Lattice topological field theories in two dimensions

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und	Prof. Dr. J. Fuchs

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Prof. Dr. M. Hinze
Leiter des Fachbereichs Mathematik

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1. Introduction

In this thesis, a state sum model for two-dimensional field theories on oriented surfaces with a framing or an r-spin structure is developed. In the following I will give a brief account of the main results and the general context serving as motivation to study this problem.

A combinatorial model for r-spin structures

With regard to spin structures on manifolds, the case of dimension two is special: The universal covering group of SO_n is a two-fold covering for $n > 2$, but the cover of SO_2 is $\mathbb{R}$. Correspondingly, oriented surfaces not only have spin-structures (which by convention refer again to the two-fold covering), but r-spin structures, where $r \in \{1, 2, 3, \dots\} \cup \{\infty\}$ and r-spin refers to the r-fold covering of the structure group SO_2.[1] Then $r = 1$ corresponds to oriented surfaces and $r = 2$ to ordinary spin surfaces.

There is a second reason why the two-dimensional case is special: An ∞-spin structure, i.e. a lift of the structure group of the oriented frame bundle to its universal cover, is the same as a framing. We expand on this in Section 3.1.

The combinatorial model for r-spin surfaces is obtained as follows. Pick a smooth triangulation of a given oriented surface Σ. A *marking* (Section 4.2) of the triangulation is a choice of orientation for each edge, and a choice of a preferred edge for each triangle (this amounts to a total ordering of its edges via the orientation of Σ). An *r-spin triangulated surface* Λ (Definition 4.6) is an r-spin surface, together with

- a choice of triangulation and marking of the underlying oriented surface $\underline{\Lambda}$,
- a choice of r-spin homomorphisms χ_σ from the standard triangle with standard r-spin structure, Δ, for each triangle σ of the triangulation.

By forgetting the r-spin structure, from Λ we can obtain the underlying oriented surface $\underline{\Lambda}$ together with a triangulation, a marking, and a choice of diffeomorphisms $\underline{\Delta} \to \underline{\Lambda}$ for each triangle. This collection of data will also be denoted by $\underline{\Lambda}$.

By comparing the r-spin homomorphisms χ_σ and $\chi_{\sigma'}$ for adjacent triangles σ, σ', one may assign an element of

$$\mathbb{Z}_r := \begin{cases} \mathbb{Z}/r\mathbb{Z} & : \ 0 < r < \infty \\ \mathbb{Z} & : \ r = \infty \end{cases}$$

[1] There is also some merit in writing "0-spin structure" instead of "∞-spin structure". After all, in passing from the universal cover to the r-spin case, we quotient the fundamental group $\mathbb{Z}$ of SO_2 by the ideal $r\mathbb{Z}$, so that $r = 0$ refers to the universal cover itself. Still, "∞-spin structure" seems more natural as it better reflects the number of choices possible in various stages of the construction

to each edge. We refer to these as *edge indices* (Sections 4.3 and 4.4). Conversely, one can use the edge indices and the marked triangulated surface $\underline{\Lambda}$ to define an r-spin structure on $\underline{\Lambda}$ minus the vertices of the triangulation. If the r-spin structure extends to the vertices, we call the edge indices *admissible* (Section 4.8). Denote by $S(\underline{\Lambda}, s)$ the r-spin surface obtained from an admissible choice s of edge indices. The edge indices s_Λ obtained from an r-spin triangulated surface Λ by the above procedure are admissible by construction.

The combinatorial model encodes the r-spin structure completely, since:

Theorem 4.17. Let Λ be an r-spin triangulated surface. The r-spin structures Λ and $S(\underline{\Lambda}, s_\Lambda)$ on $\underline{\Lambda}$ are isomorphic.

In giving this combinatorial model for an r-spin structure Λ, a number of choices have been made:

1. A triangulation of the underlying oriented surface $\underline{\Lambda}$,
2. a marking on the triangulation,
3. a choice of diffeomorphisms from the standard triangle $\underline{\Delta}$ to $\underline{\Lambda}$ for each triangle, compatible with marking and orientation,
4. a lift of the diffeomorphism in choice 3 to an r-spin homomorphism $\Delta \to \Lambda$.

The requirement of being compatible with the marking in choice 3 means that the preferred edge of the standard triangle is mapped to the preferred edge of the image triangle. This makes the space of choices in 3 contractible, and so choice 3 does not influence the edge indices. Changes in choices 2 and 4 do leave the triangulation invariant. The transformation of the edge indices under modification of these choices is fairly straightforward and is worked out in Section 4.6. The behaviour of the edge indices under Pachner moves needed for choice 1 (Section 4.9), on the other hand, is quite intricate and deriving its description is a lengthy technical argument.

In the above outline, the treatment of surfaces with boundaries is not explicitly mentioned. It is, however, crucial for the definition of a topological field theory. The corresponding discussion of r-spin parametrised boundaries, the definition of gluing, and the behaviour of the combinatorial model under gluing is given in Sections 3.7 and 4.5.

State sum construction of r-spin TFTs

Quite generally, state sum constructions of topological field theories start from a combinatorial description of the manifolds, say via a simplicial decomposition plus possibly some extra data. Then a prescription is given – depending on the choice of a symmetric monoidal category $\mathcal{S}$ and suitable algebraic datum therein – to translate the combinatorial model into a morphism in $\mathcal{S}$. Finally, one proves that thanks to the properties of the algebraic datum, this morphism is independent of the specific choices made in the combinatorial description.

Fix a symmetric monoidal category $\mathcal{S}$, which we will take to be strict for simplicity. The aim here is to define a topological field theory for r-spin surfaces. We will find, under two assumptions listed in Section 5.3, that the required algebraic datum is a Δ-separable Frobenius algebra in $\mathcal{S}$ whose Nakayama automorphism N satisfies $N^r = \mathrm{id}$. Let me describe this in more detail.

A *Frobenius algebra* A in $\mathcal{S}$ is an associative unital algebra and a coassociative counital coalgebra, such that the comultiplication is a bimodule morphism (see Section 5.3 for details). Let μ, η be the product and unit of A, and Δ, ε be the coproduct and counit. The resulting pairing b and copairing c are $b := \varepsilon \circ \mu$ and $c := \Delta \circ \eta$. The *Nakayama automorphism* is defined as (see also Equations (5.16) and (5.17))

$$N := (b \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes \sigma_{A,A}) \circ (\mathrm{id}_A \otimes c) \ : \ A \longrightarrow A \ .$$

Here $\sigma_{A,A}$ denotes the symmetric structure on $\mathcal{S}$. N is an automorphism of Frobenius algebras (Section 5.3). A Frobenius algebra is called *symmetric* if $b \circ \sigma_{A,A} = b$, or equivalentely $N = \mathrm{id}_A$. We call a Frobenius algebra Δ*-separable* if $\mu \circ \Delta = \mathrm{id}_A$.

In Section 5.3, a prescription T_A – depending on a Frobenius algebra A as above – is given that assigns to an r-spin triangulated surface Σ a morphism $T_A(\Sigma)$ in $\mathcal{S}$. The main result of this thesis is then:

Theorem 5.10. Let A be a Frobenius algebra in a symmetric strict monoidal category $\mathcal{S}$, such that A is Δ-separable and its Nakayama automorphism N satisfies $N^r = \mathrm{id}$. Then $T_A(\Sigma)$ is independent of the choice of r-spin triangulation of the r-spin surface Σ and $T_A(\Sigma) = T_A(\Sigma')$ for isomorphic r-spin surfaces Σ and Σ'.

As a corollary, for $r = 1$, i.e. in the oriented case, the Frobenius algebra has to be symmetric. This agrees with the results for the state sum construction of oriented TFTs which is reviewed in Section 2.4.

In Section 5.4 we prove that T_A is compatible with gluing, thus defining a topological field theory for r-spin surfaces. The state spaces associated to a circle are analysed in Section 5.6. Since there are r different r-spin structures on an annulus, there are r different state spaces to compute. They can be described as images of idempotents. In Equation (5.48) we define, for $\lambda \in \mathbb{Z}_r$,

$$P^\lambda = \mu \circ \sigma_{A,A} \circ (\mathrm{id} \otimes N^{1-\lambda}) \circ \Delta \ : \ A \longrightarrow A \ .$$

These are idempotents (Lemma 5.12) and under the assumption that they split, the state spaces of the r-spin TFT defined by A are the images of the P^λ, see also Section 5.6.

Context and outlook

The motivation for the work carried out in this thesis is two-fold. Firstly, the results of this thesis add a new state sum construction of topological field theories to the list of already known ones. Secondly, this thesis provides the starting point to construct r-spin quantum field theories – topological or not – from oriented theories with defects. Let me give more details on these two points.

State sum constructions: State sums arise in statistical mechanics, where they describe a classical system, say a crystal in two dimensions, in thermal equilibrium in the canonical ensemble. Suppose the crystal can be modelled by an $N \times N$ lattice where one assigns one of a finite number of possible states to each vertex. Let us call such an assignment a *configuration* and denote it by σ. Each configuration σ has an energy $E(\sigma)$. The partition function of the crystal is then the sum over all states,

$$Z = \sum_{\sigma} e^{-\beta E(\sigma)} \ ,$$

with β the inverse temperature. The result will depend on the shape and size of the lattice. A very difficult question in statistical mechanics is what happens in the so-called continuum limit, i.e. the limit of passing to smaller and smaller lattice spacings, or, equivalently, the limit of using $N \times N$ lattices for larger and larger sizes N.

State sum constructions of *topological* field theories provide a toy model for such continuum limits, since there the partition function will be *invariant* if one refines the lattice (while keeping the boundary fixed). One may thus hope to study some simple properties of continuum limits (such as dependence on the global topology of the lattice) in these models, while other important aspects (such as phase transitions) are not visible.

Mathematically, state sum constructions are interesting because they provide a relatively "hands-on" and conceptually straightforward approach to topological field theories. For oriented theories in two dimensions, state sum models were first studied in [BP, FHK] and in three dimensions in [TV, BW]. It is a natural next step to try to obtain state sum construction for topological field theories on manifolds with more geometric structures. For example, the manifold could be equipped with a principal G-bundle for some finite group G (such theories are subsumed in the class of homotopy topological field theories, see [Tu1, Tu2]), or with a spin structure. In two dimensions, a state sum construction for surfaces equipped with a principal G-bundle is given in [Tu3], and state sum constructions for spin surfaces have been given very recently and independently in [BT, NR].

The work [NR] was developed in the course of this thesis project. Parts of it make up Section 6 but the bulk of it is not repeated in this thesis. Instead, the construction of [NR] has been generalised to the r-spin case. Nonetheless, the structure of Sections 3–5 closely follows that of [NR] as most results and constructions have a direct generalisation from spin to r-spin. Still, on the technical level the generalisation from spin to r-spin is quite involved, as the geometric construction becomes more complicated.

Combinatorial models for spin manifolds in any dimension became available recently in [Bu]. It would be very interesting to use these to give a state sum construction for three-dimensional spin topological field theories.

Spin from defects: Field theories with defects have been attracting much attention recently in the context of duality relations between supersymmetric field theories in different dimensions (see e.g. [DNG]), in the description of topological phases of matter (see e.g. [KK, FS]), or as a useful invariant when comparing different realisations of a given field theory (see e.g. [DRCR]).

Defects in oriented topological field theories in two and in three dimensions have been studied in [DKR, FSV]. For non-topological quantum field theories, the most detailed results are known in rational two-dimensional oriented conformal field theory [FRS1, FrFRS1]. There, two-dimensional conformal field theories are described as boundary theories of a three-dimensional topological field theory. To study properties of a CFT on a surface Σ one evaluates the 3d TFT on $\Sigma\times[-1,1]$ with a surface defect placed at $\Sigma\times\{0\}$ [FRS1, KS]. In [FRS1], this surface defect is described by a network of one-dimensional ribbons placed on the surface $\Sigma \times \{0\}$. This network of ribbons is constructed by the same rules as in the state sum construction of an oriented two-dimensional topological field theory.

In [FrFRS2, CR], the construction of [FRS1] was given a purely two-dimensional interpretation as a generalised orbifolding procedure formulated in terms of defect lines. Topological defects in a two-dimensional rational CFT form a pivotal tensor category [FrFRS1] and according to [FrFRS2, CR], a defect in this category describes a generalised orbifold if it is equipped with the structure of a Δ-separable Frobenius algebra which is symmetric in the sense that its Nakayama automorphism is the identity (see e.g. [FSt, CR] for the corresponding definitions in non-symmetric categories). Some applications where the condition on the Nakayama automorphism is not imposed have been considered in [BCP].

It is now an evident goal to use the combinatorial model developed in this thesis, together with the corresponding algebraic relations, to define 2d r-spin CFTs in terms of oriented 2d CFTs with defects. The defect now should be equipped with the structure of a Δ-separable Frobenius algebra whose Nakayama automorphism N satisfies $N^r = \mathrm{id}$. Such a construction would describe spin CFTs for $r = 2$, parafermionic CFTs for higher values of r, and framed CFTs for $r = \infty$.

The present thesis can be seen as laying the foundation for this research programme.

This thesis is organised as follows. Chapter 2 contains a brief introduction to topological field theories, focusing on the two-dimensional case. In Chapter 3 some background on r-spin structures is given. The combinatorial model and its properties are described in Chapter 4. Chapter 5 contains the state sum construction of r-spin TFTs itself, and in Chapter 6 these results are applied to the most familiar case of spin surfaces. Here some comparison to previous results in the literature can be made.

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2. Introduction to topological field theories

2.1. An axiomatic framework for topological field theories

Topological field theories originate in physics. Indeed, starting with Witten's influential works [Wi1, Wi2] many examples of topological field theories have first been introduced in the physics literature. Nevertheless for a more systematic treatment the axiomatic framework has proved to be essential. It was introduced by Atiyah [At], following Segal [Se] and rephrased later in a more categorical setting by Quinn [Qu]. We mostly follow a version given by Kock in [Ko].

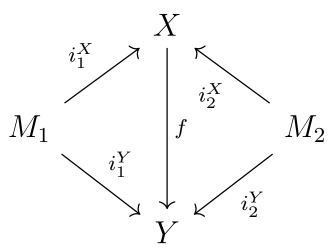
Bord$_n$ as a symmetric monoidal category

The heart of the categorical framework for topological quantum field theories is the definition of the bordism category. We give it in some detail for oriented bordisms, which is the simplest case. Unoriented bordisms, or bordisms with extra structure – e.g. framing or spin structure – can also be considered and will be relevant later in this work.

Objects are $(n-1)$-dimensional smooth oriented closed manifolds.

Definition 2.1. An n-bordism from M_1 to M_2 is a tuple (M_1, X, M_2) of an oriented n-manifold X with boundary $\partial X = \partial X_1 \sqcup \partial X_2$, together with diffeomorphisms $i_{1,2} : M_{1,2} \to \partial X_{1,2}$. We require i_1 to be orientation preserving, i_2 to be orientation reversing, where ∂X is oriented according to the inward pointing normal.

An isomorphism of n-bordisms (M_1, X, M_2) and (M_1, Y, M_2) is an orientation preserving diffeomorphism $f : X \to Y$ such that



(2.1)

commutes.

Morphisms from M_1 to M_2 are isomorphism classes of bordisms (M_1, X, M_2). The identity id_M is the class containing the cylinder $M \times [0,1]$.

Composition of bordisms (M_1, X, M_2), (M_2, Y, M_3) is given by gluing along M_2:

$$(M_2, Y, M_3) \circ (M_1, X, M_2) = (M_1, X \sqcup_{M_2} Y, M_3). \tag{2.2}$$

For details regarding the subtleties of gluing smooth manifolds see [Ko, Ch.1.3], in particular Theorem 1.3.12 there.

The monoidal product is the disjoint union, with $\emptyset$ as the tensor unit. The symmetrical structure is given by the twist bordism,

$$\tau_{M_1,M_2} = (M_1 \sqcup M_2, (M_1 \sqcup M_2) \times [0,1], M_2 \sqcup M_1) , \tag{2.3}$$

with the natural embeddings (See Figure 2.1 for a sketch in the two-dimensional case).

We thus have obtained the symmetric monoidal category $\mathbf{Bord}_n$.

Even after taking equivalence classes of bordisms, $\mathbf{Bord}_n$ is not strict. The associators, however, are uninteresting and due to coherence we can and will ignore this issue and shall treat all monoidal categories as if they were strict.

Remark 2.2. In dimension 2 – the case we will be most concerned with – manifolds are diffeomorphic if and only if they are homeomorphic (or piecewise linear equivalent). In this case one could study the topological or piecewise linear bordism categories without any difference.

The axiomatic definition of topological field theories

We can now give a – remarkably simple – definition of topological field theories and study some general properties.

Definition 2.3. Let $\mathcal{S}$ be a symmetric monoidal category. A *topological field theory* in $\dim n$ with target $\mathcal{S}$ is a symmetric monoidal functor

$$T : \mathbf{Bord}_n \to \mathcal{S} . \tag{2.4}$$

Example. Let $\mathcal{S} = \mathbf{Bord}_n$. The identity functor $\mathrm{id} : \mathbf{Bord}_n \to \mathbf{Bord}_n$ is a topological field theory with target $\mathbf{Bord}_n$.

We will see more interesting examples in dimension two later in the text.

Remark 2.4. Above, we defined the identity morphisms as cylinders. This is not absolutely essential. We could instead have added in identity morphisms "by hand". The cylinders are then merely idempotents. If idempotents in the target category split then it is always possible to restrict to the images of the cylinder idempotents and produce a topological field theory in the above sense.

Dualisability

Recall the

Definition 2.5. An object X in a symmetric monoidal category $\mathcal{S}$ is dualisable if and only if there exists an object X^*, the dual of X, and maps

$$\begin{aligned} \mathrm{ev}_X &: X \otimes X^* \to \mathbf{1} \ , \\ \mathrm{coev}_X &: \mathbf{1} \to X^* \otimes X \end{aligned} \tag{2.5}$$

such that

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X \otimes \mathrm{coev}_X} & X \otimes X^* \otimes X \\ & \searrow^{=} & \downarrow{\scriptstyle \mathrm{ev}_X \otimes \mathrm{id}_X} \\ & & X \end{array} \quad \text{and} \quad \begin{array}{ccc} X^* & \xrightarrow{\mathrm{coev}_X \otimes \mathrm{id}_{X^*}} & X^* \otimes X \otimes X^* \\ & \searrow^{=} & \downarrow{\scriptstyle \mathrm{id}_{X^*} \otimes \mathrm{ev}_X} \\ & & X^* \end{array} \tag{2.6}$$

commute.

It turns out that

Lemma 2.6. Every object M in $\mathbf{Bord}_n$ is dualisable with dual M^* given by M with the opposite orientation.

Proof. Consider the bordisms $\mathrm{ev}_X = (M \sqcup M^*, M \times [0,1], \emptyset)$ and $\mathrm{coev}_X = (\emptyset, M \times [0,1], M^* \sqcup M)$. □

This almost trivial result has interesting consequences. Since dualisability transports along monoidal functors, for a topological field theory $T : \mathbf{Bord}_n \to \mathcal{S}$ we get:
$T(M)$ is dualisable for every M.
This has an important consequence, namely the following

Corollary 2.7. Let $\mathcal{Z} : \mathbf{Bord}_n \to \mathbf{Vect}_k$ be a topological field theory with target category $\mathbf{Vect}_k$, the category of vector spaces over a field k. Then all state spaces $\mathcal{Z}(M)$ are finite dimensional.

We have seen that by studying properties of the bordism category general results about topological field theories can be derived. We continue to exploit this in two directions: In Section 2.2 we study $\mathbf{Bord}_2$ in more detail, in fact it can be presented via generators and relations, which allows to then classify the topological field theories. In Section 2.3 we briefly describe extended topological field theories which exploit the dualisability concept even more by changing the bordism category to an n-category. We can then apply the main result about extended topological field theories, the cobordism hypothesis to dimension two. In Section 2.4 we describe the lattice construction for two-dimensional topological field theories as a more hands-on alternative to the extended topological field theory framework, giving the general setting of this thesis.

2.2. Two-dimensional topological field theory via generators and relations

In dimension two, the category of bordisms can be described explicitly in terms of generators and relations. This analysis leads directly to the well-known classification result in Theorem 2.8 below, which we now review.

We give an informal description of the generating data for $\mathbf{Bord}_2$ and refer the reader to [Ko] for proofs. General results and definitions of generating data for a monoidal category can also be found in [JS]. As there is a single oriented circle (up to orientation-preserving diffeomorphism), objects in $\mathbf{Bord}_2$ are finite disjoint unions of circles. Generating bordisms are collected in Figure 2.1.

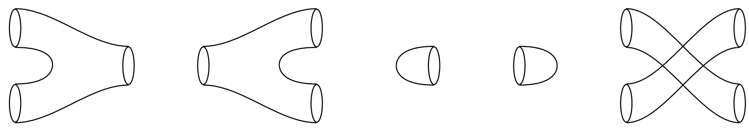
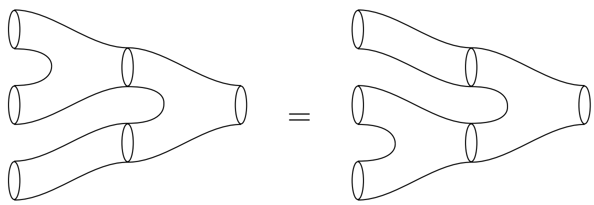


Figure 2.1.: Generators for $\mathbf{Bord}_2$. Ingoing boundaries are to the left. The rightmost bordism represents the symmetric structure of $\mathbf{Bord}_2$.

Compositions of these elementary bordisms may be equivalent, giving rise to relations. These can be derived for example via Morse theory. We refer to [Ko, Ch. 1.4] for the full list, here we just illustrate the general principle by giving two ways of obtaining the four-holed sphere with three ingoing boundaries and one outgoing boundary:



(2.7)

For simplicity, we specialise the target category to $\mathcal{S} = \mathbf{Vect}_k$. Recall that a *Frobenius algebra* over k is an associative unital algebra with non-degenerate invariant pairing (see Section 5.3 for more details). In particular, a Frobenius algebra is necessarily finite-dimensional.

The classification result for two-dimensional topological field theories compares two categories. The first is the category $\mathcal{T}_k$ of two-dimensional TFTs with values in $\mathbf{Vect}_k$, i.e. of symmetric monoidal functors $\mathbf{Bord}_2 \to \mathbf{Vect}_k$ and natural monoidal transformations. The second is the category $\mathcal{F}_k$ of commutative Frobenius algebras over k and algebra homomorphisms respecting the pairing. It is easy to check that in $\mathcal{T}_k$ and $\mathcal{F}_k$, morphisms are necessarily invertible. We have

Theorem 2.8. The functor $\mathcal{T}_k \to \mathcal{F}_k$, which assigns to $\mathcal{Z} \in \mathcal{T}_k$ the Frobenius algebra on the vector space $\mathcal{Z}(S^1)$ with unit, multiplication and pairing given by evaluating $\mathcal{Z}$ on the obvious bordisms, is an equivalence.

In this form, Theorem 2.8 was proved in [Ab], see also [Ko, Thm. 3.3.2] for a detailed exposition. The identification of 2d TFTs and commutative Frobenius algebras itself goes back to [Di].

2.3. Two-dimensional topological field theories via the cobordism hypothesis

Another way to analyse topological field theories by studying the bordism category is via the cobordism hypothesis. Since the techniques used in the statement and proof of the cobordism hypothesis are mostly different from those used in this thesis, not all terms will be explained. Nevertheless we can get some intuition and compare to the results obtained by applying the cobordism hypothesis.

The main idea is to allow cutting manifolds not only along hypersurfaces (yielding bordisms) but also along submanifolds of higher codimension, up to the point. This turns $\mathbf{Bord}_n$ into a higher category, more precisely an (∞, n)-category. Objects are then collections of (oriented or framed) points, 1-morphisms are bordisms between these, 2-morphisms are bordisms between the 1-morphisms and so on. On the other side, the target is now a symmetric monoidal (∞, n)-category.

Using these we state the (slightly reformulated)

Theorem 2.9 (Cobordism hypothesis (framed version) [Lu] 2.4.6)**.** Let $\mathbf{C}$ be a symmetric monoidal (∞, n)-category. Then the evaluation functor $Z \mapsto Z(*)$, mapping a (fully extended) topological field theory to its value on the point, induces an equivalence

$$\mathrm{Fun}^{\otimes}(\mathbf{Bord}_n^{\mathrm{fr,ext}}, \mathbf{C}) \to \mathbf{C}^{\mathrm{fd}} \ . \tag{2.8}$$

Here $\mathbf{Bord}_n^{\mathrm{fr,ext}}$ is the n-dimensional extended framed bordism category and $\mathbf{C}^{\mathrm{fd}}$ is the ∞-groupoid of fully dualisable objects in $\mathbf{C}$.

We are interested mainly in the two-dimensional case. A convenient target bicategory is $\mathbf{Alg}_2$:

- Objects are finite dimensional k-algebras. [1]
- 1-Morphisms are bimodules of finite rank.
- 2-Morphisms are bimodule intertwiners.

To study oriented – in contrast to framed – topological field theories in principle we should use the oriented version of the cobordism hypothesis. The additionally used homotopy SO_2-action is however trivial for the target category $\mathbf{Alg}_2$, see [Da, Proposition 3.2.8].

Fully extended topological field theories with target $\mathbf{Alg}_2$ are thus classified by fully dualisable objects in $\mathbf{Alg}_2$. One can show that

[1] For the following statements we also require that char $k = 0$ and k is algebraically closed.

Proposition 2.10 ([Da], [SP])**.** Fully dualisable objects in $\mathbf{Alg}_2$ are strongly separable Frobenius algebras.

Recall that a Frobenius algebra A is *separable* if the multiplication $\mu : A \otimes A \to A$ has a right inverse $f : A \to A \otimes A$ which is also an A-A-bimodule map. It is *strongly separable* if the element $f(1)$ is symmetric, i.e. $f(1) = \tau_{A,A} \circ f(1)$, where $\tau_{A,A}(a \otimes b) = b \otimes a$ is the symmetric structure in **Vect**.

A better computable approach using the same input data, strongly separable Frobenius algebras, and producing the same extendible (but not extended) topological field theory is the lattice construction described in the next section for oriented surfaces and later for framed and r-spin surfaces. In [Da] the geometrical relation between the fully extended field theories as described above and the lattice construction is made explicit.

2.4. The lattice construction of two-dimensional topological field theory as a state sum

We give a brief description of the lattice construction as a state sum model, the algebraic data and resulting topological field theories, originally due to [BP, FHK]. The target category here is $\mathbf{Vect}_k$. This has been extended to arbitrary symmetric monoidal target categories in [LP]. In the context of this thesis, these TFTs are a special case (namely $r = 1$) of the main result of this thesis, Theorem 5.10.

We present the lattice construction for oriented surfaces with triangulation. The state sum model will assign states to "edges with normal direction" and weights to edges and triangles. Given a finite set of states $\mathbf{I}$ we pick plaquette weights $C_{ijk} \in k$ for $i, j, k \in \mathbf{I}$ and edge weights $g^{ij} \in k$ for $i, j \in \mathbf{I}$. We require the weights to be cyclically invariant, i.e. $C_{ijk} = C_{kij}$ and $g^{ij} = g^{ji}$. To compute the state sum for an oriented surface, proceed as follows:

1. Triangulate the surface.

2. For every triangle Δ assign a state to each of its edges. (In particular every inner edge has two states assigned to it). Such an assignment of states is called a *configuration*.

3. The weight of such a configuration is given by

$$\prod_{\text{triangles } \Delta} C_{\langle \Delta \rangle} \prod_{\text{edges } e} g^{\langle e \rangle} \ . \tag{2.9}$$

   Here $\langle \Delta \rangle$ is the triple of states assigned to Δ in counterclockwise order and $\langle e \rangle$ is the pair of states assigned to e.

   A boundary edge e only carries one index from $\mathbf{I}$, so we still need to specify what $g^{\langle e \rangle}$ means in this case: We understand the above product as a function $|\mathbf{I}|^{\#\text{ boundary edges}} \to k$ by assigning a second state to each boundary edge.

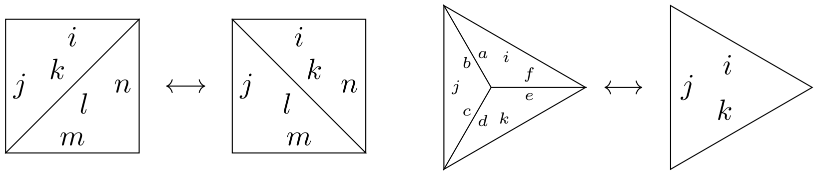


Figure 2.2.: The two-dimensional Pachner moves: 2-2 move (left) and 3-1/1-3 move (right).

4. Finally we obtain the state sum by summing over all configurations of states:

$$\mathcal{Z} = \sum_{\text{configurations}} \prod_{\text{triangles } \Delta} C_{\langle\Delta\rangle} \prod_{\text{edges } e} g^{\langle e\rangle} \ . \tag{2.10}$$

   Again, this is a function $|\mathbf{I}|^{\#\text{ boundary edges}} \to k$.

The key idea is to show that for suitable choices of C and g, $\mathcal{Z}$ is independent of the choice of triangulation. This is explained in the following.

Let now $\mathbf{H} = \mathrm{span}_k(\mathbf{I})$. For any invertible $T \in \mathrm{End}(\mathbf{H})$ the state sum is invariant under the transformation

$$\begin{aligned} C_{ijk} &\mapsto (T^{-1})^l_i (T^{-1})^m_j (T^{-1})^n_k C_{lmn} \\ g^{ij} &\mapsto T^i_l T^j_k g^{kl} \ . \end{aligned} \tag{2.11}$$

We can thus treat C_{ijk} as coordinates of $C \in \mathbf{H}^{\otimes 3}$ and g^{ij} as coordinates of a bilinear form $g : \mathbf{H}^{\otimes 2} \to k$.

In addition from now on we assume g to be nondegenerate (if it were degenerate we could always divide out its nullspace). We denote the components of the inverse of g as g_{ij}.

By a result of Pachner [Pa] any two triangulations are related by a finite sequence of Pachner moves. The moves in dimension two are shown in Figure 2.2. We ensure independence of the triangulation by requiring independence under Pachner moves. This gives additional constraints on the tensors C and g:

$$C_{ijk} g^{kl} C_{lmn} = C_{nik} g^{kl} C_{ljm} \ , \tag{2.12}$$

$$C_{fia} g^{ab} C_{bjc} g^{cd} C_{dke} g^{ef} = C_{ijk} \ . \tag{2.13}$$

Here (2.12) follows from the 2-2-move and (2.13) from the 3-1 move. Defining the map $m : \mathbf{H} \otimes \mathbf{H} \to \mathbf{H}$ in components as $m_i^{jk} := g^{mj} C_{min} g^{nk}$ we see that (2.12) requires the algebra $(\mathbf{H}, m)$ to be associative. Condition (2.13) is a bit more subtle. Assuming $(\mathbf{H}, m)$ is unital, (2.13) implies that that $(\mathbf{H}, m)$ is strongly separable with the inverse of g as separability idempotent (see Section 5.3). The cyclicity of C implies the invariance of g with respect to the product m:

$$g \circ (\mathrm{id} \otimes m) = g \circ (m \otimes \mathrm{id}) \tag{2.14}$$

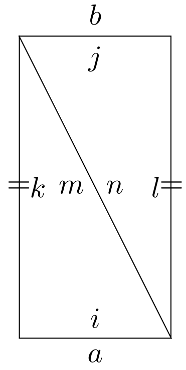


Figure 2.3.: Triangulation of the cylinder. The left and right boundaries are identified. The lower (ingoing) and upper (outgoing) boundaries have their state labels fixed

Together we see that $(\mathbf{H}, m, g)$ is a strongly separable symmetric (but not necessarily commutative) Frobenius algebra. Surfaces with boundary are treated in the more general context later, but we can already compute the state sum on the cylinder $S^1 \times [0, 1]$.

Triangulating[2] the cylinder as in Figure 2.3 we get

$$\mathcal{Z}(a, b) = C_{imk} C_{nlj} g^{mn} g^{kl} g^{ai} g^{bj} \ . \tag{2.15}$$

We use invertibility of g for the boundary labeled j to turn the state sum into a linear map $\mathcal{Z} : \mathbf{H} \to \mathbf{H}$.

$$\mathcal{Z}^i_j = C_{imk} C_{nlj} g^{mn} g^{kl} g^{ip} \ . \tag{2.16}$$

It can easily be checked that $\mathcal{Z}$ is a projector to the centre of $(\mathbf{H}, m, g)$. $\mathcal{Z}$ corresponds to P^λ in (5.48), and we omit the computation here, referring to the general results in lemmata 5.12 and (6.6).

By its nature the centre of $(\mathbf{H}, m, g)$ is always a commutative Frobenius algebra and it corresponds to the one in the classification result in Theorem 2.8. In $\mathbf{Vect}_\mathbb{C}$ the strongly separable algebras are direct sums of matrix algebras (see also Example 3). Since their centre is again strongly separable we see in particular that not all 2d TFTs can be described via state sums.

[2]This is actually not a triangulation, as the intersection of the two triangles does not only consist of a sub-simplex of either of them, but this does not modify the result and simplifies the present discussion. In chapters 4–6 we will deal with proper triangulations.

3. Surfaces with spin structures, r-spin structures and framing

A *surface* is an oriented, smooth, two-dimensional real manifold, possibly with boundary. Maps between surfaces are smooth and orientation-preserving. We identify the complex plane $\mathbb{C}$ with $\mathbb{R}^2$ and sometimes use complex coordinates, but maps between subsets of $\mathbb{C}$ need not be holomorphic.

3.1. Spin structures, r-spin structures and framing

Let $r \in \mathbb{N} \setminus \{0\} \cup \{\infty\}$. Denote by Spin_2^r the connected r-fold cover of SO_2, with $\mathrm{Spin}_2^\infty = \mathbb{R}$ denoting the universal cover. For $r \neq \infty$ these groups are explicitly given by

$$\mathrm{Spin}_2^r = \mathbb{R}/r\mathbb{Z} \ , \tag{3.1}$$

i.e. $x \sim y \Leftrightarrow x - y = n \cdot r$ for $n \in \mathbb{Z}$. Thinking of SO_2 as the unit circle in $\mathbb{C}$ we can now give the covering maps explicitly:

$$\begin{aligned} p_{SO}^r &: \mathrm{Spin}_2^r \to SO_2, \\ p_{SO}^r &: x \mapsto e^{2\pi i x}. \end{aligned} \tag{3.2}$$

Then p_{SO}^1 gives an isomorphism of Lie groups from $\mathbb{R}/\mathbb{Z}$ to $SO_2 \subset \mathbb{C}$ and $\mathrm{Spin}_2^2 = \mathrm{Spin}_2$ is the usual spin group in two dimensions.

Analogously to the definition of spin structures by Milnor [Mi] we define an r-spin structure on an SO_2-principal bundle as follows:

Definition 3.1. Let $\zeta : P_{SO} \to M$ be an SO_2-principal bundle over a manifold M. A *r-spin structure* on ζ is a pair (η, p), consisting of a Spin_2^r-principal bundle $\eta : P_{\mathrm{Spin}} \to M$ and a map $p : P_{\mathrm{Spin}} \to P_{SO}$ such that the following diagram commutes:

$$\begin{array}{ccccc} P_{\mathrm{Spin}} \times \mathrm{Spin}_2^r & \xrightarrow{R_{\mathrm{Spin}_2^r}} & P_{\mathrm{Spin}} & \overset{\eta}{\searrow} & \\ \Big\downarrow{\scriptstyle p \times p_{SO}^r} & & \Big\downarrow{\scriptstyle p} & & M \\ P_{SO} \times SO_2 & \xrightarrow{R_{SO_2}} & P_{SO} & \underset{\zeta}{\nearrow} & \end{array} \ , \tag{3.3}$$

where R_G denotes the right action of a group on its principal bundle. An *isomorphism of r-spin structures* is a map of principal bundles $f : P_{\mathrm{Spin}} \to P'_{\mathrm{Spin}}$ such that $p' \circ f = p$.

The map $p : P_{\mathrm{Spin}} \to P_{SO}$ is then automatically an r-fold cover.

We denote the bundle of oriented orthonormal frames over an oriented Riemannian manifold M by $F_{SO}(M)$. The right action of $g \in SO_n$ on a frame $(v_1, \ldots, v_n)$ is given by

$$(v_1, \ldots, v_n).g := (v_1', \ldots, v_n') \quad \text{with} \quad v_i' = \sum_j v_j g_{ji} \ . \tag{3.4}$$

A *r-spin surface* is a surface equipped with a Riemannian metric and an r-spin structure on its associated bundle of oriented orthonormal frames.

The case $r = \infty$ is special as it corresponds to a framing. A *framing* is a homotopy class of sections (i.e. trivialisations) of the oriented frame bundle. To each such trivialisation one has a canonical ∞-spin structure (in fact a r-spin structure for any r). Conversely, think of the ∞-spin structure as being obtained by gluing open patches with trivial oriented frame bundle. Since the universal cover $\mathbb{R}$ of SO_2 is contractible, one can find a homotopy such that all gluing functions are just the identity, giving a trivialisation.

3.2. r-spin structures and framing without metric

In order to construct a topological field theory later, we want to consider surfaces without metric. We start by discussing the relevant groups involved. Let $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$ for $r \neq \infty$ and $\mathbb{Z}_\infty = \mathbb{Z}$. First, consider the exact sequence of groups

$$0 \longrightarrow \mathbb{Z}_r \longrightarrow \mathrm{Spin}_2^r \xrightarrow{p_{SO}^r} SO_2 \longrightarrow 0 \ . \tag{3.5}$$

Let GL_2^+ be the group of orientation preserving linear automorphisms of $\mathbb{R}^2$. The inclusion $i : SO_2 \to GL_2^+$ is a homotopy equivalence by the QR-decomposition. By covering theory (see for example [Ha], in particular Prop. 1.31) the above sequence therefore extends to a commutative diagram of Lie groups,

$$\begin{array}{ccccccc} & & \mathrm{Spin}_2^r & \xrightarrow{p_{SO}^r} & SO_2 & & \\ & \nearrow & \Big\downarrow{\scriptstyle\delta} & & \Big\downarrow{\scriptstyle i} & \searrow & \\ 0 \longrightarrow \mathbb{Z}_r & & & & & & 0 \ , \\ & \searrow & & & & \nearrow & \\ & & \widetilde{GL}_2^r & \xrightarrow{p_{GL}^r} & GL_2^+ & & \end{array} \tag{3.6}$$

such that $\widetilde{GL}_2^r$ is the connected r-fold covering of GL_2, and the embedding

$$\delta : \mathrm{Spin}_2^r \to \widetilde{GL}_2^r \tag{3.7}$$

is the lift of i mapping the unit of Spin_2^r to the unit of $\widetilde{GL}_2^r$. An explicit construction of $\widetilde{GL}_2^2$, the spin case, is provided in Appendix A, but is not used in what follows.

By representing the elements in $\delta(\mathbb{Z}_r)$ as loops in GL_2^+ and using that $-1 \in GL_2^+$ is central one sees that integer as well as half integer elements of Spin_2^r get mapped to the centre of $\widetilde{GL}_2^r$,

$$\delta\left(\mathbb{Z}_r \cup (\mathbb{Z}_r + \frac{1}{2})\right) \subset Z(\widetilde{GL}_2^r) \ . \tag{3.8}$$

Having discussed the covering groups of GL_2 we now state the following

Definition 3.2. Let Σ be an surface. Let $\zeta : F_{GL^+} \to M$ be its bundle of oriented frames. A *r-spin structure without metric* on Σ is a pair (η, p), consisting of a $\widetilde{GL}_2^r$-principal bundle $\eta : P_{\widetilde{GL}} \to \Sigma$ and a map $p : P_{\widetilde{GL}} \to F_{GL^+}$ such that the following diagram commutes:

$$\begin{array}{ccccc} P_{\widetilde{GL}} \times \widetilde{GL}_2^r & \xrightarrow{R_{\widetilde{GL}_2^r}} & F_{\widetilde{GL}} & \xrightarrow{\eta} & \\ \downarrow{\scriptstyle p\times p^r_{GL}} & & \downarrow{\scriptstyle p} & & M \\ F_{GL^+} \times GL_2^+ & \xrightarrow{R_{GL_2^+}} & F_{GL^+} & \xrightarrow{\zeta} & \end{array} \quad . \tag{3.9}$$

An *isomorphism of r-spin structures without metric* is a map of principal bundles $f : P_{\widetilde{GL}} \to P'_{\widetilde{GL}}$ such that $p' \circ f = p$.

Definition 3.3. An *r-spin surface without metric* Σ is a surface $\underline{\Sigma}$, together with an r-spin structure without metric $P_{\widetilde{GL}}(\Sigma)$.

Remark 3.4. The groupoids of r-spin structures without metric and ordinary r-spin structures on a given surface with metric are equivalent: Let Σ be an r-spin surface without metric. If we equip it with a metric, we can obtain a spin surface as follows: A metric surface comes with a bundle of oriented orthonormal frames, and an embedding $i : F_{SO} \to F_{GL^+}$ into its bundle of oriented frames. We obtain an r-spin structure on Σ by pulling pack the $\widetilde{GL}_2$ bundle along this inclusion map. Conversely we can obtain an r-spin structure without metric from an r-spin structure by taking the associated $\widetilde{GL}_2$ bundle. These constructions are functorial, and inverse to each other.

Definition 3.5. A *morphism* (or *map*) *of r-spin surfaces without metric* is given by a map of $\widetilde{GL}_2^r$ bundles $\tilde{f} : P_{\widetilde{GL}}(\Sigma) \to P_{\widetilde{GL}}(\Sigma')$ such that the diagram

$$\begin{array}{ccc} P_{\widetilde{GL}}(\Sigma) & \xrightarrow{\tilde{f}} & P_{\widetilde{GL}}(\Sigma') \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p'} \\ F_{GL^+}(\Sigma) & \xrightarrow{df_*} & F_{GL^+}(\Sigma') \\ \downarrow & & \downarrow \\ \underline{\Sigma} & \xrightarrow{f} & \underline{\Sigma}' \end{array} \tag{3.10}$$

commutes. Here $f : \underline{\Sigma} \to \underline{\Sigma}'$ denotes the underlying map of surfaces, and df_* the map induced on the bundle of oriented frames by the derivative of f.

We will sometimes write $\tilde{f} : \Sigma \to \Sigma'$ for a map between r-spin surfaces as an abbreviation of $\tilde{f} : P_{\widetilde{GL}}(\Sigma) \to P_{\widetilde{GL}}(\Sigma')$.

Remark 3.6. Note that an isomorphism of r-spin structures as in Definition 3.2 is required to be the identity on the underlying surface, while a map of r-spin surfaces may even relate r-spin structures with different underlying surfaces. In fact, to give a map $\tilde{f} : \Sigma \to \Sigma'$ between r-spin surfaces without metric is the same as to give isomorphism of r-spin structures from $P_{\widetilde{GL}}(\Sigma)$ to the pullback r-spin structure $\tilde{f}^* P_{\widetilde{GL}}(\Sigma') = F_{GL^+} \times_{F'_{GL^+}} P_{\widetilde{GL}}(\Sigma')$.

From now on we will write "r-spin structure" (resp. "r-spin-surface") for r-spin structure without metric (resp. r-spin-surfaces without metric).

Let Σ be an r-spin surface. By right action with the an element $k \in \mathbb{Z}_r$ of the kernel of $p^r_{GL} : \widetilde{GL}^r_2 \to GL^+_2$ – see Equation (3.6) – we obtain a natural automorphism $\omega_k : \Sigma \to \Sigma$, the *leaf exchange automorphism*.

3.3. QR-decompositions for $\widetilde{GL}^r_2$

We start from the QR-decomposition of an invertible matrix into an orthogonal and an upper triangular part. If we require that the upper triangular matrix has nonnegative diagonal entries, the QR-decomposition is unique, and we get smooth[1] maps

$$\mathbf{Q} : GL^+_2 \to SO_2 \quad , \qquad \mathbf{R} : GL^+_2 \to T_2 \ , \tag{3.11}$$

such that $\mathbf{Q}(g)\,\mathbf{R}(g) = g$ for $g \in GL^+_2$. Here, T_2 is the space of upper triangular matrices with positive diagonal entries. Notice, however, that $\mathbf{Q}$ and $\mathbf{R}$ are not group homomorphisms.

We now describe a related decomposition for $\widetilde{GL}^r_2$. Since T_2 is contractible, the preimage $(p^r_{GL})^{-1}(T_2)$ has r connected components. Let $\tilde{T}_2$ be the connected component of the identity.

Lemma 3.7. The QR-decomposition lifts to $\widetilde{GL}^r_2$, i.e. there are unique smooth maps

$$\tilde{\mathbf{Q}} : \widetilde{GL}^r_2 \to \delta(\mathrm{Spin}^r_2) \quad , \qquad \tilde{\mathbf{R}} : \widetilde{GL}^r_2 \to \tilde{T}_2 \ , \tag{3.12}$$

such that $\tilde{\mathbf{Q}}(g)\,\tilde{\mathbf{R}}(g) = g$ for all $g \in \widetilde{GL}^r_2$, and such that

$$p^r_{GL} \circ \tilde{\mathbf{Q}} = \mathbf{Q} \circ p^r_{GL} \quad , \qquad p^r_{GL} \circ \tilde{\mathbf{R}} = \mathbf{R} \circ p^r_{GL} \ . \tag{3.13}$$

Proof. Since T_2 is contractible the inclusion map $i : T_2 \to GL^+_2$ admits lifts to $\widetilde{GL}^r_2$,

$$\tilde{\imath} : T_2 \to \widetilde{GL}^r_2 \quad . \tag{3.14}$$

[1] Writing out the QR-decomposition in components one easily sees that the only possible issue is at $\det g = 0$.

If we require that $\tilde{\imath}(1) = \delta(0)$ then $\tilde{\imath}$ is uniquely determined and a group homomorphism. To see the last statement consider $a, b \in T_2$. Pick paths $\gamma_a, \gamma_b, \gamma_{ab} : [0,1] \to T_2$ such that $\gamma_a(0) = \gamma_b(0) = \gamma_{ab}(0) = 1$ and $\gamma_a(1) = a$, $\gamma_b(1) = b$, $\gamma_{ab}(1) = ab$. Starting with ab we can now go γ_{ab} backwards and then $\gamma_a\gamma_b$ forwards to obtain a closed, contractible loop with basepoint ab in T_2. Using $\tilde{\imath}$ to lift this loop we obtain a path from $\tilde{\imath}(ab)$ to $\tilde{\imath}(a)\tilde{\imath}(b)$. Since the base loop was contractible the lift has to be closed. A similar argument shows $\tilde{\imath}(a^{-1}) = (\tilde{\imath}(a))^{-1}$.

We define $\tilde{\mathbf{R}} := \tilde{\imath} \circ \mathbf{R} \circ p^r_{GL}$. Using that $p^r_{GL} \circ \tilde{\imath} = \mathrm{id}_{T_2}$ and that $\tilde{\imath}$ is a group homomorphism, one checks that $p^r_{GL}(g\,\tilde{\mathbf{R}}(g)^{-1}) = \mathbf{Q}(p^r_{GL}(g)) \in SO(2)$ for all $g \in \widetilde{GL}_2^r$. Hence we can define $\tilde{\mathbf{Q}}(g) := g\,\tilde{\mathbf{R}}(g)^{-1} \in \delta(\mathrm{Spin}_2^r)$. It is then immediate that $\tilde{\mathbf{Q}}(g)\,\tilde{\mathbf{R}}(g) = g$ and that (3.13) holds.

Next we turn to the uniqueness of the QR-decomposition in $\widetilde{GL}_2^r$. Suppose that $g \in \widetilde{GL}_2^r$ has been written as $g = \delta(q) \cdot t$ with $q \in \mathrm{Spin}_2^r$ and $t \in \tilde{T}_2$. Applying p^r_{GL} and using uniqueness of the QR-decomposition of GL_2^+, we see that $\delta(q) = \tilde{\mathbf{Q}}(g)\delta(-k)$ and $t = \tilde{\mathbf{R}}(g)\delta(k)$ with $k \in \mathbb{Z}_r$. But for $k \neq 0$, $\tilde{\mathbf{R}}(g)\delta(k) \notin \tilde{T}_2$ since $\delta(k) \notin \tilde{T}_2$. □

We will refer to the decomposition $g = \tilde{\mathbf{Q}}(g)\,\tilde{\mathbf{R}}(g)$ of an element g of $\widetilde{GL}_2^r$ as $\widetilde{QR}$-decomposition.

Upper triangular matrices preserve the standard flag in $\mathbb{R}^2$. The QR-decomposition can thus be used to study how a given linear map acts on these subspaces. For example, $g \in GL_2^+$ lies in T_2 if and only if it preserves the subspace $\mathbb{C}e_1$, and in this case $\mathbf{R}(g) = g$ and consequently $\mathbf{Q}(g) = \mathbb{1}$. Later we need to look at rotated bases and thus need a rotated QR-decomposition. For $\alpha \in SO_2$ define the maps

$$\mathbf{Q}_\alpha : GL_2^+ \to SO_2\ , \qquad \tilde{\mathbf{Q}}_\alpha : \widetilde{GL}_2^r \to \delta(\mathrm{Spin}_2^r)\ , \qquad (3.15)$$
$$\mathbf{R}_\alpha : GL_2^+ \to T_2\ , \qquad \tilde{\mathbf{R}}_\alpha : \widetilde{GL}_2^r \to \tilde{T}_2$$

as, for $g \in GL_2^+$ and $\tilde{g} \in \widetilde{GL}_2^r$,

$$\mathbf{Q}_\alpha(g) = \alpha\mathbf{Q}(\alpha^{-1}g\alpha)\alpha^{-1} = \mathbf{Q}(\alpha^{-1}g\alpha)\ , \qquad \mathbf{R}_\alpha(g) = \alpha\mathbf{R}(\alpha^{-1}g\alpha)\alpha^{-1}\ , \qquad (3.16)$$
$$\tilde{\mathbf{Q}}_\alpha(\tilde{g}) = \tilde{\alpha}\tilde{\mathbf{Q}}(\tilde{\alpha}^{-1}\tilde{g}\tilde{\alpha})\tilde{\alpha}^{-1} = \tilde{\mathbf{Q}}(\tilde{\alpha}^{-1}\tilde{g}\tilde{\alpha})\ , \qquad \tilde{\mathbf{R}}_\alpha(\tilde{g}) = \tilde{\alpha}\tilde{\mathbf{R}}(\tilde{\alpha}^{-1}\tilde{g}\tilde{\alpha})\tilde{\alpha}^{-1}\ .$$

Here $\tilde{\alpha} \in \widetilde{GL}_2^r$ is a lift of α, and since elements of $\delta(\mathbb{Z}_r)$ are in the centre of $\widetilde{GL}_2^r$, the definition does not depend on the choice of $\tilde{\alpha}$. Clearly, we still have $g = \mathbf{Q}_\alpha(g)\mathbf{R}_\alpha(g)$ and $\tilde{g} = \tilde{\mathbf{Q}}_\alpha(\tilde{g})\tilde{\mathbf{R}}_\alpha(\tilde{g})$. Furthermore, if g leaves the subspace $\mathbb{C}\alpha e_1$ invariant, then $\mathbf{R}_\alpha(g) = g$ (since $\alpha^{-1}g\alpha$ leaves $\mathbb{C}e_1$ invariant) and hence also $\mathbf{Q}_\alpha(g) = \mathbb{1}$.

Lemma 3.8. 1. Let $g, h \in GL_2^+$ and $\alpha, \beta \in SO_2$. Then

$$\mathbf{Q}_\alpha(\beta g) = \beta\,\mathbf{Q}_\alpha(g), \qquad (3.17)$$
$$\mathbf{Q}_\alpha(g) = \mathbf{Q}_\alpha(g\,\mathbf{Q}_\alpha(h)^{-1}h) = \mathbf{Q}_\alpha(g\,h\mathbf{Q}_\alpha(h^{-1})).$$

2. Let $g, h \in \widetilde{GL}_2^r$ and $\alpha \in SO_2$, $\beta \in \mathrm{Spin}_2^r$. Then

$$\begin{aligned} \tilde{\mathbf{Q}}_\alpha(\delta(\beta)g) = \delta(\beta)\tilde{\mathbf{Q}}_\alpha(g), \\ \tilde{\mathbf{Q}}_\alpha(g) = \tilde{\mathbf{Q}}_\alpha(g\,\tilde{\mathbf{Q}}_\alpha(h)^{-1}h) = \tilde{\mathbf{Q}}_\alpha(g\,h\tilde{\mathbf{Q}}_\alpha(h^{-1}))\ . \end{aligned} \tag{3.18}$$

Proof. It suffices to show part 2. Part 1 then follows by applying p_{GL}^r. Furthermore, the case for general α follows straightforwardly once we verified the claims for $\alpha = 1$. Let thus $g, h \in \widetilde{GL}_2^r$ for $\alpha = 1$.

For the first equality in (3.18), compose $g = \tilde{\mathbf{Q}}(g)\,\tilde{\mathbf{R}}(g)$ with $\delta(\beta)$ to get $\delta(\beta)g = qr$, with $q = \delta(\beta)\tilde{\mathbf{Q}}(g)$ and $r = \tilde{\mathbf{R}}(g)$. From the uniqueness of the $\widetilde{QR}$-decomposition in Lemma 3.7, it follows that $\tilde{\mathbf{Q}}(\delta(\beta)g) = \delta(\beta)\tilde{\mathbf{Q}}(g)$. For the second equality, start with

$$\tilde{\mathbf{R}}(h) = \tilde{\mathbf{Q}}(h)^{-1}h\ . \tag{3.19}$$

Multiplying both sides with g gives $\tilde{\mathbf{Q}}(g)\tilde{\mathbf{R}}(g)\tilde{\mathbf{R}}(h) = g\tilde{\mathbf{Q}}(h)^{-1}h$. Since $\tilde{T}_2$ is a subgroup and by uniqueness of the $\widetilde{QR}$-decomposition, the second equality in (3.18) follows. Now invert Equation (3.19) and replace h with its inverse:

$$\tilde{\mathbf{R}}(h^{-1})^{-1} = h\,\tilde{\mathbf{Q}}(h^{-1})\ . \tag{3.20}$$

Multiplying by g gives $\tilde{\mathbf{Q}}(g)\tilde{\mathbf{R}}(g)\tilde{\mathbf{R}}(h^{-1})^{-1} = gh\tilde{\mathbf{Q}}(h^{-1})$. For the same reason as above, this shows the third equality in (3.18). □

3.4. Example: r-spin structures on $\mathbb{C}^\times$

Since every r-spin structure on $\mathbb{C}^\times$ is in particular an r-fold cover of the oriented frame bundle, $\mathbb{C}^\times \times GL_2^+$, we know that to an r-spin structure corresponds an element of $\mathrm{Hom}\big(\pi_1(\mathbb{C}^\times \times GL_2^+), \mathbb{Z}_r\big) \cong \mathbb{Z}_r \times \mathbb{Z}_r$, describing the lifting properties of curves on $\mathbb{C}^\times \times GL_2^+$. Lifting properties of curves in the second factor (GL_2^+) are fixed since fibrewise the projection from the r-spin bundle is the connected cover by p_{GL}^r. Thus there are at most r different r-spin structures on $\mathbb{C}^\times$ and we proceed to describe these explicitly.

As a first step notice that the maps $p_{SO}^r : \mathrm{Spin}_2^r \to SO_2$ extend to $\mathbb{C}^\times$:

$$\begin{aligned} p_{\mathbb{C}^\times}^r : \mathbb{C}/r\mathbb{Z} &\to \mathbb{C}^\times\ , \\ z &\mapsto e^{2\pi i z}\ . \end{aligned} \tag{3.21}$$

As before p_{SO}^1, $p_{\mathbb{C}^\times}^1$ is an isomorphism.

Let $i : \mathbb{C}^\times \to GL_2^+$ be given by

$$\begin{aligned} i_{\mathbb{C}^\times} : \mathbb{C}^\times &\to GL_2^+ \\ z &\mapsto \begin{pmatrix} \mathrm{Re}\,z & -\mathrm{Im}\,z \\ \mathrm{Im}\,z & \mathrm{Re}\,z \end{pmatrix} \end{aligned} \tag{3.22}$$

As in Equation (3.6) we can lift the inclusion map $i_{\mathbb{C}^\times} : \mathbb{C}^\times \to GL_2^+$ along p^r_{GL} and $p^r_{\mathbb{C}^\times}$ and thus extend the map δ to $\mathbb{C}/r\mathbb{Z}$:

$$\begin{array}{ccc} \mathbb{C}/r\mathbb{Z} & \xrightarrow{p^r_{\mathbb{C}^\times}} & \mathbb{C}^\times \\ \big\downarrow \delta & & \big\downarrow i_{\mathbb{C}^\times} \\ \widetilde{GL}_2^r & \xrightarrow{p^r_{GL}} & GL_2^+ \end{array} . \tag{3.23}$$

In the following will not write out the embedding $i_{\mathbb{C}^\times}$ explicitly. For $\lambda \in \mathbb{Z}$ we define an r-spin surface $\mathbb{C}^\lambda$ as follows: As a $\widetilde{GL}_2^r$-bundle it is given by the trivial principal bundle $P_{\widetilde{GL}}(\mathbb{C}^\lambda) = \mathbb{C}^\times \times \widetilde{GL}_2^r$. The right action of $\widetilde{GL}_2^r$ is given by right multiplication on the second component. The projection to the (trivial) frame bundle on $\mathbb{C}^\times$ is

$$\begin{aligned} p^\lambda : & P_{\widetilde{GL}}(\mathbb{C}^\lambda) \to \mathbb{C}^\times \times GL_2^+ \\ & (z,g) \mapsto (z, z^\lambda p^r_{GL}(g)) \ . \end{aligned} \tag{3.24}$$

The correspondence between oriented frames and elements of GL_2^+ is by taking the two basis vectors as the two column vectors of the 2×2-matrix, cf. (3.4). One quickly checks that $p^\lambda(z,g)p^r_{GL}(h) = p^\lambda(z,gh)$. We have thus defined an r-spin surface $\mathbb{C}^\lambda$ with underlying surface $\underline{\mathbb{C}^\lambda} = \mathbb{C}^\times$ for every $\lambda \in \mathbb{Z}$. In general some of these will be isomorphic; we investigate this with a simple path based argument.

Lemma 3.9. i) For $r \in \mathbb{N} \setminus \{0\}$, the map

$$\begin{aligned} \tilde{q} : & \mathbb{C}^\lambda \to \mathbb{C}^{\lambda - r} \\ & (z,g) \mapsto \left(z, \delta\left(r\left(p^1_{\mathbb{C}^\times}\right)^{-1}(z)\right) g\right) \ . \end{aligned} \tag{3.25}$$

is an isomorphism of r-spin structures over $\mathbb{C}^\times$.
ii) For $r \in \mathbb{N} \setminus \{0\}$, $\mathbb{C}^\lambda$ and $\mathbb{C}^{\lambda'}$ are isomorphic as r-spin structures over $\mathbb{C}^\times$ if and only if $\lambda \equiv \lambda' \mod r$. For $r = \infty$, $\mathbb{C}^\lambda$ and $\mathbb{C}^{\lambda'}$ are isomorphic if and only if $\lambda = \lambda'$.

Proof. We first check that $\tilde{q}$ is indeed an isomorphism of r-spin structures. The underlying map $q : \mathbb{C}^\times \to \mathbb{C}^\times$ is the identity, and $\tilde{q}$ commutes with the right action of $\widetilde{GL}_2^r$. It is well defined since it does not matter which preimage of z we pick. We verify that it is compatible with the projections p^λ and $p^{\lambda-r}$. First note that $p^r_{\mathbb{C}^\times}\left(r\left(p^1_{\mathbb{C}^\times}\right)^{-1}(z)\right) = z^r$. Using this we compute

$$\begin{aligned} p^{\lambda-r}(q(z,g)) &= p^{\lambda-r}(z, \delta\left(r\left(p^1_{\mathbb{C}^\times}\right)^{-1}(z)\right) g) = (z, z^{\lambda-r} p^r_{GL}(\delta\left(r\left(p^1_{\mathbb{C}^\times}\right)^{-1}(z)\right) g)) \\ &= (z, z^{\lambda-r} z^r p^r_{GL}(g)) = (z, z^\lambda p^r_{GL}(g)) = p^\lambda(z,g) \ . \end{aligned} \tag{3.26}$$

As a morphism of principle bundles $\tilde{q}$ is therefore an isomorphism. We now use a simple path-based argument to show that for $\lambda \not\equiv \lambda' \mod r$ the r-spin surfaces are indeed different. In the following we allow $r = \infty$. Let

$$\begin{aligned} \hat{\zeta} : [0,1] &\to \mathbb{C}^\times \times GL_2^+ \ , \\ \hat{\zeta}(t) &= (e^{2\pi i t}, e^{2\pi i t}) \ . \end{aligned} \tag{3.27}$$

The lift to $\mathbb{C}^\lambda$, starting at $(1, \delta(x_0))$ for $x_0 \in \mathbb{Z}_r$, is given by

$$\begin{aligned} \tilde{\zeta} : [0,1] \to \mathbb{C}^\lambda \ , \\ \tilde{\zeta}(t) = (e^{2\pi i t}, \delta(t(1-\lambda) + x_0)) \ . \end{aligned} \tag{3.28}$$

Notably $\tilde{\zeta}(1) = (1, \delta(1 - \lambda + x_0))$. Let now $\tilde{f} : \mathbb{C}^\lambda \to \mathbb{C}^{\lambda'}$ be an isomorphism of r-spin structures. Let y_0 such that $(1, \delta(y_0)) = \tilde{f}(1,1)$. Then $\tilde{f}(\tilde{\zeta}(1)) = \tilde{f}(1, \delta(1-\lambda+x_0)) = \tilde{f}(1,1).\delta(1-\lambda+x_0) = (1, \delta(1-\lambda+x_0+y_0))$. On the other hand we know that $\tilde{f} \circ \tilde{\zeta}$ is a lift of $\hat{\zeta}$ along $p^{\lambda'}$ that starts at $(1, \delta(x_0+y_0))$. Thus $\tilde{f}(\tilde{\zeta}(1)) = (1, \delta(1-\lambda'+x_0+y_0))$. Comparing these we see that $\lambda = \lambda'$ in $\mathbb{Z}_r$. □

We have thus found a representative for every r-spin surface with underlying surface $\mathbb{C}^\times$. The r-spin structure $\mathbb{C}^0$ (and hence also all $\mathbb{C}^{mr}, m \in \mathbb{Z}$) extends to the whole of $\mathbb{C}$ and we will therefore use the notation $\mathbb{C}^0$ for the unpunctured complex plane with the (unique up to isomorphism) r-spin structure $\mathbb{C}^0 = \mathbb{C} \times \widetilde{GL}_2^r$.

3.5. Lifting properties of maps

Lemma 3.10. Let Σ, Σ' be two r-spin surfaces and $f : \underline{\Sigma} \to \underline{\Sigma}'$ a map between the underlying surfaces. Suppose that $\underline{\Sigma}$ is contractible. Then there exist precisely r maps $\tilde{f}_i : \Sigma \to \Sigma'$ of r-spin surfaces with underlying map f; these are related by $\tilde{f}_i = \tilde{f}_j \circ \omega_{i-j}$, where ω_{i-j} is the leaf exchange automorphism corresponding to $i - j \in \mathbb{Z}_r$.

Proof. By Remark 3.6, to give a map $\tilde{f} : \Sigma \to \Sigma'$ is equivalent to giving an isomorphism $P_{\widetilde{GL}}(\Sigma) \to f^* P_{\widetilde{GL}}(\Sigma')$ of r-spin structures. Such an isomorphism exists, since $\underline{\Sigma}$ is contractible and so there is only one isomorphism class of r-spin structures on $\underline{\Sigma}$. Finally, any such lifts are either equal or related by an element in the kernel of p^r since $\underline{\Sigma}$ is in particular connected. □

Lemma 3.11. Let $\tilde{f} : \Sigma \to \Sigma'$ be a morphism of r-spin surfaces with underlying map f. Let $H : [0,1] \times \underline{\Sigma} \to \underline{\Sigma}'$ be a smooth homotopy, i.e. H is continuous and H_t is smooth for all $t \in [0,1]$. Assume $H_0 = f$. Then there is a unique lift $\tilde{H} : [0,1] \times \Sigma \to \Sigma'$ such that $\tilde{H}_0 = \tilde{f}$ and such that $\tilde{H}_t$ is a map of r-spin surfaces for each t.

Proof. Taking derivatives of H at fixed times t, we obtain a lift of H to the bundle of oriented frames. The result then follows from the homotopy lifting property of $p : P_{\widetilde{GL}}(\Sigma') \to F_{GL^+}(\Sigma')$. □

In Section 3.4 we saw that an r-spin structure on $\mathbb{C}^\times$ can be extended to $\mathbb{C}$ iff the path $\hat{\zeta}$ acts on the fibre $\mathbb{Z}_r$ as $x \mapsto x + 1$. We now extend this argument to r-spin structures on arbitrary surfaces. Let $\underline{\Sigma}$ be a surface. We denote by $\pi_T : F_{GL^+}(\underline{\Sigma}) \to T\underline{\Sigma}$ the projection that picks the first vector of a frame. A *(smooth) simple closed curve* is a closed path, that is a smooth embedding when considered as a map of S^1 into the surface. Such a curve $\gamma : [0,1] \to \underline{\Sigma}$ induces a closed curve $d\gamma : [0,1] \to T\underline{\Sigma}$ by taking the derivative. The curve $d\gamma$ always lifts along π_T by completing the frame. (The derivative

$d\gamma$ is non-zero everywhere by definition since γ is an embedding.) Any two such lifts will be homotopic, as they only differ by right multiplication with a curve in T_2.

Lemma 3.12. Let $\underline{\Sigma}$ be a surface and $v \in \underline{\Sigma}$, together with an r-spin structure on $\underline{\Sigma} \setminus \{v\}$. Let $\gamma : [0,1] \to \underline{\Sigma}$ be a contractible smooth simple closed curve encircling v counterclockwise (i.e. it is the boundary of a disk around v). Then for any $\hat{\gamma} : [0,1] \to F_{GL^+}(\underline{\Sigma})$ with $\pi_T(\hat{\gamma}) = d\gamma$ the following are equivalent:

1. The r-spin structure on $\underline{\Sigma} \setminus \{v\}$ extends to $\underline{\Sigma}$.

2. $\hat{\gamma}$ acts on the fibre, $p^{-1}(\hat{\gamma}(0)) \cong \mathbb{Z}_r$ as a shift by $+1$.

The second condition implies that an r-spin lift of $\hat{\gamma}$ is not closed unless $r = 1$; for $r = 2$ the second condition is equivalent to $\hat{\gamma}$ being not closed.

Proof. We find a chart $\psi : \Sigma \to U$, orientation preserving, $U \subset \mathbb{C}$ open, in which $\psi(\gamma)$ bounds the unit disk and $\psi(v) = 0$. By assumption $\psi \circ \gamma$ is isotopic to $\zeta : [0, 2\pi] \to \mathbb{C}$, $t \mapsto e^{it}$. Pulling back the spin structure on Σ along ψ^{-1} we see that it isomorphic to the one induced by $\mathbb{C}^0$ if and only if $\psi^*\gamma$ acts on fibres by a shift of 1. $\square$

3.6. Surfaces with parametrised boundary

We first define a set of collars around $S^1 \subset \mathbb{C}$

$$\begin{aligned} \mathcal{A} &:= \{A_{r,R} \subset \mathbb{C} : A_{r,R} = \{z \in \mathbb{C} : r < |z| < R\}; 0 < r < 1 < R\} \;, \\ \mathcal{A}_{\geq 1} &:= \{A_{r,R} \cap \{z \in \mathbb{C} : |z| \geq 1\} : A_{r,R} \in \mathcal{A}\} \;. \end{aligned} \tag{3.29}$$

Definition 3.13. A *surface with parametrised boundary* is a compact surface Σ together with smooth orientation preserving embeddings $\varphi_i : U_i \to \Sigma$, $i = 1, \ldots, B$, where $U_i \in \mathcal{A}_{\geq 1}$ and B is the number of connected components of the boundary $\partial\Sigma$ of Σ. We require that $\bigcup_{i=1}^{B} \varphi_i(\partial U_i) = \partial\Sigma$ and that the images $\varphi_i(U_i)$, $i = 1, \ldots, B$ are pairwise disjoint. A *diffeomorphism between surfaces with parametrised boundary* is a diffeomorphism between the surfaces compatible with the germs of the boundary embeddings.

Such a parametrisation in particular induces a linear order on the boundary components which will be used later. Unless otherwise indicated, in the following "surface" will stand for "surface with parametrised boundary".

Boundary components of surfaces can be glued using the parametrisation. To do this in a unique way we fix the gluing diffeomorphism

$$s : \mathbb{C}^\times \to \mathbb{C}^\times \qquad , \quad z \mapsto z^{-1} \;. \tag{3.30}$$

Definition 3.14 (Glueing of parameterised surfaces)**.** Let $(\Sigma, (\varphi_i)_i)$ be a surface and (i,j), $i \neq j$ be a pair of boundary components. The glued surface $\Sigma_{i\#j}$ is obtained by identifying points via the diffeomorphism $\varphi_i \circ s \circ \varphi_j^{-1}|_{\partial\Sigma_j} : \partial\Sigma_j \to \partial\Sigma_i$. After possibly restricting the maps φ_i and φ_j to smaller collars, also denoted by U_i and U_j, we get an embedding $\varphi_{i,j} : \mathcal{A} \ni U_i \cup s(U_j) \to \Sigma_{i\#j}$ given by

$$\varphi_{i,j}(z) = \begin{cases} \varphi_i(z), \text{ if } |z| \geq 1, \\ \varphi_j(s^{-1}(z)), \text{ if } |z| < 1. \end{cases} \tag{3.31}$$

The differentiable structure on the glued surface $\Sigma_{i\#j}$ is the one compatible with the differentiable structure on $\Sigma \backslash (\partial\Sigma_i \cup \partial\Sigma_j)$ and the differentiable structure induced by $\varphi_{i,j}$ on its image.

The definition of the glued surface is symmetric, $\Sigma_{i\#j} = \Sigma_{j\#i}$.

3.7. r-spin-surfaces with parametrised boundary

As above we first define sets $\mathcal{A}^\lambda$ of r-spin collars for $\lambda \in \mathbb{Z}_r$,

$$\mathcal{A}^\lambda := \left\{ \mathbb{C}^\lambda|_{\underline{U}} \,\middle|\, \underline{U} \in \mathcal{A} \right\} \quad , \qquad \mathcal{A}^\lambda_{\geq 1} := \left\{ \mathbb{C}^\lambda|_{\underline{U}} \,\middle|\, \underline{U} \in \mathcal{A}_{\geq 1} \right\} . \tag{3.32}$$

Definition 3.15. An *r-spin surface with parametrised boundary* is a compact r-spin surface Σ together with a collection $(\tilde{\varphi}_i)_{i=1,\dots,B}$, of r-spin embeddings

$$\tilde{\varphi}_i : U_i \to \Sigma \tag{3.33}$$

with $U_i \in \bigsqcup_{\lambda \in \mathbb{Z}_r} \mathcal{A}^\lambda_{\geq 1}$ (disjoint union), and such that the tuple $(\underline{\Sigma}, (\varphi_i)_{i=1,\dots,B})$ of underlying surface and parametrisation is a surface (with parametrised boundary). We call a boundary component i of type λ if $U_i \in \mathcal{A}^\lambda$.

r-spin surfaces with parametrised boundary are also treated in [Ra] albeit in a slightly different formalism. A description of the action of Dehn twists on r-spin surfaces is given in [GP].

As for surfaces, in the following we will write "r-spin surface" for "r-spin surface with parametrised boundary" unless stated otherwise.

By taking its derivative the diffeomorphism s from (3.30) induces a map $ds_* : \mathbb{C}^\times \times GL_2^+ \to \mathbb{C}^\times \times GL_2^+$,

$$ds_* : (z, g) \mapsto \left(\frac{1}{z}, -\frac{1}{z^2}\, g \right) . \tag{3.34}$$

We now verify that the map ds_* has r lifts $\tilde{s}^\varepsilon : \mathbb{C}^\lambda \to \mathbb{C}^{\lambda'}$ if $\lambda + \lambda' - 2 = mr$ for some $m \in \mathbb{Z}$ and no lifts otherwise: Let first $m = 0$ and $\varepsilon \in \mathbb{Z}_r$; we specify the lifts of ds_* as

$$\tilde{s}^\varepsilon(z, g) = \left(\frac{1}{z}, \delta\left(\frac{1}{2} + \varepsilon + \lambda - 1 \right) g \right) . \tag{3.35}$$

The remaining lifts (for $m \neq 0$) can then be obtained by composing with $\tilde{q}^m$, where $\tilde{q} : \mathbb{C}^\lambda \to \mathbb{C}^{\lambda+r}$ was introduced in Lemma 3.9.

Definition 3.16. Let $(\Sigma, (\tilde{\varphi}_i)_i)$ be an r-spin surface. We call a triple (i, j, ε), with i, j distinct boundary components and $\varepsilon \in \mathbb{Z}_r$ *spin gluing data*. Here the boundaries i and j have to be of type λ and $2 - \lambda + mr$ respectively (for $\lambda \in \mathbb{Z}_r$). We define the *glued r-spin surface* $\Sigma^{\varepsilon}_{i\#j}$ by identifying points along the boundary via the homeomorphism $\tilde{\varphi}_i \circ \tilde{s}^{\varepsilon} \circ \tilde{q}^m \circ \tilde{\varphi}_j^{-1}|_{\partial\Sigma_j}$, analogous to Definition 3.14, and use the maps $\tilde{\varphi}_i$ and $\tilde{\varphi}_j \circ (\tilde{s}^{\varepsilon} \circ \tilde{q}^m)^{-1}$ to define the differential structure and the r-spin structure. The bundle projection and right action commute with the gluing maps, and thus are defined on $\Sigma^{\varepsilon}_{i\#j}$ in the obvious way.

Since $(\tilde{s}^{\varepsilon})^{-1} = \tilde{s}^{-\varepsilon-1}$, the gluing operation is not symmetric, but instead satisfies

$$\Sigma^{\varepsilon}_{i\#j} = \Sigma^{-\varepsilon-1}_{j\#i} \ . \tag{3.36}$$

4. A combinatorial model for r-spin surfaces

4.1. Smooth triangulations with boundary

Below, we will make use of combinatorial surfaces and smooth triangulations. To fix conventions, in Appendix B we briefly review simplicial complexes, smooth maps from simplices to manifolds, combinatorial surfaces and smooth triangulations. Here let me just mention that a combinatorial surface is a simplicial complex $\mathcal{C}$ such that the polytope $|\mathcal{C}|$ is homeomorphic to a two-manifold, and that a smooth triangulation of a surface Σ is a homeomorphism $|\mathcal{C}| \to \Sigma$ which is a smooth embedding when restricted to a simplex. We take our combinatorial surfaces to be oriented (see Definition B.4) and our orientation convention is given in Figure 4.1.

The standard triangle $\underline{\Delta}$ is the convex hull of the vertices $\{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\} \subset \mathbb{C}$. We consider it as a simplicial complex with the usual simplices.

Definition 4.1. A *combinatorial surface with parametrised boundary* (or *combinatorial surface* for short) is a combinatorial surface $\mathcal{C}$ together with injective simplicial maps $f_i : \partial\underline{\Delta} \to \partial\mathcal{C}$, where i runs from 1 to the number of boundary components, such that the boundary $\partial\mathcal{C}$ is the disjoint union of all $f_i(\partial\underline{\Delta})$. The f_i have to be orientation reversing in the sense that the induced orientation on an edge of $\underline{\Delta}$ is mapped to the opposite orientation of the boundary edge in $\partial\mathcal{C}$ as induced by the adjacent triangle.

Via this definition we impose in particular that each boundary component of $\mathcal{C}$ consists of precisely three edges and three vertices. The orientation convention is such that when using f_i to glue the triangle into $\mathcal{C}$ one obtains an oriented simplicial complex.

Triangulated surfaces can be glued. To formulate the gluing procedure, we need the map $s_{\mathcal{C}} : \partial\underline{\Delta} \to \partial\underline{\Delta}$, $z \mapsto \bar{z}$. It acts on vertices as

$$s_{\mathcal{C}} : \partial\underline{\Delta} \to \partial\underline{\Delta} \quad , \quad 1 \mapsto 1 \ , \quad e^{\frac{2\pi i}{3}} \mapsto e^{\frac{4\pi i}{3}} \ , \quad e^{\frac{4\pi i}{3}} \mapsto e^{\frac{2\pi i}{3}} \ . \tag{4.1}$$

Let $i \neq j$ label two boundary components of a combinatorial surface $\mathcal{C}$. We can glue the surface as an abstract simplicial complex by identifying simplices along the map $f_i \circ s_{\mathcal{C}} \circ f_j^{-1}$. If we obtain a simplicial complex this way, we call (i, j) *simplicial gluing data*. The resulting simplicial complex is denoted as $\mathcal{C}_{i\#j}$ and it is again a combinatorial surface.

A simple example to illustrate that it is necessary to restrict to simplicial gluing data is as follows: take $\mathcal{C}$ to be the disjoint union of two standard triangles $\underline{\Delta}$ with boundaries

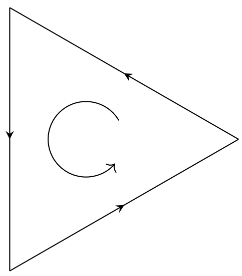


Figure 4.1.: Orientation convention for a triangle in a combinatorial surface. The circular arrow gives the order on the vertices defining the orientation. The arrows on the edges give the edge orientation induced by the orientation of the triangle. When comparing orientation of simplices to orientations of surfaces, our convention is that the above orientation matches that of the paper plane, thought of as $\mathbb{R}^2$ with its standard orientation.

parametrised by $s_{\mathcal{C}}$. The two boundary components cannot be glued since the result would not be a simplicial complex, i.e. $(1,2)$ is not simplicial gluing data.

However, for a given combinatorial surface and arbitrary gluing data (i,j), it is always possible to choose a subdivision of $\mathcal{C}$, fixing the boundary triangulation, such that (i,j) becomes simplicial gluing data.

To triangulate surfaces with parametrised boundary we first define a canonical triangulation of the unit circle S^1,

$$\varphi_S : |\partial\underline{\Delta}| \to S^1 \quad , \quad z \mapsto \frac{z}{|z|} \ . \tag{4.2}$$

Definition 4.2. A *triangulated surface with parametrised boundary* (or *triangulated surface* for short) is a tuple $((\mathcal{C}, f_i), \varphi, (\Sigma, \varphi_i))$, where $(\mathcal{C}, f_i)$ is a combinatorial surface, (Σ, φ_i) is a surface, and $\varphi : \mathcal{C} \to \Sigma$ is a triangulation such that $\varphi \circ f_i = \varphi_i \circ \varphi_S$.

Since $s_{\mathcal{C}}(z) = \bar{z}$, the diagram

$$\begin{array}{ccc} |\partial\underline{\Delta}| & \xrightarrow{\varphi_S} & S^1 \\ \downarrow{\scriptstyle |s_{\mathcal{C}}|} & & \downarrow{\scriptstyle s} \\ |\partial\underline{\Delta}| & \xrightarrow{\varphi_S} & S^1 \end{array} \tag{4.3}$$

commutes. This allows us to make the following

Definition 4.3. Let $\Sigma = ((\mathcal{C}, f_i), \varphi, (\Sigma, \varphi_i))$ be a triangulated surface and (i,j) be simplicial gluing data. The *glued triangulated surface* is

$$\Sigma_{i\#j} := \big((\mathcal{C}_{i\#j}, \hat{f}_k), \varphi_{i\#j}, (\Sigma_{i\#j}, \hat{\varphi}_k)\big) \ , \tag{4.4}$$

with $\hat{f}_k$, $\hat{\varphi}_k$ being the remaining boundary parametrisations and $\varphi_{i\#j}$ the quotient of the original triangulating map φ.

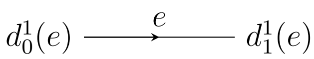


Figure 4.2.: An edge e with orientation from $d_0^1(e)$ to $d_1^1(e)$.

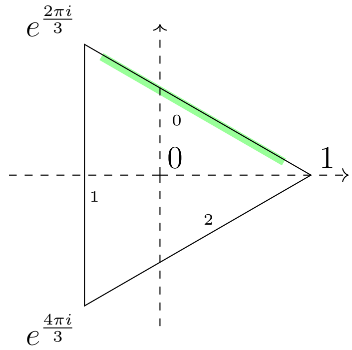


Figure 4.3.: Standard triangle $\underline{\Delta}$; the small numbers 0, 1, 2 indicate the numbering of the edges. The first edge has also been marked with a fat green line.

4.2. Markings on combinatorial surfaces

The combinatorial description of r-spin surfaces requires some extra data. Let $\mathcal{C}$ be a combinatorial surface. The first piece of data is an orientation on the edges of $\mathcal{C}$. We encode this by choosing for each edge $e \in \mathcal{C}_1$ a vertex $d_0^1(e)$ on the boundary of e. This determines a second map $e \mapsto d_1^1(e)$ by picking the other boundary vertex at each edge. We think of an edge as being oriented from $d_0^1(e)$ to $d_1^1(e)$, see Figure 4.2.

The second piece of data is a "starting edge" for each triangle in $\mathcal{C}$, that is, for each $\sigma \in \mathcal{C}_2$ we choose an edge $d_0^2(\sigma)$ of σ. This induces two further maps $\sigma \mapsto d_1^2(\sigma)$ and $\sigma \mapsto d_2^2(\sigma)$ by choosing the next and next-to-next edge counterclockwise. For the standard triangle $\underline{\Delta}$ this is illustrated in Figure 4.3, which also gives our numbering convention for the edges of $\underline{\Delta}$.

To summarise:

Definition 4.4. A *marked combinatorial surface with parametrised boundary* (or *marked combinatorial surface* for short) is a combinatorial surface $\mathcal{C}$ together with maps $d_0^1 : \mathcal{C}_1 \to \mathcal{C}_0$ and $d_0^2 : \mathcal{C}_2 \to \mathcal{C}_1$ such that:

- $d_0^1(e) \in \mathcal{B}(e)$ for all $e \in \mathcal{C}_1$,
- $d_0^2(\sigma) \in \mathcal{B}(\sigma)$ for all $\sigma \in \mathcal{C}_2$.

In addition, for boundary edges $e \in (\partial\mathcal{C})_1$ we require that they are directed in accordance with the boundary orientation as imposed by the parametrisation maps, see Figure 4.4. A *marked triangulated surface (with parametrised boundary)* is a triangulated surface together with a marking on its combinatorial surface.

Note that despite the suggestive notation we do not require any compatibility between the maps d_i^2 and the maps d_i^1 as would be the case in e.g. a Δ-set.

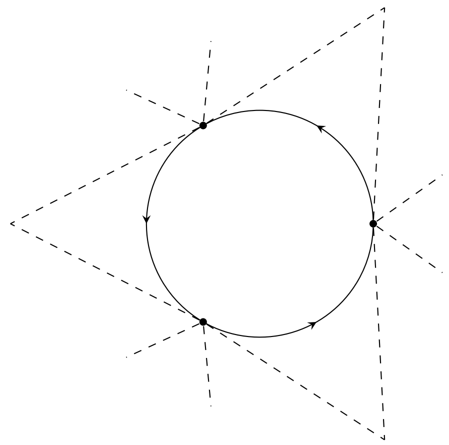


Figure 4.4.: Marking of boundary edges: The marking has to be such that the orientation induced on the edge as in Figure 4.2 agrees with the orientation induced by the parametrisation maps, see Definition 4.1. Equivalently, the orientation of boundary edges induced by the marking is opposite to that induced by the adjacent triangle via Figure 4.1.

Definition 4.5 (Glueing of markings)**.** Let $(\mathcal{C}, f_i)$ be a marked combinatorial surface with parametrised boundary, (i, j) simplicial gluing data and $\mathcal{C}_{i\#j}$ the glued surface. For $\sigma \in \mathcal{C}$ we denote by $[\sigma] \in \mathcal{C}_{i\#j}$ the image of σ under the quotient map. The marking on $\mathcal{C}_{i\#j}$ is defined as follows:

- For $e \in \mathcal{C}_1 \setminus (\mathrm{im}(f_i) \cup \mathrm{im}(f_j))$: $d_0^1([e]) = [d_0^1(e)]$.
- For $e \in (\mathrm{im}(f_j))_1$: $d_0^1([e]) = [d_0^1(e)]$.
- For $e \in (\mathrm{im}(f_i))_1$: $d_0^1([e]) = [d_0^1\left((f_j \circ s_{\mathcal{C}}^{-1} \circ f_i^{-1})(e)\right)]$.
- For $\sigma \in \mathcal{C}_2$: $d_0^2([\sigma]) = [d_0^2(\sigma)]$.

In a marked simplicial surface $\mathcal{C}$, for each face $\sigma \in \mathcal{C}_2$ there is a unique affine linear isomorphism $\check{\chi}_\sigma : \underline{\Delta} \to \sigma$ which maps the marked edge of the standard triangle to $d_0^2(\sigma)$. Consequently, in a marked triangulated surface $(\mathcal{C}, \varphi, \Sigma)$ there is a canonical smooth embedding $\chi_\sigma : \underline{\Delta} \to \Sigma$, $\chi_\sigma := \varphi \circ \check{\chi}_\sigma$ for each $\sigma \in \mathcal{C}_2$.

Recall from Section 3.4 that $\mathbb{C}^0$ is the (unpunctured) complex plane with r-spin structure $\mathbb{C}^0 = \mathbb{C} \times \widetilde{GL}_2$. We define Δ to be the triangle $\underline{\Delta}$ with r-spin structure $\mathbb{C}^0|_{\underline{\Delta}}$. The r-spin structure on $\underline{\Delta}$ is unique up to isomorphism.

Definition 4.6. An *r-spin triangulated surface (with parametrised boundary)* Σ is an r-spin surface Σ and a marked triangulated surface $((\mathcal{C}, f_i), \varphi, (\underline{\Sigma}, \varphi_i))$ together with a choice of r-spin lift $\tilde{\chi}_\sigma : \Delta \to \Sigma$ of the map $\chi_\sigma : \underline{\Delta} \to \underline{\Sigma}$ for every face $\sigma \in \mathcal{C}_1$.

Since simplices are connected and simply connected, an r-spin lift of χ_σ always exists and is uniquely determined by giving its value at one point. For every face of the triangulation there are r possible choices for the spin lift of the characteristic map, one for each element in $\mathbb{Z}_r$.

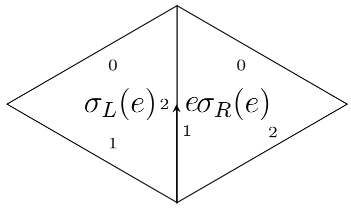


Figure 4.5.: An inner edge with left and right adjacent faces. In the configuration above $k_L = 2$ and $k_R = 1$.

4.3. Edge indices for inner edges

Given an r-spin triangulated surface, our next aim is to give a combinatorial description of the r-spin structure. This will be achieved by assigning indices to the edges of the triangulation. The definition of these indices and the description of their behaviour under changes of the triangulation and under gluing will be the main input into the algebraic treatment of lattice r-spin topological field theory in Section 5.

Let $\mathcal{C}$ be a marked combinatorial surface. Recall from Figure 4.2 that the boundary maps d_0^1, d_1^1 give each edge $e \in \mathcal{C}_1$ a (1-)orientation. For any inner edge e, we denote by $\sigma_L(e)$ the adjacent face that induces this orientation on e, and by $\sigma_R(e)$ the face that induces the opposite orientation, see Figure 4.5. Furthermore let $k_L(e)$ and $k_R(e)$ be such that $d^2_{k_L(e)}(\sigma_L(e)) = e = d^2_{k_R(e)}(\sigma_R(e))$. We say e is the $k_L(e)$'th edge of $\sigma_L(e)$ and the $k_R(e)$'th edge of $\sigma_R(e)$. If the edge e is clear from the context we will often drop the argument in $\sigma_{L/R}$ and $k_{L/R}$.

Now fix an edge e in a marked triangulated surface $(\mathcal{C}, \varphi, \Sigma)$. Let p be a point on the $k_R(e)$'th edge of Δ. Then the derivative $d(\chi_{\sigma_L}^{-1} \circ \chi_{\sigma_R})_p \in GL_2^+$ rotates a tangent vector in the direction of the edge by $e^{2\pi i(k_L/3 - k_R/3 + 1/2)}$. This can be written in terms of the QR-decomposition of GL_2^+ as

$$\mathbf{Q}_\alpha(d(\chi_{\sigma_L}^{-1} \circ \chi_{\sigma_R})_p) = e^{2\pi i(k_L/3 - k_R/3 + 1/2)} \quad , \quad \text{where} \quad \alpha = e^{2\pi i\left(\frac{k_R}{3} + \frac{5}{12}\right)} . \tag{4.5}$$

The constant $\frac{2\pi 5}{12} = 150°$ is the angle the edge labeled 0 in the standard triangle forms with the real axis (Figure 4.3). Thus α is the angle between the edge labeled k_R and the real axis. To avoid having to write out the uninteresting constant angle, we abbreviate

$$\alpha_0 := e^{2\pi i \frac{5}{12}} , \tag{4.6}$$

such that $\alpha = e^{\frac{2\pi i k_R}{3}} \alpha_0$.

For an r-spin map $\tilde{f} : \mathbb{C}^0 \to \mathbb{C}^0$ and a point $p \in \mathbb{C}$ we denote by $g_p(\tilde{f}) \in \widetilde{GL}_2$ the element such that

$$\tilde{f}(p, g) = (f(p), g_p(\tilde{f}) \cdot g) . \tag{4.7}$$

Definition 4.7 (Edge indices for inner edges)**.** Let e be an inner edge of an r-spin triangulated surface and let p be a point on the $k_R(e)$'th edge of Δ. The *edge index* $s(e) \in \mathbb{Z}_r$ for the edge e is defined via

$$\delta(s(e)) = \tilde{\mathbf{Q}}_\alpha\left(g_p(\tilde{\chi}_{\sigma_L}^{-1} \circ \tilde{\chi}_{\sigma_R})\right) \cdot \delta\left(-(k_L/3 - k_R/3 + 1/2)\right) , \tag{4.8}$$

where $\alpha = e^{\frac{2\pi i k_R}{3}}\alpha_0$.

By continuity of $\tilde{\mathbf{Q}}_\alpha$, $s(e)$ does not depend on the choice of the point p so that it makes sense not to include p in the notation. Because $\tilde{\mathbf{Q}}(\dots)$ and $\delta(\dots)$ commute in $\widetilde{GL}_2$, and by employing Lemma 3.8, we may also write

$$\delta(s(e)) = \tilde{\mathbf{Q}}_\alpha \left(\delta\left(-(k_L/3 - k_R/3 + 1/2)\right)\, g_p(\tilde{\chi}_{\sigma_L}^{-1} \circ \tilde{\chi}_{\sigma_R})\right) \ . \tag{4.9}$$

4.4. Edge indices for boundary edges

Let $\mathcal{C}$ be a marked combinatorial surface. Let $e \in \mathcal{C}_1$ be a boundary edge. Therefore it will have only a single adjacent face, which – by convention – is on the right side and denoted as $\sigma_R(e)$ as in Section 4.3. The edge e is then the $k_R(e)$'th edge of $\sigma_R(e)$. We define the index $k_L(e)$ by the condition that under the boundary parametrisation $f_i : \partial\Delta \to \mathcal{C}$, e is the k_L'th edge of Δ.

The point $z = \chi_{\sigma_R}^{-1} \circ \varphi_i(e^{2\pi i(\frac{k_L}{3}+\frac{1}{6})})$ lies on the $k_R(e)$'th edge of Δ. The derivative $d(\varphi_i^{-1} \circ \chi_{\sigma_R})_z \in GL_2^+$ rotates a tangent vector in the direction of the edge by an angle $e^{2\pi i\left(\frac{k_L}{3} - \frac{k_R}{3} + \frac{1}{2}\right)}$. In terms of the QR-decomposition this can be written as

$$\mathbf{Q}_\alpha(d(\varphi_i^{-1} \circ \chi_{\sigma_R})_z) = e^{2\pi i\left(\frac{k_L}{3} - \frac{k_R}{3} + \frac{1}{2}\right)} \ , \tag{4.10}$$

where $\alpha = e^{\frac{2\pi i k_R}{3}}\alpha_0$. We can now state the definition of edge indices for boundary edges. The definition needs some justification which will be provided in the lemma following the definition.

Definition 4.8 (Edge indices for boundary edges)**.** Let e be a boundary edge of an r-spin triangulated surface, and let z and α be as above. For a boundary of type λ, the *edge index* $s(e)$ is defined via

$$\begin{aligned}\delta(s(e)) &= \tilde{\mathbf{Q}}_\alpha\big(g_z(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_R})\big) \cdot \delta\left(\frac{k_L(\lambda-1)+k_R}{3} - \frac{1}{2} + \frac{\lambda}{6}\right) \\ &\overset{\text{Lem. 3.8}}{=} \tilde{\mathbf{Q}}_\alpha\Big(\delta\left(\frac{k_L(\lambda-1)+k_R}{3} - \frac{1}{2} + \frac{\lambda}{6}\right) \cdot g_z(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_R})\Big) \ .\end{aligned} \tag{4.11}$$

Lemma 4.9. The right hand side of (4.11) defines an element in $\delta(\mathbb{Z}_r) \subset \widetilde{GL}_2$.

Proof. Pick a small neighbourhood U of z and an appropriate neighbourhood V of $e^{2\pi i(\frac{k_L}{3}+\frac{1}{6})}$. Then the following diagram commutes:

$$\begin{array}{ccc} U \times \widetilde{GL}_2 & \xrightarrow{\tilde{\varphi}_i^{-1}\circ\tilde{\chi}_{\sigma_R}} & V \times \widetilde{GL}_2 \\ \big\downarrow{\scriptstyle p^0} & & \big\downarrow{\scriptstyle p^\lambda} \\ U \times GL_2^+ & \xrightarrow{d(\varphi_i^{-1}\circ\chi_{\sigma_R})} & V \times GL_2^+ \end{array} \ . \tag{4.12}$$

Applying this to (z, e) and inserting p_0, p_λ from (3.24) gives

$$e^{2\pi i\lambda\left(\frac{k_L}{3}+\frac{1}{6}\right)} p_{GL}\big(g_z(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_R})\big) = d(\varphi_i^{-1} \circ \chi_{\sigma_R})_z \ . \tag{4.13}$$

If one applies $\mathbf{Q}_\alpha$ to both sides and uses Lemma 3.8 one arrives at

$$e^{2\pi i\lambda\left(\frac{k_L}{3}+\frac{1}{6}\right)} \mathbf{Q}_\alpha\big(p_{GL}\big(g_z(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_R})\big)\big) = e^{2\pi i\left(\frac{k_L}{3}-\frac{k_R}{3}+\frac{1}{2}\right)} \ . \tag{4.14}$$

This is equivalent to $p_{GL}\Big(\delta\left(\frac{k_L(\lambda-1)+k_R}{3} - \frac{1}{2} + \frac{\lambda}{6}\right) \tilde{\mathbf{Q}}_\alpha\left(g_z\left(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_R}\right)\right)\Big) = 1$. □

We have defined all geometric and combinatorial ingredients needed for our combinatorial model of r-spin structures:

- *surface* (Σ, φ_i) – Definition 3.13,
  Σ – compact surface; φ_i – boundary parametrisation.

- *marked triangulated surface* $((\mathcal{C}, f_i, d_0^1, d_0^2), \varphi, (\Sigma, \varphi_i))$ – Definition 4.4,
  $\mathcal{C}$ – combinatorial surface; f_i – boundary parametrisation of $\mathcal{C}$; d_0^1, d_0^2 – marking; $\varphi : |\mathcal{C}| \to \Sigma$ – smooth triangulation; (Σ, φ_i) – surface.

- *r-spin surface* $(\Sigma, \tilde{\varphi}_i)$ – Definition 3.15,
  Σ: compact r-spin surface; $\tilde{\varphi}_i$: boundary parametrisation via r-spin maps.

- *r-spin triangulated surface* $((\mathcal{C}, f_i, d_0^1, d_0^2), (\varphi, \tilde{\chi}_\sigma), (\Sigma, \tilde{\varphi}_i))$ – Definition 4.6,
  $(\Sigma, \tilde{\varphi}_i)$ – r-spin surface; $((\mathcal{C}, f_i, d_0^1, d_0^2), \varphi, (\underline{\Sigma}, \varphi_i))$ – marked triangulated surface; $\tilde{\chi}_\sigma$ – r-spin lift of $\chi_\sigma : \Delta \to \underline{\Sigma}$.

- *edge indices* $s(e)$ – Definitions 4.7 and 4.11.

4.5. Behaviour of edge indices under gluing

Lemma 4.10. Let Σ be an r-spin triangulated surface and (i, j, ε) be r-spin gluing data such that (i, j) is simplicial gluing data. Let λ_i, λ_j be the types of the i'th and j'th boundary respectively. Let e_i, e_j be edges on the i'th and j'th boundary respectively which are to be glued together, i.e. $e_i = \left(f_i \circ s_{\mathcal{C}} \circ f_j^{-1}\right)(e_j)$. Let $e = [e_i] = [e_j]$ be the glued edge. Then $s(e) = s(e_j) - s(e_i) + \varepsilon$.

Proof. We introduce the following abbreviations: $\sigma_i = \sigma_R(e_i)$ and $\sigma_j = \sigma_R(e_j)$, as well as

- $k_L = k_L(e) = k_R(e_i)$,

- $k_i = k_L(e_i)$,

- $k_R = k_R(e) = k_R(e_j)$,

- $k_j = k_L(e_j)$.

We first describe $\tilde{q}:\mathbb{C}^\lambda \to \mathbb{C}^{\lambda-r}$ from Lemma 3.9 explicitly for $z \in S^1$. Let $\alpha \in \mathbb{R}$ and $z = e^{2\pi i\alpha}$. Then $\tilde{q}(z,g) = (z, \delta(r\alpha)g)$. Recall that $m \in \mathbb{Z}$ is defined by the equation $\lambda_i + \lambda_j - 2 = mr$. Let now z be a point on the k_R'th edge of Δ and

$$z_1 = (\varphi_j^{-1} \circ \chi_{\sigma_j})(z) \quad , \qquad z_2 = (s \circ \varphi_j^{-1} \circ \chi_{\sigma_j})(z) \ . \tag{4.15}$$

We have

$$\begin{aligned} g_z\Big(\tilde{\chi}^{-1}_{[\sigma_i]} \circ \tilde{\chi}_{[\sigma_j]}\Big) &= g_z\big(\tilde{\chi}^{-1}_{\sigma_i} \circ \tilde{\varphi}_i \circ \tilde{s}^\varepsilon \circ \tilde{\varphi}_j^{-1} \circ \tilde{\chi}_{\sigma_j}\big) \\ &= g_{z_2}\big(\tilde{\chi}^{-1}_{\sigma_i} \circ \tilde{\varphi}_i\big)\ g_{z_1}(\tilde{s}^\varepsilon \circ \tilde{q}^m)\ g_z\big(\tilde{\varphi}_j^{-1} \circ \tilde{\chi}_{\sigma_j}\big) \ . \end{aligned} \tag{4.16}$$

We pick z such that $z_1 = e^{2\pi i\left(\frac{k_j}{3}+\frac{1}{6}\right)}$, so that in particular

$$g_{z_1}(\tilde{s}^\varepsilon \circ \tilde{q}^m) = \delta\left(\frac{1}{2} + \varepsilon + \lambda_j - 1 + mr\left(\frac{k_j}{3} + \frac{1}{6}\right)\right) \ . \tag{4.17}$$

Let $\alpha = e^{\frac{2\pi i k_R}{3}}\alpha_0$.

$$\begin{aligned} \delta(s(e)) &\overset{(4.9)}{=} \tilde{\mathbf{Q}}_\alpha\left(\delta\left(-\left(\frac{k_L}{3} - \frac{k_R}{3} + \frac{1}{2}\right)\right) g_z\left(\tilde{\chi}^{-1}_{[\sigma_i]} \circ \tilde{\chi}_{[\sigma_j]}\right)\right) \\ &\overset{(4.16)}{=} \tilde{\mathbf{Q}}_\alpha\left(\delta\left(-\left(\frac{k_L}{3} - \frac{k_R}{3} + \frac{1}{2}\right)\right) g_{z_2}\big(\tilde{\chi}^{-1}_{\sigma_i} \circ \tilde{\varphi}_i\big)\, g_{z_1}(\tilde{s}^\varepsilon \circ \tilde{q}^m)\, g_z\big(\tilde{\varphi}_j^{-1} \circ \tilde{\chi}_{\sigma_j}\big)\right) \\ &= \delta(s(e_j))\,\tilde{\mathbf{Q}}_\alpha\Big(\delta\left(-\left(\frac{k_L}{3} - \frac{k_R}{3} + \frac{1}{2}\right)\right)\ g_{z_2}\big(\tilde{\chi}^{-1}_{\sigma_i} \circ \tilde{\varphi}_i\big)\ g_{z_1}\left(\tilde{s}^\varepsilon \circ \tilde{q}^m\right) \\ &\qquad \delta\left(-s(e_j)\right)\ g_z\big(\tilde{\varphi}_j^{-1} \circ \tilde{\chi}_{\sigma_j}\big)\Big) \ . \end{aligned} \tag{4.18}$$

In the last step we used the first equality in (3.18) and that $\delta(s(e_j))$ is central in $\widetilde{GL}_2^r$, see (3.8). Now insert the identity

$$\delta(s(e_j))^{-1} \overset{(4.11)}{=} \left(\tilde{\mathbf{Q}}_\alpha\left(g_z(\tilde{\varphi}_j^{-1} \circ \tilde{\chi}_{\sigma_j})\right)\delta\left(\frac{k_j(\lambda_j - 1) + k_R}{3} - \frac{1}{2} + \frac{\lambda_j}{6}\right)\right)^{-1} \tag{4.19}$$

at the second occurrence of $s(e_j)$. One can apply Lemma 3.8 to remove the combination $\tilde{\mathbf{Q}}_\alpha\big(g_z(-)\big)^{-1} g_z(-)$. This leads to

$$\begin{aligned} &\text{Eqn. (4.18)} \\ &= \delta(s(e_j))\tilde{\mathbf{Q}}_\alpha\Big(\delta\left(-(\frac{k_L}{3} - \frac{k_R}{3} + \frac{1}{2})\right) g_{z_2}\big(\tilde{\chi}^{-1}_{\sigma_i} \circ \tilde{\varphi}_i\big)\, g_{z_1}(\tilde{s}^\varepsilon \circ \tilde{q}^m) \\ &\qquad \delta\left(\frac{k_j(1-\lambda_j) - k_R}{3} + \frac{1}{2} - \frac{\lambda_j}{6}\right)\Big) \\ &\overset{(4.17)}{=} \delta(s(e_j))\tilde{\mathbf{Q}}_\alpha\Big(\delta\left(-(\frac{k_L}{3} - \frac{k_R}{3} + \frac{1}{2})\right) g_{z_2}\big(\tilde{\chi}^{-1}_{\sigma_i} \circ \tilde{\varphi}_i\big) \end{aligned}$$

$$\delta\left(\frac{1}{2}+\varepsilon+\lambda_j-1+mr\left(\frac{k_j}{3}+\frac{1}{6}\right)+\frac{k_j(1-\lambda_j)-k_R}{3}+\frac{1}{2}-\frac{\lambda_j}{6}\right)\Big)$$
$$\overset{\alpha'=\alpha_0 e^{2\pi i k_L/3}}{=} \delta(s(e_j))\tilde{\mathbf{Q}}_{\alpha'}\Big(\delta\left(-\frac{1}{2}\right)g_{z_2}\big(\tilde{\chi}_{\sigma_i}^{-1}\circ\tilde{\varphi}_i\big)$$
$$\delta\Big(1+\varepsilon+\lambda_j-1+\frac{k_j(mr-\lambda_j+1)-k_L}{3}+\frac{mr-\lambda_j}{6}\Big)\Big)$$
$$=\delta(s(e_j)-s(e_i)+\varepsilon+\lambda_j-1)\tilde{\mathbf{Q}}_{\alpha'}\Big(g_{z_2}\big(\tilde{\chi}_{\sigma_i}^{-1}\circ\tilde{\varphi}_i\big)\,\tilde{\mathbf{Q}}_{\alpha'}\,\big(g_z(\tilde{\varphi}_i^{-1}\circ\chi_{\sigma_i})\big)$$
$$\delta\left(\frac{k_j(mr-\lambda_j+1)+k_i(\lambda_i-1)}{3}+\frac{mr-\lambda_j+\lambda_i}{6}\right)\Big)$$
$$\overset{k_i+k_j=2}{=} \delta(s(e_j)-s(e_i)+\varepsilon)\ . \tag{4.20}$$

In the second to last step a replacement analogous to (4.19) was used for $s(e_i)$. □

In the notation of Lemma 4.10: Recall from Definition 4.5 that the edge e carries the orientation induced by e_j, which is opposite to the one induced by e_i. The rule $s(e)=s(e_j)-s(e_i)+\varepsilon$ is compatible with the observation $\Sigma^{\varepsilon}_{i\#j}=\Sigma^{-\varepsilon-1}_{j\#i}$ made for glued r-spin surfaces below Definition 3.16. To see this note that changing the order of (i,j) changes the direction of the edge e and thus the induced r-spin index. Changing the direction of an edge is one of the local moves discussed in the next section.

4.6. Moves leaving the triangulation invariant

In this section and Section 4.9 below we investigate how the edge indices behave under change of marking and triangulation. For the calculations below we fix a lift $\tilde{\alpha}_0$ of the angle α_0 defined in (4.6). We choose

$$\tilde{\alpha}_0 := \frac{5}{12}\ , \tag{4.21}$$

which satisfies $p_{GL}(\delta(\tilde{\alpha}_0))=\alpha_0$, as required. We will also denote with $[n]_3\in\{0,1,2\}$ the representative of n mod 3.

The following lemma details how the edge indices behave under change of marking, see Figure 4.6 for an illustration.

Lemma 4.11. Let $\Sigma=((\mathcal{C},f_i),(\varphi,\tilde{\chi}_\sigma),(\Sigma,\tilde{\varphi}_i))$ be an r-spin triangulated surface and let $s(e)$ be the corresponding edge indices.

1. Let σ be a face of $\mathcal{C}$. Change $\tilde{\chi}_\sigma$ by precomposing with the leaf exchange automorphism, i.e.
$$\tilde{\chi}_{\text{new},\sigma}=\tilde{\chi}_{\text{old},\sigma}\circ\omega_k\ .$$
Pick an edge $e\in\mathcal{B}(\sigma)$. Then $s(e)\mapsto s(e)+k$ if $\sigma=\sigma_R(e)$ and $s(e)\mapsto s(e)-k$ if $\sigma=\sigma_L(e)$.

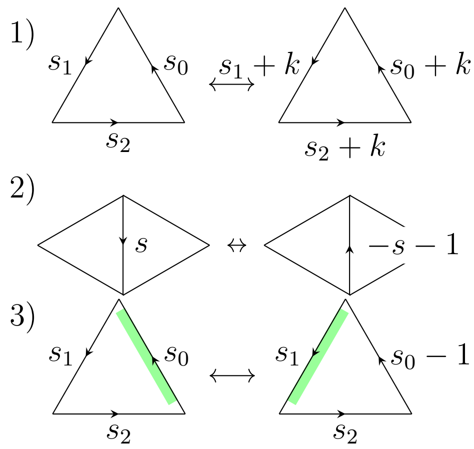


Figure 4.6.: The moves 1–3 of Lemma 4.11. Where the marking is not explicitly specified it is arbitrary but fixed.

2. Let e be an inner edge of $\mathcal{C}$. Exchange the marking $d_0^1(e)$, i.e.

$$d^1_{\text{new},0}(e) = d^1_{\text{old},1}(e) \ .$$

Then $s(e) \mapsto -s(e) - 1$.

3. Let σ be a face of $\mathcal{C}$. Change the marking $d_0^2(\sigma)$ by picking the next edge counter-clockwise, i.e.

$$d^2_{\text{new},0}(\sigma) = d^2_{\text{old},1}(\sigma) \ .$$

Then there is a choice for the $\tilde{\chi}_{\text{new},\sigma}$ such that only the edge indices on the previously marked edge of σ changes. In more detail, the only change is

$$s_{\text{new}}(d^2_{\text{new},2}(\sigma)) = s_{\text{new}}(d^2_{\text{old},0}(\sigma)) = s_{\text{old}}(d^2_{\text{old},0}(\sigma)) - 1$$

if σ is to the left of $d^2_{\text{new},2}(\sigma)$ and

$$s_{\text{new}}(d^2_{\text{new},2}(\sigma)) = s_{\text{new}}(d^2_{\text{old},0}(\sigma)) = s_{\text{old}}(d^2_{\text{old},0}(\sigma)) + 1$$

otherwise. (Here, s_{new} is evaluated with respect to the lifts $\tilde{\chi}_{\text{new},\sigma}$.)

Proof. For a given edge e we will abbreviate $k_L = k_L(e)$, $\sigma_L = \sigma_L(e)$, etc. Primed quantities indicate the new choice of data / the resulting edge indices.

(1) If e is inner, then composition with the leaf exchange automorphism just multiplies $g_z(\tilde{\chi}^{-1}_{\sigma_L} \circ \tilde{\chi}_{\sigma_R})$ with $\delta(k)$ or resp. $\delta(k)^{-1} = \delta(-k)$, which can be pulled out of the QR-decomposition since it is central. If e is a boundary edge, then the same reasoning applies for $g_z\big(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_R}\big)$.

(2) Writing out $\tilde{\mathbf{Q}}_\alpha$ in formula (4.9) in terms of (3.16), we get

$$\delta(s(e)) = \tilde{\mathbf{Q}} \left(\delta\big(-(\frac{k_L}{3} + \frac{1}{2} + \tilde{\alpha}_0)\big) g_p\big(\tilde{\chi}^{-1}_{\sigma_L} \circ \tilde{\chi}_{\sigma_R}\big)\, \delta\big(\frac{k_R}{3} + \tilde{\alpha}_0\big) \right) . \tag{4.22}$$

Note that since the possible lifts of α_0 differ by $\delta(k)$ with $k \in \mathbb{Z}_r$, which is central in $\widetilde{GL}$, the value $\delta(s(e))$ is actually independent of the choice of $\tilde{\alpha}_0$ we made in (4.21). The same applies to the calculations below.

After changing the direction of e the new edge index is given by exchanging $L \leftrightarrow R$:

$$\delta(s'(e)) = \tilde{\mathbf{Q}}\left(\delta\big(-(\frac{k_R}{3}+\frac{1}{2}+\tilde{\alpha}_0)\big)g_q\big(\tilde{\chi}_{\sigma_R}^{-1}\circ\tilde{\chi}_{\sigma_L}\big)\,\delta\big(\frac{k_L}{3}+\tilde{\alpha}_0\big)\right)\ , \tag{4.23}$$

where $q = \chi_{\sigma_L}^{-1} \circ \chi_{\sigma_R}(p)$. The sum $s(e) + s'(e)$ is then determined by

$$\begin{aligned}\delta(s(e)+s'(e)) &= \tilde{\mathbf{Q}}\left(\delta\big(s'(e)-(\frac{k_L}{3}+\frac{1}{2}+\tilde{\alpha}_0)\big)g_p\big(\tilde{\chi}_{\sigma_L}^{-1}\circ\tilde{\chi}_{\sigma_R}\big)\,\delta\big(\frac{k_R}{3}+\tilde{\alpha}_0\big)\right)\\ &= \tilde{\mathbf{Q}}\left(\delta\left(-1+s'(e)\right)\left(\delta\big(-(\frac{k_R}{3}+\frac{1}{2}+\tilde{\alpha}_0)\big)g_q\big(\tilde{\chi}_{\sigma_R}^{-1}\circ\tilde{\chi}_{\sigma_L}\big)\,\delta\big(\frac{k_L}{3}+\tilde{\alpha}_0\big)\right)^{-1}\right)\end{aligned} \tag{4.24}$$

where we used that $g_p = g_q^{-1}$ and that $\delta(1/2)$ is central in $\widetilde{GL}_2$, see (3.8). Substituting (4.23) for $s'(e)$ in the above expression gives

$$\delta(s(e)+s'(e)) = \tilde{\mathbf{Q}}\left(\delta\big(-1\big)\right) = \delta(-1)\ . \tag{4.25}$$

(3) Rotating the marked edge of σ counterclockwise means replacing χ_σ with $\chi'_\sigma := \chi_\sigma \circ (z \mapsto e^{2\pi i/3}z)$. We then choose $\tilde{\chi}'_\sigma := \tilde{\chi}_\sigma \circ ((z,g) \mapsto (e^{2\pi i/3}z, \delta(1/3)g))$. Let e be the k-th edge of σ before the change of marking. It becomes the k'-th edge of σ, with $k' = [k-1]_3$. We will first treat the case that e is an inner edge and that σ is to the right of e. Let $\alpha = e^{2\pi i\frac{k}{3}}\alpha_0$ and $\alpha' = e^{2\pi i\frac{k'}{3}}\alpha_0$. Then

$$\begin{aligned}\delta\big(s'(e)\big) &= \tilde{\mathbf{Q}}_{\alpha'}\left(\delta\Big(-\Big(\frac{k_L}{3}-\frac{k'}{3}+\frac{1}{2}\Big)\Big)g(\tilde{\chi}_{\sigma_L(e)}^{-1}\circ\tilde{\chi}'_\sigma)\right)\\ &= \tilde{\mathbf{Q}}_{\alpha}\left(\delta\Big(-\Big(\frac{k_L}{3}-\frac{k'}{3}-\frac{1}{3}+\frac{1}{2}\Big)\Big)g(\tilde{\chi}_{\sigma_L(e)}^{-1}\circ\tilde{\chi}'_\sigma)\delta\Big(-\frac{1}{3}\Big)\right)\\ &= \delta\Big(\frac{k'-k+1}{3}\Big)\tilde{\mathbf{Q}}_{\alpha}\left(\delta\Big(-\Big(\frac{k_L}{3}-\frac{k}{3}+\frac{1}{2}\Big)\Big)g(\tilde{\chi}_{\sigma_L(e)}^{-1}\circ\tilde{\chi}_\sigma)\right)\\ &= \delta\Big(\frac{k'-k+1}{3}\Big)\delta\big(s(e)\big)\ .\end{aligned} \tag{4.26}$$

If $k = 0$ then $k' - k + 1 = 3$ and thus $s'(e) = s(e) + 1$. Otherwise $k' - k + 1 = 0$ and $s'(e) = s(e)$. If σ is to the left of e we use move (2) to reverse the direction of e, apply the above argument and reverse it again to obtain $s'(e) = s(e) - 1$ for $k = 0$.

If e is a boundary edge, then in the above calculation replace $\tilde{\chi}_{\sigma_L(e)}$ with $\tilde{\varphi}_i$ and use the phase as stated in (4.11). □

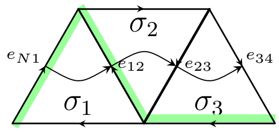


Figure 4.7.: A part of a configuration of N adjacent triangles. The first three triangles are drawn. For that configuration we have $k_1 = 0$, $k_2 = 0$ and $k_3 = 2$, as well as $\mu_{N1} = -1$, $\mu_{12} = -1$, $\mu_{23} = 1$, $\mu_{34} = 1$ and $\eta_1 = -1$, $\eta_2 = 1$, $\eta_3 = -1$.

4.7. Lifting properties of simple closed curves

In Section 3.5 we examined the lifting properties of smooth simple closed curves to the r-spin bundle. We will now determine how the lifting behaviour depends on the edge indices.

To treat inner and boundary edges on the same footing, we use the boundary parametrisation $\varphi_i : U_i \to \Sigma$ to enlarge the surface Σ to a new surface Σ^+ obtained by gluing on little collars: pick any $0 < r < 1$, define the open sets U_i^+ as $U_i \cup \{ z \in \mathbb{C} \,|\, r<|z|\leq 1 \}$ and set

$$\Sigma^+ = \Sigma \sqcup U_1^+ \sqcup \cdots \sqcup U_B^+ / \sim \, , \tag{4.27}$$

where $\sim$ identifies $\varphi_i(z) \in \Sigma$ with $z \in U_i$. It is easy to see that the following are equivalent: a) the structure of an r-spin surface (with parametrised boundary) on Σ, and b) an r-spin structure on Σ^+ which on U_i^+ is equal to $\mathbb{C}^\lambda|_{U_i^+}$, depending on the type of the i'th boundary component of Σ. We will use description b).

We will have need for r-spin structures on a triangulated surface minus its vertices.

Definition 4.12. Let $(\mathcal{C}, \Sigma, \varphi)$ be a triangulated surface. A *punctured r-spin structure* on $(\mathcal{C}, \Sigma, \varphi)$ is an r-spin structure on $\Sigma^+ \setminus \varphi(\mathcal{C}_0)$ which on the glued-on collars

$$U_i^+ \cap \{z \in \mathbb{C} | |z| \leq 1\} \setminus \{1, e^{2\pi i/3}, e^{4\pi i/3}\} \tag{4.28}$$

is equal to the restriction of $\mathbb{C}^\lambda$. A *punctured r-spin triangulated surface* is a marked triangulated surface together with a punctured r-spin structure and a choice of r-spin lift $\tilde{\chi}_\sigma : \Delta \setminus \Delta_0 \to \Sigma$ for each triangle $\sigma \in \mathcal{C}_2$. The boundary parametrisation maps $\varphi_i : U_i \to \Sigma$ extend naturally to embeddings $U_i^+ \to \Sigma^+$ and for a punctured r-spin triangulated surface we get an r-spin lift for $\varphi_i|_{\{|z|\leq 1\}}$. By slight abuse of notation we call this r-spin lift $\tilde{\varphi}_i$.

We will see in Section 4.8 how to construct a punctured r-spin structure on a marked triangulated surface with an arbitrary assignment of edge indices. On the other hand one can define edge indices for punctured r-spin triangulated surfaces in the same way as for r-spin triangulated surfaces since Definitions 4.7 and 4.8 do not rely on the extendibility of the r-spin structure. In the following let $(\mathcal{C}, \tilde{\varphi}, \Sigma)$ be a punctured r-spin triangulated surface.

4.7.1. Paths transversing inner edges

To describe the paths we want to lift to the r-spin bundle explicitly, we need a little preliminary setup. Let $(\sigma_i)_{i\in\mathbb{Z}_N}$, $\sigma_i \in \mathcal{C}_2$ be a sequence of N distinct triangles such that each two consecutive triangles σ_i, σ_{i+1} intersect in an edge $e_{i,i+1}$. We describe the marking on (σ_i) and $(e_{i,i+1})$ explicitly:

- $k_i \in \{0,1,2\}$ is the position of the edge $e_{i-1,i}$ in σ_i, $d_2^{k_i}(\sigma_i) = e_{i-1,i}$. In other words, the path enters the triangle σ_i through the edge $d_2^{k_i}(\sigma_i)$.
- $\eta_i = \pm 1$ describes the position of the edge $e_{i,i+1}$ relative to the edge $e_{i-1,i}$ in σ_i, $d_2^{[k_i+\eta_i]_3}(\sigma_i) = e_{i,i+1}$. In other words, the path exits through the edge $d_2^{[k_i+\eta_i]_3}(\sigma_i)$.
- $\mu_{i,i+1} = \pm 1$ describes the direction of the edge $e_{i,i+1}$: $\mu_{i,i+1} = 1$ if σ_i is to the right of $e_{i,i+1}$ and $\mu_{i,i+1} = -1$ otherwise.

An example of such a configuration is shown in Figure 4.7. We will in the following abbreviate

$$s_{i,i+1} = s(e_{i,i+1}) \ . \tag{4.29}$$

Using Lemma 4.11(2) we change the edge orientations, such that the new $\mu_{i,i+1} = 1$ for all i. The new edge indices are then

$$\hat{s}_{i,i+1} = \begin{cases} s_{i,i+1} \text{ if } \mu_{i,i+1} = 1, \\ -s_{i,i+1} - 1 \text{ if } \mu_{i,i+1} = -1. \end{cases} \tag{4.30}$$

.

Lemma 4.13. Given a configuration as above let $\gamma : [0,1] \to \underline{\Sigma}$ be a smooth simple closed curve that can be written as the composition [1] of paths $\gamma_0, \ldots, \gamma_{N-1}$,

$$\gamma = \gamma_0 \star \gamma_1 \star \cdots \star \gamma_{N-1}. \tag{4.31}$$

Here $\gamma_i : [0,1] \to \Sigma$ are smooth paths such that:

- $\operatorname{Im} \gamma_i \subset \operatorname{Im} \varphi(\sigma_i)$.
- $\gamma_i(0) \in \operatorname{Im} \varphi(e_{i-1,i})$.
- $\gamma_i(1) \in \operatorname{Im} \varphi(e_{i,i+1})$.

Let $\hat{\gamma} : [0,1] \to F_{GL^+}(\underline{\Sigma})$ be a lift of $d\gamma$ along the projection $\pi_T : F_{GL^+}(\underline{\Sigma}) \to T\underline{\Sigma}$ mapping a frame to its first component vector (cf. Section 3.5). Then $\hat{\gamma}$ acts on the fibre $P_{\widetilde{GL}}(\Sigma) \ni p^{-1}(\gamma(0)) \cong \mathbb{Z}_r$ as

$$x \mapsto x + \sum_{i\in\mathbb{Z}_N} \omega_{i,i+1} \tag{4.32}$$

with $\omega_{i,i+1}$ given by

$$\omega_{i,i+1} = \hat{s}_{i,i+1} + \frac{k_i + \eta_i - [k_i+\eta_i]_3}{3} + \frac{1}{2} - \frac{\eta_i}{2} \ . \tag{4.33}$$

Figure 4.8.: a) The paths $\chi_i \circ \zeta_i$ can intersect an edge in an arbitrary angle and thus the composition may not be differentiable. b) The modified paths turn left upon entering σ_i and right before leaving. They intersect all edges tangentially and thus up to a reparametrisation a smooth composition is possible.

Proof. Any two curves $\gamma, \gamma' : [0,1] \to \Sigma$ that satisfy the assumptions of Lemma 4.13 are isotopic: We can find an isotopy triangle by triangle. By the reasoning in Section 3.5 the corresponding paths in the frame bundle are homotopic. It is then sufficient to examine the action of one such curve.

• *Explicit construction of a suitable curve in the frame bundle.* We start by defining a simple closed curve $\zeta : [0,1] \to \underline{\Sigma}$. Let

$$\begin{aligned} \zeta_i : [0,1] &\to \underline{\Delta} \\ t &\mapsto z_i - \eta_i r_0 \alpha_0 e^{2\pi i \left(-\frac{\eta_i t}{6} + \frac{k_i}{3}\right)} , \end{aligned} \tag{4.34}$$

where $\alpha_0 = e^{2\pi i \frac{5}{12}}$ as in (4.6). The centre point $z_i = e^{\frac{2\pi i}{3}\left(k_i + \frac{1}{2}(1+\eta_i)\right)}$ is the preimage under $\check{\chi}_{\sigma_i}$ of the vertex $e_{i-1,i}$ and $e_{i,i+1}$ intersect in. The radius $r_0 = \frac{\sqrt{3}}{2}$ is chosen such that ζ_i starts and ends on the midpoints of the edges. This ensures that

$$\left(\chi_{\sigma_i} \circ \zeta_i\right)(1) = \left(\chi_{\sigma_{i+1}} \circ \zeta_{i+1}\right)(0), \tag{4.35}$$

and thus we obtain the simple closed curve

$$\zeta := \left(\chi_{\sigma_0} \circ \zeta_0\right) \star \left(\chi_{\sigma_1} \circ \zeta_1\right) \star \cdots \star \left(\chi_{\sigma_{N-1}} \circ \zeta_{N-1}\right) . \tag{4.36}$$

The curve ζ may not be differentiable at the edges, even if reparametrisation (with non-vanishing velocity) is taken into account. We fix this as depicted in Figure 4.8: We change ζ_i slightly at the edges, such that it intersects the edge tangentially. By reparametrisation of the paths, the composition can then be made a smooth simple closed curve. The differentials of these paths then have a lift to the frame bundle which are (up to a homotopy that leaves the base path fixed) composable. We describe these lifts in the limit of changing the initial paths ζ_i minimally. The change at the edges can then be described by rotations in the frame bundle: Let

$$\begin{aligned} \hat{\zeta}_i &: [0,1] \to \underline{\Delta} \times GL_2^+ \ , & t &\mapsto \left(\zeta_i(t), \frac{2\pi i}{6} r_0 \alpha_0 e^{2\pi i \left(-\frac{\eta_i t}{6} + \frac{k_i}{3}\right)}\right) , \\ \hat{\zeta}_i^L &: [0,1] \to \underline{\Delta} \times GL_2^+ \ , & t &\mapsto \left(\zeta_i(0), \frac{2\pi i}{6} r_0 \alpha_0 e^{2\pi i \left(\frac{t-1}{4} + \frac{k_i}{3}\right)}\right) , \end{aligned} \tag{4.37}$$

[1]Composition is defined such that $\gamma \star \gamma'$ starts at $\gamma(0)$.

$$\hat{\zeta}_i^R : [0,1] \to \underline{\Delta} \times GL_2^+ \quad , \qquad t \mapsto \left(\zeta_i(1), \frac{2\pi i}{6} r_0 \alpha_0 e^{2\pi i \left(-\frac{t}{4} - \frac{\eta_i}{6} + \frac{k_i}{3}\right)} \right).$$

The paths $\hat{\zeta}_i$ are lifts of the unmodified paths $d\zeta_i$ and $\hat{\zeta}_i^L, \hat{\zeta}_i^R$ represent the left/right rotation. Let

$$\hat{\zeta}_i^0 := \hat{\zeta}_i^L \star \hat{\zeta}_i \star \hat{\zeta}_i^R. \tag{4.38}$$

We next verify explicitly that the paths $d\chi_{\sigma_i *}\hat{\zeta}_i^0$ are composable up to homotopy: Let $\alpha = e^{2\pi i \frac{k_i+\eta_i}{3}} \alpha_0$. Using an α-rotated QR-decomposition, see (3.16), we obtain that

$$d\left(\chi_{\sigma_i+1}^{-1} \circ \chi_{\sigma_i}\right)|_{\zeta_i(1)} = e^{2\pi i \left(\frac{k_{i+1}}{3} - \frac{k_i+\eta_i}{3} + \frac{1}{2}\right)} \alpha t_i \alpha^{-1}, \tag{4.39}$$

with $t_i \in \mathbf{T}_2$ an upper triangular matrix. Therefore

$$\begin{aligned}
&\left(d\chi_{\sigma_{i+1}}^{-1} \circ d\chi_{\sigma_i}\right)_* \hat{\zeta}_i^R(1) \\
&= \left(\chi_{\sigma_{i+1}^{-1}} \circ \chi_{\sigma_i}\left(\zeta_i(1)\right) \ , \ \frac{2\pi}{6} e^{2\pi i \left(\frac{k_{i+1}}{3} - \frac{k_i+\eta_i}{3} + \frac{1}{2}\right)} \alpha t_i \alpha^{-1} r_0 \alpha_0 e^{2\pi i \left(-\frac{\eta_i}{6} + \frac{k_i}{3}\right)}\right) \\
&= \left(\zeta_{i+1}(0)\, , \ \frac{2\pi}{6} r_0 e^{2\pi i \left(\frac{k_{i+1}}{3} + \frac{1}{2} - \frac{\eta_i}{2}\right)} \alpha_0 t_i\right) \\
&= \hat{\zeta}_{i+1}^L(0).t_i.
\end{aligned} \tag{4.40}$$

Here in the first step we used Equation (4.39), in the second step we use that $e^{-\pi i \eta_i}$ is in the centre of GL_2^+. In the last step we use the right action of GL_2^+ on $F_{GL^+}(\underline{\Delta})$. The desired homotopy is then given by right action with a path from t_i to the identity matrix. Using these homotopies and composing we finally obtain a closed curve

$$\hat{\zeta} : [0,1] \to F_{GL^+}(\zeta) \tag{4.41}$$

that is homotopic to a lift of the original curve $d\gamma$ to $F_{GL^+}(\zeta)$.

• *Lifting properties of the curve* $\hat{\zeta}$. Next we determine how this curve lifts to the r-spin bundle of Σ. We first pick lifts of the paths $\hat{\zeta}_i$, $\hat{\zeta}_i^L$ and $\hat{\zeta}_i^R$: Let $\tilde{\zeta}_i, \tilde{\zeta}_i^L, \tilde{\zeta}_i^R : [0,1] \to P_{\widetilde{GL}}(\Sigma)$ be given by

$$\tilde{\zeta}_i(t) = \left(\zeta_i(t)\, , \ \delta\left(\tilde{r}_0 + \tilde{\alpha}_0 - \frac{\eta_i t}{6} + \frac{1}{4} + \frac{k_i}{3}\right)\right) \tag{4.42}$$

$$\tilde{\zeta}_i^L(t) = \left(\zeta_i(0)\, , \ \delta\left(\tilde{r}_0 + \tilde{\alpha}_0 + \frac{t}{4} + \frac{k_i}{3}\right)\right) \tag{4.43}$$

$$\tilde{\zeta}_i^R(t) = \left(\zeta_i(1)\, , \ \delta\left(\tilde{r}_0 + \tilde{\alpha}_0 + \frac{1-t}{4} - \frac{\eta_i}{6} + \frac{k_i}{3}\right)\right), \tag{4.44}$$

with $\tilde{r}_0$ a lift of $\frac{2\pi}{6} r_0$ along $p_{\mathbb{C}^\times}^r$, defined in Equation (3.21), and $\tilde{\alpha}_0$ as defined in (4.21). The lifts $\tilde{\zeta}_i, \tilde{\zeta}_i^L, \tilde{\zeta}_i^R$ are chosen such that the compositions $\tilde{\zeta}_i^0 := \tilde{\zeta}_i^L \star \tilde{\zeta}_i \star \tilde{\zeta}_i^R$ exist. Solving (4.9) for $\tilde{\mathbf{Q}}_\alpha(\dots)$, we write the r-spin transition functions as

$$g_{\zeta_i(1)}\left(\tilde{\chi}_{\sigma_{i+1}}^{-1} \circ \tilde{\chi}_{\sigma_i}\right) = \delta\left(\hat{s}_{i,i+1} + \frac{k_{i+1}}{3} - \frac{[k_i+\eta_i]_3}{3} + \frac{1}{2} + \tilde{\alpha}\right) \tilde{t}_i \delta\left(\tilde{\alpha}\right)^{-1}, \tag{4.45}$$

with $\delta(\tilde{\alpha})$ being an r-spin lift of $\alpha = \alpha_0 e^{2\pi i \left(\frac{k_i+\eta_i}{3}\right)}$, and $\tilde{t}_i \in \tilde{\mathbf{T}}_2$ the lift of t_i. Using this, we compute

$$
\begin{aligned}
&\left(\tilde{\chi}_{\sigma_{i+1}}^{-1} \circ \tilde{\chi}_{\sigma_i}\right)\left(\tilde{\zeta}_i^R(1)\right) \\
&= \left(\zeta_{i+1}(0)\,,\, \delta\left(\tilde{r}_0 + \hat{s}_{i,i+1} + \frac{k_{i+1} - [k_i+\eta_i]_3}{3} + \frac{1}{2} + \tilde{\alpha}\right)\tilde{t}_i \delta\left(-\tilde{\alpha} + \tilde{\alpha}_0 - \frac{\eta_i}{6} + \frac{k_i}{3}\right)\right) \\
&= \left(\zeta_{i+1}(0)\,,\, \delta\left(\tilde{r}_0 + \frac{k_{i+1}}{3} + \hat{s}_{i,i+1} + \frac{k_i + \eta_i - [k_i+\eta_i]_3}{3} + \frac{1-\eta_i}{2} + \tilde{\alpha}_0\right)\tilde{t}_i\right) \\
&= \tilde{\zeta}_{i+1}^L(0).(\delta(\omega_{i,i+1})\tilde{t}_i)\ ,
\end{aligned}
\tag{4.46}
$$

where in the second step we used that $\delta\left(-\frac{\eta_i}{2}\right)$ is central, see (3.8). $\omega_{i,i+1} \in \mathbb{Z}_r$ is given by (4.33). We remove the $\tilde{t}_i$ by using lifts of the homotopies used before. The path segments then compose up to a shift by $\omega_{i,i+1}$. Summing these up then gives the total action of the curve $\hat{\zeta}$. □

We now evaluate Lemma 4.13 for circular paths around vertices.

Corollary 4.14. Let v be an inner vertex and $(\sigma_i)_{i=0,\ldots,N-1}$ be the triangles containing v, ordered counterclockwise starting with an arbitrary triangle σ_1. Let D be the number of triangles σ_i such that $k_i = 0$ (i.e. the path enters σ_i through the marked edge). Then the r-spin structure on Σ extends to v if and only if

$$\sum_{i\in\mathbb{Z}_N} \hat{s}_{i,i+1} \equiv D - N + 1 \mod r\ . \tag{4.47}$$

In particular, $\hat{s}_{i,i+1} = s_{i,i+1}$ if the edge $e_{i,i+1}$ is pointing away from v.

Proof. We pick a differentiable simple closed curve γ around the vertex v as in Figure 4.9 that fulfils the decomposition conditions of Lemma 4.13. It is now a simple counting problem to reformulate Equation (4.32). We go term by term through the factors in Equation (4.33), the definition of $\omega_{i,i+1}$: Since we chose the sequence of triangles counterclockwise we have $\eta_i = -1$ for all $i = \mathbb{Z}_N$ and thus get a shift by $+1$ from the $\frac{1-\eta_i}{2}$ term for each edge. We get an extra shift by -1 from the term $(k_i + \eta_i - [k_i+\eta_i]_3)/3$ if $k_i = 0$ or equivalently $\chi_{\sigma_i}(1) = v$. Collecting these together, we see that the total action of the curve is

$$x \mapsto x + \sum_{i\in\mathbb{Z}_N} \hat{s}_{i,i+1} + N - D\ . \tag{4.48}$$

Since the curve γ also fulfils the conditions of Lemma 3.12 the r-spin structure can be extended if and only if γ acts by a shift of $+1$ on the fibre. □

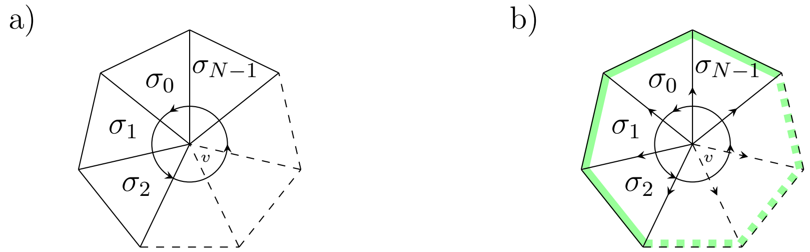


Figure 4.9.: a) Simple closed curve around a vertex v; the triangles containing v are ordered counter-clockwise. b) A possible marking of the triangles; for this choice, the statement of Corollary 4.14 becomes particularly easy: the r-spin structure extends iff $\sum_{i\in\mathbb{Z}_N} s_{i,i+1} = -N+1$.

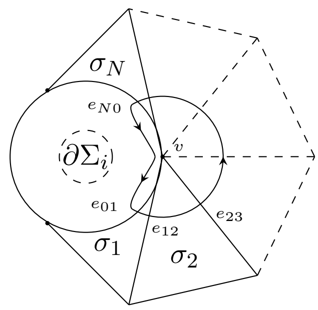


Figure 4.10.: The labeling of triangles and edges around a boundary vertex v. The triangles $(\sigma_i)_{i=1,\dots,N}$ are ordered counterclockwise, starting at the boundary. Edges are given by $e_{i,i+1} = \sigma_i \cap \sigma_{i+1}$ for $i = 1,\dots,N-1$. $e_{0,1}$ and $e_{N,0}$ are boundary edges of σ_1 and σ_N respectively. Also shown is a sketch of the simple closed curve we use.

4.7.2. Paths at the boundary

We aim to get a rule similar to Corollary 4.14 for boundary vertices. For v a boundary vertex we label the surrounding edges and triangles as in Figure 4.10. Let $s_{i,i+1}$ and $\hat{s}_{i,i+1}$ be as in Equations (4.29), (4.30). Since the orientation of the boundary is fixed, we have $\hat{s}_{0,1} = -s_{0,1} - 1$ and $\hat{s}_{N,0} = s_{N,0}$.

Lemma 4.15. Let v be a vertex on the boundary component i of type λ. Let D_{inn} be the number of triangles σ_i with $\chi_{\sigma_i}(1) = v$ or equivalently $d_2^0(\sigma_i) = e_{i-1,i}$. Let $D_{bnd} = 1$ if $\varphi_i(1) = v$ and $D_{bnd} = 0$ otherwise. The r-spin structure on Σ^+ extends to v if and only if

$$\sum_{i\in\mathbb{Z}_{N+1}} \hat{s}_{i,i+1} \equiv D_{inn} + (1-\lambda)D_{bnd} - N \mod r\ . \tag{4.49}$$

Proof. We first describe the marking on the relevant edges and triangles more explicitly:

- $k_i \in \{0,1,2\}$ is fixed by $e_{i-1,i} = d_2^{k_i}(\sigma_i)$ for $i = 1,\dots,N$ as in Section 4.7.1.
- $k_0 \in \{0,1,2\}$ is the position of $e_{N,0}$ on the boundary under the boundary inclusion map f_i, i.e. $f_i(e_{k_0}) = e_{N,0}$, where e_0, e_1, e_2 are the edges of $\underline{\Delta}$.

- $\eta_i = -1$ for $i = 1, \dots, N$ which agrees with the definition in Section 4.7.1 since we ordered the triangles counterclockwise.
- $\mu_{i,i+1} = \pm 1$ for $i = 1, \dots, N$ are as defined in Section 4.7.1. The convention for boundary edge directions then implies $\mu_{0,1} = -1$ and $\mu_{N,0} = 1$. In particular $\mu_{i,i+1} = 1$ if the edge $e_{i,i+1}$ is pointing away from v.

As was the case for inner vertices, we determine the lifting behaviour of a specific curve from the frame bundle to the r-spin bundle via the edge indices. We then relate it to the known r-spin lifting behaviour of contractible simple closed curves.

• *Construction of a suitable curve in the frame bundle.* As in the proof of Lemma 4.13 we first construct a simple closed curve around the vertex v. For the boundary part we need a new path segment. Let

$$\begin{aligned} \zeta_0 : [0,1] &\to U_i^+ \\ t &\mapsto e^{2\pi i\left(-\frac{t}{3}+\frac{1}{6}+\frac{k_0}{3}\right)}. \end{aligned} \tag{4.50}$$

The path ζ_0 now intersects $\varphi_i^{-1}(v)$, the preimage of the vertex v. To avoid this we exchange it by an isotopic path in the unit disc that still lies in $U_i^+ \cap \{z||z| \leq 1\}$. Its explicit form is not relevant as we are only interested in lifting properties of the path segments. The segments $\zeta_i : [0,1] \to \underline{\Delta}$ for $i = 1, \dots, N$ are as in Equation (4.34). The composition

$$\zeta := (\varphi_i \circ \zeta_0) \star (\chi_{\sigma_1} \circ \zeta_1) \star \cdots \star (\chi_{\sigma_N} \circ \zeta_N)\,. \tag{4.51}$$

is then a simple closed curve. By the same procedure as in the proof of Lemma 4.13 (see also Figure 4.8) we obtain a smooth simple closed curve. Its velocity curve has a lift to the frame bundle and we describe a curve homotopic to this lift explicitly by implementing the kinks as rotations in the frame bundle. Let $\hat{\zeta}_i$, $\hat{\zeta}_i^0$, $\hat{\zeta}_i^{L/R}$ $(i = 1, \dots, N-1)$ and $\hat{\zeta}_N^L$ be as in the proof of Lemma 4.13. Let

$$\hat{\zeta}_N^{R\prime} : [0,1] \to \underline{\Delta} \times GL_2^+ \quad , \quad t \mapsto \left(\zeta_N(1), \frac{2\pi i}{6} r_0 \alpha_0 e^{2\pi i\left(\frac{t}{4} - \frac{\eta_N}{6} + \frac{k_N}{3}\right)}\right) \tag{4.52}$$

and set $\hat{\zeta}_N^{0\prime} := \hat{\zeta}_N^L \star \hat{\zeta}_N \star \hat{\zeta}_N^{R\prime}$ (note that $\hat{\zeta}_N^{R\prime}$ rotates in the opposite direction relative to $\hat{\zeta}_N^R$). For the boundary part we let

$$\hat{\zeta}_0 : [0,1] \to U_i^+ \times GL_2^+ \quad , \quad t \mapsto \left(\zeta_0(t), -\frac{2\pi i}{3} e^{2\pi i\left(-\frac{t}{3}+\frac{1}{6}+\frac{k_0}{3}\right)}\right)\,. \tag{4.53}$$

The isotopy we used to make ζ_0 avoid $\varphi_i^{-1}(v)$ lifts to the frame-bundle, so we can change $\hat{\zeta}_0$ in the same way.

By Equation (4.40) we already know that the paths $(d\chi_{\sigma_i})_*\hat{\zeta}_i$ and $(d\chi_{\sigma_{i+1}})_*\hat{\zeta}_{i+1}$ are composable up to right action by an upper triangular matrix for $i = 1, \dots, N-1$. We proceed to show that the same holds for $(d\chi_{\sigma_N})_*\hat{\zeta}_N'$ and $(d\varphi_i)_*\hat{\zeta}_0$, as well as for $(d\varphi_i)_*\hat{\zeta}_0$ and $(d\chi_{\sigma_1})_*\hat{\zeta}_1$.

Let $\alpha = \alpha_0 e^{2\pi i\left(\frac{k_N+\eta_N}{3}\right)}$. Note that $d\left(\varphi_i^{-1}\circ\chi_{\sigma_N}\right)_*$ rotates the tangent to $e_{N,0}$ by $e^{2\pi i\left(\frac{k_0}{3}-\frac{k_N+\eta_N}{3}+\frac{1}{2}\right)}$ with $\eta_N=-1$.

$$
\begin{aligned}
&d\left(\varphi_i^{-1}\circ\chi_{\sigma_N}\right)_*\left(\hat{\zeta}_N^{R\prime}(1)\right)\\
&=\left(\varphi_i^{-1}\circ\chi_{\sigma_N}\left(\zeta_N(1)\right),\frac{2\pi}{6}e^{2\pi i\left(\frac{k_0}{3}-\frac{k_N+\eta_N}{3}+\frac{1}{2}\right)}\alpha t_N\alpha^{-1}r_0\alpha_0 i^2 e^{2\pi i\left(-\frac{\eta_N}{6}+\frac{k_N}{3}\right)}\right)\\
&=\left(\zeta_0(0),\frac{2\pi}{6}r_0 e^{2\pi i\left(\frac{k_0}{3}+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}\right)}t_N\right)\\
&=\left(\zeta_0(0),-\frac{2\pi i}{3}e^{2\pi i\left(\frac{k_0}{3}+\frac{1}{6}\right)}\right).\frac{r_0 t_N}{2}=\hat{\zeta}_0(0).\frac{r_0 t_N}{2}.
\end{aligned}
\tag{4.54}
$$

Here we used the explicit value of $\alpha_0 = e^{2\pi i\left(\frac{1}{4}+\frac{1}{6}\right)}$ in the second step, and introduced a matrix $t_N\in\mathbf{T}_2$. Now let $\alpha=\alpha_0 e^{2\pi i\frac{k_1}{3}}$.

$$
\begin{aligned}
&d\left(\varphi_i^{-1}\circ\chi_{\sigma_1}\right)\left(\hat{\zeta}_1^L(0)\right)=\\
&=\left(\left(\varphi_i^{-1}\circ\chi_{\sigma_1}\right)\left(\zeta_1(0)\right),\frac{2\pi}{6}e^{2\pi i\left(\frac{k_0-1}{3}-\frac{k_1}{3}+\frac{1}{2}\right)}\alpha t_0\alpha^{-1}r_0\alpha_0 e^{2\pi i\frac{k_1}{3}}\right)\\
&=\left(\zeta_0(1),-\frac{2\pi i}{3}e^{2\pi i\left(\frac{k_0-1}{3}+\frac{1}{6}\right)}\right).\frac{r_0 t_0}{2}=\hat{\zeta}_0(1).\frac{r_0 t_0}{2}.
\end{aligned}
\tag{4.55}
$$

Here we used again the value of α_0, and introduced a matrix $t_0\in\mathbf{T}_2$. As in Section 4.7.1 we pick paths in $\mathbf{T}_2$ from t_i, $i=0,\ldots,N$ to the identity matrix, as well as from $r_0/2$ to 1, and use these to obtain a closed curve

$$
\hat{\zeta}:[0,1]\to F_{GL^+}(\Sigma).
\tag{4.56}
$$

• *Lifting $\hat{\zeta}$ to the r-spin bundle.* In order to give an explicit lift of $\hat{\zeta}$ to the r-spin bundle of Σ, we pick lifts of $\hat{\zeta}_i,\hat{\zeta}_i^L$ and $\hat{\zeta}_i^R$ as in Equations (4.42)–(4.44), and of $\hat{\zeta}_N^{R\prime}$ and $\hat{\zeta}_0$ as follows:

$$
\tilde{\zeta}_N^{R\prime}(t)=\left(\zeta_N(1)\ ,\ \delta\left(\tilde{r}_0+\tilde{\alpha}_0+\frac{t}{4}+\frac{1}{4}-\frac{\eta_N}{6}+\frac{k_N}{3}\right)\right)\ .
\tag{4.57}
$$

The r-spin lift of $\hat{\zeta}_0$ depends on the given r-spin structure on the boundary component i, see Section 3.4. We choose

$$
\tilde{\zeta}_0^{\lambda}(t)=\left(\zeta_0(t)\,,\,\delta\left(\frac{\ln\frac{2\pi}{3}}{2\pi i}-\frac{1}{4}+(1-\lambda)\left(-\frac{t}{3}+\frac{1}{6}+\frac{k_0}{3}\right)\right)\right)\ .
\tag{4.58}
$$

By Equation (4.46) the composition of the r-spin lifts of $\left(\tilde{\chi}_{\sigma_i}\right)_*\tilde{\zeta}_i$ yields numbers $\omega_{i,i+1}$ for $i=1,\ldots,N-1$.

Using (4.11), we write the r-spin transition function at $e_{N,0}$ as

$$
g_{\zeta_N(1)}\left(\tilde{\varphi}_i^{-1}\circ\tilde{\chi}_{\sigma_N}\right)=\delta\left(s_{N,0}+\frac{k_0(1-\lambda)}{3}-\frac{[k_N-1]_3}{3}+\frac{1}{2}-\frac{\lambda}{6}\right)\delta(\tilde{\alpha})\tilde{t}_N\delta(\tilde{\alpha})^{-1},
\tag{4.59}
$$

with $\delta(\tilde{\alpha})$ being an r-spin lift of $\alpha = \alpha_0 e^{2\pi i\left(\frac{k_N-1}{3}\right)}$ and $\tilde{t}_N \in \tilde{\mathbf{T}}_2$ the lift of t_N. Then

$$\begin{aligned}
&\left(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_N}\right)\left(\tilde{\zeta}_N^{R\prime}(1)\right) = \\
&= \Bigg(\zeta_0(0)\,,\, \delta\bigg(s_{N,0} + \frac{k_0(1-\lambda)}{3} - \frac{[k_N-1]_3}{3} + \frac{1}{2} - \frac{\lambda}{6} + \tilde{\alpha}\bigg)\,\tilde{t}_N \\
&\qquad \delta\bigg(-\tilde{\alpha} + \tilde{r}_0 + \tilde{\alpha}_0 + \frac{1}{2} + \frac{1}{6} + \frac{k_N}{3}\bigg)\Bigg) \\
&= \Bigg(\zeta_0(0)\,,\, \delta\bigg(\frac{\ln\frac{2\pi}{3}}{2\pi i} - \frac{1}{4} + (1-\lambda)\left(\frac{k_0}{3} + \frac{1}{6}\right) + 1 + s_{N,0} + \frac{k_N - 1 - [k_N-1]_3}{3} - \frac{\ln\frac{2\pi}{3}}{2\pi i}\bigg) \\
&\qquad \tilde{t}_N \delta(\tilde{r}_0 + 1)\Bigg) \\
&= \Bigg(\zeta_0(0)\,,\, \delta\bigg(\frac{\ln\frac{2\pi}{3}}{2\pi i} - \frac{1}{4} + (1-\lambda)\left(\frac{k_0}{3} + \frac{1}{6}\right)\bigg)\,\delta\left(2 + s_{N,0} + \frac{k_N - 1 - [k_N-1]_3}{3}\right)\Bigg) \\
&\qquad .\left(\delta\left(\tilde{r}_0 - \frac{\ln\frac{2\pi}{3}}{2\pi i}\right)\tilde{t}_N\right) \\
&= \tilde{\zeta}_0^{\lambda}(0).\delta(\omega_{N,0}).\left(\delta\left(\tilde{r}_0 - \frac{\ln\frac{2\pi}{3}}{2\pi i}\right)\tilde{t}_N\right)\ .
\end{aligned} \tag{4.60}$$

Here $\omega_{N,0}$ is given by

$$\omega_{N,0} = 2 + s_{N,0} + \frac{k_N - 1 - [k_N-1]_3}{3} = 2 + \hat{s}_{N,0} + \frac{k_N - 1 - [k_N-1]_3}{3}\ . \tag{4.61}$$

We write the r-spin transition function at $e_{0,1}$ as

$$g_{\zeta_1(0)}\left(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_1}\right) = \delta\bigg(s_{0,1} + (1-\lambda)\frac{[k_0-1]_3}{3} - \frac{k_1}{3} + \frac{1}{2} - \frac{\lambda}{6}\bigg)\,\delta(\tilde{\alpha})\tilde{t}_0\delta(\tilde{\alpha})^{-1} \tag{4.62}$$

with $\delta(\tilde{\alpha})$ an r-spin lift of $\alpha = \alpha_0 e^{2\pi i \frac{k_1}{3}}$ and $\tilde{t}_0 \in \tilde{\mathbf{T}}_2$ the r-spin lift of t_0. Then

$$\begin{aligned}
&\left(\tilde{\varphi}_i^{-1} \circ \tilde{\chi}_{\sigma_1}\right)\left(\tilde{\zeta}_1^{L}(0)\right) \\
&= \Bigg(\zeta_0(1)\,,\, \delta\bigg(s_{0,1} + (1-\lambda)\frac{[k_0-1]_3}{3} - \frac{k_1}{3} + \frac{1}{2} - \frac{\lambda}{6} + \tilde{\alpha}\bigg)\,\tilde{t}_0\,\delta\bigg(-\tilde{\alpha} + \tilde{r}_0 + \tilde{\alpha}_0 + \frac{k_1}{3}\bigg)\Bigg) \\
&= \Bigg(\zeta_0(1)\,,\, \delta\bigg(s_{0,1} + (1-\lambda)\frac{[k_0-1]_3}{3} + \frac{1}{2} - \frac{\lambda}{6} + \tilde{\alpha}_0\bigg)\,\tilde{t}_0\,\delta(\tilde{r}_0)\Bigg) \\
&= \Bigg(\zeta_0(1)\,,\, \delta\bigg(\frac{\ln\frac{2\pi}{3}}{2\pi i} - \frac{1}{4} + (1-\lambda)\left(\frac{1}{6} + \frac{k_0-1}{3}\right) - \omega_{0,1}\bigg)\,\tilde{t}_0\,\delta\bigg(\tilde{r}_0 - \frac{\ln\frac{2\pi}{3}}{2\pi i}\bigg)\Bigg) \\
&= \tilde{\zeta}_0^{\lambda}(1).\delta(\omega_{0,1})^{-1}.\left(\delta\left(\tilde{r}_0 - \frac{\ln\frac{2\pi}{3}}{2\pi i}\right)\tilde{t}_0\right)\ .
\end{aligned} \tag{4.63}$$

Here $\omega_{0,1}$ is given by

$$\begin{aligned}\omega_{0,1} &= -s_{0,1} - 1 + (1-\lambda)\left(\frac{(k_0-1)-[k_0-1]_3}{3}\right) \\ &= \hat{s}_{0,1} + (1-\lambda)\left(\frac{(k_0-1)-[k_0-1]_3}{3}\right) .\end{aligned} \tag{4.64}$$

The action of $\hat{\zeta}$ on the fibre is then given by $x \mapsto x + \varepsilon$ with

$$\varepsilon = \sum_{i \in \mathbb{Z}_{N+1}} \omega_{i,i+1} . \tag{4.65}$$

Adding these numbers as in the proof of Lemma 4.15 we can reformulate Equation (4.65) as

$$\varepsilon = \sum_{i \in \mathbb{Z}_{N+1}} \hat{s}_{i,i+1} + N + 1 - D_{inn} - (1-\lambda) D_{bnd} . \tag{4.66}$$

Since $\hat{\zeta}$ is homotopic to a lift of the derivative of a differentiable simple closed curve we can apply Lemma 3.12. We then get that the r-spin structure can be extended if and only if the action of $\hat{\zeta}$ on the fibre is a shift by 1, i.e. $\varepsilon = 1$. □

4.8. Spin structures and admissible edge indices

For this section, let us fix a marked triangulated surface

$$\Sigma = ((\mathcal{C}, f_i, d_0^1, d_0^2), (\varphi, \chi), (\Sigma, \varphi_i)) .$$

Definition 4.16. An edge index assignment $s : \mathcal{C}_1 \to \mathbb{Z}_r$ is called *admissible* if

1. condition (4.47) is satisfied at each inner vertex,
2. for each boundary component i, all three vertices on that boundary component satisfy (4.49) for λ_i-type boundary conditions, where $\lambda_i \in \mathbb{Z}_r$ is fixed for each boundary component i.

We then call the boundary component i *of λ_i-type*.

Given admissible edge indices s, we are going to construct an r-spin structure on Σ in two steps. First, we use the edge indices to construct the r-spin structure on $\Sigma \setminus \{\text{vertices}\}$ – this part works without conditions on the edge indices s. Then we use the admissibility condition to extend the r-spin structure to the vertices of the triangulation. We will denote the resulting r-spin structure as

$$S(\Sigma, s) . \tag{4.67}$$

By construction, $S(\Sigma, s)$ is an r-spin triangulated surface.

We now give the detailed construction. For each face $\sigma \in \mathcal{C}_2$, pick a smooth extension χ_σ^+ of χ_σ to some (σ-dependent) open neighbourhood $\underline{\Delta}^{+,\sigma}$ of the standard triangle $\underline{\Delta}$.

Open cover with only trivial triple intersections: Let $\mathcal{I} = \mathcal{C}_2 \cup \{1, \dots, B\}$. Recall from Section 4.7 the construction of Σ^+ and that $\varphi(\mathcal{C}_0) \subset \Sigma^+$ is the set of images of the vertices under the triangulation map φ. We will construct an open cover $(V_\alpha)_{\alpha\in\mathcal{I}}$ of $\Sigma^+ \setminus \varphi(\mathcal{C}_0)$ such that non-empty intersections $V_\alpha \cap V_\beta$ are contractible, and such that $V_\alpha \cap V_\beta \cap V_\gamma = \emptyset$ whenever α, β, γ are pairwise distinct.

Around the image in Σ^+ of each edge $e \in \mathcal{C}_1$ minus its endpoints choose a contractible open neighbourhood W_e not containing any vertices. By shrinking W_e if necessary, we may assume that

- $W_e \cap W_{e'} = \emptyset$ for $e \neq e'$,
- for each face σ and each edge e on the boundary of σ, W_e is contained in the image of the extended map χ_σ^+,
- for each boundary edge e on the i'th boundary component, W_e is contained in U_i^+ (considered as a subset of the quotient surface Σ^+).

For $\sigma \in \mathcal{C}_2$ we set

$$V_\sigma = \mathring{\underline{\Delta}} \cup \bigcup_{e\in\mathcal{B}(\sigma)} (\chi_\sigma^+)^{-1}(W_e) \ . \tag{4.68}$$

Here, $\mathring{\underline{\Delta}}$ denotes the interior of $\underline{\Delta}$. For $i \in \{1, \dots, B\}$ we set

$$V_i = \{z \in \mathbb{C} | r < |z| < 1\} \cup \bigcup_{e\in\mathrm{im}(f_i)} \varphi_i^{-1}(W_e) \ , \tag{4.69}$$

where r is as in Section 4.7. We will identify the V_σ and V_i with their images in Σ^+. The V_σ and V_i then give a cover with the desired properties.

r-spin structure on the collared surface less the vertices: We define an r-spin structure on $\Sigma^+ \setminus \varphi(\mathcal{C}_0)$ via the atlas $(V_\alpha)_{\alpha\in\mathcal{I}}$. For $\sigma \in \mathcal{C}_2$ fix $\tilde{V}_\sigma = \mathbb{C}^0|_{V_\sigma}$. For $i \in \{1, \dots, B\}$ we take $\tilde{V}_i = \mathbb{C}^{\lambda_i}|_{V_i}$.

For an inner edge $e \in \mathcal{C}_1$ fix the r-spin lift of the transition function

$$f_e : (\chi_{\sigma_R}^+)^{-1}(W_e) \to (\chi_{\sigma_L}^+)^{-1}(W_e) \tag{4.70}$$

to be $\tilde{f}_e(z, g) = (f_e(z), g_z g)$ where $g_z \in \widetilde{GL}_2$ is uniquely determined by the requirement that (i) $p_{GL}(g_z) = (df_e)_z$, and that (ii) on a point p on the preimage of the edge e we have

$$\delta(s(e)) = \tilde{\mathbf{Q}}_\alpha(g_p) \cdot \delta\left(-(k_L/3 - k_R/3 + 1/2)\right) \ , \tag{4.71}$$

i.e. the rule in (4.8) is satisfied.

For a boundary edge e on the i'th boundary component we fix the r-spin lift of the transition function $f_e : (\chi_{\sigma_R}^+)^{-1}(W_e) \to \varphi_i^{-1}(W_e)$ to be $\tilde{f}_e(z, g) = (f_e(z), g_z g)$, where now g_z is characterised as follows. Let $p = \chi_{\sigma_R}^{-1} \circ \varphi_i(e^{2\pi i(\frac{k_L}{3}+\frac{1}{6})})$ as in Section 4.4. The transformation $g_z \in \widetilde{GL}_2$ is uniquely determined by $p_{GL}(g_z) = (df_e)_z$ and by demanding that at the point p we have

$$\delta(s(e)) = \tilde{\mathbf{Q}}_\alpha(g_p) \cdot \delta\left(\frac{k_L(\lambda-1)+k_R}{3} - \frac{1}{2} + \frac{\lambda}{6}\right) . \tag{4.72}$$

This is the rule stated in (4.11).

Since there are no non-trivial triple overlaps, there is no cocycle condition on the r-spin transition functions. Hence the above assignment defines an r-spin structure on $\Sigma^+ \setminus \varphi(\mathcal{C}_0)$.

Extending the r-spin structure to the entire collared surface: If the extension of the r-spin structure to the vertices $\varphi(\mathcal{C}_0)$ exists, it is unique. The conditions for extendibility are stated in Corollary 4.14 and Lemma 4.15. Since we assumed that s it admissible, we do indeed obtain an r-spin structure on Σ.

Let now

$$\Lambda = ((\mathcal{C}, f_i, d_0^1, d_0^2), (\varphi, \tilde{\chi}_\sigma), (\Lambda, \tilde{\varphi}_i))$$

be an r-spin triangulated surface. Denote the underlying marked triangulated surface by $\underline{\Lambda}$. Let s_Λ be the edge indices for Λ from Definitions 4.7 and 4.11.

Theorem 4.17. Let Λ be an r-spin triangulated surface. The r-spin structures Λ and $S(\underline{\Lambda}, s_\Lambda)$ on $\underline{\Lambda}$ are isomorphic.

Proof. This is evident from the explicit construction of $S(\underline{\Lambda}, s_\Lambda)$. In terms of the atlas $(V_\alpha)_{\alpha\in\mathcal{I}}$ considered above, the isomorphism of r-spin structures $S(\underline{\Lambda}, s_\Lambda) \to \Lambda$ is simply given by $\tilde{\chi}_\sigma^+$ on V_σ ($\sigma \in \mathcal{C}_2$) and by the identity on V_i ($i \in \{1, \dots, B\}$). □

Isomorphism classes of r-spin structures on a triangulated surface can be parametrised by equivalence classes of markings and admissible edge indices. In more detail, fix a triangulated surface $\Sigma = ((\mathcal{C}, f_i), \varphi, (\Sigma, \varphi_i))$ (without marking). Consider the set $\mathcal{M}_\Sigma := \{((d_0^1, d_0^2), s)\}$ of pairs of markings (d_0^1, d_0^2) on Σ and allowed edge indices s on the resulting marked triangulated surface. The moves described in Lemma 4.11 leave the underlying triangulation fixed and just operate on markings and edge signs. They generate an equivalence relation on $\mathcal{M}_\Sigma$ which we denote by $\sim_{\text{fix}}$.

Theorem 4.18. The assignment $((d_0^1, d_0^2), s) \mapsto S\big((\Sigma \text{ with } d_0^1, d_0^2)\, ,\, s\big)$ induces a bijection

$$\mathcal{M}_\Sigma / \sim_{\text{fix}} \quad\longrightarrow\quad (r\text{-spin structures on } \Sigma \text{ up to isom. of } r\text{-spin str.}) . \tag{4.73}$$

Proof. Well-definedness on equivalence classes: Let (d, s) and (d', s') be two pairs in $\mathcal{M}_\Sigma$ linked by a move from Lemma 4.11. Let Λ be the r-spin triangulated surface $S((\Sigma, d), s)$, and let Λ' be the new r-spin triangulated surface resulting from the move as in Lemma 4.11. Then Λ and Λ' have the same underlying r-spin surface and differ only in marking and choice of spin lifts, in particular $\underline{\Lambda} = (\Sigma, d)$ and $\underline{\Lambda}' = (\Sigma, d')$. By construction, $s_{\Lambda'} = s'$, and so by Theorem 4.17 we have $\Lambda' \cong S((\Sigma, \underline{d}'), s')$.

Surjectivity: Immediate from Theorem 4.17.

Injectivity: Let $(d,s),(d',s') \in \mathcal{M}_\Sigma$ be such that $S((\Sigma,d),s)$ and $S((\Sigma,d'),s')$ are isomorphic r-spin structures on Σ. The moves in Lemma 4.11 relate any two markings on Σ, and so by Lemma 4.11, the equivalence class of (d',s') contains elements with marking d, i.e. there is s'' such that $(d',s') \sim_{\text{fix}} (d,s'')$. Let $\Lambda := S((\Sigma,d),s)$ and $\Lambda'' := S((\Sigma,d),s'')$ be the corresponding r-spin triangulated surfaces. By well-definedness of (4.73), this is an isomorphism $f : \Lambda \to \Lambda''$ of r-spin structures. Write $\tilde{\chi}_\sigma$ and $\tilde{\chi}''_\sigma$ for the r-spin lifts in Λ and Λ'', respectively, of the embedding χ_σ (which is the same for Λ and Λ'' as their marking agrees). Then $f \circ \tilde{\chi}_\sigma$ and $\tilde{\chi}''_\sigma$ are related by the action of the leaf exchange automorphism ω_k for some $k \in \mathbb{Z}_r$ (Lemma 3.10). Applying sheet exchanges where necessary and changing the edge indices s'' according to Lemma 4.11 produces a new pair (d,s''') in the equivalence class of (d',s') with the property that $\Lambda''' := S((\Sigma,d),s''')$ has isomorphic r-spin structure to Λ, and that the isomorphism f can be chosen such that $f \circ \tilde{\chi}_\sigma = \tilde{\chi}'''_\sigma$ for all triangles. By definition of the edge indices, it then follows that $s = s'''$. Thus $(d,s) \sim_{\text{fix}} (d,s''') \sim_{\text{fix}} (d',s')$. □

We stress that the above theorem classifies r-spin structures up to isomorphism of r-spin structures as in Definition 3.2, not up to isomorphism of r-spin surfaces as in Definition 3.5. For example, if Σ is a torus with empty boundary and $r = 2$ (the spin case), $\mathcal{M}_\Sigma / \sim_{\text{fix}}$ has four elements, explicit representatives of which will be given in Section 6.4.

4.9. Pachner moves

Recall that any two finite combinatorial manifolds that are PL-homeomorphic, can be transformed into each other by a finite sequence of Pachner moves. In two dimensions, there are the 2-2 and the 3-1 Pachner move (and its inverse). We want to examine the effect of these moves on the edge indices.

A 2d-Pachner move on a complex $\mathcal{C}$ changes at most three adjacent triangles. We say two r-spin triangulated surfaces $(\mathcal{C},\varphi,\Sigma)$ and $(\mathcal{C}',\varphi',\Sigma')$ *are related by a Pachner move*, if the underlying complexes $\mathcal{C}$ and $\mathcal{C}'$ are related by a Pachner move, and if the r-spin lifts $\tilde{\chi}$ and markings are not affected away from the triangles changed by the Pachner move.

Proposition 4.19. Let (C,φ,Σ) and (C',φ',Σ) be r-spin triangulated surfaces related by a Pachner 2-2 move such that the configuration of markings on the affected subcomplex is as in Figure 4.11. If the lifts of χ_{σ_3} and χ_{σ_4} are such that $s' = s$ and $s'_A = s_A$, then the remaining edge indices are related as

$$s'_B = s_B + s + 1 \quad , \quad s'_C = s_C + 2s + 1 \quad , \quad s'_D = s_D + s + 1 \; . \tag{4.74}$$

The other choices can be obtained by composing $\tilde{\chi}_3$ or $\tilde{\chi}_4$ with the leaf exchange automorphism; the corresponding edge indices are given by Lemma 4.11.

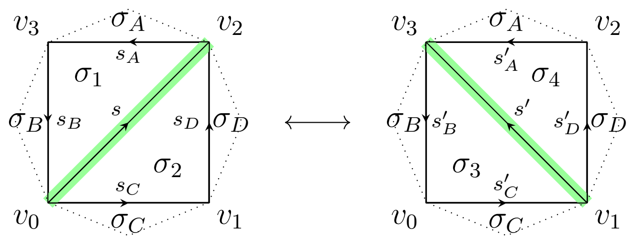


Figure 4.11.: Configuration of markings and labels for the Pachner 2-2 move. $v_0, \dots, v_3$ are vertices in $\mathcal{C}$ and in $\mathcal{C}'$. $\sigma_A, \dots, \sigma_D$ are faces in $\mathcal{C}$ and $\mathcal{C}'$, while σ_1, σ_2 belong to $\mathcal{C}$ and σ_3, σ_4 belong to $\mathcal{C}'$. $s_A, \dots, s_D$ and s as well as the primed version are the edge indices of the respective edges. The orientation of the edges is as indicated by the arrows. The green lines indicate the marked edges of the triangles, e.g. $\{v_0, v_2\}$ for σ_1.

Proof. First assume all vertices $v_0, \dots, v_3$ are inner. We use Corollary 4.14 at each of those vertices and count the difference in the numbers D and N; $\Delta D = D' - D$ and $\Delta N = N' - N$. For v_1 we have

$$\hat{s}_C + \hat{s}_D = \hat{s}'_C + \hat{s}' + \hat{s}'_D - \Delta D + \Delta N. \tag{4.75}$$

Simple counting yields $\Delta D = 1$ and $\Delta N = 1$. Adjusting for the edge directions we get

$$\begin{aligned} \hat{s}_C &= -s_C - 1 \ , & \hat{s}'_C &= -s'_C - 1 \ , \\ \hat{s}_D &= s_D \ , & \hat{s}'_D &= s'_D \ , \\ & & \hat{s}' &= s' \ . \end{aligned} \tag{4.76}$$

In total this yields

$$(-s_C - 1) + s_D = (-s'_C - 1) + s' + s'_D \ . \tag{4.77}$$

The similar counting argument for the other vertices yields

$$\begin{aligned} v_2 &: \quad s_A + (-s - 1) + (-s_D - 1) = s'_A + (-s'_D - 1) \\ v_3 &: \quad (-s_A - 1) + s_B = (-s'_A - 1) + (-s' - 1) + s'_B \\ v_0 &: \quad (-s_B - 1) + s + s_C = (-s'_B - 1) + s'_C \end{aligned} \tag{4.78}$$

Assuming $s = s'$ and $s'_A = s_A$, the above set of equations has (4.74) as unique solution.

By Lemma 4.15 the vertex rule at the boundary has the same dependence on the numbers N and D_{inn} (replacing D) and on the edge directions. The parameter D_{bnd} does not affect the Pachner move. Thus the result holds for these cases, too. □

Proposition 4.20. Let (C, φ, Σ) and (C', φ', Σ) be related by a Pachner 3-1 move. Let the configuration of markings on the affected subcomplex be as in Figure 4.12. If the r-spin lift of χ_σ is such that $s'_A = s_A$, then the remaining edge indices are related by:

$$s'_B = s_B + s_{12} \quad , \quad s'_C = s_C - s_{31} - 1 \quad , \quad s_{12} + s_{23} + s_{31} = -1 \ , \tag{4.79}$$

where s_{12}, s_{23}, s_{31} are arbitrary, subject to the last condition.

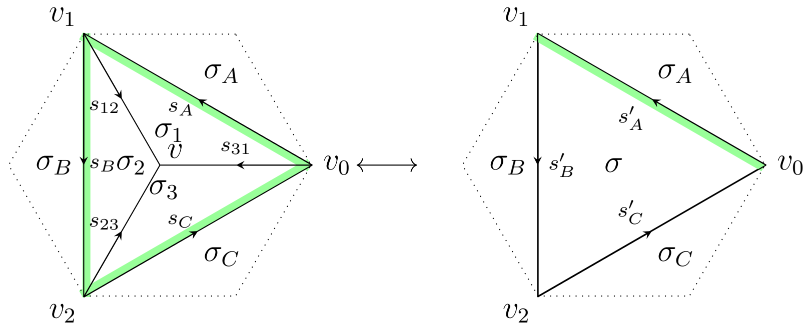


Figure 4.12.: Configuration of markings and labels for the Pachner 3-1 move and its inverse. v_0, v_1, v_2 are vertices in $\mathcal{C}$ and $\mathcal{C}'$, and v is a vertex in $\mathcal{C}$. $\sigma_1, \sigma_2, \sigma_3$ are faces of $\mathcal{C}$, σ is a face of $\mathcal{C}'$, and $\sigma_A, \sigma_B, \sigma_C$ are faces in $\mathcal{C}$ and $\mathcal{C}'$. s_{12}, s_{23}, s_{31} and s_A, s_B, s_C as well as the primed version are the edge indices of the respective edges. The orientation of the edges is as indicated by the arrows. The green lines indicate the marked edges of the triangles.

Proof. We will assume v_0, v_1, v_2 are inner. The argument in case some vertices belong to the boundary is the same. As in Proposition 4.19, we use either Corollary 4.14 or Lemma 4.15 at each of those vertices to get the relations

$$\begin{aligned} v &: \quad (-s_{12}-1)+(-s_{23}-1)+(-s_{31}-1)+3=1 \\ v_0 &: \quad s_A+s_{31}+(-s_C-1)=s'_A+(-s'_C-1)-1 \\ v_1 &: \quad (-s_A-1)+s_{12}+s_B=(-s'_A-1)+s'_B \\ v_2 &: \quad (-s_B-1)+s_{23}+s_C=(-s'_B-1)+s'_C \end{aligned} \tag{4.80}$$

We can choose $\tilde{\chi}_\sigma$ such that $s'_A = s_A$. Then the unique solution to the equations for v_0, v_1, v_2 is $s'_B = s_B + s_{12}$ and $s'_C = s_C - s_{31} - 1$. □

5. Two-dimensional lattice topological field theory

5.1. Preliminaries about Graphs

We first recall some graph theoretic notions used in [JS]. All graphs are finite. A *graph with boundary*, in the following just *graph*, $\Gamma = (\Gamma, \partial\Gamma)$ is a graph Γ together with a set $\partial\Gamma$ of univalent vertices. Elements of $\partial\Gamma$ are called *outer* vertices, and vertices not in $\partial\Gamma$ are called *inner* vertices. A graph Γ is *directed* if each edge is equipped with an orientation. For a directed graph and a vertex v, the set of ingoing edges is denoted as $\mathrm{in}(v)$ and that of outgoing edges as $\mathrm{out}(v)$. A *polarised graph* is a directed graph together with a choice of linear order on $\mathrm{in}(v)$ and $\mathrm{out}(v)$ for each inner vertex v. A *progressive graph* is a directed graph with no (oriented) circuits. The *domain* $\mathrm{dom}(\Gamma)$ (resp. *codomain* $\mathrm{cod}(\Gamma)$) of a progressive graph is the union of $\mathrm{out}(v)$ (resp. $\mathrm{in}(v)$) over all $v \in \partial\Gamma$. An *anchored progressive graph* Γ is a progressive graph together with linear orders on both $\mathrm{dom}(\Gamma)$ and $\mathrm{cod}(\Gamma)$.

Starting from a marked combinatorial surface $\mathcal{C}$ (Definition 4.4) we produce a progressive polarised graph $\Gamma(\mathcal{C})$ as follows:

1. Take the 1-skeleton of the Poincaré dual of $\mathcal{C}$. This yields a graph Γ' with only univalent and trivalent vertices. (The univalent vertices sit at the end of edges dual to edges of $\mathcal{C}$ that lie on the boundary.) Let $\partial\Gamma$ be the set of all univalent vertices of Γ'.

2. Put an additional vertex on each edge. The resulting graph $(\Gamma, \partial\Gamma)$ has bi- and trivalent inner vertices, with each inner edge bounding one bi- and one trivalent vertex. (An edge is inner if none of its bounding vertices is in $\partial\Gamma$.) Note that every edge bounds exactly one bivalent vertex.

3. Turn the graph Γ into a directed graph by orienting each edge away from the bivalent vertex.

4. For every trivalent vertex v, order the set $\mathrm{in}(v)$ as depicted in Figure 5.1 a), using the boundary maps d^2 of $\mathcal{C}$. (The set $\mathrm{out}(v)$ is empty by the orientation choice in 3.)

5. For every bivalent vertex v, order the set $\mathrm{out}(v)$ as depicted in Figure 5.1 b), using the boundary maps d^1 of $\mathcal{C}$.

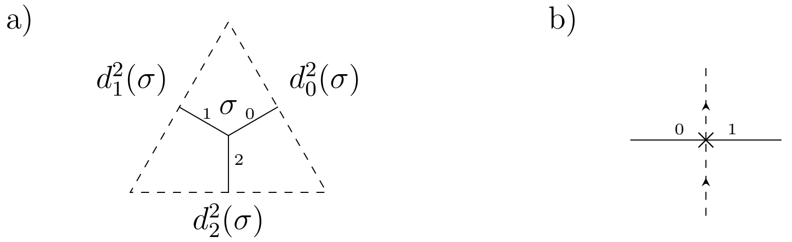


Figure 5.1.: The polarisation of the graph Γ. (a) A single triangle σ (in dashed lines) together with the dual graph. The linear order of the edges at the vertex in the centre is indicated by small numbers 0, 1, 2 which corresponds to the linear order given on the edges of σ by the maps d_0^2, d_1^2 and d_2^2. (b) A single edge of the triangulation as a dashed line, together with its orientation. The dual edge is drawn solid, and in its centre a new vertex has been placed. The linear order on the edges of that vertex is indicated by the numbers 0 and 1.

Since every edge starts at a bivalent vertex, the graph $\Gamma(\mathcal{C})$ constructed above has no circuits, so that it is indeed a polarised progressive graph. By the same argument, $\mathrm{dom}(\Gamma(\mathcal{C})) = \emptyset$.

To anchor this graph $\Gamma(\mathcal{C})$, we have to give an ordering on $\mathrm{cod}(\Gamma)$. We first order the elements of $\mathrm{cod}(\Gamma)$ by the number of the boundary component of $\mathcal{C}$ they start from. We then order the three edges for each boundary component according to the order on $\underline{\Delta}$ (see Figure 4.3), transported to the i'th boundary via the parametrisation map $f_i : \partial\underline{\Delta} \to \partial\mathcal{C}_i$.

Let $\mathcal{S}$ be a symmetric monoidal category. We will assume $\mathcal{S}$ to be strict monoidal in order to simplify notation. (But we will nonetheless think of **Vect** and **SVect** as examples, leaving it to the reader to add associators in the relevant places.) The symmetric structure will be denoted as

$$\sigma_{U,V} : U \otimes V \to V \otimes U \ . \tag{5.1}$$

We will briefly sketch how to pass from an anchored polarised progressive graph Γ to a morphism in $\mathcal{S}$, for details see [JS, Ch. 2]. Fix a *valuation* on Γ, that is, to each edge of Γ assign an object of $\mathcal{S}$, and to each vertex v a morphism in $\mathcal{S}$ compatible with the objects and linear order on $\mathrm{in}(v)$ and $\mathrm{out}(v)$. Since the graph is progressive, one can fix an order on the vertices of Γ such that for $v \geq v'$, there are no edges directed from v to v' ("all edges go up"). The graph Γ together with the valuation is called a *diagram in* $\mathcal{S}$.

Now compose all the morphisms assigned to the vertices in the chosen order, using the symmetric structure of $\mathcal{S}$ to match the in– and outgoing objects of the morphisms as dictated by the edges and tensoring with identity morphisms where necessary. This results in a morphism in $\mathcal{S}$ from the tensor product of the objects assigned to $\mathrm{dom}(\Gamma)$ in the chosen order to the corresponding product for $\mathrm{cod}(\Gamma)$. The resulting morphism is called the *value* of the diagram. By [JS, Cor. 2.3], the value of a diagram is independent of the ordering chosen on the vertices [JS, Ch. 2].[1]

We proceed to define a valuation for anchored polarised progressive graphs of the form $\Gamma(\mathcal{C})$ as described above. Choose $A \in \mathrm{Ob}(\mathcal{S})$ and morphisms

$$c_k : \mathbf{1}_\mathcal{S} \to A \otimes A \ , \qquad t : A \otimes A \otimes A \to \mathbf{1}_\mathcal{S} \ , \tag{5.2}$$

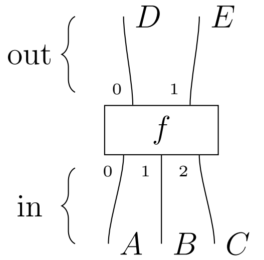
in $\mathcal{S}$ for $k \in \mathbb{Z}_r$.

Let $\Lambda = ((\mathcal{C}, f_i, d_0^1, d_0^2), (\varphi, \tilde{\chi}_\sigma), (\Sigma, \tilde{\varphi}_i))$ be an r-spin triangulated surface (Definition 4.6). To each bivalent vertex in $\Gamma(\mathcal{C})$ corresponds an (inner or boundary) edge e in the triangulation of Σ, with an edge index $s(e)$. To every such bivalent vertex assign the map $c_{s(e)}$. To every trivalent vertex assign the map t. This defines a valuation on $\Gamma(\mathcal{C})$ and therefore a morphism in $\mathcal{S}$. Let B be the number of boundary components of Σ. Then $|\mathrm{cod}(\Gamma)| = 3B$ while $\mathrm{dom}(\Gamma)$ is empty. We denote the value of the diagram by

$$T_{\mathrm{triang}}(\Lambda) : \mathbf{1}_\mathcal{S} \longrightarrow (A^{\otimes 3})^{\otimes B} \ . \tag{5.3}$$

5.2. Local moves

The local moves from Section 4.6 and 4.9 relate r-spin triangulations and consequently the corresponding diagrams. From these moves we will derive a sufficient set of relations on the maps c_k and t such that the resulting morphism T_{triang} is invariant under the moves. Some of these conditions are easiest presented in the standard graphical notation for morphisms in a tensor category, see e.g. [BK]. This is basically the language of diagrams where the vertices are drawn as boxes labeled by the morphism assigned by the valuation. The separation into $\mathrm{in}(v)$ and $\mathrm{out}(v)$ as well as the linear order is encoded by how the lines attach to the boxes. For example, a morphism $f : A \otimes B \otimes C \to D \otimes E$ is drawn as:



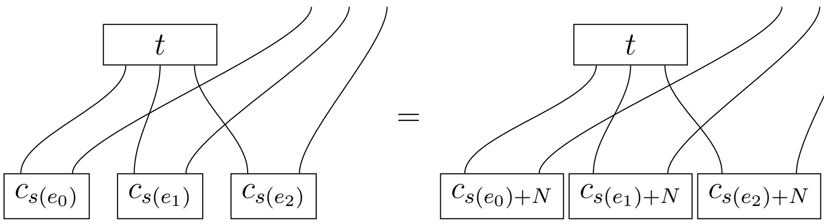
(5.4)

1. *Edge orientation change.* By Lemma 4.11 (2), replacing the orientation of a single edge e corresponds to a change of the edge index $s(e)$ to $-s(e)-1$. This corresponds to exchanging c_k and c_{-k-1} on the corresponding bivalent vertex v, together with a change of linear order on the outgoing edges. We thus require that

$$c_k = \sigma_{A,A} \circ c_{-k-1} \ . \tag{5.5}$$

[1] Strictly speaking at some point we have to take the geometrical realisation of the graph, since [JS] deals with topological graphs. Different choices of the geometrical realisation lead however to isomorphic diagrams, which due to [JS, Cor. 2.3] represent identical morphisms.

2. *Leaf exchange automorphism on a single triangle.* Let σ be a triangle that is to the left of each bounding edge. The other cases of edge orientations can be deduced from relation (5.5). By Lemma 4.11 (1), changing the r-spin lift of the characteristic map χ_σ for a single triangle σ corresponds to adding $N \in \mathbb{Z}_r$ to the edge indices on its three bounding edges e_0, e_1, e_2. We thus require that



(5.6)

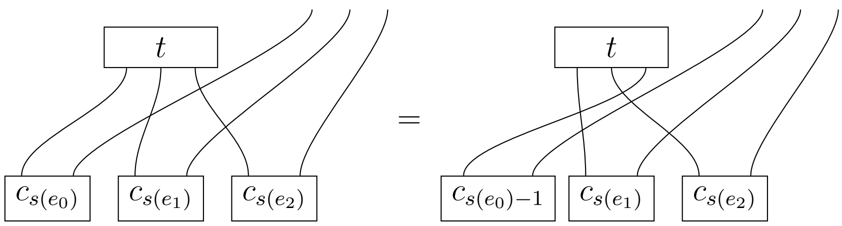
holds for all $N \in \mathbb{Z}_r$. Written out explicitly, this identity reads

$$f(s(e_0), s(e_1), s(e_2)) = f(s(e_0) + N, s(e_1) + N, s(e_2) + N) \ , \tag{5.7}$$

where

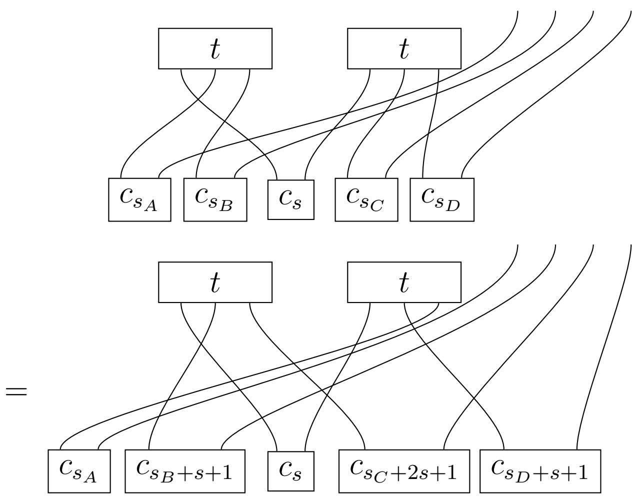
$$f(\alpha, \beta, \gamma) = \big(t \otimes \mathrm{id}_{A\otimes A\otimes A}\big) \circ \big(\mathrm{id}_{A\otimes A} \otimes c_{A,A} \otimes \mathrm{id}_{A\otimes A}\big) \\ \circ \big(\mathrm{id}_A \otimes c_{A,A} \otimes c_{A,A} \otimes \mathrm{id}_A\big) \circ \big(c_\alpha \otimes c_\beta \otimes c_\gamma\big) \ . \tag{5.8}$$

3. *Cyclic permutation of boundary edges for a single triangle.* As in 2., let σ be a triangle that is to the left of each bounding edge. By Lemma 4.11 (3), the value of the diagram will not change under such a cyclic permutation if we require the identity



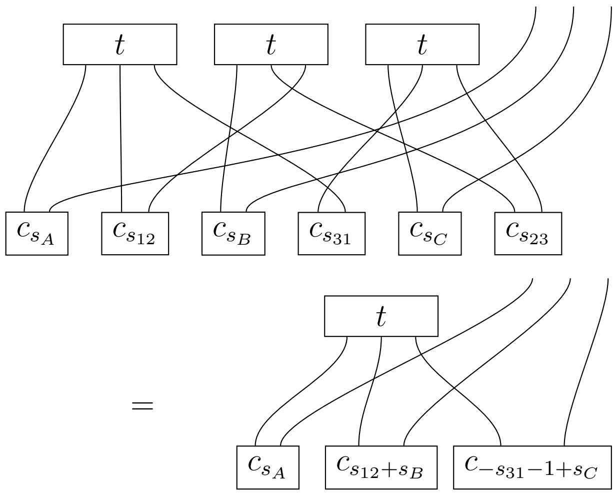
(5.9)

4. *Pachner 2-2 move.* By Proposition 4.19, a sufficient condition for invariance is



(5.10)

5. *Pachner 3-1 move and its inverse.* By Proposition 4.20, a sufficient condition for invariance is



(5.11)

whenever $s_{12} + s_{23} + s_{31} = -1$.

Proposition 5.1. Let A and t, c_k satisfy relations 1–5 above. Let Λ_1 and Λ_2 be two r-spin triangulated surfaces with the same underlying r-spin surface Σ. Then $T_{\text{triang}}(\Lambda_1) = T_{\text{triang}}(\Lambda_2)$.

Proof. For $\alpha = 1, 2$, let $\Lambda_\alpha = ((\mathcal{C}_{[\alpha]}, f_{[\alpha]i}, d^1_{[\alpha]0}, d^2_{[\alpha]0}), (\varphi_{[\alpha]}, \tilde{\chi}_{[\alpha]\sigma}), (\Sigma, \tilde{\varphi}_i))$. To prove the assertion, we will modify Λ_2 in several steps, each one leaving T_{triang} invariant.

By [Mu, Cor. 10.13] we can approximate $\varphi_{[2]}$ by a triangulation $\varphi_{[2a]} : |\mathcal{C}_{[2a]}| \to \underline{\Sigma}$ such that $\varphi_{[1]}^{-1} \circ \varphi_{[2a]}$ is piecewise-linear and $\mathcal{C}_{[2a]}$ is a subdivision of $\mathcal{C}_{[2]}$. We construct an r-spin triangulated surface Λ_{2a} from Λ_2 in two steps. First we pass from $\mathcal{C}_{[2]}$ to $\mathcal{C}_{[2a]}$ via a series of Pachner 2-2, 3-1 and 1-3 moves, and carry out the r-spin lifts of this sequence of moves as chosen in Section 4.9 on the r-spin triangulation Λ_a. Relations 4 and 5 guarantee that this does not change T_{triang}. Then carry out the small deformation from the resulting map $\varphi : |\mathcal{C}_{[2a]}| \to \underline{\Sigma}$ to $\varphi_{[2a]}$, along with a lift of the deformation to the r-spin lifts $\tilde{\chi}$. This does not affect the combinatorial data, and hence not T_{triang}. Altogether,

$$T_{\text{triang}}(\Lambda_2) = T_{\text{triang}}(\Lambda_{2a}) \ . \tag{5.12}$$

Since the triangulations $\varphi_{[1]}$ and $\varphi_{[2a]}$ now have a common subdivision, we can pass from $\varphi_{[2a]}$ to $\varphi_{[1]}$ by a sequence of Pachner moves. Let Λ_{2b} be the r-spin triangulation resulting from the r-spin lift of this sequence, so that again by Relations 4 and 5 we have

$$T_{\text{triang}}(\Lambda_{2a}) = T_{\text{triang}}(\Lambda_{2b}) \ . \tag{5.13}$$

At this point we have $\mathcal{C}_{[1]} = \mathcal{C}_{[2b]}$, $\varphi_{[1]} = \varphi_{[2b]}$ and $f_{[1]i} = f_{[2b]i}$, so that Λ_1 and Λ_{2b} differ at most in the marking d^2_0, d^1_0 and in the choice of r-spin lifts $\tilde{\chi}$.

Denote by s_1, resp. s_{2b} the edge indices resulting from Λ_1, resp. Λ_{2b}. Since Λ_1 and Λ_{2b} are r-spin triangulations of the same r-spin surface Σ, by Theorem 4.17 we have $S(\underline{\Lambda}_1, s_1) \cong S(\underline{\Lambda}_{2b}, s_{2b})$ as r-spin structures. Since the underlying triangulations already

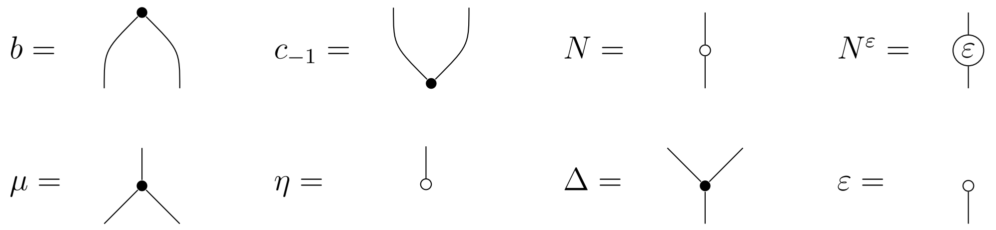


Figure 5.2.: Abbreviated graphical notation for frequently used morphisms. The pairing b and copairing c_{-1} give a duality on A, see (5.15). The Nakayama automorphism N is defined in (5.16). The product μ is given in (5.21), the existence of the unit η as assumed in Assumption 2, the coproduct Δ and counit ε are defined in (5.27).

agree, by Theorem 4.18 we must have $(d_{[1]}, s_1) \sim_{\text{fix}} (d_{[2b]}, s_{2b})$, where $d_{[1]} := (d^1_{[1]0}, d^2_{[1]0})$ and dito for $d_{[2b]}$. By Relations 1–3, T_{triang} is constant on equivalence classes for $\sim_{\text{fix}}$. Thus finally

$$T_{\text{triang}}(\Lambda_{2b}) = T_{\text{triang}}(\Lambda_1) \ . \tag{5.14}$$

□

5.3. Analysis of the algebraic structure

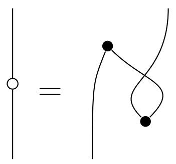
In the previous section we described a set of relations between the morphisms c_k and t, which guarantee invariance of the T_{triang} under the local moves. To further analyse these relations we will make two additional assumptions. The first one is:

> **Assumption 1:** The copairing c_{-1} is nondegenerate, i.e. there is a map $b : A \otimes A \to \mathbf{1}$ such that
>
> $$(b \otimes \text{id}_A) \circ (\text{id}_A \otimes c_{-1}) = \text{id}_A = (\text{id}_A \otimes b) \circ (c_{-1} \otimes \text{id}_A) \ . \tag{5.15}$$

By (5.5), Assumption 1 implies that c_0 also is nondegenerate with corresponding pairing $b \circ \sigma_{A,A}$. Furthermore the map b in Assumption 1 is unique. Define

$$N = (b \otimes \text{id}_A) \circ (\text{id}_A \otimes \sigma_{A,A}) \circ (\text{id}_A \otimes c_{-1}) \ : \ A \longrightarrow A \ . \tag{5.16}$$

In graphical notation, this reads



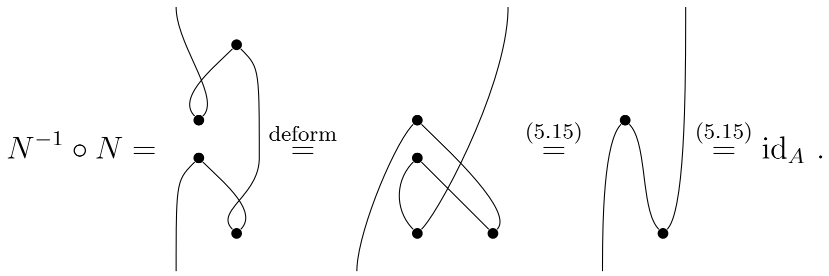
$$, \tag{5.17}$$

where we have started to use shorthand graphical symbols for some morphisms that will appear frequently. These abbreviations are collected in Figure 5.2.

The map N is invertible, with inverse given by

$$N^{-1} = (\text{id}_A \otimes b) \circ (\sigma_{A,A} \otimes \text{id}_A) \circ (c_{-1} \otimes \text{id}_A) \ . \tag{5.18}$$

One of the two computations to verify this is a follows:



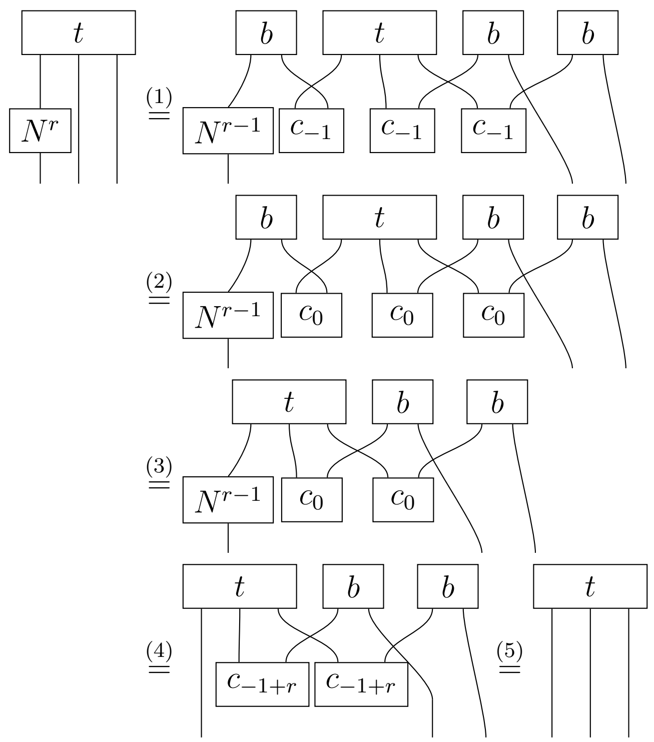
(5.19)

The map N has an additional important property.

Lemma 5.2. For $r < \infty$, the map N^r acts trivially on the map t:

$$t \circ (N^r \otimes \mathrm{id}_{A \otimes A}) = t \circ (\mathrm{id}_A \otimes N^r \otimes \mathrm{id}_A) = t \circ (\mathrm{id}_{A \otimes A} \otimes N^r) = t \ . \tag{5.20}$$

Proof. We verify the first equality. The other cases then follow from the cyclic property of t stated in (5.9). We have:



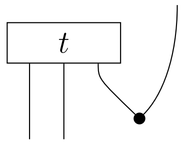
In step 1 we used the non-degeneracy of c_{-1} (Assumption 1) to insert two pairs b, c_{-1}. We also replaced one of the N's by its definition in (5.16). Step 2 is the leaf exchange (5.6). In step 3 we use the edge exchange, Equation (5.5), on the leftmost b_1, c_0 pair and replace it by id_A using Assumption 1. In step 4 the steps 2 and 3 are replicated $r - 1$ more times. Finally, in step 5 the remaining pairs b_1, c_1 are cancelled via Assumption 1. □

We would like to cast the data t, c_k into a more standard algebraic form, namely that of a Frobenius algebra. We recall that a *Frobenius algebra* (in a monoidal category) is a unital associative algebra and a counital coassociative coalgebra such that the coproduct is a map of bimodules.

We start by introducing a product: Let

$$\mu = (t \otimes \mathrm{id}_A) \circ (\mathrm{id}_{A\otimes A} \otimes c_{-1}) \ : \ A \otimes A \to A \ . \tag{5.21}$$

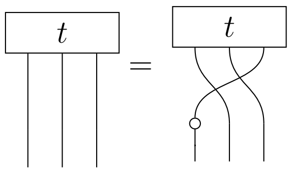
In graphical notation, this reads

$\mu =$  . (5.22)

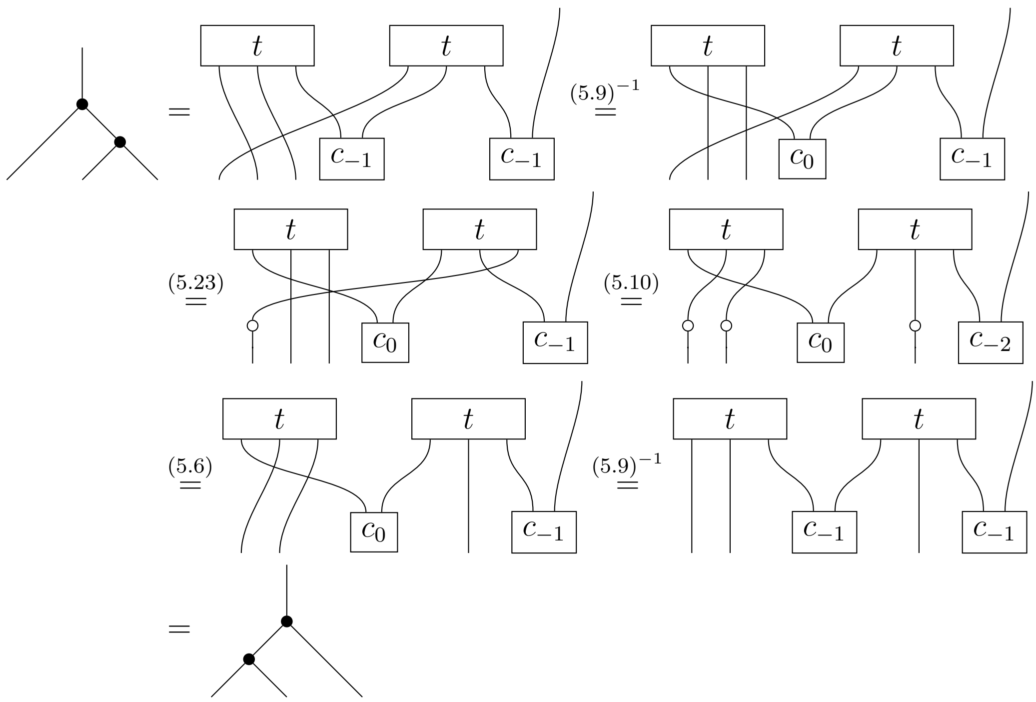
For μ we will use the graphical shorthand listed in Figure 5.2.

Lemma 5.3. The map $\mu : A \otimes A \to A$ is associative.

Proof. Using non-degeneracy of c_{-1} and (5.5), we can rewrite the cyclicity property (5.9) as an identity of morphisms $A^{\otimes 3} \to \mathbf{1}_S$:

 (5.23)

To see this write out N and add pairs of c_{-1}, b on the right hand side. The same can be done for the Pachner moves (5.10) and (5.11); we omit the details. We then compute:

 (5.24)

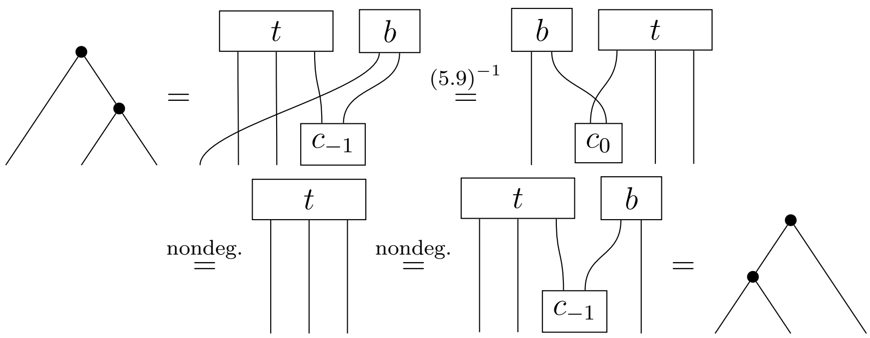
□

In the second and third equality we used the cyclicity property (in two different versions). The fourth equality is the relation from the Pachner 2-2 move: Start with the right hand side, expand the Ns and then use (5.10). The remaining cs are removed by (5.5) and nondegeneracy. The fifth equality is (5.6) used on both ts with $N = 1$.

Lemma 5.4. The pairing b is invariant with respect to the product μ, i.e.

$$b \circ (\mu \otimes \mathrm{id}_A) = b \circ (\mathrm{id}_A \otimes \mu) \ . \tag{5.25}$$

Proof. By direct computation:



(5.26)

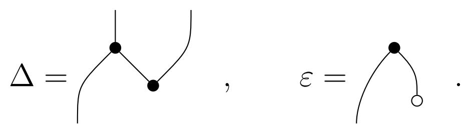
□

To proceed, we need to make our second assumption:

Assumption 2: The algebra (A, μ) has a unit $\eta : \mathbf{1}_{\mathcal{S}} \to A$.

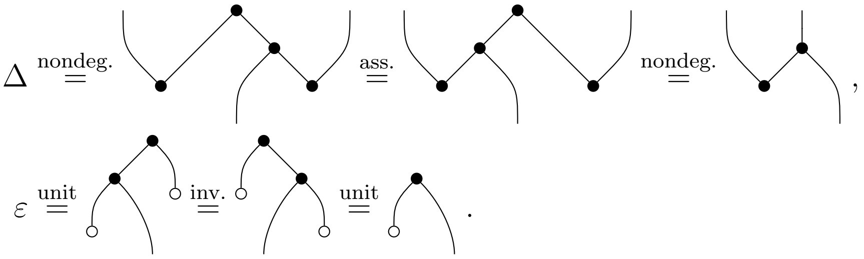
The graphical notation we use for the unit is listed in Figure 5.2. By Lemma 5.4 and Assumption 2, A together with the (non-degenerate) pairing b is a Frobenius algebra.

One may now define a coalgebra structure on the Frobenius algebra A in the standard way, so that the coproduct is a map of A-A-bimodules. Explicitly, the coproduct $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to \mathbf{1}$ are given by



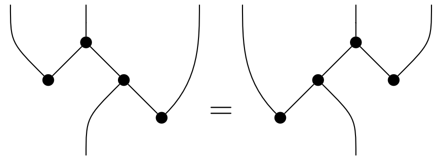
(5.27)

The asymmetry in this definition is only apparent, since

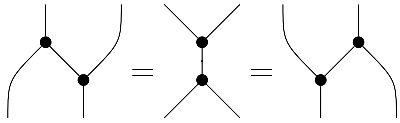


(5.28)

It is now trivial to see that ε is indeed a counit. Coassociativity is easily checked by combining the two expressions for Δ:

$(\Delta \otimes \mathrm{id}_A) \circ \Delta =$  $= (\mathrm{id}_A \otimes \Delta) \circ \Delta$. (5.29)

The Frobenius property, which states that Δ is a bimodule map, namely

 , (5.30)

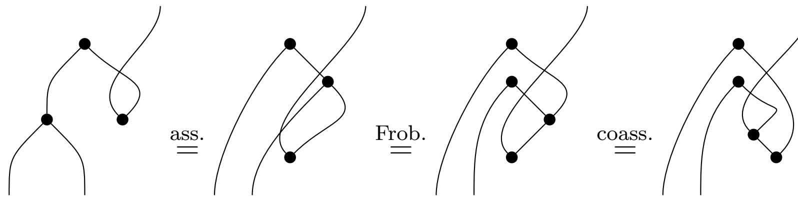
is equally straightforward to check. We omit the details. Finally, note that

$$b = \varepsilon \circ \mu \quad , \quad c_{-1} = \Delta \circ \eta \ . \tag{5.31}$$

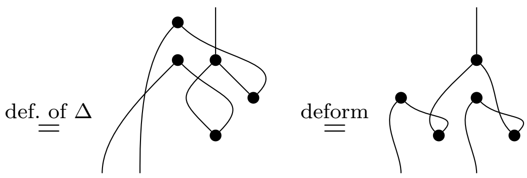
From hereon we consider A as a Frobenius algebra with structure morphisms $\mu, \eta, \Delta, \varepsilon$ as described above. By definition, the morphism N defined in (5.16) is the Nakayama automorphism of A, see e.g. [FSt]. For completeness we state

Proposition 5.5. The Nakayama automorphism is a unital algebra automorphism and a counital coalgebra automorphism of a Frobenius algebra.

Proof. That $N \circ \eta = \eta$ and $\varepsilon \circ N = \varepsilon$ is straightforward. Compatibility with the product follows from

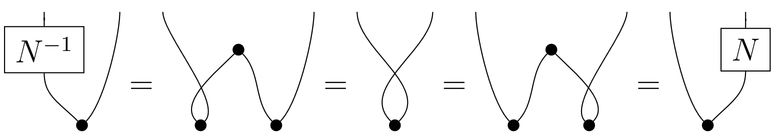
$N \circ \mu =$ 

(5.32)



$= \mu \circ (N \otimes N)$.

To see compatibility with the coproduct, first note that



. (5.33)

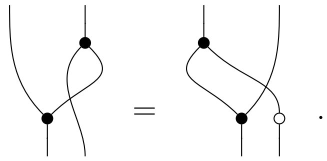
When combining this with the definition of the coproduct in terms of the product and copairing in (5.27), the compatibility of N with the coproduct follows from the already established result that N is an algebra homomorphism. □

The following identity will be used frequently in the calculations below:

Lemma 5.6. Let A be a Frobenius algebra and N its Nakayama automorphism. Then

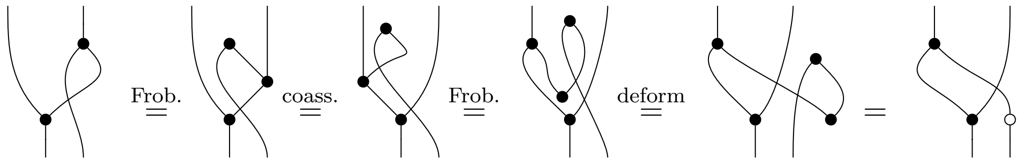
$$\big[\mathrm{id}\otimes(\mu\circ\sigma_{A,A})\big]\circ\big[\Delta\otimes\mathrm{id}\big]=\big[\mu\otimes\mathrm{id}\big]\circ\big[\mathrm{id}\otimes\sigma_{A,A}\big]\circ\big[\Delta\otimes N\big]\ . \tag{5.34}$$

Graphically, this reads



(5.35)

Proof. By direct calculation:



□

Proposition 5.5 and Lemma 5.6 hold in general. For the Frobenius algebra A constructed above from t, c_k we have in addition:

Lemma 5.7. For $r<\infty$, the Nakayama automorphism of A satisfies $N^r=\mathrm{id}_A$.

Proof. We use the unit property, nondegeneracy and Lemma 5.2:

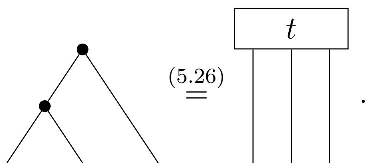
$$N^r=(t\otimes\mathrm{id}_A)\circ(N^r\otimes\eta\otimes c_{-1})\overset{5.2}{=}(t\otimes\mathrm{id}_A)\circ(\mathrm{id}_A\otimes\eta\otimes c_{-1})=\mathrm{id}_A\ . \tag{5.36}$$

□

We call a Frobenius algebra Δ-*separable* if $\mu\circ\Delta\circ\eta=\eta$, i.e. $\Delta\circ\eta$ is a separability idempotent. We have:

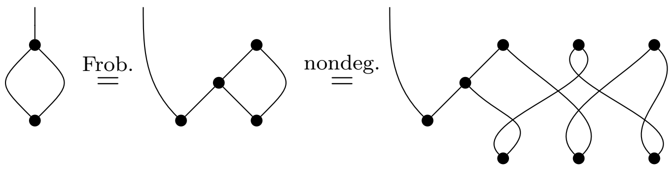
Lemma 5.8. A is Δ-separable.

Proof. First note that

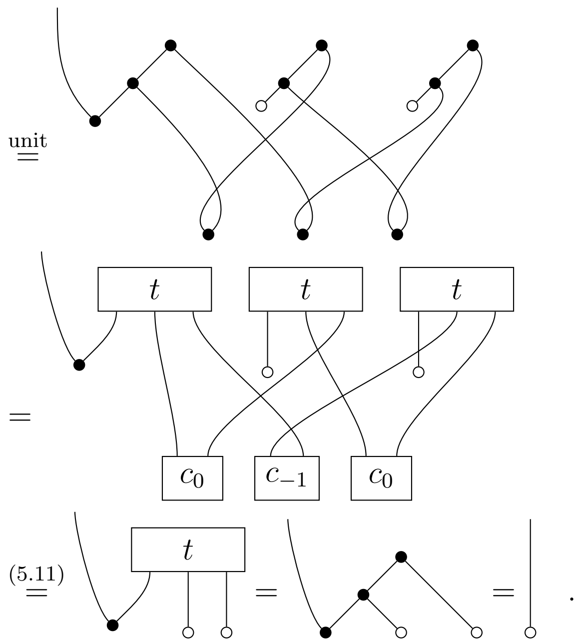


(5.37)

The statement follows from the identities



(5.38)



□

We have now arrived at the desired algebra structure encoding t, c_k and their properties. As described above, under Assumptions 1 and 2, the data t, c_k, subject to relations 1–5 in Section 5.2, give rise to a Δ-separable Frobenius algebra whose Nakayama automorphism fulfils $N^r = \mathrm{id}$ for $r < \infty$. The following result shows that the converse holds as well; the proof can be found in Appendix C.

Proposition 5.9. Let A be a Δ-separable Frobenius algebra whose Nakayama automorphism is an involution. Set

$$t = \varepsilon \circ \mu \circ (\mu \otimes \mathrm{id}_A) \ , \quad c_k = (\mathrm{id}_A \otimes N^{k+1}) \circ \Delta \circ \eta \ . \tag{5.39}$$

Then t, c_k fulfil relations 1–5 in Section 5.2.

With the tools assembled so far, we can prove the main result of this thesis. To state the result, we need a little bit more notation. Let B be the number of boundary components of the given r-spin surface. We would like to think of the morphism assigned to this r-spin surface as a "correlator", that is, we prefer to write it as a morphism $A^{\otimes 3B} \to \mathbf{1}_{\mathcal{S}}$ rather than the other way around as is the case for T_{triang} in (5.3). We use the map b to achieve this and define

$$T_A(\Sigma) := (b^{\otimes 3B}) \circ \tau \circ \left(\mathrm{id}_{A^{\otimes 3B}} \otimes T_{\text{triang}}(\mathcal{C}, \tilde{\varphi}, \Sigma)\right) \ , \tag{5.40}$$

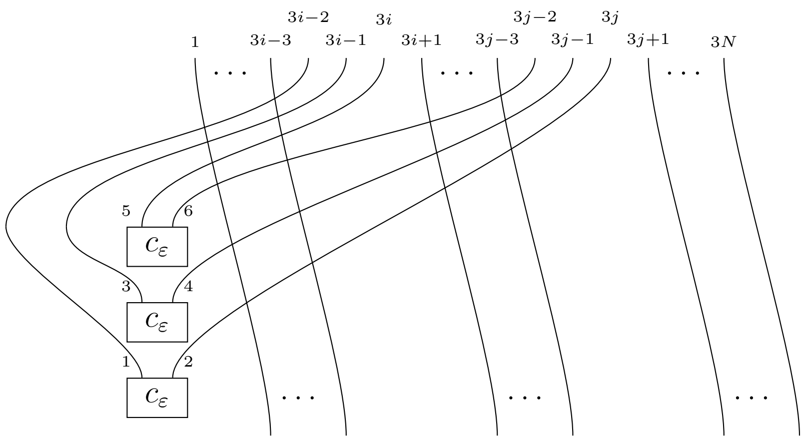
where τ is a permutation $(A^{\otimes 3B} \otimes A^{\otimes 3B}) \to (A^{\otimes 2})^{\otimes 3B}$, which connects the i'th factor of A in the first (resp. second) copy of $A^{\otimes 3B}$ in the source object to the first (resp. second) copy of A in the i'th factor of $A^{\otimes 2}$ in the target object.

Theorem 5.10. Let A be a Frobenius algebra in a symmetric strict monoidal category $\mathcal{S}$, such that A is Δ-separable and – in the case $r < \infty$ – its Nakayama automorphism N satisfies $N^r = \mathrm{id}_A$. Then $T_A(\Sigma)$ is independent of the choice of r-spin triangulation of the r-spin surface Σ and $T_A(\Sigma) = T_A(\Sigma')$ for isomorphic r-spin surfaces Σ and Σ'.

Proof. Let $\Lambda = ((\mathcal{C}, f_i, d_0^1, d_0^2), (\varphi, \tilde{\chi}_\sigma), (\Sigma, \tilde{\varphi}_i))$ be an r-spin triangulation of Σ. By Proposition 5.9, $t, c_{\pm 1}$ as defined via A satisfy relations 1–5 in Section 5.2. By Proposition 5.1, this implies that $T_{\mathrm{triang}}(\Lambda)$ is independent of the choice of r-spin triangulation. Given an isomorphism $\tilde{f} : \Sigma \to \Sigma'$ of r-spin surfaces, we obtain an r-spin triangulation $f_*\Lambda := ((\mathcal{C}, f_i, d_0^1, d_0^2), (f \circ \varphi, \tilde{f} \circ \tilde{\chi}_\sigma), (\Sigma', \tilde{\varphi}_i'))$ of Σ' which produces the same edge indices as Λ. Hence, T_{triang} gives the same morphism in $\mathcal{S}$ for Λ and $f_*\Lambda$. □

5.4. Behaviour of the morphisms under gluing of r-spin surfaces

Let Σ be an r-spin triangulated surface, and (i, j, ε) be r-spin gluing data such that (i, j) is simplicial gluing data (see Section 4.1) Consider the morphism

$\Gamma_{i,j,\varepsilon} =$ 

. (5.41)

from $A^{3(B-2)}$ to A^{3B} with B the number of boundary components of Σ. In formulas, this reads

$$\Gamma_{i,j,\varepsilon} = \tau \circ \left(c_\varepsilon^{\otimes 3} \otimes \mathrm{id}_A^{\otimes 3(B-2)} \right) , \tag{5.42}$$

where $\tau : A^{\otimes 3B} \to A^{\otimes 3B}$ represents the permutation that connects

- the first output to the $(3i-2)$'th input,
- the second output to the $(3j)$'th input,
- the third output to the $(3i-1)$'th input,
- the fourth output to the $(3j-1)$'th input,
- the fifth output to the $(3i)$'th input and

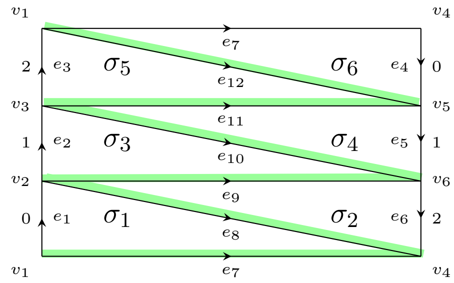


Figure 5.3.: A triangulation of the cylinder together with markings and labels. Triangles are labeled by $\sigma_1, \dots, \sigma_6$, edges are labeled by $e_1, \dots, e_{12}$ and vertices are labeled by $v_1, \dots, v_6$. The bottom and top edge – both labeled by e_7 – are identified. The marked edge $d_0^2(\sigma_i)$ for each triangle σ_i is indicated by a fat green line. The positions of the boundary edges on the respective boundary are given by small numbers $0, 1, 2$. The boundaries are ordered from left to right.

- the sixth output to the $(3j-2)$'th input.

τ keeps the order of the remaining tensor factors fixed.

We now come to an important property of T_A: the behaviour under gluing.

Proposition 5.11. Let Σ be a r-spin triangulated surface, and (i, j, ε) be spin gluing data. Then

$$T_A(\Sigma^{\varepsilon}_{i\#j}) = T_A(\Sigma) \circ \Gamma_{i,j,\varepsilon} \ . \tag{5.43}$$

Proof. Label the edges of the i-th respectively j-th boundary by i_0, i_1, i_2 and j_0, j_1, j_2. Untangling the definitions of T_A, T_{triang} and $\Gamma_{i,j,\varepsilon}$ a straightforward calculation shows that the composition $T_A(\Sigma) \circ \Gamma_{i,j,\varepsilon}$ is the same as T_A for the glued triangulation with edge indices for the glued edges as

$$\begin{aligned} &s_{j_0} - s_{i_2} + \varepsilon \ , \\ &s_{j_1} - s_{i_1} + \varepsilon \ , \\ &s_{j_2} - s_{i_0} + \varepsilon \ . \end{aligned} \tag{5.44}$$

These are the same as given by Lemma 4.10 for the glued surface. □

5.5. Evaluation of the TFT on the cylinder

Let C be an r-spin cylinder, i.e. an r-spin surface such that $\underline{C} \cong S^1 \times [0,1]$. Figure 5.3 gives an explicit triangulation of $\underline{C}$ with markings and labels.

By the construction in Section 4.8, an admissible edge index configuration determines an r-spin structure. We proceed to find the admissible edge index configurations. Let $s_i := s(e_i)$ be the edge indices, which are yet to be determined. By applying Lemma 4.11 (1) for the faces $\sigma_1, \dots, \sigma_6$, we can choose some of the edge indices to our liking. Specifically, we will set

$$s_4 = s_5 = s_6 = s_8 = s_{10} = s_{12} = 0 \ . \tag{5.45}$$

Next we evaluate Lemma 4.15, the boundary vertex rule at $v_1, \ldots, v_6$. Let λ_L be the type of the boundary with vertices v_1, v_2, v_3 and λ_R be that of the boundary with vertices v_4, v_5, v_6. We obtain $D_{bnd} = 1$ for v_1 and v_4 and $D_{bnd} = 0$ else, $N = 3$ for all vertices, $D_{inn} = 2$ for v_1, v_2, v_3, and $D_{inn} = 0$ for v_4, v_5, v_6, so that

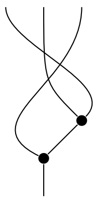
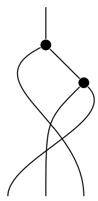
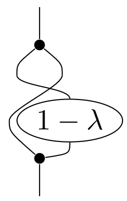
$$\begin{aligned} &v_1 : \ s_1 - s_3 + s_7 = 1 - \lambda_L \ , && v_4 : \ -s_7 = 1 - \lambda_R \ , \\ &v_2 : \ -s_1 + s_2 + s_9 = 0 \ , && v_5 : \ s_{11} = 0 \ , \\ &v_3 : \ -s_2 + s_3 + s_{11} = 0 \ , && v_6 : \ s_9 = 0 \ . \end{aligned} \tag{5.46}$$

Solutions exist if and only if $\lambda_L + \lambda_R - 2 = 0$ in $\mathbb{Z}_r$. All solutions to this set of equations are then given by

$$s_1 = s_2 = s_3 = \varepsilon + 1 \quad , \qquad s_9 = s_{11} = 0 \ , \qquad s_7 = 1 - \lambda_L \ , \tag{5.47}$$

for a parameter $\varepsilon \in \mathbb{Z}_r$. Setting $\lambda := \lambda_L$ let us denote the resulting r-spin cylinders by $C^{\varepsilon}_{\lambda}$.

The calculation of the corresponding morphism $T_A(C^{\varepsilon}_{\lambda})$ is given in Appendix D. To express the resulting morphisms, it is helpful to define the maps

$\iota^{13} =$  , $\pi^{31} =$  , $P^{\lambda} =$ 

. (5.48)

In terms of these, the cylinder morphisms read

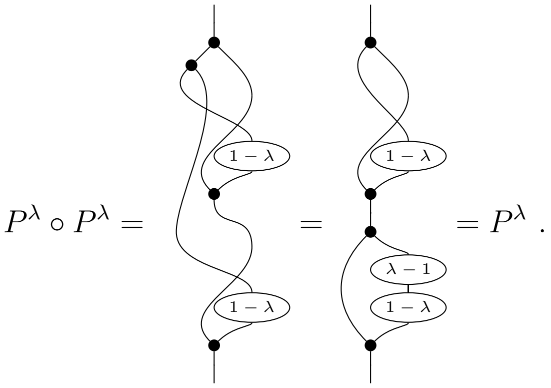
$$T_A(C^{\varepsilon}_{\lambda}) = b \circ (P^{\lambda} \otimes \mathrm{id}_A) \circ (N^{\varepsilon} \otimes \mathrm{id}_A) \circ (\pi^{31} \otimes \pi^{31}) \ . \tag{5.49}$$

5.6. Cylinder projections and state spaces

Lemma 5.12. For $\lambda \in \mathbb{Z}_r$,

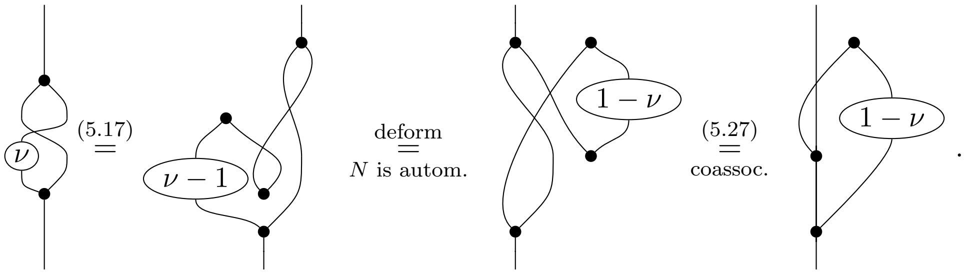
1. $P^{\lambda} \circ P^{\lambda} = P^{\lambda}$.
2. $P^{\lambda} \circ N = N \circ P^{\lambda}$.
3. $P^{\lambda} \circ N^{\gcd(\lambda-1,r)} = P^{\lambda}$ for $r < \infty$ and $P^{\lambda} \circ N^{\lambda-1} = P^{\lambda}$ for $r = \infty$.

Proof. For part 1 one computes



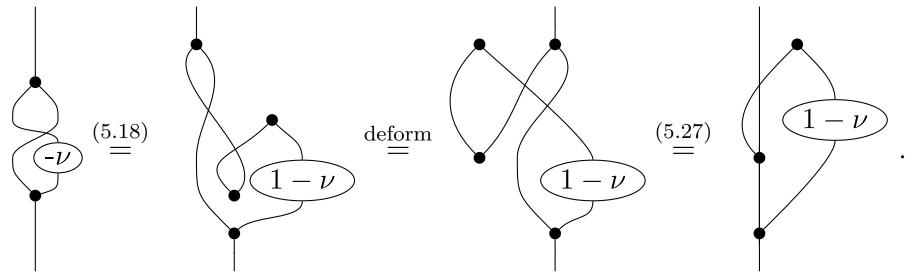
(5.50)

Part 2 follows since N is an automorphism of Frobenius algebras (Proposition 5.5). For part 3 first note that



(5.51)

On the other hand, inserting $N^{-\nu}$ on the right and using the explicit expression (5.18) for N^{-1} we get



(5.52)

which is equal to (5.51). This implies $P^\lambda \circ N^{\lambda-1} = P^\lambda$. The claim for $r < \infty$ follows since $N^r = \mathrm{id}_A$. □

Next we turn the cylinder morphisms into endomorphisms of $A^{\otimes 3}$ via the "copairing" Γ we defined in (5.42). First observe that

$$(\pi^{31} \otimes \mathrm{id}_{A^{\otimes 3}}) \circ \Gamma_{1,2,-1} = (\mathrm{id}_A \otimes \iota^{13}) \circ \Delta \circ \eta \ . \tag{5.53}$$

Using this, one arrives at

$$f_\lambda^\varepsilon := (T_A(C_\lambda^\varepsilon) \otimes \mathrm{id}_{A^{\otimes 3}}) \circ \Gamma_{2,3,-1} = \iota^{13} \circ P^\lambda \circ N^\varepsilon \circ \pi^{31} \ . \tag{5.54}$$

Since $\pi^{31} \circ \iota^{13} = \mathrm{id}_A$, together with Lemma 5.12 it follows that f_λ^ε are idempotents if $2\varepsilon = m \cdot \gcd(\lambda - 1, r)$. In particular f_λ^0 and $f_\lambda^{\lambda-1}$ are always idempotents.

To speak about state spaces, we need a further assumption:

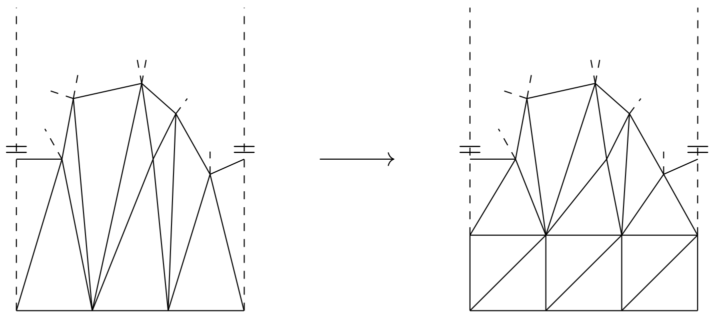


Figure 5.4.: Adding the cylinder triangulation from Figure 5.3 at a boundary component by pushing the existing triangulation inwards.

Assumption 3: The idempotents P^λ are split.

Let us denote the image of P^λ by $Z^\lambda(A)$ and write $\iota^\lambda : Z^\lambda(A) \to A$ and $\pi^\lambda : A \to Z^\lambda(A)$ for the embedding and restriction maps. We will call $Z^\lambda(A)$ the λ-*type state space* of the r-spin TFT associated to A. The morphism $T_A(\Sigma)$ assigned to an r-spin surface Σ factors through the state spaces in the following sense:

Proposition 5.13. Let Σ be an r-spin surface and let $\lambda_i \in \mathbb{Z}_r$ be the type of the i'th of the B boundary components of Σ. Then

$$T_A(\Sigma) = T_A(\Sigma) \circ \bigotimes_{i=1}^{B} \left(\iota^{13} \circ \iota^{\lambda_i} \circ \pi^{\lambda_i} \circ \pi^{31}\right) . \tag{5.55}$$

Proof. Choose an r-spin triangulation Λ of Σ. Consider the triangulation close to the i'th boundary component of Σ. We may assume that none of the boundary edges are marked edges of the adjacent triangle, and that the edge index is 0 for all boundary edges. From this build a new r-spin triangulation Λ' by "pushing the triangulation Λ slightly inward" and adding a cylinder as in Figure 5.3 with $\varepsilon = 0$. This is illustrated in Figure 5.4.

The edge indices of the initial r-spin triangulation Λ are not affected by the above procedure. By Theorem 5.10, the r-spin triangulations Λ and Λ' produce the same morphism $T_A(\Sigma)$. Comparing the morphisms produced by these two triangulations, this amounts to the identity

$$T_A(\Sigma) = \left(T_A(C^0_{\lambda_i}) \otimes T_A(\Sigma)\right) \circ \Gamma_{2,i,-1} . \tag{5.56}$$

Substituting $f^0_{\lambda_i}$ from (5.54), one sees that the right hand side is equal to $T_A(\Sigma) \circ (\mathrm{id} \otimes \cdots \otimes f^0_{\lambda_i} \otimes \cdots \otimes \mathrm{id})$, where $f^0_{\lambda_i}$ is inserted on the i'th tensor factor $A^{\otimes 3}$. Repeating this for each i and using $f^0_{\lambda_i} = \iota^{13} \circ \iota^{\lambda_i} \circ \pi^{\lambda_i} \circ \pi^{31}$ proves the claim. □

Remark 5.14. Let $\lambda \in \mathbb{Z}_r$. The Nakayama automorphism of A induces an automorphism, $\pi^\lambda \circ N \circ \iota^\lambda$, on $Z^\lambda(A)$, which by abuse of notation we still call N, or $N|_{Z^\lambda}$ if we want to be more specific. In addition $N|^r_{Z^{\gcd(\lambda-1,r)}} = \mathrm{id}$, as follows from $\pi^\lambda \circ \iota^\lambda = \mathrm{id}_{Z^\lambda(A)}$, $\iota^\lambda \circ \pi^\lambda = P^\lambda$ and Lemma 5.12.

6. Spin lattice TFT

After having developed the general theory for framed and r-spin surfaces we now turn to spin surfaces, the special – and particularly interesting case of $r = 2$. The algebraic structure simplifies in that the Nakayama automorphism is now an involution. On the geometric side all oriented surfaces admit a spin structure type and 0 and 1 boundaries can be glued to themselves. This helps a more detailed analysis of the structure of the topological field theory constructed. We will first review and analyse the structure of the topological field theory for this special case, notably the field theory on the three pointed sphere and the torus. Next we will be able to compare to the ample literature about spin topological field theories in two dimensions: In [MS] spin topological field theories with and without boundaries in two dimensions are discussed. In [BT] state sum models on closed spin surfaces are addressed. [Gu] discusses spin topological field theories in two dimensions using the cobordism hypothesis. In [BCP] a similar algebraic structure arises in the study of orbifolds of the topological twist of $N = (2, 2)$ supersymmetric field theories.

6.1. Overview of the simplifications in the case $r = 2$

As stated above there are several simplifications for the special case of spin surfaces. Let us focus on the geometry first:

6.1.1. Boundary conditions and glueing

There are 2 boundary conditions. In accordance with string theory literature we label these as NS and R boundary conditions. Here the NS spin structure on the circle is the one that extends to the disc, corresponding to $\lambda = 0$. In Definition 3.16 glueing of an r-spin surfaces along two boundary components with boundary conditions λ_1, λ_2 is defined if $\lambda_1 + \lambda_2 - 2 \equiv 0 \mod r$. We conclude that NS and R boundaries can be glued to themselves as expected. The glued indices are computed according to Lemma 4.10 as before.

6.1.2. Moves changing the marking

Since most of the moves no longer depend on edge directions, Lemma 6.1 simplifies to

Lemma 6.1. Let $\Sigma = ((\mathcal{C}, f_i), (\varphi, \tilde{\chi}_\sigma), (\Sigma, \tilde{\varphi}_i))$ be a spin triangulated surface and let $s(e)$ be the corresponding edge indices.

1. Let σ be a face of $\mathcal{C}$. Change $\tilde{\chi}_\sigma$ by precomposing with the leaf exchange automorphism, i.e.

$$\tilde{\chi}_{\mathrm{new},\sigma} = \tilde{\chi}_{\mathrm{old},\sigma} \circ \omega_k \ .$$

   Pick an edge $e \in \mathcal{B}(\sigma)$. Then $s(e) \mapsto s(e) + 1$.

2. Let e be an inner edge of $\mathcal{C}$. Exchange of the marking $d_0^1(e)$, i.e.

$$d^1_{\mathrm{new},0}(e) = d^1_{\mathrm{old},1}(e) \ .$$

   Then $s(e) \mapsto s(e) + 1$.

3. Let σ be a face of $\mathcal{C}$. Change the marking $d_0^2(\sigma)$ by picking the next edge counterclockwise, i.e.

$$d^2_{\mathrm{new},0}(\sigma) = d^2_{\mathrm{old},1}(\sigma) \ .$$

   Then there is a choice for the $\tilde{\chi}_{\mathrm{new},\sigma}$ such that only the edge indices on the previously marked edge of σ changes. In more detail, the only change is

$$s_{\mathrm{new}}(d^2_{\mathrm{new},2}(\sigma)) = s_{\mathrm{new}}(d^2_{\mathrm{old},0}(\sigma)) = s_{\mathrm{old}}(d^2_{\mathrm{old},0}(\sigma)) + 1$$

   (Here, s_{new} is evaluated with respect to the lifts $\tilde{\chi}_{\mathrm{new},\sigma}$.)

6.1.3. Vertex rules

The vertex rules, Corollary 4.14 and Lemma 4.15 simplify to

Corollary 6.2. Let v be an inner vertex and $(\sigma_i)_{i=1,\dots,N}$ be the triangles containing v, ordered counterclockwise starting with an arbitrary triangle σ_1. Let D be the number of triangles σ_i such that $k_i = 0$. Let K be the number of edges $e_{i,i+1}$ pointing away from v. Then the spin structure on Σ extends to v if and only if

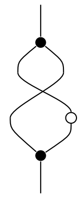
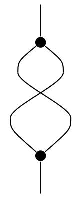
$$\sum_{i=1}^{N} s_{i,i+1} \equiv D + K + 1 \mod 2. \tag{6.1}$$

Lemma 6.3. Let v be a vertex on the boundary component i of type λ, i.e. $\lambda = 0$ for an NS-type boundary and $\lambda = 1$ for an R-type boundary. Let D_i be the number of triangles σ_i with $\chi_{\sigma_i}(1) = v$ or equivalently $d_2^0(\sigma_i) = e_{i-1,i}$. Let $D_b = 1$ if $\varphi_i(1) = v$ and $D_b = 0$ otherwise. Let K be the number of edges $e_{i,i+1}$ pointing away from v, counting the boundary edges. The r-spin structure on Σ^+ extends to v if and only if

$$\sum_{i\in\mathbb{Z}_{N+1}} s_{i,i+1} \equiv D_i + (1-\lambda)D_b + K + 1 \mod 2 \ . \tag{6.2}$$

6.1.4. Projectors on R and NS sectors

Since there are only two different boundary components the same holds for projectors and state spaces. We have

$P^{NS} := P^0 =$  $P^R := P^1 =$  (6.3)

The state spaces $Z^{NS} = Z^0(A)$ and $Z^R = Z^1(A)$ are then images of the projectors $f_0^0 = i^{31} \circ P^{NS} \circ \pi^{31}$ and $f^{0_1} = i^{31} \circ P^R \circ \pi^{31}$ respectively.

6.2. Comparison to earlier work

Remark 6.4. At this point we can make some connections to Lurie's description of topological field theories as fully dualisable objects [Lu] (see also [SP, Da] for discussions of the two-dimensional case). In the symmetric monoidal bicategory of algebras, bimodules, and bimodule morphisms (over some algebraically closed field), the fully dualisable objects are finite-dimensional semisimple algebras A. The dual is A^{op}, the algebra with opposite product. Write $A^e = A \otimes A^{op}$. Then the dualising bimodules are ${}_{A^e}A_{\mathbf{1}}$ and ${}_{\mathbf{1}}A_{A^e}$. The homotopy SO(2) action is given by the Serre-automorphism, i.e. by tensoring with the A-A-bimodule A^*. To pass from framed to spin TFTs, we need the Serre-automorphism to be an involution.

Let now A be a Frobenius algebra as in Theorem 5.10. Since A is separable, it is semi-simple. Since $A^* \cong A_N$ as bimodules (where A_N is the A-A-bimodule A with right action twisted by the Nakayama automorphism), and $A_N \otimes_A A_N \cong A_{N^2}$, we see that the Serre-automorphism is indeed an involution.

Remark 6.5. (i) In [BT], state sum models on spin surfaces are addressed independently of our work. Let us briefly point out some similarities in the algebraic structures considered in [BT] and here. In [BT], the algebraic datum is a separable Frobenius algebra on a symmetrically braided vector space, subject to further conditions, see [BT, Def. 4.1]. In particular, it is imposed that the Nakayama automorphism is an involution. Only closed surfaces are considered in [BT], so that the question of glueing and of state spaces does not arise. Nonetheless, in evaluating the model on higher genus surfaces, the projectors P^{NS} and P^R onto the NS and R state spaces appear, see [BT, Lem. 4.5].
(ii) The projectors P^{NS} and P^R also appear in [BCP], see Section 3.2 there, where they are called $\pi_A^{(\mathrm{c,c})}$ and π_A^{RR}, respectively. In [BCP], the authors are not concerned with spin TFTs, but instead consider "generalised twisted sectors" in orbifolds of 2d TFTs. Accordingly, in [BCP] no restriction on the Nakayama automorphism is imposed. The generalised twisted sectors are described as the images of $\pi_A^{(\mathrm{c,c})}$ and π_A^{RR}. In the case that the TFT arises as the topological twist of an $N = (2, 2)$ supersymmetric field theory, the

construction of [BCP] recovers the (c,c)-ring and the R-ground states of the orbifolded theory, hence the names.

6.3. Pair of pants and multiplication

We proceed to evaluate the spin TFT on the genus 0 surface with 3 boundaries, $\Sigma^{0,3}$. The results are collected in Lemmas 6.6 and 6.7 below. The computations going into the proofs are slightly tedious and therefore collected in Appendix E. The general procedure, however, is as in Section 5.5: We fix a triangulation for the surface, determine admissible edge index configurations and then proceed to calculate the morphism $T_A(\Sigma^{0,3})$.

Lemma 6.6. The surface $\Sigma^{0,3}$ admits spin structures only if the spin structures on the boundary are NS-NS-NS or NS-R-R (in any order). If so, then up to isomorphism of spin structures, there are exactly 4 spin structures.

We parametrise the corresponding spin surfaces by numbers $\varepsilon_1, \varepsilon_2 \in \{0,1\}$ and denote them by $\Sigma^{0,3}_{\delta_1,\delta_2,\delta_3,\varepsilon_1,\varepsilon_2}$ with $\delta_i \in \{NS, R\}$. The definition of these spin structures in terms of admissible edge indices is given in Appendix E. One can check that up to diffeomorphisms of spin surfaces, there is just one spin surface with boundaries of type NS-NS-NS and two such surfaces with boundary types NS-R-R.

Lemma 6.7. The values of the TFT on the above spin surfaces are

$$\begin{aligned} T_A\left(\Sigma^{0,3}_{\delta_1,\delta_2,\delta_3,\varepsilon_1,\varepsilon_2}\right) = b \circ (\mathrm{id}_A \otimes \mu) \circ \left(P^{\delta_1} \otimes P^{\delta_2} \otimes P^{\delta_3}\right) \\ \circ \left(N_{\varepsilon_1} \otimes N_{\varepsilon_2} \otimes \mathrm{id}_A\right) \circ (\pi^{31})^{\otimes 3} \ . \end{aligned} \tag{6.4}$$

We want to use the morphisms (6.4) to define an algebra structure on the state space. To do so, it is convenient to add another assumption to our list:

Assumption 4: The symmetric strict monoidal category $\mathcal{S}$ is additive.

Recall the definition of the NS- and R-type state spaces $Z^{NS/R}(A)$ from the previous section. Under the above assumption, we can now define the *total state space*

$$Z(A) := Z^{NS}(A) \oplus Z^{R}(A) \tag{6.5}$$

of the spin TFT. In the following, we will define an associative, $\mathbb{Z}_2$-graded product on $Z(A)$ and investigate some of its properties. In particular, we will show that the product agrees with the morphisms (6.4).

To start with, we need to know how the projectors P^{NS} and P^{R} interact with the structure maps of A.

Lemma 6.8. Let $\nu_1, \nu_2, \nu_3 \in \{0,1\}$ such that $\nu_1 + \nu_2 + \nu_3 = 1$. Then,

$$\begin{aligned} P^{\nu_1} \circ \mu \circ (P^{\nu_2} \otimes P^{\nu_3}) &= P^{\nu_1} \circ \mu \circ (P^{\nu_2} \otimes \mathrm{id}_A) \\ &= P^{\nu_1} \circ \mu \circ (\mathrm{id}_A \otimes P^{\nu_3}) \end{aligned} \tag{6.6}$$

$$= \mu \circ (P^{\nu_2} \otimes P^{\nu_3}) \ .$$

The unit and counit of A satisfy

$$P^{NS} \circ \eta = \eta \quad , \qquad \varepsilon \circ P^{NS} = \varepsilon \ . \tag{6.7}$$

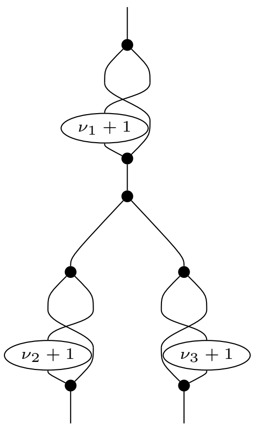
Proof. The identities (6.7) are a consequence of

$$b \circ \sigma_{A,A} \circ (N \otimes \mathrm{id}_A) = b \tag{6.8}$$

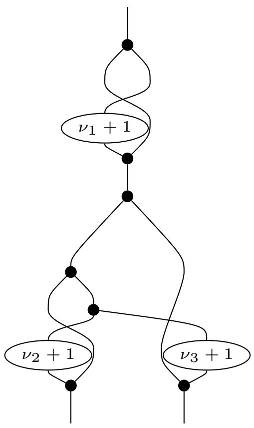
and Δ-separability. We show the first equality in Equation (6.6):

$$P^{\nu_1} \circ \mu \circ (P^{\nu_2} \otimes P^{\nu_3}) \overset{\text{Lem. 5.12}}{=} P^{\nu_1} \circ \mu \circ \big(P^{\nu_2} \otimes (P^{\nu_3} \circ N^{\nu_3+1})\big) \tag{6.9}$$

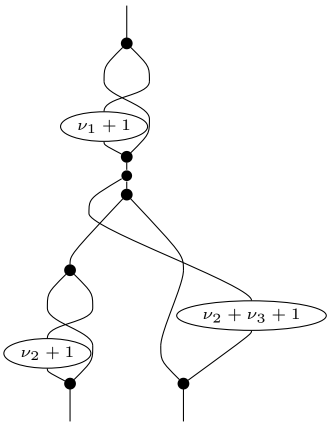
$=$



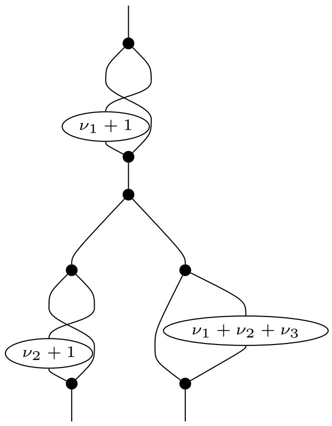
$\overset{\text{assoc.}}{=}$



$\underset{\text{assoc.}}{\overset{(5.35)}{=}}$



$\underset{\text{assoc.}}{\overset{\text{Frob.}}{=}}$

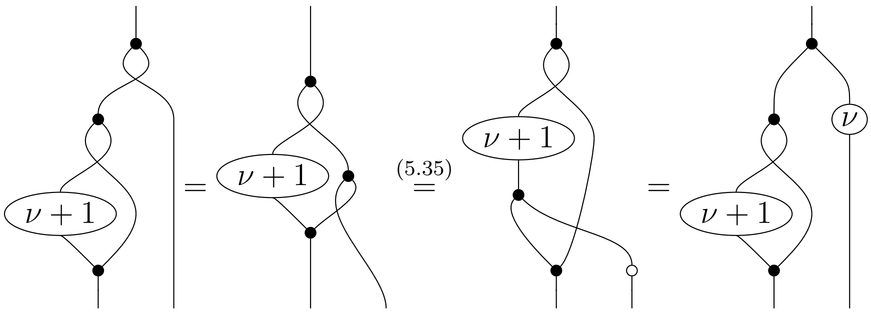


$$= P^{\nu_1} \circ \mu \circ (P^{\nu_2} \otimes \mathrm{id}_A) \ .$$

In the last step Δ-separability and the assumption $\nu_1 + \nu_2 + \nu_3 = 1$ are used. The other cases are analogous. □

Lemma 6.9. Let $\nu \in \{0, 1\}$. We have $\mu \circ \sigma_{A,A} \circ (P^\nu \otimes \mathrm{id}_A) = \mu \circ (P^\nu \otimes N^\nu)$.

Proof. By direct calculation:



(6.10)

□

Recall that in a symmetric monoidal category, a *centre of an algebra A* is an object C together with a morphism $c : C \to A$ such that

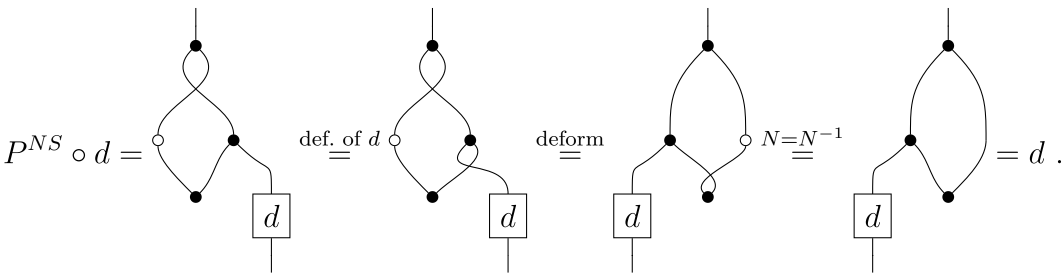
1. $\mu \circ \sigma_{A,A} \circ (c \otimes \mathrm{id}) = \mu \circ (c \otimes \mathrm{id})$, and

2. the following universal property holds: for all $d : D \to A$ such that $\mu \circ \sigma_{A,A} \circ (d \otimes \mathrm{id}) = \mu \circ (d \otimes \mathrm{id})$, there is a unique morphism $f : D \to C$ such that $d = c \circ f$.

It is easy to see that c is necessarily a monomorphism. Since a centre (C, c) is unique up to unique isomorphism, we will speak of "the" centre.

Lemma 6.10. $Z^{NS}(A)$, together with $\iota^{NS} : Z^{NS}(A) \to A$, is the centre of A.

Proof. That ι^{NS} satisfies the property 1 follows from $\iota^{NS} = p^{NS} \circ \iota^{NS}$ and Lemma 6.9. To check property 2, let $d : D \to A$ satisfy $\mu \circ \sigma_{A,A} \circ (d \otimes \mathrm{id}) = \mu \circ (d \otimes \mathrm{id})$. Then

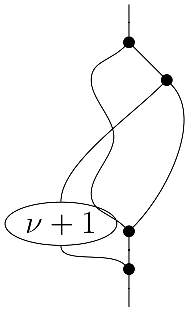


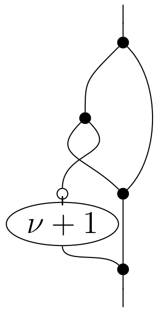
In property 2 we can hence choose $f = \pi^{NS} \circ d$, since $\iota^{NS} \circ f = P^{NS} \circ d = d$. This shows existence. For uniqueness, note that $d = \iota^{NS} \circ f$ and $d = \iota^{NS} \circ f'$ implies $f = f'$ since ι^{NS} is mono. □

Lemma 6.11. 1. $\mu \circ (P^{\nu} \otimes \mathrm{id}) \circ \Delta = \mu \circ (P^{\nu+1} \otimes \mathrm{id}) \circ \Delta$, where $\nu \in \{0, 1\}$,

2. $\mu \circ (P^{NS} \otimes P^{NS}) \circ \Delta \circ \eta = \mu \circ (P^{R} \otimes P^{R}) \circ \Delta \circ \eta$.

Proof. We prove part 1 by direct calculation:

$$\mu\circ(P^\nu\otimes\mathrm{id})\circ\Delta \overset{\text{assoc.}}{\underset{\text{coassoc.}}{=}}$$


$$\overset{(5.35)}{=}$$


$$\underset{\text{coassoc.}}{=} \mu\circ(P^{\nu+1}\otimes\mathrm{id})\circ\Delta\ . \qquad (6.11)$$

For part 2 first note that

$$(P^\nu\otimes P^\nu)\circ\Delta\circ\eta = (P^\nu\otimes\mathrm{id})\circ\Delta\circ\eta\ . \tag{6.12}$$

This follows from Lemma 6.8: replace $\eta = P^{NS}\circ\eta$ and omit one of the P^ν idempotents. Using this, we compute

$$\begin{aligned}\mu\circ(P^{NS}\otimes P^{NS})\circ\Delta\circ\eta &\overset{(6.12)}{=} \mu\circ(P^{NS}\otimes\mathrm{id})\circ\Delta\circ\eta \\ &\overset{\text{part 1}}{=} \mu\circ(P^{R}\otimes\mathrm{id})\circ\Delta\circ\eta \overset{(6.12)}{=} \mu\circ(P^{R}\otimes P^{R})\circ\Delta\circ\eta\ .\end{aligned} \tag{6.13}$$

□

Let us drop the A from the state spaces for brevity: $Z = Z^{NS}\oplus Z^R$. We use both the notations Z^{NS}/Z^R and Z^0/Z^1. Define the embedding and projection maps

$$\iota_Z^\nu : Z^\nu \to Z\ ,\quad \pi_Z^\nu : Z\to Z^\nu\ . \tag{6.14}$$

We abbreviate, for $\nu\in\{0,1\}$,

$$e_\nu = \big[Z \xrightarrow{\pi_Z^\nu} Z_\nu \xrightarrow{\iota^\nu} A\big]\quad ,\qquad f_\nu = \big[A \xrightarrow{\pi^\nu} Z_\nu \xrightarrow{\iota_Z^\nu} Z\big]\ . \tag{6.15}$$

One quickly checks that $e_\nu\circ f_\nu = P^\nu$ and that $\sum_{\nu=0}^1 f_\nu\circ e_\nu = \mathrm{id}_Z$. Finally, define the morphisms

$$\begin{aligned}\mu_Z &= \sum_{\alpha,\beta\in\{0,1\}}\big[Z\otimes Z \xrightarrow{e_\alpha\otimes e_\beta} A\otimes A \xrightarrow{\mu} A \xrightarrow{f_{\alpha\beta}} Z\big]\ , &\qquad \eta_Z &= \big[\mathbf{1}\xrightarrow{\eta} A \xrightarrow{f_0} Z\big] \\ \Delta_Z &= \sum_{\alpha,\beta\in\{0,1\}}\big[Z \xrightarrow{e_{\alpha\beta}} A \xrightarrow{\Delta} A\otimes A \xrightarrow{f_\alpha\otimes f_\beta} Z\otimes Z\big]\ , &\qquad \varepsilon_Z &= \big[Z\xrightarrow{e_0} A \xrightarrow{\varepsilon}\mathbf{1}\big]\end{aligned} \tag{6.16}$$

Recall the involution $N|_{Z^\delta}$ from Remark 5.14. We obtain an involution N_Z on Z, which in the present notation reads

$$N_Z = \sum_{\nu=0}^{1}\big[Z\xrightarrow{e_\nu} A \xrightarrow{N} A \xrightarrow{f_\nu} Z\big]\ . \tag{6.17}$$

Proposition 6.12. 1. The morphisms (6.16) define the structure of a $\mathbb{Z}_2$-graded Frobenius algebra on Z with graded components Z^0 and Z^1.

2. N_Z is the Nakayama automorphism of Z. It satisfies $(N_Z)^2 = \mathrm{id}_Z$ and $N_Z \circ \iota_Z^0 = \iota_Z^0$.

3. The product of Z is graded commutative in the sense that

$$\mu_Z \circ \sigma_{Z,Z} \circ (\iota_Z^\alpha \otimes \iota_Z^\beta) = \mu_Z \circ (N_Z^{\alpha\beta} \otimes \mathrm{id}_Z) \circ (\iota_Z^\alpha \otimes \iota_Z^\beta) \\ = \mu_Z \circ (\mathrm{id}_Z \otimes N_Z^{\alpha\beta}) \circ (\iota_Z^\alpha \otimes \iota_Z^\beta) \ . \tag{6.18}$$

Proof. Part 1: That Z is a Frobenius algebra follows in a straightforward way from Lemma 6.8. We illustrate this for the associativity relation:

$$\begin{aligned} \mu_Z \circ (\mathrm{id}_Z \otimes \mu_Z) &\overset{(1)}{=} \sum_{\alpha,\beta,\gamma\in\mathbb{Z}_2} f_{\alpha+\beta+\gamma} \circ \mu \circ (id_A \otimes P^{\beta+\gamma}) \circ (\mathrm{id}_A \otimes \mu) \circ (e_\alpha \otimes e_\beta \otimes e_\gamma) \\ &\overset{(2)}{=} \sum_{\alpha,\beta,\gamma\in\mathbb{Z}_2} f_{\alpha+\beta+\gamma} \circ \mu \circ (\mathrm{id}_A \otimes \mu) \circ (e_\alpha \otimes e_\beta \otimes e_\gamma) \\ &\overset{(3)}{=} \sum_{\alpha,\beta,\gamma\in\mathbb{Z}_2} f_{\alpha+\beta+\gamma} \circ \mu \circ (\mu \otimes \mathrm{id}_A) \circ (e_\alpha \otimes e_\beta \otimes e_\gamma) \\ &\overset{(4)}{=} \mu_Z \circ (\mu_Z \otimes \mathrm{id}_Z) \ . \end{aligned} \tag{6.19}$$

In step 1 we substituted the definition of μ_Z and used $e_{\beta+\gamma} \circ f_{\beta+\gamma} = P^{\beta+\gamma}$. For step 2 note that $f_\nu = f_\nu \circ P^\nu$ and $e_\nu = P^\nu \circ e_\nu$. Therefore, each of the two products can be surrounded by three idempotents P, and by Lemma 6.8 we can omit one of these. We choose to omit $P^{\beta+\gamma}$. Step 3 is just associativity of μ, and in step 4 one carries out steps 1 and 2 backwards.

That Z is graded is clear from the definition of the structure maps in (6.16).

Part 2: That N_Z is the Nakayama automorphism of Z follows from substituting (6.16) into the definition (5.17) of the Nakayama automorphism and using Lemma 6.8. That N_Z is an involution is immediate from (6.17), together with Lemma 5.12 (2). Furthermore, from Remark 5.14 one concludes that $N_Z \circ \iota_Z^0 = \iota_Z^0$.

Part 3: To establish graded commutativity, first compute

$$\begin{aligned} \mu_Z \circ \sigma_{Z,Z} \circ (\iota_Z^\alpha \otimes \iota_Z^\beta) &\overset{\text{def.}\,\mu_Z}{=} f_{\alpha+\beta} \circ \mu \circ \sigma_{A,A} \circ (\iota^\alpha \otimes \iota^\beta) \\ &\overset{\iota^\alpha = P^\alpha \circ \iota^\alpha}{=} f_{\alpha+\beta} \circ \mu \circ \sigma_{A,A} \circ (P^\alpha \otimes P^\beta) \circ (\iota^\alpha \otimes \iota^\beta) \\ &\overset{\text{Lem. 6.9}}{=} f_{\alpha+\beta} \circ \mu \circ (\mathrm{id}_A \otimes N^\alpha) \circ (\iota^\alpha \otimes \iota^\beta) \\ &\overset{\text{def.}\,\mu_Z, N_Z}{=} \mu_Z \circ (\mathrm{id}_Z \otimes N_Z^\alpha) \circ (\iota_Z^\alpha \otimes \iota_Z^\beta) \\ &\overset{N_Z\ \text{autom.}}{=} N_Z^\alpha \circ \mu_Z \circ (N_Z^\alpha \otimes \mathrm{id}_Z) \circ (\iota_Z^\alpha \otimes \iota_Z^\beta) \ . \end{aligned} \tag{6.20}$$

Now use the last two lines to compare to (6.18) in all four cases $\alpha, \beta \in \{0, 1\}$. For example, consider the last line with $\alpha = \beta = 1$: The multiplication μ_Z has image in $Z^{\alpha+\beta} = Z^0$. By part 2, N_Z is the identity on Z^0, and so the last line reads $\mu_Z \circ (N_Z \otimes \mathrm{id}_Z) \circ (\iota_Z^1 \otimes \iota_Z^1)$, as required. □

We still need to relate the product defined in (6.16) to the amplitude the spin TFT assigns to three-holed spheres in (6.4). This is done in the following proposition whose proof is immediate from Lemma 6.7.

Proposition 6.13. We have

$$\begin{aligned} &T_A(\Sigma^{0,3}_{\delta_1,\delta_2,\delta_3,\varepsilon_1,\varepsilon_2}) \circ \bigotimes_{i=1}^{3} \left(\iota^{13} \circ \iota^{\delta_i}\right) \\ &= \varepsilon_Z \circ \mu_Z \circ (\mu_Z \otimes \mathrm{id}_Z) \circ (N_Z^{\varepsilon_1} \otimes N_Z^{\varepsilon_2} \otimes \mathrm{id}_Z) \circ (\iota^{\delta_1} \otimes \iota^{\delta_2} \otimes \iota^{\delta_3}) \ . \end{aligned} \tag{6.21}$$

Remark 6.14. Let us compare the properties of the total state space Z to the algebraic description of spin TFTs given by Moore and Segal in [MS, Sect. 2.6]. They consider $\mathcal{S} = \mathbf{SVect}$ and require N_Z to be the parity involution. By Remark 5.14, Z^0 must then be purely even and graded commutativity (6.18) translates into ordinary commutativity of Z, considered as an algebra in **Vect**: $\mu_Z \circ c^{\mathbf{Vect}}_{Z,Z} = \mu_Z$. In addition, Moore and Segal require that the NS- and R-type "Euler characters" agree, $\chi_{ns} = \chi_r$. In our notation, this can be expressed as follows. Write $P^\nu = \iota_Z^\nu \circ \pi_Z^\nu : Z \to Z$ for the projection onto the subobject Z^ν of Z. Then

$$\begin{aligned} \chi_{ns} &= \mu^Z \circ (P^0 \otimes P^0) \circ \Delta^Z \circ \eta^Z \ , \\ \chi_r &= \mu^Z \circ (P^1 \otimes P^1) \circ \Delta^Z \circ \eta^Z \ . \end{aligned} \tag{6.22}$$

Substituting the definitions, we can rewrite $\chi_{ns} = \mu \circ (P^{NS} \otimes P^{NS}) \circ \Delta \circ \eta$ and $\chi_r = \mu \circ (P^R \otimes P^R) \circ \Delta \circ \eta$. These two are indeed equal by Lemma 6.11 (2).

6.4. Value of the TFT on spin tori

Let $\delta \in \{NS, R\}$ and consider the cylinder C^ε_δ as defined in Section 5.5. Define the spin torus T^ε_δ by gluing together the two boundary components

$$T^\varepsilon_\delta := (C^\varepsilon_\delta)^1_{1\#2} \ , \tag{6.23}$$

where the marking and edge indices of the glued spin triangulated surface are determined via Lemma 4.10. The choice of "1" over "0" in the above gluing is a convention; it can be absorbed into the value of ε.

As it stands, the above gluing is actually ill-defined as it does not produce a spin triangulated surface. We should instead first subdivide the triangulation before gluing the boundaries. However, this does not make a difference to the value of the TFT, and we take the liberty to work with (6.23), even though it is not a spin triangulated surface.

The cell decomposition of the surface $\underline{T}^{\varepsilon}_{\delta}$ is as in Figure 5.3, except that the left vertical circle (consisting of edges e_1, e_2, e_3) is replaced by the right one (consisting of edges e_4, e_5, e_6). The new edge indices are

$$s(e_4) = s_3{+}s_4{+}1 = 1{+}\varepsilon \ , \quad s(e_5) = s_2{+}s_5{+}1 = 1{+}\varepsilon \ , \quad s(e_6) = s_1{+}s_6{+}1 = 1{+}\varepsilon \ . \tag{6.24}$$

The remaining edge indices are as in Section 5.5. Let $\nu = 0$ if $\delta = NS$ and $\nu = 1$ if $\delta = R$. Then

$$s(e_7) = 1 + \nu \quad , \quad s(e_8) = s(e_9) = s(e_{10}) = s(e_{11}) = s(e_{12}) = 0 \ . \tag{6.25}$$

As an application of Lemma 4.13 we can now investigate the lifting properties of the two simple closed curves γ_h, which we define to run horizontally through σ_3 and σ_4, and γ_v, which we define to run vertically through all triangles $\sigma_1, \dots, \sigma_6$.

For γ_h: The sum of the edge indices is $s(e_{10}) + s(e_5) = 1 + \varepsilon$, all $\mu_{i,i+1}$ are 1, $k_3 = 2$, $k_4 = 0$, $\eta_3 = -1$, $\eta_4 = +1$. By (4.32) a lift of γ_h shifts the fibre by $1 + \varepsilon(0 + 1) = \varepsilon$. Thus γ_h has a closed lift iff $\varepsilon = 0$.

For γ_v: The sum of the edge indices is $1 + \nu$, all $\mu_{i,i+1}$ are 1, $k_1, \dots, k_6 = 0$, $\eta_1 = \eta_3 = \eta_5 = 1$, $\eta_2 = \eta_4 = \eta_6 = -1$. By (4.32) we get a shift by $1 + \nu - 3 + 3 = 1 + \nu$. As we already knew from the type of the boundary components, for $\delta = NS$ we get 1, so that the curve does not have a closed lift, and for $\delta = R$ we get 0, so that the curve does have a closed lift.

Let us now compute $T_A(T^{\varepsilon}_{\delta})$. According to Proposition 5.11, $T_A(T^{\varepsilon}_{\delta}) = T_A(C^{\varepsilon}_{\delta}) \circ \Gamma_{1,2,1}$. Combining (5.49) and (5.53), we find

$$\begin{aligned} T_A(T^{\varepsilon}_{\delta}) &= b \circ (P^{\delta} \otimes \mathrm{id}_A) \circ (N_{-\varepsilon} \otimes \mathrm{id}_A) \circ (\pi^{31} \otimes \pi^{31}) \circ \Gamma_{1,2,1} \\ &= \varepsilon \circ \mu \circ \big[(P^{\delta} \circ N^{1+\varepsilon}) \otimes \mathrm{id}_A \big] \circ \Delta \circ \eta \ . \end{aligned} \tag{6.26}$$

Since by Lemma 5.12, $P^{NS} \circ N = P^{NS}$, we see $T_A(T^0_{NS}) = T_A(T^1_{NS})$. From Lemma 6.11 we learn $T_A(T^1_{NS}) = T_A(T^1_R)$. Thus, on a spin torus, T_A can take at most two different values,

$$T_A(T^0_{NS}) = T_A(T^1_{NS}) = T_A(T^1_R) \qquad \text{and} \qquad T_A(T^0_R) \ , \tag{6.27}$$

in agreement with the action of spin diffeomorphisms.

6.5. Examples

Example 1. Let $A = k^{1|1} \in \mathbf{SVect}(k)$ for $\mathrm{char}(k) \neq 2$. Define the product

$$\mu \,:\, A \otimes A \to A \quad , \quad \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \mapsto \begin{pmatrix} x_0 y_0 + x_1 y_1 \\ x_0 y_1 + x_1 y_0 \end{pmatrix} \tag{6.28}$$

and the unit/counit

$$\eta : k \to A \quad , \quad k \ni \lambda \mapsto \lambda \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad , \qquad \varepsilon : A \to k \quad , \quad \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto 2x_0 \ . \tag{6.29}$$

It is straightforward to verify that these maps turn A into a Frobenius algebra. The pairing $b : A \otimes A \to k$ is then given by

$$b\left(\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}\right) = 2(x_0 y_0 + x_1 y_1). \tag{6.30}$$

Since $\mathrm{char}(k) \neq 2$, b is nondegenerate with copairing

$$c_{-1} : k \to A \otimes A \quad , \quad 1 \mapsto \frac{1}{2}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) . \tag{6.31}$$

From this one computes the Nakayama automorphism of A to be

$$N : A \to A \quad , \quad \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \mapsto \begin{pmatrix} x_0 \\ -x_1 \end{pmatrix} . \tag{6.32}$$

Thus $N^2 = \mathrm{id}_A$. The coproduct can be computed from the copairing as

$$\begin{aligned} \Delta \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} &= \frac{1}{2}\left(\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) \\ &= \frac{1}{2}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} x_1 \\ x_0 \end{pmatrix}\right) . \end{aligned} \tag{6.33}$$

From this it is immediate that A is Δ-separable. One also easily computes the idempotents $P^{NS/R}$ to be

$$P^{NS} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_0 \\ 0 \end{pmatrix} \quad , \quad P^{R} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ x_1 \end{pmatrix} . \tag{6.34}$$

Therefore, the state spaces are given by $Z_{NS} = k^{1|0}$ and $Z_R = k^{0|1}$, and from the formulas for the structure maps in (6.16) we see that in fact $Z = A$ as Frobenius algebras.

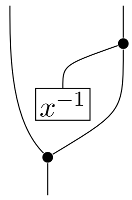
Evaluating the TFT for A on $T^{\varepsilon}_{NS/R}$ according to (6.26) gives

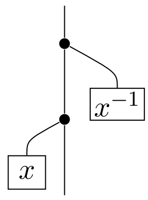
$$T_A(T^{0/1}_{NS}) = T_A(T^1_R) = 1 \quad , \quad T_A(T^0_R) = -1 . \tag{6.35}$$

Example 2. Let $(A, \mu, \eta, \varepsilon)$ be a symmetric Frobenius algebra in an additive symmetric strict monoidal category $\mathcal{S}$. Let $x \in \mathrm{Hom}\,(\mathbf{1}, A)$ be invertible with respect to the algebra product μ. We denote its inverse by $x^{-1} \in \mathrm{Hom}\,(\mathbf{1}, A)$. Let

$$\varepsilon_x := \varepsilon \circ \mu \circ (x \otimes \mathrm{id}_A) . \tag{6.36}$$

Then $A_x = (A, \mu, \eta, \varepsilon_x)$ is again a Frobenius algebra, see e.g. [FSt, Lemma 19]. In the following we draw the original Frobenius algebra morphisms as in Figure 5.2. The coproduct and Nakayama automorphism of A_x are given by

$\Delta_x =$ 

, $N_x =$ 

. (6.37)

The Nakayama automorphism N_x is thus the inner automorphism generated by x. It satisfies $N_x^2 = \mathrm{id}_A$ iff $x^2 := \mu\circ(x\otimes x)$ is central in A, i.e. if $\mu\circ\sigma_{A,A}\circ(x^2\otimes\mathrm{id}) = \mu\circ(x^2\otimes\mathrm{id})$. By definition, A_x is Δ-separable iff $\mu\circ(\mathrm{id}\otimes\mu)\circ(\mathrm{id}\otimes x^{-1}\otimes\mathrm{id})\circ\Delta = \mathrm{id}$ holds in A.

It turns out that in this example, the TFT does not actually depend on the spin structure. Namely, let $R_x := \mu\circ(\mathrm{id}_A\otimes x)$ be right multiplication by x. One quickly checks that $P^{NS} = R_x^{-1}\circ P^R\circ R_x$, so that the NS- and R-state spaces are isomorphic. This identity furthermore implies that $P^R\circ N = P^R$ (in addition to $P^{NS}\circ N = P^{NS}$, which holds by Lemma 5.12). The latter observation implies independence of the spin structure on closed surfaces, cf. expression (6.26) for the torus.

From the point of view of fully extended TFTs this is not too surprising, since a fully dualisable object in the symmetric monoidal bicategory of algebras does not involve the pairing as a piece of data. Hence, if there exists a symmetric pairing on A, the resulting TFT will be independent of the spin structure.

The next example makes this more explicit in $\mathbf{Vect}_k$.

Example 3. Let $A = M_n(k)$ be the algebra of $n\times n$ matrices for some integer $n>0$ and a field k. Let E_{ij} be the $n\times n$ matrix with zero entries everywhere but in place (i,j), where it has entry 1. It satisfies $E_{ij}E_{kl} = \delta_{j,k}E_{il}$ and consequently $\mathrm{tr}(E_{ij}E_{kl}) = \delta_{i,l}\delta_{j,k}$. Thus, the trace pairing on A is non-degenerate (independent of the characteristic of k) and we can use it to turn A into a (symmetric) Frobenius algebra. Concretely, the counit and coproduct are

$$\varepsilon(M) = \mathrm{tr}(M) \quad , \quad \Delta(M) = \sum_{i,j=1}^{n}(ME_{ij})\otimes E_{ji} = \sum_{i,j=1}^{n} E_{ij}\otimes(E_{ji}M) \ . \tag{6.38}$$

Now choose $X\in GL_n(k)$ such that, for some $\lambda\in k^\times$,

1. $X^2 = \lambda\mathbf{1}$, i.e. $X^{-1} = \lambda^{-1}X$, and
2. $\mathrm{tr}(X) = \lambda$.

From Example 2 we obtain a new Frobenius algebra A_X by twisting the counit with X. Condition (1) shows $(N_X)^2 = \mathrm{id}_A$ (since X^2 is central), and condition (2) implies that A_X is Δ-separable:

$$\mu(\Delta_X(M)) = \sum_{i,j=1}^{n} E_{ij}X^{-1}E_{ji}M = \mathrm{tr}(X^{-1})\cdot M = \lambda^{-1}\mathrm{tr}(X)\cdot M \ . \tag{6.39}$$

Thus, A_X is an example of a Δ-separable Frobenius algebra whose Nakayama automorphism is an involution. The projectors $P^{NS/R}$ are straightforward to compute:

$$P^{NS}(M) = \lambda^{-1}\mathrm{tr}(MX)\cdot\mathbf{1} \quad , \quad P^R(M) = \lambda^{-1}\mathrm{tr}(M)\cdot X \ . \tag{6.40}$$

Thus, the state spaces $Z^{NS/R}$ are one-dimensional and given by $Z^{NS} = k\,\mathbf{1}$, $Z^R = k\,X$.

From Example 2 we know that the TFT for A_X is independent of the spin structure. For example, evaluating the TFT on $T^{\varepsilon}_{NS/R}$ according to (6.26) gives $T_{A_X}(T^{\pm}_{NS}) = T_{A_X}(T^{\pm}_{R}) = 1$.

A simple example would be to take k of characteristic 3, and $n = 3$, $\lambda = 1$, $X = \mathrm{diag}(1, 1, -1)$.

A. A construction of the two-fold cover of GL_2^+

For the spin case, $r = 2$ we give an explicit construction of the group $\widetilde{GL}_2^{\,2}$ used; in this chapter $\widetilde{GL}_2 = \widetilde{GL}_2^{\,2}$. It is not explicitly used in the rest of this thesis, but may be useful as a check in calculations. We give a construction of $\widetilde{GL}_2$, parallel to the construction of the metaplectic group in [LV, Sect. I.1.8]. GL_2^+ acts on the complex upper half plane $\mathbb{H} = \{z \,|\, \mathrm{Im}(z) > 0\}$ as

$$g.z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.z = \frac{az+b}{cz+d} \ . \tag{A.1}$$

The denominator will be denoted by $j_g(z) = cz + d$; it satisfies $j_{g_1 g_2}(z) = j_{g_1}(g_2.z)\, j_{g_2}(z)$. We define $\widetilde{GL}_2$ as

$$\widetilde{GL}_2 := \big\{ (g, \varepsilon) \,\big|\, g \in GL_2^+,\ \varepsilon : \mathbb{H} \to \mathbb{C} \text{ holomorphic s.t. } \varepsilon(z)^2 = j_g(z) \big\} \ . \tag{A.2}$$

Composition and inverse are given by

$$\begin{aligned} (g_1, \varepsilon_1) \circ (g_2, \varepsilon_2) &:= (g_1 g_2, \varepsilon) \ , \qquad && \text{with} \quad \varepsilon(z) = \varepsilon_1(g_2.z)\, \varepsilon_2(z) \ , \\ (g, \varepsilon)^{-1} &:= (g^{-1}, \tilde\varepsilon) \ , && \text{with} \quad \tilde\varepsilon(z) = \varepsilon(g^{-1}.z)^{-1} \ . \end{aligned} \tag{A.3}$$

The unit is $e = (\mathbb{1}, 1)$. The map $p_{GL} : \widetilde{GL}_2 \to GL_2^+$ is given by

$$p_{GL} : (g, \varepsilon) \mapsto g \ . \tag{A.4}$$

Notice that for an element $(g, \varepsilon) \in \widetilde{GL}_2$, the function ε is uniquely determined by giving its value at a single point $z \in \mathbb{H}$, e.g. at $z = i$.

B. Simplicial complexes and smooth triangulations

In this appendix we collect some standard definitions regarding simplicial complexes and smooth maps from such complexes to smooth manifolds.

Definition B.1 ([Mu, Def. 7.1], [Pa, pp. 129])**.** A (geometrical) *simplicial complex* $\mathcal{C}$ is a collection of (closed) simplices in $\mathbb{R}^n$ such that

1. Every face of a simplex of $\mathcal{C}$ is in $\mathcal{C}$.
2. The intersection of two simplices of $\mathcal{C}$ is a face of each of them.
3. Each point of $|\mathcal{C}| := \bigcup \mathcal{C}$ has a neighbourhood intersecting only finitely many simplices of $\mathcal{C}$.

The set of all n-dimensional simplices is denoted by $\mathcal{C}_n$.

For a simplex σ the *boundary complex* is denoted by $\mathcal{B}(\sigma)$ and we define $\mathcal{F}(\sigma) := \mathcal{B}(\sigma) \cup \{\sigma\}$ to be the complex of all faces of σ.

Definition B.2 ([Pa])**.** Let $\mathcal{C}$ be a simplicial complex. Let $A \in \mathcal{C}$ be a face. Then

$$\mathrm{st}(A;\mathcal{C}) := \{B \in \mathcal{C} : A \subset B\} \quad \text{“(open) star”}, \tag{B.1}$$

$$\mathrm{clst}(A;\mathcal{C}) := \cup\{\mathcal{F}(B) : B \in \mathrm{st}(A;\mathcal{C})\} \quad \text{“(closed) star”}, \tag{B.2}$$

$$\mathrm{ast}(A;\mathcal{C}) := \{B \in \mathcal{C} : B \cap A = \emptyset\} \quad \text{“antistar”}, \tag{B.3}$$

$$\mathrm{link}(A;\mathcal{C}) := \mathrm{ast}(A;\mathcal{C}) \cap \mathrm{clst}(A;\mathcal{C}). \tag{B.4}$$

Definition B.3 ([Hu, Sect. I.5])**.** A *combinatorial n-manifold (with boundary)* is a simplicial complex such that the link of each vertex is a p.l. $(n-1)$-sphere or $(n-1)$-ball.

Definition B.4 ([Le, Sect. 5])**.**

1. An *orientation of a simplex* is an equivalence class of total orders of its vertices. We consider two orders equivalent, if they are obtained from each other by an even permutation of the vertices.
2. Given an n-dimensional oriented simplex S, the *induced orientation* on an $(n-1)$-dimensional face $F \in S$ consists of all total orders on the vertices of F, such that adding the unique vertex in S but not in F as smallest element gives a total order in the class of the orientation on S.

3. An *orientation of a combinatorial n-manifold* is an orientation on each of its n-dimensional simplices, such that for each $(n-1)$-dimensional simplex the orientations induced by the adjacent n-dimensional simplices are opposite.

Next we turn to smooth maps on arbitrary subsets of $\mathbb{R}^n$.

Definition B.5 ([Mu, Def. 1.2])**.** Let $A \subset \mathbb{R}^n$ be any set. A map $f : A \to \mathbb{R}^m$ is called smooth, if for every point $a \in A$, there exists an open neighbourhood $U \subset \mathbb{R}^n$ of a and a smooth extension $\tilde{f} : U \to \mathbb{R}^m$, of $f|_{U \cap A}$ to U.

Let $f : A \to \mathbb{R}^m$ be a smooth map as in Definition B.5 above. If there is an open set U such that $U \subset A \subset \bar{U}$, then the derivative $Df_a : \mathbb{R}^n \to \mathbb{R}^m$ at a point $a \in A$ is uniquely defined as the derivative of an (arbitrary) smooth extension [Mu, Ex. 1.2 (b)]. While smoothness is defined by demanding extendibility locally at each point, if A is compact this is equivalent to the extendibility of the map as a whole:

Lemma B.6. Let $A \subset \mathbb{R}^n$ be compact and $f : A \to \mathbb{R}^m$ be smooth. Then there exists $V \subset \mathbb{R}^n$ open, $A \subset V$, and a smooth map $\tilde{f} : V \to \mathbb{R}^n$ such that $\tilde{f}|_A = f$.

Proof. Choose a local smooth extension $\tilde{f}_x : U_x \to \mathbb{R}^n$ for every $x \in A$. The U_x cover A and thus by compactness there is a finite subcover $(U_i)_{i=1,\dots,N}$ together with smooth maps $\tilde{f}_i : U_i \to \mathbb{R}^n, i = 1, \dots, N$. These can be fitted together by choosing a smooth partition of unity subordinate to the finite subcover. □

Definition B.7. Let $A \subset \mathbb{R}^n$ with $U \subset A \subset \bar{U}$ for some $U \subset \mathbb{R}^n$ open. Let $f : A \to \mathbb{R}^m$ be a smooth map. Then

- f is called an immersion if Df_a has rank n for all $a \in A$.
- f is called an embedding if f is an immersion and a homeomorphism onto its image.

We can now define smooth triangulations. Let M be a smooth manifold. Given a simplicial complex $\mathcal{C}$ and a k-dimensional simplex $\sigma \in \mathcal{C}$, pick a k-dimensional simplex S in $\mathbb{R}^k$ and an affine linear isomorphism $L : S \to \sigma$. We call a map $f : \sigma \to M$ *smooth* if the composition $f \circ L : S \to M$ is smooth in the sense of Definition B.5. The rank of f at a point $x \in \sigma$ is the rank of $f \circ L$ at the point $L^{-1}(x)$.

Definition B.8 ([Mu, Defs. 8.1, 8.3 & Thm. 8.4])**.** Let $\mathcal{C}$ be a simplicial complex and M a smooth manifold. A map $f : |\mathcal{C}| \to M$ is called *smooth relative to* $\mathcal{C}$ if $f|_\sigma$ is smooth for each simplex σ of $\mathcal{C}$. It is called *non-degenerate*, if $f|_\sigma$ has rank equal to the dimension of σ for all $\sigma \in \mathcal{C}$. A non-degenerate smooth homeomorphism is called a *(smooth) triangulation.*

C. Proof of Proposition 5.9

Relation (1): We start with the r.h.s. and compute

$$\sigma_{A,A} \circ c_{-k-1} \overset{(5.39)}{=} \sigma_{A,A} \circ \big(\mathrm{id}_A \otimes N^{-k}\big) \circ \Delta \circ \eta = \big(N^{-k} \otimes \mathrm{id}_A\big) \circ \sigma_{A,A} \circ \Delta \circ \eta$$
$$\overset{\text{def. of } N}{=} \big(N^{-k-1} \otimes \mathrm{id}_A\big) \circ \Delta \circ \eta \overset{N \text{ is an autom.}}{=} \big(\mathrm{id}_A \otimes N^{k+1}\big) \circ \Delta \circ \eta = c_k \ . \tag{C.1}$$

Relation (2): Follows immediately from the fact that N is an automorphism of Frobenius algebras (Proposition 5.5).

Relation (3): Applying the pairing b to each leg shows that (5.9) is equivalent to

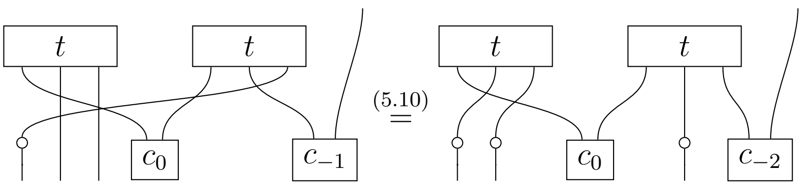
$$t \circ (N^{-s(e_0)-1} \otimes N^{-s(e_1)-1} \otimes N^{-s(e_2)-1}) = t \circ c_{A,A\otimes A} \circ (N^{-s(e_0)} \otimes N^{-s(e_1)-1} \otimes N^{-s(e_2)-1}) \ . \tag{C.2}$$

Canceling the Nakayama automorphisms gives the following reformulation of (5.9):

$$t \circ (N^{-1} \otimes \mathrm{id}_{A\otimes A}) = t \circ c_{A,A\otimes A} \ . \tag{C.3}$$

To see that this equality holds, first substitute $t = b \circ (\mathrm{id}_A \otimes \mu)$ and then use $b \circ (N^{-1} \otimes \mathrm{id}_A) = b \circ c_{A,A}$. This last identity follows by first composing the definition of N in (5.17) with b to get $b \circ (\mathrm{id}_A \otimes N) = b \circ c_{A,A}$ and then noting that $b \circ (\mathrm{id}_A \otimes N) = b \circ (N^{-1} \otimes \mathrm{id}_A)$.

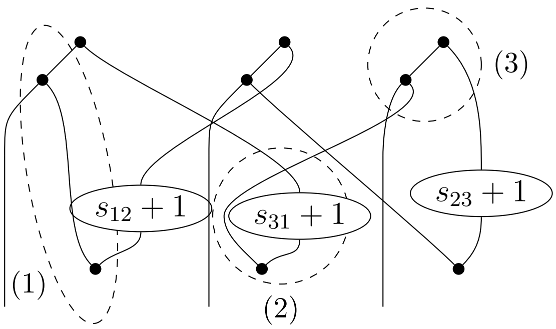
Relation (4): Recall the calculation in (5.24) which was used to establish associativity. Remove the equal sign labeled by Equation (5.10) and instead use associativity of μ to equate the first and last expression. Since we have already established Relations 2 and 3, that is, Equations (5.6) and (5.9) and implicitly Equation (5.9), this reformulation of the calculation in (5.24) shows that the equality labeled by (5.10) in (5.24) holds:



(C.4)

This equality proves a special case of relation (4), i.e. of (5.10): Using c_0 to turn the three in-going legs into out-going legs we get (5.10) for $s_A = s_B = s_C = -1$, $s_D = -2$ and $s = 1$. The remaining cases are established by composing with Nakayama automorphisms as appropriate.

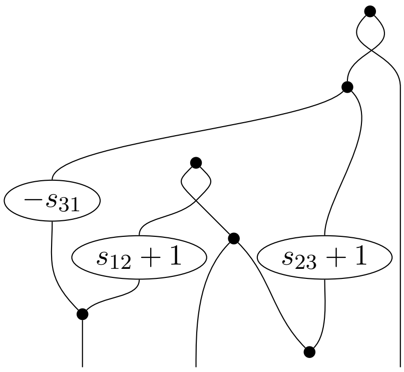
Relation (5): We have to show the identity (5.11). By composing with Nakayama automorphisms as appropriate, we may assume $s_A = s_B = s_C = -1$. Using b to turn all out-going legs into in-going ones and substituting the definitions of t and $c_{\pm 1}$, we see that (5.11) is equivalent to



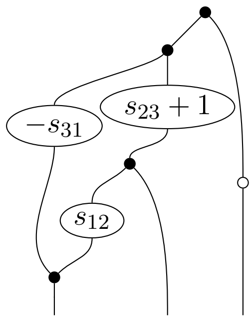
$= \varepsilon \circ \mu \circ (\mu \otimes \mathrm{id}_A) \circ (\mathrm{id}_A \otimes N^{s_{12}} \otimes N^{-s_{31}-1})$. (C.5)

To prove this identity, start from the left hand side. Inside the dashed circle 1, convert product and copairing to a coproduct by substituting (5.27). In dashed circle 2, remove the braiding by replacing $N_{s_{31}+1}$ by $N_{-s_{31}}$. Then one can use the duality properties to cancel c_{-1} against b. In dashed circle 3, apply associativity. Deforming the resulting string diagram slightly gives the first equality in:

lhs. of (C.5) $\overset{(1)}{=}$

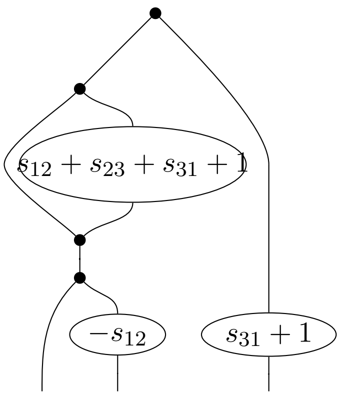


$\overset{(2)}{=}$



(C.6)

$\overset{(3)}{=}$



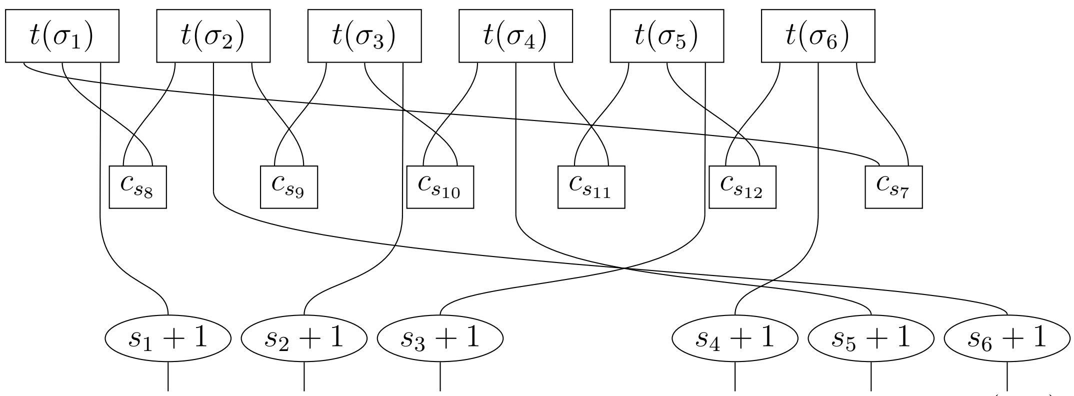
$\overset{(4)}{=}$ rhs. of (C.5) .

In the second equality, $b \circ c_{A,A} = b \circ (\mathrm{id}_A \otimes N)$ is used twice, and after one use of associativity, a pairing has been canceled against a copairing. Step 3 is associativity and the fact that N is an algebra automorphism. Equality 4 uses that $s_{12}+s_{23}+s_{31}=-1$ and Δ-separability of A.

D. Evaluation of the TFT on the cylinder

In this appendix we give some details of how to calculate the morphism $T_A(C^{\pm}_{NS/R})$ defined in Section 5.5. We start with the triangulation of the cylinder given in Figure 5.3. The dual triangulation is depicted in Figure D.1, and the corresponding graph $\Gamma(\mathcal{C})$ in Figure D.2. We label the graph in Figure D.2 according to the construction in Section 5.1 and then turn it into a correlator as in Equation (5.40). This gives the morphism T_A as a string diagram in $\mathcal{S}$:

$T_A(\Sigma) =$



(D.1)

The σ_i in $t(\sigma_i)$ is just a reference to which triangle the map comes from in order to make it easier for the reader to verify; the map is in all cases the same map $t : A^{\otimes 3} \to \mathbf{1}$.

Next we replace the morphisms t and c_k by structure maps of the Frobenius algebra

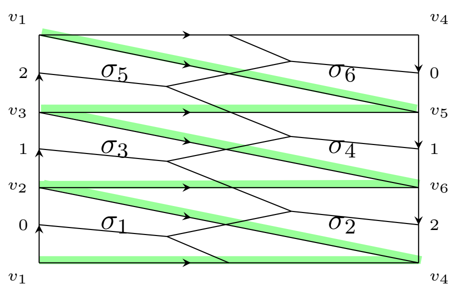


Figure D.1.: The triangulation in Figure 5.3 together with its dual.

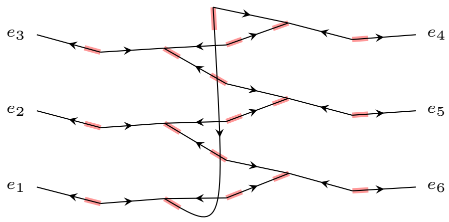
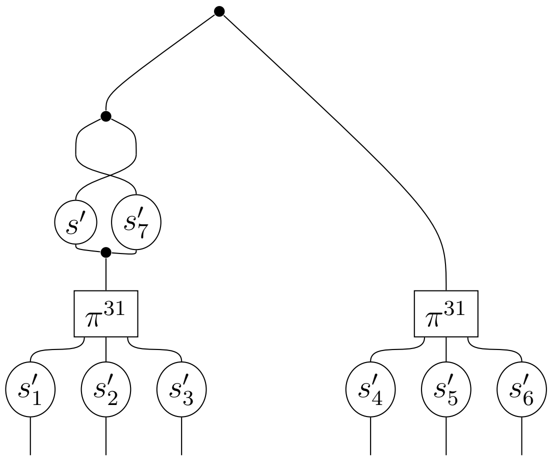


Figure D.2.: The resulting graph $\Gamma(\mathcal{C})$. We give the polarisation by marking the first leg (i.e. the leg number 0, see (5.4)) of each vertex in red. The remaining legs are labeled counterclockwise for edges in $\mathrm{in}(v)$ and clockwise for edges in $\mathrm{out}(v)$, see again (5.4).

as in Proposition 5.9. After a tedious but straightforward calculation one arrives at

$T_A(\Sigma) =$ (D.2)



with the indices $s', s'_1, \ldots, s'_7 \in \mathbb{Z}_r$ given by

$$\begin{aligned} s'_1 &= s_1 \ , & s'_4 &= s_4 + 1 \ , \\ s'_2 &= s_2 + s_8 + s_9 \ , & s'_5 &= s_5 - s_{11} - s_{12} + 1 \ , \\ s'_3 &= s_3 + s_8 + s_9 + s_{10} + s_{11} \ , & s'_6 &= s_6 - s_9 - s_{10} - s_{11} - s_{12} + 1 \ , \\ s'_7 &= s_7 \ , & s' &= -s_8 - s_9 - s_{10} - s_{11} - s_{12} \ . \end{aligned} \tag{D.3}$$

Evaluating this for the signs s_i given in (5.45) and (5.47) then yields (5.49).

E. Evaluation of the TFT on the pair of pants

In this appendix we compute the value of the spin TFT on the surface $\Sigma^{0,3}$. We demand that the i'th boundary component B_i is of type δ_i, where $i = 1, 2, 3$ and $\delta_i \in \{NS, R\}$. Our starting point is the triangulation and marking given in Figure E.1. We determine the possible spin structures with the given boundary types by computing all admissible edge signs (see Section 4.8). Since $r = 2$ we can cut down on the notation a little by writing $\mathbb{Z}_2$ multiplicatively.

To reduce the number of parameters, use Lemma 4.11(1) to set an edge sign to 1 for each of the triangles $\sigma_1, \ldots, \sigma_{11}$:

triangle	σ_1	σ_2	σ_3	σ_4	σ_5	σ_6	σ_7	σ_8	σ_9	σ_{10}	σ_{11}
edge fixed	e_{10}	e_{11}	e_4	e_6	e_{13}	e_7	e_8	e_{16}	e_9	e_5	e_{18}

Let $s_i := s(e_i)$. We thus have

$$s_4 = s_5 = s_6 = s_7 = s_8 = s_9 = s_{10} = s_{11} = s_{13} = s_{16} = s_{18} = 1 . \tag{E.1}$$

We now have to evaluate the vertex rules at vertices $v_1, \ldots, v_9$. These depend on the spin structure on the boundaries. Let

$$\nu_i = \begin{cases} 1 & \text{if } B_i \text{ is of } NS\text{-type} \\ -1 & \text{if } B_i \text{ is of } R\text{-type} \end{cases} \tag{E.2}$$

for $i = 1, 2, 3$. The conditions at the vertices can then be evaluated to

$$\begin{array}{lll} v_1 : s_1 s_3 s_{14} s_{15} = -\nu_1 \ , & v_2 : s_1 s_2 s_{12} = -1 \ , & v_3 : s_2 s_3 s_{19} s_{20} = -1 \ , \\ v_4 : s_{12} = \nu_2 \ , & v_5 : s_{17} s_{21} = -1 \ , & v_6 : s_{19} = -1 \ , \\ v_7 : s_{14} s_{21} = -\nu_3 \ , & v_8 : s_{15} = -1 \ , & v_9 : s_{17} s_{20} = -1 \ . \end{array} \tag{E.3}$$

From these equations it follows that

$$\nu_1 \nu_2 \nu_3 = 1 \ . \tag{E.4}$$

If this is the case, let $\alpha_1, \alpha_2 \in \{1, -1\}$. Then all solutions to these equations are given by

i	1	2	3	12	14	15	17	19	20	21
s_i	α_1	$-\nu_2\alpha_1$	$\nu_1\alpha_1\alpha_2$	ν_2	α_2	-1	$\nu_3\alpha_2$	-1	$-\nu_3\alpha_2$	$-\nu_3\alpha_2$

. (E.5)

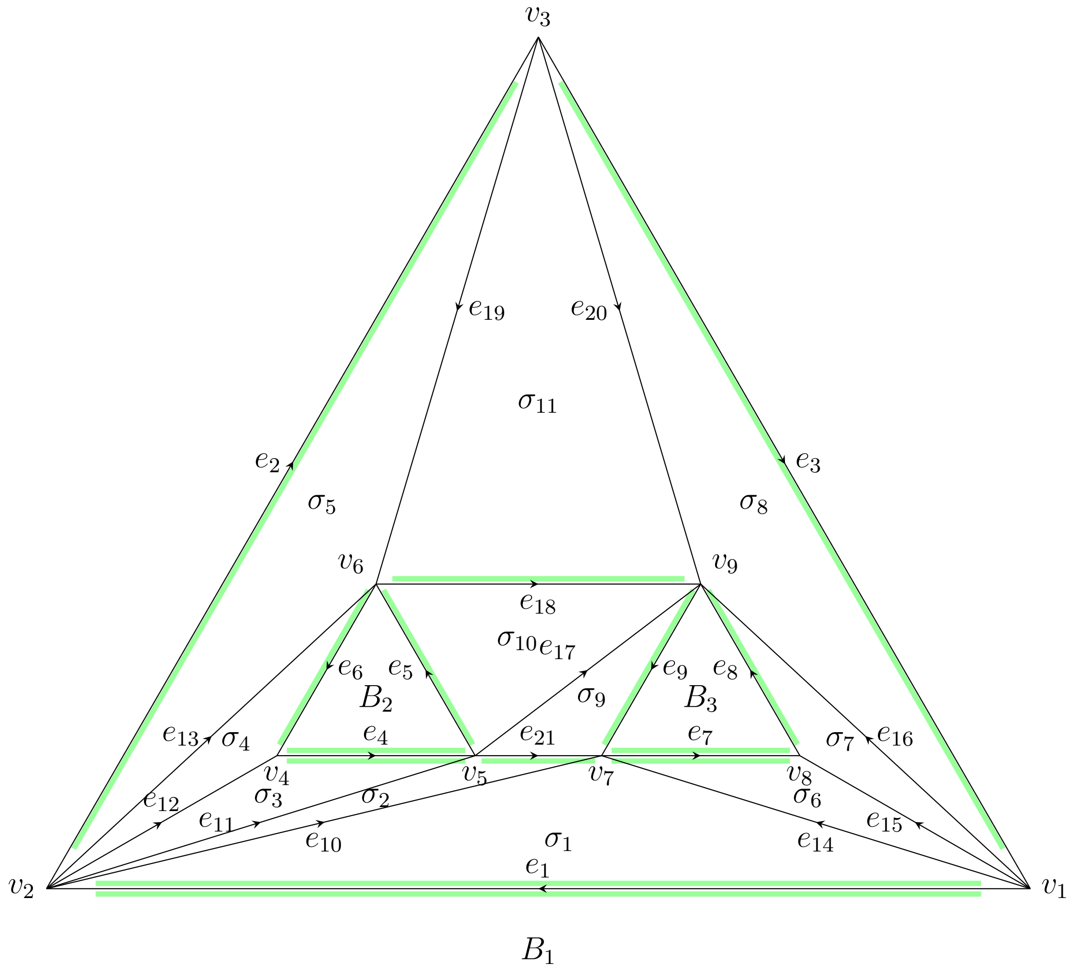


Figure E.1.: A triangulation of the genus 0 surface with 3 boundaries with markings and labels. Boundaries are labeled by B_1, B_2, B_3 and correspondingly the edges e_1, …,e_9 are boundary edges.

The result of translating the triangulation in Figure E.1 into a string diagram as in Section 5.1 and Equation (5.40) is shown in Figure E.2. We now replace the maps t and $c_{\pm 1}$ as in Proposition 5.9. After a tedious but straightforward calculation one arrives at

$$T_A = b \circ (\mathrm{id}_A \otimes \mu) \circ \big(q_{\nu_1} \otimes q_{\nu_2} \otimes q_{\nu_3}\big) \circ \big(N_{\alpha_1} \otimes \mathrm{id}_A \otimes N_{-\eta_1\alpha_2}\big) \circ (\pi^{31})^{\otimes 3} \ . \qquad \text{(E.6)}$$

Two of the identities used to get this result are worth pointing out: firstly, Lemma 6.8 has been used to insert an additional q to make the expression more symmetric; secondly, the Nakayama automorphism satisfies $q_\nu \circ N_{-\nu} = q_\nu$ (Lemma 5.12).

We now turn to the proofs of Lemmas 6.6 and 6.7 from Section 6.3.

Proof of Lemma 6.6. A spin structure with boundary types $\delta_1, \delta_2, \delta_3$ exists if and only if there are admissible edge signs on the marked triangulation given in Figure E.1. The necessary (and sufficient) condition for this stated in (E.4) proves the first part of the lemma.

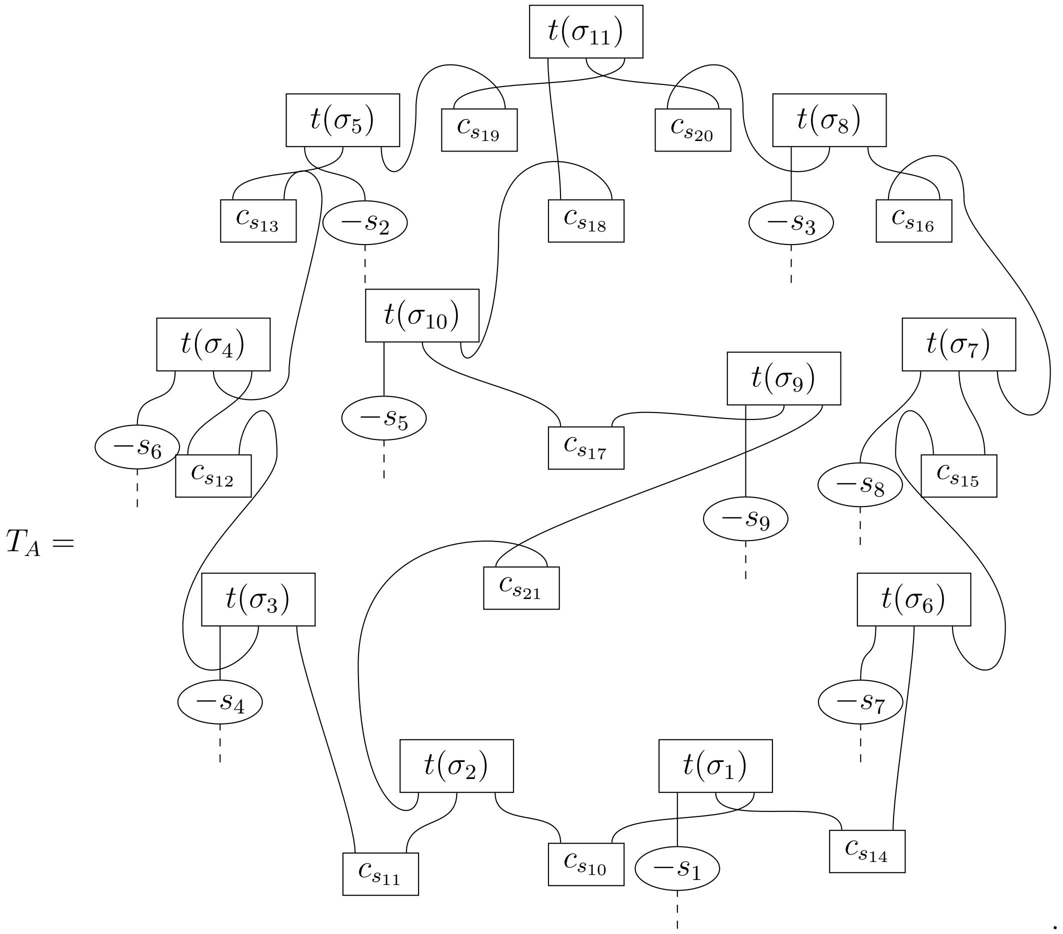


Figure E.2.: The string diagram resulting from the triangulation in Figure E.1. Here the dotted ingoing lines have to be ordered from $1, \ldots, 9$ according to the edge e_i they correspond to. As in (D.1) we write $t(\sigma_i)$ to indicate which triangle the map comes from, but the map is $t : A^{\otimes 3} \to \mathbf{1}$ in all cases.

For the second statement we need to check that up to isomorphism there are exactly four spin structures, and that representatives of these are provided by the four sets of admissible edge signs found above.

All possible spin structures are produced from any one spin structure by composing a boundary parametrisation with a leaf exchange. This gives a transitive action of $(\mathbb{Z}_2)^3$ on the set of spin structures. Since the surface is connected, the only non-trivial automorphism of the spin structure in the interior of $\Sigma^{0,3}$ is leaf exchange. On the boundary, this induces the diagonal $\mathbb{Z}_2$-action. The quotient of $(\mathbb{Z}_2)^3$ by the diagonal $\mathbb{Z}_2$ thus acts transitively and faithfully, showing that there are four spin structures (with parametrised boundary). Finally, since changing α_1 and α_2 amounts to precomposing two of the three boundaries with a leaf exchange, the four values of (α_1, α_2) precisely give the four possible spin structures. □

Proof of Lemma 6.7. Given the condition in (E.4), we get four spin structures parametrised by (α_1, α_2). The value of the TFT on the corresponding spin surface is given in (E.6). Now substitute

$$\alpha_1 = \varepsilon_1\, \varepsilon_2 \quad , \qquad \alpha_2 = -\eta_1\, \varepsilon_2 \ , \tag{E.7}$$

as well as $\mathrm{id} = N_{\varepsilon_2} \circ N_{\varepsilon_2}$. One can then remove one factor of N_{ε_2} from each leg by moving it through $b \circ (\mu \otimes \mathrm{id})$. This results in the expression stated in the lemma. □

The above proof also determines the spin structure of $\Sigma^{0,3}_{\delta_1,\delta_2,\delta_3,\varepsilon_1,\varepsilon_2}$ to be those obtained from the marked triangulation equipped with the edge signs (E.1) and (E.5), where $\alpha_{1,2}$ have been replaced as in (E.7).

English and German summaries

Summary

In this thesis I constructed a combinatorial model for r-spin, in particular spin, and framed surfaces. It is based on triangulations plus extra combinatorial data and describes closed surfaces as well as surfaces with parametrised boundary. Using this model I constructed a two-dimensional lattice topological quantum field theory (tqft) on r-spin and on framed surfaces. The algebraic input data to this tqft then consists of a Δ-separable Frobenius algebra. For r-spin tqfts its Nakayama automorphism N must satisfy $N^r = \mathrm{id}$, for framed surfaces there is no condition on N. The lattice construction is also compared to results from the cobordism hypothesis, a comparison made more interesting in this case as framed surfaces on the lattice side can be considered. The lattice construction used in this thesis is closely related to defect networks, applicable to general 2d-qfts. It is expected that translating the method to defect networks allows constructing (r-)spin-qfts from ordinary qfts and this thesis is indeed the foundation for that program. A completion of the above mentioned defect-networks program should shed more light on this connection.

Zusammenfassung

Die Arbeit besteht aus zwei wesentlichen Teilen. Im ersten Teil wird ein kombinatorisches Modell für Flächen mit r-Spinstruktur und Flächen mit Rahmung konstruiert. Das verwendete Modell besteht aus Triangulierungen mit Zusatzdaten und schließt auch Flächen mit parametrisiertem Rand mit ein. Im zweiten Teil wird das kombinatorische Modell zur Konstruktion einer topologischen Gitterquantenfeldtheorie auf diesen Flächen verwendet. Die notwendigen algebraischen Eingangsdaten sind dann eine Δ-separable Frobeniusalgebra. Im r-spin Fall muss deren Nakayamaautomorphismus zur r-ten Potenz die Identitätsabbildung sein während es im Fall von gerahmten Flächen keine Bedingung für N gibt. Die konstruierte Gitterfeldtheorie kann mit Resultaten aus der Kobordismushypothese verglichen werden. Dies ist vor allem deshalb interessant weil in der Gitterkonstruktion Flächen mit Rahmung betrachtet werden können. Die Gitterkonstruktion ist auch nah verwandt mit Defektnetzwerken, welche in zweidimensionalen Quantenfeldtheorien verwendet werden können. Diese Arbeit legt die Grundlagen um solche Defektnetzwerke zur Konstruktion von (r-)Spin-Quantenfeldtheorien aus anderen Quantenfeldtheorien zu verwenden. Eine weitergehende Untersuchung der genannten Defektnetzwerke sollte diese Verbindung noch besser klären.

Bibliography

[Ab] L. Abrams, *Two-dimensional topological quantum field theories and Frobenius algebras*, J. Knot Theory Ramifications **5** (1996) 569–587.

[At] M. Atiyah, *Topological quantum field theories*, Publ. Math. Inst. Hautes Études Sci. **68** (1988) 175–186.

[BCP] I. Brunner, N. Carqueville, D. Plencner, *Orbifolds and Topological Defects*, Comm. Math. Phys. **332** (2014) 669–712. [1307.3141 [hep-th]].

[BK] B. Bakalov, A. Kirillov, *Lectures on Tensor Categories and Modular Functors*, Univ. Lecture Ser. **21** (2001), Amer. Math. Soc., Providence, RI.

[BP] C. Bachas, P.M.S. Petropoulos, *Topological models on the lattice and a remark on string theory cloning*, Comm. Math. Phys. **152** (1993) 191–202 [hep-th/9205031].

[BT] J. Barrett, S. Tavares, *Two-dimensional state sum models and spin structures*, Comm. Math. Phys. **336 53–100** [1312.7561 [math.QA]].

[Bu] R. Budney, *Combinatorial spin structures on triangulated manifolds*, [1306.4841 [math.GT]].

[BW] J.W. Barrett, B.W. Westbury, *Invariants of piecewise linear three manifolds*, Trans. Amer. Math. Soc. **348** (1996) 3997–4022 [hep-th/9311155].

[CR] N. Carqueville, I. Runkel, *Orbifold completion of defect bicategories*, [1210.6363 [math.QA]].

[Da] O. Davidovich, *State sums in two dimensional fully extended topological field theories*, Doctoral Thesis, Univ. Texas, Austin, 2011.

[DKR] A. Davydov, L. Kong, I. Runkel, *Field theories with defects and the centre functor*, in Hisham Sati, Urs Schreiber (eds.): *Mathematical Foundations of Quantum Field theory and Perturbative String Theory*, Proc. Sympos. Pure Math., AMS, **83** (2011) 71–128, [1107.0495 [math.QA]].

[DRCR] A. Davydov, A.R. Camacho, I. Runkel, *N=2 minimal conformal field theories and matrix bifactorisations of* x^d, [1409.2144 [math.QA]].

[Di] R. Dijkgraaf, *A geometrical approach to two-dimensional conformal field theory*, Doctoral Thesis, Utrecht University, Utrecht, 1989.

[DNG] N. Drukker, D. Gaiotto, J. Gomis, *The virtue of defects in 4D gauge theories and 2D CFTs*, J. High Energy Phys. **2011:25** (2011) 1–54, [1003.1112 [hep-th]].

[FHK] M. Fukuma, S. Hosono, H. Kawai, *Lattice topological field theory in two-dimensions*, Comm. Math. Phys. **161** (1994) 157–176 [hep-th/9212154].

[FrFRS1] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert, *Duality and defects in rational conformal field theory*, Nuclear Phys. B **763** (2007) 354–430, [hep-th/0607247].

[FrFRS2] J. Fröhlich, J. Fuchs, I. Runkel, C. Schweigert, *Defect lines, dualities, and generalised orbifolds*, conference proceedings "XVI International Congress on Mathematical Physics" (Prague, August 2009), [0909.5013 [math-ph]].

[FRS1] J. Fuchs, I. Runkel, C. Schweigert, *TFT construction of RCFT correlators. I: Partition functions*, Nuclear Phys. B **646** (2002) 353–497, [hep-th/0204148].

[FS] J. Fuchs, C. Schweigert, *A note on permutation twist defects in topological bilayer phases*, Lett. Math. Phys. **104** (2014) 1385–1405, [1310.1329 [hep-th]].

[FSV] J. Fuchs, C. Schweigert, A. Valentino, *Bicategories for boundary conditions and for surface defects in 3-d TFT*, Comm. Math. Phys. **321** (2013) 543–575, [1203.4568 [hep-th]].

[FSt] J. Fuchs, C. Stigner, *On Frobenius algebras in rigid monoidal categories*, Arab. J. Sci. Eng. **33-2C** (2008) 175–191 [0901.4886 [math.CT]].

[GP] H. Geiges, J.G. Pérez, *Generalised spin structures on 2-dimensional orbifolds*, Osaka J. Math. **49(2)** (2012) 449–470, [1004.1979 [math.GT]]

[Gu] S. Gunningham, *Spin Hurwitz numbers and topological quantum field theory*, [1201.1273 [math.QA]].

[Ha] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[Hu] J. Hudson, *Piecewise linear topology*, W.A. Benjamin, Inc., 1969.

[JS] A. Joyal, R. Street, *The geometry of tensor calculus, I*, Adv. Math. **88** (1991) 55–112.

[KK] A. Kitaev, L. Kong, *Models for gapped boundaries and domain walls*, Comm. Math. Phys. **313** (2012) 351–373, [1104.5047 [cond-mat.str-el]].

[KS] A. Kapustin, N. Saulina, *Surface operators in 3d topological field theory and 2d rational conformal field theory*, in Hisham Sati, Urs Schreiber (eds.): *Mathematical Foundations of Quantum Field theory and Perturbative String Theory*, Proc. Sympos. Pure Math., AMS, **83** (2011) 175–198, [1012.0911 [hep-th]].

[Ko] J. Kock, *Frobenius Algebras and 2-D Topological Quantum Field Theories*, Cambridge University Press, 2004.

[Le] J. Lee, *Introduction to topological manifolds*, Springer, 2000 (first edition, not the second one).

[LM] H.B. Lawson, M.L. Michelsohn, *Spin Geometry*, Princeton University Press, 1989.

[LP] A.D. Lauda, H. Pfeiffer, *State sum construction of two-dimensional open-closed topological quantum field theories*, J. Knot Theory Ramifications **16** (2007) 1121–1163 [math/0602047 [math.QA]].

[Lu] J. Lurie, *On the classification of topological field theories*, Current developments in mathematics **2008** (2009) 129–280 [0905.0465 [math.CT]].

[LV] G. Lion, M. Vergne, *The Weil representation, Maslov index and Theta series*, Springer, 1980.

[Mi] J. Milnor, *Spin structures on manifolds*, Enseign. Math. (2) **9** (1963) 198–203.

[MS] G. Moore and G. Segal, *D-branes and K-theory in 2D topological field theory*, in "Dirichlet Branes and Mirror Symmetry", Clay Mathematics Monographs **4** (2009) 27–108 [hep-th/0609042].

[Mu] J. Munkres, *Elementary differential topology*, Princeton University Press, 1966.

[NR] S. Novak and I. Runkel, *State sum construction of two-dimensional topological quantum field theories on spin surfaces*, J. Knot Theory Ramifications **24** (2015) [1402.2839 [math.QA]].

[Pa] U. Pachner, *P.L. homeomorphic manifolds are equivalent by elementary shellings*, European J. Combin. **12** (1991) 129–145.

[Qu] F. Quinn, *Lectures on axiomatic topological quantum field theory*, "in Geometry and Quantum Field Theory", Amer. Math. Soc. (1995) 323–453.

[Ra] O. Randal-Williams, *Homology of the moduli spaces and mapping class groups of framed, r-Spin and Pin surfaces*, J. Topol. **7** (2014) 155-186, [1001.5366 [math.GT]].

[Se] G. B. Segal, *The definition of conformal field theory*, in "Differential geometrical methods in theoretical physics", Springer (1988) 165–171.

[SP] C. Schommer-Pries, *The classification of two-dimensional extended topological quantum field theories*, Doctoral Thesis, UC Berkeley, 2009, [1112.1000 [math.AT]].

[Tu1] V. Turaev, *Homotopy field theory in dimension 2 and group-algebras*, [math/9910010].

[Tu2] V. Turaev, *Homotopy field theory in dimension 3 and crossed group-categories*, [math/0005291].

[Tu3] V. Turaev, *Sections of fiber bundles over surfaces*, [0904.2692 [math.GT]].

[TV] V.G. Turaev, O.Y. Viro, *State sum invariants of 3-manifolds and quantum 6j-symbols*, Topology **31** (1992) 865–902.

[Wi1] E. Witten, *Topological quantum field theory*, Comm. Math. Phys. **117** (1988) 353–386.

[Wi2] E. Witten, *Quantum field theory and the Jones polynomial*, Comm. Math. Phys. **121** (1989) 351–386.