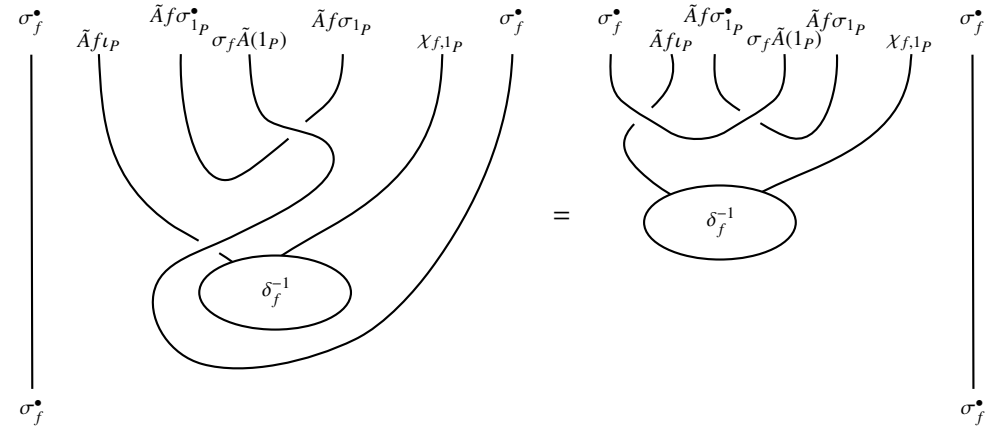
The axiom is easily verified by adjoining a triangle identity.

- Third tritransformation axiom for $\sigma\colon \tilde{A} \Rightarrow A$: For objects $P, Q \in \mathcal{P}$, the following equation is required:



The axiom is easily proved by a triangle identity, naturality of interchange cells w.r.t. the counit $AfE_{1_P}\colon (Af\sigma_{1_P}) * (Af\sigma^\bullet_{1_P}) \to 1_{AfA1_P}$ cf. (1.12) and (1.13) and invertibility of interchange cells.

- Third tritransformation axiom for $\sigma^\bullet\colon A \Rightarrow \tilde{A}$: For objects $P, Q \in \mathcal{P}$, the following equation is required:



The axiom is easily proved by adjoining a triangle identity and using Lemma 2.6.

Finally, one defines invertible modifications $E\colon \sigma\sigma^\bullet \Rrightarrow 1_A$ and $H^{-1}\colon \sigma^\bullet\sigma \Rrightarrow 1_{\tilde{A}}$ with identity components and invertible modifications E_{PQ} and H^{-1}_{PQ} and respectively. To spell out the trimodification axioms, one first has to the invertible modifications Π and M of the composite tritransformations $\sigma\sigma^\bullet$ and $\sigma^\bullet\sigma$.

The modification $M^{\sigma\sigma^\bullet}_P$ is given by—see [27, p. 142] (the components α_a, β_a in Gurski's diagram are identities):

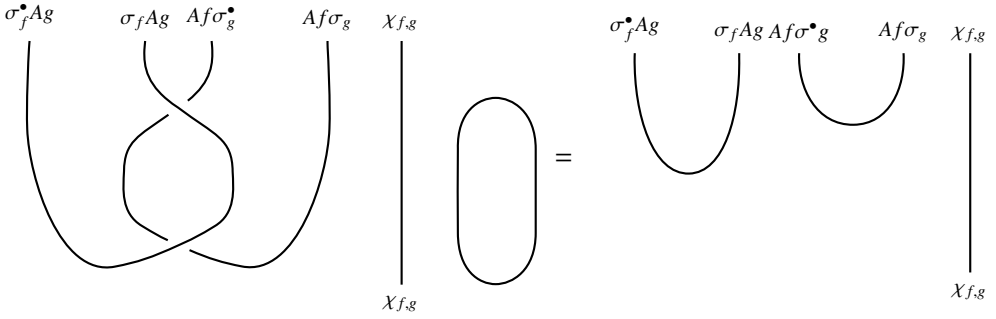
$$M_P \diamond (1_{j^*_P(\sigma_{PP})} * M^\bullet_P) = M_P$$

because $M^\bullet$ is the identity modification. Similarly,

$$M_P^{\sigma^\bullet\sigma} = M_P^\bullet \diamond (1_{j_P^*(\sigma^\bullet_{PP})} * M_P) = (1_{j_P^*(\sigma^\bullet_{PP})} * M_P) \ .$$

The component of the modification $\Pi^{\sigma\sigma^\bullet}$ at composable 1-cells f and g in $\mathcal{P}$ is defined via an interchange cell $\Sigma^{-1}_{\sigma_f,\sigma^\bullet_g}$ and $\Pi^\sigma_{f,g}$ and $\Pi^{\sigma^\bullet}_{f,g}$. Similarly, the component of the modification $\Pi^{\sigma^\bullet\sigma}$ is defined via an interchange cell $\Sigma^{-1}_{\sigma^\bullet_f,\sigma_g}$ and $\Pi^{\sigma^\bullet}_{f,g}$ and $\Pi_{f,g}$.

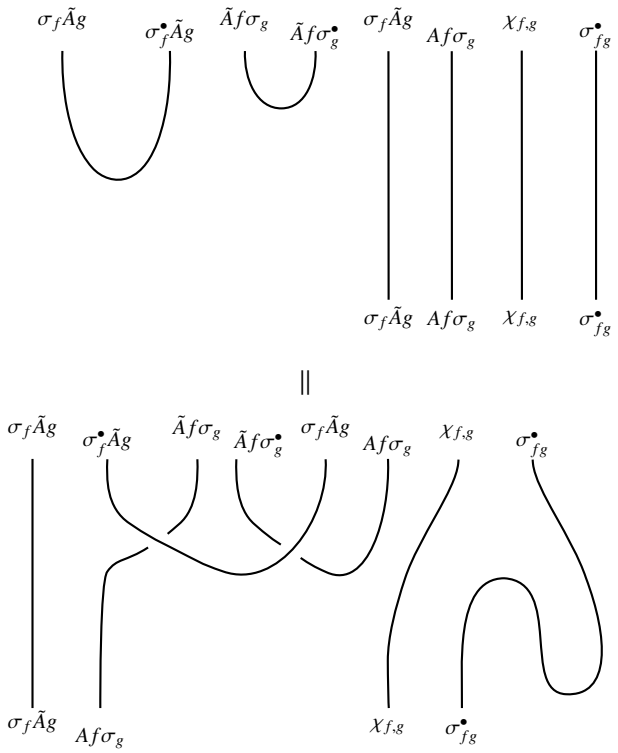
- The first trimodification axiom for E has the form:



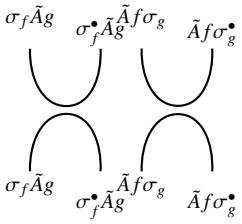
That the equation is satisfied follows from invertibility of interchange cells, which allows to cancel the two crossings on the left hand side—this operation corresponds to a Reidemeister move—and invertibility of the counit $E\colon\ \sigma_{fg} * \sigma^\bullet_{fg} \Rightarrow 1_{A(fg)}$, which allows to eliminate the loop on the left hand side.

The second trimodification axiom holds by the definition of M^σ.

- The first trimodification axiom for H^{-1} has the form:



The equation is most easily seen to be satisfied by adjoining the identity 2-cell



on the right hand side. Equality then follows from Lemma 2.6, naturality of interchange cells with respect to the unit $\tilde{A}fH\colon\ 1_{\tilde{A}f\tilde{A}g} \Rightarrow (\tilde{A}f\sigma_g) * (\tilde{A}f\sigma^\bullet_g)$ cf. (1.12) and (1.13) from 1.1.4, and triangle identities.

The second trimodification axiom holds by the definition of M^σ and a triangle identity.

By invertibility of units and counits for the adjoint equivalences σ_{PQ}, it is easy to see that the invertible modifications of the composite trimodifications $E * E^{-1}$, $E^{-1} * E$, $H * H^{-1}$, and $H^{-1} * H$ are all given by identity modifications. We have thus described $\tilde{A}$ and a biequivalence $A \cong \tilde{A}$ (with the identity perturbation). □

As a first application of Theorem 2.5, we have the following trivial corollary.

Corollary 2.5. *Given a trihomomorphism $A\colon \mathcal{P} \to \mathcal{L}$ of $\mathcal{G}ray$-categories such that each local functor A_{PQ} is equivalent to a strict functor, then A is biequivalent to a locally strict trihomomorphism in $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$.* □

In particular, notice that any functor $F\colon X \to Y$ of 2-categories with domain a discrete 2-category is equivalent to the strict functor $\tilde{F}\colon X \to Y$ that agrees with F on objects. In fact, there is an invertible icon between these functors. Together with Theorem 2.5 this observation proves the following corollary.

Corollary 2.6. *Let $A\colon \mathcal{C} \to \mathcal{L}$ be a trihomomorphism between a category $\mathcal{C}$ and a $\mathcal{G}ray$-category $\mathcal{L}$. Then A is biequivalent to a locally strict trihomomorphism $\tilde{A}\colon \mathcal{C} \to \mathcal{L}$.* □

On the other hand, we know from Example 0.1 that not every pseudofunctor is equivalent to a strict functor. Thus Theorem 2.5 cannot be applied in general, and indeed the example may be extended to show that not every trihomomorphism is biequivalent to a locally strict trihomomorphism.

Example 2.2. The strict monoidal category I from Example 0.1 can be given a unique structure of a braided strict monoidal category, where all braidings are given by identities (so I is in fact symmetric). Similarly, st$I$ is trivially braided such that $f\colon I \to \mathrm{st}I$ is a braided monoidal functor cf. Joyal and Street [32, Ex. 2.4]. This braided monoidal functor is not braided monoidal isomorphic to a strict one. Since a braided monoidal transformation is just a monoidal transformation between braided functors, this cannot be the case because we have shown in Example 0.1 that f is not isomorphic to a strict monoidal functor.

Moreover, f considered as a trihomomorphism, f is not biequivalent to a locally strict functor. The latter would have to be locally given by a strict functor $f'\colon I \to \mathrm{st}I$ as in Example 0.1. A tritransformation $f \Rightarrow f'$ must have the identity i.e. x as its component, and its adjoint equivalence must have component a string h in g's of length greater or equal 1 with naturality 2-cell $h \cdot g = h \cdot f(g) \Rightarrow f'(g) \cdot h = h$, but we have shown above that such a transformation cannot be an equivalence, so there is no such tritransformation.

Correspondingly, we have provided an example of a trihomomorphism between $\mathcal{G}ray$-categories that is not biequivalent to a locally strict trihomomorphism, and thus in particular not biequivalent to a $\mathcal{G}ray$-functor.

Corollary 2.7. *Let $A\colon \mathcal{P} \to \mathcal{L}$ be a trihomomorphism of $\mathcal{G}ray$-categories where $\mathcal{P}$ is locally small and $\mathcal{L}$ is locally cocomplete. Then A is internally biequivalent to a locally-strict trihomomorphism A' in the $\mathcal{G}ray$-category $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})$.*

Proof. This follows from the theorem above together with the coherence result from two-dimensional monad theory mentioned in the Introduction and the fact that an equivalence in a bicategory may be replaced by an adjoint equivalence, see for example [24, A.1.3]. □

The following theorem is now a consequence of Corollary 2.7, Theorem 1.8, and the fact that the functor $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})(-, i(B))$ (cf. [27, Lemma 9.7]) sends the biequivalence $A' \cong A$ to a biequivalence $\mathcal{T}ricat(\mathcal{P}, \mathcal{L})(A, i(B)) \cong \mathcal{T}ricat(\mathcal{P}, \mathcal{L})(A', i(B))$.

Theorem 2.6. *Let $A\colon \mathcal{P} \to \mathcal{L}$ be a trihomomorphism of $\mathcal{G}ray$-categories where $\mathcal{P}$ is small and locally small and $\mathcal{L}$ is cocomplete and locally cocomplete. Then A is internally biequivalent to the $\mathcal{G}ray$-functor $L(A')\colon \mathcal{P} \to \mathcal{L}$ in the $\mathcal{G}ray$-category $\mathcal{T}ricat(\mathcal{P})$ (i.e. to $iL(A')$). Given a $\mathcal{G}ray$-functor $B\colon \mathcal{P} \to \mathcal{L}$, there is a biequivalence*

$$\mathcal{T}ricat(\mathcal{P}, \mathcal{L})(A, i(B)) \cong [\mathcal{P}, \mathcal{L}](LA', B)$$

of the hom 2-category of tritransformations, trimodifications, and perturbations with the hom 2-category of $\mathcal{G}ray$-natural transformations. □

Remark 2.16. The $\mathcal{G}ray$-category $\mathcal{G}ray$ of small 2-categories is itself clearly not locally cocomplete since $[I, X] \cong X$ is not cocomplete for any non-cocomplete 2-category X. On the other hand, the tricategory $\mathcal{B}icat$ of bicategories is triequivalent to the full sub-$\mathcal{G}ray$-category $\mathcal{G}ray'$ of $\mathcal{G}ray$ determined by the 2-categories which are strictly biequivalent to the strictification $\mathrm{st}B$ of a bicategory B. However, $\mathcal{G}ray'$ is again not locally cocomplete as the following argument shows.

In Example 0.1 in the Introduction, we described a small 2-category I such that $f\colon I \to \mathrm{st}I$ is not equivalent to a strict functor. In particular, this example also shows that the strictification of a bicategory is in general not cocomplete. Otherwise, as a functor between a small and a cocomplete 2-category, f would have to be equivalent to a strict functor. Thus since $\mathrm{st}I$ is not cocomplete, the example also shows that $\mathcal{G}ray'$ can not be locally cocomplete.

2.4 The inverse of the cubical Yoneda embedding preserves representables up to biequivalence

Let $\mathcal{C}$ be a cubical tricategory—i.e. a category enriched in the monoidal tricategory $\mathcal{G}ray$. One obtains the notion of a cubical tricategory from the definition of a tricategory by requiring all hom bicategories to be 2-categories and all unit and composition pseudofunctors to be cubical cf. 1.1.4.

Recall that for a cubical tricategory $\mathcal{C}$ there is a cubical Yoneda embedding

$$y\colon \mathcal{C} \to \mathcal{T}ricat(\mathcal{C}^{\mathrm{op}}, \mathcal{G}ray)\ ,$$

which is locally given by strict biequivalences functors with strict biequivalence inverses cf. Gurski [27, Th. 9.11, 9.12, and Prop. 9.14]. The obvious prescriptions associated with the cubical Yoneda embedding can also be read off from what we present in A.2.

Definition 2.17. We denote by $\mathcal{C}'$ the full sub-$\mathcal{G}ray$-category of $\mathcal{T}ricat(\mathcal{C}^{\mathrm{op}}, \mathcal{G}ray)$ determined by the objects of the form $y(c)$ in $\mathcal{G}ray$ for all $c \in \mathrm{ob}\mathcal{C}$.

The cubical Yoneda embedding y factorizes through $\mathcal{C}'$ by definition, and there is a trihomomorphism $w\colon \mathcal{C}' \to \mathcal{C}$ forming a triequivalence-inverse to y. In fact, y and w give rise to a biequivalence in the tricategory $\mathcal{T}ricat_3$ of tricategories, trihomomorphisms, pseudo-icons, pseudo-icon modifications [18].

Theorem 2.7. *Let $\mathcal{C}'$ and $\mathcal{C}$ be cubical tricategories, and let $(y,w)\colon \mathcal{C}\to\mathcal{C}'$ be a biequivalence in $\mathcal{T}ricat_3$. Then given an object P in $\mathcal{C}$, there is a biequivalence of trihomomorphisms*

$$\omega_P\colon \mathcal{C}'(-,y(P))\to\mathcal{C}(-,P)w^{\mathrm{op}}\colon \mathcal{C}'^{\mathrm{op}}\to\mathcal{G}ray \tag{2.44}$$

with component at P' in $\mathcal{C}'$ given by

$$w_{P'P}\colon \mathcal{C}'(P',y(P))\to\mathcal{C}(w(P'),P)\ .$$

Proof. Since y and w are bijective on objects, we may use the same names for P and $y(P)$ and $w(y(P))$ when this is convenient.

The adjoint equivalence

$$\Omega\colon [1,w_{P'P}]\mathcal{C}'(-,y(P))_{P''P'}\Rightarrow[w_{P''P},1]\mathcal{C}(-,P)_{P''P'}w_{P'P''} \tag{2.45}$$

has component at the object $\alpha\colon P'\to P''$ in $\mathcal{C}'(P',P'')$ an adjoint equivalence

$$\Omega_\alpha\colon w_{P'P}\mathcal{C}'(\alpha,y(P))\to\mathcal{C}(w_{P'P''}(\alpha),P)w_{P''P} \tag{2.46}$$

in $[\mathcal{C}'(P'',P),\mathcal{C}(P',P)]$. At an object β in $\mathcal{C}'(P'',y(P))$, (2.46) has component

$$w_{P'P}(\beta\alpha)\to w_{P''P}(\beta)w_{P'P''}(\alpha)\ ,$$

which we define by $(\Omega_\alpha)_\beta=(\chi_w)^\bullet_{\beta\alpha}$. Let now $m\colon\beta\to\beta'$ be a 1-cell in $\mathcal{C}'(P'',y(P))$. We have to provide naturality 2-cells

$$(\Omega_\alpha)_m\colon (\chi_w)^\bullet_{\beta'\alpha}*w_{P'P}(m1_\alpha)\to(w_{P''P}(m)w_{P'P''}(1_\alpha))*(\chi_w)^\bullet_{\beta\alpha}\ ,$$

for which we can take the naturality 2-cell $(\chi_w)^\bullet_{m1_\alpha}$ of $(\chi_w)^\bullet$. Respect for composition at $m\colon\beta\to\beta'$ and $n\colon\beta'\to\beta''$ in $\mathcal{C}'(P'',y(P))$ is then apparently tantamount to respect for composition of $(\chi_w)^\bullet$ at $(m,1_\alpha)$ and $(n,1_\alpha)$. Likewise, respect for units at β is tantamount to respect for units of $(\chi_w)^\bullet$ at (β,α). Namely, for $m=1_\beta$, $(\chi_w)^\bullet_{m1_\alpha}=(\chi_w)^\bullet_{1_\beta1_\alpha}=1_{(\chi_w)^\bullet_{\beta\alpha}}$. Finally, naturality with respect to the 2-cell $\tau\colon m\Rightarrow m'$ in $\mathcal{C}'(P'',y(P))$ i.e.

$$(\chi_w)^\bullet_{m'1_\alpha}w_{P'P}(\tau1_{1_\alpha})=w_{P''P}(\tau)w_{P'P''}(1_{1_\alpha})(\chi_w)^\bullet_{m1_\alpha}$$

is tantamount to naturality with respect to $(\tau,1_{1_\alpha})$ of $(\chi_w)^\bullet$.

We next need to show that these Ω_α are indeed the components of pseudonatural transformations. First, given a 1-cell $k\colon\alpha\to\alpha'$ in $\mathcal{C}'(P',P'')$ we need to provide the corresponding naturality 2-cells in $[\mathcal{C}'(P'',y(P)),\mathcal{C}(P',P)]$ i.e. modifications

$$\Omega_{\alpha'}*([1,w_{P'P}](\mathcal{C}'(k,y(P))))\lll([w_{P''P},1](\mathcal{C}(w_{P'P''}(k),P)))*\Omega_\alpha \tag{2.47}$$

At an object β in $\mathcal{C}'(P'',P)$, the component of such a modification must have the form

$$(\Omega_{\alpha'})_\beta*(w_{P'P}(1_\beta k))\Rightarrow(w_{P'',P}(1_\beta)w_{P'P''}(k))*(\Omega_\alpha)_\beta$$

i.e.

$$(\chi_w)^\bullet_{\beta\alpha'} * (w_{P'P}(1_\beta k)) \Rightarrow (w_{P'',P}(1_\beta) w_{P'P''}(k)) * (\chi_w)^\bullet_{\beta\alpha} \ ,$$

for which we can correspondingly take the naturality 2-cell $(\chi_w)^\bullet_{1_\beta k}$ of $(\chi_w)^\bullet$. To write down the modification axiom, we first have to determine the naturality 2-cells of domain and codomain of (2.47). The naturality 2-cell of $\mathcal{C}'(k, y(P))$ at the 1-cell $m\colon \beta \to \beta'$ in $\mathcal{C}'(P'', y(P))$ is given by the interchange cell

$$\Sigma^{-1}_{m,k}\colon\ (1_{\beta'} k) * (m 1_\alpha) \Rightarrow (m 1_{\alpha'}) * (1_\beta k)$$

(which is shorthand for $\mathcal{M}_{\mathcal{C}'}(\Sigma^{-1}_{m,k})$) cf. Lemma A.1 or [27, Lemma 9.7]. Thus $[1, w_{P'P}](\mathcal{C}'(k, y(P)))$ has naturality 2-cell $w_{P'P}(\mathcal{M}_{\mathcal{C}'}(\Sigma^{-1}_{m,k}))$ at $m\colon \beta \to \beta'$.

Similarly, $[w_{P'',P}, 1](\mathcal{C}(w_{P'P''}(k), P))$ has naturality 2-cell $\mathcal{M}_{\mathcal{C}}(\Sigma^{-1}_{w_{P''P}(m), w_{P'P''}(k)})$ at $m\colon \beta \to \beta'$.

The naturality 2-cells of domain and codomain of (2.47) are then determined by the definition of the naturality 2-cells of a horizontal composite of 1-cells in $\mathcal{G}ray$, that is, horizontal composition of pseudonatural transformations. Hence, the naturality 2-cells are

$$\begin{aligned}&((\Omega_{\alpha'})_m * 1) \diamond (1 * (w_{P'P}(\mathcal{M}_{\mathcal{C}'}(\Sigma^{-1}_{m,k}))))\\ &= ((\chi_w)^\bullet_{(m,1_{\alpha'})} * 1) \diamond (1 * (w_{P'P}(\mathcal{M}_{\mathcal{C}'}(\Sigma^{-1}_{m,k}))))\end{aligned}$$

for the domain, and

$$\begin{aligned}&((\mathcal{M}_{\mathcal{C}}(\Sigma^{-1}_{w_{P''P}(m), w_{P'P''}(k)})) * 1) \diamond (1 * (\Omega_\alpha)_m)\\ &= ((\mathcal{M}_{\mathcal{C}}(\Sigma^{-1}_{w_{P''P}(m), w_{P'P''}(k)})) * 1) \diamond (1 * (\chi_w)^\bullet_{(m,1_\alpha)})\end{aligned}$$

for the codomain. The modification axiom thus reads

$$\begin{aligned}&(\mathcal{M}_{\mathcal{C}}(\Sigma^{-1}_{w_{P''P}(m), w_{P'P''}(k)}) * 1) \diamond (1 * (\chi_w)^\bullet_{(m,1_\alpha)}) \diamond ((\chi_w)^\bullet_{1'_\beta k} * 1_{(w_{P''P}(m), w_{P'P''}(1_\alpha))})\\ &= (1_{w_{P'P}(m 1_{\alpha'})} * (\chi_w)^\bullet_{1_\beta k}) \diamond ((\chi_w)^\bullet_{(m,1_{\alpha'})} * 1) \diamond (1 * (w_{P'P}(\mathcal{M}_{\mathcal{C}'}(\Sigma^{-1}_{m,k})))) \ ,\end{aligned}$$

but this is just naturality of $(\chi_w)^\bullet$ at the interchange 2-cell $\Sigma^{-1}_{m,k}$ and respect for composition of $(\chi_W)^\bullet$:

$$\begin{aligned}&\mathcal{M}_{\mathcal{C}}((w_{P''P} \otimes w_{P'P''})(\Sigma^{-1}_{m,k})) \diamond (\chi_w)^\bullet_{(1'_\beta, k) * (m, 1_\alpha)}\\ &= (\chi_w)^\bullet_{(m, 1_{\alpha'}) * (1_\beta, k)} \diamond w_{P'P}(\mathcal{M}_{\mathcal{C}'}(\Sigma^{-1}_{m,k})) \ .\end{aligned}$$

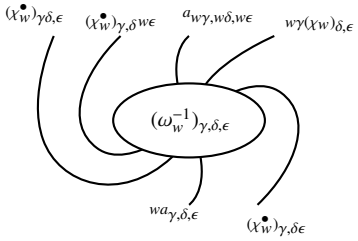
Finally, we have to check that Ω satisfies the axioms for a pseudonatural transformation. These are equations of modifications. Since modifications coincide iff their components do, the axioms follow from the corresponding axioms for $(\chi_w)^\bullet$, now in the other variable: First, respect for composition is tantamount to respect for composition of $(\chi_w)^\bullet$ (at $(1_\beta, k)$ and $(1_\beta, k')$). Second, respect for units at α is tantamount to respect

for units of $(\chi_w)^\bullet$ at (β, α). Third, naturality with respect to $\sigma\colon\ k \Rightarrow k'$ is tantamount to naturality of $(\chi_w)^\bullet$ at $(1_{1_\beta}, \sigma)$).

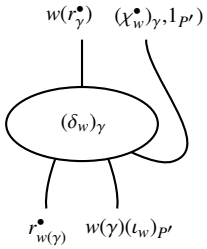
By similar arguments, one shows that χ_w gives rise to a right adjoint $\mathbf{\Omega}^\bullet$, so that $(\mathbf{\Omega}, \mathbf{\Omega}^\bullet)$ forms an adjoint equivalence as required.

Next, we have to provide invertible modifications Π and M as in the definition of a (cubical) tritransformation. Since the target *Gray* is a *Gray*-category these actually have the same form as for a Gray transformation cf. Def. 1.10 from 1.5.1. To spell this out, note that the target trihomomorphism $\mathcal{C}(-, P)w^{\mathrm{op}}$ of the tritransformation (2.44) is a composite of trihomomorphisms cf. A.3, and the representables $\mathcal{C}'(-, y(P))$ and $\mathcal{C}(-, P)$ are described in Lemma A.1, cf. also Gurski [27, Lem. 9.7].

In effect, the component at the object γ in $\mathcal{C}'(P''', y(P))$ of the component at (δ, ϵ) of Π where δ and ϵ are objects in $\mathcal{C}'(P'', P''')$ and $\mathcal{C}'(P', P'')$, is given by the mate of $(\omega_w^{-1})_{\gamma,\delta,\epsilon}$ depicted in the string diagram below.



Similarly, the component at the object γ in $\mathcal{C}'(P', y(P))$ of the single component of M is be given by the mate of $(\delta_w)_\gamma$ depicted in the string diagram below.



As for the tritransformations axioms, these are again of the same form as for homomorphisms of *Gray*-categories, and the components of the axioms clearly lift to the free functor tricategory associated with w and map to identities in the free *Gray*-category on the 2-locally discretized underlying category enriched 2-graph of $\mathcal{C}$ (they clearly map to identities in the free strict 3-category on the 2-locally discretized underlying category enriched 2-graph of $\mathcal{C}$), the target of w, and thus the axioms are fulfilled by coherence for the trihomomorphism w cf. Gurski [27, Th. 10.13, p. 174].

The components $w_{P'P}$ of the tritransformation ω_P that we have just described are biequivalences by assumption, but then ω_P must be a biequivalence of trihomomorphisms by Prop. A.16 from A.7. □

A straightforward extension of the proof just given proves the following result by coherence for trihomomorphisms.

Theorem 2.8. *Let $\mathcal{S}$ and $\mathcal{T}$ be tricategories, S an object in $\mathcal{S}$, and let $F\colon S \to T$ be a triequivalence. Then there is a biequivalence of trihomomorphisms*

$$\Gamma\colon \mathcal{S}(-,S) \Rightarrow \mathcal{T}(-,F(S))F^{\mathrm{op}}\colon \mathcal{S} \to \mathcal{B}icat$$

with component at S' in $\mathcal{S}$ given by

$$F_{S',S}\colon \mathcal{S}(S',S) \to \mathcal{T}(F(S'),F(S)) \ .$$

□

The proof of the following theorem constitutes the main technical difficulty in this section.

Theorem 2.9. *The biequivalence* (2.44) *in Theorem 2.7 forms the component ω_P at P of a tritransformation*

$$\omega\colon y^{\mathcal{C}'}y \Rightarrow w^* y^{\mathcal{C}}\colon \mathcal{C} \to \mathcal{T}ricat(\mathcal{C}'^{\mathrm{op}}, \mathcal{G}ray) \ , \tag{2.48}$$

where $y^{\mathcal{C}}$ and $y^{\mathcal{C}'}$ refer to the Yoneda embeddings, and where w^* is whiskering on the right with the trihomomorphisms w cf. A.6.

Proof. We have to show that the biequivalences ω_P form the components of a tritransformation between trihomomorphisms

$$\mathcal{C} \to \mathcal{T}ricat(\mathcal{C}'^{\mathrm{op}}, \mathcal{G}ray)$$

of cubical tricategories. First, we have to provide adjoint equivalences

$$\mathcal{T}ricat(\mathcal{C}'^{\mathrm{op}}, \mathcal{G}ray)(1,\omega_Q)y^{\mathcal{C}'}_{P,Q}y_{P,Q} \Rightarrow \mathcal{T}ricat(\mathcal{C}'^{\mathrm{op}}, \mathcal{G}ray)(\omega_P,1)(w^{\mathrm{op}})^*_{\mathcal{C}(-,P),\mathcal{C}(-,Q)}y^{\mathcal{C}}_{P,Q} \ , \tag{2.49}$$

where the y without the superscript concerns the inverse of the biequivalence w in $\mathcal{T}ricat_3$. At an object σ in $\mathcal{C}(P,Q)$, the component of this adjoint equivalence is a trimodification. The component at $P' \in \mathcal{C}'$ of this trimodification is a 2-cell in $\mathcal{G}ray$ i.e. a pseudonatural transformation. The component of such a pseudonatural transformation at the object $\gamma \in \mathcal{C}'(P',y(P))$ is defined by

$$(\eta_\sigma w(\gamma)) * (\chi^\bullet_w)_{y(\sigma),\gamma}\colon w(y(\sigma)\gamma) \to \sigma w(\gamma) \ , \tag{2.50}$$

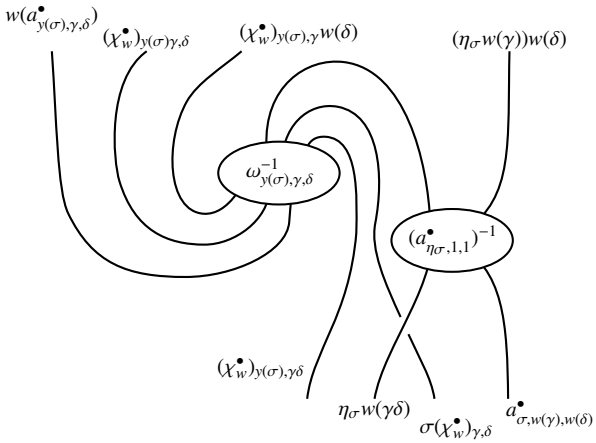
where η_σ denotes the component of the equivalence pseudo-icon $\eta\colon wy \Rightarrow 1_{\mathcal{C}}$. The naturality 2-cell at a 1-cell $l\colon \gamma \to \gamma'$ in $\mathcal{C}'(P',y(P))$ is defined by

$$(\Sigma_{\eta,w(l)} * 1_{(\chi^\bullet_w)_{y(\sigma),\gamma}}) \diamond (1_{\eta_\sigma w(\gamma')} * (\chi^\bullet_w)_{1,l}) \ . \tag{2.51}$$

One easily checks the transformation axioms for this data.

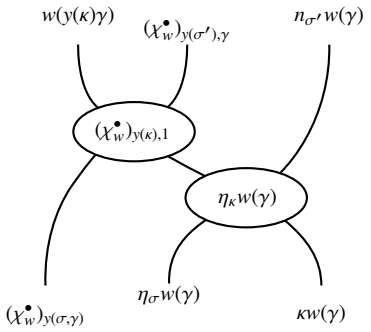
Next, one has to define the invertible modifications of the trimodification. The component at the object δ in $\mathcal{C}'^{\mathrm{op}}(P',P'') = \mathcal{C}'(P'',P')$ of such an invertible modification is

an invertible 3-cell in $\mathcal{G}ray$ i.e. an invertible modification itself. Its component at the object $\gamma \in \mathcal{C}'(P', y(P))$ is depicted in the string diagram below.



The two modification axioms are checked by calculation. The two trimodification axioms are checked by two substantial calculations.

The 2-cell of the adjoint equivalence (2.49) at a 1-cell $\kappa\colon\ \sigma \to \sigma'$ in $\mathcal{C}(P, Q)$ is a perturbation with component at P' an invertible 3-cell in $\mathcal{G}ray$ i.e. an invertible modification with component at the object $\gamma \in \mathcal{C}'(P', y(P))$ defined by the string diagram below.



The modification axiom is quickly proved from naturality of interchange cells and respect for composition and naturality of $\chi^{\bullet}_w$ with respect to an interchange cell. For the perturbation axiom, one has to determine the invertible modification in trimodifications that are the image of κ under the domain and codomain pseudofunctors of the adjoint equivalence (2.49). The components of the invertible modifications in the trimodifications arising from the cubical Yoneda embedding are given by naturality 2-cells of $a^{\bullet}$. The pseudofunctor on the left hand side of (2.49) involves the whiskering of such a trimodification with the tritransformation ω_Q on the left. The invertible modification of this whiskering involves an interchange cell in $\mathcal{G}ray$ with the component pseudonatural transformation of the adjoint equivalence of ω_Q, hence, a naturality 2-cell of $\chi^{\bullet}$. The pseudofunctor on the right hand side of (2.49) involves the whiskering of a trimodification from the cubical Yoneda embedding with the tritransformation ω_P on the right. The

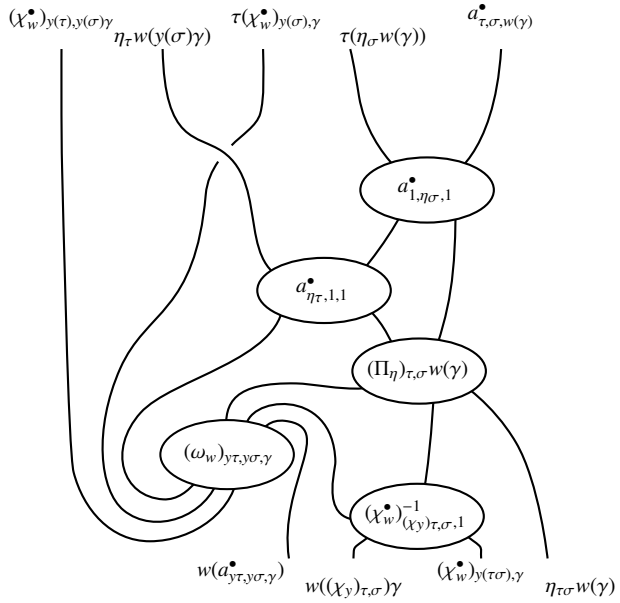
corresponding interchange cell is a naturality 2-cell of the component transformation of the trimodification from the Yoneda embedding, which is given by an interchange cell.

Proving the perturbation axiom is then a simple calculation involving the modification axiom for ω_w, naturality of interchange cells, and respect for composition and naturality of $a^\bullet$.

The transformation axioms for the adjoint equivalence (2.49) follow directly from the transformation axioms for $\chi_w^\bullet$ and η.

It remains to construct the invertible modifications Π and M and to prove the three tritransformation axioms.

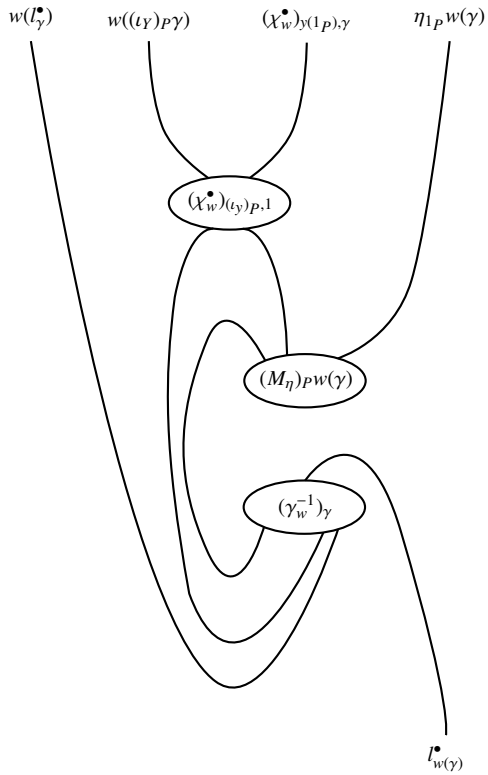
The invertible modification Π has components at the objects $\sigma \in \mathcal{C}(P,Q)$ and $\tau \in \mathcal{C}(Q,R)$ invertible perturbations with component at $P' \in \mathcal{C}'$ a modification with component at the object $\gamma \in \mathcal{C}'(P', y(P))$ defined by the string diagram below (note that in the target trihomomorphism of (2.48) w^* is a $\mathcal{G}ray$-functor cf. Theorem A.6, and the Yoneda embeddings have adjoint equivalence χ with component a trimodification with component the transformation $a^\bullet$).



The perturbation axiom follows from the first trihomomorphism axiom for w. The modification axiom is proved by a straightforward calculation.

The single component of the invertible modification M is an invertible perturbations with component at $P' \in \mathcal{C}'$ a modification with component at the object $\gamma \in \mathcal{C}'(P', y(P))$

defined by the string diagram below.



The perturbation axiom follows from the equation for a trihomomorphism corresponding to **(LAA3)** in 1.2.1. The modification axiom is proved by a straightforward calculation.

The tritransformation axioms are proved by three substantial calculations.

□

2.5 Preservation of representables under change of base with respect to st

For the proof of the following two theorems one has to recall the precise construction of the local strictification $\mathrm{st}\mathcal{P}$ cf. Gurski [27, 8.1]. The proofs thus becomes very referential and we suggest that they may be skipped by the reader, although they are basically straightforward, once everything has been spelled out. Given the monoidal structure of strictification Gurski [28], we think that the result is very natural as it corresponds to a comparison of representables under change of base.

Theorem 2.10. *There is a biequivalence of trihomomorphisms*

$$\mathrm{st} \circ \mathcal{P}(-, P) \circ e^{\mathrm{op}} \cong \mathrm{st}\mathcal{P}(-, f(P)) \colon \mathrm{st}\mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$$

with identity components. In fact, this is an equivalence pseudo-icon with pseudo-icon components that are themselves invertible icons. In other words, this is an invertible ico-icon [18].

Proof. Recall that $f\colon \mathcal{P} \to \mathrm{st}\mathcal{P}$ and $e\colon \mathrm{st}\mathcal{P} \to \mathcal{P}$ are the identity on objects. It is immediately clear that the two trihomomorphisms agree on objects. We now have to provide adjoint equivalences of the local functors,

$$\zeta_{P'',P'}\colon \mathrm{st}_{\mathcal{P}(P',P),\mathcal{P}(P'',P)} \circ \mathcal{P}(-,P)_{P'',P'} \circ e_{P',P''} \cong \mathrm{st}\mathcal{P}(-,P)_{P'',P'} \ , \tag{2.52}$$

i.e. of functors

$$\mathrm{st}\mathcal{P}(P',P'') \to [\mathrm{st}\mathcal{P}(P'',P), \mathrm{st}\mathcal{P}(P',P)] \ .$$

Since $e_{P',P''}$ is the identity on objects, and since $\mathrm{st}(\mathcal{P}(g,P))$ agrees with $\mathcal{P}(g,P)$ on objects, given an object $h \in \mathrm{ob}(\mathrm{st}\mathcal{P}(P'',P)) = \mathrm{ob}\mathcal{P}(P'',P)$, we have $\mathrm{st}(\mathcal{P}(g,P))(h) = hg$.

Similarly, since composition in $\mathrm{st}\mathcal{P}$ is given via transport of structure, change of composition, and change of units[7] as $\mathrm{st}(\otimes_{\mathcal{P}})\hat{\mathrm{st}}$ where $\hat{\mathrm{st}}$ is the cubical functor from [27, Prop. 8.5], see [27, Proof of Th. 8.4, p. 133], and since $\hat{\mathrm{st}}$ is the identity on objects, and $\mathrm{st}(\otimes_{\mathcal{P}})$ agrees with $\otimes_{\mathcal{P}}$ on objects, we have $\mathrm{st}\mathcal{P}(g, f(P))(h) = hg$. This means $\mathrm{st}\mathcal{P}(g, f(P))$ and $\mathrm{st}(\mathcal{P}(g,P))$ agree on objects.

Let now

$$j = j_1 \cdot j_2 \cdot \ldots \cdot j_n\colon\ h \to h' \tag{2.53}$$

be a 1-cell in $\mathrm{st}\mathcal{P}(P'',P)$. By the definition of st on 1-cells i.e. functors,

$$\begin{aligned}
&\mathrm{st}(\mathcal{P}(g,P))(j_1 \cdot j_2 \cdot \ldots \cdot j_n)\\
=&\mathcal{P}(g,P)(j_1) \cdot \mathcal{P}(g,P)(j_2) \cdot \ldots \cdot \mathcal{P}(g,P)(j_n)\\
=&(j_1 1_g) \cdot (j_2 1_g) \cdot \ldots \cdot (j_n 1_g) \ .
\end{aligned}$$

On the other hand, by definition we have

$$\hat{\mathrm{st}}(j_1 \cdot j_2 \cdot \ldots \cdot j_n, \emptyset_g) = (j_1, 1_g) \cdot (j_2, 1_g) \cdot \ldots \cdot (j_n, 1_g)$$

and

$$\mathrm{st}(\otimes_{\mathcal{P}})((j_1, 1_g) \cdot (j_2, 1_g) \cdot \ldots \cdot (j_n, 1_g)) = (j_1 1_g) \cdot (j_2 1_g) \cdot \ldots \cdot (j_n 1_g) \ ,$$

from which it follows that

$$\mathrm{st}\mathcal{P}(g, f(P))(j_1 \cdot j_2 \cdot \ldots \cdot j_n) = (j_1 1_g) \cdot (j_2 1_g) \cdot \ldots \cdot (j_n 1_g) \ .$$

Hence, $\mathrm{st}\mathcal{P}(g, f(P))$ and $\mathrm{st}(\mathcal{P}(g,P))$ agree on 1-cells—since these are strict functors, they agree on the identity 1-cell of h i.e. the empty string $\emptyset_h$. Finally, consider a 2-cell

$$\varphi\colon\ j_1 \cdot j_2 \cdot \ldots \cdot j_n \Rightarrow j'_1 \cdot j'_2 \cdot \ldots \cdot j'_m$$

in $\mathrm{st}\mathcal{P}(P',P)$ given by a 2-cell

$$\varphi\colon\ ((j_1 * j_2) * \ldots) * j_n \Rightarrow ((j'_1 * j'_2) * \ldots) * j'_m$$

[7] change of units does not change the composition

in $\mathcal{P}(P', P)$. By definition, $\mathrm{st}(\mathcal{P}(g, P))(\varphi)$ is given by

$$\begin{aligned} & e((j_1 1_g) \cdot (j_2 1_g) \cdot \ldots \cdot (j_n 1_g)) \\ \cong & e(j_1 \cdot j_2 \cdot \ldots \cdot j_n) 1_g \\ \xRightarrow{\varphi 1_{1_g}} & e(j'_1 \cdot j'_2 \cdot \ldots \cdot j'_m) 1_g \\ \cong & e((j'_1 1_g) \cdot (j'_2 1_g) \cdot \ldots \cdot (j'_m 1_g)) \end{aligned}$$

where the unlabeled 2-cell isomorphisms are the unique constraints given by coherence.

Similarly, by definition, $\hat{\mathrm{st}}(\varphi, 1)$ is given by

$$\begin{aligned} & e((j_1, 1_g) \cdot (j_2, 1_g) \cdot \ldots \cdot (j_n, 1_g)) \\ = & (e(j_1 \cdot j_2 \cdot \ldots \cdot j_n), e(1_g^{\cdot n})) \\ \xRightarrow{(\varphi, \gamma_{n,m})} & (e(j'_1 \cdot j'_2 \cdot \ldots \cdot j'_m), e(1_g^{\cdot m})) \\ = & e((j'_1, 1_g) \cdot (j'_2, 1_g) \cdot \ldots \cdot (j'_m, 1_g)) \ , \end{aligned}$$

where $\gamma_{n,m} \colon\ e(1_g^{\cdot n}) \cong e(1_g^{\cdot m})$ is the unique constraint 2-cell isomorphism given by coherence. Correspondingly, $\mathrm{st}\mathcal{P}(g, f(P))(\varphi) = \mathrm{st}(\otimes_{\mathcal{P}})(\hat{\mathrm{st}}(\varphi, 1))$ is given by

$$\begin{aligned} & e((j_1 1_g) \cdot (j_2 1_g) \cdot \ldots \cdot (j_n 1_g)) \\ \cong & \otimes_{\mathcal{P}} (e(j_1 \cdot j_2 \cdot \ldots \cdot j_n), e(1_g^{\cdot n})) \\ \xRightarrow{\otimes_{\mathcal{P}}(\varphi, \gamma_{n,m})} & \otimes_{\mathcal{P}} (e(j'_1 \cdot j'_2 \cdot \ldots \cdot j'_m), e(1_g^{\cdot m})) \\ \cong & e((j'_1 1_g) \cdot (j'_2 1_g) \cdot \ldots \cdot (j'_m 1_g)) \end{aligned}$$

where the unlabeled 2-cell isomorphisms are the unique constraints given by coherence. If we denote by δ_g the unique 2-cell isomorphism $e(1_g^{\cdot n}) \cong 1_g$ given by coherence, we can rewrite the above using local functoriality of $\otimes_{\mathcal{P}}$ as follows:

$$\begin{aligned} & e((j_1 1_g) \cdot (j_2 1_g) \cdot \ldots \cdot (j_n 1_g)) \\ \cong & \otimes_{\mathcal{P}} (e(j_1 \cdot j_2 \cdot \ldots \cdot j_n), 1_g) \\ \xRightarrow{\otimes_{\mathcal{P}}(\varphi, \delta_m \diamond \gamma_{n,m} \diamond \delta_n^{-1})} & \otimes_{\mathcal{P}} (e(j'_1 \cdot j'_2 \cdot \ldots \cdot j'_m), 1_g) \\ \cong & e((j'_1 1_g) \cdot (j'_2 1_g) \cdot \ldots \cdot (j'_m 1_g)) \end{aligned}$$

where the unlabeled 2-cell isomorphisms are the unique constraints given by coherence. Observing that $\delta_m \diamond \gamma_{n,m} \diamond \delta_n^{-1} = 1_{1_g}$ by coherence, this is clearly exactly the same 2-cell as $\mathrm{st}(\mathcal{P}(g, P))(\varphi)$. Thus indeed, the strict functors $\mathrm{st}\mathcal{P}(g, f(P))$ and $\mathrm{st}(\mathcal{P}(g, P))$ are literally the same.

We now have to determine the effect of the functors in (2.52) on a 1-cell

$$k = k_1 \cdot k_2 \cdot \ldots \cdot k_l \colon\ g \to g'$$

in $\mathrm{st}\mathcal{P}(P', P'')$ (and on the empty string $\emptyset_g$ if $g = g'$). Thus we have to compare the pseudonatural transformations $\mathrm{st}(\mathcal{P}(e(k), P))$ and $(\mathrm{st}\mathcal{P})(k, f(P))$ of strict functors

$\mathrm{st}\mathcal{P}(P'',P) \to \mathrm{st}\mathcal{P}(P',P)$. Consider again an object $h \in \mathrm{ob}(\mathrm{st}\mathcal{P}(P'',P)) = \mathrm{ob}\mathcal{P}(P'',P)$. The transformation $\mathrm{st}(\mathcal{P}(e(k),P))$ has the same component as $\mathcal{P}(e(k),P)$ at h i.e. $1_h e(k)$, which is a string of length 1. On the other hand, the component of $(\mathrm{st}\mathcal{P})(k, f(P))$ at h is given by

$$\mathrm{st}(\otimes_{\mathcal{P}})(\hat{\mathrm{st}}(\emptyset_h, k)) = (1_h k_1) \cdot (1_h k_2) \cdot \ldots \cdot (1_h k_l) \ .$$

Thus for the component at h of the icon-component ζ_k—a modification in $[\mathrm{st}\mathcal{P}(P'',P), \mathrm{st}\mathcal{P}(P',P)]$—of the pseudo-icon component $\zeta_{P''P'}$—i.e. the component at h of the ico-icon component, we have to determine a 2-cell

$$(\zeta_k)_h \colon\ 1_h e(k) \Rightarrow (1_h k_1) \cdot (1_h k_2) \cdot \ldots \cdot (1_h k_l)$$

in $\mathrm{st}\mathcal{P}(P',P)$ given by a 2-cell

$$e(1_h e(k)) \Rightarrow e((1_h k_1) \cdot (1_h k_2) \cdot \ldots \cdot (1_h k_l))$$

in $\mathcal{P}(P',P)$, for which we take the unique invertible constraint 2-cell given by coherence. First, we have to explain why this gives rise to the component of a modification. Thus we have to prove the modification axiom for every 1-cell $j \colon h \to h'$ in $\mathrm{st}\mathcal{P}(P'',P)$ as in (2.53). According to the definition of the strictification st on 2-cells i.e. pseudonatural transformations cf. 0.20 and [27, p. 135], the naturality 2-cell of the pseudonatural transformation $\mathrm{st}\mathcal{P}(e(k),P)$ is given by unique invertible 2-cells coming from coherence and the naturality 2-cell of the pseudonatural transformation $\mathcal{P}(e(k),P))$ at $e(h)$. Now according to the definition A.2, the naturality 2-cell $\mathcal{P}(e(k),P))$ at $e(h)$ is given by the 'interchange' 2-cell $\Sigma^{-1}_{e(j),e(k)}$ of $\mathcal{P}$. More precisely, the naturality 2-cell in question is given by the 2-cell

$$\begin{aligned} & e((1_{h'} e(k)) \cdot (j_1 1_g) \cdot (j_2 1_g) \cdot \ldots \cdot (j_n 1_g)) \\ \cong & (1_{h'} e(k)) * (e(j) 1_g) \\ \xRightarrow{\Sigma^{-1}_{e(j),e(k)}} & (e(j) 1_{g'}) * (1_h e(k)) \\ \cong & e((j_1 1_{g'}) \cdot (j_2 1_{g'}) \cdot \ldots \cdot (j_n 1_{g'}) \cdot (1_h e(k))) \end{aligned}$$

in $\mathcal{P}(P',P)$, where the unlabeled invertible 2-cells are the unique 2-cells given by coherence.

According to the definition [27, p. 145] cf. A.2, the naturality 2-cell of the pseudonatural transformation $(\mathrm{st}\mathcal{P})(k, fP))$ at h is given by the interchange 2-cell $\Sigma^{-1}_{j,k}$ of $\mathrm{st}\mathcal{P}$ i.e. the inverse of the single nontrivial composition constraint of the cubical composition functor $\mathrm{st}(\otimes_{\mathcal{P}})\hat{\mathrm{st}}$ at $(j, 1_g)$ and $(1_h, k)$. Since $\mathrm{st}(\otimes_{\mathcal{P}})$ is strict, this is simply given by the image under $\mathrm{st}(\otimes_{\mathcal{P}})$ of the corresponding constraint of $\hat{\mathrm{st}}$. The latter is an invertible 2-cell

$$\begin{aligned} & (1_{h'}, k_1) \cdot (1_{h'}, k_2) \cdot \ldots \cdot (1_{h'}, k_l) \cdot (j_1, 1_g) \cdot (j_2, 1_g) \cdot \ldots \cdot (j_n, 1_g) \\ \Rightarrow & (j_1, 1_{g'}) \cdot (j_2, 1_{g'}) \cdot \ldots \cdot (j_n, 1_{g'}) \cdot (1_h, k_1) \cdot (1_h, k_2) \cdot \ldots \cdot (1_h, k_l) \end{aligned}$$

in $\mathrm{st}\mathcal{P}(P',P)$ given by the unique invertible 2-cell

$$\epsilon_{j,k} \colon\ (e(1^{\cdot l}_{h'} \cdot j), e(k \cdot 1^{\cdot n}_{g})) \Rightarrow (e(j \cdot 1^{\cdot l}_{h}), e(1^{\cdot n}_{g'} \cdot k))$$

in $\mathcal{P}(P', P)$ arising from coherence. Therefore, the naturality 2-cell of $(\mathrm{st}\mathcal{P})(k, fP))$ at h is given by the 2-cell

$$\begin{aligned} & e((1_{h'}k_1)\cdot(1_{h'}k_2)\cdot\ldots\cdot(1_{h'}k_l)\cdot(j_1 1_g)\cdot(j_2 1_g)\cdot\ldots\cdot(j_n 1_g)) \\ \cong & \otimes_{\mathcal{P}}\,(e(1_{h'}^{\cdot l}\cdot j), e(k\cdot 1_g^{\cdot n})) \\ \xRightarrow{\otimes_{\mathcal{P}}(\epsilon_{j,k})} & \otimes_{\mathcal{P}}\,(e(j\cdot 1_h^{\cdot l}), e(1_{g'}^{\cdot n}\cdot k)) \\ \cong & e((j_1 1_{g'})\cdot(j_2 1_{g'})\cdot\ldots\cdot(j_n 1_{g'})\cdot(1_h k_1)\cdot(1_h k_2)\cdot\ldots\cdot(1_h k_l)) \end{aligned}$$

in $\mathcal{P}(P', P)$, where the unlabeled invertible 2-cells are the unique 2-cells arising from coherence. The modification axiom now follows from coherence.

If $k = \emptyset_g$, the component of $\mathrm{st}(\mathcal{P}(e(k), P))$ at h, given by the corresponding component of $\mathcal{P}(e(k), P)$, looks formally the same $1_h e(k) = 1_h 1_g$. On the other hand, the component of $(\mathrm{st}\mathcal{P})(k, f(P))$ is given by

$$\mathrm{st}(\otimes_{\mathcal{P}})(\hat{\mathrm{st}}(\emptyset_h, \emptyset_g)) = \emptyset_{hg}\ .$$

Thus, we set the icon component

$$(\zeta_{\emptyset_g})_h\colon\ 1_h 1_g \Rightarrow \emptyset_{hg}$$

to be given by the unique invertible 2-cell

$$e(1_h 1_g) = 1_h 1_g \Rightarrow 1_{hg} = e(\emptyset_{hg})$$

arising from coherence. As above, the 2-cell of $\mathrm{st}\mathcal{P}(e(\emptyset_g), P)$ at h is given by $\Sigma^{-1}_{e(j),e(\emptyset_g)} = \Sigma^{-1}_{e(j),1_g}$ (which is in fact not the identity since $\mathcal{P}$ is not a *Gray*-category—source and target do not even coincide since horizontal composition with $(1_{h'}1_k)$ and $(1_h 1_g)$ is not the identity) and coherence isomorphisms.

On the other hand, the naturality 2-cell of $(\mathrm{st}\mathcal{P})(\emptyset_g, fP))$ at h is given by the interchange 2-cell $\Sigma^{-1}_{j,\emptyset_g}$ of $\mathrm{st}\mathcal{P}$, which must be the identity since the composition functor $\mathrm{st}(\otimes_{\mathcal{P}})\hat{\mathrm{st}}$ of $\mathrm{st}\mathcal{P}$ is cubical. Just as above, the modification axiom is easily seen to follow from coherence for the tricategory $\mathcal{P}$, while it is automatic for $j = \emptyset_h$.

Next, one easily proves the two icon axioms for ζ_k and naturality. The composition constraint is of the form

$$(1_h e(k'))\cdot(1_h e(k)) \Rightarrow 1_h(e(k'\cdot k))\ .$$

This is given by the unique invertible 2-cell

$$e((1_h e(k'))\cdot(1_h e(k))) = (1_h e(k'))*(1_h e(k)) \Rightarrow 1_h(e(k'\cdot k)) = e(1_h(e(k'\cdot k)))$$

arising from coherence. The unit constraint is of the form

$$\emptyset_{hg} \Rightarrow 1_h 1_g = 1_h e(\emptyset_g)$$

This is given by the unique invertible 2-cell

$$e(\emptyset_{hg}) = 1_{hg} \Rightarrow 1_h 1_g = e(1_h 1_g)$$

arising from coherence—i.e. the unit constraint $(\otimes_{\mathcal{P}})_{h,g}$ of the composition functor of $\mathcal{P}$.

The composition and unit constraints are identities for $(\mathrm{st}\mathcal{P})(-, f(P))$ since this functor is just given by the partial functor of the cubical composition functor of $\mathrm{st}\mathcal{P}$, which means that it is strict. It is again easily seen that the two icon axioms follow from coherence.

For naturality, let $\varphi\colon\ k \Rightarrow k'\colon\ g \to g'$ be a 2-cell in $\mathrm{st}(P', P'')$. Note that the naturality axiom is an equation of modifications, which holds if and only if it holds on components. We now have to determine

$$\mathrm{st}(\otimes_{\mathcal{P}})(\hat{\mathrm{st}}(1_{\emptyset_h}, \varphi)) \ .$$

The 2-cell $\hat{\mathrm{st}}(1_{\emptyset_h}, \varphi)$ is given by $(\gamma_{l,l'}, \varphi)$ in $\mathcal{P}(P', P)$. The proof of naturality is then basically as the one above involving $\gamma_{n,m}$ and δ_n and δ_m, only in the other variable.

Finally, we have to provide invertible 2-cells between the components of the adjoint equivalences χ and ι corresponding to $\mathrm{st} \circ \mathcal{P}(-, P) \circ e$ and $(\mathrm{st}\mathcal{P})(-, f(P))$ respectively. According to [27, p. 146] , the component of the adjoint equivalence $\chi_{g,g'}$ is the pseudonatural transformation a of $\mathrm{st}\mathcal{P}$ for $(\mathrm{st}\mathcal{P})(-, f(P))$ (with two variables held fixed)—since everything is the identity on objects, the components of this a are the same as for $\mathcal{P}$.

Similarly, for $\mathrm{st}\circ\mathcal{P}(-, P)\circ e$, the component of χ is a pseudonatural transformation with component given by a component of a of $\mathcal{P}$ (because χ has identity component for e, and identity component for st [27, p. 137]). In fact, since this has the same components as the pseudonatural transformation a for $\mathrm{st}\mathcal{P}$ with two fixed variables, they must actually coincide since the naturality 2-cells both arise from unique invertible 2-cells given by coherence. This means we can take the identity for the component of the modification $\Pi_{g,g'}$. If this is a modification, it must be the identity modification, and the modification axiom is exactly the requirement that the naturality 2-cells of the source and target transformations coincide. Since both transformations are given by a, this is clear.

Similarly, ι is be given by $r^\bullet$ for $\mathrm{st}\mathcal{P}$ in both cases (according to [27, p. 146] cf. A.2). This means we can take the identity for the component of the modification M_g itself. If this is a modification, it must be the identity modification, and the modification axiom is exactly the requirement that the naturality 2-cells of the source and target transformations coincide. Since both transformations are given by $r^\bullet$, this is clear.

$\Pi_{g,g'}$ and M_g form the components of modifications Π and M, which must thus be identity modifications: if $\chi_F \neq \chi_G$, $\iota_F \neq \iota_G$, one has to prove the modification axiom, in other words, that the naturality 2-cells of the source and target transformations coincide, but this follows from coherence.

The tritransformation axioms involve the modifications Π and M, for which we were able to choose the identity modifications, but they do involve naturality 2-cells of the adjoint equivalences of the tritransformations i.e. the ico-icon components. To see that these axioms really hold, first note that the tritransformation axioms simplify considerably because they only contain tricategory constraints for the target $\mathcal{G}ray$ i.e. a $\mathcal{G}ray$-category, so these are trivial. Also, the interchange cells are trivial since the target

is a $\mathcal{G}ray$-category and since the components of the adjoint equivalence of the tritransformation (the ico-icon) are identities.

It is straightforward to determine that all the cells turning up in the tritransformation axioms are given by constraints of $\mathcal{P}$, and thus these follow from coherence.

□

Theorem 2.11. *The biequivalence of Theorem 2.10 is natural in P i.e. forms the component of a tritransformation of functors $\mathcal{P} \to \mathcal{T}ricat(\mathrm{st}\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)$.*

Proof. We have to provide adjoint equivalences

$$\begin{aligned}&\mathcal{T}ricat(\mathrm{st}\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(1, \zeta_Q)\mathcal{T}ricat(e^{\mathrm{op}}, \mathcal{G}ray)_{\mathrm{st}\circ\mathcal{P}(-,P),\mathrm{st}\circ\mathcal{P}(-,Q)}\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathrm{st})_{\mathcal{P}(-,P),\mathcal{P}(-,Q)}y^{\mathcal{P}}_{P,Q}\\ \Rightarrow&\mathcal{T}ricat(\mathrm{st}\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(\zeta_P, 1)y^{\mathrm{st}\mathcal{P}}_{f(P),f(Q)}f_{P,Q}\end{aligned} \tag{2.54}$$

of functors

$$\mathcal{P}(P, Q) \to \mathcal{T}ricat(st\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(\mathrm{st} \circ \mathcal{P}(-, P) \circ e^{\mathrm{op}}, \mathrm{st}\mathcal{P}(-, f(Q))) \ .$$

Let g be an object in $\mathcal{P}(P, Q)$. Then the component at g of this adjoint equivalence is a trimodification. Since e^{op} is the identity on objects, the component at $P' \in \mathrm{st}\mathcal{P}$ of the tritransformation which is the image under the left hand functor, is the functor $\mathrm{st}\mathcal{P}(P', g)\colon \mathrm{st}\mathcal{P}(P', P) \to \mathrm{st}\mathcal{P}(P', P)$ —which sends the object $h \in \mathrm{ob}(\mathrm{st}\mathcal{P}(P', P)) = \mathrm{ob}\mathcal{P}(P', P)$ to gh since $\mathrm{st}\mathcal{P}(P', g)$ and $\mathcal{P}(P', g)$ agree on objects.

On the other hand, since the hom functors of f are the identity on objects, we have to consider the composite of the ico-icon $\zeta_P\colon \mathrm{st} \circ \mathcal{P}(-, P) \circ e^{\mathrm{op}} \Rightarrow \mathrm{st}\mathcal{P}(-, f(P))$ with the tritransformation $\mathrm{st}\mathcal{P}(-, g)\colon \mathrm{st}\mathcal{P}(-, f(P)) \Rightarrow \mathrm{st}\mathcal{P}(-, f(Q))$ for the image of g under the right hand functor. The component at $P' \in \mathrm{st}\mathcal{P}$ is the functor $(\mathrm{st}\mathcal{P})(P', g)\colon \mathrm{st}\mathcal{P}(P', P) \to \mathrm{st}\mathcal{P}(P', Q)$—sending the object $h \in \mathrm{ob}(\mathrm{st}\mathcal{P}(P', P)) = \mathrm{ob}\mathcal{P}(P', P)$ to gh.

We maintain that the two functors $\mathrm{st}(\mathcal{P}(P', g))$ and $(\mathrm{st}\mathcal{P})(P', g)$ coincide. Indeed, this follows from the argument at the beginning of the proof of Theorem 2.10 upon replacing $\mathcal{P}$ by its dual $\mathcal{P}^{\mathrm{op}}$. Thus the two tritransformations have equal components. As the component of a trimodification we thus take the identity pseudonatural transformation.

We next have to provide invertible modifications between the adjoint equivalences of the two tritransformations. As the adjoint equivalences of the ico-icons are themselves icons i.e. have identity components, the components of these modifications must be 2-cells between the components of the adjoint equivalences of the two image tritransformations of g.

According to [27, p. 147] cf. A.2, the adjoint equivalence of the tritransformation $\mathrm{st}\mathcal{P}(-, g)$ has component given by $a^{\bullet}$ of $\mathrm{st}\mathcal{P}$ with two variables held constant. As above, this is given by $a^{\bullet}$ of $\mathcal{P}$ because e and f are given by the identity on objects and 1-cells. Similarly, the adjoint equivalence of the tritransformation $(1_{\mathrm{st}} * \mathcal{P}(-, g)) * 1_{e^{\mathrm{op}}}$ has component given by $a^{\bullet}$ of $\mathcal{P}$. Thus for the component at h of the component at i—itself a modification—of the invertible modification in question, we can take the identity 2-cell. And since both of the pseudonatural transformations above are given, up to coherence, by the adjoint equivalence $a^{\bullet}$ with two variables fixed, we can in fact take the identity modification between them. That these identity modifications form the components

of modifications as in the definition of a trimodification follows from coherence—the modification axiom is just the equality of naturality 2-cells, these modifications must thus again be the identity modifications.

Finally, we have to check the two trimodification axioms—this is just the verification of the axioms for the identity trimodification. Summarizing, the alleged adjoint equivalence (2.54) has the identity trimodifications as component—Recall that $\mathcal{T}ricat(\mathrm{st}\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)$ is a $\mathcal{G}ray$-category, and that its hom categories are 2-categories in particular. Thus the adjoint equivalence (2.54) must in fact be an icon.

We next have to provide the naturality 2-cell of the adjoint equivalence (2.54) i.e. its icon component at a 1-cell $k\colon\ g \to g'$ in $\mathcal{P}(P,Q)$. These are perturbations between the two image trimodifications.

First, consider the trimodification $1_{\zeta_Q} \otimes ((1_{1_{\mathrm{st}}} * \mathcal{P}(-,k)) * 1_{1_{e^{\mathrm{op}}}})$. Its component at $P'$is the pseudonatural transformation $1_{(\zeta_Q)'_P} \otimes \mathrm{st}\mathcal{P}(P',k) = 1_{(\zeta_Q)'_P} * \mathrm{st}\mathcal{P}(P',k) = \mathrm{st}\mathcal{P}(P',k)$ since $(\zeta_Q)'_P$ is the identity functor.On the other hand, since $f_{P,Q}$ includes k as the string of length 1, we have to consider the trimodification $(\mathrm{st}\mathcal{P})(-,k) \otimes 1_{\zeta_P}\colon\ (\mathrm{st}\mathcal{P})(-,g) \otimes \zeta_P \Rrightarrow (\mathrm{st}\mathcal{P})(-,g') \otimes \zeta_P$. The latter has component at P' the pseudonatural transformation $(\mathrm{st}\mathcal{P})(P',k) \otimes 1_{(\zeta_P)_{P'}} = (\mathrm{st}\mathcal{P})(P',k) * 1_{(\zeta_P)_{P'}} = (\mathrm{st}\mathcal{P})(P',k)$.

We maintain that the identity 2-cell of this length-1-string $1_h k$ gives rise to the identity modification $\mathrm{st}(\mathcal{P}(P',k)) \Rrightarrow (\mathrm{st}\mathcal{P})(P',k)$, which forms the components of a perturbation as wanted. Indeed, upon replacing $\mathcal{P}$ by $\mathcal{P}^{\mathrm{op}}$, we have provided such a modification in the proof of Theorem 2.10 in the general case of a string k of length n. In the case of a string of length 1, this is in fact a modification between the same pseudonatural transformations (i.e. with the same components and naturality 2-cells) with identity components, i.e. the identity modification. The perturbation axiom for a perturbation with identity components is nothing else than the requirement that the invertible modifications of the source and target trimodifications coincide, and this follows from coherence again. This means the adjoint equivalence (2.54) is in fact the identity adjoint equivalence.

Finally, we have to provide invertible modifications Π and M. These are just invertible modifications between the adjoint equivalences χ and ι of the source and target trihomomorphisms of the alleged tritransformation

$$\mathcal{T}ricat(e^{\mathrm{op}}, \mathcal{G}ray)\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathrm{st})y^{\mathcal{P}} \Rightarrow y^{\mathrm{st}\mathcal{P}} f \tag{2.55}$$

It is clear that the components of these χ are both given by $a^\bullet$ for $\mathcal{P}$. Their naturality 2-cells then match by coherence.

According to [27, p. 148] cf. A.2, the adjoint equivalence χ for the Yoneda embedding $y^{\mathrm{st}\mathcal{P}}$ is given by $a^\bullet$ for $\mathrm{st}\mathcal{P}$ with two variables held fixed, which has the component of $a^\bullet$ for $\mathcal{P}$ as its component. Since χ for f has identity component, this means, χ for the right hand side has component a component of $a^\bullet$ for $\mathcal{P}$ with two variables held fixed. The component of the adjoint equivalence χ for the left hand side is the same as the component of the adjoint equivalence for $y^{\mathcal{P}}$ because e and st are the identity on objects and 1-cells. Now the component of χ for $y^{\mathcal{P}}$ must also be given by a component of $a^\bullet$ for $\mathcal{P}$ A.2. This means the adjoint equivalences χ are the same for source and target of (2.55). Similar arguments apply to the adjoint equivalences ι. This means we can take Π and M to be the identity modifications. □

2.6 The Yoneda lemma for tricategories

With the results on functor tricategories from Appendix A, we are now ready to prove the Yoneda lemma for tricategories. Given a tricategory $\mathcal{P}$, the 3-globular set

$$\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat) \tag{2.56}$$

of trihomomorphism, tritransformations, trimodifications, and perturbations locally has the structure of a bicategory: it is not hard no write down the horizontal composition of trimodifications and the horizontal and vertical composition of perturbations cf. Theorem A.1.

Since whiskering on the left with strictification st_* gives rise to local biequivalences cf. Proposition A.6, one can give $\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)$ the structure of a tricategory via the Transport of structure theorem [27, Theorem 7.22] as we do it in Theorem A.4.

Theorem 2.12 (The Yoneda lemma for tricategories)**.** *Given a small tricategory $\mathcal{P}$ and a trihomomorphism $A\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{B}icat$, evaluation of the component at P at the identity induces a biequivalence*

$$\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)(\mathcal{P}(-, P), A-) \cong AP\ , \tag{2.57}$$

which is natural in P meaning that these biequivalences form the components of a biequivalence

$$\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)(-, A)Y^{\mathrm{op}} \Rightarrow A \tag{2.58}$$

in $\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)$ where $Y\colon \mathcal{P} \to \mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)$ is the Yoneda embedding cf. Theorem A.5.

Proof. The existence of biequivalences as in (2.57) is proved by the following chain of biequivalences, where $\mathcal{P}'$ denotes the triequivalent $\mathcal{G}ray$-category obtained from $\mathcal{P}$ by its local strictification $\mathrm{st}\mathcal{P}$ and the cubical Yoneda embedding, and where (e, f) and (y, w) denote the corresponding triequivalences. Note in particular that e and f are the identity on objects and locally given by the functors of the same name between $\mathcal{P}(P, Q)$ and its strictification $\mathrm{st}\mathcal{P}(P, Q)$.

$$\begin{aligned}
&\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)(\mathcal{P}(-, P), A)\\
\cong&\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(\mathrm{st}\mathcal{P}(-, P), \mathrm{st}A)\\
\cong&\mathcal{T}ricat((\mathcal{P}')^{\mathrm{op}}, \mathcal{G}ray)(\mathrm{st}\mathcal{P}(-, P)e^{\mathrm{op}}w^{\mathrm{op}}, \mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})\\
\cong&\mathcal{T}ricat((\mathcal{P}')^{\mathrm{op}}, \mathcal{G}ray)(\mathrm{st}\mathcal{P}(-, P)e^{\mathrm{op}}w^{\mathrm{op}}, (\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})_{\mathrm{ls}})\\
\cong&\mathcal{T}ricat_{\mathrm{ls}}((\mathcal{P}')^{\mathrm{op}}, \mathcal{G}ray)(i(\mathrm{st}\mathcal{P})(-, f(P))w^{\mathrm{op}}, (\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})_{\mathrm{ls}})\\
\cong&\mathcal{T}ricat_{\mathrm{ls}}((\mathcal{P}')^{\mathrm{op}}, \mathcal{G}ray)(i\mathcal{P}'(-, y(f(P))), (\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})_{\mathrm{ls}})\\
\cong&\mathcal{T}ricat_{\mathrm{ls}}((\mathcal{P}')^{\mathrm{op}}, \mathcal{G}ray)(i\mathcal{P}'(-, y(f(P))), iL(\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})_{\mathrm{ls}})\\
\cong&(\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})_{\mathrm{ls}}(y(f(P)))\\
\cong&(\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})(y(f(P)))\\
\cong&AP\ .
\end{aligned}$$

The first biequivalence is induced by the construction of the functor tricategory $\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)$ by transport of structure from the biequivalences

$$\mathrm{st}_*\colon\ \mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)(\mathcal{P}(-,P),A) \to \mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(\mathrm{st}\mathcal{P}(-,P), \mathrm{st}A) \tag{2.59}$$

given by whiskering on the left with the strictification functor st, cf. Theorem A.4. Its naturality in P,

$$\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)(-,A)Y^{\mathrm{op}} \Rightarrow \mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(-, \mathrm{st}A)\mathrm{st}_*^{\mathrm{op}}Y^{\mathrm{op}}\ , \tag{2.60}$$

follows from naturality of the comparison of representables

$$\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)(-,A) \Rightarrow \mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(-, \mathrm{st}A)\mathrm{st}_*^{\mathrm{op}} \tag{2.61}$$

cf. Theorem 2.8.

The second biequivalence is the biequivalence $(ew)^*$ given by whiskering on the right with the triequivalence $ew\colon\ \mathcal{P}' \cong \mathcal{P}$ cf. Theorem A.9 . For its naturality in P,

$$\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{G}ray)(-, \mathrm{st}A)\mathrm{st}_*^{\mathrm{op}}Y^{\mathrm{op}} \Rightarrow \mathcal{T}ricat((\mathcal{P}')^{\mathrm{op}}, \mathcal{G}ray)(-, \mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})((ew)^*)^{\mathrm{op}}\mathrm{st}_*^{\mathrm{op}}Y^{\mathrm{op}}\ ,$$

note that $(ew)^*$ is a $\mathcal{G}ray$-functor, see Theorem A.6. Hence, naturality in P follows from naturality of the obvious comparison of representables of $\mathcal{G}ray$-categories.

For the third biequivalence, note that we can replace each local functor of $\mathrm{st}Ae^{\mathrm{op}}\colon\ \mathrm{st}\mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$ by an equivalent strict functor. The same then applies to $\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}}\colon\ \mathrm{st}\mathcal{P}^{\mathrm{op}} \to \mathcal{G}ray$ because w is locally strict and $\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}}$ denotes the biequivalent locally strict functor obtained from change of local functors cf. Theorem 2.5. Replacing $\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}}$ by $(\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})$ gives rise to a biequivalence as required, which is clearly natural in P.

The fourth biequivalence and its naturality in P follow from Theorems 2.10 and 2.11.

The fifth biequivalence and its naturality in P follow from Theorems 2.7 and 2.7

The sixth biequivalence follows from the fact that the unit of the adjunction $L \dashv i$ from Chapter 1 is a biequivalence cf. Theorem 1.8. Its naturality in P is obvious.

The seventh biequivalence is Theorem 2.3—the Yoneda lemma for $\mathcal{G}ray$-categories and locally strict trihomomorphisms.

In the eighth biequivalence we go back from the biequivalent $(\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})_{\mathrm{ls}}$ obtained from change of local functors to $\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}}$ again.

The ninth biequivalence follows from the fact that $\mathrm{in}\,\mathrm{st}\colon\ \mathcal{B}icat \to \mathcal{B}icat$ is biequivalent to the identity trihomomorphism, where in is inclusion.

Recalling how the strictification acts on functors, pseudonatural transformations, and modifications, noting that e and w are the identity on objects and recalling that $(\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}})_{\mathrm{ls}}$ agrees with $\mathrm{st}Ae^{\mathrm{op}}w^{\mathrm{op}}$ on objects, shows that evaluation of the component at P at the identity as in Lemma 2.11 induces a biequivalence as well. □

Chapter 3

A three-dimensional descent construction

In this chapter we give a new definition of the descent construction for a trihomomorphism

$$\mathfrak{X}\colon\ C^{\mathrm{op}} \to \mathcal{B}icat\ , \tag{3.1}$$

where C is a small category considered as a locally discrete tricategory, and where $\mathcal{B}icat$ is the tricategory of bicategories. Such a trihomomorphism has been called a presheaf in bicategories in Nikolaus and Schweigert [61]. The descent construction for $\mathfrak{X}$ is another such trihomomorphism

$$\mathrm{Desc}(\mathfrak{X})\colon\ [\Delta_\delta^{\mathrm{op}}, C]^{\mathrm{op}} \to \mathcal{G}ray \tag{3.2}$$

from the dual of the category of simplicial objects in C i.e. the functor category $[\Delta_\delta^{\mathrm{op}}, C]$. Note here that we here use the subcategory Δ_δ of the simplex category Δ spanned by the coface maps in order to compare the construction with the descent construction of Nikolaus and Schweigert [61].

On a simplicial object Γ, the descent construction would usually be defined by singling out data with coherence conditions from the pseudocosimplicial bicategory given by the composite trihomomomorphism $\mathfrak{X}\Gamma^{\mathrm{op}}$. The cocycle condition for transition functions of a bundle on three-fold intersections is an example of this. In this case, we would take C to be the category $\mathcal{M}an$ of smooth manifolds. Then an open covering provides a simplicial space via multiple intersections. When $\mathfrak{X}$ is the presheaf of smooth functions into the structure group, $\mathrm{Desc}(\mathfrak{X})$ should associate to this simplicial space a bicategory related to principal bundles in terms of transition functions for this open covering. Expanding on such ideas an explicit definition of $\mathrm{Desc}(\mathfrak{X})$ has been given in [61]. Starting with the presheaf of principal bundles, the authors have shown that the descent construction leads to the notion of bundle gerbes via a plus construction. However, the definition suppresses coherence isomorphisms and equivalences, which is why we propose a new definition.

3.1 The definition of the descent construction

Let $*\colon \Delta_\delta \to \mathcal{B}icat$ denote the constant trihomomorphism at the unit 2-category I.

Definition 3.1. Given a trihomomorphism $\mathfrak{X}\colon C^{\mathrm{op}} \to \mathcal{B}icat$ on the dual of a small category C, the descent construction for $\mathfrak{X}$,

$$\mathrm{Desc}(\mathfrak{X})\colon [\Delta_\delta^{\mathrm{op}}, C]^{\mathrm{op}} \to \mathcal{B}icat \tag{3.3}$$

is defined to be the composite of the trihomomorphism

$$\mathfrak{X}_*\colon [\Delta_\delta, C^{\mathrm{op}}] \to \mathcal{T}ricat(\Delta_\delta, \mathcal{B}icat) \tag{3.4}$$

and the trihomomorphism

$$\mathcal{T}ricat(\Delta_\delta, \mathcal{B}icat)(*, -)\colon \mathcal{T}ricat(\Delta_\delta, \mathcal{B}icat) \to \mathcal{B}icat\ , \tag{3.5}$$

where $\mathfrak{X}_*$ is whiskering on the left with the trihomomorphism $\mathfrak{X}$ cf. A.4 and $\mathcal{T}ricat(\Delta_\delta, \mathcal{B}icat)(*, -)$ is the representable of the functor tricategory $\mathcal{T}ricat(\Delta_\delta, \mathcal{B}icat)$ cf. A.2.

In particular, given a simplicial object $\Gamma\colon \Delta_\delta^{\mathrm{op}} \to C$ in C, the *bicategory of descent data*,

$$\mathrm{Desc}(\mathfrak{X})(\Gamma) = \mathcal{T}ricat(\Delta_\delta, \mathcal{B}icat)(*, \mathfrak{X}\Gamma^{\mathrm{op}})\ , \tag{3.6}$$

is the bicategory of tritransformations $* \Rightarrow \mathfrak{X}\Gamma^{\mathrm{op}}$.

Remark 3.1. Several remarks on this definition have to be made. First, note that the ordinary functor category $[\Delta_\delta^{\mathrm{op}}, C]^{\mathrm{op}} \cong [\Delta_\delta, C^{\mathrm{op}}]$ coincides with the functor tricategory $\mathcal{T}ricat(\Delta_\delta, C^{\mathrm{op}})$. Second, we prove in Theorem A.3 that whiskering on the left gives rise to a trihomomorphism as required, where $\mathcal{T}ricat(\Delta_\delta, \mathcal{B}icat)$ inherits its tricategory structure from the $\mathcal{G}ray$-category $\mathcal{T}ricat(\Delta_\delta, \mathcal{G}ray)$ via transport of structure cf. Theorem A.4. Third, the definition involves the composite of trihomomorphisms cf. A.3.

Just at the set of natural transformations between two fixed ordinary functors can be described as an end, the bicategories of tritransformations (3.6) can be understood as a simple tricategorical limit. The concept of such a 'trilimit' is introduced in Power [65, Def. 7.3] as a 'trirepresentation' much like the concept of a weighted limit in enriched category theory cf. (1.26). Indeed we will see in the next section that the bicategories of descent data strictify to 2-categories of $\mathcal{G}ray$-natural transformations cf. (3.9), and the latter constitute a simple $\mathcal{G}ray$-enriched limit. This is why we say that the descent construction $\mathrm{Desc}(\mathfrak{X})$ is given by a simple tricategorical limit. Correspondingly, we call the constant trihomomorphism $*\colon \Delta_\delta \to \mathcal{B}icat$ at the unit 2-category I the *weight* of the three-dimensional descent construction.

3.2 Strictification of the descent construction

In order to compare Definition 3.1 with the descent construction from Nikolaus and Schweigert [61], we have to make the transition to the *Gray*-enriched context.

On objects, it is obvious how we have to strictify the bicategories of descent data (3.1). Namely, whiskering on the left with the strictification functor $\mathrm{st}\colon \mathcal{B}icat \to \mathcal{G}ray$ gives rise to biequivalences

$$\mathrm{st}_*\colon \mathcal{T}ricat(\Delta_\delta, \mathcal{B}icat)(*, \mathfrak{X}\Gamma^{\mathrm{op}}) \to \mathcal{T}ricat(\Delta_\delta, \mathcal{G}ray)(\mathrm{st}*, \mathrm{st}\mathfrak{X}\Gamma^{\mathrm{op}}) \ . \tag{3.7}$$

Now note that the strict functor $e\colon \mathrm{st}I \to I$ has a strict biequivalence-inverse given by the strict functor that sends the single object in I to the single object in the strictification $\mathrm{st}I$ of I. It follows that the constant *Gray*-functors $\mathrm{st}*$ and $*$ are biequivalent in the functor tricategory $\mathcal{T}ricat(\Delta_\delta, \mathcal{G}ray)$. On the other hand, the composite trihomomorphism $\mathrm{st}\mathfrak{X}$ is biequivalent to a locally strict trihomomorphism $(\mathrm{st}\mathfrak{X})_{\mathrm{ls}}$ by Corollary 2.6 from Chapter 2. Hence, we apply the hom adjunction of the left adjoint L to the inclusion $i\colon [\Delta_\delta, \mathcal{G}ray] \to \mathcal{T}ricat_{\mathrm{ls}}(\Delta_\delta, \mathcal{G}ray)$ from Chapter 1 and the fact that its unit is a strict biequivalence to obtain the isomorphism below

$$\mathcal{T}ricat(\Delta_\delta, \mathcal{G}ray)(*, i(L(\mathrm{st}\mathfrak{X})_{\mathrm{ls}}\Gamma^{\mathrm{op}})) \cong [\Delta_\delta, \mathcal{G}ray](L*, L(\mathrm{st}\mathfrak{X})_{\mathrm{ls}}\Gamma^{\mathrm{op}}) \ . \tag{3.8}$$

Summing up, this gives rise to the biequivalence

$$\mathrm{Desc}(\mathfrak{X})(\Gamma) \cong [\Delta_\delta, \mathcal{G}ray](L*, L(\mathrm{st}\mathfrak{X})_{\mathrm{ls}}\Gamma^{\mathrm{op}}). \tag{3.9}$$

Since the composition st of the strictification $\mathrm{st}\colon \mathcal{B}icat \to \mathcal{G}ray$ and the inclusion $\mathrm{in}\colon \mathcal{G}ray \to \mathcal{B}icat$ is biequivalent to the identity, a technical argument[1] shows that (3.9) is appropriately natural in Γ to give rise to a strictification of the descent construction itself:

$$\mathrm{Desc}(\mathfrak{X}) \cong \mathrm{in}[\Delta_\delta, \mathcal{G}ray](L*, -)[1, L(\mathrm{st}\mathfrak{X})_{\mathrm{ls}}] \ . \tag{3.10}$$

Remark 3.2. The representable $[\Delta_\delta, \mathcal{G}ray](L*, -)$ and $[1, L(\mathrm{st}\mathfrak{X})_{\mathrm{ls}}]$ are both *Gray*-functors, where the latter is given by postcomposition with $L(\mathrm{st}\mathfrak{X})_{\mathrm{ls}}$ on objects. The strictification (3.10) motivates the following definition of a descent construction for *Gray*-functors.

Definition 3.2. Given a *Gray*-functor $\mathfrak{X}\colon C^{\mathrm{op}} \to \mathcal{G}ray$ on the dual of a small category C, the descent construction for $\mathfrak{X}$ is defined by the *Gray*-functor:

$$\mathrm{Desc}(\mathfrak{X}) \coloneqq [\Delta_\delta, \mathcal{G}ray](L*, -)[1, \mathfrak{X}]\colon [\Delta_\delta^{\mathrm{op}}, C]^{\mathrm{op}} \to \mathcal{G}ray \tag{3.11}$$

In particular, given a simplicial object $\Gamma\colon \Delta_\delta^{\mathrm{op}} \to C$ in C, the 2-*category of descent data*,

$$\mathrm{Desc}(\mathfrak{X})(\Gamma) = [\Delta_\delta, \mathcal{G}ray](L*, \mathfrak{X}\Gamma^{\mathrm{op}}) \ , \tag{3.12}$$

is the 2-category of *Gray*-natural transformations $L* \Rightarrow \mathfrak{X}\Gamma^{\mathrm{op}}$.

[1] involving Theorems 2.8, A.6, and Prop. A.16

Recall that the left adjoint L is given on objects by forming the codescent object of the associated codescent diagram. Since $*$ is a $\mathcal{G}ray$-functor, this is the codescent object of a strict codescent diagram as we have analyzed it in Chapter 2. In the next section 3.3 we identify the codescent object $L*$ explictly. This allows us to justify our definition of the descent construction in Section 3.4, where we prove that the strict descent construction of Definition 3.2 coincides with the descent construction from Nikolaus and Schweigert [61].

3.3 The codescent object of the weight of the descent construction

Let again $I\colon \mathcal{C}at \to \mathcal{G}pd$ be the groupoidification functor cf. Def. 2.2, and let $I_\bullet$ be its composition with the inclusions $\Delta_\delta \to \mathcal{C}at$ and $\mathcal{G}pd \to 2\mathcal{C}at$. In this section we consider a slight variation of the strictification $\mathrm{st}X$ of a small 2-category X, where the length-1-strings in an identity 1-cell of X are identitified with the corresponding empty string i.e. the identity 1-cell in the 2-category $\mathrm{st}X$. Obviously, this identification may be carried out by adjoining equations of 1-cells for each object of X. This process has the form of a pushout over the small set of objects of the 2-category X—a colimit whose existence is ensured by the cocompleteness of $2\mathcal{C}at$. We denote the resulting 2-category by $\mathrm{st}'X$. It is obvious that st' can be extended to normalized pseudofunctors, pseudonatural transformations of normalized pseudofunctors (for respect of units we need that X is a 2-category), and modifications of the latter.

The strictification $\mathrm{st}'X$ classifies normalized pseudofunctors: a 2-functor $\mathrm{st}'X \to Y$ corresponds precisely to a normalized pseudofunctor $X \to Y$. In fact, there is a natural isomorphism of 2-categories

$$[\mathrm{st}'X, Y] \cong \mathcal{B}icat_{\mathrm{n}}(X, Y)\ , \tag{3.13}$$

where on the right, we have the 2-category of normalized pseudofunctors, pseudonatural transformations, and modifications.

Theorem 3.1. *The codescent object X for the codescent diagram $X_\bullet$ associated with the constant $\mathcal{G}ray$-functor $*\colon \Delta_\delta \to \mathcal{G}ray$ is biequivalent in $[\Delta_\delta, \mathcal{G}ray]$ to the composite $\mathcal{G}ray$-functor $\mathrm{st}'I_\bullet\colon \Delta_\delta \to \mathcal{G}ray$.*

Proof. The proof is organized as follows. We first produce a $\mathcal{G}ray$-natural transformation $\upsilon\colon X_0 \to \mathrm{st}'I_\bullet$ which satisfies the assumptions in the first part of the universal property of the codescent object, and so gives rise to a $\mathcal{G}ray$-natural transformation $\varphi\colon X \to \mathrm{st}'I_\bullet$ such that $\varphi x = \upsilon$ cf. Lemma 3.1 below. Next, we construct a $\mathcal{G}ray$-natural transformation $\psi\colon \mathrm{st}'I_\bullet \to X$ such that, firstly,

$$\varphi\psi = 1_{\mathrm{st}'I_\bullet} \tag{3.14}$$

cf. Lemma 3.2 below. Secondly, $\psi\varphi$ and 1_X satisfy the conditions in the second part of the universal property of X cf. Lemma 3.3 below such that we can employ Lemma 2.5 from 2.1 to show that there is an adjoint equivalence between $\psi\varphi$ and 1_X in $[\mathrm{ob}\Delta_\delta, \mathcal{G}ray]^T \cong [\Delta_\delta, \mathcal{G}ray]$. This completes the proof that X and $\mathrm{st}'I_\bullet$ are biequivalent in $[\Delta_\delta, \mathcal{G}ray]$. □

Construction of υ. Recall that $X_0 = T*$ has the following form on objects cf. (1.38) from 1.3.2:

$$X_0([j]) = \Sigma_{k\in\mathbf{N}_0}\Delta_\delta([k],[j]) \otimes I \cong \Sigma_{k\in\mathbf{N}_0}\Delta_\delta([k],[j]) \ .$$

In particular, the *Gray*-functor X_0 factorizes through the sub-*Gray*-category of discrete 2-categories. For $j = 0$, notice that $\Delta_\delta([k],[0])$ is isomorphic to I for $k = 0$ (with single object the identity of [0]) and empty otherwise. On the other hand $\mathrm{st}'I_0 = [0]$. Hence, we define $\upsilon_0 \colon X_0([0]) \to \mathrm{st}'I_0$ by

$$\upsilon_0(1_{[0]}) = 0 \in \mathrm{st}'([0]) = [0] \ . \tag{3.15}$$

Given a morphism $\delta \colon [0] \to [j]$ in Δ_δ, observe that $X_0(\delta)(1_{[0]}) = \delta 1_{[0]} = \delta$. Hence, we define

$$\upsilon_j(\delta) = (\mathrm{st}'I_\bullet)(\delta)(0) = \delta(0) \tag{3.16}$$

in $\mathrm{st}'I_j$ using that $(\mathrm{st}'I_\bullet)(\delta)$ agrees with δ on objects. Notice that we have first defined the component of υ at [0] as a 2-functor $I \to \mathrm{st}'[0] = [0]$ by specifying its image on the single object $1_{[0]}$. The component of υ at $[j]$ is given by a 2-functor

$$\Sigma_{k\geq 0}\Delta([k],[j]) \to \mathrm{st}'I_j \tag{3.17}$$

of which we defined the component at $k = 0$,

$$\Delta([0],[j]) \to \mathrm{st}'I_j \ , \tag{3.18}$$

by specifying its image in $\mathrm{ob}(\mathrm{st}'I_j) = \mathrm{ob}[j]$ on each of the objects δ in such a way that this prescription is natural with respect to δ considered as a morphism in Δ_δ.

We now define it on a general morphism $\epsilon \colon [k] \to [j]$ by

$$\upsilon_j(\epsilon) \coloneqq \upsilon_j(\epsilon\delta_k\delta_{k-1}...\delta_1) \ , \tag{3.19}$$

where δ_j is the coface map in Δ_δ. That is, $\delta_j \colon [n] \to [n+1]$ is the unique order-preserving injection whose image does not contain j, so $\delta_k\delta_{k-1}...\delta_1 \colon [0] \to [k]$ is the unique order preserving injection which sends 0 to 0. This prescription is in fact *Gray*-natural since for $\gamma \colon [j] \to [l]$ in Δ we have:

$$\begin{aligned}
&\upsilon_l((X_0(\gamma))(\epsilon)) \\
=&\upsilon_l(\gamma\epsilon) \\
=&\upsilon_l(\gamma\epsilon\delta_k\delta_{k-1}...\delta_1) \\
=&\gamma\epsilon\delta_k\delta_{k-1}...\delta_1(0) \\
=&\gamma(\upsilon_j(\epsilon\delta_k\delta_{k-1}...\delta_1)) \\
=&(\mathrm{st}'I_\bullet)(\gamma)(\upsilon_j(\epsilon)) \ .
\end{aligned}$$

Lemma 3.1. *The Gray-natural transformation υ satisfies the conditions in the first part of the universal property for X.*

Proof. We have to define an adjoint equivalence 1-cell $\bar{\epsilon}\colon \upsilon d_1 \Rightarrow \upsilon d_0$ in the hom 2-category $[\Delta_\delta, \mathcal{G}ray](X_1, \mathrm{st}'I_\bullet)$ of the functor $\mathcal{G}ray$-category. In general, there are two conditions on the components of such a 1-cell cf. Corollary 1.5 from 1.5.3. However, the condition (1.63) on interchange cells is vacuous since the domain Δ_δ is a category. Hence, there remains the naturality condition (1.62). The components

$$\bar{\epsilon}_j\colon \upsilon_j(d_1)_j \Rightarrow \upsilon_j(d_0)_j$$

of $\bar{\epsilon}$ are pseudonatural transformations. Let $\delta\colon [k] \to [j]$ and $\delta'\colon [j] \to [l]$ be morphisms in Δ, then the naturality condition requires that

$$(\bar{\epsilon}_l)_{\delta',\delta} = (\bar{\epsilon}_l)_{X_1(\delta')(1_{[j]},\delta)} = (\mathrm{st}'I_\bullet)(\delta')((\bar{\epsilon}_j)_{1_{[j]},\delta})\ .$$

A specification of adjoint equivalences $(\bar{\epsilon}_j)_{1_{[j]},\delta}$ for all j, k, and all $\delta\colon [k] \to [j]$ hence determines a 1-cell $\bar{\epsilon}\colon \upsilon d_1 \to \upsilon d_0$ as wanted ($(\bar{\epsilon}_j)_{1_{[j]},\delta}$ is trivially a pseudonatural transformation for fixed j because there are no nontrivial 1- and 2-cells in $X_1([j])$).

According to our definition (3.19) and (3.15) of υ and since $d_1(1_{[j]},\delta) = 1_{[j]}$ and $d_0(1_{[j]},\delta) = \delta$, we have to specify adjoint equivalences

$$(\bar{\epsilon}_j)_{1_{[j]},\delta}\colon \delta_j\delta_{j-1}...\delta_1(0) \to \delta\delta_k\delta_{k-1}...\delta_1(0) \tag{3.20}$$

i.e.

$$(\bar{\epsilon}_j)_{1_{[j]},\delta}\colon 0 \to \delta(0) \tag{3.21}$$

in $\mathrm{st}'I_j$. We choose (3.20) to be the shortest adoint equivalence in $\mathrm{st}'I_j$—i.e. the length-1-string in the unique isomorphism in I_j. Note that this is the identity i.e. empty string in case that $\delta(\delta_k\delta_{k-1}...\delta_1(0)) = \delta(0) = 0$. For example,if $j = 1$ and $k = 0$ and $\delta = \delta_0$, this means that

$$(\bar{\epsilon}_1)_{1_{[1]},\delta_0} = 01\colon 0 \to 1$$

as a length-1-string in $\mathrm{st}'I_1$. The 1-cell $\bar{\epsilon}$ has an apparent right adjoint equivalence-inverse $\bar{\epsilon}\colon \upsilon d_0 \Rightarrow \upsilon d_1$.

We now have to construct invertible 2-cells

$$\bar{M}\colon (\bar{\epsilon}1_{d_0}) * (\bar{\epsilon}1_{d_2}) \Rightarrow (\bar{\epsilon}1_{d_1})$$

and

$$\bar{U}\colon \bar{\epsilon}1_{s_0} \Rightarrow 1_\upsilon$$

in the hom 2-categories $[\Delta, \mathcal{G}ray](X_2, \mathrm{st}'I_\bullet)$ and $[\Delta, \mathcal{G}ray](X_0, \mathrm{st}'I_\bullet)$ respectively. Such 2-cells are given by component-2-cells $\bar{M}_j$ and $\bar{U}_j$ in $\mathcal{G}ray$—i.e. modifications for each $[j]$ in Δ—which are subject to a naturality condition cf. Corollary 1.5 from 1.5.3. Let $\delta\colon [k] \to [j]$, $\delta'\colon [j] \to [l]$, and $\delta''\colon [l] \to [m]$ be morphisms in Δ. Then the naturality conditions require that

$$(\bar{M}_m)_{\delta'',\delta',\delta} = (\bar{M}_m)_{X_2(\delta'')(1_{[l]},\delta',\delta)} = (\mathrm{st}'I_\bullet)(\delta'')((\bar{M}_l)_{1_{[l]},\delta',\delta}) \tag{3.22}$$

and

$$(\bar{U}_j)_\delta = (\bar{U}_j)_{X_0(\delta)(1_{[k]})} = (\mathrm{st}'I_\bullet)(\delta)((\bar{U}_k)_{1_{[k]}}) \ . \tag{3.23}$$

Since by definition of $\bar{\epsilon}_k$ and υ_k we have

$$(\bar{\epsilon}_k)_{s_0(1_{[k]})} = (\bar{\epsilon}_k)_{1_{[k]},1_{[k]}} = 1_{\delta_k\delta_{k-1}\ldots\delta_1(0)} = 1_{\upsilon_k(1_{[k]})} \ ,$$

we can take

$$(\bar{U}_k)_{1_{[k]}} \colon (\bar{\epsilon}_k)_{s_0(1_{[k]})} \to 1_{\upsilon_k(1_{[k]})}$$

to be the identity 2-cell. For fixed k, this is trivially a modification (the identity modification) and the prescription is extended to a 2-cell in $[\Delta, \mathcal{G}ray](X_1, \mathrm{st}'I_\bullet)$ according to (3.23).

For

$$(\bar{M}_l)_{1_{[l]},\delta',\delta} \colon ((\mathrm{st}'I_\bullet)(\delta')((\bar{\epsilon}_j)_{1_{[j]},\delta})) * (\bar{\epsilon}_l)_{1_{[l]},\delta'} \Rightarrow (\bar{\epsilon}_l)_{1_{[l]},\delta'\delta}$$

we take the unique invertible 2-cell between those parallel 1-cells in $\mathrm{st}'I_l$. For fixed l, these trivially give rise to modifications, and this prescription is extended to a 2-cell in $[\Delta, \mathcal{G}ray](X_3, \mathrm{st}'I_\bullet)$ according to (3.22).

The three axioms in the first part of the universal property of the codescent object reduce to equations of 2-cells in some $\mathrm{st}'I_n$, which must hold since any equation of parallel 2-cells holds in $\mathrm{st}'I_n$. □

Thus by the first part of the universal property of the codescent object X, there is a unique 1-cell i.e. $\mathcal{G}ray$-natural transformation $\varphi \colon X \to \mathrm{st}'I_\bullet$ such that $\varphi x = \upsilon$, $1_\varphi \epsilon = \bar{\epsilon}$, $1_\varphi M = \bar{M}$, and $1_\varphi U = \bar{U}$.

Construction of ψ: Let $\iota \colon [0] \to [j]$ be the unique (order preserving) function sending 0 to i where $0 \le i \le j$. Then $(\mathrm{st}'I_\bullet)(\iota)$ is the unique 2-functor $\mathrm{st}'[0] = [0] \to \mathrm{st}'[j]$ sending 0 to i. Thus each component

$$\alpha_j \colon \mathrm{st}'I_j \to X([j])$$

of a $\mathcal{G}ray$-natural transformation $\alpha \colon \mathrm{st}'I_\bullet \to X$ is determined on objects by its component α_0 at [0]. Namely, for each $i \in \mathrm{ob}(\mathrm{st}'I_j) = \mathrm{ob}[j]$ we have

$$\alpha_j(i) = \alpha_j((\mathrm{st}'I_\bullet)(\iota)(0)) = X(\iota)(\alpha_0(0)) \ . \tag{3.24}$$

Thus the components of ψ are determined on objects by the prescription

$$\psi_0(0) = x_0(1_{[0]}) \in X([0]) \ . \tag{3.25}$$

Note that by $\mathcal{G}ray$-naturality of x we have

$$\begin{aligned}
&\psi_j(i) \\
=&X(\iota)(\psi_0(0)) && \text{(by (3.24))} \\
=&X(\iota)(x_0(1_{[0]})) \\
=&x_j(X_0(\iota)(1_{[0]})) && \text{(by nat. of } x\text{)} \\
=&x_j(\iota 1_{[0]}) \\
=&x_j(\iota) \ .
\end{aligned}$$

In particular, for an object of $X_0([j])$ i.e. a morphism $\delta\colon [k] \to [j]$ in Δ_δ we have

$$\begin{aligned} &\psi_j(\upsilon_j(\delta)) \\ =&\psi_j(\delta\delta_k\delta_{k-1}...\delta_1(0)) \\ =&x_j(\delta\delta_k\delta_{k-1}...\delta_1) \end{aligned} \tag{3.26}$$

because $\delta\delta_k\delta_{k-1}...\delta_1$ clearly sends 0 to $\delta\delta_k\delta_{k-1}...\delta_1(0)$.

Next we define ψ_j on 1-cells. Let f be a 1-cell in $\mathrm{st}'I_j$. This is a string $f_1 \cdot f_2 \cdot ... \cdot f_n$ of length greater or equal to 1 of morphisms in I_j (the empty string was identified with the length-1-string in the identity 1-cell). Since ψ_j is supposed to be a 2-functor, we define it on length-1-strings and extend this prescription to general strings by strictness. Hence, let f be a length-1-string given by a 1-cell f of the same name in I_j. The latter is either a morphism from $[j]$ or its inverse in I_j. Assume that the former is the case. Note that f is necessarily an adjoint equivalence in $\mathrm{st}'(I_j)$. Let $\zeta\colon [1] \to [j]$ be the unique order preserving function sending 01 to f where $0 \leq i < j$. Then $\mathrm{st}'I_\bullet(\zeta)$ is the unique 2-functor $\mathrm{st}'(I_1) \to \mathrm{st}'(I_j)$ sending the free-living adjoint equivalence 01 in $\mathrm{st}'(I_1)$ to the adjoint equivalence f in $\mathrm{st}'(I_j)$.

Again, if ψ is to be $\mathcal{G}ray$-natural, we must have

$$\psi_j(f) = \psi_j((\mathrm{st}'I_\bullet)(\zeta)(01)) = X(\zeta)(\psi_1(01)) \ .$$

Hence, specifying the effect of ψ_1 on 01 appropriately determines the component 2-functors ψ_j of ψ on arbitrary 1-cells. We define $\psi_1\colon \mathrm{st}'I_1 \to X([1])$ to be the 2-functor on the free living adjoint equivalence corresponding to the normalized pseudofunctor $\tilde{\psi}_1\colon I_1 \to X([1])$ determined by sending the nontrivial isomorphism 01 to the following adjoint equivalence in $X([1])$. We know that $\psi_1(0) = x_1(\delta_1)$, $\psi_1(1) = x_1(\delta_0)$, $d_0(1_{[1]}, \delta_1) = 1_{[1]}\delta_1 = \delta_1$ and similarly $d_0(1_{[1]}, \delta_0) = \delta_0$, and $d_1(1_{[1]}, \delta_1) = 1_{[1]}$ and $d_1(1_{[1]}, \delta_0) = 1_{[1]}$. Hence, we define

$$\psi_1(01) \coloneqq (\epsilon_1)_{1_{[1]},\delta_0} * (\epsilon_1^\bullet)_{1_{[1]},\delta_1} \ . \tag{3.27}$$

Next note that the component 2-functor $\psi_j\colon \mathrm{st}'I_j \to X([j])$ corresponds to a normalized pseudofunctor $\tilde{\psi}_j\colon I_j \to X([j])$. Naturality of ψ is tantamount to naturality of $\tilde{\psi}\colon I_\bullet \to X$ where $I_\bullet$ and X are considered as ordinary functors $\Delta \to 2\mathcal{C}at_{\mathrm{ps}}$.

Let now f and g be composable 1-cells in I_j, $j > 2$. Then there is a unique order preserving function $\xi\colon [2] \to [j]$ such that fg is the image under $I_\bullet(\xi)$ of a unique composite $f'g'$ in I_2.

If $\tilde{\psi}$ is to be natural, we must have the following equality of composition constraints

$$(\tilde{\psi}_j)_{f,g} = (\tilde{\psi}_j)_{I_\bullet(\xi)(f'),I_\bullet\xi(g')} = X(\xi)((\tilde{\psi}_2)_{f',g'}) \ .$$

Thus since we have already determined $\tilde{\psi}_j$ on objects and 1-cells and since I_j only has identity 2-cells, we will have defined the data of the normalized pseudofunctor $\tilde{\psi}_j$ as soon as we determine the composition constraints for $\tilde{\psi}_2$. Then it only remains to prove the axioms of a normalized pseudofunctor.

Next note that the composition constraints $(\tilde{\psi}_2)_{01,10}$, $(\tilde{\psi}_2)_{12,21}$, and $(\tilde{\psi}_2)_{02,20}$ are already determined by naturality and the definition of $\tilde{\psi}_1$. In fact, the only constraint that we have to specify is

$$(\tilde{\psi}_2)_{12,01}\colon\ X(\delta_0)(\psi_1(01)) * X(\delta_2)(\psi_1(01)) \Rightarrow X(\delta_1)(\psi_1(01))\ . \tag{3.28}$$

By naturality of $\epsilon\colon\ xd_1 \to xd_0\colon\ X_1 \to X$,

$$\begin{aligned} &X(\delta_1)(\psi_1(01)) \\ =&X(\delta_1)((\epsilon_1)_{1_{[1]},\delta_0} * (\epsilon_1^\bullet)_{1_{[1]},\delta_1}) \\ =&(\epsilon_1)_{\delta_1,\delta_0} * (\epsilon_1^\bullet)_{\delta_1,\delta_1}\ , \end{aligned}$$

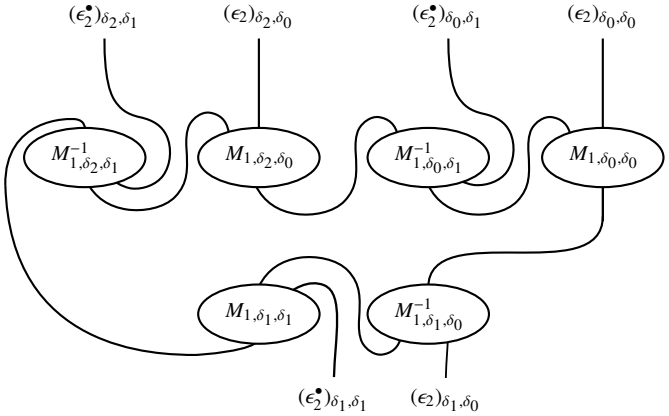
and similarly,

$$\begin{aligned} &(\tilde{\psi}_2)_{12,01}\colon\ X(\delta_0)(\psi_1(01)) * X(\delta_2)(\psi_1(01)) \\ =&(\epsilon_1)_{\delta_0,\delta_0} * (\epsilon_1^\bullet)_{\delta_0,\delta_1} * (\epsilon_1)_{\delta_2,\delta_0} * (\epsilon_1^\bullet)_{\delta_2,\delta_1}\ . \end{aligned}$$

Hence, (3.28) has the form

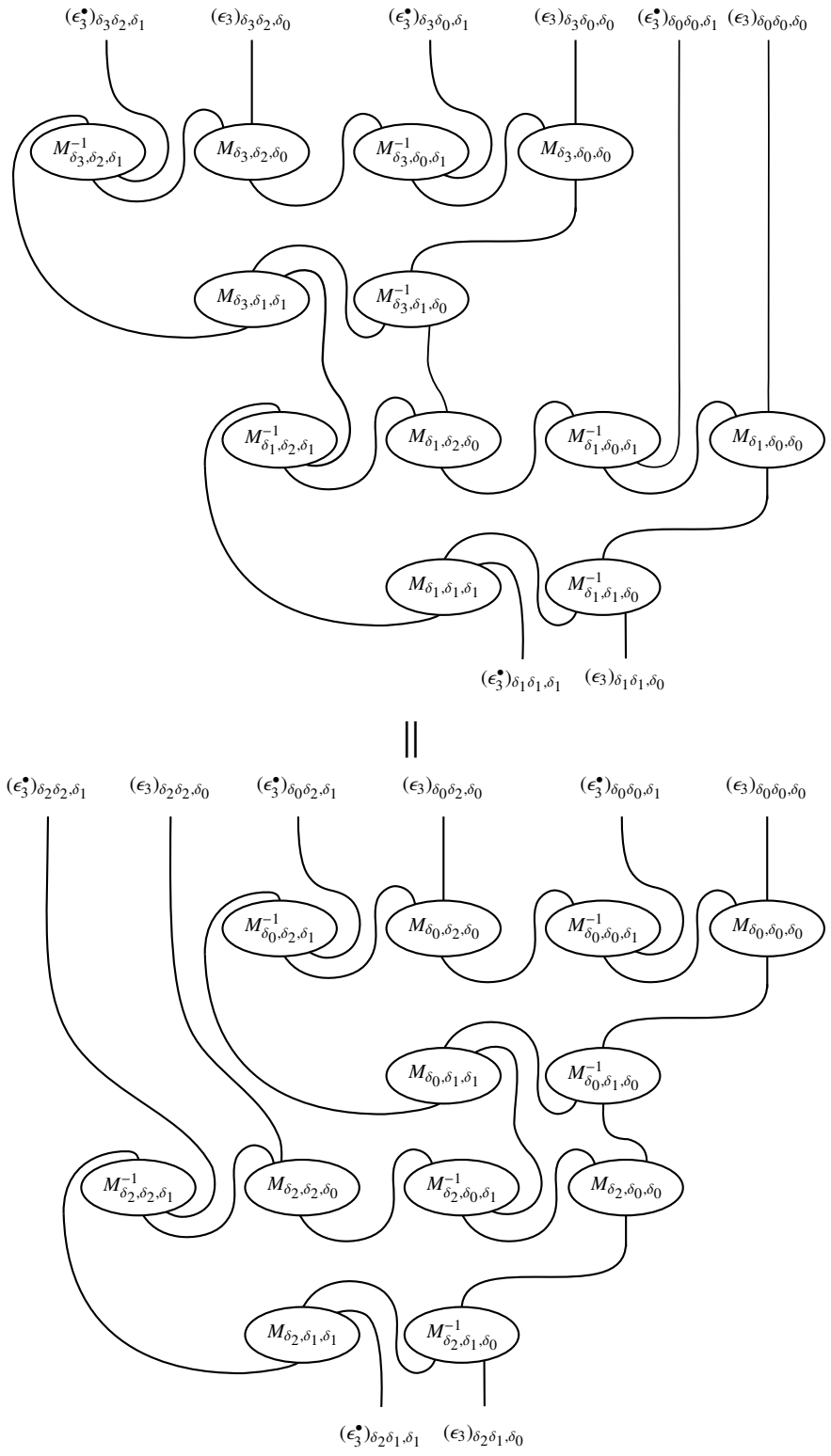
$$(\tilde{\psi}_2)_{12,01}\colon\ (\epsilon_1)_{\delta_0,\delta_0} * (\epsilon_1^\bullet)_{\delta_0,\delta_1} * (\epsilon_1)_{\delta_2,\delta_0} * (\epsilon_1^\bullet)_{\delta_2,\delta_1} \Rightarrow (\epsilon_1)_{\delta_1,\delta_0} * (\epsilon_1^\bullet)_{\delta_1,\delta_1}\ . \tag{3.29}$$

The constraint (3.29) is defined by the string diagram below.

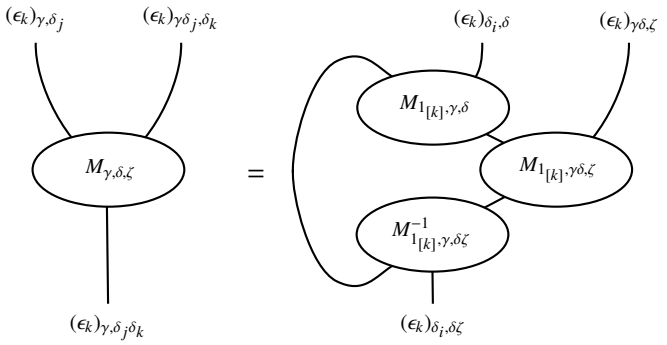


By naturality, we only have to prove the pseudofunctor axiom for composition constraints for $\tilde{\psi}_3$ depicted in Figure 3.3. To prove this axiom, one has to transform it into a form, where one can use the simplicial identities. To do so, we apply at each node of the left hand side of the axiom the relation (3.30) below (or its inverse), which is a

Figure 3.1.: Pseudofunctor axiom



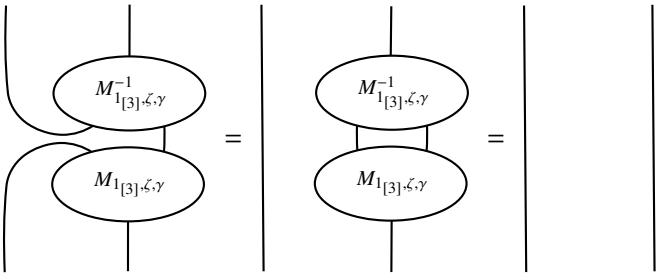
direct consequence of the first axiom (2.1) of the codescent object.



(3.30)

This results in the 2-cell depicted in Figure 3.3.

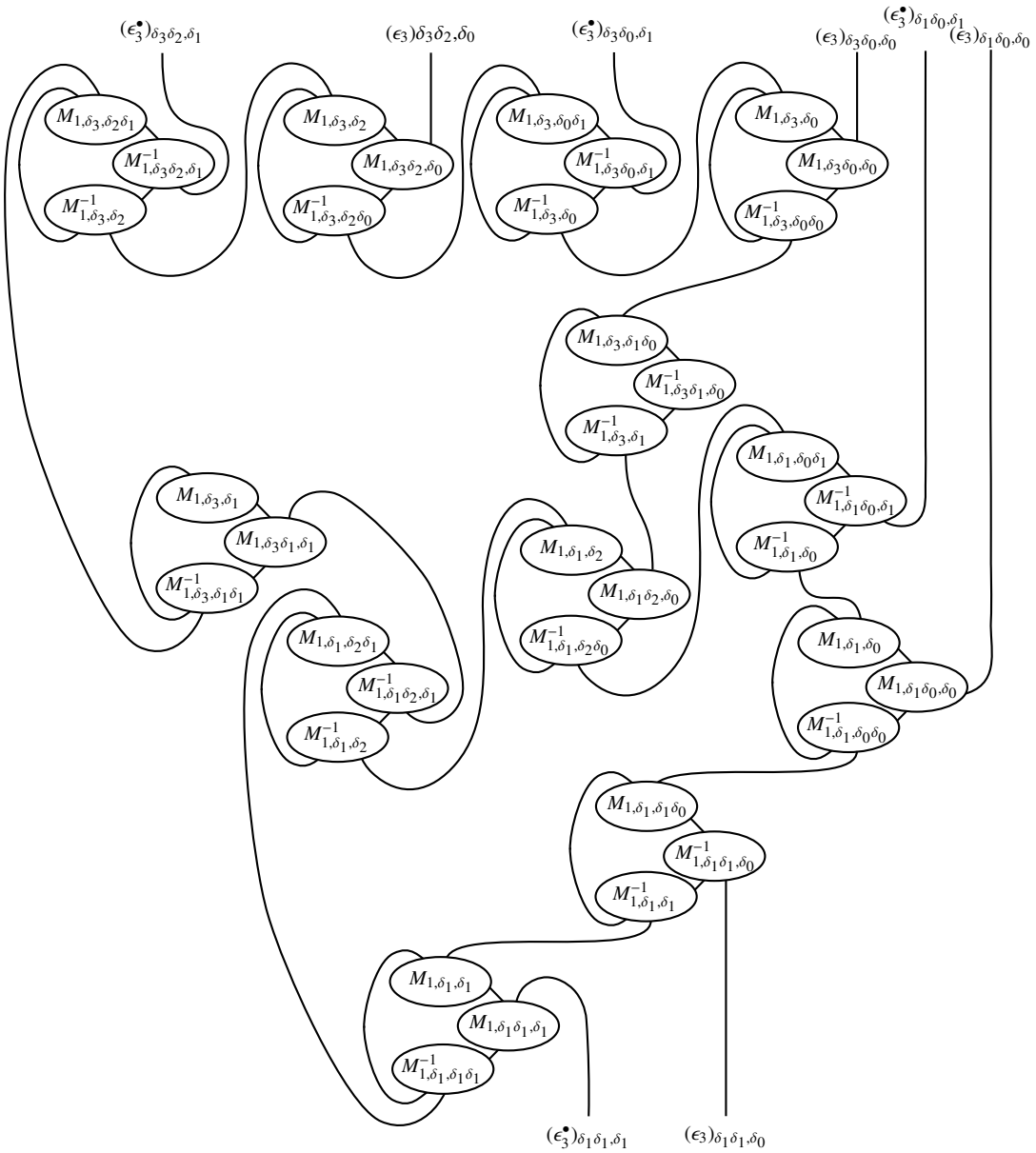
Due to the simplicial identities, we are now able to cancel many nodes by relations such as the one depicted below.

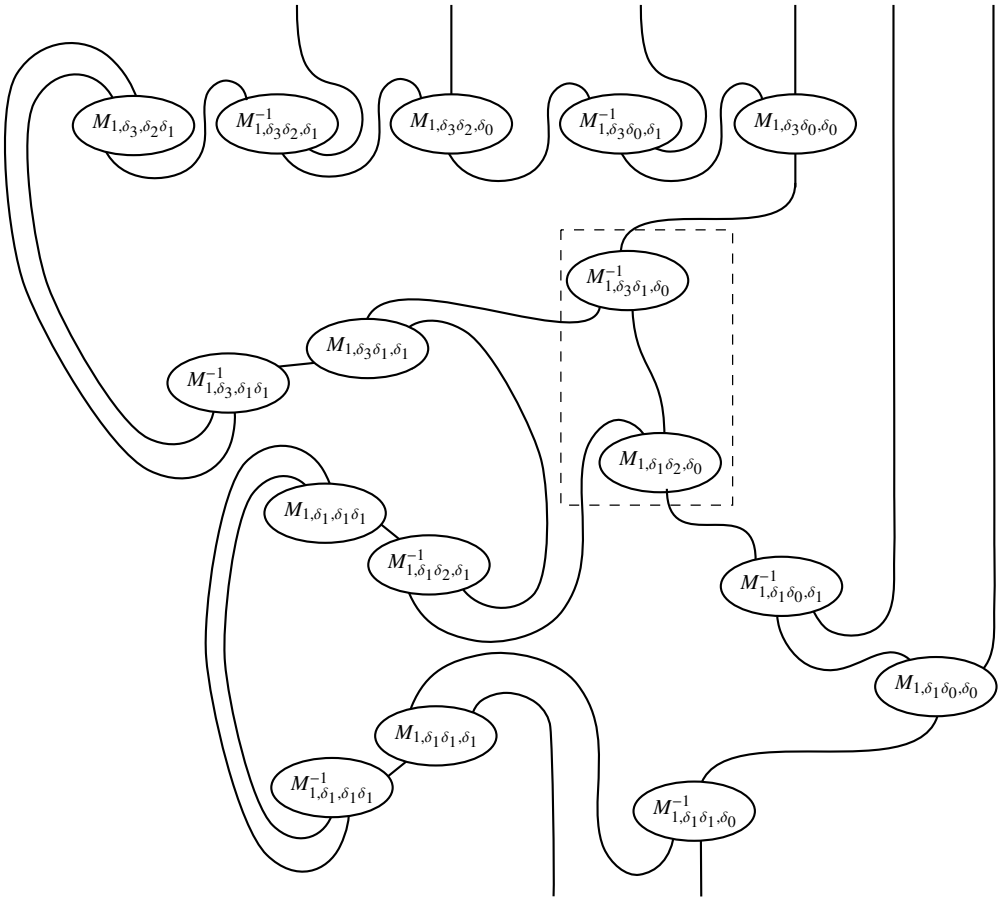


(3.31)

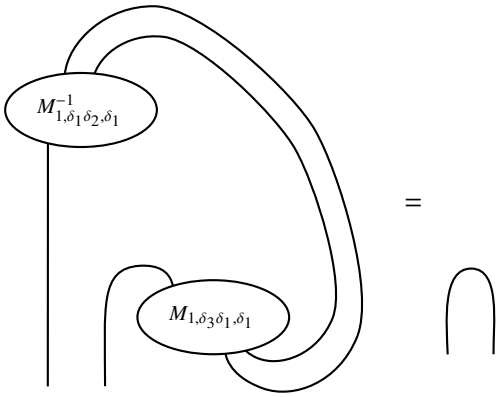
Applying such a relation in between the blow-ups once everywhere where it is possible, we obtain the cell below, where a similar relation can in turn be applied to the dashed region. For simplicity, we have omitted labels of 1-cells.

Figure 3.2.: Blow-up of the left hand side of 3.3

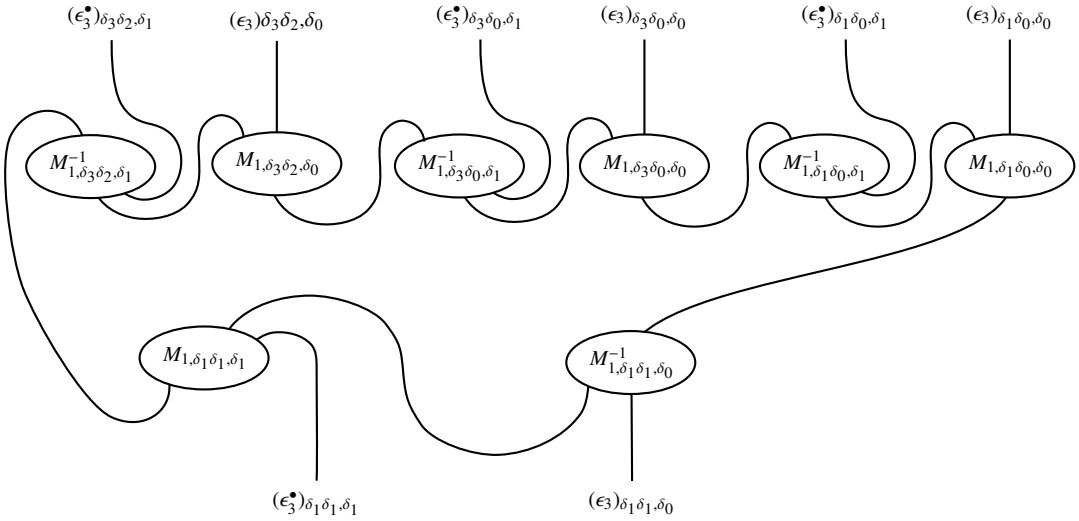




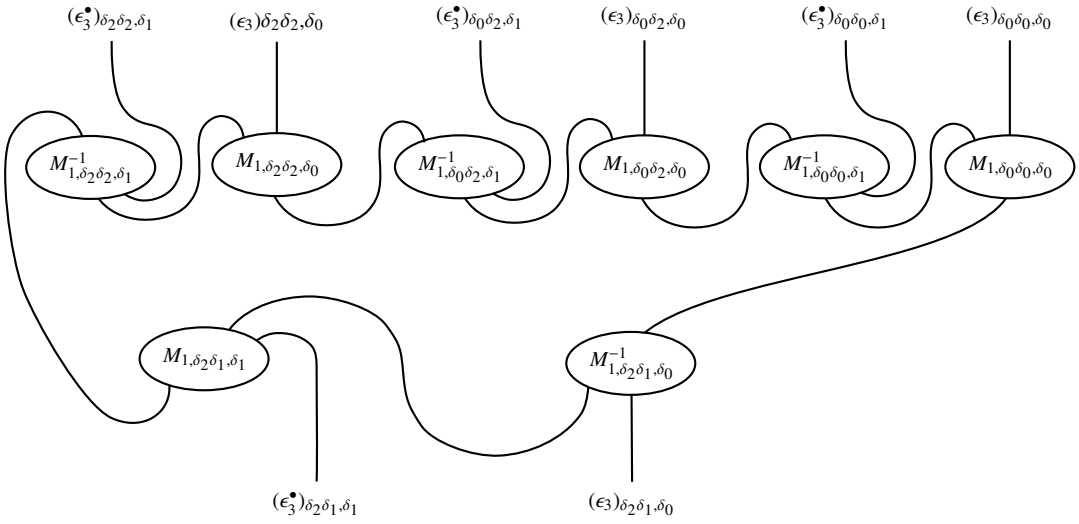
We then obtain a diagram, where due to simplicial identities, we can apply a relation similar to the following three times on the left of the diagram.



We finally arrive at the cell below, where at each node, simplicial identities apply.



Namely, this is clearly equal to the cell below, which is obtained from the right hand side of the axiom 3.3 by a similar computation.



Lemma 3.2. *ψ satisfies equation* (3.14)*.*

Proof. We show that $\varphi_j\tilde{\psi}_j\colon I_j \to \mathrm{st}'I_j$ is the unit of the adjunction $f\colon I_j \to \mathrm{st}'I_j$ (the pseudofunctor that is given by the inclusion as length-1-strings). Then by naturality, this means $\varphi_j\psi_j = 1_{\mathrm{st}'I_j}$. The components of ψ were determined on objects by the prescription (3.25) for the component $\psi_0\colon \mathrm{st}'I_0 = [0] \to X([0])$. By naturality, this means that (3.14) is satisfied on components on objects because clearly, it is satisfied for the zero-component by (3.15):

$$\varphi_0(\psi_0(0)) = \varphi_0(x_0(1_{[0]})) = \upsilon_0(1_{[0]}) = 0\ .$$

Similarly, by naturality, we only have to check that $\varphi_1\tilde{\psi}_1 = f_1 \colon I_1 \to \mathrm{st}' I_j$:

$$\begin{aligned}
&\varphi_1(\tilde{\psi}_1(01)) \\
=&\varphi_1((\epsilon_1)_{1_{[1]},\delta_0} * (\epsilon_1^\bullet)_{1_{[1]},\delta_1}) \\
=&((1_\varphi \epsilon)_1)_{1_{[1]},\delta_0} * ((1_\varphi \epsilon^\bullet)_1)_{1_{[1]},\delta_1} \\
=&(\bar{\epsilon}_1)_{1_{[1]},\delta_0} * (\bar{\epsilon}_1^\bullet)_{1_{[1]},\delta_1} \\
=&01 * 00 \\
=&01 \ .
\end{aligned}$$

This is indeed all that we need to show. The constraints must be the same as those of f_j because there are unique such cells in $\mathrm{st}' I_j$. □

Lemma 3.3. *The* 1*-cells* $1_X, \varphi\psi \colon X \to X$ *satisfy the conditions in the second part of the universal property of the codescent object* X*—i.e. there is a* 1*-cell of cocones* $(\alpha, \Gamma) \colon (X, x, \epsilon, M, U) \to (X, \varphi\psi x, \varphi\psi\epsilon, \varphi\psi M, \varphi\psi U)$*—and* α *is part of an adjoint equivalence.*

Proof. The component of the adjoint equivalence $\epsilon \colon xd_1 \Rightarrow xd_0$ at j is an adjoint equivalence $\epsilon_j \colon (xd_1)_j \Rightarrow (xd_0)_j$ of 2-functors. Hence, since $d_0(\delta, \delta_k\delta_{k-1}...\delta_1) = \delta\delta_k\delta_{k-1}...\delta_1$ and $d_1(\delta, \delta_k\delta_{k-1}...\delta_1) = \delta$, its component at $(\delta, \delta_k\delta_{k-1}...\delta_1)$ is an adjoint equivalence

$$(\epsilon_j)_{\delta,\delta_k\delta_{k-1}...\delta_1} \colon x_j(\delta) \to x_j(\delta\delta_k\delta_{k-1}...\delta_1) \ . \tag{3.32}$$

We take (3.32) to be the component of $\alpha \colon x \Rightarrow \psi\upsilon = \psi\varphi x$ at δ, cf. (3.26). Correspondingly, we take its adjoint $(\epsilon_j)_{\delta,\delta_k\delta_{k-1}...\delta_1}$ for the component of $\alpha^\bullet \colon \psi\varphi x \Rightarrow x$). These trivially gives rise to pseudonatural transformations, but this is also follows from naturality of ϵ_j and $\epsilon_j^\bullet$. Similarly, α are $\alpha_\bullet$ are $\mathcal{G}ray$-natural as a consequence of ϵ and $\epsilon^\bullet$ being $\mathcal{G}ray$-natural. Similar arguments apply to the unit and counit, and thus α is indeed part of an adjoint equivalence.

We next have to provide an invertible 2-cell

$$\Gamma \colon (\alpha 1_{d_0}) * (1_{1_X}\epsilon) \Rightarrow (1_{\psi\varphi}\epsilon) * (\alpha 1_{d_1}) \ .$$

Let $\delta \colon [k] \to [j]$ and $\delta' \colon [j] \to [l]$ be morphisms in Δ, then this has components

$$(\Gamma_l)_{\delta',\delta} \colon (\alpha_l)_{\delta'\delta} * (\epsilon_l)_{\delta',\delta} \Rightarrow (\psi_l)((\bar{\epsilon}_l)_{\delta',\delta}) * (\alpha_l)_{\delta'} \ .$$

Spelled out, this is

$$(\epsilon_l)_{\delta'\delta,\delta_k\delta_{k-1}...\delta_1} * (\epsilon_l)_{\delta',\delta} \Rightarrow (\psi_l)((\bar{\epsilon}_l)_{\delta',\delta}) * (\epsilon_l)_{\delta',\delta_j\delta_{j-1}...\delta_1} \ .$$

For $\delta' = 1_{[j]}$ and $\delta(0) \neq 0$, this has the form

$$\epsilon_{\delta,\delta_k\delta_{k-1}...\delta_1} * \epsilon_{1_{[j]},\delta} \Rightarrow X(\zeta)((\epsilon_1)_{1_{[1]},\delta_0} * (\epsilon_1^\bullet)_{1_{[1]},\delta_1}) * \epsilon_{1_{[j]},\delta_j\delta_{j-1}...\delta_1} \ .$$

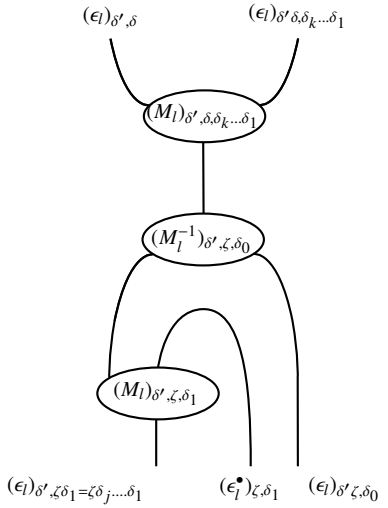
where $\zeta \colon [1] \to [j]$ be the unique order preserving function sending 01 to the unique isomorphism

$$0 = \delta_j\delta_{j-1}...\delta_1(0) \to \delta\delta_k\delta_{k-1}...\delta_1(0) = \delta(0)$$

in I_j. By naturality of ϵ, this has the form

$$(\epsilon_j)_{\delta,\delta_k\delta_{k-1}\ldots\delta_1} * (\epsilon_j)_{1_{[j]},\delta} \Rightarrow (\epsilon_j)_{\zeta,\delta_0} * (\epsilon_j^\bullet)_{\zeta,\delta_1} * (\epsilon_j)_{1_{[j]},\delta_j\delta_{j-1}\ldots\delta_1}$$

Note that we must have $\zeta\delta_1(0) = \zeta(0) = 0$, hence $\zeta\delta_1 = \delta_j\delta_{j-1}\ldots\delta_1$ as morphisms $[0] \to [j]$ in Δ. Similarly, we have $\zeta\delta_0 = \delta\delta_k\delta_{k-1}\ldots\delta_1$. In the case that $\delta' = 1_{[j]}$ and $\delta(0) \neq 0$, the component $(\Gamma_j)_{1_{[j]},\delta}$ of Γ is defined to be the cell represented by the string diagram below.



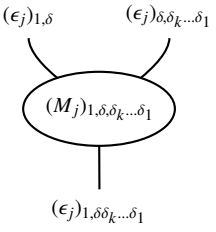
In the case that $\delta' = 1_{[j]}$ and $\delta(0) = 0$, we have $(\bar{\epsilon}_l)_{1_{[j]},\delta} = 1$ cf. (3.21). Hence,

$$(\Gamma_j)_{1_{[j]},\delta} \colon\ \epsilon_{\delta,\delta_k\delta_{k-1}\ldots\delta_1} * \epsilon_{1_{[j]},\delta} \Rightarrow X(\zeta)((\bar{\epsilon}_l)_{1_{[j]},\delta}) * \epsilon_{1_{[j]},\delta_j\delta_{j-1}\ldots\delta_1} \tag{3.33}$$

has the form

$$(\Gamma_j)_{1_{[j]},\delta} \colon\ \epsilon_{\delta,\delta_k\delta_{k-1}\ldots\delta_1} * \epsilon_{1_{[j]},\delta} \Rightarrow \epsilon_{1_{[j]},\delta_j\delta_{j-1}\ldots\delta_1} \tag{3.34}$$

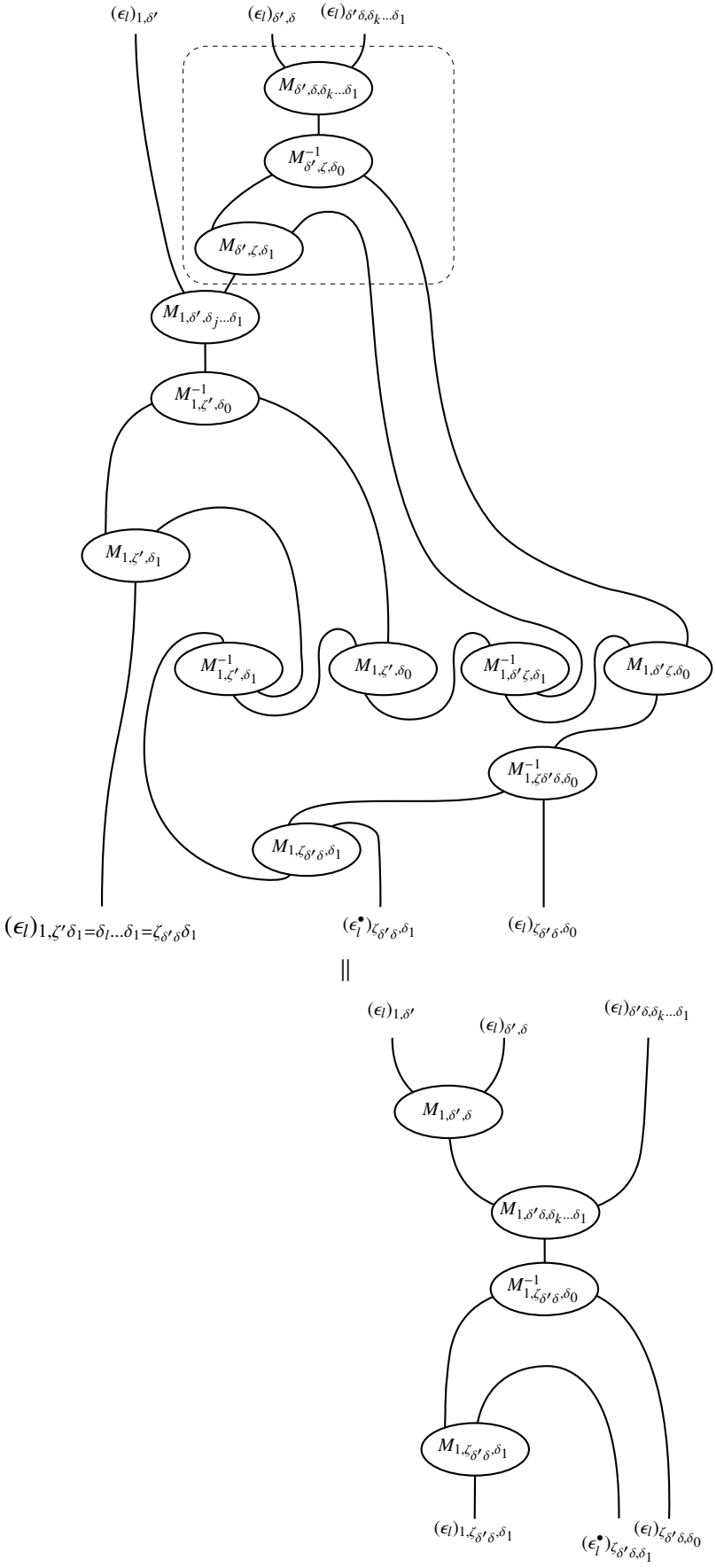
where $\delta_j\delta_{j-1}\ldots\delta_1 = \delta\delta_k\delta_{k-1}\ldots\delta_1$ because $\delta\delta_k\delta_{k-1}\ldots\delta_1(0) = \delta(0) = 0$. Therefore, we define the component $(\Gamma_j)_{1_{[j]},\delta}$ of Γ to be $(M_j)_{1,\delta,\delta_k\delta_{k-1}\ldots\delta_1}$:



These definitions are then extended to the case $\delta' \neq 1_{[j]}$ by naturality.

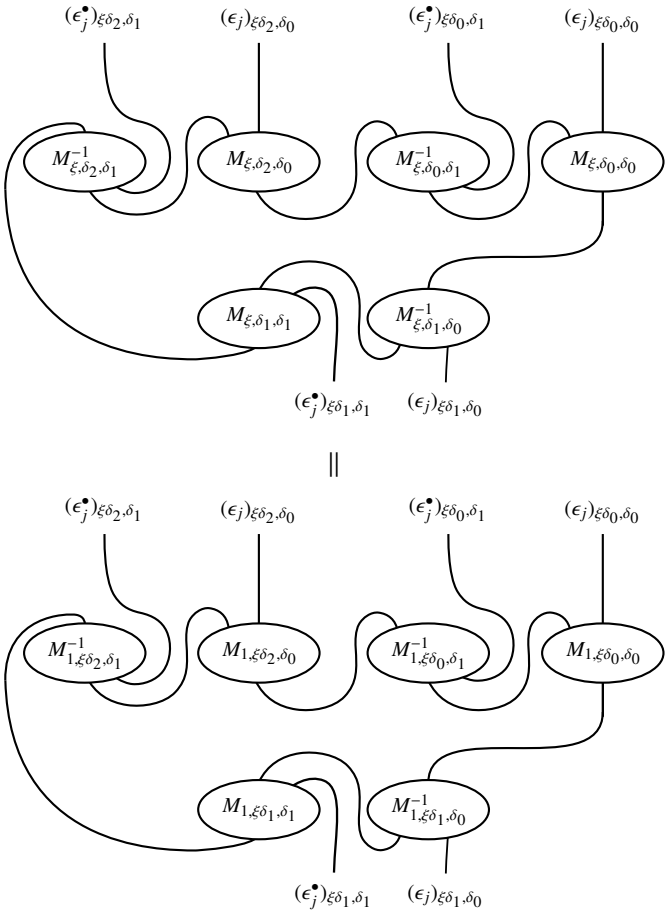
The invertible 2-cell Γ has to satisfy the two axioms in the second part of the universal property of the codescent object. This is proved by a case analysis. The first axiom for

Γ in case that $\delta'(0) \neq 0$ and $\delta(0) \neq 0$ is depicted in the string diagram below.



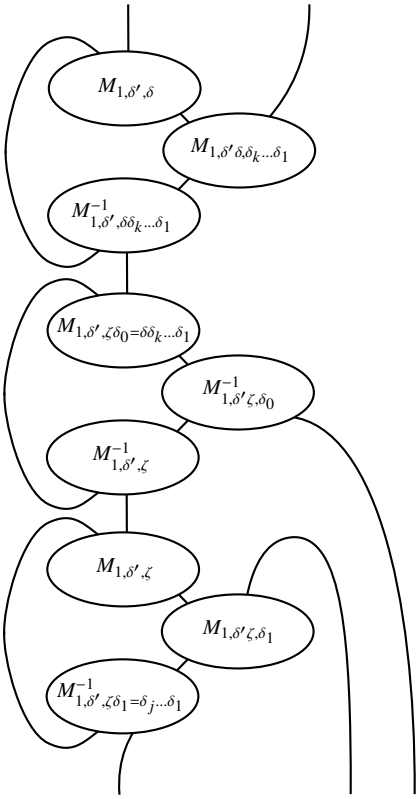
Here, we have already transformed the lower right hand part according to the following Lemma.

Lemma 3.4.

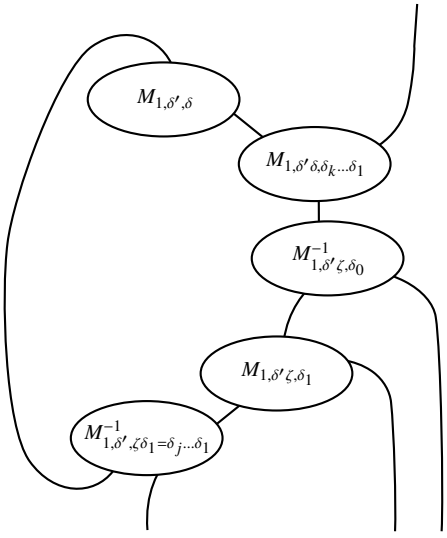


Proof. By blow-up according to (3.30) as above. □

By blowing up each of the three nodes in the dashed region according to (3.30), the cell in the dashed region is transformed into the one depicted below.

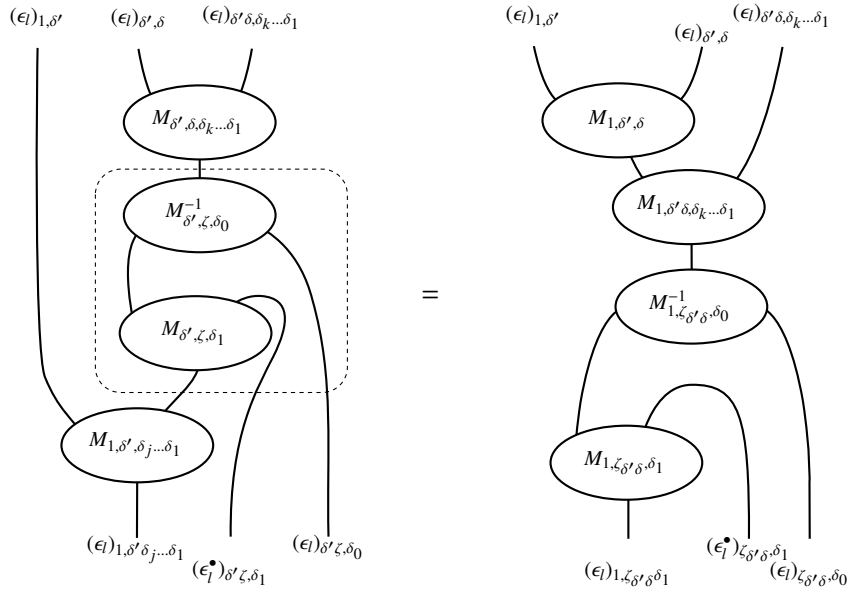


Employing relations as in (3.31), this is clearly the same as the following.



Implementing this transformation in the left hand side of the axiom, one easily transforms it into the right hand side of the axiom by employing similar relations.

The first axiom for Γ in case that $\delta'(0) = 0$ and $\delta(0) \neq 0$ has the form below.



To see that domain and codomain match on both sides, note that $\zeta_{\delta'\delta} = \delta'\zeta$ and that $\zeta_{\delta'\delta}\delta_1 = \delta'\delta_j\delta_{j-1}\ldots\delta_1$ (because both map 0 to 0). Blowing-up the two nodes in the dashed region according to (3.30), the axiom is easily seen to be satisfied by applying the first axiom for the codescent object on the right hand side.

In the case that $\delta'(0) \neq 0$ and $\delta(0) = 0$, the first axiom for Γ follows directly from the first axiom for the codescent object.

Finally, in the case that $\delta'(0) = 0$ and $\delta(0) = 0$, the first axiom for Γ is literally the first axiom for the codescent object.

Observing that $1_{\psi\varphi}U = 1_\psi \bar{U} = 1$, the second axiom for Γ, cf. (2.11), follows from the second axiom for the codescent object. □

3.4 Identification with descent data

Lemma 3.5. *Given a Gray-functor $\mathfrak{X}\colon C^{\mathrm{op}} \to \mathcal{G}ray$ with values in $\mathcal{G}ray$ on a category C, and a simplicial object $\Gamma\colon \Delta_\delta^{\mathrm{op}} \to C$, an object in the 2-category*

$$[\Delta_\delta, \mathcal{G}ray](\mathrm{st}' I_\bullet, \mathfrak{X}\Gamma^{\mathrm{op}}) \tag{3.35}$$

is given by

- *an object $\mathcal{G}$ in $\mathfrak{X}(\Gamma_0)$,*
- *an adjoint equivalence*

$$P\colon d_0^*\mathcal{G} \to d_1^*\mathcal{G} \tag{3.36}$$

in the 2-category $\mathfrak{X}(\Gamma_1)$, and

- *an invertible* 2*-cell*

$$\mu\colon\ d_2^* P * d_0^* P \Rightarrow d_1^* P \tag{3.37}$$

satisfying the following equation

$$d_2^*\mu \diamond (1 * d_0^*\mu) = d_1^*\mu \diamond (d_3^*\mu * 1)\ . \tag{3.38}$$

Proof. An object in the hom 2-category (3.35) of the functor $\mathcal{G}ray$-category is a $\mathcal{G}ray$-natural transfromation

$$\alpha\colon\ \mathrm{st}' I_\bullet \Rightarrow \mathfrak{X} \circ \Gamma^{\mathrm{op}}\colon\ \Delta \to \mathcal{G}ray\ . \tag{3.39}$$

Since $\mathrm{st}' I_0 = \mathrm{st}'[0] = [0] = I$, the component $\alpha_{[0]}$ of α at $[0] \in \mathrm{ob}\Delta$ is a 2-functor $\alpha_0\colon\ I \to \mathfrak{X}(\Gamma_0)$, which is uniquely determined by its image on the single object 0 in $[0] = I$, and so we define

$$\mathcal{G} \coloneqq \alpha_{[0]}(0)\ . \tag{3.40}$$

Next, the 2-category $\mathrm{st}' I_1$ is exactly the free living adjoint equivalence. Therefore, the component $\alpha_{[1]}$ of α at $[1] \in \mathrm{ob}\Delta$, which is a 2-functor $\alpha_{[1]}\colon\ \mathrm{st}' I_1$, corresponds precisely to an adjoint equivalence in $\mathfrak{X}(\Gamma_1)$. Since $\delta_0\colon\ [0] \to [1]$ is the (order-preserving) injection whose image does not contain 0, the source of this adjoint equivalence is $\delta_0(0)$, while its target is similarly given by $\delta_1(0)$. We thus define P to be the opposite adjoint equivalence. By $\mathcal{G}ray$-naturality P indeed has source

$$\alpha_{[1]}(1) = \alpha_{[1]}(\delta_0(0)) = d_0^*(\alpha_{[0]}(0)) = d_0^*\mathcal{G} \tag{3.41}$$

and similarly target

$$\alpha_{[1]}(\delta_1(0)) = d_1^*\mathcal{G}\ . \tag{3.42}$$

Next, the component of α at $[2] \in \mathrm{ob}\Delta$, which is a 2-functor $\alpha_{[2]}\colon\ \mathrm{st}' I_2 \to \mathfrak{X}(\Gamma_2)$, corresponds to a normalized pseudofunctor

$$\tilde{\alpha}_{[2]}\colon\ [2] \to \mathfrak{X}(\Gamma_2)\ . \tag{3.43}$$

By $\mathcal{G}ray$-naturality of $\tilde{\alpha}$, this normalized pseudofunctor is already determined on objects and 1-cells, and thus since [2] is locally discrete, as a normalized pseudofunctor, $\tilde{\alpha}_{[2]}$ is already completely determined on the cells of [2]. Also notice that, by $\mathcal{G}ray$-naturality, $\tilde{\alpha}_{[2]}$ sends any morphism in [2] to an adjoint equivalence, and thus its corresponding composition constraints are already determined by unit and counit of these. In effect, the only composition constraint which gives rise to new data is, by $\mathcal{G}ray$-naturality,

$$\mu \coloneqq (\tilde{\alpha}_{[2]})_{10,21}\colon\ d_2^* P * d_0^* P \Rightarrow d_1^* P\ . \tag{3.44}$$

(The rest of the composition constraints are determined from (3.44), units, and counits by the pseudofunctor axiom for composition constraints. The pseudofunctor axioms

for the (trivial) unit constraints force the corresponding composition constraints to be identity 2-cells).

Finally, there is one equation enforced on this data by *Gray*-naturality and the pseudofunctor axiom for composition constraints for the pseudofunctor $\tilde{\alpha}_{[3]}\colon I_3 \to \mathfrak{X}(\Gamma_3)$ corresponding to the component $\alpha_{[3]}\colon \mathrm{st}'I_3 \to \mathfrak{X}(\Gamma_3)$. This equation is precisely (3.38).

The components $\alpha_{[j]}$ for $j > 3$ are completely determined by *Gray*-naturality and the components for $j \leq 3$. □

Remark 3.3. Apart from the fact that P is required to be an adjoint equivalence and not merely an equivalence, this is precisely the definition of a Γ-equivariant object given in Nikolaus and Schweigert [61].

Also note that a mere equivalence can be replaced by an adjoint equivalence cf. Gurski [24, Th. A.1.10, p. 145].

Lemma 3.6. *Let* $(\mathcal{G}, P, \mu)$ *and* $(\mathcal{G}', P', \mu')$ *be objects in* (3.35)*. Then a* 1*-cell*

$$(\mathcal{G}, P, \mu) \to (\mathcal{G}', P', \mu') \tag{3.45}$$

is given by

- *a* 1*-cell*

$$A\colon\ \mathcal{G} \to \mathcal{G}' \tag{3.46}$$

  in $\mathfrak{X}(\Gamma_0)$*, and*

- *an invertible* 2*-cell*

$$\alpha\colon\ P' * d_0^* A \Rightarrow d_1^* A * P \tag{3.47}$$

  satisfying the following equation

$$(1 * \mu') \diamond (d_2^*\alpha * 1) \diamond (1 * d_0^*\alpha) = d_1^*\alpha \diamond (\mu * 1)\ . \tag{3.48}$$

Proof. By similar arguments as above. We only note that equation (3.48) is exactly respect for composition of a pseudonatural transformation. □

Lemma 3.7. *A* 2*-cell* $(A, \alpha) \Rightarrow (A', \alpha')$ *in* (3.35) *is given by*

- *a* 2*-cell*

$$\beta\colon\ A \to A' \tag{3.49}$$

  in $\mathfrak{X}(\Gamma_0)$ *satisfying the following equation*

$$\alpha' \diamond (1 * d_0^*\beta) = (d_1^*\beta * 1) \diamond \alpha\ . \tag{3.50}$$

Proof. By similar arguments as above. We only note that equation (3.50) is exactly the modification axiom. □

By reference to Theorem 3.1 we have thus proved the following theorem.

Theorem 3.2. *The descent construction from Nikolaus and Schweigert [61] coincides with the three-dimensional descent construction for a* $\mathcal{G}ray$*-functor* $\mathfrak{X}\colon C^{\mathrm{op}} \to \mathcal{G}ray$*:*

$$[\Delta_\delta, \mathcal{G}ray](L*, -)[1, \mathfrak{X}] \cong \mathcal{T}ricat_{\mathrm{ls}}(\Delta_\delta, \mathcal{G}ray)(*, -)[1, \mathfrak{X}]\colon\ [\Delta_\delta^{\mathrm{op}}, C]^{\mathrm{op}} \to \mathcal{G}ray\ .$$

□

To draw the connection to the theory of bundle gerbes, we consider again the $\mathcal{G}ray$-functor $\mathcal{B}un'_{U(1)}\colon \mathcal{M}an^{\mathrm{op}} \to \mathcal{G}ray$ on the category $\mathcal{M}an$ of smooth manifolds from the Introduction. To a manifold M, the functor $\mathcal{B}un'_{U(1)}$ assigns the one-object 2-category given by the monoidal category of principal smooth $U(1)$-bundles over M in terms of open coverings and smooth transition functions. The monoidal structure is defined by the tensor product of principal bundles. As explained in the Introduction, this is in fact a strict monoidal category, and pullback of principal $U(1)$-bundles gives rise to a $\mathcal{G}ray$-functor as required.

Let $Y \to M$ be a surjective submersion of smooth manifolds. Then there is an associated simplicial object $Y^{[\bullet]}$ in $\mathcal{M}an$ assigning to $[n]$ the $(n+1)$-fold fiber product $Y^{[n]} = Y \times_M Y \times_M \ldots \times_M Y$ of Y with itself over M. A bundle gerbe over M is exactly given by a surjective submersion $Y \to M$ and a $Y^{[\bullet]}$-equivariant object [61, 4.1]. Thus the theorem above implies the following.

Corollary 3.1. *A bundle gerbe over a manifold* M *corresponds to a tritransformation*

$$* \Rightarrow \mathcal{B}un'_{U(1)}(Y^{[\bullet]})^{\mathrm{op}}\colon\ \Delta_\delta \to \mathcal{G}ray \tag{3.51}$$

for a surjective submersion $Y \to M$*.* □

3.5 Outlook

The identification of the three-dimensional descent construction with the explicit one in terms of descent data given in Nikolaus and Schweigert [61] is of course not the end of the story. We expect that this comparison allows us to construct the stackification of a trihomomorphism

$$\mathfrak{X}\colon\ C^{\mathrm{op}} \to \mathcal{B}icat\ . \tag{3.52}$$

Therefore one has to specify generalizations of the sheaf conditions on such a trihomomorphism. This is done by specifying a collection τ of coverings in C, which should technically form a Grothendieck pretopology. A covering $y \to m$ gives rise to a simplicial object $y_\bullet$ in C. There is a projection to the trivial simplicial object m at the base m of the covering, and the generalized sheaf conditions—the stack conditions—require the induced functors

$$\mathfrak{X}(m) \xrightarrow{\cong} \mathrm{Desc}(\mathfrak{X})(m) \to \mathrm{Desc}(\mathfrak{X})(y_\bullet) \tag{3.53}$$

to be biequivalences [61].

There should be a plus construction stackifying a given presheaf with appropriate properties. Since we are working with bicategories, there is a varietey of conditions, which one could impose on (3.53), e.g. that for any covering it is 2-locally fully faithful, locally an equivalence, a biequivalence, or even an isomorphism of bicategories. Correspondingly, there should be a variety of plus constructions which sheafify or stackify a given presheaf with appropriate properties. A tentative idea is that this somewhat resembles the situation in two- and three-dimensional monad theory, where there are new kinds of algebras turning up, but these can be studied by reference to the strict algebras. For example, the 'exchange of homotopy limits' in [61, Prop. 7.4] may be reduced to the usual exchange of limits in enriched category theory via the strictification described in 3.2.

Apart from exchange of limits, a second property of the descent construction enters critically in the proof that one can stackify a given presheaf, namely that $\mathsf{Desc}(\mathfrak{X})$ preserves pointwise biequivalences. However, from its tricategorical description as the composition of a representable and whiskering with $\mathfrak{X}$ on the left $\mathsf{Desc}(\mathfrak{X})$ clearly has the structure of a trihomomorphism. The latter in particular implies that it preserves biequivalences: thus maps a pointwise biequivalence to a biequivalence cf. Prop. A.16.

Finally, we comment briefly on another aspect of the theory. There is a canonical functor $\bullet\colon C \to [\Delta^{\mathrm{op}}, C]$ sending an object c in C to the trivial simplicial object at c. In fact, this gives rise to a biequivalence

$$\mathfrak{X} \overset{\cong}{\Rightarrow} \mathsf{Desc}(\mathfrak{X}) \circ \bullet^{\mathrm{op}} \tag{3.54}$$

of trihomomorphisms. Note that this is implied by the hypothetical result that the bicategory of tritransformations between constant trihomomorphisms is biequivalent to the relevant hom bicategory in the target. The corresponding result for ordinary categories is trivial: the axiom of a natural transformation between constant functors just requires all components to be the same, but this component can be an arbitrary morphism between the corresponding objects in the target. Thus we have a bijection between the set of natural transformations between constant functors and the corresponding hom set of the target category. For bicategories, the argument is already slightly more complicated. First, domain and target are replaced by their strictifications. The resulting category of pseudonatural transformations and modifications is equivalent to the original one. Next, the induced functors between the strictifications can be replaced by the strictifications of the original constant functors, and these are again constant functors. This reduces the question to the case of 2-categories. For tricategories, we can not yet mirror this argument since this requires more knowledge about functor tricategories than we are able to give in Appendix A. For the descent construction in terms of its explicit identification from 3.4, on the other hand, a straightforward calculation proves that (3.54) is a biequivalence.

Appendix A

Functor tricategories

Given tricategories R and S, the notions of trihomomorphisms $F, F' \colon R \to S$, tritransformations $\alpha, \alpha' \colon F \Rightarrow F'$ between such trihomomorphism, trimodifications $m, m' \colon \alpha \Rrightarrow \alpha'$, and perturbations $\sigma \colon m \Lrightarrow m'$ clearly give rise to a 3-globular set $\mathcal{T}ricat(R, S)$. The question if this supports the structure of a tricategory has been answered positively in case that the target S is $\mathcal{G}ray$-category in Gurski [27, Theorem 9.4], in which case $\mathcal{T}ricat(R, S)$ is actually a $\mathcal{G}ray$-category. In case that the domain R is a category considered as a locally discrete tricategory and the target is the tricategory $\mathcal{B}icat$ of bicategories such a functor tricategory has been described by Carrasco et al. [11, 2.2].

In this section, we follow a strategy sketched in Gordon et al. [21, Cor. 8.3, p.74] to show that $\mathcal{T}ricat(R, \mathcal{B}icat)$ inherits the structure of a tricategory from the $\mathcal{G}ray$-category $\mathcal{T}ricat(R, \mathcal{G}ray)$ via transport of structure, where we use the $\mathcal{G}ray$-category structure of $\mathcal{T}ricat(R, \mathcal{G}ray)$ described by Gurski. Namely, we first show how whiskering on the left with a trihomomorphism gives rise to morphisms of 3-globular sets. Then we show that whiskering with the strictification $\mathrm{st} \colon \mathcal{B}icat \to \mathcal{T}ricat$ gives rise to local biequivalences

$$\mathrm{st}_* \colon \mathcal{T}ricat(S, \mathcal{B}icat)(F, F') \xrightarrow{\cong} \mathcal{T}ricat(S, \mathcal{G}ray)(\mathrm{st} \circ F, \mathrm{st} \circ F') \ . \tag{A.1}$$

The implementation of this strategy for general target S depends on a complete description of whiskering on the left, cf. Conjecture A.2, as we present it below for whiskering on the right cf. A.6 in case that the target is a $\mathcal{G}ray$-category.

To make sense of (A.1), one first has to describe the bicategory structure of the domain and target. This has been carried out in Gurski [27, 9.1], and we only describe the straightforward structure here.

Theorem A.1. *Given tricategories R, S and trihomomorphisms $F, F' \colon R \to S$, the following prescriptions give the 2-globular set*

$$\mathcal{T}ricat(R, S)(F, F') \tag{A.2}$$

the structure of a bicategory.

The vertical composite

$$\sigma' \diamond \sigma \colon m \Lrightarrow m'' \tag{A.3}$$

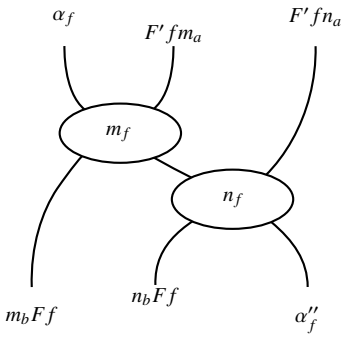
of two perturbations $\sigma\colon m \Rrightarrow m'\colon \alpha \Rightarrow \alpha'$ *and* $\sigma'\colon m' \Rrightarrow m''\colon \alpha \Rightarrow \alpha'$ *has component at an object* a *in* R *the vertical composite* $\sigma'_a \diamond \sigma_a$.

The identity perturbation at the trimodification m *has component at* a *the identity* 2*-cell* 1_{m_a}.

The horizontal composite

$$n * m\colon \alpha \Rrightarrow \alpha'' \tag{A.4}$$

of two trimodifications $m\colon \alpha \Rrightarrow \alpha'$ *and* $n\colon \alpha' \Rrightarrow \alpha''$ *has component at* a *the horizontal composite* $n_a * m_a$. *If the target* S *is a cubical tricategory, the component at a* 1*-cell* $f\colon a \to b$ *in* R *of the invertible modification of the trimodification* $n * m$ *is defined by the string diagram below.*



(A.5)

If the target is not cubical, one has to supplement this diagram in the source and the target with a constraint for the composition functor of S.

The horizontal composite

$$\tau * \sigma\colon n * m \Rrightarrow n' * m' \tag{A.6}$$

of two perturbations $\sigma\colon m \Rrightarrow m'\colon \alpha \Rightarrow \alpha'$ *and* $\tau\colon n \Rrightarrow n'\colon \alpha' \Rrightarrow \alpha''$ *has component at the object* a *in* R *given by the horizontal composite of* 2*-cells*

$$\tau_a * \sigma_a \tag{A.7}$$

in the bicategory $S(Fa, F'a)$.

The constraints of the bicategory (A.2) *are perturbations with components given by the corresponding constraints of the hom bicategories of the target* S.

Proof. See Gurski [27, Theorem 9.1]. □

Corollary A.1. *Given a tricategory* R, *a* $\mathcal{G}ray$*-category* $\mathcal{L}$, *and trihomomorphisms* $F, F'\colon R \to S$, *then the bicategory*

$$\mathcal{T}ricat(R, \mathcal{L})(F, F') \tag{A.8}$$

described in Theorem A.1 is a 2*-category.*

Proof. Immediate from the description of the constraints of the bicategory in Theorem A.1. □

A.1 The functor tricategory for target a Gray-category

Theorem A.2. *Let $\mathcal{L}$ be a $\mathcal{G}ray$-category and R be a tricategory. Then the prescriptions below give the 2-category enriched graph*

$$\mathcal{T}ricat(R, \mathcal{L}) \tag{A.9}$$

of trihomomorphisms, tritransformations, trimodifications, and perturbations the structure of a $\mathcal{G}ray$-category.

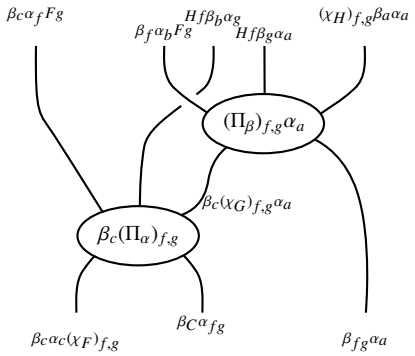
The composite $\beta\alpha$ of two tritransformations $\alpha\colon F \Rightarrow G$ and $\beta\colon G \Rightarrow H$ has component at the object $a \in R$ given by the composite

$$\beta_a \alpha_a \tag{A.10}$$

of 1-cells in the $\mathcal{G}ray$-category $\mathcal{L}$. The adjoint equivalence is defined by the obvious composite adjoint equivalence with component at the 1-cell $f\colon a \to b$ in R given by

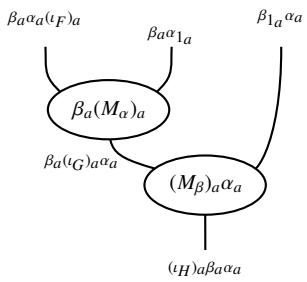
$$(\beta_f \alpha_a) * (\beta_b \alpha_f)\colon \beta_b \alpha_b F f \to H f \beta_a \alpha_a \ . \tag{A.11}$$

The component of the invertible modification Π at composable 1-cells f and g in R is given by the string diagram below.



(A.12)

The single component of the invertible modification M at the object $a \in R$ is given by the string diagram below.



(A.13)

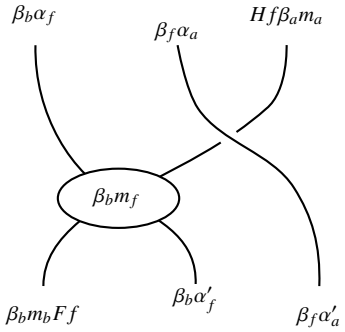
The whiskering

$$\mathcal{M}(1,m) = \beta m\colon\ \beta\alpha \Rrightarrow \beta\alpha' \tag{A.14}$$

of a trimodification $m\colon\ \alpha \Rightarrow \alpha'\colon\ F \to G$ *with a tritransformation* $\beta\colon\ G \Rightarrow H$ *on the left is a trimodification with component at the object* $a \in R$ *given by the whiskering*

$$\mathcal{M}_{\mathcal{L}}(1,m_a) = \beta_a m_a \tag{A.15}$$

in the target **Gray***-category* $\mathcal{L}$*. The invertible modification of the trimodification* (A.14) *has component at a* 1*-cell* $f\colon\ a \to b$ *in* R *defined by the string diagram below.*



(A.16)

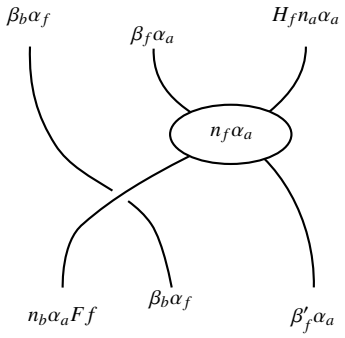
The whiskering

$$\mathcal{M}(n,1) = n\alpha\colon\ \beta\alpha \Rrightarrow \beta'\alpha \tag{A.17}$$

of a trimodification $n\colon\ \beta \Rightarrow \beta'\colon\ G \to H$ *with a tritransformation* $\alpha\colon\ F \Rightarrow G$ *on the right is a trimodification with component at the object* $a \in R$ *given by the whiskering*

$$\mathcal{M}_{\mathcal{L}}(n_a,1) = n_a\alpha_a \tag{A.18}$$

in the target **Gray***-category* $\mathcal{L}$*. The invertible modification of the trimodification* (A.17) *has component at a* 1*-cell* $f\colon\ a \to b$ *in* R *defined by the string diagram below.*



(A.19)

The whiskering

$$M(1,\sigma) = \beta\sigma\colon\ \beta m \Rrightarrow \beta m' \tag{A.20}$$

of a perturbation $\sigma\colon\ m \Rrightarrow m'\colon\ \alpha \Rightarrow \alpha'$ *with a tritransformation* $\beta\colon\ G \Rightarrow H$ *on the left is a perturbation with component at the object* $a \in R$ *given by the whiskering*

$$M_{\mathcal{L}}(1, \sigma_a) = \beta_a \sigma_a \tag{A.21}$$

in the target $\mathcal{G}ray$*-category* $\mathcal{L}$.

The whiskering

$$M(\tau, 1) = \tau\alpha\colon\ n\alpha \Rrightarrow n'\alpha \tag{A.22}$$

of a perturbation $\tau\colon\ n \Rrightarrow n'\colon\ \beta \Rightarrow \beta'$ *with a tritransformation* $\alpha\colon\ F \Rightarrow G$ *on the right is a perturbation with component at the object* $a \in R$ *given by the whiskering*

$$M_{\mathcal{L}}(\tau_a, 1) = \tau_a \alpha_a \tag{A.23}$$

in the target $\mathcal{G}ray$*-category* $\mathcal{L}$.

The interchange cell $\Sigma_{n,m}$ *at trimodifications* $n\colon\ \beta \Rrightarrow \beta'$ *and* $m\colon\ \alpha \Rrightarrow \alpha'$ *is a perturbation* $(n\alpha') * (\beta m) \Rrightarrow (\beta' m) * (n\alpha)$ *with component at the object* $a \in R$ *given by the interchange cell* Σ_{n_a, m_a} *in the target* $\mathcal{L}$.

Proof. See Gurski [27, Theorem 9.4, P. 143]. □

A.2 Representables

Lemma A.1. *For every object* P *in a tricategory* $\mathcal{P}$*, there is a trihomomorphism*

$$\mathcal{P}(-, P)\colon\ \mathcal{P}^{\mathrm{op}} \to \mathcal{B}icat$$

whose value at the object Q *in* $\mathcal{P}$ *is the bicategory* $\mathcal{P}(Q, P)$.

Proof. We first have to define $\mathcal{P}(-, P)$ on the hom bicategories $\mathcal{P}(Q, Q')$. Given an object f in $\mathcal{P}(Q, Q')$, we have to construct a functor

$$\mathcal{P}(f, P)\colon\ \mathcal{P}(Q', P) \to \mathcal{P}(Q, P)$$

of bicategories. Let g, h be objects, $\alpha, \beta\colon\ g \to h$ be 1-cells, and $\Gamma\colon\ \alpha \Rightarrow \beta$ be a 2-cell in $\mathcal{P}(Q', P)$. Then $\mathcal{P}(f, P)$ is defined on this data by whiskering with f on the right in $\mathcal{P}$, e.g.

$$\mathcal{P}(f, P)(g) = gf\ , \quad \mathcal{P}(f, P)(\alpha) = \alpha 1_f\ , \quad \mathcal{P}(f, P)(\Gamma) = \Gamma 1_{1_f}\ . \tag{A.24}$$

Since the composition functors for $\mathcal{P}$ are functorial on 2-cells, so is the given prescription. Given composable 1-cells $\alpha\colon\ g \to h$ and $\beta\colon\ h \to i$ in $\mathcal{P}(Q', P)$, the composition constraint is defined as the unique invertible 2-cell

$$\mathcal{P}(f, P)_{\beta, \alpha}\colon\ (\beta 1_f) * (\alpha 1_f) \Rightarrow (\beta * \alpha) 1_f$$

arising from coherence for the tricategory $\mathcal{P}$. Naturality with respect to 2-cells $\Gamma\colon\ \alpha \Rightarrow \alpha'$ and $\Delta\colon\ \beta \Rightarrow \beta'$ can be easily deduced from the comparison of the free tricategory and the free *Gray*-category on the target.

The unit constraint is defined as the unique invertible 2-cell

$$\mathcal{P}(f,P)_g = \otimes_{g,f}\colon\ 1_{gf} \Rightarrow 1_g 1_f$$

arising from coherence for the tricategory $\mathcal{P}$. The three pseudofunctor axioms follow from coherence for the tricategory $\mathcal{P}$.

Next, given a 1-cell $\alpha\colon\ f \to f'$ in $\mathcal{P}(Q,Q')$, we have to specify a pseudonatural transformation

$$\mathcal{P}(\alpha,P)\colon\ \mathcal{P}(f,P) \Rightarrow \mathcal{P}(f',P)\colon\ \mathcal{P}(Q',P) \to \mathcal{P}(Q,P)\ . \tag{A.25}$$

Given an object g in $\mathcal{P}(Q',P)$, the component of (A.25) is defined by

$$\mathcal{P}(\alpha,P)_g = 1_g\alpha\colon\ gf \to gf'\ .$$

Given a 1-cell $\beta\colon\ g \to g'$ in $\mathcal{P}(Q',P)$, the naturality 2-cell of (A.25) is defined by the unique invertible 2-cell

$$\mathcal{P}(\alpha,P)_\beta\colon\ (1_{g'}\alpha) * (\beta 1_f) \Rightarrow (\beta 1_{f'}) * (1_g\alpha)$$

arising from coherence for the tricategory $\mathcal{P}$. Naturality with respect to a 2-cell $\Gamma\colon\ \beta \Rightarrow \beta'$ can be easily deduced from the comparison of the free tricategory and the free *Gray*-category on the target. The two transformation axioms follow from coherence for the tricategory $\mathcal{P}$.

Finally, given a 2-cell $\Gamma\colon\ \alpha \Rightarrow \alpha'\colon\ f \to f'$ in $\mathcal{P}(Q',P)$, we have to specify a modification

$$\mathcal{P}(\Gamma,P)\colon\ \mathcal{P}(\alpha,P) \Rrightarrow \mathcal{P}(\alpha',P)\colon\ \mathcal{P}(f,P) \Rightarrow \mathcal{P}(f',P)\colon\ \mathcal{P}(Q',P) \to \mathcal{P}(Q,P)\ . \tag{A.26}$$

Given an object g in $\mathcal{P}(Q',P)$, the component of (A.26) is defined by

$$\mathcal{P}(\Gamma,P)_g = 1_{1_g}\Gamma\colon\ 1_g\alpha \Rightarrow 1_g\alpha'\colon\ gf \to gf'\ .$$

The modification axiom can easily be deduced from the comparison of the free tricategory and the free *Gray*-category on the target.

We now have to check that these prescriptions support the structure of a trihomomorphism. First, we check that they give rise to pseudofunctors on the hom bicategories.

Since the composition functors for $\mathcal{P}$ are functorial on 2-cells, so is $\mathcal{P}(-,P)$ i.e. for composable 2-cells Γ and Γ', we have an equality of modifications

$$\mathcal{P}(\Gamma',P) \diamond \mathcal{P}(\Gamma,P) = \mathcal{P}(\Gamma' \diamond \Gamma,P)\ ,$$

and $\mathcal{P}(1_\alpha,P)$ is the identity modification.

Now for the composition constraint, given composable 1-cells $\alpha\colon\ f \to f'$ and $\alpha'\colon\ f' \to f''$, we have to specify an invertible modification

$$\mathcal{P}(-,P)_{\alpha',\alpha}\colon\ \mathcal{P}(\alpha',P) * \mathcal{P}(\alpha,P) \Rrightarrow \mathcal{P}(\alpha' * \alpha,P)\ . \tag{A.27}$$

The component of (A.27) at an object g in $\mathcal{P}(Q', P)$ is the unique invertible 2-cell

$$(\mathcal{P}(-,P)_{\alpha',\alpha})_g \colon (1_g\alpha') * (1_g\alpha) \Rightarrow 1_g(\alpha' * \alpha) \ ,$$

arising from coherence for the tricategory $\mathcal{P}$, and the modification axiom follows from coherence for the tricategory $\mathcal{P}$. Naturality with respect to 2-cells $\Gamma \colon \alpha \Rightarrow \beta$ and $\Delta \colon \alpha' \Rightarrow \beta'$ can be easily verified on components from the comparison of the free functor tricategory and the free *Gray*-category on the target.

For the unit constraint at f we have to specify an invertible modification

$$\mathcal{P}(-,P)_f \colon 1_{\mathcal{P}(f,P)} \Rrightarrow \mathcal{P}(1_f, P) \ . \tag{A.28}$$

The component of (A.28) at an object g in $\mathcal{P}(Q', P)$ is the unique invertible 2-cell

$$(\mathcal{P}(-,P)_f)_g = \otimes_{g,f} \colon 1_{gf} \Rightarrow 1_g 1_f$$

arising from coherence for the tricategory $\mathcal{P}$, and the modification axiom follows from coherence for the tricategory $\mathcal{P}$. The three pseudofunctor axioms are verified on components by reference to coherence for the tricategory $\mathcal{P}$.

Next, we have to specify an adjoint equivalence

$$\chi \colon \otimes_{\mathcal{B}icat}(\mathcal{P}(-,P)_{Q',Q} \times \mathcal{P}(-,P)_{Q'',Q'}) \Rightarrow \mathcal{P}(-,P)_{Q'',Q} \otimes_{\mathcal{P}^{\mathrm{op}}} \ . \tag{A.29}$$

At the objects $f \in \mathcal{P}(Q, Q')$ and $f' \in \mathcal{P}(Q', Q'')$, (A.29) has component an adjoint equivalence

$$\chi_{f,f'} \colon \mathcal{P}(f,P)\mathcal{P}(f',P) \Rightarrow \mathcal{P}(f'f, P) \ . \tag{A.30}$$

with component

$$(\chi_{f,f'})_g = a_{g,f',f} \colon (gf')f \to g(f'f) \ .$$

at the object g in $\mathcal{P}(Q'', P)$, and naturality 2-cell $(\chi_{f,f'})_\beta = a_{\beta,1_f,1_{f'}}$ at $\beta \colon g \to g'$. Naturality follows from naturality of a, the transformation axioms from the corresponding axioms for a and its naturality (with respect to bicategory constraints). The simpler argument is to appeal to coherence again. The naturality 2-cell of (A.29) at 1-cells $\alpha \colon f \to e$ and $\alpha' \colon f' \to e'$ is a modification with component $\chi_{\alpha,\alpha'} = a_{1_g,\alpha,\alpha'}$ at the object g. The modification axiom follows from coherence. Naturality is verified on components by naturality of a, and the transformation axioms follow from coherence.

Analogously, the pseudonatural transformation $a^\bullet$ gives rise to a pseudonatural transformation $\chi^\bullet_{f',f}$, and unit and counit of the adjoint equivalence $(a, a^\bullet)$ give rise to unit and counit for an adjoint equivalence $(\chi_{f,f'}, \chi^\bullet_{f,f'})$.

The adjoint equivalence

$$\iota \colon I_{\mathcal{P}(Q,P)} \Rightarrow \mathcal{P}(-,P)_{Q,Q} I_Q \tag{A.31}$$

consists of a single component which is an adjoint equivalence

$$\iota_Q \colon 1_{\mathcal{P}(Q,P)} \Rightarrow \mathcal{P}(1_Q, P)$$

with component

$$(\iota_Q)_g = r_g^\bullet \colon\ g \to g1_Q$$

at the object g in $\mathcal{P}(Q,P)$, and naturality 2-cell

$$(\iota_Q)_\beta = r_\beta^\bullet \colon\ r_{g'}^\bullet * \beta \Rightarrow (\beta 1_{1_Q}) * r_g^\bullet$$

at the 1-cell $\beta\colon\ g \to g'$. That is, ι_Q is precisely given by the adjoint equivalence $r^\bullet$. Hence, the opposite adjoint equivalence $(r^\bullet, r)$ gives rise to an adjoint equivalence $(\iota_Q, \iota_Q^\bullet)$ as required.

The invertible modifications ω, γ, and δ of a trihomomorphism have components invertible modifications the components of which are all given by the unique invertible 2-cells arsing from coherence. The two trihomomorphism axioms follow from coherence. □

Remark A.1. The proof is just a generalization by coherence of the proof in the case that $\mathcal{P}$ is is a *Gray*-category [27]. It is however to be noted that the Yoneda embedding from [27] led itself to a coherence theorem, which is why coherence was not employed ibidem.

Lemma A.2. *Given a* 1*-cell* $f\colon\ P \to P'$ *in a tricategory* $\mathcal{P}$*, there is a tritransformation*

$$\mathcal{P}(-,f)\colon\ \mathcal{P}(-,P) \Rightarrow \mathcal{P}(-,P')\colon\ \mathcal{P}^{\mathrm{op}} \to \mathcal{B}icat \tag{A.32}$$

of representables whose component at the object Q *in* $\mathcal{P}$ *is a pseudofunctor*

$$\mathcal{P}(Q,f)\colon\ \mathcal{P}(Q,P) \to \mathcal{P}(Q,P')\ , \tag{A.33}$$

whose component at a 1*-cell* $g\colon\ Q \to P$ *in* $\mathcal{P}$ *is the composite* fg *in* $\mathcal{P}$*.*

Proof. The proof is again an extension by coherence of the proof in the case that $\mathcal{P}$ is a *Gray*-category cf. Gurski [27, Lemma 9.8]. For this reason, we will just outline the data of the tritransformation.

The pseudofunctor (A.33) is defined similarly to the definition of the pseudofunctor $\mathcal{P}(f,P)$ appearing in the proof of Lemma A.1: Let g, h be objects, $\alpha, \beta\colon\ g \to h$ be 1-cells, and $\Gamma\colon\ \alpha \Rightarrow \beta$ be a 2-cell in $\mathcal{P}(Q,P)$. Then $\mathcal{P}(Q,p)$ is defined on this data by whiskering on the left with f in $\mathcal{P}$, e.g.

$$\mathcal{P}(Q,f)(g) = fg\ ,\quad \mathcal{P}(Q,f)(\alpha) = 1_f\alpha\ ,\quad \mathcal{P}(Q,f)(\Gamma) = 1_{1_f}\Gamma\ . \tag{A.34}$$

This is made into a pseudofunctor by coherence. The adjoint equivalence

$$\mathcal{B}icat(1, \mathcal{P}(R,f))\mathcal{P}(-,P)_{QR} \Rightarrow \mathcal{B}icat(\mathcal{P}(Q,f),1)\mathcal{P}(-,P')_{Q,R}\ . \tag{A.35}$$

has component at the object $g \in \mathcal{P}(Q,R)$ a 1-cell in $\mathcal{B}icat$ i.e. a pseudonatural transformation

$$\mathcal{P}(R,f)\mathcal{P}(g,P) \Rightarrow \mathcal{P}(g,P')\mathcal{P}(Q,f) \tag{A.36}$$

with component at the object $j \in \mathcal{P}(Q,P)$, the component

$$a^{\bullet}_{f,j,g}\colon\ f(jg) \to (fj)g \tag{A.37}$$

of the right adjoint $a^{\bullet}$ of the associativity adjoint equivalence a for $\mathcal{P}$. The naturality 2-cell of (A.36) at $\alpha\colon\ j \to j'$ is exactly the naturality 2-cell $a^{\bullet}_{1,\alpha,1}$, but the naturality 2-cell of (A.35) contains further constraints due to the whiskering in $\mathcal{B}icat$. The invertible modifications Π and M are determined by coherence, from which the tritransformation axioms follow automatically. □

Lemma A.3. *Given a 2-cell $\alpha\colon\ f \Rightarrow f'\colon\ P \to P'$ in a tricategory $\mathcal{P}$, there is a trimodification*

$$\mathcal{P}(-,\alpha)\colon\ \mathcal{P}(-,f) \Rrightarrow \mathcal{P}(-,f') \tag{A.38}$$

whose component at the object Q in $\mathcal{P}$ is a pseudonatural transformation

$$\mathcal{P}(Q,\alpha)\colon\ \mathcal{P}(Q,f) \to \mathcal{P}(Q,f')\ , \tag{A.39}$$

whose component at a 1-cell $g\colon\ Q \to P$ in $\mathcal{P}$ is the whiskering $\alpha 1_g$ in $\mathcal{P}$ and whose naturality 2-cell at a 2-cell $\beta\colon\ g \Rightarrow g'$ is the 'interchange' cell $\Sigma_{\alpha,\beta}$.

Proof. The invertible modification for the trimodification $\mathcal{P}(-,\alpha)$ is determined by coherence, from which the trimodification axioms follow automatically. □

Lemma A.4. *Given a 3-cell $\Gamma\colon\ \alpha \Rightarrow \alpha'\colon\ f \Rightarrow f'\colon\ P \to P'$ in a tricategory $\mathcal{P}$, there is a perturbation*

$$\mathcal{P}(-,\Gamma)\colon\ \mathcal{P}(-,\alpha) \Rrightarrow \mathcal{P}(-,\alpha') \tag{A.40}$$

whose component at the object Q in $\mathcal{P}$ is a modification

$$\mathcal{P}(Q,\Gamma)\colon\ \mathcal{P}(Q,\alpha) \to \mathcal{P}(Q,\alpha')\ , \tag{A.41}$$

whose component at a 1-cell $g\colon\ Q \to P$ in $\mathcal{P}$ is the whiskering $\Gamma 1_{1_g}$ in $\mathcal{P}$.

Proof. The perturbation axiom follows from the comparison of the free tricategory and the free $\mathcal{G}ray$-category on the underlying category-enriched 2-graph of $\mathcal{B}icat$. □

Remark A.2. After we will have defined a tricategory structure on the 3-globular set $\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)$ in Theorem A.4, we will prove in Theorem A.5 below that the prescriptions from Lemmata A.1-A.4 give rise to a trihomomorphism—the Yoneda embedding for tricategories.

A.3 Composition of trihomomorphisms

Below we recollect the critical bits and pieces of the definition of the composite of trihomomorphisms introduced in Gurski [27, 7.1].

Definition A.1. Let R, S, and T be tricategories, and let $H\colon R \to S$ and $J\colon S \to T$ be trihomomorphisms. The following prescriptions define the composite trihomomorphism $JH\colon R \to T$.

- On objects, JH is given by the composition of the object-functions for J and H.
- Locally, JH is given by the composition of the hom functors of J and H i.e., for objects a, b in R,

$$(JH)_{a,b}\colon R(a,b) \xrightarrow{H_{a,b}} S(Ha,Hb) \xrightarrow{J_{Ha,Hb}} T(JHa,THb)\ .$$

- For objects a, b, c in R, the adjoint equivalences

$$\chi_{JH}\colon \otimes_T (JH \times JH) \Rightarrow JH\otimes_R\colon R(b,c)\times R(a,b) \to T(a,c)$$

  are defined as the composite adjoint equivalences

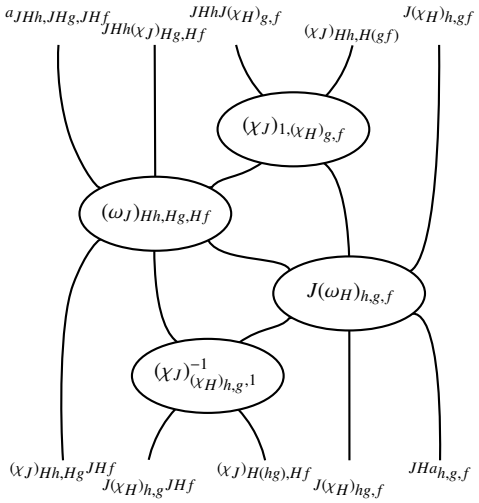
$$\otimes_T(J\times J)(H\times H) \xrightarrow{\chi_J 1_{H\times H}} J\otimes_S (H\times H) \xrightarrow{1_J\chi_H} JH\otimes_R$$

  where identifiers for objects have been omitted for readability.

- For an object a in R, the adjoint equivalences ι_{JH} are defined as the composite adjoint equivalences

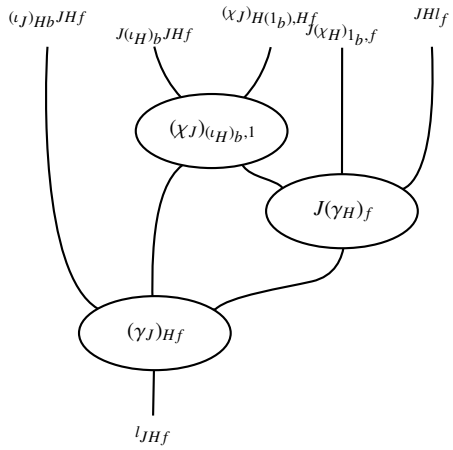
$$j_T \xrightarrow{\iota_J} Jj_S \xrightarrow{1_J\iota_H} JHj_R\colon I \to T(JHa,JHa)\ .$$

- For cubical trihomomorphisms, the component of the invertible modification ω_{JH} at appropriate 1-cells (h,g,f) in R is represented by the string diagram below.



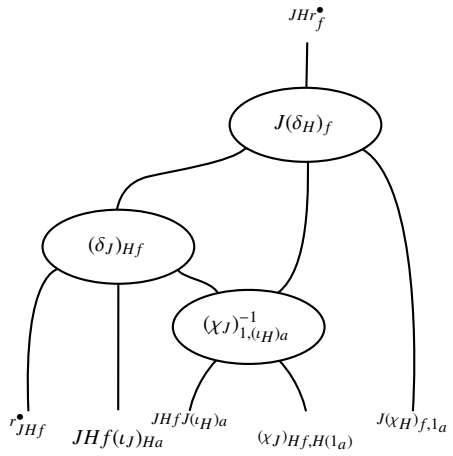
It is not hard to supplement this diagram to cover the general case cf. also Gurski [27, 7.1, p. 108].

- For cubical trihomomorphisms, the component of the invertible modification γ_{JH} at a 1-cell f in $\boldsymbol{R}$ is represented by the string diagram below.



It is not hard to supplement this diagram to cover the general case cf. also Gurski [27, 7.1, p. 109].

- For cubical trihomomorphisms, the component of the invertible modification δ_{JH} at a 1-cell f in $\boldsymbol{R}$ is represented by the string diagram below.



It is not hard to supplement this diagram to cover the general case.

Proposition A.1. *The prescriptions in Definition A.1 give rise to a trihomomorphism as required.*

Proof. See [27, Prop. 7.1]. □

A.4 Whiskering with a trihomomorphism on the left

Let R, S, and T be tricategories, and let $F\colon R \to S$ and $G\colon S \to T$ be trihomomorphisms.

Proposition A.2. *Let R, S, and T be tricategories, let $F, F'\colon R \to S$ and $G\colon S \to T$ be trihomomorphisms, and let $\alpha\colon F \Rightarrow F'$ be a tritransformation. Then the following prescriptions define a tritransformation*

$$G_*\alpha\colon GF \Rightarrow GF' \ .$$

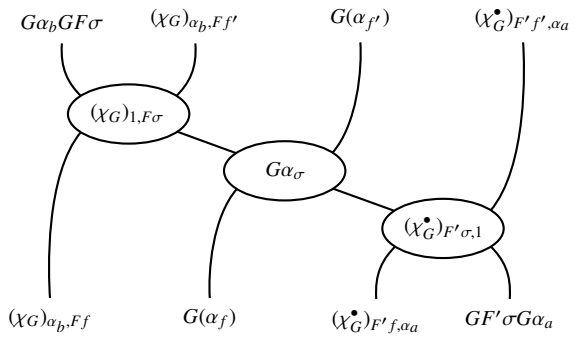
The tritransformation $G_*\alpha$ has component at an object a in R given by

$$G\alpha_a\colon GFa \to GF'a \ .$$

Its adjoint equivalence is defined by the evident composite of adjoint equivalences with component at a 1-cell $f\colon a \to b$ in R given by

$$(\chi_G^\bullet)_{F'f,\alpha_a} * G(\alpha_f) * (\chi_G)_{\alpha_b,Ff}$$

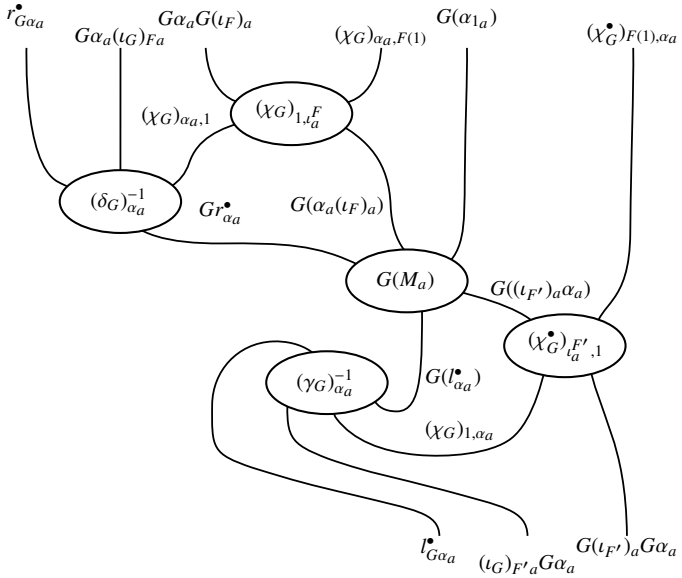
(where we have chosen no particular association), and its naturality 2-cell at a 2-cell $\sigma\colon f \to f'$ in R is represented by the following string diagram



where the cell $G\alpha_\sigma$ has to be interpreted correctly if G is not cubical.

The invertible modification $M_{G_*\alpha}$ has component at the object a in R given by the string

diagram below.



Proof. See Gurski [27, Prop. 7.5]. □

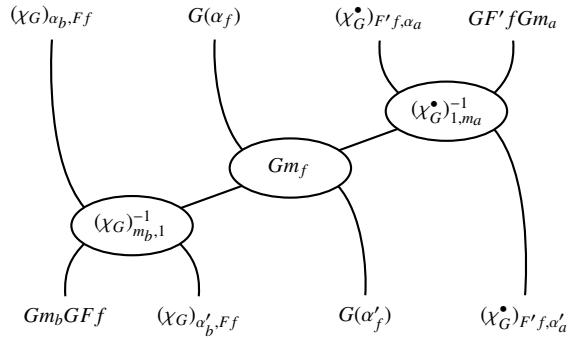
Proposition A.3. *Let R, S, and T be tricategories, let $F, F' \colon R \to S$ and $G \colon S \to T$ be trihomomorphisms, let $\alpha, \alpha' \colon F \Rightarrow F'$ be tritransformations, and let $m \colon \alpha \Rrightarrow \alpha'$ be a trimodification. Then the following prescriptions define a trimodification*

$$G_*m \colon G_*\alpha \Rrightarrow G_*\alpha' \ .$$

The trimodification G_*m has component at an object a in R given by

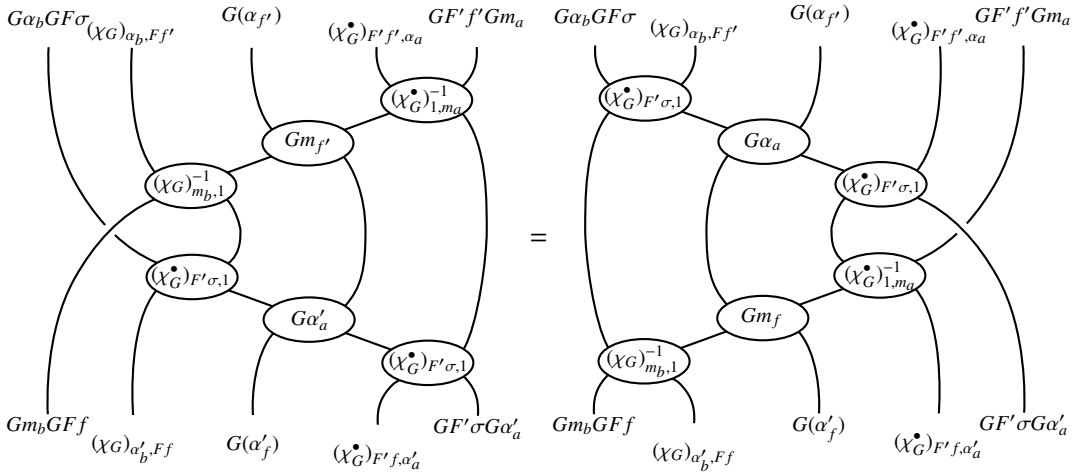
$$Gm_a \colon G\alpha_a \to G\alpha'_a \ .$$

The component at a 1-cell $f \colon a \to b$ in R of its invertible modification is represented by the following string diagram



where the cell Gm_f has to be interpreted correctly if G is not cubical.

Proof. The modification axiom is given by the equation of string diagrams below.



It is a pleasing exercise in the diagrammatical calculus to verify this axiom, in particular if one assumes for simplicity that G and the target are cubical. The proof is only mildly more complicated if G and the target are not assumed to be cubical. □

Proposition A.4. *Let R, S, and T be tricategories, let $F, F' \colon R \to S$ and $G \colon S \to T$ be trihomomorphisms, let $\alpha, \alpha' \colon F \Rightarrow F'$ be tritransformations, $m, m' \colon \alpha \Rrightarrow \alpha'$ trimodifications, and let $\sigma \colon m \lll n$ be a perturbation. Then the following prescriptions define a perturbation*

$$G_*\sigma \colon G_*m \lll G_*m' \ .$$

The perturbation $G_*\sigma$ has component at an object a in R given by

$$G\sigma_a \colon Gm_a \to Gm'_a \ .$$

Proof. The perturbation axiom is easily verified by naturality of $(\chi_G^\bullet)^{-1}$ and $(\chi_G)^{-1}$, and the perturbation axiom for σ (in the general non-cubical case, one also has to use naturality of the constraints of the hom functors of G). □

Proposition A.5. *Let R, S, and T be tricategories, and let $F, F' \colon R \to S$ and $G \colon S \to T$ be trihomomorphisms. Then the prescriptions in Propositions A.1-A.4 support the structure of local functors of functor tricategories*

$$(G_*)_{F,F'} \colon \mathcal{T}\mathit{ricat}(S,T)(F,F') \to \mathcal{T}\mathit{ricat}(S,T)(GF,GF') \ . \tag{A.42}$$

Given tritransformations $\alpha, \alpha', \alpha'' \colon F \Rightarrow F'$ and trimodifications $m \colon \alpha \Rrightarrow \alpha'$ and $m' \colon \alpha' \Rrightarrow \alpha''$, the composition constraint of this pseudofunctor (A.42) *is a perturbation*

$$(G_*m') * (G_*m) \lll G_*(m' * m)$$

with component at an object a in R the composition constraint

$$G_{m'_a,m_a}\colon\ Gm'_a * Gm_a \Rightarrow G(m'_a * m_a) \tag{A.43}$$

of the hom functor

$$G_{Fa,F'a}\colon\ S(Fa,F'a) \to T(GFa,GF'a) \tag{A.44}$$

of G.

The unit constraint of (A.42) *at the tritransformation α is a perturbation*

$$1_{G_*\alpha} \Rrightarrow G_* 1_\alpha$$

with component at an object a in R the unit constraint

$$G_{\alpha_a}\colon\ 1_{G_{\alpha_a}} \Rightarrow G(1_{\alpha_a}) \tag{A.45}$$

of (A.44).

Proof. It is straightforward to check that the component prescriptions in (A.43) and (A.45) give rise to perturbations as required by spelling out the composite trimodifications. The pseudofunctor axioms for $(G_*)_{F,F'}$—as equations of perturbations—are verified on components where they follow from the pseudofunctor axioms for the hom functor $G_{Fa,F'a}$. □

Lemma A.5. *Let R, S, and T be tricategories, let $F, F', F''\colon R \to S$ and $G\colon S \to T$ be trihomomorphisms, and let $\alpha\colon F \Rightarrow F'$ and $\beta\colon F' \Rightarrow F''$ be tritransformations. Then there is an adjoint equivalence*

$$\chi_{\beta,\alpha}\colon\ (G_*\beta)(G_*\alpha) \Rrightarrow G_*(\beta\alpha)$$

in $\mathcal{T}ricat(R,T)(GF,GF'')$ with component at an object a in R given by the adjoint equivalence

$$(\chi_G)_{\beta_a,\alpha_a}\colon\ (G\beta_a)(G\alpha_a) \to G(\beta_a\alpha_a)\ .$$

This adjoint equivalence forms the component of an adjoint equivalence

$$\chi\colon\ \otimes_{\mathcal{T}ricat(R,T)}(G_* \times G_*) \Rightarrow G_* \otimes_{\mathcal{T}ricat(R,S)} \tag{A.46}$$

in the situations where the tricategory structure of $\mathcal{T}ricat(R,T)$ and $\mathcal{T}ricat(R,S)$ is known. For example, this is the case when T and S are either a $\mathcal{G}ray$-category (in particular, a category) or $\mathcal{B}icat$ cf. Theorem A.4.

Proof. The component of the invertible modification of the trimodification with component $(\chi_G)_{\beta_a,\alpha_a}$ can be chosen as the unique constraint given by coherence for the trihomomorphism G. The trimodification axioms then follow from the definition of the composition of tritransformations and coherence for G. By similar arguments, one checks

that $\chi_G^\bullet$ gives rise to a right adjoint $\chi^\bullet$. The component of the invertible modification of the latter is in fact the mate of the one for the left adjoint.

The proof that the naturality 2-cells for χ_G give rise to naturality 2-cells for (A.46), which involves proving a perturbation axiom, proceeds essentially by coherence for G and the definition of the (tricategorical) composite of trimodifications. The axioms for a pseudonatural transformation follow from those for χ_G. That unit and counit induce perturbations follows again from coherence for G, and the modification axioms follows from the modification axiom for unit and counit of the adjoint equivalence χ_G. □

Lemma A.6. *Let R, S, and T be tricategories, let $F, F' \colon R \to S$ and $G \colon S \to T$ be trihomomorphisms, and let $\alpha \colon F \Rightarrow F'$ be a tritransformations. Then there is an adjoint equivalence*

$$\iota_\alpha \colon 1_{G_*\alpha} \Rrightarrow G_*(1_\alpha)$$

in $\mathcal{T}ricat(R,T)(GF, GF')$ with component at an object a in R given by the adjoint equivalence

$$(\iota_G)_{\alpha_a} \colon 1_{G\alpha_a} \to G(1_{\alpha_a}) \ .$$

Proof. By similar arguments. □

Conjecture A.1. *Let R, S, and T be tricategories, and let $G \colon S \to T$ be a trihomomorphisms. Then the prescriptions above give rise to a trihomomorphism*

$$G_* \colon \mathcal{T}ricat(R,S) \to \mathcal{T}ricat(R,T) \ . \tag{A.47}$$

Theorem A.3. *Given tricategories R, S and T and a trihomomorphism $G \colon S \to T$, where S and T are respectively either a $\mathcal{G}ray$-category (in particular, a category) or $\mathcal{B}icat$, then the prescriptions above give rise to a trihomomorphism*

$$G_* \colon \mathcal{T}ricat(R,S) \to \mathcal{T}ricat(R,T) \ . \tag{A.48}$$

Proof. What remains to be proved is to construct the invertible modifications ω, γ, and δ, and show that they satisfy the trihomomorphisms axioms. The component of Π at composable tritransformations $\alpha \colon F \Rightarrow F' \colon R \to S$ and $\alpha' \colon F' \Rightarrow F'' \colon R \to S$ are perturbations with component at $a \in \mathrm{ob}R$ given by $(\omega_G)_{\alpha'_a, \alpha_a}$. The perturbation axiom follows from coherence for G. Similarly, one defines the invertible modifications γ and δ. The two trihomomorphism axioms then follow from the trihomomorphism axioms for G. Notice that the constraints of the functor tricategories have components induced by the constraints of the target. If the target is a $\mathcal{G}ray$-category, these are trivial. If the target is $\mathcal{B}icat$, these are the unique constraint cells given by coherence in the relevant bicategory. □

Conjecture A.2. *Given tricategories R, S, and T, then whiskering on the left supports the structure of a trihomomorphism*

$$(-)_* \colon \mathcal{T}ricat(S,T) \to \mathcal{T}ricat(\mathcal{T}ricat(R,S), \mathcal{T}ricat(R',T)) \ . \tag{A.49}$$

A.5 Whiskering on the left with strictification

Proposition A.6. *Whiskering on the left with the strictification functor* st *is a biequivalence on hom bicategories*

$$\mathrm{st}_*\colon \mathcal{T}ricat(S, \mathcal{B}icat)(F, F') \xrightarrow{\cong} \mathcal{T}ricat(S, \mathcal{G}ray)(\mathrm{st}\circ F, \mathrm{st}\circ F') \ . \tag{A.50}$$

Proof. Let $\alpha\colon \mathrm{st}\circ F \Rightarrow \mathrm{st}\circ F'$ be a tritransformation. We aim to construct a tritransformation $\bar{\alpha}\colon F \Rightarrow F'$ such that $\mathrm{st}_*\bar{\alpha}$ is equivalent to α in $\mathcal{T}ricat(S, \mathcal{G}ray)(F, F')$.

The component of $\bar{\alpha}$ at an object a in S is a pseudofunctor defined by the diagram below

$$\begin{array}{ccc} \mathrm{st}F(a) & \xrightarrow{\alpha_a} & \mathrm{st}F'(a) \\ {\scriptstyle f}\uparrow & & \downarrow{\scriptstyle e} \\ F(a) & \xrightarrow{\bar{\alpha}_a} & F'(a) \end{array} \tag{A.51}$$

which commutes up to an invertible icon ω. In particular, $\bar{\alpha}_a$ agrees with α_a on objects.

The adjoint equivalence of the alleged tritransformation $\bar{\alpha}$ has component at a 1-cell $f\colon a \to b$ the pseudonatural transformation given by the adjoint equivalence α_f whiskered with f and e and the icon

$$\begin{array}{ccccc} \mathrm{st}F(a) & \xrightarrow{\alpha_a} & \mathrm{st}F'(a) & \xrightarrow{\mathrm{st}F'(f)} & \mathrm{st}F'(b) \\ {\scriptstyle f}\uparrow & = & \downarrow{\scriptstyle e} & \Downarrow\omega & \downarrow{\scriptstyle e} \\ F(a) & \xrightarrow{\bar{\alpha}_a} & F'(a) & \xrightarrow{F(f)} & F'(b) \end{array}$$

noting that

$$\begin{array}{ccccc} \mathrm{st}F(a) & \xrightarrow{\mathrm{st}F(f)} & \mathrm{st}F(b) & \xrightarrow{\alpha_b} & \mathrm{st}F'(b) \\ {\scriptstyle f}\uparrow & = & {\scriptstyle f}\uparrow & = & \downarrow{\scriptstyle e} \\ F(a) & \xrightarrow{F(f)} & F(b) & \xrightarrow{\bar{\alpha}_b} & F'(b) \end{array} \ .$$

In particular, the component of $\bar{\alpha}_f$ at an object k in $F(a)$ is given by $e((\alpha_f)_k)$ (up to whiskerings with identities coming from the icons).

The naturality 2-cell of the adjoint equivalence $\bar{\alpha}$ at a 2-cell $\gamma\colon f \Rightarrow f'$ is an invertible modification

$$\bar{\alpha}_{f'} * (1_{\bar{\alpha}_b} F(\gamma)) \Rrightarrow (F'(\gamma) 1_{\bar{\alpha}_a}) * \bar{\alpha}_f$$

given as follows. Since up to coherence isomorphism

$$\omega * (1_e \mathrm{st}F'(\gamma)) = (F'(\gamma) 1_e) * \omega \tag{A.52}$$

and since clearly,

$$\mathrm{st}F(\gamma)1_f = 1_f F(\gamma) \tag{A.53}$$

up to coherence isomorphism[1], we can define the naturality 2-cells of $\bar{\alpha}$ as

$$1_{1_e}(\alpha_\gamma 1_{1_f}) \ .$$

Transformation axioms: For naturality, we have to argue why, for a 3-cell $\Theta\colon\ \gamma \Rrightarrow \gamma'$, $1_{\alpha_{f'}} * (1_{1_{\bar{\alpha}_b}} F(\Theta))$ and $(F'(\Theta)1_{1_{\bar{\alpha}_a}}) * 1_{\bar{\alpha}_f}$ commute appropriately with the coherence isomorphism (A.52) and the equality (A.53). Spelling out these commutation conditions, they are easily seen to follow from comparison of the free tricategory and the free *Gray*-category on *Bicat* cf. also A.8. Naturality then follows from naturality of the adjoint equivalence α. Respect for composition and units follows from the very same for α noting that the coherence isomorphisms in between have to cancel by coherence. Also observe that this involves composition and unit constraints for the local functors of F and F', and note that these coincide with the corresponding constraints for the local functors of $\mathrm{st} \circ F$ and $\mathrm{st} \circ F'$ up to coherence isomorphisms on components. This is because composition and unit constraints of the local functors of st are given by coherence isomorphisms on components.

Invertible Modifications Π and M. These have components invertible modifications. Note that since the components of the component transformations of the adjoint equivalences are given by $e((\alpha_f)_k)$, domain and codomain of Π and M only differ in a coherence isomorphism from domain and codomain for Π and M for α on components. Therefore, the components of the component modifications of the invertible modifications Π and M for $\bar{\alpha}$ can be defined in terms of the corresponding components of the component modifications of Π and M for α respectively and coherence isomorphisms.

Modification axioms: We have to verify the modification axiom for the components and the alleged modifications Π and M for $\bar{\alpha}$ themselves. This follows from the very same axioms for the component modifications of Π and M for α and the latter themselves by similar arguments as above.

Tritransformation axioms: The axioms are verified on components of the component modifications. Again, coherence isomorphisms in between must cancel by coherence, so the axioms follow from the tritransformation axioms for α, indeed are equivalent.

Locally essentially surjective: given a trimodification m between two tritransformations $\mathrm{st}_*\alpha$ and $\mathrm{st}_*\beta$, it has components which are transformations m_a, and we take the transformation $\bar{m}_a$ as in Gurski [27, p. 136] which has components $e(m_a)$ and naturality 2-cells $e(m_f)$ up to coherence isomorphisms. Since the adjoint equivalences χ for the trihomomorphism st are induced by icons cf. Gurski [27, p. 137], they have identity components, and the transformations $\mathrm{st}(\alpha_f)$ and $\mathrm{st}(\beta_f)$ have length 1 components coinciding with the ones for α_f and β_f. From this it follows that, up to coherence isomorphism, we can take the same components as for the invertible modification of m.

Modification axiom: The 2-cells of the adjoint equivalences of the tritransformations $\mathrm{st}\alpha$ and $\mathrm{st}\beta$, at a 2-cell σ, are given by naturality 2-cells of χ for st—which are given by

[1]meaning up to an invertible modification with components coherence isomorphisms

coherence isomorphisms—and $\mathrm{st}(\alpha_\sigma)$ and $\mathrm{st}(\beta_\sigma)$. The latter are strictifications of modifications, which have components represented by the same 2-cells. The modification axiom thus follows from coherence and the modification axiom for the invertible modification of m.

Trimodification axioms: These follow again from the trimodification axioms for m because the coherence isomorphisms in between have to cancel.

2-locally fully faithful: Given a perturbation it has as its component a modification between strictifications of transformations. By restriction this gives rise to a unique modification the strictification of which coincides with the original modification cf. Gurski [27, p. 136]. The perturbation axiom directly follows from the one for the original perturbation.

□

Theorem A.4. *The bicategory enriched graph $\mathcal{T}ricat(S, \mathcal{B}icat)$ of trihomomorphisms, tritransformations, trimodifications, and perturbations has the structure of a tricategory such that whiskering on the left with the strictification* $\mathrm{st}\colon \mathcal{B}icat \to \mathcal{G}ray$ *gives rise to a triequivalence*

$$\mathrm{st}_*\colon \mathcal{T}ricat(S, \mathcal{B}icat) \to \mathcal{T}ricat(S, \mathcal{G}ray)' \tag{A.54}$$

with the sub-$\mathcal{G}ray$-category $\mathcal{T}ricat(S, \mathcal{G}ray)'$ of $\mathcal{T}ricat(S, \mathcal{G}ray)$ given by the (essential) image of st_*.

Proof. Replacing the local biequivalences from Proposition A.6 by biadjoint biequivalences cf. Gurski [26, Theorem 4.5], the theorem follows from Transport of structure cf. Gurski [27, Th. 7.22, p. 124] and the $\mathcal{G}ray$-category structure of $\mathcal{T}ricat(S, \mathcal{G}ray)$. □

Theorem A.5. *The prescriptions from Lemmata A.1-A.4 support the structure of a trihomomorphism*

$$y\colon \mathcal{P} \to \mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)\ . \tag{A.55}$$

Proof. Given a 2-cell $\alpha\colon f \Rightarrow f'\colon P \to P'$ in P, note that $y(1_\alpha) = \mathcal{P}(-, 1_\alpha)$ is a perturbation with component at Q, a modification $\mathcal{P}(Q, 1_\alpha)$ with component at the 1-cell $g\colon Q \to P$ in $\mathcal{P}$ the whiskering $1_\alpha 1_{1_g}$. Since the composition in $\mathcal{P}$ is given by pseudofunctors this is the identity 2-cell $1_{\alpha 1_g}$ in $\mathcal{P}(Q, P')$. Thus the modification $\mathcal{P}(Q, 1_\alpha)$ must be the identity modification, and hence the perturbation $\mathcal{P}(-, 1_\alpha)$ must be the identity modification. Similarly, for composable 3-cells Γ and Γ', one sees that $y(\Gamma') \diamond y(\Gamma) = y(\Gamma' \diamond \Gamma)$ because $(\Gamma' 1_{1_g}) \diamond (\Gamma 1_{1_g}) = (\Gamma' \diamond \Gamma) 1_{1_g}$. We next have to argue why y locally has the structure of a pseudofunctor. The constraint $y(\alpha') * y(\alpha) \Rightarrow y(\alpha' * \alpha)$ is a perturbation with component given by the unique coherence isomorphism $(\alpha' 1_g) * (\alpha 1_g) \cong (\alpha' * \alpha) 1_g$ arising from coherence for $\mathcal{P}$. Similarly, one determines the unit constraints of the local pseudofunctors.

For the adjoint equivalence χ, its component at composable 1-cells $f\colon Q \to R$ and $g\colon R \to S$ in $\mathcal{P}$ is a trimodification with component at P a modification with component at the 1-cell $h\colon P \to Q$ in $\mathcal{P}$ given by $a^{\bullet}_{f,g,h}\colon f(gh) \to (fg)h$. We remark that although the composition in $\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)$ is obscured by its definition

through Transport of structure, it is still true that the composite of tritransformations $\mathcal{P}(-,f)\mathcal{P}(-,g)$ has component a pseudofunctor with value on h given by $f(gh)$ because $e((\mathrm{st}\mathcal{P}(Q,f)\mathrm{st}\mathcal{P}(Q,g))(f(h))) = \mathcal{P}(Q,f)(\mathcal{P}(Q,g)(h)) = f(gh)$ cf. (A.51).

Similarly, the single component of the adjoint equivalence ι at S is a trimodification with component at R a modification with component at $h\colon R \to S$ given by $l^{\bullet}_{h}$.

The invertible modifications ω, γ, and δ, and the rest of the data of the trihomomorphism are determined by coherence, from which the trihomomorphism axioms follow automatically. □

Remark A.3. Determining the domain and codomain for the components of the invertible modifications ω, γ, and δ in the proof above involves whiskerings of trimodifications in $\mathcal{T}ricat(\mathcal{P}^{\mathrm{op}}, \mathcal{B}icat)$. Tracing through the proof of Prop. (A.6), one sees that the pseudonatural transformations which form the components of these whiskerings have the same components as the corresponding whiskerings in $\mathcal{B}icat$ (of the component-transformation of the trimodification with the component-functor of the tritransformation). Therefore, the components of the component modifications of the invertible modifications ω, γ, and δ are given by mates of components of the invertible modifications π^{-1}, λ, and ρ^{-1} of the tricategory $\mathcal{P}$ respectively.

A.6 Whiskering with a trihomomorphism on the right

Proposition A.7. *Given a $\mathcal{G}ray$-category $\mathcal{L}$, tricategories $\mathcal{R}$ and $\mathcal{S}$, a trihomomorphism $F\colon \mathcal{R} \to \mathcal{S}$, and a tritransformation $\beta\colon G \to G'\colon \mathcal{S} \to \mathcal{L}$ of trihomomorphisms, there is a tritransformation*

$$F^{*}\beta\colon GF \Rightarrow G'F$$

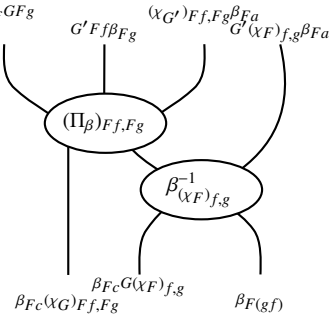
The tritransformation $F^{*}\beta$ has component at an object a in $\mathcal{R}$ given by

$$\beta_{Fa}\colon GFa \to G'Fa\ .$$

Its adjoint equivalence is defined by the whiskering $F^{*}\beta$ of the adjoint equivalence of β. That is, its component at a 1-cell $f\colon a \to b$ in $\mathcal{R}$ is given by

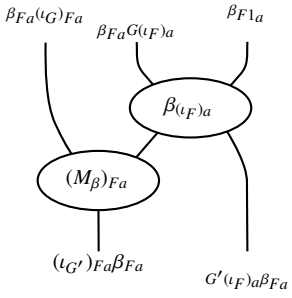
$$\beta_{Ff}\ .$$

The component of the invertible modification $\Pi_{F^*\beta}$ *at composable* 1*-cells* f, g *in* $\mathbf{R}$ *is given by the string diagram below.*



(A.56)

The component of the invertible modification $M_{F^*\beta}$ *at an objet* a *in* $\mathbf{R}$ *is given by the string diagram below.*



(A.57)

Proof. See Gurski [27, Prop. 7.5]. □

Proposition A.8. *Given a* Gray*-category* $\mathcal{L}$*, (cubical) tricategories* $\mathbf{R}$ *and* $\mathbf{S}$*, a (cubical) trihomomorphism* $F\colon \mathcal{R} \to \mathcal{S}$*, tritransformations* $\beta, \beta'\colon G \to G'\colon \mathcal{S} \to \mathcal{L}$ *of (cubical) trihomomorphisms, and a trimodification* $n\colon \beta \Rrightarrow \beta'$*, the following prescriptions define a trimodification*

$$F^*n\colon F^*\beta \Rightarrow F^*\beta' \ .$$

The trimodification F^*n *has component at an object* a *in* $\mathbf{R}$ *given by*

$$n_{Fa}\colon \beta_{Fa} \to \beta'_{Fa} \ .$$

Its invertible modification is defined by the whiskering F^*n *of the invertible modification of* n*. That is, its component at a* 1*-cell* $f\colon a \to b$ *in* $\mathbf{R}$ *is given by*

$$m_{Ff}\colon (1_{G'Ff}m_{Fa}) * \beta_{Ff} \Rightarrow \beta'_{Ff} * (m_{Fb}1_{GFf}) \ .$$

Proof. We have to prove the two trimodification axiom. The first trimodification axiom is easily verified from the first trimodification axiom for n and the modification axiom for its invertible modification at $(\chi_F)_{f,g}$. Similarly, the second trimodification axiom is verified from the corresponding axiom for n and the modifications axiom for its invertible modification at $(\iota_F)_a$. □

Proposition A.9. *Given a $\mathcal{G}ray$-category $\mathcal{L}$, (cubical) tricategories R and S, a (cubical) trihomomorphism $F\colon \mathcal{R} \to \mathcal{S}$, tritransformations $\beta,\beta'\colon G \to G'\colon \mathcal{S} \to \mathcal{L}$ of (cubical) trihomomorphisms, trimodifications $n,n'\colon \beta \Rrightarrow \beta'$, and a perturbation $\tau\colon n \Rrightarrow n'$, there is a perturbation*

$$F^*\tau\colon F^*n \Rightarrow F^*n' \ .$$

with component at an object a in R given by

$$\tau_{Fa}\colon n_{Fa} \to n'_{Fa} \ .$$

Proof. It is immediate that this data satisfies the perturbation axiom since this is literally the perturbation axiom for τ at Fa. □

Theorem A.6. *Given a $\mathcal{G}ray$-category $\mathcal{L}$ and tricategories R and S and a trihomomorphism $F\colon R \to S$, then whiskering on the right with F has the structure of a $\mathcal{G}ray$-functor*

$$F^*\colon \mathcal{T}ricat(S,\mathcal{L}) \to \mathcal{T}ricat(R,\mathcal{L}) \tag{A.58}$$

of functor tricategories.

Proof. First, note that F^*1_G is the identity tritransformation 1_{FG}. It clearly has identity components and identity adoint equivalences. Since the identity tritransformation 1_G has identity adjoint equivalences, the naturality 2-cells of the latter are identities. This implies that the invertible modifications of F^*1_G described in Proposition A.7 are given by identity modifications as required.

Second, for composable tritransformations $\beta\colon G \Rightarrow G'$ and $\beta'\colon G' \Rightarrow G''$, note that $F^*(\beta'\beta) = (F^*\beta')(F^*\beta)$, as is easily verified from the definition of the composite of two tritransformations with 0-cell target a $\mathcal{G}ray$-category cf. Th. A.2.

Third, it similarly follows that F^* preserves the two compositions of trimodifications, and it is easy to to see that F^* preserves the three compositions of perturbations as these are given by the three compositions of 3-cells in the target.

Fourth, it is easy to see that F^* preserves interchange cells. □

Proposition A.10. *Given a $\mathcal{G}ray$-category $\mathcal{L}$, tricategories R, S, T, and trihomomorphisms $F\colon R \to S$ and $F'\colon S \to T$, then we have an identity of $\mathcal{G}ray$-functors*

$$(F'F)^* = F^*F'^*\colon \mathcal{T}ricat(T,\mathcal{L}) \to \mathcal{T}ricat(R,\mathcal{L}) \ . \tag{A.59}$$

and the whiskering with the identity trihomomorphism is the identity on functor tricategories i.e. we have an identity of $\mathcal{G}ray$-functors

$$(1_T)^* = 1\colon \mathcal{T}ricat(T,\mathcal{L}) \to \mathcal{T}ricat(T,\mathcal{L}) \tag{A.60}$$

Proof. That $(1_S)^*$ is the identity $\mathcal{G}ray$-functor is nearly immediate from the precriptions defining it—verifying this only involves respect for units of the adjoint equivalences of a tritransformation β (using that both source and target are locally strict) to show that $\Pi_{(1_s)^*(\beta)} = \Pi_\beta$ and $M_{(1_s)^*(\beta)} = M_\beta$ cf. (A.56) and (A.57).

To prove the equality (A.59) note that for a trihomomorphism $G\colon T \to \mathcal{L}$, we have $G(F'F) = (GF')F$ because the target $\mathcal{L}$ is a $\mathcal{Gray}$-category—this is not true in general, rather these agree up to an identity-component tritransformation cf. Gurski [27, Prop. 7.6] or ico-icon cf. Garner and Gurski [18, pp. 561-562,].

□

Remark A.4. Notice that the identities (A.59) and (A.60) only hold true if we restrict to locally strict trihomomorphisms. In general (for target a $\mathcal{Gray}$-category), there is an identity component tritransformation

$$(F'F)^*G \cong F^*F'^*G \tag{A.61}$$

where the adjoint equivalences are the identity adjoint equivalences, and where Π and M are basically given by the composition constraints of a hom functor of G. These tritransformations give rise to a $\mathcal{Gray}$-natural isomorphism

$$(F'F)^* \cong F^*F'^* \tag{A.62}$$

since the prescriptions clearly coincide on trimodifications and perturbations. The composition of the tritransformation/ico-icon above with the tritransformations $(F'F)^*\beta$ and $F^*F'^*\beta$ inject exactly the composition and unit constraints in the composite Π and M to make them coincide.

Similarly, there is an identity component tritransformation (invertible ico-icon)

$$(1_T)^*G = G1_T \cong G \tag{A.63}$$

with identity adjoint equivalences and Π and M given by unit constraints of a hom functor of G. These constraints are exactly forming the component of a $\mathcal{Gray}$-natural isomorphism

$$(1_T)^* \cong 1\ . \tag{A.64}$$

Hence we have the following.

Proposition A.11. *Given a $\mathcal{Gray}$-category $\mathcal{L}$, tricategories R, S, T, and trihomomorphisms $F\colon R \to S$ and $F'\colon S \to T$, then there are $\mathcal{Gray}$-natural isomorphisms*

$$(F'F)^* \cong F^*F'^*\colon \mathcal{Tricat}(T, \mathcal{L}) \to \mathcal{Tricat}(R, \mathcal{L}) \tag{A.65}$$

and

$$(1_T)^* \cong 1\colon \mathcal{Tricat}(T, \mathcal{L}) \to \mathcal{Tricat}(T, \mathcal{L}) \tag{A.66}$$

□

Proposition A.12. *Given a $\mathcal{Gray}$-category $\mathcal{L}$, tricategories R and S, and a tritransformation $\alpha\colon F \Rightarrow F'\colon R \to S$, the following prescriptions define a tritransformation*

$$\alpha^*\colon F^* \Rightarrow F'^*\colon \mathcal{Tricat}(S, \mathcal{L}) \to \mathcal{Tricat}(R, \mathcal{L})\ . \tag{A.67}$$

The component of α^* at a trihomomorphism $G\colon S \to \mathcal{L}$ is defined to be the tritransformation

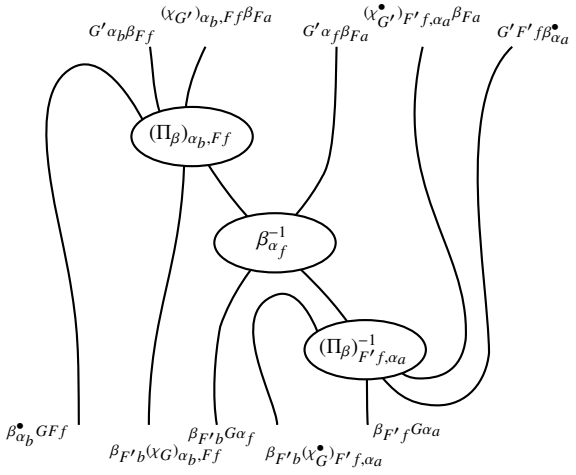
$$G_*\alpha\colon GF \Rightarrow GF' \tag{A.68}$$

from Proposition A.2 with component $G\alpha_a$ at $a \in \mathrm{ob}R$. Given a tritransformation $\beta\colon G \Rightarrow G'\colon S \to \mathcal{L}$, the component of the corresponding adjoint equivalence of α^* is a trimodification with component at $a \in \mathrm{ob}R$ defined by the 2-cell component

$$\beta^{\bullet}_{\alpha_a}\colon G'\alpha_a\beta_{Fa} \to \beta_{F'a}G\alpha_a \tag{A.69}$$

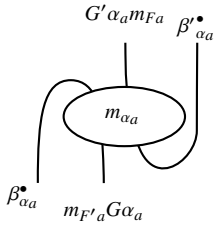
of the adjoint equivalence $\beta^{\bullet}$ of β.

The invertible modification of this trimodification has component at a 1-cell $f\colon a \to b$ in R given by the string diagram below.



(A.70)

The naturality 2-cell at a trimodification $m\colon \beta \Rrightarrow \beta'\colon G \Rightarrow G'$ of the adjoint equivalence of α^* is a perturbation with component at $a \in \mathrm{ob}R$ given by the mate of the component m_{α_a} of the invertible modification of m depicted in the string diagram below.



(A.71)

Proof. The hard bit in the proof is the verification of the two trimodification axioms for (A.70), which requires substantial calculation. □

Proposition A.13. *Given a Gray-category $\mathcal{L}$, tricategories R and S, and a trimodification*

$$n^*\colon \alpha \Rrightarrow \alpha'\colon F \Rightarrow F'\colon R \to S\ ,$$

the following descriptions define a trimodification

$$n^* \colon \alpha^* \lll \alpha'^* \ . \tag{A.72}$$

The component of the trimodification n^* at a trihomomorphism $G \colon S \to \mathcal{L}$ is defined to be the trimodification

$$G_* n \colon G_* \alpha \Rightarrow G_* \alpha' \tag{A.73}$$

from Proposition A.3 with component Gn_a at $a \in \mathrm{ob}R$. The component at a tritransformation $\beta \colon G \Rightarrow G' \colon S \to \mathcal{L}$ of the invertible modification of the trimodification n^* is a perturbation with component at $a \in \mathrm{ob}R$ given by the naturality 2-cell of the adjoint equivalence $\beta^\bullet$ below.

$$(\beta^\bullet_{n_a})^{-1} \colon (1_{\beta F' a} G n_a) * \beta^\bullet_{\alpha_a} \Rightarrow \beta^\bullet_{\alpha'_a} * (G' n_a 1_{\beta_{Fa}}) \tag{A.74}$$

Proof. The perturbation axiom at a 1-cell $f \colon a \to b$ in R is proved by a straightforward calculation involving the modification axioms for Π and n. The modification axiom at a trimodification $m \colon \beta \lll \beta' \colon G \Rightarrow G'$ is an equation of perturbations. At $a \in \mathrm{ob}R$ it is tantamount to the modification axiom for m at n_a. Noting that F^* and F'^* are $\mathcal{G}ray$-functors cf. Th. A.6, the first trimodification axiom at composable tritransformations β and β' is just the definition of the adjoint equivalence for the composite tritransformation $\beta'\beta$. The second trimodification axiom is trivially satisfied. □

Proposition A.14. *Given a $\mathcal{G}ray$-category $\mathcal{L}$, tricategories R and S, and a perturbation*

$$\sigma \colon m \lll m' \colon \alpha \lll \alpha' \colon F \Rightarrow F' \colon R \to S \ ,$$

there is a perturbation

$$\sigma^* \colon m^* \Rightarrow n'^* \tag{A.75}$$

with component at a trihomomorphism $G \colon S \to \mathcal{L}$ given by the perturbation

$$G_* \sigma \colon G_* m \Rightarrow G_* m' \tag{A.76}$$

from Proposition A.4 with component $G\sigma_a$ at $a \in \mathrm{ob}R$.

Proof. The perturbation axiom is verified on components, where it is tantamount to naturality of the adjoint equivalence $\beta^\bullet$ at σ_a. □

Theorem A.7. *Given a $\mathcal{G}ray$-category $\mathcal{L}$ and tricategories R and S, then whiskering on the right supports the structure of a morphism of 3-globular sets*

$$\mathcal{T}ricat(R, S) \to \mathcal{T}ricat(\mathcal{T}ricat(S, \mathcal{L}), \mathcal{T}ricat(R, \mathcal{L})) \ . \tag{A.77}$$

□

Proposition A.15. *Given a* **Gray***-category* $\mathcal{L}$*,tricategories* R *and* S*, and trihomomorphisms* $F, F' \colon R \to S$*, whiskering on the right supports the structure of local pseudofunctors*

$$(-)^*_{F,F'} \colon \mathcal{T}ricat(R,S)(F,F') \to \mathcal{T}ricat(\mathcal{T}ricat(S,\mathcal{L}),\mathcal{T}ricat(R,\mathcal{L}))(F^*,F'^*) \ . \tag{A.78}$$

Proof. We have to specify the constraints of the alleged pseudofunctor. These are perturbations with component at $G \colon S \to \mathcal{L}$ the perturbation $(G_*m') * (G_*m) \Rrightarrow G_*(m' * m)$ with component $G_{m'_a,m_a}$ at $a \in \mathsf{ob}S$ for composable trimodifications $m', m \colon \alpha \Rrightarrow \alpha' \colon F \Rightarrow F'$ and $1_{G_*\alpha} \Rrightarrow G_* 1_\alpha$ with component G_{α_a} respectively cf. Prop. A.5.

The perturbation axioms at $\beta \colon G \Rightarrow G' \colon S \to \mathcal{L}$ follow from respect for composition and units of the adjoint equivalence $\beta^\bullet$ of β, cf. the definition of the invertible modifications of the trimodification m^* in A.13.

The pseudofunctor axioms, as equations of perturbations, follow from the corresponding axioms for $G_{Fa,F'a}$. □

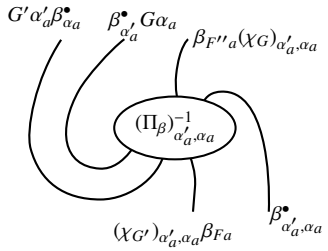
Lemma A.7. *Given a* **Gray***-category* $\mathcal{L}$*, tricategories* R *and* S*, trihomomorphisms* $F, F', F'' \colon R \to S$*, and tritransformations* $\alpha \colon F \Rightarrow F'$ *and* $\alpha' \colon F' \Rightarrow F''$*, then there is a trimodification*

$$\chi_{\alpha',\alpha} \colon \alpha'^*\alpha^* \Rrightarrow (\alpha'\alpha)^* \colon F^* \Rightarrow F''^* \colon \mathcal{T}ricat(S,\mathcal{L}) \to \mathcal{T}ricat(R,\mathcal{L}) \tag{A.79}$$

with component at a trihomomorphism $G \colon S \to T$ *the adjoint equivalence* $\chi_{\alpha',\alpha}$ *from Lemma A.5. This trimodification is itself part of an adjoint equivalence in* $\mathcal{T}ricat(\mathcal{T}ricat(S,L),\mathcal{T}ricat(R,L))$*, which forms the components of an adjoint equivalence*

$$\chi \colon \ \otimes_{\mathcal{T}ricat(\mathcal{T}ricat(S,\mathcal{L}),\mathcal{T}ricat(R,\mathcal{L}))} ((-)^* \times (-)^*) \Rightarrow (-)^* \otimes_{\mathcal{T}ricat(R,S)} \ . \tag{A.80}$$

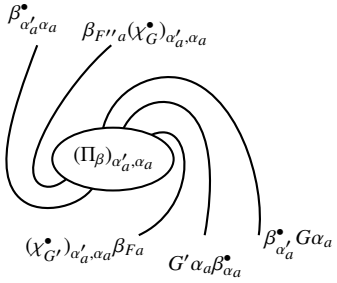
Proof. The component at a tritransformation $\beta \colon G \Rightarrow G' \colon S \to \mathcal{L}$ of the invertible modification of the trimodification (A.79) is a perturbation with component at an object a in R given by the mate of $(\Pi_\beta)^{-1}_{\alpha'_a,\alpha_a}$ depicted in the string diagram below.



(A.81)

The hard bit of the proof is the verification of the perturbation axiom at a 1-cell $f \colon a \to b$, which follows from a substantial calculation. The modification axiom at a trimodification $m \colon \beta \Rrightarrow \beta' \colon G \Rightarrow G'$ follows from the first trimodification axiom for m. The trimodification axioms follow from the definition of Π for the composite of two tritransformations (with 0-cell target a **Gray**-category).

On the other hand, there is a trimodification with component the right adjoint $\chi^{\bullet}_{\alpha',\alpha}$. The component at a tritransformation $\beta\colon\ G \Rightarrow G'\colon\ S \to \mathcal{L}$ of the invertible modification of this trimodification is a perturbation with component at an object a in R given by the mate of $(\Pi_\beta)\alpha'_a, \alpha_a$ depicted in the string diagram below.



(A.82)

The perturbation axiom is already implied by the perturbation axiom for (A.81).

That the unit and counit of this adjoint equivalence satisfy the perturbation axiom—thus give rise to an adjoint equivalence as required—is proved by a short calculation, where one has to cancel Π and Π^{-1}.

The naturality 2-cells for (A.80) are perturbations $\chi_{\alpha'_2,\alpha_2} * (n'^* n^*) \Rrightarrow (n'n)^* * \chi_{\alpha'_1,\alpha_1}$ with components given by the naturality 2-cells of the adjoint equivalence (A.46) from Lemma A.5 with components naturality 2-cells of χ_G at n'_a and n_A. The axioms for a pseudo-natural transformation thus in the end follow from those for the adjoint equivalence χ_G. The perturbation axiom at $\beta\colon\ G \Rightarrow G'\colon\ R \to S$ follows from the modification axiom for Π_β. Similarly, we have a right adjoint $\chi^{\bullet}$ to (A.80), where the modification axiom for unit and counit follow from the modification axiom for unit and counit of the adjoint equivalence $(\chi_G, \chi^{\bullet}_G)$. □

Lemma A.8. *Given a* Gray*-category $\mathcal{L}$, tricategories R and S, and a tritransformation $\alpha\colon\ F \Rightarrow F'\colon\ R \to S$, there is a trimodification*

$$\iota_\alpha\colon\ 1_{\alpha^*} \Rrightarrow (1_\alpha)^*\colon\ F^* \Rightarrow F'^*\colon\ \mathcal{T}\mathit{ricat}(S, L) \to \mathcal{T}\mathit{ricat}(R, L) \tag{A.83}$$

with component at a trihomomorphism $G\colon\ S \to T$ the adjoint equivalence ι_α from Lemma A.6, and this trimodification is itself part of an adjoint equivalence in $\mathcal{T}\mathit{ricat}(\mathcal{T}\mathit{ricat}(S, L), \mathcal{T}\mathit{ricat}(R, L))$.

Proof. By similar arguments. □

Theorem A.8. *Given* Gray*-categories $\mathcal{L}$ and $\mathcal{S}$ and a tricategory R, then whiskering on the right supports the structure of a trihomomorphism*

$$(-)^*\colon\ \mathcal{T}\mathit{ricat}(R, S) \to \mathcal{T}\mathit{ricat}(\mathcal{T}\mathit{ricat}(S, \mathcal{L}), \mathcal{T}\mathit{ricat}(R, \mathcal{L}))\ . \tag{A.84}$$

Proof. What remains to be proved is to construct the three invertible modifications ω, γ, and δ of the alleged trihomomorphism and to prove that these satisfy the two trihomomorphism axioms. The components of these invertible modifications are perturbations

with component at $G\colon S \to \mathcal{L}$ given by the perturbations ω, γ, and δ for G_* cf. Th. A.3 with components given by components of ω, γ, and δ for G. The trihomomorphism axioms thus follow from the trihomomorphism axioms for G. The modification axioms follow from the modification axioms for ω, γ, and δ for G, and the perturbation axioms at a tritransformation $\beta\colon G \Rightarrow G'\colon S \to \mathcal{L}$ correspond precisely to the three tritransformation axioms for β. □

Conjecture A.3. *Given a* **Gray***-category $\mathcal{L}$ and tricategories R and S, then whiskering on the right supports the structure of a strict trihomomorphism*

$$(-)^*\colon \mathcal{T}ricat(R,S) \to \mathcal{T}ricat(\mathcal{T}ricat(S,\mathcal{L}),\mathcal{T}ricat(R,\mathcal{L}))\ . \tag{A.85}$$

Theorem A.9. *Given a* **Gray***-category $\mathcal{L}$ and a triequivalence $F\colon R \to S$ of tricategories R and S, whiskering on the right with F gives rise to a triequivalence of functor tricategories:*

$$F^*\colon \mathcal{T}ricat(S,\mathcal{L}) \cong \mathcal{T}ricat(R,\mathcal{L})\ . \tag{A.86}$$

Proof. If S is a **Gray**-category, this follows directly from Theorem A.8. In the general case, we have already determined enough structure of the alleged trihomomorphism in Conjecture A.3 to prove this explicitly: Let $G\colon S \to R$ be a triequivalence inverse of F, and let $\alpha\colon FG \Rightarrow 1$ and $\beta\colon 1 \Rightarrow GF$ be biequivalence tritransformations with biequivalence inverses $\alpha'\colon 1 \Rightarrow FG$ and $\beta'\colon GF \Rightarrow 1$ and equivalences $m\colon \alpha\alpha' \cong 1_{1_S}$, $m'\colon \alpha'\alpha \cong 1_{FG}$, $n\colon \beta\beta' \cong 1_{GF}$, and $n'\colon \beta'\beta \cong 1_{1_R}$. With the **Gray**-natural isomorphisms from Prop. A.11, we aim to show that α gives rise to a biequivalence

$$G^*F^* \cong (FG)^* \xrightarrow{\alpha^*} (1_S)^* \cong 1_{\mathcal{T}ricat(S,(L))} \tag{A.87}$$

with biequivalence inverse

$$1_{\mathcal{T}ricat(S,(L))} \cong (1_S)^* \xrightarrow{(\alpha')^*} (FG)^* \xrightarrow{\iota_{1_S}} G^*F^* \tag{A.88}$$

and similarly for β.

Since the transformations from Prop. A.11 are **Gray**-natural isomorphisms, we only have to show that α^* is a biequivalence with biequivalence inverse $(\alpha')^*$. This follows from the fact that the composite of equivalences

$$\alpha^*(\alpha')^* \xrightarrow{\chi_{\alpha,\alpha'}} (\alpha\alpha')^* \xrightarrow{m^*} (1_{1_S})^* \xrightarrow{\iota^\bullet_{1_S}} 1_{(1_S)^*} \tag{A.89}$$

is an equivalence, where we use that m^* is an equivalence because m is an equivalence cf. Prop. A.5.

□

A.7 Biequivalences of trihomomorphisms

Let $\mathcal{S}$ and $\mathcal{T}$ be tricategories. Under the assumption that $\mathcal{S}$ is small, there is surely a forgetful functor

$$\mathcal{T}ricat(\mathcal{S},\mathcal{T}) \to \prod_{\mathrm{ob}\mathcal{S}} \mathcal{T}$$

sending a trihomomorphism to its image on objects, a tritransformation to its components, a trimodification to its components, and a perturbation to its components. This is obviously a strict trihomomorphism.

Proposition A.16. *Let $\mathcal{S}$ be a small tricategory, and let $\mathcal{T}$ be a cocomplete $\mathcal{G}ray$-category or the tricategory $\mathcal{B}icat$. Then the forgetful functor*

$$\mathcal{T}ricat(\mathcal{S},\mathcal{T}) \to \prod_{\mathrm{ob}\mathcal{S}} \mathcal{T}$$

reflects internal biequivalences.

Proof. Suppose $\sigma\colon F \Rightarrow G\colon \mathcal{S} \to \mathcal{T}$ is a tritransformation such that its components are internal biequivalences in the tricategory $\mathcal{T}$. We have to show that σ is itself an internal biequivalence in $\mathcal{T}ricat(\mathcal{S},\mathcal{T})$.

To begin with, we replace $\mathcal{T}$ by $\mathcal{T}' = \mathcal{G}ray$ if $\mathcal{T} = \mathcal{B}icat$ (otherwise $\mathcal{T}' = \mathcal{T}$), and then $\mathcal{S}$ by the triequivalent $\mathcal{G}ray$-category $\mathcal{S}'$ obtained from the local strictification $\mathrm{st}\mathcal{S}$, which is a cubical tricategory, and the cubical Yoneda embedding [27, Th. 8.4 andTh. 9.12]: $ew\colon \mathcal{S}' \cong \mathrm{st}\mathcal{S} \cong \mathcal{S}$. These replacements give rise to a triequivalence

$$\mathcal{T}ricat(\mathcal{S},\mathcal{T}) \to \mathcal{T}ricat(\mathcal{S}',\mathcal{T}')\ ,$$

cf. Theorems A.4 and A.9, and thus a tritransformation $\sigma'\colon F' \Rightarrow G'\colon \mathcal{S}' \to \mathcal{T}'$ with components that are again internal biequivalences in $\mathcal{T}'$. In the case that $\mathcal{T} = \mathcal{B}icat$, we have $F' = \mathrm{st}Few$ and $G' = \mathrm{st}Gew$. Since $\mathcal{S}t\mathcal{S}$ is locally given by the strictification $\mathrm{st}\mathcal{S}(S,S')$ of the hom bicategories of $\mathcal{S}$, the local functors of the trihomomorphisms $\mathrm{st}Fe$ and $\mathrm{st}Ge$ are biequivalent to strict functors. The same then holds for F' and G' as the inverse w of the triequivalence induced by the cubical Yoneda embedding is locally strict, see [27, Prop. 9.14]. By Theorem 2.5 from Chapter 2, F' and G' are biequivalent to locally strict trihomomorphisms F'' and G'' (which agree with F' and G' on objects), thus giving rise to a tritransformation $\sigma''\colon F'' \Rightarrow G''$ in $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{S}',\mathcal{T}')$ with components that are biequivalences again. Since by Theorem 1.7 from Chapter 1, $\mathcal{T}ricat_{\mathrm{ls}}(\mathcal{S}',\mathcal{T}')$ is the $\mathcal{G}ray$-category of pseudo algebras for the $\mathcal{G}ray$-monad on $[\mathrm{ob}\mathcal{S}',\mathcal{T}']$ generated by the Kan adjunction, by [27, Lemma 15.11], the forgetful functor

$$U\colon \mathcal{T}ricat_{\mathrm{ls}}(\mathcal{S}',\mathcal{T}') \to [\mathrm{ob}\mathcal{S}',\mathcal{T}'] \cong \prod_{\mathrm{ob}\mathcal{S}'} \mathcal{T}'$$

reflects internal biequivalences. As a consequence σ'' must be a biequivalence in $\mathcal{T}ricat_{ls}(\mathcal{S}',\mathcal{T}')$, which implies that σ must be a biequivalence in $\mathcal{T}ricat(\mathcal{S},\mathcal{T})$.

□

A.8 Internal coherence for $\mathcal{B}icat$

The cells of the tricategory $\mathcal{B}icat$ of bicategories have constraints themselves: an object i.e. a bicategory has associators and unitors, a 1-cell i.e. a (pseudo)functor has composition and unit constraints, and a pseudonatural transformation has naturality 2-cells. These constraints can in fact be recovered solely from the tricategory structure of $\mathcal{B}icat$ as we briefly outline below[2]. We refer to this phenomenon as *internal coherence* for the tricategory $\mathcal{B}icat$.

To recover the internal constraints of $\mathcal{B}icat$, one represents objects b of a bicategory B by strict functors $b\colon I \to B$, 1-cells $f\colon b \to b'$ by pseudonatural transformations $f\colon b \Rightarrow b'\colon I \to B$ (the single naturality 2-cell is determined by respect for units and is composed out of unitors, hence gives no new data), and 2-cells by modifications of such transformations. For example, the associators of B are then recovered from associators of compositions of pseudonatural transformations i.e. associators of the hom bicategories of $\mathcal{B}icat$. The image of a 1-cell f in B under a functor $F\colon B \to C$ is given (up to an identity 1-cell) by whiskering on the left $1_F * f$ of the pseudonatural transformation f with F in $\mathcal{B}icat$, and so the constraints of F are recovered as constraints of the composition in $\mathcal{B}icat$ (since we have already recovered unitors, this is really true, and we can eliminate the extra constraint coming from the additional identities above). The component at $b \in B$ of a pseudonatural transformation $\sigma\colon F \Rightarrow G\colon B \to C$ is given (up to the image of an identity 1-cell under the target pseudofunctor G—for the usual convention for composition of pseudonatural transformations, otherwise its the source functor) by whiskering on the right $\sigma * 1_b$ with $b\colon I \to B$ in $\mathcal{B}icat$, and thus the naturality 2-cells are recovered as interchange constraints for the composition in $\mathcal{B}icat$ (up to unitors and unit constraints, which we have already recovered, and which we can thus eliminate).

As a first observation note that one can recover coherence results for cells in $\mathcal{B}icat$ from coherence for $\mathcal{B}icat$: From the coherence theorem for bicategories, we know that any diagram of constraints for a bicategory commutes, but since the constraints of this bicategory can be expressed as constraints of $\mathcal{B}icat$, this also follows from coherence for the tricategory $\mathcal{B}icat$—note that no interchange cells in $\mathcal{B}icat$ are involved. Similarly, we recover coherence for functors. More interestingly, we get a coherence result for pseudonatural transformations saying that any diagram (in the image of the free tricategory) composed out of naturality 2-cells and constraints of the source and target functors commutes.

Consider a trihomomorphism $\mathcal{P} \to \mathcal{B}icat$. The image of this trihomomorphism is composed out of bicategories, functors, pseudonatural transformations, and modifications, yet the constraints associated with those bicategories, functors, and pseudonatural transformations are not constraints of the trihomomorphism. By the argument above they are however recovered as constraints of the target $\mathcal{B}icat$. Thus, as a consequence of coherence for the trihomomorphism one concludes that any diagram of constraints (in the image of the free functor tricategory) composed out of the images of constraints from the source $\mathcal{P}$, constraints of the trihomomorphism, and constraints of the target $\mathcal{B}icat$

[2]This was also observed by Benjamín Alarcón Heredia.

commutes, and the latter includes all constraints associated with the cells in the image of the trihomomorphism.

Bibliography

[1] Paul F. Baum and William Browder. The cohomology of quotients of classical groups. *Topology*, 3:305–336, 1965. ISSN 0040-9383.

[2] Jean Bénabou. Introduction to bicategories. In *Reports of the Midwest Category Seminar*, pages 1–77. Springer, Berlin, 1967.

[3] Renato Betti, Aurelio Carboni, Ross Street, and Robert Walters. Variation through enrichment. *J. Pure Appl. Algebra*, 29(2):109–127, 1983. ISSN 0022-4049. doi: 10.1016/0022-4049(83)90100-7. URL http://dx.doi.org/10.1016/0022-4049(83)90100-7.

[4] Robert Blackwell, G. M. Kelly, and John Power. Two-dimensional monad theory. *J. Pure Appl. Algebra*, 59(1):1–41, 1989. ISSN 0022-4049. doi: 10.1016/0022-4049(89)90160-6. URL http://dx.doi.org/10.1016/0022-4049(89)90160-6.

[5] Armand Borel. Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts. *Ann. of Math. (2)*, 57:115–207, 1953. ISSN 0003-486X.

[6] Armand Borel. Topology of Lie groups and characteristic classes. *Bull. Amer. Math. Soc.*, 61:397–432, 1955. ISSN 0002-9904.

[7] John Bourke. *Codescent objects in* 2*-dimensional universal algebra*. PhD thesis, University of Sydney, 2010.

[8] John Bourke. Two-dimensional monadicity. *Adv. Math.*, 252:708–747, 2014. ISSN 0001-8708. doi: 10.1016/j.aim.2013.11.007. URL http://dx.doi.org/10.1016/j.aim.2013.11.007.

[9] Peter Bouwknegt, Alan L. Carey, Varghese Mathai, Michael K. Murray, and Danny Stevenson. Twisted K-theory and K-theory of bundle gerbes. *Comm. Math. Phys.*, 228(1):17–45, 2002. ISSN 0010-3616. doi: 10.1007/s002200200646. URL http://dx.doi.org/10.1007/s002200200646.

[10] Alan L. Carey, Diarmuid Crowley, and Michael K. Murray. Principal bundles and the Dixmier Douady class. *Comm. Math. Phys.*, 193(1):171–196, 1998. ISSN 0010-3616. doi: 10.1007/s002200050323. URL http://dx.doi.org/10.1007/s002200050323.

[11] Pilar Carrasco, Antonio M. Cegarra, and Antonio R. Garzón. Classifying spaces for braided monoidal categories and lax diagrams of bicategories. *Adv. Math.*, 226(1):419–483, 2011. ISSN 0001-8708. doi: 10.1016/j.aim.2010.06.027. URL http://dx.doi.org/10.1016/j.aim.2010.06.027.

[12] Antonio M. Cegarra and Benjamín A. Heredia. Comparing geometric realizations of tricategories. *Algebr. Geom. Topol.*, 14(4):117–184, 2014. ISSN 1472-2747. doi: 10.2140/agt.2014.14.117. URL http://dx.doi.org/10.2140/agt.2014.14.117.

[13] Sjoerd E. Crans. A tensor product for Gray-categories. *Theory Appl. Categ.*, 5:No. 2, 12–69 (electronic), 1999. ISSN 1201-561X.

[14] Eduardo J. Dubuc. *Kan extensions in enriched category theory.* Lecture Notes in Mathematics, Vol. 145. Springer-Verlag, Berlin-New York, 1970.

[15] Jürgen Fuchs, Thomas Nikolaus, Christoph Schweigert, and Konrad Waldorf. Bundle gerbes and surface holonomy. In *European Congress of Mathematics*, pages 167–195. Eur. Math. Soc., Zürich, 2010. doi: 10.4171/077-1/8. URL http://dx.doi.org/10.4171/077-1/8.

[16] Richard Garner. Understanding the small object argument. *Appl. Categ. Structures*, 17(3): 247–285, 2009. ISSN 0927-2852. doi: 10.1007/s10485-008-9137-4. URL http://dx.doi.org/10.1007/s10485-008-9137-4.

[17] Richard Garner. Homomorphisms of higher categories. *Adv. Math.*, 224(6):2269–2311, 2010. ISSN 0001-8708. doi: 10.1016/j.aim.2010.01.022. URL http://dx.doi.org/10.1016/j.aim.2010.01.022.

[18] Richard Garner and Nick Gurski. The low-dimensional structures formed by tricategories. *Math. Proc. Cambridge Philos. Soc.*, 146(3):551–589, 2009. ISSN 0305-0041. doi: 10.1017/S0305004108002132. URL http://dx.doi.org/10.1017/S0305004108002132.

[19] Richard Garner and Michael Shulman. Enriched categories as a free cocompletion, 2013. arXiv:1301.3191.

[20] Krzysztof Gawedzki and Nuno Reis. Basic gerbe over nonsimply connected compact groups. *J.Geom.Phys.*, 50:28, 2004. doi: 10.1016/j.geomphys.2003.11.004. URL http://dx.doi.org/10.1016/j.geomphys.2003.11.004. arXiv:math/0307010 [math.DG].

[21] Robert Gordon, John Power, and Ross Street. Coherence for tricategories. *Mem. Amer. Math. Soc.*, 117(558):vi+81, 1995. ISSN 0065-9266. doi: 10.1090/memo/0558. URL http://dx.doi.org/10.1090/memo/0558.

[22] John W. Gray. *Formal category theory: adjointness for* 2*-categories.* Lecture Notes in Mathematics, Vol. 391. Springer-Verlag, Berlin-New York, 1974.

[23] Alexandre Grothendieck. *Groupes de monodromie en géométrie algébrique. I.* Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim.

[24] Nick Gurski. *An algebraic theory of tricategories.* ProQuest LLC, Ann Arbor, MI, 2006. ISBN 978-0542-71041-4. URL http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:3219520. Thesis (Ph.D.)–The University of Chicago.

[25] Nick Gurski. Loop spaces, and coherence for monoidal and braided monoidal bicategories. *Adv. Math.*, 226(5):4225–4265, 2011. ISSN 0001-8708. doi: 10.1016/j.aim.2010.12.007. URL http://dx.doi.org/10.1016/j.aim.2010.12.007.

[26] Nick Gurski. Biequivalences in tricategories. *Theory Appl. Categ.*, 26:No. 14, 349–384, 2012. ISSN 1201-561X.

[27] Nick Gurski. *Coherence in three-dimensional category theory*, volume 201 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. ISBN 978-1-107-03489-1. doi: 10.1017/CBO9781139542333. URL http://dx.doi.org/10.1017/CBO9781139542333.

[28] Nick Gurski. The monoidal structure of strictification. *Theory Appl. Categ.*, 28:No. 1, 1–23, 2013. ISSN 1201-561X.

[29] Nick Gurski and Angélica M. Osorno. Infinite loop spaces, and coherence for symmetric monoidal bicategories. *Adv. Math.*, 246:1–32, 2013. ISSN 0001-8708. doi: 10.1016/j.aim.2013.06.028. URL http://dx.doi.org/10.1016/j.aim.2013.06.028.

[30] George Janelidze and G. M. Kelly. A note on actions of a monoidal category. *Theory Appl. Categ.*, 9(61-91):02, 2001.

[31] J. F. Jardine. Supercoherence. *J. Pure Appl. Algebra*, 75(2):103–194, 1991. ISSN 0022-4049. doi: 10.1016/0022-4049(91)90122-I. URL http://dx.doi.org/10.1016/0022-4049(91)90122-I.

[32] André Joyal and Ross Street. Braided tensor categories. *Adv. Math.*, 102(1):20–78, 1993. ISSN 0001-8708. doi: 10.1006/aima.1993.1055. URL http://dx.doi.org/10.1006/aima.1993.1055.

[33] G. M. Kelly. On MacLane's conditions for coherence of natural associativities, commutativities, etc. *J. Algebra*, 1:397–402, 1964. ISSN 0021-8693.

[34] G. M. Kelly. An abstract approach to coherence. In *Coherence in categories*, pages 106–147. Lecture Notes in Math., Vol. 281. Springer, Berlin, 1972.

[35] G. M. Kelly. Coherence theorems for lax algebras and for distributive laws. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 281–375. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[36] G. M. Kelly. On clubs and doctrines. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 181–256. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[37] G. M. Kelly. A unified treatment of transfinite constructions for free algebras, free monoids, colimits, associated sheaves, and so on. *Bull. Austral. Math. Soc.*, 22(1):1–83, 1980. ISSN 0004-9727. doi: 10.1017/S0004972700006353. URL http://dx.doi.org/10.1017/S0004972700006353.

[38] G. M. Kelly. *Basic concepts of enriched category theory*, volume 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1982. ISBN 0-521-28702-2.

[39] G. M. Kelly and Saunders MacLane. Coherence in closed categories. *J. Pure Appl. Algebra*, 1(1):97–140, 1971. ISSN 0022-4049.

[40] G. M. Kelly and Saunders MacLane. Closed coherence for a natural transformation. In *Coherence in categories*, pages 1–28. Lecture Notes in Math., Vol. 281. Springer, Berlin, 1972.

[41] G. M. Kelly and Ross Street. Review of the elements of 2-categories. In *Category Seminar (Proc. Sem., Sydney, 1972/1973)*, pages 75–103. Lecture Notes in Math., Vol. 420. Springer, Berlin, 1974.

[42] Joachim Kock. *Frobenius algebras and 2D topological quantum field theories*, volume 59 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004. ISBN 0-521-83267-5; 0-521-54031-3.

[43] Stephen Lack. A coherent approach to pseudomonads. *Adv. Math.*, 152(2):179–202, 2000. ISSN 0001-8708. doi: 10.1006/aima.1999.1881. URL http://dx.doi.org/10.1006/aima.1999.1881.

[44] Stephen Lack. Codescent objects and coherence. *J. Pure Appl. Algebra*, 175(1-3):223–241, 2002. ISSN 0022-4049. doi: 10.1016/S0022-4049(02)00136-6. URL http://dx.doi.org/10.1016/S0022-4049(02)00136-6. Special volume celebrating the 70th birthday of Professor Max Kelly.

[45] Stephen Lack. A Quillen model structure for 2-categories. *K-Theory*, 26(2):171–205, 2002. ISSN 0920-3036. doi: 10.1023/A:1020305604826. URL http://dx.doi.org/10.1023/A:1020305604826.

[46] Stephen Lack. Bicat is not triequivalent to Gray. *Theory Appl. Categ.*, 18:No. 1, 1–3, 2007. ISSN 1201-561X.

[47] Stephen Lack. A 2-categories companion. In *Towards higher categories*, volume 152 of *IMA Vol. Math. Appl.*, pages 105–191. Springer, New York, 2010. doi: 10.1007/978-1-4419-1524-5_4. URL http://dx.doi.org/10.1007/978-1-4419-1524-5_4.

[48] Stephen Lack. Icons. *Appl. Categ. Structures*, 18(3):289–307, 2010. ISSN 0927-2852. doi: 10.1007/s10485-008-9136-5. URL http://dx.doi.org/10.1007/s10485-008-9136-5.

[49] Stephen Lack and Michael Shulman. Enhanced 2-categories and limits for lax morphisms. *Adv. Math.*, 229(1):294–356, 2012. ISSN 0001-8708. doi: 10.1016/j.aim.2011.08.014. URL http://dx.doi.org/10.1016/j.aim.2011.08.014.

[50] Stephen Lack and Ross Street. Triangulations, orientals, and skew monoidal categories. *Adv. Math.*, 258:351–396, 2014. ISSN 0001-8708. doi: 10.1016/j.aim.2014.03.003. URL http://dx.doi.org/10.1016/j.aim.2014.03.003.

[51] Tom Leinster. Basic bicategories, 1998. arXiv:math.CT/9810017.

[52] Tom Leinster. *Higher operads, higher categories*, volume 298 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2004. ISBN 0-521-53215-9. doi: 10.1017/CBO9780511525896. URL http://dx.doi.org/10.1017/CBO9780511525896.

[53] Fred E. J. Linton. Relative functorial semantics: Adjointness results. In *Category Theory, Homology Theory and their Applications, III (Battelle Institute Conference, Seattle, Wash., 1968, Vol. Three)*, pages 384–418. Springer, 1969.

[54] Saunders MacLane. Natural associativity and commutativity. *Rice Univ. Studies*, 49(4): 28–46, 1963. ISSN 0035-4996.

[55] Saunders MacLane. *Categories for the working mathematician*. Springer-Verlag, New York-Berlin, 1971. Graduate Texts in Mathematics, Vol. 5.

[56] Eckhard Meinrenken. The basic gerbe over a compact simple lie group, 2003. arXiv:math/0209194v3 [math.DG].

[57] Ieke Moerdijk and Janez Mrčun. *Introduction to foliations and Lie groupoids*, volume 91 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003. ISBN 0-521-83197-0. doi: 10.1017/CBO9780511615450. URL http://dx.doi.org/10.1017/CBO9780511615450.

[58] Michael K. Murray. Bundle gerbes. *J. London Math. Soc. (2)*, 54(2):403–416, 1996. ISSN 0024-6107. doi: 10.1093/acprof:oso/9780199534920.003.0012. URL http://dx.doi.org/10.1093/acprof:oso/9780199534920.003.0012.

[59] Michael K. Murray and Daniel Stevenson. Bundle gerbes: stable isomorphism and local theory. *J. London Math. Soc. (2)*, 62(3):925–937, 2000. ISSN 0024-6107. doi: 10.1112/S0024610700001551. URL http://dx.doi.org/10.1112/S0024610700001551.

[60] Michael K. Murray and Daniel Stevenson. Higgs fields, bundle gerbes and string structures. *Comm. Math. Phys.*, 243(3):541–555, 2003. ISSN 0010-3616. doi: 10.1007/s00220-003-0984-4. URL http://dx.doi.org/10.1007/s00220-003-0984-4.

[61] Thomas Nikolaus and Christoph Schweigert. Equivariance in higher geometry. *Adv. Math.*, 226(4):3367–3408, 2011. ISSN 0001-8708. doi: 10.1016/j.aim.2010.10.016. URL http://dx.doi.org/10.1016/j.aim.2010.10.016.

[62] John Power. Coherence for bicategories with finite bilimits. I. In *Categories in computer science and logic (Boulder, CO, 1987)*, volume 92 of *Contemp. Math.*, pages 341–347. Amer. Math. Soc., Providence, RI, 1989. doi: 10.1090/conm/092/1003207. URL http://dx.doi.org/10.1090/conm/092/1003207.

[63] John Power. A general coherence result. *J. Pure Appl. Algebra*, 57(2):165–173, 1989. ISSN 0022-4049. doi: 10.1016/0022-4049(89)90113-8. URL http://dx.doi.org/10.1016/0022-4049(89)90113-8.

[64] John Power. Why tricategories? *Inform. and Comput.*, 120(2):251–262, 1995. ISSN 0890-5401. doi: 10.1006/inco.1995.1112. URL http://dx.doi.org/10.1006/inco.1995.1112.

[65] John Power. Three dimensional monad theory. In *Categories in algebra, geometry and mathematical physics*, volume 431 of *Contemp. Math.*, pages 405–426. Amer. Math. Soc., Providence, RI, 2007. doi: 10.1090/conm/431/08282. URL http://dx.doi.org/10.1090/conm/431/08282.

[66] John Power and Edmund Robinson. A characterization of pie limits. *Math. Proc. Cambridge Philos. Soc.*, 110(1):33–47, 1991. ISSN 0305-0041. doi: 10.1017/S0305004100070092. URL http://dx.doi.org/10.1017/S0305004100070092.

[67] Daniel Schäppi. The formal theory of Tannaka duality. *Astérisque*, (357):viii+140, 2013. ISSN 0303-1179.

[68] Christopher J. Schommer-Pries. The classification of two-dimensional extended topological field theories, 2014. arXiv:1112.1000v2 [math.AT].

[69] Michael A. Shulman. Not every pseudoalgebra is equivalent to a strict one. *Adv. Math.*, 229(3):2024–2041, 2012. ISSN 0001-8708. doi: 10.1016/j.aim.2011.01.010. URL http://dx.doi.org/10.1016/j.aim.2011.01.010.

[70] Carlos Simpson. Homotopy types of strict 3-groupoids, 2011. arXiv:math/9810059 [math.CT].

[71] Danny Stevenson. *The geometry of bundle gerbes*. PhD thesis, University of Adelaide, 2000.

[72] Danny Stevenson. Décalage and Kan's simplicial loop group functor. *Theory Appl. Categ.*, 26:768–787, 2012. ISSN 1201-561X.

[73] Ross Street. The formal theory of monads. *J. Pure Appl. Algebra*, 2(2):149–168, 1972. ISSN 0022-4049.

[74] Ross Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8(2): 149–181, 1976. ISSN 0022-4049.

[75] Ross Street. Categorical and combinatorial aspects of descent theory. *Appl. Categ. Structures*, 12(5-6):537–576, 2004. ISSN 0927-2852. doi: 10.1023/B:APCS.0000049317.24861.36. URL `http://dx.doi.org/10.1023/B:APCS.0000049317.24861.36`.

[76] Konrad Waldorf. *Algebraic structures for bundle gerbes and the Wess-Zumino term in conformal field theory*. PhD thesis, Universität Hamburg, 2007.

[77] Konrad Waldorf. More morphisms between bundle gerbes. *Theory Appl. Categ.*, 18:No. 9, 240–273, 2007. ISSN 1201-561X.

[78] Harvey Wolff. V-cat and V-graph. *J. Pure Appl. Algebra*, 4:123–135, 1974. ISSN 0022-4049.

Summary

TOPICS IN THREE-DIMENSIONAL DESCENT THEORY

This thesis is concerned with the concept of strictification in three-dimensional category theory. Every tricategory is triequivalent to a *Gray*-category. However, the theory of tricategories is not equivalent to *Gray*-enriched category theory. Nevertheless their connection may be strained to allow for substantial results such as a tricategorical Yoneda lemma as we show in this thesis.

A major impetus to analyze three-dimensional strictification comes from mathematical physics. Motivated by the construction of bundle gerbes in terms of descent, we analyze coherence in the context of the three-dimensional monadicity of the *Gray*-enriched functor category. We identify the *Gray*-category of pseudo algebras for this monad with the *Gray*-category of locally strict trihomomorphisms, tritransformations, trimodifications, and perturbations, and show that coherence results from three-dimensional monad theory apply to establish an adjunction between the latter tricategorical functor category and the *Gray*-enriched functor category.

As an application, we are able to prove that any locally strict trihomomorphism with domain a small and codomain a cocomplete *Gray*-category is biequivalent to a *Gray*-functor. The Yoneda lemma for tricategories is then proved via the *Gray*-enriched Yoneda lemma. The concept of a codescent object takes a special position in the argument and also underlies the coherence theorem from three-dimensional monad theory that we employ. The codescent object is a certain *Gray*-enriched colimit, which we analyze on the free algebras of a *Gray*-monad. We show that the codescent object of the codescent diagram of a free algebra is strictly biequivalent to the free algebra itself. Since representables are in particular also free algebras for the *Gray*-monad of the *Gray*-enriched functor category, we are thus able to prove a Yoneda lemma for *Gray*-categories and locally strict trihomomorphisms, which enables us to specify an explicit strictification of a locally strict trihomomorphism with values in *Gray*. To prove a Yoneda lemma for tricategories, we analyze the invariance of functor tricategories and their representables, which broadens the basic knowledge of the theory of tricategories.

Finally, we introduce a new three-dimensional descent construction as a simple tricategorical limit. We show that the strictification of this construction coincides with a known explicit strict descent construction. This involves the identification of yet another codescent object up to strict biequivalence, namely the codescent object of the weight of the tricategorical limit, and the proof that the known strict descent construction coincides exactly with the resulting *Gray*-enriched limit. These results enable us both to give new explanations for the properties of the strict descent construction as well as to apply the latter on the three-dimensional descent construction in the general non-strict context. Finally, a new perspective on bundle gerbes is accomplished in that we show that they can be understood as tritransformations.

Zusammenfassung

THEMEN DREIDIMENSIONALER ABSTIEGSTHEORIE

Diese Arbeit beschäftigt sich mit dem Konzept der Striktifizierung in dreidimensionaler Kategorientheorie. Jede Trikategorie ist triäquivalent zu einer *Gray*-Kategorie. Andererseits kann man die Theorie der Trikategorien nicht einfach mit *Gray*-angereicherter Kategorientheorie gleichsetzen. Ihre Verflechtung ist allerdings eng genug, um entscheidende Resultate wie ein Yoneda-Lemma für Trikategorien zu ermöglichen, wie wir in dieser Arbeit zeigen.

Das Interesse an dreidimensionaler Striktifizierung findet seine Begründung in Fragen, die durch die mathematische Physik aufgeworfen werden. Motiviert durch die Abstiegskonstruktion von Bündelgerben untersuchen wir Kohärenz im Zusammenhang dreidimensionaler Monadizität der *Gray*-angereicherten Funktorkategorie. Wir identifizieren die *Gray*-Kategorie der Pseudoalgebren für diese Monade mit der *Gray*-Kategorie von lokal strikten Trihomomorphismen, Tritransformationen, Trimodifikiationen und Perturbationen und zeigen, daß die bekannten Kohärenzresultate für dreidimensionale Monadentheorie damit einen Vergleich über eine Adjunktion zwischen der letztgenannten trikategorientheoretischen Funktorkategorie und der *Gray*-angereicherten Funktorkategorie ermöglichen.

Damit können wir das folgende Striktifizierungsresult beweisen: ein beliebiger lokal strikter Trihomomorphismus zwischen einer kleinen und einer kovollständigen *Gray*-Kategorie ist biäquivalent zu einem *Gray*-Funktor. Das Yoneda-Lemma für Trikategorien lässt sich dann von dem *Gray*-angereicherten Yoneda-Lemma ableiten. Dafür wird das Konzept eines Koabstiegsobjekts benötigt, das auch dem bekannten Kohärenztheorem dreidimensionaler Monadentheorie unterliegt. Das Koabstiegsobjekt is ein bestimmter *Gray*-angereicherter Kolimes, den wir auf freien Algebren für *Gray*-Monaden untersuchen. Wir zeigen, daß das Koabstiegsobjekt des Koabstiegsdiagramms einer freien Algebra strikt biäquivalent zu der freien Algebra selbst ist. Da darstellbare Funktoren insbesondere auch freie Algebren für die oben beschriebene Monade sind, lässt sich damit ein Yoneda-Lemma für *Gray*-Kategorien und lokal strikte Trihomomorphismen beweisen, das eine explizite Striktifizierung lokal strikter Trihomomorphismen mit Werten in der *Gray*-Kategorie *Gray* ermöglicht. Um ein Yoneda-Lemma für Trikategorien zu beweisen, wird die Invarianz von Funktortrikategorien und darstellbaren Trihomomorphismen untersucht, was eine Erweiterung der grundlegenden Kenntnisse über die Theorie der Trikategorien darstellt.

Schließlich führen wir eine neue dreidimensionale Abstiegskonstruktion als einen einfachen trikategorientheoretischen Limes ein. Wir zeigen, daß die Striktifizierung dieser Abstiegskonstruktion mit einer bekannten expliziten strikten Abstiegskonstruktion übereinstimmt. Dafür identifizieren wir abermals ein Koabstiegsobjekt, nämlich jenes des Gewichts der neu eingeführten Abstiegskonstruktion bis auf strikte Biäquivalenz und zeigen, daß die bekannte Konstruktion gerade mit dem resultierenden *Gray*-angereicherten Limes übereinstimmt. Diese Erkenntnisse ermöglichen sowohl neue Erklärungen für die Eigenschaften der bekannten Abstiegskonstruktion als auch die Anwendung letztgenann-

ter auf die neu eingeführte Abstiegskonstruktion im allgemeinen nicht-strikten Kontext. Schließlich ermöglichen wir eine neue Perspektive auf Bündelgerben, indem wir zeigen daß man Bündelgerben als Tritransformationen verstehen kann.

Das erste Kapitel dieser Arbeit basiert auf der folgenden Vorveröffentlichung des Autors:

Lukas Buhné. Homomorphisms of Gray-categories as pseudo algebras. arXiv Preprint.arXiv:1408.3481 [math.CT], 2014.