MULTIPLICATIVE STRUCTURES AND INVOLUTIONS ON ALGEBRAIC K–THEORY

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und Prof. Dr. Bjørn Ian Dundas.

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To Cedric

Introduction

Algebraic K-Theory encompasses a variety of mathematical disciplines, natural settings, questions and tools, and thus also motivations. As an algebraic topologist I primarily use the motivation by chromatic homotopy theory. I refer the interested reader to [Ro] for a general introduction to the red-shift conjecture, while focussing on the aspects which centrally motivate my thesis here.

Since algebraic topology formed as an independent mathematical discipline, a major aspect of it is the study of invariants associated to topological spaces. The most prominent and easily defined of these invariants are the homotopy groups associated to a topological space. The higher homotopy groups, i.e., for $n \geq 2$, give a countable family of abelian groups associated to a space. These however suffer from the defect of being very hard to compute. Even more drastically, it is usually a very hard question to establish, if a homotopy group of some fixed degree is trivial or non-trivial for some given space, unless elementary arguments force its triviality.

Much easier to compute, but harder to define are (singular) homology groups, defined by a variety of constructions up to the 1950s, proved to coincide by an axiomatic approach by Eilenberg and Steenrod in 1945. In the 1950s people found that there are more constructions satisfying all except one of the axioms, thus establishing topological K-theory and bordism as "extraordinary" homology theories.

The Atiyah-Hirzebruch spectral sequence in principle makes it possible to compute any homology theory on any space, given only its singular homology with $\mathbb{Z}$-coefficients as input. However, chromatic homotopy theory establishes that singular homology is the least complex of all homology theories. In particular, it gives a conceptual reason for the difficulties one encounters, when one tries to actually fully calculate the Atiyah-Hirzebruch spectral sequence for specific spaces. Cobordism is at the other extreme, having chromatic complexity ∞ in a conceptually satisfying sense.

In particular one could hope for an iterative approach to understanding invariants on a topological space by starting with singular homology on this space, thus at complexity 0, and then iterating from complexity n to $n+1$. Chromatic red-shift as described for instance in [Ro] is the conjecture that one way to produce such intermediary theories of increasing complexity is given by iterating algebraic K-theory on the associated spectra.

Suslin has established in 1984 that for any separably closed field F its algebraic K-theory completed at primes p (other than its characteristic) is equivalent to topological complex K-theory. Thus the singular homology with coefficients in F being a theory of complexity 0 is transformed into a theory of complexity 1. By the results of Christian Ausoni, specifically the ones in [A-Kku] we know that $K(ku_p)$ is the spectrum associated to a cohomology theory of complexity 2. In general chromatic red-shift predicts that this is part of a pattern, saying that algebraic K-theory raises chromatic complexity by one.

This thesis is specifically concerned with the involutive structures present on these spectra. Singular homology arises as the Eilenberg-MacLane spectrum of a discrete rig category, while [R] shows that on algebraic K-theory objects we always have an involution induced by transposing and inverting matrices. On complex K-theory this involution specialises to the natural involution induced by complex conjugation. One chromatic step higher this involution describes the operation on 2 vector bundles [BDR], which conjugates each transitional vector bundle. On K-theory of complex K-theory we can describe this as the involution induced by transposition and inversion in both iterations of K-theory.

In Chapter 1 I recall the most prominent combinatorial models for connective spectra given by ringlike categories, specifically bimonoidal and bipermutative categories, while in addition recalling the results of [EM] delooping their classifying spaces. I rewrite the construction of [EM] in such a way that I can easily generalise it in Chapter 3 to bicategories.

In Chapter 2 I establish a multiplicative structure on a combinatorial model for modules over a bipermutative category as already studied additively by Angélica Osorno in [Os]. I tie in the multiplicative structure with her additive structure in a manner that makes this module bicategory a ringlike object again.

In Chapter 3 – the technical core of this thesis – I set up a multiplicative delooping analogous to [EM] for permutative bicategories, which generalises the one of [Os], while allowing a multiplication to be induced by the tensor product defined in chapter 2.

In Chapter 4 I offer a few partial results, essentially summaries of known results in various papers, pertaining to the uniqueness of such structures. The delooping of a module category of a permutative category is sufficiently unique to fix the spectrum by minimal data as observed by May and Thomason in 1978 [MT]. However, I was unfortunately unable to prove multiplicative uniqueness of this delooping, which would as a corollary imply that the multiplicative structure of chapter 3 is the same as the one obtained by iterating the construction of [EM] twice. I do outline two arguments by which one might approach this conjectural uniqueness.

Chapters 5 and 6 are concerned with the motivating calculational example $K(ku)$. In particular, since the calculations of [A-Kku] are done by trace methods, i.e., by computations along the natural map $K \to THH$, I recall the definition of topological Hochschild homology in chapter 5. Fixing conventions along the way, specifically how an involution on a ring spectrum induces one on its topological Hochschild homology, we find that the trace is compatible with the involutions defined in chapters 2 and 5.

Compare this to the introduction of Dundas [D1], where he states that the construction of the trace map in the context of [D1] is compatible with involutions induced by the appropriate functors. Thus the result in chapter 5 establishes that we have internalised the involution in chapter 2 on K-theory and in chapter 5 on topological Hochschild homology in a compatible way.

Finally in Chapter 6 I retrace the calculations of [A-Kku] to the extent that I can establish the effects of the involution on $K(ku)$ on classes, which are not in the kernel of the trace map.

Acknowledgements

Obviously such a thesis cannot be written in a social vacuum, so I want to take this opportunity to thank a few people.

First and foremost obviously my advisor, Birgit Richter: In addition to creating a safety, from which I could explore the multiplicative delooping in chapters 2 and 3, she quite regularly helped me with indispensable advice when I became disoriented mathematically or socially.

I gratefully acknowledge the funding and environment of the Research Training Group 1670. In addition to the fact that I was able to focus exclusively on my research in the first three years of my postgraduate studies, being a part of a big group of PhD students made many experiences seem less lonely, which would have been frustrating otherwise. Furthermore being entrusted with the position of spokesperson for the PhD students I gained valuable insight in how to obtain a group opinion efficiently and also had an in-depth look at the process of application talks, which made them much more transparent, hence in particular less frightening to me.

It was plain fortunate that in 2012 at the Arolla conference I had the opportunity to meet with and talk to Robert Bruner, who explained to me the Adams spectral sequence and how to effectively calculate with it so pleasantly and patiently that I still remember each minute of it fondly. Also I want to thank Christian Ausoni for giving me a nice reader's guide to his papers, which helped a lot in reading them correctly and efficiently, thus kickstarting my knowledge of $K(ku)$.

This section could never be complete without mentioning Stephanie Ziegenhagen. I agree it was a perfect match, when our paths crossed in Algebraic Topology I, which is now 8 years ago. Since then we have been through so many things together, and experienced a lot of things so integrally together that I could never imagine them without you. As this is probably the last thesis, in which we address acknowledgements to each other, I want to close the circle with: *May the force of the universal property always save us from coordinates.* :)

Finally I want to extend special thanks to people, who read the draft in various stages of completion: My parents for catching numerous errors and typos, Birgit Richter in particular for pruning the nonsense I had written, Fabian Kirchner for his careful check of the mathematical prose, and finally Stephanie Ziegenhagen for her fail-safe instinct to find undefined and confusing parts, and attach the criticism down to the word or letter that causes the confusion.

Contents

1 Permutative Categories and Connective Spectra

This chapter is just a summary of known techniques to combinatorially model (connective) spectra by permutative categories. In particular until 1.1.4 there is nothing original in this chapter. The examples are borrowed from [MayE_∞, pp. 160-167] and [R, pp. 337+338]. However I deviate quite a bit from May's notation. Furthermore I want to warn the reader that I am close to an erroneous sequence of lemmas in [MayE_∞] (VI.2.3, VI.2.6, VI.4.4) (cf. [May2, p. 321]).

The claimed result in [MayE_∞] can be stated informally as: Bipermutative categories yield maximally homotopy commutative ring-spectra. The error is combinatorial, in how the multiplication and addition ought to interact, governed by the notion of an "operad pair". But the claimed "operad pair" in [MayE_∞] is not an operad pair as defined there. The result still holds [May2] (and its accompanying papers), [EM], but the techniques employed differ quite a bit from the planned proof in [MayE_∞].

1.1 Delooping Permutative Categories

Permutative categories seen through a modern eye are a categorified version of abelian groups with just as much strictness as generality would allow - compare the classical strictification result 1.1.2.

Definition 1.1.1. A **permutative category** $(\mathcal{A}, +, 0, c_+)$ is a category $\mathcal{A}$ together with a functor $+\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, a strict additive unit $0 \in \mathcal{A}$, and a twist natural transformation (for $T\colon \mathcal{A} \times \mathcal{A} \to \mathcal{A} \times \mathcal{A}$ the exchange of factors):

$$c_+\colon (+ \circ T) \Rightarrow +,$$

satisfying the following conditions:

1. $+$ is strictly associative:
$$+ \circ (+ \times id) = + \circ (id \times +),$$
2. The unit 0 is a strict unit for $+$:
$$0 + \_ = \_ + 0 = id_{\mathcal{A}},$$
3. The twist is trivial at 0: For every object $a \in \mathcal{A}$ we have the identity
$$c_+ = id : a = 0 + a \to a + 0 = a,$$
which is natural in a.
4. The twist is its own inverse: For every two objects $a, b \in \mathcal{A}$ we have the commutative diagram:
$$\begin{tikzcd} a+b \arrow[rr, Rightarrow, no head] \arrow[dr, "c_+"'] & & a+b \\ & b+a. \arrow[ur, "c_+"'] & \end{tikzcd}$$
5. The twist is associative: For each triple of objects $a, b, c \in \mathcal{A}$ we have the commutative diagrams:
$$\begin{tikzcd} a+b+c \arrow[r, "c_+"] \arrow[dr, "id+c_+"'] & c+a+b \\ & a+c+b, \arrow[u, "c_++id"'] \end{tikzcd} \qquad \begin{tikzcd} a+b+c \arrow[r, "c_+"] \arrow[dr, "c_++id"'] & b+c+a \\ & b+a+c. \arrow[u, "id+c_+"'] \end{tikzcd}$$

I have no need in this thesis for the most general symmetric monoidal categories given for instance by module categories. Nonetheless the following statement shows that the structure of permutative categories is sufficiently general:

Theorem 1.1.2. *For any symmetric monoidal category $\mathcal{C}$ there is a symmetrically monoidally equivalent permutative category $Str(\mathcal{C})$ with a natural equivalence $Str(\mathcal{C}) \to \mathcal{C}$.*

There are many ways to obtain this result. A brute force way is to consider words in objects, have the empty word be the strict unit, and add in morphisms accordingly [MayE_∞, Prop VI.3.2,cf. pp.155-157]. This proof has as a corollary that a small symmetric monoidal category of cardinality $\aleph$ yields a permutative category of size smaller than $\aleph^\omega$ (for $\omega = |\mathbb{N}|$). In particular the theorem stays true with "small" added in everywhere.

The high-tech way to show this result is given by the Yoneda Lemma in bicategories [Le, 2.3]. One considers a monoidal category $\mathcal{C}$ as a one-point bicategory $\Sigma\mathcal{C}$, embeds it into the equivalent one-point 2-category given by the essential image of the Yoneda embedding

$$Y\colon \Sigma\mathcal{C} \to Fun(\Sigma\mathcal{C}^{op}, Cat_1)$$

and the symmetry just comes along. Then sizes are limited by the bicategorical Yoneda Lemma.

Regarding enrichments we find that a topological, simplicial, etc. symmetric monoidal category strictifies to a permutative category of the same kind.

1.1.1 Bimonoidal Categories

Since this thesis is about multiplicative structures, I want to introduce the types of multiplication on permutative categories right away. Furthermore, I prove a convenient lemma, so that I do not have to bore the reader with pages of coherence diagrams. The following two concepts are directly copied from [EM] for instance, although a look into [MayE_∞] shows that at least the E_∞-version (i.e., bipermutative categories) was known to be a fruitful concept for much longer. Like the concept of permutative categories these concepts are strictified versions of general ringlike objects in 1-categories. Laplaza has shown in 1972 [Lap] that the analogous strictification result to 1.1.2 above holds for these structures. Thus the following definitions represent no loss of generality.

Definition 1.1.3. A **ring category** $(\mathcal{R}, +, \cdot, 0, 1, c_+)$ is given by a permutative structure $(\mathcal{R}, +, 0, c_+)$ and a strictly associative and strictly unital monoidal structure $(\mathcal{R}, \cdot, 1)$, which interact by two natural isomorphisms:

$$\lambda\colon ab + ab' \to a(b + b'), \quad \rho\colon ab + a'b \to (a + a')b,$$

such that the following properties hold:

1. 0 is a strict zero for multiplication $\cdot$:

$$0 \cdot a = a \cdot 0 = 0 \quad \forall a \in \mathcal{R},$$

2. $+$-associativity of distributors:
$$\lambda \circ (\lambda + \mathrm{id}) = \lambda \circ (\mathrm{id} + \lambda), \quad \rho \circ (\rho + \mathrm{id}) = (\mathrm{id} + \rho) \circ \rho,$$
3. additive symmetry of distributors:
$$(c_+ \cdot \mathrm{id}) \circ \lambda = \lambda \circ c_+, \quad (\mathrm{id} \cdot c_+) \circ \rho = \rho \circ c_+,$$
4. $\cdot$-associativity of distributors:
$$\lambda = (\mathrm{id} \cdot \lambda) \circ \lambda, \quad \rho = (\rho \cdot \mathrm{id}) \circ \rho,$$
5. middle $\cdot$-associativity of distributors:
$$(\mathrm{id} \cdot \rho) \circ \lambda = (\lambda \cdot \mathrm{id}) \circ \rho,$$
6. mixed associativity of distributors:
$$\lambda \circ (\rho + \rho) = \rho \circ (\lambda + \lambda) \circ (\mathrm{id} + c_+ + \mathrm{id}).$$

It is nice to have a multiplicative structure on any given object, but it is genuinely hard to produce ring categories, which are not also commutative up to some degree or an infinity of degrees. In particular, I do not investigate plain ring categories in this thesis. So I define the E_∞-multiplicative version next, also directly following [MayE_∞, EM], but explicitly with no strictness assumptions on either distributor. This type of category is the central object of study in the first three chapters.

Definition 1.1.4. A **bipermutative category** $(\mathcal{R}, +, \cdot, 0, 1, c_+, c.)$ is a ring category, where the multiplicative category $(\mathcal{R}, \cdot, 1)$ is also permutative with twist $c.$, and where the distributors are interrelated via the multiplicative twist as follows:

$$\begin{tikzcd} ab + ab' \arrow[r, "\lambda"] \arrow[d, "c.+c."] & a(b+b') \arrow[d, "c."] \\ ba + b'a \arrow[r, "\rho"] & (b+b')a. \end{tikzcd}$$

Remark 1.1.5. Do note that although I am following [EM] as well, I want the distributivity transformations to be isomorphisms! This is essential in defining the bicategory of matrices in Chapter 2.

I do not fix either distributor to be the identity intentionally: Since I want to investigate multiplicative structures interacting with involutions, having both distributors general isomorphisms not forced to be identities, makes it meaningful to speak of the multiplicatively opposite bimonoidal/bipermutative category.

1.1.2 Bipermutative Structures on Finite Sets

As an illustration that bipermutative categories are natural things to consider, I give a lemma which applies in a variety of cases, where the category is a skeletal version of some category with coproducts and products (or tensor products). I have denoted the following lemma analogous to the second monoidal structure being the product. However, I want to explicitly emphasise that I do not assume π to be a product-functor or the unit $* \in \mathcal{C}$ to be terminal.

Lemma 1.1.6. *Let $\mathcal{C}$ be a small category with coproducts, which is furthermore a permutative closed category with monoidal structure*

$$\pi\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

which is strictly associative and has unit $* \in \mathcal{C}$. Assume π has a right adjoint:

$$Hom(-,-)\colon \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}.$$

Assume furthermore a chosen functor representing coproducts:

$$\sqcup\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

which is strictly associative, and a chosen representative $\emptyset$ for the initial object, thus the unit for $\sqcup$.

Then this can be endowed with unique natural transformations $c_\sqcup, c_\pi, \lambda, \rho$, such that the resulting tuple $(\mathcal{C}, \sqcup, \pi, \emptyset, *, c_\sqcup, c_\pi)$ is a bipermutative category.

Proof. The proof is just a repeated application of universal properties. Because $\pi(-,c)$ has a right adjoint, it commutes with coproducts, in particular we have $\pi(\emptyset, c) = \emptyset, \ \ \forall c \in \mathcal{C}$. Consider the additive symmetry: For $T\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ the symmetry of the product on categories both $\sqcup$ and $\sqcup \circ T$ represent a coproduct-functor on $\mathcal{C}$, hence by the universal property of the coproduct, we get a unique natural transformation

$$c_\sqcup\colon \ \sqcup \circ T \Rightarrow \sqcup.$$

By uniqueness of the natural isomorphism $c \sqcup d \to c \sqcup d$, for every pair $c, d \in \mathcal{C}$, we also get that $c_\sqcup$ is a symmetry:

$$id_\sqcup = c_\sqcup^2\colon \sqcup = \sqcup \circ T^2 \Rightarrow \sqcup.$$

The other natural isomorphisms are constructed much the same way.

For the interaction of the natural isomorphisms consider for instance the relation $\lambda = c_\pi \circ \rho \circ (c_\pi \sqcup c_\pi)$. Both are natural isomorphisms between the functors:

$$\sqcup \circ (\pi \times \pi) \circ (id \times T \times id) \circ (\Delta \times id) \Rightarrow \pi \circ (id \times \sqcup),$$

hence again by uniqueness of those natural isomorphisms we have equality. Every other diagram commutes by the same reasoning. $\square$

This lemma illustrates well, why ring categories which are just associative but admit no multiplicative symmetries, are a bit harder to come by. The easy (closed) monoidal constructions usually come from symmetric universal properties: product, tensor product, smash product, etc.. What follows are the prototypical examples which provide the structural morphisms in my examples of interest.

Example 1.1.7. In everything ringlike that follows, the category with objects the non-negative integers $\{\mathbf{n} = \{1, \ldots, n\} | \quad n \in \mathbb{N}_0\}$ and with morphisms the symmetric groups $\Sigma_*(\mathbf{n}, \mathbf{n}) = \Sigma_n$, i.e., $\Sigma_* = \coprod_n \Sigma_n$, features prominently. In particular categories and bicategories of the form "free modules over $\mathcal{R}$" have their structural natural transformations given by permutations.

Define the following functors:

$$+, \cdot \colon \Sigma_* \times \Sigma_* \to \Sigma_*$$

on objects:

$$\mathbf{n} + \mathbf{m} := \{1, \ldots, n+m\}, \mathbf{n} \cdot \mathbf{m} := \{1, \ldots, nm\},$$

and more interestingly on morphisms:

$$(f+g)(i) := \begin{cases} f(i), & i \leq n, \\ g(i-n)+n, & i \geq n+1, \end{cases} \quad \text{for} \quad f \in \Sigma_n, g \in \Sigma_m,$$

and

$$(fg)((i-1)m+j) := (f(i)-1)m + g(j) \quad (i = 1, \ldots, n; j = 1, \ldots m).$$

Note that this implicitly fixes a choice of bijections $n \times m \to nm$, in this case given by $(i, j) \mapsto (i-1)m + j$. Two easy calculations show that $+$ and $\cdot$ defined this way are strictly associative. To use Lemma 1.1.6 we need to exhibit $+$ as representing coproducts, so consider the embedding

$$\Sigma_* \to \mathrm{Fin}.$$

That is, we embed the skeletal category of finite sets and bijections into the skeletal category of finite sets and all maps. Together with the canonical injections $\mathbf{n} \to \mathbf{n}+\mathbf{m}$ and $\mathbf{m} \to \mathbf{n}+\mathbf{m}$, which are part of the category Fin, we have that $+$ represents coproducts. Hence Fin is a bipermutative category by Lemma 1.1.6. The structural maps we get are all isomorphisms, so restricting to Σ_* again makes Σ_* a bipermutative category. Furthermore we can restrict to its subcategory on all objects with just epimorphisms and get the bipermutative category Epi. Also we can restrict to the subcategory on all objects with just injections to get the bipermutative category Inj.

The induced additive symmetry $c_+ \colon n+m \to m+n$ is given by:

$$c_+(i) := \begin{cases} i+m, & i \leq n, \\ i-n, & n+1 \leq i, \end{cases}$$

and we have the multiplicative symmetry $c_\cdot \colon nm \to mn$ given by:

$$c_\cdot((i-1)m+j) = (j-1)n+i.$$

Recall that the distributivity transformations of bipermutative categories determine each other

$$\begin{tikzcd} ab+ac \arrow[r, "\lambda"] \arrow[d, "c+c"'] & a(b+c) \arrow[d, "c"] \\ ba+ca \arrow[r, "\rho"] & (b+c)a. \end{tikzcd}$$

May shows in full generality [MayE_∞, p. 155, Proposition 3.5] that one can always strictify one distributivity to be the identity. For finite sets this corresponds to ordering a product of finite sets either lexicographically or anti-lexicographically. This way we find for general bipermutative categories two one-sidedly strict cases:

$$\lambda = id \Rightarrow \rho_{b,c;a} = c_{a,b+c} \circ (c_{b,a} + c_{c,a}),$$

$$\rho = id \Rightarrow \lambda_{a;b,c} = c_{b+c,a} \circ (c_{a,b} + c_{a,c}).$$

The choice of bijection for products considered above forces $\lambda = id$ and a non-trivial right distributivity, so the first case. This makes Σ_* into a bipermutative category. The opposite choice with $\rho = id$ is given for instance in [MayE_∞, p. 161, Example 5.1].

Example 1.1.8. We can just repeat the argument above to find the pointed analogues of the above categories, hence we get $\mathrm{Inj}_+, \mathrm{Epi}_+, \mathrm{Fin}_+$. For definiteness let me reemphasise the bipermutative structure on these categories:

The coproduct (for Inj_+) is the pointed sum, hence we can define a strictly associative functor representing it by:

$$\mathbf{n}_+ + \mathbf{m}_+ := \mathbf{n}_+ \vee \mathbf{m}_+ = (\mathbf{n}+\mathbf{m})_+,$$

with the obvious extension to morphisms.

Fixing a choice of associative bijections $\bar{\omega}_{n,m} \colon \mathbf{n} \times \mathbf{m} \to \mathbf{nm}$ induces associative bijections for the smash product by:

$$\mathbf{n}_+ \wedge \mathbf{m}_+ = (\mathbf{n} \times \mathbf{m})_+ \to (\mathbf{nm})_+ = \mathbf{nm}_+.$$

Hence the symmetries and distributors are given by adjunction of basepoints to the relevant morphisms above. In other words, given the bijections ω, I fix the pointed structures such that

$$(\cdot)_+ \colon \mathrm{Fin} \to \mathrm{Fin}_+$$

becomes a strictly bipermutative functor with respect to disjoint union and pointed sum, and cartesian product and smash product.

1.1.3 Bicategories - Notation for this thesis

In 1.1.4 I use functors between bicategories with non-trivial coherence 2-cells, so I fix the notations and conventions for bicategories here.

Definition 1.1.9. A small **bicategory** with strict identities $\mathcal{C}$ is given by a set of objects $\mathcal{C}_0 = Ob\mathcal{C}$, a set of 1-categories $\mathcal{C}_1 = Mor\mathcal{C}$, and the following maps:

- source and target
$$s, t \colon \mathcal{C}_1 \to \mathcal{C}_0,$$
where we call objects $a, b \in \mathcal{C}_1$ with $t(a) = s(b)$ composable,

- identity objects
$$\mathrm{id}_{\_} \colon \mathcal{C}_0 \to \mathcal{C}_1,$$
with $s \circ \mathrm{id}_{\_} = t \circ \mathrm{id}_{\_} = id_{\mathcal{C}_0}$,

- a composition functor
$$\square\colon \mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1,$$
where the pullback is to be understood as *composable pairs* in the sense described above,

- a natural associativity isomorphism
$$\alpha\colon (\_ \square \_) \circ (\_ \square \_ \times id_{\mathcal{C}_1}) \Rightarrow (\_ \square \_) \circ (id_{\mathcal{C}_1} \times \_ \square \_).$$

These satisfy:

- The identity 1-cells are strict units, i.e., we have strict equalities of functors ($\mathcal{C}_0 \times \mathcal{C}_1 \to \mathcal{C}_1$ or $\mathcal{C}_1 \times \mathcal{C}_0 \to \mathcal{C}_1$ respectively):
$$(\_ \square \_) \circ (\mathrm{id}_{\_} \times id_{\mathcal{C}_1}) = (\_ \square \_) \circ (id_{\mathcal{C}_1} \times \mathrm{id}_{\_}) = pr_{\mathcal{C}_1},$$
where $pr_{\mathcal{C}_1}$ denotes the respective projection onto the $\mathcal{C}_1$-factor.

- The transformation α is the identity, if any factor is $\mathrm{id}_{\_}$.

- The transformation α satisfies Mac Lane's pentagon identity, i.e., we have a unique associator on fourfold $\square$-composites, hence by induction on composites of arbitrary length.

Remark 1.1.10. Most of the definition is standard apart from the fact that for this thesis I can get away with strict identities, so I took them as part of the definition.

For clarity: I stick to the convention that the associator always transforms expressions with left-biased bracketing $(ab)c$ into expressions with right-biased bracketing $a(bc)$.

Further notation: I will refer to morphisms in (the disjoint union of) the categories $\mathcal{C}_1$ as 2-cells, and refer to them globally as $\mathcal{C}_2$. Furthermore I denote the component of $\mathcal{C}_1$, which is (s,t)-over objects $a, b \in \mathcal{C}_0$ as $\mathcal{C}_1(a,b) =: \mathcal{C}(a,b)$, i.e., the full subcategories of $\mathcal{C}_1$ with objects:
$$\mathcal{C}(a,b)_0 = \{f \in \mathcal{C}_1 | s(f) = a \text{ and } t(f) = b\}.$$

Moreover I will always refer to the objects of the bicategory as objects, the objects of the morphism categories as 1-cells, thus in particular I refrain from calling 1-cells "objects" of their respective morphism categories.

I only stick to the notation $\square$ here to stress the difference between the composition functor and composition in $\mathcal{C}_1$. In the example of interest $\mathcal{M}(\mathcal{R})$ the 1-cells are given by matrices, and hence the composition functor $\square$ is matrix multiplication, which I denote by $\cdot$ or drop from notation altogether, while the composition of 2-cells is just composition in a product 1-category, so I refer to that as $\circ$. Also just for this section I denote identity 1-cells by id_a to stress that they are usually in fact natural basepoints for the categories $\mathcal{C}(a, a)$, but usually are not defined as maps of the type $x \mapsto x$. Again in $\mathcal{M}(\mathcal{R})$ these are the identity matrices, but the way these matrices represent "linear maps of modules" is at the very least not obvious. After this section I trust the context is sufficient to infer which type of identities I refer to.

Example 1.1.11. A category enriched in categories, i.e., a 2-category, is a bicategory with $\alpha = id$. In particular, every 1-category is a bicategory with discrete morphism categories and hence trivial associator as well.

Any monoidal category $(\mathcal{C}, \otimes, 1)$ can be understood as a one-point bicategory $\Sigma\mathcal{C}$: set $\Sigma\mathcal{C}_0 = \{*\}$ and $\Sigma\mathcal{C}_1 = \Sigma\mathcal{C}(*, *) = \mathcal{C}$. Composition is given by $\otimes$, the associator hence by the associator for $\otimes$.

Conversely the endomorphism category of any object $a \in \mathcal{A}_0$ in a bicategory $\mathcal{A}$ yields the monoidal category $\mathcal{A}(a, a)$, occasionally denoted by $\Omega_a\mathcal{A}$. In particular we have the trivial equality

$$\Omega_*\Sigma(\mathcal{C}, \otimes) = (\mathcal{C}, \otimes),$$

and for each object $a \in \mathcal{A}_0$ the strict inclusion functor

$$\eta_a \colon \Sigma\Omega_a\mathcal{A} = \Sigma\mathcal{A}(a, a) \to \mathcal{A}$$

with $\eta_a(*) = a$.

Given two bicategories there are adequate 1-cells between them, but the designations in the literature vary quite a bit - specifically compare the classical "Introduction to Bicategories" of Bénabou [Ben] with the more recent overview in [CCG]. On pages 9 and 10 of [CCG] the authors provide an excellent dictionary of the common terms for morphisms. A more detailed exposition can be found in Ross Street's "Categorical Structures" [Str1], in particular section 9. Furthermore its references are a nice guide to the literature up to 1993.

I fix my notation here, and only define the types of functors that appear in this thesis.

Definition 1.1.12. For two bicategories $\mathcal{C}, \mathcal{D}$ a **pseudofunctor** $F\colon \mathcal{C} \to \mathcal{D}$ consists of the following maps:

- A map on objects $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$.
- For each pair of objects $a, b \in \mathcal{C}_0$ a functor

  $$F_1\colon \mathcal{C}(a,b) \to \mathcal{C}(F_0 a, F_0 b),$$

  that is pointed at identities, i.e., for every $a \in \mathcal{C}_0$:

  $$F_1(\mathrm{id}_a) = \mathrm{id}_{F_0 a}\,.$$

- For each triple of objects $a, b, c \in \mathcal{C}_0$ a natural isomorphism, which I refer to as *compositor 2-cell*,

  $$F_2\colon (\_\,\Box_{\mathcal{D}}\,\_)(F_1 \times F_1) \Rightarrow F_1 \times (\_\,\Box_{\mathcal{C}}\,\_),$$

  which is trivial at identities, i.e., $F_2 = id_{pr_{\mathcal{D}_1}}$ at:

  $$pr_{\mathcal{D}_1} = (\_\,\Box_{\mathcal{D}}\,\_) \circ (F_1 \times F_1) \circ (id_{\mathcal{C}_1} \times \mathrm{id}_\_) \Rightarrow F_1 \circ (\_\,\Box_{\mathcal{C}}\,\_) \circ (id_{\mathcal{C}_1} \times \mathrm{id}_\_) = pr_{\mathcal{D}_1},$$

  and similarly for the other argument.

The compositor satisfies associativity, i.e., for every composable triple f, g, h of 1-cells in $\mathcal{C}_1$ the diagram

$$\begin{tikzcd}
(F_1 f \,\Box_{\mathcal{D}}\, F_1 g) \,\Box_{\mathcal{D}}\, F_1 h \arrow[r, "F_2 \Box_{\mathcal{D}} id_{\mathcal{D}_1}"] \arrow[d, "\alpha_{\mathcal{D}}"'] & F_1(f \,\Box_{\mathcal{C}}\, g) \,\Box_{\mathcal{D}}\, F_1 h \arrow[r, "F_2"] & F_1((f \,\Box_{\mathcal{C}}\, g) \,\Box_{\mathcal{C}}\, h) \arrow[d, "F_1(\alpha_{\mathcal{C}})"] \\
F_1 f \,\Box_{\mathcal{D}}\, (F_1 g \,\Box_{\mathcal{D}}\, F_1 h) \arrow[r, "id_{\mathcal{D}_1} \Box_{\mathcal{D}} F_2"'] & F_1 f \,\Box_{\mathcal{D}}\, (F_1(g \,\Box_{\mathcal{C}}\, h)) \arrow[r, "F_2"'] & F_1(f \,\Box_{\mathcal{C}}\, (g \,\Box_{\mathcal{C}}\, h))
\end{tikzcd}$$

commutes.

A pseudofunctor is called strict, if furthermore $F_2 = id$ as morphisms in $\mathcal{D}_1$, i.e., $F_1(fg) = F_1 f F_1 g$ and $F_1 \alpha = \alpha$ and for every pair of composable 1-cells $f, g \in \mathcal{C}_1$ we have

$$F_2 = id_{F_1(fg)}\colon F_1 f \,\Box_{\mathcal{D}}\, F_1 g \Rightarrow F_1(f \,\Box_{\mathcal{C}}\, g).$$

Remark 1.1.13. Occassionally – cf. [CCG, pp. 9–10] – one refers to pseudofunctors as *strong normal* functors, where strong refers to the fact that the involved 2-cell is an isomorphism and normality refers to strictly fixing identity 1-cells, which I carry through this thesis as a permanently standing assumption.

For emphasis I occassionally refer to functors, which strictly respect identity 1-cells and composition of 1-cells, as *strict normal* functors.

Remark 1.1.14. I only study bicategories arising from "finitely generated free module"-constructions, and thus by standard assumptions for K-theory only consider isomorphism subcategories of permutative 1-categories. For a clarifying mathematical reason to restrict to isomorphisms see in particular [GGN, Proposition 8.14]. Since I can restrict to exclusively isomorphism 2-cells, I have no need for less rigid functors between bicategories.

Pseudofunctors satisfy a preservation property on 1-cells, which (op)lax functors of bicategories do not satisfy in general.

Proposition 1.1.15. *A pseudofunctor $F\colon \mathcal{C} \to \mathcal{D}$ sends equivalence 1-cells of $\mathcal{C}$ to equivalences in $\mathcal{D}$. In particular, isomorphism 1-cells in $\mathcal{C}$ are sent to equivalences in $\mathcal{D}$.*

Proof. Let $f\colon a \to b$ be an equivalence in $\mathcal{C}$, i.e., there is a 1-cell $g\colon b \to a$ and isomorphism 2-cells $\varepsilon\colon gf \Rightarrow \mathrm{id}_a$, $\eta\colon \mathrm{id}_b \Rightarrow fg$. Then Ff is an equivalence inverted by Fg by the following diagrams:

$$
\begin{tikzcd}
& {} \arrow[dr, "Ff"] \arrow[d, phantom, "\Downarrow F_2"] & \\
{} \arrow[ur, "Fg"] \arrow[rr, "F(gf)"] \arrow[rr, bend right=50, "\mathrm{id}"'] \arrow[rr, phantom, bend right=25, "\Downarrow F(\varepsilon)"] & & {}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
{} \arrow[rr, bend left=50, "\mathrm{id}"] \arrow[rr, phantom, bend left=25, "\Downarrow F(\eta)"] \arrow[rr, "F(fg)"] \arrow[dr, "Ff"'] & & {} \\
& {} \arrow[ur, "Fg"'] \arrow[u, phantom, "\Downarrow F_2^{-1}"] &
\end{tikzcd}
.
$$

Hence we need the compositor two-cell and its inverse, and find that Ff is an equivalence. In particular, if F is not a strict functor, F sends isomorphisms to equivalences with F_2 as a non-trivial isomorphism to the identity.

(Furthermore note that I have implicitly used the assumption that F is normal in the diagrams by $\mathrm{id} = F\,\mathrm{id}$.) □

Definition 1.1.16. Let F, G be two pseudofunctors of bicategories:

$$(F_0, F_1, F_2), (G_0, G_1, G_2)\colon \mathcal{C} \to \mathcal{D}.$$

Then a (strong) **pseudonatural transformation** $\sigma\colon F \Rightarrow G$ consists of chosen 1-cells

$$\sigma^0\colon \mathcal{C}_0 \to \mathcal{D}_1$$

with $s(\sigma^0(a)) = F_0 a$ and $t(\sigma^0(a)) = G_0 a$, as well as coherence isomorphism 2-cells chosen for every pair $a, b \in \mathcal{C}_0$ and every 1-cell $f \in \mathcal{C}(a,b)$:

$$\begin{tikzcd}
F_0a \arrow[r, "F_1 f"] \arrow[d, "\sigma^0 a"'] \arrow[dr, phantom, "\swArrow \sigma^1_f"] & F_0b \arrow[d, "\sigma^0 b"] \\
G_0a \arrow[r, "G_1 f"'] & G_0b,
\end{tikzcd}$$

which are appropriately natural and are compatible with the compositors, i.e. we have for all objects $a, b, c \in \mathcal{C}_0$ and all 1-cells $f, g \in \mathcal{C}_1$:

$$\begin{tikzcd}
F_0a \arrow[r, "F_1 f"] \arrow[rr, bend left, "F_1(gf)"] \arrow[d, "\sigma_0"'] \arrow[dr, phantom, "\swArrow \sigma^1_f"] & F_0b \arrow[r, "F_1 g"] \arrow[d, "\sigma_0"] \arrow[dr, phantom, "\swArrow \sigma^1_g"] \arrow[u, phantom, "\Downarrow F_2^{-1}"] & F_0c \arrow[d, "\sigma_0"] \\
G_0a \arrow[r, "G_1 f"] \arrow[rr, bend right, "G_1(gf)"'] & G_0b \arrow[r, "G_1 g"] \arrow[d, phantom, "\Downarrow G_2"] & G_0c
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
F_0a \arrow[r, "F_1(gf)"] \arrow[d, "\sigma_0"'] \arrow[dr, phantom, "\swArrow \sigma^1_{gf}"] & F_0c \arrow[d, "\sigma_0"] \\
G_0a \arrow[r, "G_1(gf)"'] & G_0c.
\end{tikzcd}$$

Furthermore for bicategories with strict units we want $\sigma^1_{\mathrm{id}_a} = id_{\sigma^0_a}$ for each object $a \in \mathcal{C}_0$. If in addition the two-cells σ^1 are identities we call σ a strict natural transformation.

Remark 1.1.17. Since I restrict attention to bicategories with only isomorphism two-cells or at least functors and transformations with isomorphism two-cells, I do not need to introduce the concepts of lax and oplax natural transformations.

It is classical for 1-categories that a natural transformation $\eta\colon F \Rightarrow G$ is the same thing as a functor $\mathcal{C} \times I \to \mathcal{D}$ for $I = [0 < 1]$ the interval category. The analogous statement for bicategories holds true as well, which I want to isolate into a proposition for emphasis and reference.

Proposition 1.1.18. *A pseudonatural transformation η of functors $F, G\colon \mathcal{C} \to \mathcal{D}$ consists of the same data as a pseudofunctor $\eta\colon \mathcal{C} \times I \to \mathcal{D}$, while the coherence of 2-cells is equivalent to the pseudonaturality of the transformation.*

Given sufficient experience with 1-categories one would expect that I introduced all types of morphisms, but it should not be surprising that the extra-level of morphisms in bicategories (i.e., two-cells) introduces a higher type of morphisms between "natural transformations", called modifications.

Definition 1.1.19. Given two pseudonatural transformations σ, τ between the same two pseudofunctors F, G, a **modification** $\xi\colon \sigma \Rightarrow \tau$ consists of a choice of isomorphism two-cells

$$\xi\colon \mathcal{C}_0 \to \mathcal{D}_2,$$

making the following two diagrams of two-cells equal:

$$\begin{tikzcd} {} \arrow[r, "Ff"] \arrow[d, bend right=60, "\tau^0 a"'] \arrow[d, bend left=60, "\sigma^0 a"] \arrow[d, phantom, "\overset{\Leftarrow}{\xi_a}"] & {} \arrow[d, "\sigma^0 b"] \arrow[dl, phantom, "\swarrow \sigma^1"] \\ {} \arrow[r, "Gf"'] & {} \end{tikzcd} = \begin{tikzcd} {} \arrow[r, "Ff"] \arrow[d, "\tau^0 a"'] & {} \arrow[d, bend right=60, "\tau^0 b"'] \arrow[d, bend left=60, "\sigma^0 b"] \arrow[d, phantom, "\overset{\Leftarrow}{\xi_b}"] \arrow[dl, phantom, "\tau^1 \swarrow"] \\ {} \arrow[r, "Gf"'] & {} \end{tikzcd}.$$

Remark 1.1.20. Do note that on the level of 2-cells the diagrams above are only commutative squares resembling naturality in the context of 1-categories.

With the relevant morphisms in place I can introduce equivalences of bicategories.

Definition 1.1.21. A pseudofunctor of bicategories $F\colon \mathcal{C} \to \mathcal{D}$ is an **equivalence of bicategories**, if there is a pseudofunctor $G\colon \mathcal{D} \to \mathcal{C}$ and there are two pseudonatural equivalences $\eta\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$ and $\zeta\colon GF \Rightarrow \mathrm{id}_{\mathcal{C}}$.

The following proposition is particularly useful in chapter 3, nonetheless its appropriate context is abstract nonsense about bicategories, so this section. I repeat the proof in particular to convince the reader that it holds for bicategories with enriched morphism categories.

Proposition 1.1.22. *A pseudonatural transformation of pseudofunctors of small bicategories is an equivalence if and only if all its* 1*-cells are equivalences.*

Proof. It is clear that an inverse equivalence establishes each component 1-cell as an equivalence in the target category. So we have to establish that having all 1-cells equivalences is sufficient.

Let $\eta\colon F \Rightarrow G$ be a pseudonatural transformation comprised of 1-cells $\eta^1\colon \mathcal{C}_0 \to \mathcal{D}_1$, and for each pair of objects a, b in $\mathcal{C}$ a natural transformation:

$$\begin{tikzcd} \mathcal{C}(a,b) \arrow[r, bend left, "\eta^1_{b\,*}\circ F_1"] \arrow[r, bend right, "{\eta^1_a}^*\circ G_1"'] \arrow[r, phantom, "\Downarrow\eta^2_{a,b}"] & \mathcal{D}(Fa, Gb). \end{tikzcd}$$

By assumption each 1-cell η^1_a is an equivalence, so by the axiom of choice choose for each a an inverse equivalence ζ^1_a and isomorphism two-cells $\sigma\colon \zeta^1\eta^1 \to \mathrm{id}$ and $\tau\colon\ \mathrm{id} \to \eta^1\zeta^1$.

With these choices in place we can make ζ into a pseudonatural transformation by choosing its two-cells as indicated by the following diagram:

$$\begin{tikzcd} \zeta GA = \zeta GA\,\mathrm{id} \arrow[r, "\tau"] & (\zeta GA)\cdot(\eta\zeta) \arrow[d, "\alpha"] & (\zeta\eta)(FA\zeta) \arrow[r, "\sigma"] & \mathrm{id}\, FA\zeta = FA\zeta. \\ & \zeta((GA\eta)\zeta) \arrow[r, "\eta^2"] & \zeta((\eta FA)\zeta) \arrow[u, "\alpha"] & \end{tikzcd}$$

Since τ and σ are chosen objectwise, we get a natural transformation. Since each arrow is an isomorphism 2-cell, the 1-cells ζ along with the two-cells indicated above compromise a pseudonatural transformation. It is compatible with the compositors of G and F because η is, and hence ζ is an inverse equivalence to η. The relevant modifications are by construction given by τ and σ. □

Remark 1.1.23. For this proposition bicategories are much nicer than the stricter theory of 2-categories. Even if the pseudo-natural transformation strictly satisfies naturality, its inverse equivalence might have non-trivial 2-cells.

Remark 1.1.24. Do note that despite the fact this proposition is the analogue to the 1-category statement that a natural transformation is a natural isomorphism if and only if each component is an isomorphism, its truth is (ZF-)axiomatically equivalent to the statement that a functor is an equivalence of categories if and only if it is essentially surjective and fully faithful. Hence it is stronger because of the missing uniqueness for the inverse 1-cells.

The following proposition is an immediate generalisation from the context of 1-categories. I want to exhibit that the proof works in the context I define above. Thus it reassures us that the definitions are consistently chosen.

Proposition 1.1.25. *A pseudofunctor $F = (F_0, F_1, F_2)$ of (small) bicategories is an equivalence of bicategories if and only if F_0 is surjective up to equivalence and each functor F_1 is an equivalence of 1-categories.*

Proof. Given a pseudofunctor $F\colon \mathcal{C} \to \mathcal{D}$ that satisfies the conditions, we can by the axiom of choice find a map $G_0\colon Ob\mathcal{D} = \mathcal{D}_0 \to \mathcal{C}_0$ such that there is an equivalence $\eta_d\colon F(G_0(d)) \to d$ for each $d \in \mathcal{D}_0$. Fix that equivalence and an inverse κ^d together with the isomorphisms $\eta_d\kappa^d \cong \mathrm{id}$ and $\kappa^d\eta_d \cong \mathrm{id}$ for each $d \in \mathcal{D}_0$, it is the system of 1-cells for the natural equivalence $FG \simeq \mathrm{id}_{\mathcal{D}}$ we need.

By assumption we have for each pair d_1, d_2 an equivalence of categories

$$F_1\colon \mathcal{C}(G_0d_1, G_0d_2) \to \mathcal{D}(FG_0d_1, FG_0d_2).$$

Fix an inverse equivalence for each such pair G^{d_1,d_2}, then we make G into a functor of bicategories by the following assignment on morphism categories:

$$\mathcal{D}(d_1, d_2) \xrightarrow{\eta^*_{d_1}} \mathcal{D}(FG_0d_1, d_2) \xrightarrow{\kappa^{d_2}_*} \mathcal{D}(FG_0d_1, FG_0d_2) \xrightarrow{G^{d_1,d_2}} \mathcal{C}(G_0d_1, G_0d_2).$$

Without loss of generality make G into a functor pointed at the identity 1-cells.

This is an inverse equivalence to F by construction; it is a pseudofunctor with compositor 2-cell given by the isomorphisms $\eta\kappa \cong \mathrm{id}$ chosen above and with the natural equivalence on one side given by η with inverse κ and on the other by $G\eta$ and $G\kappa$. □

It is reassuring to know that bicategories can still be strictified to 2-categories. (This is wrong for tricategories!)

Lemma 1.1.26. *(cf. [Le]) Each bicategory is equivalent to a 2-category.*

Remark 1.1.27. From here I drop the properly emphasised but clumsy notation, and denote $\Box$ in a bicategory by $\cdot$ or do not denote it at all, while composition of 2-cells is denoted by $\circ$, as it is usually the composition in some product of 1-categories in my examples.

For functors I refer to (F_0, F_1) generically as F and I denote the compositor in uppercase greek letters Φ, thus referring to a pseudofunctor (F, Φ) for instance.

I stick to the following conventions for elements in a general bicategory: objects are denoted by lowercase latin letters $a, b, c, \ldots \in \mathcal{C}_0$, which in $\mathcal{M}(\mathcal{R})$ are natural numbers, but I do not want to restrict to that case unnecessarily. In $\mathcal{M}(\mathcal{R})$ the 1-cells are matrices, hence I denote 1-cells by uppercase latin letters $A, B, C, \ldots \in \mathcal{C}_1$, and finally 2-cells by lowercase greek letters $\varphi, \psi, \ldots$ which are morphisms in products of the coefficient category $\mathcal{R}$ for $\mathcal{M}(\mathcal{R})$.

Remark 1.1.28. I refer to bicategories $\mathcal{C}$ as *enriched in* topological spaces, simplicial sets, categories,... if the morphism categories $\mathcal{C}_1$ are enriched in these monoidal categories.

Example 1.1.29. A category enriched in topologically or simplicially enriched categories is a bicategory enriched in topological spaces or simplicial sets respectively. A topological/simplicial monoidal category gives rise to a one-point bicategory enriched in topological spaces/simplicial sets. Do note that the propositions before work in the enriched cases as well, i.e. for enriched bicategories, and enriched pseudofunctors, since the axiom of choice was only involved objectwise.

The rest of this chapter – apart from the section 1.2 - can safely be skipped by the reader familiar with the delooping in [EM]. For ease of reference I rewrite their delooping in the following sections, so that the delooping in bicategories 3 can be read in close analogy with the case in 1-categories.

1.1.4 A Delooping of Permutative Categories

Permutative categories provide a classical useful tool to model connective spectra, hence are valuable in stable homotopy theory. Even more than that Thomason proved [Th1] that "Symmetric monoidal categories model all connective spectra". Thomason was also driven by the desire to provide a nice model for a smash product of spectra: "[I]n June 1993 [...] I used this alternate model of stable homotopy to give the first known construction of a smash product which is associative and commutative up to coherent natural isomorphism in the model category." Since [MMSS] showed "all" models for symmetric monoidal categories of spectra yield (Quillen-)equivalent results, Thomason's construction of a smash product has lost attention.

I elaborate on the construction $\mathcal{C}^+$ in [Th2] in excessive detail, so I can refer back to its details for the analogous construction in bicategories 3. Warning on notation: The notational conventions for bicategories described in 1.1.27 do not apply here, because they would clash with the natural interpretations.

The following results are each found in section 4 of [Th2]. In particular, I repeat his results and definitions in order to fix the notations I mimic for bicategories in chapter 3.

Definition 1.1.30. Let $(\mathcal{C}, +, 0, c_+)$ be a permutative category and $f\colon n_+ \to m_+$

a map of pointed sets $n_+ = \{0, 1, \ldots, n\}$. Define the following functor:

$$\begin{aligned} f_* \colon \mathcal{C}^{\times n} &\to \mathcal{C}^{\times m} \\ (c_1, \ldots, c_n) &\mapsto \Big(\sum_{i \in f^{-1}j} c_i\Big)_{j=1,\ldots,m}, \end{aligned}$$

where the empty sum is defined to be the zero object (and its identity). Note that we have to use the induced ordering $f^{-1}j \subset (n, \leq)$ and sum the c_i accordingly.

Remark 1.1.31. Note in particular that this gives a left-action of the symmetric groups on the respective powers: $\sigma_*(c_1, \ldots, c_n)_j = \sum_{i \in \sigma^{-1}j} c_i = c_{\sigma^{-1}j}$.

The fact that we have to choose an ordering on the fibres of f is precisely what breaks the strictness of the functor $(\cdot)_*\mathrm{Fin}_+ \to \mathit{Cat}$, which on morphisms is the assignment $f \mapsto f_*$ according to the above definition. I isolate this fact into the following lemma.

Lemma 1.1.32. *Given pointed maps $f \colon n_+ \to m_+$ and $g \colon m_+ \to l_+$ there is a natural isomorphism of functors $\varphi_{g,f} \colon (gf)_* \Rightarrow g_* f_*$.*

Proof. We can consider this componentwise, so without loss of generality let $g \colon m_+ \to 1_+ = \{0, 1\}$ the unique map with $g^{-1}0 = \{0\}$. Then the summation according to $g_* f_*$ looks as follows:

$$(g_* f_*)(c) = \sum_{i=1}^{m} (f_* c)_i = \sum_{i=1}^{m} \sum_{j \in f^{-1}i} c_j,$$

whereas the summation of $(gf)_*$ is according to the linear order on n given by

$$(gf)_*(c) = \sum_{i \in (gf)^{-1}1} c_i = \sum_{i=1}^{n} c_i.$$

Then there is a unique additive symmetry giving the isomorphism:

$$((gf)_* c)_j = \textstyle\sum_{k \in (gf)^{-1}j} c_k \xrightarrow{c^+_{g,f}} \sum_{i \in g^{-1}j} \sum_{k \in f^{-1}i} c_k = \sum_{i \in g^{-1}j} (f_* c)_i = g_*(f_*(c)),$$

which yields a natural isomorphism of functors:

$$(gf)_* \Rightarrow g_* f_*.$$

$\square$

Remark 1.1.33. Take special note of the following composites, which reappear in the delooping constructions and the distributivity axioms of bimonoidal categories. Define

$$f\colon 4_+ \to 2_+ : f(1) = f(3) = 1, f(2) = f(4) = 2$$

and

$$g\colon 4_+ \to 2_+ : g(1) = g(2) = 1, g(3) = g(4) = 2$$

. We have the unique map $q\colon 2_+ \to 1_+$ with $q^{-1}0 = \{0\}$ for $n_+ = \{0, 1, \ldots, n\}$ pointed at 0. We also have a unique map $c\colon 4_+ \to 1_+$ with $c^{-1}0 = \{0\}$. Then we have $qf = qg = c$, and hence for a permutative category a unique isomorphism:

$$q_* g_* = c_* = (qf)_* \Rightarrow q_* f_*,$$

which is given by the symmetry:

$$1 + c^+ + 1\colon a + b + c + d \to a + c + b + d.$$

In more detail: The compositor for q and g is the identity: $\varphi_{q,g} = id$, and for q and f is the symmetry $\varphi_{q,f} = 1 + c^+ + 1$.

More generally: Since the action of the symmetric groups on the respective powers of $\mathcal{C}$ is strict, we get $\varphi_{\sigma_1,\sigma_2} = id$, for each $n \in \mathbb{N}$ and $\sigma_1, \sigma_2 \in \Sigma_n$. Consider the composite of a permutation $\sigma \in \Sigma_n$ with the unique map $q\colon n_+ \to 1_+$, with $q^{-1}0 = \{0\}$. Then we get: $q_*(c_1, \ldots, c_n) = \sum_{i=1}^n c_i$, and $q_*\sigma_*(c_1, \ldots, c_n) = \sum_{i=1}^n c_{\sigma^{-1}i}$, hence $\varphi_{q,\sigma} = c_\sigma^+$, for c_σ^+ the unique natural additive symmetry in $\mathcal{C}$ between these sums.

We can define a "classifying" pseudofunctor into the 2-category Cat for a permutative category.

Proposition 1.1.34. *Given a permutative category* $(\mathcal{C}, +, 0, c_+)$*, the assignment*

$$\begin{array}{rcl} B_{\mathcal{C}}\colon Fin_+ & \to & Cat \\ n_+ & \mapsto & \mathcal{C}^{\times n}, \\ f\colon n_+ \to m_+ & \mapsto & f_*\colon \mathcal{C}^{\times n} \to \mathcal{C}^{\times m} \end{array}$$

defines a pseudofunctor of 2*-categories, where we consider* Fin_+ *as a* 2*-category with discrete morphism categories.*

Proof. We have to prove the coherence:

$$\begin{tikzcd}
(hgf)_* \arrow[r, "\varphi_{hg,f}"] \arrow[d, "\varphi_{h,gf}"] & (hg)_* f_* \arrow[d, "\varphi_{h,g} id"] \\
h_*(gf)_* \arrow[r, "id\varphi_{g,f}"] & h_* g_* f_*,
\end{tikzcd}$$

but we know by [Lap] that the additive symmetry defining the transformation

$$(hgf)_* \Rightarrow h_* g_* f_*$$

is uniquely determined by the ordering of summands the composite $h_* g_* f_*$ induces, so the diagram commutes. Furthermore we obviously have $id_* = id$, so $B_{\mathcal{C}}$ is a normal functor. □

Remark 1.1.35. Observe that giving a covariant pseudofunctor Epi $\to$ Cat, which on objects is the assignment $n \mapsto \mathcal{C}^n$, already defines a symmetric monoidal product on $\mathcal{C}$, which is strictly associative and includes a symmetry but does not have a unit or unitors. In order to take care of the zero object we need a strong normal functor Fin $\to$ Cat. Choosing pointed sets as an index category induces projections in the additive Grothendieck construction below associated to the maps:

$$\rho^i \colon n_+ \to 1_+$$

with

$$\rho^i(j) = \begin{cases} 0 & j \neq i \\ 1 & j = i, \end{cases}$$

which is easily identified as: $(\rho^i)_* = pr_i \colon \mathcal{C}^{\times n} \to \mathcal{C}$.

This association of a monoidal category to a functor is sufficiently natural for the following lemma to hold:

Lemma 1.1.36. *A (pointed) functor $F\colon (\mathcal{C}, +) \to (\mathcal{D}, +)$ together with a natural transformation $\lambda\colon F(\_) + F(\_) \Rightarrow F(\_ + \_)$ is strongly symmetrically monoidal if and only if the induced map of pseudofunctors $B_{\mathcal{C}} \Rightarrow B_{\mathcal{D}}$ is a pseudonatural transformation. For $\mathcal{C}$ and $\mathcal{D}$ permutative F is a strictly additive functor if and only if the induced map is a strict natural transformation $B_{\mathcal{C}} \Rightarrow B_{\mathcal{D}}$.*

Additionally a natural transformation of symmetric monoidal functors is symmetrically monoidal if and only if it induces a modification of the respective induced transformations.

Proof. That a strong symmetric functor defines a pseudonatural transformation is given in [Th2, Paragraph 4.1.4]. The converse follows from the observation that we can reconstruct F by restriction to U-level 1: $F = B_F|_{\{1_+\}} \colon \mathcal{C} \cong B_{\mathcal{C}}(1_+) \to B_{\mathcal{D}}(1_+) \cong \mathcal{D}$. The pseudonaturality 2-cell of B_F for the unique map $q\colon 2_+ \to 1_+$ with $q^{-1}(0) = 0$, gives the diagram:

$$\begin{tikzcd} B_{\mathcal{C}}(2_+) \arrow[r, "B_F^1"] \arrow[d, "q"'] \arrow[dr, phantom, "\swarrow B_F^2"] & B_{\mathcal{D}}(2_+) \arrow[d, "q"] \\ B_{\mathcal{C}}(1_+) \arrow[r, "B_F^1"] & B_{\mathcal{D}}(1_+), \end{tikzcd}$$

which identifies λ as the pseudonaturality 2-cell B_F^2. It is coherently associative and symmetric because of the appropriate pseudonaturality diagrams in higher degrees.

Comparing the diagram above with the diagram in definition 1.1.19 yields the properly analogous claim for monoidal natural transformations. $\square$

I do not give the full generality of Grothendieck constructions, but only define the resulting category of the Grothendieck construction on $B_{\mathcal{C}}$ with respect to the permutative category $(\mathcal{C}, +, 0, c_+)$. It destills the complexity of the functor into an ordinary 1-category, so I do not refer to bicategories again until 2. For a compatible general exposition of the Grothendieck construction as considered by Grothendieck compare pages 47–49 in [Ben].

Definition 1.1.37. Define the category $\mathcal{C}^+$ as follows: Its objects are

$$Ob\mathcal{C}^+ := \coprod_n \mathcal{C}^{\times n},$$

its morphisms:

$$\mathcal{C}^+((c_1, \ldots, c_n), (d_1, \ldots, d_m)) = \coprod_{f \in \mathrm{Fin}_+(n_+, m_+)} \mathcal{C}^m(f_* c, d).$$

The identities are given by the identities in Fin_+ and $\mathcal{C}^m$, the composition is given as follows: $c = (c_1, \ldots, c_n), d = (d_1, \ldots, d_m), e = (e_1, \ldots, e_l)$:

$$\begin{tikzcd}
\mathcal{C}^+(d,e) \times \mathcal{C}^+(c,d) \arrow[d, equal] \\
\coprod_{g,f} \mathcal{C}^l(g_* d, e) \times \mathcal{C}^m(f_* c, d) \arrow[d, "\coprod id \times g_*"] \\
\coprod_{g,f} \mathcal{C}^l(g_* d, e) \times \mathcal{C}^l(g_* f_* c, g_* d) \arrow[d, "comp_{\mathcal{C}^l}"] \\
\coprod \mathcal{C}^l(g_* f_* c, e) \arrow[d, "\varphi^*_{g,f}"] \\
\coprod \mathcal{C}^l((gf)_* c, e) \subset \mathcal{C}^+(c,e).
\end{tikzcd}$$

It is associative precisely because φ is, and if $(\mathcal{C}, +, 0, c_+)$ carries an enrichment such that $+$ is an enriched functor, then $\mathcal{C}^+$ is enriched over the same category.

Call this the **additive Grothendieck construction** on a permutative category $(\mathcal{C}, +)$.

Remark 1.1.38. The construction $\mathcal{C}^+$ is already given by Thomason in [Th2, Definition 2.1.2]. Let me summarise the idea of the construction. Given a monoidal product on a category: $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, we can define for each map $f\colon n_+ \to m_+$ of finite sets a functor $\mathcal{C}^{\times n} \to \mathcal{C}^{\times m}$ (in the same direction). This assembles into a pseudofunctor $\mathrm{Fin}_+ \to Cat$. The Grothendieck construction associated to it is $\mathcal{C}^+$. I find it useful to describe the delooping constructions of [EM, Os, Se] in this one syntax.

Note that by the description of the composition given above we have an enriched additive Grothendieck construction for an enriched permutative category - specifically if the monoidal functor $+\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is enriched, then the enrichment carries over to $\mathcal{C}^+$.

Example 1.1.39. Consider an ordinary category, that is one enriched in sets, then we can write morphisms between tuples as pairs

$$(f, (a_1, \ldots, a_m))\colon (c_1, \ldots, c_n) \to (d_1, \ldots, d_m),$$

where $f\colon n_+ \to m_+$ and $a_i\colon \sum_{j \in f^{-1} i} c_j \to d_i$, where we understand the empty sum as the zero object.

Then composition looks as follows (with $f\colon n \to m, g\colon m \to l$):

$$(g,(b_1,\dots,b_l)) \circ (f,(a_1,\dots,a_m)) = (gf,(b_1,\dots,b_l) \circ g_*(a_1,\dots,a_m) \circ \varphi_{g,f}).$$

I consider the two extreme cases: The composites of symmetries are strict. So we have trivial compositors here, and thus (with $\bar{a}^i = (a_1^i,\dots,a_n^i)$ n-tuples of morphisms in $\mathcal{C}$):

$$(\sigma,\bar{a}^3) \circ (\tau,\bar{a}^2) \circ (\rho,\bar{a}^1) = (\sigma\tau\rho,\bar{a}^3 \circ \sigma.\bar{a}^2 \circ (\sigma\tau).\bar{a}^1).$$

The other extreme case is given for the composite of any symmetry $\sigma \in \Sigma_n$ with a map $q\colon n_+ \to 1_+$ with $q^{-1}0 = \{0\}$, there is a unique additive symmetry c_σ^+ in $\mathcal{C}$:

$$c_\sigma^+\colon q_*(c_1,\dots,c_n) = \sum_{i=1}^n c_i \to q_*(\sigma.(c_1,\dots,c_n)) = \sum_{i=1}^n c_{\sigma^{-1}i}.$$

Thus we get:

$$\begin{aligned}(q,a) \circ (\sigma,(b_1,\dots,b_n)) &= (q\sigma, a \circ q_*(b_1,\dots,b_n) \circ \varphi_{\sigma,q})\\ &= (q, a \circ (\textstyle\sum_i b_i) \circ c_\sigma^+)\end{aligned}$$

and by the naturality of the additive twist we can identify this with:

$$\begin{aligned}&= (q, a \circ c_\sigma^+ \circ (\textstyle\sum_i b_{\sigma^{-1}i}))\\ &= (q, a \circ c_\sigma^+ \circ q_*(\sigma.(b_1,\dots,b_n)))\\ &= (q, a \circ c_\sigma^+) \circ (id, \sigma.(b_1,\dots,b_n)).\end{aligned}$$

In particular this Grothendieck construction incorporates structural morphisms for permutations between n-tuples that project down to the ordinary additive symmetry in $\mathcal{C}$.

Remark 1.1.40. Note that in particular the category $\mathcal{C}^+$ has structural morphisms $(f,id)\colon c \to f_*c$, for $c = (c_1,\dots,c_n)$ and $f\colon n_+ \to m_+$. We call these morphisms the *discrete component* of morphisms $c \to d$. This is a forgetful functor $U\colon \mathcal{C}^+ \to \mathrm{Fin}_+$, which is important for the delooping construction.

Example 1.1.41. Building on the previous example let us reconsider the maps from 1.1.33. We set

$$f\colon 4_+ \to 2_+ : f(1) = f(3) = 1, f(2) = f(4) = 2,$$

$$g\colon 4_+ \to 2_+ : g(1) = g(2) = 1, g(3) = g(4) = 2,$$

$$q\colon 2_+ \to 1_+ : q(2) = q(1) = 1,$$

and write $c\colon 4_+ \to 1_+$ for the unique map with $c^{-1}0 = 0$. Then the compositor $\varphi_{q,f}$ is given by $1 + c_+ + 1$, whereas the compositor $\varphi_{q,g}$ is the identity. So we find:

$$\begin{aligned}(q,a)\circ(g,(b_1,b_2)) &= (qg, a\circ q_*(b_1,b_2))\\ &= (c, a\circ(b_1+b_2))\\ (q,a)\circ(f,(b_1,b_2)) &= (qf, a\circ(b_1+b_2))\\ &= (c, a\circ(b_1+b_2)\circ(1+c_++1)).\end{aligned}$$

Hence the following diagram represents a commutative square in $\mathcal{C}^+$:

$$\begin{tikzcd}(c_1,c_2,c_3,c_4) \arrow[rr, "{(f,\mathrm{id})}"] \arrow[d, "{(g,\mathrm{id})}"] & & (c_1+c_3,c_2+c_4) \arrow[d, "{(q,\mathrm{id})}"] \\ (c_1+c_2,c_3+c_4) \arrow[r, "{(q,\mathrm{id})}"] & (c_1+c_3+c_2+c_4) & (c_1+c_2+c_3+c_4), \arrow[l, "\varphi_{q,f}"']\end{tikzcd}$$

because φ is part of the composition law.

The additive Grothendieck construction is naturally associated to the pseudo-functors $B_\bullet$, thus leading to the following naturality:

Lemma 1.1.42. *A strong symmetric monoidal functor $(F,\mu)\colon (\mathcal{C},+) \to (\mathcal{D},+)$ induces a canonical functor F^+ on additive Grothendieck constructions as follows: It assigns tuples componentwise $F^+(c_1,\dots,c_n) = (Fc_1,\dots,Fc_n)$. For morphisms consider the case with morphism sets. The map*

$$(f,(\varphi_1,\dots,\varphi_m))\colon (c_1,\dots,c_n) \to (d_1,\dots,d_m)$$

is sent to:

$$\begin{tikzcd}(Fc_1,\dots,Fc_n) \arrow[d, "{(f,id)}"] & \\ (\sum_{j\in f^{-1}i} Fc_j)_i \arrow[r, "{(id,(\mu)_i)}"] & (F(\sum_{j\in f^{-1}i} c_j))_i \arrow[d, "{(id,(F(\varphi_1),\dots,F(\varphi_m)))}"] \\ & (Fd_1,\dots,Fd_m).\end{tikzcd}$$

This is a functor precisely because (F,μ) is symmetrically monoidal.

A monoidal natural transformation $\eta\colon (F,\mu) \Rightarrow (G,\nu)$ induces a natural transformation $\eta^+\colon F^+ \Rightarrow G^+$.

Proof. I only comment on the natural transformation. Again consider the case with morphism sets, then we set η^+ as tuples with the appropriate instances of

η and no discrete component. This is natural in morphisms $(f, (\mu)_i)$ trivially, because η is monoidal, and it is natural in morphisms $(id, (F(\varphi_1), \ldots, F(\varphi_m)))$ because it is a product of natural transformations $\eta\colon F \Rightarrow G$.

Do note that it is not meaningful for a natural transformation to be symmetrically monoidal; there is no additional compatibility for η to be satisfied. $\square$

Given any symmetric monoidal category $\mathcal{C}$ we can restrict to its subcategory of isomorphisms $\mathcal{C}^{iso}$, which is a symmetric monoidal subcategory of $\mathcal{C}$, so this gives an inclusion on their Grothendieck constructions.

Corollary 1.1.43. *There is a natural inclusion* $I\colon (\mathcal{C}^{iso})^+ \to \mathcal{C}^+$.

The following definition describes the index categories relevant for the delooping. Specifically, we consider the comma category of finite pointed sets under A_+, for A_+ not necessarily an object of Fin_+.

Definition 1.1.44. For an arbitrary finite pointed set A_+ define the index category $A_+ \downarrow \mathrm{Fin}_+$ as follows: Objects are pointed maps $p\colon A_+ \to n_+$ and morphisms $f\colon p \to q$ are commutative triangles under A_+:

$$\begin{tikzcd} A_+ \arrow[r, "p"] \arrow[dr, "q"'] & n_+ \arrow[d, "f"] \\ & m_+. \end{tikzcd}$$

Definition 1.1.45. Call the morphisms ρ^i with $\rho^i(j) = *$ for $j \neq i$ and $\rho^i(i) = 1$. They fit into the diagram:

$$\begin{tikzcd} A_+ \arrow[r, "f"] \arrow[dr, "\chi_{f^{-1}i}"'] & n_+ \arrow[d, "\rho^i"] \\ & 1_+, \end{tikzcd}$$

for each f and each non-empty preimage $f^{-1}i \neq \emptyset$.

Remark 1.1.46. Note that the functors $(\rho^i)_*\colon \mathcal{C}^n \to \mathcal{C}$ are strictly equal to the projection onto the ith factor.

Remark 1.1.47. We have a target functor, i.e., $T\colon A_+ \downarrow \mathrm{Fin}_+ \to \mathrm{Fin}_+$.

For $f\colon A_+ \to B_+$ a pointed map we have a (covariant) functor $f^*\colon B_+ \downarrow \mathrm{Fin}_+ \to A_+ \downarrow \mathrm{Fin}_+$, which is a functor over T, i.e., the diagram

$$\begin{tikzcd} B_+ \downarrow \mathrm{Fin}_+ \arrow[r, "f^*"] \arrow[dr, "T"'] & A_+ \downarrow \mathrm{Fin}_+ \arrow[d, "T"] \\ & \mathrm{Fin}_+ \end{tikzcd}$$

commutes.

Remark 1.1.48. The categories $A_+ \downarrow \mathrm{Fin}_+$ have a "full", actually discrete, subcategory on objects the characteristic maps $\chi_a \colon A_+ \to 1_+$ with $\chi_a(x) = +$ for $x \neq a$ and $\chi_a(a) = 1$. This yields a functor $j_A \colon A^\delta \to A_+ \downarrow \mathrm{Fin}_+$, which includes the unpointed set A as a discrete subcategory A^δ into $A_+ \downarrow \mathrm{Fin}_+$, by identifying each element a with its characteristic map χ_a.

It is not strictly necessary to define the delooping construction for $n = 1$ (1.1.5), but the essential mathematics happen here. The cases for $n > 1$ are then reductions to this case.

Definition 1.1.49. Given a permutative category $(\mathcal{C}, +)$ and a finite pointed set A_+ define its **first delooping category** $\mathcal{C}(A_+, 1)$ as the category of functors lifting $T \colon A_+ \downarrow \mathrm{Fin}_+ \to \mathrm{Fin}_+$ through $U \colon (\mathcal{C}^{iso})^+ \to \mathrm{Fin}_+$, i.e., the dashed arrows in the diagram:

$$\begin{tikzcd} & (\mathcal{C}^{iso})^+ \arrow[d, "U"] \\ A_+ \downarrow \mathrm{Fin}_+ \arrow[ur, dashed] \arrow[r, "T"'] & \mathrm{Fin}_+, \end{tikzcd}$$

where T is the target functor given in 1.1.47, and U is the functor sending each morphism to its discrete component as in 1.1.40. Its morphisms are the natural transformations of the functors pushed forward with $I \colon (\mathcal{C}^{iso})^+ \to \mathcal{C}^+$, i.e., natural transformations but with arbitrary components, not just isomorphisms.

Remark 1.1.50. The delooping of a permutative category given in [EM] can be understood as a categorified version of the usual classifying space construction for abelian groups (cf. [May1, pp.87+88 and Theorem 23.2]). For the functoriality of the delooping construction in finite pointed sets and arbitrary maps it is more convenient to consider all maps of finite pointed sets as structural morphisms. But one should think of the structural Epi_+-morphisms as fundamental, whereas non-surjective morphisms just keep track of zeroes.

Remark 1.1.51. By contravariant functoriality of the indexing categories over T we get that $\mathcal{C}(\_, 1)$ defines a covariant functor $\mathrm{Fin}_+ \to \mathit{Cat}$. Furthermore restriction along $j_A \colon A^\delta \to A_+ \downarrow \mathrm{Fin}_+$ is a functor $R \colon \mathcal{C}(A_+, 1) \to \mathcal{C}^{\times A}$.

Proposition 1.1.52. *Every functor $F \colon A_+ \downarrow \mathrm{Fin}_+ \to \mathcal{C}^+$ lifting T through U is isomorphic to a unique* strict representative. *i.e., a functor, which assigns to commutative triangles of $A_+ \downarrow \mathrm{Fin}_+$ only morphisms with discrete components and additive symmetries and appropriate identities in the second component. In* *particular any two functors restricting to the same A-tuple along R are naturally isomorphic.*

Proof. To build the *strict representative* F^{st} proceed as follows: Choose a bijection $\sigma_A\colon A_+ \to |A|_+$ and set $F^{st}(\sigma_A) = (F(\chi_a\colon A_+ \to 1_+))_{a\in A}$. Any other object of $A_+ \downarrow \mathrm{Fin}_+$ has a unique morphism from σ_A. So for $p \in A_+ \downarrow \mathrm{Fin}_+$ set $F^{st}(p) = (p\sigma_A^{-1})_*(F^{st}(\sigma_A)) \in \mathcal{C}^{\times|T(p)|} \subset \mathcal{C}^+$. I drop the σ_A^{-1} from the notation immediately, since it is only there to make coherent choices for all maps of finite sets at once. For a commutative triangle under A_+:

$$\begin{tikzcd} A_+ \arrow[r, "p"] \arrow[dr, "qp"'] & n_+ \arrow[d, "q"] \\ & m_+ \end{tikzcd}$$

we need a morphism $F^{st}(p) = p_*(F^{st}(\sigma_A)) \to q_*p_*(F^{st}(\sigma_A)) \to (qp)_*(F^{st}(\sigma_A)) = F^{st}(qp)$, which we can choose to be $(q, \varphi^{q,p})$; in particular it only has the claimed components. Furthermore it obviously projects down to q in Fin_+ by the forgetful functor $U\colon \mathcal{C}^+ \to \mathrm{Fin}_+$, so it is a lift of T through U. Since the second component is always a symmetry we also trivially have a functor $F^{st}\colon A_+ \downarrow \mathrm{Fin}_+ \to (\mathcal{C}^{iso})^+$, hence an object of $\mathcal{C}(A_+, 1)$.

For the isomorphism first consider the following diagram:

$$\begin{tikzcd} A_+ \arrow[r, "\sigma_A"] \arrow[dr, "\chi_a"'] & {|A|_+.} \arrow[d, "\rho^a"] \\ & 1_+ \end{tikzcd}$$

By definition F sends ρ^a to a morphism (U-)over ρ^a, hence of the form (ρ^a, f_a) with $f_a\colon (\rho^a)_*F\sigma_A = F(\sigma_A)_a \to F(\chi_a)$ an isomorphism in $\mathcal{C}$. These assemble to an isomorphism $F\sigma_A \to (F\chi_a)_{a\in A}$ in $\mathcal{C}^{\times A}$.

We can uniquely write each $Fq\colon Fp \to F(qp)$ as

$$Fp \xrightarrow{(q,\mathrm{id})} q_*Fp \xrightarrow{(\mathrm{id},F^{\mathcal{C}}(q))} F(qp),$$

with $F^{\mathcal{C}}$ a morphism in a product category $\mathcal{C}^k$ for $q\colon A_+ \to k_+$.

In particular we get a canonical morphism:

$$Fp \xrightarrow{(F^{\mathcal{C}}(p\sigma_A^{-1}))^{-1}} p_*F\sigma_A \xrightarrow{(\mathrm{id},p_*((F\chi_a)_{a\in A}))} p_*F^{st}\sigma_A = F^{st}(p),$$

which assembles to a natural transformation $F \Rightarrow F^{st}$ whose components are isomorphisms by the assumption on F. So we have a canonical isomorphism for each functor to its strict representative, and the strict representative only depends on the restriction of F along j_A. $\square$

This immediately has the following corollary, which is useful for constructing such lifting functors more easily.

Corollary 1.1.53. *Each functor F lifting T through U is uniquely determined up to natural isomorphism by its restriction to $A_+ \downarrow \mathrm{Epi}_+$. In particular we can assume without loss of generality that for each $p\colon A_+ \to k_+$ with $k > |A|$ the object Fp is given by padding with zeroes from a bijection $F\sigma_A$, and the morphisms accordingly only have identities in zero-components.*

Remark 1.1.54. Restricting to epimorphisms is extremely convenient. Given any finite set A_+ we know a priori that the index-categories $A_+ \downarrow Epi_+$ are finite, i.e., have finitely many objects.

Since for finite A_+ there is always a non-unique maximal surjection, i.e. a bijection $A_+ \to |A|_+$, any other object of $A_+ \downarrow \mathrm{Fin}_+$ can be written relative to a chosen bijection. In particular for $n > |A|$ there is an injection $|A|_+ \to n_+$, which is unique, if we choose the injection strictly monotonous and with minimal maximal element.

Let me reemphasise the uniqueness clause of the strict representative to the canonical delooping statement:

Proposition 1.1.55. *For $(\mathcal{C}, +)$ a permutative category the delooping category $\mathcal{C}(A_+, 1)$ is naturally equivalent to a product category by restriction along j_A. I.e., we have an equivalence:*

$$\mathcal{C}(A_+, 1) \simeq \mathcal{C}^A.$$

Proof. The construction of the strict representative given above can also be used to promote each A-tuple to a lifting functor, which gives the inverse equivalence to restriction along j_A. The natural isomorphism on the left was given above, on $\mathcal{C}^A$ these functors strictly compose to the identity. $\square$

Remark 1.1.56. It is true, but inessential and uninstructive to prove, that the delooping category $\mathcal{C}(A_+, 1)$ is actually naturally **isomorphic** to the Segal construction on a permutative category $\mathcal{C}$ as defined in [EM, Construction 4.1, Theorem 4.2].

The isomorphism $\mathcal{C}(A_+, 1) \to \mathcal{C}^{Seg}$ is given by sending a lifting functor F to its restriction on characteristic functions for subsets $\chi_S\colon A_+ \to 1_+$. The additors $\rho_{S,T}$ are given by the maps associated to the factorisation in $A_+ \downarrow \mathrm{Fin}_+$:

$$\begin{tikzcd} A_+ \arrow[r, "\chi_{(S,T)}"] \arrow[dr, "\chi_{S\cup T}"'] & 2_+ \arrow[d] \\ & 1, \end{tikzcd}$$

for S, T disjoint subsets of A.

The associativity of the additors given in the diagrams of [EM, Construction 4.1] follows from the fact that the map $\chi_{(S,T,U)}\colon A_+ \to 3_+$ in particular has maps in $A_+ \downarrow \mathrm{Fin}_+$ to $\chi_{(S\cup T,U)}$ and $\chi_{(S,T\cup U)}$, which both map to $\chi_{S\cup T\cup U}$, giving a commutative square in $A_+ \downarrow \mathrm{Fin}_+$, thus one in $\mathcal{C}^+$ for each lifting functor.

1.1.5 The Construction $\mathcal{C}(A_+, n)$ for Permutative 1-Categories

I want to describe the delooping construction in [EM, Construction 4.4] in the same way that I just described the Segal construction $\mathcal{C}(A_+, 1)$. To that end I consider the Segal construction with n-fold products of finite sets and maps flattened into the additive Grothendieck construction $\mathcal{C}^+$, such that the case before is $n = 1$.

As in 1.1.8 fix a smash product functor on Fin_+. Then we can define a symmetric monoidal structure on $\mathcal{C}^+$ when given a bimonoidal structure on $\mathcal{C}$.

Proposition 1.1.57. *Consider the additive Grothendieck construction $\mathcal{C}^+$ for a bimonoidal category $(\mathcal{C}, +, \cdot)$, then we have an induced monoidal structure on $\mathcal{C}^+$, which makes the forgetful functor $U\colon \mathcal{C}^+ \to \mathrm{Fin}_+$ strictly monoidal with respect to the induced multiplication on $\mathcal{C}^+$ and the smash-product functor on Fin_+. If moreover the multiplication of $(\mathcal{C}, +, \cdot)$ makes $\mathcal{C}$ a bipermutative category, the induced monoidal structure is symmetric, and the functor U is strictly symmetric monoidal.*

Proof. Given a smash product functor on Fin_+ we have fixed pointed bijections $(n\times m)_+ = n_+ \wedge m_+ \to nm_+$, hence also $\omega_{n,m}\colon n\times m \to nm$, which are associative. For two objects $c = (c_1, \ldots, c_n), d = (d_1, \ldots, d_m)$ in $\mathcal{C}^+$ set their product to be:

$$c \boxtimes d = (c_i d_j)_{\omega(i,j)},$$

where I have written points in the indexing set $\{1, \ldots, nm\}$ as images $\omega(i, j)$.

The essential subtlety is the fact that this can be made into a functor. Consider the following two objects in $\mathcal{C}^+$:

$$(f \wedge g)_*(c \boxtimes \bar{c})_{\omega(i,j)} = \sum_{\omega(k,l) \in (f \times g)^{-1}(\omega(i,j))} c_k \bar{c}_l,$$

and analogously:

$$((f_* c) \boxtimes (g_* \bar{c}))_{\omega(i,j)} = (f_* c)_i \cdot (g_* \bar{c})_j = \left(\sum_{k \in f^{-1} i} c_k \right) \left(\sum_{l \in g^{-1} j} \bar{c}_l \right).$$

By 1.1.3 and [Lap] we find that there is a unique composite of distributors and additive symmetries comparing these objects. For instance by first reducing summands on the left, then on the right, we get:

$$(f \wedge g)_*(c \boxtimes \bar{c}) = \sum_{k,l} c_k \bar{c}_l \to \sum_k \left(c_k \left(\sum_l \bar{c}_l \right) \right)$$

$$\to \left(\sum_{k \in f^{-1} i} c_k \right) \left(\sum_{l \in g^{-1} j} \bar{c}_l \right) = f_* c \boxtimes g_* \bar{c}.$$

Hence there is a unique structural morphism $D^{f,g}$ determined by the summations f and g induce. Thus for two maps in $\mathcal{C}^+$ in the case of hom-sets with structure:

$$(f, (a_1, \dots, a_{m_1})) \colon c = (c_1, \dots, c_{n_1}) \to (d_1, \dots, d_{m_1}) = d,$$

$$(g, (b_1, \dots, b_{m_2})) \colon \bar{c} = (\bar{c}_1, \dots, \bar{c}_{n_2}) \to (\bar{d}_1, \dots, \bar{d}_{m_2}) = \bar{d},$$

we set their product to be the composite:

$$c \boxtimes \bar{c} = (c_i \bar{c}_j) \xrightarrow{(f \wedge g)_*} \left(\sum c_k \bar{c}_l\right) \xrightarrow{D^{f,g}} \left(\sum c_k\right) \left(\sum \bar{c}_l\right) \xrightarrow{a_i b_j} (d_i \bar{d}_j).$$

Analogously define the product for general enriched bimonoidal categories as follows: Consider $D^{f,g} \circ (f \wedge g)_*$ on the (f,g)-component of the morphisms on the product of the additive Grothendieck construction on $\mathcal{C}$, i.e., $(\mathcal{C}^+ \times \mathcal{C}^+)((c, \bar{c}), (d, \bar{d})) = \coprod_{(p,q)} \mathcal{C}^{|\bar{c}|}(p_* c, \bar{c}) \times \mathcal{C}^{|\bar{d}|}(q_* d, \bar{d})$, and postcompose with the monoidal product $\cdot$ of $\mathcal{C}$, which in its $\omega(i,j)$th component pairs the ith factor in the first product with the jth factor in the second product.

This assignment evidently sends identities to identities, and it respects composites, because $\cdot$ is part of a bimonoidal/bipermutative structure on $\mathcal{C}$, hence we can always interchange distributors $D^{f,g}$ and tuples of genuine $\mathcal{C}$-morphism $(a_i b_j)$ by summing up and reducing the appropriate components. In summary we obtain a functor:

$$\boxtimes\colon \mathcal{C}^+ \times \mathcal{C}^+ \to \mathcal{C}^+.$$

Since we have chosen $\wedge$ to be a strictly unital functor on Fin_+ and $\cdot$ strictly unital on $\mathcal{C}$, the 1-tuple $(1) \in \mathcal{C}^+$ with entry the multiplicative unit of $\mathcal{C}$ is a strict unit for $\boxtimes$. Since $\wedge$ is strictly associative and $\cdot$ is strictly associative, $\boxtimes$ is strictly associative as well.

Finally for $\cdot$ not just a monoidal, but a braided or symmetric monoidal structure with symmetry $c^{\cdot}$, consider the symmetry in Fin_+ for $\wedge$, and call it χ. Then a multiplicative symmetry for $\boxtimes$ on $\mathcal{C}^+$ is given by:

$$c \boxtimes d = (c_i d_j)_{\omega(i,j)} \xrightarrow{\chi} (c_i d_j)_{\omega(j,i)} \xrightarrow{c^{\cdot}} (d_j c_i)_{\omega(j,i)} = d \boxtimes c.$$

It squares to the identity if $c^{\cdot}$ does, and satisfies the braiding coherence diagrams for triple products that $c^{\cdot}$ satisfies. Hence yields a braided/symmetric monoidal structure, if $(C, \cdot)$ is braided/symmetric monoidal and bimonoidal as $(C, +, \cdot)$. We have evidently constructed the symmetric monoidal structure just so that U becomes a strictly (braided/symmetric) monoidal functor. $\square$

I do not intend to get back to this multiplicative structure until chapter 3, but wanted to explicitly state it for 1-categories. It emphasises that the multiplicative structure exhibited in chapter 3 can be built easily here as well.

To define the higher delooping categories $\mathcal{C}(A_+, n)$ we need to consider target functors for the product categories $(A_+ \downarrow \mathrm{Fin}_+)^n$.

Proposition 1.1.58. *There is a forgetful functor $T_n\colon (A_+ \downarrow \mathrm{Fin}_+)^n \to \mathrm{Fin}_+$ for each $n \geq 1$, which is faithful away from the basepoint, and can moreover be chosen to be associative, i.e., the diagram*

$$\begin{tikzcd}
(A_+ \downarrow \mathrm{Fin}_+)^n \times (A_+ \downarrow \mathrm{Fin}_+)^m \times (A_+ \downarrow \mathrm{Fin}_+)^l \arrow[r, "T_{n+m}\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times T_{m+l}"'] & \mathrm{Fin}_+ \times (A_+ \downarrow \mathrm{Fin}_+)^l \arrow[d, "\mathrm{id}\times T_l"] \\
(A_+ \downarrow \mathrm{Fin}_+)^n \times \mathrm{Fin}_+ \arrow[d, "T_n\times\mathrm{id}"'] & \mathrm{Fin}_+ \times \mathrm{Fin}_+ \arrow[d, "\wedge"] \\
\mathrm{Fin}_+ \times \mathrm{Fin}_+ \arrow[r, "\wedge"] & \mathrm{Fin}_+,
\end{tikzcd}$$

commutes. Moreover the functors T_n can be chosen symmetric, i.e., the functors

$$(A_+ \downarrow \mathrm{Fin}_+)^n \times (A_+ \downarrow \mathrm{Fin}_+)^m \xrightarrow{T_n \times T_m} \mathrm{Fin}_+ \times \mathrm{Fin}_+ \xrightarrow{\wedge} \mathrm{Fin}_+$$

and

$$(A_+ \downarrow \mathrm{Fin}_+)^{n+m} \xrightarrow{tw^{\times}_{n,m}} (A_+ \downarrow \mathrm{Fin}_+)^{m+n} \xrightarrow{T_m \times T_n} \mathrm{Fin}_+ \times \mathrm{Fin}_+ \xrightarrow{\wedge} \mathrm{Fin}_+$$

are naturally isomorphic by exchanging priority of the smash components, i.e., $\chi^{\wedge}$ *on* Fin_+.

Proof. Simply set T_n to be the following functor:

$$(A_+ \downarrow \mathrm{Fin}_+)^n \xrightarrow{(T)^n} \mathrm{Fin}_+^n \xrightarrow{\wedge} \mathrm{Fin}_+,$$

where T is the target functor of $A_+ \downarrow \mathrm{Fin}_+$, and $\wedge$ is the n-fold smash, which is defined because $\wedge$ is strictly associative. Then the T_n inherit associativity and symmetry as claimed, and are just as faithful as T and $\wedge$. Hence for maps with $f^{-1}+ = \{+\}$ we get injectivity on hom-sets. □

These functors should give the reader a reasonable hunch how I define $\mathcal{C}(A_+, n)$ such that $\mathcal{C}(A_+, 1)$ considered above trivially becomes the case $n = 1$.

Definition 1.1.59. The **higher delooping category** of a permutative category $\mathcal{C}(A_+, n)$ for $n \in \mathbb{N}$ and A_+ a finite pointed set, is given as the category of functors lifting T_n through U, i.e., the dashed arrows in the diagram:

$$\begin{tikzcd} & (\mathcal{C}^{iso})^+ \arrow[d, "U"] \\ (A_+ \downarrow \mathrm{Fin}_+)^n \arrow[r, "T_n"] \arrow[ur, dashed] & \mathrm{Fin}_+. \end{tikzcd}$$

Its morphisms are the natural transformations of functors pushed forward with the inclusion $(\mathcal{C}^{iso})^+ \to \mathcal{C}$, i.e., natural transformations with arbitrary components, not just isomorphisms.

Example 1.1.60. Consider again (1.1.33) the prototypical surjections $f, g\colon 4_+ \to 2_+$, with $f(1) = f(3) = 1$,$f(2) = f(4) = 2$, $g(1) = g(2) = 1$,$g(3) = g(4) = 2$,and $q\colon 2_+ \to 1_+$ again. Choosing the bijection ω for the smash product gives the following commutative cube:

$$\begin{tikzcd}
(2\times 2)_+ \arrow[rr, "id\times q"] \arrow[dr, "\omega"] \arrow[dd, "q\times id"'] & & (2\times 1)_+ \arrow[dr, "\omega"] \arrow[dd, "q\times id"'] & \\
& 4_+ \arrow[rr, "f"] \arrow[dd, "g"'] & & 2_+ \arrow[dd, "q"] \\
(1\times 2)_+ \arrow[rr, "id\times q"] \arrow[dr, "\omega"'] & & (1\times 1)_+ \arrow[dr, "\omega"] & \\
& 2_+ \arrow[rr, "q"'] & & 1_+ .
\end{tikzcd}$$

Flattening this cube makes a pentagon with the additional arrow given by $1 + c^+ + 1$, which appears for instance in axiom (5) of [EM, Construction 4.4].

Proposition 1.1.61. *Each permutation $\sigma \in \Sigma_n$ induces a functor*

$$\sigma\colon (A_+ \downarrow \mathrm{Fin}_+)^{\times n} \to (A_+ \downarrow \mathrm{Fin}_+)^{\times n}.$$

Precomposing with this permutation of the components and post-composing a functor with the symmetry χ of the smash-product on Fin_+*, induces a natural Σ_n-action on $\mathcal{C}(A_+, n)$.*

Proof. The statement concerning the symmetry warrants some explanation. By 1.1.58 we know that we can choose the target functors T_n as $(A_+ \downarrow \mathrm{Fin}_+)^n \to \mathrm{Fin}_+^n \to \mathrm{Fin}_+$, hence the symmetry isomorphism from T_n to $\sigma^* T_n$, for $\sigma \in \Sigma_n$ can be pushed forward to Fin_+ by the appropriate symmetry $\chi^{\wedge}_{\sigma}$ of the smash-product on Fin_+. Then by pushing forward the permuted functor in $\mathcal{C}^+$ with the same symmetry we get a functor lifting T_n again. □

Remark 1.1.62. To state the following proposition conveniently I introduce a standard simplifying assumption. For $\mathcal{C}$ a permutative category we can without loss of generality assume that $0 \in \mathcal{C}$ is an isolated object, i.e., it has at most non-trivial endomorphisms, but no maps in $\mathcal{C}$ from or to different objects. This makes $\mathcal{C} \setminus \{0\} \sqcup \{0\}$ a decomposition of $\mathcal{C}$ by full subcategories.

For $\mathcal{C}$ with isolated zero 0 as above we can define the smash product of $\mathcal{C}$ with a finite pointed set $A_+ = A \sqcup \{*\}$ as:

$$A_+ \wedge \mathcal{C} := End(0) \coprod_{x\in A} (\mathcal{C} \setminus \{0\}).$$

The assumption is without loss of generality since we can attach to each category $\mathcal{C}$ a disjoint basepoint $\mathcal{C}_+ := \mathcal{C} \sqcup \{*\}$. For $\mathcal{C}$ permutative we extend the monoidal functor by letting $*$ act as a strict unit, thus in particular the object $0 \in \mathcal{C}$ is not neutral in $\mathcal{C}_+$. On classifying spaces we get $|N\mathcal{C}_+| = |N\mathcal{C}| \sqcup \{*\}$, i.e., we only added a disjoint basepoint to the classifying space as well.

Finally note that for $\mathcal{C}$ permutative with an isolated additive unit 0, any finite pointed set A_+ and any natural number n, the higher delooping categories have an isolated zero as well: The category $\mathcal{C}(A_+, n)$ has a basepoint given by the functor

$$O\colon (A_+ \downarrow Epi_+)^{\times n} \to \mathcal{C}^+$$

with $O((f_1, \ldots, f_n)\colon (A_+, \ldots, A_+) \to (k^1_+, \ldots, k^n_+)) = 0$ the $k^1 \cdot \ldots \cdot k^n$-tuple consisting only of the additive unit, and each morphism is sent to its appropriate discrete component with $\mathcal{C}^{\times \bullet}$-components only identities. If the additive unit in $\mathcal{C}$ is isolated, then this zero functor O is an isolated basepoint of $\mathcal{C}(A_+, n)$ as well, and we can identify the smash as:

$$A_+ \wedge \mathcal{C}(A_+, n) = End(O) \sqcup \coprod_{x \in A} (\mathcal{C}(A_+, n) \setminus \{O\}) \times \{x\}.$$

Take note that the disjoint union is over all non-basepoints in A_+, hence all elements of A. The fact that O is isolated ensures that each $\mathcal{C}(A_+, n) \setminus \{O\}$ forms a (sub)category.

The extension functors are a bit obscured by the fact that I chose to reduce the arguments in the delooping-construction of [EM] to n equal inputs A_+ only, but it coalesces nicely.

Proposition 1.1.63. *We have a natural inclusion of categories:*

$$e\colon A_+ \wedge \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, 1 + n).$$

Furthermore this inclusion is Σ_n-equivariant, where on $\mathcal{C}(A_+, 1+n)$ the action is given by restriction along the inclusion $\Sigma_n = \Sigma_1 \times \Sigma_n \to \Sigma_{1+n}$, i.e., letting n-permutations act on the indices $\{2, \ldots, n+1\}$.

Proof. As above we see if the additive unit in $\mathcal{C}$ is isolated, then the zero functor O is an isolated basepoint of $\mathcal{C}(A_+, n)$, and we can identify the smash as:

$$A_+ \wedge \mathcal{C}(A_+, n) = End(O) \sqcup \coprod_{x \in A} (\mathcal{C}(A_+, n) \setminus \{O\}) \times \{x\}.$$

Hence I can describe the extension functor on each component seperately. We set $e(O) = O$ and send an endomorphism of $O \in \mathcal{C}(A_+, n)$ to the endomorphism of $O \in \mathcal{C}(A_+, 1+n)$, which we obtain by appropriately extending with identities.

More interestingly consider a summand $\mathcal{C}(A_+, n) \setminus \{0\} \times \{x\}$. Then we have to define $e(F, x)(p_1, \ldots, p_{n+1})$ for each $(n+1)$-tuple of maps $p_i \colon A_+ \to k_i$. We set:

$$e(F, x)(p_1, \ldots, p_{n+1}) = \begin{cases} 0 & p_1 \neq (\rho^x \colon A_+ \to 1_+), \\ F(p_2, \ldots, p_{n+1}) & p_1 = \rho^x. \end{cases}$$

Accordingly, $e(F, x)$ is the identity on the additive unit for all $(n+1)$-tuples of morphisms, which do not have $id_{1_+} = id_{\rho^x}$ as its first component. On the tuples, where the last component is id_{1_+} and target and source have last component ρ^x we can use F on the first n maps.

This is obviously a functor, natural in $\mathcal{C}$ and A_+. Equivariance follows by our choice of singling out the first component. Changing the marked component changes the inclusion $\Sigma_n \to \Sigma_{1+n}$ but still yields equivariance as claimed. □

Again we have the result making $\mathcal{C}(A_+, n)$ a delooping of $\mathcal{C}$.

Theorem 1.1.64. *For $(\mathcal{C}, +)$ a permutative category we have a natural equivalence of categories*

$$\mathcal{C}(A_+, n) \simeq Set(A, \mathcal{C}(A_+, n-1)),$$

with the map $\mathcal{C}(A_+, n) \to Set(A, \mathcal{C}(A_+, n-1))$ given by restriction of functors along $(\mathrm{id}_{n-1}, j_A) \colon (A_+ \downarrow \mathrm{Fin}_+)^{n-1} \times A^\delta \to (A_+ \downarrow \mathrm{Fin}_+)^n$. Inductively we get a natural equivalence:

$$\mathcal{C}(A_+, n) \simeq \mathcal{C}^{A^{\times n}}.$$

Proof. The start of the induction is the case $n = 1$ displayed above. The same argument with *strict representatives* can be made to prove the equivalence above by making a functor in $\mathcal{C}(A_+, n)$ only consist of discrete components and additive symmetries one component at a time. □

I give the construction of the Eilenberg-Mac Lane spectrum based on this delooping in chapter 3 in the maximal generality I need it. The case of permutative 1-categories then follows by considering them as permutative bicategories with discrete morphism categories. The maximal generality in this thesis is motivated by the principal example of interest $K(ku)$. The next chapter is thus concerned with the module bicategory of a bimonoidal category. For the spectrum ku I want to fix the relevant models and maps next.

1.2 Models for ku

Assuming that the delooping given by $\mathcal{C}(A_+, n)$ yields an E_∞ symmetric ring spectrum $H\mathcal{C}$, which I prove in chapter 3, we get models for connective K-theory with an E_∞-multiplication by considering nicely explicit bipermutative categories.

Example 1.2.1. Given a commutative ring k, consider its (skeletal) category of finitely generated free modules $\mathcal{M}_k$ on objects:

$$\mathrm{Ob}\mathcal{M}_k = \mathrm{Ob\ Fin} = \{\mathbf{n} | n \in \mathbb{N}\}.$$

Consider the unpointed sets $\mathbf{n}$ as ranks of finitely generated free modules over k. To establish its bipermutative structure first consider the morphism sets in a bigger category $\mathcal{M}^L$ (compare 1.1.7):

$$\mathcal{M}^L_k(\mathbf{n}, \mathbf{m}) := Hom_k(k\{1, \dots, n\}, k\{1, \dots, m\})$$

of all k-linear maps of free modules on the unpointed sets $\mathbf{n}, \mathbf{m}$.

Fixing the direct sum functor as the linear extension of disjoint union gives a strictly associative coproduct-functor for $\mathcal{M}^L$:

$$k\{1, \dots, n\} \oplus k\{1, \dots, m\} := k\{\mathbf{n} + \mathbf{m}\} = k\{1, \dots, n, n+1, \dots, n+m\},$$

with the obvious extension to morphisms by linearly extending the description of Fin on basis elements.

The product functor chosen on finite sets (specifically by fixing associative bijections $\omega = \omega_{n,m} \colon \mathbf{n} \times \mathbf{m} \to \mathbf{nm}$) extends to the tensor-product of free modules:

$$k\{1, \dots, n\} \otimes k\{1, \dots, m\} := k\{1, \dots, nm\},$$

where we define the tensor-product of linear maps represented as quadratic matrices $f \in M_n(k), g \in M_m(k)$ as follows:

$$(f \otimes g)(e_{\omega(i,j)}) := \sum_{\omega(s,t) \in nm} f_{si} g_{tj} e_{\omega(s,t)}.$$

For the entries of the representing matrix for $f \otimes g$ with respect to the ordering on $\mathbf{nm}$ fixed by $\omega \colon \mathbf{n} \times \mathbf{m} \to \mathbf{nm}$ we get:

$$(f \otimes g)_{\omega(i_1,j_1),\omega(i_2,j_2)} = f_{i_1,i_2} \cdot g_{j_1,j_2},$$

which is a strictly associative representation of the tensor-product. We know it is left-adjoint to the Hom_k-functor, hence Lemma 1.1.6 applies, and we get a bipermutative structure on the $\mathcal{M}_k^L$.

Analogous to finite sets we can restrict to injections, surjections and isomorphisms. The case of isomorphisms is the one of primary interest in this thesis, so I define *the* module category $\mathcal{M}_k$ of a commutative ring as:

$$\mathcal{M}_k(n,m) = \begin{cases} GL_n(k) & n = m \\ \emptyset & n \neq m. \end{cases}$$

For $k = \mathbb{R}, \mathbb{C}$ we have a topological version of the above example.

Example 1.2.2. The same constructions as in the example above describe continuous functors with respect to the topologies on $GL_n\mathbb{R}$ and $GL_n\mathbb{C}$ as subspaces of $M_n\mathbb{R} \cong \mathbb{R}^{n^2}$ and $M_n\mathbb{C} \cong \mathbb{C}^{n^2}$. Call the category with objects the natural numbers and morphism spaces GL_nk considered as a topologically enriched category $\mathcal{M}_k^c$, where the upper index is a reminder for continuity. Call the analogous discrete category with morphism sets GL_nk and their discrete topology $\mathcal{M}_k^\delta$.

For real and complex coefficients we can restrict to the respective compact subgroups:

$$O_n \to GL_n(\mathbb{R}), \text{ and } U_n \to GL_n(\mathbb{C}).$$

These inclusions define subcategories of the topological as well as the discrete module categories. Denote the topological subcategories by $\mathcal{V}_k^c \subset \mathcal{M}_k^c$ and analogously the discrete subcategories by $\mathcal{V}_k^\delta \subset \mathcal{M}_k^\delta$. Since the symmetries and distributors are unitary morphisms as well, the canonical inclusion functors are bipermutative:

$$\mathcal{V}_\mathbb{R} \to \mathcal{M}_\mathbb{R},$$

$$\mathcal{V}_\mathbb{C} \to \mathcal{M}_\mathbb{C},$$

for the topological as well as the discrete versions.

A continuous inverse is given by the Gram-Schmidt process, which I denote by $r\colon \mathcal{M}_k \to \mathcal{V}_k$. This is compatible with direct sum by considering the sum as orthogonal. It can be promoted to a homeomorphism as follows (cf. [MT, pp.33-35]): We have a natural map

$$g_n\colon O_nk \times H_n^+k \to GL_nk$$

with $g_n(U,B) = UB$ for O_nk the orthogonal group for $k = \mathbb{R}$ and the unitary group for $k = \mathbb{C}$, and H_n^+k the space of symmetric/hermitian positive definite matrices. For each n the map g_n is a homeomorphism with inverse:

$$h_n \colon GL_nk \to O_nk \times H_n^+k$$

given by:

$$h_n(A) = (A\sqrt{(A^*A)}^{-1}, \sqrt{(A^*A)}).$$

The map g_n is compatible with direct sums. For tensor-products consider the following diagram:

$$\begin{tikzcd}
O_n \times H_n^+ \times O_m \times H_m^+ \arrow[r, "g_n \times g_m"] \arrow[d] & GL_n \times GL_m \arrow[dd, "\otimes"] \\
O_n \times O_m \times H_n^+ \times H_m^+ \arrow[d, "\otimes\times\otimes"'] & \\
O_{nm} \times H_{nm}^+ \arrow[r, "g_{nm}"'] & GL_{nm},
\end{tikzcd}$$

which needs to commute for g to induce a strictly multiplicative functor $\mathcal{V}_k \to \mathcal{M}_k$. Fixing associative bijections ω write:$(A \otimes B)_{ij} = A_{i_1j_1} \cdot B_{i_2j_2}$.

Then we have:

$$\begin{aligned}
(g_n(U,B) \otimes g_m(V,C))_{ij} &= g_n(U,B)_{i_1j_1} g_m(V,C)_{i_2j_2} \\
&= (UB)_{i_1j_1}(VC)_{i_2j_2} = \sum_{k,l} U_{i_1k}B_{kj_1}V_{i_2l}C_{lj_2},
\end{aligned}$$

while the other side is given by:

$$\begin{aligned}
g_{nm}(U \otimes V, B \otimes C)_{ij} &= ((U \otimes V)(B \otimes C))_{ij} \\
&= \sum_p (U \otimes V)_{ip}(B \otimes C)_{pj} = \sum_p U_{i_1p_1}V_{i_2p_2}B_{p_1j_1}C_{p_2j_2}.
\end{aligned}$$

The terms thus agree by commutativity of addition and multiplication in k, so we get a bipermutative inclusion $I \colon \mathcal{V}_k \to \mathcal{M}_k$ as well as a bipermutative retraction $R \colon \mathcal{M}_k \to \mathcal{V}_k$ given by g. Specifically we get topologically enriched functors $R \colon \mathcal{M}_k^c \to \mathcal{V}_k^c$, i.e., functors, which are continuous on the morphism spaces, and the same assignments define functors on the discrete versions $R \colon \mathcal{M}_k^\delta \to \mathcal{V}_k^\delta$.

Remark 1.2.3. The tensor-product structure on categories of the form $\mathcal{M}_k$ has a more natural interpretation: Choose an euclidean/hermitian scalar product in each dimension n for $k = \mathbb{R}, \mathbb{C}$ or any non-degenerate bilinear form for an arbitrary field k - say $\langle \cdot, \cdot \rangle$. By basic linear algebra we know that any bilinear form b yields a uniquely determined linear map f such that $\langle f\cdot, \cdot \rangle = b(\cdot, \cdot)$. If moreover the bilinear form b is non-degenerate as well, then f is an isomorphism. Hence fixing (any) non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ we get an isomorphism:

$$Bil_+(V) \cong GL(V)$$

of non-degenerate bilinear-forms on V and linear automorphisms of V, which is a homeomorphism when meaningful.

In light of this consider the following tensor product of bilinear forms: For two bilinear forms $b^V \colon V \otimes V \to k$ and $b^W \colon W \otimes W \to k$, define their tensor product $b^{V \otimes W} \colon (V \otimes W)^{\otimes 2} \to k$ on generators:

$$b^{V \otimes W}(v_1 \otimes w_1, v_2 \otimes w_2) = b^V(v_1, v_2) \cdot b^W(w_1, w_2),$$

and extend bilinearly. This is non-degenerate if both forms above are non-degenerate. It is symmetric/hermitian if both forms are symmetric/hermitian.

On representing matrices we get the following:

$$b^{V \otimes W}(e_i \otimes e_j, e_k \otimes e_l) = b^V(e_i, e_k) b^W(e_j, e_l) = b^V_{ik} b^W_{jl},$$

hence precisely the coefficients given in 1.2.1 for the tensor-product of the linear maps associated to the representing matrices.

Finally I summarise the canonical functors between the examples:

Example 1.2.4. For each commutative ring k we have a canonical inclusion functor

$$k[\cdot] \colon \mathrm{Fin} \to \mathcal{M}^L_k$$

given by sending each finite set to its associated free module, and each map to its linear extension. This is a strictly additive as well as multiplicative functor, moreover it strictly respects the symmetries and distributors. We can obviously restrict to Inj, Epi, Σ and restrict the codomain to the appropriate linear maps, i.e., monomorphisms, epimorphisms, or isomorphisms.

We also have the analogous canonical inclusion functor given by reduced free modules:

$$\tilde{k}[\cdot] \colon \mathrm{Fin}_+ \to \mathcal{M}^L_k$$

sending each finite set to the associated free module with the basepoint divided out. It is again strictly symmetric monoidal with respect to pointed sum and smash product, and respects the distributivity transformations as well.

Finally for $k = \mathbb{R}, \mathbb{C}$ these functors even have image in the categories $\mathcal{V}_k$, since the described maps are obviously orthogonal/unitary.

Definition 1.2.5. Since we want to model connective complex K-theory we consider $k = \mathbb{C}$ in the example above, and find that $\mathcal{V}_{\mathbb{C}}^{c}$ also becomes a topological bipermutative category in that case. The delooping of $\mathcal{V}_{\mathbb{C}}^{c}$ is the prototypical model for connective complex K-theory. In particular I denote by $ku := H\mathcal{V}_{\mathbb{C}}^{c}$ the delooping spectrum of the topological category of finitely generated complex vector spaces with morphism spaces the unitary isomorphisms.

Preliminaries on Discrete Models for ku

For each odd prime p there are in addition "discrete models" for ku, which are a suitable replacement when studying its $H\mathbb{F}_p$-homology.

Fix a prime p for which we want an $H\mathbb{F}_p$-approximation of ku, i.e., a spectrum E with a map $E \to ku$, which induces an isomorphism on $H\mathbb{F}_p$-homology. Following Quillen [Q1] we want to approximate ku by algebraic K-theory of the algebraic closure $\bar{\mathbb{F}}_p$ of the finite field with p elements. In fact it is sufficient to restrict to a subfield of $\bar{\mathbb{F}}_p$ which contains the appropriate roots of unity, which we construct here.

For that we need to choose a prime that is a generator of $(\mathbb{Z}/p^2)^\times$. Observe that necessarily the existence of just one generator implies that p is odd, because for $p = 2$ we have $(\mathbb{Z}/2^k)^\times \cong \{\pm 1\} \times \mathbb{Z}/2^{k-2}\mathbb{Z}$, where for $k \geq 3$ the second factor is always generated by the powers of 5 [Gauß, Art. 91, p. 89 - Latin edn.]. For p odd the group of units has a decomposition $(\mathbb{Z}/p^k)^\times \cong \mathbb{Z}/(p-1) \times \mathbb{Z}/p^{k-1}$, and is thus a product of two cyclic groups of coprime order [Gauß, Art. 84, p. 82 - Latin edn.]. Given an integer g reducing to a multiplicative generator of the units of $\mathbb{Z}/p$ we know by Fermat's little theorem $g^{p-1} = 1 \bmod p$, so $g^{p-1} = 1 + lp$ for some $l \in \mathbb{Z}$. Then we have in $\mathbb{Z}/p^2$:

$$(g+p)^{p-1} = g^{p-1} + (p-1)g^{p-2}p \bmod p^2 \neq g^{p-1} \bmod p^2,$$

so at least one of the integers $\{g, g+p\}$ satisfies

$$g^{p-1} \neq 1 \bmod p^2 \quad \text{or} \quad (g+p)^{p-1} \neq 1 \bmod p^2.$$

An integer of $\{g, g+p\}$ satisfying the above inequality (multiplicatively) generates the units of $\mathbb{Z}/p^k$ for each $k \geq 2$, in particular it generates the units of the p-adic integers $\mathbb{Z}_p$ topologically.

We use Dirichlet's Theorem on arithmetic progressions in the following form:

Theorem 1.2.6 (Dirichlet). *For a natural number $n \geq 2$ and a unit $a \in (\mathbb{Z}/n)^\times$ consider the class of primes $P_a = \{\ p \in \mathbb{N} \mid p \text{ prime and } \ p = a \bmod n\}$. Then each class P_a has "logarithmic density" $\frac{1}{\varphi(n)}$ in the set of all primes, for $\varphi(n)$ the number of units in $\mathbb{Z}/n$.*

Remark 1.2.7. I do not need the concept of logarithmic density again, so I only give a vague description: The intuition is that it is an adapted way to measure subsets of countable sets (such as the set of all prime numbers), such that the measure is 0 for finite subsets.

One proof of the theorem by complex analysis involves the Dirichlet L-series associated to a homomorphism $(\mathbb{Z}/n)^\times \to \mathbb{C}^\times$. For the trivial homomorphism which sends everything to $1 \in \mathbb{C}$ the L-series has a singularity in 1. This forces the L-series of every non-trivial character to be bounded, but non-zero, in 1. This gives the following comparison of divergence around $s = 1$:

$$\sum_{p \equiv a \bmod n} p^{-s} = \frac{1}{\varphi(a)} \log \frac{1}{s-1} \pm C.$$

In words: The sum over all primes, which are in P_a, taken with the exponent $-s$ diverges like $\log \frac{1}{s-1}$ in 1 (up to a constant $C \in \mathbb{R}$). In particular there are infinitely many such primes.

We can use the theorem in particular to specialise to a generator $a \in \mathbb{Z}/p^2$, and find a prime q with $q = a \bmod p^2$, which generates the units of each $\mathbb{Z}/p^k$ by the considerations before.

Example 1.2.8. I want to exhibit valid choices for all primes below 100. I organised the table by smallest multiplicative generator q for $\mathbb{Z}/p^2$:

q	p
2	$3, 5, 11, 13, 19, 29, 37, 53, 59, 61, 67, 83,$
3	$7, 17, 31, 43, 79, 89,$
5	$23, 47, 73, 97,$
7	$41, 71.$

We want to approximate ku by algebraic K-theory applied to a suitable tower of field extensions. Start with the prime field with q elements $\mathbb{F}_q$. Since q generates the units of $\mathbb{Z}/p^2$ it generates $\mathbb{Z}/p^\times$ as well. So the cyclotomic polynomial of degree $p-1$

$$\varphi_p(X) = \sum_{i=0}^{p-1} X^i$$

is irreducible over $\mathbb{F}_q$. Thus we have the extension of fields

$$\ell_0 := \mathbb{F}_q \to \mathbb{F}_q[X]/\varphi_p \cong \mathbb{F}_{q^{p-1}} = \ell_0(\zeta_p) =: k_0$$

for ζ_p some chosen primitive pth root of unity. Since q and p are trivially coprime by the assumptions, each element in $\mathbb{F}_q$ has a pth root in ℓ_0. However, the units of k_0 have order $q^{p-1}-1$. Because q is a unit in $\mathbb{Z}/p$, we get

$$q^{p-1} - 1 = 0 \bmod p,$$

so taking the pth power is not injective, hence not surjective. So there is an element $a \in k_0$, which does not have a pth root. There is an obvious candidate: ζ_p. Because of our assumption on q and p we have $q^{p-1}-1 = 0 \bmod p$, but $q^{p-1} - 1 \neq 0 \bmod p^2$, because $p-1$ is strictly smaller than the order of the units of $\mathbb{Z}/p^2$, so the units of $\ell_0(\zeta_p)$ decompose as:

$$\ell_0(\zeta_p)^\times \cong \mathbb{Z}/p\langle\zeta_p\rangle \times \mathbb{Z}/s,$$

for some s coprime to p. In particular we find that the kernel of $(\cdot)^p$ is contained in the $\mathbb{Z}/p$-summand, while the image is contained in the $\mathbb{Z}/s$-factor, hence ζ_p does not have a pth root in $k_0 = \ell_0(\zeta_p)$. Since p is odd, we get that $f_m(X) = X^{p^m} - \zeta_p$ is irreducible for each $m \geq 1$ if and only if ζ_p has no pth root in k_0, which we just established. So inductively call $\alpha_i = \sqrt[p]{\alpha_{i-1}}$ with $\alpha_0 = \zeta_p$. More explicitly for each $i \geq 1$ we choose a primitive p^ith root of ζ_p, and call it α_i.

For exposition let me choose a presentation. We can write k_i for $i \geq 1$ as:

$$\mathbb{F}_q[X, Y_i]/\left(Y_i^{p^i} - X, \sum_{k=0}^{p-1} X^k\right) \cong \mathbb{F}_q[Y_i]/\left(\sum_{k=0}^{p-1} Y^{p^i k} = \varphi_p(Y_i^{p^i})\right).$$

Then the field $\ell_i \subset k_i$ is given as the fixed-set under the Galois-action of the factor $\mathbb{Z}/p-1$, which stems from the cyclotomic extension.

In the fields k_i we trivially have the inclusions $k_i \to k_{i+1}$ with $Y_i \mapsto Y_{i+1}^p$, i.e., identifying Y_{i+1} as a pth root of Y_i. This yields the following diagram

$$\begin{tikzcd}
k_0 \arrow[r] & k_1 \arrow[r] & \dots \arrow[r] & k_i \arrow[r] & \dots \\
\ell_0 \arrow[r] \arrow[u] & \ell_1 \arrow[r] \arrow[u] & \dots \arrow[r] & \ell_i \arrow[r] \arrow[u] & \dots,
\end{tikzcd}$$

where the horizontal arrows signify $\mathbb{Z}/p$-Galois-extensions, while the vertical arrows are $\mathbb{Z}/(p-1)$-Galois-extensions. Hence on colimits $K = \operatorname{colim}_i k_i$ and $L = \operatorname{colim}_i \ell_i$ we get the $\mathbb{Z}/(p-1)$-extension: $L \to K$, which by the presentations given above is of the form

$$L \to K = L[X]/\varphi_p = L(\zeta_p).$$

Remark 1.2.9. Let me emphasise that the prime p defining the degree of the extension with Galois group $\mathbb{Z}/p-1$ is structurally important, while q only serves to ensure the existence of this extension and its particular choice is irrelevant to the construction.

Example 1.2.10. Building on the previous example the extension of L by a primitive pth root of unity ζ_p, as above $L \to L(\zeta_p)$, induces a map of bipermutative categories $\mathcal{V}_L \to \mathcal{V}_{L(\zeta_p)}$. Hence the delooping of these bipermutative categories provides a map

$$H(\mathcal{V}_L) \to H(\mathcal{V}_{L(\zeta_p)}),$$

which is a map of E_∞ symmetric ring spectra. Again referring to chapter 3 we see that these spectra are models for the algebraic K-theory of their respective fields and hence we understand this map as:

$$K(L) \to K(L(\zeta_p)).$$

Consider the Galois group of the extension $L \to L(\zeta_p)$: $G = Gal(L(\zeta_p)/L)$. For any homology theory h_* with $p-1 = |G| = |\mathbb{Z}/p-1|$ a unit in its coefficients the map $K(L) \to K(L(\zeta_p))$ induces an isomorphism on h-homology groups:

$$h_*K(L) = (h_*K(L(\zeta_p)))^G.$$

In particular for $h = H\mathbb{F}_p$ we get an equivalence of p-completed spectra:

$$K(L)^\wedge_p \simeq (K(L(\zeta_p))^\wedge_p)^{hG}.$$

Comparison of the Models

One essential insight that led Quillen to the definition of algebraic K-theory [Q1, Q2, Q3] was the fact that he could compute the full Algebraic K-theory of all finite fields by comparison to fibres of Adams operations on BU, so in essence by comparison to ku. I want to exhibit the map involved, which is established by the Brauer lift, but I shall defer the proofs to the relevant sources.

Again by the construction in 1.2 we can easily fix a homomorphism:

$$\mu\colon L(\zeta_p)^\times \subset \langle \zeta_p \rangle \times \bigoplus_{l\ \text{prime} \neq p} \mathbb{Z}/l^\infty \to \mathbb{C}^\times$$

by setting $\mu(\zeta_p) = \exp(\frac{2\pi i}{p})$. For a summand indexed by a prime l choose a primitive lth root of unity in $\mathbb{C}^\times$ as $\exp(\frac{2\pi i}{l})$ as well as the primitive l^jth roots of unity $\exp(\frac{2\pi i}{l^j}) = \zeta_{l,j}$. This yields coherent homomorphisms for each l

$$\mu_l\colon \mathbb{Z}/l^j \to \mathbb{C}^\times, \quad \text{hence on the colimit} \quad \mu_l\colon \mathbb{Z}/l^\infty \to \mathbb{C}^\times.$$

With these choices fixed we can use the following theorem (cf. [Ros, p. 283, Theorem 5.3.4]):

Theorem 1.2.11. *Let G be a finite group, and let $\rho\colon G \to GL_n\overline{\mathbb{F}}_q$ be a finite-dimensional representation of G over the algebraic closure of $\mathbb{F}_q$. Let $\{\xi_g^i \mid i = 1,\dots,n\}$ be the eigenvalues of ρ_g with multiplicities, so that the trace of ρ_g is given as: $tr(\rho_g) = \sum_i \xi_g^i$.*

The function $f^\rho\colon G \to \mathbb{C}$ with $f^\rho(g) = \sum_i \mu(\xi_g)$ is a class-function, hence by basic complex representation theory (cf. [Se, Part I, Chapters 1-3]) a linear combination of characters of complex G-representations. Call f^ρ the Brauer character of ρ.

In fact we have integral coefficients, so f uniquely determines a complex virtual representation of G, called the Brauer lift *of ρ - denote it $F(\rho)$. Furthermore the Brauer lift is additive, i.e., for a short exact sequence of $\overline{\mathbb{F}}_q[G]$-modules:*

$$0 \to U \to V \to W \to 0$$

the lifts satisfy $F(V) = F(U) + F(W)$.

Remark 1.2.12. Observe that the Brauer lift consists of representations of G that have eigenvalues on the circle $\mathbb{S}^1 \subset \mathbb{C}$, because the group has finite order.

I want to exhibit the idea of how this induces a comparison map, but I gloss over quite a few details, which are in part explained in [Ros, pp. 284–285] and much more in the original [Q1, Q2].

We want to consider the homomorphism $i_n\colon GL_n(L(\zeta_p)) \to GL_n(\overline{\mathbb{F}}_q)$. Since $L(\zeta_p)$ is a colimit of finite fields the general linear groups $GL_n(l_i)$ are finite groups, which yield $GL_n(L(\zeta_p))$ as their colimit. The Brauer Lift is evidently stable in the colimit over the fields l_i, since the eigenvalues of the matrices do not change.

We can determine the virtual dimension of the Brauer Lift by the trace of the identity: We get $\xi^i(id_n) = 1$ for $1 \leq i \leq n$, so $\sum_i \mu(\xi^i) = \sum_i \mu(1) = \sum_i 1 = n$. This is obviously not stable in n; thus subtract the trivial $GL_n(\mathbb{C})$ representation of $GL_n l_i$, and consider the lift of i_n minus n. Obviously the trivial complex representation of $GL_n l_i$ of dimension n is a lift for the trivial $GL_n l_i$-representation over $\overline{\mathbb{F}}_q$, so we get $F(i_n - n) = F(i_n) - n$ and hence a stable class of a virtual representation of dimension zero giving a map

$$BGL(L(\zeta_p)) \to BGL^\delta(\mathbb{C}) \to BGL(\mathbb{C}) \simeq BU.$$

This map induces homology isomorphisms with $\mathbb{F}_m$-coefficients for any prime m other than q, thus induces an equivalence of completed spaces at each prime $m \neq q$:

$$BGL(L(\zeta_p))^\wedge_m \to BU^\wedge_m,$$

given as theorems 1.6 and 4.7 by Quillen in [Q1].

If the Brauer Lift happened to be not just a virtual representation but indeed a genuine homomorphism $\Phi\colon GL(L(\zeta_p)) \to GL(\mathbb{C})$ we could try to restrict to the GL_n again, and induce a map of bipermutative categories $\mathcal{V}_{L(\zeta_p)} \to \mathcal{V}_\mathbb{C}$, which would give an infinite loop map, i.e., a map of spectra $K(L(\zeta_p)) \to K(\mathbb{C}) = ku$, and furthermore of E_∞-ring spectra. But calculating in low dimensions for $GL_2(\mathbb{F}_q)$ shows that the Brauer Lift has a genuine negative component. I suspect this approach could be repaired with a “ring-complete” version of $\mathcal{V}_\mathbb{C}$ as given by [BDRR2], but the result has been established long before that by other methods.

The additivity directly yields that the Brauer Lift is an E_∞-map with respect to the E_∞-structure on $BGL(L(\zeta_p))^+$ and BU^+ induced by direct sum of matrices. Furthermore in “E_∞ Ring Spaces and E_∞ Ring Spectra” May shows [MayE_∞, pp. 212-222] that the induced map is also an E_∞-map for the E_∞-structure induced by tensor-products. With all this in place we have an equivalence of E_∞ ring spectra at p:

$$K(L(\zeta_p))^\wedge_p \to K(\mathbb{C}) = ku^\wedge_p.$$

The equivalence of spectra is given in [MayE_∞, pp. 217+218, Corollary VIII.2.7, Theorem VIII.2.8], the compatibility with the E_∞ structures on these spectra is [MayE_∞, pp. 219-222, Theorem VIII.2.11].

The Involutions

In this thesis I want to investigate the induced involution on the algebraic K-theory of ku as well, so as the final comment on the models I establish which involution is induced on $K(l(\zeta_p))$ by the Brauer lift.

Proposition 1.2.13. *For any multiplicative embedding $\mu\colon l(\zeta_p)^\times \to \mathbb{C}^\times$ we have the following relation for involutions:*

$$\mu \circ (\cdot)^{-1} = (\cdot)^{-1} \circ \mu = \overline{(\cdot)} \circ \mu.$$

That is, we have on $\mathbb{C}^\times$ that multiplicative inversion $(\cdot)^{-1}$ and complex conjugation $\overline{(\cdot)}$ coincide on the image of the embedding, and the embedding is a monoid homomorphism, thus compatible with multiplicative inversion.

Proof. Since we have $\mu(1) = 1$ it commutes with inverting elements, which is a homomorphism because the involved groups are commutative. But since the order of every element of $l(\zeta_p)^\times$ is finite, we know that $\mu(l(\zeta_p)^\times) \subset \mathbb{S}^1$. Hence inverting and complex conjugation coincide. □

Proposition 1.2.14. *For any representation of a finite group $\rho\colon G \to GL_n\overline{\mathbb{F}}_q$ and the induced representation of the group G^{op} with opposed multiplication given by $\rho \circ (\cdot)^{-1}\colon G^{op} \to GL_n\overline{\mathbb{F}}_q$ we have the following relation for the Brauer characters:*

$$\overline{f^\rho} = f^{\rho\circ(\cdot)^{-1}}.$$

Proof. For $g \in G$ calculate the Brauer character with ξ_g^i again the eigenvalues of ρ_g with multiplicities:

$$\overline{\sum_i \mu\xi_g^i} = \sum_i \overline{\mu\xi_g^i} = \sum_i (\mu\xi_g^i)^{-1} = \sum_i \mu((\xi_g^i)^{-1}) = \sum_i \mu(\xi_{g^{-1}}^i),$$

hence follows the claim. □

Finally, we would like to induce this Brauer character by some virtual representation which only explicitly depends on ρ and starts from the same group G instead of the one with opposed multiplication, but we do not want to cancel out the inversion. Note that the target is a general linear group. For $G = GL_nR$ the group hence comes equipped with a second isomorphism from G to G^{op} given by transposition.

Theorem 1.2.15. *For any representation of a finite group $\rho\colon G \to GL_n\overline{\mathbb{F}}_q$ let $\rho^\dagger$ be the representation induced by considering the composition:*

$$G \xrightarrow{(\cdot)^{-1}} G^{op} \xrightarrow{\rho} GL_n\overline{\mathbb{F}}_q^{op} \xrightarrow{(\cdot)^t} GL_n\overline{\mathbb{F}}_q$$

Then their Brauer characters satisfy:

$$f^{\rho^\dagger} = \overline{f^\rho},$$

and hence by uniqueness of the associated (virtual) representation we find

$$F(f^{\rho^\dagger}) = \overline{(\cdot)} \circ F(f^\rho).$$

Proof. Obviously transposing matrices does not change the eigenvalues involved in the definition of the Brauer character, so the preceding proposition directly yields the claimed result. □

For ease of reference I summarise Quillen's approximation [Q1] by the Brauer lift with respect to its multiplicative and involutive structure in one theorem.

Theorem 1.2.16. *The Brauer lift at any prime $p \geq 3$ is a map of E_∞ ring spectra $K(L(\zeta_p)) = H(\mathcal{M}(L(\zeta_p))) \to H(\mathcal{M}(\mathbb{C})) = ku$, which is an equivalence of E_∞-ring spectra after completion at p: $K(L(\zeta_p))^\wedge_p \to ku^\wedge_p$.*

Furthermore the involution on ku given by complex conjugation is approximated by $(\cdot)^t \circ (\cdot)^{-1}$ on $L(\zeta_p)$, in particular the involution as induced on ku by 3.5.5 from complex conjugation cancels out to give the approximation of E_∞-ring spectra with involution:

$$(K(L(\zeta_p))^\wedge_p, \mathrm{id}) \to (ku^\wedge_p, \overline{(\cdot)}_*).$$

2 Bicategories of Matrices

2.1 Osorno's Delooping of $\mathcal{M}(\mathcal{R})$

Given a permutative category that has a compatible associative multiplication, one can define its module bicategory [Os]. Furthermore, Osorno provides a delooping of this bicategory by considering block sums of matrices and organising these into a Γ-structure on the module bicategory. This leads to an associated spectrum given a permutative bicategory.

I extend Osorno's result in a multiplicative manner. This means I define the bicategory-analogue of bipermutative categories in this chapter and adapt Osorno's delooping in a manner that it has an induced multiplicative structure in the next chapter, leading to an E_∞ symmetric ring spectrum.

Given a bimonoidal category $(\mathcal{R}, \oplus, \otimes)$ one can define its bicategory of matrices as follows:

Definition 2.1.1. The **bicategory of matrices** $\mathcal{M}(\mathcal{R})$ associated to a bimonoidal category $\mathcal{R}$ is given as follows. It has as objects the natural numbers $n \in \mathbb{N}_0$, and its morphism categories are:

$$\mathcal{M}(\mathcal{R})(n,m) := \begin{cases} GL_n\mathcal{R}, & n = m, \\ \emptyset, & \text{else,} \end{cases}$$

with $GL_n\mathcal{R}$ the categories of weakly invertible $n \times n$-matrices over the bimonoidal coefficients $\mathcal{R}$ (cf. [Os, R, BDR]).

For this category to have an associator it is vital that the distributivity morphisms of $\mathcal{R}$ are *isomorphisms* (cf. equation (4) on p.323 of [R])!

In what follows I need that for each bimonoidal category the bipermutative category Σ_* is part of the bicategory $\mathcal{M}(\mathcal{R})$ in a well-behaved way:

Proposition 2.1.2. *Consider the map*

$$E_\bullet \colon \Sigma_n \to GL_n\mathcal{R} \quad \sigma \mapsto E_\sigma,$$

with $(E_\sigma)_{ij} := \delta_{i,\sigma j}$. *This map satisfies* $E_{\sigma\tau} = E_\sigma E_\tau$. *It is a faithful functor between monoidal categories.*

Furthermore there is an action on general matrices:

$$(E_\sigma A E_\tau)_{ij} = A_{\sigma^{-1}i,\tau j},$$

hence in particular:

$$(E_{\sigma^{-1}} A E_\sigma)_{ij} = A_{\sigma i,\sigma j}.$$

Structurally more satisfactory we get the following embedding.

Proposition 2.1.3. *The category of finite sets as described in Example 1.1.7 includes into the bicategory of modules for each coefficient category* $\mathcal{R}$*:*

$$\Sigma_* \to \mathcal{M}(\mathcal{R}).$$

More explicitly: Consider Σ_* *as a bicategory with discrete morphism categories, then for each coefficient category* $\mathcal{R}$ *we get a strict normal functor* $E_\bullet \colon \Sigma_* \to \mathcal{M}(\mathcal{R})$, *i.e.,* $E_\bullet$ *strictly respects identities and compositions 1.1.13.*

Proof. I only give the indication of why this is true in my setup. The essential point is the strictness of 0 and 1 in the coefficient category as units, as well as the strict equality $0 \cdot a = 0$. □

This has the following extremely convenient corollary.

Corollary 2.1.4. *For each (small) coefficient category* $\mathcal{R}$ *its module bicategory* $\mathcal{M}(\mathcal{R})$ *has a sub-bicategory which is the faithful image of* $E_\bullet$, *in particular, this sub-bicategory is a* 2*-category, so the associator restricts to the identity there.*

Remark 2.1.5. With these results it is just a minor abuse of notation to identify permutations with their images in $\mathcal{M}(\mathcal{R})$, hence I write $\sigma = E_\sigma$. In particular the identity matrix of an object $n \in \mathcal{M}(\mathcal{R})$ is in the image of E and I write $E_n = E_{id_n}$ for the unit matrix.

In [Os] Osorno established that the direct sum of matrices equips this bicategory with a well-behaved permutative structure [Os, Theorem 4.7], and the main result of the paper [Os, Theorem 3.6] states that this can be delooped just as the classical case [Se].

Theorem 2.1.6 ([Os, Theorem 4.7])**.** *The bicategory of matrices $\mathcal{M}(\mathcal{R})$ is strictly symmetric monoidal with respect to the* ***block sum*** *of matrices:*

$$\begin{aligned} \boxplus\colon \mathcal{M}(\mathcal{R}) \times \mathcal{M}(\mathcal{R}) &\to \mathcal{M}(\mathcal{R}) \\ (n,m) &\mapsto n+m \\ (A,B) &\mapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right). \end{aligned}$$

The symmetry is just the one given by the functor $E_\bullet$ defined in Proposition 2.1.2 from Σ_* (cf. Example 1.1.7), i.e.:

$$\Sigma_{n+m} \ni c^+_{n,m} = c_+ \colon n \boxplus m \to m \boxplus n.$$

This symmetric monoidal structure exhibits the classifying space of $\mathcal{M}(\mathcal{R})$ as an infinite loop space.

Theorem 2.1.7 ([Os, Theorem 3.6])**.** *Let $\mathcal{M}$ be a strict symmetric monoidal bicategory. Then there is a special Γ-bicategory $\widehat{\mathcal{M}}$ such that:*

$$\widehat{\mathcal{M}}(1) \cong \mathcal{M}.$$

Therefore the classifying space $|N\mathcal{M}|$ is an infinite loop space upon group completion.

I elaborate on the permutative structure and the delooping further in 2.3 once my multiplicative matters are in place. In particular my main result of this chapter is the following.

Theorem 2.1.8. *Given a bipermutative coefficient category $(\mathcal{R}, \oplus, \otimes)$ there are two permutative structures $\boxplus, \boxtimes$ on its module bicategory $\mathcal{M}(\mathcal{R})$ that can be arranged into a bipermutative bicategory.*

This bipermutative structure can then be fitted onto the delooping of $\mathcal{M}(\mathcal{R})$, such that the result is an E_∞-ring spectrum. This is the content of the next chapter 3.

Theorem 2.1.9. *There is an E_∞ symmetric ring spectrum $H\mathcal{M}(\mathcal{R})$, which is weakly equivalent to the spectrum of Osorno's Γ-space $|N\widehat{\mathcal{M}(\mathcal{R})}|$, with multiplication induced by the multiplicative structure of $\mathcal{M}(\mathcal{R})$.*

Remark 2.1.10. The embeddings of the symmetric groups are compatible with direct sum of matrices in the nicest possible way:

$$\begin{tikzcd} \Sigma_n \times \Sigma_m \arrow[r, "\sqcup"] \arrow[d] & \Sigma_{m+n} \arrow[d] \\ GL_n\mathcal{R} \times GL_m\mathcal{R} \arrow[r, "\boxplus"] & GL_{n+m}\mathcal{R}. \end{tikzcd}$$

So we have $\sigma \boxplus \tau = \sigma \sqcup \tau$, i.e., considering permutations as matrices yields that their direct sum is the disjoint union.

2.2 Definition - Symmetric Monoidal and Permutative Bicategories

In what follows I need two types of symmetric monoidal structures. One is the E_∞-structure which we deloop, thus the one thought of as additive. The other one gives the induced E_∞-multiplication on the delooping.

For convenience I use the shorthand $Ob\mathcal{C} = \mathcal{C}_0$. For 1-cells, i.e., objects of morphism categories, when I do not want to refer to their source and target I use $Mor\mathcal{C} = \mathcal{C}_1$.

By weakening the definition of 2-categories to bicategories one has an assortment of ways how monoidality can be defined for a bicategory. Apart from the weakenings of unit axioms, one can impose associativity up to isomorphism, and varying degrees of symmetry. For bicategories there is one more degree of symmetry in addition to "associative, braided," and "symmetric", which is called "sylleptic". For a detailed discussion of these notions, as well as a guide to the 7 sources, which incrementally built the notion of "monoidal bicategory" in a fully weakened version, I defer to the PhD thesis of Christopher Schommer-Pries [SP]. The original fully weakened definition of braided monoidal bicategories goes back to Kapranov and Voevodsky in [KV] as braided Gray monoids.

Definition 2.2.1. A **permutative bicategory** $(\mathcal{C}, +, 0, c_+)$ is a bicategory with strict identities, a strict normal 1.1.13 functor

$$+ \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

a chosen additive unit $0 \in \mathcal{C}_0$ and a strict natural transformation

$$c_+ \colon \; + \circ tw \Rightarrow +$$

for $tw\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ the strict isomorphism, which exchanges factors.

These satisfy the following identities:

- Adding 0 is strictly equal to the identity functor on $\mathcal{C}$:

$$0 + \_ = \_ + 0 = id_{\mathcal{C}}.$$

- Addition is strictly associative, i.e., we have an equality of functors

$$(\_ + \_) \circ ((\_ + \_) \times id) = (\_ + \_) \circ (id \times (\_ + \_)),$$

giving a well-defined strict normal n-fold sum functor for each $n \geq 0$:

$$\sum_n \colon \mathcal{C}^{\times n} \to \mathcal{C}.$$

Additionally the additive twist has to make the following diagrams strictly commutative for every $a, b, c \in \mathcal{C}_0$:

$$\begin{tikzcd}
a+b+c \arrow[r, "c_+ + id_c"] \arrow[dr, "c_+"'] & b+a+c \arrow[d, "id_b + c_+"] \\
& b+c+a
\end{tikzcd}
\qquad
\begin{tikzcd}
a+b+c \arrow[r, "id_a + c_+"] \arrow[dr, "c_+"'] & a+c+b \arrow[d, "c_+ + id_b"] \\
& c+a+b
\end{tikzcd}$$

$$\begin{tikzcd}
a+b \arrow[rr, equal] \arrow[dr, "c_+"'] & & a+b \\
& b+a, \arrow[ur, "c_+"'] &
\end{tikzcd}$$

giving a unique (strict) natural transformation for every $n \in \mathbb{N}_0$ and $\sigma \in \Sigma_n$:

$$c_\sigma\colon \sum_n \circ \left(\sigma\colon \mathcal{C}^{\times n} \to \mathcal{C}^{\times n}\right) \Rightarrow \sum_n$$

built from composites of c_+.

Remark 2.2.2. This is a maximally strictified version of the definition of "strict symmetric monoidal" Angélica Osorno uses in [Os, Definition 3.1]. She considers more generally a monoidal product, which is just a pseudofunctor, as well as a symmetry, which is just pseudonatural. Thus in addition to the assumption of strict identities, the strict functoriality of + and the strict naturality are stronger conditions to impose.

Definition 2.2.3. A **symmetric monoidal bicategory** $(\mathcal{C}, \cdot, 1, c.)$ consists of a bicategory with strict units $\mathcal{C}$, a pseudofunctor

$$(\cdot, \Phi) \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

a chosen unit-object $1 \in \mathcal{C}_0$, a strong pseudonatural transformation

$$(c^1_\cdot, c^2_\cdot) \colon \; \cdot \circ tw \Rightarrow \cdot,$$

satisfying the following identities:

- Multiplying with 1 is strictly equal to the identity functor on $\mathcal{C}$:

$$1 \cdot \_ = \_ \cdot 1 = id_{\mathcal{C}}.$$

- Multiplication is strictly associative, i.e., we have a strict equality of functors (and their compositors):

$$(\_ \cdot \_) \circ ((\_ \cdot \_) \times id) = (\_ \cdot \_) \circ (id \times (\_ \cdot \_)),$$

  giving a well-defined pseudofunctor:

$$(\prod_n, \Phi^{\Pi_n}) \colon \mathcal{C}^{\times n} \to \mathcal{C}.$$

  In addition the multiplicative twist makes the following diagrams strictly commute for every $a, b, c \in \mathcal{C}_0$:

$$\begin{tikzcd} abc \arrow[r, "c.\cdot id_c"] \arrow[dr, "c."'] & bac \arrow[d, "id_b \cdot c."] \\ & bca \end{tikzcd} \quad \begin{tikzcd} abc \arrow[r, "id_a \cdot c."] \arrow[dr, "c."'] & acb \arrow[d, "c.\cdot id_b"] \\ & cab \end{tikzcd} \quad \begin{tikzcd} ab \arrow[r, equal] \arrow[dr, "c."'] & ab \\ & ba, \arrow[u, "c."'] \end{tikzcd}$$

  which means in more detail that the functors as well as their compositors coincide. In particular $c^1.$ squares to the identity transformation with identity two-cell, hence $c^2.$ has to square to the identity as well. Again this implies that we have a unique strong pseudonatural transformation for every $n \in \mathbb{N}_0, \sigma \in \Sigma_n$:

$$c_\sigma \colon \prod_n \circ \left(\sigma \colon \mathcal{C}^{\times n} \to \mathcal{C}^{\times n}\right) \Rightarrow \prod_n$$

  built from composites of $(c^1_\cdot, c^2_\cdot)$.

Remark 2.2.4. The above notion is precisely the notion of "strict symmetric monoidal" Osorno gives in [Os] apart from my standing assumption on strict identity 1-cells in the underlying bicategory. Since I consider no other symmetric monoidal structures on bicategories than the ones given by block sum and tensor-product on $\mathcal{M}(\mathcal{R})$, I choose to drop the attribute "strict", since the essential difference to "permutative" is the **non-strictness** of the monoidal functor $(\cdot, \Phi)$.

Remark 2.2.5. Do note that the condition that c^1 squares to the identity implies that it is an isomorphism 1-cell, not just an equivalence as one might guess.

Remark 2.2.6. In chapter 3 I can much more easily generalise the Grothendieck construction as I defined it in 1.1.37, since the additive structure of $\mathcal{M}(\mathcal{R})$ is even a bit stricter than Osorno axiomatised, making her delooping apply to more general monoidal bicategories than mine does.

I chose the symbols before incorporating the intuition that I think of permutative structures as additive structures, which we deloop, while symmetric monoidal structures can potentially give superimposed multiplications on the delooping.

2.3 The Multiplicative Structure on $\mathcal{M}(\mathcal{R})$

Just as in the classical case of commutative rings one should expect the module category of a bipermutative category to have a multiplicative structure analogous to the tensor product of modules. Since for combinatorial reasons I decided to restrict to a coordinatised version of modules, given by ranks and matrices, the tensor product has to be one of matrices as well.

Let me reiterate that the distributivity morphisms for a bipermutative category are isomorphisms in this thesis! (Compare Remark 1.1.5.)

Given a bipermutative coefficient category $(\mathcal{R}, +, \cdot)$ we want to define a tensor product on its bicategory of matrices $\mathcal{M}(\mathcal{R})$. Choose an associative bijection $\omega_{n,m} \colon \mathbf{n} \times \mathbf{m} \to \mathbf{nm}$, defining a strictly associative monoidal product on Fin, which represents the cartesian product of finite sets - cf. Example 1.1.7. For definiteness I set $\omega_{n,m}(i,j) := (i-1)\cdot m+j$ with inverse $\theta_{n,m}(i) = (((i-1) \text{ div } m)+1, ((i-1) \bmod m+1)$, where $i \text{ div } m := \lfloor \frac{i}{m} \rfloor$ is the integer part of division of i by m, while $i \bmod m$ is the remainder r for $i = qm + r$ the Euclidean division of i by m.

Recall that this is consistent with the associative smash product on Fin_+ described in Example 1.1.8.

Definition 2.3.1. Given a choice of associative bijections $\omega\colon \mathbf{n}\times\mathbf{m} \to \mathbf{nm}$ define the tensor product as follows:

$$\begin{array}{rlcl} \boxtimes\colon & \mathcal{M}(\mathcal{R}) & \times & \mathcal{M}(\mathcal{R}) \longrightarrow \mathcal{M}(\mathcal{R}) \\ & (n,m) & \mapsto & nm \\ & (A,B) & \mapsto & (A\boxtimes B)_{\omega(i_1,j_1),\omega(i_2,j_2)} := A_{i_1,i_2}\cdot B_{j_1,j_2}. \end{array}$$

The same description applies to the tensor product of 2-cells.

The rest of the section is devoted to proving that $\mathcal{M}(\mathcal{R})$ equipped with this monoidal structure satisfies the axioms given in 2.2.3.

Remark 2.3.2. Obviously my choice of ω is dictated by the choices I fixed in 1.1.7 so that I can establish $E_\bullet$ as a bipermutative functor.

Example 2.3.3. For clarity consider the following small example. Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

and

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

then with the bijections chosen above we have:

$$A\boxtimes B = \left(\begin{array}{cc|cc} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ \hline A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{array}\right).$$

Remark 2.3.4. Following this example it is easy to see that the tensor product as defined in 2.3.1 respects weakly invertible matrices with coefficients in a bipermutative category. In particular, if we considered a tensor-product of matrices given by columnwise or linewise ordering (as opposed to blockwise), any entry $a_{ij} = 0$ would produce a full zero column or line, hence definitely not a (weakly) invertible matrix.

Remark 2.3.5. As I indicated before the structure induced by direct sum of matrices on $\mathcal{M}(\mathcal{R})$ yields a permutative structure, but the tensor-product does not. The fact that $(\mathcal{M}(\mathcal{R}), \boxplus)$ is permutative is shown in the proof of [Os, Theorem 4.7]. The tensor-product already fails at the first strictness of a permutative bicategory, because $\boxtimes$ is not a strict functor, i.e., it only respects composition up to an isomorphism two-cell, which is strict if and only if the multiplicative symmetry of the coefficients is trivial, hence only for $\mathcal{R}$ an ordinary ring (or rig as in the case of $\mathbb{N}$).

2.3.1 The Matrix Tensor Product is Symmetric Monoidal

This subsection is devoted to proving that the tensor product equips $\mathcal{M}(\mathcal{R})$ with a symmetric monoidal structure in all detail. It can be skipped safely by the reader without losing any essential information. I am sure that an elegant short proof by exploiting the functor $E_\bullet \colon \Sigma_* \to \mathcal{M}(\mathcal{R})$ can be devised, but I want to exhibit the additional strictness the tensor-product on $\mathcal{M}(\mathcal{R})$ satisfies in explicit detail.

Lemma 2.3.6. *The assignment $\boxtimes$ of definition 2.3.1 is a pseudofunctor of bicategories.*

In addition we have the following natural identities on 1*-cells:*

$$A \boxtimes B = (A \boxtimes id)(id \boxtimes B)$$

as well as strict compositors:

$$(A^1 A^2) \boxtimes id = (A^1 \boxtimes id)(A^2 \boxtimes id),$$

$$id \boxtimes (B^1 B^2) = (id \boxtimes B^1)(id \boxtimes B^2).$$

Proof. Normality is obvious: the identity matrices E_n and E_m are sent to

$$E_n \boxtimes E_m = E_{nm},$$

by strictness of 0 and 1 in $\mathcal{R}$. The interesting aspect is the compositor

$$\Phi^{\boxtimes} \colon (A^1 \circ A^2) \boxtimes (B^1 \circ B^2) \Rightarrow (A^1 \boxtimes B^1) \circ (A^2 \boxtimes B^2).$$

Composition of 1-cells in $\mathcal{M}(\mathcal{R})$ is given by matrix multiplication, hence

$$(A^1 \circ A^2) \boxtimes (B^1 \circ B^2)_{\omega(i_1,j_1),\omega(i_2,j_2)}$$

$$
\begin{aligned}
&= (A^1 \circ A^2)_{i_1,i_2}(B^1 \circ B^2)_{j_1,j_2} \\
&= \left(\sum_k A^1_{i_1,k}A^2_{k,i_2}\right)\left(\sum_l B^1_{j_1,l}B^2_{l,j_2}\right) \\
&\Rightarrow^{\rho^{-1}} \sum_k A^1_{i_1,k}A^2_{k,i_2}\left(\sum_l B^1_{j_1,l}B^2_{l,j_2}\right) \\
&\Rightarrow^{\sum_k \lambda^{-1}} \sum_{(k,l)} A^1_{i_1,k}A^2_{k,i_2}B^1_{j_1,l}B^2_{l,j_2} \\
&\Rightarrow^{\sum_{(k,l)} id\cdot c^{\mathcal{R}}_{\otimes}\cdot id} \sum_{(k,l)} A^1_{i_1,k}B^1_{j_1,l}A^2_{k,i_2}B^2_{l,j_2} \\
&= \sum_{(k,l)} (A^1 \boxtimes B^1)_{\omega(i_1,j_1),\omega(k,l)}(A^2 \boxtimes B^2)_{\omega(k,l),\omega(i_2,j_2)} \\
&= (A^1 \boxtimes B^1) \circ (A^2 \boxtimes B^2)_{\omega(i_1,j_1),\omega(i_2,j_2)}.
\end{aligned}
$$

So define $\Phi^{\boxtimes} := (id \cdot c_{\otimes} \cdot id) \circ \lambda^{-1} \circ \rho^{-1}$ in the manner described above for each component (with summations suppressed because of the appropriate coherences in the coefficient category). It is natural, because the involved morphisms are natural in $\mathcal{R}$. It is obvious if either $A^1A^2 = id$ or $B^1B^2 = id$ then the involved natural isomorphisms are forced to be identities, hence follow the strict identities claimed above.

To see that Φ is associative I refer the reader to [Lap]: Given a morphism

$$((A^1 \circ A^2) \circ A^3) \boxtimes ((B^1 \circ B^2) \circ B^3) \Rightarrow (A^1 \circ (A^2 \circ A^3)) \boxtimes (B^1 \circ (B^2 \circ B^3))$$

comprised only of structural (iso)morphisms of the bipermutative category $\mathcal{R}$ there is a unique structural morphism between the given source and target. Since the associator of $\mathcal{M}(\mathcal{R})$ is given by structural morphisms of $\mathcal{R}$ and the compositor $\Phi^{\boxtimes}$ of $\boxtimes$ is given by structural morphisms of $\mathcal{R}$ as well, this gives that $\Phi^{\boxtimes}$ is associative in the appropriate manner (cf. [Le, p. 4]). □

Remark 2.3.7. Since this is the first proof of this type let me emphasise that it is sufficient to consider the compatibilities on 1-cells, because the 2-cells are any type of $n \times n$-matrix with no additional condition. So the calculations on 1-cells are "always" strictly natural with respect to 2-cells.

Remark 2.3.8. The additional strict identities show that the compositor of $\boxtimes$ is a result of the natural isomorphism: $(id\boxtimes B)(A\boxtimes id) \Rightarrow (A\boxtimes id)(id\boxtimes B) = A\boxtimes B$.

Remark 2.3.9. Consistently with the Deligne conjecture for Algebraic K-theory we see that we need at least a braiding (i.e., an E_2-structure) on the coefficient category $\mathcal{R}$ to define an $E_1 = A_\infty$-multiplication on its module category. Cf. for instance [Ba, Example 3.9] and [Lu2, Remarks after C.6.3.5.17]. For a more thorough survey of the Deligne conjecture on Hochschild cohomology as well as a survey of its proofs see Section 16 of [MSm].

Lemma 2.3.10. *The functor $\boxtimes$ is strictly associative.*

Proof. For this proof I fix the specific associative bijections from the beginning of this section. For $A \in GL_n\mathcal{R}, B \in GL_m\mathcal{R}, C \in GL_l\mathcal{R}$ we have:

$$\begin{aligned}
&((A \boxtimes B) \boxtimes C)_{(i_1-1)ml+(j_1-1)l+k_1,(i_2-1)ml+(j_2-1)l+k_2} \\
&= (A \boxtimes B)_{(i_1-1)m+j_1,(i_2-1)m+j_2} \cdot C_{k_1,k_2} \\
&= A_{i_1,i_2} \cdot B_{j_1,j_2} \cdot C_{k_1,k_2} \\
&= A_{i_1,i_2} \cdot (B \boxtimes C)_{(j_1-1)l+k_1,(j_2-1)l+k_2} \\
&= (A \boxtimes (B \boxtimes C))_{(i_1-1)ml+(j_1-1)l+k_1,(i_2-1)ml+(j_2-1)l+k_2}.
\end{aligned}$$

□

Lemma 2.3.11. *The object* 1 *with its identities is a strict unit for $\boxtimes$.*

Proof. We have:

$$(A \boxtimes 1)_{(i_1-1)1+j_1,(i_2-1)1+j_2} = A_{i_1,i_2}$$

for $i_1, i_2 = 1, \dots, |A|$ and $j_1 = j_2 = 1$, analogously $1 \boxtimes A = A$. □

The following statement can also be thought of as a convention: Just as the empty matrix is a strictly neutral element for $\boxplus$, it is a strict zero for $\boxtimes$.

Lemma 2.3.12. *The object* 0 *with its identity considered as the empty matrix (of objects and morphisms respectively) is a strict zero for $\boxtimes$.*

I needed some commutativity to show that $\boxtimes$ is a functor, it should be much less surprising that it is necessary for commutativity of $\boxtimes$.

Lemma 2.3.13. *The bicategory of matrices $\mathcal{M}(\mathcal{R})$ over a bipermutative coefficient category $\mathcal{R}$ is symmetric monoidal with respect to $\boxtimes$.*

Proof. At this point I borrow the bipermutative structure from Σ_* (cf. 2.1.2), let $A \in GL_n\mathcal{R}$, $B \in GL_m\mathcal{R}$, and consider:

$$(c_{n,m}(A \boxtimes B))_{(i_1-1)n+j_1,(i_2-1)m+j_2} = (A \boxtimes B)_{(j_1-1)m+i_1,(i_2-1)m+j_2} = A_{j_1,i_2}B_{i_1,j_2},$$

$$((B \boxtimes A)c_{n,m})_{(i_1-1)n+j_1,(i_2-1)m+j_2} = (B \boxtimes A)_{(i_1-1)n+j_1,(j_2-1)n+i_2} = B_{i_1,j_2}A_{j_1,i_2},$$

these can obviously be transformed into each other by the multiplicative twist of $\mathcal{R}$, so the symmetry has as one-cells $c_{n,m}\colon nm \to mn$ and two-cells $(C^{\boxtimes})_{ij} = c^{\mathcal{R}}_{\cdot}\ \ \forall i, j$. □

Example 2.3.14. Consider this again on 2×2-matrices, i.e., a diagram:

$$\begin{tikzcd}
2 \cdot 2 \arrow[r, "A \boxtimes B"] \arrow[d, "c_{2,2}"'] \arrow[dr, phantom, "\swarrow c^{\mathcal{R}}"] & 2 \cdot 2 \arrow[d, "c_{2,2}"] \\
2 \cdot 2 \arrow[r, "B \boxtimes A"'] & 2 \cdot 2.
\end{tikzcd}$$

Use the identification $c_{2,2} = (23)$ to calculate:

$$\begin{aligned}
(B \boxtimes A)E_{(23)} &= \begin{pmatrix} B_{11}A_{11} & B_{11}A_{12} & B_{12}A_{11} & B_{12}A_{12} \\ B_{11}A_{21} & B_{11}A_{22} & B_{12}A_{21} & B_{12}A_{22} \\ B_{21}A_{11} & B_{21}A_{12} & B_{22}A_{11} & B_{22}A_{12} \\ B_{21}A_{21} & B_{21}A_{22} & B_{22}A_{21} & B_{22}A_{22} \end{pmatrix} \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \\
&= \begin{pmatrix} B_{11}A_{11} & B_{12}A_{11} & B_{11}A_{12} & B_{12}A_{12} \\ B_{11}A_{21} & B_{12}A_{21} & B_{11}A_{22} & B_{12}A_{22} \\ B_{21}A_{11} & B_{22}A_{11} & B_{21}A_{12} & B_{22}A_{12} \\ B_{21}A_{21} & B_{22}A_{21} & B_{21}A_{22} & B_{22}A_{22} \end{pmatrix}
\end{aligned}$$

and the other side:

$$\begin{aligned}
E_{(23)}(A \boxtimes B) &= \begin{pmatrix} 1 & & & \\ & & 1 & \\ & 1 & & \\ & & & 1 \end{pmatrix} \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix} \\
&= \begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix},
\end{aligned}$$

thus we have the 2-cell given by the multiplicative twist of $\mathcal{R}$ in each component:

$$\begin{pmatrix} B_{11}A_{11} & B_{12}A_{11} & B_{11}A_{12} & B_{12}A_{12} \\ B_{11}A_{21} & B_{12}A_{21} & B_{11}A_{22} & B_{12}A_{22} \\ B_{21}A_{11} & B_{22}A_{11} & B_{21}A_{12} & B_{22}A_{12} \\ B_{21}A_{21} & B_{22}A_{21} & B_{21}A_{22} & B_{22}A_{22} \end{pmatrix}$$

$$\Big\downarrow c^{\mathcal{R}}$$

$$\begin{pmatrix} A_{11}B_{11} & A_{11}B_{12} & A_{12}B_{11} & A_{12}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} & A_{22}B_{11} & A_{22}B_{12} \\ A_{11}B_{21} & A_{11}B_{22} & A_{12}B_{21} & A_{12}B_{22} \\ A_{21}B_{21} & A_{21}B_{22} & A_{22}B_{21} & A_{22}B_{22} \end{pmatrix}.$$

Let me summarise these results into one big lemma:

Lemma 2.3.15. *The module bicategory $\mathcal{M}(\mathcal{R})$ of a bipermutative category $\mathcal{R}$ is strictly symmetric monoidal with respect to the tensor product of matrices $\boxtimes$, i.e., we have:*

- $(\boxtimes, \Phi^{\boxtimes})$ *is a pseudofunctor:*

$$\boxtimes\colon \mathcal{M}(\mathcal{R}) \times \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{R}),$$

- $\boxtimes$ *is strictly associative, i.e.,*

$$\boxtimes \circ (\boxtimes \times id) = \boxtimes \circ (id \times \boxtimes),$$

- $\boxtimes$ *has a strict unit* 1*, i.e.,*

$$\boxtimes \circ (id \times 1) = \boxtimes \circ (1 \times id) = id_{\mathcal{M}(\mathcal{R})},$$

- $\boxtimes$ *has a strong symmetry transformation* $(c^{\Sigma}_{\cdot}, c^{\mathcal{R}}_{\cdot})$.

Additionally the symmetry satisfies the following coherences strictly:

$$\begin{tikzcd} l \cdot m \cdot n \arrow[r, "c^{\Sigma}_{l,m} \boxtimes 1_n"] \arrow[dr, "c^{\Sigma}_{l,mn}"'] & m \cdot l \cdot n \arrow[d, "1_m \boxtimes c^{\Sigma}_{l,n}"] \\ & m \cdot n \cdot l \end{tikzcd} \qquad \begin{tikzcd} l \cdot m \cdot n \arrow[r, "1_l \boxtimes c^{\Sigma}_{m,n}"] \arrow[dr, "c^{\Sigma}_{lm,n}"'] & l \cdot n \cdot m \arrow[d, "c^{\Sigma}_{n,l} \boxtimes 1_m"] \\ & n \cdot l \cdot m. \end{tikzcd}$$

Furthermore the symmetry is its own inverse:

$$\begin{tikzcd} n \cdot m \arrow[rr, equal] \arrow[dr, "c^{\Sigma}_{n,m}"'] & & n \cdot m \\ & m \cdot n. \arrow[ur, "c^{\Sigma}_{m,n}"'] & \end{tikzcd}$$

Proof. Each of the properties that do not follow from the previous lemmas is just promoted to $\mathcal{M}(\mathcal{R})$ from Σ_* by the functor $E_\bullet$, so there is nothing new to prove. □

In summary I have proved that $(\mathcal{M}(\mathcal{R}), \boxtimes, 1, c_\cdot)$ is a strict symmetric monoidal bicategory in the sense also used by [Os, Definition 3.1]. Since the tensor product of matrices satisfies these strict axioms, it is sufficient for me to consider this type of symmetric monoidal bicategory, although it is very probable that this class does not cover all equivalence classes of the most general type of bicategories with a symmetric monoidal structure one could devise.

With the tensor-product in place I can state the second strict monoidality the functor E satisfies, which quite trivially follows from the fact that I chose the same bijection for the tensor-product as I did for the product in Σ_*.

Proposition 2.3.16. *For each bipermutative coefficient category $\mathcal{R}$ the inclusion*

$$E\colon \Sigma_* \to \mathcal{M}(\mathcal{R})$$

is strictly symmetric monoidal with respect to $\times$ on Σ_* and $\boxtimes$ on $\mathcal{M}(\mathcal{R})$.

Remark 2.3.17. With the symmetric monoidal structures on $\mathcal{M}(\mathcal{R})$ settled the remark that everything works enriched as well is obligatory. The calculations before extend to 2-cells, because they are defined merely as part of the appropriate product-categories with no additional conditions, thus reordering 2-cells as indicated by the 1-cells is compatible with the enrichment.

2.4 The Bimonoidal Structure on $\mathcal{M}(\mathcal{R})$

Osorno has proved that $(\mathcal{M}(\mathcal{R}), \boxplus, 0, c_+)$ is a permutative bicategory (see Theorem 2.1.6), and in Section 2.3 I elaborate on the fact that $(\mathcal{M}(\mathcal{R}), \boxtimes, 1, c_\cdot)$ is a second symmetric monoidal bicategory structure on $\mathcal{M}(\mathcal{R})$. One would want these to interact in a manner analogous to bipermutative 1-categories. This section is devoted to making the analogy precise, and establishing $\mathcal{M}(\mathcal{R})$ as a bipermutative bicategory.

In 1.1.7 the choice of a strictly associative functor representing the product made the left-distributor strict in Σ_*, i.e., we have $\lambda = id$. Here I used the same bijection that fixes this for the tensor-product structure in 2.3.1. It should be intuitive that this makes $E_\bullet$ into a well-behaved bipermutative functor. I elaborate on that after the appropriate definition for bicategories.

The distributors of the bipermutative structure on Σ_* promote to natural transformations in $\mathcal{M}(\mathcal{R})$ without using two-cells.

Proposition 2.4.1. *We have strict equalities of one-cells for $A \in GL_n\mathcal{R}$, $B \in GL_m\mathcal{R}$, $C \in GL_l\mathcal{R}$:*

$$(A \boxplus B) \boxtimes C = A \boxtimes C \boxplus A \boxtimes C$$

and

$$A \boxtimes (B \boxplus C) c^{\Sigma}_{m+l,n}(c^{\Sigma}_{n,m} \boxplus c^{\Sigma}_{n,l}) = c^{\Sigma}_{m+l,n}(c^{\Sigma}_{n,m} \boxplus c^{\Sigma}_{n,l})((A \boxtimes B) \boxplus (A \boxtimes C)).$$

Proof. I only comment on the strictness, which is a result of the fact that the multiplicative twist enters twice as a two-cell, hence cancels out. □

Example 2.4.2. Let me elaborate on $l = n = 2, m = 1$, so we get:

$$c_{m+l,n} = c_{3,2},$$

which is

$$c_{3,2}((i-1)3 + j) = (j-1)2 + i$$

i.e., $c_{3,2}$ is the cycle (2453). We have

$$c_{2,1} + c_{2,2} = id + (23) = (45),$$

hence

$$c_{3,2}(c_{2,1} + c_{2,2}) = (2453)(45) = (432).$$

On one side we find:

$$
\begin{aligned}
&A \boxtimes (B \boxplus C)E_{(432)} \\
&= \begin{pmatrix}
A_{11}b & & & A_{12}b & & \\
& A_{11}C_{11} & A_{11}C_{12} & & A_{12}C_{11} & A_{12}C_{12} \\
& A_{11}C_{21} & A_{11}C_{22} & & A_{12}C_{21} & A_{12}C_{22} \\
A_{21}b & & & A_{22}b & & \\
& A_{21}C_{11} & A_{21}C_{12} & & A_{22}C_{11} & A_{22}C_{12} \\
& A_{21}C_{21} & A_{21}C_{22} & & A_{22}C_{21} & A_{22}C_{22}
\end{pmatrix}
\begin{pmatrix}
1 & & & & & \\
& & 1 & & & \\
& & & 1 & & \\
& 1 & & & & \\
& & & & 1 & \\
& & & & & 1
\end{pmatrix} \\
&= \begin{pmatrix}
A_{11}b & A_{12}b & & & & \\
& & A_{11}C_{11} & A_{11}C_{12} & A_{12}C_{11} & A_{12}C_{12} \\
& & A_{11}C_{21} & A_{11}C_{22} & A_{12}C_{21} & A_{12}C_{22} \\
A_{21}b & A_{22}b & & & & \\
& & A_{21}C_{11} & A_{21}C_{12} & A_{22}C_{11} & A_{22}C_{12} \\
& & A_{21}C_{21} & A_{21}C_{22} & A_{22}C_{21} & A_{22}C_{22}
\end{pmatrix}.
\end{aligned}
$$

On the other side we have:

$$
\begin{aligned}
&E_{(432)}(A \boxtimes B \boxplus A \boxtimes C) \\
&= \begin{pmatrix}
1 & & & & & \\
& & 1 & & & \\
& & & 1 & & \\
& 1 & & & & \\
& & & & 1 & \\
& & & & & 1
\end{pmatrix}
\begin{pmatrix}
A_{11}b & A_{12}b & & & & \\
A_{21}b & A_{22}b & & & & \\
& & A_{11}C_{11} & A_{11}C_{12} & A_{12}C_{11} & A_{12}C_{12} \\
& & A_{11}C_{21} & A_{11}C_{22} & A_{12}C_{21} & A_{12}C_{22} \\
& & A_{21}C_{11} & A_{21}C_{12} & A_{22}C_{11} & A_{22}C_{12} \\
& & A_{21}C_{21} & A_{21}C_{22} & A_{22}C_{21} & A_{22}C_{22}
\end{pmatrix} \\
&= \begin{pmatrix}
A_{11}b & A_{12}b & & & & \\
& & A_{11}C_{11} & A_{11}C_{12} & A_{12}C_{11} & A_{12}C_{12} \\
& & A_{11}C_{21} & A_{11}C_{22} & A_{12}C_{21} & A_{12}C_{22} \\
A_{21}b & A_{22}b & & & & \\
& & A_{21}C_{11} & A_{21}C_{12} & A_{22}C_{11} & A_{22}C_{12} \\
& & A_{21}C_{21} & A_{21}C_{22} & A_{22}C_{21} & A_{22}C_{22}
\end{pmatrix} = A \boxtimes (B \boxplus C)E_{(432)}.
\end{aligned}
$$

Because of this strictness I define what a bipermutative bicategory is in close analogy with 1-categories.

Definition 2.4.3. A **bipermutative bicategory** $\mathcal{R}$ is a bicategory (with strict identities) with two monoidal structures $\boxplus, \boxtimes$, an additive symmetry $c_{\boxplus}$, making

$(\mathcal{R}, \boxplus)$ into a permutative bicategory (Definition 2.2.1), a multiplicative symmetry $c_\boxtimes$, making $(\mathcal{R}, \boxtimes)$ into a symmetric monoidal bicategory (Definition 2.2.3), and strictly natural distributivity isomorphisms (strictly invertible 1-cells):

$$\lambda \colon a \boxtimes b \boxplus a \boxtimes b' \to a \boxtimes (b \boxplus b'),$$

$$\rho \colon a \boxtimes b \boxplus a' \boxtimes b \to (a \boxplus a') \boxtimes b,$$

satisfying the following strict identities of 1-cells:

1. strict zero:
$$0 \boxtimes a = a \boxtimes 0 = 0 \quad \forall a \in \mathcal{R},$$
2. $\boxplus$-associativity of distributors:
$$\lambda(\lambda \boxplus id) = \lambda(id \boxplus \lambda),$$
$$\rho(\rho \boxplus id) = \rho(id \boxplus \rho),$$
3. additive symmetry of distributors:
$$(c_\boxplus \boxtimes id)\lambda = \lambda \circ c_\boxplus,$$
$$(id \boxtimes c_+)\rho = \rho c_+,$$
4. $\boxtimes$-associativity of distributors:
$$\lambda = \lambda \circ (\lambda \boxtimes id),$$
$$\rho = \rho \circ (id \boxtimes \rho),$$
5. middle associativity of distributors:
$$\lambda \circ (id \boxtimes \rho) = \rho \circ (\lambda \boxtimes id),$$
6. mixed associativity of distributors:
$$\lambda(\rho \boxplus \rho) = \rho(\lambda \boxplus \lambda)(1 \boxtimes c_\boxplus \boxtimes 1),$$
7. multiplicative symmetry of distributors:
$$c_\boxtimes \circ \lambda = \rho \circ (c_\boxtimes \boxplus c_\boxtimes).$$

Remark 2.4.4. Let me emphasise that I have modelled this definition of bipermutative bicategory in such a way that the only thing left to show given the two symmetric monoidal structures $\boxplus$ and $\boxtimes$ on $\mathcal{M}(\mathcal{R})$ is: There are distributors λ and ρ, they are strict natural transformations and they satisfy the coherences above. There is no additional data in the form of coherence 2-cells involved.

Definition 2.4.5. Define the distributivity 1-cells for $\mathcal{M}(\mathcal{R})$ as follows:

$$\lambda := \mathrm{id} = E_{\lambda}, \quad \text{and} \quad \rho := E_{\rho^{\Sigma}} = c^{\Sigma}_{m+l,n}(c^{\Sigma}_{n,m} \boxplus c^{\Sigma}_{n,l}),$$

with identities as 2-cells.

With these distributivity 1-cells I can easily prove the following theorem, which I use to summarise all explicit details about the bipermutative structure of $\mathcal{M}(\mathcal{R})$, because most of it is part of the lemmas already proven above.

Theorem 2.4.6. *For $\mathcal{R}$ a bipermutative 1-category (possibly enriched over the symmetric monoidal categories Top, Cat, sSet), the following is a bipermutative bicategory $\mathcal{M}(\mathcal{R})$ (with 2-cells in the same enrichment):*

- $Ob\mathcal{M}(\mathcal{R}) = \mathbb{N}_0$,
- $\mathcal{M}(\mathcal{R})(n,m) = \begin{cases} GL_n\mathcal{R}, & n = m, \\ \emptyset, & n \neq m, \end{cases}$
- $(A \boxplus B)_{i,j} = \begin{cases} A_{i,j}, & 1 \leq i,j \leq |A|, \\ B_{i-|A|,j-|A|}, & 1 \leq i-|A|, j-|A| \leq |B|, \\ 0, & \end{cases}$
- $\boxplus$ *is a strict normal 1.1.13 functor, i.e.,* $(A_1A_2 \boxplus B_1B_2) = (A_1 \boxplus B_1)(A_2 \boxplus B_2)$ *and* $\mathrm{id}_n \boxplus \mathrm{id}_m = \mathrm{id}_{n+m}$,
- $(A \boxtimes B)_{(i_1-1)|B|+j_1,(i_2-1)|B|+j_2} := A_{i_1,i_2}B_{j_1,j_2}$,
- $\boxtimes$ *is a pseudofunctor, i.e.,* $(\mathrm{id}_n \boxtimes \mathrm{id}_m) = \mathrm{id}_{nm}$ *and there is a natural isomorphism 2-cell* $(A_1 \boxtimes B_1)(A_2 \boxtimes B_2) \Rightarrow (A_1A_2) \boxtimes (B_1B_2)$ *given by the adequate composition of (both) $\mathcal{R}$-distributors and its multiplicative symmetry $c^{\mathcal{R}}$ (cf. 2.3.6),*

- *the matrix E_{c^+} for the additive twist in Σ_*:*

$$c^+_{n,m}(i) = \begin{cases} i+m, & i \leq n, \\ i-n, & i \geq n+1, \end{cases}$$

  yields the additive twist with $(C^+)_{i,j} = \delta_{i,c^+_{n,m}(j)}$, which is a strict natural transformation $\boxplus \circ tw \Rightarrow \boxplus$, i.e., a pseudonatural transformation with coherence 2-cells identities,

- *the bijections $c_{n,m}((i-1)m+j) = i+(j-1)n$ yield the multiplicative twist with $C_{i,j} = \delta_{i,c_{n,m}(j)}$, which is a strong pseudonatural transformation with 2-cell given by the $c^{\mathcal{R}}$ in each component.*

the distributors are given as:

- *$\lambda = \mathrm{id}\colon nm + nl \to n(m+l)$, and*

- *$\rho = c_{n,m+l}(c_{m,n} + c_{l,n})\colon mn + ln \to (m+l)n$,*

and satisfy the coherences of 2.4.3.

Proof. The only thing left to prove is the fact that the distributors satisfy the claimed coherences. For that consider the functor

$$E\colon \Sigma_* \to \mathcal{M}(\mathcal{R})$$

again. I already established that it is strictly symmetric monoidal with respect to $\boxplus$, but given $\boxtimes$ as in 2.3.1 and $\times$ as in 1.1.7 it is obvious that E is also strictly symmetric monoidal with respect to these structures. Take particular note that the coherence 2-cell of $c_\boxtimes$ does not feature here because the multiplicative symmetry of $\mathcal{R}$ is forced to be the identity for the product $0 \cdot 0 = 0 \cdot 1 = 0$ and $1 \cdot 1$ by the axioms of symmetric monoidal categories for the first and third case and the additional zero-axiom for bipermutative categories.

The distributors are defined as part of the image of E explicitly, so obviously we have $E_\lambda = \lambda$ and $E_\rho = \rho$, but E is a strict functor of bicategories (and even 2-categories, if we restrict our attention to its image), so all coherences these distributors satisfy in Σ_* directly promote to $\mathcal{M}(\mathcal{R})$ for arbitrary bipermutative coefficients $\mathcal{R}$.

In particular, E is a strict functor of bipermutative bicategories, which is strictly additive, strictly multiplicative, and strictly satisfies $E(c^+) = C^\boxplus, E(c^\cdot) = C^\boxtimes$ as well as $E\lambda = \lambda, E\rho = \rho$. $\square$

Remark 2.4.7. Let me emphasise that I can get away with such a strict structure, because the 2-cells in $\mathcal{M}(\mathcal{R})$ are just parts of the appropriate product categories with no additional compatibility condition among them. If one were to impose "weak invertibility" on the matrices of 2-cells for instance, I do not know, if we still get such a strict bipermutative structure.

2.5 Transposition and Involutions

In commutative rings we are well aware of the formula:

$$(AB)^t = B^t A^t.$$

In preparation for involutions on module bicategories I want to isolate how this formula behaves with genuine bipermutative categories as coefficients.

Definition 2.5.1. For any bicategory $\mathcal{C}$ consider the 1-opposed bicategory $\mathcal{C}^{op_1}$, which has the same objects, 1-cells, 2-cells, but opposed composition of 1-cells, which I denote by $\circ$, while I do not denote the composition of 1-cells in $\mathcal{C}$, just as usual for ordinary matrix multiplication. The associator is then the inverse of the original associator:

$$A \circ (B \circ C) = (CB)A \xrightarrow{\alpha^{-1}} C(BA) = (A \circ B) \circ C.$$

For a bimonoidal 1-category $\mathcal{R}$ we also consider the μ-opposed category $\mathcal{R}^\mu$ with the same objects and morphisms, opposed multiplication, and hence exchanged distributors.

Proposition 2.5.2. *For bimonoidal coefficients transposition is a strict normal 1.1.13 functor:*

$$(\cdot)^t \colon \mathcal{M}(\mathcal{R}^\mu) \to \mathcal{M}(\mathcal{R})^{op_1}.$$

Proof. We calculate on 1-cells:

$$\begin{aligned}
(A^t \circ_1 B^t)_{ij} &= (B^t A^t)_{ij} \\
&= \sum_k B^t_{ik} A^t_{kj} \\
&= \sum_k A^t_{kj} \circ B^t_{ik} \\
&= \sum_k A_{jk} \circ B_{ki} = (AB)_{ji} = (AB)^t_{ij}.
\end{aligned}$$

So transposition strictly respects composition of 1-cells and strictly respects identities. □

Remark 2.5.3. For bipermutative coefficients we could use the multiplicative twist to suppress the μ-opposition. However to make the book keeping of oppositions more transparent I do not use that.

I want to consider involutions on the coefficient category $\mathcal{R}$ as considered by Richter in [R, Definition 3.1].

Definition 2.5.4. An anti-involution on a bipermutative category $\mathcal{R}$ is given by a self-inverse strictly symmetric monoidal functor $T\colon (\mathcal{R},+) \to (\mathcal{R},+)$ with respect to $(\mathcal{R},+,0,c_+^{\mathcal{R}})$ together with a natural isomorphism:

$$t\colon T(a)T(b) \to T(ba).$$

Satisfying:

- (T,t) strictly respects the unit, i.e., $T(1) = 1$ and

$$t = id\colon T(a)1 = 1T(a) = T(1)T(a) \to T(a),$$

- t is associative with respect to multiplication

$$\begin{tikzcd}
T(a)T(b)T(c) \arrow[r, "tT(c)"] \arrow[d, "T(a)t"] & T(ba)T(c) \arrow[d, "t"] \\
T(a)T(cb) \arrow[r, "t"] & T(cba).
\end{tikzcd}$$

- (T,t) is symmetric with respect to multiplication:

$$\begin{tikzcd}
T(a)T(b) \arrow[r, "t"] \arrow[d, "c"] & T(ba) \arrow[d, "T(c)"] \\
T(b)T(a) \arrow[r, "t"] & T(ab).
\end{tikzcd}$$

- the involution commutes with the distributors:

$$\begin{tikzcd}
T(a)T(b) + T(a)T(c) \arrow[r, "\lambda"] \arrow[d, "t+t"] & T(a)T(b+c) \arrow[d, "t"] \\
T(ba) + T(ca) \arrow[r, "T(\rho)"] & T((b+c)a),
\end{tikzcd}$$

and

$$\begin{tikzcd}
T(a)T(c) + T(b)T(c) \arrow[r, "\rho"] \arrow[d, "t+t"] & T(a+b)T(c) \arrow[d, "t"] \\
T(ca) + T(cb) \arrow[r, "T(\lambda)"] & T(c(a+b)).
\end{tikzcd}$$

Remark 2.5.5. It is quite obvious that in the bipermutative case, the way I consider it in this thesis, one of the compatibilities with distributors implies the other, but the exposition is more transparent this way.

Remark 2.5.6. Richter proceeds in [R] to define an induced involution on the bar construction of the monoidal categories $GL_n\mathcal{R}$. I define this involution as induced on matrix bicategories.

Proposition 2.5.7. *Let* $(F, \varphi)\colon \mathcal{R} \to \mathcal{A}$ *be a strictly additive functor of strictly bimonoidal categories, i.e.*

$$F(0) = 0, F(r+s) = F(r) + F(s),$$

furthermore let F *be strictly unital* $F(1) = 1$*, then*

- *a lax transformation* $\varphi\colon F(a)F(b) \to F(ab)$ *promotes to a lax normal functor*
$$\mathcal{M}F\colon \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{A}),$$
- *if furthermore* $\varphi\colon F(a)F(b) \to F(ab)$ *is a natural isomorphism, so* F *is strongly multiplicative, then* (F, φ) *promotes to a pseudofunctor*
$$\mathcal{M}F\colon \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{A}).$$

Proof. Again the interesting point is what happens on 1-cells:

$$\begin{aligned}
(\mathcal{M}FA \cdot \mathcal{M}FB)_{ij} &= \sum_k FA_{ik} \cdot FB_{kj} \\
&\Rightarrow^{\varphi} \sum_k F(A_{ik}B_{kj}) = F(\sum_k A_{ik}B_{kj}) \\
&= F(AB_{ij}) = \mathcal{M}F(AB)_{ij},
\end{aligned}$$

obviously the functor $(\mathcal{M}F, \varphi)$ then is just as good as the constraint φ of F. $\square$

Lemma 2.5.8. *An anti-involution on a bimonoidal category $\mathcal{R}$ is a strictly additive, strongly multiplicative functor from $\mathcal{R}$ to its multiplicative opposition $\mathcal{R}^\mu$*

$$T\colon \mathcal{R} \to \mathcal{R}^\mu.$$

Consequently an anti-involution induces a pseudofunctor of module bicategories:

$$\mathcal{M}T\colon \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{R}^\mu).$$

□

This is as far as I can come in the bicategory setting without appealing to classifying spaces, so let me summarise what the involution induces on module bicategories.

Lemma 2.5.9. *Composing transposition and an anti-involution on coefficients gives a pseudofunctor*

$$\mathcal{M}T \circ (\cdot)^t\colon \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{R})^{op_1}.$$

2.5.1 Involution and Tensor-products

One aim of this chapter on module bicategories is to get a combinatorial insight on how the involution on the coefficient category and the E_∞-structure on its module bicategory interact. Fortunately this is easily described on the level of bipermutative bicategories.

It is obvious that the induced involution strictly respects direct sum:

Lemma 2.5.10. *For a bimonoidal category $\mathcal{R}$ with involution (T, t) the induced involution on module bicategories is strictly additive and symmetric:*

$$\mathcal{M}T(A \boxplus B) = \mathcal{M}TA \boxplus \mathcal{M}TB,$$

and

$$\mathcal{M}T(c^+_{m,n}) = c^+_{m,n}.$$

The tensor product structure, if defined, is also easily seen to be compatible with the coordinatewise involution:

Lemma 2.5.11. *For a bipermutative category $\mathcal{R}$ with involution (T, t) we have a strictly natural isomorphism of functors:*

$$t\colon \boxtimes \circ \mathcal{M}T \times \mathcal{M}T \Rightarrow \mathcal{M}T \circ \boxtimes,$$

each considered as functors $\mathcal{M}(\mathcal{R}) \times \mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{R}^\mu)$.

Proof. This is a simple calculation, again consider 1-cells:

$$\begin{aligned}(\mathcal{M}T(A) \boxtimes \mathcal{M}T(B))_{(i_1-1)|B|+j_1,(i_2-1)|B|+j_2} &= \mathcal{M}T(A)_{i_1,i_2} \circ \mathcal{M}T(B)_{j_1,j_2} \\ &= T(A_{i_1,i_2}) \circ T(B_{j_1,j_2}) \\ &\Rightarrow^t T(A_{i_1,i_2} B_{j_1,j_2}) \\ &= \mathcal{M}T((A \boxtimes B)_{(i_1-1)|B|+j_1,(i_2-1)|B|+j_2}) \\ &= \mathcal{M}T(A \boxtimes B)_{(i_1-1)|B|+j_1,(i_2-1)|B|+j_2}.\end{aligned}$$

□

So we can summarise:

Theorem 2.5.12. *For a bipermutative category $\mathcal{R}$ with involution (T, t) the coordinatewise involution on the bicategory of matrices $\mathcal{M}(\mathcal{R})$ induces a strong bipermutative functor:*

$$\mathcal{M}T\colon (\mathcal{M}(\mathcal{R}), \boxplus, \boxtimes) \to (\mathcal{M}(\mathcal{R}^\mu), \boxplus, \boxtimes),$$

in the sense that it is strictly additive, and strongly multiplicative with respect to $\boxtimes$.

Proof. I only need to elaborate on the multiplicative symmetry of $\mathcal{M}T$, which is a consequence of the compatibility on coefficients:

$$\begin{tikzcd} T(a)T(b) \arrow[r, "t"] \arrow[d, "c"] & T(ba) \arrow[d, "Tc"] \\ T(b)T(a) \arrow[r, "t"] & T(ab). \end{tikzcd}$$

□

In particular the fact that $\mathcal{M}T$ is strictly additive with respect to $\boxplus$ implies that it induces a map of Γ-spaces, i.e., an infinite loop map on classifying spaces as follows:

$$B\mathcal{M}T\colon B\mathcal{M}(\mathcal{R}) \to B\mathcal{M}(R^\mu),$$

cf. 2.1.7. In what follows I want to define an internal involution on $B\mathcal{M}(\mathcal{R})$, and refine Osorno's delooping of Theorem 2.1.7 to one that allows us to induce a multiplicative structure more easily. Thus as the last compatibility, which is directly visible on the level of bicategories of matrices, we see that transposition and tensor-product strictly commute.

Proposition 2.5.13. *For any bipermutative category $\mathcal{R}$ transposition induces a strictly additive and strictly multiplicative strict normal 1.1.13 functor on its bicategory of matrices $\mathcal{M}(\mathcal{R})$:*

$$(\cdot)^t\colon (\mathcal{M}(\mathcal{R}^\mu), \boxplus, \boxtimes^\mu) \to (\mathcal{M}(\mathcal{R})^{op_1}, \boxplus, \boxtimes^{op}),$$

where we consider the opposite multiplication on $\mathcal{M}(\mathcal{R})^{op_1}$, i.e. fully reversed $A \boxtimes^{op} B = B \boxtimes A$.

Proof. The fact that transposition is a strict normal functor is Proposition 2.5.2. The strict additivity is obvious as $(A \boxplus B)^t = A^t \boxplus B^t$. For the multiplicativity consider the following sequence of equations:

$$\begin{aligned}
(A \boxtimes^\mu B)^t_{\omega(i_1,i_2),\omega(j_1,j_2)} &= (A \boxtimes^\mu B)_{\omega(j_1,j_2),\omega(i_1,i_2)}\\
&= A_{j_1,i_1} \circ B_{j_2,i_2}\\
&= B^t_{i_2,j_2} A^t_{i_1,j_1}\\
&= (B^t \boxtimes A^t)_{\omega(i_2,i_1),\omega(j_2,j_1)} \quad = (A^t \boxtimes^{op} B^t)_{\omega(i_1,i_2),\omega(j_1,j_2)}.
\end{aligned}$$

Thus transposition and the coordinatised tensor-product 2.3.1 commute up to one exchange of factors, yielding the claimed compatibility. $\square$

Thus we see that by strict additivity of transposition we get an infinite loop map of classifying spaces as:

$$B\mathcal{M}T \circ (\cdot)^t\colon B\mathcal{M}(\mathcal{R}) \to B\mathcal{M}(R)^{op_1}.$$

2.6 Basics on Nerves of Bicategories

On page 2 of [CCG] one can see various constructions of nerves, thus classifying spaces for bicategories, all homotopy equivalent after realisation. In previous versions of this thesis I considered the "Segal Nerve" as for instance in [CCG, p.21, Definition 5.2]. I finally noticed that in this bisimplicial set associated to a bicategory one of the simplicial directions only consists of homotopy equivalences [CCG, Theorem 6.2.]. Hence I can restrict to one simplicial direction, simplifying the exposition.

Definition 2.6.1. The nerve of a bicategory $\mathcal{C}$ (with only isomorphism 2-cells) is the simplicial set with n-simplices pseudofunctors:

$$N\mathcal{C}_n := \mathbf{NorHom}([n], \mathcal{C}).$$

Let me be more explicit about this, I consider the ordered set $[n] = \{0 < 1 < \ldots < n-1 < n\}$ as a 1-category, which is a bicategory with only identity 2-cells. Then a pseudofunctor

$$(F, \varphi)\colon [n] \to \mathcal{C}$$

is the same thing as a collection of objects $F_i \in Ob\mathcal{C}$, and for each pair $0 \leq i < j \leq n$ a choice of 1-cell $A_{i<j}\colon F_i \to F_j \in Ob\mathcal{C}(F_i, F_j)$ (where normality corresponds to the fixed choice $A_{i \leq i} = id_{F_i}$), and for each triple $i < j < k$ a 2-cell $\varphi_{i<j<k}\colon A_{jk}A_{ij} \to A_{ik}$, assembling to the compositor φ of F, which is hence associative in the appropriate sense.

Compare page 22 of [LP], where there is also a condition on identities I do not need, because I only consider normal functors.

So for a bicategory $\mathcal{C}$ (possibly enriched) we get a simplicial set:

$$N\mathcal{C}\colon \Delta^{op} \to Set.$$

I want to elaborate on the simplicial operators, let

$$\Phi\colon [n] \to [m]$$

be a monotone map: The effect on an n-simplex $F\colon [n] \to \mathcal{C}$ is then given as:

$$\Phi^* F(i) := F \circ \Phi(i),$$

on 1-cells we have:

$$\Phi^* A_{ij} := A_{\Phi(i),\Phi(j)}\colon F_{\Phi(i)} \to F_{\Phi(j)},$$

which is the identity, if $\Phi(i) = \Phi(j)$ according to the normality condition on F. Finally on compositors we get:

$$\Phi^* \varphi_{i<j<k} := \varphi_{\Phi(i),\Phi(j),\Phi(k)},$$

which is the identity if any two of the three indices coincide. This is coherent, because I only consider bicategories with strict identity 1-cells.

Remark 2.6.2. The terminology varies, which is partly due to the fact that there are at least 10 reasonable ways to define a nerve for bicategories (cf. the diagram [CCG, p. 2]). This particular construction is called the "unitary geometric nerve" in [CCG], where more generally all lax functors are considered. These coincide with pseudofunctors for bicategories with just isomorphism 2-cells. In particular the warning after Theorem 6.5 in [CCG] does not apply for bicategories with all 2-cells isomorphisms.

We are used to the fact that natural transformations of functors on 1-categories induce homotopies. For bicategories the same argument yields that an arbitrary pseudonatural transformation induces a homotopy. The observation is not original, but in the presence of 10 different nerve constructions I want to exhibit this fact specifically for the one I use.

Proposition 2.6.3. *A pseudonatural transformation η of pseudofunctors*

$$F, G\colon \mathcal{C} \Rightarrow \mathcal{D}$$

is equivalent to a pseudofunctor $\mathcal{C} \times I \to \mathcal{D}$, hence induces a map:

$$N(\mathcal{C} \times I) \cong N\mathcal{C} \times NI \to N\mathcal{D}.$$

Proof. This is plainly the universal property of the product in bicategories, i.e., $Fun(\mathcal{A}, \mathcal{C} \times \mathcal{D}) = Fun(\mathcal{A}, \mathcal{C}) \times Fun(\mathcal{A}, \mathcal{D})$, where Fun can be any of the classes of functors between bicategories. Thus in particular for $\mathcal{A} = [n]$ and Fun the class of normal pseudofunctors we get the claimed natural isomorphism. □

2.6.1 Opposition of a Bicategory and its Nerve

As I alluded to at the end of section 2.5, I want to induce an involution on the nerve of a bicategory. For that I need one last preparation.

Definition 2.6.4. Let the reversal functor

$$r\colon \Delta \to \Delta$$

be given as the identity on objects and on morphisms $\Phi\colon [n] \to [m]$ define:

$$r(\Phi)(i) := m - \Phi(n - i).$$

Given a simplicial object in any category $X\colon \Delta^{op} \to \mathcal{C}$, set $X^{op} := X \circ r$, analogously for cosimplicial objects.

The special point special about *Top* as a target category is the cosimplicial object that defines geometric realisation. (The analogous isomorphism in chain complexes *Ch* is given by only a sign depending on the chain degree.)

Lemma 2.6.5. *Let $\Delta^\bullet \colon \Delta \to Top$ be the cosimplicial space defined as usual:*

$$\Delta^n = \{(t_0, \dots, t_n) \in I^{n+1} | \sum t_i = 1\},$$

with cosimplicial operators:

$$\delta^i(t_0, \dots, t_{n-1}) = (t_0, \dots, t_{i-1}, 0, t_i, \dots, t_{n-1})$$

and

$$\sigma^i(t_0, \dots, t_n) = (t_0, \dots, t_{i-1}, t_i + t_{i+1}, t_{i+2}, \dots, t_n).$$

Then we have an isomorphism of cosimplicial topological spaces:

$$\Gamma \colon \Delta^\bullet \to \Delta^\bullet \circ r.$$

Proof. Define $\Gamma(t_0, \dots, t_n) = (t_n, \dots, t_0)$, this is obviously a degreewise homeomorphism, and the identities:

$$\Gamma \circ \delta^i = \delta^{n-i} \circ \Gamma$$

$$\Gamma \circ \sigma^i = \sigma^{n-i} \circ \Gamma$$

give that Γ is an isomorphism of cosimplicial objects. $\square$

Lemma 2.6.6. *Let $X \colon \Delta^{op} \to Top$ be a simplicial space (in particular sets with discrete topology), then consider geometric realisation as a coend:*

$$X \otimes_\Delta \Delta^\bullet = |X|.$$

Then we have the identity

$$X \otimes (\Delta^\bullet \circ r) = (X \circ r) \otimes \Delta^\bullet,$$

and hence a natural homeomorphism of realisations:

$$X \otimes \Gamma \colon |X| \to |X \circ r|.$$

Proof. The identity
$$X \otimes (\Delta^\bullet \circ r) = (X \circ r) \otimes \Delta^\bullet$$
can be seen as follows. Both objects are quotients of the object
$$\coprod_n X_n \times \Delta^n,$$
because r does not change anything on the simplices. The identification according to $X \otimes (\Delta^\bullet \circ r)$ then is:
$$[x, \delta^{n-i} t] = [d_i x, t],$$
and in $(X \circ r) \otimes \Delta^\bullet$ it is:
$$[d_{n-i} x, t] = [x, \delta^i t].$$
So the idenfications are just listed in a different order, but the equivalence relation we divide out is the same.

As a consequence the natural homeomorphism spells out:
$$\begin{aligned} X \otimes \Gamma \colon |X| &\to |X \circ r| \\ [x, (t_0, \dots, t_n)] &\mapsto [x, (t_n, \dots, t_0)]. \end{aligned}$$
□

With the simplicial considerations in place I define the classifying space of a bicategory as follows:

Definition 2.6.7. Given a bicategory $\mathcal{C}$ consider its nerve, which is a simplicial set as defined before:
$$N\mathcal{C} \colon \Delta^{op} \to Set.$$
The geometric realisation of this simplicial set then defines the classifying space of $\mathcal{C}$:
$$B\mathcal{C} := |N\mathcal{C}|.$$

We can understand opposing 1-cells as the opposition of simplicial objects by precomposition with the reversal functor $r \colon \Delta \to \Delta$.

Lemma 2.6.8. *The nerve of the bicategory $\mathcal{C}^{op_1}$ with reversed composition of 1-cells is isomorphic to the r-reversed simplicial set $NC \circ r$:*
$$NC \circ r \cong N(C^{op_1}).$$

Proof. This is immediate from the definition. The core point is that opposing functors $[n] \to \mathcal{C}$ does change the direction of 1-cells, but does not change the direction of 2-cells, just their indexing. $\square$

Hence we find that the homeomorphism $B\mathcal{C} \cong B\mathcal{C}^{op}$ extends to bicategories:

Lemma 2.6.9. *The isomorphism* $\Gamma\colon \Delta^\bullet \to \Delta^\bullet \circ r$ *extends to a natural homeomorphism:*

$$\Gamma\colon B\mathcal{C} \to B\mathcal{C}^{op_1}.$$

So the homeomorphism interprets a sequence of n 1-cells in $\mathcal{C}$ as an n-sequence of the opposed 1-cells.

Definition 2.6.10. For $\mathcal{R}$ a bimonoidal category with involution T define the induced involution on its module bicategory as follows:

$$B\mathcal{M}(\mathcal{R}) \xrightarrow{B\mathcal{M}(T)} B\mathcal{M}(\mathcal{R}^\mu) \xrightarrow{B(\cdot)^t} B\mathcal{M}(\mathcal{R})^{op_1} \xrightarrow{\Gamma} B\mathcal{M}(\mathcal{R}).$$

Recall that transposition and the involution are covariant with respect to the 2-cells, so the 1-cells of the functors are opposed twice, but the 2-cells are never opposed, so the constraints of F, G, H (for $n \geq 2$) do not change their direction.

Remark 2.6.11. Chasing through the definitions and taking into account the homeomorphism

$$B\mathcal{M}(\mathcal{R}) \cong \coprod_n |BGL_n\mathcal{R}|,$$

where $BGL_n\mathcal{R}$ is the bar construction on the monoidal category $GL_n\mathcal{R}$ as defined in [BDR, Definition 3.8], it is easy to see that this is precisely the same involution as defined in [R].

2.7 Examples for Nerves of Bicategories

There is an integral class in degree 3, from which I can bootstrap my calculations of the involution on $V(1)_*K(ku)$. I can describe it easily as induced from a functor $\Sigma^2\mathbb{S}^1 \to \mathcal{M}(\mathcal{V}_\mathbb{C})$, thus induced by a map $\mathbb{S}^3 \to K(\mathbb{Z}, 3) \to K(ku)$. For this I want to prepare some preliminaries on the nerve of bicategories. For this section recall that we can understand the totally ordered set $[n] = [0 < 1 < \ldots < n]$ as a 1-category, thus as a bicategory with discrete morphism categories.

Proposition 2.7.1. *Let $\mathcal{C}$ be an arbitrary bicategory, and $F\colon [n] \to \mathcal{C}$ a strong normal functor. Then F is uniquely determined by its restriction to all 2-faces, i.e., by all restrictions*

$$[2] \to [n] \to \mathcal{C}.$$

Such a system of pseudofunctors $[2] \to \mathcal{C}$ determines a (unique) pseudofunctor $[n] \to \mathcal{C}$ if and only if each four compatible 2-faces can be extended over $[3]$

$$\begin{tikzcd} (\partial\Delta^3 =) \coprod_4 [2] \arrow[r] \arrow[d] & \mathcal{C} \\ {[3],} \arrow[ur] & \end{tikzcd}$$

where the boundary can obviously not be made into a (bi)category, but we can still express it as functors on the disjoint union subject to the appropriate compatibility on 1-cells.

Proof. This follows by inspecting the definition of pseudofunctor carefully. The data given by functors $[2] \to \mathcal{C}$ precisely gives the compositor 2-cells, and the condition on extending a functor on the boundary of $[3]$ to all of $[3]$ is precisely the associativity condition on compositors 1.1.12. □

The following result is classical and implicit in Section 5 of [Str2], where Street even more generally considers nerves of n-categories for each n. However the exposition is quite dated, so I want to phrase the specific result I need in the context I set up here.

Proposition 2.7.2. *The bicategory $\Sigma^2 A$ with A an abelian (possibly topological) group with one object $*$, one 1-cell id_* and $\Sigma^2 A(\mathrm{id}_*, \mathrm{id}_*) = A$ yields as classifying space a double delooping of A, i.e., there is a homotopy equivalence*

$$\Omega^2 |N\Sigma^2 A| = A.$$

Thus define $B^2 A = |N\Sigma^2 A|$.

In particular, if A is a discrete group we get $B^2 A = K(A, 2)$, and for $A = \mathbb{S}^1$ we have $B^2\mathbb{S}^1 = K(\mathbb{Z}, 3)$, so $\Sigma^2\mathbb{S}^1$ is a bicategory modelling a $K(\mathbb{Z}, 3)$.

Proof. We see immediately from the definition $N\Sigma^2 A_0 = N\Sigma^2 A_1 = \{*\}$, as well as $N\Sigma^2 A_2 = A$. The functors $r_{a,b,a+b}\colon [0 < 1 < 2 < 3] \to \mathcal{C}$ with $r_{a,b,a+b}(012) = a, r_{a,b,a+b}(023) = b, r_{a,b,a+b}(123) = a + b, r_{a,b,a+b}(013) = \mathrm{id}_{\mathrm{id}_*}$ introduce the relations of the Bar complex, thus we get that $N\Sigma^2 A$ is a model for the double delooping as claimed. □

The Prototypical Class in $H(\mathcal{MV}_{\mathbb{C}}) = K(ku)$

Definition 2.7.3. Consider the topologically enriched 1-category SX for X an arbitrary topological space defined as

$$0 \xrightarrow{X} 1.$$

It is a category for arbitrary X, because no non-trivial compositions need to be defined. The classifying space is the suspension of X, hence in particular we can realise spheres by $X = \mathbb{S}^n$, giving $BSX = \mathbb{S}^{n+1}$. Call it the **directed suspension**.

Example 2.7.4. Consider the categories $\mathcal{V}_{\mathbb{C}}$ and $\mathcal{M}_{\mathbb{C}}$ and the directed suspension of the topological circle $S\mathbb{S}^1$. The functor $u\colon S\mathbb{S}^1 \to \mathcal{V}_{\mathbb{C}} \subset \mathcal{M}_{\mathbb{C}}$ with $u(0) = u(1) = 1$ and the identity on morphisms realises the Bott class on classifying spaces

$$\mathbb{S}^2 \to \coprod_n BGL_n\mathbb{C} \to \Omega B\left(\coprod_n BGL_n\mathbb{C}\right) \simeq BU \times \mathbb{Z},$$

since the Bott class can be represented as

$$\Sigma\mathbb{S}^1 = \Sigma U(1) \to BU(1) \simeq \mathbb{C}P^\infty \to BU_\otimes.$$

By the fact that the objects 0 and 1 are sent to the same object we get a factorisation over the one-point suspension $\Sigma\mathbb{S}^1$, because $\mathbb{S}^1$ is an associative monoid:

$$\begin{tikzcd} S\mathbb{S}^1 \arrow[r] \arrow[dr] & \mathcal{V}_{\mathbb{C}} \\ & \Sigma\mathbb{S}^1, \arrow[u] \end{tikzcd}$$

which on classifying spaces realises:

$$\begin{tikzcd} \mathbb{S}^2 \arrow[r] \arrow[dr] & \coprod_n BGL_n\mathbb{C} \arrow[r, dashed] & BU \times \mathbb{Z} \\ & B\mathbb{S}^1 \simeq K(\mathbb{Z},2). \arrow[u] & \end{tikzcd}$$

We can suspend these categories to bicategories analogously. Consider a 1-category $\mathcal{C}$, and define its directed suspension bicategory $S\mathcal{C}$ by

$$0 \xrightarrow{\mathcal{C}} 1,$$

where again we do not need a composition for 1-cells, hence objects of $\mathcal{C}$. It realises the (unreduced) suspension on classifying spaces, i.e., $BS\mathcal{C} = \Sigma B\mathcal{C}$.

In particular we get the following example.

Example 2.7.5. We can suspend the category $S\mathbb{S}^1$ to the bicategory $SS\mathbb{S}^1 = S^2\mathbb{S}^1$ realising $\mathbb{S}^3$ on classifying spaces. Thus we get a directed suspension of the "Bott functor" above

$$Su\colon S^2\mathbb{S}^1 \to S\mathcal{V}_{\mathbb{C}}.$$

In general the directed suspension of a bipermutative category includes into the bicategory of matrices $j\colon S\mathcal{R} \to \mathcal{M}(\mathcal{R})$ by the inclusion: on objects $j_0(0) = j_0(1) = 2$, on 1-cells: $j_1(r) = \begin{pmatrix} 1 & r \\ & 1 \end{pmatrix}$, and the identification $\mathcal{M}(\mathcal{R})(j_1 r, j_1 s) \cong \mathcal{R}(1,1)^{\times 2} \times \mathcal{R}(0,0) \times \mathcal{R}(r,s)$, yields the inclusion $j_2\colon \mathcal{R}(r,s) \to \{\mathrm{id}_1\}^{\times 2} \times \{\mathrm{id}_0\} \times \mathcal{R}(r,s) \subset \mathcal{R}(1,1)^{\times 2} \times \mathcal{R}(0,0) \times \mathcal{R}(r,s)$.

Example 2.7.6. For the suspended Bott functor we get

$$S^2\mathbb{S}^1 \to S\mathcal{V}_{\mathbb{C}} \to \mathcal{M}(\mathcal{V}_{\mathbb{C}}),$$

with the additional factorisation over the double one-point suspension $\Sigma^2\mathbb{S}^1$, which is a bicategory because $\Sigma\mathbb{S}^1$ is a monoidal category, because $\mathbb{S}^1$ is an abelian group. So we get

$$\begin{tikzcd} S^2\mathbb{S}^1 \arrow[r] \arrow[dr] & \mathcal{M}(\mathcal{V}_{\mathbb{C}}) \\ & \Sigma^2\mathbb{S}^1 \arrow[u] \end{tikzcd}$$

and thus on classifying spaces:

$$\begin{tikzcd} \mathbb{S}^3 \arrow[r] \arrow[dr] & B\mathcal{M}(\mathcal{V}_{\mathbb{C}}) \arrow[r, dashed] & \Omega B\left(\coprod BGL_n\mathcal{V}_{\mathbb{C}}\right) \\ & B^2\mathbb{S}^1 \simeq K(\mathbb{Z},3). \arrow[u] & \end{tikzcd}$$

This is the class representing the *Dirac Monopole* in $\mathbb{R}^3$ considered as a 2 vector bundle as described in [ADR].

We can simplify the discussion of the suspended Bott class as given in the following section. Since it is given by a class over the multiplicative unit 1, its 1×1-matrix is weakly invertible, and we do not need to extend to 2×2-matrices.

2.7.1 The Involution on the Monopole $\mathbb{S}^3 \to \mathcal{M}(\mathcal{V}_\mathbb{C})$

The examples above determine a class $a \in \pi_3 H\mathcal{M}(\mathcal{V}_\mathbb{C}) = \pi_3 K(ku)$, the Dirac Monopole (cf. [ADR]), thus we can understand the involution on it.

Lemma 2.7.7. *Consider the class $\bar{a} \in \pi_3 K(ku) = K_3(ku)$ represented by the functor:*

$$S_1 u\colon S^2\mathbb{S}^1 \to \mathcal{M}(\mathcal{V}_\mathbb{C})$$

with S_1 the functor which assigns matrix rank 1 to the objects $j(0) = j(1) = 1$, and considers the 1-cells of $S^2\mathbb{S}^1$ both as the 1×1-matrix (1) in $\mathcal{M}(\mathcal{V}_\mathbb{C})_1$ It obviously commutes with complex conjugation on 2-cells, while transposition has no effect. So the diagram:

$$\begin{tikzcd} S^2\mathbb{S}^1 \arrow[r, equal] \arrow[d, "S_1 u"] & S^2\mathbb{S}^1 \arrow[r, "\overline{(\cdot)}_2"] \arrow[d, "S_1 u"] & S^2\mathbb{S}^1 \arrow[d, "S_1 u"] \\ \mathcal{M}(\mathcal{V}_\mathbb{C}) \arrow[r, "(\cdot)^t"] & \mathcal{M}(\mathcal{V}_\mathbb{C})^{op_1} \arrow[r, "\overline{(\cdot)}_2"] & \mathcal{M}(\mathcal{V}_\mathbb{C})^{op_1} \end{tikzcd}$$

strictly commutes.

Additionally observe that the functor S_1 is oblivious to opposition of 1-cells, because in a directed suspension this only amounts to relabelling source and target object. In summary we find:

Theorem 2.7.8. *On the class $a \in \pi_3 K(ku)$ the internalised involution induced by $\mathcal{M}(\mathcal{V}_\mathbb{C})$ is represented as the composite:*

$$|NS^2\mathbb{S}^1| \xrightarrow{|N\overline{(\cdot)}_2|} |NS^2\mathbb{S}^1| \xrightarrow{\cong} |N(S^2\mathbb{S}^1)^{op_1}| = |\widetilde{N(S^2\mathbb{S}^1)}| \xrightarrow{\Gamma} |N(S^2\mathbb{S}^1)|.$$

In particular, the outer maps induce multiplication by -1 on a, thus the involution induces the identity $a \mapsto a$.

Proof. As above we see that the double directed suspension of $\mathbb{S}^1$ realises to $\Sigma^2\mathbb{S}^1 = \mathbb{S}^3$. The conjugation represents a reflection along an equator, thus has degree -1.

For Γ consider non-degenerate simplices of maximal degree. These are precisely given by functors $[0 < 1 < 2] \to S^2\mathbb{S}^1$ assigning for example 01 to the initial 1-cell, 12 to the identity 1-cell, 02 to the terminal 1-cell, and choosing any $x \in \mathbb{S}^1$ as compositor 2-cell. Abusively call such a functor x as well. Then the maximal cells are parametrised as $[x, (t_0, t_1, t_2)]$. Here Γ acts as: $[x, (t_0, t_1, t_2)] \mapsto [x, (t_2, t_1, t_0)]$. In particular it has degree given by the sign of the transposition (02), which is thus -1. $\square$

3 Multiplicative Delooping of Bipermutative Bicategories

In the previous chapters I convinced the reader that bipermutative bicategories exist, and that they occur when one wants to study algebraic K-theory of a bipermutative (1-)category. In particular, the primary example of this thesis $K(ku)$ can be described as the Eilenberg-MacLane-spectrum of the bicategory of finitely generated free modules of finite-dimensional complex vector spaces:

$$K(ku) = H\mathcal{M}(\mathcal{V}_{\mathbb{C}}).$$

To tie this in with the calculations made by Christian Ausoni in the papers [A-THH, AR1, A-Kku] we need a combinatorial handle on the E_∞-structure on $K(ku)$ induced by the tensor-product on $\mathcal{M}(\mathcal{V}_{\mathbb{C}})$. To this end I modify the delooping given by Angélica Osorno in [Os] in a manner analogous to [EM] (cf. in particular the paragraph after Definition 4.3.) so the resulting construction allows an induced multiplication by the multiplicative structure of a bipermutative bicategory. I do restrict to the case of E_∞-structures, and also use a specific E_∞-operad, the Barratt-Eccles-operad in a tentative multicategory of permutative bicategories.

The reader should compare the delooping of this chapter to the delooping in section 6 of [GJOs]. The authors, however, are driven by the desire to generalise [Th1] to permutative bicategories, thus their emphasis is different from mine. This makes the deloopings differ in a few ways, which I suspect are inessential.

3.1 The Additive Grothendieck Construction

This section is where the work in 1.1.4 to rewrite the delooping construction of [EM] becomes fruitful. Since the additive symmetric monoidal structure on $\mathcal{M}(\mathcal{R})$ as described by [Os] is sufficiently strict that the Grothendieck construction 1.1.37 can be used for symmetric monoidal bicategories as well. So we need to study which functors are adequate for the analogous construction of $\mathcal{C}(A_+, n)$ such that the delooping given by Osorno is the case $n = 1$ and such that we get pairings 3.3.3

$$\mathcal{C}(A_+, n) \times \mathcal{C}(A_+, m) \to \mathcal{C}(A_+, n+m),$$

which induces an E_∞-multiplication on the resulting spectrum 3.3.16.

I again use the shorthand $Ob\mathcal{C} = \mathcal{C}_0$ and for 1-cells when I do not want to refer to their source and target I write $Mor\mathcal{C} = \mathcal{C}_1$.

Remark 3.1.1. As indicated before $\boxplus$, i.e., block sum of matrices turns $\mathcal{M}(\mathcal{R})$ into a permutative bicategory. The tensor-product does not.

Symmetric monoidal bicategories of the strict type of permutative bicategories allow for the same construction of an associated $\mathcal{C}^+$ as in 1.1.37.

Lemma 3.1.2. *A permutative bicategory $\mathcal{C}$ has an associated pseudofunctor*

$$B_\mathcal{C} \colon \mathrm{Fin}_+ \to Bicat,$$

given by $B_\mathcal{C}(n_+) = \mathcal{C}^{\times n}$ and $F_\mathcal{C}(f \colon n_+ \to m_+) = f_* \colon \mathcal{C}^{\times n} \to \mathcal{C}^{\times m}$, where f_* is the strict functor $f_*(c_1, \dots, c_n)_j = \sum_{i \in f^{-1} j} c_i$, with compositors given by the additive symmetry: $\varphi^{f,g} \colon f_* \circ g_* \Rightarrow (fg)_*$ as a strict natural transformation of strict functors.

Proof. The proofs in 1.1.4 transfer without any problems when I restrict the target to be the 2-category of bicategories with strict functors as 1-cells and strictly natural transformations as 2-cells, which works because of the strictness of permutative bicategories. $\square$

In particular the construction 1.1.37 translates literally:

Definition 3.1.3. Given a permutative bicategory $(\mathcal{C}, +, 0, c_+)$ define its additive Grothendieck construction $\mathcal{C}^+$ as follows: It has objects:

$$\mathcal{C}_0^+ = \coprod_{n \geq 0} \mathcal{C}^{\times n}$$

and morphism categories:

$$\mathcal{C}^+((c_1, \dots, c_n), (d_1, \dots, d_m)) = \coprod_{f \in \mathrm{Fin}_+(n_+, m_+)} \mathcal{C}^{\times m}(f_*(c_1, \dots, c_n), (d_1, \dots, d_m)),$$

with composition functors given for a triple of objects $a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_m), c = (c_1, \ldots, c_l)$ as:

$$\begin{array}{c}
\mathcal{C}^+(b,c) \times \mathcal{C}^+(a,b) \\
\Big\| \\
\coprod_{f,g} \mathcal{C}^{\times l}(g_* b, c) \times \mathcal{C}^{\times m}(f_* a, b) \\
\Big\downarrow {\scriptstyle \coprod id \times g_*} \\
\coprod_{f,g} \mathcal{C}^{\times l}(g_* b, c) \times \mathcal{C}^{\times l}(g_* f_* a, g_* b) \\
\Big\downarrow {\scriptstyle \coprod comp_{\mathcal{C}^l}} \\
\coprod_{f,g} \mathcal{C}^{\times l}(g_* f_* a, c) \\
\Big\downarrow {\scriptstyle \varphi^*} \\
\coprod_{f,g} \mathcal{C}^{\times l}((gf)_* a, c) \subset \mathcal{C}^+(a,c).
\end{array}$$

Identities are given by pairs $(id, (id))$ for $id \colon n_+ \to n_+$ and $(id) \colon (a_1, \ldots, a_n) \to (a_1, \ldots, a_n)$ the n-tuple of identities. In particular the identities are strict identities because the ones in $\mathcal{C}$ are strict.

The associator is given as follows: In each product bicategory $\mathcal{C}^l$ we have an associator given by the l-tuple with the appropriate instances of the $\mathcal{C}$-associator. For this paragraph call this α_l. Then consider the following two ways of forming a three-fold composite for a, b, c as above and $d = (d_1, \ldots, d_k)$:

$$\begin{tikzcd}
\mathcal{C}^+(c,d) \times \mathcal{C}^+(b,c) \times \mathcal{C}^+(a,b) \arrow[r] \arrow[d] & \mathcal{C}^+(b,d) \times \mathcal{C}^+(a,b) \arrow[d] \\
\mathcal{C}^+(b,d) \times \mathcal{C}^+(a,b) \arrow[r] & \mathcal{C}^+(a,d).
\end{tikzcd}$$

So we need a natural transformation of the two composition-functors:

$$\mathcal{C}^+(c,d) \times \mathcal{C}^+(b,c) \times \mathcal{C}^+(a,b) \Rightarrow \mathcal{C}^+(a,d),$$

which is defined on the category $\mathcal{C}^+(a,d) = \coprod_f \mathcal{C}^k(f_* a, d)$, hence it has components α_k.

Remark 3.1.4. Let me issue a warning here: I have no idea what happens, when one tries to apply the same construction to less strict symmetric monoidal categories - even of the type I defined in 2.2.3. I strongly suspect, it involves a lot more care.

In particular this construction is not naturally set up with respect to the context of bicategories, in that it is the Grothendieck construction of a functor into the 2-category of bicategories with strict functors and strict natural transformations as morphisms. Not every strong functor of symmetric monoidal 1-categories can be made strict - consider for instance the strictification functor $\varepsilon\colon (\mathcal{C},+)^{st} \to (\mathcal{C},+)$, which augments the permutative strictification of an arbitary symmetric monoidal category 1.1.2. Its suspension to bicategories 1.1.11 hence is an example of a strong normal functor/pseudofunctor, which is not equivalent to a strict functor.

Remark 3.1.5. If the input bicategory $\mathcal{C}$ is in fact a 2-category, i.e., a bicategory with associator-cells only identities, then $\mathcal{C}^+$ is a 2-category as well.

Remark 3.1.6. Do note that the subcategory of morphisms with only discrete components is a strict 2-category, because its composition is just the one in Fin_+.

Furthermore, just as in the 1-categorical case, we have that each morphism 1-cell in $\mathcal{C}^+$ can be written uniquely as:

$$\mathbf{c} = (c_1, \ldots, c_n) \xrightarrow{(f,\mathrm{id})} f_*\mathbf{c} \xrightarrow{(\mathrm{id}_m,(A_1,\ldots,A_m))} (d_1, \ldots, d_m).$$

3.2 A Multiplicative Delooping for Bicategories

Reminder 3.2.1. Because it features prominently in this chapter let me recall the concept of an *equivalence* in a bicategory. It is a 1-cell, say $A\colon a \to b$, which has a 1-cell in the other direction $B\colon b \to a$ such that $AB \cong \mathrm{id}_b$ and $BA \cong \mathrm{id}_a$. By fixing directions and only demanding morphisms instead of isomorphism 2-cells we arrive at the concept of *adjoint* 1-cells, but I do not need that here.

Example 3.2.2. Adjoint as well as equivalence 1-cells in our primary example of interest $\mathcal{M}(\mathcal{M}_k)$ are just permutation matrices. To see this first note that we consider only isomorphism 2-cells, because we only have isomorphisms in $\mathcal{M}_k$, hence being adjoint and being equivalent is the same. Furthermore since $\mathcal{M}_k$ is skeletal the existence of an isomorphism $AB \to \mathrm{id}_n$ already gives an equality of the 1-cells, i.e., matrices, $AB = \mathrm{id}_n$, analogously for BA. Hence A and B are strictly invertible matrices, both in $GL(\mathbb{N})$, thus permutation matrices.

The natural analogue for bicategories of 1.1.4 is the following proposition:

Proposition 3.2.3. *There is a natural inclusion $(\mathcal{C}^{eq_1})^+ \to \mathcal{C}^+$, where the bicategory $\mathcal{C}^{eq_1}$ is the bicategory with the same objects as $\mathcal{C}$, and morphism categories on* 1*-cells only the equivalences and all* 2*-cells.*

Furthermore there is a natural inclusion $(\mathcal{C}^{eq_1,iso_2})^+ \to (\mathcal{C}^{eq_1})^+ \to \mathcal{C}^+$ where we restrict to just equivalence 1*-cells and isomorphism* 2*-cells.*

The Construction $\mathcal{C}(A_+, 1)$ for Permutative Bicategories

Many of the next considerations would probably work for bicategories with not just isomorphism 2-cells, if one modifies the results appropriately. Most of the time this means changing equivalences into adjunctions, which probably introduces a lot more thought about when these 1-cells compose appropriately. But since my emphasis is on delooping to a K-theory spectrum, in this chapter I only consider bicategories with $\mathcal{C} = \mathcal{C}^{iso_2}$, hence also $\mathcal{C}^+ = (\mathcal{C}^{iso_2})^+$. This in particular makes every adjunction in $\mathcal{C}$ already an equivalence.

To arrange the delooping bicategories as functor bicategories we need to understand the forgetful functor $\mathcal{C}^+ \to \mathrm{Fin}_+$ again, which yields a stronger assertion for bicategories than for 1-categories.

Proposition 3.2.4. *The forgetful functor $U\colon \mathcal{C}^+ \to \mathrm{Fin}_+$, which assigns each tuple of objects in $\mathcal{C}^+$, i.e., $(c_1, \dots, c_n)$ to its finite pointed set n_+ and on morphisms $(f, (A_1, \dots, A_m))$ forgets down to the discrete component in finite pointed sets $f\colon n_+ \to m_+$, is a strict functor of bicategories.*

Proof. This is trivially true, since the target is a 1-category, thus does not support non-trivial compositor 2-cells. □

Recall the “comma categories” introduced in 1.1.44: For an arbitrary finite set A with added disjoint basepoint $\{*\} \sqcup A = A_+$ consider the category of pointed maps under it, i.e., $A_+ \downarrow \mathrm{Fin}_+$ with objects maps from A_+ to a natural number $n_+ = \{0, 1, \dots, n\} = \{*\} \sqcup \{1, \dots, n\}$, and morphisms commutative triangles under A_+.

Recall that the index categories also have forgetful functors $T\colon A_+ \downarrow \mathrm{Fin}_+ \to \mathrm{Fin}_+$, which send each object to its target and forgets the commutativity of triangles under A_+.

This way I can define $\mathcal{C}(A_+, 1)$ for permutative bicategories:

Definition 3.2.5. The bicategory $\mathcal{C}(A_+, 1)$ has objects strong normal functors that lift T through U

$$\begin{tikzcd} & (\mathcal{C}^{eq_1,iso_2})^+ \arrow[d, "U"] \\ A_+ \downarrow \mathrm{Fin}_+ \arrow[r, "T"] \arrow[ur, dashed] & \mathrm{Fin}_+, \end{tikzcd}$$

i.e., send maps of finite sets under A_+ to equivalences in $\mathcal{C}^+$.

The category of morphisms between two such lifts F, G is given by the morphism category $\mathrm{Bicat}(J_*F, J_*G)$, with $J\colon (\mathcal{C}^{eq_1})^+ \to \mathcal{C}^+$ the natural inclusion. That is, we consider the morphism category with strong pseudonatural transformations, i.e., with isomorphism 2-cells but arbitrary 1-cells of $\mathcal{C}$, and modifications between those comprised of isomorphism 2-cells.

Remark 3.2.6. Recall that a map of finite based sets $f\colon A_+ \to B_+$ induces a map of the indexing categories in the opposite direction $f^*\colon B_+ \downarrow \mathrm{Fin}_+ \to A_+ \downarrow \mathrm{Fin}_+$, which is a functor over $T\colon B_+ \downarrow \mathrm{Fin}_+ \to \mathrm{Fin}_+$. By restricting lifting functors from $A_+ \downarrow \mathrm{Fin}_+$ to $B_+ \downarrow \mathrm{Fin}_+$ along f^* we thus get lifting functors from $B_+ \downarrow \mathrm{Fin}_+$, so in summary a strict normal functor $f_*\colon \mathcal{C}(A_+, 1) \to \mathcal{C}(B_+, 1)$ in the same direction as f.

Since in what follows the subbicategory of $\mathcal{C}^+$ with just equivalence 1-cells is the central object, I reduce the notation to $\mathcal{C}^{eq+}$ to refer to the additive Grothendieck construction on the subbicategory of equivalence 1-cells and isomorphism 2-cells of a permutative bicategory $(\mathcal{C}, +)$.

Remark 3.2.7. Again consider in $A_+ \downarrow \mathrm{Fin}_+$ the "full", actually discrete, subcategory given by characteristic functions $\chi_a\colon A_+ \to 1_+$ with $\chi_a(x) = *$ for $x \neq a$ and $\chi_a(a) = 1$. This yields a natural inclusion:

$$\chi_\bullet\colon A^\delta \to A_+ \downarrow \mathrm{Fin}_+.$$

On the other hand we have the subcategory of $A_+ \downarrow \mathrm{Fin}_+$ given by objects the bijections and morphisms between them. This gives an inclusion of the translation category associated to the bijections of A, or equivalently pointed bijections of A_+

$$E\Sigma_A \to A_+ \downarrow \mathrm{Fin}_+.$$

This inclusion embeds the full subcategory of initial objects of $A_+ \downarrow \mathrm{Fin}_+$, since each map under A_+ can be factored uniquely through a bijection. In particular

I do not refer to these initial objects as initial again, because the isomorphisms between them are prominent in the delooping.

The delooping is supposed to be a generalisation of the classical delooping of (topological) abelian groups, so we should expect the objects to be determined by their "summands". The following proposition should thus not be surprising, parallel to the analogous statement in 1.1.4.

Proposition 3.2.8. *Any pseudofunctor lifting T through U has a unique up to equivalence* strict representative*. More precisely: Any two functors with the same restrictions along $\chi_\bullet$ are naturally equivalent in $\mathcal{C}(A_+, 1)$.*

Proof. Choose a total ordering on A, hence a bijection $\sigma^A\colon A_+ \to |A|_+$, and consider a lifting functor $F\colon A_+ \downarrow \mathrm{Fin}_+ \to (\mathcal{C}^{eq})^+$.

By the assumption that F sends 1-cells to equivalences we have an equivalence in the product bicategory $\mathcal{C}^{\times|A|}$ of the form $F\sigma_A \to (F\chi_a)_{a\in A_+}$ given by the components associated to the diagrams in $A_+ \downarrow \mathrm{Fin}_+$:

$$\begin{tikzcd} A_+ \arrow[r, "\sigma_A"] \arrow[dr, "\chi_a"'] & {|A|_+} \arrow[d, "\rho^a"] \\ & 1_+. \end{tikzcd}$$

So the equivalence is given by $(\mathrm{id}_{|A|}, (F^{\mathcal{C}}\rho_a)_{a\in A})$ in $\mathcal{C}^{\times|A|} \subset \mathcal{C}^+$, for $F^{\mathcal{C}}$ the $\mathcal{C}$-1-cells of the equivalence without their discrete components ρ^a in $\mathcal{C}^+$. Choose an inverse to this equivalence in the product category, hence $\zeta_a F^{\mathcal{C}}(\rho_a) \cong \mathrm{id}_{F\chi_a}$ with the analogous isomorphism for the other composition of ζ_a with $F^{\mathcal{C}}(\rho_a)$.

Build the *strict representative* as follows: $F^{st}(\sigma_A) := (F\chi_a)_{a\in A}$. Any other object of $A_+ \downarrow \mathrm{Fin}_+$ has a unique morphism coming from σ_A, so for $p \in A_+ \downarrow \mathrm{Fin}_+$ set $F^{st}(p) := (p \circ \sigma_A^{-1})_*(F^{st}(\sigma_A)) \in \mathcal{C}^{\times|Tp|}$. Again drop σ_A^{-1} from the notation for instance by assuming A_+ totally ordered, thus a unique element of Fin_+ itself. For a commutative triangle under A_+:

$$\begin{tikzcd} A_+ \arrow[r, "p"] \arrow[dr, "qp"'] & n_+ \arrow[d, "q"] \\ & m_+ \end{tikzcd}$$

we need to have a morphism

$$F^{st}(p) = p_*(F^{st}(\sigma_A)) \to q_*p_*(F^{st}\sigma_A) \to (qp)_*(F^{st}(\sigma_A)) = F^{st}(qp),$$

which by construction of $\mathcal{C}^+$ we can take to be $(q, \varphi^{q,p})$, and this is obviously a morphism over q in Epi_+. So we have constructed F^{st} as a lift of T through U, which sends each commutative triangle in $A_+ \downarrow \mathrm{Fin}_+$ to morphisms in $\mathcal{C}^+$ with just discrete components and additive symmetries. In particular we can choose F^{st} with identity 2-cells, and thus have a strict normal functor, because the additive symmetries were assumed to be strictly natural for permutative bicategories.

By the decomposition of 1-cells in $\mathcal{C}^+$, we can uniquely write the map $F(p \circ \sigma_A^{-1})\colon F(\sigma_A) \to F(p)$ as its discrete component followed by a 1-cell with discrete component the identity

$$F\sigma_A \xrightarrow{(p,\mathrm{id})} p_* F\sigma_A \xrightarrow{(\mathrm{id}, F^{\mathcal{C}}(p\circ\sigma_A^{-1}))} Fp\,.$$

So we have in $\mathcal{C}^+$ with the equivalence 1-cells ζ_a as chosen before:

$$F^{st}p = p_*((F\chi_a)_{a\in A}) \xrightarrow{(\mathrm{id}, p_*((\zeta_a)_a))} p_* F\sigma_A \xrightarrow{(\mathrm{id}, F^{\mathcal{C}}(p\circ\sigma_A^{-1}))} Fp,$$

which we can promote to a pseudonatural transformation by choosing as the naturality 2-cells the inverses of the adequate compositor 2-cells F. This transformation then trivially commutes with the strict compositor of F^{st} and the ones of F, and has as 1-cells equivalences by construction. So we have established a pseudonatural equivalence $F^{st} \simeq F$, which only depended on data coming from F, while F^{st} even only depended on the restriction of F along $A^\delta \to A_+ \downarrow \mathrm{Fin}_+$, hence is as unique as claimed. □

Remark 3.2.9. At this point let me informally compare this construction to the one displayed in the proof of Theorem 3.6 in [Os]. The objects as described there are strict functors $(A_+ \downarrow \mathrm{Fin}_+)^{op} \to \mathcal{C}^+$, so precisely the chosen inverse equivalences ζ I just described. For notational convenience let me treat them as if the functors in [Os] were written down as covariant functors $A_+ \downarrow \mathrm{Fin}_+ \to \mathcal{C}^+$.

The passage to the strict representative as I indicated above shows that each pseudofunctor $A_+ \downarrow \mathrm{Fin}_+ \to \mathcal{C}^{eq+}$ lifting T through U is naturally equivalent to one that is not just a strict functor, but also just comprised of discrete components. So we can include the functors described by Osorno into $\mathcal{C}(A_+, 1)$ and find that the strict representative is of the kind described in [Os], so we get a surjection up to equivalence, which by inspection of the 1- and 2-cells described in that same proof is also an equivalence on the morphism categories. (Most specifically she describes the construction on $\underline{\mathbf{n}}$, which is n_+ in my convention, so a subbicategory of $\mathcal{C}(n_+, 1)$.)

I chose the morphism categories in the delooping construction just so that this equivalence is true.

The passage to the strict representative is sufficiently natural that the typical delooping result is an easy corollary:

Corollary 3.2.10. *We have a natural equivalence of bicategories:*

$$(\cdot)^{st} \colon \mathcal{C}(A_+, 1) \to \mathcal{C}^A.$$

Proof. We know that $\mathcal{C}^A$ is strictly equal to the bicategory of functors $A^\delta \to \mathcal{C}^+$ and we can restrict each functor to its components on $(\chi_a)_{a \in A}$, which is precisely A^δ as a full subcategory of $A_+ \downarrow \mathrm{Fin}_+$. So the inclusion of the product bicategory by the functor, which sends each tuple to a lifting functor which is its own strict representative, is an inverse equivalence for $(\cdot)^{st}$. On the left we find that the natural equivalence to the identity is just the one described at the end of the proof before. On the right we have: Making a tuple into a strict functor and then restricting to its χ_a-summands is strictly equal to the identity functor on $\mathcal{C}^A$. $\square$

Remark 3.2.11. This is the initial step in the induction to prove the analogous equivalence for the higher delooping bicategories $\mathcal{C}(A_+, n)$.

3.2.1 The Construction $\mathcal{C}(A_+, n)$ for Permutative Bicategories

This section is where the rewriting of the delooping constructions of [EM] and [Os] in 1.1.4 and the section before comes to fruition, because this way it easily generalises to using $(A_+ \downarrow \mathrm{Fin}_+)^{\times n}$ as the index category, and letting coherence 2-cells take care of themselves by using pseudofunctors.

Recall from 1.1.4 the target functors $T_n \colon (A_+ \downarrow \mathrm{Fin}_+)^n \to \mathcal{C}_+$ given in 1.1.58: We have the analogous definition of $\mathcal{C}(A_+, n)$ for bicategories.

Definition 3.2.12. For a permutative bicategory $(\mathcal{C}, +)$ the delooping bicategory $\mathcal{C}(A_+, n)$ has as objects strong normal functors F lifting T_n through U:

$$\begin{tikzcd} & (\mathcal{C}^{eq})^+ \arrow[d, "U"] \\ (A_+ \downarrow \mathrm{Fin}_+)^n \arrow[ur, dashed, "F"] \arrow[r, "T_n"'] & \mathrm{Fin}_+. \end{tikzcd}$$

Its morphism bicategories are again the pseudonatural transformations and modifications of the functors after including by $(\mathcal{C}^{eq})^+ \to \mathcal{C}^+$.

Remark 3.2.13. Do note that as in the case in 1.1.4 the delooping bicategories $\mathcal{C}(A_+, n)$ have a canonical basepoint object given by the functor O_n, which has as objects the adequate zero-tuples of each degree and morphisms consisting of the adequate discrete components with id_0 as its second component.

Remark 3.2.14. Since for $n = 1$ we do not have to choose bijections for the smash product in Fin_+ the functor T_1 is strictly the same as T in the section before, in particular I described the same bicategory of functors.

It is consistent to set $\mathcal{C}(A_+, 0) = \mathcal{C}$, since $(A_+ \downarrow \mathrm{Fin}_+)^0 = *$ is the one-point category.

Remark 3.2.15. Apart from an opposition of the indexing category $A_+ \downarrow \mathrm{Fin}_+$ this definition would read "strict normal" functors in the delooping considered by [Os], which works well there because the additive structure is strict enough. Since I want to induce a multiplicative structure from a symmetric monoidal structure as given by $\boxtimes$ in 3, which prominently features a pseudofunctor which is usually not strict, I need to consider more generally all strong normal pseudofunctors with possibly non-trivial compositor 2-cells.

It is possible to give an explicit construction of the delooping bicategories of a permutative bicategory along the lines of [EM] and [Os]. However, since the tensor functor I describe in 2.3.1 has a non-trivial isomorphism 2-cell I cannot restrict to strict additors the way Osorno does in [Os, Proof of Theorem 3.6, p. 11], but have to allow potentially non-trivial isomorphism 2-cells. This becomes unwieldy in the explicit construction, so I arranged the delooping by functor bicategories, analogous to the rewriting of [EM] I present in 1.1.4 above. In particular the construction runs parallel to the 1-categorical case, with equivalences inserted where there are isomorphisms for permutative 1-categories.

Remark 3.2.16. At this point an informal comparison to [EM] is convenient. For the case of bicategories with discrete morphism categories (or actually topological spaces or simplicial sets, which are discrete as 1-categories, but possibly non-discrete as spaces,) we can compare to Construction 4.4. on page 19 of [EM]. The systems described there are indexed over arbitrary product categories $(A_1 \downarrow \mathrm{Epi}_+) \times \ldots \times (A_n \downarrow \mathrm{Epi}_+)$. By passing their based systems of subsets $S = (S_1, \ldots, S_n)$ to their unbased components we can associate to each such subset a characteristic map, and thus an object in $A_i \downarrow \mathrm{Epi}_+$. The additors ρ described there are then given under the condition that we can factor a characteristic map over 2_+, and hence we get an associated map for this $2_+ \to 1_+$-component.

Condition (1) is then the fact that the wedge at the basepoints of the index-categories $A_i \downarrow \mathrm{Epi}_+$, which is given by tuples of maps, which have any component mapping to 0_+, smashes to 0_+ and is hence sent to 0_+ by T_n. Condition (2) is the normality of the functor, i.e., strictly respecting identities. Condition (3) expresses the fact that the choice of which subset to map to which element in 2_+ should not matter. Condition (4) is simply the functoriality, as in respecting composites strictly, because the context in [EM] are enriched 1-categories, and last condition (5) is by construction of $\mathcal{C}^+$ and by the remark 1.1.41, which still applies for the additive Grothendieck construction on even general bicategories, also just strictly respecting composites. Here $1 + c_+ + 1$ is the twist needed to express the following trivially commutative diagram in $\mathrm{Epi}_+^{\times 2}$:

$$\begin{tikzcd} 2_+ \wedge 2_+ \arrow[r] \arrow[d] & 1_+ \wedge 2_+ \arrow[d] \\ 2_+ \wedge 1_+ \arrow[r] & 1_+ \wedge 1_+, \end{tikzcd}$$

by flattening it with the bijections ω chosen before, such that we have get indexing sets appropriate for summations. Morphisms are the appropriate stricter version of the ones considered in [Os] as well, so the same remarks apply.

In particular do note that $\mathcal{C}(A_+, n)$ could easily be generalised to n different indexing categories, but since the resulting spectrum is defined by inserting $\mathbb{S}^1$ for each A_+, I chose to reduce to the case with equal inputs.

As indicated at the end of the last section I prove the equivalence of $\mathcal{C}(A_+, n)$ to the appropriate product bicategory by displaying the inductive step in constructing the equivalence. For this let me emphasise that for arbitrary bicategories (possibly enriched) we have the following simple case of the exponential law, for A, B sets considered as discrete categories, hence bicategories:

$$Fun(A, Fun(B, \mathcal{C})) \cong Fun(A \times B, \mathcal{C}) \cong \mathcal{C}^{A \times B}.$$

In particular I can reduce the index juggling quite a bit by proving this form of the following theorem.

Theorem 3.2.17. *For $(\mathcal{C}, +)$ a permutative bicategory we have the following natural equivalence of bicategories:*

$$\mathcal{C}(A_+, n) \simeq Set(A, \mathcal{C}(A_+, n-1)).$$

Inductively we find the natural equivalence:

$$\mathcal{C}(A_+, n) \simeq \mathcal{C}^{A^{\times n}}.$$

Proof. The functor $\mathcal{C}(A_+, n) \to Set(A, \mathcal{C}(A_+, n-1))$ is – as in the case of permutative 1-categories – given by restricting along $(A_+ \downarrow \mathrm{Fin}_+)^{n-1} \times A^\delta \to (A_+ \downarrow \mathrm{Fin}_+)^n$ with one component the inclusion of A as the full discrete subcategory of characteristic functions χ_a in $A_+ \downarrow \mathrm{Fin}_+$.

The same reasoning as for $\mathcal{C}(A_+, 1)$ before yields for each functor in $\mathcal{C}(A_+, n)$ a strict representative, which in this case means strict with respect to one of the $A_+ \downarrow \mathrm{Fin}_+$-factors, for instance the last one as described above. So the restriction has an inverse functor given by extending an A-tuple of functors in $\mathcal{C}(A_+, n-1)$, by sending the maps in the last factor to the appropriate discrete components in $\mathcal{C}^+$, and thus summing up the functors according to the chosen bijections. □

Furthermore we have the following generalisations of the analogous results in 1.1.4.

Proposition 3.2.18. *For each $n \in \mathbb{N}$ we have a strictly natural strict Σ_n-action on $\mathcal{C}(A_+, n)$, given by permuting the inputs and pushing forward with the induced symmetry $\chi^\wedge$ of* Fin_+ *in $\mathcal{C}^+$.*

Proposition 3.2.19. *For each pointed finite set A_+ we have natural strict extension functors*

$$A_+ \wedge \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, 1+n),$$

which are $\Sigma_1 \times \Sigma_n$-equivariant.

Both proofs essentially proceed as the case for 1-categories, where the strictness of the functors is a consequence of the fact that they only use the discrete components in $\mathcal{C}^+$, which are part of the included 2-category on all objects but just discrete morphisms.

Remark 3.2.20. The extension functors as well as the Σ_n-action of the propositions above strictly respect the basepoint functors O. For the extension this is obvious, since we extend functors by zeroes. For the symmetric action observe that in particular the assignment on objects gives constant tuples, which are hence invariant under permutations.

The delooping construction $\mathcal{C}(A_+, n)$ is directly comparable to Osorno's delooping [Os] by restricting the source of the lifting functors.

Proposition 3.2.21. *For two finite pointed sets A_+, B_+ we have a functor*

$$A_+ \downarrow \mathrm{Fin}_+ \times B_+ \downarrow \mathrm{Fin}_+ \to (A_+ \wedge B_+) \downarrow \mathrm{Fin}_+,$$

which by the canonical identification $A_+ \wedge B_+ \cong (A \times B)_+$ *is an indexing category for the construction* $\mathcal{C}(\_, 1)$*.*

Proof. By choosing an identification $k_+ \wedge l_+ \cong kl_+$, i.e., a total ordering on binary products, for instance the lexicographic order, we get a smash product functor on Fin_+. Thus we map a pair of morphisms $f\colon k_+ \to l_+$, $g\colon m_+ \to n_+$ to $f \wedge g\colon km_+ \to ln_+$.

Analogously on objects, for a pair $p\colon A_+ \to k_+$ and $q\colon B_+ \to l_+$, we can consider their product $p \times q\colon A_+ \times B_+ \to k_+ \times l_+$ composed with the canonical projection $k_+ \times l_+ \to kl_+$ to the smash product. This factors over $(A \times B)_+ = A_+ \wedge B_+$.

Thus the above assignments define a functor, since the smash product on Fin_+ is a functor. □

Corollary 3.2.22. *For any finite pointed set* A_+ *we have a natural functor*

$$(A_+ \downarrow \mathrm{Fin}_+)^{\times n} \to {A_+}^{\wedge n} \downarrow \mathrm{Fin}_+.$$

Proposition 3.2.23. *Restricting a functor* $F \in \mathcal{C}((A_+)^{\wedge n}, 1)$ *along the functor* $S\colon (A_+ \downarrow \mathrm{Fin}_+)^{\times n} \to {A_+}^{\wedge n} \downarrow \mathrm{Fin}_+$ *gives an element of* $\mathcal{C}(A_+, n)$*.*

Proof. Reconsider the definition of T_n as in 1.1.58: For $T\colon A_+ \downarrow \mathrm{Fin}_+ \to \mathrm{Fin}_+$ the forgetful functor assigning to each object $p\colon A_+ \to n_+$ its target n_+ and to each commutative triangle $f\colon p \to fp$ the map f we set T_n to be

$$(A_+ \downarrow \mathrm{Fin}_+)^n \to \mathrm{Fin}_+^n \to \mathrm{Fin}_+$$

with first map $(T)^n$ and second map the n-fold smash product.

This factors as:

$$\begin{tikzcd}
(A_+ \downarrow \mathrm{Fin}_+)^n \arrow[r, "(T)^n"] \arrow[d, "S"'] & \mathrm{Fin}_+^n \arrow[d, "\wedge"] \\
(A_+)^{\wedge n} \downarrow \mathrm{Fin}_+ \arrow[r, "T"] & \mathrm{Fin}_+.
\end{tikzcd}$$

Thus a functor $F\colon {A_+}^{\wedge n} \downarrow \mathrm{Fin}_+ \to \mathcal{C}^+$ satisfying $UF = T$ trivially satisfies $UFS = TS$, which by the commutative square above gives $UFS = TS = T_n$, so FS is an element of $\mathcal{C}(A_+, n)$. □

Finally to tie the delooping categories in with the delooping constructed in [Os] we need the following equivalence:

Theorem 3.2.24. *Smashing the source category* $(A_+ \downarrow \mathrm{Fin}_+)^n \to (A_+)^{\wedge n} \downarrow \mathrm{Fin}_+$ *induces a natural strict restriction functor*

$$S^*\colon \mathcal{C}((A^{\times n})_+, 1) \to \mathcal{C}(A_+, n),$$

which is a natural equivalence of bicategories.

Proof. Recall the statement and proof of corollary 3.2.10: Specifically for the delooping bicategories $\mathcal{C}((A^{\times n})_+, 1)$ and $\mathcal{C}(A_+, n)$ we find that the natural inclusions ${(A^{\times n})_+}^\delta \to (A_+ \downarrow \mathrm{Fin}_+)^n$ and ${(A^{\times n})_+}^\delta \to (A_+)^{\wedge n} \downarrow \mathrm{Fin}_+$ – each identifying ${(A^{\times n})_+}^\delta$ as a discrete subcategory of the respective indexing categories – fit into a commutative triangle:

$$\begin{tikzcd}
 & {(A^{\times n})_+}^\delta \arrow[dl] \arrow[dr] & \\
(A_+ \downarrow \mathrm{Fin}_+)^n \arrow[rr, "S"] & & (A^{\times n})_+ \downarrow \mathrm{Fin}_+.
\end{tikzcd}$$

Since restriction along the diagonal arrows of this triangle each give equivalences of bicategories by 3.2.10, we find that S^* also is an equivalence of bicategories. □

3.3 The Multiplicative Structure on $\mathcal{C}(A_+, n)$

Let me reiterate that I only consider multiplicative structures as induced on matrices for a bipermutative coefficient category $\mathcal{R}$. Hence I restrict to the E_∞-case and by choosing the Barratt-Eccles-operad as in [EM, p. 16, Theorem 3.7] I can avoid constructing a multi(bi)category-structure for permutative bicategories.

To understand how a bipermutative structure induces a multiplication on the delooping bicategories $\mathcal{C}(A_+, n)$ I have to fix a multiplication on $\mathcal{C}^+$ and compatible target functors T_n. The induced multiplication on $\mathcal{C}^+$ is a direct generalisation from the case of 1-categories, so I repeat the proof to keep track of strictnesses and genuine 2-cells.

Theorem 3.3.1. *Consider a bipermutative bicategory (see 2.4.3)* $(\mathcal{C}, +, \cdot)$*. Fix a smash product on* Fin_+*, then we have a symmetric monoidal structure* $\boxtimes$ *on* $\mathcal{C}^+$ *making the forgetful functor strictly symmetric monoidal*

$$U\colon (\mathcal{C}^+, \boxtimes) \to (\mathrm{Fin}_+, \wedge).$$

Proof. Do the same on objects as in 1.1.4:

$$(c_1, \ldots, c_n) \boxtimes (d_1, \ldots, d_m) = (c_i d_j)_{\omega(i,j)},$$

where the multiplication on the right is the multiplicative structure of the bipermutative bicategory $(\mathcal{C}, +, \cdot)$. Again the subtlety is the definition on morphisms. For this consider first the following objects in $\mathcal{C}^+$:

$$(f \times g)_*(c \boxtimes \bar{c})_{\omega(i,j)} = \sum_{\omega(k,l) \in (f\times g)^{-1}(\omega(i,j))} c_k \bar{c}_l,$$

and analogously:

$$(f_* c \boxtimes g_* \bar{c})_{\omega(i,j)} = (f_* c)_i \cdot (g_* \bar{c})_j = \left(\sum_{k \in f^{-1} i} c_k \right) \left(\sum_{l \in g^{-1} j} \bar{c}_l \right).$$

By 2.4.3 we find a unique structural map $D^{f,g}$ comprised of isomorphism 1-cells (given for instance here by first all left reductions, then all right reductions)

$$(f \times g)_*(c \boxtimes \bar{c}) = \sum_{k,l} c_k \bar{c}_l \to \sum_k \left(c_k \left(\sum_l \bar{c}_l \right) \right)$$

$$\to \left(\sum_{k \in f^{-1} i} c_k \right) \left(\sum_{l \in g^{-1} j} \bar{c}_l \right) = f_* c \boxtimes g_* \bar{c},$$

which is given by composites of distributors, and uniquely determined by the summations of f and g. Hence for two maps in $\mathcal{C}^+$:

$$(f, (a_1, \ldots, a_{m_1}))\colon c = (c_1, \ldots, c_{n_1}) \to d = (d_1, \ldots, d_{m_1}),$$

$$(g, (b_1, \ldots, b_{m_2}))\colon \bar{c} = (\bar{c}_1, \ldots, \bar{c}_{n_2}) \to \bar{d} = (\bar{d}_1, \ldots, \bar{d}_{m_2}),$$

we set their product to be the following composite:

$$c \boxtimes \bar{c} = (c_i \bar{c}_j) \xrightarrow{(f\times g)_*} \left(\sum c_k \bar{c}_l\right) \xrightarrow{D^{f,g}} \left(\sum c_k\right) \left(\sum \bar{c}_l\right) \xrightarrow{a_i \cdot b_j} (d_i \cdot \bar{d}_j).$$

The map $\boxtimes$ respects identity 1-cells strictly, because $\cdot$ was assumed to be normal 2.4.3. Since f, g are part of the data of morphisms in $\mathcal{C}^+$, the structural morphism $(f \times g)_*$ is strictly natural. Since the distributors in 2.4.3 are strict natural transformations, $D^{f,g}$ strictly commutes with genuine morphisms of $\mathcal{C}$ as well. Thus only the appropriate products of compositor 2-cells for $\cdot$ yield the compositor for $\boxtimes$ with no additional 2-cells introduced by either $(f \times g)_*$ or $D^{f,g}$. So we have a strong normal functor

$$\boxtimes \colon \mathcal{C}^+ \times \mathcal{C}^+ \to \mathcal{C}^+.$$

Since 1_+ is a strict unit object for $\wedge$ on Fin_+, the 1-tuple $(1) \in \mathcal{C}^+$ with entry the multiplicative unit of $\mathcal{C}$ yields a strict unit object in $\mathcal{C}^+$.

The functor $\boxtimes$ is strictly associative: visibly on objects precisely because the bijections for the smash-product in Fin_+ are chosen that way, and because the multiplication on $\mathcal{C}$ was assumed strictly associative. Because of the strict identity of functors for triple products that $\cdot$ on $\mathcal{C}$ satisfies by assumption, we get strict associativity for $\boxtimes$ as a strict functor identity on $\mathcal{C}^+$.

Finally the multiplicative symmetry transformation is given as follows. Let the symmetry in Fin_+ with respect to $\wedge$ be χ, then the 1-cell for the symmetry of $\boxtimes$ is the composite:

$$c \boxtimes d = (c_i d_j)_{\omega(i,j)} \xrightarrow{\chi} (c_i d_j)_{\omega(j,i)} \xrightarrow{c^{\cdot}_1} (d_j c_i)_{\omega(j,i)} = d \boxtimes c.$$

Since the symmetry χ introduces no 2-cell, the 2-cell for the symmetry of $\boxtimes$ is thus given as the appropriate product of the symmetry 2-cells of $\cdot$ in $\mathcal{C}$.

The symmetry squares to the identity strictly, since the symmetries of $(\mathrm{Fin}_+, \wedge)$ and $(\mathcal{C}, \cdot)$ do. It satisfies the two diagrams for triple products strictly for the same reason.

For the final claim we only need to observe that the discrete components of the functor $\boxtimes$ and its symmetry $c^{\boxtimes}$ are modelled just so that the forgetful functor $U \colon \mathcal{C}^+ \to \mathrm{Fin}_+$ is strictly symmetric monoidal. □

Remark 3.3.2. Since I have established that a bipermutative structure gives a functor over the smash-product functor $\wedge\colon \mathrm{Fin}_+ \times \mathrm{Fin}_+ \to \mathrm{Fin}_+$ I strongly conjecture that this could be used to make bimonoidal and bipermutative categories much more explicit in the context of ∞-categories. Compare this for instance to (p. 149) Definition 2.1.3.7 in [Lu2] and more directly to (p. 136) Definition 2.0.0.7, where Lurie defines a symmetric monoidal ∞-category just so that by adding in all morphisms of Fin_+ into $\mathcal{C}^+$ the map $U \colon \mathcal{C}^+ \to \mathrm{Fin}_+$ exhibits its nerve $N\mathcal{C}^+$ as a symmetric monoidal ∞-category, and by 2.1.3.7 $\boxtimes$ as a symmetric monoidal functor.

The multiplication on $\mathcal{C}^+$ induces a pairing of the delooping bicategories.

Theorem 3.3.3. *Given a bipermutative bicategory $(\mathcal{C}, +, \cdot)$ the delooping bicategories $\mathcal{C}(A_+, n)$ have a pairing pseudofunctor:*

$$\mu_{n,m} \colon \mathcal{C}(A_+, n) \times \mathcal{C}(A_+, m) \to \mathcal{C}(A_+, n+m),$$

which is strictly natural in A_+, and $\Sigma_n \times \Sigma_m$-equivariant.

Furthermore the pairing strictly satisfies $\mu(O_n, \_) = \mu(\_, O_m) = O_{n+m}$, i.e., pairing with the zero-functor yields the constant map to the zero-functor $O_{n+m} \in \mathcal{C}(A_+, n+m)$.

Proof. By the propositions before we know that we can pair two lifting functors $F_n \colon (A_+ \downarrow \mathrm{Fin}_+)^n \to \mathcal{C}^+$ and $G_m \colon (A_+ \downarrow \mathrm{Fin}_+)^m \to \mathcal{C}^+$ by the symmetric monoidal structure on $\mathcal{C}^+$ to give $\boxtimes_*(F_n, G_m) \colon (A_+ \downarrow \mathrm{Fin}_+)^{n+m} \to \mathcal{C}^+ \times \mathcal{C}^+ \to \mathcal{C}^+$, which is evidently compatible with the Σ_n-operation on F_n and the Σ_m- operation on G_m independently, thus with $\Sigma_n \times \Sigma_m$ as a whole. Furthermore evidently $\boxtimes_*(O_n, \_) = \boxtimes_*(\_, O_m) = O_{n+m}$, since the zero-functor acts as a strict zero for $\boxtimes$, which is induced by $\cdot$ on the bipermutative bicategory.

Since the symmetric monoidal structure of $\mathcal{C}^+$ is defined over the forgetful functor $U \colon \mathcal{C}^+ \to \mathrm{Fin}_+$ such that it becomes strictly symmetric monoidal, the resulting functor lifts the map

$$(A_+ \downarrow \mathrm{Fin}_+)^n \times (A_+ \downarrow \mathrm{Fin}_+)^m \xrightarrow{T_n \times T_m} \mathrm{Fin}_+ \times \mathrm{Fin}_+ \xrightarrow{\wedge} \mathrm{Fin}_+,$$

which by 1.1.58 is the same as T_{n+m}. Hence the resulting functor is in $\mathcal{C}(A_+, n+m)$.

The same description applies to 1- and 2-cells, since I did not need to refer to 1-equivalences to define the symmetric monoidal structure on $\mathcal{C}^+$. Hence we have a strict symmetric monoidal inclusion $(\mathcal{C}^{eq})^+ \to \mathcal{C}^+$, and can extend the product to 1- and 2-cells. By applying the compositor of $\cdot$ appropriately componentwise we get the compositor 2-cell for μ.

Strict naturality in the pointed set is a consequence of the fact that a map of pointed finite sets $f \colon A_+ \to B_+$ induces a strict normal 1.1.13 functor $\mathcal{C}(A_+, n) \to \mathcal{C}(B_+, n)$ by pulling the category $B_+ \downarrow \mathrm{Fin}_+$ back along f to $A_+ \downarrow \mathrm{Fin}_+$ and then pulling back functors along this pullback. In particular it is restriction of the source category, hence the multiplication is strictly natural in A_+. □

Remark 3.3.4. Do note that by convention $\mathcal{C}(A_+, 0) = \mathcal{C}$, so for $(\mathcal{C}, +, \cdot)$ bipermutative we trivially have a map $\eta_0 \colon * \to \mathcal{C}(A_+, 0)$ sending the object to 1 and its identity.

Furthermore by the extension $A_+ \times \mathcal{C}(A_+, 0) \to \mathcal{C}(A_+, 1)$ we get a map $\eta_1 \colon A_+ \cong A_+ \times \{1\} \to \mathcal{C}(A_+, 1)$, which hence sends a pair $(a, 1)$ to the functor, which is the tuple (1) at $\rho^a \colon A_+ \to 1_+$ with zeroes adequately added everywhere else.

In particular for $(\mathcal{C}, +, \cdot)$ we can rewrite the extension maps $A_+ \times \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, 1+n)$ as the multiplication with η_1:

$$A_+ \times \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, 1) \times \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, 1+n).$$

The pairing inherits strict associativity from the strict associativity of $(\mathcal{C}, \cdot)$.

Proposition 3.3.5. *For $(\mathcal{C}, +, \cdot)$ a bipermutative bicategory the pairing of 3.3.3 is strictly associative, i.e.,*

$$\begin{tikzcd}
\mathcal{C}(A_+, l) \times \mathcal{C}(A_+, m) \times \mathcal{C}(A_+, n) \arrow[r] \arrow[d] & \mathcal{C}(A_+, l) \times \mathcal{C}(A_+, m+n) \arrow[d] \\
\mathcal{C}(A_+, l+m) \times \mathcal{C}(A_+, n) \arrow[r] & \mathcal{C}(A_+, l+m+n)
\end{tikzcd}$$

is strictly commutative for each pointed finite set A_+ and natural numbers l, m, n.

Proof. This follows from the strict associativity of the chosen bijections $\omega \colon n \times m \to nm$ making the monoidal structure on $\mathcal{C}^+$ with $((c_i), (d_j)) \mapsto (c_i \cdot d_j)_{\omega(i,j)}$ strictly associative, because $\cdot$ is strictly associative. $\square$

By rewriting the extension maps as above we get the following corollary:

Corollary 3.3.6. *The pairings commute with extension maps strictly, i.e., there is a unique pairing of the form*

$$\mathcal{C}(A_+, n) \wedge A_+ \wedge \mathcal{C}(A_+, m) \longrightarrow \mathcal{C}(A_+, n+1+m).$$

Definition 3.3.7. For $\sigma \in \Sigma_n$ define the map:

$$\mu_\sigma \colon \mathcal{C}(A_+, k_1) \times \ldots \times \mathcal{C}(A_+, k_n) \to \mathcal{C}(A_+, \sum_i k_{\sigma^{-1}(i)})$$

as the composite of the symmetry in $(Bicat, \times)$

$$c_\sigma^\times \colon \prod_i \mathcal{C}(A_+, k_i) \to \prod_i \mathcal{C}(A_+, k_{\sigma^{-1}(i)})$$

followed by the n-fold pairing (uniquely determined by the proposition above):

$$\prod_i \mathcal{C}(A_+, k_{\sigma^{-1}(i)}) \to \mathcal{C}(A_+, \sum_i k_{\sigma^{-1}(i)}).$$

Remark 3.3.8. By definition of $\mu_\bullet$ we have a functor

$$\Sigma_n \times \left(\coprod_{k_1+\ldots+k_n=N} \mathcal{C}(A_+, k_1) \times \ldots \times \mathcal{C}(A_+, k_n) \right) \to \mathcal{C}(A_+, N)$$

with Σ_n considered as a discrete category, which factors as

$$\Sigma_n \times_{\Sigma_n} \left(\coprod_{k_1+\ldots+k_n=N} \mathcal{C}(A_+, k_1) \times \ldots \times \mathcal{C}(A_+, k_n) \right) \to \mathcal{C}(A_+, N).$$

Furthermore since we established that binary multiplication becomes the constant map to the zero-functor, if one parameter is the zero-functor, we find that each $\mu_\bullet$ becomes the constant map if one parameter is restricted to the zero-functor.

Finally I want to state the E_∞-commutativity in its binary form for clarity before summarising the E_∞-structure in 3.3.16.

Proposition 3.3.9. *For a bipermutative bicategory $(\mathcal{C}, +, \cdot)$, a finite pointed set A_+ and two natural numbers n, m the two pairings $\mu_{\mathrm{id}}, \mu_{(12)} \colon \mathcal{C}(A_+, n) \wedge \mathcal{C}(A_+, m) \to \mathcal{C}(A_+, n+m)$ are pseudonaturally isomorphic.*

Furthermore the pseudonatural isomorphisms inherent the coherence of the $\cdot$-symmetry in that for each two $\sigma, \tau \in \Sigma_N$ there is a unique composite pseudonatural isomorphism $\mu_\sigma \Rightarrow \mu_\tau$ of pairings

$$\mathcal{C}(A_+, n_1) \wedge \ldots \wedge \mathcal{C}(A_+, n_N) \to \mathcal{C}(A_+, \sum_i n_i).$$

Proof. The 1-cells of the pseudonatural transformation consist of $(\chi_{n,m}, c_\cdot^1)$ with $\chi_{n,m}$ the block permutation shifting the first n elements of $n+m$ past the last m elements, and $c_\cdot^1$ the 1-cell of the pseudonatural symmetry for $(\mathcal{C}, \cdot)$.

In particular by 2.2.3 we already see that each 1-cell is a strict isomorphism, which squares to the identity, thus we only need pseudonaturality. The pseudonaturality 2-cell is given accordingly by $(\mathrm{id}, c_\cdot^2)$.

Furthermore we see immediately that the coherence of $c_\cdot$ promotes to the coherence claimed above. $\square$

Reminder 3.3.10. For 1-categories $\mathcal{C}, \mathcal{D}$ that natural transformations $\eta\colon F \Rightarrow G$ are in a natural one-to-one correspondence with functors $H\colon \mathcal{C} \times I \to \mathcal{D}$.

Specifically, the functors F, G are restrictions of H to the objects $0, 1 \in I$ respectively, while the components of the natural transformation η are the arrows $H(c, 0 \to 1) = \eta_c$. Naturality of η is then equivalent to H being a functor, because $\mathcal{C} \times I$ is a product-category.

Remark 3.3.11. For $\mathcal{C}, \mathcal{D}$ bicategories the above correspondence generalises to pseudonatural transformations, which are in one-to-one correspondence with pseudofunctors. However, since for pseudofunctors the compositor 2-cells fill triangles, while the 2-cell involved in the pseudonaturality condition fills a square, we actually get (at least) two correspondences by fixing one or the other triangle in the diagramme

$$\begin{tikzcd} H(c,0) \arrow[r] \arrow[d] \arrow[dr] & H(d,0) \arrow[d] \\ H(c,1) \arrow[r] & H(d,1) \end{tikzcd}$$

to be filled with the compositor 2-cell.

The same correspondence establishes that a pseudonatural transformation is strictly natural, i.e., has only identity 2-cells, if and only if its associated pseudofunctor is a strict functor.

To introduce the specific E_∞-coherences for the pairing on the delooping bicategories, recall the Barratt-Eccles operad in 1-categories (cf. [EM, p. 15]).

Definition 3.3.12. For any discrete set M define its translation category EM as follows: Its objects are the elements of M, its arrow set is $M \times M$, where $(s,t)\colon s \to t$ with composition $(t,u) \circ (s,t) = (s,u)$ and identities (u,u) for an object $u \in M$. By definition each object is initial and terminal, hence the classifying space of EM is contractible for any M.

Moreover for G a group we have a canonical action $EG \times G \to EG$ by the assignment $(s,t).g := (sg, tg)$.

We can by coherence of the pseudonatural isomorphisms in 3.3.9 extend the map

$$\Sigma_n \times_{\Sigma_n} \left(\coprod_{k_1 + \ldots + k_n = N} \mathcal{C}(A_+, k_1) \times \ldots \times \mathcal{C}(A_+, k_n) \right) \to \mathcal{C}(A_+, N)$$

over $E\Sigma_n$ as follows:

Corollary 3.3.13. *The n-fold pairing*

$$\mu\colon \Sigma_n \times_{\Sigma_n} \left(\coprod_{k_1+\ldots+k_n=N} \mathcal{C}(A_+, k_1) \times \ldots \times \mathcal{C}(A_+, k_n) \right) \to \mathcal{C}(A_+, N)$$

extends to a pairing

$$\mu^E\colon E\Sigma_n \times_{\Sigma_n} \left(\coprod \mathcal{C}(A_+, k_1) \times \ldots \times \mathcal{C}(A_+, k_n) \right) \to \mathcal{C}(A_+, N).$$

Proof. Since we want to extend the assignment μ, we can define μ^E at each object of $E\Sigma_n$ by μ. Locally, i.e., for each arrow $(s,t) \in E\Sigma_n$, we set $\mu^E(\_, (s,t))$ to be the canonical pseudonatural 1-cells for $\mu_s \Rightarrow \mu_t$ as established in 3.3.9.

By analogy with 3.3.11 fill in the upper right triangle with the pseudonaturality 2-cell for the canonical pseudonatural isomorphism $\mu_s \Rightarrow \mu_t$.

This assignment defines a normal functor (i.e., one pointed at identity 1-cells) because $c^1_{\cdot}$ strictly squares to the identity by 2.2.3. The compositor 2-cells are coherent, because the two diagrams:

$$\begin{tikzcd}
\mu^E(s, F_1, \ldots, F_n) \arrow[r] \arrow[d] \arrow[dr] & \mu^E(s, G_1, \ldots, G_n) \arrow[d] \\
\mu^E(t, F_1, \ldots, F_n) \arrow[r] \arrow[d] \arrow[dr] & \mu^E(t, G_1, \ldots, G_n) \arrow[d] \\
\mu^E(u, F_1, \ldots, F_n) \arrow[r] & \mu^E(u, G_1, \ldots, G_n)
\end{tikzcd}$$

$$\begin{tikzcd}
\mu^E(s, F_1, \ldots, F_n) \arrow[r] \arrow[d] \arrow[dr] & \mu^E(s, G_1, \ldots, G_n) \arrow[d] \\
\mu^E(u, F_1, \ldots, F_n) \arrow[r] & \mu^E(u, G_1, \ldots, G_n)
\end{tikzcd}$$

with each upper right triangle filled by the pseudonaturality 2-cells express that there is a unique $\cdot$-symmetry from an s-permuted input to an u-permuted input. In particular, the composite twist factored over a t-permuted input produces the same multiplicative twist. Do note that all the other triangles are filled with identities, including the ones expressing the equalities $E((t,u), \_) \,\square\, E((s,t), \_) = E((s,u), \_)$, so that the above prism with base a triangle degenerates to just three (potentially) non-trivial 2-cells. $\square$

Reminder 3.3.14. Since the Σ_*-module $(E\Sigma_n)_n$ in fact is an operad, we have a multiassociative, Σ_*-equivariant, and unital multicomposition:

$$E\Sigma_N \times E\Sigma_{k_1} \times \ldots \times E\Sigma_{k_N} \to E\Sigma_{\sum_i k_i}.$$

I refer to this as *block sum composition* as it is given by application of the functor E to the multicomposition

$$\Sigma_N \times \Sigma_{k_1} \times \ldots \times \Sigma_{k_N} \to \Sigma_{\sum_i k_i},$$

which can be described as

$$(\sigma, \tau_1, \ldots, \tau_N) \mapsto \sigma\langle k_1, \ldots, k_N\rangle \circ (\tau_1 \boxplus \ldots \boxplus \tau_N)$$

for $\sigma\langle k_1, \ldots, k_N\rangle$ the permutation that permutes the N blocks of length k_i by exchanging the blocks according to σ, and $\boxplus$ a disjoint union functor on finite sets as in 1.1.7.

Proposition 3.3.15. *The $E\Sigma_*$-extensions of the pairings of delooping bicategories for a bipermutative bicategory $(\mathcal{C}, +, \cdot)$ of the above corollary make the following multiassociativity-diagram commute*

$$\begin{tikzcd}
E\Sigma_N \times (\prod_i E\Sigma_{k_i}) \times \left(\prod_j \mathcal{C}(A_+, l_j^i)\right) \arrow[r, "\cong"] \arrow[d] & E\Sigma_N \times \prod_i \left(E\Sigma_{k_i} \times \prod_j \mathcal{C}(A_+, l_j^i)\right) \arrow[d] \\
E\Sigma_{\sum_i k_i} \times \prod_i \prod_j \mathcal{C}(A_+, l_j^i) \arrow[dr] & E\Sigma_N \times \prod_i \mathcal{C}(A_+, \sum_j l_j^i) \arrow[d] \\
 & \mathcal{C}(A_+, \sum_{i,j} l_j^i)
\end{tikzcd}$$

for all natural numbers N, k_i, l_j^i.

Proof. By the specific structure of $E\Sigma_*$ and the definition of the pairings μ_σ by equivariance, we can reduce to the case, where each object in $E\Sigma_*$ is the identity, which is just the strict associativity of the pairings. Since morphisms in the $E\Sigma_*$ are uniquely determined by their source and target, and the pseudonatural isomorphisms of 3.3.9 are coherent, this extends to the morphisms as well. □

I want to again suppress the operadic context for the E_∞-structure and instead display what comprises the algebra structure of $\mathcal{C}(A_+, \bullet)$ over $E\Sigma_*$, including its coherences.

Theorem 3.3.16. *Given a bipermutative bicategory $(\mathcal{C}, +, \cdot)$ the resulting pairing of delooping categories from the theorem above:*

$$\mu_{n,m} \colon \mathcal{C}(A_+, n) \times \mathcal{C}(A_+, m) \to \mathcal{C}(A_+, n+m)$$

is E_∞ in the following sense (cf. the definition of a commutative symmetric ring spectrum as in [Schw2, p. 9], as well as [MayE_∞, pp. 66-68]):

- *It is strictly associative, i.e., we get a well-defined triple product for every $n, m, l \in \mathbb{N}$ as a strict identity of strong normal functors:*

$$\mu_{n,m+l} \circ (\mathrm{id} \times \mu_{m,l}) = \mu_{n+m,l} \circ (\mu_{n,m} \times \mathrm{id}).$$

- *The functor $\{*\} \to 1 \in \mathcal{C} \subset \mathcal{C}^+$ considered as an element of $\mathcal{C}(A_+, 0)$ is a strict unit, turning $\mu_{0,n} = \mu_{n,0} = \mathrm{id}_{\mathcal{C}(A_+,n)}$ into a strict identity of functors.*

- *We have a natural central map $\iota_1 \colon A_+ \to \mathcal{C}(A_+, 1)$ given by*

$$\iota_1(a)(p\colon A_+ \to k_+) = \begin{cases} 1, & \text{if } \ a \notin p^{-1}+, \\ 0, & \text{if } \ a \in p^{-1}+, \end{cases}$$

  with structural maps being given either by identities $0 + 0 = 0$ or $1 + 0 = 0 + 1 = 1$, hence strict identities. Centrality means that we have a strict equality of the functors

$$A_+ \times \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, 1) \times \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, 1+n)$$

  and

$$\begin{aligned} A_+ \times \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, n) \times A_+ &\to \mathcal{C}(A_+, n) \times \mathcal{C}(A_+, 1) \\ &\to \mathcal{C}(A_+, n+1) \to \mathcal{C}(A_+, 1+n), \end{aligned}$$

  where the final arrow here is the action by $\chi_{1,n}$ which shuffles the last input coordinate to first place without changing the order of the other inputs, and pushing forward the functor by χ so that it is a lift 1.1.61. Do note that these maps describe the extension functors $A_+ \wedge \mathcal{C}(A_+, n) \to \mathcal{C}(A_+, 1+n)$.

- *For each two multiplications of n inputs associated to two permutations $\sigma, \tau \in \Sigma_n$ there is a pseudonatural isomorphism:*

$$\begin{tikzcd} \mathcal{C}(A_+, m_1) \times \ldots \times \mathcal{C}(A_+, m_n) \arrow[rr, bend left, "\mu_\sigma"] \arrow[rr, bend right, "\mu_\tau"'] \arrow[rr, phantom, "\Downarrow c_{\overrightarrow{\sigma\tau}}"] & & \mathcal{C}(A_+, m_1 + \ldots + m_n). \end{tikzcd}$$

- *The isomorphisms are coherent in that they compose vertically as in $E\Sigma_n$: $C_{\overrightarrow{\sigma\tau}} C_{\overrightarrow{\rho\sigma}} = C_{\overrightarrow{\rho\tau}}$.*
- *The isomorphisms have a block sum composition 3.3.14, which is multiassociative and Σ_*-equivariant.*

Proof. All the claims are just summaries of the propositions above. Note that the coherence of the isomorphisms $C_{\_}$ follows from the coherence of the compositor 2-cells for μ^E, which itself follows from the coherence of the $\cdot$-symmetry for triple products in $(\mathcal{C}, \cdot)$. □

3.4 The Symmetric Spectrum from the Delooping

The passage from the delooping categories $\mathcal{C}(A_+, n)$ to the associated spectrum is fortunately very straight-forward. I fix a pointed simplicial $\mathbb{S}^1$:

Definition 3.4.1. Consider the simplicial set $\Delta_1 = \Delta(\_, [1])$ and its simplicial subset of constant maps $\partial\Delta_1$, then $\mathbb{S}^1 := \Delta_1/\partial\Delta_1$ is a pointed simplicial set.

Since the delooping bicategories $\mathcal{C}(A_+, n)$ are strict normal 1.1.13 functors in pointed finite sets we get the following:

Theorem 3.4.2. *The delooping bicategories $\mathcal{C}(A_+, n)$ are strictly simplicial bicategories by insertion of a simplicial set. In particular $\mathcal{C}(\mathbb{S}^1, n)$ is a simplicial bicategory with Σ_n-action.*

By naturality of the extension functors and understanding $\mathbb{S}^1$ as a discrete simplicial bicategory we get suspension maps as strict functors of bicategories:

$$\mathbb{S}^1 \wedge \mathcal{C}(\mathbb{S}^1, n) \to \mathcal{C}(\mathbb{S}^1, 1+n),$$

which is Σ_n-equivariant, and assembles to $\Sigma_m \times \Sigma_n$-equivariant maps

$$\mathbb{S}^m \wedge \mathcal{C}(\mathbb{S}^1, n) \to \mathcal{C}(\mathbb{S}^1, m+n).$$

For a bipermutative bicategory $\mathcal{C}$ the pairings from above assemble into strictly associative pairings of simplicial bicategories:

$$\mathcal{C}(\mathbb{S}^1, n) \wedge \mathcal{C}(\mathbb{S}^1, m) \to \mathcal{C}(\mathbb{S}^1, n+m),$$

with two strictly central unit maps

$$\iota_0 \colon \{*\} \to \mathcal{C}(\mathbb{S}^1, 0), \quad \iota_1 \colon \mathbb{S}^1 \to \mathcal{C}(\mathbb{S}^1, 1),$$

and each two permutations of higher multiplications connected by coherent simplicial pseudonatural isomorphisms.

In summary: Set $HC_n = |N\mathcal{C}(\mathbb{S}^1, n)|$, then the Eilenberg-Mac Lane spectrum H to a permutative bicategory $\mathcal{C}$ inherits a natural E_∞-ring spectrum structure from a bipermutative structure on $\mathcal{C}$.

By applying the nerve and geometric realisation we get:

Theorem 3.4.3. *A bipermutative bicategory $\mathcal{C}$ yields an E_∞-symmetric ring spectrum, which is level-equivalent to the spectrum defined in [Os]. In particular it is semi-stable, because it is equivalent to the symmetric spectrum of a Γ-space.*

Proof. By 3.2.24 we know that restriction of lifting functors along smashing $(A_+ \downarrow \mathrm{Fin}_+)^n \to (A^n)_+ \downarrow \mathrm{Fin}_+$ gives a natural equivalence $\mathcal{C}((A_+)^{\wedge n}, 1) \to \mathcal{C}(A_+, n)$, which by inserting $\mathbb{S}^1$ yields a level-equivalence:

$$\mathcal{C}((\mathbb{S}^1)^{\wedge n}, 1) = \mathcal{C}(\mathbb{S}^n, 1) \to \mathcal{C}(\mathbb{S}^1, n),$$

so in particular on realisation of nerves we get a map of symmetric spectra, which is a level-equivalence. □

3.5 The Induced Involution

We have already seen that an involution on bipermutative coefficients $\mathcal{R}$ induces a strictly additive functor on the bicategories of matrices $\mathcal{M}(\mathcal{R}) \to \mathcal{M}(\mathcal{R}^\mu)$. By transposition we can remove the μ-opposition, still strictly additively, but at the expense of opposing 1-cells and the tensor-product. Finally the comparison $B\mathcal{C} \cong B\mathcal{C}^{op_1}$ is what we need to study with respect to additivity and multiplicativity.

Lemma 3.5.1. *Given a bicategory $\mathcal{C}$ and its 1-opposition, i.e., with respect to 1-cells, the homeomorphism Γ of their nerves is strictly natural with respect to functors $F\colon \mathcal{C} \to \mathcal{D}$. So we have a commutative diagram:*

$$\begin{tikzcd} |N\mathcal{C}| \arrow[r, "\Gamma"] \arrow[d, "|NF|"'] & |N\mathcal{C}^{op}| \arrow[d, "|NF|"] \\ |N\mathcal{D}| \arrow[r, "\Gamma"] & |N\mathcal{D}^{op}|. \end{tikzcd}$$

□

In particular the homeomorphism is compatible with the pairings established above.

Corollary 3.5.2. *The pairing of delooping bicategories commutes with opposition:*

$$
\begin{tikzcd}
{|N\mathcal{C}(A_+,n)| \times |N\mathcal{C}(A_+,m)|} \arrow[r, "\Gamma\times\Gamma"] \arrow[d] & {|N\mathcal{C}^{op_1}(A_+,n)| \times |N\mathcal{C}^{op_1}(A_+,m)|} \arrow[d] \\
{|N(\mathcal{C}(A_+,n) \times \mathcal{C}(A_+,m))|} \arrow[r, "\Gamma"] \arrow[d] & {|N(\mathcal{C}^{op_1}(A_+,n) \times \mathcal{C}^{op_1}(A_+,m))|} \arrow[d] \\
{|N(\mathcal{C}(A_+,n+m))|} \arrow[r, "\Gamma"] & {|N(\mathcal{C}^{op_1}(A_+,n+m))|.}
\end{tikzcd}
$$

Proof. The only thing left to emphasise is that the homeomorphism $|X| \times |Y| \to |X \times Y|$ is natural as well (given the compactly generated topology on the product). □

Remark 3.5.3. A minor warning is in order about the notation $\mathcal{C}^{op_1}(A_+, n)$. Since $(\mathcal{C}^{op_1})^+ \neq (\mathcal{C}^+)^{op_1}$ this is potentially ambiguous, thus I intend to mean the delooping bicategory of $\mathcal{C}^{op_1}$.

Since the induced involution is defined in [R] and analogously in 2.6.10 as the composite:

$$B\mathcal{M}(\mathcal{R}) \xrightarrow{B\mathcal{M}(T)} B\mathcal{M}(\mathcal{R}^\mu) \xrightarrow{B(\cdot)^t} B\mathcal{M}(\mathcal{R})^{op_1} \xrightarrow{\Gamma} B\mathcal{M}(\mathcal{R}),$$

and we have already established that Γ strictly commutes with the pairings, we only have to establish the effect of coordinatewise involution and subsequent transposition. Both functors strictly commute with the direct sum of matrices, thus induce functors on the delooping bicategories, but in 2.5.13 we saw that transposition fully reverses the monoidal structure given by tensor product. So we have to consider the following situation. For emphasis I suppress the commutativity of the multiplication and call it a bimonoidal bicategory.

Proposition 3.5.4. *Given a bimonoidal bicategory $(\mathcal{C}, +, \cdot)$ and its multiplicative opposition $(\mathcal{C}, +, \circ)$ the induced monoidal structure on $\mathcal{C}^+$ by $\circ$ is strictly naturally isomorphic to the opposite monoidal structure on $\mathcal{C}^+$ induced by $\cdot$.*

Proof. By retracing the construction in 3.3.1 we find on objects that $\circ$ induces

$$(c_1, \ldots, c_n) \circ_* (d_1, \ldots, d_m) = (c_i \circ d_j)_{\omega(i,j)} = (d_j \cdot c_i)_{\omega(i,j)}.$$

Thus the isomorphism is given by using the smash symmetry χ on Fin_+ to exchange the indices:

$$(c_1, \ldots, c_n) \circ_* (d_1, \ldots, d_m) = (d_j \cdot c_i)_{\omega(i,j)} \xrightarrow{(\chi_{n,m}, \mathrm{id})} (d_j \cdot c_i)_{\omega(j,i)} = d \cdot_* c = c \cdot_*^{op} d.$$

□

In particular we find that the involution and subsequent transposition strictly oppose the multiplicative structure on the delooping bicategories.

Corollary 3.5.5. *The induced involution is a functor* $I\colon (\mathcal{C}, +, \cdot) \to (\mathcal{C}^{op_1}, +, \cdot^{op})$, *thus composition with* Γ *induces a map of* E_∞*-spectra:*

$$\Gamma \circ |NI|\colon (H\mathcal{C}, \mu) \to (H\mathcal{C}, \mu^{opp}).$$

4 Partial Uniqueness Results

Evidently, I am interested if the multiplication on the delooping I describe in chapter 3 describes a new structure or the known E_∞-structure on the K-theory of an E_∞-ring spectrum.

Given this I can avoid the complicated constructions of the trace map described for example in [BHM, S] (compare also [DGM]). By [BGT1] there is an essentially unique map of symmetric multifunctors from K-theory to any other (symmetric) multifunctor F that satisfies $F(\mathbb{S}) = \mathbb{S}$. For topological Hochschild homology the equality $THH(\mathbb{S}) = \mathbb{S}$ is immediate from the definitions, thus there is a unique E_∞-map $K \to THH$, which by [BGT1, Theorem 1.9] is the trace map. Specifically the essential uniqueness proven in [BGT1] entails that the trace map is a point in the space of E_∞-maps $K \to THH$, which is contractible.

However, for the multiplicative uniqueness I have to concede conjecture status, while the additive uniqueness, i.e., the fact that there is essentially only one delooping of the nerve of a symmetric monoidal category by its E_∞-structure is classical [MT], which can be rephrased nicely with the results of [GGN].

Convention: The Language of ∞-Categories

To conveniently state and prove results about uniqueness of E_∞-structures we evidently need some organisational language in which to compare them. As shown in [EM] multicategories can be used in absence of symmetric monoidal structures on a category to define operad-structures. However, comparison theorems in this setting seem too restrictive to expect. Since E_∞-structures are a coherently weakened notion of commutativity, we would not expect different E_∞-structures from potentially different E_∞-operads to be comparable by strict multifunctors. This would probably prove to be even worse for general operads.

Given the recent popularisation of $(\infty, 1)$-categories for instance by [Ber1, Ber2, J, JT, Lu1, Lu2] advancing the theory of quasi-categories as a convenient model for $(\infty, 1)$-categories, the authors have made such comparison results easier to state and prove in a satisfactory manner. Hence following my principal sources [GGN, BGT1] for the uniqueness results I use the language of $(\infty, 1)$-categories, and refer to specific results about quasi-categories in [Lu1, Lu2] where I need them. For a nice survey I specifically recommend [Ber2] and an accompanying talk [Ber3].

4.1 Uniqueness of the Spectrum $\mathcal{R} \mapsto H\mathcal{R}$

I can directly (partially) quote this result from [GGN] with the only alteration that, as in [BDRR1], I refer to the associated spectrum of a permutative category $\mathcal{C}$ as its Eilenberg-MacLane-spectrum $H\mathcal{C}$, for instance given by the delooping of [EM].

Theorem 4.1.1 (First part of Prop. 8.2. in [GGN])**.** *The Eilenberg-MacLane-spectrum functor* $H\colon \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \to Sp$ *is lax symmetric monoidal.*

This statement does not obviously incorporate ∞-categories. Their use is implicit in the symmetric monoidal structure on $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$, which probably does not exist, when we consider $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$ as a 1-category with objects small symmetric monoidal 1-categories and symmetric monoidal functors as morphisms – cf. [K] for a specific construction of a product on symmetric monoidal categories with strictly symmetric monoidal functors, which fails to have a unit object, and does not produce bimonoidal/bipermutative categories as its monoids. However, if we relax to considering $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$ as a 2-category with the same objects and 1-cells, but additionally monoidal natural transformations as 2-cells, we can see in [Schm] that $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$ does in fact support the structure of a "symmetric monoidal bicategory". But his construction has the wrong (E_∞) monoids, thus at most serves as a proof that $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$ can equipped with a symmetric monoidal bicategory structure at all.

I strongly conjecture that a symmetric monoidal product on the bicategory of symmetric monoidal categories with the right monoids can be constructed: Specifically the construction of a "classifying pseudofunctor" as in 1.1.34, which for $(\mathcal{C}, +)$ symmetric monoidal gives a pseudofunctor $B\mathcal{C}\colon \mathrm{Fin}_+ \to Cat$. The additive Grothendieck construction as in 1.1.37 is the Grothendieck construction over this functor. When we restrict $B\mathcal{C}$ to the source category with just surjections Epi_+ however, the Grothendieck construction $\mathcal{C}^{+,epi}$ over this functor admits a natural adjunction

$$\mathcal{C}^{+,epi} \rightleftarrows \mathcal{C}$$

and thus the classifying spaces are homotopy equivalent. We can easily consider the product $B\mathcal{C} \times B\mathcal{D}$, and the Grothendieck construction over this functor yields a candidate for a "smash product" of permutative categories. However I do not know, which of the specific "averaging" processes described by the Grothendieck construction yields the most convenient smash product. We could consider the product $B\mathcal{C} \times B\mathcal{D}$ and pull back by the diagonal $\mathrm{Fin}_+ \to \mathrm{Fin}_+ \times \mathrm{Fin}_+$. Alternatively we could directly consider the Grothendieck construction restricted to $\mathrm{Epi}_+ \times \mathrm{Epi}_+$. Since this becomes unwieldy quite fast, I have not established the properties this product might have. I am quite sure it is unital and associative "up to adjunction", symmetric up to isomorphism, however, one would have to establish the appropriate coherences of adjunctions and isomorphisms.

The theorem quoted above reframes the results in [EM] that the associated Eilenberg-MacLane-spectrum is a symmetric multifunctor from permutative categories to spectra, where in [EM] the multicategory-structure is precisely the concession needed for the fact that the 1-category of symmetric monoidal categories (probably) does not support a symmetric monoidal product.

The results of [GGN] make it unnecessary to construct a symmetric monoidal product on $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$ on the 2-categorical level.

Recall from [Lu2, GGN] that given an ∞-category, we have naturally associated to it its category of pointed objects $\mathcal{C}_*$, its category of E_∞-monoids $\mathrm{Mon}_{E_\infty}\mathcal{C}$, modelled as Γ-objects in $\mathcal{C}$, its category of grouplike E_∞-monoids $\mathrm{Grp}_{E_\infty}\mathcal{C}$, and its stabilisation, for instance given as the category of spectrum objects in $\mathcal{C}$, denoted $Sp(\mathcal{C})$. Moreover, each pointed object $c \in \mathcal{C}$ gives rise to a free E_∞-monoid hc given by $hc(n_+) = c^{\times n}$ with the evident maps induced by pointed maps in $\mathrm{Fin}_+ = \Gamma$. Group completion provides a free grouplike object associated to an arbitrary E_∞-monoid in the sense that it is a left-adjoint to the forgetful functor in the opposite direction. Finally, a grouplike Γ-object gives rise to an associated spectrum for instance by insertion of simplicial spheres $\mathbb{S}^n$ (cf. 3.4.2).

Theorem 4.1.2 (Theorem 5.1 [GGN])**.** *Let $\mathcal{C}^\otimes$ be a closed symmetric monoidal presentable ∞-category $\mathcal{C}$. The ∞-categories $\mathcal{C}_*$, $\mathrm{Mon}_{E_\infty}\mathcal{C}$, $\mathrm{Grp}_{E_\infty}\mathcal{C}$, $Sp(\mathcal{C})$ all admit closed symmetric monoidal structures, which are uniquely determined by the requirement that the respective free functors from $\mathcal{C}$ are symmetric monoidal.*

Moreover each of the functors

$$\mathcal{C}_* \to \mathrm{Mon}_{E_\infty}\mathcal{C} \to \mathrm{Grp}_{E_\infty}\mathcal{C} \to Sp(\mathcal{C})$$

uniquely extends to a symmetric monoidal functor.

By [GGN] the ∞-category of symmetric monoidal categories thus has a symmetric monoidal ∞-category-structure by the following argument of Theorem 5.1 in [GGN] specialised for $\mathcal{C}$ the ∞-category of 1-categories with cartesian product as its closed symmetric monoidal structure. The ∞-category of symmetric monoidal categories is identified as a smashing ∞-localisation with the E_∞-monoids in spaces $\mathrm{Mon}_{E_\infty}(Top)$ [GGN, Theorem 4.6]. So they get

$$\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} = \mathrm{Mon}_{E_\infty}(Cat_1) \simeq Cat_1 \otimes \mathrm{Mon}_{E_\infty}(Top),$$

where $\otimes$ denotes the tensor-product of presentable ∞-categories as defined in [Lu2]. Hence arguing the same way as for Bousfield localisations of symmetric monoidal model categories, one only has to establish that $\_ \otimes \mathrm{Mon}_{E_\infty}(Top)$ respects local equivalences with respect to Mon_{E_∞}. This is trivial for a smashing localisation in ∞-categories, because of the equivalence $\_ \otimes \mathrm{Mon}_{E_\infty}(Top) \otimes \mathrm{Mon}_{E_\infty}(Top) \simeq \_ \otimes \mathrm{Mon}_{E_\infty}(Top)$, which makes the smashing functor idempotent up to a chosen coherent equivalence, hence a localisation. The equivalence $\mathrm{Mon}_{E_\infty}(\mathrm{Mon}_{E_\infty}(\_)) \simeq \mathrm{Mon}_{E_\infty}(\_)$ is just an elaboration on the Eckmann-Hilton-argument, two compositions which are homomorphisms with respect to each other and have neutral elements are equal.

Replace in [GGN, p. 19] the expression "algebraic K-theory" with "Eilenberg-MacLane spectrum"; this identifies the Eilenberg-MacLane spectrum functor as the composition of functors

$$\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \xrightarrow{(\cdot)^{iso}} \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \xrightarrow{N(\cdot)} \mathrm{Mon}_{E_\infty}(\mathcal{T}) \longrightarrow \mathrm{Grp}_{E_\infty}(\mathcal{T}) \longrightarrow Sp,$$

for $\mathcal{T}$ some ∞-category of spaces and Sp modelled by any model category of spectra as seen in [MMSS] or directly by stabilisation of an ∞-category of spaces as in [Lu2, Section 1.4.3]. Additionally by [Lu2, p. 624] the ∞-category Sp admits an essentially unique (i.e., parametrised by a contractible Kan complex) symmetric monoidal structure characterised by

1. $Sp \times Sp \to Sp$ preserves colimits in each argument,

2. the unit is the sphere spectrum $\mathbb{S}$.

Even more drastically, we have by [Lu2, Corollary 6.3.2.19] that each stable presentable ∞-category $\mathcal{C}$ with a symmetric monoidal product, which preserves colimits in each argument, admits a unique (up to contractible choice) symmetric monoidal functor $Sp \to \mathcal{C}$ (in particular pointed $\mathbb{S} \to 1_{\mathcal{C}}$), which preserves small colimits.

Thus Sp is not only equipped with a unique symmetric monoidal structure, but it is initial among stable presentable ∞-categories with a symmetric monoidal product preserving colimits. Hence Sp is unique with these properties.

Returning to the Eilenberg-MacLane spectrum we find that the functors

$$\mathrm{Mon}_{E_\infty}(sSet) \longrightarrow \mathrm{Grp}_{E_\infty}(sSet) \longrightarrow Sp,$$

are uniquely symmetric monoidal by [GGN, Theorem 5.1], thus we can reduce multiplicative uniqueness to the claim

$$\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \xrightarrow{(\cdot)^{iso}} \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \xrightarrow{N(\cdot)} \mathrm{Mon}_{E_\infty}(sSet),$$

is uniquely symmetric monoidal. Passing from a category to its maximal subgroupoid $(\cdot)^{iso}$ is strictly symmetric monoidal on the level of 1-categories, and it is the unique such structure on ∞-categories by [GGN, Corollary 5.5 (i)] on the subdiagram for $\mathcal{C} = Cat_1$ the ∞-category of small 1-categories with cartesian product as symmetric monoidal structure and $\mathcal{D} = Gpd$ the ∞-category of small 1-groupoids with the same symmetric monoidal structure:

$$\begin{tikzcd} Cat_1 \arrow[r] \arrow[d, "(\cdot)^{iso}"'] & \mathrm{Mon}_{E_\infty} Cat_1 \arrow[d, dashed] \\ Gpd \arrow[r] & \mathrm{Mon}_{E_\infty} Gpd. \end{tikzcd}$$

Here the corollary precisely says that the unique symmetric monoidal structure given on $\mathrm{Mon}_{E_\infty}\mathcal{C}$ for a symmetric monoidal ∞-category $\mathcal{C}$ allows a unique symmetric monoidal extension of the dashed arrow in the diagram. (Compare however Warning 5.2 in [GGN]: The E_∞-monoids are defined with respect to the cartesian product. Their monoidal product is the extension of a potentially different monoidal structure on $\mathcal{C}$.) The identification $\mathrm{Mon}_{E_\infty} Cat_1 = \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$ gives uniqueness for $(\cdot)^{iso}$.

The same argument for $\mathcal{C} = Gpd$ and $\mathcal{D} = sSet$ yields the unique extension of the nerve $N(\cdot)$ to a symmetric monoidal functor, because the nerve is strictly product-preserving, hence extends uniquely to a symmetric monoidal functor on the respective E_∞-monoids. Since the Eilenberg-MacLane spectrum $H\mathcal{C}$ has zero-level $H\mathcal{C}(\mathbb{S}^0) = N\mathcal{C}$, we find that it is this unique extension.

Thus by these arguments we get the following classical (cf. [MT]) theorem:

Theorem 4.1.3. *There is a unique functor extending the classifying space construction $|N\cdot|\colon Cat_1 \to Top$ to their E_∞-monoids, thus inducing a commutative diagram, where the horizontal arrows are the respective free functors:*

$$\begin{tikzcd}
Cat_1 \arrow[r] \arrow[d, "|N\cdot|"'] & Mon_{E_\infty}(Cat_1) \arrow[r, equal] \arrow[d] & \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \\
Top \arrow[r] & Mon_{E_\infty}(Top) \arrow[r, equal] & Top^{\Gamma}.
\end{tikzcd}$$

Note that the equality below is by definition, while the equality above is the identification of symmetric monoidal 1-categories with Γ-objects in 1-categories, which can be found originally in [Th1, Th2].

4.2 Conjectural Multiplicative Uniqueness

The theorems of [GGN] establish the functor H as the unique functor from symmetric monoidal groupoids to connective spectra with the group completion property as classically identified in [MT].

Apart from the claim in [May2] on page 321 in a footnote "I now have a sketch proof that looks convincing." I could not find a result immediately stating the fact that the constructions in [EM, May2, GGN] each produce the same E_∞-structure on $H\mathcal{R}$. Unfortunately I could not prove this uniqueness, i.e., that the multifunctor-structure of [EM] is essentially the only multifunctor-structure on $H(\cdot)$. I suspect the essential uniqueness can analogously to [BGT2, BGT1, GGN] be specified to say the space of multifunctor-structures on $H(\cdot)$ is actually contractible.

The following elaboration on [GGN, p. 19] would identify "my" E_∞-structure on $H\mathcal{M}(\mathcal{R})$, as described in the previous chapter, with the E_∞-structure on the Eilenberg-MacLane spectrum of the category of matrices with coefficients in the Eilenberg-MacLane spectrum $H(\mathcal{M}(H\mathcal{R}))$ given by using [EM] twice.

In the sequence

$$\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \xrightarrow{(\cdot)^{iso}} \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \xrightarrow{N(\cdot)} \mathrm{Mon}_{E_\infty}(\mathcal{T}) \longrightarrow \mathrm{Grp}_{E_\infty}(\mathcal{T}) \longrightarrow Sp,$$

the first, third and fourth functor admit unique multifunctor-structures by [GGN].

Thus the appropriate conjecture establishing multiplicative uniqueness of H is:

Conjecture 4.2.1. The passage from symmetric monoidal categories to Γ-spaces has a unique multifunctor-structure.

I have tried two avenues, which can probably be made to work: Given the symmetric monoidal structures on Γ-spaces, and the unique symmetric monoidal structure on $\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}$ extending the product on Cat asserted by [GGN], one should be able to make the identification of $Mon_{E_\infty}(N\cdot)\colon \mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at} \to Mon_{E_\infty}(sSet)$ as the tensor-unit in a symmetric monoidal sub-∞-category of the functors $Fun(\mathcal{S}\mathrm{ym}\mathcal{M}\mathrm{on}\mathcal{C}\mathrm{at}, Mon_{E_\infty}(sSet))$, but I simply do not know how to approach finding the appropriate subcategory systematically.

Furthermore each nerve N is defined as a right adjoint. Specifically choose a cosimplicial category. Then functors from this category associate a simplical object to categories. Considering instead $Ex^2 \circ N$ one can promote this to the right adjoint in a Quillen equivalence of categories with the Thomason model structure to simplicial sets in the Kan model structure [Th3].

But the symmetric monoidality established in [GGN] is canonical only for left-adjoint functors, thus the nerve is not easily a canonical extension. This might feel trivial, but since the canonical structures in [GGN] are established by specific left-adjoint functors, their compatibility with right-adjoints is not straight-forward.

This is parallel to the transfer of model structures: Presentable ∞-categories are nerves of combinatorial simplicial model categories. In particular these are cofibrantly generated with generating (trivial) cofibrations satisfying compactness with respect to their category. The dual notion of "cosmall/cocompact" turns out to be too trivial. For instance in $Sets$ the only cosmall objects are the empty set and any one-point-set. Thus most categories of the type "sets with structure" do not have many candidates for generating (trivial) fibrations, much less a model structure determined by them. On presentable categories we see this on objects: The opposite of a presentable category is not presentable in general.

In conclusion: The first avenue is quite ambitious, since in particular it yields a comparison of the E_∞-structures produced by [EM, May2, GGN].

The modest approach runs into a similar problem, i.e., a map in the "wrong" direction. In principle it should not be too hard to produce an E_∞-map of symmetric spectra $H(\mathcal{M}(\mathcal{R})) \to H(\mathcal{M}(H\mathcal{R}))$ with the E_∞-structures of chapter 3 and [EM]. I modelled the delooping in chapter 3 explicitly as a careful generalisation of [EM]. It would be conceptual to pass through strictification to 2-categories: The canonical monoidality map of strictification is an inclusion $(\mathcal{C}\times\mathcal{D})^{st}\to\mathcal{C}^{st}\times\mathcal{D}^{st}$ given by including words of pairs to pairs of words of equal length. It is an equivalence. Any inverse would need to be coherently associative and symmetric with respect to the product, which is not possible without introducing an adequate tricategory-structure on permutative bicategories. This hinges on the fact that the canonical map is only pseudonatural, and the anchor equivalence $\mathcal{C}^{st}\to\mathcal{C}$ is only lax monoidal up to an invertible transformation.

Consider instead strictification by Yoneda embedding into $Fun(\mathcal{C}^{op}, Cat)$. This produces a similar problem in addition to several new ones. Canonically we get a map in the wrong direction. One could try to resolve this by a Day-type convolution, but this introduces new problems. What is a small diagram over which to take the colimit? In addition the target 2-category as well as the source tricategory make a variety of lax colimits conceivable. Finally given a Day-type convolution, does it make the Yoneda-embedding an appropriate multifunctor?

Another problem, which I assume is much simpler to resolve, is the fact that strictification cannot respect the functor-strictnesses of addition and tensor product the way I axiomatised it for permutative and symmetric monoidal bicategories. Thus even if one could pass to a bimonoidally equivalent 2-category, the result would not have addition and tensor product given by strict 2-functors. Thus some clever construction of an equivalent 2-category, for which these functors are strict would yield a canonical map to the construction of [EM], but I did not find such a 2-category. So here the essential question is how a bipermutative bicategory can strictify to a bipermutative 2-category, thus a particular example of a simplicially enriched bipermutative 1-category suited to the machine of [EM].

Finally, I want to state a guess, why these problems arise: K-theory constructions involve passing to categories of isomorphisms first. By [GGN, Proposition 8.14] there is a structural reason for this: Group completion adds objectwise monoidal inverses, but consequently turns every morphism into an equivalence by an Eckmann-Hilton-type argument. I have not found a good way to impose invertibility of morphisms productively, but it has to enter in an essential way.

Furthermore this seems to conflict heavily with the principal example $\mathcal{M}(\mathcal{R})$. There the equivalence 1-cells are automatically isomorphisms, since the 2-cells are only products of isomorphisms. For instance for $Ob\mathcal{R}=\mathbb{N}$ we only get permutation matrices as equivalence 1-cells, which only incorporates the endomorphism-objects $\mathcal{R}(0,0)$ and $\mathcal{R}(1,1)$ as 2-cells. This "delooping" does not satisfy the comparison of [BDRR1], i.e., it does not deloop the appropriate space in order to be equivalent to $K(H\mathcal{R})$. I have no idea how to resolve this conflict. I have to concede however that the conflict might be my illusion, and [GGN, Proposition 8.14] only enforces isomorphism 2-cells in $\mathcal{M}(\mathcal{R})$, which is consistent with the assumption in [BDRR1]. In particular this perspective makes the assumption in [BDRR1] that each translation functor $X \oplus \_$ be a faithful functor appear as an inessential peculiarity of the Grayson-Quillen-completion, while discarding all non-invertible morphisms is essential to the construction.

5 THH and the Trace Map

Since my construction of the delooping of a bimonoidal bicategory naturally yields a symmetric spectrum and I do not want to switch contexts too much, I refer the reader to [Sh] for Topological Hochschild Homology in symmetric spectra, as well as [AnR] for a careful study of the algebraic structure present for nice input spectra. I also want to mention the nicely model independent paper [BFV], which states in general that THH of an E_n-ring spectrum is an E_{n-1}-ring spectrum by careful analysis of the involved operads. They do not need to fix their context of a model category of spectra, since any category tensored over topological spaces or simplicial sets will do, and they all are.

The generalisation of Hochschild homology to ring spectra by way of "functors with smash products" is due to Bökstedt and part of a paper, which is notorious for its preprint-stage [B1]. It is however highly influential, in particular, since Bökstedt subsequently fully calculated $THH(\mathbb{Z}/p)$ as well as $THH(\mathbb{Z})$ in [B2], he provided the foundations for many subsequent calculations of topological Hochschild homology.

5.1 THH with Coefficients

The algebra $V(1)_*THH(ku)$ that Christian Ausoni describes in [A-THH] is quite unwieldy — in particular it is of good use to have descriptions of easier objects with clear relations to $THH(ku)$ the way he describes it in [A-THH]. By introducing the appropriate analog of logarithmic structures on ring spectra, Sagave and Schlichtkrull provided a localisation sequence, which makes the extension $\ell \to ku$ tamely ramified in a well-defined way [SaS]. In particular, they describe the algebra $V(1)_*THH(ku)$ as a square-zero extension of the $V(1)$-homotopy of $THH^{log}(ku)$. However, since I introduce THH purely for analysis of $V(1)_*K(ku)$ by trace methods along the lines of [A-THH, A-Kku], the introduction of logarithmic structures and the resulting cofibre sequence on topological Hochschild homology seems a drastic detour, which I do not want to present here.

Even before it was known that there are symmetric monoidal categories of spectra, people have studied THH in that then hypothetical category [MS] or like Bökstedt in auxiliary categories, which turned out to be equivalent to the model categories of spectra [MMSS]. In particular, given the construction of Hochschild homology for discrete algebras as presented in [Lo] for instance, the generalisation to spectra is straightforward.

Definition 5.1.1. [EKMM, Definition IX.2.1] Given an associative $\mathbb{S}$-algebra A with product $\mu\colon A \wedge A \to A$ and unit $\eta\colon \mathbb{S} \to A$, and an A-bimodule M with actions $M \wedge A \to M$ and $A \wedge M \to M$, which I denote by μ as well, let $THH_\bullet(A, M)$ be the following simplicial spectrum: In degree n we have the $(n+1)$-fold smash product:

$$THH_n(A, M) := M \wedge A^{\wedge n},$$

with face maps

$$d_i\colon M \wedge A^{\wedge n} \to M \wedge A^{\wedge n-1}$$

defined as

$$d_i = \begin{cases} id^{\wedge i-1} \wedge \mu \wedge id^{\wedge n-i} & 0 \leq i \leq n \\ (\mu \wedge id^{n-1}) \circ t & i = n+1, \end{cases}$$

for t the symmetry of the smash product that exchanges factors as follows:

$$M \wedge A_1 \wedge A_2 \wedge \ldots \wedge A_n \xrightarrow{t} A_n \wedge M \wedge A_1 \wedge A_2 \wedge \ldots \wedge A_{n-1}.$$

The degeneracies are given by insertion of units at all places after the module M: $s_i\colon M \wedge A^{\wedge n} \to M \wedge A^{\wedge n+1}$ for $0 \leq i \leq n$:

$$s_i = id^{i+1} \wedge \eta \wedge id^{n-i},$$

where I have notationally suppressed the unit isomorphism $A \cong \mathbb{S} \wedge A$. Call this the simplicial Hochschild spectrum of A with coefficients in M.

For M the algebra itself we set $THH_\bullet(A) := THH_\bullet(A, A)$ and call the resulting simplicial spectrum the simplicial Hochschild spectrum of A.

This construction can be defined for arbitrary $\mathbb{S}$-algebras A and A-bimodules M. However, for it to be of topological significance, in particular for $THH_\bullet(A, M)$ to be homotopy invariant, we impose a technical condition.

Remark 5.1.2. If the unit of the algebra $\mathbb{S} \to A$ is a cofibration in the model structure on a chosen model category of spectra, and A and M are cofibrant $\mathbb{S}$-modules, the simplicial spectrum $THH_\bullet(A, M)$ is *proper* [EKMM, Theorem VII.6.7]. This means that for each simplicial degree n the inclusion

$$sTHH_n(A, M) \to THH_n(A, M)$$

of $sTHH_n(A, M)$ the image of all degeneracies with target degree n is a cofibration [EKMM, p. 182].

This in particular implies that a weak equivalence $M \to M'$ of A-bimodules and a weak equivalence $A \to A'$ of algebras, with M' and A' again cofibrant $\mathbb{S}$-modules with the unit $\mathbb{S} \to A'$ a cofibration, gives rise to a levelwise weak equivalence $THH_\bullet(A, M) \to THH_\bullet(A', M')$, and since both spectra are proper this induces an equivalence on the realisations as well.

For a nicely short exposition of this compare to section 7 of [BLPRZ].

Definition 5.1.3. Let A be an associative $\mathbb{S}$-algebra, which is a cofibrant $\mathbb{S}$-module, and for which the unit $\mathbb{S} \to A$ is a cofibration. Let furthermore M be an A-bimodule, which is a cofibrant $\mathbb{S}$-module. The *topological Hochschild homology* of A with coefficients in M is defined as the realisation of the (proper) simplicial spectrum $THH_\bullet(A, M)$. To give this unambiguous meaning, understand this as the coend

$$THH(A, M) := |THH_\bullet(A, M)| = \int^{\Delta} THH_q(A, M) \wedge (\Delta_q)_+,$$

where we use the tensored structure of a model category of spectra over topological spaces to form $\_ \wedge (\Delta_q)_+$ and then form the coend.

Analogously, with $A = M$ define $THH(A) := |THH_\bullet(A)| = |THH_\bullet(A, A)|$.

We care how multiplicative opposition affects this construction and find that it opposes the simplicial structure as expected.

Remark 5.1.4. For an associative $\mathbb{S}$-algebra A denote its multiplicative opposition A^μ. In particular, I do not want to use the notation A^{op}, since this notation implies the wrong idea for the Eilenberg-MacLane-spectrum of a bipermutative (bi)category $H\mathcal{C}$: No opposition of morphisms is involved, only the symmetry of the smash product in spectra.

Proposition 5.1.5. *(cf.[Lo, 5.2.1], [Lan, Chapter 5]) For an $\mathbb{S}$-algebra A and an A-bimodule M we see that simplicial opposition on the Hochschild spectrum* *is isomorphic to multiplicative opposition, i.e.*

$$\iota \colon THH_\bullet(A^\mu, M^\mu) \cong \widetilde{THH_\bullet}(A, M)$$

where on the left we oppose the multiplication on A and exchange the left- and right-action on M, while on the right we reverse the simplicial direction as in 2.6.8, i.e., $\widetilde{THH_\bullet}(A, M) = THH_\bullet(A, M) \circ r$ for $r \colon \Delta \to \Delta$ the reversion functor.

Proof. The relevant isomorphism in discrete algebra reads $m \otimes a_1 \otimes \ldots \otimes a_n \mapsto m \otimes a_n \otimes \ldots \otimes a_1$. This can be generalised to spectra by the appropriate twists of the smash product. □

The lemma 2.6.6 literally applies, because we used the tensored structure over Top. This gives the following isomorphism of spectra:

Proposition 5.1.6. *We have the identification given by reversing the simplex coordinates in realisations:*

$$\Gamma \colon |\widetilde{THH(A)}| \to |THH(A)|.$$

Given these two structural maps, we can easily induce an involution on THH of a ring spectrum with anti-involution.

Definition 5.1.7. Let (A, μ, τ) be an associative $\mathbb{S}$-algebra with anti-involution τ, i.e., a self-inverse $\mathbb{S}$-algebra map: $\tau \colon (A, \mu) \to (A, \mu^{opp})$. Then we call the following sequence of maps:

$$THH(A) \xrightarrow{\tau} THH(A^\mu) \xrightarrow{\iota} \widetilde{THH(A)} \xrightarrow{\Gamma} THH(A)$$

the induced involution of τ on $THH(A)$.

5.2 The Bökstedt Spectral Sequence

The Bökstedt spectral sequence, calculating topological from algebraic Hochschild homology, is the essential tool Bökstedt uses in [B2] to calculate $THH(\mathbb{Z})$ and $THH(\mathbb{Z}/p)$. For a published reference see [EKMM, Theorem IX.2.9].

The induced maps of τ and ι are simplicial by the results before, so for the spectral sequence associated to the simplicial filtration, which yields the Bökstedt spectral sequence under flatness assumptions, we can deduce the following result. For technical convenience assume we have arranged for $THH_\bullet(A, M)$ to be proper by the conditions mentioned above in Remark 5.1.2.

Theorem 5.2.1. *Let h be a generalised homology theory, then the simplicial filtration of $|THH(A)|$ yields a spectral sequence*

$$E^1_{*,n} = C^{cell}_*(THH_n(A), THH_{n-1}(A), h_*) \Rightarrow h_*(THH(A)).$$

If h satisfies the Künneth-formula on A, i.e., $h_*(A \wedge A) \cong h_*(A) \otimes_{h_*(\mathbb{S})} h_*(A)$, then we can identify the E^2-term with the algebraic Hochschild-homology of $h_*(A)$, i.e.:

$$E^2 \cong HH(h_*(A)).$$

The induced involution given above is compatible with the simplicial filtration, thus induces a map of spectral sequences. In particular we find on E^2-terms:

$$HH(h_*A) \xrightarrow{\tau} HH(h_*(A^\mu)) = HH((h_*A)^\mu) \xrightarrow{\iota} \widetilde{HH}(h_*A) \xrightarrow{\Gamma} HH(h_*A).$$

Remark 5.2.2. Do note that τ and ι can be induced on the simplicial level, while Γ is a map of chain complexes given by introducing the adequate sign associated to the map

$$\Delta^n/\partial\Delta^n \to \Delta^n/\partial\Delta^n$$

with $[t_0, t_1, \ldots, t_n] \mapsto [t_n, t_{n-1}, \ldots, t_0]$, which is given by the sign of the permutation that fully inverts the set $\{0, 1, \ldots, n\}$, i.e.,

$$(0\ n)(1\ n-1)\ldots(\left\lfloor \frac{n}{2} \right\rfloor \left\lceil \frac{n}{2} \right\rceil),$$

which is $(-1)^{\frac{n(n+1)}{2}}$.

The Bökstedt spectral sequence generalises to the case with coefficients in a bimodule as well by the same filtration:

Theorem 5.2.3. *Let h be a generalised homology theory, A an associative $\mathbb{S}$-algebra, M an A-bimodule; additionally assume that h satisfies the Künneth-isomorphisms $h_*(A \wedge M) = h_*(A) \otimes_{h_*(\mathbb{S})} h_*(M)$, $h_*(M \wedge A) = h_*(M) \otimes_{h_*(\mathbb{S})} h_*(A)$, and $h_*(A \wedge A) = h_*(A) \otimes_{h_*(\mathbb{S})} h_*(A)$, then we have a Bökstedt spectral sequence of the form:*

$$HH(h_*(A), h_*(M)) \Rightarrow h_*(THH(A, M)).$$

If additionally for $M_* = h_*(M)$ and $A_* = h_*(A)$ the algebra A_* is projective over $k = h_*\mathbb{S}$ then we can understand the E^2-term as a derived functor:

$$Tor^{A_* \otimes_k {A_*}^{op}}(M_*, A_*).$$

Remark 5.2.4. I have suppressed convergence discussions in spectral sequences, because the homology theories and spectra I consider only give modules and algebras with non-negative grading, thus we have strongly convergent first quadrant spectral sequences.

5.3 The Multiplicative Structure of the Involution

Structural Example: The Product for the Commutative Case

Most sources discuss the product on Hochschild homology for the chain complex associated to the simplicial module formed by the Hochschild complex (cf. [Lo, Lemma 1.6.11] or [McL, Theorem VIII.8.8]). However this introduces the complications associated to dealing with the sum of shuffles in the Eilenberg-Zilber map, so I prefer the simplicial structure of the product in algebraic modules as a template for spectra.

Remark 5.3.1. For A a commutative k-algebra let M, N be A-modules and consider the simplicial modules $C_n(M, A) = M \otimes A^{\otimes n}$ and $C_n(N, A) = N \otimes A^{\otimes n}$. In general we have the pairing $M \otimes A^{\otimes n} \otimes N \otimes A^{\otimes n} \to M \otimes N \otimes (A \otimes A)^{\otimes n}$, which is given by reordering the tensor factors according to the two-rail rail fence cipher, i.e., by pairing M with N and the ith A-factor of the first $A^{\otimes n}$ with the ith A-factor in the second $A^{\otimes n}$. This assembles to a simplicial isomorphism:

$$C_\bullet(M, A) \otimes C_\bullet(N, A) \to C_\bullet(M \otimes N, A \otimes A).$$

Hochschild homology is a functor in the module variable for arbitrary maps linear over the appropriate algebra. Furthermore it is a functor in the algebra argument for algebra maps. For commutative A we have that $\mu\colon A \otimes A \to A$ is an A-linear map of A-modules as well as a map of k-algebras, thus for $M = N = A$ we get a map

$$C_\bullet(A, A) \otimes C_\bullet(A, A) \to C_\bullet(A \otimes A, A \otimes A) \to C_\bullet(A, A).$$

Analogously, we can define for A a commutative $\mathbb{S}$-algebra a pairing of the simplicial spectrum $THH_\bullet A$ with itself as

$$THH_\bullet(A) \wedge THH_\bullet(A) \to THH_\bullet(A \wedge A) \to THH_\bullet(A).$$

Since this map is simplicial, it is obviously compatible with the filtration giving the Bökstedt spectral sequence, and thus introduces the structure of a spectral sequence of differential graded algebras.

Remark 5.3.2. To use [BGT2] I need to diverge into spectral categories, i.e., categories enriched over a symmetric monoidal model category of spectra: The above discussion applies to spectral categories as follows. Consider two small spectral categories $\mathcal{C}, \mathcal{D}$, and set their smash product to be the category with objects $Ob\mathcal{C} \times Ob\mathcal{D}$ with smash-product on morphism spectra $\mathcal{C} \wedge \mathcal{D}((c_1, d_1), (c_2, d_2)) = \mathcal{C}(c_1, c_2) \wedge \mathcal{D}(d_1, d_2)$.

For a small spectral category with cofibrant morphism spectra $\mathcal{C}$ define its simplicial Hochschild spectrum as (compare [BGT1, p. 73]):

$$THH_q(\mathcal{C}) := \bigvee \mathcal{C}(c_{q-1}, c_q) \wedge \mathcal{C}(c_{q-2}, c_{q-1}) \wedge \ldots \wedge \mathcal{C}(c_0, c_1) \wedge \mathcal{C}(c_q, c_0).$$

Take special note of the last factor, making it a "circle of degree q" in $\mathcal{C}$. The analogous face- and degeneracy-maps make it a simplicial spectrum, call its realisation the topological Hochschild homology of $\mathcal{C}$.

The indicated rail-fence shuffle above induces an isomorphism of simplicial spectra:

$$THH_\bullet(\mathcal{C}) \wedge THH_\bullet(\mathcal{D}) \to THH(\mathcal{C} \wedge \mathcal{D}),$$

which by coherence of the symmetry of $\wedge$ in spectra is strongly symmetric monoidal in the sense that the following diagram strictly commutes:

$$\begin{tikzcd} THH_\bullet(\mathcal{C}) \wedge THH_\bullet(\mathcal{D}) \arrow[r] \arrow[d, "c_\wedge"'] & THH_\bullet(\mathcal{C} \wedge \mathcal{D}) \arrow[d, "c_\wedge"] \\ THH_\bullet(\mathcal{D}) \wedge THH_\bullet(\mathcal{C}) \arrow[r] & THH_\bullet(\mathcal{D} \wedge \mathcal{C}). \end{tikzcd}$$

Since realisation and smash product commute by naturality of the product isomorphism $|X \times Y| \cong |X| \times |Y|$ in compactly generated Hausdorff spaces for X, Y simplicial spaces, this descends to the same multifunctoriality after realisation.

Proposition 5.3.3. *Consider the category of small spectral categories Cat_{Sp} enriched over a chosen symmetric monoidal model category of spectra. By [Tab] we* *know that by cofibrantly replacing a small spectral category in the model structure on Cat_{Sp} we obtain a weakly equivalent small spectral category with cofibrant morphism spectra.*

On the category of small spectral categories with cofibrant morphism spectra consider topological Hochschild homology $THH_\bullet(\cdot)$. This is a strong symmetric monoidal and simplicially enriched functor

$$THH_\bullet(\cdot)\colon Cat_{Sp} \to sSp.$$

In particular, it is a multifunctor by considering the multicategory-structures induced by the symmetric monoidal products on Cat_{Sp} and sSp.

By passing to realisations we get a strong symmetric monoidal and simplicially enriched functor, hence a multifunctor:

$$THH(\cdot)\colon Cat_{Sp} \to Sp.$$

Proof. The structure of a multifunctor on THH is an elaboration on the transformation $THH(\mathcal{C}) \wedge THH(\mathcal{D}) \to THH(\mathcal{C} \wedge \mathcal{D})$, which is strictly symmetric, strictly unital, and coherently associative, hence yields a multifunctor of the described type. □

Remark 5.3.4. I strongly recommend the preprint of Bjørn Ian Dundas establishing multifunctoriality of topological cyclic homology as well, and moreover proving that the trace maps are also natural transformations for these multifunctor-structures [D2].

In absence of an explicit symmetric monoidal product on symmetric monoidal categories (cf. however [GGN] for a monoidal structure on the ∞-category of symmetric monoidal categories) the authors in [EM] describe a multicategory-structure on permutative categories instead, which has bipermutative categories as its E_∞-monoids, and establish that the Eilenberg-MacLane spectrum of permutative categories (which [EM] call "K-theory") can be given the structure of a multifunctor. Thus by the natural equivalence $K(H\mathcal{R}) = H(\mathcal{M}(\mathcal{R}))$ established in [BDRR1] we can consider an induced multifunctor-structure on $H(\mathcal{M}_{\text{-}}) = H \circ \mathcal{M}_{\text{-}}$. In particular, in absence of a comparison of the multiplicative structure I describe in chapter 3, I refer to the E_∞-structures induced by the multifunctor-structure as **the** E_∞-structure.

I summarise the appropriate identifications of [BDRR1, BGT2] in the following theorem – to summarise the known results.

Theorem 5.3.5. *The algebraic K-theory space $B\mathcal{M}(\mathcal{R})$ of a permutative category $\mathcal{R}$ is naturally equivalent to the algebraic K-theory space of its associated Eilenberg-MacLane spectrum $K(H\mathcal{R})$ [BDRR1].*

The algebraic K-theory functor from small spectral categories to spectra can be given the structure of a (symmetric) multifunctor in an essentially unique way [BGT2, Theorem 1.5].

There is an essentially unique natural transformation of (symmetric) multifunctors $K \Rightarrow THH$, which by [BGT2, Theorem 6.3] is the trace map *from algebraic K-theory to topological Hochschild homology [BGT2, Theorem 1.9]. Thus in particular we get a unique natural multiplicative map $K(H\mathcal{R}) \to THH(H\mathcal{R})$ for Eilenberg-MacLane spectra.*

Remark 5.3.6. As seen in 4 I have to concede that I do not know if the uniqueness of [BGT2] forces the E_∞-structure of chapter 3 on $H(\mathcal{M}(\mathcal{R}))$ to agree with the canonical structure on $K(H\mathcal{R})$ asserted by [BGT2]. If, however, one were able to prove that the Eilenberg-MacLane spectrum functor admits an essentially unique multifunctor-structure, or more modestly to produce E_∞-equivalences $K(H\mathcal{R}) \to H\mathcal{M}(\mathcal{R})$ and $H\mathcal{M}(\mathcal{R}) \to K(H\mathcal{R})$, then either of these would imply that $H\mathcal{M}(\mathcal{R})$ has the unique multifunctor-structure given by composition of the unique structure on $K\colon Cat_{Sp} \to Sp$ and the conjecturally unique structure on $H\colon PermCat \to Sp$, while the second approach obviously directly gives the claimed equivalence.

Since I introduce the trace map $K \to THH$ by its multiplicative universality as proven in [BGT2], I want to emphasise the equivalence of E_∞ and commutative structures in most model categories of spectra.

Remark 5.3.7. In orthogonal and symmetric spectra with their positive stable model structure as well as in the category of $\mathbb{S}$-modules we have that commutative ring spectra model all E_∞-ring spectra – [MMSS, Lemma 15.5] and also [EKMM, Theorem 5.1, Chapter III]. However, since I already use the reference [EM] often in preceding chapters, I follow the setup of their Theorem 1.4; in particular I restrict to the case of symmetric and orthogonal spectra in the positive stable model structure for this remark.

Considering the multicategory-structure on symmetric (or orthogonal) spectra induced by the smash-product, it is meaningful to speak of multifunctor-categories $MFun(\mathbb{M}, Sp^\Sigma)$, where we consider a simplicially enriched multicategory $\mathbb{M}$ and symmetric spectra with their natural simplicial mapping spaces.

Given an enriched multifunctor $f\colon \mathbb{M}' \to \mathbb{M}$ we have an induced restriction functor:

$$f^*\colon MFun(\mathbb{M}', Sp^\Sigma) \to MFun(\mathbb{M}, Sp^\Sigma)$$

by precomposition with f, which by [EM, Theorem 1.4] is the right adjoint in a Quillen adjunction – with the left-adjoint given by extension in the appropriate manner (cf. p.56 of [EM]).

If f is an equivalence of simplicially enriched multicategories, i.e., $\pi_0 f$ is an equivalence of ordinary 1-categories, and for each set of objects we have a weak equivalence of simplicial sets $M(a_1, \dots, a_n; b) \to M'(fa_1, \dots, fa_n; fb)$, then Theorem 1.4 of [EM] furthermore yields that this adjunction is a Quillen equivalence (compare also [Ber1]). To my knowledge it has not been established that these equivalences of multicategories are weak equivalences in a model structure on small multicategories, which preferably would extend the Bergner model structure on simplicially enriched categories.

When we consider the Barratt-Eccles operad $E\Sigma_*$ as a one-point multicategory and map it to the terminal multicategory Com, we have an underlying isomorphism of (one-point) 1-categories. Since each multimorphism-category of $E\Sigma_*$ is equivalent to the one-point category we get a weak equivalence of its nerve to a point as well, giving a Quillen equivalence:

$$\mathbb{P}\colon MFun(E\Sigma_*, Sp^\Sigma) \rightleftarrows MFun(Com, Sp^\Sigma)\colon U,$$

where I consider the functors given by extension and restriction as a prolongation functor $\mathbb{P}$ and a forgetful functor U. In particular for a cofibrant E_∞-ring spectrum A in Sp^Σ we have a natural E_∞-map $\eta\colon A \to UP(A)$ to a stably equivalent commutative symmetric ring spectrum.

5.3.1 The Opposite E_∞-Structure

Following Section 9 of [EM] define the following map of operads.

Definition 5.3.8. For each $k \geq 0$ set $r_k\colon \{1, \dots, k\} \to \{1, \dots, k\}$ to be $r_k(j) = k + 1 - j$, i.e., the map consisting of only transpositions $(1\ k)$, $(2\ k-1)$ until the centre. In other words r_k fully reverses the set $\{1, \dots, k\}$.

Lemma 5.3.9. *The symmetric sequences $(\Sigma_n)_{n\in\mathbb{N}}$ and $(E\Sigma_n)_{n\in\mathbb{N}}$ of categories, where Σ_n is the discrete category on objects $\sigma \in \Sigma_n$, while $E\Sigma_n$ is the translation* *category of Σ_n with objects $\sigma \in \Sigma_n$ and a unique morphism between each pair of objects, form the associative and the Barratt-Eccles-operad.*

Both maps $op\colon \Sigma_* \to \Sigma_*$, $op\colon E\Sigma_* \to E\Sigma_*$ defined as $op(\sigma) = r_k \circ \sigma$ for $\sigma \in \Sigma_k$, and extended to a covariant functor in the unique way on $E\Sigma_*$, are maps of operads. Specifically, op respects units, is equivariant with respect to the obvious right Σ_*-action by functors, and preserves multicomposition, i.e., block sum followed by permutation of blocks.

Proof. The claim is explicit after Definition 9.1.11 of [EM], however, the proof is "left to the reader". Since in particular the compatibility with multicompositions can be confusing, I want to elaborate on that.

Unitality of *op* is obvious, since $r_1 = id_{\{1\}}$. Equivariance is obvious as well, since we defined *op* by a left-action and the equivariance-condition involves the right-action of Σ_n on itself.

For compatibility with multicompositions I first reduce the condition we have to show drastically: For Θ the multicomposition on Σ_* and $E\Sigma_*$ we can reduce an expression $\Theta(r_n \circ \sigma_n; r_{k_1} \circ \sigma_{k_1}, \ldots, r_{k_n} \circ \sigma_{k_n})$ by using the right-Σ_*-action on the resulting product: $\Theta(r_n \circ \sigma_n; r_{k_1} \circ \sigma_{k_1}, \ldots, r_{k_n} \circ \sigma_{k_n}) = \Theta(r_n \circ \sigma_n; r_{k_1}, \ldots, r_{k_n}).(\sigma_{k_1} \oplus \ldots \oplus \sigma_{k_n})$. Thus we can, without loss of generality, consider a multiproduct $\Theta(r_n \circ \sigma_n; r_{k_1}, \ldots, r_{k_n})$. But by the "inner" equivariance condition on an operad, we can replace σ_n on the left by the appropriate permutation on the right, giving an expression $\Theta(r_n; r_{l_1}, \ldots, r_{l_n})$. As the final reduction, note that by multiassociativity of the multicomposition Θ it suffices to show $\Theta(r_2; r_n, r_m) = r_{n+m}$, which yields the higher compatibilities by an easy induction.

Note that $\Theta(r_2; \sigma_n, \sigma_m) = \chi^+_{n,m} \circ (\sigma_n \oplus \sigma_m)$, for $\chi^+_{n,m}$ the symmetry of the sum in Fin and $\sigma_i \in \Sigma_i$. Thus we have to show $\chi^+_{n,m} \circ (r_n \oplus r_m) = r_{n+m}$. This is an easy calculation, recall

$$\chi^+_{n,m}(i) = \begin{cases} i+m, & 1 \le i \le n, \\ i-n, & n+1 \le i \le n+m, \end{cases}$$

and for convenience

$$(r_n \oplus r_m)(i) = \begin{cases} r_n(i), & 1 \le i \le n, \\ r_m(i-n)+n, & n+1 \le i \le n+m. \end{cases}$$

The block sum trivially preserves the condition on i, thus the composite $\varphi :=$

$\chi^+_{n,m} \circ (r_n \oplus r_m)$ is given by

$$\varphi(i) = \chi^+_{n,m} \circ (r_n \oplus r_m)(i) = \begin{cases} r_n(i) + m, & 1 \leq i \leq n \\ r_m(i-n), & n+1 \leq i \leq n+m. \end{cases}$$

Clearly we have $\varphi(1) = r_n(1) + m = n + m$ and $\varphi(n+m) = r_m(m) = 1$, so $\varphi = (1\ n+m) \circ \bar{\varphi}$ and $\bar{\varphi}$ is of the form $\Theta(r_2; r_{n-1}, r_{m-1})$, so we are done by induction. □

In particular we can define what the opposite algebra for an associative and an E_∞-algebra are, when we restrict to $E\Sigma_*$ as the E_∞-operad.

Definition 5.3.10. An associative algebra in spectra determines and is determined by a multifunctor $\Sigma_* \to Sp$, which sends the unique object of the multicategory Σ_* to the algebra. In particular, we define its opposite algebra as the composition $\Sigma_* \to \Sigma_* \to Sp$ with the first map being the opposition *op* above. This multifunctor determines and is determined by the algebra with opposed multiplication, which is associative if and only if the original multiplication is associative.

Analogously an E_∞-algebra in spectra is a multifunctor $E\Sigma_* \to Sp$, which is simplicially enriched with respect to the usual simplicial structure on spectra and $E\Sigma_*$ made simplicial by arity-wise application of the nerve. Since *op* is in particular a map of sets in each arity, it extends uniquely to a functor $E(op)\colon E\Sigma_* \to E\Sigma_*$ in each arity, thus to a simplicially enriched multifunctor. The opposite E_∞-algebra is given by precomposition with $E(op)$.

In particular the underlying associative algebra of an $E\Sigma_*$-algebra is given by restriction along $\Sigma_* \to E\Sigma_*$ the inclusion of objects, and the underlying associative algebra of the opposed E_∞-structure is the opposed associative algebra.

We can thus define what an anti-involution on an $E\Sigma_*$-spectrum is.

Definition 5.3.11. An anti-involution $\tau\colon A \to A$ for A an $E\Sigma_*$-algebra is an $E\Sigma_*$-map with respect to the $E\Sigma_*$-structure on the source, and the opposed $E\Sigma_*$-structure on the target, with $\tau^2 = \mathrm{id}$.

Remark 5.3.12. Recall the Quillen-equivalence of E_∞-ring spectra and commutative ring spectra for instance in the positive stable model structure on symmetric spectra [EM, MMSS], or the model structure on $\mathbb{S}$-modules as exhibited in [EKMM]. With the notation as in Remark 5.3.7 we find that for A a cofibrant

$E\Sigma_*$-algebra the prolonged involution $\mathbb{P}\tau\colon \mathbb{P}A \to \mathbb{P}A^\mu$ is a map of commutative spectra. Since $\mathbb{P}(A^\mu) = \mathbb{P}(A)^\mu = \mathbb{P}A$ by strict commutativity, we find that $\mathbb{P}\tau$ is a self-inverse algebra map of commutative algebras, which by $\mathbb{P}(A)^\mu = \mathbb{P}A$ becomes an endomorphism.

5.3.2 Induced Multiplications on THH and the Involution

I introduced the trace by its multiplicative universality as proven in [BGT2], thus we need to see that the induced involution of 5.1.7 opposes multiplication to find that the trace map commutes with the involutions on K-theory and topological Hochschild homology.

The identification $N\mathcal{C}^{op} = \widetilde{N\mathcal{C}}$ extends to the cyclic nerve defining THH as we see above, and is strictly symmetric monoidal.

Proposition 5.3.13. *The natural isomorphism*

$$\iota\colon THH(A^\mu, M^\mu) \to TH\widetilde{H(A, M)}$$

is strictly symmetric monoidal with respect to the smash product.

Proof. The rail-fence isomorphism indicated above and ι are instances of the symmetry of the smash product, thus the coherence of the smash-symmetry yields the claim. □

The natural homeomorphism Γ is symmetric monoidal as well.

Proposition 5.3.14. *The natural homeomorphism* $\Gamma\colon |\cdot| \Rightarrow |\tilde{\cdot}|$ *is symmetric monoidal with respect to cartesian as well as smash-product.*

Proof. This is a bit easier to see by considering the symmetric monoidal structure of realisation oplax, i.e., $|X \times Y| \cong |X| \times |Y|$. The natural map in this case is given by realisation of pr_X and pr_Y respectively. In particular it is induced on simplicial objects. The natural transformation Γ, however, operates on the realisation coordinates, thus the transformations strictly commute. □

These propositions assemble to the following:

Theorem 5.3.15. *For an E_∞-ring spectrum A with anti-involution $T\colon (A,\mu) \to (A, \mu^{opp})$, consider the internal involution induced on THH by*

$$THH(A) \xrightarrow{T} THH(A^\mu) \xrightarrow{\iota} \widetilde{THH}(A) \xrightarrow{\Gamma} THH(A).$$

This is a natural E_∞-map with respect to the induced E_∞-structure on the source and the opposed E_∞-structure on the target $THH(A)$.

Proof. This is just assembling the last three propositions, where we have analysed each of the maps individually. In particular, the multiplicative opposition induced from T is not changed by ι and Γ, thus follows the claim. □

We can reuse [BGT2] to establish that the trace map commutes with the induced involutions.

Theorem 5.3.16. *The unique natural transformation of (simplicially) enriched multifunctors $tr\colon K \Rightarrow THH$ commutes with the involution on K induced as in 3.5.5 and induced on THH as above.*

Proof. In the diagram

$$\begin{tikzcd} K(A) \arrow[r] \arrow[d, "T_*"] & THH(A) \\ K(A) \arrow[r] & THH(A) \arrow[u, "T_*"'] \end{tikzcd}$$

both the upper horizontal map as well as the map given by composing the induced involutions with the trace give an E_∞-map $K(A) \to THH(A)$. The first E_∞-structure is directly asserted by [BGT2], the second follows from the fact, that the E_∞-structure is opposed twice by the respective involutions. Thus by uniqueness of the multiplicative trace [BGT2] we get that the threefold composite describes the trace as well. □

Remark 5.3.17. With just a conjectural identification of the E_∞-structures on $H(\mathcal{M}(H\mathcal{R}))$ and $H(\mathcal{M}(\mathcal{R}))$ the reference to 3.5.5 in the theorem is more informal than I intended. Formally, one could, however, set up exactly the same procedure I describe in chapter 3 to establish that the involution opposes the E_∞-structure on $H(\mathcal{M}(H\mathcal{R}))$ as well. This amounts to mostly rewriting [EM] the way I describe in 1.1.4 but explicitly establishing the multiplicative structure for topologically enriched permutative categories the way I do for bicategories in chapter 3. This is not an immediate specialisation of chapter 3, because I implicitly assume discrete sets of 1-cells in this thesis, but I expect no essential difficulties in generalising chapter 3 to bicategories with morphism categories internal to topological spaces.

Remark 5.3.18. I think, the calculations in chapter 6 are more easily readable if I declare for the reader how I think about the three maps involved in inducing the involution on THH, and the analogous sequence on K-theory 3.5.5, i.e. the classifying spaces of the bicategory of matrices:

$$THH(A) \xrightarrow{T} THH(A^\mu) \xrightarrow{\iota} \widetilde{THH(A)} \xrightarrow{\Gamma} THH(A),$$

$$B\mathcal{M}(\mathcal{R}) \xrightarrow{T} B\mathcal{M}(\mathcal{R}^\mu) \xrightarrow{(\cdot)^t} B\mathcal{M}(\mathcal{R})^{op_1} \xrightarrow{\Gamma} B\mathcal{M}(\mathcal{R}).$$

In both cases we directly induce a multiplicatively opposing map using the involution. The transposition on matrices and the map ι allow to identify the simplices in the nerve of the multiplicative opposition with simplices in the simplicially opposed nerve. Finally Γ, in both cases, modifies the simplices in the realisation by a map of degree ± 1 only depending on the simplicial degree of the simplex, which internalises the involution.

In summary: $\Gamma \circ \iota$ as well as $\Gamma \circ (\cdot)^t$ consist of degrees and a preferred identification of simplices, thus are usually easily analysed.

5.4 A Useful Subspectrum of THH

In section 3 of [MS] the authors identify a simplicial subspectrum of $THH_\bullet(A)$ which is naturally included for any A. Since then the functor homology interpretations of Hochschild homology (cf. for instance [Lo, PR]) via the Loday functor $\mathcal{L}(A, M)\colon \mathrm{Fin}_+ \to k{-}\mathrm{Mod}$ have been established, providing a nicely clean, natural interpretation of this subspectrum. I want to elaborate on this in this section.

This section deserves an emphasised *special acknowledgement*: Since Stephanie Ziegenhagen and I have been close ever since our own modest beginnings in Algebraic Topology, I have also observed her conception of her thesis [Z] in quite some detail. If I had not been a test-case for numerous “functor co*homology”-talks provided in her own trademark-clarity, I am quite sure I would never have understood and probably not even bothered to understand that context. Thus the essential clarifications in this section rest firmly on her shoulders.

Non-Commutative Sets

The existence of the subspectrum identified in [MS, Section 3] does not depend on any additional structure on an associative ring spectrum A. In particular, it is not relevant if A happens to be commutative or not.

To properly identify the simplicial topological Hochschild homology spectrum for an associative algebra object however, one obviously needs to keep track of the order with respect to which one multiplies. The category of non-commutative sets $\mathrm{Fin}_+^{\mathrm{As}}$ does just that. I follow the exposition in the sections 1.2-1.4 in [PR], but instead immediately consider pointed sets, called $\Gamma(as)$ in [PR].

Definition 5.4.1. (cf. [PR, Section 1.2]) The category $\mathrm{Fin}_+^{\mathrm{As}}$ has objects pointed finite sets $n_+ = \{*, 1, \ldots, n\}$, and morphisms pointed maps $f\colon n_+ \to m_+$ with chosen total orderings on the fibres $f^{-1}j$ for every $j \in m_+$ (including the base-point). For maps composable in finite pointed sets, i.e., $f\colon n_+ \to m_+$, $g\colon m_+ \to l_+$ the underlying map is the composite in finite sets gf with total orderings on the fibres given as indicated by:

$$(gf)^{-1}i = f^{-1}g^{-1}i = \coprod_{j \in g^{-1}i} f^{-1}j.$$

Explicitly, the ordering of elements $j \in g^{-1}i$ provides an ordering of the summands, while each summand is ordered with the order chosen for f.

Lemma 5.4.2. *[PR, Lemma 1.1] Any morphism $f\colon n_+ \to m_+$ in $\mathrm{Fin}_+^{\mathrm{As}}$ has a unique decomposition $\Delta_f \circ \sigma_f$, where $\Delta_f\colon n_+ \to m_+$ is order-preserving and pointed, and $\sigma_f\colon n_+ \to n_+$ is a bijection, usually not pointed.*

Proof. The proof of [PR] is for the unpointed case, so I want to retrace the decomposition for pointed maps. Given a map $f\colon n_+ \to m_+$ we find a unique order-preserving map Δ_f isomorphic to it over m_+, i.e., with

$$\begin{tikzcd} n_+ \arrow[r, "f"] \arrow[d, "\sigma_f"'] & m_+ \\ n_+. \arrow[ur, "\Delta_f"'] & \end{tikzcd}$$

More explicitly Δ_f is the unique order-preserving map with the same fibre-cardinalities as f, i.e., $|\Delta_f^{-1}\{i\}| = |f^{-1}\{i\}|$ for every $i \in m_+$. In particular there is a unique bijection $n_+ \to n_+$ which makes the fibres of f into intervals in n_+ while order preserving on the fibres. The total ordering chosen on the fibres of f then fixes a unique fibre-wise bijection $n_+ \to n_+$ of the order induced by f to the order induced by the total order on n_+. The composite is the unique bijection σ_f.

In particular we find that σ_f is pointed if and only if the base-point is minimal in the chosen order of $f^{-1}\{*\}$. $\square$

Remark 5.4.3. Since σ_f is not pointed in general the decomposition is not internal to pointed sets. However, the bijection necessarily still satisfies $\sigma_f(*) \in \Delta_f^{-1}(*)$, because the composite is a pointed map.

This category makes it possible to define a Loday functor $\mathcal{L}(A, M)\colon \mathrm{Fin}_+^{\mathrm{As}} \to Sp$ for an associative $\mathbb{S}$-algebra A and A-bimodule M. I want to specifically elaborate on the dependence of $\mathcal{L}$ on the symmetric monoidal structure with respect to which it is defined. Hence I consider a general symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, c_\otimes)$ an associative $\otimes$-algebra A in $\mathcal{C}$ and an A-bimodule M, keeping the designations usual for monoids and bimodules in $(Sp, \wedge, \mathbb{S}, c_\wedge)$.

Definition 5.4.4. (cf. [PR, Section 1.3]) The Loday functor $\mathcal{L}(A, M)\colon \mathrm{Fin}_+^{\mathrm{As}} \to \mathcal{C}$ is given on objects as $n_+ \mapsto M \otimes A^{\otimes n}$. For a morphism $f\colon n_+ \to m_+$ consider its unique decomposition $f = \sigma \circ \delta$. Then σ describes a unique symmetry of the monoidal structure:

$$c_\sigma\colon M \otimes A^{\otimes n} \to A^{\otimes |f^{-1}\{*\}<*|} \otimes M \otimes A^{\otimes |f^{-1}\{*\}>*|} \otimes A^{\otimes |f^{-1}\{1\}|} \otimes \ldots \otimes A^{\otimes |f^{-1}\{m\}|}.$$

From this reordered object we consider the map

$$A^{\otimes |f^{-1}\{*\}<*|} \otimes M \otimes A^{\otimes |f^{-1}\{*\}>*|} \otimes A^{\otimes |f^{-1}\{1\}|} \otimes \ldots \otimes A^{\otimes |f^{-1}\{m\}|} \to M \otimes A^{\otimes m},$$

composed of the left- and right-action of A on M, i.e., $A^{\otimes |f^{-1}\{*\}<*|} \otimes M \otimes A^{\otimes |f^{-1}\{*\}>*|} \to M$ and $|f^{-1}i|$-fold products $A^{\otimes |f^{-1}\{i\}|} \to A$. Call this map δ_*, then define $f_* = \delta_* \circ c_\sigma$. This makes $\mathcal{L}(A, M)$ a functor by uniqueness of the decomposition 5.4.2.

To relate this functor to the simplicial topological Hochschild homology spectrum I need to elaborate on the simplicial circle a bit more. More specifically, we need to know that it is in fact a simplicial associative pointed set.

Proposition 5.4.5. *(cf. [PR, Section 1.4]) Recall the simplicial set $\Delta_1 = \Delta(\_, [1])$ with its boundary $\partial\Delta^1$ identified as the constant maps $f\colon [n] \to [1]$. The quotient $\mathbb{S}^1 = \Delta_1/\partial\Delta_1$ is a pointed simplicial set by 3.4.1, which can be promoted to a pointed associative simplicial set $\mathbb{S}^1_{As}\colon \Delta^{op} \to \mathrm{Fin}_+^{\mathrm{As}}$.*

Given an associative $\mathbb{S}^1$, we can interpret topological Hochschild homology for associative algebras. Recall the spectral Loday functor for an associative $\mathbb{S}$-algebra A and A-bimodule M:

$$\mathcal{L}(A, M)\colon \mathrm{Fin}_+^{\mathrm{As}} \to Sp.$$

More generally for an associative algebra and bimodule in a symmetric monoidal category $(\mathcal{C}, \otimes, \mathbb{1}, c_\otimes)$ with the assignment on objects: $\mathcal{L}(A,M)(n_+) = M \otimes A^{\otimes n}$. The total order on the fibres precisely makes this a well-defined functor for associative (as opposed to commutative) objects. We can easily identify the simplicial Hochschild spectrum of A with coefficients in M as the composite of the Loday functor $\mathcal{L}(A,M)\colon \mathrm{Fin}_+^{\mathrm{As}} \to Sp$ with the pointed associative circle 5.4.5: $\mathbb{S}^1_{As}\colon \Delta^{op} \to \mathrm{Fin}_+^{\mathrm{As}}$.

Proposition 5.4.6. *(cf. [Lo, p. 213]) The composite:*

$$\Delta^{op} \xrightarrow{\mathbb{S}^1_{As}} \mathrm{Fin}_+^{\mathrm{As}} \xrightarrow{\mathcal{L}(A,M)} Sp$$

is strictly equal to the simplicial Hochschild spectrum of A with coefficients in M as defined in 5.1.1. In particular, the identification is natural in maps of algebras and bimodules.

Proof. Objectwise the identification is clear, the example above should convince the reader that I fixed the choices of orderings and basepoints in 5.4.5 just so that one can identify the face maps and degeneracies in the standard complex easily with the maps induced on the pointed associative circle. □

With these considerations in place we can find that the subspectrum identified in [MS, Section 3] is inherent to the Loday-functor.

Proposition 5.4.7. *[MS, Section 3] Let A be an associative algebra, and M an A-bimodule with a map of A-bimodules $A \to M$. In particular both have a fixed unit map from the sphere spectrum $\mathbb{S} \to A \to M$.*

Then we have two factorisations of the identity at M:

$$\begin{tikzcd} M \cong M \wedge \mathbb{S} \arrow[rr] \arrow[d] & & M \wedge A \arrow[d] \\ M \vee A \arrow[r] & M \vee M = (M \wedge \mathbb{S}) \vee (\mathbb{S} \wedge M) \arrow[r] & M, \end{tikzcd}$$

with the analogous factorisations holding for the left-module action and the algebra structure of A.

Thus in particular for A an $\mathbb{S}$-algebra, and M an A-algebra, we can define the Loday-functor with respect to coproducts $\mathcal{L}^\vee(A,M)(n_+) = M \vee A^{\vee n}$. By universal property of the coproduct to describe a map $\mathcal{L}^\vee(A,M)(n_+) = M \vee A^{\vee n} \to M \wedge A^{\wedge n} = \mathcal{L}(A,M)(n_+)$, we need to describe it on each summand. Thus for M

consider $M \cong M \wedge \mathbb{S}^{\wedge n} \to M \wedge A^{\wedge n}$, while the ith A-summand is mapped to the ith smash factor by the analogous description.

The factorisation above gives that this is a natural transformation

$$\mathcal{L}^{\vee}(A, M) \Rightarrow \mathcal{L}(A, M),$$

which is moreover natural in the algebra A and the A-algebra M appropriately.

Proposition 5.4.8. *(cf. [MS, Lemma 3.3]) The coproduct Loday-functor on an $\mathbb{S}$-algebra A, and an A-algebra M, evaluated on the associative circle is naturally isomorphic to the simplicial spectrum $M \vee ((\mathbb{S}^1) \wedge A)$, for $\mathbb{S}^1 = \Delta_1/\partial\Delta_1$ the simplicial circle.*

In particular, the geometric realisation of the natural transformation above yields a natural map:

$$|\mathcal{L}^{\vee}(A, M)(\mathbb{S}^1_{As})| = |M \vee ((\mathbb{S}^1) \wedge A)| = M \vee \Sigma A \to THH(A, M).$$

I used the description of topological Hochschild homology as a Loday functor on the associative circle to facilitate the following identification:

Theorem 5.4.9. *The Loday functor evaluated on the opposite associative circle $\mathbb{S}^1_{As} \circ r\colon \Delta^{op} \to \mathrm{Fin}_+^{\mathrm{As}}$ is naturally isomorphic to the Loday functor on the opposite algebra and opposed bimodule:*

$$\mathcal{L}(A, M)(\mathbb{S}^1_{As} \circ r) \cong \mathcal{L}(A^{\mu}, M^{\mu})(\mathbb{S}^1_{As}).$$

In particular, for M an A-algebra, the natural transformation

$$\mathcal{L}^{\vee}(A, M) \Rightarrow \mathcal{L}(A, M),$$

commutes with this isomorphism, as does its geometric realisation.

We can draw the following corollary, which I use repeatedly in the following chapter. To not confuse opposition of module actions with opposition of multiplications I only assume an appropriate map $A \to M$, which we need for the map $\mathcal{L}^{\vee}(A, M) \to \mathcal{L}(A, M)$.

Theorem 5.4.10. *Given an associative $\mathbb{S}$-algebra A with anti-involution $T\colon A \to A^{\mu}$, and an left- and right-A-linear map $A \to M$ we have the following commu-*

tative diagram:

$$\begin{tikzcd}
M \vee \Sigma A \arrow[r] \arrow[d] & M \vee \Sigma A \arrow[r] \arrow[d] \arrow[rr, bend left, "1\vee(-1)"] & M \vee \widetilde{\Sigma} A \arrow[r] \arrow[d] & M \vee \Sigma A \arrow[d] \\
THH(A,M) \arrow[r, "{(T,\mathrm{id}_M)}"] & THH(A^\mu, M^\mu) \arrow[r, "\iota"] & |\widetilde{THH}(A,M)| \arrow[r, "\Gamma"] & THH(A,M).
\end{tikzcd}$$

More explicitly: The bimodule includes at simplicial degree 0*, thus each of the lower three maps restricts to the identity. The suspended algebra is the realisation of the simplicial spectrum* $\mathbb{S}^1 \wedge A$*, hence includes at degree* 1*, so that* ι *and* Γ *together induce a sign.*

Remark 5.4.11. I want to place the appropriate emphasis on this subspectrum. Despite the fact that these are the "obvious" classes in $THH(A,M)$, they are usually not trivial. Instead one can usually use the suspension $\Sigma A \to THH(A,M)$, inducing a map $h_* A \to h_{*+1} THH(A,M)$, and multiplicative structures on THH to exhaust the classes of interest. Good examples of this are [MS] and [A-THH], the second of which we study in detail in the next chapter.

6 The Involution on $V(1)_*K(ku_p)$

In this chapter we can finally use the results of all preceding chapters to revisit the calculations of Christian Ausoni to analyse them along the induced involutions - primarily the ones in [A-THH, A-Kku]. For this I go through many details of the calculations and recall the ones, which I need for establishing the involution on classes in homology or homotopy groups. However I have written this chapter under the assumption that the reader has the sources [A-THH, A-Kku] close. In particular the effect of the involution has clearer emphasis, when I consider the hard calculations of [A-THH, A-Kku] as given.

Inherent to the calculations of [A-THH, A-Kku] is the restriction to odd primes, for a partial picture at $p = 2$ see [AnHL] computing the homotopy groups $\pi_* THH(\ell)$ and $\pi_* THH(ko)$ locally at 2.

6.1 The Involutions on ℓ and ku

Preliminaries

The model provided by algebraic K-theory of an algebraic closure of a finite field $K(\overline{\mathbb{F}}_q)$, which comes with a ring map given by the Brauer lift $K(\overline{\mathbb{F}}_q) \to KU$, as well as the connective cover given by $H\mathcal{M}_{\mathbb{C}} = ku \to KU$ yield the same E_∞-structure, when completed at p for which $q \in (\mathbb{Z}/p^2)^\times$ is a generator, by [BR2]. The homology theories in [A-THH, A-Kku] are insensitive to p-completion, because the involved homology theories are $H\mathbb{F}_p$-local (in the sense of Bousfield-localisation), hence I switch between the models for ku whenever convenient for a clearer exposition.

Do note that by the homotopy limit involved in the definition of topological cyclic homology and the fact that completion can for instance be described as a homotopy colimit we cannot expect completion and cyclic homology to commute.

On K-theory we can trace the analogous failure back to the fact that a ring usually has fewer units than its p-completion, thus analysing $BGL(R^\wedge_p)$ and

$BGL(R)^\wedge_p$ are usually two different problems. However topological Hochschild homology does commute with colimits (given cofibrant spectra, as we assumed above 5.1.2), thus as long as I rely on [A-THH] for the determination of $V(1)_*THH(ku)$ and do not refer to $TC(ku)$ and $K(ku)$, it is not ambiguous, if I do not specify, if ku denotes its integral, p-local or p-completed version.

By the description of ku given above 1.2 we know that at an odd prime p the spectrum ku has a direct summand ℓ called the Adams summand, first identified by Adams through operations on vector bundles - see Lecture 4 of [Ad2]. The inclusion of fields $L \to L(\zeta_p)$ induces a map of E_∞-ring spectra $i\colon K(L) \to K(L(\zeta_p))$, and I fix these completed at p as a model for the inclusion $i\colon \ell_p \to ku_p$ 1.2.

By basic obstruction theory (cf. [EKMM, Proposition 3.1], [MayE_∞, p. 36, Lemma 2.12]) we can realise the map $ku \to H\mathbb{Z} = H(\pi_0 ku)$ as a map of E_∞-ring spectra, as well as the map $ku \to H\mathbb{Z} \to H\mathbb{Z}/p$. This induces in particular a map of E_∞-ring spectra $\ell \to H\mathbb{Z}/p$, which by [Ad1, Lemma 16.8] realises the inclusion:

$$H_*(\ell;\mathbb{Z}/p) = \mathbb{Z}/p[\xi_1,\xi_2,\ldots] \otimes E(\tau_2,\tau_3,\ldots)$$
$$\to A_* = H_*(H\mathbb{Z}/p,\mathbb{Z}/p) = \mathbb{Z}/p[\xi_1,\xi_2,\ldots] \otimes E(\tau_0,\tau_1,\ldots)$$

of the indicated subalgebra of the dual Steenrod algebra at p (cf. for instance [Ko, pp. 51-53]), with generators in degrees $|\xi_i| = 2p^i - 2$ and $|\tau_i| = 2p^i - 1$. Do note that the map $H\mathbb{Z}_p \to H\mathbb{Z}/p$ induces an injection on $H\mathbb{Z}/p$-homology as well with image the full polynomial algebra, and all of the exterior algebra apart from τ_0, i.e., the dual of the Bockstein element in the Steenrod-algebra, which is the $H\mathbb{F}_{p*}$-Bockstein map.

Since the map $\ell \to ku \to \mathbb{Z}$ can be realised on bipermutative categories as $\mathcal{M}_L \to \mathcal{M}_{L(\zeta_p)} \to \mathbb{N}^\delta$, with $\mathbb{N} = \mathbb{N}^\delta$ considered as a discrete bipermutative category with its obvious rig structure, we see that the involutions are compatible as follows:

$$\begin{tikzcd}
\ell \arrow[r] \arrow[d, "\tau_\ell"] & ku \arrow[r] \arrow[d, "\tau_{ku}"] & H\mathbb{Z} \arrow[d, equal] \\
\ell \arrow[r] & ku \arrow[r] & H\mathbb{Z},
\end{tikzcd}$$

independently of what involutions we chose compatibly on ku and ℓ.

In particular we find that any involution on ℓ given by an involution on $\mathcal{M}_L$ induces the identity on its homology:

Proposition 6.1.1. *Let $(T, t)\colon \mathcal{M}_L \to \mathcal{M}_L^\mu$ be an involution of bipermutative categories with $H\mathcal{M}_L = \ell$, then the involution induced on homology with $\mathbb{Z}/p$-coefficients is trivial.*

Proof. Let me repeat the core of the argument: If the involution arises on bipermutative categories, then we can map to the discrete rig $\mathbb{N}$, and this induces the monomorphism $H_*\ell \to H_*H\mathbb{Z}$. Since $\mathbb{N}$ is discrete it only supports the trivial involution. $\square$

Recall [A-THH, Theorem 2.5]:

Theorem 6.1.2. *There is an isomorphism of A_*-comodule algebras:*

$$H_*(ku, \mathbb{F}_p) = H_*(\ell, \mathbb{F}_p) \otimes P_{p-1}(x)$$

with $x \in H_2(ku, \mathbb{F}_p)$ the Hurewicz-image of $u \in \pi_2 ku$, where $P_{p-1}(x)$ denotes the polynomial algebra on x truncated by x^{p-1}.

Furthermore $H_*\ell$ can be identified as the inclusion of $\mathbb{Z}/(p-1)$- fixed points under the action by the Galois group of $L \to L(\zeta_p)$, hence in particular we have:

$$\ell \simeq ku^{h\mathbb{Z}/(p-1)}.$$

Proof. Each of the statements is found on pp. 1268–1269 of [A-THH]. Compare also for the fixed point statement the corresponding statements on THH and K-theory on p. 1307 of [A-THH]. While the fixed point statement is immediate from the homological fixed point spectral sequence (cf. for instance [BrRo]) which collapses at E^2, because the relevant group homology is acyclic, since the order of the group is a unit $p - 1 = -1 \in \mathbb{F}_p^\times$ – cf. [Rot, p. 156]. $\square$

Theorem 6.1.3. *The involution on H_*ku is completely determined by the effect on $u \in \pi_2 ku$, thus also on its image under the Hurewicz map $x \in H_2(ku, \mathbb{F}_p)$.*

Explicitly: For the map of commutative $\mathbb{S}$-algebras $\tau\colon ku \to ku$ induced by strictifying $H\mathcal{V}_\mathbb{C}$ and complex conjugation along the Quillen equivalence of commutative and E_∞-ring spectra (recalled in 5.3.7), we get

$$\tau_*(u^n) = (-1)^n u^n.$$

Since on $\mathcal{V}_\mathbb{C}$ complex conjugation and transposition-inversion agree, this is also the effect of the involution induced by the identity.

Proof. We have the canonical map $ku \to KU$, and by a classical result of Snaith [Sn] we know, that we can obtain KU as the suspension spectrum of the infinite complex projective space by inverting the Bott class $u \in \pi_2\Sigma^\infty_+\mathbb{C}P^\infty$. For a modernised account in motivic spectra compare Gepner-Snaith [GSn].

But this class arises as the suspension of $u\colon \mathbb{S}^2 \to \mathbb{C}P^\infty$ on space level. In particular we can choose to realise it as the inclusion $\Sigma U(1) \to BU(1) \simeq \mathbb{C}P^\infty$, where complex conjugation evidently acts on $\Sigma U(1) \cong \mathbb{S}^2$ by a reflection along one equator, hence has degree -1. □

We have seen at 1.2 that the involution induced 3.5.5 on ℓ is strictly equal to the identity. In particular for commutative models inverting and transposing agrees with complex conjugation, thus I consider the effect of complex conjugation as fundamental.

Corollary 6.1.4. *Transposition-inversion induces the identity on $(H\mathbb{F}_p)_*\ell$, thus on $H\mathbb{F}_p$-homology of ku it is given as $x \mapsto -x$ and the identity on $H_*\ell$.*

Proof. I already presented above that the fact that $\ell \to H\mathbb{F}_p$ induces a monomorphism on homology, forces any self-map on bipermutative categories to be visible on $\mathbb{N}^\delta$, thus trivial. □

Corollary 6.1.5. *Complex conjugation on $\pi_*ku = ku_* \cong \mathbb{Z}[u]$ induces the map: $u^n \mapsto (-1)^n u^n$ by 6.1.3. Thus for a prime $p \geq 3$ the map $\ell \to ku$ realising the inclusion $\mathbb{Z}_{(p)}[v] \mapsto \mathbb{Z}_{(p)}[u]$ with $v \mapsto u^{p-1}$, identifies the effect of conjugation on ℓ as the identity.*

Proceeding in following [A-THH] we consider topological Hochschild homology of ku with coefficients in $H\mathbb{Z}_p$, which is a ku-module by the canonical maps $ku \wedge H\mathbb{Z}_p \to H\mathbb{Z}_p \wedge H\mathbb{Z}_p \to H\mathbb{Z}_p$.

Theorem 6.1.6 ([A-THH, pp. 1282–1287, Proposition 5.6])**.** *There is an isomorphism of A_*-comodule algebras*

$$H_*(THH(ku, H\mathbb{Z}_p), \mathbb{F}_p) = H_*(H\mathbb{Z}_p, \mathbb{F}_p) \otimes E([\sigma x], [\sigma\xi_1]) \otimes P([y]).$$

with degrees $|\sigma x| = |x| + 1 = 3$, $|\sigma\xi_1| = 2p - 2 + 1 = 2p - 1$, and $|y| = 2p$.

The Bökstedt spectral sequence for $THH(ku, H\mathbb{Z}_p)$:

$$E^2 = HH^{\mathbb{F}_p}(H_*(ku, \mathbb{F}_p), H_*(H\mathbb{Z}_p, \mathbb{F}_p)) \Rightarrow H_*(THH(ku, H\mathbb{Z}_p); \mathbb{F}_p)$$

has E^2-term:

$$HH_*(H_*ku, H_*H\mathbb{Z}_p) \cong H_*H\mathbb{Z}_p \otimes E(\sigma x, \sigma\xi_1, \sigma\xi_2, \ldots) \otimes \Gamma(y, \sigma\tau_2, \sigma\tau_3, \ldots),$$

where Γ denotes the divided power algebra over $\mathbb{F}_p$ on the given generators. The spectral sequence collapses at E^{2p} and thus has E^∞-term:

$$E^{2p} = E^\infty = H_*H\mathbb{Z}_p \otimes E(\sigma x, \sigma\xi_1) \otimes P_p(y, \sigma\tau_2, \sigma\tau_3, \ldots)$$

with multiplicative extensions $[y]^p = [\sigma\tau_2]$ and $[\sigma\tau_i]^p = [\sigma\tau_{i+1}]$.

Proof. This is all explicit in [A-THH] at the given pages. □

Remark 6.1.7. Note in particular that the divided power algebra $\Gamma(y)$ in E^2-terms of the Bökstedt spectral sequence actually gives rise to a polynomial algebra in 6.1.6.

For the Adams summand the analogous computational result was already published in 1991 by McClure and Staffeldt. In absence of the complications introduced by the truncated polynomial algebra $P_{p-1}(x)$ I can directly quote the result for $THH(\ell)$.

Theorem 6.1.8 ([MS, Proposition 4.2; p.22], cf. also [A-THH, Theorem 5.9])**.** *For any prime $p \geq 3$ there is an isomorphism of $\mathbb{F}_p$-algebras:*

$$H_*(THH(\ell), \mathbb{F}_p) = H_*(\ell, \mathbb{F}_p) \otimes E([\sigma\xi_1], [\sigma\xi_2]) \otimes P([\sigma\tau_2]).$$

Since the computation of $THH(\ell)$ involves fewer complications than the analogous one for $THH(ku)$ I can immediately determine the full effect of the involution here.

Proposition 6.1.9. *An endomorphism of ℓ, which induces the identity on homology $H\mathbb{F}_{p*}\ell$, induces the identity on the tensor factor $H\mathbb{F}_{p*}\ell$ of $H\mathbb{F}_{p*}THH(\ell)$ as well. On the suspension classes the homeomorphism Γ induces the sign -1.*

Proof. It is easily seen that the tensor factor $(H\mathbb{F}_p)_*\ell$ stems from simplicial degree 0, thus ι and Γ are identities there. The suspended classes $[\sigma x]$ can be represented by classes $1 \otimes x$ in the Bökstedt spectral sequence. In particular we see that since these classes are of simplicial degree 1, the simplicial inversion is the identity, while the homeomorphism Γ introduces the sign of one transposition, thus -1. □

I opted to present these results before the calculational cornerstone of [A-THH], which is fundamental to Ausoni's calculations, hence also to mine. In [MS] the choice of ℓ as their object of focus facilitates the calculations performed by McClure and Staffeldt, particular the absence of the truncated polynomial algebra in $H\mathbb{F}_p$-homology of ℓ. In [A-THH] Christian Ausoni traces the effect of this algebra in $H\mathbb{F}_{p_*}ku$ on homology carefully, isolating its Hochschild homology in [A-THH, Proposition 3.3], which I reduce here to isolating the acyclic resolution and the resultant cycles. More generally for k a commutative ring with unit the Hochschild homology of algebras $A = k[X]/f$ with f a monic polynomial was quite generally calculated in [GGRSV], however I am following the exposition of Ausoni specialised to $f = X^n$ and $k = \mathbb{F}_p$. As far as I know the original source for this resolution is [MN, Section 3].

Proposition 6.1.10. *Let $A = P_h(x)$ be the polynomial algebra truncated by the ideal x^h over $k = \mathbb{F}_p$, then there is an acyclic resolution of A as an $A \otimes A^{op} = A^e$-module with underlying graded module:*

$$X = A^e \otimes E(\sigma x) \otimes \Gamma(\tau)$$

with A^e in resolution degree 0, σx of degree $(1, |x|)$, τ of degree $(2, h|x|)$. For h a unit in $\mathbb{F}_p$ the cycles in $A \otimes_{A^e} X$ consist of the A-submodule $\Gamma(\tau) \otimes \{\sigma x\} \oplus (\tau) \otimes \{x\}$ for $(\tau) \subset \Gamma(\tau)$ the ideal of positive divided powers in $\Gamma(\tau)$.

Proof. This is all explicit in [A-THH, Proposition 3.3]. Note that X in fact describes a $\mathbb{Z}$-resolution of $\mathbb{Z}[x]/x^h$, identifying the cycles in $A \otimes_{A^e} X$ however is less clean for $k = \mathbb{Z}$. □

Proposition 6.1.11. *Consider on $A = P_h(x)$ the morphism of commutative $k = \mathbb{F}_p$-algebras given by $x \mapsto -x$. On generators in Hochschild homology it induces: $x \mapsto -x, \sigma x \mapsto -\sigma x, \tau \mapsto \tau$.*

Proof. In the acyclic resolution given above with $A^e = P_h(x) \otimes P_h(y)$ we have $d(\sigma x) = x - y, d(\tau) = \frac{x^h - y^h}{x - y} \sigma x$ [A-THH, Proposition 3.3]. Thus a lift of the given map is given by $\sigma x \mapsto -\sigma x$ and $\tau \mapsto \tau$, giving the claimed effect. □

The analogous consideration works immediately for $H\mathbb{F}_{p_*}THH(ku, H\mathbb{Z}_p)$.

Theorem 6.1.12. *Complex conjugation on ku induces the identity on the tensor factor $H\mathbb{F}_{p_*}(H\mathbb{Z}_p) \subset H\mathbb{F}_{p_*}(THH(ku, H\mathbb{Z}_p)) = H\mathbb{F}_{p_*}H\mathbb{Z}_p \otimes E(\sigma x, \sigma\xi_1) \otimes P(\tau)$.*

The involution induced by $THH(A,M) \to \widetilde{THH(A,M)} \to THH(A,M)$, *with first map the anti-involution composed with simplicial reversion, and second map the homeomorphism* Γ *on realisations – cf. 5.1.7 – induces the following maps:* $\sigma x \mapsto \sigma x, \sigma\xi_1 \mapsto -\sigma\xi_1, \tau \mapsto -\tau$.

In more detail: The map induced by $x \mapsto -x$ *gives* $\sigma x \mapsto -\sigma x$ *and the identity on* $\sigma\xi_1$ *and* τ, *while the homeomorphism* Γ *induces a sign* -1 *on all three classes.*

Proof. We see in the resolution X chosen above that ι can be represented as the identity, since it is also a resolution of A as an $(A^e)^{op} = A^{\otimes 2}$-module, since A is commutative. Thus only Γ introduces an additional effect as a sign dependent on resolution degree, which is 1 for the suspensions, and 2 for τ, thus we get -1 in both cases 5.2.2. □

6.2 Increasing Chromatic Complexity – Reduction by p

As indicated I use the modules, which are easiest to describe, whenever possible. For the next calculational steps of [A-THH] I thus need to introduce "mod p homotopy". The idea is quite simple, instead of considering the prime p as a self-map on $H\mathbb{Z}$ or $H\mathbb{Z}_p$, we can consider it as a self-map of the sphere spectrum, giving the cofibre sequence of spectra:

$$\mathbb{S} \xrightarrow{p} \mathbb{S} \longrightarrow V(0) \longrightarrow \Sigma\mathbb{S}.$$

In particular one could hope that $V(0)$ gives a better approximation to homotopy groups than $H\mathbb{F}_p$, thus its homology theory is usually called *mod p homotopy.* In other words the spectrum $V(0)$ is the two-cell spectrum $\mathbb{S}^0 \cup_p \mathbb{D}^1 = V(0)$.

It is classical that the spectrum $V(0)$ at a prime $p \geq 3$ admits a multiplication, which is part of an A_{p-1}-structure, which cannot be extended to A_p. For $p = 2$ we do not have a multiplication, while for $p = 3$ the multiplication is not associative even up to homotopy. For a good survey of this I refer to [Schw1]. In particular in [Schw1, Theorem 2.5] we see the obstruction to extending the A_{p-1}-structure to an A_p-structure.

The construction [Schw1, Definition 2.1] works by introducing levels of extended powers $D_nX = X^{\wedge n} \wedge_{\Sigma_n} E\Sigma_{n+}$. As a consequence, a coherent $M =$

$\mathbb{S}^2 \cup_p \mathbb{D}^3$-module structure as defined in [Schw1, Definition 2.1] entails degrees of commutativity as well. Specifically in [Schw1, Example 2.4] the "tautological" coherent module structure on $V(0)$ is defined up to degree $p-1$. So for $p \geq 5$, the second extended power $D_2V(0) = V(0)^{\wedge 2} \wedge_{\Sigma_2} E\Sigma_{2+}$ is part of the module structure. Thus the multiplication admits the following factorisation:

$$\begin{tikzcd} V(0) \wedge V(0) \arrow[r, "tw"] \arrow[d, "\mu"'] & V(0) \wedge V(0) \arrow[d, "\mu"'] \\ D_2V(0) \arrow[r, "(\_\wedge(12))"] & D_2V(0), \end{tikzcd}$$

where the transposition (12) acts on the factor $E\Sigma_2$, and thus is canonically simplicially homotopic to the identity. Thus the multiplication on $V(0)$ is homotopy commutative for $p \geq 5$, by analogously considering the third extended power, which is also part of the module structure for $p \geq 5$, we get homotopy associativity as well.

For the classical interpretations in particular of the obstruction class I defer to the references of [Schw1], in particular the three Toda references, and the reference to Ravenel.

6.2.1 The Involution on $V(0)_*THH(ku, H\mathbb{Z}_p)$

Referring to [A-THH, Proposition 10.1] I focus exclusively on ku from here. The appropriate restriction for ℓ follows by the observation, that the Galois group acts by maps on coefficients of the approximating bipermutative categories. In particular the action strictly commutes with the involution on ku, thus the induced involution for ℓ can be recovered by restricting to the submodule of fixed points under the action of the Galois group by [A-THH, Proposition 10.1].

Introducing $V(0)$-coefficients makes the topological Hochschild homology of $H\mathbb{Z}_p$ easier to understand. By the equivalence $V(0) \wedge H\mathbb{Z}_p \simeq H\mathbb{F}_p$ the resulting module admits an $\mathbb{F}_p$-algebra structure for all odd primes $p \geq 3$.

Proposition 6.2.1. *[A-THH, Theorem 5.7] For any prime $p \geq 3$ there is an isomorphism of $\mathbb{F}_p$-algebras*

$$V(0)_*THH(H\mathbb{Z}_p) \cong E(\lambda_1) \otimes P(\mu_1),$$

with degrees $|\lambda_1| = 2p-1$ and $|\mu_1| = 2p$. Moreover the Hurewicz homomorphism is an injection

$$V(0)_*THH(H\mathbb{Z}_p) \to (H\mathbb{F}_p)_*(V(0) \wedge THH(H\mathbb{Z}_p))$$

with $\lambda_1 \mapsto [\sigma\xi_1]$ *and* $\mu_1 \mapsto [\sigma\tau_1] - \tau_0[\sigma\xi_1]$.

The fact that the Hurewicz is an injection immediately yields the following corollary.

Corollary 6.2.2. *The involution induced on* $V(0)_*THH(H\mathbb{Z}_p)$ *by the identity as in 5.1.7 is* $\lambda_1 \mapsto -\lambda_1$ *and* $\mu_1 \mapsto -\mu_1$.

Proof. Arguing as in the propositions for ℓ we identify the claimed effects as the effect of ι and Γ on suspension classes. □

For $THH(ku, H\mathbb{Z}_p)$ the equivalence $V(0) \wedge H\mathbb{Z}_p \simeq H\mathbb{F}_p$ yields an $\mathbb{F}_p$-algebra structure on the mod p homotopy for all $p \geq 3$, giving the result:

Proposition 6.2.3. *[A-THH, Theorem 6.8] There is an isomorphism of* $\mathbb{F}_p$*-algebras for any prime* $p \geq 3$*:*

$$V(0)_*THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1) \otimes P(\mu_1)$$

with degrees $|z| = 3, |\lambda_1| = 2p - 1$ *and* $|\mu_1| = 2p$.

The Hurewicz homomorphism is an injection

$$V(0)_*THH(ku, H\mathbb{Z}_p) \to H\mathbb{F}_{p*}(V(0) \wedge THH(ku, H\mathbb{Z}_p))$$

with $z \mapsto [\sigma x], \lambda_1 \mapsto [\sigma\xi_1]$*, and* $\mu_1 \mapsto [\tau] - \tau_0[\sigma\xi_1]$.

Remark 6.2.4. Let me note in particular that the 0th Postnikov section $j\colon ku \to H\mathbb{Z}_p$ induces a map of $H\mathbb{Z}_p$-algebras, which on mod p homotopy gives $j_*(\sigma x) = 0, j_*(y) = \sigma\tau_1$, and $j_*(\sigma\xi_1) = \sigma\xi_1$. Thus the only class we have not analysed with respect to the involution is σx.

Corollary 6.2.5. *The involution on* $V(0)_*THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1) \otimes P(\mu_1)$ *is given as follows:* $z \mapsto z, \lambda_1 \mapsto -\lambda_1, \mu_1 \mapsto -\mu_1$.

Proof. The Hurewicz homomorphism, as well as the map induced by $j\colon ku \to H\mathbb{Z}_p$ give monomorphisms in the degrees relevant to λ_1 and μ_1, giving the claim for them. For z simply note that it is the mod p reduction of the integral class σx considered before. □

6.3 Reducing by $\alpha_1\colon \Sigma^{2p-2}V(0) \to V(0)$

In [A-THH, Sections 7+8] Ausoni proceeds to identify the mod p homotopy of $THH(ku)$ by considering a Bockstein spectral sequence associated to the Bott class $u\colon ku \to ku$. In mod p coefficients the resulting algebra $V(0)_*THH(ku)$ however has infinitely many generators and infinitely many relations for any presentation [A-THH, Corollary 7.11]. Thus even for purely presentational reasons it is convenient to introduce one further reduction here.

Recall that the obstruction to extending the A_{p-1}-structure on $V(0)$ to an A_p-structure is the Adams self-map usually called $\alpha_1\colon \Sigma^{2p-2}V(0) \to V(0)$. In particular in the coherent module structures as considered in [Schw1] it appears as a non-trivial obstruction to unitality of a non-existent A_p-structure on the Moore spectrum (cf. [Schw1, Theorem 2.5]).

Consider the cofibre sequence defining $V(1)$:

$$\Sigma^{2p-2}V(0) \xrightarrow{\alpha_1} V(0) \longrightarrow V(1) \longrightarrow \Sigma^{2p-1}V(0).$$

Compare specifically to page 58 of [Toda]. In particular [Toda, Theorem 4.1] fixes $V(1)$ as the unique (up to homotopy equivalence) spectrum with 4 cells with attaching maps as indicated: $V(1) = (\mathbb{S} \cup_p C\mathbb{S}) \cup_{\alpha_1} C(\mathbb{S}^{2p-2} \cup_p C\mathbb{S}^{2p-2})$. It is usual to call its homology theory $V(1)$-*homotopy*, I do so as well in what follows.

By considering $V(0)$ and $V(1)$ as part of a family of spectra $V(a)$ with inclusions $V(a) \to V(b)$ for $a < b$ Toda identifies multiplicative pairings of the form $V(a) \wedge V(b) \to V(c)$ with $a, b \leq c$. In particular for $V(1)$ included into $V(1\frac{1}{2}), V(2\frac{1}{4}), V(2\frac{3}{4})$, and $V(3)$, we find that the first three cases of [Toda, Theorem 4.4] give a multiplication on $V(1)$ for $p \geq 11, p = 7$ and $p = 5$ respectively. Furthermore [Toda, Theorem 6.3] gives in particular that for $p = 3$ such a multiplication cannot exist, while [Toda, Theorem 6.1] explicitly establishes that a spectrum of the analogous type of $V(1)$ does not exist at $p = 2$ at all. In particular since this consideration has naturally led us to primes with $p \geq 5$ we can apply [Oka] to find that the obstruction to homotopy-commutativity is always 2-torsion, while the obstruction to homotopy-associativity is always 3-torsion. Thus both vanish in the coefficients of $V(1)$.

Since I wanted to introduce $V(1)$ as late as possible with respect to the calculations in [A-THH], I only partially quoted the result of [MS] regarding $THH(\ell)$. Here it can thus serve to convince the reader that $V(1)$ simplifies the modules considerably.

Proposition 6.3.1. *([MS], cf. [A-THH, Theorem 5.9]) For any prime $p \geq 3$ there is an isomorphism of $\mathbb{F}_p$-algebras:*

$$(H\mathbb{F}_p)_*(THH(\ell)) \cong (H\mathbb{F}_p)_*\ell \otimes E(\sigma\xi_1, \sigma\xi_2) \otimes P(\sigma\tau_2).$$

The $V(1)$-homotopy $V(1)_*THH(\ell)$ maps by an injective Hurewicz homomorphism to $(H\mathbb{F}_p)_*(V(1)\wedge THH(\ell))$, with image generated as an algebra by $[\sigma\xi_1], [\sigma\xi_2]$ and $[\sigma\tau_2] - \tau_0[\sigma\xi_2]$, yielding an isomorphism of $\mathbb{F}_p$-algebras:

$$V(1)_*THH(\ell) \cong E(\lambda_1, \lambda_2) \otimes P(\mu_2),$$

with degrees $|\lambda_1| = 2p-1, |\lambda_2| = 2p^2-1,$ and $|\mu_2| = 2p^2$, with generators defined as the preimages of $[\sigma\xi_1], [\sigma\xi_2]$ and $[\sigma\tau_2] - \tau_0[\sigma\xi_2]$ respectively.

In particular since the generators are preimages of suspension classes by an injective Hurewicz homomorphism, the involutions are determined by 6.1.9, giving the following cleaner statement.

Corollary 6.3.2. *For the $V(1)$-homotopy of $THH(\ell)$:*

$$V(1)_*THH(\ell) \cong E(\lambda_1, \lambda_2) \otimes P(\mu_2),$$

the homeomorphism Γ (cf. 2.6.6) induces a sign -1 on each generator. Thus the induced involution 5.1.7 on $V(1)_*THH(\ell)$ is given as: $\lambda_1 \mapsto -\lambda_1, \lambda_2 \mapsto -\lambda_2, \mu_2 \mapsto -\mu_2$.

6.3.1 The Homology and $V(1)$-Homotopy of $THH(ku)$

People familiar with the computations in [A-THH] know that the algebras on homology $H\mathbb{F}_{p*}THH(ku)$ and $V(1)_*THH(ku)$ contain big subalgebras Ω_* and Ξ_* respectively on $(p-1)^2+1 = p^2-2p$ generators with quite a few relations - cf. [A-THH, Definition 9.9, Definition 9.13]. Essentially this stems from the truncated polynomial algebra in $H\mathbb{F}_{p*}ku$ [A-THH, Proposition 2.3].

In order to understand the involution on $H_*THH(ku)$ and $V(1)_*THH(ku)$ however I do not need to display the relations, it suffices to understand the effect on the cycles $(x) \otimes (\tau) \oplus (\sigma x) \otimes \Gamma(\tau) \to HH_*(P_{p-1}(x))$.

To determine the involution on $V(1)_*THH(ku)$ we shall use the topological Hochschild homology of ku with coefficients in $H\mathbb{Z}_p$, i.e., $V(1)_*THH(ku, H\mathbb{Z}_p)$ as an anchor.

Lemma 6.3.3. *([A-THH, p. 1305]) The isomorphism on mod p homotopy $V(0)_*THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1) \otimes P(\mu_1)$ implies the algebra isomorphism:*

$$V(1)_*THH(ku, H\mathbb{Z}_p) \cong E(z, \lambda_1, \varepsilon) \otimes P(\mu_1)$$

with ε of degree $2p-1$.

Proof. The Hurewicz from $V(0)_*THH(ku, H\mathbb{Z}_p)$ to $H\mathbb{F}_{p*}(V(0) \wedge THH(ku, H\mathbb{Z}_p))$ is a monomorphism [A-THH, Theorem 6.8]. So the Adams map $\alpha_1 \colon \Sigma^{2p-2}V(0) \to V(0)$ induces the zero map on $V(0)_*THH(ku, H\mathbb{Z}_p)$, because it induces the trivial map in $H\mathbb{F}_p$-homology. Thus the $V(1)$-homotopy of $THH(ku, H\mathbb{Z}_p)$ consists of two shifted copies of its mod p homotopy, which we can parametrise by a formal generator ε of degree $2p-1$. □

Proposition 6.3.4. *The induced involution 5.1.7 on $V(1)_*THH(ku, H\mathbb{Z}_p)$ is given on generators as:*

$$z \mapsto z, \lambda_1 \mapsto -\lambda_1, \mu_1 \mapsto -\mu_1, \varepsilon \mapsto \varepsilon.$$

Proof. Since ε in essence describes the connecting homomorphism of the cofibre sequence:

$$\Sigma^{\bullet}V(0) \to V(0) \to V(1) \to \Sigma^{\bullet}V(0),$$

and thus acts on the coefficients it commutes with the involution induced on THH. On the other elements the involution is the one given in 6.2.5 □

As seen above I do not need to display the relations of the algebras Ξ_* and Ω_* defined in [A-THH, Definition 9.9, Definition 9.13], thus I use the following simplified description – which amounts to ignoring the relations introduced by boundaries in the Hochschild complex.

Proposition 6.3.5. *The $P_{p-1}(u)$-algebras Ξ_* and Θ_* admit a surjection of the $P_{p-1}(u)$-module $\Gamma(\tau)\{\sigma u\} \oplus (\gamma_1\tau)\{u\} \oplus \mathbb{F}_p \otimes \{\mu_2\}$ with $(\gamma_1\tau) \subset \Gamma(\tau)$ the ideal of positive divided powers in $\Gamma(\tau)$.*

Specifically for Ξ_* we assign $\gamma_i\tau \cdot \sigma u \mapsto \bar{z}_i$ and $\gamma_i\tau \cdot u \mapsto \bar{y}_j$.

Proof. The generators of Ξ_* and Ω_* arise from the E^2-term of the Bökstedt spectral sequence $HH_*(H\mathbb{F}_{p*}ku) \Rightarrow H\mathbb{F}_{p*}THH(ku)$, which are images of the generators given by the inclusion $HH_*(P_{p-1}(u)) \to HH_*(H\mathbb{F}_{p*}ku)$. For Hochschild homology of this truncated polynomial algebra we have determined the claimed surjection in 6.1.10. The class μ_2 maps to $[\sigma\tau_2]$ in $H\mathbb{F}_{p*}THH(ku)$ and to the class with the same name in $V(1)_*THH(ku)$. □

This description is sufficient to determine the effect of the maps defining the involution on $THH(ku)$.

Theorem 6.3.6. *For the isomorphism of [A-THH, Proposition 9.10]*

$$H\mathbb{F}_{p*}THH(ku) \cong H\mathbb{F}_{p*}\ell \otimes E([\sigma\xi_1]) \otimes \Xi_*$$

and the analogous isomorphism in $V(1)$-homotopy of [A-THH, Theorem 9.15]: $V(1)_*THH(ku) = E(\lambda_1) \otimes \Theta_*$ we can determine the involutions as follows. On μ_2 we have $\mu_2 \mapsto -\mu_2$ as visible in $THH(\ell)$ 6.1.9,6.3.2. On $a_i \in \Omega_*$ we have $a_i \mapsto (-1)^i a_i$, analogously on $\bar{z}_i \in \Xi_*$: $\bar{z}_i \mapsto (-1)^i \bar{z}_i$. For $b_i \in \Omega_*$ we have $b_i \mapsto (-1)^{i+1} b_i$, analogously for $\bar{y}_i \in \Xi_*$: $\bar{y}_i \mapsto (-1)^{i+1}\bar{y}_i$.

Proof. The classes a_i and $\bar{z}_i$ arise as infinite cycles in the Bökstedt spectral sequence represented by the classes $\sigma x \gamma_i \tau$ giving the claim by Theorem 6.1.12. Analogously the classes b_i and $\bar{y}_i$ are cycles associated to the classes $x\gamma_i\tau$ for $i \geq 1$, thus giving the claim again by Theorem 6.1.12. □

6.4 Results on the Involution on $V(1)_*K(ku_p)$

Since the calculations in [A-Kku], as well as [AR2, AR3] rely on trace methods, my intended approach was to determine the involution on $K(ku_p)$ and $K(\ell_p)$ by using the trace $tr\colon K \Rightarrow THH$. As shown above it commutes with the involutions 5.3.16 defined on K as in 2.6.10 and on THH as in 5.1.7. A few of the classes allow more direct approaches, so I prefer these for the exposition below.

From this point on I explicitly denote the completions at p of ℓ and ku. In particular since the computations in [AR2, A-Kku] rely partly on comparison to the integers, thus on the results of [BHM] and more specifically [BM] it would not be reasonable to expect global information given our limited state of knowledge about $K(\mathbb{Z})$. Instead we rely on the computations for $K(\mathbb{Z}_p)$.

6.4.1 The Module $V(1)_*K(ku_p)$ and its Traces

For reference in the sections below I directly quote the main result of [A-Kku]. I examine a few classes individually below, so I quote only the identification of the module given in [A-Kku, Theorem 8.1].

Theorem 6.4.1. *[A-Kku, Theorem 8.1]There is an isomorphism of $P(b)$-modules*

$$\begin{aligned} V(1)_*K(ku_p) \cong & P(b) \otimes E(\lambda_1, a_1) \\ & \oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{\sigma_n |\ 1 \leq n \leq p-2\} \\ & \oplus P(b) \otimes \mathbb{F}_p\{\partial\lambda_1, \partial b, \partial a_1, \partial\lambda_1 a_1\} \\ & \oplus P(b) \otimes E(a_1) \otimes \mathbb{F}_p\{t^d\lambda_1 |\ 0 < d < p\} \\ & \oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{t^{p^2-p}\lambda_2\} \\ & \oplus \mathbb{F}_p\{s\}. \end{aligned}$$

With regard to the traces of the classes, I draw the following corollary, which is less precise than the determination in [A-Kku, Theorem 8.1] but sufficient for determining the involution on the first direct summand.

Corollary 6.4.2. *to [A-Kku, Theorem 8.1] In the direct sum decomposition above, the third to sixth summand are contained in the kernel of the map induced by the trace $V(1)_*K(ku_p) \to V(1)_*THH(ku_p)$.*

Furthermore the traces of the classes σ_n are contained in the $P_{p-1}(u)$-subalgebra of $V(1)_*THH(ku_p)$ generated by $a_0 \in V(1)_3THH(ku_p)$, while the traces of the first summand are part of the $P_{p-1}(u)$-subalgebra generated by a_1, b_1, and λ_1.

We can use basic linear algebra over $\mathbb{F}_p$ to actually find that the trace can be described as a direct sum of maps with "orthogonal" images in low degrees. Again this follows directly from reading [A-Kku, Theorem 8.1] and [A-THH, Theorem 9.15] appropriately.

Corollary 6.4.3. *to [A-Kku, Theorem 8.1] The trace corresponds to a direct sum of maps with respect to the direct sum decomposition of $V(1)_*K(ku_p)$ above and the direct sum decomposition of $V(1)_*THH(ku_p)$ (as a $P_{p-1}(u)$-module) given by the generators $\lambda_1, a_i, b_i, \mu_2$.*

In particular, in low degrees it is injective on the classes λ_1, a_1 of the first summand, with image having trivial intersection with the other summands.

6.4.2 The Involution on $E(\lambda_1, a_1) \subset V(1)_*K(ku_p)$.

By the corollary above we only need to determine the involution on λ_1, and a_1 in $V(1)_*THH(ku)$, which implies we have the same effect on $V(1)_*K(ku_p)$ by injectivity in those degrees. So I isolate the part of 6.3.6 relevant to $K(ku_p)$ here.

Theorem 6.4.4. *The involution on the $P(b)$-subalgebra*

$$E(\lambda_1, a_1) \subset V(1)_*K(ku_p)$$

is given by $\lambda_1 \mapsto -\lambda_1$ and $a_1 \mapsto -a_1$.

Proof. This follows directly from the theorems on $V(1)_*THH(ku)$. Specifically recall that λ_1 stems from the class of $\sigma\xi_1$ in $H\mathbb{F}_{p*}THH(H\mathbb{Z}_p)$ with ξ_1 the generator in the dual of the Steenrod algebra. Thus the involution induced by conjugation is trivial, Γ induces a sign -1.

The class a_1 can be traced back to the class $\sigma x\gamma_1\tau = \sigma x\tau$ in the Bökstedt spectral sequence. The class σx has simplicial degree 1, τ has simplicial degree 2. The map induced by conjugation induces a sign -1 on σx and the identity on τ, the homeomorphism Γ induces a sign -1 on both, yielding the claim. $\square$

Remark 6.4.5. We have the involution determined on this full subalgebra after determining the effect on b as well below.

6.4.3 The Suspended Bott Classes σ_n

By restricting the isomorphism of [A-Kku, Proposition 5.2] we get an inclusion identifying the classes $\sigma_n \in V(1)_*K(ku_p)$. In fact they are global, i.e., $\sigma_n \in V(1)_*K(ku)$.

Specifically they arise as follows: We have the inclusion $BBU_\otimes \subset BGL_1ku$, where the units of an E_∞-ring spectrum are defined as the (homotopy) pullback:

$$\begin{array}{ccc} GL_1A & \longrightarrow & \Omega^\infty A \\ \downarrow & & \downarrow \\ GL_1(\pi_0 A) & \longrightarrow & \pi_0 A, \end{array}$$

which is a sufficient notion of "units" for our purposes. For ku we have $GL_1\pi_0 ku = GL_1\mathbb{Z} = \{\pm 1\}$, hence $GL_1ku = BU \times \{\pm 1\}$. Restricting to the "index" $+1$, we get the inclusion $BBU \to BGL_1ku$. By delooping the topologically enriched permutative category on objects natural numbers with endomorphisms GL_nku along the lines of [EM] recalled in 1.1.4 we get a model for K-theory of ku. In particular its underlying infinite loop space is the group completion: $\Omega B(\coprod_n BGL_nku) = \Omega^\infty K(ku)$. The canonical inclusion $GL_1ku \to \coprod_n GL_nku$ fits in the sequence of maps

$$BBU \to BGL_1ku \to \coprod_n BGL_nku \to \Omega^\infty K(ku).$$

Considering units in a stricter setting, for instance with ku as a symmetric ring spectrum, we can find a strictly associative model for $BU_\otimes$. Thus we have a one-point suspension category $\Sigma BU_\otimes$. Embed this into its free permutative category $\mathbb{P}\Sigma BU_\otimes = \coprod_n E\Sigma_n \times_{\Sigma_n} (\Sigma BU_\otimes)^n$. The induced map of permutative categories $\mathbb{P}\Sigma BU_\otimes \to \coprod_n GL_n ku$ delooped as in 1.1.4 induces a map:

$$H\mathbb{P}\Sigma BU_\otimes \to H(\coprod_n GL_n ku),$$

which we can identify as

$$\omega\colon \Sigma^\infty BBU_\otimes \to K(ku)$$

as given in [A-Kku, p. 627].

Trivially the map $BU \to BU \times \mathbb{Z}$ identifying BU as a connected cover of $BU \times \mathbb{Z}$ is an isomorphism in positive degrees, thus we have classes $y_n \in \pi_{2n}BU$ for $n \geq 1$ which map to the powers of the Bott class u^n.

Suspending these once gives classes $\sigma y_n \in \pi_{2n+1}\Sigma BU$ giving classes:

$$\Sigma^\infty \mathbb{S}^{2n+1} \to \Sigma^\infty \Sigma BU \to \Sigma^\infty BBU \to K(ku).$$

Call these σ_n in agreement with [A-Kku, Definition 3.2]. Then we have:

Proposition 6.4.6. *(cf. [A-Kku, Proposition 5.2]) The classes $\sigma_n \in \pi_{2n+1}K(ku)$ are non-trivial for $1 \leq n \leq p-2$.*

Remark 6.4.7. I have to concede that I am not certain about non-triviality for the higher σ_n, however the traces of the σ_n are $u^{n-1}a_0$ by [A-Kku, Theorem 8.1]. Thus one should probably consider the upper limit $p-2$ as an artefact of the relations for Θ_* and Ξ_* in $V(1)_*THH(ku)$. At least on the subspectrum $\Sigma(ku) \to THH(ku)$ one can identify these globally in homotopy groups, thus in particular removing the bound $p-2$. However establishing their non-triviality would entail non-trivial calculations in the homotopy groups of $THH(ku)$.

We can identify the involution on these suspended classes as follows:

Theorem 6.4.8. *The involution on the classes $\sigma_n \in \pi_{2n+1}K(ku)$ for $n \geq 1$ is given as $\sigma_n \mapsto (-1)^{n+1}\sigma_n$.*

Proof. Consider part of the inclusion into $\Omega^\infty K(ku)$:

$$BBU \to BGL_1 ku \to \coprod_n BGL_n ku.$$

Then the classes of the σ_n are given by $\Sigma\mathbb{S}^{2n} \to BBU \to BGL_1ku$. Thus conjugation acts as it does on u^n, giving a sign $(-1)^n u^n$. Transposition has no effect on GL_1ku. Finally Γ acts on simplicial degree 1 here, thus reverses the signs to give $(-1)^{n+1}\sigma_n$. □

6.4.4 The Higher Bott Class $b \in V(1)_{2p+2}K(ku)$

The class of major interest in $K(ku)$ is a class in degree $2p+2$ of the $V(1)$-homotopy of $K(ku)$, which is a non-trivial root of $v_2 \in \pi_*V(1)$, thus in particular establishing $K(ku)$ as the representing spectrum of a homology theory of chromatic type 2.

Moreover: By the calculations of Ausoni in [A-Kku], in particular Theorem 8.1 as recalled above in Theorem 6.4.1, we know that apart from a sporadic class the module $V(1)_*K(ku)$ is a free module over the polynomial algebra on b.

Remark 6.4.9. To be consistent in denoting $H\mathbb{F}_p$-homology on the left, I refer to classes in degree n as $z \in H\mathbb{F}_{p,n}X$ in the following proposition.

Recall the construction of the element b:

Proposition 6.4.10. *Consider the homology algebra of $\mathbb{C}P^\infty \simeq K(\mathbb{Z},2)$, which is a divided power algebra $\Gamma(y) \cong H\mathbb{F}_{p_*}K(\mathbb{Z},2)$.*

Then in the spectral sequence in $H\mathbb{F}_p$-homology associated to the bar filtration of $K(\mathbb{Z},3) = B(K(\mathbb{Z},2))$ the class $y^{p-1} \otimes y$ is an infinite non-bounded cycle of degree $(2,2p)$. By [A-Kku, Lemma 2.3] we have in particular an exact sequence $H\mathbb{F}_{p,5}(K(\mathbb{Z},3)) \to V(1)_{2p+2}(K(\mathbb{Z},3)) \to H\mathbb{F}_{p,2p+2}(K(\mathbb{Z},3)) \to H\mathbb{F}_{p,0}(K(\mathbb{Z},3))$. Since $H\mathbb{F}_{p,5}(K(\mathbb{Z},3)) = 0$, and $(P^1)^*(\gamma_{p-1}(y)) = 0$ the last map and the first group are zero, so we have a unique class $V(1)_{2p+2}K(\mathbb{Z},3)$ which maps to the class of $y^{p-1} \otimes y$ by the Hurewicz $V(1) \to H\mathbb{F}_p$. Finally by considering the embedding $K(\mathbb{Z},3) = BBU(1) \to BBU \to \coprod_n BGL_n(ku)$ as before we get the higher Bott element $b \in V(1)_{2p+2}K(ku)$.

Theorem 6.4.11. *The involution on the higher Bott element is trivial, specifically the algebra map $ku \to ku$ induced by conjugation acts as a sign -1, and the homeomorphism Γ acts as a sign -1.*

Proof. By [A-Kku, p. 623] we know $\Sigma K(\mathbb{Z},2) \to B_2 \to \Sigma^2(K(\mathbb{Z},2)^{\wedge 2})$ induces an injective map in degree $2p+2$: $V(1)_{2p+2}B_2 \to V(1)_{2p+2}(\Sigma^2(K(\mathbb{Z},2)^{\wedge 2}))$ with $B_2 \subset K(\mathbb{Z},3)$ the image of the 2-skeleton.

In particular the map $B_2 \to K(\mathbb{Z},3) \to BBU \to \coprod_n BGL_n(ku)$ again is a class associated to $GL_1 ku$, thus transposition has no effect. Furthermore it is a class of simplicial degree 2 by definition, thus Γ acts as a sign -1. Finally we have to determine how complex conjugation acts on b. For this consider the representative of the bar spectral sequence $y^{p-1} \otimes y$. On homology of $K(\mathbb{Z},2) = BU(1)$ complex conjugation acts by a group homomorphism on $U(1)$ as $y^n \mapsto (-1)^n y^n$. In particular we get: $y^{p-1} \otimes y \mapsto (-1)^p y^{p-1} \otimes y = -y^{p-1} \otimes y$.

In summary Γ and the conjugation cancel out, which is visible on B_2. $\square$

6.4.5 Summary of the Induced Involution

Here I want to summarise the above results. Recall the isomorphism of [A-Kku]:

$$\begin{aligned} V(1)_*K(ku_p) \cong & P(b) \otimes E(\lambda_1, a_1) \\ & \oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{\sigma_n |\ 1 \le n \le p-2\} \\ & \oplus P(b) \otimes \mathbb{F}_p\{\partial\lambda_1, \partial b, \partial a_1, \partial\lambda_1 a_1\} \\ & \oplus P(b) \otimes E(a_1) \otimes \mathbb{F}_p\{t^d \lambda_1 |\ 0 < d < p\} \\ & \oplus P(b) \otimes E(\lambda_1) \otimes \mathbb{F}_p\{t^{p^2-p}\lambda_2\} \\ & \oplus \mathbb{F}_p\{s\}. \end{aligned}$$

In the section above we have established that b is invariant under the induced involution. In the first section we have determined the involution on λ_1 and a_1 to each be given by a sign. In the second section we found that the involution induces $\sigma_n \mapsto (-1)^{n+1}\sigma_n$.

6.4.6 The TC-Classes

For the third to the sixth summand in the above decomposition, one would have to establish, if there is an involution on topological cyclic homology, which is compatible with the cyclotomic trace. Specifically the classes $t^d\lambda_1$ and $t^{p^2-p}\lambda_2$ are composites of the eponymous classes $\lambda_i \in V(1)_*THH(ku)$ and powers of the classes $t \in H_*(\mathbb{Z}/p^n) \to H_*(\mathbb{S}^1)$ arising from the homotopy fixed point spectral sequences from THH to TC. In particular, one would have to establish how the circle action on THH behaves with respect to the involution.

Regrettably I have to say that this is beyond the scope of this thesis. In particular I cannot even offer a conjecture on what map the involution induces on the other summands. Specifically I do not know the effect of the involution on ∂, s

and t, which are each part of the remaining classes, and each most transparently appear in $V(1)_*TC(ku_p)$.

I hope to get back to a full description of the involution on $V(1)_*K(ku_p)$ in future work. Most probably the description of an involution on TC can be bypassed for this case, since the class ∂ is an artifact of p-adic completion, which is already present in $V(1)_*K(\mathbb{Z}_p)$, and the classes t^d are visible in the homotopy fixed point spectral sequence for $THH(ku)^{h\mathbb{S}^1}$.

Bibliography

[Ad1] J. Adams, *Stable Homotopy and Generalised Homology,* Univ. of Chicago Press, 1974, Reissued Edition 1995

[Ad2] J. Adams, *Lectures on Generalised Cohomology,* Lec. Notes Math. **99**, 1969

[AnHL] V. Angeltveit, M. Hill, T. Lawson, *The Topological Hochschild Homology of ℓ and ko*, Am. J. Math. **132**(2), 2010, `arxiv:0710.4368`

[AnR] V. Angeltveit, J. Rognes, *Hopf Algebra Structure on Topological Hochschild Homology*, Algebr. Geom. Topol. **5**, 2005, 1223-1290, `arxiv:0502195`

[A-Kku] C. Ausoni, *On the Algebraic K-Theory of the Complex K-Theory Spectrum*, Inventiones mathematicae **180**(3), 2010, 611–668, `arxiv:0902.2334`

[A-THH] C. Ausoni *Topological Hochschild Homology of Connective Complex K-Theory*, Am. J. Math. **127**(6), 2005, 1261–1313

[AR1] C. Ausoni, J. Rognes, *Rational Algebraic K-Theory of Topological K-Theory*, Geom. Topol. **16**, 2012, 2037–2065, `arxiv:0708.2160`

[AR2] C. Ausoni, J. Rognes, *Algebraic K-Theory of Topological K-Theory,* Acta Math. **188**, 2002, 1–39

[AR3] C. Ausoni, J. Rognes, *Algebraic K-Theory of the First Morava K-Theory,* J. Eur. Math. Soc. **14**, 2011, 1041–1079, `arxiv:1006.3413`

[ADR] C. Ausoni, B. Dundas, J. Rognes, *Divisibility of the Dirac Magnetic Monopole as a Two-Vector Bundle over the Three-Sphere,* Doc. Math. **13**, 2008, 795–801

[BDR] N. Baas, B. Dundas, J. Rognes, *Two-Vector Bundles and Forms of Elliptic Cohomology*, London Mathematical Society Lecture Note Series **308**, 2004, 18–44, `arxiv:0306027`

[BDRR1] N. Baas, B. Dundas, B. Richter, J. Rognes, *Stable Bundles over Rig Categories*, Journal of Topology **4**(3), 2011, 623–640, `arxiv:0909.1742`

[BDRR2] N. Baas, B. Dundas, B. Richter, J. Rognes, *Ring Completion of Rig Categories*, Journal für die reine und angewandte Mathematik (Crelles Journal) **674**, 2013, 43–80, `arxiv:0706.0531`

[BR1] A. Baker, B. Richter, Γ-*Cohomology of Rings of Numerical Polynomials and E_∞-Structures on K-Theory*, Comment. Math. Helv. **80** (4), 2005, 691–723, `arxiv:0304473`

[BR2] A. Baker, B. Richter, *Uniqueness of E_∞-Structures for Connective Covers*, Proc. Amer. Math. Soc. **136**, (2008), 707–714, `arxiv:0506422`

[Ba] C. Barwick, *Multiplicative Structures on Algebraic K-Theory*, preprint `arxiv:1304.4867`, (2013)

[Ben] J. Bénabou, *Introduction to Bicategories*, in Reports of the Midwest Category Seminar, Springer Berlin, 1967

[Ber1] J. Bergner, *A Model Category Structure on the Category of Simplicial Categories*, Trans. Amer. Math. Soc. **359**, 2007, 2043–2058, `arxiv:0406507`

[Ber2] J. Bergner, *A Survey of $(\infty,1)$-Categories*, `arxiv:0610239`

[Ber3] J. Bergner, *Models for Homotopical Higher Categories*, Talk at the MSRI (Jan. 2014), https://www.youtube.com/watch?v=q8J0bbaZnRM

[BGT1] A. Blumberg, D. Gepner, G. Tabuada, *A Universal Characterization of Higher Algebraic K-theory*, Geometry and Topology **17**, 2013, 733–838, `arxiv:1001.2282`

[BGT2] A. Blumberg, D. Gepner, G. Tabuada, *Uniqueness of the Multiplicative Cyclotomic Trace*, to appear in Advances in Mathematics, `arxiv:1103.3923`, (2012)

[BV] J. Boardman, R. Vogt, *Homotopy Invariant Algebraic Structures on Topological Spaces,* Springer Verlag Berlin, 1973

[BLPRZ] I. Bobkova, A. Lindenstrauss, K. Poirier, B. Richter, I. Zakharevich, *On the Higher Topological Hochschild Homology of* $\mathbb{F}_p$ *and Commutative* $\mathbb{F}_p$*-Group Algebras,* Proceedings of the BIRS workshop *WIT: Women in Topology*, 2014

[B1] M. Bökstedt, *Topological Hochschild Homology,* Unpublished

[B2] M. Bökstedt, *The Topological Hochschild Homology of* $\mathbb{Z}$ *and* $\mathbb{Z}/p$, Unpublished

[BM] M. Bökstedt, I. Madsen, *Topological Cyclic Homology of the Integers,* Astérisque **226**, 1994, 57–143

[BHM] M. Bökstedt, W. Hsiang, I. Madsen, *The Cyclotomic Trace and Algebraic* K*-Theory of Spaces*, Inventiones mathematicae **111**(1), 1993, 465–539

[BFV] M. Brun, Z. Fiedorowicz, R. Vogt, *On the Multiplicative Structure of Topological Hochschild Homology*, Algebraic & Geometric Topology **7**, 2007, 1633-1650, `arxiv:0410367`

[BrRo] R. Bruner, J. Rognes, *Differentials in the Homological Homotopy Fixed Point Spectral Sequence,* Alg. Geom. Topol. **5**, 2005, 653–690, `arxiv:0406081`

[CCG] P. Carrasco, A. Cegarra, A. Garzón, *Nerves and Classifying Spaces for Bicategories*, Algebraic and Geometric Topology **10** (1), 2010, 219–274, `arxiv:0903.5058`

[D1] B. Dundas, *The Cyclotomic Trace for Symmetric Monoidal Categories,* Geometry and Topology Århus 1998, Contemp. Math. **258**, 2000, AMS, 121–143

[D2] B. Dundas, *The Cyclotomic Trace Preserves Operad Actions,* 2014, preprint

[DGM] B. Dundas, T. Goodwillie, R. McCarthy, *The Local Structure of Algebraic K-Theory*, Springer London, 2012

[EKMM] A. Elmendorf, I. Kriz, M. Mandell, J.P. May, *Rings, Modules, and Algebras in Stable Homotopy Theory*, AMS Surveys and Monographs **47**, 1995

[EM] A. Elmendorf, M. Mandell, *Rings, Modules, and Algebras in Infinite Loop Space Theory*, Adv. Math. **205**, 2006, 163–228, `arxiv:0403403`

[Gauß] C.F. Gauß, *Disquisitiones Arithmeticae*, 1801, Latin http://edoc.hu-berlin.de/ebind/hdok2/h284_gauss_1801/pdf/h284_gauss_1801.pdf, English revised translation 1986, Springer New York

[GGN] D. Gepner, M. Groth, T. Nikolaus, *Universality of Multiplicative Infinite Loop Space Machines*, preprint, `arxiv:1305.4550`

[GSn] D. Gepner, V. Snaith, *Motivic Spectra Representing Cobordism and K-Theory,* Documenta Math. **14**, 2009, 359–396, `arxiv:0712.2817`

[GHOsT] P. Goerss, M. Hopkins, *Moduli Spaces of Commutative Ring Spectra,* London Math. Soc. Lecture Notes **315**, 2004, 151–200

[GJ] P. Goerss, J. Jardine, *Simplicial Homotopy Theory*, Birkhäuser Basel, 2009 reprint

[GGRSV] Jo. Guccione, Ju. Guccione, M. Redondo, A. Solotar, O. Villamayor, *Cyclic Homology of Algebras with One Generator,* K-Theory **5**, 1991, 51–69

[GJOs] N. Gurski, N. Johnson, A. Osorno, *K-Theory for 2-Categories*, preprint, `arxiv:1503.07824`

[J] A. Joyal, *Quasi-Categories and Kan Complexes*, J. Pure Appl. Algebra **175**, 2002, 207–222

[JT] A. Joyal, M. Tierney, *Quasi-Categories vs. Segal Spaces*, Contemp. Math. **431**, 2007, 277–326, `arxiv:0607820`

[KV] M. Kapranov, V. Voevodsky, *Braided Monoidal 2-Categories and Manin-Schechtman Higher Braid Groups,* J. Pure Appl. Algebra **92**, 1994, 241–267

[Ko] S. Kochman, *Bordism, Stable Homotopy and Adams Spectral Sequences,* Fields Institute Monographs, AMS, 1996

[K] H. König, *The Segal Model as a Ring Completion and a Tensor Product of Permutative Categories*, PhD thesis, 2011, http://ediss.sub.uni-hamburg.de/volltexte/2011/5032

[LP] S. Lack, S. Paoli, *2-Nerves for Bicategories*, Journal of K-theory **38(2)**, 2008, 153–175, `arxiv:0607271`

[Lan] M. Lange, *Examples of Involutions on Algebraic K-Theory of Bimonoidal Categories*, 2011, Diploma Thesis, available at http://www.math.uni-hamburg.de/home/richter/DA-Lange.pdf

[Lap] M. Laplaza, *Coherence for Categories with Associativity, Commutativity and Distributivity*, Bull. AMS **78**, 1972, 220–222

[Le] T. Leinster, *Basic Bicategories*, `arxiv:9810017`, 1998

[Lo] J.-L. Loday, *Cyclic Homology*, Grundlehren **301**, Springer, 1992

[Lu1] J. Lurie, *Higher Topos Theory*, Princeton University Press **170**, 2009, `arxiv:0608040`

[Lu2] J. Lurie, *Higher Algebra*, (August 2012) Preprint at http://www.math.harvard.edu/~lurie/papers/higheralgebra.pdf

[Lu] J. Lurie, *Stable Infinity Categories*, `arxiv:0608228`

[McL] S. Mac Lane, *Homology,* 1995 reprint of the 1975 edition

[MMSS] M. Mandell, J.P. May, S. Schwede, B. Shipley *Model Categories of Diagram Spectra*, Proceedings of the London Mathematical Society, **82**(2), 2001, 441–512

[MN] T. Masuda, T. Natsume, *Cyclic Cohomology of Certain Affine Schemes,* Publ. RIMS, Kyoto Univ. **21**, 1985, 1261–1279

[MayE_∞] J.P. May, *E_∞-ring spaces and E_∞-ring spectra*, Lect. Notes Math. **577**, Springer Heidelberg, 1977

[May1] J.P. May, *Simplicial objects in algebraic topology*, Chicago Lect. in Math., Univ. Chicago Press, 1967

[May2] J.P. May, *The Construction of E_∞-Ring Spaces from Bipermutative Categories*, Geometry & Topology Monographs **16**, 2009, 283–330, `arxiv:0903.2818`

[MT] J.P. May, R. Thomason, *The Uniqueness of Infinite Loop Space Machines*, Journal of Topology **17**(3), 1978, 205–224

[MSm] J. McClure, J. Smith, *Operads and Cosimplicial Objects: An Introduction,* 2004, `arxiv:0402117`

[MS] J. McClure, R. Staffeldt, *On the Topological Hochschild Homology of bu* I, Am. J. Math., 1993, 1–45

[MT] M. Mimura, H. Toda, *Topology of Lie Groups I*, Transl. of Math. Mon. AMS, 2000

[Oka] S. Oka, *Multiplicative Structure of Finite Ring Spectra and Stable Homotopy of Spheres,* Alg. Top. Århus 1982, Lecture Notes in Math. **1051**, 1984, 418–441

[Os] A. Osorno, *Spectra Associated to Symmetric Monoidal Bicategories*, Algebraic and Geometric Topology, **12**(1), 2012, 307–342, `arxiv:1005.2227`

[PR] T. Pirashvili, B. Richter, *Hochschild and Cyclic Homology via Functor Homology,* K-Theory **25** (1), 2002, 39–49

[Q1] D. Quillen, *The Adams Conjecture*, Journal of Topology **10**(1), 1971, 67–80

[Q2] D. Quillen, *On the Cohomology and K-theory of the General Linear Groups over a Finite Field*, Annals of Mathematics, 1972, 552–586

[Q3] D. Quillen, *Higher algebraic K-theory: I*, Higher K-theories, 1973, 85–147

[R] B. Richter, *An Involution on the K-theory of Bimonoidal Categories with Anti-Involution*, Algebraic and Geometric Topology **10**, (2010), 315–342, `arxiv:0804.0401`

[Ro] J. Rognes, *Chromatic Redshift,* `arxiv:1403.4838`

[RSS] J. Rognes, S. Sagave, C. Schlichtkrull, *Logarithmic Topological Hochschild Homology of Topological K-Theory Spectra*, preprint, `arxiv:1410.2170`, (2014)

[Ros] J. Rosenberg, *Algebraic K-Theory and its Applications*, 1994

[Rot] J. Rotman, *An Introduction to Homological Algebra*, Springer New York, second edition 2009

[Ru] Y. Rudyak, *On Thom Spectra, Orientability, and Cobordism*, Springer Berlin Heidelberg, 1998

[SaS] S. Sagave, C. Schlichtkrull, *Localization Sequences for Logarithmic Topological Hochschild Homology*, preprint, `arxiv:1402.1317`, (2014)

[S] C. Schlichtkrull, *The Cyclotomic Trace for Symmetric Ring Spectra*, Geometry & Topology Monographs **16**, 2009, 545–592, `arxiv:0903.3495`

[Schm] V. Schmitt, *Tensor Product of Symmetric Monoidal Categories*, 2007, `arxiv:0711.0324`

[SP] C. Schommer-Pries, *The Classification of Two-Dimensional Extended Topological Field Theories,* PhD thesis, UC Berkeley 2009, `arxiv:1112.1000`

[Schw1] S. Schwede, *The Stable Homotopy Category is Rigid,* Ann. Math. **166**, 2007, 837–863

[Schw2] S. Schwede, *An untitled book project about symmetric spectra (version 2012)*, http://www.math.uni-bonn.de/people/schwede/SymSpec.pdf

[Se] G. Segal, *Categories and Cohomology Theories*, Topology **13**(3), 1974, 293–312

[Se] J.-P. Serre, *Linear Representations of Finite Groups*, Springer New York, 1977

[Sh] B. Shipley, *Symmetric Ring Spectra and Topological Hochschild homology*, K-Theory **19**(2), 2000, 155-183, `arxiv:9801079`

[Sn] V. Snaith, *Algebraic Cobordism And K-Theory,* Mem. Amer. Math. Soc. **221**, 1979

[Str1] R. Street, *Categorical Structures*, in Handbook of Algebra **1**, ed. M. Hazewinkel, Elsevier, 1996

[Str2] R. Street, *The Algebra of Oriented Simplexes*, J. Pure Appl Alg. **49**(3), 1987, 283–335

[Sw] R. Switzer, *Algebraic Topology - Homology and Homotopy*, Classics in Math., Springer Berlin Heidelberg, 2002 reprint

[Tab] G. Tabuada, *Homotopy Theory of Spectral Categories*, Adv. Math. **221(4)**, 2009, 1122–1143, `arxiv:0801.4524`

[Th1] R. Thomason, *Symmetric Monoidal Categories Model All Connective Spectra*, Theory and Applications of Categories **1.5**, 1995, 78–118

[Th2] R. Thomason, *Homotopy Colimits in the Category of Small Categories*, Math. Proc. Cambridge Philos. Soc. **85.1**, 1979, 91–109

[Th3] R. Thomason, *Cat as a Closed Model Category,* Cahiers Topologie Géom. Différentielle **21**(3), 1980, 305–324

[Toda] H. Toda, *On Spectra Realising Exterior Parts of the Steenrod Algebra,* Topology **10**, 1971, 53–65

[W] C. Weibel, *The K-Book: An Introduction to Algebraic K-Theory,* Graduate Studies in Math. **145**, AMS, 2013

[Z] S. Ziegenhagen, *E_n-Cohomology as Functor Cohomology and Additional Structures,* PhD thesis, 2014, http://ediss.sub.uni-hamburg.de/volltexte/2014/6950

Summary

In this thesis I investigate the interaction of multiplicative and involutive structures on algebraic K-theory of E_∞-ring spectra. Algebraic K-theory is associated classically to discrete rings by a definition of Quillen [Q3] with preceding approaches by Grothendieck, Bass and Milnor, who defined K_0R, K_1R and K_2R respectively. Quillen identified algebraic K-theory as the homotopy groups of a space naturally associated to a ring, unifying the first three definitions, and providing a definition of K_nR for all natural numbers. One approach to computations is the generalisation of K-theory to more ringlike objects, yielding more induced structures on K-theory, for instance multiplicative structures as described in [EM, GGN, BGT2, May2] and involutions as defined in [R].

Driven by the main example, algebraic K-theory of the connective complex K-theory spectrum ku, and building on the computations of Christian Ausoni in [A-Kku, A-THH], I focus on the induced multiplicative structure on $K(ku)$ in chapters 1 to 3. Building on the delooping of permutative bicategories as developed by Angélica Osorno in [Os], I exhibit a tensor product on a bicategory of matrices $\mathcal{M}(\mathcal{R})$ associated to a bipermutative category $\mathcal{R}$ in chapter 2. By modifying the delooping of Osorno in chapter 3 I find an induced E_∞-ring spectrum structure on $K(ku)$ by identifying this spectrum as the Eilenberg-MacLane-spectrum associated to the bicategory of matrices over finite-dimensional complex vector spaces. The involution as defined in [R] easily generalises to this setting, so I can exhibit the interaction of the involution with the multiplication easily.

Since the calculations in [A-Kku, A-THH] rely on trace methods, i.e., are obtained by careful comparison of $K(ku)$ to topological Hochschild homology $THH(ku)$ along the trace map, I can use the compatibility of the trace map with the multiplicative structures defined on both as a universal property by the results of [BGT1, BGT2]. This in particular implies that the trace map is compatible with the involution defined on K-theory as in [R] and on topological Hochschild homology analogous to [Lo], giving the main result of chapter 5.

Finally in chapter 6 I investigate the involution on mod (p, v_1) homotopy groups of $K(ku)$ as calculated in [A-Kku]. Specifically there is a subalgebra of $V(1)_*K(ku)$, which can be understood purely in terms of the trace map $K(ku) \to THH(ku)$, and there are special classes σ_n as well as the "higher Bott element" b. For all of these I describe the effect induced by complex conjugation on ku, and thus the induced involution on algebraic K-theory on $V(1)_*K(ku)$.

Zusammenfassung

In dieser Dissertation untersuche ich Multiplikationen und Involutionen auf algebraischer K-Theorie von E_∞-Ringspektren. Klassisch ist algebraische K-Theorie eine Invariante diskreter Ringe definiert von Quillen in [Q3]. Quillen vereinheitlicht Definitionen von Grothendieck, Bass und Milnor der Gruppen K_iR für $i = 0, 1, 2$ in dieser Reihenfolge. Er definiert algebraische K-Theorie als Homotopiegruppen eines natürlich zu einem Ring R assoziierten Raumes $BGL(R)^+$. Diese Definition vereinheitlicht zugleich die vorher genannten Definitionen und gibt eine Definition für alle natürlichen Zahlen. Ein Zugang zu Berechnungen ist die Verallgemeinerung von K-Theorie auf ringartige Objekte, was insbesondere Aufschluss gibt über induzierte Strukturen auf K-Theorie wie etwa Multiplikationen [EM, GGN, BGT2, May2] und Involutionen [R].

In den Kapiteln 1 bis 3 untersuche ich die multiplikative Struktur von $K(ku)$, die durch die Identifikation entlang [BDRR1] als Delooping einer Bikategorie von Matrizen $\mathcal{M}(\mathcal{R})$ induziert wird. Genauer definiere ich in Kapitel 2 ein Tensorprodukt, das eine Multiplikation auf $\mathcal{M}(\mathcal{R})$ induziert, die sich mit der additiven Struktur, die Angélica Osorno [Os] definiert, verträgt. In Kapitel 3 beschreibe ich eine Variante ihres Deloopings [Os], die durch das Tensorprodukt die Struktur eines E_∞-Ringspektrums erhält. Die von Birgit Richter in [R] beschriebene Involution lässt sich auf $\mathcal{M}(\mathcal{R})$ erweitern, und wir erhalten, dass die Involution diese Multiplikation opponiert.

Die Berechnungen der mod (p, v_1) Homotopiegruppen von Christian Ausoni in [A-Kku, A-THH] basieren grundlegend auf Spurmethoden, also einem sorgsamen Vergleich algebraischer K-Theorie mit topologischer Hochschildhomologie. Die vergleichende Abbildung ist die Spur $K(ku) \to THH(ku)$, die ich mithilfe der Resultate aus [BGT1, BGT2] als die universelle multiplikative natürliche Transformation $K \Rightarrow THH$ in Kapitel 5 definieren kann. Aus dieser Universalität folgt das Hauptresultat von Kapitel 5, dass sich die Spurabbildung auch mit der induzierten Involution auf K-Theorie wie in [R] beschrieben und der auf topologischer Hochschildhomologie zu der in [Lo] analogen Involution verträgt.

In Kapitel 6 untersuche ich die Involution auf mod (p, v_1) Homotopiegruppen von $K(ku)$, wie sie aus der Berechnung von [A-Kku] hervorgehen. Es gibt eine Unteralgebra in $V(1)_*K(ku)$, die sich vollständig über die Spur in $V(1)_*THH(ku)$ verstehen lässt, sowie spezielle Elemente σ_n und das "höhere Bott-Element" b in diesem Modul. Für diese Klassen beschreibe ich die Involution auf $V(1)_*K(ku)$.

Publications/Publikationen

I have not published anything based on this thesis.
Es gibt keine Veröffentlichungen, die aus dieser Arbeit hervorgegangen sind.

Eidesstattliche Versicherung

Ich versichere an Eides statt, dass ich die Arbeit selbstständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen benutzt habe.

Marc Lange, Hamburg, 24.7.2015