Arithmetic Local Coordinates And Applications To Arithmetic Self-Intersection Numbers

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Per a la meva dona Estela i el meu fill Pau.

Contents

Introduction

Arakelov theory, also known as *arithmetic intersection theory*, is used to study number theoretic problems from a geometrical point of view.
More than 100 years ago, many mathematicians with Dedekind and Weber in [DW] leading the way, observed that there is an analogy between number fields and function fields.
In the 1960's, Grothendieck defined the notion of schemes, which turned out to be the right framework for an algebraic geometry over number fields as well as over function fields. This was the beginning of the idea of the vague formula

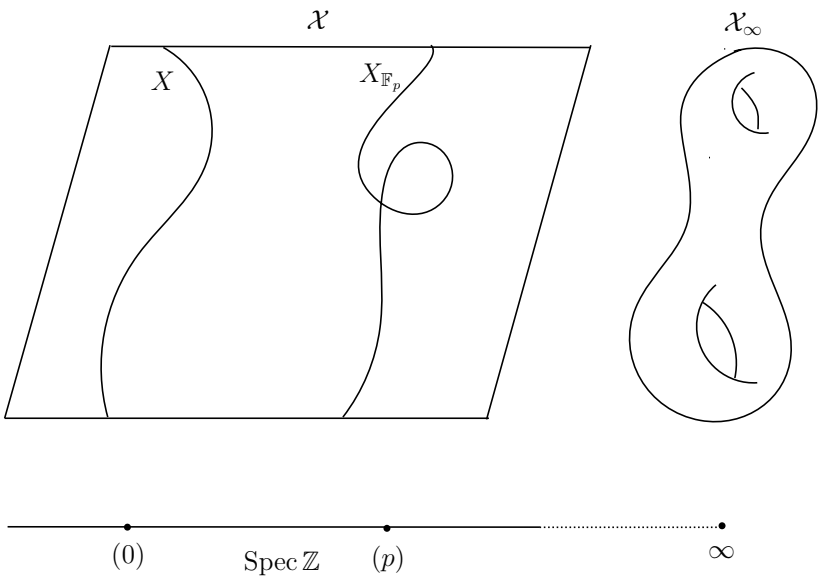
$$\boxed{\text{algebraic number theory}} + \boxed{\text{algebraic geometry}} = \boxed{\text{arithmetic algebraic geometry}}$$

In 1974, S. Ju. Arakelov defined in [Ar] an intersection theory on arithmetic surfaces over the ring of integers of a number field. He showed that geometry over number fields in addition with differential geometry on some corresponding complex manifolds behaves like geometry over a compact variety. Hence the idea behind Arakelov theory is

$$\boxed{\text{arithmetic algebraic geometry}} + \boxed{\text{complex differential geometry}} = \boxed{\text{Arakelov theory}}$$

In 1987, P. Deligne generalized in [De] the arithmetic intersection theory of Arakelov. Indeed, until then, the intersection theory was only defined for arithmetic divisors with admissible Green's functions. Deligne discarded this condition and therefore he opened the way to a higher dimensional generalization.
In 1991, H. Gillet and C. Soulé in [GS2] extended the arithmetic intersection theory to higher dimensions by translating the theory of Green's functions to the more manageable notion of Green's currents.

Now we give a motivation why it is of interest to calculate arithmetic self-intersection numbers. For simplicity we reduce to the case that the arithmetic variety is of dimension two. An example of an arithmetic surface $\mathcal{X}$ is visualized in the following picture.



Let $\widehat{\mathrm{Z}}^1(\mathcal{X})$ be the group of arithmetic divisors $(\mathcal{D}, g_\mathcal{D})$, i.e. $\mathcal{D}$ is a divisor on $\mathcal{X}$ and $g_\mathcal{D}$ is a Green's function for $\mathcal{D}$. Moreover, let $\widehat{\mathrm{CH}}^1(\mathcal{X})$ be the arithmetic Chow group of $\mathcal{X}$. Then the arithmetic intersection number is a pairing from $\widehat{\mathrm{Z}}^1(\mathcal{X}) \times \widehat{\mathrm{Z}}^1(\mathcal{X})$ to $\mathbb{R}$, which factors through $\widehat{\mathrm{CH}}^1(\mathcal{X}) \times \widehat{\mathrm{CH}}^1(\mathcal{X})$. As a special case, for an arithmetic horizontal prime divisor $(\mathcal{P}, g_\mathcal{P}) \in \widehat{\mathrm{Z}}^1(\mathcal{X})$, its *arithmetic self-intersection number* is defined by

$$(\mathcal{P}, g_\mathcal{P})^2 := (\mathcal{P}, \mathcal{P} - \mathrm{div}(f))_{\mathrm{fin}} + \frac{1}{2}\left(\left(g_\mathcal{P} + \log|f|^2\right)[\mathcal{P}(\mathbb{C})] + \int_{\mathcal{X}_\infty} g_\mathcal{P} \cdot \omega_{g_\mathcal{P}} \right) \in \mathbb{R}, \qquad (1)$$

where $f \in k(\mathcal{X})^\times$ is a rational function such that $\mathcal{P} - \mathrm{div}(f)$ and $\mathcal{P}$ have no common horizontal components. The differential form $\omega_{g_\mathcal{P}}$ denotes the smooth $(1,1)$-form on the induced complex manifold $\mathcal{X}_\infty$, which is given by $\mathrm{dd}^c g_\mathcal{P}$ outside $\mathcal{P}(\mathbb{C})$.
For a hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|) \in \widehat{\mathrm{Pic}}\,(\mathcal{X})$, its first Chern class $\widehat{c}_1(\overline{\mathcal{L}}) \in \widehat{\mathrm{CH}}^1(\mathcal{X})$ is an element in the arithmetic Chow group of $\mathcal{X}$ and is given by $\widehat{c}_1(\overline{\mathcal{L}}) = \left[\mathrm{div}(l), -\log\|l\|^2\right]$, where l is a non-trivial rational section of $\mathcal{L}$. The *arithmetic self-intersection number of* $\overline{\mathcal{L}}$ is defined by

$$\overline{\mathcal{L}}^2 := \left(\mathrm{div}(l), -\log\|l\|^2\right)^2.$$

This number is of great interest in Arakelov theory. An interesting example of a hermitian line bundle is $\overline{\omega}_{\mathcal{X},\mathrm{Ar}} = \left(\omega_{\mathcal{X}/\mathrm{Spec}\,\mathbb{Z}}, \|\cdot\|_{\mathrm{Ar}}\right)$, where $\omega_{\mathcal{X}/\mathrm{Spec}\,\mathbb{Z}}$ is the dualizing sheaf of $\mathcal{X}$ and $\|\cdot\|_{\mathrm{Ar}}$ is the Arakelov metric, see [Sz]. Indeed, explicit upper and lower bounds for $\overline{\omega}^2_{\mathcal{X},\mathrm{Ar}}$ imply great conjectures in number theory. For instance, an explicit nontrivial lower bound

for $\overline{\omega}^2_{\mathcal{X},\mathrm{Ar}}$ implies an effective version of Bogomolov conjecture, and an explicit upper bound for $\overline{\omega}^2_{\mathcal{X}_P,\mathrm{Ar}}$, where $\{\mathcal{X}_P \to \mathcal{Y}\}$ is a certain family of morphisms of arithmetic surfaces, prove an effective version of Mordell's conjecture, see [Zh, Conjecture 1.4.1] and [La, Appendix by P. Vojta]. Here we also note that if $\mathcal{X}$ comes from a Shimura variety, then it is conjectured that the arithmetic self-intersection number $\overline{\omega}^2_{\mathcal{X},\mathrm{Ar}}$ is essentially given by logarithmic derivatives of L-functions, see [Kü1].

In order to calculate the arithmetic self-intersection number of a hermitian line bundle on an arithmetic surface $\mathcal{X}$ we have to choose a rational function $f \in k(\mathcal{X})^\times$ as in equation (1). In general, it is hard to find a rational function f such that the geometric intersection number $(\mathcal{P}, \mathcal{P} - \mathrm{div}(f))_{\mathrm{fin}}$ is computable. Moreover, in some situations, it is not clear how to evaluate the Green's function $g_{\mathcal{P}} + \log|f|^2$ for $\mathcal{P} - \mathrm{div}(f)$ at $\mathcal{P}(\mathbb{C})$. It is only known how $g_{\mathcal{P}}$ looks like in a local coordinate $z_{\mathcal{P}}$ in $\mathcal{P}(\mathbb{C})$, as $z_{\mathcal{P}}$ tends to zero.

The main result of this thesis is the following: We define a new analytic object, which is called an *arithmetic local coordinate*. With this notion we transfer the problem of finding a suitable rational function $f \in k(\mathcal{X})^\times$ to the calculation of an arithmetic local coordinate $z_{\mathcal{P}}$ in $\mathcal{P}$. We show that the arithmetic self-intersection number of an arithmetic divisor $(\mathcal{P}, g_{\mathcal{P}})$ can be written as a limit formula using an arithmetic local coordinate. Similar techniques have been used by B. Gross and D. Zagier in order to obtain the famous Gross-Zagier theorem. We apply the theory of arithmetic local coordinates and recover well-known arithmetic self-intersection numbers. Apart from the application of arithmetic local coordinates to arithmetic self-intersection numbers, we show that these coordinates appear in other interesting fields of number theory, e.g. arithmetic local coordinates in CM points are related to periods in the theory of Taylor expansions of modular forms. However, instead of going too deep into detail, this thesis is focused on the construction of the general theory of arithmetic local coordinates and their generalization to higher dimensional arithmetic varieties.
We also apply the idea of arithmetic local coordinates to the computation of arithmetic self-intersection numbers and generalized arithmetic self-intersection numbers on arithmetic varieties, and compare this new theory with the arithmetic intersection theory of H. Gillet and C. Soulé in [GS2] and with the generalized arithmetic intersection theory of J. I. Burgos Gil, J. Kramer and U. Kühn in [BGKK].

In Chapter 1 we define a particular choice of a local coordinate, namely an *arithmetic local coordinate* $z_{\mathcal{P}}$ in a horizontal prime divisor $\mathcal{P}$. That is for all rational functions $f \in k(\mathcal{X})^{\times}$ defined by the equation

$$(\mathrm{div}(f) - \mathrm{ord}_{\mathcal{P}}(f)\mathcal{P}, \mathcal{P})_{\mathrm{fin}} = \lim_{\mathbf{Q}\to\mathbf{P}} \big(\log|f(\mathbf{Q})| - \mathrm{ord}_{\mathcal{P}}(f) \log|z_{\mathcal{P}}(\mathbf{Q})| \big),$$

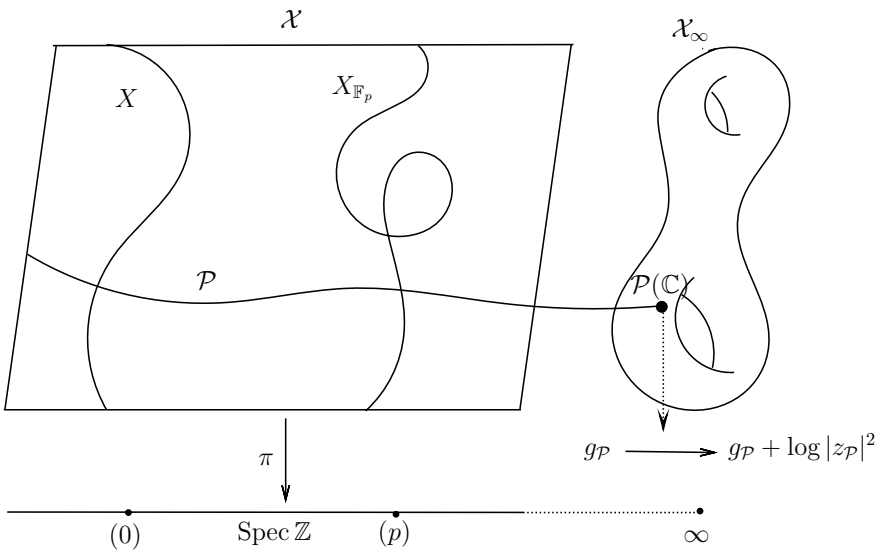
where $\mathbf{Q}$ is a family of points on $\mathcal{X}_{\infty}$ converging analytically to $\mathbf{P} := \mathcal{P}(\mathbb{C})$. With this notion we can work directly with improper intersections. Indeed, for this let $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ be a proper morphism of arithmetic surfaces. For instance, one can choose $\mathcal{Y} = \mathbb{P}^1_{\mathbb{Z}}$ because there are plenty of arithmetic surfaces which cover $\mathbb{P}^1_{\mathbb{Z}}$. Then compute an arithmetic local coordinate in some horizontal prime divisor $\mathcal{P}_{\mathcal{Y}} \in \varphi_*\mathcal{P}$ on $\mathcal{Y}$. Using functoriality we can compute an arithmetic local coordinate $z_{\mathcal{P}}$ in $\mathcal{P} \in \varphi^*\mathcal{P}_{\mathcal{Y}}$ on $\mathcal{X}$. Then we can compute the arithmetic self-intersection number of an arithmetic divisor $(\mathcal{P}, g_{\mathcal{P}})$ by the formula in the following theorem, which is one of the main theorems of the first chapter:

Theorem (Thm 1.25) *Let $\mathcal{P}$ be a horizontal prime divisor on the arithmetic surface $\mathcal{X}$ and let $z_{\mathcal{P}}$ be an arithmetic local coordinate in $\mathcal{P}$. For an arithmetic divisor $(\mathcal{P}, g_{\mathcal{P}}) \in \widehat{\mathrm{Z}}^1(\mathcal{X})$, its arithmetic self-intersection number is given by*

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \lim_{\mathbf{Q}\to\mathbf{P}} \left(\log|z_{\mathcal{P}}(\mathbf{Q})| + \frac{1}{2} g_{\mathcal{P}}(\mathbf{Q}) \right) + \frac{1}{2} \int_{\mathcal{X}_{\infty}} g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}}, \tag{2}$$

where $\omega_{g_{\mathcal{P}}} := \mathrm{dd}^{\mathrm{c}} g_{\mathcal{P}}$ outside $\mathbf{P}$.

Thus instead of constructing the Green's function $g_{\mathcal{P}} + \log|f|^2$, we only have to evaluate the function $g_{\mathcal{P}} + \log|z_{\mathcal{P}}|^2$ at $\mathcal{P}(\mathbb{C})$. An example is visualized in the following picture.



We explore some important properties of arithmetic local coordinates and show that they are equivalent to the normalized tangent vectors in the *intersection theory with a tangent vector* by B. Gross and D. Zagier in [GZ]. Note that they used this new appproch in the proof of the famous Gross-Zagier theorem.

We calculate examples of arithmetic local coordinates in horizontal prime divisors on the arithmetic surface $\mathbb{P}^1_{\mathbb{Z}}$ with the usual holomorphic structure on $\mathbb{P}^1_{\mathbb{C}}$. Moreover, we calculate examples of arithmetic local coordinates in the cusp and in CM points on the modular curve $\mathcal{X}(1)$ over $\mathbb{Z}$ associated to the modular group $\Gamma(1) := \mathrm{PSL}_2(\mathbb{Z})$. For instance, we show that the canonical local coordinate $q = \mathrm{e}^{2\pi i\tau}$ in the cusp $\mathcal{S}_\infty$ of $\mathcal{X}(1)$ is an arithmetic local coordinate. Moreover, if $\mathcal{P}_{\tau_0}$ is the horizontal prime divisor in $\mathcal{X}(1)$ coming from a CM point, then we show that

$$z_{\mathcal{P}_{\tau_0}} = \left\| \frac{\mathrm{E}_4^2\mathrm{E}_6}{\Delta}(\tau_0) \right\|_{\mathrm{Pet}} \cdot \frac{\tau - \tau_0}{\tau - \overline{\tau_0}}$$

is an arithmetic local coordinate in a non-elliptic CM point τ_0, and we show that

$$z_{\mathcal{P}_i} = 2^4 3^3 \left\| E_4(i) \right\|_{\mathrm{Pet}} \cdot \left(\frac{\tau - i}{\tau + i} \right)^2 , \text{resp. } z_{\mathcal{P}_\rho} = 2^6 \left\| \mathrm{E}_6(\rho) \right\|_{\mathrm{Pet}} \cdot \left(\frac{\tau - \rho}{\tau - \overline{\rho}} \right)^3 ,$$

is an arithmetic local coordinates in the elliptic CM point $i = \sqrt{-1}$, resp. $\rho = \frac{1+\sqrt{-3}}{2}$. Here $\|\cdot\|_{\mathrm{Pet}}$ denotes the Petersson metric for a non-holomorphic modular form, Δ denotes the modular discriminant and E_4, resp. E_6, is the classical Eisenstein series of weight 4, resp. 6.

Using these arithmetic local coordinates we calculate as an application of our main theorem two previously known arithmetic self-intersection numbers with the help of equation (2): On $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ the arithmetic self-intersection number of the Serre twist $\mathcal{O}_{\mathcal{X}}(1)$ equipped with the Fubini-Study metric $\|\cdot\|_{\mathrm{FS}}$ and on $\mathcal{X} = \mathcal{X}(1)$ the arithmetic self-intersection number of the line bundle of modular forms $\mathcal{M}_k(\Gamma(1)) := \mathcal{O}_{\mathcal{X}}\left(\mathcal{S}_\infty\right)^{\otimes k/12}$ equipped with the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$, where $k \in \mathbb{N}$ with $12|k$.

At the end of this chapter we define a new analytic object, which we call an *adjusted Green's function*. This can be seen as a global version of an arithmetic local coordinate. Indeed, adjusted Green's functions are Green's functions $g_{\mathcal{P}}$ for a horizontal prime divisor $\mathcal{P}$, characterized by the simple property

$$\lim_{\mathbf{Q}\to\mathbf{P}} \left(g_{\mathcal{P}}(\mathbf{Q}) + \log\left|z_{\mathcal{P}}(\mathbf{Q})\right|^2\right) = 0,$$

where $z_{\mathcal{P}}$ is an arithmetic local coordinate in $\mathcal{P}$.
We compare the properties of arithmetic local coordinates with those of adjusted Green's functions and find the following theorem, which will be useful in Chapter 3, where the Green's functions $g_{\mathcal{P}}$ are of log-log-type:

Theorem (Thm 1.33) *Let $\mathcal{X}$ be an arithmetic surface and let $(\mathcal{P}, g_{\mathcal{P}}) \in \widehat{\mathrm{Z}}^1(\mathcal{X})$ be an arithmetic divisor for a horizontal prime divisor $\mathcal{P}$. Then its arithmetic self-intersection number is given by*

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(g_{\mathcal{P}} \cdot \omega_{\alpha_{\mathcal{P}}} - \alpha_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}} + g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}}\right), \tag{3}$$

where $\alpha_{\mathcal{P}}$ is an adjusted Green's function for $\mathcal{P}$.

In Chapter 2 we apply the ideas from the first chapter to higher dimensional arithmetic varieties $\mathcal{X}$. First we recall the Arakelov theory on arithmetic varieties, which is due to Gillet and Soulé in [GS2]. We discuss the *arithmetic intersection number* in detail. More explicitly, the arithmetic intersection number of $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^q(\mathcal{X})$ with $p + q = \dim(\mathcal{X})$ is given by

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) = \mathrm{ht}_{[Y,g_Y]}(Z) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_Y} \wedge g_Z,$$

where $\mathrm{ht}_{[Y,g_Y]}(Z)$ is the *height of Z with respect to* $[Y, g_Y]$. We define a special class of Green's forms, namely *Z-adjusted Green's forms $\alpha_{Y,Z}$ for Y*, which are characterized by the equation

$$\mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z) = 0.$$

We calculate examples of adjusted Green's forms associated to a point lying in a hypersurface in the arithmetic variety $\mathbb{P}^d_{\mathbb{Z}} = \operatorname{Proj}\mathbb{Z}[x_0, \ldots, x_d]$ of dimension $d+1$. For instance, we show that for the cycles $Y := \{7 \cdot x_0 = 0\}$ and $Z := \{7 \cdot x_0 = x_1 = \cdots = x_{d-2} = 5 \cdot x_{d-1} = 0\}$ a Z-adjusted Green's forms for Y is given by

$$\alpha_{Y,Z} := g_Y - 2 \cdot \log(5 \cdot 7),$$

where g_Y is the Levine form for Y, and that a Y-adjusted Green's forms for Z is given by

$$\alpha_{Z,Y} := g_Z - \left(2 \cdot \log(7) + \sum_{m=1}^{d-1} \sum_{n=1}^{m} \frac{1}{n}\right) \cdot \omega_{\mathrm{FS}}^{d-1},$$

where g_Z is the Levine form for Z and ω_{FS} denotes the Fubini Study form on $\mathbb{P}^d_{\mathbb{C}}$.

We define a modification of the $*$-product between two Green's forms g_Y and g_Z by

$$g_Y \bullet g_Z := \omega_{\alpha_{Y,Z}} \wedge g_Z - \omega_{g_Z} \wedge \alpha_{Y,Z} + \omega_{g_Z} \wedge g_Y, \tag{4}$$

where $\alpha_{Y,Z}$ is a Z-adjusted Green's form for Y. We show $[0, g_Y \bullet g_Z] \in \widehat{\mathrm{CH}}^{\dim(\mathcal{X})}(\mathcal{X})$ and that its arithmetic degree equals the arithmetic intersection number of $[Y, g_Y]$ and $[Z, g_Z]$, i.e.

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) = \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \left(\omega_{\alpha_{Y,Z}} \wedge g_Z - \omega_{g_Z} \wedge \alpha_{Y,Z} + \omega_{g_Z} \wedge g_Y\right). \tag{5}$$

We generalize the definition of Z-adjusted Green's form for Y to a family of Z-adjusted Green's forms for a family of cycles $Y_1, \ldots, Y_n$ and compute examples on $\mathbb{P}^d_{\mathbb{Z}}$ and on the arithmetic 3-fold $\mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{X}(1)$.
With the use of a family of adjusted Green's forms we find a description of the arithmetic self-intersection number of a hermitian line bundle on $\mathcal{X}$, similar to equation (5). Therewith we compute on the arithmetic variety $\mathcal{X} = \mathbb{P}^d_{\mathbb{Z}}$ the well-known arithmetic self-intersection number of the Serre twist $\mathcal{O}_{\mathcal{X}}(1)$ equipped with the Fubini-Study metric $\|\cdot\|_{\mathrm{FS}}$.

In the last part of this chapter we generalize the definition of arithmetic local coordinates to higher dimensional arithmetic varieties $\mathcal{X}$. For this we consider irreducible and reduced

cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^q(\mathcal{X})$ with $p+q = \dim(\mathcal{X})$. Let $\pi : \widetilde{\mathcal{X}_\infty} \longrightarrow \mathcal{X}_\infty$ be a desingularization of $\mathcal{X}_\infty$ along $Y(\mathbb{C})$ and assume $\pi^{-1}(Y(\mathbb{C})) = \{z = 0\}$ for an equation $z = 0$ up to a null set. Then the equation $z = 0$ is called a *Z-adjusted equation for Y* if the equation

$$(\mathrm{div}(f) - \mathrm{ord}_Y(f)Y, Z)_{\mathrm{fin}} = \lim_{t\to 0}\left(\int_{\mathcal{X}_\infty} \log|f|\,\delta_{Z_t(\mathbb{C})} - \mathrm{ord}_Y(f)\int_{\widetilde{\mathcal{X}_\infty}} \alpha \log|z| \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))}\right)$$

holds for all K_1-chains f such that $\mathrm{div}(f) - \mathrm{ord}_Y(f)Y$ and Z intersect generically properly. For all $t > 0$, the cycles $Z_t(\mathbb{C}) \in \mathrm{Z}^q(\mathcal{X}_\infty)$ have to intersect $\mathrm{div}(f)(\mathbb{C})$ properly and have to fulfill the property $\lim_{t\to 0} Z_t(\mathbb{C}) = Z(\mathbb{C})$. Moreover, the real, smooth, ∂- and $\overline{\partial}$-closed form α is defined by $\pi_*\left(\alpha \wedge \delta_{\pi^{-1}(Y(\mathbb{C}))}\right) = \delta_{Y(\mathbb{C})}$.
We calculate examples of adjusted equations associated to a point lying in a hypersurface in the arithmetic variety $\mathbb{P}^d_{\mathbb{Z}} = \mathrm{Proj}\,\mathbb{Z}[x_0, \ldots, x_d]$. For instance, we show that for $Y := \{7 \cdot x_0 = 0\}$ and $Z := \{7 \cdot x_0 = x_1 = \cdots = x_{d-2} = 5 \cdot x_{d-1} = 0\}$ the equation $5 \cdot 7 \cdot \frac{x_0}{x_d} = 0$ is both a Z-adjusted equation for Y on $U := \{(x_0, \ldots, x_d) \in \mathbb{P}^d_{\mathbb{C}} \mid x_d \neq 0\}$ and also a Y-adjusted equation for Z on $V := \{(x_0, \ldots, x_d, y_0, \ldots, y_{d-1}) \in \widetilde{\mathcal{X}_\infty} \mid x_d \neq 0, y_0 \neq 0\}$, where $\widetilde{\mathcal{X}_\infty}$ denotes the blow up of $\mathbb{P}^d_{\mathbb{C}}$ along $Z(\mathbb{C})$.
As a generalization of equation (2) we show the following theorem:

Theorem (Thm 2.66) *Consider the situation as above. Then the arithmetic intersection number $\widehat{\deg}_{\mathcal{X}}([Y, g_Y] \cdot [Z, g_Z])$ can be written as*

$$\lim_{t\to 0}\left(\int_{\widetilde{\mathcal{X}_\infty}} \alpha \log|z| \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))} + \frac{1}{2}\int_{\mathcal{X}_\infty} g_Y \wedge \delta_{Z_t(\mathbb{C})}\right) + \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{g_Y} \wedge g_Z,$$

where $z = 0$ is a Z-adjusted equation for Y.

Since arithmetic local coordinates are equivalent to the tangent vector in the intersection theory by B. Gross and D. Zagier, it would be interesting to know if adjusted equations have an analogue in the intersection theory of J. H. Bruinier, B. Howard and T. Yang in [BHY], where they generalize the idea of the intersection theory with a tangent vector to Shimura varieties of orthogonal type. It might be of use for the article [BY2], where J. H. Bruinier and T. Yang conjecture relations between arithmetic self-intersection numbers of CM cycles with derivatives of automorphic L-functions.

In Chapter 3 we discuss the generalized arithmetic intersection theory.
First we restrict ourselves to the case that $\mathcal{X}$ is an arithmetic surface. Note that the generalized arithmetic intersection theory is very useful, for example in the case when the arithmetic surface comes from a modular curve, where the canonical and interesting Green's functions have singularities of log-log-type.
Recall that for a horizontal prime divisor $\mathcal{P}$ a Green's function $g_{\mathcal{P}}$ with singularities of log-log-type along $\mathcal{P}(\mathbb{C})$ has an expansion

$$g_{\mathcal{P}}(\mathbf{Q}) = -2\eta_{g_{\mathcal{P}}} \log\left(-\log|t(\mathbf{Q})|^2\right) - 2\log|t(\mathbf{Q})| - 2\log\left(\varphi_{g_{\mathcal{P}}}(\mathbf{Q})\right)$$

for a local coordinate t in $\mathbf{P} := \mathcal{P}(\mathbb{C})$, where $\eta_{g_{\mathcal{P}}} \in \mathbb{R}$ and the function $\varphi_{g_{\mathcal{P}}}$ satisfies some extra conditions, see [Kü2]. Because of equation (3) we define the *generalized arithmetic self-intersection number* of such an arithmetic divisor $(\mathcal{P}, g_{\mathcal{P}})$ by

$$(\mathcal{P}, g_{\mathcal{P}})^2 := \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(g_{\mathcal{P}} \cdot \omega_{\alpha_{\mathcal{P}}} - \alpha_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}} + g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}}\right), \tag{6}$$

where $\alpha_{\mathcal{P}}$ is an adjusted Green's function for $\mathcal{P}$. A main result is the following:

Theorem (Thm 3.4) *Let $\mathcal{P}$ be a horizontal prime divisor and let $g_{\mathcal{P}}$ be a log-log Green's function for $\mathcal{P}$, which is locally given by*

$$g_{\mathcal{P}}(\mathbf{Q}) = -2\eta_{g_{\mathcal{P}}} \log\left(-\log|z_{\mathcal{P}}(\mathbf{Q})|^2\right) - 2\log|z_{\mathcal{P}}(\mathbf{Q})| - 2\log\left(\varphi_{g_{\mathcal{P}}}(\mathbf{Q})\right),$$

where $z_{\mathcal{P}}$ is an arithmetic local coordinate in $\mathcal{P}$. Then the generalized arithmetic self-intersection number of $(\mathcal{P}, g_{\mathcal{P}})$ is given by

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \eta_{g_{\mathcal{P}}} - \log\left(\varphi_{g_{\mathcal{P}}}(\mathbf{P})\right) - \lim_{\varepsilon \to 0}\left(\eta_{g_{\mathcal{P}}} \log\left(-\log \varepsilon^2\right) - \frac{1}{2}\int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}}\right), \tag{7}$$

where the integral is taken over the complex manifold $\mathcal{X}_\varepsilon := \mathcal{X}_\infty \backslash \{x \in \mathcal{X}_\infty \,|\, |z_{\mathcal{P}}(x)| < \varepsilon\}$.

As an application of equation (7) we consider the following situation: Let $\pi_\Gamma : \mathcal{X}(\Gamma) \longrightarrow \mathcal{X}(1)$ be a proper map, where $\mathcal{X}(\Gamma)$ is a modular curve over $\mathcal{O}_K$ associated to a congruence subgroup $\Gamma < \Gamma(1) := \mathrm{PSL}_2(\mathbb{Z})$. Moreover, let $\mathcal{S}_j$ be a cusp of $\mathcal{X}(\Gamma)$. Then we recover on $\mathcal{X}(\Gamma)$ the known arithmetic self-intersection number of the line bundle $\mathcal{O}_{\mathcal{X}(\Gamma)}(\mathcal{S}_j)$ equipped

with metric $\|\cdot\|_{\mathrm{hyp}}$, which is associated to the hyperbolic Green's function for $\mathcal{S}_j$.

In the last part of the discussion of the generalized arithmetic intersection theory on arithmetic surfaces we show that the modified version (6) of the generalized arithmetic intersection number with the use of an adjusted Green's function can be generalized to two arbitrary arithmetic divisors with log-log Green's functions $(\mathcal{D}_1, g_{\mathcal{D}_1})$ and $(\mathcal{D}_2, g_{\mathcal{D}_2})$. More precisely, we define the *generalized arithmetic intersection number of* $(\mathcal{D}_1, g_{\mathcal{D}_1})$ *and* $(\mathcal{D}_2, g_{\mathcal{D}_2})$ by

$$(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) := \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \left(g_{\mathcal{D}_2} \cdot \omega_{\alpha_{\mathcal{D}_1}} - \alpha_{\mathcal{D}_1} \cdot \omega_{g_{\mathcal{D}_2}} + g_{\mathcal{D}_1} \cdot \omega_{g_{\mathcal{D}_2}} \right), \tag{8}$$

where $\alpha_{\mathcal{D}_1}$ is an adjusted Green's function for $\mathcal{D}_1$ depending on $\mathcal{D}_2$. We show that this modified version of the generalized arithmetic intersection number is well-defined and coincides with the generalized arithmetic intersection number due to Kühn in [Kü2].

When $\mathcal{X}$ is a higher-dimensional arithmetic variety we have to consider the generalized arithmetic intersection theory of J. I. Burgos Gil, J. Kramer and U. Kühn in [BGKK] with Green's forms of log-log-type. For instance, this leads to a well-defined intersection theory on compactifications of non-compact Shimura varieties, where the natural Green's currents have singularities of log-log-type.
If the cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^q(\mathcal{X})$ with $p+q = \dim(\mathcal{X})$ intersect generically properly and if g_Y and g_Y are Green's forms for Y and Z of log-log-type along a normal crossing divisor $\mathcal{S}_\infty$, then the generalized arithmetic intersection number of $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty)$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^q(\mathcal{X}, \mathcal{S}_\infty)$ is in [BGKK] defined by

$$(Y, Z)_{\mathrm{fin}} + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_Y * g_Z, \tag{9}$$

where $g_Y * g_Z$ is the $*$-product between g_Y and g_Z, which is given by

$$g_Y * g_Z = \mathrm{dd}^c(\sigma_{ZY} g_Z) \wedge g_Y + \omega_{g_Y} \wedge \sigma_{YZ} g_Z.$$

Here $\{\sigma_{Y,Z}, \sigma_{Z,Y}\}$ is a partition of unity to the cover $\mathcal{X}_\infty \backslash Z(\mathbb{C})$ and $\mathcal{X}_\infty \backslash Y(\mathbb{C})$ of $\mathcal{X}_\infty$. Because of equation (4) we set

$$g_Y \bullet g_Z := \omega_{\alpha_{Y,Z}} \wedge g_Z - \omega_{g_Z} \wedge \alpha_{Y,Z} + \omega_{g_Z} \wedge g_Y, \tag{10}$$

where $\alpha_{Y,Z}$ is a Z-adjusted Green's form for Y. Then we define a modified version of the generalized arithmetic intersection number of $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty)$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^q(\mathcal{X}, \mathcal{S}_\infty)$ by

$$\frac{1}{2} \int_{\mathcal{X}_\infty} g_Y \bullet g_Z. \tag{11}$$

An investigation of the difference $g_Y \bullet g_Z - g_Y * g_Z$ shows the following important fact:

Theorem (Thm 3.31) *Assume that the cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^q(\mathcal{X})$ with $p + q = \dim(\mathcal{X})$ intersect generically properly. Then the modified version of the generalized arithmetic intersection number of $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty)$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^q(\mathcal{X}, \mathcal{S}_\infty)$ in (11) coincides with the generalized arithmetic intersection number due to Burgos-Kramer-Kühn in (9), i.e.*

$$\frac{1}{2} \int_{\mathcal{X}_\infty} g_Y \bullet g_Z = (Y, Z)_{\mathrm{fin}} + \frac{1}{2} \int_{\mathcal{X}_\infty} g_Y * g_Z.$$

Moreover, it can be shown that the currents $g_Y * g_Z$ and $g_Y \bullet g_Z$ are also well-defined when Y and Z do not intersect generically properly. Because of this we make some notes about the case $Y = Z$. In particular we show the crucial fact that

$$\int_{\mathcal{X}_\infty} g_Y \bullet g_Y = \int_{\mathcal{X}_\infty} g_Y * g_Y$$

if g_Y is a Y-adjusted Green's form for Y.

In the last part of this thesis we generalize the modified $*$-product (10) to a family of Green's forms $g_{Y_1}, \ldots, g_{Y_n}$ and find an alternative description of the generalized arithmetic self-intersection number of a *good hermitian line bundle* (for a definition see [BGKK]). As an application we compute on the arithmetic 3-fold $\mathcal{X} = \mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{X}(1)$ the known arithmetic self-intersection number of $\overline{\mathcal{L}}(k) := p_1^* \overline{\mathcal{M}}_k(\Gamma(1)) \otimes p_2^* \overline{\mathcal{M}}_k(\Gamma(1))$, where $k \in \mathbb{N}$ with $12|k$, $p_i : \mathcal{X} \longrightarrow \mathcal{X}(1)$ is the projection onto the i-th factor and $\overline{\mathcal{M}}_k(\Gamma(1)) := \left(\mathcal{O}_{\mathcal{X}(1)}(\mathcal{S}_\infty)^{\otimes k/12}, \|\cdot\|_{\mathrm{Pet}}\right)$ is the line bundle of modular forms equipped with the Petersson metric.

Chapter 1

Arithmetic Local Coordinates On Arithmetic Surfaces

In this chapter we start with a review of the relevant definitions of arithmetic intersection theory on arithmetic surfaces $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_K$. We define *arithmetic local coordinates* in horizontal prime divisors. We show that they always exist and that they satisfy a functoriality property. We calculate examples of arithmetic local coordinates on the arithmetic surfaces $\mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{X}(1)$. In the case that the horizontal prime divisor on $\mathcal{X}$ is induced by a K-rational point on the generic fibre X we show that the concept of arithmetic local coordinates is equivalent to the *intersection theory with a tangent vector* by B. Gross and D. Zagier in [GZ]. As an application we compare the tangent vectors for Heegner points on $\mathcal{X}(1)$ calculated in [GZ] with our arithmetic local coordinates on $\mathcal{X}(1)$. We prove that the arithmetic self-intersection number of a hermitian line bundle on an arithmetic surface can be written as a limit formula with the use of an arithmetic local coordinate. With this new formula we recover classical examples on $\mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{X}(1)$. In the last part of this chapter we show that arithmetic local coordinates can be defined using a special kind of Green's functions, called *adjusted Green's functions*. These have a simpler characterization than the arithmetic local coordinates and thereby are easier to handle with. Using adjusted Green's functions we prove another version of the arithmetic self-intersection number of a hermitian line bundle, where we only have to integrate a smooth differential form.

1.1 Review of Arakelov theory on arithmetic surfaces

Let K be a number field and let $\mathcal{O}_K$ be the ring of integers of K. An *arithmetic surface* $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ is a reduced, 2-dimensional, regular scheme, which is projective and flat over $\operatorname{Spec} \mathcal{O}_K$. Moreover, we assume that the generic fibre

$$X = \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} K$$

is geometrically connected, i.e. $\mathcal{X}$ is a regular model of X over $\operatorname{Spec} \mathcal{O}_K$. For a closed point $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$, let $\mathcal{X}_\mathfrak{p} = \mathcal{X} \times_{\operatorname{Spec} \mathcal{O}_K} \operatorname{Spec} \mathbb{F}_\mathfrak{p}$ denote the special fibre of $\mathcal{X}$ above $\mathfrak{p}$. Let $\mathcal{X}_\infty = X(\mathbb{C})$ be the set of complex-valued points of the generic fibre X considered as a scheme over $\mathbb{Q}$. Indeed, $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ can be seen as a *Stein factorization of* $\mathcal{X} \longrightarrow \operatorname{Spec} \mathbb{Z}$, i.e. a morphism from $\mathcal{X}$ to $\operatorname{Spec} \mathcal{O}_K$ which has connected fibres. The existence of a Stein factorization of $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} \mathbb{Z}$ follows from [Ha, Corollary 11.5]. Note that $\mathcal{X}_\infty$ is a compact, 1-dimensional, complex manifold. Actually we have a disjoint decomposition $\mathcal{X}_\infty = \coprod_{\sigma : K \hookrightarrow \mathbb{C}} \mathcal{X}_\sigma(\mathbb{C})$ into the connected components coming from the complex curves

$$\mathcal{X}_\sigma = X \times_{\operatorname{Spec} K, \sigma} \operatorname{Spec} \mathbb{C}.$$

For a smooth function $f \in \mathcal{C}^\infty(\mathcal{X}_\infty)$ we define the 1-forms $\partial f = \frac{\partial f}{\partial z} \mathrm{d}z$ and $\overline{\partial} f = \frac{\partial f}{\partial \overline{z}} \mathrm{d}\overline{z}$ with respect to a local coordinate $z = x + iy \in \mathbb{C}$, where as usual

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Moreover, let us define the real operators $\mathrm{d} = \partial + \overline{\partial}$ and $\mathrm{d}^c = \frac{i}{4\pi}(\overline{\partial} - \partial)$, so that $\mathrm{d}\mathrm{d}^c = \frac{i}{2\pi}\partial\overline{\partial}$. In polar coordinates $z = r\mathrm{e}^{i\varphi}$ we can write

$$\mathrm{d}f = \frac{\partial f}{\partial \varphi}\mathrm{d}\varphi + \frac{\partial f}{\partial r}\mathrm{d}r \quad \text{and} \quad \mathrm{d}^c f = \frac{1}{4\pi}\frac{\partial f}{\partial r} r\mathrm{d}\varphi - \frac{1}{4\pi}\frac{1}{r}\frac{\partial f}{\partial \varphi}\mathrm{d}r,$$

see [La, p.11]. Let P be a point on $\mathcal{X}_\infty$. A *Green's function g_P for P* is a real-valued smooth function $g_P \in \mathcal{C}^\infty(\mathcal{X}_\infty \backslash P)$ outside P such that
i) for a local coordinate t in P we have an expansion

$$g_P(Q) = -\log |t(Q)|^2 + \varphi(Q)$$

near P, where $\varphi \in \mathcal{C}^\infty(\mathcal{X}_\infty)$ is smooth and
ii) the $(1,1)$-form ω_{g_P} on $\mathcal{X}_\infty$, which is uniquely defined by $\omega_{g_P}\big|_{\mathcal{X}_\infty \backslash P} = \mathrm{d}\mathrm{d}^c g_P$ outside

P, is a *normalized volume form*. This means that ω_{g_P} is a $(1,1)$-form, locally given by $f(z)i\mathrm{d}z\wedge\mathrm{d}\bar{z}$ where f is a positive, real-valued, smooth function, satisfying the normalization $\int_{\mathcal{X}_\infty}\omega_{g_P}=1$.
Here we note that Green's functions are not uniquely determined. Indeed, if g_P is a Green's function for P, then for all real numbers $\alpha\in\mathbb{R}$, the function $g_P+\alpha$ is also a Green's function for P.

Let $\mathrm{Z}^1(\mathcal{X})$ denote the group of divisors $\mathcal{D}$ on $\mathcal{X}$. Any divisor $\mathcal{D}$ is a linear combination of prime divisors, i.e. irreducible and reduced divisors. There are two types of prime divisors: horizontal and vertical prime divisors. A prime divisor $\mathcal{D}$ is called *horizontal* if $\mathcal{D}$ is the Zariski closure in $\mathcal{X}$ of a closed point on the generic fibre X, and is called *vertical* if $\mathcal{D}$ is an irreducible component of a special fibre $\mathcal{X}_\mathfrak{p}$, hence $\pi(\mathcal{D})=\mathfrak{p}\in\operatorname{Spec}\mathcal{O}_K$.
A horizontal divisor $\mathcal{D}$ on $\mathcal{X}$ is a linear combination of horizontal prime divisors and induces a divisor $\mathcal{D}(\mathbb{C})=\sum\operatorname{ord}_{P_i}(\mathcal{D}(\mathbb{C}))P_i$ on $\mathcal{X}_\infty$. With this notion, let g_{P_i} be a Green's function for P_i. Then $g_\mathcal{D}:=\sum\operatorname{ord}_{P_i}(\mathcal{D}(\mathbb{C}))g_{P_i}$ will be called a *Green's function for* $\mathcal{D}$.

Let $\widehat{\mathrm{Z}}^1(\mathcal{X})$ be the group of arithmetic divisors $(\mathcal{D},g_\mathcal{D})\in\widehat{\mathrm{Z}}^1(\mathcal{X})$, where $\mathcal{D}$ is a divisor and $g_\mathcal{D}$ is a Green's function for $\mathcal{D}$. For two arithmetic divisors $(\mathcal{D}_1,g_{\mathcal{D}_1})$ and $(\mathcal{D}_2,g_{\mathcal{D}_2})$ such that $\mathcal{D}_1$ and $\mathcal{D}_2$ have no common components, their *arithmetic intersection number* $(\mathcal{D}_1,g_{\mathcal{D}_1})\cdot(\mathcal{D}_2,g_{\mathcal{D}_2})\in\mathbb{R}$ is defined by

$$(\mathcal{D}_1,g_{\mathcal{D}_1})\cdot(\mathcal{D}_2,g_{\mathcal{D}_2})=(\mathcal{D}_1,\mathcal{D}_2)_{\mathrm{fin}}+\frac{1}{2}g_{\mathcal{D}_1}[\mathcal{D}_2(\mathbb{C})]+\frac{1}{2}\int\limits_{\mathcal{X}_\infty}g_{\mathcal{D}_2}\cdot\omega_{g_{\mathcal{D}_1}},\tag{1.1}$$

where $g_{\mathcal{D}_1}[\mathcal{D}_2(\mathbb{C})]:=\sum\operatorname{ord}_{P_i}(\mathcal{D}_1(\mathbb{C}))\operatorname{ord}_{Q_j}(\mathcal{D}_2(\mathbb{C}))g_{P_i}(Q_j)$ for $\mathcal{D}_1(\mathbb{C})=\sum\operatorname{ord}_{P_i}(\mathcal{D}_1(\mathbb{C}))P_i$ and $\mathcal{D}_2(\mathbb{C})=\sum\operatorname{ord}_{Q_j}(\mathcal{D}_2(\mathbb{C}))Q_j$. Moreover, the differential form $\omega_{g_{\mathcal{D}_1}}$ denotes the smooth $(1,1)$-form on $\mathcal{X}_\infty$ given by $\mathrm{d}\mathrm{d}^c g_{\mathcal{D}_1}$ outside $\mathcal{D}_1(\mathbb{C})$ and $(\mathcal{D}_1,\mathcal{D}_2)_{\mathrm{fin}}$ denotes the *geometric intersection number of* $\mathcal{D}_1$ *and* $\mathcal{D}_2$. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are prime divisors with no common component, the geometric intersection number equals

$$(\mathcal{D}_1,\mathcal{D}_2)_{\mathrm{fin}}=\sum_{x\in\mathcal{X}}\log\#\big(\mathcal{O}_{\mathcal{X},x}/(\mathcal{D}_{1,x},\mathcal{D}_{2,x})\big),$$

where $\mathcal{D}_{i,x}$ $(i=1,2)$ are local equations for $\mathcal{D}_i$ $(i=1,2)$ at the point $x\in\mathcal{X}$ and the sum runs through the closed points x in $\mathcal{X}$. For linear combinations of prime divisors, the intersection number can be defined by bilinearity. For the case that the prime divisors have common vertical components, we refer to [Li2, Theorem 1.12. p.381]. If the divisors

$\mathcal{D}_1$ and $\mathcal{D}_2$ have common horizontal components, we have to move $\mathcal{D}_1$ by the divisor of a rational function $f \in k(\mathcal{X})^\times$ such that $\mathcal{D}_1 - \mathrm{div}(f)$ and $\mathcal{D}_2$ have no common horizontal components. Indeed, let $\widehat{\mathrm{R}}^1(\mathcal{X}) \subset \widehat{\mathrm{Z}}^1(\mathcal{X})$ be the group of arithmetic divisors of the form $(\mathrm{div}(f), -\log|f|^2)$ with $f \in k(\mathcal{X})^\times$. The quotient

$$\widehat{\mathrm{CH}}^1(\mathcal{X}) = \widehat{\mathrm{Z}}^1(\mathcal{X}) \big/ \widehat{\mathrm{R}}^1(\mathcal{X})$$

is the *arithmetic Chow group of* $\mathcal{X}$. Then we have the following

Theorem 1.1. (Arakelov, Deligne et al. [So2, théorème 1, p.329])
There exists a bilinear, symmetric pairing

$$\begin{aligned} \widehat{\mathrm{CH}}^1(\mathcal{X}) \times \widehat{\mathrm{CH}}^1(\mathcal{X}) &\longrightarrow \mathbb{R} \\ ([\mathcal{D}_1, g_{\mathcal{D}_1}], [\mathcal{D}_2, g_{\mathcal{D}_2}]) &\longmapsto (\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) \,. \end{aligned}$$

Moreover, there exists a bilinear pairing

$$\begin{aligned} \mathrm{ht} : \widehat{\mathrm{CH}}^1(\mathcal{X}) \times \mathrm{Z}^1(\mathcal{X}) &\longrightarrow \mathbb{R} \\ ([\mathcal{D}_1, g_{\mathcal{D}_1}], \mathcal{D}_2) &\longmapsto \mathrm{ht}_{[\mathcal{D}_1, g_{\mathcal{D}_1}]}(\mathcal{D}_2), \end{aligned}$$

which satisfies the equation

$$(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = \mathrm{ht}_{[\mathcal{D}_1, g_{\mathcal{D}_1}]}(\mathcal{D}_2) + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_{\mathcal{D}_2} \cdot \omega_{g_{\mathcal{D}_1}} .$$

The number $\mathrm{ht}_{[\mathcal{D}_1, g_{\mathcal{D}_1}]}(\mathcal{D}_2) \in \mathbb{R}$ is called the *height of* $\mathcal{D}_2$ *with respect to* $[\mathcal{D}_1, g_{\mathcal{D}_1}]$. For an arithmetic divisor $(\mathcal{D}, g_{\mathcal{D}}) \in \widehat{\mathrm{Z}}^1(\mathcal{X})$, the *arithmetic self-intersection number of* $(\mathcal{D}, g_{\mathcal{D}})$ is given by

$$(\mathcal{D}, g_{\mathcal{D}})^2 = (\mathcal{D}, \mathcal{D} - \mathrm{div}(f))_{\mathrm{fin}} + \frac{1}{2} \left(g_{\mathcal{D}} + \log|f|^2 \right) [\mathcal{D}(\mathbb{C})] + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_{\mathcal{D}} \cdot \omega_{g_{\mathcal{D}}}. \tag{1.2}$$

Here $f \in k(\mathcal{X})^\times$ is chosen such that $\mathcal{D}$ *intersects* $\mathcal{D} - \mathrm{div}(f)$ *properly on the generic fibre* X. By this we mean that $\mathcal{D}(\mathbb{C}) \cap (\mathcal{D} - \mathrm{div}(f))(\mathbb{C}) = \emptyset$. By Chow's Moving Lemma (see [Li2, Corollary 1.10, p.379]) we can always find such an f.

Let $\widehat{\mathrm{Pic}}\,(\mathcal{X})$ denotes the group of isomorphism classes of *hermitian line bundles* $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$. This means that $\mathcal{L}$ is an invertible sheaf on $\mathcal{X}$ and $\|\cdot\|$ defines a continuous

hermitian metric on $\mathcal{L}_\infty = \mathcal{L} \otimes_{\mathcal{O}_K} \mathbb{C}$ over $\mathcal{X}_\infty$. For a section s of $\mathcal{L}_\infty$ and a local coordinate t in $P \in \mathcal{X}_\infty$ we can write

$$\|s\|(t) = |t|^{\mathrm{ord}_P(s)}\, \psi(t),$$

where ψ is non-vanishing and continuous.
By [So1, Proposition 1, p.67] all arithmetic divisors arise from the isomorphism

$$\widehat{c}_1 : \widehat{\mathrm{Pic}}(\mathcal{X}) \xrightarrow{\sim} \widehat{\mathrm{CH}}^1(\mathcal{X}), \quad (\mathcal{L}, \|\cdot\|) \longmapsto \left[\mathrm{div}(l), -\log \|l\|^2\right],$$

where l is any non-trivial rational section of $\mathcal{L}$. The inverse map of $\widehat{c}_1$ is given by $[\mathcal{D}, g_\mathcal{D}] \longmapsto (\mathcal{O}_\mathcal{X}(\mathcal{D}), \|\cdot\|)$. The metric $\|\cdot\|$ is defined by $\|1_\mathcal{D}\|^2 = \exp(-g_\mathcal{D})$, where $1_\mathcal{D}$ denotes the canonical 1-section of $\mathcal{O}_\mathcal{X}(\mathcal{D})$. Note that this isomorphism is compatible with arithmetic intersection numbers. More explicitly, consider two hermitian line bundles $\overline{\mathcal{L}}$ and $\overline{\mathcal{M}}$. Then their arithmetic intersection number is defined by

$$\begin{aligned}\overline{\mathcal{L}}.\overline{\mathcal{M}} :&= \left(\mathrm{div}(l), -\log\|l\|^2\right) \cdot \left(\mathrm{div}(m), -\log\|m\|^2\right) \\ &= (\mathrm{div}(l), \mathrm{div}(m))_{\mathrm{fin}} - \log\|m\|\,[\mathrm{div}(l)] - \int_{\mathcal{X}_\infty} \log\|l\| \cdot c_1(\overline{\mathcal{M}}),\end{aligned}$$

where l and m are non-trivial rational sections of $\mathcal{L}$ and $\mathcal{M}$ resp., whose induced divisors on $\mathcal{X}_\infty$ have no points in common, and $c_1(\overline{\mathcal{M}})$ denotes the *first Chern form of* $\overline{\mathcal{M}}$ *on* $\mathcal{X}_\infty$. For a non-trivial rational section m of $\mathcal{M}$ on $\mathcal{X}_\infty$, the first Chern form $c_1(\overline{\mathcal{M}})$ is given by $-\mathrm{dd}^c(\log\|m\|^2)$ outside $\mathrm{div}(m)$ on $\mathcal{X}_\infty$.

Moreover, we have the *height of* $\mathcal{P}$ *with respect to* $\overline{\mathcal{L}} = (\mathcal{L}, \|\cdot\|)$, which is given by

$$\mathrm{ht}_{\overline{\mathcal{L}}}(\mathcal{P}) = (\mathrm{div}(l), \mathcal{P})_{\mathrm{fin}} - \log\|l\|\,[\mathcal{P}(\mathbb{C})] \in \mathbb{R},$$

where l is any non-trivial rational section of $\mathcal{L}$ having divisor disjoint from $\mathcal{P}$ on the generic fibre. The *arithmetic self-intersection number of* $\overline{\mathcal{L}}$ is given by

$$\overline{\mathcal{L}}^2 = \left(\mathrm{div}(l), -\log\|l\|^2\right)^2,$$

where l is a non-trivial rational section of $\mathcal{L}$.

1.2 Arithmetic local coordinates

In the definition of the arithmetic self-intersection number of an arithmetic prime divisor $(\mathcal{P}, g_\mathcal{P})$ we have to choose a rational function $f \in k(\mathcal{X})^\times$ such that $\mathcal{P}$ and $\mathcal{P} - \mathrm{div}(f)$ have

no common components, see equation (1.2). Since no rational function $f \in k(\mathcal{X})^\times$ used to move $\mathcal{P}$ as above should be preferred over the others, we may ask whether there is an analytic shadow of $\mathcal{P}$ that replaces the geometric intersection number at the finite places by an analytic datum on the complex manifold $\mathcal{X}_\infty$. This leads to the definition of an arithmetic local coordinate.

Definition 1.2. Let $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ be an arithmetic surface and let P be a closed point on the generic fibre X with residue field $k(P)$. Moreover, let $\mathcal{P}$ be the horizontal prime divisor given by the Zariski closure of P in $\mathcal{X}$. Over the curve $\mathcal{X}_\sigma$ for $\sigma : K \hookrightarrow \mathbb{C}$, the induced divisor $\mathcal{P}\big|_{\mathcal{X}_\sigma} = \sum P_{i,\sigma}$ is the sum of $d_P = [k(P) : K]$ points ($i = 1, \ldots, d_P$). An *arithmetic local coordinate $z_\mathcal{P}$ in $\mathcal{P}$* is a family of local coordinates $\left(z_{P_{i,\sigma}}\right)_{i,\sigma}$ in the induced complex-valued points on $\mathcal{X}_\infty$ such that for all rational functions $f \in k(\mathcal{X})^\times$ the following equation holds:

$$(\operatorname{div}(f) - \operatorname{ord}_\mathcal{P}(f)\mathcal{P}, \mathcal{P})_{\text{fin}} = \sum_{\sigma : K \hookrightarrow \mathbb{C}} \sum_{i=1}^{d_P} \lim_{Q_{i,\sigma} \to P_{i,\sigma}} \left(\log |f_\sigma(Q_{i,\sigma})| - \operatorname{ord}_{P_{i,\sigma}}(f_\sigma) \log \left| z_{P_{i,\sigma}}(Q_{i,\sigma}) \right| \right).$$

Here, for each i and σ, the point $Q_{i,\sigma}$ converges to $P_{i,\sigma}$ in the complex topology on $\mathcal{X}_\infty$.

Remark 1.3. To simplify the notation we set $|\alpha| := \prod_{i,\sigma} |\alpha_{i,\sigma}|$ for a family of real numbers $\alpha = (\alpha_{i,\sigma})_{i,\sigma}$. Moreover, the equation in Definition 1.2 will be written as

$$(\operatorname{div}(f) - \operatorname{ord}_\mathcal{P}(f)\mathcal{P}, \mathcal{P})_{\text{fin}} = \lim_{Q \to P} \left(\log |f(Q)| - \operatorname{ord}_\mathcal{P}(f) \log |z_\mathcal{P}(Q)| \right). \tag{1.3}$$

Remark 1.4. Note that a local coordinate in a point P on $\mathcal{X}_\infty$ is a pair (U, z_P). Hence arithmetic local coordinates do not only depend on the divisor $\mathcal{P}$ but also on the open neighborhood U of $\mathcal{P}(\mathbb{C})$.

Now we will see that arithmetic local coordinates are independent of the rational function f used in the defining equation (1.3) and that they do exist.

Proposition 1.5. *Suppose $\mathcal{P}$ is a horizontal prime divisor on an arithmetic surface $\mathcal{X}$. Then we have:*
i) There exists an arithmetic local coordinate $z_\mathcal{P}$ in $\mathcal{P}$;
ii) If $z_\mathcal{P}$ and $w_\mathcal{P}$ are both arithmetic local coordinates in $\mathcal{P}$, then there is a family $\gamma = (\gamma_{i,\sigma})_{i,\sigma}$ of complex valued functions $\gamma_{i,\sigma}$ on $\mathcal{X}_\sigma$ with $|\gamma(P)| := \prod |\gamma_{i,\sigma}(P_{i,\sigma})| = 1$ such that

$$\gamma w_\mathcal{P} := \left(\gamma_{i,\sigma} \cdot w_{P_{i,\sigma}}\right)_{i,\sigma} = z_\mathcal{P}.$$

Proof. *i)* We will show that given a family of local coordinates $z_{\mathcal{P}} = \left(z_{P_{i,\sigma}}\right)_{i,\sigma}$ in the points $\mathcal{P}(\mathbb{C})$, there exists a family $\alpha = (\alpha_{i,\sigma})_{i,\sigma}$ of positive real numbers $\alpha_{i,\sigma} \in \mathbb{R}_{>0}$ such that the family $\alpha z_{\mathcal{P}} = \left(\alpha_{i,\sigma} z_{P_{i,\sigma}}\right)_{i,\sigma}$ is an arithmetic local coordinate in $\mathcal{P}$.

Let P be a closed point on the generic fibre X such that $\mathcal{P}$ is the Zariski closure of P in $\mathcal{X}$ and set $n = [k(P) : \mathbb{Q}]$, where $k(P)$ is the residue field of P. Let $z_{\mathcal{P}}$ be a family of local coordinates in the points $\mathcal{P}(\mathbb{C})$. Moreover, fix a rational function $f \in k(\mathcal{X})^{\times}$ with $\mathrm{ord}_{\mathcal{P}}(f) \neq 0$. Then there exists a family of real numbers $\alpha = (\alpha_{i,\sigma})_{i,\sigma} \in \mathbb{R}^n$ such that

$$(\mathrm{div}(f) - \mathrm{ord}_{\mathcal{P}}(f)\mathcal{P}, \mathcal{P})_{\mathrm{fin}} - \lim_{Q \to P} (\log|f(Q)| - \mathrm{ord}_{\mathcal{P}}(f) \log|z_{\mathcal{P}}(Q)|) = -\mathrm{ord}_{\mathcal{P}}(f) \log|\alpha| \,.$$

Obviously the numbers $\alpha_{i,\sigma}$ can be chosen to be positive and the family of local coordinates $\alpha z_{\mathcal{P}} = \left(\alpha_{i,\sigma} z_{P_{i,\sigma}}\right)_{i,\sigma}$ satisfies the equation

$$(\mathrm{div}(f) - \mathrm{ord}_{\mathcal{P}}(f)\mathcal{P}, \mathcal{P})_{\mathrm{fin}} = \lim_{Q \to P} (\log|f(Q)| - \mathrm{ord}_{\mathcal{P}}(f) \log|\alpha z_{\mathcal{P}}(Q)|) \,.$$

Now we show that any family of local coordinates $z_{\mathcal{P}}$ in $\mathcal{P}$ on $\mathcal{X}_{\infty}$ which satisfies the equation (1.3) for one function $f \in k(\mathcal{X})^{\times}$ with $\mathrm{ord}_{\mathcal{P}}(f) \neq 0$, satisfies the same equation for all function $g \in k(\mathcal{X})^{\times}$ and hence is an arithmetic local coordinate in $\mathcal{P}$.
First note that the equation (1.3) is satisfied for any rational function $f \in k(\mathcal{X})^{\times}$ with $\mathrm{ord}_{\mathcal{P}}(f) = 0$. Indeed, because of the product formula

$$(\mathrm{div}(f), \mathcal{P})_{\mathrm{fin}} = \log|f(P)| \,.$$

Now consider a rational function $f \in k(\mathcal{X})^{\times}$ with $\mathrm{ord}_{\mathcal{P}}(f) \neq 0$. Then there are functions $t \in k(\mathcal{X})^{\times}$ with $\mathrm{ord}_{\mathcal{P}}(t) = 1$ and $h \in k(\mathcal{X})^{\times}$ with $\mathrm{ord}_{\mathcal{P}}(h) = 0$ such that f can be written as $f = h \cdot t^{\mathrm{ord}_{\mathcal{P}}(f)}$. Calculating the geometric intersection number yields

$$\begin{aligned}(\mathrm{div}(f) - \mathrm{ord}_{\mathcal{P}}(f)\mathcal{P}, \mathcal{P})_{\mathrm{fin}} &= (\mathrm{ord}_{\mathcal{P}}(f)\mathrm{div}(t) + \mathrm{div}(h) - \mathrm{ord}_{\mathcal{P}}(f)\mathcal{P}, \mathcal{P})_{\mathrm{fin}} \\ &= \mathrm{ord}_{\mathcal{P}}(f)(\mathrm{div}(t) - \mathcal{P}, \mathcal{P})_{\mathrm{fin}} + (\mathrm{div}(h), \mathcal{P})_{\mathrm{fin}}.\end{aligned}$$

On the other hand, there is the analytic limit

$$\begin{aligned}&\lim_{Q \to P} (\log|f(Q)| - \mathrm{ord}_{\mathcal{P}}(f) \log|z_{\mathcal{P}}(Q)|) = \\ &\lim_{Q \to P} (\mathrm{ord}_{\mathcal{P}}(f) \log|t(Q)| - \mathrm{ord}_{\mathcal{P}}(f) \log|z_{\mathcal{P}}(Q)|) + \log|h(P)| \,.\end{aligned} \tag{1.4}$$

Thus dividing both sides of equation (1.4) by $\mathrm{ord}_{\mathcal{P}}(f)$ and applying the product formula, we see that the function f satisfies the equation (1.3) if and only if the function t satisfies

equation (1.3). It follows that (1.3) is satisfied for every rational function $g \in k(\mathcal{X})^\times$ if and only if it is satisfied for one function $f \in k(\mathcal{X})^\times$ such that $\mathrm{ord}_\mathcal{P}(f) \neq 0$, since we can always find a function $h' \in k(\mathcal{X})^\times$ with $\mathrm{ord}_\mathcal{P}(h') = 0$ and $g = h' \cdot t^{\mathrm{ord}_\mathcal{P}(f)}$. This proves the first assertion of the proposition.
ii) The second assertion is trivial. □

The next proposition shows how arithmetic local coordinates transform under pullbacks. This yields a method of constructing arithmetic local coordinates. For this let $K \subset L$ be an extension of number fields and consider two arithmetic surfaces $\mathcal{Y} \longrightarrow \operatorname{Spec} \mathcal{O}_L$ and $\mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$. Let $\varphi : \mathcal{Y} \longrightarrow \mathcal{X}$ be a surjective proper morphism of arithmetic surfaces over $\operatorname{Spec} \mathcal{O}_K$. Let $\mathcal{P}$ be a horizontal prime divisor on $\mathcal{X}$ with generic point P and let $z_\mathcal{P} = \left(z_{P_{i,\sigma}}\right)_{i,\sigma}$ be an arithmetic local coordinate in $\mathcal{P}$, where $\sigma : K \hookrightarrow \mathbb{C}$ and $i = 1, \ldots, [k(P) : K]$. Moreover, let $\varphi^*\mathcal{P} = \sum b_\nu \mathcal{P}_\nu + \mathcal{V}$ be the decomposition of $\varphi^*\mathcal{P}$ into horizontal prime divisors $\mathcal{P}_\nu$ with generic point P_ν, and a vertical divisor $\mathcal{V}$ on $\mathcal{Y}$.

Proposition 1.6. *Consider the situation as above. Then for each horizontal prime divisor $\mathcal{P}_\nu \subseteq \varphi^*(\mathcal{P})$, there is a family of real numbers $\alpha_\nu = (\alpha_{j,\tau})_{j,\tau}$ for $\tau : L \hookrightarrow \mathbb{C}$ and $j = 1, \ldots, [k(P_\nu) : L]$ such that*

$$z_{\mathcal{P}_\nu} = \alpha_\nu \cdot (\varphi^* z_\mathcal{P})^{1/b_\nu} \tag{1.5}$$

is an arithmetic local coordinate in $\mathcal{P}_\nu$. More explicitly,

$$\log |\alpha_\nu| = -\frac{1}{b_\nu} \left(\sum_{i \neq \nu} b_i \mathcal{P}_i + \mathcal{V}, \mathcal{P}_\nu \right)_{\mathcal{Y},\mathrm{fin}}.$$

Proof. Let $f \in k(\mathcal{X})^\times$ have $\mathrm{ord}_\mathcal{P}(f) = 1$. Since the morphism $\varphi : \mathcal{Y} \longrightarrow \mathcal{X}$ is surjective, we obtain an inclusion of function fields $k(\mathcal{X}) \hookrightarrow k(\mathcal{Y})$ and f induces a rational function $\varphi^* f$ in $k(\mathcal{Y})^\times$ such that $\mathrm{ord}_{\mathcal{P}_\nu}(\varphi^* f) = b_\nu$. For two divisors $\mathcal{D}_\mathcal{X} \in \mathrm{Z}^1(\mathcal{X})$ and $\mathcal{D}_\mathcal{Y} \in \mathrm{Z}^1(\mathcal{Y})$, where $\varphi_* \mathcal{D}_\mathcal{X}$ and $\mathcal{D}_\mathcal{Y}$ do not have any common component, the projection formula $(\varphi^* \mathcal{D}_\mathcal{Y}, \mathcal{D}_\mathcal{X})_{\mathcal{X},\mathrm{fin}} = (\mathcal{D}_\mathcal{Y}, \varphi_* \mathcal{D}_\mathcal{X})_{\mathcal{Y},\mathrm{fin}}$ holds. Observe that we have $\varphi_* \mathcal{P}_\nu = [k(P_\nu) : k(P)]\mathcal{P}$ and $\varphi^* \mathrm{div}(f) = \mathrm{div}(\varphi^* f)$. Using the definition of the arithmetic local coordinate $z_\mathcal{P}$ in $\mathcal{P}$,

we find

$$\begin{aligned}
&[k(P_\nu):k(P)]\lim_{Q\to P}(\log|f(Q)|-\log|z_{\mathcal{P}}(Q)|)=\\
&[k(P_\nu):k(P)]\,(\operatorname{div}(f)-\mathcal{P},\mathcal{P})_{\mathcal{X},\mathrm{fin}}=\\
&\qquad\Big(\operatorname{div}(\varphi^*f)-\sum b_i\mathcal{P}_i-\mathcal{V},\mathcal{P}_\nu\Big)_{\mathcal{Y},\mathrm{fin}}=\\
&\qquad(\operatorname{div}(\varphi^*f)-b_\nu\mathcal{P}_\nu,\mathcal{P}_\nu)_{\mathcal{Y},\mathrm{fin}}-\sum_{i\neq\nu}(b_i\mathcal{P}_i+\mathcal{V},\mathcal{P}_\nu)_{\mathcal{Y},\mathrm{fin}}\,.
\end{aligned}$$

Therefore the equation

$$\begin{aligned}
(\operatorname{div}(\varphi^*f)-b_\nu\mathcal{P}_\nu,\mathcal{P}_\nu)_{\mathcal{X},\mathrm{fin}}=&[k(P_\nu):k(P)]\lim_{Q\to P}(\log|f(Q)|-\log|z_{\mathcal{P}}(Q)|)+\\
&\sum_{i\neq\nu}(b_i\mathcal{P}_i+\mathcal{V},\mathcal{P}_\nu)_{\mathcal{Y},\mathrm{fin}}
\end{aligned}$$

holds. By equation (1.5) we have

$$\begin{aligned}
\lim_{Q_\nu\to P_\nu}(\log|\varphi^*f(Q_\nu)|-b_\nu\log|z_{\mathcal{P}_\nu}(Q_\nu)|)=&\lim_{Q_\nu\to P_\nu}(\log|\varphi^*f(Q_\nu)|-\log|\varphi^*z_{\mathcal{P}}(Q_\nu)|)-\\
&b_\nu\log|\alpha_\nu|
\end{aligned}$$

on $\mathcal{Y}_\infty$. It remains to show

$$\lim_{Q_\nu\to P_\nu}(\log|\varphi^*f(Q_\nu)|-\log|\varphi^*z_{\mathcal{P}}(Q_\nu)|)=[k(P_\nu):k(P)]\lim_{Q\to P}(\log|f(Q)|-\log|z_{\mathcal{P}}(Q)|)\,.$$

But this is clear, since the valuations of $\varphi^*f(Q_\nu)=f\circ\varphi(Q_\nu)=f(Q)$ and $\varphi^*z_{\mathcal{P}}(Q_\nu)=z_{\mathcal{P}}(Q)$ do not depend on the extensions $\tau:k(P_\nu)\hookrightarrow\mathbb{C}$ of $\sigma:k(P)\hookrightarrow\mathbb{C}$. □

Remark 1.7. Here we note a crucial fact about how arithmetic local coordinates transform under base change morphisms: Let $\mathcal{P}$ be a horizontal prime divisor on an arithmetic surface $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_K$ such that the generic point P is K-rational. Let $z_{\mathcal{P}}$ be an arithmetic local coordinate in $\mathcal{P}$. For a number field $L\supset K$ let $\widetilde{\mathcal{X}}$ be the arithmetic surface over $\operatorname{Spec}\mathcal{O}_L$ defined by a desingularization of $\mathcal{X}\times_{\operatorname{Spec}\mathcal{O}_K}\operatorname{Spec}\mathcal{O}_L$. Assume that the divisor $\widetilde{\mathcal{P}}$ on $\widetilde{\mathcal{X}}$, which is induced by $\mathcal{P}$, is a horizontal prime divisor. Then by Proposition 1.6, $z_{\mathcal{P}}$ is also an arithmetic local coordinate in $\widetilde{\mathcal{P}}$.

Remark 1.8. In the next proposition we will calculate arithmetic local coordinates in horizontal prime divisors on $\mathbb{P}^1_{\mathbb{Z}}$. Then we can use Proposition 1.6 to construct arithmetic local coordinates in horizontal prime divisors on arithmetic surfaces which cover $\mathbb{P}^1_{\mathbb{Z}}$.

Proposition 1.9. *Let $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}\mathbb{Z}[x_0, x_1]$ be equipped with the usual homogeneous coordinates x_0 and x_1. Then the following hold:*
i) Let $\mathcal{P} = (a : b) \in \mathcal{X}(\mathbb{Z})$ with $b \neq 0$. On the chart $U = \{z = x_0/x_1 \in \mathbb{C}\}$ on the projective line $\mathcal{X}_\infty = \mathbb{P}^1_{\mathbb{C}}$, an arithmetic local coordinate $z_\mathcal{P}$ in $\mathcal{P}$ is given by

$$z_\mathcal{P} = b^2 \left(z - \frac{a}{b}\right).$$

ii) For a number field K, let $\mathbb{P}^1_{\mathcal{O}_K} := \operatorname{Proj}\mathcal{O}_K[x_0, x_1]$ and let $\mathcal{P} = (\alpha : \beta) \in \mathbb{P}^1_{\mathcal{O}_K}(\mathcal{O}_K)$ with $\beta \neq 0$. For $\sigma : K \hookrightarrow \mathbb{C}$ let $|\cdot|_\sigma$ denotes the σ-adic valuation, hence $|\alpha|_\sigma = |\sigma(\alpha)|$ for $\alpha \in K$. On the chart $U = \{z = x_0/x_1 \in \mathbb{C}\}$ on $\mathbb{P}^1_{\mathbb{C}}$, an arithmetic local coordinate $z_\mathcal{P}$ in $\mathcal{P}$ is given by

$$z_\mathcal{P} = \left(|\beta|^2_\sigma \cdot (z - P_\sigma)\right)_\sigma,$$

where $\sum P_\sigma$ denotes the divisor on U, which is induced by $\mathcal{P}(\mathbb{C})$.
Moreover, let P be the generic point of $\mathcal{P}$ and assume that $\sum P_\sigma$ defines a closed point on $\mathbb{P}^1_{\mathbb{Q}}$. Let $\mathcal{P}_\sum$ denote the Zariski closure of $\sum P_\sigma$ in $\mathbb{P}^1_{\mathbb{Z}}$. Then

$$z_{\mathcal{P}_\sum} = (\alpha_\sigma \cdot (z - P_\sigma))_\sigma, \text{ where } \alpha_\sigma := |\beta|^2_\sigma \cdot \prod_{\nu=2}^{n} |\alpha_\nu \beta - \beta_\nu \alpha|_\sigma$$

is an arithmetic local coordinate in $\mathcal{P}_\sum$ on $\mathbb{P}^1_{\mathbb{Z}}$, where $\mathcal{P}_\sum(\mathbb{C}) = \sum_{\nu=1}^n (\alpha_\nu : \beta_\nu)$ is the decomposition of $\mathcal{P}_\sum$ over $\mathbb{P}^1_{\mathbb{C}}$ with $\alpha_1 := \alpha$ and $\beta_1 := \beta$.

Proof. *i)* The divisor $\mathcal{P}$ corresponds to the polynomial $bx_0 - ax_1$. Taking a rational function $f = \frac{bx_0 - ax_1}{dx_0 - cx_1}$ with $bc \neq ad$, the geometric intersection number $(\operatorname{div}(f) - \mathcal{P}, \mathcal{P})_{\text{fin}}$ equals $-\log|R(P,Q)|$, where $R(P,Q)$ is the resultant of the polynomials $P = (dx_0 - cx_1)$ and $Q = (bx_0 - ax_1)$, see [Li1, Theorem 2.1.1]. Thus we have

$$(\operatorname{div}(f) - \mathcal{P}, \mathcal{P})_{\text{fin}} = -\log|bc - ad|.$$

The induced rational function f on $\mathcal{X}_\infty$ is given by $f = \frac{bz-a}{dz-c}$ on the chart U. Considering a point $Q_\infty = \frac{a}{b} + \varepsilon$ close to $\mathcal{P}(\mathbb{C}) = P_\infty = \frac{a}{b}$, we have to calculate the limit

$$\lim_{Q_\infty \to P_\infty} \left(\log|f(Q_\infty)| - \log|z_\mathcal{P}(Q_\infty)|\right).$$

The above expression equals

$$\log\left|\frac{b\left(\frac{a}{b} + \varepsilon\right) - a}{d\left(\frac{a}{b} + \varepsilon\right) - c}\right| - \log\left|b^2\left(\frac{a}{b} + \varepsilon - \frac{a}{b}\right)\right| = \log\left|\frac{b^2\varepsilon}{da - cb + bd\varepsilon}\right| - \log\left|b^2\varepsilon\right|$$

as $\varepsilon \longrightarrow 0$. Cancelling $b^2\varepsilon$ and setting $\varepsilon = 0$, we see that the term equals the geometric intersection number $-\log|bc-ad|$.
ii) We show that

$$z_{\mathcal{P}} = \left(|\beta|_\sigma^2 \cdot (z - P_\sigma)\right)_\sigma$$

is an arithmetic local coordinate in $\mathcal{P}$ on the arithmetic surface $\mathbb{P}^1_{\mathcal{O}_K}$. The divisor $\mathcal{P}$ is given by the polynomial $\beta x_0 - \alpha x_1$. If we take the function $f = \frac{\beta x_0 - \alpha x_1}{\delta x_0 - \gamma x_1}$ for $\gamma, \delta \in \mathcal{O}_K$ with $\beta\gamma \neq \alpha\delta$, then the equation

$$(\operatorname{div}(f) - \mathcal{P}, \mathcal{P})_{\text{fin}} = -\log\left|\operatorname{Nm}_{K|\mathbb{Q}}(\beta\gamma - \alpha\delta)\right|$$

holds, see [La, p.56]. Considering $z_{\mathcal{P}}$ induced by the rational function $z_P := \frac{\beta(\beta x_0 - \alpha x_1)}{x_1}$, where P is the generic point of $\mathcal{P}$, then over $\left(\mathbb{P}^1_{\mathcal{O}_K}\right)_\infty$ we have

$$\lim_{Q\to P}\left(\log|f(Q)| - \log|z_{\mathcal{P}}(Q)|\right) = \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log\left|\frac{f}{z_P}(P)\right|_\sigma = -\sum_{\sigma:K\hookrightarrow\mathbb{C}} \log|\beta\gamma - \alpha\delta|_\sigma .$$

The definition of the σ-adic valuation and the norm of algebraic numbers shows the claim. Now we consider the divisor $\mathcal{P}_\sum$. Over $\mathbb{P}^1_{\mathcal{O}_K}$ we have the decomposition $\mathcal{P}_\sum = \sum_{\nu=1}^n \mathcal{P}_\nu$, where each divisor $\mathcal{P}_\nu := (\alpha_\nu : \beta_\nu)$ is a conjugate of the divisor $(\alpha : \beta)$. Now using the fact that

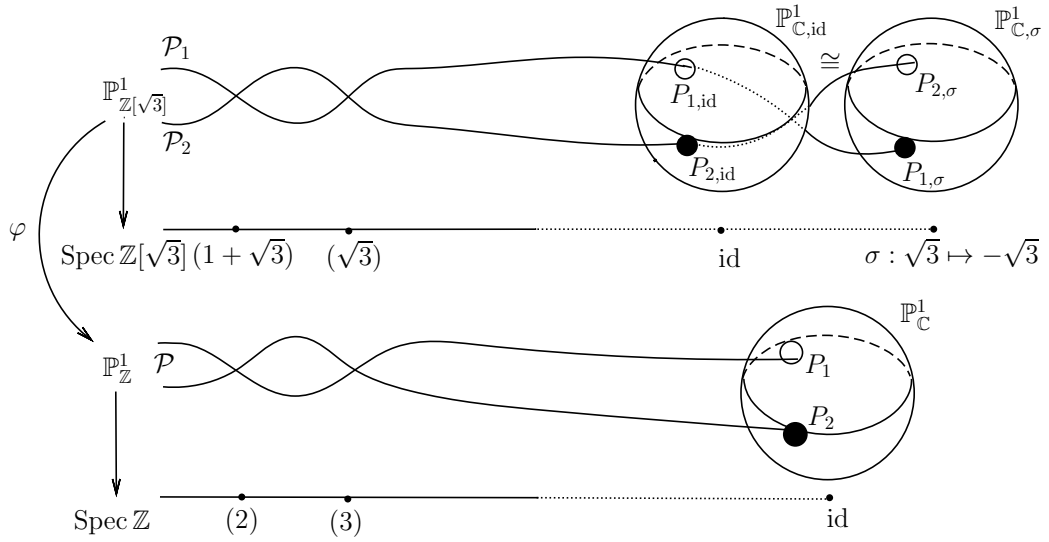
$$(\mathcal{P}_\nu, \mathcal{P})_{\text{fin}} = \log\left|\operatorname{Nm}_{K|\mathbb{Q}}(\alpha_\nu\beta - \beta_\nu\alpha)\right|$$

holds for $\nu \neq 1$, Proposition 1.6 shows that

$$z_{\mathcal{P}_\sum} = (\alpha_\sigma \cdot (z - P_\sigma))_\sigma , \text{ where } \alpha_\sigma := |\beta|_\sigma^2 \cdot \prod_{\nu=2}^n |\alpha_\nu\beta - \beta_\nu\alpha|_\sigma ,$$

is an arithmetic local coordinate in $\mathcal{P}_\sum$ on $\mathbb{P}^1_{\mathbb{Z}}$. □

Example 1.10. We consider the arithmetic surface $\mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}\mathbb{Z}[x_0, x_1]$ and the divisor $\mathcal{P}$, given by the polynomial $x_0^2 - 3x_1^2$. Over $\mathbb{P}^1_{\mathbb{Z}[\sqrt{3}]}$ we have the decomposition $\mathcal{P} = \mathcal{P}_1 + \mathcal{P}_2$, where $\mathcal{P}_1$ is associated to $x_0 - \sqrt{3}x_1$ and $\mathcal{P}_2$ to $x_0 + \sqrt{3}x_1$, respectively. Moreover, let $\varphi : \mathbb{P}^1_{\mathbb{Z}[\sqrt{3}]} \longrightarrow \mathbb{P}^1_{\mathbb{Z}}$ denotes the projection. This situation is visualized in the following picture.



Now if $z_{\mathcal{P}} = (z_{P_1}, z_{P_2})$ is an arithmetic local coordinate in $\mathcal{P}$ on $\mathbb{P}^1_{\mathbb{Z}}$, where z_{P_1} and z_{P_2} are local coordinates on $\mathbb{P}^1_{\mathbb{C}}$, then the pullback of z_{P_1} under φ defines local coordinates $z_{P_{1,\mathrm{id}}}$ and $z_{P_{1,\sigma}}$ in the points $P_{1,\mathrm{id}} = \left(\sqrt{3}, 1\right)$ and $P_{1,\sigma} = \left(-\sqrt{3}, 1\right)$ on the isomorphic Riemann surfaces $\mathbb{P}^1_{\mathbb{C},\mathrm{id}} \cong \mathbb{P}^1_{\mathbb{C},\sigma}$. Up to constants α_{id} and α_σ, depending on the geometric intersection number of $\mathcal{P}_1$ and $\mathcal{P}_2$, this defines an arithmetic local coordinate $z_{\mathcal{P}_1} = \left(\alpha_{\mathrm{id}} z_{P_{1,\mathrm{id}}}, \alpha_\sigma z_{P_{1,\sigma}}\right)$ in $\mathcal{P}_1$ on $\mathbb{P}^1_{\mathbb{Z}[\sqrt{3}]}$.

1.3 Examples on the modular curve $\mathcal{X}(1)$

A good introduction into this topic can be found in [Sh2].
Let $\mathbb{H} = \{\tau = x + iy \in \mathbb{C} \big| y > 0\}$ denote the upper half plane and Γ a finite index subgroup of the full modular group

$$\Gamma(1) = \mathrm{PSL}_2(\mathbb{Z}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \Big| a, b, c, d \in \mathbb{Z} \text{ with } ad - bc = 1 \right\} / \pm \mathbb{1}.$$

Important examples are given by *congruence subgroups*, i.e. subgroups $\Gamma < \Gamma(1)$ such that there exists $N \in \mathbb{N}_{>0}$ with

$$\Gamma(N) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma(1) \Big| \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \equiv \mathbb{1} \mod N \right\} \subset \Gamma.$$

The modular curve $X(\Gamma)$ is the quotient $\Gamma \backslash \mathbb{H} \cup \mathbb{P}^1_{\mathbb{Q}}$, given by the Möbius transformations

$$\gamma(\tau) = \frac{a\tau + b}{c\tau + d} \quad \text{and} \quad \gamma\left(\frac{m}{n}\right) = \frac{am + bn}{cm + dn}$$

with $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $\tau \in \mathbb{H}$ and $\frac{m}{n} \in \mathbb{P}^1_{\mathbb{Q}}$. Its natural topology is induced by this quotient. For $\tau \in \mathbb{H} \cup \mathbb{P}^1_{\mathbb{Q}}$ let $\Gamma_\tau = \{\gamma \in \Gamma \big| \gamma(\tau) = \tau\}$ denote the stabilizer of τ in Γ and $n_\tau = |\Gamma_\tau|$ its order. A local coordinate in a point P_{τ_0} on $X(\Gamma)$ that corresponds to the orbit of $\tau_0 \in \mathbb{H}$ is given by

$$t_{\tau_0} = \omega_{\tau_0}^{n_{\tau_0}}, \text{ where } \omega_{\tau_0} = \frac{\tau - \tau_0}{\tau - \overline{\tau_0}}. \tag{1.6}$$

If $n_{\tau_0} > 1$, the point P_{τ_0} will be called *elliptic*, otherwise *non-elliptic*. The cusps S_{τ_0} on $X(\Gamma)$ correspond to the orbits of $\tau_0 \in \mathbb{P}^1_{\mathbb{Q}}$ and a local coordinate in S_{τ_0} is given by $q_{\tau_0} = \mathrm{e}^{2\pi i \gamma(\tau)/b}$, where $\gamma \in \Gamma$ satisfies $\gamma(\tau_0) = i\infty$ and $b = [\Gamma(1)_{\tau_0} : \Gamma_{\tau_0}]$ denotes the width of the cusp S_{τ_0}.

Let $\mathcal{M}_k(\Gamma)$ denote the space of meromorphic modular forms of weight $k \in \mathbb{Z}$ associated to a finite index subgroup $\Gamma < \Gamma(1)$. Thus an element $f \in \mathcal{M}_k(\Gamma)$ is a meromorphic function $f : \mathbb{H} \to \mathbb{C}$ that satisfies the functional equation

$$f(\tau)|_k\gamma := (c\tau + d)^{-k} f(\gamma\tau) = f(\tau) \quad \text{for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$$

and is meromorphic at the cusps of Γ. More precisely, for the local coordinates $q_{\tau_0} = \mathrm{e}^{2\pi i\gamma(\tau)/b}$ in the cusps of Γ we have $f(\tau) = \sum_{n=n_0}^{\infty} a_n(f) q_{\tau_0}^n$ where $n_0 \in \mathbb{Z}$ and $a_n(f) \in \mathbb{C}$. Here we note that meromorphic modular forms are rational sections of a line bundle on $X(\Gamma)$, which is called the line bundle of meromorphic modular forms and will also be denoted by $\mathcal{M}_k(\Gamma)$.

Furthermore, consider the Shimura-Maass differential operator ∂ defined by

$$\partial f = \frac{1}{2\pi i}\left(\frac{df}{d\tau} + \frac{kf}{\tau - \overline{\tau}}\right)$$

for $f \in \mathcal{M}_k(\Gamma)$. Inductively we set $\partial^0 f = f$ and $\partial^n f = \partial^{n-1} \circ \partial f$ for $n > 0$. The advantage of this differential operator is that if $f \in \mathcal{M}_k(\Gamma)$ is a meromorphic modular form of weight k, then ∂f transforms like a meromorphic modular form of weight $k+2$, namely the equation $\partial f(\tau)|_{k+2}\gamma = \partial f(\tau)$ holds for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, see [Za, p.51].

Let $\widetilde{\mathcal{M}}_k(\Gamma)$ be the space consisting of the meromorphic modular forms $f \in \mathcal{M}_k(\Gamma)$ and the elements of the form $\partial^r f$ where $f \in \mathcal{M}_{k-2r}(\Gamma)$. For $f \in \widetilde{\mathcal{M}}_k(\Gamma)$, the Petersson metric $\|\cdot\|_{\mathrm{Pet}}$ is defined by

$$\|f(\tau)\|^2_{\mathrm{Pet}} = |f(\tau)|^2 (4\pi \mathrm{Im}(\tau))^k.$$

Note that the Petersson metric defines a hermitian metric on the line bundle of meromorphic modular forms, which is singular only at the elliptic points and the cusps of $X(\Gamma)$, see [Kü3, Proposition 4.9.].

Now, note that $X(\Gamma)$ is an algebraic curve over some number field K. Thus we can calculate arithmetic local coordinates in horizontal prime divisors on a regular model $\mathcal{X}(\Gamma)$ of $X(\Gamma)$.
We start with the cusp of the modular curve $X(\Gamma(1))$.

Proposition 1.11. *Let $X(1) = X(\Gamma(1)) = \mathbb{P}^1_{\mathbb{Q}}$ and let $\mathcal{X}(1)$ denote the regular model $\mathbb{P}^1_{\mathbb{Z}}$ of $X(1)$. Let S_∞ denote the cusp of $X(1)$ and let $\mathcal{S}_\infty$ denote the closure of S_∞ in $\mathcal{X}(1)$. Then an arithmetic local coordinate $z_{\mathcal{S}_\infty}$ in $\mathcal{S}_\infty$ is given by*

$$z_{\mathcal{S}_\infty} = q = \mathrm{e}^{2\pi i\tau}.$$

Proof. Let j be the modular invariant. The Fourier-expansion of j is given by $j(q) = \frac{1}{q} + 744 + \sum_{n>0} a_n q^n$ for some $a_n \in \mathbb{Z}$. Since $\mathcal{X}(1)$ is the compactification of the j-line $\mathcal{Y} = \operatorname{Spec}\mathbb{Z}[j]$, we have $j \in k(\mathcal{X}(1))^\times$. The order of j in S_∞ is given by $\operatorname{ord}_{S_\infty}(j) = -1$. Let $\mathcal{P}_\rho$ denotes the Zariski closure of the elliptic point P_ρ of order 3 in $\mathcal{X}(1)$. Because P_ρ has potential good reduction, see [Si, Proposition 5.5, p.181], it follows that $\mathcal{P}_\rho$ does not intersect $\mathcal{S}_\infty$ on $\mathcal{X}(1)$. Hence we obtain

$$(\operatorname{div}(j) + \mathcal{S}_\infty, \mathcal{S}_\infty)_{\mathrm{fin}} = (\mathcal{P}_\rho, \mathcal{S}_\infty)_{\mathrm{fin}} = 0.$$

With the help of the Fourier series of j we find

$$\lim_{\tau\to i\infty} (\log|j(\tau)| + \log|z_{\mathcal{S}_\infty}(\tau)|) = \lim_{q\to 0}(\log|j(q)| + \log|q|) = \lim_{q\to 0}(\log|1 + \mathcal{O}(q)|) = 0,$$

which proves the proposition. □

Now we will come to the calculation of arithmetic local coordinates in non-cuspidal points on $X(1)$.

Proposition 1.12. *Let $P_{\tau_0} \in X(1)$ be a non-elliptic point defined over some number field K. Let $\mathcal{P}_{\tau_0}$ be the Zariski closure of P_{τ_0} in $\mathcal{X}(1)_{\mathcal{O}_K} := \mathcal{X}(1) \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathcal{O}_K$ and set $\mathcal{P}_{\tau_0}(\mathbb{C}) = \sum_{i=0}^{[K:\mathbb{Q}]-1} P_{\tau_i}$. Then there is a family of complex numbers $\alpha = (\alpha_i)_i$ with $\log|\alpha| = (\mathcal{S}_\infty, \mathcal{P}_{\tau_0})_{\mathcal{X}(1)_{\mathcal{O}_K},\mathrm{fin}}$ such that*

$$z_{\mathcal{P}_{\tau_0}} = \left(\alpha_i \left\|\partial j(\tau_i)\right\|_{\mathrm{Pet}} t_{\tau_i}\right)_i$$

is an arithmetic local coordinate in $\mathcal{P}_{\tau_0}$, where t_{τ_i} is defined by equation (1.6).

Proof. Consider the rational function $j(\tau) - j(\tau_0)$ on $\mathcal{X}(1)_{\mathcal{O}_K}$. Since $\operatorname{div}(j(\tau) - j(\tau_0)) = \mathcal{P}_{\tau_0} - \mathcal{S}_\infty$, where $\mathcal{S}_\infty$ is the Zariski closure of the cusp S_∞ on $X(1)_K := X(1) \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} K$, there exists a family of algebraic numbers α such that

$$(\operatorname{div}(j(\tau) - j(\tau_0)) - \mathcal{P}_{\tau_0}, \mathcal{P}_{\tau_0})_{\text{fin}} = (-\mathcal{S}_\infty, \mathcal{P}_{\tau_0})_{\text{fin}} = -\log|\alpha|\,. \tag{1.7}$$

On the induced complex manifold $\mathcal{X}(1)_{\mathcal{O}_K}(\mathbb{C})$ we have to compute the limit

$$\lim_{\tau \to \tau_0} \left(\log|j(\tau) - j(\tau_0)| - \log\left|z_{\mathcal{P}_{\tau_0}}(\tau)\right| \right),$$

which is given by

$$\sum_i \lim_{\tau \to \tau_i} \left(\log|j(\tau) - j(\tau_i)| - \log|\alpha_i \, \|\partial j(\tau_i)\|_{\text{Pet}} \, t_{\tau_i}| \right).$$

It suffices to show that $\log|j(\tau) - j(\tau_i)|$ equals $\log\left| \|\partial j(\tau_i)\|_{\text{Pet}} \frac{\tau - \tau_i}{\tau - \overline{\tau_i}} \right|$ in the limit $\tau \to \tau_i$, or equivalently

$$\log\left| \lim_{\tau \to \tau_i} \frac{j(\tau) - j(\tau_i)}{\tau - \tau_i} \right| = \log\left| \frac{\|\partial j(\tau_i)\|_{\text{Pet}}}{\tau_i - \overline{\tau_i}} \right|. \tag{1.8}$$

The definition of the Petersson metric $\|\partial j(\tau_i)\|_{\text{Pet}} = 4\pi \operatorname{Im}(\tau_i)\,|\partial j(\tau_i)|$ and the Shimura-Maass operator $\partial j = \frac{1}{2\pi i} \frac{\mathrm{d}j}{\mathrm{d}\tau}$ yield $\|\partial j(\tau_i)\|_{\text{Pet}} = \left|(\tau_i - \overline{\tau_i}) \frac{\mathrm{d}j}{\mathrm{d}\tau}(\tau_i)\right|$ and hence (1.8). □

Important algebraic points on $X(1)$ are Heegner points of discriminant D, see the definition in [GZ]. Note that the number of Heegner points $P_{\tau_0} \in X(1)$ of discriminant D equals the class number $h(D) = \#Cl_K$ of $K = \mathbb{Q}(\sqrt{D})$ and that the residue field of P_{τ_0} is isomorphic to the Hilbert class field $H = K(j(\tau_0))$ of K.

Corollary 1.13. *Consider the situation as in Proposition 1.12, where P_{τ_0} is a non-elliptic Heegner point and H is the Hilbert class field. Then we can set $\alpha = (\alpha_i = 1)_i$, where $i = 1, \ldots, [H : \mathbb{Q}]$. In other words*

$$z_{\mathcal{P}_{\tau_0}} = \left(\|\partial j(\tau_i)\|_{\text{Pet}} \, t_{\tau_i} \right)_i$$

is an arithmetic local coordinate in the closure $\mathcal{P}_{\tau_0}$ in $\mathcal{X}(1)_{\mathcal{O}_H}$ of the point P_{τ_0}.

Proof. By [Si, Proposition 5.5, p.181] and [Za, Corollary, p.71] any Heegner point has potential good reduction. It follows that $\mathcal{P}_{\tau_0}$ does not intersect the cusp $\mathcal{S}_\infty$ on $\mathcal{X}(1)_{\mathcal{O}_H}$. Hence equation (1.7) vanishes. □

Now we want to calculate the derivatives of meromorphic modular forms more explicitly. Note that Rankin showed in [Ra, Theorem 4.3.4] that any meromorphic modular form of even weight $f \in \mathcal{M}_{2k}(\Gamma)$ associated to a finite index subgroup $\Gamma < \Gamma(1)$ can be written as

$$f(\tau) = (\partial j(\tau))^k \frac{P(j(\tau))}{Q(j(\tau))},$$

where P and Q are polynomials in $\mathbb{C}[j(\tau)]$.

Proposition 1.14. *Let $\mathrm{E}_4(\tau)$ and $\mathrm{E}_6(\tau)$ denote the classical Eisenstein series of weight 4 and 6 respectively, and let $\Delta(\tau)$ denotes the modular discriminant (of weight 12). Then*

$$\partial j(\tau) = -\left(\frac{\mathrm{E}_4^2\mathrm{E}_6}{\Delta}\right)(\tau).$$

In particular, an arithmetic local coordinate in the Zariski closure $\mathcal{P}_{\tau_0}$ of a non-elliptic Heegner point P_{τ_0} is given by

$$z_{\mathcal{P}_{\tau_0}} = \left(\left\|\frac{\mathrm{E}_4^2\mathrm{E}_6}{\Delta}(\tau_i)\right\|_{\mathrm{Pet}} t_{\tau_i}\right)_i.$$

Proof. The function $\frac{\mathrm{E}_4^2\mathrm{E}_6}{\Delta}(\tau)$ is the unique meromorphic modular form in $\mathcal{M}_2(\Gamma(1))$ that is holomorphic in $\mathbb{H}$ and has a Fourier series of the form $\frac{1}{q} + \mathbb{Z}[[q]]$, see [AKN, Section 3]. Since ∂j is holomorphic in $\mathbb{H}$ and has a Fourier expansion $(\partial j)(\tau) = -\frac{1}{q} + \mathbb{Z}[[q]]$, the result follows using Corollary 1.13. □

The case that the Heegner points P_{τ_0} are elliptic can be treated similarly.

Proposition 1.15. *Let $\mathcal{P}_i$ and $\mathcal{P}_\rho$ be the horizontal prime divisors on $\mathcal{X}(1)$ coming from the elliptic Heegner points P_i and P_ρ, respectively. Then*

$$z_{\mathcal{P}_i} = \frac{1}{2}\left\|\partial^2 j(i)\right\|_{\mathrm{Pet}} t_i \quad \text{and} \quad z_{\mathcal{P}_\rho} = \frac{1}{6}\left\|\partial^3 j(\rho)\right\|_{\mathrm{Pet}} t_\rho$$

are arithmetic local coordinates in $\mathcal{P}_i$ and $\mathcal{P}_\rho$, resp.
Moreover, we can write

$$z_{\mathcal{P}_i} = 2^4 3^3 \left\|E_4(i)\right\|_{\mathrm{Pet}} t_i \quad \text{and} \quad z_{\mathcal{P}_\rho} = 2^6 \left\|\mathrm{E}_6(\rho)\right\|_{\mathrm{Pet}} t_\rho.$$

Proof. Because the Heegner points P_i and P_ρ have potential good reduction ([Si, Proposition 5.5, p.181]) and because of $j(i) = 12^3$ and $j(\rho) = 0$, we are left to prove

$$\begin{aligned} 0 &= \lim_{\tau\to i}\left(\log\left|j(\tau)-12^3\right| - \log\left|\frac{1}{2}\left\|\partial^2 j(i)\right\|_{\mathrm{Pet}} t_i\right|\right) \text{ and} \\ 0 &= \lim_{\tau\to\rho}\left(\log|j(\tau)| - \log\left|\frac{1}{6}\left\|\partial^3 j(\rho)\right\|_{\mathrm{Pet}} t_\rho\right|\right). \end{aligned} \tag{1.9}$$

Since j is holomorphic in ω_i and ω_ρ (see [Ra, equation 4.1.17]), it follows from [Za, Proposition 17] that the values $\partial j(i)$, $\partial j(\rho)$ and $\partial^2 j(\rho)$ vanish. Then the usual derivation shows

$$\begin{aligned} \lim_{\tau\to i}\frac{j(\tau)-12^3}{(\tau-i)^2} &= \frac{1}{2}\frac{\mathrm{d}^2 j}{\mathrm{d}\tau^2}(i) = \frac{(2\pi i)^2}{2}\partial^2 j(i) \text{ and} \\ \lim_{\tau\to\rho}\frac{j(\tau)}{(\tau-\rho)^3} &= \frac{1}{6}\frac{\mathrm{d}^3 j}{\mathrm{d}\tau^3}(\rho) = \frac{(2\pi i)^3}{6}\partial^3 j(\rho). \end{aligned}$$

The definition of the Petersson metric yields

$$\left\|\partial^2 j(i)\right\|_{\mathrm{Pet}} = \left|2\frac{\mathrm{d}^2 j}{\mathrm{d}\tau^2}(i)\right| \text{ and } \left\|\partial^3 j(\rho)\right\|_{\mathrm{Pet}} = \left|2\mathrm{Im}(\rho)\frac{\mathrm{d}^3 j}{\mathrm{d}\tau^3}(\rho)\right|.$$

The desired limit formulas follow from the local coordinates $t_i = \left(\frac{\tau-i}{\tau+i}\right)^2$ and $t_\rho = \left(\frac{\tau-\rho}{\tau-\overline{\rho}}\right)^3$. To calculate the higher derivatives of the modular invariant $j(\tau)$, we can use the definition $\partial^n j = \partial\circ\partial^{n-1} j$, where $\partial j = -\mathrm{E}_4^2\mathrm{E}_6/\Delta$. After a long calculation, using the classical derivation of the Eisenstein series

$$\frac{1}{2\pi i}\frac{\mathrm{dE}_2}{\mathrm{d}\tau} = \frac{\mathrm{E}_2^2-\mathrm{E}_4}{12}, \quad \frac{1}{2\pi i}\frac{\mathrm{dE}_4}{\mathrm{d}\tau} = \frac{\mathrm{E}_2\mathrm{E}_4-\mathrm{E}_6}{3}, \quad \frac{1}{2\pi i}\frac{\mathrm{dE}_6}{\mathrm{d}\tau} = \frac{\mathrm{E}_2\mathrm{E}_6-\mathrm{E}_4^2}{2},$$

where E_2 is the holomorphic Eisenstein series of weight 2 (see [Za, Proposition 7] and [Za, Proposition 15]) and using the equations $\mathrm{E}_4^3 = \Delta j$ and $\mathrm{E}_6^2 = \Delta(j-1728)$, we derive the following expressions:

$$\begin{aligned} \partial^2 j &= \frac{1}{6}\mathrm{E}_2^*\partial j + \mathrm{E}_4\left(\frac{7}{6}j - 1152\right) \text{ and} \\ \partial^3 j &= \frac{1}{6}\mathrm{E}_2^*\partial^2 j + \frac{1}{72}\left(84\mathrm{E}_4 + \left(\mathrm{E}_2^*\right)^2\right)\partial j + \left(\mathrm{E}_2^*\mathrm{E}_4 - \mathrm{E}_6\right)\left(\frac{7}{18}j - 384\right). \end{aligned}$$

Here $\mathrm{E}_2^* = \mathrm{E}_2 - \frac{3}{\pi\mathrm{Im}(\tau)}$ is the modular Eisenstein series of weight 2. We see immediately that $\partial^2 j(i) = 2^5 3^3 E_4(i)$ and $\partial^3 j(\rho) = 2^7 3\mathrm{E}_6(\rho)$ hold, since $\partial j(i)$, $\partial j(\rho)$ and $\partial^2 j(\rho)$ vanish. □

In the final part of this section we want to give an application of arithmetic local coordinates, namely a normalization condition in the theory of power series expansion of meromorphic modular forms.

Proposition 1.16. *Let P_{τ_0} be a non-elliptic Heegner point on $X(1)$ and consider the local coordinate $z_{P_{\tau_0}} = \|\partial j(\tau_0)\|_{\mathrm{Pet}}\, t_{\tau_0}$ in P_{τ_0}. Then there exists a constant $\Omega_K \in \mathbb{C}^\times$ depending only on $K = \mathbb{Q}(\tau_0)$ such that all meromorphic modular forms $f \in \mathcal{M}_k(\Gamma)$ associated to a congruence subgroup $\Gamma < \Gamma(1)$, which have algebraic Fourier coefficients and are holomorphic in the point $\tau_0 \in \mathbb{H}$, can be written as*

$$\left(\frac{\tau - \overline{\tau_0}}{\tau_0 - \overline{\tau_0}}\right)^k f(\tau) = \sum_{n=0}^{\infty} \alpha_n z_{P_{\tau_0}}^n,$$

where $\alpha_n \in \Omega_K^k \overline{\mathbb{Q}}$ are algebraic numbers up to the factor Ω_K^k.
In particular the coefficients α_n of modular functions ($k = 0$) are algebraic.

Proof. It was shown in [Za, Proposition 17] and [OR, equation 3.10] that

$$\left(\frac{\tau - \overline{\tau_0}}{\tau_0 - \overline{\tau_0}}\right)^k f(\tau) = \sum_{n=0}^{\infty} \frac{(-4\pi \mathrm{Im}(\tau_0))^n}{n!} (\partial^n f)(\tau_0) t_{\tau_0}^n$$

for all $f \in \mathcal{M}_k(\Gamma)$, which are holomorphic in the point $\tau_0 \in \mathbb{H}$. Using the definition of the local coordinate $z_{P_{\tau_0}}$, we obtain

$$\left(\frac{\tau - \overline{\tau_0}}{\tau_0 - \overline{\tau_0}}\right)^k f(\tau) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{(\partial^n f)(\tau_0)}{|\partial j(\tau_0)|^n} z_{P_{\tau_0}}^n.$$

By [Sh1, Main Theorem I] there exists a constant $\Omega_K \in \mathbb{C}^\times$, depending only on K such that

$$\frac{(\partial^n f)(\tau_0)}{\Omega_K^{k+2n}} \in \overline{\mathbb{Q}}$$

for all $n \in \mathbb{N}$ and all $f \in \mathcal{M}_k(\Gamma)$, which have algebraic Fourier coefficients. For instance, we can define Ω_K as the Chowla-Selberg period (see [Za, equation 97]), i.e. the product of suitable gamma factors. Thus for these $f \in \mathcal{M}_k(\Gamma)$ there is a constant $\Omega_K \in \mathbb{C}^\times$ such that

$$\frac{(\partial^n f)(\tau_0)}{|\partial j(\tau_0)|^n} \in \Omega_K^k \overline{\mathbb{Q}}$$

for all $n \in \mathbb{N}$. The proof follows. □

1.4 Intersection theory with a tangent vector

In this section we review the *intersection theory with a tangent vector*, which can be found in [Gr], [GZ] and [Co], and compare it with the theory of arithmetic local coordinates.
In the remainder, let $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ be an arithmetic surface with generic fibre X and let $\mathcal{P}$ always be the horizontal prime divisor coming from a K-rational point $P \in X(K)$.

Definition 1.17. Let $f \in k(\mathcal{X})^\times$ be a rational function and let $z_P \in \widehat{\mathcal{O}}_{\mathcal{X},P}$ be a formal local parameter in P. Then the *modified value of f in P* is the number defined by

$$f_{z_P}[P] := \frac{f}{z_P^{\operatorname{ord}_{\mathcal{P}}(f)}}(P) \in K^\times.$$

Remark 1.18. To be more concret, let us consider a local parameter $t_P \in k(\mathcal{X})^\times$ in P. Then we have an isomorphism $\widehat{\mathcal{O}}_{\mathcal{X},P} \cong K[[t_P]]$ and the formal local parameter z_P is a power series $z_P = \alpha t_P + \sum_{i\geq 2} \alpha_i t_P^i$ in t_P with $\alpha \in K^\times$ and $\alpha_i \in K$. It is clear that the modified value of f in P only depends on the first coefficient of the power series expansion in t_P. In other words, taking the formal local parameter $z_P = \alpha t_P + \sum_{i\geq 2} \alpha_i t_P^i$ yields $f_{z_P}[P] = f_{t_P}[P]$ if and only if $\alpha = 1$.

For a closed point $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ let $|\cdot|_\mathfrak{p}$ be the $\mathfrak{p}$-adic valuation $|\alpha|_\mathfrak{p} = \operatorname{Nm}(\mathfrak{p})^{-\nu_\mathfrak{p}(\alpha)}$ for $\alpha \in K$, normalized such that the product formula holds. Moreover, let $(\cdot,\cdot)_\mathfrak{p}$ denotes the local arithmetic intersection number over the special fibre $\mathcal{X}_\mathfrak{p}$. This is for two horizontal prime divisors $\mathcal{P}_1, \mathcal{P}_2 \in \widehat{\mathrm{Z}}^1(\mathcal{X})$ with $\mathcal{P}_1(\mathbb{C}) \cap \mathcal{P}_2(\mathbb{C}) = \emptyset$ given by

$$(\mathcal{P}_1, \mathcal{P}_2)_\mathfrak{p} = \sum_{\substack{x \in \mathcal{X} \\ \pi(x) = \mathfrak{p}}} \log \#\big(\mathcal{O}_{\mathcal{X},x} \big/ (\mathcal{P}_{1,x}, \mathcal{P}_{2,x})\big),$$

where $\mathcal{P}_{1,x}$ (resp. $\mathcal{P}_{2,x}$) is a local equation for $\mathcal{P}_1$ (resp. $\mathcal{P}_2$) at the closed point $x \in \mathcal{X}$. So we have

$$(\mathcal{P}_1, \mathcal{P}_2)_{\mathrm{fin}} = \sum_{(0)\neq\mathfrak{p}\in\operatorname{Spec}\mathcal{O}_K} (\mathcal{P}_1, \mathcal{P}_2)_\mathfrak{p}.$$

Definition 1.19. For a formal local parameter $z_P \in \widehat{\mathcal{O}}_{\mathcal{X},P}$ in P we define the *z_P-regularized local self-intersection number $(\mathcal{P}, \mathcal{P})_{\mathfrak{p},z_P}$ of $\mathcal{P}$* by

$$(\mathcal{P}, \mathcal{P})_{\mathfrak{p},z_P} := (\mathcal{P} - \operatorname{div}(f), \mathcal{P})_\mathfrak{p} - \log |f_{z_P}[P]|_\mathfrak{p},$$

where $f \in k(\mathcal{X})^\times$ is a local parameter in P.

Note that the definition is independent of the rational function f by the basic properties of the local arithmetic intersection number. Moreover, consider two formal local parameters z_P and z'_P such that $z_P = \alpha z'_P + \mathcal{O}(z'^2_P)$. Then the equality

$$(\mathcal{P}, \mathcal{P})_{\mathfrak{p},z_P} = (\mathcal{P}, \mathcal{P})_{\mathfrak{p},z'_P} + \log |\alpha|_\mathfrak{p}$$

holds. Thus the definition of the $\mathfrak{p}$-adic valuation shows $(\mathcal{P}, \mathcal{P})_{\mathfrak{p},z_P} = (\mathcal{P}, \mathcal{P})_{\mathfrak{p},z'_P}$ for all maximal $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ if and only if $\alpha \in \mathcal{O}_K^\times$.

Remark 1.20. Note that the z_P-regularized local self-intersection number permits formal local parameters $z_P \in \widehat{\mathcal{O}}_{\mathcal{X},P}$ that don't necessarily arise from local parameters $t_P \in k(\mathcal{X})^\times$. This generalization is useful for calculating the regularized local self-intersection number explicitly, see Example 1.24 below. However, setting $f = t_P \in k(\mathcal{X})^\times$ in Definition 1.19 yields the simple identity

$$(\mathcal{P},\mathcal{P})_{\mathfrak{p},t_P} = (\mathcal{P} - \operatorname{div}(t_P),\mathcal{P})_{\mathfrak{p}}.$$

Remark 1.21. Using the notation

$$(\mathcal{P},\mathcal{P})_{\mathrm{fin},z_P} := \sum_{(0)\neq\mathfrak{p}\in\operatorname{Spec}\mathcal{O}_K} (\mathcal{P},\mathcal{P})_{\mathfrak{p},z_P},$$

the product formula on K shows

$$(\mathcal{P},\mathcal{P})_{\mathrm{fin},z_P} = (\mathcal{P} - \operatorname{div}(f),\mathcal{P})_{\mathrm{fin}} + \log|f_{z_P}[P]|,$$

where we used the notation $\log|f_{z_P}[P]| = \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log|f_{z_P}[P]|_\sigma$. Thus the arithmetic self-intersection number of an arithmetic divisor $(\mathcal{P},g) \in \widehat{\mathrm{Z}}^1(\mathcal{X})$ equals

$$\begin{aligned}(\mathcal{P},g)^2 &= (\mathcal{P},\mathcal{P})_{\mathrm{fin},z_P} - \log|f_{z_P}[P]| + \frac{1}{2}\left(\left(\log|f|^2 + g\right)[\mathcal{P}(\mathbb{C})] + \int\limits_{\mathcal{X}_\infty} g\cdot\omega_g\right)\\ &= (\mathcal{P},\mathcal{P})_{\mathrm{fin},z_P} + \frac{1}{2}\left(\left(\log|z_P|^2 + g\right)[\mathcal{P}(\mathbb{C})] + \int\limits_{\mathcal{X}_\infty} g\cdot\omega_g\right).\end{aligned}$$

Now we want to normalize the formal local parameter z_P to simplify the z_P-regularized local self-intersection number. For this let $T(\mathcal{X})$ denote the relative tangent bundle of the arithmetic surface $\pi : \mathcal{X} \longrightarrow \operatorname{Spec}\mathcal{O}_K$ and let $\mathcal{P} : \operatorname{Spec}\mathcal{O}_K \longrightarrow \mathcal{X}$ denote the section associated to the K-rational point $P \in X(K)$. Pulling back the relative tangent bundle $T(\mathcal{X})$ under $\mathcal{P}$ we obtain a line bundle $T_{\mathcal{P}}(\mathcal{X}) := \mathcal{P}^*T(\mathcal{X})$ over $\operatorname{Spec}\mathcal{O}_K$, since $\mathcal{P}(\operatorname{Spec}\mathcal{O}_K)$ lies in the relative smooth locus of $\mathcal{X}$ over $\operatorname{Spec}\mathcal{O}_K$. We will call $T_{\mathcal{P}}(\mathcal{X})$ the *tangent space to* $\mathcal{X}$ *at* $\mathcal{P}$. Let $T(X)$ be the relative tangent bundle of the generic fibre $\pi_K : X \longrightarrow \operatorname{Spec}K$. Since the tangent space $T_P(X)$ is given by the pullback of the tangent bundle $T(X)$ via the point $P : \operatorname{Spec}K \longrightarrow X$, we obtain an isomorphism

$$T_P(X) \cong T_{\mathcal{P}}(\mathcal{X}) \otimes_{\mathcal{O}_K} K$$

using the natural isomorphism $T(X) \cong T(\mathcal{X}) \otimes_{\mathcal{O}_K} K$.
Thus, if $\partial_{z_P} \in T_P(X)$ is a tangent vector associated to a formal local parameter $z_P \in \widehat{\mathcal{O}}_{X,P}$, there exists $\alpha \in K^\times$, unique up to a unit in $\mathcal{O}_K$ such that $\alpha\partial_{z_P}$ induces a section of $T_{\mathcal{P}}(\mathcal{X})$.

Proposition 1.22. *Let $z_P \in \widehat{\mathcal{O}}_{\mathcal{X},P}$ be a formal local parameter in P such that ∂_{z_P} induces a section of $T_{\mathcal{P}}(\mathcal{X})$. Then for all $(0) \neq \mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$, the z_P-regularized local self-intersection number vanishes, i.e. $(\mathcal{P},\mathcal{P})_{\mathfrak{p},z_P} = 0$. In particular,*

$$(\mathcal{P},\mathcal{P})_{\mathrm{fin},z_P} = 0.$$

Proof. Let $t_P \in k(\mathcal{X})^\times$ be a local parameter in $P \in X(K)$ such that $\partial_{t_P} = \partial_{z_P}$ in $T_{\mathcal{P}}(\mathcal{X})$. In other words $t_P = \alpha z_P + \mathcal{O}(z_P^2)$ with $\alpha \in \mathcal{O}_K^\times$. Because of the adjunction isomorphism $T_{\mathcal{P}}(\mathcal{X}) \cong \mathcal{P}^*\mathcal{O}_{\mathcal{X}}(\mathcal{P})$ (see [Li2, Lemma 9.1.36]) we see that t_P vanishes to order 1 along $\mathcal{P}$. It follows that $\mathcal{P} - \operatorname{div}(t_P)$ does not intersect $\mathcal{P}$ on $\mathcal{X}$, i.e. $(\mathcal{P},\mathcal{P})_{\mathfrak{p},t_P} = 0$ for all maximal $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$. Because of $(\mathcal{P},\mathcal{P})_{\mathfrak{p},z_P} = (\mathcal{P},\mathcal{P})_{\mathfrak{p},t_P}$ the z_P-regularized local self-intersection number $(\mathcal{P},\mathcal{P})_{\mathfrak{p},z_P}$ vanishes for all maximal $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$. □

In the following proposition we will show that the intersection theory with a tangent vector can be viewed as a part of the arithmetic intersection theory using arithmetic local coordinates. Indeed, the intersection theory with a tangent vector only permits divisors $\mathcal{P}$ coming from K-rational points, whereas the definition of arithmetic local coordinates allows more general divisors.

Proposition 1.23. *A tangent vector $\partial_{z_P} \in T_P(X)$ in a point $P \in X(K)$ induces a section of $T_{\mathcal{P}}(\mathcal{X})$ if and only if for all rational functions $f \in k(\mathcal{X})^\times$ the following equation holds:*

$$(\operatorname{div}(f) - \operatorname{ord}_{\mathcal{P}}(f)\mathcal{P},\mathcal{P})_{\mathrm{fin}} = \lim_{Q \to P} \left(\log|f(Q)| - \operatorname{ord}_{\mathcal{P}}(f) \log|z_P(Q)|\right). \tag{1.10}$$

In other words the induced family $(z_{P_\sigma})_{\sigma: K \hookrightarrow \mathbb{C}}$ is an arithmetic local coordinate in $\mathcal{P}$.

Proof. First consider the case where the tangent vector ∂_{z_P} is induced by a local parameter $z_P \in k(\mathcal{X})^\times$.
Let $\widetilde{z}_P$ be a local parameter in P such that $\partial_{\widetilde{z}_P}$ induces a section of $T_{\mathcal{P}}(\mathcal{X})$. If we write $z_P = \alpha\widetilde{z}_P + \mathcal{O}(\widetilde{z}_P^2)$, where $\alpha \in K^\times$, we obtain for all $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ the equation

$$\log|\alpha|_{\mathfrak{p}} = \log|\alpha|_{\mathfrak{p}} + (\mathcal{P},\mathcal{P})_{\mathfrak{p},\widetilde{z}_P} = (\mathcal{P},\mathcal{P})_{\mathfrak{p},z_P} = (\mathcal{P},\mathcal{P} - \operatorname{div}(z_P))_{\mathfrak{p}}.$$

Since $\log|\alpha|_{\mathfrak{p}}$ vanishes for all $(0) \neq \mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ if and only if $\alpha \in \mathcal{O}_K^\times$, summing over all $\mathfrak{p} \neq (0)$ shows that the geometric intersection number $(\mathcal{P},\mathcal{P} - \operatorname{div}(z_P))_{\mathrm{fin}}$ vanishes if and only if α is a unit in $\mathcal{O}_K$, that means that z_P induces a section of $T_{\mathcal{P}}(\mathcal{X})$. By the product formula on K we know that equation (1.10) is satisfied if and only if (1.10) holds for one rational function $f \in k(\mathcal{X})^\times$ with $\operatorname{ord}_P(f) \neq 0$. Since we have shown this for $f = z_P$, the

proof follows when $z_P \in k(\mathcal{X})^\times$.
For a general tangent vector ∂_{z_P}, choose a local parameter t_P such that $z_P = \alpha t_P + \mathcal{O}(t_P^2)$ and $\alpha \in \mathcal{O}_K^\times$. Then ∂_{z_P} induces a section of $T_{\mathcal{P}}(\mathcal{X})$ if and only if ∂_{t_P} induces a section of $T_{\mathcal{P}}(\mathcal{X})$. The equation (1.10) follows from the fact that $\log\left|\mathrm{Nm}_{K|\mathbb{Q}}(\alpha)\right| = \sum_{\sigma:K\hookrightarrow\mathbb{C}} \log|\alpha|_\sigma$ vanishes if and only if $\alpha \in \mathcal{O}_K^\times$. □

Example 1.24. In this example we compare the tangent vectors for Heegner points on the modular curve $X(1)$ calculated in [GZ, p.263] with the arithmetic local coordinates in Proposition 1.14 and Proposition 1.15. For this let us consider the differential $\omega = \eta^4(q)\mathrm{d}q/q = 2\pi i\eta^4(\tau)d\tau$ with the eta-function $\eta(\tau) = q^{1/24}\prod(1-q^n)$ as in [GZ]. It determines a tangent vector $\partial_{z_{\tau_0}}$ associated to the formal local parameter $z_{\tau_0} = 2\pi i\eta^4(\tau_0)(\tau-\tau_0)$ in the non-elliptic Heegner point P_{τ_0} on $X(1)$. It follows that there exists an arithmetic local coordinate $z_{\mathcal{P}_{\tau_0}} = (\alpha_{\tau_i}(\tau_i - \overline{\tau_i})2\pi i\eta^4(\tau_i)t_{\tau_i})_i$ in the closure $\mathcal{P}_{\tau_0}$ in $\mathcal{X}(1)$ of P_{τ_0}, where the constants α_{τ_i} are as in [GZ]. Here we set $\mathcal{P}_{\tau_0}(\mathbb{C}) = \sum P_{\tau_i}$. Using the arithmetic local coordinate as in Lemma 1.14, we obtain

$$\left\|\frac{\mathrm{E}_4^2\mathrm{E}_6}{\Delta}\right\|_{\mathrm{Pet}}(\tau_i) \equiv \alpha_{\tau_i}(\tau_i - \overline{\tau_i})2\pi i\eta^4(\tau_i),$$

where $\equiv$ means equality up to functions γ_i satisfying $\prod\gamma_i(\tau_i) = 1$. Setting $\gamma_i = 1$ yields

$$\left|\frac{\mathrm{E}_4^2\mathrm{E}_6}{\Delta}(\tau_i)\right| = \left|\alpha_{\tau_i}\eta^4(\tau_i)\right|.$$

Together with the equations $\mathrm{E}_4^3 = \Delta j$ and $\mathrm{E}_6^2 = \Delta(j-1728)$, this implies

$$\alpha_{\tau_i} = j(\tau_i)^{2/3}(j(\tau_i)-1728)^{1/2}$$

up to a 6th root of unity. Note that this constant was calculated in [GZ] with a different approach.
Since the arithmetic local coordinates are also well-defined in the elliptic Heegner points P_i and P_ρ, we can calculate the arithmetic local coordinate

$$z_{\mathcal{P}_{\tau_0}} = \alpha_{\tau_0}\left((\tau_0 - \overline{\tau_0})2\pi i\eta^4(\tau_0)\right)^{n_{\tau_0}} \cdot t_{\tau_0}$$

with the help of the equations (1.15). Hence we have to calculate the constants α_{τ_0} in

$$\left|\frac{(4\pi y_0)^{n_{\tau_0}}}{n_{\tau_0}!}\partial^{n_{\tau_0}}j(\tau_0)\right| = \left|\alpha_{\tau_0}\left((\tau_0-\overline{\tau_0})2\pi i\eta^4(\tau_0)\right)^{n_{\tau_0}}\right|.$$

Explicitly, the latter equation is given by $2^4 3^3 \left\|\mathrm{E}_4\right\|_{\mathrm{Pet}}(i) = |\alpha_i(2i2\pi i\eta^4(i))^2|$ for the point $\tau_0 = i$ and is given by $2^6 \left\|\mathrm{E}_6\right\|_{\mathrm{Pet}}(\rho) = \left|\alpha_\rho(\sqrt{3}\, i2\pi i\eta^4(\rho))^3\right|$ for $\tau_0 = \rho$, respectively. It follows

$$\alpha_i = 2^6 3^4 \quad \text{and} \quad \alpha_\rho = 2^9 3^{3/2}$$

as in [GZ, p.263].

1.5 Arithmetic self-intersection numbers

Now we come to the proof of one main theorem mentioned in the introduction, but in a more general version using different Green's functions.

Theorem 1.25. *Let $\mathcal{X}$ be an arithmetic surface and let $z_{\mathcal{P}}$ be an arithmetic local coordinate in a horizontal prime divisor $\mathcal{P}$ on $\mathcal{X}$. Given two arithmetic divisors $(\mathcal{P}, g_1)$,$(\mathcal{P}, g_2) \in \widehat{\mathrm{Z}}^1(\mathcal{X})$, their arithmetic intersection number equals*

$$(\mathcal{P}, g_1) \cdot (\mathcal{P}, g_2) = \lim_{Q \to P} \left(\log |z_{\mathcal{P}}(Q)| + \frac{1}{2} g_1(Q) \right) + \frac{1}{2} \int_{\mathcal{X}_\infty} g_2 \cdot \omega_{g_1},$$

where the equality $\mathrm{dd}^c g_1 = \omega_{g_1}$ holds outside $\mathcal{P}(\mathbb{C})$.

Proof. If we take a rational function $f \in k(\mathcal{X})^\times$ with $\mathrm{ord}_{\mathcal{P}}(f) = 1$ we obtain

$$\begin{aligned}(\mathcal{P}, g_1) \cdot (\mathcal{P}, g_2) &= (\mathcal{P} - \mathrm{div}(f), \log|f|^2 + g_1) \cdot (\mathcal{P}, g_2) \\ &= (\mathcal{P} - \mathrm{div}(f), \mathcal{P})_{\mathrm{fin}} + \frac{1}{2} \left(\left(\log|f|^2 + g_1 \right) [\mathcal{P}(\mathbb{C})] + \int_{\mathcal{X}_\infty} g_2 \cdot \omega_{g_1} \right),\end{aligned}$$

since $\mathrm{dd}^c \log|f|^2$ vanishes outside $\mathrm{div}(f)$ on $\mathcal{X}_\infty$. The definition of an arithmetic local coordinate yields

$$\begin{aligned}(\mathcal{P}, g_1) \cdot (\mathcal{P}, g_2) = &\lim_{Q \to P} \left(\log|z_{\mathcal{P}}(Q)| - \log|f(Q)| \right) \\ &+ \lim_{Q \to P} \left(\log|f(Q)| + \frac{1}{2} g_1(Q) \right) + \frac{1}{2} \int_{\mathcal{X}_\infty} g_2 \cdot \omega_{g_1}.\end{aligned}$$

Cancelling the term $\log|f(Q)|$, the proof follows. □

In section 1.6 we give a different proof of Theorem 1.25. In the remainder of this section, we use Theorem 1.25 to compute some important arithmetic self-intersection numbers on $\mathbb{P}^1_{\mathbb{Z}}$.

Proposition 1.26. *Consider the arithmetic surface $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ and the hermitian line bundle $\overline{\mathcal{L}} = (\mathcal{O}_{\mathcal{X}}(1), \|\cdot\|_{\mathrm{FS}})$ with the Fubini-Study metric $\|\cdot\|_{\mathrm{FS}}$. Then its arithmetic self-intersection number is given by*

$$\overline{\mathcal{L}}^2 = \frac{1}{2}.$$

Proof. First note that this proposition is a classical fact, see for instance [Bo, p.957]. Consider the arithmetic divisor $(\mathcal{P}, g_{\mathcal{P}}) \in \widehat{\mathrm{Z}}^1(\mathcal{X})$ given by the divisor $\mathcal{P} = (a : b) \in \mathcal{X}(\mathbb{Z})$, where $b \neq 0$, and the Green's function $g_{\mathcal{P}} = -\log \|bx_0 - ax_1\|^2_{\mathrm{FS}}$ associated to the Fubini-Study metric $\|\cdot\|_{\mathrm{FS}}$. Then we have

$$\overline{\mathcal{L}}^2 = (\mathcal{P}, g_{\mathcal{P}})^2 .$$

In Proposition 1.9 we calculated the arithmetic local coordinate $z_{\mathcal{P}} = b^2\left(z - \frac{a}{b}\right)$ in $\mathcal{P}$ on the chart $U = \{z = \frac{x_0}{x_1} \in \mathbb{C}\}$. On this chart the Green's function is given by $g_{\mathcal{P}} = -\log\left(\frac{|bz-a|^2}{1+|z|^2}\right)$. With the Fubini-Study form $\omega_{\mathrm{FS}} = \omega_{g_{\mathcal{P}}} = \mathrm{dd}^c \log\left(1 + |z|^2\right)$ Theorem 1.25 implies

$$\begin{aligned}
\overline{\mathcal{L}}^2 &= \lim_{Q \to P} \left(\log |z_{\mathcal{P}}(Q)| + \frac{1}{2} g_{\mathcal{P}}(Q) \right) + \frac{1}{2} \int_{\mathbb{C}} g_{\mathcal{P}} \cdot \omega_{\mathrm{FS}} \\
&= \lim_{z \to \mathcal{P}(\mathbb{C})} \left(\log \left| \frac{b^2\left(z - \frac{a}{b}\right)}{bz - a} \right| + \frac{1}{2} \log\left(1 + |z|^2\right) \right) \\
&\quad - \frac{1}{2} \int_{\mathbb{C}} \log \left(\frac{|bz-a|^2}{1+|z|^2} \right) \mathrm{dd}^c \log\left(1 + |z|^2\right).
\end{aligned}$$

One can calculate directly (or see [CM, Corollaire 2.2.2]) that

$$\int_{\mathbb{C}} \log \left(\frac{|bz-a|^2}{1+|z|^2} \right) \mathrm{dd}^c \log\left(1 + |z|^2\right) = \log\left(a^2 + b^2\right) - 1$$

holds. Thus we get

$$\overline{\mathcal{L}}^2 = \log|b| + \frac{1}{2}\log\left(1 + \left|\frac{a}{b}\right|^2\right) - \frac{1}{2}\log\left(a^2 + b^2\right) + \frac{1}{2} = \frac{1}{2}.$$

□

In the next proposition we calculate the arithmetic self-intersection number of the line bundle of modular forms equipped with the Petersson metric $\overline{\mathcal{M}}_k(\Gamma(1))$. Note that the

arithmetic intersection number $\overline{\mathcal{M}}_k(\Gamma(1))^2$ is a priori not defined because the Petersson metric is not smooth in the cusp of the modular curve $X(1)$. This problem will be solved in Chapter 3 using a generalized arithmetic intersection theory. Indeed, in Proposition 3.6 we will see that the generalized arithmetic self-intersection number $\overline{\mathcal{M}}_k(\Gamma(1))^2 = (\mathcal{P}, g_{\mathcal{P}})^2$ coincides with the arithmetic self-intersection number in Theorem 1.25 if for the divisor $\mathcal{P}$, the set $\mathcal{P}(\mathbb{C})$ does not contain the cusp of $X(1)$.

Proposition 1.27. *Let $\mathcal{X} = \mathcal{X}(1) = \mathbb{P}^1_{\mathbb{Z}}$. For a natural number $k \in \mathbb{N}$ with $12|k$, the arithmetic self-intersection number of the line bundle of modular forms equipped with the Petersson metric $\overline{\mathcal{M}}_k(\Gamma(1)) = \left(\mathcal{O}_{\mathcal{X}}\left(\mathcal{S}_\infty\right)^{\otimes k/12}, \|\cdot\|_{\mathrm{Pet}}\right)$ is given by*

$$\overline{\mathcal{M}}_k(\Gamma(1))^2 = k^2 \zeta_{\mathbb{Q}}(-1)\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right),$$

where $\zeta_{\mathbb{Q}}$ denotes the Riemann zeta function.

Proof. A similar proof of this result can be found in [Kü3].
We start with the case $k = 12$. Consider the arithmetic divisor

$$(\mathcal{P}, g_{\mathcal{P}}) = \left(\mathrm{div}\left(\Delta(\tau)j(\tau)\right), -\log\|\Delta(\tau)j(\tau)\|^2_{\mathrm{Pet}}\right).$$

Then $\mathcal{P}(\mathbb{C}) = P_\rho$ is the elliptic point of order 3 and

$$\overline{\mathcal{M}}_{12}(\Gamma(1))^2 = (\mathcal{P}, g_{\mathcal{P}})^2.$$

By Proposition 1.15, an arithmetic local coordinate in $\mathcal{P}$ is given by $z_{\mathcal{P}} = \frac{1}{6}\left\|\partial^3 j(\rho)\right\|_{\mathrm{Pet}} t_\rho$. It follows

$$\begin{aligned}\overline{\mathcal{M}_{12}}(\Gamma(1))^2 = &\lim_{\tau\to\rho}\left(\log\left|\frac{1}{6}\left\|\partial^3 j(\rho)\right\|_{\mathrm{Pet}} t_\rho(\tau)\right| - \log\|\Delta(\tau)j(\tau)\|_{\mathrm{Pet}}\right) - \\ &\frac{1}{2}\int\limits_{\mathcal{X}(1)_\infty} \log\|\Delta(\tau)j(\tau)\|^2_{\mathrm{Pet}}\cdot\omega_{\mathrm{Pet}}.\end{aligned}$$

Here $\omega_{\mathrm{Pet}} = \omega_{g_{\mathcal{P}}} = \frac{\mu}{4\pi}$ denotes the Petersson normalized hyperbolic $(1,1)$-form on $X(1)$, where μ is induced by the $\Gamma(1)$-invariant hyperbolic $(1,1)$-form

$$\frac{i}{2}\frac{\mathrm{d}z\wedge\mathrm{d}\overline{z}}{(\mathrm{Im}(z))^2} = \frac{\mathrm{d}x\wedge\mathrm{d}y}{y^2}, \quad z = x+iy$$

on $\mathbb{H}$, see [Sh1, Proposition 2.18.]. It is well-known (see [Kü2]) that

$$-\frac{1}{2}\int\limits_{\mathcal{X}(1)_\infty} \log\|\Delta(\tau)j(\tau)\|^2_{\mathrm{Pet}}\cdot\omega_{\mathrm{Pet}} = \log\|\Delta(\rho)\|_{\mathrm{Pet}} + 12^2\zeta_{\mathbb{Q}}(-1)\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right).$$

Moreover, using equation (1.9), we find

$$\lim_{\tau\to\rho}\left(\log\left|\frac{1}{6}\left\|\partial^3 j(\rho)\right\|_{\mathrm{Pet}} t_\rho(\tau)\right| - \log\left\|\Delta(\tau)j(\tau)\right\|_{\mathrm{Pet}}\right) = -\log\left\|\Delta(\rho)\right\|_{\mathrm{Pet}}.$$

Therefore,

$$\overline{\mathcal{M}}_{12}(\Gamma(1))^2 = 12^2\zeta_{\mathbb{Q}}(-1)\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right).$$

The general case $k \neq 12$ follows from the bilinearity of the arithmetic intersection number

$$\overline{\mathcal{M}}_k(\Gamma(1))^2 = \frac{k^2}{12^2}\overline{\mathcal{M}}_{12}(\Gamma(1))^2 = k^2\zeta_{\mathbb{Q}}(-1)\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right).$$

□

In section 3.1 we give a different proof of Proposition 1.27 using the arithmetic divisor $(\mathcal{P}, g_{\mathcal{P}}) = \left(\mathrm{div}\left(\Delta(\tau)\right), -\log\left\|\Delta(\tau)\right\|^2_{\mathrm{Pet}}\right)$. Therewith it is necessary to calculate the arithmetic self-intersection number using the generalized arithmetic intersection theory, which will be defined in Chapter 3.

1.6 Adjusted Green's functions

In this section we rewrite the notion of arithmetic local coordinates and the corresponding arithmetic intersection theory of arithmetic divisors in the terminology of Green's functions and hermitian line bundles. At this point we would like to thank Ben Howard who described in [Ho] an arithmetic local coordinate in terms of the height function with respect to a hermitian line bundle.

Definition 1.28. Let $\mathcal{X}$ be an arithmetic surface and let $\mathcal{P}$ be a horizontal prime divisor on $\mathcal{X}$. Let $z_{\mathcal{P}}$ be an arithmetic local coordinate in $\mathcal{P}$. A Green's function $\alpha_{\mathcal{P}}$ for $\mathcal{P}$ is called an *adjusted Green's function for* $\mathcal{P}$, if $\alpha_{\mathcal{P}}$ is locally given by

$$\alpha_{\mathcal{P}}(Q) = -\log\left|z_{\mathcal{P}}(Q)\right|^2 + \varphi(Q)$$

with the property that $\varphi(\mathcal{P}(\mathbb{C})) := \sum \varphi(P_i) = 0$, where $\mathcal{P}(\mathbb{C}) = \sum P_i$.

Recall that a Green's function $g_{\mathcal{P}}$ for $\mathcal{P}$ can locally by written as

$$g_{\mathcal{P}}(Q) = -\log\left|z_{\mathcal{P}}(Q)\right|^2 + \varphi(Q),$$

where $z_{\mathcal{P}}$ is an arithmetic local coordinate in $\mathcal{P}$ and $\varphi \in \mathcal{C}^\infty(\mathcal{X}_\infty)$ is a smooth function. Then, it is clear that $g_{\mathcal{P}}$ induces an adjusted Green's function $\alpha_{\mathcal{P}}$ for $\mathcal{P}$ by setting

$$\alpha_{\mathcal{P}} := g_{\mathcal{P}} - \varphi(\mathcal{P}(\mathbb{C})).$$

Example 1.29. *i)* As in Proposition 1.26 let $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, x_1]$ and consider the Green's function $g_{\mathcal{P}}$ and the arithmetic local coordinate $z_{\mathcal{P}}$ in $\mathcal{P} = (a : b) \in \mathcal{X}(\mathbb{Z})$, where $b \neq 0$, given by

$$g_{\mathcal{P}}(z) = -\log\left(\frac{|bz - a|^2}{1 + |z|^2}\right) \text{ and } z_{\mathcal{P}} = b^2\left(z - \frac{a}{b}\right).$$

It follows that

$$\alpha_{\mathcal{P}}(z) = -\log\left(\frac{|bz - a|^2}{1 + |z|^2}\right) - \log\left(1 + \left|\frac{a}{b}\right|^2\right) - 2\log|b|$$

is an adjusted Green's function for $\mathcal{P}$.
ii) A Green's function for the cusp $\mathcal{S}_\infty$ on $\mathcal{X}(1) = \mathbb{P}^1_{\mathbb{Z}}$ is given by

$$g_{\mathcal{S}_\infty}(q) = -\log \|\Delta(q)\|^2_{\mathrm{FS}} = -\log\left(\frac{|\Delta(q)|^2}{|\mathrm{E}_4(q)|^6 + |\Delta(q)|^2}\right),$$

where $\Delta(q)$ denotes the modular discriminant. It was shown in Proposition 1.11 that $z_{\mathcal{S}_\infty} = q$ is an arithmetic local coordinate in $\mathcal{S}_\infty$. Hence $g_{\mathcal{S}_\infty}$ already is an adjusted Green's function for $\mathcal{S}_\infty$, because

$$\lim_{q \to 0}\left(|\mathrm{E}_4(q)|^6 + |\Delta(q)|^2\right) = \lim_{q \to 0}\left(1 + \mathcal{O}(q)\right) = 1.$$

Proposition 1.30. *Let $g_{\mathcal{P}}$ be a Green's function for $\mathcal{P}$. Then $g_{\mathcal{P}}$ is an adjusted Green's function for $\mathcal{P}$ if and only if the height $\mathrm{ht}_{[\mathcal{P}, g_{\mathcal{P}}]}(\mathcal{P})$ vanishes.*

Proof. Recall that the height defines a bilinear pairing on $\widehat{\mathrm{CH}}^1(\mathcal{X}) \times \mathrm{Z}^1(\mathcal{X})$. Take a rational function $f \in k(\mathcal{X})^\times$ with $m := -\mathrm{ord}_{\mathcal{P}}(f) \neq 0$ and a Green's function $g_{\mathcal{P}} = -\log|z_{\mathcal{P}}|^2 + \varphi$, where $z_{\mathcal{P}}$ is an arithmetic local coordinate in $\mathcal{P}$. Then by the definition of $z_{\mathcal{P}}$ we have

$$\begin{aligned}
m \cdot \mathrm{ht}_{[\mathcal{P}, g_{\mathcal{P}}]}(\mathcal{P}) =& \mathrm{ht}_{[m\mathcal{P}, m g_{\mathcal{P}}]}(\mathcal{P}) = (m\mathcal{P} + \operatorname{div}(f), \mathcal{P})_{\mathrm{fin}} + \frac{1}{2}\lim_{Q \to P}\left(m g_{\mathcal{P}}(Q) - \log|f(Q)|^2\right) \\
=& (\operatorname{div}(f) - \mathrm{ord}_{\mathcal{P}}(f)\mathcal{P}, \mathcal{P})_{\mathrm{fin}} - \frac{1}{2}\lim_{Q \to P}\left(\log|f(Q)|^2 + \mathrm{ord}_{\mathcal{P}}(f) g_{\mathcal{P}}(Q)\right) \\
=& (\operatorname{div}(f) - \mathrm{ord}_{\mathcal{P}}(f)\mathcal{P}, \mathcal{P})_{\mathrm{fin}} - \lim_{Q \to P}\left(\log|f(Q)| - \mathrm{ord}_{\mathcal{P}}(f)\log|z_{\mathcal{P}}(Q)|\right) \\
& - \frac{\mathrm{ord}_{\mathcal{P}}(f)}{2}\varphi(\mathcal{P}(\mathbb{C})) \\
=& \frac{m}{2}\varphi(\mathcal{P}(\mathbb{C})).
\end{aligned}$$

Hence the height $\mathrm{ht}_{[\mathcal{P}, g_{\mathcal{P}}]}(\mathcal{P})$ vanishes if and only if $\varphi(\mathcal{P}(\mathbb{C}))$ vanishes, which is the definition of an adjusted Green's function for $\mathcal{P}$. □

Remark 1.31. The definition of the height shows that a Green's function $g_{\mathcal{P}}$ for $\mathcal{P}$ is an adjusted Green's function for $\mathcal{P}$ if and only if the equation

$$(\mathrm{div}(f) - \mathrm{ord}_{\mathcal{P}}(f)\mathcal{P}, \mathcal{P})_{\mathrm{fin}} = \lim_{Q \to P} \left(\log |f(Q)| + \frac{1}{2} \mathrm{ord}_{\mathcal{P}}(f) g_{\mathcal{P}}(Q) \right)$$

holds for all rational functions $f \in k(\mathcal{X})^{\times}$.

Using adjusted Green's functions we can give a second proof of Theorem 1.25:

Proof of Theorem 1.25. By the definition of the arithmetic intersection number we have

$$(\mathcal{P}, g_1) \cdot (\mathcal{P}, g_2) = \mathrm{ht}_{[\mathcal{P}, g_1]}(\mathcal{P}) + \frac{1}{2} \int_{\mathcal{X}_\infty} g_2 \cdot \omega_{g_1}.$$

For an adjusted Green's function $\alpha_{\mathcal{P}}$ for $\mathcal{P}$ the equation $\mathrm{ht}_{[\mathcal{P}, \alpha_{\mathcal{P}}]}(\mathcal{P}) = 0$ holds by Proposition 1.30. Since the height is linear on $\widehat{\mathrm{CH}}^1(\mathcal{X})$, it follows that

$$\begin{aligned} \mathrm{ht}_{[\mathcal{P}, g_1]}(\mathcal{P}) &= \mathrm{ht}_{[\mathcal{P}, g_1]}(\mathcal{P}) - \mathrm{ht}_{[\mathcal{P}, \alpha_{\mathcal{P}}]}(\mathcal{P}) = \mathrm{ht}_{[0, g_1 - \alpha_{\mathcal{P}}]}(\mathcal{P}) \\ &= \frac{1}{2} \lim_{Q \to P} (g_1(Q) - \alpha_{\mathcal{P}}(Q)) = \lim_{Q \to P} \left(\log |z_{\mathcal{P}}(Q)| + \frac{1}{2} g_1(Q) \right), \end{aligned}$$

which shows that

$$(\mathcal{P}, g_1) \cdot (\mathcal{P}, g_2) = \lim_{Q \to P} \left(\log |z_{\mathcal{P}}(Q)| + \frac{1}{2} g_1(Q) \right) + \frac{1}{2} \int_{\mathcal{X}_\infty} g_2 \cdot \omega_{g_1}$$

and hence proves Theorem 1.25. □

Remark 1.32. *i)* In terms of an adjusted Green's function $\alpha_{\mathcal{P}}$ for $\mathcal{P}$, Theorem 1.25 can also be written as

$$(\mathcal{P}, g_1) \cdot (\mathcal{P}, g_2) = \frac{1}{2} \lim_{Q \to P} (g_1(Q) - \alpha_{\mathcal{P}}(Q)) + \frac{1}{2} \int_{\mathcal{X}_\infty} g_2 \cdot \omega_{g_1}.$$

Moreover, note that in the special case where $g_1 = g_2 = \alpha_{\mathcal{P}}$ is an adjusted Green's function for $\mathcal{P}$, we obtain the arithmetic self-intersection number

$$(\mathcal{P}, \alpha_{\mathcal{P}})^2 = \frac{1}{2} \int_{\mathcal{X}_\infty} \alpha_{\mathcal{P}} \cdot \omega_{\alpha_{\mathcal{P}}}.$$

For instance, consider $\mathcal{P}$ and $\alpha_{\mathcal{P}}$ as in Example 1.29 i). Then the arithmetic self-intersection number of $(\mathcal{P}, \alpha_{\mathcal{P}})$ is given by

$$(\mathcal{P}, \alpha_{\mathcal{P}})^2 = \frac{1}{2} - \log\left(a^2 + b^2\right).$$

ii) In terms of hermitian line bundles we have the following:
For a horizontal prime divisor $\mathcal{P}$ let $\mathcal{O}_{\mathcal{X}}(\mathcal{P})$ be the corresponding line bundle and consider two hermitian metrics $\|\cdot\|_1$ and $\|\cdot\|_2$ on $\mathcal{O}_{\mathcal{X}}(\mathcal{P})$. Then Theorem 1.25 can be translated into

$$\begin{aligned}&(\mathcal{O}_{\mathcal{X}}(\mathcal{P}),\|\cdot\|_1)\,.\,(\mathcal{O}_{\mathcal{X}}(\mathcal{P}),\|\cdot\|_2)\\ &=\lim_{Q\to P}\left(\log\|1_{\mathcal{P}}\|_{z_{\mathcal{P}}}(Q)-\log\|1_{\mathcal{P}}\|_2(Q)\right)-\int\limits_{\mathcal{X}_\infty}\log\|1_{\mathcal{P}}\|_1\cdot c_1(\mathcal{O}_{\mathcal{X}}(\mathcal{P}),\|\cdot\|_2),\end{aligned}$$

where $1_{\mathcal{P}}$ denotes the canonical section of $\mathcal{O}_{\mathcal{X}}(\mathcal{P})$ and $\|\cdot\|_{z_{\mathcal{P}}}$ is any hermitian metric on $\mathcal{O}_{\mathcal{X}}(\mathcal{P})$ such that $\lim_{Q\to P}\left(\log\|1_{\mathcal{P}}\|_{z_{\mathcal{P}}}(Q)-\log|z_{\mathcal{P}}(Q)|\right)=0$.

The next theorem will be useful when studying generalized arithmetic self-intersection numbers, which we will consider in the last chapter. However, in the usual case it can be seen as a version of Theorem 1.25 without evaluation.

Theorem 1.33. *Let $\mathcal{X}$ be an arithmetic surface and $(\mathcal{P},g_{\mathcal{P}})\in\widehat{\mathrm{Z}}^1(\mathcal{X})$ an arithmetic divisor for the horizontal prime divisor $\mathcal{P}$. Then its arithmetic self-intersection number is given by*

$$(\mathcal{P},g_{\mathcal{P}})^2=\frac{1}{2}\int\limits_{\mathcal{X}_\infty}\left(g_{\mathcal{P}}\cdot\omega_{\alpha_{\mathcal{P}}}-\alpha_{\mathcal{P}}\cdot\omega_{g_{\mathcal{P}}}+g_{\mathcal{P}}\cdot\omega_{g_{\mathcal{P}}}\right),$$

where $\alpha_{\mathcal{P}}$ is an adjusted Green's function for $\mathcal{P}$.

Proof. By Theorem 1.1 the equation

$$(\mathcal{P},g_{\mathcal{P}})^2=\mathrm{ht}_{[\mathcal{P},g_{\mathcal{P}}]}(\mathcal{P})+\frac{1}{2}\int\limits_{\mathcal{X}_\infty}g_{\mathcal{P}}\cdot\omega_{g_{\mathcal{P}}}$$

holds. The definition of the adjusted Green's form $\alpha_{\mathcal{P}}$ shows $\mathrm{ht}_{[\mathcal{P},\alpha_{\mathcal{P}}]}(\mathcal{P})=0$ and hence $\mathrm{ht}_{[\mathcal{P},g_{\mathcal{P}}]}(\mathcal{P})=\mathrm{ht}_{[\mathcal{P},g_{\mathcal{P}}]}(\mathcal{P})-\mathrm{ht}_{[\mathcal{P},\alpha_{\mathcal{P}}]}(\mathcal{P})$. Since the arithmetic intersection number is symmetric, i.e.

$$(\mathcal{P},g_{\mathcal{P}})\cdot(\mathcal{P},\alpha_{\mathcal{P}})=(\mathcal{P},\alpha_{\mathcal{P}})\cdot(\mathcal{P},g_{\mathcal{P}}),$$

we obtain

$$\mathrm{ht}_{[\mathcal{P},g_{\mathcal{P}}]}(\mathcal{P})+\frac{1}{2}\int\limits_{\mathcal{X}_\infty}\alpha_{\mathcal{P}}\cdot\omega_{g_{\mathcal{P}}}=\mathrm{ht}_{[\mathcal{P},\alpha_{\mathcal{P}}]}(\mathcal{P})+\frac{1}{2}\int\limits_{\mathcal{X}_\infty}g_{\mathcal{P}}\cdot\omega_{\alpha_{\mathcal{P}}}.$$

This yields

$$\mathrm{ht}_{[\mathcal{P},g_{\mathcal{P}}]}(\mathcal{P})=\frac{1}{2}\int\limits_{\mathcal{X}_\infty}\left(g_{\mathcal{P}}\cdot\omega_{\alpha_{\mathcal{P}}}-\alpha_{\mathcal{P}}\cdot\omega_{g_{\mathcal{P}}}\right),$$

which proves the theorem. □

Chapter 2

A Generalization On Arithmetic Varieties

In this chapter we start with a short review of the basic definitions of arithmetic intersection theory on higher dimensional arithmetic varieties, which is due to H. Gillet and C. Soulé in [GS2]. In particular we describe the arithmetic intersection product and the height of a cycle explicitly.
We generalize the definition of adjusted Green's functions for divisors on arithmetic surfaces to adjusted Green's forms for a cycle on arithmetic varieties and show some important properties of these objects. More generally, we also define a family of adjusted Green's forms for a family of cycles, which depend on one fixed cycle.
On the arithmetic variety $\mathbb{P}^d_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_d]$ we calculate some examples of adjusted Green's forms for two fixed cycles Y and Z, which are defined by linear equations.
We define a modified version of the $*$-product between two Green's forms with the help of an adjusted Green's form. We demonstrate our new approach with two formulas for the arithmetic self-intersection number of a hermitian line bundle. Therewith we calculate a classical example of an arithmetic self-intersection number on $\mathbb{P}^d_{\mathbb{Z}}$.
In the last part of this chapter we define adjusted equations, generalizing the definition of arithmetic local coordinates on arithmetic surfaces. We also calculate examples of adjusted equations on $\mathbb{P}^d_{\mathbb{Z}}$. As an application we use a new version for the arithmetic intersection number of two arithmetic cycles to give an alternative proof of the previously computed arithmetic self-intersection number on $\mathbb{P}^d_{\mathbb{Z}}$.

2.1 Review of higher dimensional Arakelov theory

In this section we give the background of the higher dimensional arithmetic intersection theory. The following can be found in [BGS], [So1], [GS2] and [KMY]. First let us define the notion of arithmetic varieties, cycles and Green's forms.

Definition 2.1. An *arithmetic variety* $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ is a reduced, regular scheme $\mathcal{X}$, which is flat and projective over $\operatorname{Spec} \mathcal{O}_K$, where as usual $\mathcal{O}_K$ denotes the ring of integers of a number field K. Moreover, we assume that the generic fibre X is geometrically connected.

In the remainder let $d \in \mathbb{N}$ be the relative dimension of $\mathcal{X}$, hence $\dim(\mathcal{X}) = d + 1$. Note that if $d = 1$, the definition of an arithmetic variety coincides with the definition of an arithmetic surface $\mathcal{X}$ as in Chapter 1.

Definition 2.2. Let $\mathcal{X}_\infty = X(\mathbb{C})$ denote the set of complex points of the generic fibre X. On $\mathcal{X}_\infty$ let $A^{(p,q)}(\mathcal{X}_\infty)$ be the space of smooth (p,q)-forms endowed with the *Schwartz topology*. This means that a sequence $(\eta)_n$ in $A^{(p,q)}(\mathcal{X}_\infty)$ converges to η in $A^{(p,q)}(\mathcal{X}_\infty)$ if and only if there exists a compact set K such that for any η_n we have $\operatorname{supp}(\eta_n) \subset K$ and any derivation of η_n converges uniformly to the corresponding derivation of η. The *space of (p,q)-currents* $D^{(p,q)}(\mathcal{X}_\infty)$ is the continuous dual space of $A^{(d-p,d-q)}(\mathcal{X}_\infty)$.

We have an embedding $A^{(p,q)}(\mathcal{X}_\infty) \hookrightarrow D^{(p,q)}(\mathcal{X}_\infty)$ sending ω to $[\omega]$, which is for $\alpha \in A^{(d-p,d-q)}(\mathcal{X}_\infty)$ defined by

$$[\omega](\alpha) := \int_{\mathcal{X}_\infty} \omega \wedge \alpha.$$

Let ∂ and $\bar{\partial}$ denote the usual derivations on $A^{(p,q)}(\mathcal{X}_\infty)$. The corresponding derivations on $D^{(p,q)}(\mathcal{X}_\infty)$ will be also denoted by ∂ and $\bar{\partial}$. Explicitly they are given by

$$\partial S(\eta) = (-1)^{p+q+1} S(\partial \eta) \text{ and } \bar{\partial} S(\eta) = (-1)^{p+q+1} S(\bar{\partial} \eta),$$

where $S \in D^{(p,q)}(\mathcal{X}_\infty)$ and $\eta \in A^{(d-p,d-q)}(\mathcal{X}_\infty)$.
It follows $[\partial \eta] = \partial[\eta]$ and $[\overline{\partial} \eta] = \overline{\partial}[\eta]$ for any $\eta \in A^{(p,q)}(\mathcal{X}_\infty)$.

Definition 2.3. For an integer $p \geq 0$ let $\mathrm{Z}^p(\mathcal{X})$ be the group of cycles Z in $\mathcal{X}$ of codimension p. Any cycle $Z = \sum n_i Z_i$, where $n_i \in \mathbb{Z}$, is a formal sum of *irreducible cycles* Z_i, i.e. irreducible closed subschemes of $\mathcal{X}$. A *K_1-chain* f is a formal sum $f = \sum f_W$ with $f_W \in k(W)^\times$, where W runs through a finite set of integral subschemes of $\mathcal{X}$ of codimension $p-1$ and $k(W)$ denotes its function field.

The divisor of a K_1-chain $f = \sum f_W$ as in Definition 2.3 is an example of a codimension p cycle. It is given by $\mathrm{div}(f) = \sum \mathrm{div}(f_W) \in \mathrm{Z}^p(\mathcal{X})$. Here $\mathrm{div}(f_W) := \sum \mathrm{ord}_V(f_W) \cdot V$, where the sum runs through all integral subschemes V of $\mathcal{X}$ of codimension p.

Notation 2.4. Let $F_\infty : \mathcal{X}_\infty \longrightarrow \mathcal{X}_\infty$ be the map coming from the conjugation on $\mathbb{C}$. Then we set

$$A^{(p,p)}(\mathcal{X}) := \{\omega \in A^{(p,p)}(\mathcal{X}_\infty) |\ \omega \text{ real},\ F_\infty^*\omega = (-1)^p\,\omega\} \text{ and}$$
$$D^{(p,p)}(\mathcal{X}) := \{\omega \in D^{(p,p)}(\mathcal{X}_\infty) |\ \omega \text{ real},\ F_\infty^*\omega = (-1)^p\,\omega\},$$

where a current $\omega \in D^{(p,p)}(\mathcal{X}_\infty)$ is *real* if for any $\eta \in A^{(d-p,d-p)}(\mathcal{X}_\infty)$ we have $\omega(\overline{\eta}) = \overline{\omega(\eta)}$. Moreover, for later use we need the following:

$$\widetilde{A}^{(p,p)}(\mathcal{X}) := A^{(p,p)}(\mathcal{X})/\{\partial(u) + \overline{\partial}(v) \in A^{(p,p)}(\mathcal{X}) |\ u \in A^{(p-1,p)}(\mathcal{X}_\infty), v \in A^{(p,p-1)}(\mathcal{X}_\infty)\};$$
$$\widetilde{D}^{(p,p)}(\mathcal{X}) := D^{(p,p)}(\mathcal{X})/\{\partial(u) + \overline{\partial}(v) \in D^{(p,p)}(\mathcal{X}) |\ u \in D^{(p-1,p)}(\mathcal{X}_\infty), v \in D^{(p,p-1)}(\mathcal{X}_\infty)\}.$$

An irreducible cycle $Z \in \mathrm{Z}^p(\mathcal{X})$ defines a real current $\delta_{Z(\mathbb{C})} \in D^{(p,p)}(\mathcal{X})$ by integration along the smooth part of $Z(\mathbb{C})$. More explicitly, the current $\delta_{Z(\mathbb{C})}$ is defined by

$$\delta_{Z(\mathbb{C})}(\omega) = \int_{\widetilde{Z(\mathbb{C})}} \nu^*(\omega),$$

where $\nu : \widetilde{Z(\mathbb{C})} \longrightarrow Z(\mathbb{C})$ is a desingularization of $Z(\mathbb{C})$ along the set of singular points of $Z(\mathbb{C})$. For an arbitrary cycle Z, the current $\delta_{Z(\mathbb{C})}$ will be extended by linearity.
Note that a *desingularization along the set of singular points* (or only *desingularization*) is a morphism that satisfies the next proposition.

Proposition 2.5. Hironaka's theorem, [Hi]
Let Z be an irreducible closed subset of $\mathcal{X}_\infty$. Then there exists a proper map $\nu : \widetilde{Z} \longrightarrow Z$ such that
i) $\widetilde{Z}$ is smooth;
ii) for the set of singular points Z^{sing} of Z, the divisor $E := \nu^{-1}(Z^{\mathrm{sing}})$ has normal crossings, i.e. in local coordinates $z_1, \ldots, z_d$ given by the equation $E = \{z_1 \cdots z_k = 0\}$ for some $k \leq d$;
iii) the map $\nu : \widetilde{Z} \backslash E \longrightarrow Z \backslash Z^{\mathrm{sing}}$ is an isomorphism.

Definition 2.6. Suppose $Z \in \mathrm{Z}^p(\mathcal{X})$.
i) A *Green's current for Z* is a current $g_Z \in \mathrm{D}^{(p-1,p-1)}(\mathcal{X})$ such that

$$\mathrm{dd}^c g_Z + \delta_{Z(\mathbb{C})} = [\omega_{g_Z}]$$

for a smooth form $\omega_{g_Z} \in A^{(p,p)}(\mathcal{X})$, where as usual $\mathrm{d} := \partial + \bar{\partial}$ and $\mathrm{d}^c := (\partial - \bar{\partial})/(4\pi i)$.
ii) A *Green's form for* Z is a smooth $(p-1,p-1)$-form g_Z on $\mathcal{X}_\infty \backslash Z(\mathbb{C})$ such that the induced current $[g_Z]$ is a Green's current for Z and such that there exists a smooth projective complex variety $\widetilde{\mathcal{X}_\infty}$, a proper map $\pi : \widetilde{\mathcal{X}_\infty} \longrightarrow \mathcal{X}_\infty$ and a smooth form φ on $\widetilde{\mathcal{X}_\infty} \backslash \pi^{-1}(Z(\mathbb{C}))$ satisfying the following properties:

1. $E_\infty := \pi^{-1}(Z(\mathbb{C}))$ is a normal crossing divisor;

2. $\pi : \widetilde{\mathcal{X}_\infty} \backslash E_\infty \longrightarrow \mathcal{X}_\infty \backslash Z(\mathbb{C})$ is an isomorphism and $\pi_*(\varphi) = g_Z$;

3. for all $x \in \widetilde{\mathcal{X}_\infty}$ with local equation $(z_1, \ldots, z_k, U)$ for $Z(\mathbb{C})$ in x, there exist smooth ∂- and $\overline{\partial}$-closed forms α_i and a smooth form β on U such that

$$\varphi|_U = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \beta.$$

By a local equation $(z_1, \ldots, z_k, U)$ for $Z(\mathbb{C})$ in x we mean that $\pi^{-1}(Z(\mathbb{C})) \cap U = \{z_1 \cdots z_k = 0\}$ holds for a system of holomorphic coordinates $(z_1, \ldots, z_d)$ of U centered at x.

Theorem 2.7. [So1, Theorem 3] and [KMY, Proposition 1.2.11./1.2.12.].
i) For any cycle $Z \in \mathrm{Z}^p(\mathcal{X})$ *there exists a Green's form for* Z. *In particular, for any cycle* Z *there exists a Green's current for* Z.
ii) If g_1 *and* g_2 *are Green's currents for* $Z \in \mathrm{Z}^p(\mathcal{X})$, *then there exist* $\eta \in A^{(p-1,p-1)}(\mathcal{X})$, $u \in D^{(p-2,p-1)}(\mathcal{X}_\infty)$ *and* $v \in D^{(p-1,p-2)}(\mathcal{X}_\infty)$ *such that*

$$g_1 - g_2 = [\eta] + \partial u + \bar{\partial} v.$$

Example 2.8. Let W be an integral subscheme of $\mathcal{X}$ of codimension $p-1$ and $k(W)$ its function field. Then any function $f \in f(W)^\times$ defines a current $[\log |f|^2] \in \mathrm{D}^{(p-1,p-1)}(\mathcal{X})$ by

$$[\log |f|^2](\omega) = \int\limits_{\widetilde{W(\mathbb{C})}} \log |\widetilde{f}|^2 \cdot i^*(\omega),$$

where $\widetilde{f}$ is the restriction of f to a desingularization $\widetilde{W(\mathbb{C})}$ of the closure $\overline{W(\mathbb{C})}$ of $W(\mathbb{C})$ and i denotes the map $\widetilde{W(\mathbb{C})} \longrightarrow \mathcal{X}_\infty$ coming from the inclusion $W(\mathbb{C}) \subset \mathcal{X}_\infty$.
Then $[-\log |f|^2]$ defines a Green's current for $\mathrm{div}(f)$ such that

$$\mathrm{dd}^c[-\log |f|^2] + \delta_{\mathrm{div}(f)(\mathbb{C})} = 0,$$

see [So1] for instance.

Example 2.9. Consider the arithmetic variety $\mathcal{X} = \mathbb{P}^d_{\mathbb{Z}} = \operatorname{Proj}\mathbb{Z}[x_0, \dots, x_d]$ and the cycle

$$Z = \{\langle a^{(0)}, x\rangle = \cdots = \langle a^{(p-1)}, x\rangle = 0\}$$

of codimension p, where for $0 \leq i \leq p-1$ and $x = (x_0, \dots, x_d)$ we set $\langle a^{(i)}, x\rangle := a_0^{(i)} x_0 + \cdots + a_d^{(i)} x_d \in \mathbb{Z}[x_0, \dots, x_d]$. The *Levine form g_Z associated to Z* is defined by

$$g_Z(x) = -\log\left(\frac{\sum_{i=0}^{p-1}\left|\langle a^{(i)}, x\rangle\right|^2}{\sum_{i=0}^{d}|x_i|^2}\right)\cdot\left(\sum_{j=0}^{p-1}\left(\mathrm{dd}^c\log\left(\sum_{i=0}^{p-1}\left|\langle a^{(i)}, x\rangle\right|^2\right)\right)^j\wedge\omega_{\mathrm{FS}}^{p-1-j}\right),$$

where $\omega_{\mathrm{FS}} = \mathrm{dd}^c\log\left(|x_0|^2 + \cdots + |x_d|^2\right)$ is the Fubini-Study form on $\mathbb{P}^d_{\mathbb{C}}$, see [BY1, equation (6)]. By [BGS, Proposition 1.4.1] the Levine form g_Z associated to Z satisfies $\mathrm{dd}^c[g_Z] + \delta_{Z(\mathbb{C})} = [\omega_{\mathrm{FS}}^p]$. Hence the associated current $[g_Z]$ is an example of a Green's current.
Now let us describe g_Z locally as in [GS1]. For this let $\widetilde{\mathcal{X}_\infty}$ be the blow up of $\mathcal{X}_\infty = \mathbb{P}^d_{\mathbb{C}}$ along $Z(\mathbb{C})$. Explicitly, it is given by

$$\widetilde{\mathcal{X}_\infty} = \left\{(x_0, \dots, x_d, y_0, \dots, y_{p-1}) \in \mathbb{P}^d_{\mathbb{C}} \times \mathbb{P}^{p-1}_{\mathbb{C}} \mid \exists t \in \mathbb{C} \text{ s.t. } \langle a^{(i)}, x\rangle = t y_i \ \forall i = 0, \dots, p-1\right\}.$$

Let $\pi : \widetilde{\mathcal{X}_\infty} \longrightarrow \mathcal{X}_\infty$ be the projection onto the first factor. Then we have $\pi^{-1}(Z(\mathbb{C})) = \{t = 0\}$ and $\pi : \widetilde{\mathcal{X}_\infty}\backslash\pi^{-1}(Z(\mathbb{C})) \longrightarrow \mathcal{X}_\infty\backslash Z(\mathbb{C})$ is an isomorphism. On $\widetilde{\mathcal{X}_\infty}\backslash\pi^{-1}(Z(\mathbb{C}))$ we define for $x = (x_0, \dots, x_d)$ and $y = (y_0, \dots, y_{p-1})$ the smooth form

$$\varphi(x, y) = -\log\left(\frac{\sum_{i=0}^{p-1}\left|\langle a^{(i)}, x\rangle\right|^2}{\sum_{i=0}^{d}|x_i|^2}\right)\cdot\left(\sum_{j=0}^{p-1}\left(\mathrm{dd}^c\log\left(\sum_{i=0}^{p-1}|y_i|^2\right)\right)^j\wedge\omega_{\mathrm{FS}}^{p-1-j}\right).$$

This form satisfies $\pi_*(\varphi) = g_Z$ and because $\langle a^{(i)}, x\rangle = t y_i$ for all $i = 0, \dots, p-1$ it follows that we can write $\varphi = -\alpha\log|t|^2 + \beta$ with the smooth ∂- and $\overline{\partial}$-closed form

$$\alpha(x, y) = \sum_{j=0}^{p-1}\left(\mathrm{dd}^c\log\left(\sum_{i=0}^{p-1}|y_i|^2\right)\right)^j\wedge\omega_{\mathrm{FS}}^{p-1-j}$$

and the smooth form

$$\beta(x, y) = -\log\left(\frac{\sum_{i=0}^{p-1}|y_i|^2}{\sum_{i=0}^{d}|x_i|^2}\right)\cdot\left(\sum_{j=0}^{p-1}\left(\mathrm{dd}^c\log\left(\sum_{i=0}^{p-1}|y_i|^2\right)\right)^j\wedge\omega_{\mathrm{FS}}^{p-1-j}\right)$$

on $\widetilde{\mathcal{X}_\infty}$. This shows that g_Z is a Green's form for Z.

Remark 2.10. *i)* For any divisor $D \in \mathrm{Z}^1(\mathcal{X})$ and for any Green's current g_D for D, there exists a metric $\|\cdot\|$ on the line bundle $\mathcal{O}_{\mathcal{X}}(D)$ such that

$$g_D = [-\log \|1_D\|^2],$$

where 1_D denotes the canonical section of $\mathcal{O}_{\mathcal{X}}(D)$, see [KMY, Proposition 1.2.14.].
ii) More generally, consider a cycle $Z \in \mathrm{Z}^p(\mathcal{X})$. Then every Green's current g_Z can be represented by a Green's form $\widetilde{g_Z}$ in the sense that

$$g_Z = [\widetilde{g_Z}] \text{ in } \widetilde{D}^{(p-1,p-1)}(\mathcal{X}),$$

see [BGS, Proposition 1.3.1. ii)].

Now we come to the definition of the height of a cycle and the arithmetic intersection product. For this we need the following notation.

Notation 2.11. Let $\mathcal{X}$ be an arithmetic variety of dimension $d+1$ and let p be an integer with $0 \leq p \leq d+1$. Then we define the integer $p^\vee$ with $0 \leq p^\vee \leq d+1$ by setting $p^\vee := d+1-p$.

Definition 2.12. Let $\widehat{\mathrm{Z}}^p(\mathcal{X})$ be the group of pairs (Z, g_Z), where $Z \in \mathrm{Z}^p(\mathcal{X})$ and g_Z is a Green's current for Z. Let $\widehat{\mathrm{R}}^p(\mathcal{X})$ be the subgroup of $\widehat{\mathrm{Z}}^p(\mathcal{X})$ generated by elements of the form
i) $(0, \partial u + \bar{\partial} v)$, where $u \in D^{(p-2,p-1)}(\mathcal{X}_\infty)$ and $v \in D^{(p-1,p-2)}(\mathcal{X}_\infty)$ and
ii) $\big(\mathrm{div}(f), [-\log |f|^2]\big)$, where $f = \sum f_W$ is a K_1-chain.
Then the *p-codimensional arithmetic Chow group of* $\mathcal{X}$ is defined by

$$\widehat{\mathrm{CH}}^p(\mathcal{X}) := \widehat{\mathrm{Z}}^p(\mathcal{X}) \big/ \widehat{\mathrm{R}}^p(\mathcal{X}).$$

Theorem 2.13. [So1, Theorem 2]
There exists an associative, commutative, bilinear pairing

$$\begin{aligned} \widehat{\mathrm{CH}}^p(\mathcal{X}) \times \widehat{\mathrm{CH}}^q(\mathcal{X}) &\longrightarrow \widehat{\mathrm{CH}}^{p+q}(\mathcal{X})_{\mathbb{Q}} := \widehat{\mathrm{CH}}^{p+q}(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Q} \\ ([Y, g_Y], [Z, g_Z]) &\longmapsto [Y, g_Y] \cdot [Z, g_Z], \end{aligned}$$

which makes $\oplus_{p \geq 0} \widehat{\mathrm{CH}}^p(\mathcal{X})_{\mathbb{Q}}$ *into a graded ring.*

Definition 2.14. *i)* The pairing in Theorem 2.13 is called the *arithmetic intersection product of* $[Y, g_Y]$ *and* $[Z, g_Z]$.
ii) The graded ring $\oplus_{p \geq 0} \widehat{\mathrm{CH}}^p(\mathcal{X})_{\mathbb{Q}}$ equipped with the arithmetic intersection product will be denoted by $\widehat{\mathrm{CH}}(\mathcal{X})_{\mathbb{Q}}$ and is called the *arithmetic Chow ring of* $\mathcal{X}$.

Remark 2.15. In Theorem 2.13 the arithmetic Chow group $\widehat{\mathrm{CH}}^{p+q}(\mathcal{X})$ needs to be tensored with $\mathbb{Q}$. The reason of this is the lack of a Moving Lemma over $\mathcal{X}$. Because of this, the definition of the pairing in Theorem 2.13 involves K-theory.

By Chow's Moving Lemma over the generic fibre X (see [Li2, Corollary 1.10, p.379]) we can assume that the cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^q(\mathcal{X})$ *intersect properly on* X. This means $Y_K \cap Z_K = \emptyset$ or $\mathrm{codim}(Y_K \cap Z_K) = \mathrm{codim}(Y_K) + \mathrm{codim}(Z_K)$, where Y_K resp. Z_K denotes the restriction of Y resp. Z to the generic fibre X. Then, as in [So1, Theorem 2], it is possible to define the pairing $[Y, g_Y] \cdot [Z, g_Z] \in \widehat{\mathrm{CH}}^{p+q}(\mathcal{X})_{\mathbb{Q}}$ between $[Y, g_Y] \in \widehat{\mathrm{CH}}^{p}(\mathcal{X})$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^{q}(\mathcal{X})$. If moreover the cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^q(\mathcal{X})$ intersect properly on the whole $\mathcal{X}$, then the pairing in Theorem 2.13 can be constructed explicitly by

$$[Y, g_Y] \cdot [Z, g_Z] := [Y.Z, g_Y * g_Z].$$

Here the cycle $Y.Z \in \mathrm{Z}^{p+q}(\mathcal{X})$ denotes the *geometric intersection product*

$$Y.Z = \sum \chi^x(Y, Z) \cdot \overline{\{x\}}.$$

This means that the sum runs through the irreducible $p+q$-codimensional subschemes x of $\mathcal{X}$ lying in the intersection $Y \cap Z$ and the integer $\chi^x(Y, Z) \in \mathbb{Z}$ denotes Serre's intersection multiplicity, see [So1, Theorem 1]. The datum $g_Y * g_Z$ is the $*$-*product of the Green's currents* g_Y *and* g_Z, explicitly given by

$$g_Y * g_Z := g_Y \wedge \delta_{Z(\mathbb{C})} + \omega_{g_Y} \wedge g_Z, \tag{2.1}$$

where ω_{g_Y} is the smooth form given by $\mathrm{dd}^c g_Y + \delta_{Y(\mathbb{C})} = [\omega_{g_Y}]$. Because of the equation

$$\mathrm{dd}^c (g_Y * g_Z) + \delta_{Y.Z(\mathbb{C})} = [\omega_{g_Y} \wedge \omega_{g_Z}], \tag{2.2}$$

see [So1, Theorem 4], it follows $[Y.Z, g_Y * g_Z] \in \widehat{\mathrm{CH}}^{p+q}(\mathcal{X})$.

Remark 2.16. Note that the current $g_Y \wedge \delta_{Z(\mathbb{C})}$ in equation (2.1) is a priori not defined. With Remark 2.10 we can find a Green's form $\widetilde{g_Y}$ for Y such that $[\widetilde{g_Y}] = g_Y$. Then the current $g_Y \wedge \delta_{Z(\mathbb{C})}$ is defined by $\widetilde{g_Y} \wedge \delta_{Z(\mathbb{C})}$. More explicitly, for $\eta \in A^{(d-p-q,d-p-q)}(\mathcal{X}_\infty)$ we set

$$g_Y \wedge \delta_{Z(\mathbb{C})}(\eta) := \widetilde{g_Y} \wedge \delta_{Z(\mathbb{C})}(\eta) = \int_{\widetilde{Z(\mathbb{C})}} \pi^*(\widetilde{g_Y}) \wedge \pi^*(\eta),$$

where $\pi : \widetilde{Z(\mathbb{C})} \longrightarrow Z(\mathbb{C})$ is a desingularization of $Z(\mathbb{C})$. Note that this is indeed independent of the choice of $\widetilde{g_Y}$, see [BGS, Proposition 1.3.1. ii)].

Notation 2.17. Because of Remark 2.16 we do not distinguish between forms and currents. For instance, equation (2.2) will be written as $\mathrm{dd}^c(g_Y * g_Z) + \delta_{Y.Z(\mathbb{C})} = \omega_{g_Y} \wedge \omega_{g_Z}$ and we write g_Y for a Green's current as well as for a Green's form for Y.

Proposition 2.18. [So1, Theorem 3]
i) Let $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ be a morphism of arithmetic varieties. Then there exists a pullback morphism

$$\varphi^* : \widehat{\mathrm{CH}}^p(\mathcal{Y}) \longrightarrow \widehat{\mathrm{CH}}^p(\mathcal{X}),$$

which satisfies $\varphi^*(\alpha \cdot \beta) = \varphi^*(\alpha) \cdot \varphi^*(\beta)$.
ii) If moreover $\varphi : \mathcal{X} \longrightarrow \mathcal{Y}$ is generically smooth, then there exists a pushforward morphism

$$\varphi_* : \widehat{\mathrm{CH}}^p(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^{p-\delta}(\mathcal{Y}),$$

where $\delta = \dim(\mathcal{Y}) - \dim(\mathcal{X})$ is the relative dimension of φ.
iii) There is the projection formula

$$\varphi_*(\varphi^*(\alpha) \cdot \beta) = \alpha \cdot \varphi_*(\beta) \in \widehat{\mathrm{CH}}^{p+q-\delta}(\mathcal{X}).$$

Definition 2.19. Consider the arithmetic variety $\operatorname{Spec} \mathcal{O}_K$ of dimension 1. An element in $\widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K)$ is given by $[\sum a_i \mathfrak{p}_i, (g_\sigma)_\sigma]$, where $\mathfrak{p}_i$ is a maximal ideal of $\mathcal{O}_K$, $a_i \in \mathbb{Z}$ and $(g_\sigma)_\sigma$ is a family of real number g_σ indexed by $\sigma : K \hookrightarrow \mathbb{C}$. Then the map $\widehat{\deg} : \widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K) \longrightarrow \mathbb{R}$ given by

$$\widehat{\deg}\left(\left[\sum a_i \mathfrak{p}_i, (g_\sigma)_\sigma\right]\right) = \sum n_i \log \#\left(\mathcal{O}_K / \mathfrak{p}_i\right) + \frac{1}{2}\sum g_\sigma$$

is called the *arithmetic degree map*. Here we note that in the case that $K = \mathbb{Q}$, the arithmetic degree map defines an isomorphism from $\widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathbb{Z})$ to $\mathbb{R}$.
For an arithmetic variety $\pi : \mathcal{X} \longrightarrow \operatorname{Spec} \mathcal{O}_K$ there is a pushforward morphism $\pi_* : \widehat{\mathrm{CH}}^{d+1}(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K)$ and hence a map $\widehat{\deg}_{\mathcal{X}} := \widehat{\deg} \circ \pi_* : \widehat{\mathrm{CH}}^{d+1}(\mathcal{X}) \longrightarrow \mathbb{R}$, which will also be called the *arithmetic degree map*. Explicitly this map is given by

$$\widehat{\deg}_{\mathcal{X}}\left(\left[\sum a_i P_i, g\right]\right) = \sum a_i \log \#k(P_i) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} g,$$

where $k(P_i)$ is the residue field of the point P_i on $\mathcal{X}$ and $g \in \widetilde{A}^{(d,d)}(\mathcal{X})$.

Remark 2.20. For an arithmetic variety $\mathcal{X}$ we consider the group homomorphisms

$$\omega : \widehat{\mathrm{CH}}^p(\mathcal{X}) \longrightarrow A^{(p,p)}(\mathcal{X}),\ [Z, g_Z] \longmapsto \omega_{g_Z} \text{ and}$$
$$a : \widetilde{A}^{(p-1,p-1)}(\mathcal{X}) \longrightarrow \widehat{\mathrm{CH}}^p(\mathcal{X}),\ [\eta] \longmapsto [0, \eta].$$

With this notion we obtain for $\eta \in \widetilde{A}^{(p-1,p-1)}(\mathcal{X})$ and $y \in \widehat{\mathrm{CH}}^q(\mathcal{X})$ the useful formula

$$a(\eta) \cdot y = a(\eta \wedge \omega(y)) \tag{2.3}$$

in $\widehat{\mathrm{CH}}^{p+q}(\mathcal{X})$, see [So1, Remark 2.3.1].

Definition 2.21. For an arithmetic variety $\mathcal{X}$ of dimension $d+1$ let $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X})$, where $p^\vee$ is as in Notation 2.11. Moreover, let $(\widetilde{Y}, g_{\widetilde{Y}})$ be a representative of $[Y, g_Y]$ such that $\widetilde{Y}$ and Z intersect properly on the generic fibre X.
i) The *arithmetic intersection number of* $[Y, g_Y]$ *and* $[Z, g_Z]$ is the real number

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) \in \mathbb{R}.$$

ii) For $([Y, g_Y]|Z) := [\widetilde{Y}.Z, g_{\widetilde{Y}} \wedge \delta_{Z(\mathbb{C})}] \in \widehat{\mathrm{CH}}^{d+1}(\mathcal{X})$ the *height of* Z *with respect to* $[Y, g_Y]$ is the real number

$$\mathrm{ht}_{[Y,g_Y]}(Z) := \widehat{\deg}_{\mathcal{X}}\left([Y, g_Y]|Z\right) \in \mathbb{R}.$$

iii) If the cycles $\widetilde{Y}$ and Z intersect properly on $\mathcal{X}$ and if $\widetilde{Y}.Z = \sum a_i P_i$ with $a_i \in \mathbb{Z}$, then the *geometric intersection number of* $\widetilde{Y}$ *and* Z is the real number

$$(\widetilde{Y}, Z)_{\mathrm{fin}} := \sum a_i \log \#k(P_i) \in \log \mathbb{Q}.$$

Remark 2.22. An explicit consideration of the arithmetic degree map and the arithmetic intersection product shows that the height of Z with respect to $[Y, g_Y]$ can be written as

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y]|Z\right) = \widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z] - a\left(g_Z \wedge \omega_{g_Y}\right)\right),$$

where g_Z is any choice of Green's current for Z. It follows that we have

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) = \mathrm{ht}_{[Y,g_Y]}(Z) + \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{g_Y} \wedge g_Z \text{ with}$$
$$\mathrm{ht}_{[Y,g_Y]}(Z) = (\widetilde{Y}, Z)_{\mathrm{fin}} + \frac{1}{2}\int_{\mathcal{X}_\infty} g_{\widetilde{Y}} \wedge \delta_{Z(\mathbb{C})},$$

where $(\widetilde{Y}, g_{\widetilde{Y}})$ is a representative of $[Y, g_Y]$ such that $\widetilde{Y}$ and Z intersect properly on X.

Remark 2.23. Let $\mathcal{X}$ be an arithmetic surface. For two arithmetic divisors $(\mathcal{D}_1, g_{\mathcal{D}_1})$ and $(\mathcal{D}_2, g_{\mathcal{D}_2})$ in $\widehat{\mathrm{Z}}^1(\mathcal{X})$, the arithmetic intersection number $\widehat{\deg}_{\mathcal{X}}\left([\mathcal{D}_1, g_{\mathcal{D}_1}] \cdot [\mathcal{D}_2, g_{\mathcal{D}_2}]\right)$ coincides with the arithmetic intersection number $(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2})$ as in equation (1.1). In other words we have the following commutative diagram:

$$\begin{array}{ccccc}
\widehat{\mathrm{Z}}^1(\mathcal{X}) \times \widehat{\mathrm{Z}}^1(\mathcal{X}) & \longrightarrow & \widehat{\mathrm{CH}}^1(\mathcal{X}) \times \widehat{\mathrm{CH}}^1(\mathcal{X}) & \xrightarrow{\cdot} & \widehat{\mathrm{CH}}^2(\mathcal{X}) \\
 & \searrow^{\cdot} & \downarrow & \swarrow_{\widehat{\deg}_{\mathcal{X}}} & \downarrow{\scriptstyle \pi_*} \\
 & & \mathbb{R} & \xleftarrow[\widehat{\deg}]{} & \widehat{\mathrm{CH}}^1(\operatorname{Spec} \mathcal{O}_K)
\end{array}$$

Now we state two important facts about the height of a cycle.

Proposition 2.24. [Bo, Proposition 2.3.1 iv)]
Consider a proper morphism $\varphi : \mathcal{Y} \longrightarrow \mathcal{X}$ of arithmetic varieties. Then for $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^q(\mathcal{Y})$ with $p + q = \dim(\mathcal{Y})$ and $\dim(\varphi(Z)) = \dim(Z)$, the following equation holds:

$$\mathrm{ht}_{\varphi^*([Y,g_Y])}(Z) = \mathrm{ht}_{[Y,g_Y]}(\varphi_*(Z)).$$

Proposition 2.25. *Let $\mathcal{X}$ be an arithmetic variety. Let $[Y_1, g_{Y_1}], \ldots, [Y_n, g_{Y_n}] \in \widehat{\mathrm{CH}}^{p_i}(\mathcal{X})$ and let $Z \in \mathrm{Z}^q(\mathcal{X})$ such that $\sum p_i + q = \dim(\mathcal{X})$. If $g'_{Y_1}, \ldots, g'_{Y_n}$ is another family of Green's forms for the cycles $Y_1, \ldots, Y_n$, then the following equation holds:*

$$\begin{aligned}
&\mathrm{ht}_{\prod_{i=1}^n [Y_i, g_{Y_i}]}(Z) - \mathrm{ht}_{\prod_{i=1}^n [Y_i, g'_{Y_i}]}(Z) = \\
&\frac{1}{2} \sum_{i=1}^{n} \int_{\mathcal{X}_\infty} \omega_{g'_{Y_1}} \wedge \cdots \wedge \omega_{g'_{Y_{i-1}}} \wedge \omega_{g_{Y_{i+1}}} \wedge \cdots \wedge \omega_{g_{Y_n}} \wedge \left(g_{Y_i} - g'_{Y_i}\right) \wedge \delta_{Z(\mathbb{C})}.
\end{aligned}$$

Here $\prod$ denotes the arithmetic intersection product as in Theorem 2.13.

Proof. First note that the arithmetic Chow ring $\widehat{\mathrm{CH}}(\mathcal{X})_{\mathbb{Q}}$ of $\mathcal{X}$ is a commutative graded ring. By equation (2.3) and Remark 2.22 it remains to show that the equation

$$\prod_{i=1}^{n} [Y_i, g_{Y_i}] - \prod_{i=1}^{n} [Y_i, g'_{Y_i}] = \sum_{i=1}^{n} [0, g_{Y_i} - g'_{Y_i}] \cdot \left(\prod_{k=1}^{i-1} [Y_k, g'_{Y_k}] \cdot \prod_{k=i+1}^{n} [Y_k, g_{Y_k}] \right) \tag{2.4}$$

holds in $\widehat{\mathrm{CH}}(\mathcal{X})_{\mathbb{Q}}$. Indeed, if we use equation (2.3), we see that the right side of (2.4) is given by the cycle

$$\sum_{i=1}^{n}[0,(g_{Y_i}-g'_{Y_i})\cdot\omega_{g'_{Y_1}}\wedge\cdots\wedge\omega_{g'_{Y_{i-1}}}\wedge\omega_{g_{Y_{i+1}}}\wedge\cdots\wedge\omega_{g_{Y_n}}].$$

Applying the height of Z with respect to this cycle and using the fact that the height is linear in $\widehat{\mathrm{CH}}(\mathcal{X})_{\mathbb{Q}}$, we see that (2.4) proves Proposition 2.25.
Writing the right side of (2.4) as

$$\sum_{i=1}^{n}\left(\prod_{k=1}^{i-1}[Y_k,g'_{Y_k}]\cdot\left([Y_i,g_{Y_i}]-[Y_i,g'_{Y_i}]\right)\cdot\prod_{k=i+1}^{n}[Y_k,g_{Y_k}]\right),$$

the proof of equation (2.4) follows from the following general fact:
Let R be a commutative ring and let $a_1,\ldots,a_n,b_1,\ldots,b_n\in R$. Then

$$\prod_{i=1}^{n}a_i-\prod_{i=1}^{n}b_i=\sum_{i=1}^{n}\left(\prod_{k=1}^{i-1}b_k(a_i-b_i)\prod_{k=i+1}^{n}a_k\right). \tag{2.5}$$

The proof of (2.5) is an easy induction over n which can be seen by using the equation

$$\prod_{i=1}^{n+1}a_i-\prod_{i=1}^{n+1}b_i=\left(\prod_{i=1}^{n}a_i-\prod_{i=1}^{n}b_i\right)a_{n+1}+(a_{n+1}-b_{n+1})\prod_{i=1}^{n}b_i.$$

□

In the final part of this section we translate Definition 2.21 in the terminology of hermitian line bundles. This can be done because there is an isomorphism between the group of isomorphism classes of hermitian line bundles $\widehat{\mathrm{Pic}}\,(\mathcal{X})$ and the 1-codimensional arithmetic Chow group $\widehat{\mathrm{CH}}^1(\mathcal{X})$, see [So1, Proposition 1, p.67].

Definition 2.26. Let $\widehat{\mathrm{Pic}}\,(\mathcal{X})$ be the group of isomorphism classes of hermitian line bundles and let $\widehat{c}_1$ denotes the isomorphism $\widehat{\mathrm{Pic}}\,(\mathcal{X})\xrightarrow{\sim}\widehat{\mathrm{CH}}^1(\mathcal{X})$. Suppose $\overline{\mathcal{L}}\in\widehat{\mathrm{Pic}}\,(\mathcal{X})$.
i) The *arithmetic self-intersection number of* $\overline{\mathcal{L}}$ is given by

$$\overline{\mathcal{L}}^{d+1}:=\widehat{\deg}_{\mathcal{X}}\left(\widehat{c}_1\left(\overline{\mathcal{L}}\right)^{d+1}\right).$$

ii) For a cycle $Z\in\mathrm{Z}^p(\mathcal{X})$ the *height of* Z *with respect to* $\overline{\mathcal{L}}$ is defined by

$$\mathrm{ht}_{\overline{\mathcal{L}}}(Z):=\widehat{\deg}_{\mathcal{X}}\left(\widehat{c}_1\left(\overline{\mathcal{L}}\right)^{d+1-p}|Z\right).$$

Remark 2.27. Let us describe the arithmetic self-intersection number of a hermitian line bundle $\overline{\mathcal{L}}$ more explicit: Let (D, g_D) be an arithmetic divisor such that $[D, g_D] = \widehat{c}_1(\overline{\mathcal{L}})$. Then the height of D with respect to $\overline{\mathcal{L}}$ is given by

$$\mathrm{ht}_{\overline{\mathcal{L}}}(D) = (D_1, \ldots, D_d, D)_{\mathrm{fin}} + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_{D_1} * \cdots * g_{D_d} \wedge \delta_{D(\mathbb{C})},$$

where for $i = 1, \ldots, d$ the arithmetic cycle (D_i, g_{D_i}) is a representative of $[D, g_D]$ such that all D_i intersect pairwise properly on the generic fibre X and also intersect properly with D on X. Thus we have to choose a family $\{f_i\}_{i=1,\ldots,d}$ of rational functions $f_i \in k(\mathcal{X})^\times$, where all divisors $D - \mathrm{div}(f_1), \ldots, D - \mathrm{div}(f_d)$ and D intersect pairwise properly on X. Then the height of D with respect to $\overline{\mathcal{L}}$ is given by

$$\begin{aligned}\mathrm{ht}_{\overline{\mathcal{L}}}(D) = &(D - \mathrm{div}(f_1), \ldots, D - \mathrm{div}(f_d), D)_{\mathrm{fin}} + \\ &\frac{1}{2} \sum_{i=1}^{d} \int\limits_{\mathcal{X}_\infty} \omega_{g_D}^{i-1} \wedge \left(g_D + \log |f_i|^2\right) \wedge \delta_{(D - \mathrm{div}(f_{i+1}))(\mathbb{C})} \wedge \cdots \wedge \delta_{(D - \mathrm{div}(f_d))(\mathbb{C})} \wedge \delta_{D(\mathbb{C})}.\end{aligned}$$

It follows that the arithmetic self-intersection number of $\overline{\mathcal{L}}$ can be written as

$$\overline{\mathcal{L}}^{d+1} = \mathrm{ht}_{\overline{\mathcal{L}}}(D) + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \omega_{g_D}^d \wedge g_D.$$

2.2 Adjusted Green's forms

In this section we generalize the definition of an adjusted Green's function on arithmetic surfaces to an adjusted Green's form on higher dimensional arithmetic varieties. This is motivated by the following remark.

Remark 2.28. In [BGS, section 2.3.2.] the authors defined a special class of Green's forms g_Y for a cycle $Y \in \mathrm{Z}^p(\mathcal{X})$, which are called μ-normalized Green's currents for a Kähler form μ. These satisfy the conditions that ω_{g_Y} is harmonic with respect to μ and

$$\int\limits_{\mathcal{X}_\infty} \omega \wedge g_Y = 0$$

for all $\omega \in A^{(d-p,d-p)}(\mathcal{X})$ which are harmonic with respect to μ.

Now consider two natural numbers p, q with $p \cdot q = \dim(\mathcal{X})$. Then for all cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and all Green's forms g_Y for Y, we have

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y]^q\right) = \mathrm{ht}_{[Y,g_Y]^{q-1}}(Y) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_Y}^{q-1} \wedge g_Y.$$

So, instead of considering a condition on the integral, we could consider those Green's forms such that the height vanishes. This leads to the definition of an adjusted Green's form in the special case $Y \in \mathrm{Z}^1(\mathcal{X})$.

Definition 2.29. Let D be a divisor on the arithmetic variety $\mathcal{X}$ with $\dim(\mathcal{X}) = d + 1$. An *adjusted Green's form for* D is a Green's form g_D for D that satisfies

$$\mathrm{ht}_{[D,g_D]^d}(D) = 0.$$

Proposition 2.30. *Let D be a divisor with $\deg(D) \neq 0$ and let g_D be a Green's form for D. Then there is a unique number $\alpha \in \mathbb{R}$ such that $\alpha_D := g_D + \alpha$ is an adjusted Green's form for D. Explicitly,*

$$\alpha = -\frac{2}{d \cdot \deg(D)} \cdot \mathrm{ht}_{[D,g_D]^d}(D).$$

Proof. We have to show that the height $\mathrm{ht}_{[D,\alpha_D]^d}(D)$ vanishes. First note the equation

$$\mathrm{ht}_{[D,\alpha_D]^d}(D) = \mathrm{ht}_{[D,g_D+\alpha]^d}(D) = \mathrm{ht}_{[D,g_D]^d}(D) + d \cdot \mathrm{ht}_{\left[0,\alpha\cdot\omega_{g_D}^{d-1}\right]}(D).$$

Now the proposition follows immediately from the equation

$$2 \cdot \mathrm{ht}_{\left[0,\omega_{g_D}^{d-1}\right]}(D) = \int\limits_{\mathcal{X}_\infty} \omega_{g_D}^{d-1} \wedge \delta_{D(\mathbb{C})} = \int\limits_{\mathcal{X}_\infty} \omega_{g_D}^{d} = \deg(D).$$

□

A crucial fact about an adjusted Green's form α_D for a divisor D is the following: Let $\mathcal{Y}$ and $\mathcal{X}$ be two arithmetic varieties with $\dim(\mathcal{Y}) \geq \dim(\mathcal{X})$ such that there exists a surjective proper morphism $\varphi : \mathcal{Y} \longrightarrow \mathcal{X}$. Then for a Green's form g_D for a divisor D on $\mathcal{X}$, both numbers $\mathrm{ht}_{[D,g_D]^{d_\mathcal{X}}}(D)$ and $\mathrm{ht}_{(\varphi^*[D,g_D])^{d_\mathcal{Y}}}(\varphi^* D)$ are well-defined. Here $d_\mathcal{X}$ and $d_\mathcal{Y}$ denote the relative dimensions of $\mathcal{X}$ and $\mathcal{Y}$ resp. Hence we can use functorialities to compute an adjusted Green's form for $\varphi^* D$ in terms of an adjusted Green's form for D. For this we need the following lemma.

Lemma 2.31. *Let $\varphi : \mathcal{Y} \longrightarrow \mathcal{X}$ be a surjective proper morphism between arithmetic varieties $\mathcal{Y}$ and $\mathcal{X}$. Set $d_{\mathcal{Y}} := \dim(\mathcal{Y}) - 1$ and $d_{\mathcal{X}} := \dim(\mathcal{X}) - 1$. Let D be a divisor on $\mathcal{X}$ and let g_D be a Green's form for D. Then the following equation holds:*

$$\deg(D) \cdot \widehat{\deg}_{\mathcal{Y}} \left([\varphi^* D, \varphi^* g_D]^{d_{\mathcal{Y}}+1} \right) = \deg(\varphi^* D) \cdot \widehat{\deg}_{\mathcal{X}} \left([D, g_D]^{d_{\mathcal{X}}+1} \right). \tag{2.6}$$

Proof. First note that we can assume $\deg(D) \neq 0$. Because of $d_{\mathcal{Y}} \geq d_{\mathcal{X}}$ and the fact that the pullback is multiplicative, we obtain the equation

$$[\varphi^* D, \varphi^* g_D]^{d_{\mathcal{Y}}+1} = \varphi^* [D, g_D]^{d_{\mathcal{X}}+1} \cdot [\varphi^* D, \varphi^* g_D]^{d_{\mathcal{Y}}-d_{\mathcal{X}}}. \tag{2.7}$$

Let $\omega_{\mathcal{X}}$ be any form in $A^{(d_{\mathcal{X}}, d_{\mathcal{X}})}(\mathcal{X})$ satisfying $\int_{\mathcal{X}_\infty} \omega_{\mathcal{X}} = 2$. Then

$$[D, g_D]^{d_{\mathcal{X}}+1} = \left[0, \widehat{\deg}_{\mathcal{X}} \left([D, g_D]^{d_{\mathcal{X}}+1} \right) \omega_{\mathcal{X}} \right]$$

For instance we can take $\omega_{\mathcal{X}} := \frac{2}{\deg(D)} \cdot \omega_{g_D}^{d_{\mathcal{X}}}$. It follows

$$\varphi^* [D, g_D]^{d_{\mathcal{X}}+1} = \left[0, \widehat{\deg}_{\mathcal{X}} \left([D, g_D]^{d_{\mathcal{X}}+1} \right) \frac{2}{\deg(D)} \varphi^* \omega_{g_D}^{d_{\mathcal{X}}} \right].$$

By equation (2.3) and equation (2.7) we obtain

$$[\varphi^* D, \varphi^* g_D]^{d_{\mathcal{Y}}+1} = \left[0, \widehat{\deg}_{\mathcal{X}} \left([D, g_D]^{d_{\mathcal{X}}+1} \right) \frac{2}{\deg(D)} (\varphi^* \omega_{g_D})^{d_{\mathcal{Y}}} \right].$$

Taking the arithmetic degree map yields

$$\widehat{\deg}_{\mathcal{Y}} \left([\varphi^* D, \varphi^* g_D]^{d_{\mathcal{Y}}+1} \right) = \frac{1}{\deg(D)} \widehat{\deg}_{\mathcal{X}} \left([D, g_D]^{d_{\mathcal{X}}+1} \right) \cdot \int_{\mathcal{Y}_\infty} (\varphi^* \omega_{g_D})^{d_{\mathcal{Y}}}.$$

Hence equation (2.6) follows from $\int_{\mathcal{Y}_\infty} (\varphi^* \omega_{g_D})^{d_{\mathcal{Y}}} = \int_{\mathcal{Y}_\infty} \omega_{\varphi^* g_D}^{d_{\mathcal{Y}}} = \deg(\varphi^* D)$. □

Proposition 2.32. *Consider the situation as in Lemma 2.31 and assume that the divisor D satisfies $\deg(D) \neq 0$. Let α_D be an adjusted Green's form for D. Then the unique constant $\alpha_{\mathcal{Y}} \in \mathbb{R}$ such that $\varphi^* \alpha_D + \alpha_{\mathcal{Y}}$ is an adjusted Green's form for $\varphi^* D$ is defined by the relation*

$$\alpha_{\mathcal{Y}} \cdot d_{\mathcal{Y}} \cdot \deg(D) \cdot \deg(\varphi^* D) = \deg(D) \int_{\mathcal{Y}_\infty} \varphi^* \alpha_D \wedge (\omega_{\varphi^* \alpha_D})^{d_{\mathcal{Y}}} - \deg(\varphi^* D) \cdot \int_{\mathcal{X}_\infty} \alpha_D \wedge \omega_{\alpha_D}^{d_{\mathcal{X}}}. \tag{2.8}$$

Proof. Let α_D be an adjusted Green's form for D. Hence

$$\widehat{\deg}_{\mathcal{X}}\left([D,\alpha_D]^{d_{\mathcal{X}}+1}\right) = \mathrm{ht}_{[D,\alpha_D]^{d_{\mathcal{X}}}}(D) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \alpha_D \wedge \omega_{\alpha_D}^{d_{\mathcal{X}}} = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \alpha_D \wedge \omega_{\alpha_D}^{d_{\mathcal{X}}}.$$

In this case equation (2.6) is given by

$$\deg(D)\cdot\widehat{\deg}_{\mathcal{Y}}\left(\varphi^*\left[D,\alpha_D\right]^{d_{\mathcal{Y}}+1}\right) = \frac{\deg(\varphi^*D)}{2}\int\limits_{\mathcal{X}_\infty} \alpha_D \wedge \omega_{\alpha_D}^{d_{\mathcal{X}}}. \tag{2.9}$$

Moreover, if we use (2.22) and (2.23) we obtain the equation

$$\mathrm{ht}_{[\varphi^*D,\varphi^*\alpha_D+\alpha_{\mathcal{Y}}]^{d_{\mathcal{Y}}}}(\varphi^*D) = \mathrm{ht}_{[\varphi^*D,\varphi^*\alpha_D]^{d_{\mathcal{Y}}}}(\varphi^*D) + \frac{\alpha_{\mathcal{Y}}}{2}\cdot d_{\mathcal{Y}}\cdot\int\limits_{\mathcal{Y}_\infty}(\omega_{\varphi^*\alpha_D})^{d_{\mathcal{Y}}}. \tag{2.10}$$

Note that $\int_{\mathcal{Y}_\infty}(\omega_{\varphi^*\alpha_D})^{d_{\mathcal{Y}}} = \deg(\varphi^*D)$. Now using the equations

$$\mathrm{ht}_{[\varphi^*D,\varphi^*\alpha_D]^{d_{\mathcal{Y}}}}(\varphi^*D) = \widehat{\deg}_{\mathcal{Y}}\left([\varphi^*D,\varphi^*\alpha_D]^{d_{\mathcal{Y}}+1}\right) - \frac{1}{2}\int\limits_{\mathcal{Y}_\infty}\varphi^*\alpha_D\wedge(\omega_{\varphi^*\alpha_D})^{d_{\mathcal{Y}}}$$

and (2.9), the equation (2.10) vanishes if and only if

$$\alpha_{\mathcal{Y}} = \frac{\deg(D)\cdot\int_{\mathcal{Y}_\infty}\varphi^*\alpha_D\wedge\omega_{\varphi^*\alpha_D}^{d_{\mathcal{Y}}} - \deg(\varphi^*D)\cdot\int_{\mathcal{X}_\infty}\alpha_D\wedge\omega_{\alpha_D}^{d_{\mathcal{X}}}}{d_{\mathcal{Y}}\cdot\deg(\varphi^*D)\cdot\deg(D)}.$$

This shows the relation (2.8). □

Instead of considering only one divisor D, we can also define adjusted Green's forms associated to two cycles $Y\in\mathrm{Z}^p(\mathcal{X})$ and $Z\in\mathrm{Z}^{p^\vee}(\mathcal{X})$. For this note that the height is a bilinear pairing

$$\widehat{\mathrm{CH}}^p(\mathcal{X})\times\mathrm{Z}^{p^\vee}(\mathcal{X})\longrightarrow\mathbb{R},\ ([Y,g_Y],Z)\mapsto\mathrm{ht}_{[Y,g_Y]}(Z).$$

Definition 2.33. Let $\mathcal{X}$ be an arithmetic variety and let $Y\in\mathrm{Z}^p(\mathcal{X})$ and $Z\in\mathrm{Z}^{p^\vee}(\mathcal{X})$ be two cycles on $\mathcal{X}$, where $p^\vee$ is as in Notation 2.11. A *Z-adjusted Green's form for Y* is a Green's form $\alpha_{Y,Z}$ for Y such that

$$\mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z) = 0.$$

The subgroup of $\widehat{\mathrm{CH}}^p(\mathcal{X})$ consisting of the Z-adjusted Green's forms, where $Z\in\mathrm{Z}^{p^\vee}(\mathcal{X})$, will be called the *$p$-codimensional Z-arithmetic Chow group of $\mathcal{X}$* and is denoted by

$$\widehat{\mathrm{CH}}^p(\mathcal{X},Z) := \{\widehat{\alpha}\in\widehat{\mathrm{CH}}^p(\mathcal{X})\mid \mathrm{ht}_{\widehat{\alpha}}(Z)=0\}.$$

Remark 2.34. *i)* The definition of a Z-adjusted Green's form $\alpha_{Y,Z}$ for Y shows that the pair $(Y, \alpha_{Y,Z})$ only depends on its arithmetic Chow group. More precisely, let $\alpha_{Y,Z}$ be a Z-adjusted Green's form for Y and let $(\widetilde{Y}, \alpha_{\widetilde{Y},Z})$ be a representative of $[Y, \alpha_{Y,Z}]$. Then $\alpha_{\widetilde{Y},Z}$ is a Z-adjusted Green's form for $\widetilde{Y}$.
ii) Because the height is a function on $\widehat{\mathrm{CH}}^p(\mathcal{X}) \times \mathrm{Z}^{p^\vee}(\mathcal{X})$, we see that

$$[\mathrm{div}(f), -\log|f|^2] \in \widehat{\mathrm{CH}}^p(\mathcal{X}, Z)$$

for all $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$.
iii) The aim of Definition 2.33 is to define an arithmetic intersection number, where the cycles meet non-properly on the generic fibre X. However, Definition 2.33 also makes sense when the cycles Y and Z intersect properly on X or even intersect properly on the whole arithmetic variety $\mathcal{X}$.

Proposition 2.35. *Let $\mathcal{X}$ be an arithmetic variety and let $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ be two cycles.*
i) Let g_Y be a Green's form for $Y \in \mathrm{Z}^p(\mathcal{X})$ and let $\omega_{Y,Z} \in A^{(p-1,p-1)}(\mathcal{X})$ be a smooth form such that $\int_{\mathcal{X}_\infty} \omega_{Y,Z} \wedge \delta_{Z(\mathbb{C})} \neq 0$. Then there exists a unique real number α such that $\alpha_{Y,Z} := g_Y + \alpha \cdot \omega_{Y,Z}$ is a Z-adjusted Green's form for Y. More explicitly,

$$\alpha = -2 \cdot \frac{\mathrm{ht}_{[Y,g_Y]}(Z)}{\int_{\mathcal{X}_\infty} \omega_{Y,Z} \wedge \delta_{Z(\mathbb{C})}}$$

ii) Let $\alpha_{Y,Z}$ and $\widetilde{\alpha}_{Y,Z}$ be two Z-adjusted Green's forms for Y. Then the differential form $\omega_{Y,Z} := \alpha_{Y,Z} - \widetilde{\alpha}_{Y,Z}$ satisfies

$$\int_{\mathcal{X}_\infty} \omega_{Y,Z} \wedge \delta_{Z(\mathbb{C})} = 0.$$

Proof. *i)* We have to show that $\mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z)$ vanishes for $\alpha_{Y,Z} = g_Y + \alpha \cdot \omega_{Y,Z}$. From the bilinearity of the height we obtain

$$\mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z) = \mathrm{ht}_{[Y,g_Y]}(Z) + \alpha \cdot \mathrm{ht}_{[0,\omega_{Y,Z}]}(Z).$$

The proof follows from

$$\mathrm{ht}_{[0,\omega_{Y,Z}]}(Z) = \frac{1}{2} \int_{\mathcal{X}_\infty} \omega_{Y,Z} \wedge \delta_{Z(\mathbb{C})}$$

and the definition of the constant α.
ii) For two Z-adjusted Green's forms $\alpha_{Y,Z}$ and $\widetilde{\alpha}_{Y,Z}$ for Y we have

$$0 = \mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z) - \mathrm{ht}_{[Y,\widetilde{\alpha}_{Y,Z}]}(Z) = \mathrm{ht}_{[0,\omega_{Y,Z}]}(Z) = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{Y,Z} \wedge \delta_{Z(\mathbb{C})},$$

which proves the proposition. □

The next proposition shows a more explicit characterization of adjusted Green's forms.

Proposition 2.36. *Let $\mathcal{X}$ be an arithmetic variety and let $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ be two irreducible and reduced cycles on $\mathcal{X}$. Let g_Y be a Green's form for Y. Then g_Y is a Z-adjusted Green's form for Y if and only if for all K_1-chains f with $\mathrm{div}(f) \in \mathrm{Z}^p(\mathcal{X})$ and where $\mathrm{div}(f) - \mathrm{ord}_Y(f)Y$ and Z intersect properly on the generic fibre, the following equation holds:*

$$(\mathrm{div}(f) - \mathrm{ord}_Y(f)Y, Z)_{\mathrm{fin}} = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(\log|f|^2 + \mathrm{ord}_Y(f)g_Y\right) \wedge \delta_{Z(\mathbb{C})}. \tag{2.11}$$

Proof. First recall that the height of $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ with respect to $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ is given by

$$\mathrm{ht}_{[Y,g_Y]}(Z) = (\widetilde{Y}, Z)_{\mathrm{fin}} + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} g_{\widetilde{Y}} \wedge \delta_{Z(\mathbb{C})},$$

where $(\widetilde{Y}, g_{\widetilde{Y}})$ is a representative of $[Y, g_Y]$ such that $\widetilde{Y}$ and Z intersect properly on the generic fibre X.
Now fix an integer $m \neq 0$ and let f be a K_1-chain such that $mY + \mathrm{div}(f)$ intersect Z properly on X. Then

$$\begin{aligned} m \cdot \mathrm{ht}_{[Y,g_Y]}(Z) = \mathrm{ht}_{[mY,mg_Y]}(Z) &= (mY + \mathrm{div}(f), Z)_{\mathrm{fin}} + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(mg_Y - \log|f|^2\right) \wedge \delta_{Z(\mathbb{C})} \\ &= (\mathrm{div}(f) - \mathrm{ord}_Y(f)Y, Z)_{\mathrm{fin}} - \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(\log|f|^2 + \mathrm{ord}_Y(f)g_Y\right) \wedge \delta_{Z(\mathbb{C})}. \end{aligned}$$

Thus the height $\mathrm{ht}_{[Y,g_Y]}(Z)$ vanishes and hence g_Y is a Z-adjusted Green's form for Y if and only if the equation (2.11) holds. □

Remark 2.37. The definition of the height shows that equation (2.11) is independent of the K_1-chain f. Thus to calculate a Z-adjusted Green's form for Y, it is enough to consider only one K_1-chain f, where $\mathrm{ord}_Y(f) \neq 0$.

Proposition 2.38. *Let $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ be two cycles with the decomposition $Y = \sum \alpha_k Y_k$ and $Z = \sum \beta_l Z_l$ into their irreducible components. For all k and all l let α_{Y_k,Z_l} be a Z_l-adjusted Green's form for Y_k. Then*

$$\alpha_{Y,Z} := \sum_k \alpha_k \alpha_{Y_k}$$

is a Z-adjusted Green's form for Y, where α_{Y_k} is any Green's form for Y_k such that

$$\sum_l \beta_l \int_{\mathcal{X}_\infty} (\alpha_{Y_k} - \alpha_{Y_k,Z_l}) \wedge \delta_{Z_l(\mathbb{C})} = 0. \tag{2.12}$$

Proof. By the bilinearity of the height we obtain with $\alpha_{Y,Z} = \sum_k \alpha_k \alpha_{Y_k}$ the equation

$$\mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z) = \sum_{k,l} \alpha_k \beta_l \mathrm{ht}_{[Y_k,\alpha_{Y_k}]}(Z_l).$$

Because of $\mathrm{ht}_{[Y_k,\alpha_{Y_k},Z_l]}(Z_l) = 0$, the height $\mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z)$ is given by

$$\sum_{k,l} \alpha_k \beta_l \left(\mathrm{ht}_{[Y_k,\alpha_{Y_k}]}(Z_l) - \mathrm{ht}_{[Y_k,\alpha_{Y_k},Z_l]}(Z_l) \right).$$

By Proposition 2.25 this sum equals

$$\frac{1}{2} \sum_{k,l} \alpha_k \beta_l \int_{\mathcal{X}_\infty} (\alpha_{Y_k} - \alpha_{Y_k,Z_l}) \wedge \delta_{Z_l(\mathbb{C})}.$$

Now the relation (2.12) shows Proposition 2.38. □

Remark 2.39. It follows directly from Proposition 2.38 that adjusted Green's forms are linear in the first entry. By this we mean the following: Let $Y \in \mathrm{Z}^p(\mathcal{X})$ with the decomposition $Y = \sum \alpha_k Y_k$ into its irreducible components and let $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$. Then, if for all k the Green's form $\alpha_{Y_k,Z}$ is a Z-adjusted Green's form for Y_k, then $\alpha_{Y,Z} := \sum_k \alpha_k \alpha_{Y_k,Z}$ is a Z-adjusted Green's form for Y.

Proposition 2.40. *Let $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ be two cycles and let g_Y be a Green's form for Y and g_Z be a Green's form for Z. Consider two forms $\omega_{Y,Z} \in A^{(p-1,p-1)}(\mathcal{X})$ and $\omega_{Z,Y} \in A^{(p^\vee-1,p^\vee-1)}(\mathcal{X})$ that satisfy the relation*

$$\int_{\mathcal{X}_\infty} \left(\omega_{Y,Z} \wedge \delta_{Z(\mathbb{C})} - \omega_{Z,Y} \wedge \delta_{Y(\mathbb{C})} \right) = \int_{\mathcal{X}_\infty} \left(\omega_{g_Y} \wedge g_Z - \omega_{g_Z} \wedge g_Y \right). \tag{2.13}$$

Then $\alpha_{Y,Z} := g_Y + \omega_{Y,Z}$ is a Z-adjusted Green's form for Y if and only if $\alpha_{Z,Y} := g_Z + \omega_{Z,Y}$ is a Y-adjusted Green's form for Z.

Proof. For $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X})$ the arithmetic intersection number

$$\mathrm{ht}_{[Y,g_Y]}(Z) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_Y} \wedge g_Z$$

of $[Y, g_Y]$ and $[Z, g_Z]$ is symmetric. Hence the equation

$$\mathrm{ht}_{[Z,g_Z]}(Y) - \mathrm{ht}_{[Y,g_Y]}(Z) = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} (\omega_{g_Y} \wedge g_Z - \omega_{g_Z} \wedge g_Y) \tag{2.14}$$

holds. Now writing

$$\mathrm{ht}_{[Z,g_Z]}(Y) = \mathrm{ht}_{[Z,g_Z+\omega_{Z,Y}]}(Y) - \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{Z,Y} \wedge \delta_{Y(\mathbb{C})} \tag{2.15}$$

(also for $\mathrm{ht}_{[Y,g_Y]}(Z)$ with $\omega_{Y,Z}$ instead of $\omega_{Z,Y}$ and $\delta_{Z(\mathbb{C})}$ instead of $\delta_{Y(\mathbb{C})}$) and considering the equation (2.13), we obtain the equality

$$\mathrm{ht}_{[Z,\alpha_{Z,Y}]}(Y) = \mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z).$$

□

The next result follows directly from Proposition 2.40.

Corollary 2.41. *For $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ let $\alpha_{Y,Z}$ be a Z-adjusted Green's form for the cycle Y and let g_Z be an arbitrary Green's form for Z satisfying the equation*

$$\int\limits_{\mathcal{X}_\infty} \omega_{\alpha_{Y,Z}} \wedge g_Z = \int\limits_{\mathcal{X}_\infty} \omega_{g_Z} \wedge \alpha_{Y,Z}. \tag{2.16}$$

Then g_Z is a Y-adjusted Green's form for Z.

In the next proposition we will see that adjusted Green's forms satisfy a functoriality property. For this we consider the following situation:
Let $\mathcal{X}$ and $\mathcal{X}'$ be two arithmetic varieties with $\dim(\mathcal{X}) = \dim(\mathcal{X}')$ and consider a surjective proper morphism $\varphi : \mathcal{X} \longrightarrow \mathcal{X}'$. Moreover, consider two cycles $Y' \in \mathrm{Z}^p(\mathcal{X}')$ and $Z' \in \mathrm{Z}^{p^\vee}(\mathcal{X}')$ and two cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$. Let $\varphi^* Y' = \sum \alpha_k Y_k$ and $\varphi^* Z' = \sum \beta_l Z_l$ be the decompositions into the irreducible components of $\varphi^* Y'$ and $\varphi^* Z'$. Now let $\alpha_{Y',Z'}$ be a Z'-adjusted Green's form for Y' and let $\alpha_{Y,Z}$ be a Z-adjusted Green's form for Y with a decomposition $\varphi^* \alpha_{Y',Z'} = \sum \alpha_k \alpha_{Y_k}$ into Green's forms α_{Y_k} for Y_k.

Proposition 2.42. *Assume that the above situation holds.*
i) The property of being an adjusted Green's form is functorial. In other words $\varphi^*\alpha_{Y',Z'}$ *is a* φ^*Z'*-adjusted Green's form for* φ^*Y' *and* $\varphi_*\alpha_{Y,Z}$ *is a* φ_*Z*-adjusted Green's form for* φ_*Y*.*
ii) Fix two irreducible components $Y_\mu \in \varphi^*Y'$ *and* $Z_\nu \in \varphi^*Z'$ *on* $\mathcal{X}$*. Then there exists an explicit form* $\omega_{Z_\nu} \in A^{(p-1,p-1)}(\mathcal{X})$ *such that* $\alpha_{Y_\mu} + \omega_{Z_\nu}$ *is a* Z_ν*-adjusted Green's form for* Y_μ*. More precisely,* ω_{Z_ν} *is any form in* $A^{(p-1,p-1)}(\mathcal{X})$ *satisfying*

$$\int_{\mathcal{X}_\infty} \omega_{Z_\nu} \wedge \delta_{Z_\nu(\mathbb{C})} = \sum_{k \neq \mu} \alpha_k \left(2 \cdot \left(\widetilde{Y}_k, Z_\nu \right)_{\mathcal{X},\mathrm{fin}} + \int_{\mathcal{X}_\infty} \alpha_{\widetilde{Y}_k} \wedge \delta_{Z_\nu(\mathbb{C})} \right),$$

where $\widetilde{Y}_k := Y_k + \mathrm{div}(f_k)$ *intersect* Z_ν *generically properly and* $\alpha_{\widetilde{Y}_k} := \alpha_{Y_k} - \log|f_k|^2$ *for* K_1*-chains* f_k *with* $\mathrm{div}\,(f_k) \in \mathrm{Z}^p(\mathcal{X})$*.*

Proof. *i)* This follows from the fact that the height is functorial. Indeed, let us show that $\varphi^*\alpha_{Y',Z'}$ is a φ^*Z'-adjusted Green's form for φ^*Y'. For this note that for $\alpha \in \widehat{\mathrm{CH}}^p(\mathcal{X}')$ and $\xi \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ we have

$$\mathrm{ht}_{\varphi^*\alpha}(\xi) = \mathrm{ht}_\alpha(\varphi_*\xi), \tag{2.17}$$

see Proposition 2.24. Now consider the arithmetic cycle $\alpha = (Y', \alpha_{Y',Z'})$ and the cycle $\xi = \varphi^*Z'$. The projection formula implies $\varphi_*\varphi^*Z' = [k(\mathcal{X}) : k(\mathcal{X}')]Z'$. Hence we derive

$$\mathrm{ht}_{[\varphi^*Y',\varphi^*\alpha_{Y',Z'}]}(\varphi^*Z') = \mathrm{ht}_{[Y',\alpha_{Y',Z'}]}(\varphi_*\varphi^*Z') = [k(\mathcal{X}) : k(\mathcal{X}')]\mathrm{ht}_{[Y',\alpha_{Y',Z'}]}(Z') = 0,$$

which proves the assertion.
The push forward can be treated in a similar way by using the functoriality

$$\mathrm{ht}_\alpha(\varphi^*\xi) = \mathrm{ht}_{\varphi_*\alpha}(\xi),$$

where $\alpha \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $\xi \in \mathrm{Z}^{p^\vee}(\mathcal{X}')$ are explicitly given by $\alpha = (Y, \alpha_{Y,Z})$ and $\xi = \varphi_*Z$.
ii) Using Proposition 2.42 i) and considering the arithmetic cycle $\alpha = (Y', \alpha_{Y',Z'})$ and the cycle $\xi = Z_\nu$, we obtain the equation

$$\mathrm{ht}_{[\varphi^*Y',\varphi^*\alpha_{Y',Z'}]}(Z_\nu) = \mathrm{ht}_{[Y',\alpha_{Y',Z'}]}(\varphi_*Z_\nu) = [k(Z_\nu) : k(Z)]\mathrm{ht}_{[Y',\alpha_{Y',Z'}]}(Z') = 0.$$

Now using the Green's form $\varphi^*\alpha_{Y',Z'} = \sum \alpha_k \alpha_{Y_k}$ and the cycle $Y_\mu \in \varphi^*Y'$, it follows

$$\begin{aligned}\mathrm{ht}_{[Y_\mu,\alpha_{Y_\mu}+\omega_{Z_\nu}]}(Z_\nu) =& \mathrm{ht}_{[Y_\mu-\varphi^*Y',\alpha_{Y_\mu}-\varphi^*\alpha_{Y',Z'}+\omega_{Z_\nu}]}(Z_\nu) = \\ & \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{Z_\nu} \wedge \delta_{Z_\nu(\mathbb{C})} - \sum_{k\neq\mu} \alpha_k \mathrm{ht}_{[Y_k,\alpha_{Y_k}]}(Z_\nu).\end{aligned}$$

The vanishing of $\mathrm{ht}_{[Y_\mu,\alpha_{Y_\mu}+\omega_{Z_\nu}]}(Z_\nu)$ and hence the proposition follows from the explicit definitions of ω_{Z_ν} and $\mathrm{ht}_{[Y_k,\alpha_{Y_k}]}(Z_\nu)$. □

Proposition 2.43. *Consider n cycles $Y_1,\dots,Y_n$ with $Y_i \in \mathrm{Z}^{p_i}(\mathcal{X})$ such that all possible intersections are proper. Then the intersection cycle $Y := Y_1 \dots Y_n \in \mathrm{Z}^p(\mathcal{X})$ of $Y_1,\dots,Y_n$ is a well-defined cycle of codimension $p := \sum p_i$. For each i let g_{Y_i} be a Green's form for Y_i and let ω_{Y_i} be a smooth form in $A^{(p_i-1,p_i-1)}(\mathcal{X})$. Moreover, consider a cycle $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$. Now define the Green's form*

$$g_Y := g_{Y_1} * \cdots * g_{Y_n} + \sum_{k=1}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \omega_{Y_{i_1}} \wedge \bigwedge_{l=2}^{k} \mathrm{dd}^c\omega_{Y_{i_l}} \wedge \bigwedge_{\substack{j=1,\\ j\notin\{i_1,\cdots,i_k\}}}^{n} \omega_{g_{Y_j}} \tag{2.18}$$

for Y. Then g_Y is a Z-adjusted Green's form for Y if and only if

$$\mathrm{ht}_{\prod_{i=1}^n [Y_i,g_{Y_i}+\omega_{Y_i}]}(Z) = 0.$$

In particular, if for all i the form ω_{Y_i} is closed, then the Green's form in (2.18) is given by

$$g_Y = g_{Y_1} * \cdots * g_{Y_n} + \sum_{i=1}^{n} \omega_{Y_i} \wedge \bigwedge_{j\neq i} \omega_{g_{Y_j}}.$$

Proof. We show

$$\mathrm{ht}_{[Y,g_Y]}(Z) = \mathrm{ht}_{\prod_{i=1}^n [Y_i,g_{Y_i}+\omega_{Y_i}]}(Z). \tag{2.19}$$

Because of $[Y_1 \dots Y_n, g_{Y_1} * \cdots * g_{Y_n}] = \prod_{i=1}^n [Y_i, g_{Y_i}]$, the left side of equation (2.19) is given by $\mathrm{ht}_{\prod_{i=1}^n [Y_i,g_{Y_i}]}(Z) + \mathrm{ht}_{[0,\omega]}(Z)$, where we set

$$\omega := \sum_{k=1}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \omega_{Y_{i_1}} \wedge \bigwedge_{l=2}^{k} \mathrm{dd}^c\omega_{Y_{i_l}} \wedge \bigwedge_{\substack{j=1,\\ j\notin\{i_1,\cdots,i_k\}}}^{n} \omega_{g_{Y_j}}. \tag{2.20}$$

Now we consider the right side of (2.19). First note that $\widehat{\mathrm{CH}}(\mathcal{X})_{\mathbb{Q}}$ has a ring-structure. Now we need the following general fact: Let R be a ring and let $a_1,\dots,a_n,b_1,\dots,b_n \in R$. Then

$$\prod_{i=1}^{n}(a_i+b_i) - \prod_{i=1}^{n} a_i = \sum_{k=1}^{n} \sum_{1\le i_1<\cdots<i_k\le n} b_{i_1}\cdots b_{i_k} \prod_{\substack{j=1,\\ j\notin\{i_1,\cdots,i_k\}}}^{n} a_j.$$

Now with $a_i := [Y_i, g_{Y_i}]$ and $b_i := [0, \omega_{Y_i}]$ and because $[0, \omega_{Y_i}] \cdot [0, \omega_{Y_j}] = [0, \omega_{Y_i} \wedge \mathrm{dd}^c \omega_{Y_j}]$ holds in $\widehat{\mathrm{CH}}(\mathcal{X})_{\mathbb{Q}}$, we obtain the equality

$$\begin{aligned} \prod_{i=1}^{n} ([Y_i, g_{Y_i}] + [0, \omega_{Y_i}]) - \prod_{i=1}^{n} [Y_i, g_{Y_i}] = \\ \sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} [0, \omega_{Y_{i_1}} \wedge \mathrm{dd}^c \omega_{Y_{i_2}} \wedge \cdots \wedge \mathrm{dd}^c \omega_{Y_{i_k}}] \cdot \prod_{\substack{j=1, \\ j \notin \{i_1, \cdots, i_k\}}}^{n} [Y_j, g_{Y_j}]. \end{aligned} \tag{2.21}$$

Now the proof follows from the fact that the right side of (2.21) is given by

$$\sum_{k=1}^{n} \sum_{1 \le i_1 < \cdots < i_k \le n} [0, \omega_{Y_{i_1}} \wedge \bigwedge_{l=2}^{k} \mathrm{dd}^c \omega_{Y_{i_l}} \wedge \bigwedge_{\substack{j=1, \\ j \notin \{i_1, \cdots, i_k\}}}^{n} \omega_{g_{Y_j}}],$$

because then we obtain the equality of heights

$$\mathrm{ht}_{\prod_{i=1}^{n} [Y_i, g_{Y_i} + \omega_{Y_i}]}(Z) = \mathrm{ht}_{\prod_{i=1}^{n} [Y_i, g_{Y_i}]}(Z) + \mathrm{ht}_{[0,\omega]}(Z)$$

with ω as in (2.20).
Moreover, if for all i the form ω_{Y_i} is closed, i.e. $\mathrm{dd}^c \omega_{Y_i} = 0$, then in the definition of ω in equation (2.20), the sum vanishes of all $k \neq 1$. Hense with $n = 1$ we have $\omega = \sum_{i=1}^{n} \omega_{Y_i} \wedge \bigwedge_{j \neq i} \omega_{g_{Y_j}}$. □

Because of Proposition 2.43 we need to generalize the definition of an adjusted Green's form for one cycle to a family of adjusted Green's forms for a family of cycles. This will be useful in the study of multiple arithmetic intersections.

Definition 2.44. Fix a cycle $Z \in \mathrm{Z}^q(\mathcal{X})$ and consider n cycles $Y_1, \ldots, Y_n$ with $Y_i \in \mathrm{Z}^{p_i}(\mathcal{X})$ such that $\sum p_i + q = \dim(\mathcal{X})$. For each i let g_{Y_i} be a Green's form for Y_i. Then the family $\{g_{Y_i}\}_{i=1,\ldots,n}$ is a *family of Z-adjusted Green's forms for* $\{Y_i\}_{i=1,\ldots,n}$ if the following equation holds:

$$\mathrm{ht}_{\prod_{i=1}^{n} [Y_i, g_{Y_i}]}(Z) = 0.$$

Remark 2.45. *i)* For two cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ it is clear that a family of Z-adjusted Green's forms for $\{Y\}$ is given by $\{\alpha_{Y,Z}\}$, where $\alpha_{Y,Z}$ is a Z-adjusted Green's form for Y.
ii) Let D be a divisor on the arithmetic variety $\mathcal{X}$ with $\dim(\mathcal{X}) = d+1$. Then an adjusted Green's form for D is the same as a family $\{\alpha_D, \ldots, \alpha_D\}$ (d-times) of D-adjusted Green's forms for $\{D, \ldots, D\}$ (d-times).

Proposition 2.46. *For each $i = 1, \dots, n$ let $Y_i \in \mathrm{Z}^{p_i}(\mathcal{X})$ and let $Z \in \mathrm{Z}^q(\mathcal{X})$ such that $q + \sum p_i = \dim(\mathcal{X})$. Moreover, for each i let g_{Y_i} be a Green's form for Y_i and let ω_{Y_i} be a closed, smooth form in $A^{(p_i-1,p_i-1)}(\mathcal{X})$. Then there exist a unique real number α such that $\{g_{Y_i} + \alpha \cdot \omega_{Y_i}\}_i$ is a family of Z-adjusted Green's forms for $\{Y_i\}_i$. More explicitly,*

$$\alpha = \frac{-2\mathrm{ht}_{\prod_{i=1}^n [Y_i, g_{Y_i}]}(Z)}{\sum_{j=1}^n \int_{\mathcal{X}_\infty} \omega_{Y_j} \wedge \bigwedge_{i \neq j} \omega_{g_{Y_i}} \wedge \delta_{Z(\mathbb{C})}}.$$

Here we assume $\sum_{j=1}^n \int_{\mathcal{X}_\infty} \omega_{Y_j} \wedge \bigwedge_{i \neq j} \omega_{g_{Y_i}} \wedge \delta_{Z(\mathbb{C})} \neq 0$.

Proof. This follows directly from Proposition 2.43. Indeed, because all ω_{Y_i} are closed, we have the equation

$$\mathrm{ht}_{\prod_{i=1}^n [Y_i, g_{Y_i} + \alpha \cdot \omega_{Y_i}]}(Z) = \mathrm{ht}_{\prod_{i=1}^n [Y_i, g_{Y_i}]}(Z) + \sum_{j=1}^n \mathrm{ht}_{[0, \alpha \cdot \omega_{Y_j} \wedge \bigwedge_{i \neq j} \omega_{g_{Y_i}}]}(Z). \tag{2.22}$$

The right side of (2.22) equals

$$\mathrm{ht}_{\prod_{i=1}^n [Y_i, g_{Y_i}]}(Z) + \frac{\alpha}{2} \sum_{j=1}^n \int_{\mathcal{X}_\infty} \omega_{Y_j} \wedge \bigwedge_{i \neq j} \omega_{g_{Y_i}} \wedge \delta_{Z(\mathbb{C})}. \tag{2.23}$$

Thus with the definition of α we see that the left side of (2.22) vanishes, which shows the proposition. □

Similar to Proposition 2.42 we consider the following situation:
Let $\mathcal{X}$ and $\mathcal{X}'$ be two arithmetic varieties with $\dim(\mathcal{X}) = \dim(\mathcal{X}')$ and consider a surjective proper morphism $\varphi : \mathcal{X} \longrightarrow \mathcal{X}'$. Fix a cycle Z' on $\mathcal{X}'$ and a cycle Z on $\mathcal{X}$. Let $\{Y_i'\}_{i=1,\dots,n'}$ be a family of cycles on $\mathcal{X}'$ and let $\{Y_j\}_{j=1,\dots,n}$ be a family of cycles on $\mathcal{X}$ such that $\sum \mathrm{codim}(Y_j) + \mathrm{codim}(Z) = \sum \mathrm{codim}(Y_i') + \mathrm{codim}(Z') = \dim(\mathcal{X})$.

Proposition 2.47. *Assume that the above situation holds. Let $\{\alpha_{Y_i'}\}_{i=1,\dots,n'}$ be a family of Z'-adjusted Green's forms for $\{Y_i'\}_{i=1,\dots,n'}$ and let $\{\alpha_{Y_j}\}_{j=1,\dots,n}$ be a family of Z-adjusted Green's forms for $\{Y_j\}_{j=1,\dots,n}$. Then $\{\varphi^* \alpha_{Y_i'}\}_{i=1,\dots,n'}$ is a family of $\varphi^* Z'$-adjusted Green's forms for $\{\varphi^* Y_i'\}_{i=1,\dots,n'}$ and $\{\varphi_* \alpha_{Y_j}\}_{j=1,\dots,n}$ is a family of $\varphi_* Z$-adjusted Green's forms for $\{\varphi_* Y_j\}_{j=1,\dots,n}$.*

Proof. By Proposition 2.18 the pullback morphism is multiplicative. With the functoriality in Proposition 2.24 it follows

$$\mathrm{ht}_{\prod [\varphi^* Y_i', \varphi^* \alpha_{Y_i'}]}(\varphi^* Z') = \mathrm{ht}_{\varphi^* \prod [Y_i', \alpha_{Y_i'}]}(\varphi^* Z') = \mathrm{ht}_{\prod [Y_i', \alpha_{Y_i'}]}(\varphi_* \varphi^* Z').$$

Using $\varphi_*\varphi^*Z' = [k(\mathcal{X}) : k(\mathcal{X}')]Z'$ and the linearity of the height, the latter expression equals $[k(\mathcal{X}) : k(\mathcal{X}')]\mathrm{ht}_{\prod[Y_i', \alpha_{Y_i'}]}(Z')$ and hence vanishes by the definition of $\{\alpha_{Y_i'}\}_i$.
On the other hand we have the equation

$$\mathrm{ht}_{\prod[\varphi_*Y_j, \varphi_*\alpha_{Y_j}]}(\varphi_*Z) = \mathrm{ht}_{\varphi^*\prod[\varphi_*Y_j, \varphi_*\alpha_{Y_j}]}(Z) = \mathrm{ht}_{\prod[k(\mathcal{X}):k(\mathcal{X}')][Y_j, \alpha_{Y_j}]}(Z) = [k(\mathcal{X}) : k(\mathcal{X}')]^n \mathrm{ht}_{\prod[Y_j, \alpha_{Y_j}]}(Z) = 0,$$

which shows that $\{\varphi_*\alpha_{Y_j}\}_j$ is a family of φ_*Z-adjusted Green's forms for $\{\varphi_*Y_j\}_j$. □

2.3 Arithmetic intersection theory using adjusted Green's forms

In this section we want to define a version of the arithmetic intersection number with the help of an adjusted Green's form. First we do this without the use of the dirac current $\delta_{Z(\mathbb{C})}$. This will be done similar to Theorem 1.33 in the first chapter.

Proposition 2.48. *Let $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X})$. Moreover, let $\alpha_{Y,Z}$ be a Z-adjusted Green's form for Y and let $\alpha_{Z,Y}$ be a Y-adjusted Green's form for Z. Then we have the following arithmetic intersection numbers:*

$$\widehat{\deg}_{\mathcal{X}}([Y, \alpha_{Y,Z}] \cdot [Z, g_Z]) = \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{\alpha_{Y,Z}} \wedge g_Z;$$

$$\widehat{\deg}_{\mathcal{X}}([Y, g_Y] \cdot [Z, \alpha_{Z,Y}]) = \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{\alpha_{Z,Y}} \wedge g_Y.$$

Proof. The follows immediately from the definition of the arithmetic intersection number and the definition of an adjusted Green's form, e.g. we have the equation

$$\widehat{\deg}_{\mathcal{X}}([Y, \alpha_{Y,Z}] \cdot [Z, g_Z]) = \mathrm{ht}_{[Y, \alpha_{Y,Z}]}(Z) + \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{\alpha_{Y,Z}} \wedge g_Z = \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{\alpha_{Y,Z}} \wedge g_Z.$$

□

Definition 2.49. Suppose $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$. Let g_Y be a Green's form for Y and let g_Z be a Green's form for Z. If $\alpha_{Y,Z}$ is a Z-adjusted Green's form for Y, then we set

$$g_Y \bullet g_Z := \omega_{\alpha_{Y,Z}} \wedge g_Z - \omega_{g_Z} \wedge \alpha_{Y,Z} + \omega_{g_Z} \wedge g_Y$$

and call it the *normalized $*$-product of g_Y and g_Z*.

A priori the definition of the normalized $*$-product depends on the adjusted Green's form $\alpha_{Y,Z}$. Let us explain why it is useful not to emphasize this dependence. In the next remark we will show that $g_Y \bullet g_Z$ is integrable over $\mathcal{X}_\infty$. Moreover, in the next proposition below, we will see that this integral do not depend on $\alpha_{Y,Z}$. Hence although the normalized $*$-product $g_Y \bullet g_Z$ depends on the Z-adjusted Green's form $\alpha_{Y,Z}$, we only write $g_Y \bullet g_Z$ because we are only interested in the integral.

Remark 2.50. Let $Y \in \mathrm{Z}^p(\mathcal{X})$ and let g_Y be a Green's form for Y. Let f be a K_1-chain such that $\mathrm{div}(f) \in \mathrm{Z}^p(\mathcal{X})$. Then we set $\widetilde{Y} := Y + \mathrm{div}(f)$ and $g_{\widetilde{Y}} := g_Y - \log|f|^2$. Because $-\log|f|^2$ is a Z-adjusted Green's form for $\mathrm{div}(f)$ (see Remark 2.34 *ii)*) and because adjusted Green's forms are linear in the first entry (see Remark 2.39) we set $\alpha_{\widetilde{Y},Z} := \alpha_{Y,Z} - \log|f|^2$. It is easy to see that with this notion we have

$$g_{\widetilde{Y}} \bullet g_Z = g_Y \bullet g_Z.$$

Thus we can assume that Y and Z intersect properly on the generic fibre X. In other words we can assume $Y(\mathbb{C}) \cap Z(\mathbb{C}) = \emptyset$.
Now consider $\mathrm{dd}^c(g_Y \bullet g_Z)$. It is clear that

$$\mathrm{dd}^c(g_Y \bullet g_Z) = \omega_{g_Y} \wedge \omega_{g_Z} - \omega_{\alpha_{Y,Z}} \wedge \delta_{Z(\mathbb{C})}.$$

Moreover, note that $\omega_{\alpha_{Y,Z}} \wedge \delta_{Z(\mathbb{C})} = \omega_{\alpha_{Y,Z}} * g_Z = g_Z * \omega_{\alpha_{Y,Z}} = \omega_{\alpha_{Y,Z}} \wedge \omega_{g_Z}$ in $\widetilde{D}^{(d,d)}(\mathcal{X})$. In a similar way we find $\omega_{\alpha_{Y,Z}} \wedge \omega_{g_Z} = \omega_{g_Y} \wedge \omega_{g_Z}$ in $\widetilde{D}^{(d,d)}(\mathcal{X})$. Therefore we have

$$\mathrm{dd}^c(g_Y \bullet g_Z) = 0$$

by dimension reasons. It follows $[0, g_Y \bullet g_Z] \in \widehat{\mathrm{CH}}^{d+1}(\mathcal{X})$.

Proposition 2.51. *Let $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X})$. Then the arithmetic intersection number of $[Y, g_Y]$ and $[Z, g_Z]$ is given by*

$$\widehat{\deg}_{\mathcal{X}}([Y, g_Y] \cdot [Z, g_Z]) = \frac{1}{2} \int_{\mathcal{X}_\infty} g_Y \bullet g_Z.$$

Proof. Let $\alpha_{Y,Z}$ be a Z-adjusted Green's form for Y. Then the normalized $*$-product of g_Y and g_Z is given by $g_Y \bullet g_Z = \omega_{\alpha_{Y,Z}} \wedge g_Z - \omega_{g_Z} \wedge \alpha_{Y,Z} + \omega_{g_Z} \wedge g_Y$. By Proposition 1.30, the height $\mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z)$ vanishes. It follows from equation (2.14) that

$$\mathrm{ht}_{[Z,g_Z]}(Y) = \frac{1}{2} \int_{\mathcal{X}_\infty} \big(\omega_{\alpha_{Y,Z}} \wedge g_Z - \omega_{g_Z} \wedge \alpha_{Y,Z}\big).$$

Hence the proof follows from the definition of the arithmetic intersection number. □

The following is a version of the arithmetic intersection number with the use of the dirac current $\delta_{Z(\mathbb{C})}$.

Proposition 2.52. *Let $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X})$ be two arithmetic cycles and let $\alpha_{Y,Z}$ be a Z-adjusted Green's form for Y. Then the arithmetic intersection number of $[Y, g_Y]$ and $[Z, g_Z]$ is given by*

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) = \frac{1}{2} \int\limits_{\mathcal{X}_\infty} (g_Y - \alpha_{Y,Z}) \wedge \delta_{Z(\mathbb{C})} + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \omega_{g_Y} \wedge g_Z.$$

Proof. This follows from the fact that the equation

$$\mathrm{ht}_{[Y,g_Y]}(Z) = \mathrm{ht}_{[Y,g_Y]}(Z) - \mathrm{ht}_{[Y,\alpha_{Y,Z}]}(Z) = \mathrm{ht}_{[0,g_Y-\alpha_{Y,Z}]}(Z) = \frac{1}{2} \int\limits_{\mathcal{X}_\infty} (g_Y - \alpha_{Y,Z}) \wedge \delta_{Z(\mathbb{C})}$$

holds and from the definition of the arithmetic intersection number. □

Remark 2.53. Let us consider a special case of Proposition 2.52:
Assume that Y and Z intersect properly on the generic fibre X. Then we have

$$\begin{aligned}\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) =& \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_Y \wedge \delta_{Z(\mathbb{C})} - \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \alpha_{Y,Z} \wedge \delta_{Z(\mathbb{C})} + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \omega_{g_Y} \wedge g_Z \\ =& -\frac{1}{2} \int\limits_{\mathcal{X}_\infty} \alpha_{Y,Z} \wedge \delta_{Z(\mathbb{C})} + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_Y * g_Z.\end{aligned}$$

It follows the useful formula

$$\int\limits_{\mathcal{X}_\infty} \alpha_{Y,Z} \wedge \delta_{Z(\mathbb{C})} = -2\,(Y, Z)_{\mathrm{fin}}\,. \tag{2.24}$$

In the last part of this section we generalize Proposition 2.51 and Proposition 2.52 to a family of arithmetic cycles.

Proposition 2.54. *For $i = 1, \ldots, n+1$ let $[Y_i, g_{Y_i}] \in \widehat{\mathrm{CH}}^{p_i}(\mathcal{X})$ such that $\sum p_i = \dim(\mathcal{X})$.*

Then

$$i) \quad \widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}[Y_i, g_{Y_i}]\right) = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_{Y_1}} \wedge \cdots \wedge \omega_{g_{Y_n}} \wedge g_{Y_{n+1}} +$$
$$\frac{1}{2}\sum_{i=1}^{n}\int\limits_{\mathcal{X}_\infty} \omega_{\alpha_{Y_1}} \wedge \cdots \wedge \omega_{\alpha_{Y_{i-1}}} \wedge \omega_{g_{Y_{i+1}}} \wedge \cdots \wedge \omega_{g_{Y_n}} \wedge (g_{Y_i} - \alpha_{Y_i}) \wedge \delta_{Y_{n+1}(\mathbb{C})};$$

$$ii) \quad \widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}[Y_i, g_{Y_i}]\right) = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_{Y_{n+1}}} \wedge \cdots \wedge \omega_{g_{Y_2}} \wedge g_{Y_1} +$$
$$\frac{1}{2}\sum_{i=1}^{n}\int\limits_{\mathcal{X}_\infty} \omega_{\alpha_{Y_1}} \wedge \cdots \wedge \omega_{\alpha_{Y_{i-1}}} \wedge \omega_{g_{Y_{i+2}}} \wedge \cdots \wedge \omega_{g_{Y_{n+1}}} \wedge \left(\omega_{\alpha_{Y_i}} \wedge g_{Y_{i+1}} - \omega_{g_{Y_{i+1}}} \wedge \alpha_{Y_i}\right).$$

Here $\{\alpha_{Y_i}\}_{i=1,\ldots,n}$ *is a family of* Y_{n+1}*-adjusted Green's forms for* $\{Y_i\}_{i=1,\ldots,n}$.

Proof. *i)* We write

$$\widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}[Y_i, g_{Y_i}]\right) = \mathrm{ht}_{\prod_{i=1}^{n}[Y_i, g_{Y_i}]}(Y_{n+1}) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_{Y_1}} \wedge \cdots \wedge \omega_{g_{Y_n}} \wedge g_{Y_{n+1}}.$$

By the definition of the family $\{\alpha_{Y_i}\}_i$ of Y_{n+1}-adjusted Green's forms for $\{Y_i\}_i$ we have $\mathrm{ht}_{\prod_{i=1}^{n}[Y_i, \alpha_{Y_i}]}(Y_{n+1}) = 0$. Thus it remains to show

$$\mathrm{ht}_{\prod_{i=1}^{n}[Y_i, g_{Y_i}]}(Y_{n+1}) - \mathrm{ht}_{\prod_{i=1}^{n}[Y_i, \alpha_{Y_i}]}(Y_{n+1})$$
$$= \frac{1}{2}\sum_{i=1}^{n}\int\limits_{\mathcal{X}_\infty} \omega_{\alpha_{Y_1}} \wedge \cdots \wedge \omega_{\alpha_{Y_{i-1}}} \wedge \omega_{g_{Y_{i+1}}} \wedge \cdots \wedge \omega_{g_{Y_n}} \wedge (g_{Y_i} - \alpha_{Y_i}) \wedge \delta_{Y_{n+1}(\mathbb{C})}.$$

This was done in Proposition 2.25.

ii) The idea of the proof of the second part of Proposition 2.54 is the following: We write

$$\widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}[Y_i, g_{Y_i}]\right) = \mathrm{ht}_{\prod_{i=2}^{n+1}[Y_i, g_{Y_i}]}(Y_1) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_{Y_{n+1}}} \wedge \cdots \wedge \omega_{g_{Y_2}} \wedge g_{Y_1}.$$

Now we want to construct the height $\mathrm{ht}_{\prod_{i=1}^{n}[Y_i, \alpha_{Y_i}]}(Y_{n+1})$ from the height $\mathrm{ht}_{\prod_{i=2}^{n+1}[Y_i, g_{Y_i}]}(Y_1)$. By the bilinearity and the symmetry of the arithmetic intersection number, we have for all $i = 1, \ldots, n$ the equation

$$\left(\prod_{j=1}^{i}[Y_j, \alpha_{Y_j}] \cdot \prod_{j=i+2}^{n+1}[Y_j, g_{Y_j}]\right) \cdot [Y_{i+1}, g_{Y_{i+1}}] = \left(\prod_{j=1}^{i-1}[Y_j, \alpha_{Y_j}] \cdot \prod_{j=i+1}^{n+1}[Y_j, g_{Y_j}]\right) \cdot [Y_i, \alpha_{Y_i}].$$

It follows that for all $i = 1, \ldots, n$ we have

$$\begin{aligned}\mathrm{ht}_{\prod_{j=1}^{i}[Y_j,\alpha_{Y_j}]\cdot\prod_{j=i+2}^{n+1}[Y_j,g_{Y_j}]}(Y_{i+1}) - \mathrm{ht}_{\prod_{j=1}^{i-1}[Y_j,\alpha_{Y_j}]\cdot\prod_{j=i+1}^{n+1}[Y_j,g_{Y_j}]}(Y_i) = \\ \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{\alpha_{Y_1}} \wedge \cdots \wedge \omega_{\alpha_{Y_{i-1}}} \wedge \omega_{g_{Y_{i+1}}} \wedge \cdots \wedge \omega_{g_{Y_{n+1}}} \wedge \alpha_{Y_i} - \\ \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{\alpha_{Y_1}} \wedge \cdots \wedge \omega_{\alpha_{Y_i}} \wedge \omega_{g_{Y_{i+2}}} \wedge \cdots \wedge \omega_{g_{Y_{n+1}}} \wedge g_{Y_{i+1}}.\end{aligned}$$

The latter expression clearly equals

$$\frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{\alpha_{Y_1}} \wedge \cdots \wedge \omega_{\alpha_{Y_{i-1}}} \wedge \omega_{g_{Y_{i+2}}} \wedge \cdots \wedge \omega_{g_{Y_{n+1}}} \wedge \left(\omega_{g_{Y_{i+1}}} \wedge \alpha_{Y_i} - \omega_{\alpha_{Y_i}} \wedge g_{Y_{i+1}}\right).$$

Now using the fact that $\mathrm{ht}_{\prod_{j=1}^{n}[Y_j,\alpha_{Y_j}]}(Y_{n+1}) = 0$, the telescoping series yields

$$\begin{aligned}\mathrm{ht}_{\prod_{i=2}^{n+1}[Y_i,g_{Y_i}]}(Y_1) = \\ \sum_{i=1}^{n}\left(\mathrm{ht}_{\prod_{j=1}^{i-1}[Y_j,\alpha_{Y_j}]\cdot\prod_{j=i+1}^{n+1}[Y_j,g_{Y_j}]}(Y_i) - \mathrm{ht}_{\prod_{j=1}^{i}[Y_j,\alpha_{Y_j}]\cdot\prod_{j=i+2}^{n+1}[Y_j,g_{Y_j}]}(Y_{i+1})\right).\end{aligned}$$

The latter expression equals

$$\frac{1}{2}\sum_{i=1}^{n}\int_{\mathcal{X}_\infty} \omega_{\alpha_{Y_1}} \wedge \cdots \wedge \omega_{\alpha_{Y_{i-1}}} \wedge \omega_{g_{Y_{i+2}}} \wedge \cdots \wedge \omega_{g_{Y_{n+1}}} \wedge \left(\omega_{\alpha_{Y_i}} \wedge g_{Y_{i+1}} - \omega_{g_{Y_{i+1}}} \wedge \alpha_{Y_i}\right).$$

This proves the proposition. □

The next result is the special case of Proposition 2.54, where all cycles Y_i are given by one divisor D.

Corollary 2.55. *Let $[D, g_D] \in \widehat{\mathrm{CH}}^1(\mathcal{X})$ and let α_D be an adjusted Green's form for D. Then the following hold:*

i) $$\widehat{\deg}_{\mathcal{X}}\left([D,g_D]^{d+1}\right) = \frac{1}{2}\sum_{i+j=d-1}\int_{\mathcal{X}_\infty} \omega_{\alpha_D}^i \wedge \omega_{g_D}^j \wedge (g_D - \alpha_D) \wedge \delta_{D(\mathbb{C})} + \frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{g_D}^d \wedge g_D;$$

ii) $$\begin{aligned}\widehat{\deg}_{\mathcal{X}}\left([D,g_D]^{d+1}\right) = &\frac{1}{2}\int_{\mathcal{X}_\infty} (\omega_{\alpha_D} \wedge g_D - \omega_{g_D} \wedge \alpha_D) \wedge \left(\sum_{i+j=d-1} \omega_{\alpha_D}^i \wedge \omega_{g_D}^j\right) + \\ &\frac{1}{2}\int_{\mathcal{X}_\infty} \omega_{g_D}^d \wedge g_D;\end{aligned}$$

iii) If $g_D - \alpha_D = \alpha \in \mathbb{R}$ *then*

$$\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^{d+1}\right) = \frac{\alpha \cdot d \cdot \deg(D)}{2} + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_D}^d \wedge g_D.$$

Proof. The first two assertions are clear.
If $g_D - \alpha_D = \alpha \in \mathbb{R}$, then $\omega_{g_D} = \omega_{\alpha_D}$ and hence

$$\sum_{i+j=d-1} \int\limits_{\mathcal{X}_\infty} \omega_{\alpha_D}^i \wedge \omega_{g_D}^j \wedge (g_D - \alpha_D) \wedge \delta_{D(\mathbb{C})} = \alpha \cdot d \int\limits_{\mathcal{X}_\infty} \omega_{g_D}^{d-1} \wedge \delta_{D(\mathbb{C})} = \alpha \cdot d \cdot \deg(D).$$

□

2.4 Examples on $\operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_d]$

Let $\mathcal{X} = \mathbb{P}^d_{\mathbb{Z}} = \operatorname{Proj} \mathbb{Z}[x_0, \ldots, x_d]$ and consider the cycles $Y \in \mathrm{Z}^1(\mathcal{X})$ and $Z \in \mathrm{Z}^d(\mathcal{X})$, which are given by the equations

$$Y = \{\langle a^{(0)}, x\rangle = 0\} \text{ and } Z = \{\langle a^{(0)}, x\rangle = \cdots = \langle a^{(d-1)}, x\rangle = 0\},$$

where $\langle a^{(i)}, x\rangle := a_0^{(i)} x_0 + \cdots + a_d^{(i)} x_d$ such that the vectors $a^{(0)}, \ldots, a^{(d-1)}$ are linearly independent in $\mathbb{Z}^{d+1}$. In this section we compute some examples of adjusted Green's forms associated to these cycles. A canonical Green's form for a cycle in $\mathbb{P}^d_{\mathbb{Z}}$ is the Levine form as in Example 2.9. Recall that the Levine forms associated to Y and Z are given by

$$g_Y = -\log\left(\frac{\left|\langle a^{(0)}, x\rangle\right|^2}{|x_0|^2 + \cdots + |x_d|^2}\right) \text{ and}$$

$$g_Z = -\log\left(\frac{\sum_{i=0}^{d-1}\left|\langle a^{(i)}, x\rangle\right|^2}{\left(|x_0|^2 + \cdots + |x_d|^2\right)}\right) \cdot \left(\sum_{i=0}^{d-1}\left(\mathrm{dd}^c \log\left(\sum_{i=0}^{d-1}\left|\langle a^{(i)}, x\rangle\right|^2\right)\right)^i \wedge \omega_{\mathrm{FS}}^{d-1-i}\right),$$

where $\omega_{\mathrm{FS}} = \mathrm{dd}^c \log\left(|x_0|^2 + \cdots + |x_d|^2\right)$ is the Fubini-Study form on $\mathbb{P}^d_{\mathbb{C}}$.

Proposition 2.56. *The Green's form*

$$\alpha_{Y,Z} = g_Y - 2 \cdot \mathrm{ht}_{[Y,g_Y]}(Z)$$

is a Z*-adjusted Green's form for* Y*. More explicitly,*

$$\alpha_{Y,Z} = -\log\left(\frac{\left|\langle a^{(0)}, x\rangle\right|^2}{|x_0|^2 + \cdots + |x_d|^2}\right) - \log\left(|\det A_0|^2 + \cdots + |\det A_d|^2\right),$$

where for $i = 0, \ldots, d$ the matrices A_i are defined by $(e_{i+1}, a^{(0)}, \ldots, a^{(d-1)})$, where e_i is the i-th vector of the standard basis of $\mathbb{Z}^{d+1}$

Proof. By Proposition 2.35 and the fact that $\int_{\mathcal{X}_\infty} \delta_{Z(\mathbb{C})} = 1$ holds, it is clear that $\alpha_{Y,Z}$ is a Z-adjusted Green's form for Y.
We show $2 \cdot \mathrm{ht}_{[Y,g_Y]}(Z) = \log\left(|\det A_0|^2 + \cdots + |\det A_d|^2\right)$, where the matrices A_i are as in the proposition. For this we choose a rational function $f = \frac{\langle a^{(0)}, x\rangle}{\langle \alpha, x\rangle}$, where the vectors $\alpha, a^{(0)}, \ldots, a^{(d-1)}$ are linearly independent in $\mathbb{Z}^{d+1}$. Then

$$\mathrm{ht}_{[Y,g_Y]}(Z) = (Y - \mathrm{div}(f), Z)_{\mathrm{fin}} + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(g_Y + \log|f|^2\right) \wedge \delta_{Z(\mathbb{C})}. \tag{2.25}$$

The geometric intersection number in (2.25) is given by

$$(Y - \mathrm{div}(f), Z)_{\mathrm{fin}} = \left(\langle \alpha, x\rangle, \langle a^{(0)}, x\rangle, \ldots, \langle a^{(d-1)}, x\rangle\right)_{\mathrm{fin}} = \log|Res|\,,$$

where Res denotes the resultant of the polynomials $\langle \alpha, x\rangle, \langle a^{(0)}, x\rangle, \ldots, \langle a^{(d-1)}, x\rangle$, which is given by the determinant of the vectors $\alpha, a^{(0)}, \ldots, a^{(d-1)}$, see [Li1, Theorem 2.1.1] and [GKZ, Theorem 3.1., p.458]. Using the Green's form

$$g_Y = -\log\left|\langle a^{(0)}, x\rangle\right|^2 + \log\left(|x_0|^2 + \cdots + |x_d|^2\right)$$

and the rational function $f = \frac{\langle a^{(0)}, x\rangle}{\langle \alpha, x\rangle}$, hence $\log|f|^2 = \log\left|\langle a^{(0)}, x\rangle\right|^2 - \log|\langle \alpha, x\rangle|^2$, the integral in (2.25) is given by

$$\frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(g_Y + \log|f|^2\right) \wedge \delta_{Z(\mathbb{C})} = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(-\log|\langle \alpha, x\rangle|^2 + \log\left(|x_0|^2 + \cdots + |x_d|^2\right)\right) \wedge \delta_{Z(\mathbb{C})}.$$

Thus the height $\mathrm{ht}_{[Y,g_Y]}(Z)$ is given by

$$\log\left|\det(\alpha, a^{(0)}, \ldots, a^{(d-1)})\right| + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \left(-\log|\langle \alpha, x\rangle|^2 + \log\left(|x_0|^2 + \cdots + |x_d|^2\right)\right) \wedge \delta_{Z(\mathbb{C})}.$$

Now the proof follows from Laplace's formula for determinants, because therewith we see $\log\left|\det(\alpha, a^{(0)}, \ldots, a^{(d-1)})\right| = \log\left|\sum_{i=0}^{d} \alpha_{i+1}(-1)^i \det A_i\right|$, where α_i is the i-th entry of the vector α. Because the height is independent of the vector α, is follows that $\mathrm{ht}_{[Y,g_Y]}(Z)$ is given by $\frac{1}{2}\log\left(|\det A_0|^2 + \cdots + |\det A_d|^2\right)$. □

Proposition 2.57. *The Green's form*

$$\alpha_{Z,Y} = g_Z - \left(\log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right) + \sum_{m=1}^{d-1}\sum_{n=1}^{m} \frac{1}{n}\right)\omega_{\mathrm{FS}}^{d-1}$$

is a Y-adjusted Green's form for Z.

Proof. Using the Z-adjusted Green's form $\alpha_{Y,Z} = g_Y - 2 \cdot \mathrm{ht}_{[Y,g_Y]}(Z)$ for Y and Corollary 2.41 with $\omega_{\alpha_{Y,Z}} = \omega_{\mathrm{FS}}$ and $\omega_{\alpha_{Z,Y}} = \omega_{\mathrm{FS}}^d$, we have to show the equation

$$\int_{\mathcal{X}_\infty} \omega_{\mathrm{FS}} \wedge \left(g_Z - \left(\log \left(\left| a_0^{(0)} \right|^2 + \cdots + \left| a_d^{(0)} \right|^2 \right) + \sum_{m=1}^{d-1} \sum_{n=1}^{m} \frac{1}{n} \right) \omega_{\mathrm{FS}}^{d-1} \right)$$
$$= \int_{\mathcal{X}_\infty} \omega_{\mathrm{FS}}^d \wedge \left(g_Y - 2 \cdot \mathrm{ht}_{[Y,g_Y]}(Z) \right).$$

Note the well-known fact that

$$\int_{\mathcal{X}_\infty} \omega_{\mathrm{FS}}^d \wedge g_Y = -\log \left(\left| a_0^{(0)} \right|^2 + \cdots + \left| a_d^{(0)} \right|^2 \right) + \sum_{n=1}^{d} \frac{1}{n} \tag{2.26}$$

holds, see [CM, Corollaire 2.10]. Now, since $\int_{\mathcal{X}_\infty} \omega_{\mathrm{FS}}^d = 1$, it remains to show

$$\int_{\mathcal{X}_\infty} \omega_{\mathrm{FS}} \wedge g_Z = \sum_{m=1}^{d} \sum_{n=1}^{m} \frac{1}{n} - 2 \cdot \mathrm{ht}_{[Y,g_Y]}(Z). \tag{2.27}$$

Considering the classical hypersurface $H = \{x_1 = \cdots = x_d = 0\}$ we have

$$\int_{\mathcal{X}_\infty} \omega_{\mathrm{FS}} \wedge g_H = \sum_{m=1}^{d} \sum_{n=1}^{m} \frac{1}{n},$$

see [BGS, Proposition 1.4.1. p.922]. Because of $\mathrm{ht}_{[Y,g_Y]}(H) = 0$, the proof of this proposition follows from the equation

$$\int_{\mathcal{X}_\infty} \omega_{\mathrm{FS}} \wedge g_Z = \int_{\mathcal{X}_\infty} \omega_{\mathrm{FS}} \wedge g_H - 2 \left(\mathrm{ht}_{[Y,g_Y]}(Z) - \mathrm{ht}_{[Y,g_Y]}(H) \right).$$

□

The next result is an easy consequence of Corollary 2.30.

Corollary 2.58. *The Green's form*

$$\alpha_Y = g_Y - \frac{1}{d} \left(\log \left(\left| a_0^{(0)} \right|^2 + \cdots + \left| a_d^{(0)} \right|^2 \right) + \sum_{m=1}^{d-1} \sum_{n=1}^{m} \frac{1}{n} \right)$$

is an adjusted Green's form for Y.

Proof. This follows from Corollary 2.30 and the fact that

$$\mathrm{ht}_{[Y,g_Y]^d}(Y) = \frac{1}{2}\log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right) + \frac{1}{2}\sum_{m=1}^{d-1}\sum_{n=1}^{m}\frac{1}{n}$$

holds, see [CM, Corollaire 2.10]. □

Remark 2.59. We can give another proof of Corollary 2.58 using Proposition 2.43. Indeed, for $i = 1, \ldots, d-1$ consider the arithmetic divisors (Y_i, g_{Y_i}), where $Y_i = \{\langle a^{(i)}, x\rangle = 0\}$ and g_{Y_i} is the Levine form associated to Y_i. Note that they intersect pairwise properly on the whole $\mathcal{X}$ and are rationally equivalent to (Y, g_Y). Now consider the Green's form $g_Z := g_Y * g_{Y_1} * \cdots * g_{Y_{d-1}}$ for the intersecting cycle $Z = Y.Y_1 \ldots Y_{d-1}$. Then obviously $\mathrm{ht}_{[Z,g_Z]}(Y) = \mathrm{ht}_{[Y,g_Y]^d}(Y)$. By Proposition 2.43 it follows that $\alpha_Y := g_Y + \alpha$ is an adjusted Green's form for Y if and only if $\alpha_{Z,Y} := g_Z + d \cdot \alpha \cdot \omega_{g_Y}^{d-1}$ is a Y-adjusted Green's form for Z. Now we can show Corollary 2.58 using the Y-adjusted Green's form

$$\alpha_{Z,Y} = g_Z - \left(\log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right) + \sum_{m=1}^{d-1}\sum_{n=1}^{m}\frac{1}{n}\right)\omega_{\mathrm{FS}}^{d-1}$$

for $Z = \{\langle a^{(0)}, x\rangle = \cdots = \langle a^{(d-1)}, x\rangle = 0\}$.

Now we want to calculate an example of an arithmetic intersection number with the new formula in Corollary 2.55. More precisely, we want to calculate the arithmetic self-intersection number of the Serre twist equipped with the Fubini-Study metric.

Proposition 2.60. *Let $\mathcal{X} = \mathbb{P}^d_{\mathbb{Z}}$ and $\overline{\mathcal{L}} = (\mathcal{O}_{\mathcal{X}}(1), \|\cdot\|_{\mathrm{FS}})$. Then*

$$\overline{\mathcal{L}}^{d+1} = \frac{1}{2}\sum_{m=1}^{d}\sum_{n=1}^{m}\frac{1}{n} = \frac{d+1}{2}\sum_{i=2}^{d+1}\frac{1}{i}.$$

Proof. First note that the equation

$$\sum_{m=1}^{d}\sum_{n=1}^{m}\frac{1}{n} = (d+1)\sum_{i=2}^{d+1}\frac{1}{i}$$

is an easy induction over d. Now consider the arithmetic cycle (D, g_D), where $D = \{\langle a^{(0)}, x\rangle = 0\}$ and g_D is the Levine form associated to D. Then

$$\overline{\mathcal{L}}^{d+1} = \widehat{\deg}_{\mathcal{X}}\left([D, g_D]^{d+1}\right).$$

By Corollary 2.58 we know that

$$\alpha_D := g_D - \frac{1}{d}\left(\log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right) + \sum_{m=1}^{d-1}\sum_{n=1}^{m}\frac{1}{n}\right)$$

is an adjusted Green's form for D. Corollary 2.55 shows

$$\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^{d+1}\right) = \frac{d}{2}\deg(D)(g_D - \alpha_D) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_D}^d \wedge g_D.$$

Because of $\deg(D) = 1$, the proof follows from

$$g_D - \alpha_D = \frac{1}{d}\left(\log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right) + \sum_{m=1}^{d-1}\sum_{n=1}^{m}\frac{1}{n}\right)$$

and the fact that

$$\int\limits_{\mathcal{X}_\infty} \omega_{\mathrm{FS}}^d \wedge g_D = -\log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right) + \sum_{n=1}^{d}\frac{1}{n}$$

holds, see equation (2.26). □

In the last part of this section we verify the formulas for the arithmetic intersection number in Proposition 2.52.

• First consider the case

$$\overline{\mathcal{L}}^{d+1} = \widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) = \frac{1}{2}\int\limits_{\mathbb{P}^d_{\mathbb{C}}} (g_Y - \alpha_{Y,Z}) \wedge \delta_{Z(\mathbb{C})} + \frac{1}{2}\int\limits_{\mathbb{P}^d_{\mathbb{C}}} \omega_{\mathrm{FS}} \wedge g_Z,$$

where $\alpha_{Y,Z}$ is a Z-adjusted Green's form for Y. By Proposition 2.56 we can choose

$$\alpha_{Y,Z} = g_Y - 2 \cdot \mathrm{ht}_{[Y,g_Y]}(Z).$$

Because of

$$\int\limits_{\mathbb{P}^d_{\mathbb{C}}} \omega_{\mathrm{FS}} \wedge g_Z = \sum_{m=1}^{d}\sum_{n=1}^{m}\frac{1}{n} - 2 \cdot \mathrm{ht}_{[Y,g_Y]}(Z)$$

we clearly get

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) = \frac{1}{2}\sum_{m=1}^{d}\sum_{n=1}^{m}\frac{1}{n}.$$

• Now consider the case

$$\overline{\mathcal{L}}^{d+1} = \widehat{\deg}_{\mathcal{X}}\left([Z, g_Z] \cdot [Y, g_Y]\right) = \frac{1}{2}\int\limits_{\mathbb{P}^d_{\mathbb{C}}} (g_Z - \alpha_{Z,Y}) \wedge \delta_{Y(\mathbb{C})} + \frac{1}{2}\int\limits_{\mathbb{P}^d_{\mathbb{C}}} \omega_{\mathrm{FS}}^d \wedge g_Y,$$

where $\alpha_{Z,Y}$ is a Y-adjusted Green's form for Z. By Proposition 2.57 we can choose

$$\alpha_{Z,Y} = g_Z - \left(\log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right) + \sum_{m=1}^{d-1}\sum_{n=1}^{m}\frac{1}{n}\right)\omega_{\mathrm{FS}}^{d-1}.$$

Hence using

$$\int\limits_{\mathbb{P}^d_{\mathbb{C}}} \omega_{\mathrm{FS}}^d \wedge g_Y = \sum_{n=1}^{d}\frac{1}{n} - \log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right),$$

see equation (2.26), it follows

$$\widehat{\deg}_{\mathcal{X}}\left([Z, g_Z] \cdot [Y, g_Y]\right) = \frac{1}{2}\sum_{m=1}^{d}\sum_{n=1}^{m}\frac{1}{n}.$$

2.5 Approximation of the Dirac current with applications on arithmetic intersection numbers

In this section we will only consider irreducible and reduced cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$. Now let g_Y be a Green's form for Y. For a point $x \in Y(\mathbb{C})$ let $(z_1, \ldots, z_k, U)$ be a local equation for $Y(\mathbb{C})$ centered at x, i.e. $U \cap \pi^{-1}(Y(\mathbb{C})) = \{z_1 \cdots z_k = 0\}$ for a desingularization $\pi : \widetilde{\mathcal{X}_\infty} \longrightarrow \mathcal{X}_\infty$ of $\mathcal{X}_\infty$ along $Y(\mathbb{C})$. As in Definition 2.6 we find smooth ∂- and $\overline{\partial}$-closed forms α_i and a smooth form β on U such that

$$\pi^* g_Y\big|_U = \sum \alpha_i \log|z_i|^{-2} + \beta.$$

Assumption 2.61. In this section we fix one desingularization $\pi : \widetilde{\mathcal{X}_\infty} \longrightarrow \mathcal{X}_\infty$ and we assume that the open subset $U \subset \widetilde{\mathcal{X}_\infty}$ with $U \cap \pi^{-1}(Y(\mathbb{C})) = \{z_1 \cdots z_k = 0\}$ is dense in $\widetilde{\mathcal{X}_\infty}$. Because $\widetilde{\mathcal{X}_\infty}\backslash U$ is in this case a null set and because we are only interested in the integral of smooth differential forms, we can assume without loss of generality

$$\pi^{-1}(Y(\mathbb{C})) = \{z_1 \cdots z_k = 0\} \text{ and } \pi^* g_Y = \sum \alpha_i \log|z_i|^{-2} + \beta. \tag{2.28}$$

Moreover, we assume in this section that there exists a family $(Z_t(\mathbb{C}))_{t\in\mathbb{R}_{\geq 0}}$ of irreducible and reduced cycles $Z_t(\mathbb{C}) \in \mathrm{Z}^{p^\vee}(\mathcal{X}_\infty)$ with the property that $\lim_{t\to 0} Z_t(\mathbb{C}) = Z(\mathbb{C})$ and such that for all $t > 0$ the cycles $Z_t(\mathbb{C})$ and $Y(\mathbb{C})$ intersect properly on $\mathcal{X}_\infty$.

Definition 2.62. Let $z_1 \cdots z_k = 0$ be an equation for $\pi^{-1}(Y(\mathbb{C}))$ up to a null set. Then $z_1 \cdots z_k = 0$ is a *Z-adjusted equation for Y* if the equation

$$\begin{aligned}&(\mathrm{div}(f) - \mathrm{ord}_Y(f)Y, Z)_{\mathrm{fin}} = \\ &\lim_{t\to 0}\left(\int_{\mathcal{X}_\infty} \log|f|\,\delta_{Z_t(\mathbb{C})} - \mathrm{ord}_Y(f)\sum_i \int_{\widetilde{\mathcal{X}_\infty}} \alpha_i \log|z_i| \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))}\right)\end{aligned} \tag{2.29}$$

holds for any K_1-chain f such that for all $t > 0$ the cycles $\mathrm{div}(f)(\mathbb{C})$ and $Z_t(\mathbb{C})$ as well as the cycles $(\mathrm{div}(f) - \mathrm{ord}_Y(f)Y)(\mathbb{C})$ and $Z(\mathbb{C})$ intersect properly on $\mathcal{X}_\infty$. Here α_i are any real, smooth, ∂- and $\overline{\partial}$-closed forms that satisfy $\pi_*\left(\sum \alpha_i \wedge \delta_{E_i}\right) = \delta_{Y(\mathbb{C})}$, where E_i is given by the equation $z_i = 0$.

Remark 2.63. *i)* First note that the smooth forms α_i in Definition 2.62 exist, see [So1, Lemma 3]. Moreover, the equation $z_1 \cdots z_k = 0$ do not depend on the choice of the α_i. This follows from the fact that any choice of the forms α_i in Definition 2.62 defines a Green's current g_Y for the same cycle Y, see [So1, Theorem 3]. Then because of Assumption 2.61 we have $\pi^* g_Y = \sum \alpha_i \log|z_i|^{-2} + \beta$. Now note that for two different Green's currents $g_Y, \widetilde{g}_Y \in \widetilde{\mathrm{D}}^{(p-1,p-1)}(\mathcal{X})$ for Y, the difference $g_Y - \widetilde{g}_Y$ is represented by a smooth form. Hence with $\pi^* g_Y = \sum \alpha_i \log|z_i|^{-2} + \beta$ and $\pi^* \widetilde{g}_Y = \sum \widetilde{\alpha}_i \log|z_i|^{-2} + \widetilde{\beta}$, the integral $\int_{\widetilde{\mathcal{X}_\infty}} \pi^*(g_Y - \widetilde{g}_Y) \wedge \delta_{\pi^{-1}(Z(\mathbb{C}))}$ is well-defined for any cycle $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$. It follows that $\sum \int_{\widetilde{\mathcal{X}_\infty}} (\alpha_i - \widetilde{\alpha}_i) \log|z_i| \wedge \delta_{\pi^{-1}(Z(\mathbb{C}))}$ is well-defined for any cycle $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$, which implies that the latter integral vanishes.

ii) A priori, the equation (2.29) depends on the family $(Z_t(\mathbb{C}))_{t\in\mathbb{R}_{\geq 0}}$. But in the proof of the next proposition we will see that (2.29) is independent of the choice of the cycles $Z_t(\mathbb{C})$.

iii) In the proof of the next proposition we will also see that instead of considering the condition $\pi^{-1}(Y(\mathbb{C})) = \{z_1 \cdots z_k = 0\}$ up to a null set as in Assumption 2.61, it is enough to consider those local coordinates on an open subset $U \subset \widetilde{\mathcal{X}_\infty}$ such that $U \cap \pi^{-1}(Z(\mathbb{C})) \subset \pi^{-1}(Z(\mathbb{C}))$ is a dense subset.

iv) Instead of considering only one chart, we also could generalize Definition 2.62 to a family of local equations. By this we mean the following: Let $\widetilde{\mathcal{X}_\infty} = \bigcup_{j=1}^n U_j$ and let $z_{1,j} \cdots z_{k_j,j} = 0$ be an equation for $\pi^{-1}(Y(\mathbb{C})) \cap U_j$. Then the family $\left\{z_{1,j} \cdots z_{k_j,j} = 0\right\}_j$

could be defined as a Z-adjusted equation for Y if the equation

$$(\mathrm{div}(f) - \mathrm{ord}_Y(f)Y, Z)_{\mathrm{fin}} =$$
$$\lim_{t\to 0}\left(\int_{\mathcal{X}_\infty} \log|f|\,\delta_{Z_t(\mathbb{C})} - \mathrm{ord}_Y(f)\sum_{j=1}^{n}\sum_{i=1}^{k_j}\int_{U_j}\sigma_j\alpha_{i,j}\log|z_{i,j}|\wedge\delta_{\pi^{-1}(Z_t(\mathbb{C}))}\right)$$

holds for any K_1-chain f as in Definition 2.62. Here the real, smooth, ∂- and $\overline{\partial}$-closed forms $\alpha_{i,j}$ satisfy $\pi_*\left(\sum_i \alpha_{i,j}\wedge\delta_{E_{i,j}}\right) = \delta_{\overline{Y(\mathbb{C})\cap\pi(U_j)}}$, where $E_{i,j}$ is given by $z_{i,j}=0$ and σ_j is a integrable partition of unity to the cover $\{U_j\}_j$ of $\widetilde{\mathcal{X}_\infty}$. Note that this is independent of the partition of unity by the definition of the integral over a smooth differential form. However, in this thesis we will only consider one chart and one equation $z_1\cdots z_k = 0$.

Proposition 2.64. *Let $Y\in \mathrm{Z}^p(\mathcal{X})$ and $Z\in \mathrm{Z}^{p^\vee}(\mathcal{X})$ be irreducible and reduced cycles on an arithmetic variety $\mathcal{X}$. Let $z_1\cdots z_k = 0$ be an equation for $\pi^{-1}(Y(\mathbb{C}))$ up to a null set. Then there exists a unique positive real number α such that $\widetilde{z_1}\cdots\widetilde{z_k}=0$ is a Z-adjusted equation for Y, where for $i=1,\dots,k$ we set $\widetilde{z_i} := \alpha z_i$.*

Proof. *i) Well-definedness:* By the properties of the K_1-chain f, the geometric intersection number $(\mathrm{div}(f)-\mathrm{ord}_Y(f)Y, Z)_{\mathrm{fin}}$ is a real number. On the analytic side, let g_Y be a Green's form for Y associated to the forms α_i. Then the Green's form $-\log|f|^2 - \mathrm{ord}_Y(f)g_Y$ for $\mathrm{div}(f)-\mathrm{ord}_Y(f)g_Y$ is integrable along a desingularization of $Z(\mathbb{C})$. Hence

$$\int_{\mathcal{X}_\infty}\left(\log|f|^2 + \mathrm{ord}_Y(f)g_Y\right)\wedge\delta_{Z(\mathbb{C})}$$

is a well-defined number. It follows that the limit

$$\lim_{t\to 0}\left(\int_{\mathcal{X}_\infty}\log|f|\,\delta_{Z_t(\mathbb{C})} + \frac{\mathrm{ord}_Y(f)}{2}\int_{\mathcal{X}_\infty} g_Y\wedge\delta_{Z_t(\mathbb{C})}\right)$$

is well-defined and independent of the choice of the cycles $Z_t(\mathbb{C})$. Because $Z_t(\mathbb{C})\cap Y(\mathbb{C}) = \emptyset$ and because π is an isomorphism apart from $Y(\mathbb{C})$, the equation

$$\int_{\mathcal{X}_\infty} g_Y\wedge\delta_{Z_t(\mathbb{C})} = \int_{\widetilde{\mathcal{X}_\infty}}\pi^* g_Y\wedge\delta_{\pi^{-1}(Z_t(\mathbb{C}))}$$

holds. Now equation (2.28) shows that

$$\lim_{t\to 0}\left(\int_{\mathcal{X}_\infty}\log|f|\,\delta_{Z_t(\mathbb{C})} - \mathrm{ord}_Y(f)\sum_i\int_{\widetilde{\mathcal{X}_\infty}}\alpha_i\log|z_i|\wedge\delta_{\pi^{-1}(Z_t(\mathbb{C}))}\right)$$

is well-defined.

ii) Existence depending on f: Because of part *i)* we know that there exists a number $\alpha \in \mathbb{R}$ such that

$$(\mathrm{div}(f) - \mathrm{ord}_Y(f)Y, Z)_{\mathrm{fin}} - \lim_{t\to 0}\left(\int_{\mathcal{X}_\infty} \log|f|\,\delta_{Z_t(\mathbb{C})} - \mathrm{ord}_Y(f)\sum_i \int_{\widetilde{\mathcal{X}_\infty}} \alpha_i \log|z_i| \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))}\right)$$
$$= -\mathrm{ord}_Y(f)\cdot\log|\alpha| \sum_i \int_{\widetilde{\mathcal{X}_\infty}} \alpha_i \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))}.$$

It follows that the equation $\widetilde{z_1}\cdots\widetilde{z_k} = 0$ satisfies (2.29) for one K_1-chain f.

iii) Product-formula: Now let f be a K_1-chain, where $\mathrm{ord}_Y(f) = 0$ and such that $\mathrm{div}(f)$ and Z intersect properly on the generic fibre X. Then the formula

$$(\mathrm{div}(f), Z)_{\mathrm{fin}} = -\frac{1}{2}\int_{\mathcal{X}_\infty} \log|f|^{-2} * g_Z = \int_{\mathcal{X}_\infty} \log|f|\,\delta_{Z(\mathbb{C})}$$

holds for any choice of Green's form g_Z for Z. If we consider the cycles $Z_t(\mathbb{C})$, we obtain the equation

$$(\mathrm{div}(f), Z)_{\mathrm{fin}} = \lim_{t\to 0}\int_{\mathcal{X}_\infty} \log|f|\,\delta_{Z_t(\mathbb{C})}.$$

iv) Independence of f: Consider two K_1-chains f_1 and f_2 with the assumptions in the proposition and with $\mathrm{ord}_Y(f_1) \neq 0 \neq \mathrm{ord}_Y(f_2)$. Assume that f_1 satisfies the equation (2.29). We can find a suitable K_1-chain h with $\mathrm{ord}_Y(h) = 0 = \mathrm{ord}_Z(h)$ and such that $f_1^{\mathrm{ord}_Y(f_2)} = h\cdot f_2^{\mathrm{ord}_Y(f_1)}$. It follows the equation

$$(\mathrm{div}(f_2) - \mathrm{ord}_Y(f_2)Y, Z)_{\mathrm{fin}}$$
$$= \frac{\mathrm{ord}_Y(f_2)}{\mathrm{ord}_Y(f_1)}(\mathrm{div}(f_1) - \mathrm{ord}_Y(f_1)Y, Z)_{\mathrm{fin}} - \frac{1}{\mathrm{ord}_Y(f_1)}(\mathrm{div}(h), Z)_{\mathrm{fin}}$$
$$= \frac{\mathrm{ord}_Y(f_2)}{\mathrm{ord}_Y(f_1)}\lim_{t\to 0}\left(\int_{\mathcal{X}_\infty} \log|f_1|\,\delta_{Z_t(\mathbb{C})} - \mathrm{ord}_Y(f_1)\sum_i \int_{\widetilde{\mathcal{X}_\infty}} \alpha_i \log|z_i| \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))}\right)$$
$$-\frac{1}{\mathrm{ord}_Y(f_1)}\lim_{t\to 0}\int_{\mathcal{X}_\infty} \log|h|\,\delta_{Z_t(\mathbb{C})}$$
$$= \lim_{t\to 0}\left(\int_{\mathcal{X}_\infty} \log|f_2|\,\delta_{Z_t(\mathbb{C})} - \mathrm{ord}_Y(f_2)\sum_i \int_{\widetilde{\mathcal{X}_\infty}} \alpha_i \log|z_i| \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))}\right).$$

Thus the equation (2.29) is independent of K_1-chains.
v) Uniqueness: Because α is positive and real, it is uniquely determined. □

Proposition 2.65. *Let $z_1 \cdots z_k = 0$ be an equation for $\pi^{-1}(Y(\mathbb{C}))$ up to a null set. Then $z_1 \cdots z_k = 0$ is a Z-adjusted equation in Y if and only if*

$$\int_{\widetilde{\mathcal{X}_\infty}} \beta \wedge \delta_{\pi^{-1}(Z(\mathbb{C}))} = 0. \tag{2.30}$$

Here β is the smooth form $\pi^*\alpha_{Y,Z} + \sum \alpha_i \log |z_i|^2$, where $\alpha_{Y,Z}$ is a Z-adjusted Green's form for Y.

Proof. We will use Proposition 2.36 to prove Proposition 2.65. The right side of equation (2.11) can be written as the limit

$$\lim_{t\to 0} \left(\int_{\mathcal{X}_\infty} \log |f| \, \delta_{Z_t(\mathbb{C})} + \frac{\mathrm{ord}_Y(f)}{2} \int_{\widetilde{\mathcal{X}_\infty}} \pi^*\alpha_{Y,Z} \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))} \right).$$

With the help of Proposition 2.36 the equation

$$(\mathrm{div}(f) - \mathrm{ord}_Y(f)Y, Z)_{\mathrm{fin}} = \lim_{t\to 0} \left(\int_{\mathcal{X}_\infty} \log |f| \, \delta_{Z_t(\mathbb{C})} + \frac{\mathrm{ord}_Y(f)}{2} \int_{\widetilde{\mathcal{X}_\infty}} \pi^*\alpha_{Y,Z} \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))} \right)$$

holds. Hence to prove Proposition 2.65 it remains to show

$$\lim_{t\to 0} \left(\int_{\widetilde{\mathcal{X}_\infty}} \left(\pi^*\alpha_{Y,Z} + \sum_i \alpha_i \log |z_i|^2 \right) \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C}))} \right) = 0.$$

But this is exactly the equation (2.30). □

Theorem 2.66. *Let $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X})$. Let $z_1 \cdots z_k = 0$ be a Z-adjusted equation for Y. Then*

$$\widehat{\deg}_{\mathcal{X}}([Y, g_Y] \cdot [Z, g_Z]) = \lim_{t\to 0} \left(\sum_i \int_{\widetilde{\mathcal{X}_\infty}} \alpha_i \log |z_i| \wedge \delta_{\pi^{-1}(Z_t(\mathbb{C})} + \frac{1}{2} \int_{\mathcal{X}_\infty} g_Y \wedge \delta_{Z_t(\mathbb{C})} \right) + \frac{1}{2} \int_{\mathcal{X}_\infty} \omega_{g_Y} \wedge g_Z.$$

Proof. Because of Proposition 2.52 we have

$$\widehat{\deg}_{\mathcal{X}}\left([Y,g_Y]\cdot[Z,g_Z]\right)=\frac{1}{2}\lim_{t\to 0}\int\limits_{\mathcal{X}_\infty}(g_Y-\alpha_{Y,Z})\wedge\delta_{Z_t(\mathbb{C})}+\frac{1}{2}\int\limits_{\mathcal{X}_\infty}\omega_{g_Y}\wedge g_Z,$$

where $\alpha_{Y,Z}$ is a Z-adjusted Green's form for Y. If $\pi:\widetilde{\mathcal{X}_\infty}\longrightarrow\mathcal{X}_\infty$ is a desingularization of $\mathcal{X}_\infty$ along $Y(\mathbb{C})$ we obtain

$$\int\limits_{\mathcal{X}_\infty}\alpha_{Y,Z}\wedge\delta_{Z_t(\mathbb{C})}=\int\limits_{\widetilde{\mathcal{X}_\infty}}\pi^*\alpha_{Y,Z}\wedge\delta_{\pi^{-1}(Z_t(\mathbb{C}))}.$$

Because of Proposition 2.65 the latter equals

$$\sum_i\int\limits_{\widetilde{\mathcal{X}_\infty}}\alpha_i\log|z_i|^{-2}\wedge\delta_{\pi^{-1}(Z_t(\mathbb{C}))}+f(t)$$

with $\lim_{t\to 0}f(t)=0$. The proof follows. □

Now we compute some examples of adjusted equations.

Proposition 2.67. *Let $\mathcal{X}=\mathbb{P}^d_{\mathbb{Z}}=\operatorname{Proj}\mathbb{Z}[x_0,\dots,x_d]$. Let $Y\in\mathrm{Z}^1(\mathcal{X})$ and $Z\in\mathrm{Z}^d(\mathcal{X})$ given by the equations*

$$Y=\{\langle a^{(0)},x\rangle=0\}\text{ and }Z=\{\langle a^{(0)},x\rangle=\dots=\langle a^{(d-1)},x\rangle=0\},$$

where for $i=0,\dots,d-1$ we set $\langle a^{(i)},x\rangle:=a_0^{(i)}x_0+\dots+a_d^{(i)}x_d$ as in section 2.4. Moreover, let A_i be the matrix $(e_{i+1},a^{(0)},\dots,a^{(d-1)})$, where e_i is the i-th vector of the standard basis of $\mathbb{Z}^{d+1}$ as in Proposition 2.56 and assume $\det(A_i)\neq 0$. Then we set

$$z:=\det(A_i)\cdot\frac{\langle a^{(0)},x\rangle}{x_i}.$$

i) Let

$$U_i:=\{(x_0,\dots,x_d)\in\mathbb{P}^d_{\mathbb{C}}\mid x_i\neq 0\}.$$

Then a Z-adjusted equation for Y is on U_i given by $z=0$.
ii) Let

$$\widetilde{\mathcal{X}_\infty}=\left\{(x_0,\dots,x_d,y_0,\dots,y_{d-1})\in\mathbb{P}^d_{\mathbb{C}}\times\mathbb{P}^{d-1}_{\mathbb{C}}\mid\exists t\in\mathbb{C}\text{ s.t. }\langle a^{(i)},x\rangle=ty_i\ \forall i=0,\dots,d-1\right\}$$

be the blow up of $\mathcal{X}_\infty$ along $Z(\mathbb{C})$ and let

$$U_{i,j}:=\{(x_0,\dots,x_d,y_0,\dots,y_{d-1})\in\widetilde{\mathcal{X}_\infty}\mid x_i\neq 0,y_j\neq 0\}.$$

Then for $i\neq 0$, a Y-adjusted equation for Z is on $U_{i,0}$ given by $z=0$.

Remark 2.68. In the proof of Proposition 2.67 we will see that $U_i \subset \mathbb{P}^d_{\mathbb{C}}$ and $U_{i,0} \subset \widetilde{\mathbb{P}^d_{\mathbb{C}}}$ are dense subsets. Moreover, the condition $\det(A_i) \neq 0$ implies that $\langle a^{(0)}, x\rangle / x_i = 0$ is an equation for $U_i \cap Y(\mathbb{C})$ and also for $U_{i,0} \cap \pi^{-1}(Z(\mathbb{C}))$, where $\pi : \widetilde{\mathcal{X}_\infty} \longrightarrow \mathcal{X}_\infty$ is the projection onto the first factor. Thus the equation $z = 0$ satisfies the condition in Assumption 2.61.

Proof. *i)* Let $z := \det(A_i) \cdot \frac{\langle a^{(0)}, x\rangle}{x_i}$. By definition of a Z-adjusted equation for Y, we have to verify that the equation

$$(\operatorname{div}(f) - Y, Z)_{\text{fin}} = \lim_{t \to 0} \left(\int\limits_{\mathcal{X}_\infty} \log|f| \, \delta_{Z_t(\mathbb{C})} - \int\limits_{\mathcal{X}_\infty} \log|z| \, \delta_{Z_t(\mathbb{C})|_{U_i}} \right) \tag{2.31}$$

holds for a rational function f with $\operatorname{ord}_Y(f) = 1$ such that both sides of (2.31) are well-defined. As in Proposition 2.56 we choose $f = \frac{\langle a^{(0)}, x\rangle}{\langle \alpha, x\rangle}$, where the vectors $\alpha, a^{(0)}, \ldots, a^{(d-1)}$ are linearly independent in $\mathbb{Z}^{d+1}$. Then we have

$$(\operatorname{div}(f) - Y, Z)_{\text{fin}} = -\log\left|\det(\alpha, a^{(0)}, \ldots, a^{(d-1)})\right|.$$

For $t > 0$ consider the cycles

$$Z_t(\mathbb{C}) := \{\langle a^{(0)} + t\beta, x\rangle = \langle a^{(1)}, x\rangle = \cdots = \langle a^{(d-1)}, x\rangle = 0\},$$

where the vector β satisfies $\det\left(\beta, a^{(0)}, \cdots, a^{(d-1)}\right) \neq 0$. Therewith we have the two integrals

$$\begin{aligned}
&\int\limits_{\mathcal{X}_\infty} \log|f| \, \delta_{Z_t(\mathbb{C})} \\
&= \log\left|\det(a^{(0)}, a^{(0)} + t\beta, a^{(1)}, \ldots, a^{(d-1)})\right| - \log\left|\det(\alpha, a^{(0)} + t\beta, a^{(1)}, \ldots, a^{(d-1)})\right| \\
&= \log|t| + \log\left|\det(\beta, a^{(0)}, \ldots, a^{(d-1)})\right| - \log\left|\det(\alpha, a^{(0)}, \ldots, a^{(d-1)})\right| + \mathcal{O}(t)
\end{aligned}$$

and

$$\begin{aligned}
\int\limits_{\mathcal{X}_\infty} \log|z| \, \delta_{Z_t(\mathbb{C})|_{U_i}} &= \int\limits_{\mathcal{X}_\infty} \log\left|\frac{\langle a^{(0)}, x\rangle}{x_i}\right| \delta_{Z_t(\mathbb{C})|_{U_i}} + \log|\det(A_i)| \\
&= \log\left|\det(a^{(0)}, a^{(0)} + t\beta, a^{(1)}, \ldots, a^{(d-1)})\right| \\
&= \log|t| + \log\left|\det(\beta, a^{(0)}, \ldots, a^{(d-1)})\right|.
\end{aligned}$$

Hence, in the limit $t \longrightarrow 0$ we have

$$\lim_{t \to 0} \left(\int\limits_{\mathcal{X}_\infty} \log|f| \, \delta_{Z_t(\mathbb{C})} - \int\limits_{\mathcal{X}_\infty} \log|z| \, \delta_{Z_t(\mathbb{C})|_{U_i}} \right) = -\log\left|\det(\alpha, a^{(0)}, \ldots, a^{(d-1)})\right|,$$

which proves the claim.
ii) First note that the blow up $\widetilde{\mathcal{X}_\infty} = \bigcup_{i,j} U_{i,j}$ can be covered by the $U_{i,j}$. On these open sets there are for $i \neq j$ the local coordinates

$$\varphi_{i,j} : U_{i,j} \xrightarrow{\sim} \mathbb{C}^d,\ (x_0, \ldots, x_d, y_0, \ldots, y_{d-1}) \longmapsto \left(\frac{\langle a^{(j)}, x\rangle}{x_i}, \frac{y_0}{y_j}, \ldots, 1, \ldots, \frac{y_{d-1}}{y_j} \right),$$

see [Hu, section 2.5]. Let $\pi : \widetilde{\mathcal{X}_\infty} \longrightarrow \mathcal{X}_\infty$ be the projection onto the first factor. Then we have $\pi^{-1}(Z(\mathbb{C})) = \{t = 0\}$, where the equation $t = 0$ is as in Example 2.9. On the open set $U_{i,j}$ we have $\pi^{-1}(Z(\mathbb{C}))|_{U_{i,j}} = \{\langle a^{(j)}, x\rangle / x_i = 0\}$, see [GS1]. We claim that $z := \det(A_i)\langle a^{(0)}, x\rangle / x_i$ is a Y-adjusted equation for Z on $U_{i,0}$. For this we consider the K_1-chain f, which is on $W(\mathbb{C}) := \{\langle a^{(1)}, x\rangle = \cdots = \langle a^{(d-1)}, x\rangle = 0\}$ given by the function $\frac{\langle a^{(0)}, x\rangle}{\langle \alpha, x\rangle}$. We denote this K_1-chain by $f = \frac{\langle a^{(0)}, x\rangle}{\langle \alpha, x\rangle} \delta_{W(\mathbb{C})}$. Here the vector α has to satisfy $\det(\alpha, a^{(0)}, \ldots, a^{(d-1)}) \neq 0$. Moreover, we set $Y_t(\mathbb{C}) := \{\langle a^{(0)}, x\rangle + t x_i = 0\}$. Then we have to show the equation

$$(\mathrm{div}(f) - Z, Y)_{\mathrm{fin}} = \lim_{t \to 0} \left(\int_{\mathcal{X}_\infty} \log|f|\, \delta_{Y_t(\mathbb{C})} - \int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \log|z| \wedge \delta_{Y_t(\mathbb{C})|_{U_{i,0}}} \right),$$

where $\widetilde{\alpha} := \sum_{j=0}^{d-1} \left(\mathrm{dd}^c \log \left(\sum_{i=0}^{d-1} |y_i|^2 \right) \right)^j \wedge \omega_{\mathrm{FS}}^{d-1-j}$. Note that by [GS1, p.207] we have $\pi_* \left(\widetilde{\alpha} \wedge \delta_{\pi^{-1}(Z(\mathbb{C}))} \right) = \delta_{Z(\mathbb{C})}$. The geometric intersection number $(\mathrm{div}(f) - Z, Y)_{\mathrm{fin}}$ is given by $-\log \left| \det(\alpha, a^{(0)}, \ldots, a^{(d-1)}) \right|$. On the analytic side we have the two integrals

$$\begin{aligned}
&\int_{\mathcal{X}_\infty} \log|f| \delta_{Y_t(\mathbb{C})} \\
&\qquad = \int_{\mathcal{X}_\infty} \log \left| \frac{\langle a^{(0)}, x\rangle}{\langle \alpha, x\rangle} \right| \delta_{W(\mathbb{C})} \wedge \delta_{Y_t(\mathbb{C})} \\
&\qquad = \log \left| \det(a^{(0)}, a^{(0)} + t e_{i+1}, a^{(1)}, \ldots, a^{(d-1)}) \right| - \log \left| \det(\alpha, a^{(0)}, \ldots, a^{(d-1)}) \right| + \mathcal{O}(t) \\
&\qquad = \log|t| + \log|\det(A_i)| - \log \left| \det(\alpha, a^{(0)}, \ldots, a^{(d-1)}) \right| + \mathcal{O}(t)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \log|z| \wedge \delta_{Y_t(\mathbb{C})|_{U_{i,0}}} \\
&\qquad = \int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \log\left|\det(A_i)\frac{\langle a^{(0)}, x\rangle}{x_i}\right| \wedge \delta_{\{\frac{\langle a^{(0)},x\rangle}{x_i}+t=0\}} \\
&\qquad = \log|\det(A_i)| \cdot \int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \wedge \delta_{\{\frac{\langle a^{(0)},x\rangle}{x_i}=0\}} + \int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \log\left|\frac{\langle a^{(0)}, x\rangle}{x_i}\right| \wedge \delta_{\{\frac{\langle a^{(0)},x\rangle}{x_i}+t=0\}} + \mathcal{O}(t) \\
&\qquad = (\log|\det(A_i)| + \log|t|) \cdot \int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \wedge \delta_{\{\frac{\langle a^{(0)},x\rangle}{x_i}=0\}} + \mathcal{O}(t).
\end{aligned}$$

Now, note that $\{\frac{\langle a^{(0)},x\rangle}{x_i} = 0\} = \pi^{-1}(Z(\mathbb{C}))|_{U_{i,0}}$. Therewith we have

$$\int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \wedge \delta_{\pi^{-1}(Z(\mathbb{C}))|_{U_{i,0}}} = \int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \wedge \delta_{\pi^{-1}(Z(\mathbb{C}))} = \int_{\mathcal{X}_\infty} \delta_{Z(\mathbb{C})} = 1.$$

Hence, we have the desired condition

$$\lim_{t\to 0}\left(\int_{\mathcal{X}_\infty} \log|f|\,\delta_{Y_t(\mathbb{C})} - \int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \log|z| \wedge \delta_{Y_t(\mathbb{C})|_{U_{i,0}}}\right) = -\log\left|\det(\alpha, a^{(0)}, \ldots, a^{(d-1)})\right|.$$

□

Example 2.69. Consider the arithmetic surface $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ and the divisor $\mathcal{P} = (a : b) = \{bx_0 - ax_1 = 0\} \in \mathcal{X}(\mathbb{Z})$ with $b \neq 0$. As a special case of Proposition 2.67 we see that

$$z_{\mathcal{P}} := \det(b)(bx_0 - ax_1)|_U = b(bz - a) = b^2\left(z - \frac{a}{b}\right)$$

is a $\mathcal{P}$-adjusted equation for $\mathcal{P}$ on the chart $U = \{z = x_0/x_1 \in \mathbb{C}\}$. Note that the function $z_{\mathcal{P}}$ was calculated in the first chapter and defined there an arithmetic local coordinate in $\mathcal{P}$.

In the last part of this section we want to compute the arithmetic self-intersection number of the Serre twist equipped with the Fubini-Study metric using Theorem 2.66.

- First consider the case

$$\widehat{\deg}_{\mathcal{X}}\left([Y,g_Y]\cdot[Z,g_Z]\right)=$$
$$\lim_{t\to 0}\left(\sum_i \int_{\widetilde{\mathcal{X}_\infty}} \alpha_i \log|z_i|\wedge\delta_{\pi^{-1}(Z_t(\mathbb{C}))}+\frac{1}{2}\int_{\mathcal{X}_\infty} g_Y\wedge\delta_{Z_t(\mathbb{C})}\right)+\frac{1}{2}\int_{\mathcal{X}_\infty}\omega_{g_Y}\wedge g_Z,$$

where $z_1\cdots z_k=0$ is a Z-adjusted equation for Y. By Proposition 2.67 we can choose $k=1$, $\alpha_1=1$ and $z_1:=z=\det A_i\cdot\frac{\langle a^{(0)},x\rangle}{x_i}$ on $U_i=\{(x_0,\ldots,x_d)\in\mathbb{P}^d_{\mathbb{C}}\mid x_i\neq 0\}$. Hence we have to compute

$$\widehat{\deg}_{\mathcal{X}}\left([Y,g_Y]\cdot[Z,g_Z]\right)=$$
$$\lim_{t\to 0}\left(\int_{\mathcal{X}_\infty}\log|z|\,\delta_{Z_t(\mathbb{C})|_{U_i}}+\frac{1}{2}\int_{\mathcal{X}_\infty} g_Y\wedge\delta_{Z_t(\mathbb{C})}\right)+\frac{1}{2}\int_{\mathcal{X}_\infty}\omega_{g_Y}\wedge g_Z.$$

As in the proof of Proposition 2.67 we take

$$Z_t(\mathbb{C})=\{\langle a^{(0)}+t\beta,x\rangle=\langle a^{(1)},x\rangle=\cdots=\langle a^{(d-1)},x\rangle=0\}$$

such that the vector β satisfies $\det\left(\beta,a^{(0)},\cdots,a^{(d-1)}\right)\neq 0$. Because of

$$\frac{1}{2}\int_{\mathcal{X}_\infty}\omega_{g_Y}\wedge g_Z=\frac{1}{2}\sum_{m=1}^{d}\sum_{n=1}^{m}\frac{1}{n}-\frac{1}{2}\log\left(|\det A_0|^2+\cdots+|\det A_d|^2\right),$$

it is enough to show

$$\lim_{t\to 0}\left(\int_{\mathcal{X}_\infty}\log|z|\,\delta_{Z_t(\mathbb{C})|_{U_i}}+\frac{1}{2}\int_{\mathcal{X}_\infty} g_Y\wedge\delta_{Z_t(\mathbb{C})}\right)=\frac{1}{2}\log\left(|\det A_0|^2+\cdots+|\det A_d|^2\right).$$

Note that

$$\int_{\mathcal{X}_\infty}\log|z|\,\delta_{Z_t(\mathbb{C})|_{U_i}}=\log|t|+\log\left|\det\left(\beta,a^{(0)},\cdots,a^{(d-1)}\right)\right|.$$

Moreover, we have

$$\frac{1}{2}\int\limits_{\mathcal{X}_\infty} g_Y \wedge \delta_{Z_t(\mathbb{C})} = \frac{1}{2}\int\limits_{\mathcal{X}_\infty} g_Y|_{U_i} \wedge \delta_{Z_t(\mathbb{C})|_{U_i}} =$$

$$-\frac{1}{2}\int\limits_{\mathcal{X}_\infty} \log\left(\frac{\left|\frac{\langle a^{(0)},x\rangle}{x_i}\right|^2}{\left|\frac{x_0}{x_i}\right|^2+\cdots+\left|\frac{x_{i-1}}{x_i}\right|^2+1+\left|\frac{x_{i+1}}{x_i}\right|^2+\cdots+\left|\frac{x_d}{x_0}\right|^2}\right) \wedge \delta_{Z_t(\mathbb{C})|_{U_i}} =$$

$$\log|\det A_i| - \log\left|\det\left(a^{(0)}, a^{(0)}+t\beta, a^{(1)}, \cdots, a^{(d-1)}\right)\right| + \mathcal{O}(t) +$$

$$\frac{1}{2}\int\limits_{\mathcal{X}_\infty} \log\left(\left|\frac{x_0}{x_i}\right|^2+\cdots+\left|\frac{x_{i-1}}{x_i}\right|^2+1+\left|\frac{x_{i+1}}{x_i}\right|^2+\cdots+\left|\frac{x_d}{x_0}\right|^2\right) \wedge \delta_{Z_t(\mathbb{C})|_{U_i}}.$$

Now because of

$$\int\limits_{\mathcal{X}_\infty} \log\left(\left|\frac{x_0}{x_i}\right|^2+\cdots+\left|\frac{x_{i-1}}{x_i}\right|^2+1+\left|\frac{x_{i+1}}{x_i}\right|^2+\cdots+\left|\frac{x_d}{x_0}\right|^2\right) \wedge \delta_{Z_t(\mathbb{C})|_{U_i}} =$$

$$\log\left(|\det A_0|^2+\cdots+|\det A_d|^2\right) - \log|\det A_i|^2 + \mathcal{O}(t)$$

we obtain the arithmetic intersection number

$$\widehat{\deg}_{\mathcal{X}}\left([Y,g_Y]\cdot[Z,g_Z]\right) = \frac{1}{2}\sum_{m=1}^{d}\sum_{n=1}^{m}\frac{1}{n}.$$

- Now consider the case

$$\widehat{\deg}_{\mathcal{X}}\left([Z,g_Z]\cdot[Y,g_Y]\right) =$$

$$\lim_{t\to 0}\left(\sum_i \int\limits_{\widetilde{\mathcal{X}_\infty}} \alpha_i \log|z_i| \wedge \delta_{\pi^{-1}(Y_t(\mathbb{C}))} + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} g_Z \wedge \delta_{Y_t(\mathbb{C})}\right) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_Z} \wedge g_Y,$$

where $z_1\cdots z_k = 0$ is a Y-adjusted equation for Z. By Proposition 2.67 we can choose $k=1$, $\alpha_1 := \widetilde{\alpha} = \sum_{j=0}^{d-1}\left(\mathrm{dd}^c\log\left(\sum_{i=0}^{d-1}|y_i|^2\right)\right)^j \wedge \omega_{\mathrm{FS}}^{d-1-j}$ and $z_1 := z = \det A_i \cdot \frac{\langle a^{(0)},x\rangle}{x_i}$ on

$$U_{i,0} := \{(x_0,\ldots,x_d,y_0,\ldots,y_{d-1}) \in \widetilde{\mathcal{X}_\infty} \mid x_i \neq 0, y_0 \neq 0\}.$$

Hence we have to compute

$$\widehat{\deg}_{\mathcal{X}}\left([Z,g_Z]\cdot[Y,g_Y]\right) =$$

$$\lim_{t\to 0}\left(\int\limits_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha}\log|z| \wedge \delta_{Y_t(\mathbb{C})|_{U_{i,0}}} + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} g_Z \wedge \delta_{Y_t(\mathbb{C})}\right) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_Z} \wedge g_Y.$$

As in the proof of Proposition 2.67 we set $Y_t(\mathbb{C}) := \{\langle a^{(0)}, x\rangle + tx_i = 0\}$. First note that

$$\int_{\mathcal{X}_\infty} \omega_{g_Z} \wedge g_Y = \sum_{n=1}^{d} \frac{1}{n} - \log\left(\sum_{i=0}^{d} \left|a_i^{(0)}\right|^2\right)$$

and

$$\int_{\widetilde{\mathcal{X}_\infty}} \widetilde{\alpha} \log|z| \wedge \delta_{Y_t(\mathbb{C})|_{U_{i,0}}} = \log|t| + \log\left|\det\left(A_i\right)\right| + \mathcal{O}(t).$$

Now let α_{Z,Y_t} be the Green's form

$$g_Z - \left(\log\left(\left|a_0^{(0)}\right|^2 + \cdots + \left|a_i^{(0)} + t\right|^2 + \cdots + \left|a_d^{(0)}\right|^2\right) + \sum_{m=1}^{d-1}\sum_{n=1}^{m} \frac{1}{n}\right) \omega_{\mathrm{FS}}^{d-1}$$

for Z. Because

$$\int_{\mathcal{X}_\infty} \alpha_{Z,Y_t} \wedge \delta_{Y_t(\mathbb{C})} = -2\log\left|\det(a^{(0)}, a^{(0)} + te_{i+1}, a^{(1)}, \ldots, a^{(d-1)})\right| = -2\log|t| - 2\log\left|\det\left(A_i\right)\right|,$$

see equation (2.24), we find

$$\int_{\mathcal{X}_\infty} g_Z \wedge \delta_{Y_t(\mathbb{C})} =$$

$$-2\log|t| - 2\log\left|\det\left(A_i\right)\right| + \sum_{m=1}^{d-1}\sum_{n=1}^{m} \frac{1}{n} + \log\left(\sum_{i=0}^{d}\left|a_i^{(0)}\right|^2\right) + \mathcal{O}(t).$$

Hence we obtain the arithmetic intersection number

$$\widehat{\deg}_{\mathcal{X}}\left([Z, g_Z] \cdot [Y, g_Y]\right) = \frac{1}{2}\sum_{m=1}^{d}\sum_{n=1}^{m} \frac{1}{n}.$$

Chapter 3

Generalized Arithmetic Intersection Theory

In the first part of this chapter we discuss the generalized arithmetic intersection theory on arithmetic surfaces with log-log Green's functions. Using adjusted Green's functions we define a generalized arithmetic intersection number and show that a limit version of this number contains arithmetic local coordinates. With this new formula we compute the arithmetic self-intersection number on the modular curve $\mathcal{X}(1)$ from the first chapter. We also calculate an example of a generalized arithmetic self-intersection number on a modular curve $\mathcal{X}(\Gamma)$, which is due to U. Kühn in [Kü4]. With the use of adjusted Green's functions we define a generalized arithmetic intersection number of two arbitrary arithmetic divisors with log-log-singularities and show that this number coincides with the generalized arithmetic intersection number due to Kühn in [Kü2].
In the second part of this chapter we discuss the generalized arithmetic intersection theory on arithmetic varieties with log-log Green's forms. First we review the generalized arithmetic intersection theory, which is due to J. I. Burgos Gil, J. Kramer and U. Kühn in [BGKK]. Similar to the second chapter we define a modified $*$-product between two log-log Green's forms and therewith we define a modified version of the generalized arithmetic intersection number of arithmetic cycles with log-log Green's forms. We show that this number coincides with the generalized arithmetic intersection number due to Burgos Gil, Kramer and Kühn. In the last part of this chapter we calculate a known example of a generalized arithmetic self-intersection number on the arithmetic 3-fold $\mathbb{P}^2_{\mathbb{Z}} = \mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{X}(1)$.

3.1 Generalized arithmetic self-intersection numbers on arithmetic surfaces using arithmetic local coordinates

In this section we define a new version of the generalized arithmetic self-intersection number of an arithmetic divisor with a log-log Green's function.
The following two definitions can be found in [Kü3, Definition 3.1].

Definition 3.1. Let $\mathcal{S}_\infty = \sum S_j$ be a fixed reduced divisor on $\mathcal{X}_\infty$ and let $\mathcal{D}$ be a divisor on $\mathcal{X}$. A *log-log Green's function $g_\mathcal{D}$ for $\mathcal{D}$* is a smooth function outside $\mathcal{D}(\mathbb{C}) \cup \mathcal{S}_\infty$ such that
i) $g_\mathcal{D}$ is a Green's function for $\mathcal{D}(\mathbb{C})\backslash(\mathcal{D}(\mathbb{C}) \cap \mathcal{S}_\infty)$ on $\mathcal{X}_\infty \backslash \mathcal{S}_\infty$;
ii) for a local coordinate t in S_j we have an expansion

$$g_\mathcal{D}(Q_j) = -2\eta_{g_\mathcal{D},j} \log\left(-\log|t(Q_j)|^2\right) - 2\mathrm{ord}_{S_j}(\mathcal{D}) \log|t(Q_j)| - 2\log\left(\varphi_{g_\mathcal{D},j}(Q_j)\right)$$

near S_j, where $\eta_{g_\mathcal{D},j} \in \mathbb{R}$ and $\varphi_{g_\mathcal{D},j}$ is a positive smooth function outside S_j, satisfying the growth conditions

$$\left|\frac{\partial \varphi_{g_\mathcal{D},j}}{\partial t}(Q_j)\right| \leq \frac{\beta}{|t(Q_j)|^{1-\rho}}, \quad \left|\frac{\partial \varphi_{g_\mathcal{D},j}}{\partial \bar{t}}(Q_j)\right| \leq \frac{\beta}{|t(Q_j)|^{1-\rho}}, \quad \left|\frac{\partial^2 \varphi_{g_\mathcal{D},j}}{\partial t \partial \bar{t}}(Q_j)\right| \leq \frac{\beta}{|t(Q_j)|^{2-\rho}}$$

near S_j, where $\beta, \rho \in \mathbb{R}_+$.

Definition 3.2. Let $\widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty)$ denote the group of arithmetic divisors $(\mathcal{D}, g_\mathcal{D})$, where $g_\mathcal{D}$ is a log-log Green's function for $\mathcal{D}$. Moreover, let $\widehat{\mathrm{R}}^1(\mathcal{X})$ denotes the usual group of arithmetic divisors of the form $(\mathrm{div}(f), -\log|f|^2)$ with $f \in k(\mathcal{X})^\times$. Then

$$\widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{S}_\infty) := \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty) \big/ \widehat{\mathrm{R}}^1(\mathcal{X})$$

is the *generalized arithmetic Chow group of $\mathcal{X}$ with respect to $\mathcal{S}_\infty$*.

Now we define a new version of the generalized arithmetic self-intersection number.

Definition 3.3. Let $(\mathcal{P}, g_\mathcal{P}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty)$ be an arithmetic divisor, where $g_\mathcal{P}$ is a log-log Green's function for a horizontal prime divisor $\mathcal{P}$. We define the *generalized arithmetic self-intersection number of* $(\mathcal{P}, g_\mathcal{P})$ by

$$(\mathcal{P}, g_\mathcal{P})^2 = \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \left(g_\mathcal{P} \cdot \omega_{\alpha_\mathcal{P}} - \alpha_\mathcal{P} \cdot \omega_{g_\mathcal{P}} + g_\mathcal{P} \cdot \omega_{g_\mathcal{P}}\right),$$

where $\alpha_\mathcal{P}$ is an adjusted Green's function for $\mathcal{P}$.

Theorem 3.4. *Let $(\mathcal{P}, g_{\mathcal{P}}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty)$ be an arithmetic divisor, where $\mathcal{P}$ is a horizontal prime divisor, $g_{\mathcal{P}}$ is a log-log Green's function for $\mathcal{P}$ and $\mathcal{S}_\infty = \mathcal{P}(\mathbb{C})$. Let $t_{\mathcal{P}} = \left(t_{P_j}\right)_j$ be a fixed family of local coordinates in $\mathcal{P}(\mathbb{C}) = \sum P_j$. Write locally*

$$g_{\mathcal{P}}(Q_j) = -2\eta_{g_{\mathcal{P}},j} \log\left(-\log\left|t_{P_j}(Q_j)\right|^2\right) - 2\log\left|t_{P_j}(Q_j)\right| - 2\log\left(\varphi_{g_{\mathcal{P}},j}(Q_j)\right)$$

on $\mathcal{X}_\varepsilon := \mathcal{X}_\infty \backslash \bigcup B_\varepsilon(P_j)$ near P_j, where $B_\varepsilon(P_j) = \{x \in \mathcal{X}_\infty | \left|t_{P_j}(x)\right| < \varepsilon\}$. Then the generalized arithmetic self-intersection number of $(\mathcal{P}, g_{\mathcal{P}})$ is well-defined and as a limit given by

$$\begin{aligned}(\mathcal{P}, g_{\mathcal{P}})^2 = &\sum_j \left(\eta_{g_{\mathcal{P}},j} - \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right)\right) + \lim_{Q \to P} \left(\log|z_{\mathcal{P}}(Q)| - \log|t_{\mathcal{P}}(Q)|\right) - \\ &\lim_{\varepsilon \to 0} \left(\sum_j \eta_{g_{\mathcal{P}},j} \log\left(-\log \varepsilon^2\right) - \frac{1}{2} \int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}} \right),\end{aligned}$$

where $z_{\mathcal{P}}$ is an arithmetic local coordinate in $\mathcal{P}$.

Proof. For two log-log Green's functions $g_{\mathcal{P}}$ and $g'_{\mathcal{P}}$ for $\mathcal{P}$, the equation

$$\int\limits_{\mathcal{X}_\varepsilon} g'_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}} = -\sum_j \int\limits_{\partial B_\varepsilon(P_j)} g'_{\mathcal{P}} \cdot \mathrm{d}^c g_{\mathcal{P}} - \int\limits_{\mathcal{X}_\varepsilon} \mathrm{d}g'_{\mathcal{P}} \cdot \mathrm{d}^c g_{\mathcal{P}} \tag{3.1}$$

holds by Stokes Theorem, where the orientation of $\partial B_\varepsilon(P_j)$ is induced from the orientation of $B_\varepsilon(P_j)$. In [Kü2, equation (20)] it was shown

$$\int\limits_{\partial B_\varepsilon(P_j)} g'_{\mathcal{P}} \cdot \mathrm{d}^c g_{\mathcal{P}} = 2\left(\eta_{g_{\mathcal{P}},j} + \eta_{g'_{\mathcal{P}},j} \log\left(-\log \varepsilon^2\right) + \log\left(\varphi_{g'_{\mathcal{P}},j}(P_j)\right)\right) + f(\varepsilon), \tag{3.2}$$

where f is a continuous function with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. For an adjusted Green's function $\alpha_{\mathcal{P}}$ for $\mathcal{P}$ we have $\eta_{\alpha_{\mathcal{P}},j} = 0$ for all j. Hence we have the two terms

$$\int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{P}} \cdot \omega_{\alpha_{\mathcal{P}}} = -2\sum_j \left(\eta_{g_{\mathcal{P}},j} \log\left(-\log \varepsilon^2\right) + \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right)\right) - \int\limits_{\mathcal{X}_\varepsilon} \mathrm{d}g_{\mathcal{P}} \cdot \mathrm{d}^c \alpha_{\mathcal{P}} + f(\varepsilon) \text{ and}$$

$$\int\limits_{\mathcal{X}_\varepsilon} \alpha_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}} = -2\sum_j \left(\eta_{g_{\mathcal{P}},j} + \log\left(\varphi_{\alpha_{\mathcal{P}},j}(P_j)\right)\right) + \int\limits_{\mathcal{X}_\varepsilon} \mathrm{d}\alpha_{\mathcal{P}} \cdot \mathrm{d}^c g_{\mathcal{P}} + f(\varepsilon).$$

Using the symmetry $\mathrm{d}\alpha_{\mathcal{P}} \cdot \mathrm{d}^c g_{\mathcal{P}} = \mathrm{d}g_{\mathcal{P}} \cdot \mathrm{d}^c \alpha_{\mathcal{P}}$ we obtain

$$\begin{aligned}&\frac{1}{2} \int\limits_{\mathcal{X}_\varepsilon} \left(g_{\mathcal{P}} \cdot \omega_{\alpha_{\mathcal{P}}} - \alpha_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}}\right) = \\ &\qquad \sum_j \left(\eta_{g_{\mathcal{P}},j} + \log\left(\varphi_{\alpha_{\mathcal{P}},j}(P_j)\right) - \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right) - \eta_{g_{\mathcal{P}},j} \log\left(-\log \varepsilon^2\right)\right) + f(\varepsilon).\end{aligned}$$

Taking the limit $\varepsilon \longrightarrow 0$, it follows

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \sum_j \left(\eta_{g_{\mathcal{P}},j} + \log\left(\varphi_{\alpha_{\mathcal{P}},j}(P_j)\right) - \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right)\right) - \lim_{\varepsilon\to 0}\left(\sum_j \eta_{g_{\mathcal{P}},j}\log\left(-\log\varepsilon^2\right) - \frac{1}{2}\int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{P}}\cdot\omega_{g_{\mathcal{P}}}\right).$$

The proposition follows now from the two equations

$$\log\left(\varphi_{\alpha_{\mathcal{P}},j}(P_j)\right) = -\frac{1}{2}\lim_{Q_j\to P_j}\left(\alpha_{\mathcal{P}}(Q_j) + \log\left|t_{P_j}(Q_j)\right|^2\right) \text{ and}$$
$$0 = \lim_{Q_j\to P_j}\left(\alpha_{\mathcal{P}}(Q_j) + \log\left|z_{P_j}(Q_j)\right|^2\right).$$

□

If the family of local coordinates $t_{\mathcal{P}}$ in Theorem 3.4 is an arithmetic local coordinate in $\mathcal{P}$, then we get the following useful version of the generalized arithmetic self-intersection number:

Corollary 3.5. *Consider the situation as in Theorem 3.4, where the Green's function $g_{\mathcal{P}}$ is locally given by*

$$g_{\mathcal{P}}(Q_j) = -2\eta_{g_{\mathcal{P}},j}\log\left(-\log\left|z_{P_j}(Q_j)\right|^2\right) - 2\log\left|z_{P_j}(Q_j)\right| - 2\log\left(\varphi_{g_{\mathcal{P}},j}(Q_j)\right),$$

where $z_{\mathcal{P}} = \left(z_{P_j}\right)_j$ is an arithmetic local coordinate in $\mathcal{P}$. Then the generalized arithmetic self-intersection number of $(\mathcal{P}, g_{\mathcal{P}})$ is given by

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \sum_j \left(\eta_{g_{\mathcal{P}},j} - \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right)\right) - \lim_{\varepsilon\to 0}\left(\sum_j \eta_{g_{\mathcal{P}},j}\log\left(-\log\varepsilon^2\right) - \frac{1}{2}\int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{P}}\cdot\omega_{g_{\mathcal{P}}}\right),$$

where $\mathcal{X}_\varepsilon := \mathcal{X}_\infty \setminus \bigcup\{x \in \mathcal{X}_\infty \,|\, \left|z_{P_j}(x)\right| < \varepsilon\}$.

Now we will see that the arithmetic self-intersection number of the line bundle of modular forms equipped with the Petersson metric $\overline{\mathcal{M}}_k(\Gamma(1))$ (see Proposition 1.27) indeed could be calculated without the use of the generalized arithmetic intersection theory.

Proposition 3.6. *Let $(\mathcal{P}, g_{\mathcal{P}}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty)$ be an arithmetic prime divisor, where $g_{\mathcal{P}}$ is a log-log Green's function for $\mathcal{P}$ such that $\mathcal{P}(\mathbb{C}) \cap \mathcal{S}_\infty = \emptyset$. Then the generalized arithmetic self-intersection number of $(\mathcal{P}, g_{\mathcal{P}})$ is given by*

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \lim_{Q\to P}\left(\log\left|z_{\mathcal{P}}(Q)\right| + \frac{1}{2}g_{\mathcal{P}}(Q)\right) + \frac{1}{2}\int\limits_{\mathcal{X}_\infty} g_{\mathcal{P}}\cdot\omega_{g_{\mathcal{P}}},$$

where $z_{\mathcal{P}}$ is an arithmetic local coordinate in $\mathcal{P}$.

Proof. For $\mathcal{P}(\mathbb{C}) = \sum P_i$ and $\mathcal{S}_\infty = \sum S_j$ we set

$$\mathcal{X}_\varepsilon := \mathcal{X}_\infty \backslash \big(\bigcup_i B_\varepsilon(P_i) \cup \bigcup_j B_\varepsilon(S_j) \big).$$

Then we have to calculate the limit

$$(\mathcal{P}, g_\mathcal{P})^2 = \frac{1}{2} \lim_{\varepsilon \to 0} \int\limits_{\mathcal{X}_\varepsilon} \left(g_\mathcal{P} \cdot \omega_{\alpha_\mathcal{P}} - \alpha_\mathcal{P} \cdot \omega_{g_\mathcal{P}} + g_\mathcal{P} \cdot \omega_{g_\mathcal{P}} \right).$$

With the help of Stokes Theorem we obtain

$$\begin{aligned} \int\limits_{\mathcal{X}_\varepsilon} (g_\mathcal{P} \cdot \omega_{\alpha_\mathcal{P}} - \alpha_\mathcal{P} \cdot \omega_{g_\mathcal{P}} + g_\mathcal{P} \cdot \omega_{g_\mathcal{P}}) = & \sum_i \int\limits_{\partial B_\varepsilon(P_i)} (\alpha_\mathcal{P} \cdot \mathrm{d}^c g_\mathcal{P} - g_\mathcal{P} \cdot \mathrm{d}^c \alpha_\mathcal{P}) + \\ & \sum_j \int\limits_{\partial B_\varepsilon(S_j)} (\alpha_\mathcal{P} \cdot \mathrm{d}^c g_\mathcal{P} - g_\mathcal{P} \cdot \mathrm{d}^c \alpha_\mathcal{P}) + \int\limits_{\mathcal{X}_\varepsilon} g_\mathcal{P} \cdot \omega_{g_\mathcal{P}}. \end{aligned}$$

Since $\mathrm{ord}_{S_j}(\mathcal{P}(\mathbb{C})) = 0$, the integral

$$\int\limits_{\partial B_\varepsilon(S_j)} (\alpha_\mathcal{P} \cdot \mathrm{d}^c g_\mathcal{P} - g_\mathcal{P} \cdot \mathrm{d}^c \alpha_\mathcal{P})$$

vanishes in the limit $\varepsilon \longrightarrow 0$, see [La, p.23]. Moreover, using the calculations as in [Kü2, equation (20)], the equation

$$\int\limits_{\partial B_\varepsilon(P_i)} (\alpha_\mathcal{P} \cdot \mathrm{d}^c g_\mathcal{P} - g_\mathcal{P} \cdot \mathrm{d}^c \alpha_\mathcal{P}) = 2 \left(\log \left(\varphi_{\alpha_\mathcal{P},i}(P_i) \right) - \log \left(\varphi_{g_\mathcal{P},i}(P_i) \right) \right) + f(\varepsilon)$$

holds, where f is a continuous function with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. If we write locally

$$\begin{aligned} g_\mathcal{P}(Q_i) &= -2 \log |z_{P_i}(Q_i)| - 2 \log \left(\varphi_{g_\mathcal{P},i}(Q_i) \right) \text{ and} \\ \alpha_\mathcal{P}(Q_i) &= -2 \log |z_{P_i}(Q_i)| - 2 \log \left(\varphi_{\alpha_\mathcal{P},i}(Q_i) \right) \end{aligned}$$

near P_i, where $z_\mathcal{P} = (z_{P_i})_i$ is an arithmetic local coordinate in $\mathcal{P}$, we see that $\log \left(\varphi_{\alpha_\mathcal{P},i}(P_i) \right)$ vanishes for all i. Thus taking the limit $\varepsilon \longrightarrow 0$, it follows

$$(\mathcal{P}, g_\mathcal{P})^2 = -\sum_i \log \left(\varphi_{g_\mathcal{P},i}(P_i) \right) + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_\mathcal{P} \cdot \omega_{g_\mathcal{P}}. \tag{3.3}$$

Now with $\sum_i \log \left(\varphi_{g_\mathcal{P},i}(Q_i) \right) = -\frac{1}{2} \left(g_\mathcal{P}(Q) + 2 \log |z_\mathcal{P}(Q)| \right)$ we see that equation (3.3) can be written as

$$(\mathcal{P}, g_\mathcal{P})^2 = \lim_{Q \to P} \left(\log |z_\mathcal{P}(Q)| + \frac{1}{2} g_\mathcal{P}(Q) \right) + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_\mathcal{P} \cdot \omega_{g_\mathcal{P}}.$$

□

Remark 3.7. In the special case $\mathcal{S}_\infty = \emptyset$ we recover the arithmetic self-intersection number of an arithmetic divisor $(\mathcal{P}, g_\mathcal{P}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \emptyset) = \widehat{\mathrm{Z}}^1(\mathcal{X})$.

With Corollary 3.5 we can give a different proof of Proposition 1.27 using the generalized arithmetic self-intersection number.

Proof of Proposition 1.27. We take the arithmetic divisor

$$(\mathcal{P}, g_\mathcal{P}) = \left(\mathrm{div}\left(\Delta(\tau)\right), -\log\|\Delta(\tau)\|^2_{\mathrm{Pet}}\right).$$

The Petersson metric shows

$$\|\Delta(\tau)\|_{\mathrm{Pet}} = |\Delta(\tau)|\left(4\pi \mathrm{Im}(\tau)\right)^6 = |\Delta(\tau)|\left(-\log|q|^2\right)^6.$$

Using the Fourier expansion $\Delta(\tau) = q\prod(1-q^n)^{24}$ we obtain

$$-\log\|\Delta(\tau)\|^2_{\mathrm{Pet}} = -12\log\left(-\log|q|^2\right) - 2\log|q| + \mathcal{O}(q)$$

and hence $\eta_{g_\mathcal{P},\infty} = 6$. By Proposition 1.11, an arithmetic local coordinate in $\mathrm{div}\left(\Delta(\tau)\right) = \mathcal{S}_\infty$ is given by q. Thus Corollary 3.5 shows

$$\overline{\mathcal{M}}_{12}(\Gamma(1))^2 = 6 - \lim_{\varepsilon\to 0}\left(6\log\left(-\log\varepsilon^2\right) + \int\limits_{\mathcal{X}(1)_\varepsilon} \log\|\Delta(\tau)\|_{\mathrm{Pet}}\cdot\omega_{\mathrm{Pet}}\right),$$

where $\mathcal{X}(1)_\varepsilon := \mathcal{X}(1)_\infty\backslash\{x\in\mathcal{X}(1)_\infty|\,|q(x)|<\varepsilon\}$. As in [Kü3, Corollary 5.4.] we have the limit

$$\lim_{\varepsilon\to 0}\left(6\log\left(-\log\varepsilon^2\right) + \int\limits_{\mathcal{X}(1)_\varepsilon} \log\|\Delta(\tau)\|_{\mathrm{Pet}}\cdot\omega_{\mathrm{Pet}}\right) = 12 - 12^2\zeta'_\mathbb{Q}(-1).$$

It follows

$$\overline{\mathcal{M}}_{12}(\Gamma(1))^2 = 12^2\zeta_\mathbb{Q}(-1)\left(\frac{\zeta'_\mathbb{Q}(-1)}{\zeta_\mathbb{Q}(-1)} + \frac{1}{2}\right).$$

□

3.2 Scattering constants for congruence subgroups

In this section we give a short summary of the article [Kü4] and specialize the result given in [Kü4, Theorem 4.4] with the help of the generalized arithmetic self-intersection number in Corollary 3.5.

Definition 3.8. Let $\Gamma < \Gamma(1)$ be a subgroup of finite index, S_j a cusp of $X(\Gamma)$ and Γ_j its stabilizer. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and $\tau \in \mathbb{H}$, the *non-holomorphic Eisenstein series for the cusp* S_j is given by

$$E_{S_j}(\tau, s) = \sum_{\gamma \in \Gamma_j \backslash \Gamma} \mathrm{Im}\left(\sigma_j^{-1}\gamma(\tau)\right)^s,$$

where $\sigma_j \in \mathrm{PSL}_2(\mathbb{R})$ satisfies $\sigma_j^{-1}\Gamma_j\sigma_j = \left\{\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) \middle| m \in \mathbb{Z}\right\}$.

The following can be found in [Kü4, Properties 2.3.]:
The function $E_{S_j}(\tau, s)$ is $\Gamma(1)$-invariant in τ and has in s a meromorphic continuation on $\mathbb{C}$, with a simple pole in $s = 1$ with residue $3/(\pi[\Gamma(1) : \Gamma])$. For a cusp S_j of $X(\Gamma)$, the Fourier expansion of $E_{S_j}(\tau, s)$ in S_j is given by

$$E_{S_j}(\tau, s) = y^s + \varphi_j(s)y^{1-s} + \sum_{m \neq 0} a_m(y, s)\mathrm{e}^{2\pi i m x},$$

where we set $\tau = x + iy$. For an explicit description of the terms $\varphi_j(s)$ and $a_m(y, s)$ we refer to [Kü4, Properties 2.3.]. Here we only note that the function $\varphi_j(s)$ has a meromorphic continuation with a simple pole in $s = 1$ with residue $3/(\pi[\Gamma(1) : \Gamma])$.

Definition 3.9. The constant

$$C_j^\Gamma := \lim_{s \to 1}\left(\varphi_j(s) - \frac{3/(\pi[\Gamma(1) : \Gamma])}{s - 1}\right)$$

is called the *scattering constant for* S_j *on* $X(\Gamma)$.

Definition 3.10. For any cusp S_j of $X(\Gamma)$ we define the $\Gamma(1)$-invariant function

$$g_j(\tau) := 4\pi \lim_{s \to 1}\left(E_{S_j}(\tau, s) - \varphi_j(s)\right) - \frac{12}{[\Gamma(1) : \Gamma]}\log(4\pi).$$

The induced function on $X(\Gamma)$ will be also denoted by g_j and is called the *hyperbolic Green's function for the cusp* S_j.

Indeed, by [Kü4, Proposition 2.7.] the function g_j is always a log-log Green's function for S_j, which satisfies

$$\mathrm{dd}^c g_j = \frac{12}{[\Gamma(1) : \Gamma]}\omega$$

outside S_j, where $\omega := \omega_{\mathrm{Pet}}|_{X(\Gamma)}$ denotes the $(1,1)$-form on $X(\Gamma)$, induced by the Petersson normalized hyperbolic $(1,1)$-form ω_{Pet} on $X(1)$.
Moreover, note that we have the normalization $\int_{X(\Gamma)} \omega = \frac{[\Gamma(1):\Gamma]}{12}$, see [Kü4].

The following example is due to [Kü4, Remark 4.5].

Example 3.11. In the special case $\Gamma = \Gamma(1)$ there is only one cusp S_∞ on $X(1)$. The function φ_∞ is given by

$$\varphi_\infty(s) = \pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)},$$

where $\Gamma(s)$ denotes the gamma function. The scattering constant $C_\infty^{\Gamma(1)}$ is given by

$$\begin{aligned} C_\infty^{\Gamma(1)} &= \lim_{s\to 1}\left(\pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\frac{\zeta_{\mathbb{Q}}(2s-1)}{\zeta_{\mathbb{Q}}(2s)} - \frac{3}{\pi(s-1)}\right) \\ &= -\frac{6}{\pi}(12\zeta_{\mathbb{Q}}'(-1) - 1 + \log(4\pi)). \end{aligned}$$

Moreover, by Kronecker's limit formula, we have the equation

$$-\log\|\Delta(\tau)\|_{\mathrm{Pet}}^2 = 4\pi\lim_{s\to 1}\left(E_{S_\infty}(\tau,s) - \varphi_\infty(s)\right) - 12\log(4\pi) =: g_\infty(\tau).$$

Proposition 3.12. *Let $\Gamma < \Gamma(1)$ be a congruence subgroup and let $\mathcal{X}(\Gamma)$ be a regular model of $X(\Gamma)$ such that there exists a proper morphism $\pi_\Gamma : \mathcal{X}(\Gamma) \longrightarrow \mathcal{X}(1)$ of $\operatorname{Spec}\mathbb{Z}$-schemes. Moreover, let $\mathcal{S}_i$ denote the closure in $\mathcal{X}(\Gamma)$ of a cusp S_i on $X(\Gamma)$ with cusp width b_i. Then the family*

$$z_{\mathcal{S}_i} = q^{1/b_i}$$

defines an arithmetic local coordinate in $\mathcal{S}_i$.

Proof. The modular curve $X(\Gamma)$ defines an algebraic curve over some number field K, hence $\mathcal{X}(\Gamma)$ is an arithmetic surface over $\operatorname{Spec}\mathcal{O}_K$. Note that $z_{\mathcal{S}_i} = q^{1/b_i}$ is indeed a family of local coordinates in the cusps induced by $\mathcal{S}_i$ on $\mathcal{X}(\Gamma)_\infty \cong \coprod_{\sigma:K\hookrightarrow\mathbb{C}} X(\Gamma_\sigma)$, where Γ_σ are congruence subgroups of $\Gamma(1)$. The modular invariant j defines by pullback a rational function on $\mathcal{X}(\Gamma)$ satisfying $\operatorname{ord}_{S_i}(j) = -b_i$. Thus the ramification index of j in S_i equals the cusp width. The result follows immediately from Proposition 1.6, since the horizontal divisors associated to different cusps do not intersect on $\mathcal{X}(\Gamma)$, see [KM, Theorem 10.9.1]. □

Proposition 3.13. *Let Γ be a congruence subgroup and let $X(\Gamma)$ be defined over some number field K. Let $\mathcal{X} = \mathcal{X}(\Gamma)$ be an arithmetic surface over $\operatorname{Spec}\mathcal{O}_K$ associated to $X(\Gamma)$ such that there exists a proper morphism $\pi_\Gamma : \mathcal{X}(\Gamma) \longrightarrow \mathcal{X}(1)$ of $\operatorname{Spec}\mathbb{Z}$-schemes. Consider the hermitian line bundle $\overline{\mathcal{L}} = \left(\mathcal{O}_{\mathcal{X}}(\mathcal{S}_j), \|\cdot\|_{\mathrm{hyp}}\right)$, where $\mathcal{S}_j$ is the closure in $\mathcal{X}$ of a cusp S_j*

on $X(\Gamma)$ and the metric $\|\cdot\|_{\mathrm{hyp}}$ is associated to the hyperbolic Green's function g_j for the cusp $\mathcal{S}_j$. Then the self-intersection number of $\overline{\mathcal{L}}$ is given by

$$\overline{\mathcal{L}}^2 = -2\pi \sum_{\sigma: K \hookrightarrow \mathbb{C}} C_j^{\Gamma_\sigma} + [K:\mathbb{Q}] \frac{6 - 12\log(4\pi)}{[\Gamma(1):\Gamma]},$$

where $C_j^{\Gamma_\sigma}$ denotes the scattering constant for the cusp $\mathcal{S}_j$ on $\mathcal{X}(\Gamma)_\sigma = X(\Gamma_\sigma)$.

Proof. Since the local coordinate $z_{\mathcal{S}_j} = q^{1/b_j}$ is an arithmetic local coordinate in $\mathcal{S}_j$, the arithmetic self-intersection number in question is given by

$$\overline{\mathcal{L}}^2 = \sum_{\sigma: K \hookrightarrow \mathbb{C}} \Bigg(\eta_{g_{j,\sigma}} - \log(\varphi_{g_{j,\sigma}}(S_{j,\sigma})) - \lim_{\varepsilon \to 0} \Bigg(\eta_{g_{j,\sigma}} \log(-\log \varepsilon^2) - \frac{1}{2} \int\limits_{X(\Gamma_\sigma)_\varepsilon} g_j \cdot \frac{12}{[\Gamma(1):\Gamma]} \omega \Bigg) \Bigg),$$

where $\mathcal{S}_j(\mathbb{C}) = \sum_\sigma S_{j,\sigma}$ and for $z_{\mathcal{S}_j} = \left(z_{S_{j,\sigma}}\right)_\sigma$ we set

$$X(\Gamma_\sigma)_\varepsilon := X(\Gamma_\sigma) \backslash \{P \in X(\Gamma_\sigma) | \left| z_{S_{j,\sigma}}(P) \right| < \varepsilon\}.$$

The hyperbolic Green's function g_j can be written in the local coordinate $z_{\mathcal{S}_j}$ as

$$g_j(Q) = -\log \left| z_{\mathcal{S}_j}(Q) \right|^2 - \frac{12}{[\Gamma(1):\Gamma]} \log(-\log \left| z_{\mathcal{S}_j}(Q) \right|^2) + f_j(Q),$$

where f_j is a smooth function, which is continuous and has the special value $f_j(S_{j,\sigma}) = 0$ for all $\sigma : K \hookrightarrow \mathbb{C}$, see [Kü4, Proposition 2.7]. Thus we obtain the constants

$$\eta_{g_{j,\sigma}} = \frac{6}{[\Gamma(1):\Gamma_\sigma]} = \frac{6}{[\Gamma(1):\Gamma]} \text{ and}$$
$$\log(\varphi_{g_{j,\sigma}}(S_{j,\sigma})) = -\frac{1}{2} f_j(S_{j,\sigma}) = 0$$

for all $\sigma : K \hookrightarrow \mathbb{C}$. By [Kü4, Lemma 2.8] we have the limit

$$\lim_{\varepsilon \to 0} \left(\log(-\log |\varepsilon|^2) - \int\limits_{X(\Gamma_\sigma)_\varepsilon} g_j \cdot \omega \right) = \frac{[\Gamma(1):\Gamma]}{12} 4\pi C_j^{\Gamma_\sigma} + 2\log(4\pi).$$

Summing up the $[K:\mathbb{Q}]$ embeddings $\sigma : K \hookrightarrow \mathbb{C}$ yields

$$\begin{aligned} \overline{\mathcal{L}}^2 &= \sum_{\sigma: K \hookrightarrow \mathbb{C}} \left(\frac{6}{[\Gamma(1):\Gamma]} - 2\pi C_j^{\Gamma_\sigma} - \frac{12\log(4\pi)}{[\Gamma(1):\Gamma]} \right) \\ &= -2\pi \sum_\sigma C_j^{\Gamma_\sigma} + [K:\mathbb{Q}] \frac{6 - 12\log(4\pi)}{[\Gamma(1):\Gamma]}, \end{aligned}$$

which proves the claim. □

Example 3.14. Let $\mathcal{X} = \mathcal{X}(1)$ and let $\mathcal{S}_\infty$ be the unique cusp of $\mathcal{X}(1)$. Its scattering constant $C_\infty^{\Gamma(1)}$ is given by $-2\pi C_\infty^{\Gamma(1)} = 12(12\zeta'_{\mathbb{Q}}(-1) - 1 + \log(4\pi))$, see Example 3.11. Thus by Proposition 3.13, we obtain the arithmetic self-intersection number

$$\overline{\mathcal{O}_{\mathcal{X}}}(\mathcal{S}_\infty)^2 = 12^2 \left(\frac{1}{2}\zeta_{\mathbb{Q}}(-1) + \zeta'_{\mathbb{Q}}(-1) \right),$$

which is the same as in Proposition 1.27.

3.3 Generalized arithmetic intersection theory on arithmetic surfaces using adjusted Green's functions

In this section we show that the generalized arithmetic self-intersection number of an arithmetic divisor with a log-log Green's function $g_{\mathcal{P}}$ for a horizontal prime divisor $\mathcal{P}$ coincides with the generalized arithmetic self-intersection number in [Kü3]. Moreover, we define the generalized arithmetic intersection number for two arbitrary arithmetic divisors with log-log Green's functions with the help of adjusted Green's functions and compare it with the generalized arithmetic intersection theory in [Kü3].
The following definition is due to [Kü3, Definition 3.6].

Definition 3.15. Let $(\mathcal{D}_1, g_{\mathcal{D}_1}), (\mathcal{D}_2, g_{\mathcal{D}_2}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty)$ be two arithmetic divisors with log-log Green's functions such that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect properly on the generic fibre X. Set $\mathcal{S}_\infty = \sum S_j$. Then the *generalized arithmetic intersection number due to Kühn of* $(\mathcal{D}_1, g_{\mathcal{D}_1})$ *and* $(\mathcal{D}_2, g_{\mathcal{D}_2})$ is defined by

$$\begin{aligned}(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) := (\mathcal{D}_1, \mathcal{D}_2)_{\mathrm{fin}} + \sum_j \left(\mathrm{ord}_{S_j}(\mathcal{D}_2)\eta_{g_{\mathcal{D}_1},j} + \mathrm{ord}_{S_j}(\mathcal{D}_1)\eta_{g_{\mathcal{D}_2},j} \right) + \\ \frac{1}{2}\int_{\mathcal{X}_\infty} \left(g_{\mathcal{D}_1} \cdot \omega_{g_{\mathcal{D}_2}} + g_{\mathcal{D}_2} \cdot \omega_{g_{\mathcal{D}_1}} + \mathrm{d}g_{\mathcal{D}_2} \cdot \mathrm{d}^{\mathrm{c}} g_{\mathcal{D}_1} \right).\end{aligned}$$

Let us make some notes about the generalized arithmetic intersection number due to Kühn.
For $k = 1, 2$ write $\mathcal{D}_k(\mathbb{C}) = \sum_i \mathrm{ord}_{D_{k,i}}(\mathcal{D}_k(\mathbb{C}))D_{k,i}$. Moreover, set

$$\mathcal{X}_\varepsilon := \mathcal{X}_\infty \Big\backslash \Big(\bigcup_{D_{1,i} \notin \mathcal{S}_\infty} B_\varepsilon(D_{1,i}) \cup \bigcup_{D_{2,i} \notin \mathcal{S}_\infty} B_\varepsilon(D_{2,i}) \cup \bigcup_{S_j \in \mathcal{S}_\infty} B_\varepsilon(S_j) \Big). \tag{3.4}$$

Apart from $\mathcal{D}_1(\mathbb{C})$ we have $g_{\mathcal{D}_2} \cdot \omega_{g_{\mathcal{D}_1}} = \mathrm{d}\left(g_{\mathcal{D}_2} \cdot \mathrm{d}^c g_{\mathcal{D}_1}\right) - \mathrm{d}g_{\mathcal{D}_2} \cdot \mathrm{d}^c g_{\mathcal{D}_1}$. It follows that the generalized arithmetic intersection number due to Kühn can be written as

$$(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = (\mathcal{D}_1, \mathcal{D}_2)_{\mathrm{fin}} + \sum_j \left(\mathrm{ord}_{S_j}(\mathcal{D}_2)\eta_{g_{\mathcal{D}_1},j} + \mathrm{ord}_{S_j}(\mathcal{D}_1)\eta_{g_{\mathcal{D}_2},j}\right) + \frac{1}{2}\lim_{\varepsilon\to 0}\left(\int_{\partial\mathcal{X}_\varepsilon} g_{\mathcal{D}_2} \cdot \mathrm{d}^c g_{\mathcal{D}_1} + \int_{\mathcal{X}_\varepsilon} g_{\mathcal{D}_1} \cdot \omega_{g_{\mathcal{D}_2}}\right).$$

An explicit calculation of $\int_{\partial\mathcal{X}_\varepsilon} g_{\mathcal{D}_2} \cdot \mathrm{d}^c g_{\mathcal{D}_1}$ yields

$$\begin{aligned}(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = (\mathcal{D}_1, \mathcal{D}_2)_{\mathrm{fin}} &+ \frac{1}{2} g_{\mathcal{D}_2}\left[\mathcal{D}_1 - \sum \mathrm{ord}_{S_j}(\mathcal{D}_1) S_j\right] + \\ &\sum_{j=1}^{r} \mathrm{ord}_{S_j}(\mathcal{D}_1)\left(\eta_{g_{\mathcal{D}_2},j} - \log\left(\varphi_{g_{\mathcal{D}_2},j}(S_j)\right)\right) - \\ &\lim_{\varepsilon\to 0}\left(\sum_{j=1}^{r} \mathrm{ord}_{S_j}(\mathcal{D}_1)\eta_{g_{\mathcal{D}_2},j}\log\left(-\log\varepsilon^2\right) - \frac{1}{2}\int_{\mathcal{X}_\varepsilon} g_{\mathcal{D}_1} \cdot \omega_{g_{\mathcal{D}_2}}\right),\end{aligned} \tag{3.5}$$

see [Kü3, Lemma 3.9]. Moreover, in [Kü3, Theorem 3.11] it was shown that the generalized arithmetic intersection number due to Kühn defines a bilinear and symmetric pairing

$$\begin{aligned}\widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{S}_\infty) \times \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{S}_\infty) &\longrightarrow \mathbb{R} \\ ([\mathcal{D}_1, g_{\mathcal{D}_1}], [\mathcal{D}_2, g_{\mathcal{D}_2}]) &\longmapsto (\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}).\end{aligned}$$

Proposition 3.16. *Let $\mathcal{P}$ be a horizontal prime divisor on an arithmetic surface $\mathcal{X}$ and set $\mathcal{P}(\mathbb{C}) = \sum P_j$. For an arithmetic prime divisor $(\mathcal{P}, g_{\mathcal{P}}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{P}(\mathbb{C}))$ with a log-log Green's function $g_{\mathcal{P}}$ for $\mathcal{P}$, the generalized arithmetic self-intersection number due to Kühn coincides with the generalized arithmetic self-intersection number as in Definition 3.3.*

Proof. We have to show that the generalized arithmetic self-intersection number due to Kühn can be written as

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \sum_j \left(\eta_{g_{\mathcal{P}},j} - \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right)\right) + \lim_{Q\to P}\left(\log|z_{\mathcal{P}}(Q)| - \log|t_{\mathcal{P}}(Q)|\right) - \lim_{\varepsilon\to 0}\left(\sum_j \eta_{g_{\mathcal{P}},j}\log\left(-\log\varepsilon^2\right) - \frac{1}{2}\int_{\mathcal{X}_\varepsilon} g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}}\right),$$

where $z_{\mathcal{P}}$ is an arithmetic local coordinate in $\mathcal{P}$. For $\mathcal{S}_\infty = \mathcal{P}(\mathbb{C}) = \sum P_j$ fix a family of local coordinates $t_{\mathcal{P}} = \left(t_{P_j}\right)_j$ in $\mathcal{P}(\mathbb{C})$. Consider the two arithmetic prime divisors $(\mathcal{P}, g_{\mathcal{P}})$ and $(\mathcal{P}', g_{\mathcal{P}'})$, where $g_{\mathcal{P}}$ is a log-log Green's function and $(\mathcal{P}', g_{\mathcal{P}'})$ is defined by $\left(\mathcal{P} - \operatorname{div}(f), g_{\mathcal{P}} + \log|f|^2\right)$ for a rational function $f \in k(\mathcal{X})^\times$ such that $\mathcal{P}$ and $\mathcal{P}'$ intersect properly on the generic fibre X. Then the generalized arithmetic self-intersection number due to Kühn of $(\mathcal{P}, g_{\mathcal{P}})$ and $(\mathcal{P}', g_{\mathcal{P}'})$ is given by

$$(\mathcal{P}, \mathcal{P}')_{\text{fin}} + \sum_j \left(\eta_{g_{\mathcal{P}'},j} - \log\left(\varphi_{g_{\mathcal{P}'},j}(P_j)\right)\right) -$$

$$\lim_{\varepsilon \to 0} \left(\sum_j \eta_{g_{\mathcal{P}'},j} \log\left(-\log \varepsilon^2\right) - \frac{1}{2} \int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}'}} \right).$$

Here $\mathcal{X}_\varepsilon := \mathcal{X}_\infty \backslash \bigcup \{x \in \mathcal{X}_\infty | \left|t_{P_j}(x)\right| < \varepsilon\}$ and the log-log Green's functions $g_{\mathcal{P}}$ and $g_{\mathcal{P}'}$ have the local expansions

$$g_{\mathcal{P}}(Q_j) = -2\eta_{g_{\mathcal{P}},j} \log\left(-\log\left|t_{P_j}(Q_j)\right|^2\right) - 2\log\left|t_{P_j}(Q_j)\right| - 2\log\left(\varphi_{g_{\mathcal{P}},j}(Q_j)\right) \text{ and}$$

$$g_{\mathcal{P}'}(Q_j) = -2\eta_{g_{\mathcal{P}'},j} \log\left(-\log\left|t_{P_j}(Q_j)\right|^2\right) - 2\log\left(\varphi_{g_{\mathcal{P}'},j}(Q_j)\right)$$

near P_j. Because of $g_{\mathcal{P}'}(Q_j) = g_{\mathcal{P}}(Q_j) + 2\log|f(Q_j)|$ we obtain $\eta_{g_{\mathcal{P}'},j} = \eta_{g_{\mathcal{P}},j}$ and

$$\log\left(\varphi_{g_{\mathcal{P}'},j}(P_j)\right) = \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right) + \lim_{Q_j \to P_j} \left(\log|f(Q_j)| - \log\left|t_{P_j}(Q_j)\right|\right)$$

for all j. Moreover, note that $\omega_{g_{\mathcal{P}'}} = \omega_{g_{\mathcal{P}}}$. Now let $z_{\mathcal{P}}$ be an arithmetic local coordinate in $\mathcal{P}$. Then

$$(\mathcal{P} - \operatorname{div}(f), \mathcal{P})_{\text{fin}} = \lim_{Q \to P} \left(\log|z_{\mathcal{P}}(Q)| - \log|f(Q)|\right).$$

It follows that the generalized arithmetic self-intersection number due to Kühn of $(\mathcal{P}, g_{\mathcal{P}})$ and $(\mathcal{P}', g_{\mathcal{P}'})$ is given by

$$\sum_j \left(\eta_{g_{\mathcal{P}},j} - \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right)\right) + \lim_{Q \to P} \left(\log|z_{\mathcal{P}}(Q)| - \log|t_{\mathcal{P}}(Q)|\right) -$$

$$\lim_{\varepsilon \to 0} \left(\sum_j \eta_{g_{\mathcal{P}},j} \log\left(-\log \varepsilon^2\right) - \frac{1}{2} \int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}} \right).$$

This number coincides with the generalized arithmetic self-intersection number as in Proposition 3.4. Now the proof of Proposition 3.16 follows from $(\mathcal{P}, g_{\mathcal{P}}) \cdot (\mathcal{P}', g_{\mathcal{P}'}) = (\mathcal{P}, g_{\mathcal{P}})^2$. $\square$

Definition 3.17. Let $\mathcal{D}_1, \mathcal{D}_2 \in \mathrm{Z}^1(\mathcal{X})$ be two divisors on $\mathcal{X}$. A Green's function $\alpha_{\mathcal{D}_1,\mathcal{D}_2}$ for $\mathcal{D}_1$ is called a *$\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1$*, if the height $\mathrm{ht}_{[\mathcal{D}_1,\alpha_{\mathcal{D}_1,\mathcal{D}_2}]}(\mathcal{D}_2)$ vanishes, i.e. $\alpha_{\mathcal{D}_1,\mathcal{D}_2}$ is a $\mathcal{D}_2$-adjusted Green's form for $\mathcal{D}_1$.

In the next lemma we recall the basic properties of adjusted Green's functions. We also prove these properties although the proof can be found in the second chapter.

Lemma 3.18. *Let $\mathcal{D}_1$ and $\mathcal{D}_1'$ be divisors on $\mathcal{X}$ and let $\mathcal{D}_2$ be a divisor on $\mathcal{X}$ such that $\deg(\mathcal{D}_2) \neq 0$. Then the following properties hold:*
i) There exists a $\mathcal{D}_2$-adjusted Green's function $\alpha_{\mathcal{D}_1,\mathcal{D}_2}$ for $\mathcal{D}_1$.
ii) If $\alpha_{\mathcal{D}_1,\mathcal{D}_2}$ is a $\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1$ and $\alpha_{\mathcal{D}_1',\mathcal{D}_2}$ is a $\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1'$, then $\alpha_{\mathcal{D}_1+\mathcal{D}_1',\mathcal{D}_2} := \alpha_{\mathcal{D}_1,\mathcal{D}_2} + \alpha_{\mathcal{D}_1',\mathcal{D}_2}$ is a $\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1 + \mathcal{D}_1'$.
iii) If $\mathcal{D}_1$ intersect $\mathcal{D}_2$ properly on the generic fibre X, then any $\mathcal{D}_2$-adjusted Green's function $\alpha_{\mathcal{D}_1,\mathcal{D}_2}$ for $\mathcal{D}_1$ satisfies

$$\alpha_{\mathcal{D}_1,\mathcal{D}_2}[\mathcal{D}_2(\mathbb{C})] = -2\,(\mathcal{D}_1,\mathcal{D}_2)_{\mathrm{fin}}\,.$$

iv) If $\mathcal{D}_1 = \mathrm{div}(f)$ for a rational function $f \in k(\mathcal{X})^\times$, then the canonical Green's function $-\log|f|^2$ for $\mathrm{div}(f)$ is a $\mathcal{D}_2$-adjusted Green's function for $\mathrm{div}(f)$.

Proof. *i)* It is clear that any Green's function $g_{\mathcal{D}_1}$ for $\mathcal{D}_1$ can be rescaled by some $\beta \in \mathbb{R}$ such that $\alpha_{\mathcal{D}_1,\mathcal{D}_2} := g_{\mathcal{D}_1} + \beta$ is a $\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1$ because of the equation

$$\begin{aligned}\mathrm{ht}_{[\mathcal{D}_1,\alpha_{\mathcal{D}_1,\mathcal{D}_2}]}(\mathcal{D}_2) =& \mathrm{ht}_{[\mathcal{D}_1,g_{\mathcal{D}_1}+\beta]}(\mathcal{D}_2) = \mathrm{ht}_{[\mathcal{D}_1,g_{\mathcal{D}_1}]}(\mathcal{D}_2) + \mathrm{ht}_{[0,\beta]}(\mathcal{D}_2) = \\ & \mathrm{ht}_{[\mathcal{D}_1,g_{\mathcal{D}_1}]}(\mathcal{D}_2) + \frac{\beta}{2}\deg(\mathcal{D}_2).\end{aligned}$$

ii) The second assertion follows from the fact that the height is linear in $\widehat{\mathrm{CH}}^1(\mathcal{X})$. More precisely, the height

$$\mathrm{ht}_{[\mathcal{D}_1+\mathcal{D}_1',\alpha_{\mathcal{D}_1+\mathcal{D}_1',\mathcal{D}_2}]}(\mathcal{D}_2) = \mathrm{ht}_{[\mathcal{D}_1+\mathcal{D}_1',\alpha_{\mathcal{D}_1,\mathcal{D}_2}+\alpha_{\mathcal{D}_1',\mathcal{D}_2}]}(\mathcal{D}_2) = \mathrm{ht}_{[\mathcal{D}_1,\alpha_{\mathcal{D}_1,\mathcal{D}_2}]}(\mathcal{D}_2) + \mathrm{ht}_{[\mathcal{D}_1',\alpha_{\mathcal{D}_1',\mathcal{D}_2}]}(\mathcal{D}_2)$$

vanishes if both $\mathrm{ht}_{[\mathcal{D}_1,\alpha_{\mathcal{D}_1,\mathcal{D}_2}]}(\mathcal{D}_2)$ and $\mathrm{ht}_{[\mathcal{D}_1',\alpha_{\mathcal{D}_1',\mathcal{D}_2}]}(\mathcal{D}_2)$ vanish.
iii) If $\mathcal{D}_1$ intersect $\mathcal{D}_2$ properly on X, then the definition of the height shows

$$\mathrm{ht}_{[\mathcal{D}_1,\alpha_{\mathcal{D}_1,\mathcal{D}_2}]}(\mathcal{D}_2) = (\mathcal{D}_1,\mathcal{D}_2)_{\mathrm{fin}} + \frac{1}{2}\alpha_{\mathcal{D}_1,\mathcal{D}_2}[\mathcal{D}_2(\mathbb{C})] = 0.$$

iv) The last assertion follows immediately from the definition of the height. Since the height gives a well-defined pairing on $\widehat{\mathrm{CH}}^1(\mathcal{X}) \times \mathrm{Z}^1(\mathcal{X})$, all elements in $\widehat{\mathrm{R}}^1(\mathcal{X}) \times \mathrm{Z}^1(\mathcal{X})$ have to map to zero, hence

$$\mathrm{ht}_{[\mathrm{div}(f),-\log|f|^2]}(\mathcal{D}_2) = 0$$

for any divisor $\mathcal{D}_2 \in \mathrm{Z}^1(\mathcal{X})$. □

Definition 3.19. For two arithmetic divisors $(\mathcal{D}_1, g_{\mathcal{D}_1}), (\mathcal{D}_2, g_{\mathcal{D}_2}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty)$ with log-log Green's functions $g_{\mathcal{D}_1}$ and $g_{\mathcal{D}_2}$ for $\mathcal{D}_1$ and $\mathcal{D}_2$ we define the *generalized arithmetic intersection number of* $(\mathcal{D}_1, g_{\mathcal{D}_1})$ *and* $(\mathcal{D}_2, g_{\mathcal{D}_2})$ by

$$(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \left(g_{\mathcal{D}_2} \cdot \omega_{\alpha_{\mathcal{D}_1,\mathcal{D}_2}} - \alpha_{\mathcal{D}_1,\mathcal{D}_2} \cdot \omega_{g_{\mathcal{D}_2}} + g_{\mathcal{D}_1} \cdot \omega_{g_{\mathcal{D}_2}} \right), \tag{3.6}$$

where $\alpha_{\mathcal{D}_1,\mathcal{D}_2}$ is a $\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1$.

The next result is a more general version of Corollary 3.5 with different log-log Green's functions and follows directly from the equations (3.1) and (3.2). However, it will be useful for later calculations.

Corollary 3.20. *Let* $(\mathcal{P}, g_{\mathcal{P}}), (\mathcal{P}, g'_{\mathcal{P}}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{P}(\mathbb{C}))$ *be two arithmetic prime-divisors with log-log Green's functions* $g_{\mathcal{P}}$ *and* $g'_{\mathcal{P}}$ *for* $\mathcal{P}$*. Set* $\mathcal{P}(\mathbb{C}) = \sum P_j$ *and write*

$$g_{\mathcal{P}}(Q_j) = -2\eta_{g_{\mathcal{P}},j} \log\left(-\log\left|z_{P_j}(Q_j)\right|^2\right) - 2\log\left|z_{P_j}(Q_j)\right| - 2\log\left(\varphi_{g_{\mathcal{P}},j}(Q_j)\right)$$

near P_j *on the manifold* $\mathcal{X}_\varepsilon = \mathcal{X}_\infty \backslash \bigcup\{x \in \mathcal{X}_\infty | \left|z_{P_j}(x)\right| < \varepsilon\}$*, where* $z_{\mathcal{P}} = \left(z_{P_j}\right)_j$ *is an arithmetic local coordinate in* $\mathcal{P}$*. Then the generalized arithmetic intersection number* $(\mathcal{P}, g_{\mathcal{P}}) \cdot (\mathcal{P}, g'_{\mathcal{P}})$ *is given by*

$$\begin{aligned}(\mathcal{P}, g_{\mathcal{P}}) \cdot (\mathcal{P}, g'_{\mathcal{P}}) = &\sum_j \left(\eta_{g_{\mathcal{P}},j} - \log\left(\varphi_{g_{\mathcal{P}},j}(P_j)\right)\right) - \\ &\lim_{\varepsilon \to 0} \left(\sum_j \eta_{g_{\mathcal{P}},j} \log\left(-\log \varepsilon^2\right) - \frac{1}{2} \int\limits_{\mathcal{X}_\varepsilon} g'_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}} \right).\end{aligned}$$

Proposition 3.21. *Let* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *be divisors on an arithmetic surface* $\mathcal{X}$*, which intersect properly on the generic fibre of* $\mathcal{X}$*. For two arithmetic divisors* $(\mathcal{D}_1, g_{\mathcal{D}_1}), (\mathcal{D}_2, g_{\mathcal{D}_2}) \in \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty)$ *with log-log Green's functions* $g_{\mathcal{D}_1}$ *and* $g_{\mathcal{D}_2}$ *for* $\mathcal{D}_1$ *and* $\mathcal{D}_2$*, the generalized arithmetic intersection number (3.6) is well-defined and coincides with the generalized arithmetic intersection number due to Kühn in (3.5).*

Proof. We have to show that the generalized arithmetic intersection number can be written as

$$\begin{aligned}(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = (\mathcal{D}_1, \mathcal{D}_2)_{\text{fin}} + \frac{1}{2} g_{\mathcal{D}_2} \left[\mathcal{D}_1 - \sum \operatorname{ord}_{S_j}(\mathcal{D}_1) S_j\right] + \\ \sum_{j=1}^{r} \operatorname{ord}_{S_j}(\mathcal{D}_1) \left(\eta_{g_{\mathcal{D}_2},j} - \log\left(\varphi_{g_{\mathcal{D}_2},j}(S_j)\right)\right) - \\ \lim_{\varepsilon \to 0} \left(\sum_{j=1}^{r} \operatorname{ord}_{S_j}(\mathcal{D}_1) \eta_{g_{\mathcal{D}_2},j} \log\left(-\log \varepsilon^2\right) - \frac{1}{2} \int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{D}_1} \cdot \omega_{g_{\mathcal{D}_2}} \right),\end{aligned}$$

where $\mathcal{X}_\varepsilon$ is as in (3.4). Using Stokes Theorem and the formulas in [La, p.23] we derive for two log-log Green's functions $g_{\mathcal{D}_1}$ and $g_{\mathcal{D}_2}$ the integral

$$\begin{aligned}\int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{D}_2} \cdot \omega_{g_{\mathcal{D}_1}} =& g_{\mathcal{D}_2} \left[\mathcal{D}_1 - \sum \operatorname{ord}_{S_j}(\mathcal{D}_1) S_j\right] - \\ & \sum_{j=1}^{r} \int\limits_{\partial B_\varepsilon(S_j)} g_{\mathcal{D}_2} \cdot \mathrm{d}^c g_{\mathcal{D}_1} - \int\limits_{\mathcal{X}_\varepsilon} \mathrm{d} g_{\mathcal{D}_2} \cdot \mathrm{d}^c g_{\mathcal{D}_1} + f(\varepsilon),\end{aligned}$$

where f satisfies $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. This formula can also be found in [Kü2, equation (20)]. It was shown in the proof of [Kü3, Lemma 3.9]

$$\begin{aligned}&\sum_{j=1}^{r} \int\limits_{\partial B_\varepsilon(S_j)} g_{\mathcal{D}_2} \cdot \mathrm{d}^c g_{\mathcal{D}_1} = \\ &2 \sum_{j=1}^{r} \left(\operatorname{ord}_{S_j}(\mathcal{D}_2) \eta_{g_{\mathcal{D}_1},j} + \operatorname{ord}_{S_j}(\mathcal{D}_1) \left(\eta_{g_{\mathcal{D}_2},j} \log\left(-\log \varepsilon^2\right) + \log\left(\varphi_{g_{\mathcal{D}_2},j}(S_j)\right)\right)\right).\end{aligned}$$

Using the equation $\mathrm{d} g_{\mathcal{D}_2} \cdot \mathrm{d}^c g_{\mathcal{D}_1} = \mathrm{d} g_{\mathcal{D}_1} \cdot \mathrm{d}^c g_{\mathcal{D}_2}$ and the fact that $\eta_{\alpha_{\mathcal{D}_1,\mathcal{D}_2},j}$ vanishes for all j, the equation (3.6) is given by

$$\begin{aligned}(\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = \\ \sum_{j=1}^{r} \operatorname{ord}_{S_j}(\mathcal{D}_1) \left(\eta_{g_{\mathcal{D}_2},j} - \log\left(\varphi_{g_{\mathcal{D}_2},j}(S_j)\right)\right) + \sum_{j=1}^{r} \operatorname{ord}_{S_j}(\mathcal{D}_2) \log\left(\varphi_{\alpha_{\mathcal{D}_1,\mathcal{D}_2},j}(S_j)\right) - \\ \lim_{\varepsilon \to 0} \left(\sum_{j=1}^{r} \operatorname{ord}_{S_j}(\mathcal{D}_1) \eta_{g_{\mathcal{D}_2},j} \log\left(-\log \varepsilon^2\right) - \frac{1}{2} \int\limits_{\mathcal{X}_\varepsilon} g_{\mathcal{D}_1} \cdot \omega_{g_{\mathcal{D}_2}} \right) + \\ \frac{1}{2} \left(g_{\mathcal{D}_2} \left[\mathcal{D}_1 - \sum \operatorname{ord}_{S_j}(\mathcal{D}_1) S_j\right] - \alpha_{\mathcal{D}_1,\mathcal{D}_2} \left[\mathcal{D}_2 - \sum \operatorname{ord}_{S_j}(\mathcal{D}_2) S_j\right] \right).\end{aligned}$$

Hence it remains to show

$$(\mathcal{D}_1, \mathcal{D}_2)_{\text{fin}} = \sum_{j=1}^{r} \operatorname{ord}_{S_j}(\mathcal{D}_2) \log\left(\varphi_{\alpha_{\mathcal{D}_1,\mathcal{D}_2},j}(S_j)\right) - \frac{1}{2}\alpha_{\mathcal{D}_1,\mathcal{D}_2}\left[\mathcal{D}_2 - \sum \operatorname{ord}_{S_j}(\mathcal{D}_2) S_j\right].$$

Since $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect properly on X, the equation $(\mathcal{D}_1, \mathcal{D}_2)_{\text{fin}} = -\frac{1}{2}\alpha_{\mathcal{D}_1,\mathcal{D}_2}[\mathcal{D}_2]$ holds, see Lemma 3.18. Thus we are left to show that the sum

$$\sum_{j=1}^{r} \operatorname{ord}_{S_j}(\mathcal{D}_2)\left(\log\left(\varphi_{\alpha_{\mathcal{D}_1,\mathcal{D}_2},j}(S_j)\right) + \frac{1}{2}\alpha_{\mathcal{D}_1,\mathcal{D}_2}(S_j)\right)$$

vanishes. But this follows from the local description of $\alpha_{\mathcal{D}_1,\mathcal{D}_2}$ near S_j. Indeed, if $\operatorname{ord}_{S_j}(\mathcal{D}_2) \neq 0$ then $\operatorname{ord}_{S_j}(\alpha_{\mathcal{D}_1,\mathcal{D}_2}) = 0$ and hence

$$\alpha_{\mathcal{D}_1,\mathcal{D}_2}(S_j) = -2\log\left(\varphi_{\alpha_{\mathcal{D}_1,\mathcal{D}_2},j}(S_j)\right).$$

□

Corollary 3.22. *The generalized arithmetic intersection number defines a bilinear, symmetric pairing*

$$\begin{aligned}\widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty) \times \widehat{\mathrm{Z}}^1(\mathcal{X}, \mathcal{S}_\infty) &\longrightarrow \mathbb{R}\\ ((\mathcal{D}_1, g_{\mathcal{D}_1}), (\mathcal{D}_2, g_{\mathcal{D}_2})) &\longmapsto (\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}),\end{aligned}$$

which factors through $\widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{S}_\infty) \times \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{S}_\infty)$ *and then coincides with the generalized arithmetic intersection number due to Kühn.*

Proof. We use the properties of Lemma 3.18. Let $\alpha_{\mathcal{D}_1,\mathcal{D}_2}$ be a $\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1$ and let $\alpha_{\mathcal{D}_1',\mathcal{D}_2}$ be a $\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1'$. Because $\alpha_{\mathcal{D}_1+\mathcal{D}_1',\mathcal{D}_2} := \alpha_{\mathcal{D}_1,\mathcal{D}_2} + \alpha_{\mathcal{D}_1',\mathcal{D}_2}$ is a $\mathcal{D}_2$-adjusted Green's function for $\mathcal{D}_1 + \mathcal{D}_1'$, we easily see that

$$\left(\mathcal{D}_1 + \mathcal{D}_1', g_{\mathcal{D}_1+\mathcal{D}_1'}\right) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = (\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) + \left(\mathcal{D}_1', g_{\mathcal{D}_1'}\right) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}).$$

Thus the generalized arithmetic intersection number is linear in the first entry. Moreover, in the case that $(\mathcal{D}_1, g_{\mathcal{D}_1}) = \left(\operatorname{div}(f), -\log|f|^2\right)$ for a rational function $f \in k(\mathcal{X})^\times$, we use the $\mathcal{D}_2$-adjusted Green's function $\alpha_{\operatorname{div}(f),\mathcal{D}_2} := -\log|f|^2$ for $\operatorname{div}(f)$. Therewith we obtain the trivial generalized arithmetic intersection number

$$\left(\operatorname{div}(f), -\log|f|^2\right) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = 0.$$

Note that the right side of equation (3.6) is independent of adjusted Green's functions. Hence for all rational functions $f \in k(\mathcal{X})^\times$ the equation

$$\big(\mathcal{D}_1 + \operatorname{div}(f), g_{\mathcal{D}_1+\operatorname{div}(f)}\big) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2}) = (\mathcal{D}_1, g_{\mathcal{D}_1}) \cdot (\mathcal{D}_2, g_{\mathcal{D}_2})$$

holds. Thus we can assume that $\mathcal{D}_1$ and $\mathcal{D}_2$ intersect properly on the generic fibre X. Now the corollary follows from Proposition 3.21 and the fact that the generalized arithmetic intersection number due to Kühn defines a bilinear, symmetric pairing. □

3.4 Generalized arithmetic intersection numbers on arithmetic varieties using adjusted Green's forms

In this section we define a new version of the generalized arithmetic intersection number of two arithmetic cycles with log-log Green's forms on an arithmetic variety using adjusted Green's forms. Moreover, we prove that this generalized arithmetic intersection number coincides with the generalized arithmetic intersection number due to Burgos-Kramer-Kühn in [BGKK].

Definition 3.23. Let $\mathcal{S}_\infty$ be a normal crossing divisor on $\mathcal{X}_\infty$. A *log-log Green's form for* $Z \in \mathrm{Z}^p(\mathcal{X})$ is a differential form $g_Z \in A^{(p-1,p-1)}(\mathcal{X}_\infty \setminus (Z(\mathbb{C}) \cup \mathcal{S}_\infty))$ such that
i) g_Z is a Green's form for $Z(\mathbb{C}) \setminus (Z(\mathbb{C}) \cap \mathcal{S}_\infty)$ on $\mathcal{X}_\infty \setminus \mathcal{S}_\infty$;
ii) there exists a desingularization $\pi : \widetilde{\mathcal{X}_\infty} \longrightarrow \mathcal{X}_\infty$ of $\mathcal{X}_\infty$ along $Z(\mathbb{C})$ such that for all $x \in \widetilde{\mathcal{X}_\infty}$ with local equation $(z_1, \ldots, z_k, U)$ for $Z(\mathbb{C})$ in x there is a local expansion

$$\pi^* g_Z|_U = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \beta,$$

where α_i are smooth forms on U and β is the pullback of a *pre-log-log form along* $\mathcal{S}_\infty$ *on* $\mathcal{X}_\infty$. For a definition of a pre-log-log form see [BGKK, Definition 7.3.]. Let $\widehat{\mathrm{Z}}^p(\mathcal{X}, \mathcal{S}_\infty)$ be the group of pairs (Z, g_Z), where $Z \in \mathrm{Z}^p(\mathcal{X})$ and g_Z is a log-log Green's form for Z. Then

$$\widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty) := \widehat{\mathrm{Z}}^p(\mathcal{X}, \mathcal{S}_\infty) \big/ \widehat{\mathrm{R}}^p(\mathcal{X})$$

is called the *p-codimensional generalized arithmetic Chow group of* $\mathcal{X}$ *with respect to* $\mathcal{S}_\infty$.

Definition 3.24. Suppose $Y \in \mathrm{Z}^p(\mathcal{X})$ and $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$. Let g_Y be a log-log Green's form for Y and let g_Z be a log-log Green's form for Z.
i) Let $\alpha_{Z,Y}$ be a Y-adjusted Green's form for Z. Then we set

$$g_Y \bullet g_Z := \omega_{g_Y} \wedge g_Z + \omega_{\alpha_{Z,Y}} \wedge g_Y - \omega_{g_Y} \wedge \alpha_{Z,Y}$$

and call it the *normalized $*$-product of g_Y and g_Z.*
ii) The *generalized arithmetic intersection number of* $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty)$ *and* $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X}, \mathcal{S}_\infty)$ is defined by

$$\widehat{\deg}_{\mathcal{X}}([Y, g_Y] \cdot [Z, g_Z]) = \frac{1}{2} \int_{\mathcal{X}_\infty} g_Y \bullet g_Z.$$

Proposition 3.25. *i) The generalized arithmetic intersection number of* $[Y, g_Y]$ *and* $[Z, g_Z]$ *is well-defined.*
ii) Assume that the cycles $Y \in \mathrm{Z}^p(\mathcal{X})$ *and* $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ *intersect properly on the generic fibre* X*. Then the generalized arithmetic intersection number of* $[Y, g_Y]$ *and* $[Z, g_Z]$ *can be written as a limit by*

$$\widehat{\deg}_{\mathcal{X}}([Y, g_Y] \cdot [Z, g_Z]) = \frac{1}{2} \lim_{\varepsilon \to 0} \left(\int_{\mathcal{X}_\varepsilon} \omega_{g_Y} \wedge g_Z + \int_{\partial B_\varepsilon} (\alpha_{Z,Y} \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c \alpha_{Z,Y}) \right),$$

where $\mathcal{X}_\varepsilon := \mathcal{X}_\infty \backslash B_\varepsilon$ *with* $B_\varepsilon := B_\varepsilon(\mathcal{S}_\infty) \cup B_\varepsilon(Y(\mathbb{C})) \cup B_\varepsilon(Z(\mathbb{C}))$.

Proof. *i)* By Remark 2.50 we can assume $Y(\mathbb{C}) \cap Z(\mathbb{C}) = \emptyset$. Moreover, by definition of a pre-log-log form β on $\mathcal{X}_\infty$ along $\mathcal{S}_\infty$ both forms β and $\mathrm{dd}^c\beta$ have *log-log growth along* $\mathcal{S}_\infty$, see [BGKK, Definition 7.1.]. It follows that each term $\omega_{g_Y} \wedge g_Z$, $\omega_{\alpha_{Z,Y}} \wedge g_Y$ and $\omega_{g_Y} \wedge \alpha_{Z,Y}$ has log-log growth along $\mathcal{S}_\infty$. Because $g_Y \bullet g_Z = \omega_{g_Y} \wedge g_Z + \omega_{\alpha_{Z,Y}} \wedge g_Y - \omega_{g_Y} \wedge \alpha_{Z,Y}$ is smooth on $\mathcal{X}_\infty \backslash \mathcal{S}_\infty$, it follows that $g_Y \bullet g_Z$ is the pullback of a pre-log-log form along $\mathcal{S}_\infty$ on $\mathcal{X}_\infty$. By [BGKK, Proposition 7.6.] this form is integrable. Hence the definition of the generalized arithmetic intersection number is well-defined.
ii) On $\mathcal{X}_\varepsilon$ the following equation holds:

$$\begin{aligned} \omega_{\alpha_{Z,Y}} \wedge g_Y - \omega_{g_Y} \wedge \alpha_{Z,Y} &= \mathrm{dd}^c \alpha_{Z,Y} \wedge g_Y - \mathrm{dd}^c g_Y \wedge \alpha_{Z,Y} \\ &= \mathrm{d}\left(\mathrm{d}^c \alpha_{Y,Z} \wedge g_Z - \mathrm{d}^c g_Z \wedge \alpha_{Y,Z}\right). \end{aligned}$$

Hence the second assertion follows from Stokes Theorem. □

The next definition is due to [BGKK].

Definition 3.26. Let $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty)$ and $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X}, \mathcal{S}_\infty)$ such that Y and Z intersect properly on the generic fibre X.
i) Let $\{\sigma_{Y,Z}, \sigma_{Z,Y}\}$ be two smooth functions on $\mathcal{X}_\infty$ with the following properties:
1) $\sigma_{Y,Z} = 1$ on an open neighbourhood of $Y(\mathbb{C})$;

2) $\sigma_{Y,Z} = 0$ on an open neighbourhood of $Z(\mathbb{C})$;
3) $\sigma_{Y,Z} + \sigma_{Z,Y} = 1$ on $\mathcal{X}_\infty$.
The $*$-*product between* g_Y *and* g_Z is defined by

$$g_Y * g_Z = \mathrm{dd}^c(\sigma_{ZY} g_Z) \wedge g_Y + \omega_{g_Y} \wedge \sigma_{YZ} g_Z.$$

ii) The *generalized arithmetic intersection number due to Burgos-Kramer-Kühn of* $[Y, g_Y]$ *and* $[Z, g_Z]$ is defined by

$$\widehat{\deg}_{\mathcal{X}}([Y, g_Y] \cdot [Z, g_Z]) = (Y, Z)_{\mathrm{fin}} + \frac{1}{2} \int_{\mathcal{X}_\infty} g_Y * g_Z. \tag{3.7}$$

Proposition 3.27. [BGKK, Theorem 7.48]
The equation (3.7) defines a symmetric bilinear pairing

$$\widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty) \times \widehat{\mathrm{CH}}^q(\mathcal{X}, \mathcal{S}_\infty) \longrightarrow \mathbb{R}.$$

In particular, there exists an arithmetic degree map

$$\widehat{\deg}_{\mathcal{X}} : \widehat{\mathrm{CH}}^{d+1}(\mathcal{X}, \mathcal{S}_\infty) \longrightarrow \mathbb{R}.$$

The next proposition follows from the properties of σ_{ZY} and the use of Stokes Theorem.

Proposition 3.28. [BGKK, equation (7.34)]
The generalized arithmetic intersection number due to Burgos-Kramer-Kühn can be written as a limit by

$$\begin{aligned}&\widehat{\deg}_{\mathcal{X}}([Y, g_Y] \cdot [Z, g_Z]) \\ &\quad = (Y, Z)_{\mathrm{fin}} + \frac{1}{2} \lim_{\varepsilon \to 0} \left(\int_{\mathcal{X}_\varepsilon} \omega_{g_Y} \wedge g_Z + \int_{\partial B_\varepsilon} (\sigma_{ZY} g_Z \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c(\sigma_{ZY} g_Z)) \right),\end{aligned}$$

where $\mathcal{X}_\varepsilon := \mathcal{X}_\infty \backslash B_\varepsilon$ *with* $B_\varepsilon := B_\varepsilon(\mathcal{S}_\infty) \cup B_\varepsilon(Y(\mathbb{C})) \cup B_\varepsilon(Z(\mathbb{C}))$.

Proposition 3.29. [BGKK, Proposition 7.21. ii)]
Let g_Y *be a log-log Green's form for* $Y \in \mathrm{Z}^p(\mathcal{X})$. *Then for any form* $\alpha \in A^{(p^\vee - 1, p^\vee - 1)}(\mathcal{X})$ *the equality*

$$-\lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(Y(\mathbb{C}))} \alpha \wedge \mathrm{d}^c g_Y = \int_{\mathcal{X}_\infty} \alpha \wedge \delta_{Y(\mathbb{C})}$$

holds.

With the help of Proposition 3.29 we deduce from Proposition 3.28 the following special cases.

Corollary 3.30. [BGKK, Theorem 7.33.] and [BBGK, Theorem 1.14.]
The following hold:
i) If $Z(\mathbb{C}) \cap \mathcal{S}_\infty = \emptyset$, *then*

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) = (Y, Z)_{\mathrm{fin}} + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \left(g_Y \wedge \delta_{Z(\mathbb{C})} + \omega_{g_Y} \wedge g_Z\right).$$

ii) In the case that $Y(\mathbb{C}) \cap \mathcal{S}_\infty = \emptyset$, *we have*

$$\widehat{\deg}_{\mathcal{X}}\left([Y, g_Y] \cdot [Z, g_Z]\right) = (Y, Z)_{\mathrm{fin}} + \frac{1}{2} \int\limits_{Z(\mathbb{C})\backslash\widetilde{(Z(\mathbb{C})\cap\mathcal{S}_\infty)}} g_Y + \lim_{\varepsilon \to 0} \left(\frac{1}{2} \int\limits_{\mathcal{X}_\varepsilon} \omega_{g_Y} \wedge g_Z + \int\limits_{\partial B_\varepsilon(\mathcal{S}_\infty)} \left(g_Z \wedge \mathrm{d}^c g_Y - g_Y \wedge \mathrm{d}^c g_Z\right) \right),$$

where $\mathcal{X}_\varepsilon := \mathcal{X}_\infty \backslash B_\varepsilon(\mathcal{S}_\infty)$ *and* $Z(\mathbb{C})\backslash\widetilde{(Z(\mathbb{C})\cap\mathcal{S}_\infty)}$ *denotes a desingularization of the closure of* $Z(\mathbb{C})\backslash(Z(\mathbb{C}) \cap \mathcal{S}_\infty)$.

In the next theorem we compare the generalized arithmetic intersection number in Definition 3.24 with the generalized arithmetic intersection number due to Burgos-Kramer-Kühn in Definition 3.26.

Theorem 3.31. *Assume that the cycles* $Y \in \mathrm{Z}^p(\mathcal{X})$ *and* $Z \in \mathrm{Z}^{p^\vee}(\mathcal{X})$ *intersect properly on the generic fibre* X. *Then the generalized arithmetic intersection number of* $[Y, g_Y] \in \widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty)$ *and* $[Z, g_Z] \in \widehat{\mathrm{CH}}^{p^\vee}(\mathcal{X}, \mathcal{S}_\infty)$ *coincides with the generalized arithmetic intersection number due to Burgos-Kramer-Kühn, i.e.*

$$\frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_Y \bullet g_Z = (Y, Z)_{\mathrm{fin}} + \frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_Y * g_Z.$$

Proof. First we consider the integral

$$\frac{1}{2} \int\limits_{\mathcal{X}_\infty} g_Y \bullet g_Z = \frac{1}{2} \int\limits_{\mathcal{X}_\infty} \left(\omega_{g_Y} \wedge g_Z + \omega_{\alpha_{Z,Y}} \wedge g_Y - \omega_{g_Y} \wedge \alpha_{Z,Y}\right),$$

where $\alpha_{Z,Y}$ is a Y-adjusted Green's form for Z. Using Proposition 3.25 it follows that the latter equals

$$\frac{1}{2}\lim_{\varepsilon\to 0}\left(\int\limits_{\mathcal{X}_\varepsilon}\omega_{g_Y}\wedge g_Z+\int\limits_{\partial B_\varepsilon}\left(\alpha_{Z,Y}\wedge \mathrm{d}^c g_Y-g_Y\wedge \mathrm{d}^c\alpha_{Z,Y}\right)\right),$$

where $\mathcal{X}_\varepsilon := \mathcal{X}_\infty\backslash B_\varepsilon$ with $B_\varepsilon := B_\varepsilon(\mathcal{S}_\infty)\cup B_\varepsilon(Y(\mathbb{C}))\cup B_\varepsilon(Z(\mathbb{C}))$. Because of $Y(\mathbb{C})\cap Z(\mathbb{C}) = \emptyset$, Proposition 3.29 and equation (2.24) show

$$\frac{1}{2}\lim_{\varepsilon\to 0}\int\limits_{\partial B_\varepsilon(Y(\mathbb{C}))}\alpha_{Z,Y}\wedge \mathrm{d}^c g_Y=-\frac{1}{2}\int\limits_{\mathcal{X}_\infty}\alpha_{Z,Y}\wedge\delta_{Y(\mathbb{C})}=(Y,Z)_{\mathrm{fin}}\,.$$

Because B_ε is the disjoint union of $B_\varepsilon(Y(\mathbb{C}))$ and

$$B_\varepsilon\backslash Y(\mathbb{C}) := \left(B_\varepsilon(\mathcal{S}_\infty)\cup B_\varepsilon(Z(\mathbb{C}))\right)\setminus\left(B_\varepsilon(\mathcal{S}_\infty)\cap B_\varepsilon(Y(\mathbb{C}))\right),$$

it follows that the generalized arithmetic intersection number equals

$$(Y,Z)_{\mathrm{fin}}+\frac{1}{2}\lim_{\varepsilon\to 0}\left(\int\limits_{\mathcal{X}_\varepsilon}\omega_{g_Y}\wedge g_Z+\int\limits_{\partial(B_\varepsilon\backslash Y(\mathbb{C}))}\left(\alpha_{Z,Y}\wedge \mathrm{d}^c g_Y-g_Y\wedge \mathrm{d}^c\alpha_{Z,Y}\right)\right).$$

Now let us consider the generalized arithmetic intersection number due to Burgos-Kramer-Kühn, i.e.

$$(Y,Z)_{\mathrm{fin}}+\frac{1}{2}\int\limits_{\mathcal{X}_\infty}g_Y * g_Z=(Y,Z)_{\mathrm{fin}}+\frac{1}{2}\int\limits_{\mathcal{X}_\infty}\left(\mathrm{dd}^c\left(\sigma_{ZY}g_Z\right)\wedge g_Y+\omega_{g_Y}\wedge\sigma_{YZ}g_Z\right).$$

Because of Proposition 3.28 the latter equals

$$(Y,Z)_{\mathrm{fin}}+\frac{1}{2}\lim_{\varepsilon\to 0}\left(\int\limits_{\mathcal{X}_\varepsilon}\omega_{g_Y}\wedge g_Z+\int\limits_{\partial B_\varepsilon}\left(\sigma_{ZY}g_Z\wedge \mathrm{d}^c g_Y-g_Y\wedge \mathrm{d}^c\left(\sigma_{ZY}g_Z\right)\right)\right).$$

Using the basic properties of σ_{ZY}, the latter equals

$$(Y,Z)_{\mathrm{fin}}+\frac{1}{2}\lim_{\varepsilon\to 0}\left(\int\limits_{\mathcal{X}_\varepsilon}\omega_{g_Y}\wedge g_Z+\int\limits_{\partial(B_\varepsilon\backslash Y(\mathbb{C}))}\left(\sigma_{ZY}g_Z\wedge \mathrm{d}^c g_Y-g_Y\wedge \mathrm{d}^c\left(\sigma_{ZY}g_Z\right)\right)\right).$$

Thus to prove Theorem 3.31 it remains to show that the integral

$$\int\limits_{\partial(B_\varepsilon\backslash Y(\mathbb{C}))}\left(\alpha_{Z,Y}\wedge \mathrm{d}^c g_Y-g_Y\wedge \mathrm{d}^c\alpha_{Z,Y}-\sigma_{ZY}g_Z\wedge \mathrm{d}^c g_Y+g_Y\wedge \mathrm{d}^c\left(\sigma_{ZY}g_Z\right)\right)$$

vanishes in the limit $\varepsilon \longrightarrow 0$. If we set $\omega_{Z,Y} := \alpha_{Z,Y} - \sigma_{ZY} g_Z$, we have to show the equation

$$\lim_{\varepsilon \to 0} \int\limits_{\partial(B_\varepsilon \backslash Y(\mathbb{C}))} \omega_{Z,Y} \wedge \mathrm{d}^c g_Y = \lim_{\varepsilon \to 0} \int\limits_{\partial(B_\varepsilon \backslash Y(\mathbb{C}))} g_Y \wedge \mathrm{d}^c \omega_{Z,Y}. \tag{3.8}$$

Since both forms $\omega_{Z,Y}$ and g_Y are pre-log-log forms along $\mathcal{S}_\infty$ on $B_\varepsilon \backslash Y(\mathbb{C})$, it follows from Proposition 3.29 that both sides of (3.8) vanish. □

Remark 3.32. Let us note a crucial property about the normalized $*$-product $g_Y \bullet g_Z$ of two log-log Green's forms g_Y and g_Z for Y and Z. Assume that Y and Z intersect properly on the generic fibre X. It follows from Proposition 3.25 that

$$\int\limits_{\mathcal{X}_\infty} g_Y \bullet g_Z = \int\limits_{\mathcal{X}_\infty} g_Y * g_Z \tag{3.9}$$

holds, if the geometric intersection number $(Y, Z)_{\mathrm{fin}}$ vanishes. In the case $Y = Z$ we can formulate a similar result. For this assume that $Y \in \mathrm{Z}^p(\mathcal{X})$ with $p = p^\vee$. First note that the $*$-product $g_Y * g_Z$ in [BGKK] can also be defined when the cycles Y and Z intersect non-properly on the generic fibre. For instance, the equation

$$g_Y * g_Y = \omega_{g_Y} \wedge g_Y$$

holds, see [BGKK, Proposition 6.10]. Because the normalized $*$-product between g_Y and g_Y is also well-defined, we have

$$g_Y \bullet g_Y = \omega_{g_Y} \wedge g_Y + \omega_{\alpha_{Y,Y}} \wedge g_Y - \omega_{g_Y} \wedge \alpha_{Y,Y},$$

where $\alpha_{Y,Y}$ is a Y-adjusted Green's form for Y. Then we obtain

$$g_Y \bullet g_Y - g_Y * g_Y = \omega_{\alpha_{Y,Y}} \wedge g_Y - \omega_{g_Y} \wedge \alpha_{Y,Y}.$$

It follows from equation (2.14) that

$$\int\limits_{\mathcal{X}_\infty} g_Y \bullet g_Y = \int\limits_{\mathcal{X}_\infty} g_Y * g_Y$$

holds, if g_Y is a Y-adjusted Green's form for Y, similar to equation (3.9), where Y and Z do not intersect on the whole $\mathcal{X}$.

In the last part of this section we consider multiple generalized arithmetic intersections.

Definition 3.33. For $i = 1, \dots, n+1$ let $[Y_i, g_{Y_i}] \in \widehat{\mathrm{CH}}^{p_i}(\mathcal{X}, \mathcal{S}_\infty)$ such that $\sum p_i = \dim(\mathcal{X})$. Then we define the *generalized arithmetic intersection number of* $[Y_1, g_{Y_1}], \dots, [Y_n, g_{Y_n}]$ *and* $[Y_{n+1}, g_{Y_{n+1}}]$ by

$$\widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}[Y_i, g_{Y_i}]\right) := \frac{1}{2}\int\limits_{\mathcal{X}_\infty} \omega_{g_{Y_{n+1}}} \wedge \cdots \wedge \omega_{g_{Y_2}} \wedge g_{Y_1} + \\ \frac{1}{2}\sum_{i=1}^{n}\int\limits_{\mathcal{X}_\infty} \omega_{\alpha_{Y_1}} \wedge \cdots \wedge \omega_{\alpha_{Y_{i-1}}} \wedge \omega_{g_{Y_{i+2}}} \wedge \cdots \wedge \omega_{g_{Y_{n+1}}} \wedge \left(\omega_{\alpha_{Y_i}} \wedge g_{Y_{i+1}} - \omega_{g_{Y_{i+1}}} \wedge \alpha_{Y_i}\right),$$

where $\{\alpha_{Y_i}\}_{i=1,\dots,n}$ is a family of Y_{n+1}-adjusted Green's forms for $\{Y_i\}_{i=1,\dots,n}$.

Proposition 3.34. *The following assertions hold:*
i) The generalized arithmetic intersection number $\widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}[Y_i, g_{Y_i}]\right)$ *is well-defined and coincides with the generalized arithmetic intersection number due to Burgos-Kramer-Kühn.*
ii) The generalized arithmetic self-intersection number of $[D, g_D] \in \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{S}_\infty)$ *is given by*

$$\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^{d+1}\right) = \frac{1}{2}\int\limits_{\mathcal{X}_\infty}\left(\omega_{g_D}^d \wedge g_D + (\omega_{\alpha_D} \wedge g_D - \omega_{g_D} \wedge \alpha_D) \wedge \left(\sum_{i+j=d-1} \omega_{\alpha_D}^i \wedge \omega_{g_D}^j\right)\right),$$

where α_D *is an adjusted Green's form for* D.

Proof. *i)* Using Remark 2.50 it is easy to see that

$$\widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}[Y_i, g_{Y_i}]\right) = \widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}\left[Y_i + \mathrm{div}(f_i), g_{Y_i} - \log|f_i|^2\right]\right)$$

for K_1-chains f_i of the right codimension. Hence we can assume $\bigcap_{i=1}^{n+1} Y_i(\mathbb{C}) = \emptyset$. Because of Proposition 2.54 there exists $[Z, g_Z] \in \bigoplus_{p\geq 0} \widehat{\mathrm{CH}}^p(\mathcal{X}, \mathcal{S}_\infty) \otimes \mathbb{Q}$ such that

$$\widehat{\deg}_{\mathcal{X}}\left(\prod_{i=1}^{n+1}[Y_i, g_{Y_i}]\right) = \widehat{\deg}_{\mathcal{X}}\left([Y_1, g_{Y_1}] \cdot [Z, g_Z]\right).$$

Now the result follows from Theorem 3.31.
ii) This is the special case $[Y_1, g_{Y_1}] = \cdots = [Y_{n+1}, g_{Y_{n+1}}] = [D, g_D] \in \widehat{\mathrm{CH}}^1(\mathcal{X}, \mathcal{S}_\infty)$. □

3.5 Examples on $\mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{X}(1)$

Let $\mathcal{X}$ be the arithmetic 3-fold $\mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{X}(1)$. The divisor

$$D := \mathcal{S}_\infty \times_{\mathbb{Z}} \mathcal{X}(1) + \mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{S}_\infty,$$

where $\mathcal{S}_\infty$ denotes the cusp on $\mathcal{X}(1)$, is a normal crossing divisor, see [BGKK, Chapter 7.8]. For $i = 1, 2$ let $p_i : \mathcal{X} \longrightarrow \mathcal{X}(1)$ denote the projection onto the i-th factor. Then D can be written as $\operatorname{div}(\Delta(q_1) \cdot \Delta(q_2))$, where for $j = 1, 2$ we set $\Delta(q_j) := p_j^*(\Delta(q))$ and $\Delta(q)$ denotes the modular discriminant. A canonical Green's form for D is given by

$$g_{D,\mathrm{Pet}} = -\log \|\Delta(q_1) \cdot \Delta(q_2)\|^2_{\mathrm{Pet}} := -\log \|\Delta(q_1)\|^2_{\mathrm{Pet}} - \log \|\Delta(q_2)\|^2_{\mathrm{Pet}},$$

where $\|\cdot\|_{\mathrm{Pet}}$ is the Petersson metric. Moreover, if we write $\mathcal{X} = \mathbb{P}^2_{\mathbb{Z}}$ we see that

$$g_{D,\mathrm{FS}} = -\log \|\Delta(q_1) \cdot \Delta(q_2)\|^2_{\mathrm{FS}} := -\log \|\Delta(q_1)\|^2_{\mathrm{FS}} - \log \|\Delta(q_2)\|^2_{\mathrm{FS}}.$$

is another Green's form for D, where $\|\cdot\|_{\mathrm{FS}}$ is the Fubini-Study metric. It is easy to see that $g_{D,\mathrm{Pet}} = p_1^*(g_{\mathcal{S}_\infty,\mathrm{Pet}}) + p_2^*(g_{\mathcal{S}_\infty,\mathrm{Pet}})$ and $g_{D,\mathrm{FS}} = p_1^*(g_{\mathcal{S}_\infty,\mathrm{FS}}) + p_2^*(g_{\mathcal{S}_\infty,\mathrm{FS}})$ for $g_{\mathcal{S}_\infty,\mathrm{Pet}} := -\log \|\Delta(q)\|^2_{\mathrm{Pet}}$ and $g_{\mathcal{S}_\infty,\mathrm{FS}} := -\log \|\Delta(q)\|^2_{\mathrm{FS}}$ on $\mathcal{X}(1)$.
In this section we calculate an example of a generalized arithmetic intersection number, namely the generalized arithmetic self-intersection number of the line bundle with a logarithmically singular hermitian metric, which is associated to the class $[D, g_{D,\mathrm{Pet}}]$.

Lemma 3.35. *For $\mathcal{X}$ and D as above,*

$$-\log \|\Delta(q_1) \cdot \Delta(q_2)\|^2_{\mathrm{FS}} - \frac{1}{2}$$

is an adjusted Green's form for D.

Proof. Set $g_D := -\log \|\Delta(q_1) \cdot \Delta(q_2)\|^2_{\mathrm{FS}}$. By Corollary 2.30 an adjusted Green's form for D is given by

$$\alpha_D := g_D - \frac{2 \cdot \mathrm{ht}_{[D,g_D]^2}(D)}{d \cdot \deg(D)},$$

where d is the relative dimension of $\mathcal{X}$. With $d = 2 = \deg(D)$ it remains to show $\mathrm{ht}_{[D,g_D]^2}(D) = 1$. By the definition of the height function we have

$$\mathrm{ht}_{[D,g_D]^2}(D) = \widehat{\deg}_{\mathcal{X}}\left([D, g_D]^3\right) - \frac{1}{2} \int_{\mathcal{X}_\infty} \omega^2_{g_D} \wedge g_D.$$

For $i = 1, 2$ consider the arithmetic divisors $(D_i, g_{D_i}) := \left(\operatorname{div}(\Delta(q_i)), -\log \|\Delta(q_i)\|^2_{\mathrm{FS}}\right)$, so that $D = D_1 + D_2$ and $g_D = g_{D_1} + g_{D_2}$. It follows $\omega^2_{g_D} = 2 \cdot \omega_{g_{D_1}} \wedge \omega_{g_{D_2}}$. This yields

$$\frac{1}{2} \int\limits_{\mathcal{X}_\infty} \omega^2_{g_D} \wedge g_D = \int\limits_{\mathcal{X}_\infty} \omega_{g_{D_1}} \wedge \omega_{g_{D_2}} \wedge (g_{D_1} + g_{D_2}),$$

which is given by

$$\int\limits_{\mathcal{X}_\infty} p_1^*\left(\omega_{g_{\mathcal{S}_\infty}}\right) \wedge p_2^*\left(\omega_{g_{\mathcal{S}_\infty}}\right) \wedge \left(p_1^*\left(g_{\mathcal{S}_\infty}\right) + p_2^*\left(g_{\mathcal{S}_\infty}\right)\right),$$

where $g_{\mathcal{S}_\infty} = -\log \|\Delta(q)\|^2_{\mathrm{FS}}$. Because of $\mathcal{X} = \mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{X}(1)$ the latter equals

$$2 \int\limits_{\mathcal{X}(1)_\infty} \omega_{g_{\mathcal{S}_\infty}} \cdot \int\limits_{\mathcal{X}(1)_\infty} g_{\mathcal{S}_\infty} \wedge \omega_{g_{\mathcal{S}_\infty}}.$$

It is well-known that $\int_{\mathcal{X}(1)_\infty} \omega_{g_{\mathcal{S}_\infty}} = \int_{\mathbb{P}^1_{\mathbb{C}}} \omega_{\mathrm{FS}} = 1$. In Proposition 1.26 we calculated $\int_{\mathcal{X}(1)_\infty} g_{\mathcal{S}_\infty} \wedge \omega_{g_{\mathcal{S}_\infty}} = 2 \cdot \overline{\mathcal{L}}^2$, where $\overline{\mathcal{L}} := (\mathcal{O}_{\mathcal{X}}(1), \|\cdot\|_{\mathrm{FS}})$ with the Fubini-Study metric $\|\cdot\|_{\mathrm{FS}}$. Hence we have

$$\frac{1}{2} \int\limits_{\mathcal{X}_\infty} \omega^2_{g_D} \wedge g_D = 4 \cdot \overline{\mathcal{L}}^2.$$

Now let us compute $\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^3\right)$. Because of $[D, g_D] = p_1^*[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}] + p_2^*[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}]$ we have

$$\begin{aligned}[D, g_D]^3 =& p_1^*\left[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}\right]^3 + p_2^*\left[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}\right]^3 + \\ & 3 \cdot \left(p_1^*\left[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}\right]^2 \cdot p_2^*\left[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}\right] + p_1^*\left[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}\right] \cdot p_2^*\left[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}\right]^2\right)\end{aligned} \tag{3.10}$$

in $\widehat{\mathrm{CH}}^3(\mathcal{X}, D(\mathbb{C}))$. The first two terms of (3.10) vanish because of $\left[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}\right]^3 = 0$. By Proposition 1.26 we have $\left[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}\right]^2 = \left[0, 2 \cdot \overline{\mathcal{L}}^2 \cdot \omega_{g_{\mathcal{S}_\infty}}\right]$. Hence by equation (2.3) the last term of (3.10) equals

$$3 \cdot \left[0, 2 \cdot \overline{\mathcal{L}}^2 \cdot p_1^* \omega_{g_{\mathcal{S}_\infty}} \wedge p_2^* \omega_{g_{\mathcal{S}_\infty}}\right] + 3 \cdot \left[0, p_1^* \omega_{g_{\mathcal{S}_\infty}} \wedge 2 \cdot \overline{\mathcal{L}}^2 \cdot p_2^* \omega_{g_{\mathcal{S}_\infty}}\right]. \tag{3.11}$$

Taking the arithmetic degree map yields

$$\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^3\right) = 6 \cdot \overline{\mathcal{L}}^2 \int\limits_{\mathcal{X}(1)_\infty} \omega_{g_{\mathcal{S}_\infty}} \cdot \int\limits_{\mathcal{X}(1)_\infty} \omega_{g_{\mathcal{S}_\infty}} = 6 \cdot \overline{\mathcal{L}}^2.$$

Summarize, we obtain $\mathrm{ht}_{[D, g_D]^2}(D) = 6 \cdot \overline{\mathcal{L}}^2 - 4 \cdot \overline{\mathcal{L}}^2 = 2 \cdot \overline{\mathcal{L}}^2 = 1$. □

Proposition 3.36. *For a natural number k such that $12|k$ we consider the hermitian line bundle $\overline{\mathcal{L}}(k) = p_1^*\overline{\mathcal{M}}_k \otimes p_2^*\overline{\mathcal{M}}_k$ on the arithmetic 3-fold $\mathcal{X} = \mathcal{X}(1) \times_{\mathbb{Z}} \mathcal{X}(1)$, where $\overline{\mathcal{M}}_k$ is the line bundle of modular forms for $\Gamma(1)$ of weight k equipped with the Petersson metric. Then its generalized arithmetic self-intersection number is given by*

$$\overline{\mathcal{L}}(k)^3 = \frac{k^3}{2}\zeta_{\mathbb{Q}}(-1)\left(\frac{\zeta'_{\mathbb{Q}}(-1)}{\zeta_{\mathbb{Q}}(-1)} + \frac{1}{2}\right).$$

Proof. First note that a different proof of this proposition can be found in [BGKK, Theorem 7.61]. Because of $\overline{\mathcal{L}}(k)^3 = \frac{k^3}{12^3}\overline{\mathcal{L}}(12)^3$ it is enough to prove the proposition only in the case $k = 12$. For the arithmetic divisor $[D, g_D] := [p_1^*(\mathcal{S}_\infty) + p_2^*(\mathcal{S}_\infty), p_1^* g_{\mathcal{S}_\infty} + p_2^* g_{\mathcal{S}_\infty}]$, where $g_{\mathcal{S}_\infty} := -\log\|\Delta(q)\|^2_{\mathrm{Pet}}$, we have

$$\overline{\mathcal{L}}(12)^3 = \widehat{\deg}_{\mathcal{X}}\left([D, g_D]^3\right).$$

By Proposition 3.34

$$\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^3\right) = \frac{1}{2}\lim_{\varepsilon\to 0}\int\limits_{\mathcal{X}_\infty\setminus B_\varepsilon(D(\mathbb{C}))}\left(\omega^2_{\alpha_D}\wedge g_D + \omega_{\alpha_D}\wedge\omega_{g_D}\wedge(g_D-\alpha_D) + \omega^2_{g_D}\wedge(g_D-\alpha_D)\right),$$

where α_D is an adjusted Green's form for D. If we set

$$\alpha_{\mathcal{S}_\infty} := -\log\|\Delta(q)\|^2_{\mathrm{FS}} - \frac{1}{4},$$

then by Lemma 3.35 we see that $\alpha_D := p_1^*\alpha_{\mathcal{S}_\infty} + p_2^*\alpha_{\mathcal{S}_\infty}$ is an adjusted Green's form for D. Using the fact that $\omega^2_{g_{\mathcal{S}_\infty}}$ and $\omega^2_{\alpha_{\mathcal{S}_\infty}}$ vanish on $\mathcal{X}(1)$, it follows

$$\begin{aligned}\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^3\right) = \frac{1}{2}\lim_{\varepsilon\to 0}\int\limits_{\mathcal{X}_\infty\setminus B_\varepsilon(D(\mathbb{C}))}&\left(2p_1^*\omega_{\alpha_{\mathcal{S}_\infty}}\wedge p_2^*\omega_{\alpha_{\mathcal{S}_\infty}}\wedge(p_1^*g_{\mathcal{S}_\infty}+p_2^*g_{\mathcal{S}_\infty}) + \right.\\ &\left(p_1^*\omega_{\alpha_{\mathcal{S}_\infty}}\wedge p_2^*\omega_{g_{\mathcal{S}_\infty}} + p_2^*\omega_{\alpha_{\mathcal{S}_\infty}}\wedge p_1^*\omega_{g_{\mathcal{S}_\infty}} + 2p_1^*\omega_{g_{\mathcal{S}_\infty}}\wedge p_2^*\omega_{g_{\mathcal{S}_\infty}}\right)\wedge\\ &\left.(p_1^*g_{\mathcal{S}_\infty} + p_2^*g_{\mathcal{S}_\infty} - p_1^*\alpha_{\mathcal{S}_\infty} - p_2^*\alpha_{\mathcal{S}_\infty})\right).\end{aligned} \tag{3.12}$$

For $S_\infty := \mathcal{S}_\infty(\mathbb{C})$ we consider an ε-neighborhood $B_\varepsilon(D(\mathbb{C})) = B_\varepsilon(S_\infty)\times\mathcal{X}(1)_\infty + \mathcal{X}(1)_\infty\times B_\varepsilon(S_\infty)$ of $D(\mathbb{C})$. Then $\mathcal{X}_\infty\setminus B_\varepsilon(D(\mathbb{C})) = \mathcal{X}(1)_\infty\setminus B_\varepsilon(S_\infty)\times\mathcal{X}(1)_\infty\setminus B_\varepsilon(S_\infty)$. Using the fact that $\int_{\mathcal{X}(1)_\infty}\omega_{\alpha_{\mathcal{S}_\infty}} = 1 = \int_{\mathcal{X}(1)_\infty}\omega_{g_{\mathcal{S}_\infty}}$, the equation (3.12) is equivalent to

$$\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^3\right) = \frac{1}{2}\lim_{\varepsilon\to 0}\int\limits_{\mathcal{X}(1)_\infty\setminus B_\varepsilon(S_\infty)}\left(6\cdot\omega_{g_{\mathcal{S}_\infty}}\wedge(g_{\mathcal{S}_\infty}-\alpha_{\mathcal{S}_\infty}) + 6\cdot\omega_{\alpha_{\mathcal{S}_\infty}}\wedge g_{\mathcal{S}_\infty} - 2\cdot\omega_{\alpha_{\mathcal{S}_\infty}}\wedge\alpha_{\mathcal{S}_\infty}\right).$$

Recall that we set $g_{\mathcal{S}_\infty} = -\log \|\Delta(q)\|^2_{\mathrm{Pet}}$ and $\alpha_{\mathcal{S}_\infty} = -\log \|\Delta(q)\|^2_{\mathrm{FS}} - \frac{1}{4}$. Although it is hard to calculate the integrals

$$\int\limits_{\mathcal{X}(1)_\infty \setminus B_\varepsilon(S_\infty)} \omega_{\alpha_{\mathcal{S}_\infty}} \wedge g_{\mathcal{S}_\infty} \text{ and } \int\limits_{\mathcal{X}(1)_\infty \setminus B_\varepsilon(S_\infty)} \omega_{g_{\mathcal{S}_\infty}} \wedge \alpha_{\mathcal{S}_\infty},$$

we know from Corollary 3.20

$$\frac{1}{2} \int\limits_{\mathcal{X}(1)_\infty \setminus B_\varepsilon(S_\infty)} \left(\omega_{\alpha_{\mathcal{S}_\infty}} \wedge g_{\mathcal{S}_\infty} - \omega_{g_{\mathcal{S}_\infty}} \wedge \alpha_{\mathcal{S}_\infty}\right) = \eta_{g_{\mathcal{S}_\infty}} - \log\left(\varphi_{g_{\mathcal{S}_\infty}}(S_\infty)\right) + \log\left(\varphi_{\alpha_{\mathcal{S}_\infty}}(S_\infty)\right) - \eta_{g_{\mathcal{S}_\infty}} \log\left(-\log \varepsilon^2\right) + f(\varepsilon),$$

where $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$. With the constant $\log\left(\varphi_{g_{\mathcal{S}_\infty}}(S_\infty)\right) = 0$, the self-intersection number $\widehat{\deg}_{\mathcal{X}}\left([D, g_D]^3\right)$ equals

$$6\left(\eta_{g_{\mathcal{S}_\infty}} - \lim_{\varepsilon \to 0}\left(\eta_{g_{\mathcal{S}_\infty}} \log\left(-\log \varepsilon^2\right) - \frac{1}{2} \int\limits_{\mathcal{X}(1)_\infty \setminus B_\varepsilon(S_\infty)} \omega_{g_{\mathcal{S}_\infty}} \wedge g_{\mathcal{S}_\infty}\right)\right) + 6 \log\left(\varphi_{\alpha_{\mathcal{S}_\infty}}(S_\infty)\right) - \int\limits_{\mathcal{X}(1)_\infty} \omega_{\alpha_{\mathcal{S}_\infty}} \wedge \alpha_{\mathcal{S}_\infty}.$$

Because of Corollary 3.5 the latter equals

$$6 \cdot \overline{\mathcal{M}}_{12}\left(\Gamma(1)\right)^2 + 6 \log\left(\varphi_{\alpha_{\mathcal{S}_\infty}}(S_\infty)\right) - \int\limits_{\mathcal{X}(1)_\infty} \omega_{\alpha_{\mathcal{S}_\infty}} \wedge \alpha_{\mathcal{S}_\infty}.$$

The proof of Proposition 3.36 follows now from $\log\left(\varphi_{\alpha_{\mathcal{S}_\infty}}(S_\infty)\right) = \frac{1}{8}$ because then

$$6 \log\left(\varphi_{\alpha_{\mathcal{S}_\infty}}(S_\infty)\right) - \int\limits_{\mathcal{X}(1)_\infty} \omega_{\alpha_{\mathcal{S}_\infty}} \wedge \alpha_{\mathcal{S}_\infty} = 0.$$

□

In the last part of this section we give a different proof of Proposition 3.36 using Proposition 3.34.

Lemma 3.37. *For $\mathcal{X}$ and D as in Lemma 3.35, the family $\{p_1^*(g_{\mathrm{FS}}), p_2^*(g_{\mathrm{FS}})\}$ is a family of $p_1^*(\mathcal{S}_\infty)$-adjusted Green's forms for $\{p_1^*(\mathcal{S}_\infty), p_2^*(\mathcal{S}_\infty)\}$, where we set*

$$g_{\mathrm{FS}} := -\log \|\Delta(q)\|^2_{\mathrm{FS}}.$$

Proof. We have to show

$$\mathrm{ht}_{p_1^*[\mathcal{S}_\infty,g_{\mathrm{FS}}]\cdot p_2^*[\mathcal{S}_\infty,g_{\mathrm{FS}}]}(p_1^*\mathcal{S}_\infty)=0.$$

By the definition of the height, the last equation is equivalent to the equality

$$\widehat{\deg}_{\mathcal{X}}\left(p_1^*[\mathcal{S}_\infty,g_{\mathrm{FS}}]\cdot p_2^*[\mathcal{S}_\infty,g_{\mathrm{FS}}]\cdot p_1^*[\mathcal{S}_\infty,g_{\mathrm{FS}}]\right)=\frac{1}{2}\int\limits_{\mathcal{X}_\infty} p_1^*\omega_{g_{\mathrm{FS}}}\wedge p_2^*\omega_{g_{\mathrm{FS}}}\wedge p_1^*g_{\mathrm{FS}}. \tag{3.13}$$

By equation (3.11) we have

$$p_1^*[\mathcal{S}_\infty,g_{\mathrm{FS}}]\cdot p_2^*[\mathcal{S}_\infty,g_{\mathrm{FS}}]\cdot p_1^*[\mathcal{S}_\infty,g_{\mathrm{FS}}]=\left[0,2\cdot\overline{\mathcal{L}}^2\cdot p_1^*\omega_{g_{\mathcal{S}_\infty}}\wedge p_2^*\omega_{g_{\mathcal{S}_\infty}}\right],$$

where $\overline{\mathcal{L}}=(\mathcal{O}_{\mathcal{X}}(1),\|\cdot\|_{\mathrm{FS}})$. If we take the arithmetic degree map, we see that the left side of (3.13) is given by $\overline{\mathcal{L}}^2$. The right side of (3.13) is given by $\frac{1}{2}\int_{\mathcal{X}(1)_\infty}\omega_{g_{\mathrm{FS}}}\wedge g_{\mathrm{FS}}$. This integral is also given by $\overline{\mathcal{L}}^2$, which follows from the fact that g_{FS} is an adjusted Green's function for $\mathcal{S}_\infty$ on $\mathcal{X}(1)$. □

Second proof of Proposition 3.36. For $[D,g_D]=[p_1^*(\mathcal{S}_\infty)+p_2^*(\mathcal{S}_\infty),p_1^*g_{\mathcal{S}_\infty}+p_2^*g_{\mathcal{S}_\infty}]$ with $g_{\mathcal{S}_\infty}=-\log\|\Delta(q)\|^2_{\mathrm{Pet}}$ we have

$$\overline{\mathcal{L}}(12)^3=\widehat{\deg}_{\mathcal{X}}\left([D,g_D]^3\right).$$

The bilinearity of the arithmetic intersection number shows

$$\begin{aligned}\widehat{\deg}_{\mathcal{X}}\left([D,g_D]^3\right)=&\widehat{\deg}_{\mathcal{X}}\left(p_1^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]^3\right)+\widehat{\deg}_{\mathcal{X}}\left(p_2^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]^3\right)+\\&3\cdot\widehat{\deg}_{\mathcal{X}}\left(p_1^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]^2\cdot p_2^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]\right)+3\cdot\widehat{\deg}_{\mathcal{X}}\left(p_1^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]\cdot p_2^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]^2\right).\end{aligned}$$

Because $\widehat{\deg}_{\mathcal{X}}\left(p_1^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]^3\right)$ and $\widehat{\deg}_{\mathcal{X}}\left(p_2^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]^3\right)$ vanish and because of the symmetry of the arithmetic intersection number, it remains to show

$$\widehat{\deg}_{\mathcal{X}}\left(p_1^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]\cdot p_2^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]\cdot p_1^*[\mathcal{S}_\infty,g_{\mathcal{S}_\infty}]\right)=\overline{\mathcal{M}}_{12}\left(\Gamma(1)\right)^2.$$

By Proposition 3.34 we have to compute the integral

$$\begin{aligned}\frac{1}{2}\lim_{\varepsilon\to0}\int_{\mathcal{X}_\infty\setminus B_\varepsilon(D(\mathbb{C}))}&\big(p_1^*\omega_{g_{\mathcal{S}_\infty}}\wedge p_2^*\omega_{g_{\mathcal{S}_\infty}}\wedge p_1^*g_{\mathcal{S}_\infty}+\\&p_1^*\omega_{\alpha_{\mathcal{S}_\infty}}\wedge\left(p_2^*\omega_{\beta_{\mathcal{S}_\infty}}\wedge p_1^*g_{\mathcal{S}_\infty}-p_1^*\omega_{g_{\mathcal{S}_\infty}}\wedge p_2^*\beta_{\mathcal{S}_\infty}\right)+\\&p_1^*\omega_{g_{\mathcal{S}_\infty}}\wedge\left(p_1^*\omega_{\alpha_{\mathcal{S}_\infty}}\wedge p_2^*g_{\mathcal{S}_\infty}-p_2^*\omega_{g_{\mathcal{S}_\infty}}\wedge p_1^*\alpha_{\mathcal{S}_\infty}\right)\big),\end{aligned}$$

where $\{p_1^*\alpha_{\mathcal{S}_\infty}, p_2^*\beta_{\mathcal{S}_\infty}\}$ is a family of $p_1^*(\mathcal{S}_\infty)$-adjusted Green's forms for $\{p_1^*(\mathcal{S}_\infty), p_2^*(\mathcal{S}_\infty)\}$. By Lemma 3.37 we can choose $\alpha_{\mathcal{S}_\infty} = \beta_{\mathcal{S}_\infty} = -\log\|\Delta(q)\|^2_{\mathrm{FS}}$. Using the fact that $\int_{\mathcal{X}(1)_\infty}\omega_{\alpha_{\mathcal{S}_\infty}} = 1 = \int_{\mathcal{X}(1)_\infty}\omega_{g_{\mathcal{S}_\infty}}$, the number $\widehat{\deg}_{\mathcal{X}}\left(p_1^*[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}]\cdot p_2^*[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}]\cdot p_1^*[\mathcal{S}_\infty, g_{\mathcal{S}_\infty}]\right)$ is given by

$$\frac{1}{2}\lim_{\varepsilon\to 0}\int\limits_{\mathcal{X}(1)_\infty\setminus B_\varepsilon(S_\infty)}\left(\omega_{g_{\mathcal{S}_\infty}}\wedge g_{\mathcal{S}_\infty} + \omega_{\alpha_{\mathcal{S}_\infty}}\wedge g_{\mathcal{S}_\infty} - \omega_{g_{\mathcal{S}_\infty}}\wedge\alpha_{\mathcal{S}_\infty}\right).$$

As in the proof of Proposition 3.36, the latter integral equals

$$\log\left(\varphi_{\alpha_{\mathcal{S}_\infty}}(S_\infty)\right) + 6 - \lim_{\varepsilon\to 0}\left(6\log\left(-\log\varepsilon^2\right) - \frac{1}{2}\int\limits_{\mathcal{X}(1)_\infty\setminus B_\varepsilon(S_\infty)}\omega_{g_{\mathcal{S}_\infty}}\wedge g_{\mathcal{S}_\infty}\right).$$

Because of $\log\left(\varphi_{\alpha_{\mathcal{S}_\infty}}(S_\infty)\right) = 0$ the latter equals $\overline{\mathcal{M}}_{12}\left(\Gamma(1)\right)^2$. □

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Zusammenfassung

In dieser Arbeit definieren wir ein neues analytisches Objekt, eine *arithmetische lokale Koordinate*. Wir zeigen, dass sich mithilfe einer arithmetischen lokalen Koordinate die arithmetische Selbstschnittzahl $(\mathcal{P}, g_{\mathcal{P}})^2$ eines arithmetischen Divisors $(\mathcal{P}, g_{\mathcal{P}})$ auf einer arithmetischen Fläche $\mathcal{X}$ als einen analytischen Grenzwert schreiben lässt. Genauer gesagt zeigen wir

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \lim_{\mathbf{Q}\to\mathbf{P}} \left(\log|z_{\mathcal{P}}(\mathbf{Q})| + \frac{1}{2} g_{\mathcal{P}}(\mathbf{Q}) \right) + \frac{1}{2} \int_{\mathcal{X}_\infty} g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}},$$

wobei $z_{\mathcal{P}}$ eine arithmetische lokale Koordinate ist und $\mathbf{Q}$ eine Familie von Punkten ist, die analytisch gegen $\mathbf{P} := \mathcal{P}(\mathbb{C})$ konvergiert. Mit dieser Version der arithmetischen Selbstschnittzahl reduzieren wir das Problem Chow's Moving Lemma anzuwenden auf die Berechnung einer arithmetischen lokalen Koordinate. In der Tat müssen wir nur die Funktion $g_{\mathcal{P}} + \log|z_{\mathcal{P}}|^2$ an $\mathcal{P}(\mathbb{C})$ auswerten. Man bemerke außerdem, dass die Berechnung der geometrischen Schnittzahl an den endlichen Stellen entfällt. Des Weiteren wenden wir die Idee der arithmetischen lokalen Koordinaten auf die Berechnungen von verallgemeinerten arithmetischen Selbstschnittzahlen an, wobei die Greenschen Funktionen log-log Singularitäten aufweisen. Dies wird ermöglicht, indem wir ein neues analytisches Objekt einführen, die *angepassten Greenschen Funktionen*. Diese können als eine Art globale Version von arithmetischen lokalen Koordinaten angesehen werden.
Wir verallgemeinern arithmetische lokale Koordinaten und angepasste Greensche Funktionen auf höherdimensionale arithmetische Varietäten $\mathcal{X}$. Damit erhalten wir neue Formeln für die arithmetische Schnittzahl von zwei arithmetischen Zykeln (Y, g_Y) und (Z, g_Z). Wir definieren eine modifizierte Version des Stern Produktes

$$g_Y \bullet g_Z := \omega_{\alpha_{Y,Z}} \wedge g_Z - \omega_{g_Z} \wedge \alpha_{Y,Z} + \omega_{g_Z} \wedge g_Y$$

zwischen zwei Greenschen Formen g_Y und g_Z, wobei $\alpha_{Y,Z}$ eine angepasste Greensche Form zu Y ist, welche eine angepasste Greensche Funktion auf arithmetischen Flächen verallgemeinert. Eine Untersuchung des gewöhnlichen Stern Produktes $g_Y * g_Z$ von J. I. Burgos Gil,

J. Kramer and U. Kühn zeigt, dass sich das Integral über der komplexen Mannigfaltigkeit $\mathcal{X}_\infty$ von beiden $*$-Produkten nur um die geometrische Schnittzahl an den endlichen Stellen der Zykel Y und Z unterscheidet.

Abstract

In this thesis we define a new analytic object, which is called an *arithmetic local coordinate*. We show that the arithmetic self-intersection number $(\mathcal{P}, g_{\mathcal{P}})^2$ of an arithmetic divisor $(\mathcal{P}, g_{\mathcal{P}})$ on an arithmetic surface $\mathcal{X}$ can be written as a limit formula using an arithmetic local coordinate. Indeed, we show

$$(\mathcal{P}, g_{\mathcal{P}})^2 = \lim_{\mathbf{Q}\to\mathbf{P}} \left(\log|z_{\mathcal{P}}(\mathbf{Q})| + \frac{1}{2} g_{\mathcal{P}}(\mathbf{Q}) \right) + \frac{1}{2} \int_{\mathcal{X}_\infty} g_{\mathcal{P}} \cdot \omega_{g_{\mathcal{P}}},$$

where $z_{\mathcal{P}}$ is an arithmetic local coordinate and $\mathbf{Q}$ is a family of points converging analytically to $\mathbf{P} := \mathcal{P}(\mathbb{C})$. With this notion we reduce the problem of applying Chow's Moving Lemma to the calculation of an arithmetic local coordinate $z_{\mathcal{P}}$. Indeed, we only have to evaluate the function $g_{\mathcal{P}} + \log|z_{\mathcal{P}}|^2$ at $\mathcal{P}(\mathbb{C})$. Moreover, note that the calculation of the geometric intersection number at the finite places do not occur in this limit formula. We also apply this idea to the computation of generalized arithmetic self-intersection numbers, where the Green's functions have log-log singularities. This is done with the use of a new analytic object, which is called an *adjusted Green's function*. This can be seen as a global version of an arithmetic local coordinate.

We generalize the notions of arithmetic local coordinates and adjusted Green's functions to higher dimensional arithmetic varieties $\mathcal{X}$. We find new formulas for the arithmetic intersection number of two arithmetic cycles (Y, g_Y) and (Z, g_Z). We define a modified version of the star product

$$g_Y \bullet g_Z := \omega_{\alpha_{Y,Z}} \wedge g_Z - \omega_{g_Z} \wedge \alpha_{Y,Z} + \omega_{g_Z} \wedge g_Y$$

between two Green's forms g_Y and g_Z, where $\alpha_{Y,Z}$ is an adjusted Green's form for Y, generalizing the definition of an adjusted Green's function on an arithmetic surface. An investigation of the usual star product $g_Y * g_Z$ of J. I. Burgos Gil, J. Kramer and U. Kühn shows that the integral over the complex manifold $\mathcal{X}_\infty$ of both $*$-products only differ by the geometric intersection number at the finite places of the cycles Y and Z.

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Lebenslauf

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