The hyper-Kähler/quaternionic Kähler correspondence and the geometry of the c-map

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Abstract

This thesis deals with quaternionic pseudo-Kähler manifolds obtained from Haydys' HK/QK correspondence with particular emphasis on complete quaternionic Kähler manifolds of negative scalar curvature that appear in certain string theory constructions.

The starting point for the HK/QK correspondence is a pseudo-hyper-Kähler manifold endowed with a real-valued function fulfilling certain assumptions. In particular, the function is the Hamiltonian for a Killing vector field, which preserves one of the three complex structures while rotating the other two. The HK/QK correspondence then constructs a quaternionic pseudo-Kähler manifold of the same dimension, which is endowed with a Killing vector field. A shift of the Hamiltonian function by an additive constant leads to a one-parameter family of deformations of the resulting quaternionic pseudo-Kähler metric.

We give a new and self-contained proof that the manifolds obtained from the HK/QK correspondence are quaternionic pseudo-Kähler. We reprove the known relation between the HK/QK correspondence, conical pseudo-hyper-Kähler manifolds and the hyper-Kähler quotient construction. As a new result, we prove the compatibility of the HK/QK correspondence with the hyper-Kähler and quaternionic Kähler quotient constructions. As an example, we show that a one-parameter family of quaternionic Kähler manifolds obtained from the cotangent bundle of complex projective space via the HK/QK correspondence is locally isometric to quaternionic projective space for one choice of parameter and locally isometric to another Wolf space for a different choice of parameter.

We show that all manifolds in the image of the supergravity c-map can be obtained via the HK/QK correspondence from a manifold in the image of the rigid c-map. We also show that the shift of the Hamiltonian function in this class of examples leads to the one-loop deformed supergravity c-map. We show that in each family of quaternionic Kähler manifolds obtained from the one-loop deformed supergravity c-map, all manifolds with positive deformation parameter are pairwise isometric.

We show that for a large class of examples, the quaternionic Kähler manifolds obtained from the one-loop deformed supergravity c-map with positive deformation parameter are complete if the undeformed metric is complete. This in particular gives explicit deformations by complete quaternionic Kähler metrics of all Wolf spaces of non-compact type (except for quaternionic hyperbolic space)

and of all non-symmetric Alekseevsky spaces.

We give an explicit realization of Salamon's E-H formalism and use this to calculate the quartic symmetric tensor field determining the Riemann curvature tensor of a quaternionic Kähler manifold for all manifolds in the image of the q-map. We use this to show that the members of an explicit series of complete quaternionic Kähler manifolds that we construct from the q-map are not locally homogeneous.

Zusammenfassung

Die vorliegende Dissertation beschäftigt sich mit Haydys' HK/QK-Korrespondenz mit besonderem Augenmerk auf vollständige quaternionische Kähler-Mannigfaltigkeiten, die aus bestimmten Konstruktionen in der Stringtheorie stammen.

Den Ausgangspunkt der HK/QK-Korrespondenz bildet eine Pseudo-Hyper-Kähler-Mannigfaltigkeit, die mit einer reell-wertigen Funktion versehen ist, welche bestimmte Voraussetzungen erfüllt. Insbesondere ist die Funktion hamiltonsch bezüglich eines Killing-Vektorfeldes, welches eine der drei komplexen Strukturen erhält und die anderen beiden rotiert. Die HK/QK-Korrespondenz konstruiert dann eine quaternionische Pseudo-Kähler-Mannigfaltigkeit der selben Dimension, versehen mit einem Killing-Vektorfeld. Das Verschieben der Hamilton-Funktion um eine additive Konstante führt zu einer Ein-Parameter-Familie von Deformationen der resultierenden quaternionischen Pseudo-Kähler-Metrik.

Wir präsentieren einen neuen, eigenständigen Beweis für die Tatsache dass die durch die HK/QK-Korrespondenz konstruierten Mannigfaltigkeiten quaternionisch pseudo-Kählersch sind. Wir weisen erneut den bekannten Zusammenhang zwischen der HK/QK-Korrespondenz, konischen Pseudo-Hyper-Kähler-Mannigfaltigkeiten und der Hyper-Kähler-Quotienten-Konstruktion nach. Als ein neues Resultat zeigen wir dass die HK/QK-Korrespondenz mit den Hyper-Kähler- und Quaternionisch-Kähler-Quotienten-Konstruktionen verträglich ist. Als Beispiel zeigen wir, dass eine per HK/QK-Korrespondenz vom Kotangentialraum des komplex projektiven Raumes erhaltene Ein-Parameter-Famile von quaternionischen Kähler-Mannigfaltigkeiten für eine bestimmte Wahl des Parameters lokal isometrisch zum quaternionisch projektiven Raum und für eine andere Wahl des Parameters lokal isometrisch zu einem weiteren Wolf-Raum ist.

Wir zeigen, dass alle Mannigfaltigkeiten im Bild der Supergravitations-c-Abbildung per HK/QK-Korrespondenz aus Mannigfaltigkeiten im Bild der rigiden c-Abbildung konstruiert werden können. Desweiteren zeigen wir, dass das Verschieben der Hamilton-Funktion für diese Klasse von Beispielen zu der Ein-Schleifendeformation der Supergravitations-c-Abbildung führt. Wir zeigen dass in jeder durch die Ein-Schleifendeformation der Supergravitations-c-Abbildung erhaltenen Familie von quaternionischen Kähler-Mannigfaltigkeiten alle Mannigfaltigkeiten mit positivem Deformationsparameter paarweise isometrisch sind.

Für eine große Klasse von Beispielen zeigen wir, dass die durch die Ein-Schleifen-

deformation der Supergravitations-c-Abbildung mit positivem Deformationsparameter erhaltenen quaternionischen Kähler-Mannigfaltigkeiten vollständig sind, wenn die undeformierte Metrik vollständig ist. Dadurch erhalten wir insbesondere für alle Wolf-Räume vom nicht-kompakten Typ (bis auf den quaternionisch hyperbolischen Raum) und alle nicht-symmetrischen Alekseevsky-Räume explizite Deformationen durch vollständige quaternionische Kähler-Metriken.

Wir geben eine explizite Realisierung von Salamon's E-H-Formalismus und benutzen diese um das, den Riemann-Tensor einer jeden quaternionischen Kähler Mannigfaltigkeit bestimmende, symmetrische quartische Tensorfeld für alle Mannigfaltigkeiten im Bild der q-Abbildung zu bestimmen. Dies verwenden wir um zu zeigen, dass alle Mitglieder einer aus der q-Abbildung konstruierten Serie von vollständigen quaternionischen Kähler-Mannigfaltigkeiten nicht lokal homogen sind.

Contents

Chapter 1

Introduction

1.1 Background and motivation

Quaternionic Kähler manifolds constitute a field of study that is of strong interest to both theoretical physicists and to pure mathematicians. For already more than three decades, this field has seen vast mutual influence from physics and mathematics and has stimulated a considerable amount of interdisciplinary collaborations. Similarly to Kähler and hyper-Kähler geometry, quaternionic Kähler geometry was invented by mathematicians and later turned out to be related to supersymmetry.

In differential geometry, quaternionic Kähler manifolds are widely known for appearing on Berger's list of all possible holonomy groups of simply connected, irreducible, non-locally symmetric Riemannian manifolds [Be]. In fact, quaternionic Kähler manifolds (of dimension greater than four) can be defined as Riemannian manifolds whose holonomy group is contained in $Sp(n) \cdot Sp(1) \approx \frac{Sp(n) \times Sp(1)}{\mathbb{Z}_2}$. Alekseevsky showed that all (pseudo-)quaternionic Kähler manifolds are Einstein [A1]. In this thesis, we exclude the case of zero scalar curvature in the definition of quaternionic Kähler manifolds (a simply connected quaternionic Kähler manifold of zero scalar curvature would be hyper-Kähler, see e.g. [Sw1]). This leaves us with two very different cases: quaternionic Kähler manifolds of positive scalar curvature and quaternionic Kähler manifolds of negative scalar curvature (see the discussion below). Since quaternionic Kähler manifolds of negative scalar curvature are the ones related to supergravity, many results on quaternionic Kähler manifolds of positive scalar curvature will remain unmentioned in this thesis. See Salamon's essay [Sa2], Chapter 12 in the book [BoGal] by Boyer and Galicki, or Amann's thesis [Amann] and references therein for an overview of the results on quaternionic Kähler manifolds that can be found in the mathematical literature.

On the physics side, quaternionic Kähler manifolds play a crucial role in supergravity and string theory: They appear as the target spaces for hyper-multiplet scalar fields in three- and four-dimensional $\mathcal{N} = 2$ supergravity theories, as was shown by Bagger and Witten in [BW]. The type of target space geometry depends on the space-time dimension, on the amount of supersymmetry and on the representation chosen for the matter multiplets of the supergravity theory. Since different supergravity theories can be related by the technique of *dimensional reduction*, there often exist surprising and non-trivial relations between the corresponding target space geometries. In particular, dimensional reduction of four-dimensional $\mathcal{N} = 2$ vector multiplets to three-dimensional hyper-multiplets leads to the so-called *supergravity c-map*, which assigns a $4(n+1)$-dimensional quaternionic Kähler manifold of negative scalar curvature to each $2n$-dimensional *projective special Kähler* manifold. This construction was worked out by Ferrara and Sabharwal in [FS], which is why the quaternionic Kähler metric of manifolds in the image of the supergravity c-map is often called the Ferrara-Sabharwal metric. Similarly, the reduction of five-dimensional $\mathcal{N} = 2$ vector multiplets to four-dimensions leads to the *supergravity r-map*, which assigns a $2n$-dimensional projective special Kähler manifold to each $(n-1)$-dimensional *projective special real* manifold. The latter construction was worked out by de Wit and Van Proeyen in [DV]. The composition of the supergravity r- and c-map is called the *q-map*. The supergravity c-map is realized in the low energy limit of type II string theories compactified on a Calabi-Yau three-fold. Quantum corrections to the Ferrara-Sabharwal metric appearing in this context are investigated in much detail in the physics literature. While the full non-perturbative correction to the Ferrara-Sabharwal metric is still unknown (see [Alex] for a review or [AB] for the latest paper), the perturbative corrections in the string coupling constant g_s were fully determined in [RSV]. In this paper, Robles-Llana, Saueressig and Vandoren give an explicit expression for the *one-loop deformed Ferrara-Sabharwal metric* and argue that higher loop contributions are excluded. While the supergravity c-map, as well as the supergravity r-map, are known to preserve completeness [CHM], the question of completeness for the one-loop deformation constituted an open problem prior to this thesis (see the appendix of [ACDM]).

In this thesis, we often consider pseudo-Riemannian analogues of quaternionic Kähler manifolds that have arbitrary signature. All symmetric quaternionic pseudo-Kähler manifolds were classified by Cortés and Alekseevsky in [AC]. Their list of examples contains in particular the following three series of symmetric pseudo-quaternionic Kähler manifolds which each consist of one example for every possible dimension and signature:

$$\begin{aligned} \mathbb{H}P^{k,\ell} &:= \frac{Sp(k+1,\ell)}{Sp(1)\times Sp(k,\ell)}, \\ X(k,\ell) &:= \frac{SU(k+2,\ell)}{S[U(2)\times U(k,\ell)]}, \\ Y(k,\ell) &:= \frac{SO_0(k+4,\ell)}{SO(4)\times SO_0(k,\ell)}. \end{aligned} \tag{1.1}$$

The sign of the scalar curvature is of no particular relevance in the study of quaternionic pseudo-Kähler manifolds of arbitrary signature. In the examples chosen in the above equation, the sign of the scalar curvature is positive. It can be changed by changing the sign of the metric and, hence, inverting the signature. Since in this thesis, we will focus a lot of attention on the Riemannian case, we also introduce the following notations for quaternionic pseudo-Kähler manifolds of negative scalar curvature:

$$\mathbb{H}H^{k,\ell} := -\mathbb{H}P^{\ell,k}, \quad \tilde{X}(k,\ell) := -X(\ell,k), \quad \tilde{Y}(k,\ell) := -Y(\ell,k). \tag{1.2}$$

In the Riemannian case, the classification of symmetric quaternionic Kähler manifolds goes back to Wolf [W]. For this reason, symmetric quaternionic Kähler manifolds are called *Wolf spaces* (*of compact type* in the case of positive scalar curvature, respectively *of non-compact type* in the case of negative scalar curvature). There is one Wolf space of compact type for every compact simple Lie group, i.e. the Wolf spaces of compact type consist of the three series $\mathbb{H}P^n := \mathbb{H}P^{n,0}$, $X(n) := X(n,0)$ and $Y(n) := Y(n,0)$, and of five exceptional examples corresponding to the Lie groups G_2, F_4, E_6, E_7 and E_8. We denote the non-compact duals of $\mathbb{H}P^n$, $X(n)$ and $Y(n)$ by $\mathbb{H}H^n := \mathbb{H}H^{n,0}$, $\tilde{X}(n) := \tilde{X}(n,0)$ and $\tilde{Y}(n) := \tilde{Y}(n,0)$, respectively. Apart from $\mathbb{H}H^n$ and $\tilde{X}(n)$, all Wolf spaces of non-compact type are in the image of the q-map. $\tilde{X}(n)$ is, however, in the image of the supergravity c-map [GST, DV].

In the case of positive scalar curvature, it is conjectured by LeBrun and Salamon that all complete quaternionic Kähler manifolds are symmetric, i.e. that they are Wolf spaces of compact type [LS]. As supporting evidence, they showed that up to isometry and rescaling, there are only finitely many examples of such manifolds in each dimension. In dimension four and eight, the LeBrun-Salamon conjecture was proven by Hitchin, respectively by Poon and Salamon (see the references in [LS]). In higher dimensions, the conjecture is still open and constitutes the biggest open problem in the field of quaternionic Kähler manifolds.

In the case of negative scalar curvature the situation is very different and there are for instance examples of homogeneous, non-symmetric quaternionic Kähler manifolds, the so-called normal quaternionic Kähler manifolds or Alekseevsky spaces [A2, DV, Co]. The latter are all in the image of the q-map [DV]. LeBrun showed that in the case of negative scalar curvature, complete non-locally homogeneous quaternionic Kähler manifolds exist in abundance using deformation theory on the twistor space of quaternionic hyperbolic space [L]. His proofs are not constructive, however. Constructions of complete quaternionic Kähler metrics that are not locally homogeneous so far either use deformation theory without giving explicit metrics (see e.g. [D] and references therein), the quaternionic Kähler quotient construction (see [G2], [G3], [BCGP], etc.), or are restricted to low dimensions (see, e.g., [DFISUV]).

A rather recently discovered tool for the construction of quaternionic Kähler manifolds is the so-called *HK/QK correspondence* which was invented by Haydys in [Ha] and extended to the pseudo-Riemannian setting in [ACM, ACDM]. As opposed to the (one-loop deformed) supergravity c-map, the HK/QK correspondence can be used to construct quaternionic Kähler manifolds of both positive and negative scalar curvature. While the correspondence has already been investigated from various perspectives in [Ha, Hi4, MS1, MS2], the explicit quaternionic Kähler metric coming from this construction has only been derived and studied for very few examples and many questions about the construction are still open. Especially the question of completeness of the quaternionic Kähler manifolds obtained from the HK/QK correspondence is, up to now, entirely unexplored.

1.2 Main results and outline

The HK/QK correspondence constructs a quaternionic pseudo-Kähler manifold (endowed with a non-vanishing Killing vector field) from a pseudo-hyper-Kähler manifold endowed with a real-valued function fulfilling certain properties. In particular, the function is the Hamiltonian for a Killing vector field, which preserves one of the three complex structures while rotating the other two. The Hamiltonian function can be shifted by an additive constant, which leads to a one-parameter family of deformations of the resulting quaternionic pseudo-Kähler metric. As we will see in Chapter 4, the construction involves the choice of an S^1-bundle with connection over the initial pseudo-hyper-Kähler manifold and the choice of a certain codimension one submanifold of the S^1-bundle.

In this thesis, we give a new and self-contained proof of the fact that the manifolds obtained from the HK/QK correspondence are quaternionic pseudo-Kähler (see Theorem 4.1.2). Theorem 4.1.2 gives explicit expressions for the resulting quaternionic Kähler metric, its signature, quaternionic structure and local $Sp(1)$-connection one-forms. As a large class of examples, we apply the HK/QK correspondence to all conical pseudo-hyper-Kähler manifolds, which in particular establishes the following HK/QK correspondence:

$$\big((\mathbb{H}^n)^o,\, f = (r^2 + c)/2\big) \xmapsto[(c\neq 0)]{\text{HK/QK cor.}} \begin{cases} (\mathbb{H}P^n)^o & (c > 0) \\ \mathbb{H}H^n & (c < 0), \end{cases}$$

where the superscript ^{o} always denotes some open subset which will be specified in the main text. For $c < 0$, $(\mathbb{H}^n)^o$ is a proper subset of $\mathbb{H}^n$ and thus incomplete, while the resulting quaternionic Kähler manifold $\mathbb{H}H^n$ is complete. This phenomenon occurs in most of the examples of quaternionic Kähler manifolds with negative scalar curvature that we study in this thesis.

We show how the HK/QK correspondence is related to conical pseudo-hyper-Kähler manifolds and to the Swann bundle construction (Theorem 4.2.1 and Corollary 4.2.6) and we show a compatibility result between the HK/QK correspondence and the hyper-Kähler and quaternionic Kähler quotient constructions (Theorem 4.3.1). These findings are illustrated with an example that in particular shows the following HK/QK correspondences:

$$\Big(T^*\mathbb{C}P^n,\, f = \frac{1}{2}(c + \sqrt{1+\tilde{r}^2})\Big) \xmapsto{\text{HK/QK cor.}} \begin{cases} (\mathbb{H}P^n)^o & (c = 0) \\ (X(n))^o & (c = 1) \end{cases}$$

and

$$\Big((T^*\mathbb{C}H^n)^o,\, f = -\frac{1}{2}(c + \sqrt{1-\tilde{r}^2})\Big) \xmapsto{\text{HK/QK cor.}} \begin{cases} \mathbb{H}H^n & (c = 0) \\ (\tilde{X}(n))^o & (c = 1). \end{cases}$$

The first example shows that the one-parameter family of quaternionic Kähler metrics obtained via the HK/QK correspondence from the shift of the Hamiltonian function on a given hyper-Kähler manifold can be extendible to a compact manifold for two different choices of parameter. In this case, the resulting quaternionic Kähler manifold is locally isometric to two different Wolf spaces. The second example shows that in some cases a complete quaternionic Kähler metric can get deformed into an incomplete one. This also happens for the one-loop deformed supergravity c-map in the case of negative deformation parameter, while on the other hand, completeness is preserved for positive deformation parameter (see the discussion below).

For the supergravity c-map, we have the following results: We give a mathematical proof of the fact that the one-loop deformed Ferrara-Sabharwal metric is quaternionic Kähler by showing that it can be obtained via the HK/QK correspondence from a pseudo-hyper-Kähler manifold in the image of the so-called *rigid c-map* (Theorem 5.4.1). Concerning completeness, we show that, for positive deformation parameter, the manifolds in the image of the one-loop deformed q-map are complete, if the undeformed quaternionic Kähler manifold is complete (Corollary 6.3.8). For $\tilde{X}(n)$, we also show that the one-loop deformation is complete (Corollary 6.3.6) and we show some progress towards the general case of the one-loop deformed supergravity c-map (Proposition 6.3.10). For negative deformation parameter, the one-loop deformed Ferrara-Sabharwal metric is always incomplete [ACDM, Rem. 9]. The undeformed case corresponds to the choice of parameter $c = 0$. Note that in the context of compactifications of type II string theories on a Calabi-Yau three-fold, a positive deformation parameter corresponds to a negative Euler characteristic of the internal space in the case of type IIA string theory, respectively to a positive Euler characteristic in the case of type IIB [RSV].

These results in particular give deformations by complete quaternionic Kähler metrics of all Wolf spaces of non-compact type, except for quaternionic hyperbolic space, and of all non-symmetric Alekseevsky spaces. As opposed to [LS] and similar approaches, we can here give explicit expressions of the deformed metrics. The deformations are of the following kind: For any complete projective special Kähler manifold, we have a family of complete quaternionic Kähler metrics g^c_{FS} depending on a parameter $c \in \mathbb{R}^{\geq 0}$ on a fixed manifold $\bar{N}$, where $(\bar{N}, g^0_{FS})$ is the undeformed quaternionic Kähler manifold in the image of the supergravity c-map and all manifolds $(\bar{N}, g^c_{FS})$ with positive deformation parameter $c \in \mathbb{R}^{>0}$ are pairwise isometric (Proposition 5.5.2). For the case of the Wolf space $G^*_2/SO(4)$, we show that the deformed metric is not locally homogeneous and hence different from the undeformed metric using computer algebra software (see Remark 6.3.9).

We also construct a series of complete non-locally homogeneous quaternionic Kähler manifolds in the image of the (undeformed) q-map, i.e. we have an example in each dimension with an explicitly given metric that is not manifestly constructed via a quaternionic Kähler quotient.

The thesis is structured as follows:

Chapter 2 gives a short introduction into quaternionic pseudo-Kähler geometry including some well-known properties and discusses the pseudo-Riemannian versions of quaternionic projective and quaternionic hyperbolic space as examples. It also reviews the quaternionic Kähler quotient construction, which is illustrated by the examples

$$\mathbb{H}P^{k+1,\ell} /\!\!/\!\!/ S^1_{(\mathrm{diag.})} = X(k,\ell) \quad \text{and} \quad \mathbb{H}H^{k,\ell+1} /\!\!/\!\!/ S^1_{(\mathrm{diag.})} = \tilde{X}(k,\ell).$$

In Chapter 3, we introduce pseudo-hyper-Kähler manifolds as well as the hyper-Kähler quotient construction. As an example, we show in particular how to obtain the hyper-Kähler structure on the cotangent bundles of complex projective and complex hyperbolic space from a hyper-Kähler reduction. While in Section 3.6, we also discuss the Swann bundle construction and lifts of Killing vector fields and isometric group actions from a quaternionic pseudo-Kähler manifold to its Swann bundle, most of Chapter 3 focuses on *conical pseudo-hyper-Kähler* manifolds and their relation to quaternionic pseudo-Kähler geometry. Conical pseudo-hyper-Kähler manifolds are local versions of Swann bundles and are characterized by possessing a certain homothetic vector field.

Chapter 4 introduces, proves, analyses properties of and illustrates the HK/QK correspondence between pseudo-hyper-Kähler manifolds endowed with a certain real-valued function and quaternionic Kähler manifolds of the same dimension endowed with a non-vanishing Killing vector field.

Chapter 5 shows that all manifolds in the image of the one-loop deformed supergravity c-map can be obtained via the HK/QK correspondence from a manifold in the image of the rigid c-map. Section 5.5 then summarizes properties of the one-loop deformed supergravity c-map metric.

In Chapter 6, we study the completeness question for the manifolds in the image of the one-loop deformed supergravity c-map while in particular giving a full answer in the case of the one-loop deformed q-map.

In Chapter 7, we give an explicit (local) realization of the complex vector bundles E and H over a quaternionic Kähler manifold used in Salamon's E-H formalism introduced in [Sa1]. This gives an easy and clear way to translate between formulas in the mathematics literature and the *quaternionic vielbein* formalism used in the physics literature. Using these formulas, we calculate a quartic tensor field determining the curvature tensor for all manifolds in the image of the q-map. This is then used to study an explicit series of complete quaternionic Kähler manifolds of negative scalar curvature constructed via the q-map.

1.3 Remarks and relation to other work

The quaternionic Kähler quotient was introduced in [G1, GL]. The example $\mathbb{H}P^{n+1} /\!\!/\!\!/ S^1 = X(n)$ was the first example discussed by Galicki and Lawson and the examples in Section 2.2 are a straightforward generalization thereof. The hyper-Kähler quotient construction was introduced in [LR] and [HKLR]. The example $\mathbb{H}^{n+1} /\!\!/\!\!/ S^1 = T^*(\mathbb{C}P^n)$ in Section 3.4 was first discussed in [LR] and [Hi1] (see [BoGal, Ex. 12.8.5] and references therein).

The results in Chapter 3 about conical pseudo-hyper-Kähler manifolds and their relation to quaternionic Kähler manifolds are essentially all known from [Sw1]. Here, they are rephrased from a local point of view, which just assumes the existence of a vector field ξ, called the *Euler vector field*, such that the Levi-Civita connection ∇ fulfills $\nabla.\xi = \mathrm{Id}$. This viewpoint was also taken in [ACM] and is close to the treatment of the subject in the physics literature (see [DRV1, DRV2] and references therein). We need explicit results and formulae about conical hyper-Kähler manifolds in this formalism to motivate the HK/QK correspondence and to prove properties thereof in Chapter 4.

The account of the HK/QK correspondence presented in Chapter 4 and in particular the proof of the quaternionic Kähler property of the resulting metric only make use of an S^1-bundle over the original hyper-Kähler manifold and do not involve a higher-dimensional conical hyper-Kähler manifold. This approach was also taken in [MS1, MS2], where so called *elementary deformations* of the original hyper-Kähler metric are used to relate the HK/QK correspondence to Swann's twist formalism [Sw2]. In [ACDM] and [Ha], the proof of the quaternionic Kähler property of the resulting metric is based on the construction of a higher-dimensional conical hyper-Kähler manifold. In [Hi4], Hitchin discusses the HK/QK correspondence from the point of view of the corresponding twistor spaces.

Note that the presentation of the HK/QK correspondence in Section 4.1 is entirely self-contained. It just uses the basic facts about quaternionic Kähler geometry introduced in Section 2.1. The reader who is just interested in applying the HK/QK correspondence and in the proof that the resulting manifold is indeed quaternionic Kähler can skip Chapter 3 and go directly to Section 4.1.

Apart from Section 5.5, Chapter 5 has already appeared in a joint publication with Alekseevsky, Cortés and Mohaupt [ACDM]. The result that the rigid and the one-loop deformed supergravity c-map are related by the HK/QK correspondence previously appeared in the physics literature in [APP]. On the level of twistor spaces, the simple relation between quaternionic Kähler manifolds in the image of the undeformed supergravity c-map and the corresponding pseudo-hyper-Kähler manifolds in the image of the rigid c-map was already discovered in [RVV1, RVV2]. For a treatment of the one-loop deformed supergravity c-map on the level of twistor spaces, see [APSV] and references therein. In addition to what has already been published in [ACDM], we prove in Section 5.5 that for a given projective special Kähler manifold, the one-loop deformed Ferrara-Sabharwal metrics g^c_{FS} on $\bar{N}$ with $c > 0$ are all pairwise isometric, i.e. $(\bar{N}, g^c_{FS}) \approx (\bar{N}, g^{c'}_{FS})$ for any $c, c' \in \mathbb{R}^{>0}$. For the example[1] $G^*_2/SO(4)$, we show that $(\bar{N}, g^0_{FS}) \approx G^*_2/SO(4)$ and $(\bar{N}, g^1_{FS})$ are non-isometric using computer algebra software (see Remark 6.3.9).

While the question of completeness for the undeformed supergravity c-map was entirely answered in [CHM], the results on the completeness of the one-loop deformed c-map in Section 6.3 are new.

Our formulas in Section 7.1 for the Levi-Civita connection and the curvature of quaternionic Kähler manifolds in terms of the quaternionic vielbein formalism can also be found in [BW, FS, ACDGV]. The Levi-Civita connection and the curvature of the manifolds in the image of the supergravity c-map have been calculated, respectively stated in [FS]. In Section 7.3 we do exactly the same calculations for the case of the q-map[2]. For the Levi-Civita connection, we extend the result to the one-loop deformed case.

Although part of the work on my article [CDL] with V. Cortés and D. Lindemann was done during my time as a PhD student, the classification of complete

[1] $G^*_2/SO(4)$ is the simplest example of a manifold in the image of the q-map.

[2] While their result for the Levi-Civita connection agrees with ours, the result stated in [FS] for the curvature at least seems to be missing terms.

projective special real surfaces will only be mentioned in a remark in Chapter 6.

1.4 Outlook

$$\begin{tikzcd}[column sep=small]
 & & \mathbb{H}^{n,2}_{<0} \arrow[dll, dashed, "/\!\!/\!\!/^{\mathrm{HK}}"'] \arrow[dr, "\mathbb{H}^*"] & \\
 ? \arrow[rrr, mapsto, "\text{HK/QK cor.}"'] \arrow[d, mapsto, "/\!\!/\!\!/^{\mathrm{HK}}"'] & & & \mathbb{H}H^{n,1} \arrow[d, mapsto, "/\!\!/\!\!/^{\mathrm{QK}} S^1_{(\mathrm{diag.})}"] \\
 T^*(\mathbb{C}^{n-1,1}_{<0}) \subset \mathbb{H}^{n-1,1} \arrow[rrr, mapsto, "\text{HK/QK cor.}"'] & & & \tilde{X}(n)
\end{tikzcd}$$

As another simple application for the compatibility of the HK/QK-correspondence with the hyper-Kähler and quaternionic Kähler quotient constructions, one could construct the missing hyper-Kähler manifold in the adjacent diagram on the right by performing an appropriate hyper-Kähler quotient of $\mathbb{H}^{n,2}_{<0}$. This would reprove the HK/QK correspondence between $T^*(\mathbb{C}^{n-1,1}_{<0}) \subset \mathbb{H}^{n-1,1}$ and $\tilde{X}(n) = Gr_{0,2}(\mathbb{C}^{n,2})$. These are (up to a change of sign) the manifolds in the image of the rigid and supergravity c-map, respectively, when the underlying projective special Kähler manifold is complex hyperbolic space $\mathbb{C}H^{n-1}$. To fill in the missing manifold in the diagram, one has to identify the Killing vector field on $\tilde{X}(n)$ that is induced by the HK/QK correspondence for the c-map in Chapter 5, find a corresponding Killing vector field on $\mathbb{H}H^{n,1}$, lift this vector field to $\mathbb{H}^{n,2}_{<0}$ and then perform the corresponding hyper-Kähler quotient.

In the case $n = 1$ for the above idea, there is a natural strategy to also understand the one-loop deformed metric in this way:

$$\begin{tikzcd}[column sep=small]
 & & \mathbb{H}^{1,2}_{<0} \arrow[dll, dashed, "/\!\!/\!\!/^{\mathrm{HK}}"'] \arrow[dr, "\mathbb{H}^*"] & \\
 ? \arrow[rrr, mapsto, "\text{HK/QK cor.}"'] \arrow[d, mapsto, "/\!\!/\!\!/^{\mathrm{HK}}"'] & & & \mathbb{H}H^{1,1} \arrow[d, mapsto, "/\!\!/\!\!/^{\mathrm{QK}} ?"] \\
 T^*(\mathbb{C}^{0,1}_{<0}) \subset \mathbb{H}^{0,1} \arrow[rrr, mapsto, "\text{HK/QK cor.}"'] & & & (\bar{N}, g^c_{UH})
\end{tikzcd}$$

The one-loop deformed universal hypermultiplet metric can be expressed in terms of the hyperbolic eigenfunction ansatz for 4-dimensional quaternionic Kähler manifolds with two commuting Killing vector fields given in [CP] (see Remark 8 in the appendix of [ACDM]). The hyperbolic eigenfunction ansatz in turn can be (locally) expressed as a quaternionic Kähler quotient of $\mathbb{H}P^2$, $\mathbb{H}H^2$, or $\mathbb{H}H^{1,1}$ [BCGP]. This can be used to understand the HK/QK correspondence for the universal hypermultiplet using the compatibility of the HK/QK correspondence with the HK and QK quotient constructions using a diagram as depicted on the right.

In [K], Kronheimer constructed all asymptotically locally Euclidean hyper-Kähler four-manifolds as hyper-Kähler quotients of flat quaternionic vector spaces. In [GN], Galicki and Nitta constructed quaternionic Kähler orbifold analogues of Kronheimer's examples as quaternionic Kähler quotients of quaternionic projective spaces. Using the compatibility of the HK/QK correspondence with hyper- and quaternionic Kähler quotients, it should be possible to show that the manifolds constructed by Kronheimer are (locally) related to the orbifolds constructed by Galicki and Nitta. For this class of examples, it would be interesting to work out the respective Killing vector fields on both sides of the correspondence and to study the deformations of the quaternionic Kähler metrics obtained from a shift of the Hamiltonian function chosen on the hyper-Kähler side.

In more generality, one could try to systematically study the HK/QK correspondence for all quaternionic Kähler quotients of quaternionic projective and quaternionic hyperbolic space, or even for all quaternionic Kähler quotients of symmetric quaternionic (pseudo-)Kähler manifolds. The quaternionic Kähler quotients of symmetric quaternionic (pseudo-)Kähler manifolds were systematically studied by Grandini on the level of Lie algebras in [Gr].

In this thesis, we did not pay much attention to quaternionic Kähler manifolds of positive scalar curvature, since in this case, all examples obtained from the HK/QK correspondence are bound to be incomplete. In case the LeBrun-Salamon conjecture is wrong, it is conceivable that some example of positive scalar curvature obtained from the HK/QK correspondence can be completed to a compact quaternionic Kähler manifold that is not symmetric. Candidates for such a situation can be found by choosing a Killing vector field on a Wolf space of compact type and then studying the one-parameter family of quaternionic Kähler manifolds resulting from a free choice of Hamiltonian function on the hyper-Kähler side. This idea is highly speculative, but something similar did happen before in the case of compact irregular Sasaki-Einstein manifolds:
Gauntlett, Martelli, Sparks and Waldram constructed compact irregular Sasaki-Einstein manifolds in [GMSW2] by extending a two-parameter family of local metrics found in [GMSW1] to $S^2 \times S^3$ for certain discrete choices of the parameters. This very surprising and rather accidental finding contradicts a conjecture by Cheeger and Tian which states that all Ricci-flat Kähler cones are standard [CT].

It remains to investigate, whether the series of complete non-locally homogeneous quaternionic Kähler metrics constructed in Section 7.4 can be obtained via a quaternionic Kähler quotient from a symmetric quaternionic pseudo-Kähler manifold. If this is not the case, these examples are manifestly different from all examples that were previously discussed in the literature.

Using our curvature results from Chapter 7, we plan to study whether the quaternionic Kähler manifolds constructed in Section 7.4 (or other examples obtained from the q-map) have non-positive sectional curvature. The only complete quaternionic Kähler manifolds of non-positive sectional curvature that have appeared in the literature so far are either locally symmetric or (non-explicit) small deformations of quaternionic hyperbolic space.

The orthogonal series $Y(n)$ of Wolf spaces can locally be obtained from the HK/QK correspondence using its compatibility with the hyper-Kähler and quaternionic Kähler quotient constructions and the fact that $Y(n) = \mathbb{H}P^{n+3} /\!\!/\!\!/ Sp(1)_{(\text{diag.})}$. For a certain choice of Killing vector field on the quaternionic Kähler side, the corresponding hyper-Kähler manifold should (locally) be a hyper-Kähler quotient of flat quaternionic vector space by an $Sp(1)$-action. The family of deformations of the quaternionic Kähler metric obtained from a shift of the Hamiltonian function is also worth studying in this case. Both this and the question below can similarly be studied for the Wolf spaces of non-compact type.

It is a natural question to ask, how the exceptional Wolf spaces can be obtained from the HK/QK correspondence. This question can be studied by choosing a Killing vector field on an exceptional Wolf space, lifting it to the Swann bundle and then performing the corresponding hyper-Kähler quotient of the Swann bundle. This situation can be investigated systematically on the level of Lie algebras.

The hyper-Kähler structure on cotangent bundles of Hermitian symmetic spaces constructed by Biquard and Gauduchon constitutes a natural candidate for applying the HK/QK correspondence. Out of this large class of examples, we so far only studied the cases $\mathbb{C}P^n$ and $\mathbb{C}H^n$.

Chapter 2

Quaternionic Kähler geometry

In Section 2.1, we introduce the notion of quaternionic (pseudo-)Kähler manifold and state some well-known properties. As examples, we discuss the pseudo-Riemannian versions $\mathbb{H}P^{k,\ell}$, $\mathbb{H}H^{k,\ell}$ of quaternionic projective, respectively quaternionic hyperbolic space.

In Section 2.2, we introduce the quaternionic Kähler quotient construction which is due to Galicki and Lawson [G1, GL], and illustrate it with the example of the S^1-action on $\mathbb{H}P^{k+1,\ell}$, respectively $\mathbb{H}H^{k,\ell+1}$, induced from the diagonal S^1-action on quaternionic vector space. This leads to symmetric quaternionic pseudo-Kähler manifolds defined by complex Grassmannians.

The discussion of the Swann bundle construction is postponed to Chapter 3 and curvature properties of quaternionic Kähler manifolds are discussed in Chapter 7.

2.1 Quaternionic Kähler manifolds

Definition 2.1.1 *A **quaternionic** (pseudo-)**Kähler manifold** (M, g, Q) of $\dim_{\mathbb{R}} M > 4$ is a (pseudo-)Riemannian manifold (M, g) of non-zero scalar curvature together with a parallel rank three subbundle $Q \subset \operatorname{End} TM$ that is locally spanned by three skew-symmetric almost complex structures J_1, J_2, J_3 that fulfill*

$$J_1 J_2 = J_3.$$

The four-dimensional case is special. Here, we add an additional property to the definition. This property automatically holds for all higher-dimensional quaternionic (pseudo-)Kähler manifolds (see e.g. [AM]).

Definition 2.1.2 *A four-dimensional (pseudo-)Riemannian manifold* (M, g, Q) *with a rank three subbundle* $Q \subset \operatorname{End} TM$ *is called* ***quaternionic*** *(pseudo-)* ***Kähler*** *if it fulfills the assumptions of Definition 2.1.1 and in addition,* Q *annihilates the Riemann tensor* R *of* g*, i.e.*

$$-JR(X,Y)Z + R(X,Y)JZ + R(JX,Y)Z + R(X,JY)Z = 0 \quad (X,Y,Z \in \mathfrak{X}(M))$$

for any local section J *in* Q*.*

Definition 2.1.3 *Let* M *be a smooth manifold. A collection* (J_1, J_2, J_3) *of three almost complex structures such that* $J_1 J_2 = J_3$ *is called an* ***almost hyper-complex structure****.*

Remark 2.1.4 For any quaternionic (pseudo-)Kähler manifold (M, g, Q), we endow Q with the natural scalar product

$$\langle A, B \rangle := -\frac{1}{\dim_{\mathbb{R}} M} \operatorname{tr} AB, \quad A, B \in Q.$$

Note that a local almost hyper-complex structure (J_1, J_2, J_3) spanning Q is a local orthonormal frame in Q with respect to $\langle \cdot, \cdot \rangle$. We call a local orthonormal frame (J_1, J_2, J_3) in Q **oriented** if $J_1 J_2 = J_3$.

Remark 2.1.5 The property that Q is parallel with respect to the Levi-Civita connection, i.e. $\nabla_X(\Gamma(Q)) \subset \Gamma(Q)$ for all $X \in \mathfrak{X}(M)$, is equivalent to the equation

$$\nabla_{\cdot} J_\alpha = 2(\bar\theta_\beta(\cdot) J_\gamma - \bar\theta_\gamma(\cdot) J_\beta) \tag{2.1}$$

for every cyclic permutation (α, β, γ) of $(1,2,3)$, where (J_1, J_2, J_3) is a local oriented orthonormal frame in Q and $\bar\theta_\alpha$, $\alpha = 1,2,3$, are local one-forms. We choose the following basis for $\mathfrak{so}(3) \cong \mathfrak{sp}(1)$:

$$e_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

We call

$$\bar\theta := \sum_{\alpha=1}^{3} \bar\theta_\alpha e_\alpha \in \Omega^1(U, \mathfrak{so}(3))$$

the **local $Sp(1)$-connection one-form** with respect to the frame (J_1, J_2, J_3) over $U \subset M$. Note that with the formula

$$d\omega(X_0, \dots, X_k) = \sum_{i=0}^{k} (-1)^i (\nabla_{X_i}\omega)(X_0, \dots, \widehat{X_i}, \dots, X_k) \tag{2.3}$$

$(X_0, \dots, X_k \in \mathfrak{X}(M))$ for any torsion-free connection ∇ and any k-form $\omega \in \Omega^k(M)$, we obtain

$$d\omega_\alpha = 2(\bar\theta_\beta \wedge \omega_\gamma - \bar\theta_\gamma \wedge \omega_\beta) \tag{2.4}$$

from Eq. (2.1), where

$$\omega_\alpha := g(J_\alpha \cdot\,, \cdot) \in \Omega^2(U) \qquad (\alpha = 1, 2, 3)$$

are the **local fundamental two-forms** with respect to (J_1, J_2, J_3). The last equation implies that

$$\Omega_4^{(U)} := \sum_{\alpha=1}^{3} \omega_\alpha \wedge \omega_\alpha \in \Omega^4(U) \tag{2.5}$$

is closed. The four-form $\Omega_4^{(U)}$ is independent of the choice of orthonormal frame (J_1, J_2, J_3) in $Q\big|_U$, i.e. Eq. (2.5) defines a global four-form $\Omega_4 \in \Omega^4(M)$, which is called the **fundamental four-form** of (M, g, Q).

In dimension bigger than four, we now give a characterization of quaternionic (pseudo-)Kähler manifolds which uses the exterior derivative of the fundamental two-forms instead of the Levi-Civita connection.

Definition 2.1.6 *A (pseudo-)Riemannian manifold (M, g) of $\dim_{\mathbb{R}} M > 4$ together with a rank three subbundle $Q \subset \operatorname{End} TM$ fulfilling Definition 2.1.1, except for Q being parallel, is called an* ***almost quaternionic*** *(pseudo-)**Hermitian manifold**.*

Theorem 2.1.7 [Sw1] *Let (M, g, Q) be an almost quaternionic (pseudo-)Hermitian manifold, $\dim_{\mathbb{R}} M > 8$, such that the fundamental four-form is closed. Then (M, g, Q) is quaternionic (pseudo-)Kähler.*

Theorem 2.1.8 [Sw1] *Let (M, g, Q) be an almost quaternionic (pseudo-)Hermitian manifold, $\dim_{\mathbb{R}} M = 8$, such that the fundamental four-form is closed and* *the algebraic ideal generated by* $Q^\flat := \{g(J\cdot\,,\cdot) \mid J \in Q\} \subset \Lambda^2(T^*M)$ *is a differential ideal. Then* (M, g, Q) *is quaternionic (pseudo-)Kähler.*

Eq. (2.4) holds for all quaternionic (pseudo-)Kähler manifolds. It implies that the fundamental four-form is closed. Together with this fact, the two above theorems immediately give the following corollary:

Corollary 2.1.9 *Let* (M, g, Q) *be an almost quaternionic (pseudo-)Hermitian manifold,* $\dim_{\mathbb{R}} M > 4$*, such that for any point* $x \in M$*, there exists a neighborhood* $U \subset M$ *of* x *and an almost hyper-complex structure* (J_1, J_2, J_3) *on* U *spanning* $Q\big|_U$ *such that Eq. (2.4) is fulfilled for some one-forms* $\bar\theta_\alpha \in \Omega^1(U)$*,* $\alpha = 1, 2, 3$*. Then* (M, g, Q) *is quaternionic (pseudo-)Kähler with local* $Sp(1)$*-connection one-form* $\bar\theta = \sum \bar\theta_\alpha e_\alpha$ *with respect to* (J_1, J_2, J_3)*.*

The quaternionic Kähler property in four dimensions can often be deduced from the higher-dimensional case using the following result:

Definition 2.1.10 *Let* (M, g, Q) *be a quaternionic (pseudo-)Kähler manifold. A submanifold* $N \subset M$ *is called* ***quaternionic*** *if* Q *preserves* $TN \subset TM$*.*

Proposition 2.1.11 [M] *Let* (M, g, Q) *be a quaternionic (pseudo-)Kähler manifold and* $N \subset M$ *a quaternionic submanifold. Then* $(N, g|_N, Q|_N)$ *is quaternionic (pseudo-)Kähler.*

Remark 2.1.12 Note that all quaternionic (pseudo-)Kähler manifolds are Einstein (see e.g. [Besse]). Hence, their scalar curvature *scal* is constant. The real number

$$\nu := \frac{scal}{4n(n+2)} \qquad (\dim_{\mathbb{R}} M = 4n) \tag{2.6}$$

is called the **reduced scalar curvature**.

For later use, we cite the following well-known result by Alekseevsky:

Proposition 2.1.13 [A1] *Let* (M, g, Q) *be a quaternionic (pseudo-)Kähler manifold and let* (J_1, J_2, J_3) *be a locally defined almost hyper-complex structure spanning* Q*. Then the local fundamental two-forms are given by*

$$\frac{\nu}{2}\omega_\alpha = d\bar\theta_\alpha - 2\bar\theta_\beta \wedge \bar\theta_\gamma, \tag{2.7}$$

for every cyclic permutation (α, β, γ) *of* $(1, 2, 3)$*, where* $\bar{\theta} = \sum_{\alpha=1}^{3} \bar{\theta}_\alpha e_\alpha$ *is the local* $Sp(1)$*-connection one-form with respect to* (J_1, J_2, J_3)*.*

Now, we come to the model examples of quaternionic pseudo-Kähler manifolds with positive, respectively negative scalar curvature. Note that in the pseudo-Riemannian category, the sign of the scalar curvature loses its relevance, since if (M, g, Q) is a quaternionic pseudo-Kähler manifold of signature $(4k, 4\ell)$ and positive scalar curvature, then $(M, -g, Q)$ is a quaternionic pseudo-Kähler manifold of signature $(4\ell, 4k)$ and negative scalar curvature. Nevertheless, we make the distinction between positive and negative scalar curvature here since in later chapters we focus on positive definite quaternionic Kähler manifolds.

Example 2.1.14 For $k, \ell \in \mathbb{N}_0$, let

$$\mathbb{H}^{k+1,\,\ell}_{>0} := \{q = z + jw \in \mathbb{H}^{k+1,\,\ell} \mid \langle q, q\rangle_{(k+1,\,\ell)} > 0\}$$

be endowed with the standard flat pseudo-Riemannian metric

$$\hat{g}^{(k+1,\,\ell)}_{\text{flat}} := \sum_{I,\,J=0}^{n} I^{k+1,\,\ell}_{IJ} (dz^I d\bar{z}^J + dw_I d\bar{w}_J)$$

of signature $(4k + 4, 4\ell)$, where $I^{k+1,\,\ell} := \operatorname{diag}(\overbrace{+1, \ldots, +1}^{(k+1)-\text{times}}, \overbrace{-1, \ldots, -1}^{\ell-\text{times}})$. The invertible quaternions $\mathbb{H}^* = \mathbb{R}^{>0} \cdot Sp(1)$ act on $\mathbb{H}^{k+1,\,\ell}_{>0}$ via right-multiplication. Let $M^{(k,\,\ell)}_+ := \mathbb{H}^{k+1,\,\ell}_{>0}/\mathbb{H}^*$ be endowed with the pseudo-Riemannian metric $g^{(k,\,\ell)}_+$ such that the projection from the unit sphere in $\mathbb{H}^{k+1,\,\ell}_{>0}$ to $M^{(k,\,\ell)}_+$ is a pseudo-Riemannian submersion. Then $g^{(k,\,\ell)}_+$ is a pseudo-quaternionic Kähler metric of reduced scalar curvature $\nu = 4$. We call $(M^{(k,\,\ell)}_+, g^{(k,\,\ell)}_+)$ **quaternionic projective space** of signature (k, ℓ) and denote it by $\mathbb{H}P^{k,\,\ell}$. It is a pseudo-Riemannian symmetric space and has the following realization as a homogeneous space:

$$\mathbb{H}P^{k,\,\ell} \approx \frac{Sp(k + 1, \ell)}{Sp(1) \times Sp(k, \ell)}. \tag{2.8}$$

For any $J \in \{1, \ldots, k + 1\}$ we have a chart $U_J := \{q \in \mathbb{H}^{k+1,\,\ell}_{>0} \mid q^J \neq 0\}$ with complex coordinates $(\phi^\mu_{(J)}, \psi^{(J)}_\mu)_{\mu=1,\ldots,\hat{J},\ldots,k+\ell+1}$ defined by

$$\phi^\mu_{(J)} + j\psi^{(J)}_\mu := u^\mu_{(J)} = q^\mu (q^J)^{-1}, \quad \mu \in \{1, \ldots, k + \ell + 1\}\backslash\{J\}. \tag{2.9}$$

The quaternionic structure Q on $\mathbb{H}P^{k,\,\ell}$ can be defined by local fundamental two-forms

$$\omega_\alpha^{(J)} = \frac{1}{2}(d\bar{\theta}_\alpha^{(J)} - 2\bar{\theta}_\beta^{(J)} \wedge \bar{\theta}_\gamma^{(J)}) \tag{2.10}$$

on U_J, where

$$\bar{\theta}_1^{(J)} = \frac{1}{\underbrace{1 + \langle u_{(J)}, u_{(J)} \rangle_{(k,\,\ell)}}_{=1+\sum_{\mu,\,\nu\neq J} I_{\mu\nu}^{k+1,\,\ell} u^\mu \bar{u}^\nu}} \sum_{\mu,\,\nu\neq J} I_{\mu\nu}^{k+1,\,\ell}\,\mathrm{Im}(\bar{\phi}_{(J)}^\mu d\phi_{(J)}^\nu + \bar{\psi}_\mu^{(J)} d\psi_\nu^{(J)}), \tag{2.11}$$

$$\bar{\theta}_+^{(J)} := \bar{\theta}_2^{(J)} + i\bar{\theta}_3^{(J)} = \frac{1}{1 + \langle u_{(J)}, u_{(J)} \rangle_{(k,\,\ell)}} \sum_{\mu,\,\nu\neq J} I_{\mu\nu}^{k+1,\,\ell}(\phi_{(J)}^\mu d\psi_\nu^{(J)} - \psi_\mu^{(J)} d\phi_{(J)}^\nu).$$

Example 2.1.15 Similarly to the above example, we define a quaternionic pseudo-Kähler metric $g_-^{(k,\,\ell)}$ of reduced scalar curvature $\nu = -4$ on $M_-^{(k,\,\ell)} := \mathbb{H}_{<0}^{k,\,\ell+1}/\mathbb{H}^*$, where $\mathbb{H}_{<0}^{k,\,\ell+1} := \{q \in \mathbb{H}^{k,\,\ell+1} \mid \langle q, q \rangle_{(k,\,\ell+1)} < 0\}$. We call $(M_-^{(k,\,\ell)}, g_-^{(k,\,\ell)})$ **quaternionic hyperbolic space** of signature (k, ℓ) and denote it by $\mathbb{H}H^{k,\,\ell}$. It is a pseudo-Riemannian symmetric space and has the following realization as a homogeneous space:

$$\mathbb{H}H^{k,\,\ell} \approx \frac{Sp(k, \ell+1)}{Sp(k, \ell) \times Sp(1)}. \tag{2.12}$$

For any $J \in \{k+1, \ldots, k+\ell+1\}$, we have a chart $U_J := \{q \in \mathbb{H}_{<0}^{k,\,\ell+1} \mid q^J \neq 0\}$ with complex coordinates $(\phi_{(J)}^\mu, \psi_\mu^{(J)})_{\mu=1,\,\ldots,\,\hat{J},\,\ldots,\,k+\ell+1}$ defined as in (2.9). The quaternionic structure Q on $\mathbb{H}H^{k,\,\ell}$ can be defined by local fundamental two-forms

$$\omega_\alpha^{(J)} = -\frac{1}{2}(d\bar{\theta}_\alpha^{(J)} - 2\bar{\theta}_\beta^{(J)} \wedge \bar{\theta}_\gamma^{(J)}) \tag{2.13}$$

on U_J, where

$$\bar{\theta}_1^{(J)} = \frac{1}{\underbrace{-1 + \langle u_{(J)}, u_{(J)} \rangle_{(k,\,\ell)}}_{=-1+\sum_{\mu,\,\nu\neq J} I_{\mu\nu}^{k,\,\ell+1} u^\mu \bar{u}^\nu}} \sum_{\mu,\,\nu\neq J} I_{\mu\nu}^{k,\,\ell+1}\,\mathrm{Im}(\bar{\phi}_{(J)}^\mu d\phi_{(J)}^\nu + \bar{\psi}_\mu^{(J)} d\psi_\nu^{(J)}), \tag{2.14}$$

$$\bar{\theta}_+^{(J)} := \bar{\theta}_2^{(J)} + i\bar{\theta}_3^{(J)} = \frac{1}{-1 + \langle u_{(J)}, u_{(J)} \rangle_{(k,\,\ell)}} \sum_{\mu,\,\nu\neq J} I_{\mu\nu}^{k,\,\ell+1}(\phi_{(J)}^\mu d\psi_\nu^{(J)} - \psi_\mu^{(J)} d\phi_{(J)}^\nu).$$

Remark 2.1.16 For future reference, we note that (with a slight abuse of notation) the metric on $\mathbb{H}P^{k,\,\ell}$, respectively $\mathbb{H}H^{k,\,\ell}$ in the coordinates defined in

Examples 2.1.14 and 2.1.15 is given by

$$
\begin{aligned}
g_{\pm}^{(k,\ell)} = &\pm \frac{1}{\pm 1 + \langle u_{(J)}, u_{(J)} \rangle_{(k,\ell)}} \sum_{\mu,\nu \neq J} I_{\mu\nu} (d\phi^{\mu}_{(J)} d\bar{\phi}^{\nu}_{(J)} + d\psi^{(J)}_{\mu} d\bar{\psi}^{(J)}_{\nu}) \\
&\mp \frac{1}{(\pm 1 + \langle u_{(J)}, u_{(J)} \rangle_{(k,\ell)})^2} \Big(\Big| \sum_{\mu,\nu \neq J} (\bar{\phi}^{\mu}_{(J)} d\phi^{\nu}_{(J)} + \bar{\psi}^{(J)}_{\mu} d\psi^{(J)}_{\nu}) \Big|^2 \\
&\qquad + \Big| \sum_{\mu,\nu \neq J} (\phi^{\mu}_{(J)} d\psi^{(J)}_{\nu} - \psi^{(J)}_{\mu} d\phi^{\nu}_{(J)}) \Big|^2 \Big).
\end{aligned} \tag{2.15}
$$

The almost hyper-complex structure (J_1, J_2, J_3) on U_J defined by the fundamental two-forms given in the above examples fulfills

$$
J_1^* d\phi^{\mu}_{(J)} = i d\phi^{\mu}_{(J)}, \quad J_1^* d\psi^{(J)}_{\mu} = i d\psi^{(J)}_{\mu}, \quad J_2^* d\phi^{\mu}_{(J)} = -d\bar{\psi}^{(J)}_{\mu}. \tag{2.16}
$$

Note that J_1, J_2, J_3 are integrable complex structures on U_J.

2.2 The quaternionic Kähler quotient

For a proof of the following proposition, see for example [ACDV] or [BoGal, Prop. 12.4.1]:

Proposition 2.2.1 *Let (M, g, Q) be a quaternionic (pseudo-)Kähler manifold and let $X \in \mathfrak{X}(M)$ be a Killing vector field. Then X preserves Q and the fundamental four-form:*

$$
\mathcal{L}_X(\Gamma(Q)) \subset \Gamma(Q), \quad \mathcal{L}_X \Omega_4 = 0. \tag{2.17}
$$

Due to the above propostion, we can drop the assumption that the fundamental four-form is preserved in the next two theorems.

Theorem 2.2.2 [GL, Th. 2.4.] *Let (M, g, Q) be a quaternionic (pseudo-)Kähler manifold and let $X \in \mathfrak{X}(M)$ be a Killing vector field. Then there exists a unique section $\mu^X \in \Gamma(Q)$ such that*

$$
\nabla.\, \mu^X \big|_U = \sum_{\alpha=1}^{3} \omega_\alpha(X, \cdot) J_\alpha \tag{2.18}
$$

for each oriented orthonormal frame (J_1, J_2, J_3) in $Q\big|_U$ over an open subset $U \subset M$.

Definition 2.2.3 *Let (M, g, Q) be a quaternionic (pseudo-)Kähler manifold and $X \in \mathfrak{X}(M)$ a Killing vector field. The section $\mu^X \in \Gamma(Q)$ given by the above theorem is called the* ***quaternionic Kähler moment map*** *associated with X.*

Remark 2.2.4 Note that due to Eq. (2.1) and the fact that J_1, J_2, J_3 are linearly independent, Eq. (2.18) is equivalent to

$$\mu^X\big|_U =: \sum_{\mu=1}^{3} \mu_\alpha^X J_\alpha, \quad d\mu_\alpha^X + 2\mu_\beta^X \bar{\theta}_\gamma - 2\mu_\gamma^X \bar{\theta}_\beta = \iota_X \omega_\alpha \tag{2.19}$$

for every cyclic permutation (α, β, γ) of $(1, 2, 3)$.

Using Eqs. (2.7), (2.3) and (2.1), one can show that

$$\begin{aligned} \mathcal{L}_X \omega_\alpha &= \nu(\mu_\beta^X \omega_\gamma - \mu_\gamma^X \omega_\beta) + \nabla_X \omega_\alpha \\ &= (\nu\mu_\beta^X + 2\bar{\theta}_\beta(X))\omega_\gamma - (\nu\mu_\gamma^X + 2\bar{\theta}_\gamma(X))\omega_\beta. \end{aligned} \tag{2.20}$$

Using the scalar product $\langle A, B\rangle = -\frac{1}{\dim_{\mathbb{R}} M} \operatorname{tr} AB$ on Q, this gives the following explicit formula for the quaternionic Kähler moment map with respect to X:

$$\mu^X\big|_U = \sum_{\mu=1}^{3} \mu_\alpha^X J_\alpha, \quad \mu_\alpha^X = \frac{1}{\nu}\langle J_\beta, (\mathcal{L}_X - \nabla_X) J_\gamma\rangle. \tag{2.21}$$

The above theorem gives the existence and uniqueness of the following map:

Definition 2.2.5 *Let (M, g, Q) be a quaternionic (pseudo-)Kähler manifold and let G be a Lie group acting isometrically on (M, g). Then the (**quaternionic Kähler) moment map** μ for (M, g, Q, G) is the smooth map from M to $\mathfrak{g}^* \otimes Q$ defined by*

$$\mu^v := \langle \mu, v\rangle := \mu^{v^\sharp}, \quad v \in \mathfrak{g}, \tag{2.22}$$

where $\mu^{v^\sharp}$ is the quaternionic Kähler moment map associated with the fundamental vector field[1] $v^\sharp \in \mathfrak{X}(M)$ induced by v and $\mu^v = \langle \mu, v\rangle$ denotes the contraction of $v \in \mathfrak{g}$ with the $\mathfrak{g}^*$-factor of μ.

Theorem 2.2.6 [GL, Th. 3.1.] *Let (M, g, Q) be a quaternionic pseudo-Kähler manifold. Let G be a connected compact Lie group acting freely and isometrically*

[1]We define fundamental vector fields without an extra minus sign, i.e. $\cdot^\sharp : \mathfrak{g} \to \mathfrak{X}(M)$, $v \mapsto v^\sharp$ is a Lie algebra anti-homomorphism: $v^\sharp|_p := \frac{d}{dt}\big|_{t=0} \exp(tv) \cdot p, \quad p \in M.$

on (M, g) and such that the restriction of g to the distribution tangent to the G-orbits is non-degenerate. Let μ be the corresponding quaternionic Kähler moment map.

Then $\bar{M} := \mu^{-1}(\{0\})/G$ inherits a quaternionic pseudo-Kähler structure $(\bar{g}, \bar{Q})$ from (M, g, Q).

Definition 2.2.7 *The quaternionic (pseudo-)Kähler manifold $(\bar{M}, \bar{g}, \bar{Q})$ obtained from the above theorem is called the* ***quaternionic Kähler quotient*** *of (M, g, Q) with respect to G and we will denote it by*

$$M/\!\!/\!\!/G = (\bar{M}, \bar{g}, \bar{Q}).$$

Remark 2.2.8 In the situation of the above theorem, let $p : M_0 := \mu^{-1}(\{0\}) \to \bar{M}$ denote the projection. An orthonormal frame (J_1, J_2, J_3) in $Q\big|_U$ over some G-invariant open subset $U \subset M$ induces an orthonormal frame $(\bar{J}_1, \bar{J}_2, \bar{J}_3)$ in $\bar{Q}\big|_{\bar{U}}$ over $\bar{U} := p(U \cap \mu^{-1}(\{0\}))$. The corresponding local fundamental two-forms are related by

$$p^*\bar{\omega}_\alpha = \omega_\alpha\big|_{p^{-1}(\bar{U})}, \quad \alpha = 1, 2, 3.$$

Remark 2.2.9 In the above theorem, one can replace the assumption that G is compact and acts freely on M by the assumption that 0 is a regular value of μ and that G acts properly[2] and freely on M_0 (see, e.g., [Lee]), or just by the assumption that M_0/G is a smooth manifold of dimension

$$\dim_{\mathbb{R}}\bar{M} = \dim_{\mathbb{R}}M - 4\dim G$$

such that the projection map p is a smooth submersion.

Example 2.2.10 For $k, \ell \in \mathbb{N}_0$, we consider

$$M^+ := \mathbb{H}P^{k+1,\ell} = \mathbb{H}^{k+2,\ell}_{>0}/\mathbb{H}^*, \quad M^- := \mathbb{H}H^{k,\ell+1} = \mathbb{H}^{k,\ell+2}_{<0}/\mathbb{H}^*$$

respectively (see Examples 2.1.14 and 2.1.15), endowed with the S^1-action induced by diagonal left-multiplication of $e^{it} \in S^1$ on quaternionic vector space. Recall that we defined quaternionic projective and quaternionic hyperbolic space

[2]The G-action on M_0 is called proper if pre-images of compact subsets of $M_0 \times M_0$ under the map $G \times M_0 \to M_0 \times M_0$, $(g, q) \mapsto (g \cdot q, q)$ are compact.

via right-multiplication of $\mathbb{H}^*$. The zero level set of the corresponding quaternionic Kähler moment map is the following smooth codimension 3 submanifold:

$$M_0^{\pm} = \{[q = z + jw]_{\mathbb{H}^*} \in M^{\pm} \mid \langle z, z\rangle = \langle w, w\rangle, \langle z, \bar{w}\rangle = 0\} \subset M^{\pm}. \tag{2.23}$$

Note that the S^1-action on $M_0^{\pm}$ is free. Hence, the quaternionic Kähler quotient $\mathbb{H}P^{k+1,\,\ell} /\!/\!/ S^1$, respectively $\mathbb{H}H^{k,\,\ell+1} /\!/\!/ S^1$, induces a quaternionic Kähler metric $\bar{g}_{\pm}$ of signature $(4k, 4\ell)$ on

$$\bar{M}^{\pm} := M_0^{\pm}/S^1.$$

Let $A \in G^+ := SU(k+2, \ell)$, respectively $A \in G^- := SU(k, \ell+2)$ act on $q = z + jw \in \mathbb{H}^{k+2,\,\ell}_{>0}$ (respectively $\mathbb{H}^{k,\,\ell+2}_{<0}$) by $q \mapsto Aq = Az + j\bar{A}w$, where we consider $z, w \in \mathbb{C}^{k+\ell+2}$ as column vectors. The induced $G^{\pm}$-action on $M^{\pm}$ preserves the level set $M_0^{\pm}$, so we have an induced action on $\bar{M}^{\pm}$. This action is transitive and preserves $\bar{g}_{\pm}$. The pseudo-Riemannian manifold $(\bar{M}^{\pm}, \bar{g}_{\pm})$ is in fact symmetric and we denote it by $X(k, \ell)$, respectively $\tilde{X}(k, \ell)$. Calculating the stabilizer of a point in $\bar{M}^{\pm}$ under the $G^{\pm}$-action gives the following realization as a homogeneous space:

$$X(k,\ell) \approx \frac{SU(k+2,\ell)}{S(U(2)\times U(k,\ell))}, \quad \tilde{X}(k,\ell) \approx \frac{SU(k,\ell+2)}{S(U(k,\ell)\times U(2))}. \tag{2.24}$$

Chapter 3

Hyper-Kähler geometry

In this chapter, we discuss pseudo-hyper-Kähler manifolds and, in particular, the relation between conical pseudo-hyper-Kähler manifolds and quaternionic pseudo-Kähler manifolds. To motivate the *HK/QK correspondence* described in Chapter 4, we also discuss infinitesimal automorphisms of conical pseudo-hyper-Kähler manifolds, as well as the hyper-Kähler quotient construction. The property *conical* is defined by the existence of a vector field ξ, called the *Euler vector field*, such that the Levi-Civita connection ∇ fulfills $\nabla.\xi = \mathrm{Id}$. Conical pseudo-hyper-Kähler manifolds are locally homothetic to the Swann bundle over a quaternionic pseudo-Kähler manfold. We review the Swann bundle construction [Sw1] in the last section. All results in this chapter about conical pseudo-hyper-Kähler manifolds are essentially known from [Sw1], which uses a slightly different local characterization of the Swann bundles over quaternionic pseudo-Kähler manifolds.

All results presented in this chapter will be needed in Chapter 4 for the motivation of the HK/QK correspondence, as well as for the proofs of its properties. The examples presented in this chapter will also be reused for the discussion of examples of the HK/QK correspondence in Chapter 4.

In Section 3.1, we introduce the notion of (pseudo-)hyper-Kähler manifold and discuss the standard hyper-Kähler structure on quaternionic vector spaces. In Section 3.2, we introduce conical pseudo-hyper-Kähler manifolds, show that they admit a global hyper-Kähler potential and that they induce a quaternionic Kähler structure on an appropriately chosen codimension four submanifold. We discuss the example of an open subset of flat quaternionic vector space with quaternionic Lorentzian and positive signature, respectively, endowed with the

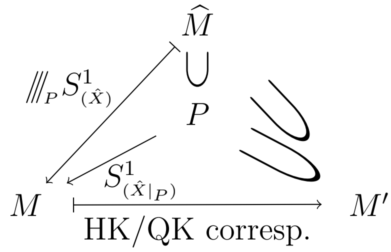


Figure 3.1: Relation between the HK/QK correspondence (Chapter 4), the hyper-Kähler quotient construction (Section 3.4) and the construction from Section 3.2.

Euler vector field induced by uniform scaling by a positive factor. By choosing a codimension four submanifold in this example, we recover quaternionic hyperbolic space and a chart in quaternionic projective space, respectively.

In Section 3.3, we consider tri-holomorphic Killing vector fields $\hat{X}$ on a conical (pseudo-)hyper-Kähler manifold $\widehat{M}$ that commute with the Euler vector field ξ. We give an explicit expression for the unique ξ-homogeneous hyper-Kähler moment map associated with $\hat{X}$. We consider a level set P with respect to a non-zero level of this homogeneous hyper-Kähler moment map. Using the results from Section 3.2, we show how geometric data on P inherited from $\widehat{M}$ induces a quaternionic (pseudo-)Kähler structure on an appropriately chosen codimension one submanifold $M' \subset P$. When $\hat{X}$ induces a free S^1-action on P, the geometric data defined on P in this section, as well as the quaternionic Kähler structure on M' are exactly reconstructed when applying the HK/QK correspondence to the hyper-Kähler quotient $M = P/S^1$, see Chapter 4 and Figure 3.1. We continue the examples discussed in Section 3.2 and consider the tri-holomorphic S^1-action defined by diagonal left-multiplication in quaternionic vector space. Choosing a codimension one submanifold in the level set P, we obtain quaternionic hyperbolic space and an open subset of quaternionic projective space in a realization different from the standard one which we obtained in Section 3.2.

In Section 3.4, we review the hyper-Kähler quotient construction from [HKLR]. As a simple example, we discuss the S^1-action on quaternionic vector space defined by left-multiplication on just one of the quaternionic coordinates. Then the hyper-Kähler quotient is again a quaternionic vector space of quaternionic dimension reduced by one, endowed with the standard flat metric. As a second example, we discuss the diagonal S^1-action on quaternionic vector space and show that when we choose a non-zero level for the homogeneous hyper-Kähler moment map, the hyper-Kähler quotient is $T^*(\mathbb{C}P^n)$ endowed with the Calabi metric [Ca, LR, Hi1]. If we start with $\{\langle q,q\rangle_{n,1} < 0\} \subset \mathbb{H}^{n,1}$ instead, we

obtain a tubular neighborhood of the zero section in $T^*(\mathbb{C}H^n)$. The hyper-Kähler structure in both cases agrees with the one constructed in [BiGau].
We show that, under appropriate assumptions, the hyper-Kähler quotient with respect to two commuting Lie group actions can be performed *in stages* and that the outcome does not depend on the order in which one performs the two respective hyper-Kähler quotients. This is needed later to show the compatibility of the HK/QK correspondence with the hyper-Kähler and quaternionic Kähler quotient constructions.

In Section 3.5, we consider isometric and tri-holomorphic Lie group actions on conical (pseudo-)hyper-Kähler manifolds that preserve the Euler vector field. We prove that in this situation the hyper-Kähler quotient with level zero for the homogeneous hyper-Kähler moment map is again conical and that the relation between conical pseudo-hyper-Kähler manifolds and quaternionic pseudo-Kähler manifolds given in Section 3.2 is compatible with the quaternionic Kähler and hyper-Kähler quotient constructions (with level zero).

In Section 3.6, we recall the Swann bundle construction [Sw1] in a formalism that does not make use of reduced frame bundles. For any quaternionic pseudo-Kähler manifold, the Swann bundle construction defines a conical pseudo-hyper-Kähler structure on the metric cone over the $SO(3)$-bundle of local oriented orthonormal frames in the quaternionic structure. In the first subsection, we show that for any Killing vector field on a quaternionic pseudo-Kähler manifold, there exists a unique tri-holomorphic lift to the Swann bundle that is Killing and commutes with the Euler vector field. We describe the norm of the lifted vector field and the relation between the homogeneous hyper-Kähler moment map associated with it and the quaternionic Kähler moment map associated with the initial vector field. The lifted vector field is non-vanishing if and only if the initial vector field and the quaternionic Kähler moment map do not vanish simultaneously. In the second subsection, we discuss the canonical lift of isometric group actions from a quaternionic Kähler manifold to the Swann bundle. Infinitesimally, the canonically lifted group action is described by the unique lifts of Killing vector fields to the Swann bundle described before.

3.1 Hyper-Kähler manifolds

Definition 3.1.1 *A (pseudo-)**Kähler manifold** (M, g, J) is a (pseudo-)Riemannian manifold (M, g) together with an almost complex structure J such that*

1. J is integrable,

2. $g(J\cdot\,, J\cdot) = g(\cdot\,,\cdot)$,

3. $d\omega = 0$, where $\omega := g(J\cdot\,,\cdot) \in \Omega^2(M)$.

*ω is called the **Kähler form**.*

Remark 3.1.2 The complex structure of a Kähler manifold is covariantly constant with respect to the Levi-Civita connection, i.e. $\nabla.J = 0$.

Definition 3.1.3 *A (pseudo-)**hyper-Kähler manifold** (M, g, J_1, J_2, J_3) is a (pseudo-)Riemannian manifold (M, g) together with three almost complex structures J_1, J_2, J_3 such that*

1. $J_1 J_2 = J_3$

2. (M, g, J_α) is Kähler for $\alpha = 1, 2, 3$.

Remark 3.1.4 The hyper-Kähler structure can be recovered from the three Kähler forms $\omega_\alpha : TM \to T^*M$, $v \mapsto g(J_\alpha v, \cdot)$:

$$g = \omega_1 \circ \omega_2^{-1} \circ \omega_3, \quad J_\alpha = g^{-1} \circ \omega_\alpha \quad (\alpha = 1, 2, 3). \tag{3.1}$$

Proposition 3.1.5 (Hitchin-Lemma) [Hi2, Lemma 6.8]
Let (M, g, J_1, J_2, J_3) be a (pseudo-)Riemannian manifold together with an almost hyper-complex structure such that $g(J_\alpha\cdot\,, J_\alpha\cdot) = g(\cdot\,,\cdot)$ and $d\omega_\alpha = 0$ for $\alpha = 1, 2, 3$. Then J_1, J_2, J_3 are integrable, i.e. (M, g, J_1, J_2, J_3) is a (pseudo-) hyper-Kähler manifold.

Remark 3.1.6 Let (M, g, J_1, J_2, J_3) be a (pseudo-)hyper-Kähler manifold. Then

$$\omega_+ := \omega_2 + i\omega_3 \in \Omega^{2,0}_{J_1}(M) \tag{3.2}$$

defines a holomorphic symplectic form on (M, J_1).

Example 3.1.7 We endow $M = \mathbb{H}^n$ with complex coordinates $(z^1, \ldots, z^n, w_1, \ldots, w_n)$ given by

$$\mathbb{H}^n \to \mathbb{C}^{2n}, \quad q = z + jw \mapsto (z, w), \tag{3.3}$$

and define a hyper-Kähler structure (g, J_1, J_2, J_3) on $\mathbb{H}^n$ by the following metric and holomorphic symplectic form:

$$g = g_{\text{flat}}^{(n,0)} = \sum_{\mu=1}^{n} (dz^\mu d\bar{z}^\mu + dw_\mu d\bar{w}_\mu), \tag{3.4}$$

$$\omega_+ = \omega_2 + i\omega_3 = \sum_{\mu=1}^{n} dz^\mu \wedge dw_\mu. \tag{3.5}$$

For future use, we describe this hyper-Kähler structure in more detail in terms of real and complex coordinates: The coordinates $(z^\mu, w_\mu)_{\mu=1,\ldots,n}$ are J_1-holomorphic and

$$J_2^* dz^\mu = -d\bar{w}_\mu, \quad \mu = 1, \ldots, n. \tag{3.6}$$

Equivalently, $J_1 \frac{\partial}{\partial z^\mu} = i \frac{\partial}{\partial z^\mu}$, $J_1 \frac{\partial}{\partial w_\mu} = i \frac{\partial}{\partial w_\mu}$, $J_2 \frac{\partial}{\partial z^\mu} = \frac{\partial}{\partial \bar{w}_\mu}$, $J_3 \frac{\partial}{\partial z^\mu} = -i \frac{\partial}{\partial \bar{w}_\mu}$. The first Kähler form $\omega_1 = g(J_1 \cdot\, , \cdot)$ is given by

$$\omega_1 = \frac{i}{2} \sum_{\mu=1}^{n} (dz^\mu \wedge d\bar{z}^\mu + dw_\mu \wedge d\bar{w}_\mu). \tag{3.7}$$

With real coordinates $(x^\mu, y^\mu, u_\mu, v_\mu)_{\mu=1,\ldots,n}$ defined by

$$\mathbb{H}^n \to \mathbb{R}^{4n}, \quad q = x + iy + ju + kv \mapsto (x, y, u, v), \tag{3.8}$$

g is the standard metric on $\mathbb{R}^{4n}$:

$$g = \sum_{\mu=1}^{n} ((dx^\mu)^2 + (dy^\mu)^2 + (du_\mu)^2 + (dv_\mu)^2). \tag{3.9}$$

The real coordinates (x, y, u, v) define an isomorphism

$$\begin{aligned} \kappa_q :\ & T_q\mathbb{H}^n \to \mathbb{H}^n, \qquad (3.10) \\ & \sum_{\mu=1}^{n} \left(a^\mu \frac{\partial}{\partial x^\mu} + b^\mu \frac{\partial}{\partial y^\mu} + c_\mu \frac{\partial}{\partial u_\mu} + d_\mu \frac{\partial}{\partial v_\mu}\right)\Big|_q \mapsto (a^\mu + ib^\mu + jc_\mu + kd_\mu)_{\mu=1,\ldots,n} \end{aligned}$$

of real vector spaces between the tangent space at a point $q \in \mathbb{H}^n$ and $\mathbb{H}^n$. Using this identification, the hypercomplex structure (J_1, J_2, J_3) is given by right-multiplication with $(i, j, -k)$:

$$J_\alpha v = \kappa_q^{-1}(\kappa_q(v) \cdot i_\alpha),\ v \in T_q\mathbb{H}^n,\ (i_\alpha) = (i, j, -k),\ \alpha = 1, 2, 3. \tag{3.11}$$

3.2 Conical hyper-Kähler manifolds

Definition 3.2.1 *A* ***conical*** *(pseudo-)****hyper-Kähler manifold*** *$(M, g, J_1, J_2, J_3, \xi)$ is a (pseudo-)hyper-Kähler manifold together with a time-like or space-like vector field $\xi \in \mathfrak{X}(M)$ such that $\nabla.\xi = \mathrm{Id}_{TM}$, where ∇ is the Levi-Civita connection. ξ is called the* ***Euler vector field****.*

Let $(M, g, J_1, J_2, J_3, \xi)$ be a conical (pseudo-)hyper-Kähler manifold. We define

$$\begin{aligned}
\sigma &:= \operatorname{sgn} g(\xi,\xi) \in C^\infty(M),\\
r^2 &:= |g(\xi,\xi)| \in C^\infty(M),\\
\theta_\alpha &:= \frac{\sigma}{r^2} g(J_\alpha\xi,\cdot) \in \Omega^1(M),\\
\hat{\theta}_\alpha &:= \frac{r^2}{2}\theta_\alpha = \frac{\sigma}{2} g(J_\alpha\xi,\cdot) \in \Omega^1(M) \qquad\qquad (\alpha = 1,2,3).
\end{aligned} \tag{3.12}$$

Proposition 3.2.2 *A global Kähler potential for all three Kähler forms is given by*

$$\hat{K} := \sigma r^2 = g(\xi,\xi). \tag{3.13}$$

More precisely,

$$\omega_\alpha = \sigma d\hat{\theta}_\alpha = \frac{1}{4} dd^c_\alpha \hat{K}, \tag{3.14}$$

where $d^c_\alpha = i(\bar{\partial}_\alpha - \partial_\alpha)$ is the d^c-operator associated with J_α for $\alpha = 1,2,3$.

***Proof*:** For $X, Y \in \mathfrak{X}(M)$, we have

$$\begin{aligned}
d\hat{\theta}_\alpha(X,Y) &= X(\hat{\theta}_\alpha(Y)) - Y(\hat{\theta}_\alpha(X)) - \hat{\theta}_\alpha([X,Y])\\
&\overset{\mathrm{Tor}(\nabla)=0}{=} X(\hat{\theta}_\alpha(Y)) - Y(\hat{\theta}_\alpha(X)) - \frac{\sigma}{2} g(J_\alpha\xi, \nabla_X Y) + \frac{\sigma}{2} g(J_\alpha\xi, \nabla_Y X)\\
&\overset{\nabla g=0}{=} X(\hat{\theta}_\alpha(Y)) - Y(\hat{\theta}_\alpha(X)) - \frac{\sigma}{2} X(g(J_\alpha\xi, Y) + \frac{\sigma}{2} g(\nabla_X(J_\alpha\xi), Y)\\
&\qquad + \frac{\sigma}{2} Y(g(J_\alpha\xi, X) - \frac{\sigma}{2} g(\nabla_Y(J_\alpha\xi), X)\\
&= + \frac{\sigma}{2} g(\nabla_X(J_\alpha\xi), Y) - \frac{\sigma}{2} g(\nabla_Y(J_\alpha\xi), X)\\
&\underset{\nabla\xi=\mathrm{Id}}{\overset{\nabla J_\alpha=0}{=}} \sigma\,\omega_\alpha(X,Y).
\end{aligned} \tag{3.15}$$

Using

$$r dr = \frac{1}{2} d(r^2) = \frac{\sigma}{2} d(g(\xi,\xi)) = \frac{\sigma}{2}\nabla.(g(\xi,\xi)) \overset{\nabla g=0}{=} \sigma g(\xi, \nabla.\xi) = \sigma g(\xi,\cdot), \tag{3.16}$$

we can show that

$$\frac{\sigma}{4} d^c_\alpha \hat{K} = -\frac{\sigma}{4} J^*_\alpha (d\hat{K}) = -\frac{r}{2} J^*_\alpha dr = -\frac{\sigma}{2} g(\xi, J_\alpha \cdot) = \hat{\theta}_\alpha. \tag{3.17}$$

□

The following lemma shows that $\xi, J_1\xi, J_2\xi, J_3\xi$ induce a local $(CO(3) = \mathbb{R}^{>0} \times SO(3))$-action on M:

Lemma 3.2.3

$$[\xi, J_\alpha \xi] = 0, \quad [J_\alpha \xi, J_\beta \xi] = -2J_\gamma \xi. \tag{3.18}$$

***Proof*:** This follows immediately from ∇ being torsion-free, from $\nabla . \xi = \mathrm{Id}_{TM}$, from $\nabla . J_\alpha = 0$ and from $J_\alpha J_\beta = -J_\beta J_\alpha = J_\gamma$. □

We split the metric and Kähler forms into a vertical part corresponding to the distribution tangent to the local $CO(3)$-action and a horizontal part corresponding to the orthogonal distribution:

Lemma 3.2.4 *The hyper-Kähler metric can be written as*

$$g = \sigma dr^2 + \sigma r^2 (\sum_{\alpha=1}^{3} (\theta_\alpha)^2 + \sigma \breve{g}), \tag{3.19}$$

where $\breve{g} \in \Gamma(\mathrm{Sym}^2 T^*M)$ is a tensor field which has four-dimensional kernel

$$\mathcal{D}^v := \mathrm{span}_\mathbb{R} \{\xi, J_1\xi, J_2\xi, J_3\xi\} \subset TM. \tag{3.20}$$

The Kähler forms are given by

$$\omega_\alpha = \sigma\, r dr \wedge \theta_\alpha + r^2 (\sigma\, \theta_\beta \wedge \theta_\gamma + \breve{\omega}_\alpha) \tag{3.21}$$

for every cyclic permutation (α, β, γ) of $(1, 2, 3)$, where

$$\breve{\omega}_\alpha := \breve{g}(J_\alpha \cdot\,, \cdot) \in \Omega^2(M). \tag{3.22}$$

Furthermore,

$$2\sigma \breve{\omega}_\alpha = d\theta_\alpha - 2\theta_\beta \wedge \theta_\gamma. \tag{3.23}$$

***Proof*:** We write the metric as

$$g = \frac{(g(\xi,\cdot))^2}{g(\xi,\xi)} + \sum_{\alpha=1}^{3} \frac{(g(J_\alpha\xi,\cdot))^2}{g(\xi,\xi)} + g(\xi,\xi)\,\breve{g} = \sigma dr^2 + \sigma r^2 (\sum_{\alpha=1}^{3} (\theta_\alpha)^2 + \sigma\breve{g}), \quad (3.24)$$

where the last equality follows from Eq. (3.16) and from the definitions in Eq. (3.12). Since $g(J_\alpha\xi, J_\alpha\xi) = g(\xi,\xi) = \sigma r^2$ and $\xi, J_1\xi, J_2\xi, J_3\xi$ are pairwise orthogonal, the symmetric tensor field $\breve{g}$ on M defined by Eq. (3.24) has $\ker\breve{g} = \mathcal{D}^v$. Eq. (3.21) follows from $J_\alpha^*\theta_\alpha = \frac{\sigma}{r^2} g(\xi,\cdot) = \frac{1}{r}dr$ and $J_\alpha^*\theta_\beta = -\theta_\gamma$ together with Eq. (3.19), while Eq. (3.23) is obtained as follows:

$$d\theta_\alpha = d\big(\frac{2}{r^2}\hat{\theta}_\alpha\big) \overset{(3.14)}{=} -\frac{2}{r} dr \wedge \theta_\alpha + \frac{2\sigma}{r^2}\omega_\alpha \overset{(3.21)}{=} 2\theta_\beta \wedge \theta_\gamma + 2\sigma\breve{\omega}_\alpha. \quad (3.25)$$

□

While the horizontal Kähler forms $\breve{\omega}_\alpha$ get rotated by the $SO(3)$-part of the local $CO(3)$-action, the horizontal metric is $CO(3)$-invariant:

Proposition 3.2.5 *The tensor field*

$$\breve{g} = \frac{1}{r^2} g - \frac{\sigma}{r^2} dr^2 - \sigma \sum_{\alpha=1}^{3} (\theta_\alpha)^2 \quad (3.26)$$

is invariant under ξ and $J_\alpha\xi$, $\alpha = 1,2,3$.

***Proof*:** Using $J_0 := \mathrm{Id}_{TM}$, we have

$$\begin{aligned}\mathcal{L}_{J_a\xi} g(X,Y) &= (\nabla_{J_a\xi} g)(X,Y) + g(\nabla_X(J_a\xi), Y) + g(X, \nabla_Y(J_a\xi)) \\ &\underset{\nabla g=0}{\overset{\nabla(J_a\xi)=J_a}{=}} g(J_a X, Y) + g(X, J_a Y)\end{aligned} \quad (3.27)$$

for $X, Y \in \mathfrak{X}(M)$, $a = 0,1,2,3$. This shows

$$\mathcal{L}_\xi g = 2g, \quad \mathcal{L}_{J_\alpha\xi} g = 0 \qquad (\alpha = 1,2,3). \quad (3.28)$$

The equations $\mathcal{L}_{J_\alpha\xi} r = dr(J_\alpha\xi) \overset{(3.16)}{=} 0$, $\mathcal{L}_{J_\alpha\xi} dr = d(\iota_{J_\alpha\xi} dr) = 0$,

$$\mathcal{L}_{J_\alpha\xi}\theta_\alpha = d(\underbrace{\iota_{J_\alpha\xi}\theta_\alpha}_{=1}) + \iota_{J_\alpha\xi} \underbrace{d\theta_\alpha}_{=2\sigma\breve{\omega}_\alpha + 2\theta_\beta\wedge\theta_\gamma} = 0 \quad (3.29)$$

$$\mathcal{L}_{J_\alpha\xi}\theta_\beta = d(\underbrace{\iota_{J_\alpha\xi}\theta_\beta}_{=0}) + \iota_{J_\alpha\xi} \underbrace{d\theta_\beta}_{=2\sigma\breve{\omega}_\beta+2\theta_\gamma\wedge\theta_\alpha} = -2\theta_\gamma, \tag{3.30}$$

and $\mathcal{L}_{J_\alpha\xi}\theta_\gamma = 2\theta_\beta$ imply $\mathcal{L}_{J_\alpha\xi}\breve{g} = 0$.

$\mathcal{L}_\xi\breve{g} = 0$ follows from $\mathcal{L}_\xi g \overset{(3.28)}{=} 2g$, $\mathcal{L}_\xi r \overset{(3.16)}{=} r$, $\mathcal{L}_\xi(dr) = d(\iota_\xi dr) = dr$ and

$$\mathcal{L}_\xi\theta_\alpha = d(\underbrace{\iota_\xi\theta_\alpha}_{=0}) + \iota_\xi \underbrace{d\theta_\alpha}_{=2\sigma\breve{\omega}_\alpha+2\theta_\beta\wedge\theta_\gamma} = 0 \qquad (\alpha = 1,2,3). \tag{3.31}$$

□

As we shall see in the next theorem the horizontal parts of the metric and Kähler forms define a quaternionic Kähler structure (g', Q) on every codimension four submanifold $M' \subset M$ that is transversal to the local $CO(3)$-action. The induced quaternionic structure Q is globally trivial. This result is essentially known from [Sw1]. Here, we prove it in the formalism of conical hyper-Kähler manifolds defined by the existence of an Euler vector field (see Definition 3.2.1) and obtain explicit expressions for the fundamental two-forms and the $Sp(1)$-connection one-form of the resulting quaternionic pseudo-Kähler manifold.

Theorem 3.2.6
Let M' be a codimension four submanifold of M that is transversal to the distribution $\mathcal{D}^v$, i.e. $TM|_{M'} = \mathcal{D}^v|_{M'} \oplus TM'$. Then

$$g' := \breve{g}\big|_{M'} \tag{3.32}$$

is a quaternionic (pseudo-)Kähler metric on M'. A compatible quaternionic structure is given by

$$Q := \mathrm{span}_{\mathbb{R}}\{J'_1, J'_2, J'_3\}, \qquad J'_\alpha := \mathrm{pr}^{\mathcal{D}^v}_{TM'} \circ J_\alpha|_{TM'} \qquad (\alpha = 1,2,3), \tag{3.33}$$

where

$$\mathrm{pr}^{\mathcal{D}^v}_{TM'} : TM|_{M'} = \mathcal{D}^v|_{M'} \oplus TM' \to TM' \tag{3.34}$$

is the projection onto the second summand (i.e. the projection onto TM' along $\mathcal{D}^v$).

Remark 3.2.7 Note that $\omega'_\alpha := \breve{\omega}_\alpha|_{M'}$, $\alpha = 1,2,3$ are the fundamental two-forms on (M', g') with respect to the frame (J'_1, J'_2, J'_3) in Q, i.e. $\omega'_\alpha = g'(J'_\alpha\cdot\,,\cdot)$, $\alpha = 1,2,3$.

Proof (of Theorem 3.2.6)**:**
The fact that $(M', g', J_1', J_2', J_3')$ is hyper-Hermitian follows from (M, g, J_1, J_2, J_3) being hyper-Hermitian and from the definitions, since $\ker \breve{g} = \mathcal{D}^v$ and since $\mathcal{D}^v$ is J_α-invariant for $\alpha = 1, 2, 3$.

From Eq. (3.23), we obtain

$$\begin{aligned} d\breve{\omega}_\alpha &= \frac{1}{2\sigma}(2\theta_\beta \wedge d\theta_\gamma - 2\theta_\gamma \wedge d\theta_\beta) \\ &= 2\theta_\beta \wedge \breve{\omega}_\gamma - 2\theta_\gamma \wedge \breve{\omega}_\beta. \end{aligned}$$

Restricting this equation to M' gives

$$d\omega_\alpha' = 2\bar{\theta}_\beta \wedge \omega_\gamma' - 2\bar{\theta}_\gamma \wedge \omega_\beta', \tag{3.35}$$

where $\bar{\theta}_\alpha := \theta_\alpha|_{M'}$. This shows that (M', g', Q) is quaternionic (pseudo-)Kähler if $\dim_{\mathbb{R}} M' > 4$ (see Corollary 2.1.9).

The four-dimensional case can be deduced from the higher-dimensional case as follows[1]:
Assume that $\dim_{\mathbb{R}} M = 8$. Let $M_0 := \mathbb{H}$ be endowed with the standard hyper-Kähler structure $(g_0, J_1^0, J_2^0, J_3^0)$ that was defined in Example 3.1.7, i.e. $g_0 = dzd\bar{z} + dwd\bar{w}$ and $\omega_+^0 = dz \wedge dw$ in complex coordinates (z, w) defined by $q = z + jw \in \mathbb{H}$. Let $\xi_0 := 2\,\mathrm{Re}(z\partial_z + w\partial_w)$ (see Example 3.2.10 below). Then

$$(\tilde{M} := M \times \mathbb{H}, \tilde{g} := g + \sigma g_0, \tilde{\xi} := \xi + \xi_0)$$

together with the product hyper-complex structure $(\tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$ is a conical (pseudo-)hyper-Kähler manifold. $\tilde{M}' := M' \times \mathbb{H} \subset \tilde{M}$ is a codimension four submanifold transversal to the distribution spanned by $\tilde{\xi}$ and $\tilde{J}_\alpha \tilde{\xi}$, $\alpha = 1, 2, 3$. According to the above proof, the 8-dimensional manifold $\tilde{M}'$ inherits a quaternionic Kähler structure defined by three fundamental two-forms $\tilde{\omega}_\alpha'$, $\alpha = 1, 2, 3$. $M' = M' \times \{0\} \subset \tilde{M}'$ is a quaternionic submanifold and, hence $\tilde{\omega}_\alpha'\big|_{M'}$ are fundamental two-forms that define a quaternionic Kähler structure on M' (see Proposition 2.1.11). They agree with ω_α', since the corresponding components of the $Sp(1)$-connection one-form are given by

$$\frac{1}{\tilde{g}(\tilde{\xi}, \tilde{\xi})}\tilde{g}(\tilde{J}_\alpha \tilde{\xi}, \cdot)\big|_{M' \subset \tilde{M}' \subset \tilde{M}} = \frac{1}{g(\xi, \xi) + z\bar{z} + w\bar{w}}\big(g(J_\alpha \xi, \cdot) + \omega_\alpha^0(\xi_0, \cdot)\big)\Big|_{M'}$$

[1] This idea is taken from [MS2, Cor. 4.2.].

$$= \frac{1}{g(\xi,\xi)} g(J_\alpha \xi, \cdot)\Big|_{M'} = \theta_\alpha\big|_{M'} = \bar{\theta}_\alpha.$$

□

Conversely, every quaternionic (pseudo-)Kähler manifold admits a canonically defined $CO(3)$-principal bundle with a conical pseudo-hyper-Kähler structure that locally inverts the above construction:

Theorem 3.2.8 [Sw1] (see Section 3.6)
For any quaternionic (pseudo-)Kähler manifold $(\bar{M}, \bar{g}, Q)$, the pseudo-Riemannian cone $(\widehat{M}, \hat{g}) = (\mathbb{R}^{>0} \times S, \sigma dr^2 + r^2 g_S)$ admits a conical pseudo-hyper-Kähler structure such that, up to scaling of the metric by a positive constant, $(\bar{M}, \bar{g}, Q)$ is locally recovered as in Theorem 3.2.6.
Here, $\pi : S \to \bar{M}$ denotes the principal $SO(3)$-bundle of local oriented orthonormal frames in Q and $g_S = \sigma \sum_{\alpha=1}^{3} (\theta_\alpha)^2 + \frac{|\nu|}{4} \pi^* \bar{g}$, where $\nu := \frac{scal}{4n(n+2)}$, $\dim_{\mathbb{R}} \bar{M} = 4n$, is the reduced scalar curvature of $(\bar{M}, \bar{g})$, $\sigma := \operatorname{sgn} \nu$ and

$$\theta = \sum \theta_\alpha e_\alpha : TS \to \mathfrak{so}(3)$$

is the principal connection one-form on S induced by the Levi-Civita connection of $(\bar{M}, \bar{g})$.

Remark 3.2.9 The conical pseudo-hyper-Kähler manifold $\widehat{M}$ in the above theorem is called the **Swann bundle** over $\bar{M}$:

$$CO(3) \hookrightarrow \widehat{M} \to \bar{M}. \tag{3.36}$$

In the following example, we choose open subsets in quaternionic vector space with positive, respectively quaternionic Lorentzian signature and a homothetic vector field ξ such that ξ is space-like, respectively time-like. ξ is induced by uniform scaling of quaternionic vector space by a positive factor. Then the construction in Theorem 3.2.6 defines a positive definite quaternionic Kähler structure on appropriately chosen codimension four submanifolds. In the case of positive definite quaternionic vector space, this construction yields a chart in quaternionic projective space and, in the case of quaternionic Lorentzian vector space, we obtain quaternionic hyperbolic space.

Example 3.2.10 We endow[2] $M_+ = \mathbb{H}^{n+1} \backslash \{0\}$, $M_- = \{\langle q, q \rangle_{(-)} < 0\} \subset \mathbb{H}^{n,1}$

[2] M_+, M_- are chosen such that ξ is a space-like, respectively time-like vector field.

with complex coordinates $(z^I, w_I)_{I=0,\dots,n}$ defined by $q = z + jw \in M_\pm$ and define a (pseudo-)hyper-Kähler structure $(g_\pm, J_1, J_2, J_3)$ on $M_\pm$ by (see Ex. 3.1.7)

$$g_\pm = \pm(dz^0 d\bar{z}^0 + dw_0 d\bar{w}_0) + \sum_{\mu=1}^{n}(dz^\mu d\bar{z}^\mu + dw_\mu d\bar{w}_\mu), \tag{3.37}$$

$$\omega_+^{(\pm)} = \omega_2^{(\pm)} + i\omega_3^{(\pm)} = \pm dz^0 \wedge dw_0 + \sum_{\mu=1}^{n} dz^\mu \wedge dw_\mu. \tag{3.38}$$

Here, $\langle\cdot,\cdot\rangle_{(\pm)}$ denotes the standard quaternion-Hermitian inner product

$$\langle q, u\rangle_{(\pm)} = \pm q^0\bar{u}^0 + \sum_{\mu=1}^{n} q^\mu \bar{u}^\mu, \quad q, u \in \mathbb{H}^{n+1}, \tag{3.39}$$

on $\mathbb{H}^{n+1}$, respectively $\mathbb{H}^{n,1}$. Together with

$$\xi = \sum_{I=0}^{n}\Big(z^I\frac{\partial}{\partial z^I} + w_I\frac{\partial}{\partial w_I} + \bar{z}^I\frac{\partial}{\partial \bar{z}^I} + \bar{w}_I\frac{\partial}{\partial \bar{w}_I}\Big), \tag{3.40}$$

$(M_\pm, g_\pm, J_1, J_2, J_3)$ is a conical (pseudo-)hyper-Kähler manifold.

For this example, the geometric data defined in Eq. (3.12) reads

$$\begin{aligned}
&\sigma = \operatorname{sgn} g_\pm(\xi,\xi) = \pm 1,\\
&r^2 = |g_\pm(\xi,\xi)| = \pm\langle q,q\rangle^2_{(\pm)} = |z^0|^2 + |w_0|^2 \pm \sum_{\mu=1}^{n}\big(|z^\mu|^2 + |w_\mu|^2\big),\\
&\hat{\theta}_0 := \frac{\sigma}{2} g_\pm(\xi,\cdot) = \frac{1}{2} r dr = \frac{1}{2}\operatorname{Re}\Big(\bar{z}^0 dz^0 + \bar{w}_0 dw_0 \pm \sum_{\mu=1}^{n}(\bar{z}^\mu dz^\mu + \bar{w}_\mu dw_\mu)\Big),\\
&\hat{\theta}_1 = \frac{\sigma}{2} g_\pm(J_1\xi,\cdot) = \frac{1}{2}\operatorname{Im}\Big(\bar{z}^0 dz^0 + \bar{w}_0 dw_0 \pm \sum_{\mu=1}^{n}(\bar{z}^\mu dz^\mu + \bar{w}_\mu dw_\mu)\Big),\\
&\hat{\theta}_2 = \frac{\sigma}{2} g_\pm(J_2\xi,\cdot) = \frac{1}{2}\operatorname{Re}\Big(z^0 dw_0 - w_0 dz^0 \pm \sum_{\mu=1}^{n}(z^\mu dw_\mu - w_\mu dz^\mu)\Big),\\
&\hat{\theta}_3 = \frac{\sigma}{2} g_\pm(J_3\xi,\cdot) = \frac{1}{2}\operatorname{Im}\Big(z^0 dw_0 - w_0 dz^0 \pm \sum_{\mu=1}^{n}(z^\mu dw_\mu - w_\mu dz^\mu)\Big).
\end{aligned} \tag{3.41}$$

Choose $M'_\pm = \{q^0 = 1\} \subset M_\pm$, i.e. $M'_+ \approx \mathbb{H}^n$, $M'_- \approx \{u \in \mathbb{H}^n \mid \|u\|^2 < 1\}$. In complex coordinates $(\phi^\mu, \psi_\mu)_{\mu=1,\dots,n}$ defined by $u = \phi + j\psi \in M'_\pm$, the quaternionic Kähler metric on $M'_\pm$ is given by (compare Eq. (2.15))

$$\begin{aligned} g'_\pm = \breve{g}_\pm|_{M'_\pm} &\overset{(3.26)}{=} \left(\frac{1}{r^2} g_\pm - \frac{4\sigma}{r^4}\sum_{a=0}^{3}(\hat{\theta}_a)^2\right)\Big|_{M'_\pm} \\ &= \frac{\sum_{\mu=1}^n (d\phi^\mu d\bar{\phi}^\mu + d\psi_\mu d\bar{\psi}_\mu)}{1 \pm (\|\phi\|^2 + \|\psi\|^2)} \\ &\quad \mp \frac{\left|\sum_{\mu=1}^n (\bar{\phi}^\mu d\phi^\mu + \bar{\psi}_\mu d\psi_\mu)\right|^2 + \left|\sum_{\mu=1}^n (\phi^\mu d\psi_\mu - \psi_\mu d\phi^\mu)\right|^2}{\left(1 \pm (\|\phi\|^2 + \|\psi\|^2)\right)^2}. \end{aligned} \tag{3.42}$$

Since J_1, J_2, J_3 preserve $TM'_\pm \subset TM_\pm$, the quaternionic structure Q on $M'_\pm$ is spanned by the standard complex structures J'_1, J'_2, J'_3 on $M'_\pm \subset \mathbb{H}^n$ (see Ex. 3.1.7).

Note that while (M'_+, g'_+) is isometric to $(\mathbb{H}P^n)^o := \{q^0 \neq 0\} \subset \mathbb{H}P^n$ and thus incomplete, (M'_-, g'_-) is isometric to the symmetric space $\mathbb{H}H^n$ and thus complete (see Examples 2.1.14 and 2.1.15). The normalization of $(M'_\pm, g'_\pm)$ is again such that the reduced scalar curvature is $\nu = \pm 4$.

3.3 Infinitesimal automorphisms of conical hyper-Kähler manifolds

Let $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ be a conical (pseudo-)hyper-Kähler manifold and let $\hat{X}$ be a tri-holomorphic Killing vector field on $\widehat{M}$ such that $[\hat{X}, \xi] = 0$. In the following, we will use the definitions in Eq. (3.12) (with a hat added to the metric, complex structures and Kähler forms) and

$$\hat{Z} := \hat{J}_1 \xi. \tag{3.43}$$

Proposition 3.3.1 (see [ACDM])
There exists exactly one hyper-Hamiltonian function $\mu^{\hat{X}} \in C^\infty(\widehat{M}, \mathbb{R}^3)$ for $\hat{X}$ such that $\xi(\mu^{\hat{X}}) = 2\mu^{\hat{X}}$. More precisely, the functions

$$\mu_\alpha^{\hat{X}} := -\sigma\hat{\theta}_\alpha(\hat{X}) = -\frac{1}{2}\hat{g}(\hat{J}_\alpha \xi, \hat{X}) \in C^\infty(\widehat{M}) \qquad (\alpha = 1, 2, 3) \tag{3.44}$$

fulfill

$$d\mu_\alpha^{\hat{X}} = \hat{\omega}_\alpha(\hat{X}, \cdot) \tag{3.45}$$

and $\xi(\mu_\alpha^{\hat{X}}) = 2\mu_\alpha^{\hat{X}}$.

***Proof*:** Since $\hat{X}$ preserves $\hat{g}$, $\hat{J}_\alpha$ and ξ, we have $\mathcal{L}_{\hat{X}}\hat{\theta}_\alpha = 0$. From this, we obtain

$$d\mu^{\hat{X}}_\alpha = -\sigma\, d(\iota_{\hat{X}}\hat{\theta}_\alpha) = -\sigma(\mathcal{L}_{\hat{X}}\hat{\theta}_\alpha - \iota_{\hat{X}}d\hat{\theta}_\alpha) \overset{(3.14)}{=} \hat{\omega}_\alpha(\hat{X},\cdot).$$

Since $\mathcal{L}_\xi \hat{g} \overset{(3.28)}{=} 2\hat{g}$ and $\mathcal{L}_\xi(\hat{J}_\alpha\xi) \overset{(3.18)}{=} 0$, we have $\mathcal{L}_\xi\hat{\theta}_\alpha = 2\hat{\theta}_\alpha$ and hence

$$\xi(\mu^{\hat{X}}_\alpha) = -\sigma\mathcal{L}_\xi(\hat{\theta}_\alpha(\hat{X})) = -2\sigma\hat{\theta}_\alpha(\hat{X}) = 2\mu^{\hat{X}}_\alpha.$$

□

Remark 3.3.2 We call the map $\mu^{\hat{X}}$ given by the above proposition the **homogeneous hyper-Kähler moment map** associated with $\hat{X}$.

From now on, we assume that $\hat{X}$ is space-like or time-like.

We consider the level set

$$P := \{\mu^{\hat{X}} = (-\sigma, 0, 0)\} \subset \widehat{M} \tag{3.46}$$

of the hyper-Kähler moment map and define the following data on P:

$$\begin{aligned}
g_P &:= \hat{g}|_P \in \Gamma(\mathrm{Sym}^2\, T^*P),\\
\theta^P_\alpha &:= \sigma\hat{\theta}_\alpha|_P = \frac{1}{2}\hat{g}(\hat{J}_\alpha\xi,\cdot)\big|_P \in \Omega^1(P) \qquad (\alpha = 1,2,3),\\
f &:= \sigma\frac{r^2}{2}\Big|_P = \frac{\hat{g}(\xi,\xi)}{2}\Big|_P \in C^\infty(P),\\
\theta^P_0 &:= \frac{1}{2}df = \frac{1}{2}\hat{g}(\xi,\cdot)\big|_P = \frac{\sigma}{2}rdr\big|_P \in \Omega^1(P),\\
X_P &:= \sigma\hat{X}|_P \in \mathfrak{X}(P),\\
\eta &:= \sigma\frac{\hat{g}(\hat{X},\cdot)}{\hat{g}(\hat{X},\hat{X})}\bigg|_P \in \Omega^1(P),\\
f_1 &:= \frac{2}{\hat{g}(\hat{X},\hat{X})}\bigg|_P \in C^\infty(P)\\
Z^P_1 &:= \hat{Z}|_P = \hat{J}_1\xi|_P \in \mathfrak{X}(P).
\end{aligned} \tag{3.47}$$

The fact that $\hat{Z} = \hat{J}_1\xi$ is tangent to P follows from

$$d\mu^{\hat{X}}_\alpha(\hat{Z}) = \mathcal{L}_{\hat{Z}}\mu^{\hat{X}}_\alpha = -2\delta_{2\alpha}\mu^{\hat{X}}_3 + 2\delta_{3\alpha}\mu^{\hat{X}}_2$$

since $\mu_2\big|_P = \mu_3\big|_P = 0$. The last equation used

$$\mathcal{L}_{\hat{J}_\alpha\xi}\mu^{\hat{X}}_\beta \underset{(3.18)}{\overset{(3.28)}{=}} -2\mu^{\hat{X}}_\gamma. \tag{3.48}$$

Remark 3.3.3 Note that, if non-empty, every level set of the hyper-Kähler moment map $\mu^{\hat{X}}$ is a smooth submanifold of codimension 3 in $\widehat{M}$, due to Eq. (3.45).

Proposition 3.3.4 *Assume that P is non-empty and let $M' \subset P$ be a codimension one submanifold that is transversal to Z_1^P. Then*

$$g' = \frac{1}{2|f|}\Big(g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2\Big)\Big|_{M'} \tag{3.49}$$

is a quaternionic (pseudo-)Kähler metric on M'.

***Proof*:** Since $\mu_1^{\hat{X}}\big|_P \neq 0$, $\xi(\mu_\alpha^{\hat{X}}) = 2\mu_\alpha^{\hat{X}}$ and (3.48) imply that $P \subset \widehat{M}$ is transversal to ξ, $\hat{J}_2\xi$, $\hat{J}_3\xi$. Hence, $M' \subset \widehat{M}$ is transversal to $\mathcal{D}^v = \mathrm{span}_{\mathbb{R}}\{\xi, \hat{J}_1\xi, \hat{J}_2\xi, \hat{J}_3\xi\}$. According to Theorem 3.2.6,

$$\begin{aligned} g' = \breve{g}\big|_{M'} &= \Big(\frac{1}{r^2}\hat{g} - \frac{\sigma}{r^2}dr^2 - \frac{4\sigma}{r^4}\sum_{\alpha=1}^{3}(\hat{\theta}_\alpha)^2\Big)\Big|_{M'\subset P} \\ &= \Big(\frac{1}{2|f|}g_P - \frac{\sigma}{f^2}(\theta_0^P)^2 - \frac{\sigma}{f^2}\sum_{\alpha=1}^{3}(\theta_\alpha^P)^2\Big)\Big|_{M'} \end{aligned}$$

is a quaternionic (pseudo-)Kähler metric on M'. □

Remark 3.3.5 Note that if $(\widehat{M}, \hat{g}, \hat{J}_\alpha, \xi)$ is the Swann bundle (see Section 3.6) over a quaternionic (pseudo-)Kähler manifold $(\bar{M}, \bar{g}, Q)$, $\hat{\pi} : \widehat{M} \to \bar{M}$, then the group $\mathbb{R}^{>0} \times SO(3)$ generated by $\xi, \hat{J}_1\xi, \hat{J}_2\xi, \hat{J}_3\xi$ acts as the standard conformal linear group $CO(3)$ on the three-dimensional vector space spanned by the functions $\mu_\alpha^{\hat{X}}$. Then Z_1^P induces a free S^1-action on P and $P/S^1_{(Z_1^P)}$ is diffeomorphic to $\bar{M}^o := \bar{M}\backslash\hat{\pi}(\{\mu^{\hat{X}} = 0\})$.

Let $\bar{\mu}^X$ be the quaternionic Kähler moment map associated with the Killing vector field $X \in \mathfrak{X}(\bar{M})$ induced by $\hat{X}$. On $\bar{M}^o$, $J := (\|\bar{\mu}^X\|^{-1}\bar{\mu}^X)|_{\bar{M}^o}$ defines an integrable complex structure (see, e.g., [Ba, Prop. 3.3.]). The quaternionic (pseudo-) Kähler stuctures defined on codimension one submanifolds in P transversal to

Z_1^P via Theorem 3.2.6 patch together to the quaternionic (pseudo-)Kähler structure $(\frac{|\nu|}{4}\bar{g}|_{\bar{M}^o}, \mathbb{R}J|_{\bar{M}^o} \oplus V)$, where $V = J^\perp \subset Q|_{\bar{M}^o} \subset \mathrm{End}\, T\bar{M}^o$ is a rank two vector bundle whose unit sphere bundle is isomorphic to the S^1-principal bundle $P \to \bar{M}^o$.

Note that in general, $\bar{M}^o$ can be equal to $\bar{M}$. In the case where $(\bar{M}, \bar{g})$ is positive definite, complete and of positive scalar curvature however, $\bar{M}^o$ must be a proper subset of $\bar{M}$ and, hence $(\bar{M}^o, \frac{|\nu|}{4}\bar{g}|_{\bar{M}^o})$ is incomplete. The latter is due to the fact that on a compact quaternionic Kähler manifold of positive scalar curvature, there exists not even a compatible almost complex structure [AMP, Th. 3.8.].

In the following example, we again treat quaternionic vector space with positive and quaternionic Lorentzian signature simultaneously (see Example 3.2.10). The choice of appropriate codimension four submanifolds $M'_\pm$ again leads to (local) realizations of quaternionic projective and quaternionic hyperbolic space. This time, we do not choose the canonical (local) sections $M'^{\text{can.}}_\pm := \{q^0 = 1\}$ in the $\mathbb{H}^*$-bundles $\mathbb{H}^{n+1}_{>0} \to \mathbb{H}P^n$, respectively $\mathbb{H}^{n,1}_{<0} \to \mathbb{H}H^n$. Instead, we choose submanifolds $M'_\pm$ that are contained in the respective level sets $P_\pm$ of the hyper-Kähler moment map associated with the diagonal S^1-action on quaternionic vector space. This allows us to establish the HK/QK correspondence between, e.g., a certain subset in $T^*(\mathbb{C}H^n)$ on the hyper-Kähler side and $\mathbb{H}H^n$ on the quaternionic Kähler side in Section 4.4.

Example 3.3.6 Let

$$\widehat{M}_+ = \mathbb{H}^{n+1}_{>0} = \mathbb{H}^{n+1}\backslash\{0\},\ \widehat{M}_- = \mathbb{H}^{n,1}_{<0} = \{\langle q, q\rangle_{(-)} < 0\} \subset \mathbb{H}^{n,1}$$

with the conical (pseudo-)hyper-Kähler structure $(\hat{g}_\pm, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ defined by Eqs. (3.37)–(3.40) as in Example 3.2.10. We consider the vector field $\hat{X}$ induced by the action $q = z + jw \mapsto e^{it}q$ of $e^{it} \in S^1$ on $\widehat{M}_\pm$ at $t = 0$, scaled by a factor of two for convenience:

$$\hat{X} = 2i \sum_{I=0}^{n} \Big(z^I \frac{\partial}{\partial z^I} - w_I \frac{\partial}{\partial w_I} - \bar{z}^I \frac{\partial}{\partial \bar{z}^I} + \bar{w}_I \frac{\partial}{\partial \bar{w}_I} \Big). \tag{3.50}$$

The components $\mu_\alpha^{\hat{X}} = -\frac{1}{2}\hat{g}(\hat{J}_\alpha \xi, \hat{X})$ of the homogeneous hyper-Kähler moment map associated with $\hat{X}$ are given by

$$\mu_1^{\hat{X}} = -(\underbrace{\langle z, z\rangle_{(\pm)}}_{\pm|z^0|^2+\sum|z^\mu|^2} - \langle w, w\rangle_{(\pm)}),$$

$$\mu_+^{\hat{X}} = \mu_2^{\hat{X}} + i\mu_3^{\hat{X}} = 2i\langle z, \bar{w}\rangle = 2i\,\big(\pm z^0 w_0 + \sum_{\mu=1}^{n} z^\mu w_\mu\big). \tag{3.51}$$

Let

$$\lambda := \langle z, z\rangle_{(\pm)} - \langle w, w\rangle_{(\pm)}, \quad \chi := \langle z, \bar{w}\rangle_{(\pm)} = \pm z^0 w_0 + \sum_{\mu=1}^{n} z^\mu w_\mu. \tag{3.52}$$

To introduce appropriate coordinates, we restrict ourselves to the chart $\{z^0 \neq 0\}$ in

$$\{\lambda > 0\} \subset \{z \neq 0\} \subset \mathbb{H}^{n+1}_{>0} = \widehat{M}_+$$

and to

$$\{\lambda < 0\} \subset \{\langle z, z\rangle_{(-)} < 0\} \subset \{z^0 \neq 0\} \subset \mathbb{H}^{n,1}_{<0} = \widehat{M}_-,$$

i.e. we consider

$$\begin{aligned} \widehat{M}^o_+ &:= \{q = z + jw \in \mathbb{H}^{n+1} \mid \langle z, z\rangle_{(+)} > \langle w, w\rangle_{(+)},\, z^0 \neq 0\}, \\ \widehat{M}^o_- &:= \{q = z + jw \in \mathbb{H}^{n,1} \mid \langle z, z\rangle_{(-)} < \langle w, w\rangle_{(-)},\, \langle q, q\rangle_{(-)} < 0\}. \end{aligned} \tag{3.53}$$

We endow $\widehat{M}^o_\pm$ with coordinates $(\lambda, \phi, \chi, \zeta^\mu, \eta_\mu)_{\mu=1,\dots,n}$, where λ, χ are given by Eq. (3.52) and

$$\phi := \arg z^0, \quad \zeta^\mu := (z^0)^{-1} z^\mu, \quad \eta_\mu := z^0 w_\mu \qquad (\mu = 1, \dots, n). \tag{3.54}$$

The coordinates are chosen such that the level sets of ϕ are transversal to the S^1-action ($\hat{X}(\phi) = 2$), while $\lambda, \chi, \zeta^\mu, \eta_\mu$ are S^1-invariant, so in these coordinates, $\hat{X}\big|_{\widehat{M}^o_\pm} = 2\frac{\partial}{\partial\phi}$. When we set $\lambda = \pm 1$ and $\chi = 0$, this induces coordinates $(\phi, \zeta^\mu, \eta_\mu)_{\mu=1,\dots,n}$ on the level set

$$P_\pm = \{\mu_1^{\hat{X}} = -\sigma = \mp 1,\, \mu_2^{\hat{X}} = \mu_3^{\hat{X}} = 0\} = \{\lambda = \pm 1,\, \chi = 0\} \subset \widehat{M}^o_\pm \tag{3.55}$$

of the homogeneous hyper-Kähler moment map associated with $\hat{X}$.

Note that in the current example, $\xi, \hat{J}_1\xi, \hat{J}_2\xi, \hat{J}_3\xi$ generate a free $\mathbb{H}^*$-action on $\widehat{M}_\pm$. Submanifolds $M'_\pm \subset \widehat{M}_\pm$ that intersect each $\mathbb{H}^*$-orbit at most once can be identified via the projection map with subsets of $\mathbb{H}P^n$, respectively $\mathbb{H}H^n$. Under this identification, the induced metric g' is independent of the choice of section according to Proposition 3.2.5.

With the choice $M'_\pm = \{\phi = 0\} \subset P_\pm$ of $(Z_1^{P_\pm} = \hat{J}_1\xi\big|_{P_\pm})$-transversal submanifold, we recover the chart $\{q^0 \neq 0\}$ in

$$
\begin{aligned}
&(\mathbb{H}P^n)^o := \\
&\quad \mathbb{H}P^n \backslash \{[q = z + jw]_{\mathbb{H}^*_{\text{right}}} \mid (z, w) \in \mathbb{C}^{2n+2}\backslash\{0\},\ \|z\|^2 = \|w\|^2,\ z \cdot w = 0\}
\end{aligned}
\tag{3.56}
$$

for the case M'_+, while for M'_-, we recover the whole symmetric space $\mathbb{H}H^n$ (see Remark 3.3.5). One way of seeing this is by Remark 3.3.5 and the fact that $\widehat{M}_\pm/\mathbb{Z}_2$ is the Swann bundle over $\mathbb{H}P^n$, respectively $\mathbb{H}H^n$. $(\mathbb{H}P^n)^o$ is the complement of the zero level set of the quaternionic Kähler moment map associated with the Killing vector field X on $\mathbb{H}P^n$ that is induced by $\hat{X}$. It is an open and everywhere dense submanifold of $\mathbb{H}P^n$. $\widehat{M}_-$ has empty intersection with the zero level set of the homogeneous hyper-Kähler moment: The reverse Cauchy-Schwarz inequality (RCS) for complex Lorentzian vector spaces gives the following implication for $q = z + jw \in \widehat{M}_- \cap \{\lambda = 0\}$:

$$
\begin{aligned}
&\langle q, q\rangle < 0,\ \langle z, z\rangle = \langle w, w\rangle \\
\Rightarrow\ &\langle z, z\rangle = \langle w, w\rangle < 0 \\
\Rightarrow\ &|\chi|^2 = |\langle z, \bar{w}\rangle|^2 \overset{RCS}{\geq} \langle z, z\rangle\langle w, w\rangle > 0.
\end{aligned}
$$

This shows that $\widehat{M}_- \cap \{\mu^{\hat{X}} = 0\} = \emptyset$.

Note that while all three almost complex structures induced on the canonical choice of submanifold $M'^{\text{can.}}_\pm = \{q^0 = 1\} \subset \widehat{M}_\pm$ (see Example 3.2.10) are integrable, the almost complex structures J'_2, J'_3 induced on $M'_\pm$ are non-integrable. J'_1 is proportional to the quaternionic Kähler moment map associated with X and hence integrable (see Remark 3.3.5 and Proposition 4.1.9).

In the following remark, we specify the image $\widehat{N}_\pm$ of the coordinate function $(\lambda, \phi, \chi, \zeta^\mu, \eta_\mu)_{\mu=1,\dots,n} : \widehat{M}^o_\pm \to \mathbb{R}^2 \times \mathbb{C}^{2n+1}$ defined in the above example and give the inverse map from $\widehat{N}_\pm$ to $\widehat{M}^o_\pm$.

Remark 3.3.7 In the above example, the coordinates $(\lambda, \phi, \chi, \zeta, \eta)$ on $\widehat{M}^o_\pm$

take their values in

$$\widehat{N}_+ := \mathbb{R}^{>0} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n,$$
$$\widehat{N}_- := \{(\lambda, \phi, \chi, \zeta, \eta) \in \mathbb{R}^{<0} \times \mathbb{R} \times \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^n \mid \|\zeta\|^2 < 1, \lambda^2 + 4\langle\hat{\zeta}, \hat{\zeta}\rangle\langle\hat{\eta}, \hat{\eta}\rangle > 0\},$$

respectively. Here, $\langle\hat{\zeta}, \hat{\zeta}\rangle := \pm|\zeta^0|^2 + \underbrace{\sum_{\mu=1}^{n} |\zeta^\mu|^2}_{=:\|\zeta\|^2}$ and $\langle\hat{\eta}, \hat{\eta}\rangle := \pm|\eta_0|^2 + \|\eta\|^2$, where

$$\zeta^0 := 1, \quad \eta_0 := \pm(\chi - \sum_{\mu=1}^{n} \zeta^\mu \eta_\mu). \tag{3.57}$$

The inverse map from $\widehat{N}_\pm$ to $\widehat{M}^o_\pm$ is given by

$$z^I = \rho_\lambda e^{i\phi} \zeta^I, \quad w_I = (\rho_\lambda)^{-1} e^{-i\phi} \eta_I \qquad (I = 0, \ldots, n), \tag{3.58}$$

where

$$\rho_\lambda := \frac{1}{\sqrt{\pm 2\langle\hat{\zeta}, \hat{\zeta}\rangle}} \sqrt{\pm\lambda + \sqrt{\lambda^2 + 4\langle\hat{\zeta}, \hat{\zeta}\rangle\langle\hat{\eta}, \hat{\eta}\rangle}}\,. \tag{3.59}$$

For future reference, we determine the differentials of z^I and w_I in terms of the coordinates $(\lambda, \phi, \chi, \zeta, \eta)$:

$$\begin{aligned} dz^I &= z^I\big(\rho_\lambda^{-1} d\rho_\lambda + i d\phi\big) + z^0 d\zeta^I, \\ dw_I &= -w_I\big(\rho_\lambda^{-1} d\rho_\lambda + i d\phi\big) + (z^0)^{-1} d\eta_I \end{aligned} \tag{3.60}$$

with $d\zeta^0 = 0$, $d\eta_0 = \pm(d\chi - \sum(\zeta^\mu d\eta_\mu + \eta_\mu d\zeta^\mu))$ and

$$\rho_\lambda^{-1} d\rho_\lambda = \pm \frac{1}{2\sqrt{\lambda^2 + 4\langle\hat{\zeta}, \hat{\zeta}\rangle\langle\hat{\eta}, \hat{\eta}\rangle}} \Big(d\lambda - \rho_\lambda{}^2\, d\langle\hat{\zeta}, \hat{\zeta}\rangle + \rho_\lambda^{-2}\, d\langle\hat{\eta}, \hat{\eta}\rangle\Big). \tag{3.61}$$

For use in Section 4.4, we explicitly determine the geometric data defined in Eq. (3.47) for Example 3.3.6:

Remark 3.3.8 We want to express the geometric data on $P_\pm = \{\lambda = \pm 1, \chi = 0\} \subset \widehat{M}^o_\pm$ defined in Eq. (3.47) for Example 3.3.6 in terms of the coordinates (ϕ, ζ, η) defined above. Using $\theta^P_a = \sigma\hat{\theta}_\alpha\big|_P$, $a = 0, \ldots, 3$, we obtain the following:

$$
\begin{aligned}
\theta_2^P + i\theta_3^P &\overset{(3.41)}{=} \frac{1}{2}\Big(\pm (z^0 dw_0 - w_0 dz^0) + \sum_{\mu=1}^{n} (z^\mu dw_\mu - w_\mu dz^\mu) \Big)\Big|_{P_\pm} \\
&\overset{(3.60)}{=} \Big(-\chi(\rho_\lambda^{-1} d\rho_\lambda + i d\phi) + \frac{1}{2} d\chi - \sum_{\mu=1}^{n} \eta_\mu d\zeta^\mu \Big)\Big|_{P_\pm} \\
&= -\sum_{\mu=1}^{n} \eta_\mu d\zeta^\mu, \qquad (3.62) \\
\theta_0^P + i\theta_1^P &\overset{(3.41)}{=} \frac{1}{2}\Big(\pm (\bar z^0 dz^0 + \bar w_0 dw_0) + \sum_{\mu=1}^{n} (\bar z^\mu dz^\mu + \bar w_\mu dw_\mu) \Big)\Big|_{P_\pm} \\
&\overset{(3.60)}{=} \frac{1}{2}\Big(\lambda(\rho_\lambda^{-1} d\rho_\lambda + i d\phi) + \sum_{\mu=1}^{n} (\rho_\lambda{}^2\, \bar\zeta^\mu d\zeta^\mu + \rho_\lambda^{-2}\, \bar\eta_\mu d\eta_\mu) \pm \rho_\lambda^{-2}\, \bar\eta_0 d\eta_0 \Big)\Big|_{P_\pm} \\
&= \pm\frac{1}{2}\rho_\pm^{-1} d\rho_\pm \pm \frac{i}{2} d\phi + \frac{1}{2}\big(\rho_\pm{}^2 \,(\partial_{\hat J_1} \langle \hat\zeta, \hat\zeta\rangle)\big|_{P_\pm} + \rho_\pm^{-2}\,(\partial_{\hat J_1}\langle \hat\eta, \hat\eta\rangle)\big|_{P_\pm}\big) \\
&\overset{(3.61)}{=} \pm \frac{1}{2\sqrt{1 \pm \tilde r^2}} \big(\langle \hat\eta, \hat\eta\rangle\, d\langle \hat\zeta, \hat\zeta\rangle + \langle \hat\zeta, \hat\zeta\rangle\, d\langle \hat\eta, \hat\eta\rangle \big) \\
&\qquad \pm \frac{i}{2} d\phi + \frac{i}{4}\big(\rho_\pm{}^2\, d^c\langle \hat\zeta, \hat\zeta\rangle + \rho_\pm^{-2}\, d^c\langle \hat\eta, \hat\eta\rangle \big) \qquad (3.63) \\
&= \pm\frac{1}{4} d\big(\sqrt{1 \pm \tilde r^2}\,\big) + i\left(\pm\frac{1}{2} d\phi + \frac{1}{4} d^c\big(\pm\sqrt{1 \pm \tilde r^2} \mp 2\log \rho_\pm \big)\right),
\end{aligned}
$$

where[3]

$$
\begin{aligned}
\rho_\pm &:= \rho_\lambda|_{P_\pm} = \frac{1}{\sqrt{\pm 2 \langle \hat\zeta, \hat\zeta\rangle}} \sqrt{1 + \sqrt{1 \pm \tilde r^2}} \in C^\infty(P_\pm) \\
\tilde r^2 &:= \pm 4 \langle \hat\zeta, \hat\zeta\rangle \langle \hat\eta, \hat\eta\rangle \in C^\infty(P_\pm), \quad \tilde r^2 \geq 0. \qquad (3.64)
\end{aligned}
$$

In Eq. (3.63), $d^c = i(\bar\partial - \partial)$ acts on the holomorphic functions $(\zeta^\mu, \eta_\mu)_{\mu=1,\dots,n}$. It will later turn out to be the d^c-operator associated with the first complex structure of the hyper-Kähler quotient $\widehat{M}_\pm /\!\!/\!\!/_{P_\pm} S^1_{(\mathrm{diag.})} \approx M'_\pm = \{\phi = 0\} \subset P_\pm$ (see Example 3.4.7).

The other geometric data defined in (3.47) can be calculated as well:

$$
f = \pm \frac{\sqrt{1 \pm \tilde r^2}}{2}, \quad f_1 = \pm \frac{1}{2\sqrt{1 \pm \tilde r^2}},
$$

[3] By abuse of notation, $\langle \hat\zeta, \hat\zeta\rangle := \langle \hat\zeta, \hat\zeta\rangle|_{P_\pm} = \pm 1 + \|\zeta\|^2 \in C^\infty(P_\pm)$ and $\langle \hat\eta, \hat\eta\rangle := \langle \hat\eta, \hat\eta\rangle|_{P_\pm} = \pm|\sum_{\mu=1}^{n} \zeta^\mu \eta_\mu|^2 + \|\eta\|^2 \in C^\infty(P_\pm)$, compare Eq. (3.57) and above.

$$
\begin{aligned}
X_P &= \pm 2\frac{\partial}{\partial\phi}, \quad Z_1^P = \frac{\partial}{\partial\phi} + 2i\sum_{\mu=1}^{n}\big(\eta_\mu\frac{\partial}{\partial\eta_\mu} - \bar{\eta}_\mu\frac{\partial}{\partial\bar{\eta}_\mu}\big),\\
\eta &= \pm\frac{1}{2}d\phi + \frac{1}{4\sqrt{1\pm\tilde{r}^2}}\big(\rho_\pm^{\,2}d^c\langle\hat{\zeta},\hat{\zeta}\rangle - \rho_\pm^{-2}d^c\langle\hat{\eta},\hat{\eta}\rangle\big)\\
&= \pm\frac{1}{2}\big(d\phi - \rho_\pm^{-1}d^c\rho_\pm\big),\\
g_P &= \pm\sqrt{1\pm\tilde{r}^2}\,(d\phi - \rho_\pm^{-1}d^c\rho_\pm)^2 + \sum_{\mu=1}^{n}\big(\rho_\pm^{\,2}\,d\zeta^\mu d\bar{\zeta}^\mu + \rho_\pm^{-2}\,d\eta_\mu d\bar{\eta}_\mu\big)\\
&\qquad \pm\rho_\pm^{-2}\,\big|d\big(\sum_{\mu=1}^{n}\zeta^\mu\eta_\mu\big)\big|^2 \mp 4\sqrt{1\pm\tilde{r}^2}\,|\rho_\pm^{-1}\,\partial\rho_\pm|^2.
\end{aligned}
\tag{3.65}
$$

From the above example, we get the following corollary. It gives the realization of quaternionic hyperbolic space that we will later obtain when we apply the HK/QK correspondence to a certain subset in $T^*(\mathbb{C}H^n)$ in Section 4.4.

Corollary 3.3.9

$$
N'_- := \{(\zeta,\eta)\in\mathbb{C}^n\times\mathbb{C}^n \mid \|\zeta\|^2 < 1, \tilde{r}^2 < 1\},
\tag{3.66}
$$

together with[4]

$$
\begin{aligned}
g'_- &= (\rho_-^{-1}d\rho_-)^2 \qquad\qquad (3.67)\\
&+ \frac{1}{\sqrt{1-\tilde{r}^2}}\Big(\sum_{\mu=1}^{n}\big(\rho_-^{\,2}\,d\zeta^\mu d\bar{\zeta}^\mu + \rho_-^{-2}\,d\eta_\mu d\bar{\eta}_\mu\big) - \rho_-^{-2}\,\big|d\big(\sum_{\mu=1}^{n}\zeta^\mu\eta_\mu\big)\big|^2\Big)\\
&+ \frac{1}{1-\tilde{r}^2}\Big(4\big|\sum_{\mu=1}^{n}\eta_\mu d\zeta^\mu\big|^2 + \frac{1}{4}\big(d\sqrt{1-\tilde{r}^2}\big)^2 + \frac{1}{4}\big(d^c(-\sqrt{1-\tilde{r}^2} + 2\log\rho_-)\big)^2\Big)
\end{aligned}
$$

is isometric to $\mathbb{H}H^n$.

***Proof*:** From the argument at the end of Example 3.3.6, we know that $M'_- \subset \widehat{M}_- = \mathbb{H}^{n,1}_{<0}$ defines a global section of the $\mathbb{H}^*$-bundle $\mathbb{H}^{n,1}_{<0} \to \mathbb{H}H^n$. By Proposition 3.2.5, (M'_-, g'_-) with the metric g'_- defined by Eq. (3.49) is isometric to $(M'^{\text{can.}}_-, g'^{\text{can.}}_-)$. Here $g'^{\text{can.}}_-$ is obtained from the canonical section $M'^{\text{can.}}_- = \{q^0 = 1\} \subset \widehat{M}_- = \mathbb{H}^{n,1}_{<0}$ via Theorem 3.2.6 (see Eq. (3.42) in Example 3.2.10). The latter is isometric to $\mathbb{H}H^n$ (see also Example 2.1.15 and

[4] Recall that on N'_-, $\tilde{r}^2 = 4(1-\|\zeta\|^2)(-|\sum_{\mu=1}^{n}\zeta^\mu\eta_\mu|^2 + \|\eta\|^2)$ and $\rho_- = \frac{1}{\sqrt{2(1-\|\zeta\|^2)}}\sqrt{1+\sqrt{1-\tilde{r}^2}}$.

Remark 2.1.16). The coordinates $(\zeta^\mu, \eta_\mu)_{\mu=1,\dots,n}$ defined in Example 3.3.6 give a diffeomorphism from

$$M'_- = \{q = z + jw \in \mathbb{H}^{n,1} \mid \langle w, w\rangle = 1 + \langle z, z\rangle,\ \arg z^0 = 0,\ \langle z, \bar{w}\rangle = 0,\ \langle q, q\rangle < 1\}$$

to $N'_- \approx \{-1\} \times \{0\} \times \{0\} \times N'_- \subset \widehat{N}_-$. In these coordinates and with the geometric data calculated in Remark 3.3.8, the metric g' defined by Eq. (3.49) reads as in Eq. (3.67). □

Remark 3.3.10 Similarly, $N'_+ := \{(\zeta, \eta) \in \mathbb{C}^n \times \mathbb{C}^n\}$ endowed with the metric

$$\begin{aligned} g'_+ = {} & -(\rho_+^{-1} d\rho_+)^2 \\ & + \frac{1}{\sqrt{1+\tilde{r}^2}} \Big(\sum_{\mu=1}^{n} \big({\rho_+}^2\, d\zeta^\mu d\bar{\zeta}^\mu + \rho_+^{-2}\, d\eta_\mu d\bar{\eta}_\mu \big) + \rho_+^{-2} \big| d\big(\sum_{\mu=1}^{n} \zeta^\mu \eta_\mu \big) \big|^2 \Big) \\ & - \frac{1}{1+\tilde{r}^2} \Big(4 \big| \sum_{\mu=1}^{n} \eta_\mu d\zeta^\mu \big|^2 + \frac{1}{4} \big(d\sqrt{1+\tilde{r}^2} \big)^2 + \frac{1}{4} \big(d^c(+\sqrt{1+\tilde{r}^2} - 2\log\rho_+) \big)^2 \Big) \end{aligned} \tag{3.68}$$

is isometric to $\{q^0 \neq 0\} \subset (\mathbb{H}P^n)^o$, where $(\mathbb{H}P^n)^o$ is the complement of the zero level set of the quaternionic Kähler moment map with respect to the diagonal S^1-action that was defined in Example 3.3.6 (see Eq. (3.56)).

3.4 The hyper-Kähler quotient

Definition 3.4.1 *Let (M, g, J_1, J_2, J_3) be a (pseudo-)hyper-Kähler manifold and let G be a Lie group acting isometrically and tri-holomorphically on M. A* ***hyper-Kähler moment map*** *μ for (M, g, J_1, J_2, J_3, G) is a smooth G-equivariant[5] map from M to $\mathfrak{g}^* \otimes \mathbb{R}^3$ such that*

$$d\mu^v = (\omega_1(v^\sharp, \cdot), \omega_2(v^\sharp, \cdot), \omega_3(v^\sharp, \cdot)), \quad v \in \mathfrak{g}. \tag{3.69}$$

Here, $\mu^v := \langle \mu, v\rangle \in C^\infty(M, \mathbb{R}^3)$ denotes the contraction of $v \in \mathfrak{g}$ with the $\mathfrak{g}^*$-factor of μ and $v^\sharp \in \mathfrak{X}(M)$ denotes the fundamental vector field[6] induced by $v \in \mathfrak{g}$. The action of G on $\mathfrak{g}^*$ is given by the coadjoint action.

Remark 3.4.2 We will also use the notation $\mu^{v^\sharp} := \mu^v$ for a fundamental

[5] If G is connected, μ is G-equivariant if and only if $d\mu^v(w^\sharp) = \mu^{[v,w]}$ for all $v, w \in \mathfrak{g}$.

[6] We define fundamental vector fields without an extra minus sign, i.e. $\cdot^\sharp : \mathfrak{g} \to \mathfrak{X}(M)$, $v \mapsto v^\sharp$ is a Lie algebra anti-homomorphism: $v^\sharp|_p := \frac{d}{dt}\big|_{t=0} \exp(tv)\cdot p, \quad p \in M$.

vector field $v^\sharp$ induced by a vector $v \in \mathfrak{g}$.

If $X \in \mathfrak{X}(M)$ is a tri-holomorphic Killing vector field on a (pseudo-)hyper-Kähler manifold, we call a function $\mu^X \in C^\infty(M, \mathbb{R}^3)$ such that

$$d\mu^X_\alpha = \omega_\alpha(X, \cdot), \quad \alpha = 1{,}2{,}3,$$

a **hyper-Hamiltonian function** or a **hyper-Kähler moment map** with respect to X.

Theorem 3.4.3 [HKLR]
Let G be a compact Lie group acting freely, isometrically and tri-holomorphically on a (pseudo-)hyper-Kähler manifold (M, g, J_1, J_2, J_3) such that the restriction of g to the distribution tangent to the G-orbits is non-degenerate. Let μ be a (pseudo-)hyper-Kähler moment map for the action of G and let $c \in Z(\mathfrak{g}^*) \otimes \mathbb{R}^3$ such that the level set $M_c := \mu^{-1}(\{c\}) \subset M$ is non-empty. Then $\bar{M} := M_c/G$ inherits a hyper-Kähler structure from M. The Kähler forms $\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3$, on $\bar{M}$ are defined by

$$p^*\bar{\omega}_\alpha = \omega_\alpha\big|_{M_c} \qquad (\alpha = 1, 2, 3), \tag{3.70}$$

where $p : M_c \to \bar{M}$ denotes the standard projection.

Definition 3.4.4 *The (pseudo-)hyper-Kähler manifold $\bar{M}$ obtained from the above theorem is called the **hyper-Kähler quotient** of M with respect to G with level c and we will denote it by*

$$\bar{M} = M /\!/\!/_{M_c} G.$$

Remark 3.4.5 In the above theorem, one can replace the assumption that G is compact and acts freely on M by the assumption that c is a regular value of μ and that G acts properly[7] and freely on M_c (see, e.g., [Lee]), or just by the assumption that M_c/G is a smooth manifold of dimension

$$\dim_{\mathbb{R}} \bar{M} = \dim_{\mathbb{R}} M - 4 \dim G$$

such that the projection map p is a smooth submersion.

In the HK/QK correspondence (see Chapter 4), we always have to choose a Kähler moment map, which is only fixed up to a constant. In the following

[7]The G-action on M_c is called proper if pre-images of compact subsets of $M_c \times M_c$ under the map $G \times M_c \to M_c \times M_c$, $(g, q) \mapsto (g \cdot q, q)$ are compact.

example, we discuss the S^1-action on $\mathbb{H}^{n,1}_{<0}$, respectively $\mathbb{H}^{n+1}_{>0}$ defined by left-multiplication on just one of the quaternionic coordinates. In the hyper-Kähler reduction for this example, we scale the Killing vector field by a factor $c \in \mathbb{R}$ or equivalently, we choose different level sets depending on c. When we apply the HK/QK correspondence in Chapter 4 to (open subsets of) flat quaternionic vector space, c will determine the choice of the Kähler moment map. The present example of a hyper-Kähler quotient will then show that the result is quaternionic hyperbolic space, respectively a chart in quaternionic projective space, irrespectively of the choice of Kähler moment map. It will also allow us to establish the HK/QK correspondence between $T^*(\mathbb{C}P^n)$ and an open subset of the symmetric space $X(n) = Gr_n(\mathbb{C}^{n+2})$ (and similarly for the non-compact duals) for different choices of the Kähler moment map.

Example 3.4.6 For $c \in \mathbb{R}^{>0}$, we consider the hyper-Kähler quotient

$$\mathbb{H}^{n+1}\backslash\{0\} /\!\!/\!\!/_{\{q^0=\sqrt{|c|}\, e^{it}\}} S^1_{(q^0)} \approx \mathbb{H}^n$$

and for $c \in \mathbb{R}^{<0}$, we consider

$$\{\hat{q} \in \mathbb{H}^{n,1} \mid \langle \hat{q}, \hat{q} \rangle < 0\} /\!\!/\!\!/_{\{q^0=\sqrt{|c|}\, e^{it}\}} S^1_{(q^0)} \approx \{q \in \mathbb{H}^n \mid \|q\|^2 < |c|\},$$

where the action of $e^{it} \in S^1$ is given by multiplication of e^{it} from the left on the zeroth quaternionic coordinate q^0. Here, we use the notation $\hat{q} = (q^0, q) \in \widehat{M}_\pm$, where $\widehat{M}_+ = \mathbb{H}^{n+1}_{>0}$ and $\widehat{M}_- = \mathbb{H}^{n,1}_{<0}$. In the case of $\mathbb{H}^{n,1}_{<0}$, the metric is taken to be negative definite in the direction of q^0, see Example 3.2.10. The level set of the hyper-Kähler moment map is chosen to be

$$P_\pm := \{\hat{q} = \hat{z} + j\hat{w} \in \widehat{M}_\pm \mid |z^0|^2 = |c|,\, w_0 = 0\}.$$

More precisely, we choose the level set $\{\mu^{\hat{X}} = (\mp 1, 0, 0)\}$ for the homogeneous hyper-Kähler moment map

$$\mu^{\hat{X}} = \frac{1}{|c|}\big(\mp (|z^0|^2 - |w_0|^2),\, \mathrm{Re}\,(\pm 2iz^0 w_0),\, \mathrm{Im}\,(\pm 2iz^0 w_0)\big)$$

associated with the tri-holomorphic Killing vector field

$$\hat{X} = \frac{2i}{|c|}\Big(z^0\frac{\partial}{\partial z^0} - w_0\frac{\partial}{\partial w_0} - \bar{z}^0\frac{\partial}{\partial \bar{z}^0} + \bar{w}_0\frac{\partial}{\partial \bar{w}_0}\Big).$$

Here, the upper and lower sign correspond to the case of $\mathbb{H}^{n+1}_{>0}$ and $\mathbb{H}^{n,1}_{<0}$, respectively. The hyper-Kähler structure on the quotient is again the standard one on quaternionic vector spaces, see Example 3.1.7.

In the next example, we obtain $T^*(\mathbb{C}P^n)$ and a tubular neighborhood of the zero section in $T^*(\mathbb{C}H^n)$ from a hyper-Kähler reduction of flat quaternionic vector space. This will allow us to apply the HK/QK correspondence to these hyper-Kähler manifolds in the next chapter.

Example 3.4.7 We continue Example 3.3.6 and perform the following hyper-Kähler quotients:

$$\mathbb{H}^{n+1} /\!\!/\!\!/_{\{\lambda=1,\,\chi=0\}} S^1_{(\text{diag.})} \approx T^*\mathbb{C}P^n,$$

$$\{q \in \mathbb{H}^{n,1} \mid \langle q, q\rangle < 0\} /\!\!/\!\!/_{\{\lambda=-1,\,\chi=0\}} S^1_{(\text{diag.})} \approx \{\tilde{r}^2 < 1\} \subset T^*\mathbb{C}H^n.$$

In the case of quaternionic vector space with positive definite signature, this can be found in [LR] and [Hi1]. In both cases, the resulting hyper-Kähler metric agrees with the one constructed by Biquard and Gauduchon in [BiGau]. They construct a complete hyper-Kähler metric on the cotangent bundle of any Hermitian symmetric space of compact type and an incomplete hyper-Kähler metric on a specific tubular neighborhood of the zero section in the cotangent bundle of any Hermitian symmetric space of non-compact type.

In this example, we will determine the hyper-Kähler structure on the chart

$$T^*(\{[z^0 : z^1 : \ldots : z^n]_{\mathbb{C}^*} \in \mathbb{C}P^n \mid z^0 \neq 0\}) \subset T^*\mathbb{C}P^n$$

and on $\{\tilde{r}^2 < 1\} \subset T^*\mathbb{C}H^n$. While the metric defined on the charts in $T^*\mathbb{C}P^n$ patches together to a complete hyper-Kähler metric on the whole cotangent bundle, the hyper-Kähler metric on $\{\tilde{r}^2 < 1\} \subset T^*\mathbb{C}H^n$ is incomplete and can not be extended [BiGau].

Let $\widehat{M}_+ = \{q = z + jw \in \mathbb{H}^{n+1} \mid z^0 \neq 0\}$ and $\widehat{M}_- = \{q \in \mathbb{H}^{n,1} \mid \langle q, q\rangle < 0\}$ be endowed with the standard (pseudo-)hyper-Kähler structure $(\hat{g}_\pm, \hat{J}_1, \hat{J}_2, \hat{J}_3)$ (see Example 3.2.10). As in Example 3.3.6, we consider the tri-holomorphic Killing vector field $\hat{X}$ generating the action $q = z + jw \mapsto e^{it}q = e^{it}z + j(e^{-it}w)$ of $e^{it} \in S^1$ on $\widehat{M}_\pm$ (scaled by a factor of two):

$$\hat{X} := 2i \sum_{I=0}^{n} \Big(z^I \frac{\partial}{\partial z^I} - w_I \frac{\partial}{\partial w_I} - \bar{z}^I \frac{\partial}{\partial \bar{z}^I} + \bar{w}_I \frac{\partial}{\partial \bar{w}_I} \Big).$$

Again, we consider the level set

$$P_\pm := \{\lambda = \pm 1,\ \chi = 0\} \subset \widehat{M}_\pm,$$

where $\lambda := \langle z, z\rangle - \langle w, w\rangle$ and $\chi := \langle z, \bar{w}\rangle := \pm z^0 w_0 + \sum_{\mu=1}^n z^\mu w_\mu$. The orbit space $P_\pm/S^1_{(diag.)}$ is diffeomorphic to the global section[8] $M'_\pm := \{\phi := \arg z^0 = 0\} \subset P_\pm$. The Kähler forms on $M'_\pm$ induced from $\widehat{M}$ are given by

$$\omega_\alpha = \hat{\omega}_\alpha\big|_{M'_\pm} \overset{(3.14)}{=} \sigma d\hat{\theta}_\alpha\big|_{M'_\pm} \overset{(3.47)}{=} d\theta^P_\alpha|_{M'}. \tag{3.71}$$

From this equation and from Eq. (3.62), we obtain that

$$\omega_+ = \omega_2 + i\omega_3 = \sum_{\mu=1}^n d\zeta^\mu \wedge d\eta_\mu \tag{3.72}$$

in complex coordinates $(\zeta^\mu := (z^0)^{-1} z^\mu, \eta_\mu := z^0 w_\mu)_{\mu=1,\dots,n}$ on $M'_\pm$. Eq. (3.63) implies that $(\zeta^\mu, \eta_\mu)_{\mu=1,\dots,n}$ are actually J_1-holomorphic coordinates and that $\omega_1^{(\pm)} = \frac{1}{4} dd^c_{J_1} K_\pm$ for the Kähler potential

$$K_\pm = \pm\sqrt{1 \pm \tilde{r}^2} \mp 2\log\rho_\pm = \pm\sqrt{1 \pm \tilde{r}^2} \mp \log\frac{1 + \sqrt{1 \pm \tilde{r}^2}}{1 \pm \|\zeta\|^2} \pm \log 2, \tag{3.73}$$

where

$$\tilde{r}^2 = 4(1 \pm \|\zeta\|^2)(\pm|\sum \zeta^\mu \eta_\mu|^2 + \|\eta\|^2),\ \rho_\pm = \frac{1}{\sqrt{2(1 \pm \|\zeta\|^2)}}\sqrt{1 + \sqrt{1 \pm \tilde{r}^2}}.$$

The coordinates $(\zeta^\mu, \eta_\mu)_{\mu=1,\dots,n}$ take their values in

$$M'_+ \approx \{(\zeta, \eta) \in \mathbb{C}^n \times \mathbb{C}^n\},\quad M'_- \approx \{(\zeta, \eta) \in \mathbb{C}^n \times \mathbb{C}^n \mid \|\zeta\|^2 < 1,\ \tilde{r}^2 < 1\}.$$

For later use, we give an explicit expression for the hyper-Kähler metric obtained from the above example:

Remark 3.4.8 Note that, using the notation

$$\langle \hat{\zeta}, \hat{\zeta}\rangle = \pm 1 + \|\zeta\|^2,\quad \langle \hat{\eta}, \hat{\eta}\rangle = \pm|\sum_{\mu=1}^n \zeta^\mu \eta_\mu|^2 + \|\eta\|^2,$$

we get the following expression for the first Kähler form from the Kähler potential

[8] Note that on P_-, we have $z^0 \neq 0$: $\|q\|^2 < 0, \lambda < 0 \Rightarrow \|z\|^2 < 0 \Rightarrow |z^0|^2 > 0$.

given in Eq. (3.73):

$$
\begin{aligned}
\omega_1^{(\pm)} &= \frac{i}{2}\partial_{J_1}\bar{\partial}_{J_1}K_\pm \qquad (3.74)\\
&= \frac{i}{2}\Bigg(\pm \frac{1}{\langle\hat{\zeta},\hat{\zeta}\rangle}\sum_{\mu=1}^{n} d\zeta^\mu\wedge d\bar{\zeta}^\mu \mp \frac{1}{(\langle\hat{\zeta},\hat{\zeta}\rangle)^2}\big(\sum_{\mu=1}^{n}\bar{\zeta}^\mu d\zeta^\mu\big)\wedge\big(\sum_{\nu=1}^{n}\zeta^\nu d\bar{\zeta}^\nu\big)\\
&\qquad \pm\frac{1}{2}\frac{\partial_{J_1}\bar{\partial}_{J_1}(\pm\tilde{r}^2)}{1+\sqrt{1\pm\tilde{r}^2}} \mp \frac{1}{4\sqrt{1\pm\tilde{r}^2}}\frac{\partial_{J_1}(\pm\tilde{r}^2)\wedge\bar{\partial}_{J_1}(\pm\tilde{r}^2)}{(1+\sqrt{1\pm\tilde{r}^2})^2}\Bigg).
\end{aligned}
$$

A direct calculation using $\pm\tilde{r}^2 = 4\langle\hat{\zeta},\hat{\zeta}\rangle\langle\hat{\eta},\hat{\eta}\rangle$ and Eq. (3.61) for $\lambda=\pm1$ gives

$$
\begin{aligned}
\omega_1^{(\pm)} &= \frac{i}{2}\Bigg(\sum_{\mu=1}^{n}(\rho_\pm^2\, d\zeta^\mu\wedge d\bar{\zeta}^\mu + \rho_\pm^{-2}d\eta_\mu\wedge d\bar{\eta}_\mu) \qquad (3.75)\\
&\qquad \pm\rho_\pm^{-2}d\big(\sum_{\mu=1}^{n}\zeta^\mu\eta_\mu\big)\wedge d\big(\sum_{\nu=1}^{n}\bar{\zeta}^\nu\bar{\eta}_\nu\big) \mp 4\rho_\pm^{-2}\sqrt{1\pm\tilde{r}^2}\,\partial_{J_1}\rho_\pm\wedge\bar{\partial}_{J_1}\rho_\pm\Bigg).
\end{aligned}
$$

The above equation leads to the following expression for the hyper-Kähler metric:

$$
\begin{aligned}
g_\pm &= \sum_{\mu=1}^{n}(\rho_\pm^2\, d\zeta^\mu d\bar{\zeta}^\mu + \rho_\pm^{-2}d\eta_\mu d\bar{\eta}_\mu) \qquad (3.76)\\
&\qquad \pm\rho_\pm^{-2}\Big|d\big(\sum_{\mu=1}^{n}\zeta^\mu\eta_\mu\big)\Big|^2 \mp 4\rho_\pm^{-2}\sqrt{1\pm\tilde{r}^2}\,|\partial_{J_1}\rho_\pm|^2.
\end{aligned}
$$

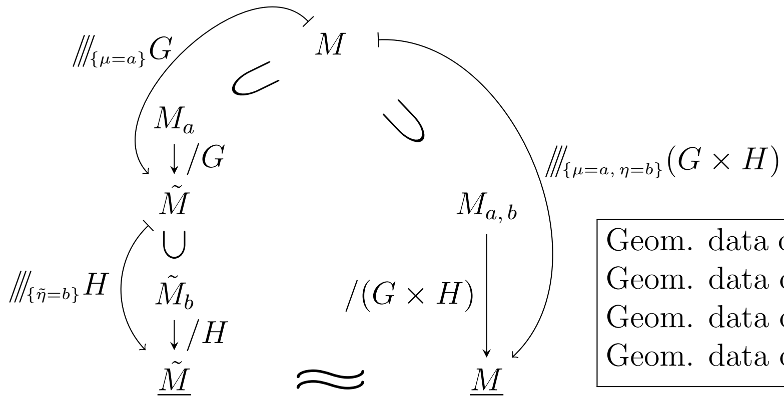
To show the compatibility of the HK/QK correspondence with the hyper-Kähler and quaternionic Kähler quotient constructions in the next chapter, we need the following rather obvious proposition. It states that the hyper-Kähler quotient with respect to two commuting Lie group actions can be performed *in stages* and since it is formulated completely symmetrically with respect to the two Lie group actions, it in particular implies that the outcome of the hyper-Kähler reduction does not depend on the order in which one performs the two respective hyper-Kähler quotients. For the reader's convenience, we include a diagram of the manifolds involved and a list of geometric data defined on them (see Figure 3.2).

Proposition 3.4.9 *Let (M,g,J_1,J_2,J_3) be a (pseudo-)hyper-Kähler manifold and let G, H be compact Lie groups acting isometrically and tri-holomorphically on M such that their actions commute and such that the action of $G\times H$ on M is free. Assume that g is non-degenerate along the G-orbits, the H-orbits and* *the $(G \times H)$-orbits in M. Let $\mu : M \to \mathfrak{g}^* \otimes \mathbb{R}^3$ and $\eta : M \to \mathfrak{h}^* \otimes \mathbb{R}^3$ such that $\mu \oplus \eta$ is a hyper-Kähler moment map for the $(G \times H)$-action on M. Let $a \in Z(\mathfrak{g}^*) \otimes \mathbb{R}^3$ and $b \in Z(\mathfrak{h}^*) \otimes \mathbb{R}^3$.*

Then we have an induced isometric, tri-holomorphic and free action of H on the hyper-Kähler quotient $M/\!/\!/_{\{\mu=a\}}G$ with hyper-Kähler moment map $\tilde{\eta}$ induced by η and

$$(M/\!/\!/_{\{\mu=a\}}G)/\!/\!/_{\{\tilde{\eta}=b\}}H \approx M/\!/\!/_{\{\mu=a,\,\eta=b\}}(G \times H)$$

as (pseudo-)hyper-Kähler manifolds.



Geom. data on M :	$g,\, \omega_\alpha,\, \mu,\, \eta$.
Geom. data on $\tilde{M}$:	$\tilde{g},\, \tilde{\omega}_\alpha,\, \tilde{\eta}$.
Geom. data on $\underline{\tilde{M}}$:	$\underline{\tilde{g}},\, \underline{\tilde{\omega}}_\alpha$.
Geom. data on $\underline{M}'$:	$\underline{g},\, \underline{\omega}_\alpha$.

Figure 3.2: Illustration and list of geometric data for the proof of Proposition 3.4.9.

***Proof*:** G acts freely on M with hyper-Kähler moment map μ, so we can consider the hyper-Kähler quotient

$$M/\!/\!/_{\{\mu=a\}}G = (\tilde{M}, \tilde{g}, \tilde{J}_1, \tilde{J}_2, \tilde{J}_3)$$

with $\tilde{M} := M_a/G$, $M_a = \mu^{-1}(\{a\})$.

Due to the $(G \times H)$-equivariance of $\mu \oplus \eta$, η is constant on the G-orbits in M and hence induces a smooth map $\tilde{\eta} : \tilde{M} \to \mathfrak{h}^* \otimes \mathbb{R}^3$ on $\tilde{M} = M_a/G$:

$$d\eta^w(v^\sharp) = v^\sharp(\eta^w) = v^\sharp((\mu \oplus \eta)^w) = (\mu \oplus \eta)^{[w,\,v]_{\mathfrak{g}\oplus\mathfrak{h}}} = 0$$

$(v \in \mathfrak{g} \subset \mathfrak{g} \oplus \mathfrak{h},\ w \in \mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{h})$. Analogously to the above equation, one shows that μ is constant on the H-orbits in M. Hence, H acts on $M_a = \mu^{-1}(\{a\}) \subset M$. Since G and H commute, H also acts on the orbit space $\tilde{M} = M_a/G$. It is straightforward to check that H acts freely on $\tilde{M}$ and that H preserves the

Kähler forms $\tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3$, i.e. that it acts isometrically and tri-holomorphically on $\tilde{M}$. The map $\tilde{\eta} : \tilde{M} \to \mathfrak{h}^* \otimes \mathbb{R}^3$ fulfills[9]

$$d\tilde{\eta}^w_\alpha = \tilde{\omega}_\alpha(w^{\tilde{\sharp}}, \cdot), \quad w \in \mathfrak{h},\ \alpha = 1, 2, 3$$

and is H-equivariant. The restriction of the metric to the H-orbits is non-degenerate. Hence, we can consider the hyper-Kähler quotient

$$\tilde{M}/\!\!/\!\!/_{\{\tilde{\eta}=b\}} H = (\underline{\tilde{M}}, \underline{\tilde{g}}, \underline{\tilde{J}_1}, \underline{\tilde{J}_2}, \underline{\tilde{J}_3})$$

with $\underline{\tilde{M}} = \tilde{M}_b/H$, $\tilde{M}_b = \tilde{\eta}^{-1}(\{b\}) \subset \tilde{M}$. Note that $\tilde{M}_b = M_{a,b}/G$, i.e. $\underline{\tilde{M}} = (M_{a,b}/G)/H$, where $M_{a,b} = \{\mu = a,\ \eta = b\} \subset M$. This can be naturally identified with $\underline{M} := M_{a,b}/(G \times H)$. Since the Kähler forms on hyper-Kähler quotients are defined purely in terms of pullbacks (see Eq. (3.70)), it is easy to check that under the above identification, the hyper-Kähler structure on $\underline{\tilde{M}}$ agrees with that on $\underline{M}$ obtained from the hyper-Kähler quotient

$$M/\!\!/\!\!/_{\{\mu=a,\,\eta=b\}}(G \times H) = (\underline{M}, \underline{g}, \underline{J_1}, \underline{J_2}, \underline{J_3}).$$

□

Remark 3.4.10 The assumptions that ensure the smoothness of the respective hyper-Kähler quotients in the above proposition can be relaxed (see Remark 3.4.5). Also note that G and H are treated entirely symmetrically. Hence, for commuting Lie group actions of G and H on a (pseudo-)hyper-Kähler manifold M, we have

$$(M/\!\!/\!\!/_\star G)/\!\!/\!\!/_\star H \approx (M/\!\!/\!\!/_\star H)/\!\!/\!\!/_\star G$$

for appropriate choices of level sets, whenever all four hyper-Kähler quotients exist.

3.5 Hyper-Kähler quotients of conical hyper-Kähler manifolds

Recall that any codimension four submanifold that is transversal to the vertical distribution in a conical pseudo-hyper-Kähler manifold inherits a quaternionic pseudo-Kähler stucture (see Theorem 3.2.6). First, we show that a vector field

[9]Here, $w^{\tilde{\sharp}} \in \mathfrak{X}(\tilde{M})$ denotes the fundamental vector field on $\tilde{M}$ induced by $w \in \mathfrak{h}$.

on a conical pseudo-hyper-Kähler manifold preserving the conical hyper-Kähler structure defines a Killing vector field on such a quaternionic Kähler submanifold and we relate the homogeneous hyper-Kähler moment map to the quaternionic Kähler moment map.

Proposition 3.5.1 *Let $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ be a conical pseudo-hyper-Kähler manifold and let $\hat{X} \in \mathfrak{X}(\widehat{M})$ be a Killing vector field such that $[\hat{X}, \xi] = 0$. Let M' be a codimension four submanifold that is transversal to the distribution $\mathcal{D}^v := \mathrm{span}\{\xi, \hat{J}_1\xi, \hat{J}_2\xi, \hat{J}_3\xi\} \subset T\widehat{M}$ and denote the projection to TM' along $\mathcal{D}^v$ by $\mathrm{pr}^{\mathcal{D}^v}_{TM'} : T\widehat{M}\big|_{M'} \to TM'$. Then*

$$X := \mathrm{pr}^{\mathcal{D}^v}_{TM'} \circ \hat{X}\big|_{M'} \in \mathfrak{X}(M') \tag{3.77}$$

is a Killing vector field with respect to the quaternionic pseudo-Kähler metric g' on M' given in Theorem 3.2.6.

Let $Q = \mathrm{span}_{\mathbb{R}}\{J'_1, J'_2, J'_3\}$ be the quaternionic structure on (M', g') given in Theorem 3.2.6. Then the quaternionic Kähler moment map $\mu^X \in \Gamma(Q)$ associated with X is given by $\mu^X := \sum_{\alpha=1}^{3} \mu^X_\alpha J'_\alpha$,

$$\mu^X_\alpha := \frac{1}{r^2} \hat{\mu}^{\hat{X}}_\alpha \Big|_{M'} \in C^\infty(M'), \tag{3.78}$$

where $\hat{\mu}^{\hat{X}}_\alpha = -\frac{1}{2}\hat{g}(\hat{J}_\alpha\xi, \hat{X}) \in C^\infty(\widehat{M})$ are the components of the homogeneous hyper-Kähler moment map associated with $\hat{X}$ given in Proposition 3.3.1.

***Proof*:** The horizontal part $\breve{g}$ of the conical pseudo-hyper-Kähler metric is invariant under ξ and $\hat{J}_\alpha\xi$ and has kernel $\mathcal{D}^v$. Since $\hat{X}$ preserves $\breve{g}$ and commutes with ξ, $\hat{J}_\alpha\xi$, $X = \mathrm{pr}^{\mathcal{D}^v}_{TM'} \circ \hat{X}\big|_{M'}$ preserves $g' = \breve{g}\big|_{M'}$.

Recall that the components of the local $Sp(1)$-connection one-form with respect to (J'_1, J'_2, J'_3) are given by $\bar{\theta}_\alpha := \theta_\alpha\big|_{TM'} \in \Omega_1(M')$ (see the proof of Theorem 3.2.6), where $\theta_\alpha = \frac{1}{\hat{g}(\xi,\xi)}\hat{g}(\hat{J}_\alpha\xi, \cdot)$ (see Eq. (3.12)). Recall that with $r^2 = |\hat{g}(\xi, \xi)|$ and $\sigma = \mathrm{sgn}\ \hat{g}(\xi, \xi)$, the Kähler forms on $\widehat{M}$ are given by

$$\hat{\omega}_\alpha = \sigma\, r dr \wedge \theta_\alpha + r^2(\sigma\, \theta_\beta \wedge \theta_\gamma + \breve{\omega}_\alpha), \tag{3.79}$$

where $\breve{\omega}_\alpha = \frac{\sigma}{2}(d\theta_\alpha - 2\theta_\beta \wedge \theta_\gamma)$ (see Lemma 3.2.4). Since $0 = \frac{\sigma}{2}\mathcal{L}_{\hat{X}}(\hat{g}(\xi,\xi)) = rdr(\hat{X})$, this shows

$$\begin{aligned} d\hat{\mu}_\alpha^{\hat{X}} &= \hat{\omega}_\alpha(\hat{X},\cdot) = -\sigma\,\theta_\alpha(\hat{X})\,rdr + \sigma\,r^2\,\theta_\beta(\hat{X})\,\theta_\gamma - \sigma\,r^2\,\theta_\gamma(\hat{X})\,\theta_\beta + r^2\,\breve{\omega}_\alpha(\hat{X},\cdot) \\ &= \frac{2}{r}\hat{\mu}_\alpha^{\hat{X}} dr - 2\,\hat{\mu}_\beta^{\hat{X}}\theta_\gamma + 2\,\hat{\mu}_\gamma^{\hat{X}}\theta_\beta + r^2\,\breve{\omega}_\alpha(\hat{X},\cdot). \end{aligned} \tag{3.80}$$

Since $\breve{\omega}_\alpha$ has kernel $\mathcal{D}^v$ and the fundamental two-forms on M' are given by $\omega'_\alpha = \breve{\omega}\big|_{M'}$ (see Remark 3.2.7), we have

$$d\mu_\alpha^X = \frac{1}{r^2}\left(-\frac{2}{r}\hat{\mu}_\alpha^{\hat{X}} dr + d\hat{\mu}_\alpha^{\hat{X}}\right)\Big|_{M'} \overset{(3.80)}{=} -2\mu_\beta^X\bar{\theta}_\gamma + 2\mu_\gamma^X\bar{\theta}_\beta + \omega'_\alpha(X,\cdot). \tag{3.81}$$

This shows that $\mu^X = \sum_{\alpha=1}^3 \mu_\alpha^X J'_\alpha$ is the quaternionic Kähler moment map associated with X (see Remark 2.2.4). □

Now, we show that for higher-dimensional Lie group actions on conical pseudo-hyper-Kähler manifolds, the homogeneous hyper-Kähler moment is automatically equivariant.

Proposition 3.5.2 *Let $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ be a conical pseudo-hyper-Kähler manifold and let G be a connected Lie group that acts on $\widehat{M}$ such that the action preserves $(\hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$. Let $\hat{\mu} : \widehat{M} \to \mathfrak{g}^* \otimes \mathbb{R}^3$ be defined by*

$$\hat{\mu}_\alpha^v = \langle \hat{\mu}_\alpha, v\rangle = -\frac{1}{2}\hat{g}(\hat{J}_\alpha\xi, v^\sharp) \quad (v \in \mathfrak{g},\ \alpha = 1,2,3).$$

Then $\hat{\mu}$ is G-equivariant and hence a hyper-Kähler moment map with respect to the G-action.

***Proof*:** Let X, Y be fundamental vector fields induced by some vectors in $\mathfrak{g}$. Since Y preserves $(\hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$, we have

$$d\hat{\mu}_\alpha^X(Y) = \mathcal{L}_Y\big(-\frac{1}{2}\hat{g}(\hat{J}_\alpha\xi, X)\big) = -\frac{1}{2}\hat{g}(\hat{J}_\alpha\xi, \mathcal{L}_Y X) = \hat{\mu}_\alpha^{[Y,X]}.$$

This shows that $\hat{\mu}$ is G-equivariant. By Proposition 3.3.1, $\hat{\mu}$ fulfills $d\hat{\mu}_\alpha^X = \hat{\omega}_\alpha(X,\cdot)$. □

Definition 3.5.3 *We call the map $\hat{\mu} : \widehat{M} \to \mathfrak{g} \otimes \mathbb{R}^3$ given by*

$$\hat{\mu}_\alpha^v = \langle \hat{\mu}_\alpha, v\rangle = -\frac{1}{2}\hat{g}(\hat{J}_\alpha\xi, v^\sharp) \quad (v \in \mathfrak{g},\ \alpha = 1,2,3)$$

*the **homogeneous hyper-Kähler moment map** associated with G.*

We now prove the compatibility of the construction in Theorem 3.2.6 with the (level zero) hyper-Kähler and quaternionic Kähler quotient constructions. The analogous statement for the Swann bundle over a quaternionic Kähler manifold was proven in [Sw1]. For a better orientation, we include Figure 3.3, which shows a digram of the manifolds involved and a list of geometric data on the respective manifolds. These are further explained in the proof of the theorem.

Theorem 3.5.4 *Let $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ be a conical pseudo-hyper-Kähler manifold and let G be a compact connected Lie group that acts freely on $\widehat{M}$ such that the action preserves $(\hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ and such that the restriction of $\hat{g}$ to the distribution tangent to the G-orbits is non-degenerate. Let $\hat{\mu} : \widehat{M} \to \mathfrak{g}^* \otimes \mathbb{R}^3$ be the homogeneous hyper-Kähler moment map associated with the G-action.*

Then ξ induces a vector field $\underline{\xi}$ on $\underline{\widehat{M}} = \widehat{M}_0/G = \hat{\mu}^{-1}(\{0\})/G$ such that the hyper-Kähler quotient

$$\widehat{M}/\!\!/\!\!/_{\{\hat{\mu}=0\}}G = (\underline{\widehat{M}}, \underline{\hat{g}}, \underline{\hat{J}_1}, \underline{\hat{J}_2}, \underline{\hat{J}_3})$$

together with $\underline{\xi}$ is a conical pseudo-hyper-Kähler manifold.

Let $M' \subset \widehat{M}$ be a G-invariant codimension four submanifold transversal to the distribution $\mathcal{D}^v = \mathrm{span}_{\mathbb{R}}\{\xi, \hat{J}_1\xi, \hat{J}_2\xi, \hat{J}_3\xi\} \subset T\widehat{M}$ and let (g', Q) denote the induced quaternionic pseudo-Kähler structure on M' (see Theorem 3.2.6). Then G acts isometrically and freely on (M', g').

Consider the quaternionic Kähler quotient $M'/\!\!/\!\!/G = (\underline{M'}, \underline{g'}, \underline{Q})$. $\underline{M'}$ can be canonically identified with a submanifold in $\underline{\widehat{M}}$ that is transversal to the distribution $\underline{\mathcal{D}}^v = \mathrm{span}_{\mathbb{R}}\{\underline{\xi}, \underline{\hat{J}_1}, \underline{\hat{J}_2}, \underline{\hat{J}_3}\} \subset T\underline{\widehat{M}}$ and the quaternionic pseudo-Kähler structure induced from $\underline{\widehat{M}}$ (via Theorem 3.2.6) is identical to $(\underline{g'}, \underline{Q})$.

***Proof*:** Since

$$d\hat{\mu}^X_\alpha(\xi) = \hat{\omega}_\alpha(X, \xi) = -\hat{g}(\hat{J}_\alpha\xi, X) = 2\hat{\mu}^X_\alpha$$

vanishes on $\widehat{M}_0$ for any fundamental vector field $X \in \mathfrak{X}(\widehat{M})$, ξ is tangent to $\widehat{M}_0$. Let $\widehat{\nabla}$ and ∇^0 denote the Levi-Civita connections of $(\widehat{M}, \hat{g})$ and of the pseudo-Riemannian submanifold $(\widehat{M}_0, \hat{g}|_{\widehat{M}_0})$, respectively. Since $\widehat{\nabla}_\cdot\xi = \mathrm{Id}_{T\widehat{M}}$, we have for $q \in \widehat{M}_0$ and $v \in T_q\widehat{M}_0 \subset T_q\widehat{M}$:

$$\nabla^0_v(\xi|_{\widehat{M}_0}) = \mathrm{pr}^\perp_{T_q\widehat{M}_0}(\widehat{\nabla}_v\xi) = \mathrm{pr}^\perp_{T_q\widehat{M}_0}\, v = v. \tag{3.82}$$

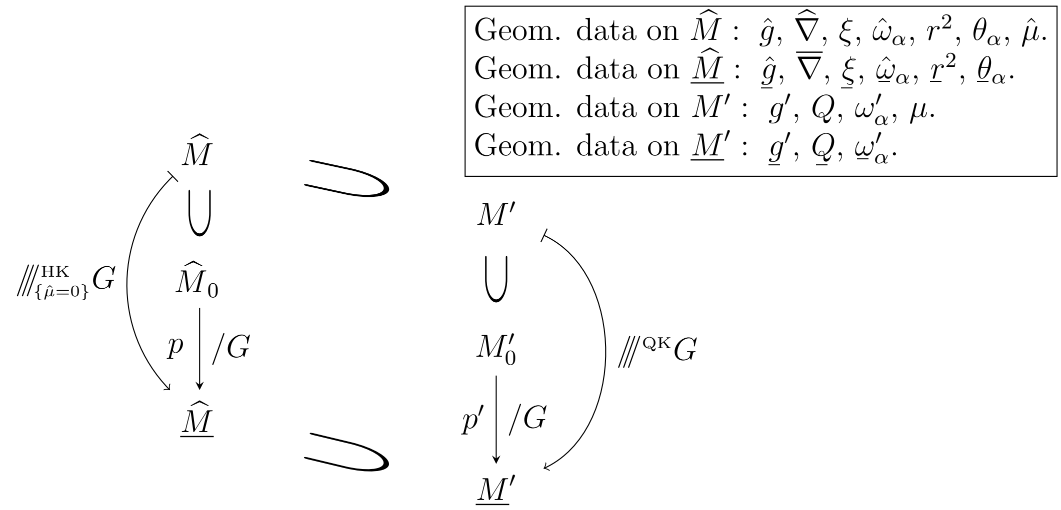


Figure 3.3: Illustration and list of geometric data for the proof of Theorem 3.5.4.

Here, $\mathrm{pr}^\perp_{T_q\widehat{M}_0} : T_q\widehat{M} = T_q\widehat{M}_0 \oplus (T_q\widehat{M}_0)^\perp \to T_q\widehat{M}_0$ denotes the orthogonal projection of $T\widehat{M}|_{\widehat{M}_0}$ to $T\widehat{M}_0$ with respect to $\hat{g}$. (See [O, Ch. 4] for the relation between the Levi-Civita connection on a pseudo-Riemannian manifold and the Levi-Civita connection on a pseudo-Riemannian submanifold.)

Let $p : \widehat{M}_0 \to \underline{\widehat{M}} = \widehat{M}_0/G$ denote the standard projection. Since $\xi|_{\widehat{M}_0} \in \mathfrak{X}(\widehat{M}_0)$ is preserved by the G-action, ξ induces a vector field $\underline{\xi} \in \mathfrak{X}(\underline{\widehat{M}})$ on $\underline{\widehat{M}} = \widehat{M}_0/G$. $\xi|_{\widehat{M}_0}$ is horizontal with respect to the decomposition

$$T\widehat{M}_0 = T^v\widehat{M}_0 \oplus^\perp T^h\widehat{M}_0, \quad T^v\widehat{M}_0 := \ker dp,$$

or in other words, $\xi|_{\widehat{M}_0}$ is orthogonal to the distribution tangent to the G-orbits. Hence, the horizontal lift $\tilde{\underline{\xi}} \in \Gamma(T^h\widehat{M}_0)$ of $\underline{\xi} \in \mathfrak{X}(\underline{\widehat{M}})$ is equal to $\xi|_{\widehat{M}_0}$. Note that $p : (\widehat{M}_0, \hat{g}|_{\widehat{M}_0}) \to (\underline{\widehat{M}}, \underline{\hat{g}})$ is a pseudo-Riemannian submersion. If $\overline{\nabla}$ denotes the Levi-Civita connection of $(\underline{\widehat{M}}, \underline{\hat{g}})$, we have

$$\widetilde{\overline{\nabla}_Y \underline{\xi}} = \mathrm{pr}_{T^h\widehat{M}_0}(\nabla^0_{\tilde{Y}}\tilde{\underline{\xi}}) = \mathrm{pr}_{T^h\widehat{M}_0}(\nabla^0_{\tilde{Y}}(\xi|_{\widehat{M}_0})) \overset{(3.82)}{=} \mathrm{pr}_{T^h\widehat{M}_0}(\tilde{Y}) = \tilde{Y},$$

for $Y \in \mathfrak{X}(\underline{\widehat{M}})$ and hence $\overline{\nabla}_\cdot \underline{\xi} = \mathrm{Id}_{T\underline{\widehat{M}}}$. This shows that $(\underline{\widehat{M}}, \underline{\hat{g}}, \underline{\hat{J}}_1, \underline{\hat{J}}_2, \underline{\hat{J}}_3, \underline{\xi})$ is a conical pseudo-hyper-Kähler manifold. (See, e.g., [FIP, Ch. 1 & Ch. 7] for the relation between the Levi-Civita connections of two pseudo-Riemannian manifolds that are related by a pseudo-Riemannian submersion.)

Since the horizontal lift of $\underline{\xi}$ is given by $\tilde{\underline{\xi}} = \xi|_{\widehat{M}_0}$, we have $\operatorname{sgn}\underline{\hat{g}}(\underline{\xi}, \underline{\xi}) = \operatorname{sgn}\hat{g}(\xi, \xi)|_{\widehat{M}_0} = \sigma$ and the radial function $\underline{r} = \sqrt{|\underline{\hat{g}}(\underline{\xi}, \underline{\xi})|}$ on $\underline{\widehat{M}}$

is related to $r = \sqrt{|\hat{g}(\xi,\xi|} \in C^\infty(\widehat{M})$ by $p^*\underline{r} = r|_{\widehat{M}_0}$. Recall that the Kähler forms $\underline{\hat{\omega}}_1, \underline{\hat{\omega}}_2, \underline{\hat{\omega}}_3$ on the hyper-Kähler quotient $\underline{\widehat{M}}$ are defined by

$$p^*\underline{\hat{\omega}}_\alpha = \hat{\omega}_\alpha|_{\widehat{M}_0} \quad (\alpha = 1, 2, 3). \tag{3.83}$$

Thus, $\underline{\xi} = dp(\xi|_{\widehat{M}_0})$ implies

$$p^*(\underline{\hat{\omega}}_\alpha(\underline{\xi}, \cdot)) = (p^*\underline{\hat{\omega}}_\alpha)(\xi|_{\widehat{M}_0}, \cdot) = \hat{\omega}_\alpha\big|_{\widehat{M}_0}(\xi|_{\widehat{M}_0}, \cdot).$$

In total, this shows that for $\theta_\alpha = \frac{\sigma}{r^2}\hat{\omega}_\alpha(\xi, \cdot) \in \Omega^1(\widehat{M})$ and $\underline{\theta}_\alpha = \frac{\sigma}{\underline{r}^2}\underline{\hat{\omega}}_\alpha(\underline{\xi}, \cdot) \in \Omega^1(\underline{\widehat{M}})$, we have

$$p^*\underline{\theta}_\alpha = \theta_\alpha\big|_{\widehat{M}_0}. \tag{3.84}$$

Recall that the quaternionic stucture on M' induced from $\widehat{M}$ is given by $Q = \mathrm{span}_\mathbb{R}\{J_1', J_2', J_3'\}$, where J_1', J_2', J_3' are almost complex structures on M' as defined in Theorem 3.2.6. The corresponding fundamental two-forms are given by

$$\omega_\alpha' = \breve{\omega}_\alpha\big|_{M'} = \frac{\sigma}{2}(d\theta_\alpha - 2\theta_\beta \wedge \theta_\gamma)\big|_{M'}. \tag{3.85}$$

According to Proposition 3.5.1, the quaternionic Kähler moment map associated with the G-action on M' is given by $\mu := \sum_{\alpha=1}^3 (\frac{1}{r^2}\hat{\mu}_\alpha)\big|_{M'} J_\alpha' \in \Gamma(Q)$. Since the almost complex structures J_α' are linearly independent, $M_0' = \mu^{-1}(\{0\}) \subset M'$ is a submanifold of $\widehat{M}_0 = \hat{\mu}^{-1}(\{0\}) \subset \widehat{M}$. It is of codimension four and transversal to $\mathcal{D}^v|_{\widehat{M}_0} \subset T\widehat{M}_0$. Hence, $\underline{M}'$ is a codimension four submanifold in $\underline{\widehat{M}}$ that is transversal to $\underline{\mathcal{D}}^v$. Let $p' : \widehat{M}_0' \to \underline{M}' = M_0'/G$ denote the standard projection and let

$$\iota_{\underline{M}'} : \underline{M}' \to \underline{\widehat{M}}, \quad \iota_{M_0'} : M_0' \to \widehat{M}_0.$$

The quaternionic structure $\underline{Q}$ on $\underline{\mathrm{M}}'$ induced by the quaternionic Kähler quotient is spanned by three almost complex structures $\underline{J}_1, \underline{J}_2, \underline{J}_3$. The corresponding fundamental two-forms are defined by (see Remark 2.2.8)

$$p'^*\underline{\omega}_\alpha' = \omega_\alpha'\big|_{M_0'} \overset{(3.85)}{=} \frac{\sigma}{2}(d\theta_\alpha - 2\theta_\beta \wedge \theta_\gamma)\big|_{M_0'} = \iota_{M_0'}^*\Big(\frac{\sigma}{2}(d\theta_\alpha - 2\theta_\beta \wedge \theta_\gamma)\big|_{\widehat{M}_0}\Big). \tag{3.86}$$

The almost pseudo-hyper-Hermitian structure on $\underline{M}'$ induced from $\underline{\widehat{M}}$ via Theorem 3.2.6 has fundamental two forms defined by

$$\underline{\breve{\omega}}_\alpha\big|_{\underline{M}'} = \frac{\sigma}{2}\iota_{\underline{M}'}^*(d\underline{\theta}_\alpha - 2\underline{\theta}_\beta \wedge \underline{\theta}_\gamma). \tag{3.87}$$

Their pullback to M'_0 via p' agrees with $p'^*\underline{\omega}'_\alpha$ (see Eq. (3.86)), since $\iota_{\underline{M}'}\circ p' = p\circ\iota_{M'_0}$ implies

$$p'^*\iota^*_{\underline{M}'}\underline{\theta}_\alpha = \iota^*_{M'_0}p^*\underline{\theta}_\alpha \overset{(3.84)}{=} \iota^*_{M'_0}\big(\theta_\alpha\big|_{\widehat{M}_0}\big).$$

This shows that the quaternionic pseudo-Kähler structures on $\underline{M}'$ induced from M' via the quaternionic Kähler quotient and from $\underline{\widehat{M}}$ via Theorem 3.2.6 are identical. □

3.6 The Swann bundle

In this section, we first review the Swann bundle construction from [Sw1] in the form presented in the author's collaboration with D.V. Alekseevsky, V. Cortés and T. Mohaupt [ACDM] and show that, up to rescaling of the metric, it is (locally) inverse to the construction of quaternionic pseudo-Kähler manifolds as submanifolds of conical pseudo-hyper-Kähler manifolds in Theorem 3.2.6. The presentation does not use the formalism of reduced frame bundles from [Sw1]. Instead the Swann bundle is directly constructed as the Riemannian cone over an $SO(3)$-bundle over a quaternionic (pseudo-)Kähler manifold, which is closer to the treatment of this topic in the physics literature (see, e.g., [DRV1, DRV2]).

In the first subsection, we show that Killing vector fields on a quaternionic (pseudo-)Kähler manifold can be uniquely lifted to tri-holomorphic Killing vector fields on the Swann bundle that commute with the Euler vector field. In the second subsection, we discuss canonical lifts of isometric group actions from quaternionic (pseudo-)Kähler manifolds to the Swann bundle.

Let (M, g, Q) be a connected quaternionic (pseudo-)Kähler manifold. Let

$$\pi : S \to M$$

denote the principal $SO(3)$-bundle of frames (J_1, J_2, J_3) in Q such that $J_1J_2 = J_3$ and $J_\alpha^2 = -\operatorname{Id}_{TM}$, $\alpha = 1, 2, 3$. The principal action of an element $A \in SO(3)$ is given by

$$\tau(A, \cdot) : S \to S, \quad s = (J_1, J_2, J_3) \mapsto \tau(A, s) := R_{A^{-1}}s := (J_1, J_2, J_3)A^{-1},$$

i.e. we consider S as a left-principal bundle. We choose the basis (e_α) of $\mathfrak{so}(3)$ given in Eq. (2.2). It corresponds to the standard basis of $\mathfrak{sp}(1) = \operatorname{Im}\mathbb{H} \cong \mathbb{R}^3$ under the canonical isomorphism $\mathfrak{sp}(1) \cong ad(\mathfrak{sp}(1)) = \mathfrak{so}(3)$. Let us denote by

$Z_\alpha \in \mathfrak{X}(M)$ the fundamental vector fields associated with (e_α):

$$Z_\alpha\Big|_s = \frac{d}{dt}\Big|_{t=0} \tau(\exp(te_\alpha), s), \quad s \in S.$$

Then

$$[e_\alpha, e_\beta] = 2e_\gamma, \quad [Z_\alpha, Z_\beta] = -2Z_\gamma \tag{3.88}$$

for every cyclic permutation (α, β, γ) of $(1,2,3)$. In the following, (α, β, γ) will always be a cyclic permutation, whenever the three letters appear in an expression.

Let $\boldsymbol{\sigma} = (J_1, J_2, J_3) \in \Gamma(U, S)$ be a local section defined over some open subset $U \subset M$ and let $\bar{\theta} = \sum_{\alpha=1}^{3} \bar{\theta}_\alpha e_\alpha$ be the local $Sp(1)$-connection one-form with respect to (J_1, J_2, J_3), i.e. the local fundamental two-forms $\omega_\alpha = g(J_\alpha, \cdot)$ fulfill

$$d\omega_\alpha = 2(\bar{\theta}_\beta \wedge \omega_\gamma - \bar{\theta}_\gamma \wedge \omega_\beta).$$

In the local trivialization $\pi^{-1}(U) \cong U \times SO(3)$ of S given by $\boldsymbol{\sigma}$, we can define an $\mathfrak{so}(3)$-valued one-form on $\pi^{-1}(U)$ by

$$\theta^{(U)} := \pi^* \bar{\theta} + \varphi,$$

where $\varphi = \sum \varphi_\alpha e_\alpha \in \Omega^1(SO(3), \mathfrak{so}(3))$ is the Maurer-Cartan form on $SO(3)$. Since $\theta^{(U)}$ is independent of the choice of section $\boldsymbol{\sigma}$, it defines an $\mathfrak{so}(3)$-valued one-form

$$\theta =: \sum_{\alpha=1}^{3} \theta_\alpha e_\alpha \in \Omega^1(S, \mathfrak{so}(3)) \tag{3.89}$$

on S. The one-form θ is in fact the connection one-form of the principal connection on S induced by the Levi-Civita connection ∇ of (M,g) (see Eq. (2.1)). Its curvature is defined by

$$\Omega := d\theta - \frac{1}{2}[\theta \wedge \theta],$$

where

$$\frac{1}{2}[\theta \wedge \theta](X,Y) := [\theta(X), \theta(Y)], \quad X, Y \in T_s S, \quad s \in S.$$

Writing $\Omega = \sum \Omega_\alpha e_\alpha$ and using Eq. (3.88), we have

$$\Omega_\alpha = d\theta_\alpha - 2\theta_\beta \wedge \theta_\gamma. \tag{3.90}$$

From the above equation and $\theta_\alpha(Z_{\alpha'}) = \delta_{\alpha\alpha'}$, we immediately get the following lemma:

Lemma 3.6.1

$$\mathcal{L}_{Z_\alpha}\theta_\alpha = 0, \quad \mathcal{L}_{Z_\alpha}\theta_\beta = -2\theta_\gamma, \quad \mathcal{L}_{Z_\alpha}\theta_\gamma = 2\theta_\beta;$$

$$\mathcal{L}_{Z_\alpha}\Omega_\alpha = 0, \quad \mathcal{L}_{Z_\alpha}\Omega_\beta = -2\Omega_\gamma, \quad \mathcal{L}_{Z_\alpha}\Omega_\gamma = 2\Omega_\beta.$$

For any local section $\boldsymbol{\sigma} = (J_1, J_2, J_3) \in \Gamma(U, S)$ over some open subset $U \subset M$, the one-forms $\boldsymbol{\sigma}^*\theta_\alpha = \bar{\theta}_\alpha$ are the components of the local $Sp(1)$-connection one-form and from (2.7), we get

$$\boldsymbol{\sigma}^*\Omega_\alpha = \frac{\nu}{2}\omega_\alpha, \tag{3.91}$$

where $\omega_1, \omega_2, \omega_3$ are the local fundamental two-forms with respect to (J_1, J_2, J_3).

We endow the manifold S with the pseudo-Riemannian metric

$$g_S = \sigma \sum_{\alpha=1}^{3} (\theta_\alpha)^2 + \frac{|\nu|}{4}\pi^* g, \tag{3.92}$$

where

$$\nu := \frac{scal}{4n(n+2)} \quad (\dim_{\mathbb{R}} M = 4n) \tag{3.93}$$

is the reduced scalar curvature of (M, g) and $\sigma := \operatorname{sgn} \nu$ is its sign.

Now, we consider the cone $\widehat{M} = \mathbb{R}^{>0} \times S$ over S with the radial coordinate $r \in \mathbb{R}^{>0}$, the **Euler vector field**

$$\xi := Z_0 := r\frac{\partial}{\partial r} \tag{3.94}$$

and the following exact two-forms:

$$\hat{\omega}_\alpha := \sigma d\hat{\theta}_\alpha \in \Omega^2(\widehat{M}), \quad \hat{\theta}_\alpha = \frac{r^2}{2}\theta_\alpha \in \Omega^1(\widehat{M}). \tag{3.95}$$

From now on, Z_α, $\alpha = 1, 2, 3$, both denotes the fundamental vector field on S, as well as its canonical extension to $\widehat{M} = \mathbb{R}^{>0} \times S$.

Using the above data, one recovers Swann's hyper-Kähler structure on $\widehat{M}$:

Theorem 3.6.2 [Sw1, ACDM]

Let (M, g, Q) be a connected (pseudo-)quaternionic Kähler manifold. Let $\widehat{M} = \mathbb{R}^{>0} \times S$ with radial coordinate $r \in \mathbb{R}^{>0}$, g_S and $\hat{\omega}_\alpha$ be defined as above. Then the cone metric

$$\hat{g} = \sigma dr^2 + r^2 g_S \tag{3.96}$$

is a pseudo-hyper-Kähler metric on $\widehat{M}$ *with Kähler forms* $\hat{\omega}_\alpha$. *Together with the Euler vector field* $\xi = r\frac{\partial}{\partial r}$, $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3)$ *is a conical pseudo-hyper-Kähler manifold. The signature of* $\hat{g}$ *is* $(4k+4, 4\ell)$ *if* $\nu > 0$ *and* $(4k, 4\ell+4)$ *if* $\nu < 0$, *where* $(4k, 4\ell)$ *is the signature of the quaternionic pseudo-Kähler metric* g *on* M.

Remark 3.6.3 The corresponding complex structures $\hat{J}_1, \hat{J}_2, \hat{J}_3$ on $\widehat{M}$ preserve the distribution $\mathcal{D}^v := \mathrm{span}_{\mathbb{R}}\{Z_a \mid a = 0, \ldots, 3\} \subset T\widehat{M}$ as well as its orthogonal complement and satisfy

$$\hat{J}_\alpha Z_0 = Z_\alpha, \quad \hat{J}_\alpha Z_\alpha = -Z_0, \quad \hat{J}_\alpha Z_\beta = Z_\gamma, \quad \hat{J}_\alpha Z_\gamma = -Z_\beta, \quad \hat{\pi}_* \circ \hat{J}_\alpha|_{(r,s)} = J_\alpha \circ \hat{\pi}_*,$$

where $r \in \mathbb{R}^{>0}$, $s = (J_1, J_2, J_3) \in S$ and $\hat{\pi} := \pi \circ \mathrm{pr}_2 : \widehat{M} = \mathbb{R}^{>0} \times S \to M$.

Definition 3.6.4 *For any quaternionic (pseudo-)Kähler manifold* (M, g, Q), *the conical pseudo-hyper-Kähler manifold* $(\hat{\pi} : \widehat{M} \to M, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ *is called the* ***Swann bundle*** *over* M.

If $M' \subset \widehat{M}$ is a codimension four submanifold that is transversal to the distribution $\mathcal{D}^v = \mathrm{span}\{\xi, \hat{J}_1\xi, \hat{J}_2\xi, \hat{J}_3\xi\} \subset T\widehat{M}$, then there is a neighborhood around every point in M' that intersects each $(\mathbb{R}^{>0} \times SO(3))$-orbit of $\widehat{M}$ at most once. So locally, M' defines a section of $\widehat{M}$ and fulfills the assumptions of the following proposition:

Proposition 3.6.5 *Let* (M, g, Q) *be a quaternionic (pseudo-)Kähler manifold and let* $(\hat{\pi} : \widehat{M} \to M, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ *be the Swann bundle over* M. *Let* $\hat{\boldsymbol{\sigma}} : U \to \widehat{M}$ *be a local section over some open subset* $U \subset M$. *Define* $M' := \hat{\boldsymbol{\sigma}}(U) \subset \widehat{M}$ *and*

$$\boldsymbol{\sigma} := \mathrm{pr}_2 \circ \hat{\boldsymbol{\sigma}} =: (J_1, J_2, J_3) \in \Gamma(U, S).$$

If $(g', Q = \mathrm{span}_{\mathbb{R}}\{J_1', J_2', J_3'\})$ *denotes the quaternionic (pseudo-)Kähler structure on* M' *obtained from Theorem 3.2.6, then* $\hat{\boldsymbol{\sigma}}$ *is an isomorphism between* $(U, \frac{|\nu|}{4} g\big|_U, J_1, J_2, J_3)$ *and* $(M', g', J_1', J_2', J_3')$.

***Proof*:** Note that the geometric data $r^2 = |\hat{g}(\xi, \xi)|$, $\sigma = \mathrm{sgn}\, \hat{g}(\xi, \xi)$, $\theta_\alpha = \frac{\sigma}{r^2}\hat{g}(\hat{J}_\alpha \xi, \cdot)$, $\hat{\theta}_\alpha = \frac{r^2}{2}\theta_\alpha$ defined on conical (pseudo-)hyper-Kähler manifolds in (3.12) agrees with the geometric data on the Swann bundle defined in this section. Recall that the fundamental two-forms on M' associated with J'_1, J'_2, J'_3 are given by (see Remark 3.2.7 and (3.23))

$$\omega'_\alpha = \frac{\sigma}{2}(d\theta_\alpha - 2\theta_\beta \wedge \theta_\gamma)\big|_{M'}.$$

The local fundamental two-forms on $U \subset M$ associated with $\boldsymbol{\sigma} = (J_1, J_2, J_3)$ fulfill

$$\frac{|\nu|}{4}\omega_\alpha \overset{(2.7)}{=} \frac{\sigma}{2}(d\bar{\theta}_\alpha - 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma),$$

where $\bar{\theta}_\alpha = \boldsymbol{\sigma}^*\theta_\alpha = \hat{\boldsymbol{\sigma}}^* \operatorname{pr}_2^* \theta_\alpha = \hat{\boldsymbol{\sigma}}^*\theta_\alpha$. (Note that in this section, we often extended geometric data from S to $\widehat{M} = \mathbb{R}^{>0} \times S$ without explicitly pointing this out, e.g., by writing $(\operatorname{pr}_2)^*$.) This shows that $\hat{\boldsymbol{\sigma}}^*\omega'_\alpha = \frac{|\nu|}{4}\omega_\alpha$. □

3.6.1 Lifts of Killing vector fields to the Swann bundle

This section was written in collaboration with Lana Casselmann.

For the proof of the next proposition, we use the following definition:

Definition 3.6.6 *The differential operator $D_\theta : \Omega^k(S) \to \Omega^{k+1}(S)$ defined by*

$$(D_\theta\eta)(Y_1, \dots, Y_{k+1}) := d\eta((Y_1)^h, \dots, (Y_{k+1})^h), \tag{3.97}$$

where $\eta \in \Omega^k(S)$ is a differential k-form on S and $(Y_i)^h \in \Gamma(\ker\theta)$ denotes the horizontal part of $Y_i \in \mathfrak{X}(S)$ with respect to the connection θ, is called the ***absolute differential defined by $\boldsymbol{\theta}$***.

Proposition 3.6.7 *Let X be a Killing vector field on (M, g). Then there exists a unique lift $\hat{X} \in \mathfrak{X}(S)$ of X to S such that $\mathcal{L}_{\hat{X}}\theta_\alpha = 0$, $\alpha = 1, 2, 3$. It is given by $\hat{X} = \tilde{X} + \sum_{\alpha=1}^3 f_\alpha Z_\alpha$, where $\tilde{X} \in \Gamma(\ker\theta)$ denotes the horizontal lift and $f := \sum f_\alpha e_\alpha \in C^\infty(S, \mathfrak{so}(3))$ is $SO(3)$-equivariant and fulfills $\boldsymbol{\sigma}^* f_\alpha = -\frac{\nu}{2}\mu_\alpha^X$ for any local section $\boldsymbol{\sigma} = (J_1, J_2, J_3) : U \to S$, where $\sum_{\alpha=1}^3 \mu_\alpha^X J_\alpha$ is the restriction to U of the quaternionic Kähler moment map associated with X.*

***Proof*:** For any vector field $X \in \mathfrak{X}(M)$, an arbitrary lift $\hat{X}$ to S is given by

$$\hat{X} = \tilde{X} + \sum_{\alpha=1}^3 f_\alpha Z_\alpha, \tag{3.98}$$

where $\tilde{X} \in \Gamma(TS)$ is the unique horizontal lift ($d\pi(\tilde{X}) = X$, $\theta_\alpha(\tilde{X}) = 0$) and f_α, $\alpha = 1, 2, 3$, are arbitrary smooth functions on S.

Lemma 3.6.8

$$\mathcal{L}_{\hat{X}}\theta_\alpha = \iota_{\tilde{X}}\Omega_\alpha + df_\alpha + 2f_\beta\theta_\gamma - 2f_\gamma\theta_\beta$$

for any cyclic permutation (α, β, γ) *of* $(1, 2, 3)$.

***Proof*:** Using Cartan's formula and the fact that $\tilde{X}$ is horizontal, we get

$$\begin{aligned} \mathcal{L}_{\tilde{X}}\theta_\alpha &= (d \circ \iota_{\tilde{X}} + \iota_{\tilde{X}} \circ d)\theta_\alpha \\ &\overset{(3.90)}{=} \iota_{\tilde{X}}(\Omega_\alpha + 2\theta_\beta \wedge \theta_\gamma) \\ &= \iota_{\tilde{X}}\Omega_\alpha. \end{aligned}$$

For $\sum f_{\alpha'} Z_{\alpha'}$, we obtain

$$\begin{aligned} \sum_{\alpha'=1}^{3} \mathcal{L}_{f_{\alpha'}Z_{\alpha'}}\theta_\alpha &= \sum_{\alpha'=1}^{3} \theta_\alpha(Z_{\alpha'})df_{\alpha'} + f_{\alpha'}\mathcal{L}_{Z_{\alpha'}}\theta_\alpha \\ &= df_\alpha + 2f_\beta\theta_\gamma - 2f_\gamma\theta_\beta \end{aligned}$$

using Lemma 3.6.1 and $\theta_\alpha(Z_{\alpha'}) = \delta_{\alpha\alpha'}$. □

According to the above lemma, $\mathcal{L}_{\hat{X}}\theta_\alpha = 0$ is equivalent to

$$df_\alpha + 2f_\beta\theta_\gamma - 2f_\gamma\theta_\beta = -\iota_{\tilde{X}}\Omega_\alpha \tag{3.99}$$

for $\alpha = 1, 2, 3$. Following the idea of the proof of [GL, Theorem 2.4.], we show that this equation has a unique solution: Applying the exterior derivative d to Eq. (3.99) gives

$$2df_\beta \wedge \theta_\gamma + 2f_\beta d\theta_\gamma - 2df_\gamma \wedge \theta_\beta - 2f_\gamma d\theta_\beta = -d\iota_{\tilde{X}}\Omega_\alpha.$$

Using Eq. (3.99) to replace df_β, df_γ in the above equation and using the expression for the curvature Ω in (3.90) yields

$$2f_\beta\Omega_\gamma - 2f_\gamma\Omega_\beta = -d\iota_{\tilde{X}}\Omega_\alpha + 2\theta_\beta \wedge \iota_{\tilde{X}}\Omega_\gamma - 2\theta_\gamma \wedge \iota_{\tilde{X}}\Omega_\beta = -\mathcal{L}_{\tilde{X}}\Omega_\alpha.$$

For the last equality, we used Cartan's formula, the fact that $\tilde{X}$ is horizontal and Eq. (3.90). Hence, the components of $f = \sum f_\alpha e_\alpha \in \Omega^0(S, \mathfrak{so}(3))$ defined by

$$[f, \Omega] = -\mathcal{L}_{\tilde{X}}\Omega \tag{3.100}$$

give the unique solution to Eq. (3.99).

Let $\boldsymbol{\sigma} = (J_1, J_2, J_3) : U \to S$ be a section of S. Since Ω_α is horizontal, we have

$$\boldsymbol{\sigma}^*(\iota_{\tilde{X}}\Omega_\alpha) = \boldsymbol{\sigma}^*(\iota_{d\boldsymbol{\sigma}(X)}\Omega_\alpha) = \iota_X \boldsymbol{\sigma}^*\Omega_\alpha \overset{(3.91)}{=} \frac{\nu}{2}\iota_X\omega_\alpha,$$

where $\omega_\alpha = g(J_\alpha \cdot, \cdot)$. Using the fact that pullback and exterior derivative commute, one obtains the following equation for the functions $(\bar{f}_\alpha) := (\boldsymbol{\sigma}^* f_\alpha)$ on $U \subset M$ by pulling back Eq. (3.99):

$$d\bar{f}_\alpha + 2\bar{f}_\beta\bar{\theta}_\gamma - 2\bar{f}_\gamma\bar{\theta}_\beta = -\frac{\nu}{2}\iota_X\omega_\alpha. \tag{3.101}$$

Up to a factor of $-\frac{\nu}{2}$, the functions $\bar{f}_\alpha$ are the coefficients of the quaternionic Kähler moment map $\mu := -\frac{2}{\nu}\sum \bar{f}_\alpha J_\alpha$ associated with X (see (2.19)).

We now define a function $f^{(U)} := \sum f^{(U)}_\alpha e_\alpha \in \Omega^0(\pi^{-1}(U), \mathfrak{so}(3))$ such that $f^{(U)}\big|_{\boldsymbol{\sigma}(U)} = \pi^*\bar{f}$ and such that $R^*_{g^{-1}} f^{(U)} = Ad_g(f^{(U)})$:

$$\sum f^{(U)}_\alpha(\boldsymbol{\sigma}(p)g^{-1})e_\alpha := \sum \bar{f}_\alpha(p) Ad_g(e_\alpha) \quad (g \in SO(3),\, p \in U).$$

In terms of the basis (e_1, e_2, e_3) of $\mathfrak{so}(3)$, Ad_g has a simple expression:
If $g = (g_{\alpha\beta})_{\alpha,\beta=1,2,3} \in SO(3)$, then

$$f^{(U)}_\alpha(\boldsymbol{\sigma}(p)g^{-1}) = \sum_{\beta=1}^{3} g_{\alpha\beta}\bar{f}_\beta(p) \quad (p \in U).$$

The absolute differential D_θ of the $SO(3)$-equivariant function $f^{(U)}$,

$$D_\theta f^{(U)} = df^{(U)} + [f^{(U)}, \theta],$$

is equivariant. In components, $D_\theta f^{(U)}$ is given by

$$D_\theta f^{(U)}_\alpha = df^{(U)}_\alpha + 2f^{(U)}_\beta\theta_\gamma - 2f^{(U)}_\gamma\theta_\beta,$$

i.e. Eq. (3.99) corresponds to $D_\theta f^{(U)} = -\iota_{\tilde{X}}\Omega$. Both sides of this equation are $SO(3)$-equivariant and horizontal. To show that $D_\theta f^{(U)} = -\iota_{\tilde{X}}\Omega$, it thus suffices to check the equation $\boldsymbol{\sigma}^*(D_\theta f^{(U)}) = -\boldsymbol{\sigma}^*(\iota_{\tilde{X}}\Omega)$, which is fulfilled by construction of $f^{(U)}$. Since Eq. (3.99) has the unique solution f, we have $f^{(U)} = f|_{\pi^{-1}(U)}$. □

Corollary 3.6.9 *Let X be a Killing vector field on (M, g). Then there exists a unique tri-holomorphic Killing vector field $\hat{X}$ on $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3)$ that is a lift of X and that commutes with the Euler vector field ξ. It is given by the canonical extension to $\widehat{M} = \mathbb{R}^{>0} \times S$ of the vector field on S obtained from Proposition 3.6.7.*

***Proof*:**

Existence: Let $\hat{X}$ be the canonical extension to $\widehat{M}$ of the vector field obtained from Proposition 3.6.7. Then $\mathcal{L}_{\hat{X}}\xi = \mathcal{L}_{\hat{X}}r = \mathcal{L}_{\hat{X}}\theta_\alpha = 0$. Consequently, $\mathcal{L}_{\hat{X}}\hat{\theta}_\alpha = \mathcal{L}_{\hat{X}}(\frac{r^2}{2}\theta_\alpha) = 0$. This shows that $\hat{X}$ preserves the Kähler forms $\hat{\omega}_\alpha = \sigma d\hat{\theta}_\alpha$ and hence that $\hat{X}$ is tri-holomorphic and Killing.

Uniqueness: An arbitrary lift of X to $\widehat{M}$ is given by

$$\hat{X} = \tilde{X} + \sum_{a=0}^{3} f_a Z_a, \tag{3.102}$$

where $\tilde{X} \in \Gamma(\widehat{M})$ is the canonical extension to $\widehat{M}$ of the horizontal lift to S and f_a, $a = 0, \ldots, 3$, are arbitrary smooth functions on $\widehat{M}$. The equation

$$0 = \mathcal{L}_{\hat{X}}\xi = [\hat{X}, \xi] = [\sum_{a=0}^{3} f_a Z_a, \xi] = -\sum_{a=0}^{3} \xi(f_a) Z_a$$

implies $\xi(f_a) = 0$ for $a = 0, \ldots, 3$.

$$0 = \mathcal{L}_{\hat{X}}(\hat{g}(\xi, \xi)) = \sigma\mathcal{L}_{\hat{X}}(r^2) = 2\sigma f_0 r^2$$

implies $f_0 = 0$. Hence, $\hat{X}$ is the canonical extension to $\widehat{M}$ of a vector field on S that is a lift of X and preserves $\theta_\alpha = \frac{\sigma}{r^2}\hat{g}(\hat{J}_\alpha\xi, \cdot)$. The latter is unique by Proposition 3.6.7. □

Proposition 3.6.10 *Let (M, g, Q) be a quaternionic (pseudo-)Kähler manifold and let $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ be the Swann bundle over M, $\hat{\pi} : \widehat{M} \to M$. Let $X \in \mathfrak{X}(M)$ be a Killing vector field on M and let $\hat{X} \in \mathfrak{X}(\widehat{M})$ be the unique lift to $\widehat{M}$ given by the above corollary. Then*

$$\hat{g}(\hat{X}, \hat{X}) = r^2 \frac{|\nu|}{4} \hat{\pi}^*(g(X, X) + \nu\|\mu^X\|^2) \tag{3.103}$$

and

$$\frac{|\nu|}{4}\hat{\pi}^*(g(X,X)) = \frac{1}{r^2}\big(\hat{g}(\hat{X},\hat{X}) - \frac{4\sigma}{r^2}\|\hat{\mu}^{\hat{X}}\|^2\big) \tag{3.104}$$

where $\mu^X \in \Gamma(Q)$ *is the quaternionic Kähler moment map associated with* X, $\|\mu^X\|^2 = -\frac{1}{\dim_{\mathbb{R}} M}\operatorname{tr}(\mu^X)^2$, *and* $\hat{\mu}^{\hat{X}} \in C^\infty(\widehat{M}, \mathbb{R}^3)$ *is the homogeneous hyper-Kähler moment map associated with* $\hat{X}$, $\|\hat{\mu}^{\hat{X}}\|^2 = \sum_{\alpha=1}^3 (\hat{\mu}_\alpha^{\hat{X}})^2$.

***Proof*:** According to Corollary 3.6.9, $\hat{X}$ is the canonical extension of $\tilde{X} + \sum_{\alpha=1}^3 f_\alpha Z_\alpha \in \mathfrak{X}(S)$ to $\widehat{M} = \mathbb{R}^{>0} \times S$, where $\tilde{X} \in \Gamma(\ker\theta)$ is the horizontal lift of X and the functions $f_\alpha \in C^\infty(S)$ are given in Proposition 3.6.7. For every local section $\sigma = (J_1, J_2, J_3) \in \Gamma(U, S)$, the restriction to $\pi^{-1}(U)$ of the $\mathfrak{so}(3)$-valued function $f = \sum_{\alpha=1}^3 f_\alpha e_\alpha \in C^\infty(S, \mathfrak{so}(3))$ is given by the $SO(3)$-equivariant extension of (see Proposition 3.6.7)

$$f\big|_{\sigma(U)} = -\frac{\nu}{2}\sum_{\alpha=1}^3 (\pi^*\mu_\alpha^X)\big|_{\sigma(U)} e_\alpha,$$

where μ_α^X are the components of the quaternionic Kähler moment map: $\mu^X\big|_U = \sum \mu_\alpha^X J_\alpha$. Consider the natural scalar product $\langle v, w\rangle = -\frac{1}{8}\operatorname{tr} vw$ on $\mathfrak{so}(3)$. For $g \in SO(3)$, it is Ad_g-invariant and it fulfills $\langle e_\alpha, e_{\alpha'}\rangle = \delta_{\alpha\alpha'}$. Hence,

$$\begin{aligned}\sum_{\alpha=1}^3 (f_\alpha)^2\Big|_{\pi^{-1}(U)} = \langle f, f\rangle\Big|_{\pi^{-1}(U)} &= \frac{\nu^2}{4}\pi^*(<\sum_\alpha \mu_\alpha^X e_\alpha, \sum_{\alpha'} \mu_{\alpha'}^X e_{\alpha'}>) \\ &= \frac{\nu^2}{4}\pi^*(\sum_\alpha (\mu_\alpha^X)^2) = \frac{\nu^2}{4}\pi^*\big(\|\mu^X\|^2\big|_U\big) \end{aligned} \tag{3.105}$$

for any section σ over U. Using the fact that the above equation holds globally, we get

$$\begin{aligned}\hat{g}(\hat{X},\hat{X}) &\overset{(3.96)}{=} r^2 g_S(\tilde{X} + \sum_{\alpha=1}^3 f_\alpha Z_\alpha, \tilde{X} + \sum_{\alpha'=1}^3 f_{\alpha'} Z_{\alpha'}) \\ &\overset{(3.92)}{=} r^2\big(\sigma\sum_{\alpha=1}^3 (f_\alpha)^2 + \frac{|\nu|}{4}\pi^*(g(X,X))\big) \\ &= r^2\pi^*\big(\sigma\frac{\nu^2}{4}\|\mu^X\|^2 + \frac{|\nu|}{4}g(X,X)\big).\end{aligned}$$

This shows (3.103), since $\sigma = \operatorname{sgn}\nu$. Recall that the components of the homogeneous hyper-Kähler moment map are given by $\mu_\alpha^{\hat{X}} = -\frac{\sigma r^2}{2}\theta_\alpha(\hat{X})$ (see Proposition

3.3.1). Eq. (3.104) then follows from

$$\|\hat{\mu}^{\hat{X}}\|^2 = \frac{r^4}{4}\sum_{\alpha=1}^{3}(f_\alpha)^2 \overset{(3.105)}{=} \frac{\nu^2 r^4}{16}\hat{\pi}^*\|\mu^X\|^2.$$

□

Example 3.6.11 Consider the quaternionic Kähler manifolds[10]

$$\begin{aligned} M_+ &= \mathbb{H}P^n = (\mathbb{H}^{n+1}\backslash\{0\})/\mathbb{H}^*, \\ M_- &= \mathbb{H}H^n = \{q \in \mathbb{H}^{n,1} \mid \langle q, q\rangle < 0\}/\mathbb{H}^*, \end{aligned}$$

and their respective Swann bundles

$$\begin{aligned} \widehat{M}_+ &= (\mathbb{H}^{n+1}\backslash\{0\})/\mathbb{Z}_2, \\ \widehat{M}_- &= \{q \in \mathbb{H}^{n,1} \mid \langle q, q\rangle < 0\}/\mathbb{Z}_2, \end{aligned}$$

$\hat{\pi} : \widehat{M}_\pm \to M_\pm$.

The S^1-action

$$e^{it}\cdot [q]_{\mathbb{Z}_2} = [e^{\frac{i}{2}t}q]_{\mathbb{Z}_2}, \quad e^{it} \in S^1,\ [q]_{\mathbb{Z}_2} \in \widehat{M}_\pm,$$

on $\widehat{M}_\pm$ is well-defined. It is free, tri-holomorphic, isometric and commutes with the $\mathbb{R}^{>0}$-action generated by the Euler vector field $\xi \in \mathfrak{X}(\widehat{M}_\pm)$. Since it commutes with the $\mathbb{H}^*/\mathbb{Z}_2$-action, it induces a well-defined isometric S^1-action on $\mathbb{H}P^n$, respectively $\mathbb{H}H^n$. The induced S^1-action on the quaternionic Kähler manifold is not free. For instance, on

$$\hat{\pi}(\{[q = z + jw]_{\mathbb{Z}_2} \in \widehat{M}_\pm \mid w = 0\}) \subset M_\pm,$$

the induced S^1-action is trivial, since left- and right-multiplication of $e^{it} \in S^1$ on $z \in \mathbb{C}^{n+1}$ are identical. Proposition 3.6.10 shows, that the S^1-action is locally free on

$$\begin{aligned} U_\pm &= \hat{\pi}(\{\hat{g}(\hat{X},\hat{X}) - \frac{4\sigma}{r^2}\|\hat{\mu}^{\hat{X}}\|^2 \neq 0\}) \\ &= \{[q = z + jw]_{\mathbb{H}^*} \in M_\pm \mid \langle z, z\rangle\langle w, w\rangle - |\langle z, \bar{w}\rangle|^2 \neq 0\} \subset M_\pm, \end{aligned}$$

where $\hat{X} \in \mathfrak{X}(\widehat{M}_\pm)$ is the vector field generating the S^1-action on $\widehat{M}_\pm$. Note

[10]The $\mathbb{H}^*$-quotient is defined by multiplication from the right and $\widehat{M}_\pm$ are endowed with the hyper-Kähler structure described in Examples 3.1.7, 3.2.10.

that zero is a regular value of the quaternionic Kähler moment map on $M_\pm$ and the zero level set is contained in $U_\pm$. Hence, we can perform the quaternionic Kähler quotient $M_\pm /\!/\!/ S^1$.

3.6.2 Lifting isometric group actions to the Swann bundle

Proposition 3.6.12 *Let $(\bar{M}, \bar{g}, Q)$ be a quaternionic (pseudo-)Kähler manifold and let G be a compact, connected Lie group acting isometrically on $(\bar{M}, \bar{g})$ such that the action is free on the zero level set $\{\mu = 0\} \subset \bar{M}$ of the quaternionic Kähler moment map μ associated with G.*

Then the action canonically lifts to an isometric and tri-holomorphic G-action on the Swann bundle $\hat{\pi} : \widehat{M} \to \bar{M}$ that commutes with the Euler vector field ξ. The action is free on $\{\hat{\mu} = 0\} \subset \widehat{M}$, where $\hat{\mu}$ is the homogeneous hyper-Kähler moment map associated with G.

***Proof*:** Due to Proposition 2.2.1, G preserves Q. It also preserves the inner product $\langle \cdot, \cdot \rangle$ and the orientation on Q. Hence, it induces an action on the $SO(3)$-bundle S of oriented orthonormal frames in Q. This action preserves the $SO(3)$-connection one-form $\theta = \sum_{\alpha=1}^{3} \theta_\alpha e_\alpha$ on S, since θ is induced by the Levi-Civita connection of $(\bar{M}, \bar{g})$. The canonical extension of the G-action to $\widehat{M} = \mathbb{R}^{>0} \times S$ commutes with the Euler vector field $\xi = r\partial_r$. Since the Kähler forms on $\widehat{M}$ are given by $\hat{\omega}_\alpha = \frac{\sigma}{2} d(r^2 \theta_\alpha)$, the action is isometric and tri-holomorphic. Since the G-action on $\{\mu = 0\} \subset \bar{M}$ is free, the lifted action is free on $\pi^{-1}(\{\mu = 0\}) = \{\hat{\mu} = 0\} \subset \widehat{M}$. $\square$

Remark 3.6.13 The above proposition shows that if we can perform the quaternionic Kähler quotient $\bar{M} /\!/\!/ G$, then we can also perform the hyper-Kähler quotient $\widehat{M} /\!/\!/_{\{\hat{\mu}=0\}} G$ with level 0 and obtain a smooth (pseudo-)hyper-Kähler manifold. $\widehat{M} /\!/\!/_{\{\hat{\mu}=0\}} G$ is again a conical (pseudo-)hyper-Kähler manifold. In fact, it is the Swann bundle over $\bar{M} /\!/\!/ G$ [Sw1] (see also Theorem 3.5.4).

Proposition 3.6.14 *Let $(\bar{M}, \bar{g}, Q)$ be a quaternionic (pseudo-)Kähler manifold with an isometric S^1-action generated by a vector field $X \in \mathfrak{X}(\bar{M})$ such that X and the quaternionic Kähler moment map μ^X associated with X do not vanish simultaneously. Then the lifted isometric tri-holomorphic S^1-action on the Swann bundle $\hat{\pi} : \widehat{M} \to \bar{M}$ that commutes with ξ is locally free.*

***Proof*:** The lifted S^1-action is obtained as in the proof of Proposition 3.6.12. Since the lift $\hat{X} \in \mathfrak{X}(\widehat{M})$ of X constructed in the last subsection is unique, it generates the lifted S^1-action on $\widehat{M}$. $\hat{X}$ is the sum of the horizontal lift of X and a vertical part that vanishes if and only if $\hat{\mu}$ vanishes. The latter happens exactly on $\hat{\pi}^{-1}(\{\mu^X = 0\})$. Since by assumption, X and μ^X do not vanish simultaneously, $\hat{X}$ vanishes nowhere on $\widehat{M}$. Hence, the S^1-action is locally free on $\widehat{M}$. □

Remark 3.6.15 The above proposition shows that if S^1 acts isometrically on a quaternionic (pseudo-)Kähler manifold, then we can perform the hyper-Kähler quotient of the Swann bundle with respect to the lifted action with an arbitrary level obtaining at most orbifold singularities.

Chapter 4

The Hyper-Kähler/quaternionic Kähler correspondence

In Section 4.1, we introduce the HK/QK correspondence. It constructs a quaternionic pseudo-Kähler manifold endowed with a Killing vector field from a pseudo-hyper-Kähler manifold of the same dimension endowed with a real-valued function. This function is the Kähler moment map (with respect to the first Kähler form) of a *rotating* Killing vector field, which means that the vector field preserves the metric and first complex structure while acting as an infinitesimal rotation on the plane spanned by the other two complex structures. The Kähler moment map can be shifted by a real constant. The choice of this constant influences the local geometry and the global topology of the resulting quaternionic pseudo-Kähler manifold. The construction is taken from the author's collaboration [ACDM] and is based on the *conification* of (pseudo-)hyper-Kähler manifolds with rotating Killing vector field introduced in [ACM]. It extends results of Andriy Haydys who discovered the HK/QK correspondence and studied the case where the initial hyper-Kähler manifold is positive definite, and the resulting quaternionic Kähler metric is positive definite and of positive scalar curvature [Ha]. In contrast to [ACDM], we will give a new and self-contained proof of the fact that the resulting metric is quaternionic pseudo-Kähler. In our account, the construction and the proof just make use of an S^1-bundle over the original pseudo-hyper-Kähler manifold and do not involve the construcion of a higher-dimensional conical hyper-Kähler manifold. In [ACDM], the proof was based on the conification construction from [ACM], which is similar to the way the quaternionic Kähler property was proven in [Ha].

We explicitly determine the signature of the metric and the local $Sp(1)$-connection one-form for all quaternionic pseudo-Kähler manifolds obtained from the HK/QK correspondence. We also determine the quaternionic Kähler moment map of the Killing vector field defined by the HK/QK correspondence. It is nowhere vanishing and thus defines a global integrable complex structure that is compatible with the quaternionic structure. This shows in particular that quaternionic Kähler manifolds that are obtained from the HK/QK correspondence can never be positive definite, of positive scalar curvature and complete.

In Subsection 4.1.1, we apply the HK/QK correspondence to an arbitrary conical pseudo-hyper-Kähler manifold $(M, g, J_1, J_2, J_3, \xi)$. The real-valued function is chosen such that the corresponding rotating Killing vector field is $J_1\xi$. Since M is conical hyper-Kähler, it is locally the Swann bundle over a quaternionic pseudo-Kähler manifold $\bar{M}$. Applying the HK/QK correspondence while leaving the parameter $c \in \mathbb{R}^*$ in the choice of ω_1-Hamiltonian function free leads to a family of quaternionic Kähler metrics which is again defined on M. This family is locally homothetic to the family of quaternionic Kähler metrics on the Swann bundle over $\bar{M}$ defined in [Sw1]. As an example, we consider quaternionic vector space with the standard positive definite hyper-Kähler metric. For $c > 0$, the HK/QK correspondence leads to a chart in quaternionic projective space and for $c < 0$, the result is isometric to quaternionic hyperbolic space.

In Section 4.2, we show that if M is obtained from a conical hyper-Kähler manifold $\widehat{M}$ via an S^1-hyper-Kähler quotient with level set P (with non-zero level) and $M' \subset P$ is an appropriate codimension one submanifold endowed with the quaternionic Kähler structure induced from $\widehat{M}$, then M and M' are related by the HK/QK correspondence. The global consideration of this result gives a reverse construction for the HK/QK correspondence (the *QK/HK correspondence*) which is a combination of the Swann bundle construction and a hyper-Kähler quotient (with non-zero level) with respect to the canonical lift of an isometric S^1-action.

In Section 4.3, we show the compatibility of the HK/QK correspondence with the hyper-Kähler and quaternionic Kähler quotient constructions.

In Section 4.4 we apply the HK/QK correspondence to a chart in $T^*(\mathbb{C}P^n)$ and to the tubular neighborhood of the zero-section in $T^*(\mathbb{C}H^n)$ on which we defined a hyper-Kähler structure via a hyper-Kähler quotient in Section 3.4. As a result, we obtain families $g'^c_\pm$ of quaternionic Kähler metrics on $M'_+ = \{(\zeta, \eta) \in \mathbb{C}^{2n}\}$, respectively[1] $M'_- = \{\|\zeta\|^2 < 1, \tilde{r}^2 < 1\} \subset M'_+$, where $c \in \mathbb{R}^{\geq 0}$ (see Eq. (4.62)). As an application of the results from Section 4.2, we show that (M'_+, g'^0_+) is isometric to[2] a chart in $(\mathbb{H}P)^o$ and that (M'_-, g'^0_-) is isometric to $\mathbb{H}H^n$. As an application of the results from Section 4.3, we show that (M'_+, g'^1_+) is isometric to a chart in a proper subset $(X(n))^o$ of the Wolf space $X(n)$ and that (M'_-, g'^1_-) is isometric to a proper subset $(\tilde{X}(n))^o$ of the Wolf space $\tilde{X}(n)$. We also give a first analysis of the case $c > 0$ and show in particular that while $(M'_-, g'^0_-) \approx \mathbb{H}H^n$ is complete, (M'_-, g'^c_-) is incomplete for all $c > 0$. Furthermore, we give supporting evidence for our expectation that $(M'_\pm, g'^c_\pm)$ is not locally symmetric for c different from zero and one.

4.1 The HK/QK correspondence

First, we review the HK/QK correspondence in a form similar to the one published in the author's collaboration [ACDM]:

Let (M, g, J_1, J_2, J_3, f) be a (pseudo-)hyper-Kähler manifold with Kähler forms $\omega_\alpha := g(J_\alpha \cdot\,, \cdot)$, $\alpha = 1, 2, 3$, together with a real-valued function $f \in C^\infty(M)$ such that $Z := -\omega_1^{-1}(df) \in \mathfrak{X}(M)$ is a time-like or space-like J_1-holomorphic Killing vector field satisfying $\mathcal{L}_Z J_2 = -2J_3$.

We assume that $\sigma := \operatorname{sgn} f$ and $\sigma_1 := \operatorname{sgn} f_1$ are constant and non-zero, where $f_1 := f - \frac{g(Z,Z)}{2} \in C^\infty(M)$. This can be achieved by restricting M to an open subset.

Let $\pi : P \to M$ be an S^1-principal bundle[3] with principal connection η whose curvature is

$$d\eta = \pi^*(\omega_1 - \frac{1}{2} d\beta) \in \Omega^2(P), \tag{4.1}$$

where

$$\beta := g(Z, \cdot) \in \Omega^1(M). \tag{4.2}$$

From now on, we will often drop π^* when pulling back covariant tensor fields from M to P.

[1] $\tilde{r} = \sqrt{4(1 \pm \|\zeta\|^2)\Big(\pm (\zeta \cdot \eta)(\bar{\zeta} \cdot \bar{\eta}) + \|\eta\|^2\Big)}$

[2] $(\mathbb{H}P^n)^o = \mathbb{H}P^n \backslash \{[q = z + jw]_{\mathbb{H}^*_{\text{right}}} \mid (z, w) \in \mathbb{C}^{2n+2} \backslash \{0\},\ \|z\|^2 = \|w\|^2,\ z \cdot w = 0\}$, see Eq. (3.56).

[3] P exists globally if $[\frac{1}{2\pi}(\omega_1 - \frac{1}{2} d\beta)] = [\frac{1}{2\pi}\omega_1] \in H^2_{dR}(M, \mathbb{Z})$ (see e.g. [Wood, Prop. 8.3.1]). Otherwise, we restrict M to an open subset.

We endow P with the (pseudo-)Riemannian metric

$$g_P := \frac{2}{f_1}\eta^2 + \pi^* g \in \Gamma(\mathrm{Sym}^2 T^*P) \tag{4.3}$$

and with the vector field

$$Z_1^P := \tilde{Z} + f_1 X_P \in \mathfrak{X}(P), \tag{4.4}$$

where $\tilde{Z} \in \Gamma(\ker\eta) \subset \mathfrak{X}(P)$ denotes the horizontal lift of Z to P and X_P denotes the fundamental vector field of the principal action of P (normalized such that $\eta(X_P) = 1$). Furthermore, we endow P with the following one-forms[4]:

$$\begin{aligned}
\theta_0^P &:= \frac{1}{2}df \\
\theta_1^P &:= \eta + \frac{1}{2}\beta \\
\theta_2^P &:= \frac{1}{2}\omega_3(Z,\cdot) \\
\theta_3^P &:= -\frac{1}{2}\omega_2(Z,\cdot).
\end{aligned} \tag{4.5}$$

Let M' be a codimension one submanifold of P which is transversal to the vector field Z_1^P, i.e. $TP|_{M'} = TM' \oplus \mathbb{R}Z_1^P$. Let

$$\mathrm{pr}_{TM'}^{Z_1^P} : TP\big|_{M'} = TM' \oplus \mathbb{R}Z_1^P \to TM' \tag{4.6}$$

denote the projection onto the first summand (i.e. the projection onto TM' along Z_1^P). Define the vector field

$$X := \mathrm{pr}_{TM'}^{Z_1^P} \circ X_P\big|_{M'} \in \mathfrak{X}(M'). \tag{4.7}$$

For any vector field $Y \in \mathfrak{X}(M)$ on M, we introduce the notation

$$Y' := \mathrm{pr}_{TM'}^{Z_1^P} \circ \widetilde{Y}\big|_{M'} \in \mathfrak{X}(M'). \tag{4.8}$$

Define $\mathcal{D}^h := \{Z, J_1Z, J_2Z, J_3Z\}^{\perp_g} \subset TM$
and $\mathcal{D}'^h := \mathrm{span}\,\{Y' \mid Y \in \Gamma(\mathcal{D}^h)\} \subset TM'$. Note that with

$$\mathcal{D}'^v := \mathrm{span}\,\{X, (J_1Z)', (J_2Z)', (J_3Z)'\} \subset TM', \tag{4.9}$$

[4]Note that in comparison to [ACDM], we changed the sign of θ_0^P.

we have the splitting

$$TM' = \mathcal{D}'^v \oplus \mathcal{D}'^h. \tag{4.10}$$

Using this splitting, we now define an almost hyper-complex structure on M':

Proposition 4.1.1 *An almost hyper-complex structure (J_1', J_2', J_3') on M' is uniquely defined by*

$$J_\alpha' X = -\frac{1}{f_1}(J_\alpha Z)', \quad J_\alpha'(J_\beta Z)' = (J_\gamma Z)' \qquad (\alpha = 1,2,3) \tag{4.11}$$

and

$$J_\alpha'(Y') = (J_\alpha Y)' \qquad \textit{for all } Y \in \Gamma(\mathcal{D}^h). \tag{4.12}$$

***Proof*:** Since J_α preserves $\mathcal{D}^h$, J_α' preserves $\mathcal{D}'^h$. It is clear that Eqs. (4.11) and (4.12) uniquely define three almost complex structures and that they preserve $\mathcal{D}'^v$. The matrices representing $J_1'\big|_{\mathcal{D}'^v}$, $J_2'\big|_{\mathcal{D}'^v}$, $J_3'\big|_{\mathcal{D}'^v}$ with respect to the frame $(X, -\frac{1}{f_1}(J_1 Z)', -\frac{1}{f_1}(J_2 Z)', -\frac{1}{f_1}(J_3 Z)')$ in $\mathcal{D}'^v$ are given by

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Together with

$$J_{\alpha_1}' J_{\alpha_2}' Y' = J_{\alpha_1}'(J_{\alpha_2} Y)' = (J_{\alpha_1} J_{\alpha_2} Y)', \qquad (\alpha_1, \alpha_2 = 1,2,3)$$

for all $Y \in \Gamma(\mathcal{D}^h)$, we obtain that (J_1', J_2', J_3') fulfill

$$J_1' J_2' = -J_2' J_1' = J_3'.$$

□

The following theorem constitutes the HK/QK correspondence:

Theorem 4.1.2 *Let (M, g, J_1, J_2, J_3) be a (pseudo-)hyper-Kähler manifold and $f \in C^\infty(M)$ such that the assumptions on f and $Z := -\omega_1^{-1}(df)$ stated above are fulfilled. Choose an S^1-bundle P with connection η and a submanifold $M' \subset P$ as above. Let*

$$Q := \operatorname{span}\{J_1', J_2', J_3'\}, \tag{4.13}$$

where J_1', J_2', J_3' are given by Proposition 4.1.1. With

$$g' := \frac{1}{2|f|}\big(g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2\big)\Big|_{M'}, \tag{4.14}$$

(M', g', Q) is a quaternionic pseudo-Kähler manifold.

The signature of g' is related to $\operatorname{sign} g = (4k, 4\ell)$ *as follows:*

$$\operatorname{sign} g' = \begin{cases} (4k-4, 4\ell+4) & \text{if } f > 0,\ f_1 < 0 \\ (4k+4, 4\ell-4) & \text{if } f < 0,\ f_1 > 0 \\ (4k, 4\ell) & \text{if } f f_1 > 0. \end{cases} \tag{4.15}$$

The local $Sp(1)$-connection one-form with respect to (J_1', J_2', J_3') is given by $\bar\theta = \sum_{\alpha=1}^{3} \bar\theta_\alpha e_\alpha$, where

$$\bar\theta_\alpha := \frac{1}{f}\theta_\alpha^P\Big|_{M'} \qquad (\alpha = 1, 2, 3). \tag{4.16}$$

Remark 4.1.3 The above relation between the (pseudo-)hyper-Kähler manifold with ω_1-Hamiltonian function (M, g, J_1, J_2, J_3, f) and the quaternionic pseudo-Kähler manifold with Killing vector field[5] (M', g', Q, X) is called the **HK/QK correspondence**. We say that (M', g', Q, X) is **obtained from** (M, g, J_1, J_2, J_3, f) **via the HK/QK correspondence with the choices** (P, η, M') or simply that (M', g', Q, X) is obtained from $(M, g, J_1, J_2, J_3, f, P, \eta, M')$ via the HK/QK correspondence.

For the proof of the above theorem, we will split the hyper-Kähler metric g on M according to the splitting $TM = \mathcal{D}^v \oplus^{\perp g} \mathcal{D}^h$, where

$$\mathcal{D}^v := \operatorname{span}_{\mathbb{R}}\{Z, J_1 Z, J_2 Z, J_3 Z\} \subset TM \tag{4.17}$$

and $\mathcal{D}^h = (\mathcal{D}^v)^{\perp g}$. Define the following one-forms on M:

$$\begin{aligned} \theta_0 &:= \frac{1}{2}df = -\frac{1}{2}\omega_1(Z, \cdot), \\ \theta_1 &:= \frac{1}{2}\beta = \frac{1}{2}g(Z, \cdot), \end{aligned}$$

[5] X_P commutes with Z_1^P, $1/(2|f|)\,(g_P - 2/f\,\sum(\theta_a^P)^2) \in \Gamma(\mathrm{Sym}^2 T^*P)$ has kernel $\mathbb{R}Z_1^P$ and is preservered by X_P and Z_1^P. Hence, $X = \mathrm{pr}_{TM'}^{Z_1^P} \circ X_P|_{M'}$ preserves g'.

$$\begin{aligned} \theta_2 &:= \frac{1}{2}\omega_3(Z,\cdot), \\ \theta_3 &:= -\frac{1}{2}\omega_2(Z,\cdot). \end{aligned} \tag{4.18}$$

Proposition 4.1.4 *The (pseudo-)hyper-Kähler metric can be written as*

$$g = \frac{4}{\beta(Z)}\sum_{a=0}^{3}(\theta_a)^2 + \breve{g}, \tag{4.19}$$

where $\breve{g} \in \Gamma(\mathrm{Sym}^2 T^*M)$ *is a tensor field that is invariant under* Z *and has four-dimensional kernel* $\ker \breve{g} = \mathcal{D}^v$.

The Kähler forms on M *are given by*

$$\omega_\alpha = \frac{4}{\beta(Z)}(\theta_0 \wedge \theta_\alpha + \theta_\beta \wedge \theta_\gamma) + \breve{\omega}_\alpha \tag{4.20}$$

for every cyclic permutation (α, β, γ) *of* $(1,2,3)$*, where*

$$\breve{\omega}_\alpha := \breve{g}(J_\alpha\cdot\,,\cdot) \in \Omega^2(M). \tag{4.21}$$

***Proof*:**
Since (J_1, J_2, J_3) is hyper-Hermitian, Z, J_1Z, J_2Z and J_3Z are pairwise orthogonal and all have squared norm equal to $g(Z,Z) = \beta(Z)$. Hence,

$$\breve{g} = g - \frac{4}{\beta(Z)}\sum_{a=0}^{3}(\theta_a)^2 = g - \frac{1}{g(Z,Z)}\left((Z^\flat)^2 + (J_1Z^\flat)^2 + (J_2Z^\flat)^2 + (J_3Z^\flat)^2\right)$$

has $\ker \breve{g} = \mathcal{D}^v$.

Since Z is Killing and fulfills $\mathcal{L}_Z J_2 = -2J_3$, we have $\mathcal{L}_Z(\beta(Z)) = 0$ and

$$\mathcal{L}_Z\theta_0 = 0, \quad \mathcal{L}_Z\theta_1 = 0, \quad \mathcal{L}_Z\theta_2 = -2\theta_3, \quad \mathcal{L}_Z\theta_3 = 2\theta_2. \tag{4.22}$$

This implies

$$\mathcal{L}_Z\breve{g} = \mathcal{L}_Z\left(g - \frac{4}{\beta(Z)}\sum_{a=0}^{3}(\theta_a)^2\right) = 0. \tag{4.23}$$

Eq. (4.20) follows directly from Eq. (4.19) and

$$J_\alpha^*\theta_0 = -\theta_\alpha, \quad J_\alpha^*\theta_\beta = -\theta_\gamma. \tag{4.24}$$

□

Proof (of Theorem 4.1.2 for $\dim_{\mathbb{R}} M > 4$)**:**
From the definition of g' (see Eq. (4.14)) and the splitting of the hyper-Kähler metric g in Eq. (4.19), we get

$$\begin{aligned} g' &= \frac{1}{2|f|}\Big(\frac{2}{f_1}\eta^2 + \pi^* g - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2\Big)\Big|_{M'} \\ &= \frac{1}{2|f|}\Big(\frac{2}{f_1}\eta^2 + \frac{4}{\beta(Z)}\big((\theta_0^P)^2 + (\theta_1^P - \eta)^2 + (\theta_2^P)^2 + (\theta_3^P)^2\big) \\ &\qquad\qquad + \pi^*\breve{g} - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2\Big)\Big|_{M'} \\ &= \frac{1}{2|f|}\Big(\frac{4f_1}{f\beta(Z)}\big((\theta_0^P)^2 + (\theta_1^P - \frac{f}{f_1}\eta)^2 + (\theta_2^P)^2 + (\theta_3^P)^2\big) + \pi^*\breve{g}\Big)\Big|_{M'} \\ &= \lambda\sigma\sigma_1\sum_{a=0}^{3}(\theta_a')^2 + \frac{1}{2|f|}\pi^*\breve{g}\big|_{M'}, \end{aligned} \tag{4.25}$$

where

$$\begin{aligned} \theta_0' &:= \frac{1}{|f|}\sqrt{\left|\frac{2f_1}{\beta(Z)}\right|}\,\theta_0^P\big|_{M'}, \quad \theta_1' := \frac{1}{|f|}\sqrt{\left|\frac{2f_1}{\beta(Z)}\right|}\,(\theta_1^P - \frac{f}{f_1}\eta)\big|_{M'}, \\ \theta_2' &:= \frac{1}{|f|}\sqrt{\left|\frac{2f_1}{\beta(Z)}\right|}\,\theta_2^P\big|_{M'}, \quad \theta_3' := \frac{1}{|f|}\sqrt{\left|\frac{2f_1}{\beta(Z)}\right|}\,\theta_3^P\big|_{M'} \end{aligned} \tag{4.26}$$

are one-forms on M' and

$$\lambda := \operatorname{sgn}\beta(Z), \quad \sigma = \operatorname{sgn} f, \quad \sigma_1 = \operatorname{sgn} f_1. \tag{4.27}$$

Note that Z_1^P lies in the kernel of θ_0^P, $\theta_1^P - f/f_1\,\eta$, θ_2^P, θ_3^P and $\pi^*\breve{g}$. Consequently, the splitting of g' given in Eq. (4.25) corresponds to the splitting $TM' = \mathcal{D}'^v \oplus \mathcal{D}'^h$ defined in the proof of Proposition 4.1.1, i.e. the first summand is non-degenerate on $\mathcal{D}'^v$ and has kernel $\mathcal{D}'^h$, while the second summand is non-degenerate on $\mathcal{D}'^h$ and has kernel $\mathcal{D}'^v$. Eq. (4.25) thus implies that the signature of g' is given by Eq. (4.15) and in particular, it shows that g' in non-degenerate.

Now, we want to show that

$$\omega'_\alpha := g'(J'_\alpha \cdot\,,\cdot) = \frac{\sigma}{2}(d\bar\theta_\alpha - 2\bar\theta_\beta \wedge \bar\theta_\gamma) \qquad (\alpha = 1,2,3). \tag{4.28}$$

Lemma 4.1.5

$$d\theta^P_\alpha = \pi^*\omega_\alpha \qquad (\alpha = 1{,}2{,}3). \tag{4.29}$$

***Proof*:** For θ^P_1, this follows from the definition of the curvature of η (see Eq. (4.1)). For θ^P_2 and θ^P_3, this is obtained from $\mathcal{L}_Z\omega_3 = 2\omega_2$ and $\mathcal{L}_Z\omega_2 = -2\omega_3$ respectively, e.g.:

$$2d\theta^P_2 = d(\iota_Z\omega_3) \overset{d\omega_\alpha=0}{=} \mathcal{L}_Z\omega_3 = 2\omega_2.$$

□

Since (J'_1, J'_2, J'_3) (see Proposition 4.1.1) agrees with (J_1, J_2, J_3) on $\mathcal{D}'^h$,

$$\pi^*\breve g\big|_{M'}(J'_\alpha \cdot\,,\cdot) = \pi^*(\breve g(J_\alpha \cdot\,,\cdot))\big|_{M'} = \pi^*\breve\omega\big|_{M'}.$$

On $\mathcal{D}'^v$, (X, J'_1X, J'_2X, J'_3X) are pairwise orthogonal with respect to $\sum_{a=0}^3(\theta'_a)^2$ and fulfill

$$\theta'_0(J'_1X) = -\theta'_1(X) = -\theta'_2(J'_3X) = \theta'_3(J'_2X) = \frac{\lambda\sigma_1}{|f|}\sqrt{\left|\frac{\beta(Z)}{2f_1}\right|} \neq 0.$$

Using the fact that (J'_1, J'_2, J'_3) is an almost hyper-complex structure, this implies

$$J'^*_\alpha\theta'_0 = -\theta'_\alpha, \quad J'^*_\alpha\theta'_\beta = -\theta'_\gamma. \tag{4.30}$$

In total, we have

$$\omega'_\alpha = \lambda\sigma\sigma_1(\theta'_0\wedge\theta'_\alpha + \theta'_\beta\wedge\theta'_\gamma) + \frac{1}{2|f|}\pi^*\breve\omega_\alpha\big|_{M'} \qquad (\alpha = 1,2,3). \tag{4.31}$$

This is equal to

$$\begin{aligned}
&\frac{\sigma}{2}(d\bar\theta_\alpha - 2\bar\theta_\beta\wedge\bar\theta_\gamma)\\
\overset{(4.16)}{\underset{(4.29)}{=}}\ &\Big(\frac{1}{2|f|}\pi^*\omega_\alpha - \frac{\sigma}{2f^2}\underbrace{df}_{2\theta^P_0}\wedge\theta^P_\alpha - \frac{\sigma}{f^2}\theta^P_\beta\wedge\theta^P_\gamma\Big)\Big|_{M'}\\
\overset{(4.20)}{=}\ &\Big(\frac{1}{2|f|}\pi^*\breve\omega_\alpha + \frac{2}{|f|\,\beta(Z)}\pi^*(\theta_0\wedge\theta_\alpha + \theta_\beta\wedge\theta_\gamma) - \frac{\sigma}{f^2}(\theta^P_0\wedge\theta^P_\alpha + \theta^P_\beta\wedge\theta^P_\gamma)\Big)\Big|_{M'}\\
=\ &\frac{1}{2|f|}\pi^*\breve\omega_\alpha\big|_{M'} + \lambda\sigma\sigma_1(\theta'_0\wedge\theta'_\alpha + \theta'_\beta\wedge\theta'_\gamma)
\end{aligned}$$

$$\overset{(4.31)}{=} \omega'_\alpha$$

and shows Eq. (4.28). In the second to last equality, we used $\theta_a^P = \pi^*\theta_a$ for $a = 0, 2, 3$ and

$$\begin{aligned}\Big(\frac{2}{|f|\,\beta(Z)}\pi^*\theta_1 - \frac{\sigma}{f^2}\theta_1^P\Big)\Big|_{M'} &= \Big(\frac{2}{|f|\,\beta(Z)}(\theta_1^P - \eta) - \frac{\sigma}{f^2}\theta_1^P\Big)\Big|_{M'} \\ = \Big(\frac{1}{|f|}\frac{2f_1}{f\,\beta(Z)}\big(\theta_1^P - \frac{f}{f_1}\eta\big)\Big)\Big|_{M'} &= \lambda\sigma\sigma_1\frac{1}{|f|}\sqrt{\Big|\frac{2f_1}{\beta(Z)}\Big|}\theta'_1. \end{aligned} \tag{4.32}$$

Eq. (4.28) shows that Q is compatible with g' and implies

$$d\omega'_\alpha = \sigma(\bar\theta_\beta \wedge d\bar\theta_\gamma - \bar\theta_\gamma \wedge d\bar\theta_\beta) = 2(\bar\theta_\beta \wedge \omega'_\gamma - \bar\theta_\gamma \wedge \omega'_\beta). \tag{4.33}$$

Together with Corollary 2.1.9, this finishes the proof for $\dim_{\mathbb{R}} M > 4$. □

Proof (of Theorem 4.1.2 for $\dim_{\mathbb{R}} M = 4$)**:**
The four-dimensional case can be deduced from the higher-dimensional case as follows[6]:
Assume that $\dim_{\mathbb{R}} M = 4$. Let $M_0 := \mathbb{H}$ be endowed with the standard hyper-Kähler structure $(g_0, J_1^0, J_2^0, J_3^0)$ that was defined in Example 3.1.7, i.e. $g_0 = dzd\bar z + dwd\bar w$ and $\omega_+^0 = dz \wedge dw$ in complex coordinates (z, w) defined by $q = z + jw \in \mathbb{H}$. Let $f^0 := w\bar w \in C^\infty(M_0)$. This defines a J_1^0-holomorphic vector field

$$Z^0 := -(\omega_1^0)^{-1}(df) = 2i(w\partial_w - \bar w\partial_{\bar w})$$

that fulfills $\mathcal{L}_{Z^0} J_2^0 = -2J_3^0$ and $f_1^0 := f^0 - \frac{1}{2}g_0(Z^0, Z^0) = -w\bar w$. Then $\eta_0^{M_0} := \frac{1}{2}\operatorname{Im}(\bar z dz - \bar w dw)$ fulfills $d\eta_0^{M_0} = \omega_1^0 - \frac{1}{2}d(\iota_{Z^0} g_0)$.

We consider $(\tilde M := M \times \mathbb{H}, \tilde g := g + g_0, \tilde f := f + f^0)$ together with the product hyper-complex structure $(\tilde J_1, \tilde J_2, \tilde J_3)$. Let $\tilde U \subset \tilde M$ be a neighborhood of $M = M \times \{0\} \subset \tilde M$, such that the signs of $\tilde f$, $\tilde f_1 := f_1 + f_1^0$ and $\tilde f - \tilde f_1$ restricted to $\tilde U$ are constant. Then the restriction of the above data from $\tilde M$ to $\tilde U$ fulfills the assumptions of the HK/QK correspondence. The restriction of $P \times \mathbb{H}$ defines an S^1-bundle $\tilde P$ over $\tilde U$ with connection $\tilde\eta = (\eta + \eta_0^{M_0})\big|_{\tilde P}$. The HK/QK correspondence with the choices $(\tilde P, \tilde\eta, \tilde M' := M' \times \mathbb{H})$ then defines a quaternionic Kähler structure $(\tilde g', \tilde Q)$ on the 8-dimensional manifold $\tilde M'$. $M' = M' \times \{0\} \subset \tilde M'$ is a quaternionic submanifold and, hence $(M', \tilde g'\big|_{M'}, \tilde Q\big|_{M'})$ is quaternionic Kähler by Proposition 2.1.11. The globally defined $Sp(1)$-connection one-form on $\tilde M'$

[6] This idea is taken from [MS2, Cor. 4.2.].

$$\begin{tikzcd} & P, X_P, Z_1^P \arrow[dl, bend right, "S^1_{(X_P)}"'] \arrow[dr, bend left, "\bar{S}^1_{(Z_1^P)}"] & \\ M, Z \arrow[rr, mapsto, "\text{HK/QK cor.}"'] & & \bar{M}, \bar{X} \end{tikzcd}$$

Figure 4.1: HK/QK correspondence (global version).

obtained from the HK/QK-correspondence restricts to $\bar{\theta} \in \Omega^1(M', \mathfrak{so}(3))$ on M', which in particular shows that $(\tilde{g}'\big|_{M'}, \tilde{Q}\big|_{M'}) = (g', Q)$. □

Remark 4.1.6 Note that if Z_1^P induces a free S^1-action (denoted by $\bar{S}^1$) on P and if $M' \subset P$ intersects each $\bar{S}^1$-orbit at most once, then M' defines a section $\bar{\boldsymbol{\sigma}} : U \to P$, $\bar{\boldsymbol{\sigma}}(U) = M'$, of $\bar{\pi} : P \to \bar{M} := P/\bar{S}^1$ over $U := \bar{\pi}(M') \subset \bar{M}$. M' can be identified with U via $\bar{\boldsymbol{\sigma}}$. The geometric data defined on such submanifolds $U \subset \bar{M}$ under this identification via the HK/QK correspondence patches together to a quaternionic (pseudo-)Kähler structure $(\bar{g}, \bar{Q})$ on $\bar{M}$ together with a Killing vector field $\bar{X} \in \mathfrak{X}(\bar{M})$ (see Figure 4.1). In this situation we also say that $(\bar{M}, \bar{g}, \bar{Q}, \bar{X})$ is obtained from the HK/QK correspondence. The quaternionic Kähler moment map $\mu^{\bar{X}}$ associated with $\bar{M}$ is nowhere vanishing on $\bar{M}$ and thus defines a global integrable complex structure $\bar{J} := \bar{J}_1 := -\frac{1}{\sqrt{\|\mu^{\bar{X}}\|^2}}\mu^{\bar{X}} \in \Gamma(\bar{M}, \bar{Q})$ on $\bar{M}$ that is compatible with $\bar{Q}$. The sign is chosen such that $\bar{J}$ locally corresponds to the complex structure J_1' on M' (see Proposition 4.1.9 below).

Remark 4.1.7 Using the well-known result by Alekseevsky [A1] that

$$\frac{\nu}{2}\omega_\alpha' = d\bar{\theta}_\alpha - 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma, \tag{4.34}$$

we obtain from Eq. (4.28) that the reduced scalar curvature of any quaternionic (pseudo-)Kähler manifold (M', g') obtained from the HK/QK correspondence is

$$\nu = \frac{scal}{4n(n+2)} = 4\sigma \qquad (\dim M' = 4n). \tag{4.35}$$

Remark 4.1.8 Note that the HK/QK correspondence can also be applied if we drop the assumption that $g(Z, Z)$ is non-vanishing. The above procedure then gives a manifold M' together with a tensor field $g' \in \Gamma(\mathrm{Sym}^2 T^*M')$. We believe that also in this situation, it is possible to show that (M', g') is quaternionic pseudo-Kähler with globally defined fundamental two-forms

$$\omega_\alpha' := \frac{\sigma}{2}(d\bar{\theta}_\alpha - 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma), \quad \bar{\theta} := (f^{-1}\theta_\alpha^P)\big|_{M'}.$$

Proposition 4.1.9 *Let $(M', g', Q = \mathrm{span}_{\mathbb{R}}\{J_1', J_2', J_3'\}, X)$ be a quaternionic (pseudo-)Kähler manifold with Killing vector field that is obtained via the HK/QK correspondence from a hyper-Kähler manifold with function f. Then the quaternionic Kähler moment map on M' associated with X is $\mu^X = -\frac{1}{2|f|}\big|_{M'} J_1' \in \Gamma(M', Q)$.*

***Proof*:** Recall that Q is defined by globally defined fundamental two-forms

$$\omega_\alpha' = \frac{\sigma}{2}(d\bar{\theta}_\alpha - 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma), \quad \bar{\theta}_\alpha = \frac{1}{f}\theta_\alpha^P\big|_{M'}.$$

Let $a \in C^\infty(P)$ such that $(X_P - aZ_1^P)\big|_{M'} \in \Gamma(TM')$. Then the Killing vector field on (M', g') is given by $X = (X_P - aZ_1^P)\big|_{M'}$. Note that

$$(\iota_X \bar{\theta}_\alpha)_{\alpha=1,2,3} = \big((f^{-1} - a)\big|_{M'}, 0, 0\big)$$

and

$$\begin{aligned}
\iota_X d\bar{\theta}_\alpha &\overset{(4.29)}{=} \big(-f^{-2}df \wedge \theta_\alpha^P + f^{-1}\pi^*\omega_\alpha\big)\big|_{M'}(X, \cdot) \\
&= \Big(f^{-2}\theta_\alpha^P(X_P - aZ_1^P)df - af^{-1}\pi^*(\iota_Z\omega_\alpha)\Big)\Big|_{M'} \\
&= \begin{cases} \big(f^{-2}(1 - af)df + f^{-1}df\big)\big|_{M'} = f^{-2}df\big|_{M'} & (\alpha = 1) \\ 2af^{-1}\theta_3^P\big|_{M'} = 2a\big|_{M'}\bar{\theta}_3 & (\alpha = 2) \\ -2af^{-1}\theta_2^P\big|_{M'} = -2a\big|_{M'}\bar{\theta}_2 & (\alpha = 3). \end{cases}
\end{aligned}$$

From this, we obtain

$$(\mathcal{L}_X\bar{\theta}_\alpha = d\iota_X\bar{\theta}_\alpha + \iota_X d\bar{\theta}_\alpha)_{\alpha=1,2,3} = \big(-da\big|_{M'}, 2a\big|_{M'}\bar{\theta}_3, -2a\big|_{M'}\bar{\theta}_2\big).$$

This implies $(\mathcal{L}_X\omega_\alpha')_{\alpha=1,2,3} = (0, 2a\big|_{M'}\omega_3', -2a\big|_{M'}\omega_2')$. From

$$\mathcal{L}_X\omega_\alpha' \overset{(2.20)}{=} (\nu\mu_\beta^X + 2\bar{\theta}_\beta(X))\omega_\gamma' - (\nu\mu_\gamma^X + 2\bar{\theta}_\gamma(X))\omega_\beta'$$

and $\nu = 4\sigma$ (see Remark 4.1.7), we then obtain that the components of the quaternionic Kähler moment map associated with X with respect to the frame (J_1', J_2', J_3') in Q are given by

$$(\mu_\alpha^X)_{\alpha=1,2,3} = \Big(-\frac{1}{2|f|}\Big|_{M'}, 0, 0\Big).$$

□

4.1.1 HK/QK correspondence for conical hyper-Kähler manifolds

Let $(M, g, J_1, J_2, J_3, \xi)$ be a conical (pseudo-)hyper-Kähler manifold. Similarly to Eq. (3.27), one checks that $J_1\xi$ is a J_1-holomorphic Killing vector field satisfying $\mathcal{L}_{J_1\xi}\omega_2 = -2\omega_3$.

For $c \in \mathbb{R}^*$, we choose $f = \frac{\lambda}{2}(r^2 + c)$, where $r^2 = |g(\xi, \xi)|$ and $\lambda = \operatorname{sgn} g(\xi, \xi)$. Then $Z = -\omega_1^{-1}(df) = J_1\xi$ and $f_1 = f - \frac{1}{2}g(Z, Z) = \frac{\lambda}{2}c$. Note that for $c < 0$, we have to restrict M to $M_>^{(c)} = \{r^2 + c > 0\} \subset M$ or to $M_<^{(c)} = \{r^2 + c < 0\} \subset M$ to fulfill the assumption on the sign of f. For simplicity, we will not write this restriction explicitly in the following.

We consider the trivial S^1-bundle $\pi = \mathrm{pr}_1 : P = M \times S^1 \to M$, endowed with the flat principal connection $\eta = ds \in \Omega^1(P)$, where s is the natural coordinate on $S^1 = \{e^{is} \mid s \in \mathbb{R}\}$. Note that with the notations from Section 3.2 (with σ replaced by λ), $\beta = g(Z, \cdot) = \lambda r^2\theta_1 = 2\lambda\hat{\theta}_1$ and hence $d\eta = 0 \overset{(3.14)}{=} \omega_1 - \frac{1}{2}d\beta$. The one-forms on P are given by

$$\begin{aligned}
\theta_0^P &= \frac{1}{2}df = \frac{\lambda}{2}rdr,\\
\theta_1^P &= \eta + \frac{1}{2}\beta = ds + \lambda\frac{r^2}{2}\theta_1 = ds + \lambda\hat{\theta}_1,\\
\theta_2^P &= \frac{1}{2}\omega_3(J_1\xi,\cdot) = \lambda\frac{r^2}{2}\theta_2 = \lambda\hat{\theta}_2,\\
\theta_3^P &= -\frac{1}{2}\omega_2(J_1\xi,\cdot) = \lambda\frac{r^2}{2}\theta_3 = \lambda\hat{\theta}_3.
\end{aligned}$$

The metric and Killing vector field on P are given by (using $\theta_0 := \frac{1}{r}dr$)

$$g_P = \frac{2}{f_1}\eta^2 + g \overset{(3.19)}{=} \frac{4\lambda}{c}ds^2 + r^2(\lambda\sum_{a=0}^{3}(\theta_a)^2 + \breve{g}), \quad Z_1^P = \tilde{Z} + f_1X_P = J_1\xi + \lambda\frac{c}{2}\partial_s.$$

Here, $\breve{g}$ is the horizontal part of the conical pseudo-hyper-Kähler metric (see Lemma 3.2.4). We choose the Z_1^P-transversal submanifold

$$M' = M \times \{1\} = \{s = 0\} \subset P.$$

The quaternionic pseudo-Kähler metric g' on $M' \approx M$ obtained from the HK/QK correspondence is then ($\sigma = \operatorname{sgn} f$)

$$g' = \frac{1}{2|f|}\Big(g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2\Big)\Big|_{M'} = \frac{\sigma r^2}{r^2+c}\left(\frac{\lambda c}{r^2+c}\sum_{a=0}^{3}(\theta_a)^2 + \breve{g}\right)$$
$$= \frac{\sigma}{r^2+c}\,g - \frac{\sigma\lambda\, r^4}{(r^2+c)^2}\sum_{a=0}^{3}(\theta_a)^2. \qquad (4.36)$$

Remark 4.1.10 Note that for $c \to \infty$ the quaternionic pseudo-Kähler metrics $cg' = \sigma cg'$ on M converge to the original conical pseudo-hyper-Kähler metric g on M. For $c \to -\infty$, the metrics $\sigma cg'$ on $M_<^{(c)}$ converge to the original metric g on M.

Remark 4.1.11 If M is the Swann bundle over a quaternionic (pseudo-)Kähler manifold $\bar{M}$, then the two-parameter family of quaternionic Kähler metrics $\frac{\sigma}{\boldsymbol{p}}g'$ on M' with the replacements $c \mapsto \boldsymbol{q}$ and $r^2 \mapsto \boldsymbol{p}r^2$, $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^*$, is identical to the metric $\boldsymbol{g_1}$ in [Sw1, Theorem 3.5] (note that the constant $\boldsymbol{c}$ in [Sw1] is related to the reduced scalar ν of $\bar{M}$ by $\boldsymbol{c} = \nu/4$). The original conical pseudo-hyper-Kähler metric on M is recovered from $\boldsymbol{g_1}$ by setting $\boldsymbol{p} = 0$, $\boldsymbol{q} = 1$.

Example 4.1.12 Let $M = \mathbb{H}^n$ be endowed with the standard flat conical[7] hyper-Kähler structure of positive definite signature and with the complex coordinates $(z^\mu, w_\mu)_{\mu=1,\dots,n}$ defined by $q = z + jw \in M$ (see Examples 3.1.7 and 3.2.10). For the current example, the metric obtained from Eq. (4.36) reads

$$g' = \frac{\sigma}{c + \|z\|^2 + \|w\|^2}\sum_{\mu=1}^{n}(dz^\mu d\bar{z}^\mu + dw_\mu d\bar{w}_\mu)$$
$$- \sigma\frac{\big|\sum_{\mu=1}^n(\bar{z}^\mu dz^\mu + \bar{w}_\mu dw_\mu)\big|^2 + \big|\sum_{\mu=1}^n(z^\mu dw_\mu - w_\mu dz^\mu)\big|^2}{(c + \|z\|^2 + \|w\|^2)^2}. \qquad (4.37)$$

For $c > 0$ ($\sigma = +1$), (M, g') is isometric to the chart $\{q^0 \neq 0\} \subset \mathbb{H}P^n$ (see Eqs. (2.15) or (3.26)). For $c < 0$, $(M_<^{(c)}, g')$ is complete and isometric to $\mathbb{H}H^n$ (recall that $\sigma = -1$ on $M_<^{(c)} = \{c + \|z\|^2 + \|w\|^2 < 0\} \subset M$). Up to restriction of the

[7] To be precise, $\mathbb{H}^n$ is not conical and does not fulfill all assumptions of the HK/QK correspondence since $g(\xi,\xi) = g(Z,Z)$ vanishes at the origin. Nevertheless, we can apply the HK/QK correspondence (see Remark 4.1.8) and we see that, in this example, the result is still quaternionic Kähler.

$$\begin{tikzcd}
 & & \widehat{M} \arrow[dll, "/\!\!/\!\!/_P S^1_{(\hat{X})}"'] & \\
 & & \cup & \\
 & & P \arrow[dll, "S^1_{(\hat{X}|_P)}"] & \\
M \arrow[rrr, mapsto, "\text{HK/QK corresp.}"'] & & & M' \arrow[ul, phantom, "\subset" description]
\end{tikzcd}$$

Figure 4.2: Relation between the HK/QK correspondence, the hyper-Kähler quotient construction and the construction of quaternionic Kähler manifolds as submanifolds of conical hyper-Kähler manifolds (see Theorem 4.2.1).

respective manifolds this establishes the following HK/QK correspondence:

$$\left(\mathbb{H}^n,\ f = (r^2+c)/2\right) \xmapsto[(c\neq 0)]{\text{HK/QK cor.}} \begin{cases} \mathbb{H}P^n & (c>0) \\ \mathbb{H}H^n & (c<0). \end{cases}$$

4.2 Reverse construction (QK/HK correspondence)

In the following theorem, we show that if M is obtained from a conical hyper-Kähler manifold $\widehat{M}$ via an S^1-hyper-Kähler quotient with level set P and $M' \subset P$ is a codimension one submanifold transversal to $\hat{J}_1\xi\big|_P$ endowed with the quaternionic Kähler structure induced from $\widehat{M}$, then M and M' are related by the HK/QK correspondence (see Figure 4.2). It is important that P is taken with respect to a non-zero level of the homogeneous hyper-Kähler moment map.

Theorem 4.2.1 *Let $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi, \hat{X})$ be a conical (pseudo-)hyper-Kähler manifold endowed with a space- or time-like tri-holomorphic Killing vector field $\hat{X}$ that commutes with the Euler vector field ξ and induces a free S^1-action on $\widehat{M}$.*

Assume that the level set $P := \{\hat{\mu}^{\hat{X}} = (-\sigma, 0, 0)\}$ ($\sigma = \operatorname{sgn}\hat{g}(\xi,\xi)$) of the homogeneous hyper-Kähler moment map $\hat{\mu}^{\hat{X}}$ associated with $\hat{X}$ is non-empty. Consider the hyper-Kähler quotient $\widehat{M}/\!\!/\!\!/_P S^1_{(\hat{X})} = (M := P/S^1_{(\hat{X}|_P)}, g, J_1, J_2, J_3)$ and define a function $f \in C^\infty(M)$ by $p^*f = \frac{\hat{g}(\xi,\xi)}{2}\big|_P$, where $p : P \to M$ denotes the projection. Assume that $\lambda := \operatorname{sgn}\left(\hat{g}(\xi,\xi) - \frac{4}{\hat{g}(\hat{X},\hat{X})}\right)\big|_P$ is constant and non-zero.

Then (M, g, J_1, J_2, J_3, f) fulfills the assumptions of the HK/QK correspondence.

Choose a codimension one submanifold $M' \subset P$ transversal to $\hat{J}_1\xi\big|_P$. Let

$(M', g', Q = \mathrm{span}_{\mathbb{R}}\{J'_1, J'_2, J'_3\})$ *be the quaternionic pseudo-Kähler manifold obtained from the HK/QK correspondence with the choices* $(P, \eta := \sigma\frac{\hat{g}(\hat{X},\cdot)}{\hat{g}(\hat{X},\hat{X})}\big|_P, M')$. *Then* g', J'_1, J'_2, J'_3 *are identical to the data on* M' *induced from* $\hat{M}$ *by Theorem 3.2.6.*

***Proof*:** P is an S^1-principal bundle over M, $p : P \to M$, with fundamental vector field[8] $X_P := \sigma\hat{X}|_P \in \mathfrak{X}(P)$.

$$\eta = \sigma\frac{\hat{g}(\hat{X},\cdot)}{\hat{g}(\hat{X},\hat{X})}\bigg|_P \in \Omega^1(P) \tag{4.38}$$

defined an S^1-principal conncetion on P, i.e. $\eta(X_P) = 1$, $\mathcal{L}_{X_P}\eta = 0$. The functions $\hat{g}(\xi,\xi)/2\big|_P$ and $2/\hat{g}(\hat{X},\hat{X})\big|_P$ are X_P-invariant, so they define functions f, $f_1 \in C^\infty(M)$ via

$$p^*f = \frac{\hat{g}(\xi,\xi)}{2}\bigg|_P, \quad p^*f_1 = \frac{2}{\hat{g}(\hat{X},\hat{X})}\bigg|_P. \tag{4.39}$$

The vector field $\hat{Z} := \hat{J}_1\xi$ is tangent to P and commutes with $\hat{X}$, so $Z_1^P := \hat{J}_1\xi|_P$ induces a vector field $Z \in \mathfrak{X}(M)$.

Recall that the Kähler forms $\omega_1, \omega_2, \omega_3$ on the hyper-Kähler quotient M are defined by

$$p^*\omega_\alpha = \hat{\omega}_\alpha\big|_P, \quad \alpha = 1, 2, 3.$$

Since $\hat{Z}$ is $\hat{J}_1$-holomorphic, Killing and fulfills $\mathcal{L}_{\hat{Z}}\hat{J}_2 = -2\hat{J}_3$, we have $\mathcal{L}_{\hat{Z}}\hat{\omega}_1 = 0$, $\mathcal{L}_{\hat{Z}}\hat{\omega}_2 = -2\hat{\omega}_3$ and $\mathcal{L}_{\hat{Z}}\hat{\omega}_3 = 2\hat{\omega}_2$. This implies $\mathcal{L}_Z\omega_1 = 0$, $\mathcal{L}_Z\omega_2 = -2\omega_3$ and $\mathcal{L}_Z\omega_3 = 2\omega_2$, from which we conclude that

$$\mathcal{L}_Z g = 0, \quad \mathcal{L}_Z J_1 = 0, \quad \mathcal{L}_Z J_2 = -2J_3, \tag{4.40}$$

i.e. Z is a J_1-holomorphic Killing vector field that rotates J_2 and J_3. Since $\hat{Z} = \hat{J}_1\xi$ is tangent to P, we have

$$\begin{aligned} p^*df &= \frac{1}{2}d(\hat{g}(\xi,\xi))\big|_P \overset{(3.16)}{=} \hat{g}(\xi,\cdot)\Big|_P = -\hat{\omega}_1(\hat{J}_1\xi,\cdot)\Big|_P \\ &= -\hat{\omega}_1\big|_P(\hat{Z}|_P,\cdot) = -(p^*\omega_1)(Z_1^P,\cdot) = -(p^*\omega_1)(\tilde{Z},\cdot), \end{aligned}$$

[8]The extra sign σ is purely conventional and chosen to match the definitions in Chapter 4.

where $\tilde{Z} \in \Gamma(\ker\eta)$ denotes the horizontal lift of $Z \in \mathfrak{X}(M)$ to P. This implies

$$df = -\omega_1(Z, \cdot), \tag{4.41}$$

i.e. $-f$ is a ω_1-Hamiltonian function for Z. By our assumptions, $\operatorname{sgn} f = \sigma$ and $\sigma_1 := \operatorname{sgn} f_1 = \operatorname{sgn} \hat{g}(\hat{X}, \hat{X})$ are constant and non-zero.

The metric g on M is related to the metric $g_P := \hat{g}|_P$ by

$$g_P = \left(\frac{1}{\hat{g}(\hat{X}, \hat{X})}(\hat{g}(\hat{X}, \cdot))^2\right)\Big|_P + p^*g = \frac{2}{p^*f_1}\eta^2 + p^*g. \tag{4.42}$$

Splitting Z_1^P into horizontal and vertical part, one gets $Z_1^P = \tilde{Z} + aX_P$, where

$$a = \eta(Z_1^P) = \sigma\frac{\hat{g}(\hat{X}, \hat{J}_1\xi)}{\hat{g}(\hat{X}, \hat{X})}\Bigg|_P = \frac{-2\sigma\hat{\mu}_1^{\hat{X}}}{\hat{g}(\hat{X}, \hat{X})}\Bigg|_P = \frac{2}{\hat{g}(\hat{X}, \hat{X})}\Bigg|_P = p^*f_1,$$

i.e.

$$Z_1^P = \tilde{Z} + (p^*f_1)X_P. \tag{4.43}$$

Since

$$p^*f = \frac{1}{2}g_P(Z_1^P, Z_1^P) \overset{(4.42)}{\underset{(4.43)}{=}} \frac{1}{2}(p^*g)(\tilde{Z}, \tilde{Z}) + p^*f_1,$$

we get

$$f_1 = f - \frac{1}{2}\beta(Z), \tag{4.44}$$

where $\beta := g(Z, \cdot)$. Note that

$$p^*(g(Z, Z)) = 2p^*(f - f_1) = \left(\hat{g}(\xi, \xi) - \frac{4}{\hat{g}(\hat{X}, \hat{X})}\right)\Big|_P, \tag{4.45}$$

i.e. $Z \in \mathfrak{X}(M)$ is space- or time-like by assumption. $p^*\beta \in \Omega^1(P)$ can be expressed as follows:

$$\begin{aligned} p^*\beta &= (p^*g)(\tilde{Z}, \cdot) = g_P(\tilde{Z}, \cdot) = g_P(Z_1^P - (p^*f_1)X_P, \cdot) \\ &= \hat{g}(\hat{J}_1\xi, \cdot)\Big|_P - \frac{2\sigma}{\hat{g}(\hat{X}, \hat{X})}\hat{g}(\hat{X}, \cdot)\Big|_P. \end{aligned}$$

Together with the fact that $\hat{\omega}_1 \overset{(3.14)}{=} \frac{1}{2}d\big(\hat{g}(\hat{J}_1\xi,\cdot)\big)$, this allows us to calculate the curvature of the principal connection η on P as follows:

$$\begin{aligned} d\eta &= \sigma d\Big(\frac{\hat{g}(\hat{X},\cdot)}{\hat{g}(\hat{X},\hat{X})}\Big)\Big|_P = \frac{1}{2}d\big(\hat{g}(\hat{J}_1\xi,\cdot)\big)\Big|_P - \frac{1}{2}d\Big(\hat{g}(\hat{J}_1\xi,\cdot) - \frac{2\sigma\hat{g}(\hat{X},\cdot)}{\hat{g}(\hat{X},\hat{X})}\Big)\Big|_P \\ &= \hat{\omega}_1\big|_P - \frac{1}{2}dp^*\beta = p^*(\omega_1 - \frac{1}{2}d\beta). \end{aligned} \tag{4.46}$$

In total, we have shown that (M, g, J_1, J_2, J_3, f) fulfills the assumptions of the HK/QK correspondence and that (P, η, M') is a valid choice of S^1-bundle with principal connection and codimension one submanifold.

Define $\theta_0^P := \frac{1}{2}p^*df = \frac{1}{2}\hat{g}(\xi,\cdot)\big|_P$ and $\theta_\alpha^P := \frac{1}{2}\hat{g}(\hat{J}_\alpha\xi,\cdot)\big|_P$. Then

$$\theta_1^P = \frac{1}{2}\hat{g}(\hat{Z},\cdot)\big|_P = \frac{1}{2}g_P(Z_1^P,\cdot) = \frac{1}{2}g_P(\tilde{Z}+f_1X_P,\cdot) = \eta+\frac{1}{2}(p^*g)(\tilde{Z},\cdot) = \eta+\frac{1}{2}p^*\beta,$$

$$\theta_2^P = \frac{1}{2}\hat{\omega}_3(\hat{Z},\cdot)\big|_P = \frac{1}{2}(p^*\omega_3)(Z_1^P,\cdot) = \frac{1}{2}(p^*\omega_3)(\tilde{Z},\cdot) = \frac{1}{2}p^*(\iota_Z\omega_3)$$

and similarly, $\theta_3^P = -\frac{1}{2}p^*(\iota_Z\omega_2)$, i.e. the definitions here (and in Section 3.3) agree with the ones from Chapter 4. Recall that the geometric data g', J_1', J_2', J_3' on M' defined by the HK/QK correspondence is uniquely determined by the components

$$\bar{\theta}_\alpha := \Big(\frac{1}{p^*f}\theta_\alpha^P\Big)\Big|_{M'} \in \Omega^1(M')$$

of the $Sp(1)$-connection one-form. The data induced from $\widehat{M}$ by Theorem 3.2.6 is also uniquely determined by the components of the $Sp(1)$-connection one-form. In this case, these are given by

$$\theta_\alpha\big|_{M'} = \Big(\frac{1}{\hat{g}(\xi,\xi)}\hat{g}(\hat{J}_\alpha\xi,\cdot)\Big)\Big|_{M'} = \Big(\frac{1}{p^*f}\theta_\alpha^P\Big)\Big|_{M'} = \bar{\theta}_\alpha.$$

This proves the theorem. □

Remark 4.2.2 Note that in the above theorem, we can drop the assumption that λ is constant and non-zero, i.e. we can allow $g(Z,Z)$ to vanish. In this case, we can still apply the HK/QK correspondence (see Remark 4.1.8) and Theorem 3.2.6 then shows that the result is still quaternionic pseudo-Kähler.

Example 4.2.3 Let $c \in \mathbb{R}^*$. For $c > 0$, we consider $\widehat{M}_+ = \mathbb{H}^{n+1}\backslash\{0\}$ and for $c < 0$, we consider $\widehat{M}_- = \mathbb{H}^{n,1}_{<0}$. We endow $\widehat{M}_\pm$ with the standard conical (pseudo-)hyper-Kähler structure (see Example 3.2.10) and with the tri-holomorphic Killing vector field

$$\hat{X} = \frac{4}{|c|}\operatorname{Im}(w_0\partial_{w_0} - z^0\partial_{z^0})$$

(see Ex. 3.4.6). We use the notation $\hat{q} = (q^0, q = (q^1, \dots, q^n)) = \hat{z} + j\hat{w} \in \widehat{M}_\pm$. The level set $P_\pm = \{\mu^{\hat{X}} = (\mp 1, 0, 0)\}$ of the homogeneous hyper-Kähler moment map associated with $\hat{X}$ reads

$$P_\pm = \{\hat{q} = \hat{z} + j\hat{w} \in \widehat{M}_\pm \mid |z^0|^2 = |c|,\, w_0 = 0\}$$

(see Example 3.4.6). The hyper-Kähler quotient $\widehat{M}_\pm /\!/\!/_{P_\pm} S^1_{(\hat{X})}$ is isomorphic to $M_\pm = \{q \in \mathbb{H}^n \mid \mp\|q\|^2 < |c|\} \subset \mathbb{H}^n$ (i.e. $M_+ = \mathbb{H}^n$). The function $f \in C^\infty(M_\pm)$ induced by $\hat{g}(\xi,\xi)/2$ reads $f = \frac{c+\|q\|^2}{2}$. Similarly to Example 3.2.10, the quaternionic Kähler structure on $M'_\pm := \{\hat{q} \in P_\pm \mid q^0 = \sqrt{|c|}\} \approx M_\pm$ induced from $\widehat{M}_\pm$ via Theorem 3.2.6 is isomorphic to the chart $\{q^0 \neq 0\} \subset \mathbb{H}P^n$ and to $\mathbb{H}H^n$, respectively. Theorem 4.2.1 then establishes the following HK/QK correspondence[9]:

$$\begin{tikzcd} & \mathbb{H}^{n+1}_{>0} \arrow[dl, mapsto, "/\!/\!/_{P_+} S^1_{(q^0)}"'] \arrow[d, phantom, "(c>0)"] \arrow[dr, bend left] & \\ \mathbb{H}^n \arrow[rr, mapsto, "f=(\|q\|^2+c)/2", "\text{HK/QK cor.}"'] & {} & \{q^0 \neq 0\} \subset \mathbb{H}P^n \end{tikzcd} \quad , \quad \begin{tikzcd} & \mathbb{H}^{n,1}_{<0} \arrow[dl, mapsto, "/\!/\!/_{P_-} S^1_{(q^0)}"'] \arrow[d, phantom, "(c<0)"] \arrow[dr, bend left] & \\ \mathbb{H}^n_{<|c|} \arrow[rr, mapsto, "f=(\|q\|^2+c)/2", "\text{HK/QK cor.}"'] & {} & \mathbb{H}H^n \end{tikzcd} .$$

This example for the HK/QK correspondence was already constructed in a direct way in Example 4.1.12.

For our purposes of studying examples, the relation between conical pseudo-hyper-Kähler manifolds and the HK/QK correspondence given in Theorem 4.2.1 is sufficient. For a more global understanding and to do the title of this section justice, we consider the situation where the conical pseudo-hyper-Kähler manifold is a Swann bundle for the rest of the section.

Remark 4.2.4 If in the above theorem, $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$, $\hat{\pi} : \widehat{M} \to \bar{M}$ is the

[9] To be precise, $\mathbb{H}^n$ does not fulfill all assumptions of the HK/QK correspondence since $\hat{g}(\xi,\xi) - \frac{4}{\hat{g}(\hat{X},\hat{X})}$ vanishes at $\{\hat{q} = (q^0, q) \in P_\pm \mid q = 0\}$. Nevertheless, we can apply the HK/QK correspondence (see Remark 4.2.2) and we see that, in this example, the result is still quaternionic Kähler.

Swann bundle over a quaternionic (pseudo-)Kähler manifold $(\bar{M}, \bar{g}, Q)$ and $\hat{X}$ the unique lift of a Killing vector field $X \in \mathfrak{X}(\bar{M})$ given by Corollary 3.6.9, then the assumption that $\hat{X}$ is space- or time-like is equivalent to the assumption that $\sigma_1 := \operatorname{sgn}(\bar{g}(X,X) + \nu\|\mu^X\|^2)$ is constant and non-zero, where $\mu^X \in \Gamma(Q)$ is the quaternionic Kähler moment map associated with X (see Proposition 3.6.10). In this situation, Z_1^P induces a free S^1-action on P and P/Z_1^P is diffeomorphic to $\bar{M}^o := \bar{M}\backslash\{\mu^X = 0\}$ (see Remark 3.3.5).

Proposition 4.2.5 *Let $(\bar{M}, \bar{g}, Q, X)$ be a quaternionic (pseudo-)Kähler manifold with a Killing vector field and $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$, $\hat{\pi} : \widehat{M} \to \bar{M}$ the Swann bundle over $\bar{M}$ and assume that the unique lift $\hat{X} \in \mathfrak{X}(\widehat{M})$ of X given by Corollary 3.6.9 is space- or time-like and induces a free S^1-action on $\widehat{M}$. Let $Z \in \mathfrak{X}(M)$ denote the vector field induced by $\hat{J}_1\xi$ on the hyper-Kähler quotient $(M := P/S^1, g, J_1, J_2, J_3)$ with level set $P = \{\hat{\mu}^{\hat{X}} = (-\sigma, 0, 0)\}$ $(\sigma = \operatorname{sgn}\hat{g}(\xi,\xi))$ of the homogeneous hyper-Kähler moment map $\hat{\mu}^{\hat{X}}$ associated with $\hat{X}$. If $p : P \to M$ denotes the projection, then*

$$\hat{\pi}\big(p^{-1}(\{g(Z,Z) = 0\})\big) = \{\bar{g}(X,X) = 0\} \subset \bar{M}. \tag{4.47}$$

***Proof*:** Note that due to the requirement that $\hat{X}$ is space- or time-like, $\bar{g}(X,X)$ and μ^X cannot vanish simultaneously. Together with (3.103) and the fact that

$$1 = \|\hat{\mu}^{\hat{X}}\|^2\big|_P = \frac{\nu^2 r^4}{16}\hat{\pi}^*\|\mu^X\|^2\big|_P,$$

Equation (4.45) implies

$$p^*(g(Z,Z)) = \sigma\, r^2 \hat{\pi}^*\Big(\frac{\bar{g}(X,X)}{\bar{g}(X,X) + \nu\|\mu^X\|^2}\Big)\Big|_P.$$

□

The following obvious corollary allows for the global understanding of Theorem 4.2.1 in the following Remark 4.2.7.

Corollary 4.2.6 *Let $(\bar{M}, \bar{g}, Q, X)$ be a connected quaternionic (pseudo-)Kähler manifold endowed with a space- or time-like Killing vector field X such that $\sigma_1 := \operatorname{sgn}(\bar{g}(X,X) + \nu\|\mu^X\|^2)$ is constant and non-zero, where $\mu^X \in \Gamma(Q)$ is the quaternionic Kähler moment map associated with X. Let $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ be the Swann bundle over $\bar{M}$ and $\sigma := \operatorname{sgn}\nu$ the sign of the scalar curvature of $\bar{g}$. Assume that the unique lift $\hat{X} \in \mathfrak{X}(\widehat{M})$ of X given by Corollary 3.6.9* *induces a free S^1-action on $\widehat{M}$. Consider the hyper-Kähler-quotient $\widehat{M}/\!/\!/_P S_1 = (M := P/S_1, g, J_1, J_2, J_3)$ with respect to the level set $P := \{\hat{\mu}^{\hat{X}} = (-\sigma, 0, 0)\}$ of the homogeneous hyper-Kähler moment map $\hat{\mu}^{\hat{X}}$ associated with $\hat{X}$ and define a function $f \in C^\infty(M)$ by $p^*f = \frac{\hat{g}(\xi,\xi)}{2}\Big|_P$, where $p : P \to M = P/S_1$ denotes the projection.*

Then (M, g, J_1, J_2, J_3, f) fulfills the assumptions of the HK/QK correspondence.

Let $M' \subset P$ be a codimension one submanifold that intersects each $(\mathbb{R}^{>0} \times SO(3))$-orbit in $\widehat{M}$ at most once, $U := \hat{\pi}(M')$. Let (M', g', Q', X') be the quaternionic pseudo-Kähler manifold with Killing vector field obtained from (M, g, J_1, J_2, J_3, f) via the HK/QK correspondence with the choices $(P, \eta := \sigma \frac{\hat{g}(\hat{X},\cdot)}{\hat{g}(\hat{X},\hat{X})}\Big|_P, M')$. Then (M, g', Q', X') is isomorphic to $(U, \frac{|\nu|}{4} g\big|_U, Q\big|_U, \sigma X\big|_U)$.

***Proof*:** $(\widehat{M}, \hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ is a conical (pseudo-)hyper-Kähler manifold. By Remark 4.2.4, the assumption that σ_1 is constant and non-zero implies that $\hat{X}$ is space- or time-like. By construction, $\hat{X}$ is tri-holomorphic and Killing and it commutes with ξ. Note that since $\widehat{M}$ is a Swann bundle, P is automatically non-empty. Otherwise, $\mu^X = 0$ everywhere on $\bar{M}$ (see Remark 3.3.5), which would imply $X = 0$ and thus contradicts the assumption that σ_1 is non-zero. The assumption that $\operatorname{sgn} \bar{g}(X, X)$ is constant and non-zero implies that $\lambda = \operatorname{sgn}\left(\hat{g}(\xi,\xi) - \frac{4}{\hat{g}(\hat{X},\hat{X})}\right)\big|_P \overset{(4.45)}{=} \operatorname{sgn} p^*(g(Z, Z))$ is constant and non-zero (see Proposition 4.2.5). Hence, we can apply Theorem 4.2.1.

By Theorem 4.2.1, (M, g, J_1, J_2, J_3, f) fulfills the assumptions of the HK/QK correspondence. The submanifold $M' \subset P$ is transversal to $\hat{J}_1\xi\big|_P$. According to Theorem 4.2.1, $g', Q' = \operatorname{span}_{\mathbb{R}}\{J'_1, J'_2, J'_3\}$ is the geometric data on M' induced from $\widehat{M}$ via Theorem 3.2.6. By Proposition 3.6.5, this agrees with $\frac{|\nu|}{4} g\big|_U, Q\big|_U$ on $U \subset \bar{M}$. The vector field X' on M' is induced by $X_P = \sigma\hat{X}\big|_P$ (see the proof of Theorem 4.2.1 and the definition of X' in Eq. (4.7)) and thus corresponds to $\sigma X\big|_U$. □

Remark 4.2.7 Note that in the situation of the above corollary,

$$\hat{\pi}\big|_P : P \to \bar{M}^o := \bar{M} \backslash \{\mu^X = 0\}$$

is an S^1-principle bundle, i.e. from the HK/QK correspondence, we globally recover $(\bar{M}^o, \bar{g}\big|_{\bar{M}^o}, Q\big|_{\bar{M}^o}, \sigma X\big|_{\bar{M}^o})$, while the zero level-set of the quaternionic Kähler moment map associated with X can not be reconstructed (see Figure 4.3, where $\widehat{M}^o := \hat{\pi}^{-1}(\bar{M}^o)$).

$$
\begin{tikzcd}
 & \widehat{M}^o \arrow[ddl, "/\!\!/\!\!/_P S^1_{(\hat{X})}"'] \arrow[ddr, "CO(3) = \mathbb{R}^{>0} \times SO(3)"] & \\
 & P \arrow[u, phantom, "\bigcup"] \arrow[dl, "S^1_{(\hat{X}|_P)}"] \arrow[dr, "\bar{S}^1_{(\hat{J}_1\xi|_P)}"'] & \\
M \arrow[rr, mapsto, "\text{HK/QK corresp.}"'] & & \bar{M}^o
\end{tikzcd}
$$

Figure 4.3: Relation between the HK/QK correspondence, the hyper-Kähler quotient construction and the Swann bundle construction (see Remark 4.2.7).

Remark 4.2.8 Note that in the above corollary, we can drop the assumption that $g(X, X)$ is non-vanishing. The condition on $g(X, X) + \nu \|\mu^X\|^2$ ensures that the lifted vector field $\hat{X}$ is space- or time-like. In this case, we can still apply the HK/QK correspondence (see Remark 4.1.8 and Proposition 4.2.5) and since we started with a quaternionic (pseudo-)Kähler manifold, we know that the result is quaternionic (pseudo-)Kähler.

4.3 Compatibility of the HK/QK correspondence with quotient constructions

In the following theorem, we show the compatibility of the HK/QK correspondence with the (level zero) hyper-Kähler and the quaternionic Kähler quotient constructions in the situation of Theorem 4.2.1 (see Figure 4.4 for an illustration).

Theorem 4.3.1 *In the situation of Theorem 4.2.1, let G be a compact connected Lie group that acts on $\widehat{M}$, preserving $\hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi, \hat{X}$ and such that $\hat{g}$ is non-degenerate along the G-orbits and along the $(S^1 \times G)$-orbits. Assume that $S^1 \times G$ acts freely on $\widehat{M}$, where S^1 is the action induced by $\hat{X}$ and assume that M' is G-invariant. Let $\hat{\kappa}$ denote the homogeneous hyper-Kähler moment map for the G-action on $\widehat{M}$ and let κ denote the induced hyper-Kähler moment map for the G-action on M. Consider the hyper-Kähler quotient $M/\!\!/\!\!/_{\{\kappa=0\}}G = (\underline{M}, \underline{g}, \underline{J}_1, \underline{J}_2, \underline{J}_3)$. Let $\underline{f} \in C^\infty(\underline{M})$ be induced by $\frac{\hat{g}(\xi,\xi)}{2} \in C^\infty(\widehat{M})$.*

Then $(\underline{M}, \underline{g}, \underline{J}_1, \underline{J}_2, \underline{J}_3, \underline{f})$ and the quaternionic pseudo-Kähler manifold $M'/\!\!/\!\!/G$ are related via the HK/QK correspondence.

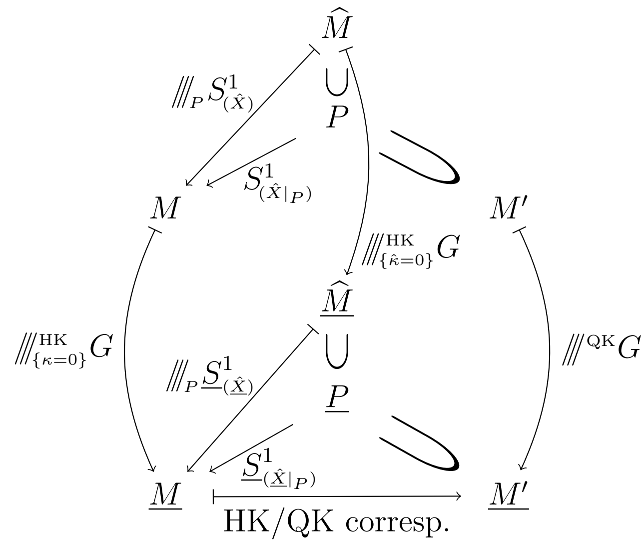


Figure 4.4: Compatibility of the HK/QK correspondence with the hyper-Kähler and quaternionic Kähler quotient constructions.

***Proof*:** The codimension four submanifold $M' \subset \widehat{M}$ is transversal to the vertical distribution $\mathcal{D}^v = \mathrm{span}_{\mathbb{R}}\{\xi, \hat{J}_1\xi, \hat{J}_2\xi, \hat{J}_3\xi\}$ (see the proof of Proposition 3.3.4). By Theorem 3.5.4, the hyper-Kähler quotient $\widehat{M}/\!/\!/_{\{\hat{\kappa}=0\}} G = (\underline{\widehat{M}}, \underline{\hat{g}}, \underline{\hat{J}_1}, \underline{\hat{J}_2}, \underline{\hat{J}_3})$ together with the induced vector field $\underline{\xi}$ is conical pseudo-hyper-Kähler and the quaternionic Kähler structure on the quaternionic Kähler quotient $M'/\!/\!/G = (\underline{M'}, \underline{g'}, \underline{Q} = \mathrm{span}_{\mathbb{R}}\{\underline{J'_1}, \underline{J'_2}, \underline{J'_3}\})$ is identical to the one induced from $\underline{\widehat{M}}$ on the codimension four submanifold $\underline{M'} \subset \underline{\widehat{M}}$. The vector field $\hat{X} \in \mathfrak{X}(\widehat{M})$ induces a tri-holomorphic Killing vector field $\underline{\hat{X}} \in \mathfrak{X}(\underline{\widehat{M}})$ that commutes with $\underline{\xi}$. By Theorem 4.2.1, the hyper-Kähler quotient of $\underline{\widehat{M}}$ with level set[10] $\underline{P} = \{\underline{\hat{\mu}}^{\underline{\hat{X}}} = (-\sigma, 0, 0)\}$ with respect to the S^1-action induced by $\underline{\hat{X}}$ (denoted by $\underline{S}^1$) is related to $M'/\!/\!/G$ via the HK/QK correspondence. The corresponding function needed in the HK/QK correspondence is induced by $\underline{\hat{g}}(\underline{\xi}, \underline{\xi})/2$, which itself is induced by $\hat{g}(\xi, \xi)/2$. According to Proposition 3.4.9 and Remark 3.4.10, $\underline{\widehat{M}}/\!/\!/_{\underline{P}} \underline{S}^1$ can be identified with $M/\!/\!/_{\{\kappa=0\}} G = (\underline{M}, \underline{g}, \underline{J_1}, \underline{J_2}, \underline{J_3})$. □

Remark 4.3.2 Note that the HK/QK correspondence given in the above theorem is performed with the choices $\underline{P} = (P \cap \{\hat{\kappa} = 0\})/G$, $\underline{\eta} = \sigma \frac{\underline{\hat{g}}(\underline{\hat{X}}, \cdot)}{\underline{\hat{g}}(\underline{\hat{X}}, \underline{\hat{X}})} \in \Omega^1(\underline{P})$ (which is induced by η) and $\underline{M'} = (M' \cap \{\hat{\kappa} = 0\})/G \subset \underline{P}$.

Remark 4.3.3 As always, the assumptions in the above theorem that ensure the smoothness of the respective quotients can be relaxed (see Remark 3.4.5).

[10]Here, the $\mathbb{R}^3$-valued function $\underline{\hat{\mu}}^{\underline{\hat{X}}}$ on $\underline{\widehat{M}}$ is the homogeneous hyper-Kähler moment map associated with $\underline{\hat{X}}$.

4.4 HK/QK correspondence for $T^*\mathbb{C}P^n$ and $T^*\mathbb{C}H^n$

Recall that for Hermitian symmetric spaces of compact type, there exists a complete hyper-Kähler metric on the cotangent bundle [BiGau]. In the non-compact case, the metric is incomplete and only defined on a certain neighborhood of the zero section in the cotangent bundle [BiGau]. The S^1-action on the (holomorphic) cotangent bundle given by fiberwise scalar multiplication of $e^{it} \in S^1$ fulfills the assumptions of the HK/QK correspondence, i.e. the fundamental vector field Z generating it is a rotating J_1-holomorphic Killing vector field.

In this section, we apply the HK/QK correspondence to the chart $T^*(\{z^0 \neq 0\}) \subset T^*\mathbb{C}P^n$ and to the neighborhood $\{\tilde{r}^2 < 1\} \subset T^*\mathbb{C}H^n$ of the zero section. The hyper-Kähler structure for these two examples has been constructed by a hyper-Kähler quotient from quaternionic vector space with positive, respectively quaternionic Lorentzian signature in Example 3.4.7.

Let

$$\begin{aligned} M_+ = \{(\zeta,\eta) \in \mathbb{C}^{2n}\} &\approx^{11} \{[z,w]_{\mathbb{C}^*} \mid z^0 \neq 0,\, z\cdot w = 0\} \\ &\subset \{[z,w]_{\mathbb{C}^*} \mid z \in \mathbb{C}^{n+1}\backslash\{0\},\, w \in \mathbb{C}^{n+1},\, z \cdot w = 0\} \approx T^*\mathbb{C}P^n \end{aligned} \tag{4.48}$$

and

$$M_- = \{\tilde{r}^2 < 1,\, \|\zeta\|^2 < 1\} \subset \{(\zeta,\eta) \in \mathbb{C}^{2n} \mid \|\zeta\|^2 < 1\} = T^*\mathbb{C}H^n, \tag{4.49}$$

where

$$\tilde{r} := \sqrt{4(1 \pm \|\zeta\|^2)\Big(\pm(\zeta\cdot\eta)(\bar{\zeta}\cdot\bar{\eta}) + \|\eta\|^2\Big)}. \tag{4.50}$$

Let J_1 be the standard complex structure on $M_- \subset M_+ = \mathbb{C}^{2n}$ such that (ζ,η) are J_1-holomorphic coordinates. We define a hyper-Kähler structure $(g^{(\pm)}, J_1, J_2^{(\pm)}, J_3^{(\pm)})$ on $M_\pm$ by $\omega_+ = \omega_2 + i\omega_3 = \sum_{\mu=1}^n d\zeta^\mu \wedge d\eta_\mu$ and $\omega_1^{(\pm)} = \frac{i}{2}\partial_{J_1}\bar{\partial}_{J_1}K_\pm$, where

$$K_\pm := \pm\sqrt{1 \pm \tilde{r}^2} \mp \log\left(\frac{1+\sqrt{1\pm\tilde{r}^2}}{1 \pm \|\zeta\|^2}\right) \tag{4.51}$$

(see Example 3.4.7).

Now, we apply the HK/QK correspondence to $(M_\pm, g^{(\pm)}, J_1, J_2^{(\pm)}, J_3^{(\pm)}, f^{(\pm)})$,

[11] $\mathbb{C}^*$-action on $\mathbb{C}^{2n+2}$: $(z,w) \mapsto (\lambda z, \lambda^{-1}w)$, $\lambda \in \mathbb{C}^*$.
Biholomorphism: $\zeta = (\frac{z^1}{z^0}, \ldots, \frac{z^n}{z^0})$, $\eta = (z^0 w_1, \ldots, z^0 w_n)$.

where

$$f^{(\pm)} := \pm\frac{\sqrt{1 \pm \tilde{r}^2} + c}{2} \tag{4.52}$$

for some $c \in \mathbb{R}$. The vector field

$$Z := 2i\sum_{\mu=1}^{n}\Big(\eta_\mu\frac{\partial}{\partial\eta_\mu} - \bar{\eta}_\mu\frac{\partial}{\partial\bar{\eta}_\mu}\Big) \tag{4.53}$$

is induced by the J_1-holomorphic and isometric action $(\zeta, \eta) \mapsto (\zeta, \mathrm{e}^{2it}\eta)$ of $\mathrm{e}^{it} \in S^1$ on $M_\pm$. Under this action, $\omega_+ \mapsto e^{2it}\omega_+$ and hence $\mathcal{L}_Z\omega_2 = -2\omega_3$. A direct calculation gives

$$\begin{aligned} -\omega_1^{(\pm)}(Z, \cdot) &= 2\,\frac{1 \pm \|\zeta\|^2}{\sqrt{1 \pm \tilde{r}^2}}\sum_{\mu=1}^{n}\mathrm{Re}\left(\bar{\eta}_\mu A_\mu \pm (\bar{\zeta}\cdot\bar{\eta})\,\zeta^\mu A_\mu\right) \\ &= \frac{1}{4}\frac{1}{\sqrt{1 \pm \tilde{r}^2}}d(\tilde{r}^2) = df^{(\pm)}, \end{aligned} \tag{4.54}$$

where for $\mu = 1, \ldots, n$,

$$A_\mu := d\eta_\mu \pm \frac{1}{1 \pm \|\zeta\|^2}\sum_{\sigma=1}^{n}(\bar{\zeta}^\mu\eta_\sigma + \bar{\zeta}^\sigma\eta_\mu)d\zeta^\sigma \in \Omega_{J_1}^{(1,0)}(M_\pm). \tag{4.55}$$

We have

$$\begin{aligned} \beta = g^{(\pm)}(Z, \cdot) &= d^c_{J_1}f^{(\pm)} = -J_1^* df^{(\pm)} \\ &= 2\,\frac{1 \pm \|\zeta\|^2}{\sqrt{1 \pm \tilde{r}^2}}\sum_{\mu=1}^{n}\mathrm{Im}\left(\bar{\eta}_\mu A_\mu \pm (\bar{\zeta}\cdot\bar{\eta})\,\zeta^\mu A_\mu\right), \end{aligned} \tag{4.56}$$

$\beta(Z) = g^{(\pm)}(Z, Z) = \frac{\tilde{r}^2}{\sqrt{1 \pm \tilde{r}^2}}$ and

$$f_1^{(\pm)} = \pm\frac{1}{2}\left(\frac{1}{\sqrt{1 \pm \tilde{r}^2}} + c\right). \tag{4.57}$$

For f and f_1 to be non-zero everywhere on $M_\pm$, we assume that $c \geq 0$. For Z to be non-zero everywhere on $M_\pm$, we would need to restrict to $\{\tilde{r} > 0\}$, but we can apply the HK/QK correspondence without this restriction (see Remark 4.1.8).

We endow the trivial S^1-principal bundle $\pi = \mathrm{pr}_1 : P := M_\pm \times S^1 \to M_\pm$ with the principal connection $\eta = ds + \eta_M$, where

$$\eta_M = \frac{1}{4} d^c_{J_1}(K_\pm - 2f^{(\pm)}) = \frac{1}{4} d^c_{J_1}(\mp \log \rho_\pm{}^2) = \mp \frac{1}{2\rho_\pm} d^c_{J_1} \rho_\pm, \tag{4.58}$$

$$\rho_\pm := \sqrt{\frac{1}{2}\frac{1+\sqrt{1\pm \tilde{r}^2}}{1 \pm \|\zeta\|^2}}\ . \tag{4.59}$$

Here, s is the natural coordinate on $S^1 = \{e^{is} \mid s \in \mathbb{R}\}$. The principal curvature is then

$$d\eta = \frac{1}{4} dd^c_{J_1} K_\pm - \frac{1}{2} dd^c_{J_1} f^{(\pm)} = \omega_1^{(\pm)} - \frac{1}{2} d\beta.$$

The one-forms on P defined in Eq. (4.5) read

$$\begin{aligned}
\theta_0^P &= \frac{1}{2} df^{(\pm)} = \pm \frac{d\sqrt{1 \pm \tilde{r}^2}}{4}, \\
\theta_1^P &= \eta + \frac{1}{2} d\beta = ds + \frac{1}{4} d^c_{J_1} K_\pm, \\
\theta_2^P &= \frac{1}{2}\omega_3(Z, \cdot) = -\sum_{\mu=1}^n \mathrm{Re}(\eta_\mu d\zeta^\mu), \\
\theta_3^P &= -\frac{1}{2}\omega_2(Z, \cdot) = -\sum_{\mu=1}^n \mathrm{Im}(\eta_\mu d\zeta^\mu).
\end{aligned}$$

The (pseudo-)Riemannian metric on P is given by

$$\begin{aligned}
g_P = \frac{2}{f_1^{(\pm)}}\eta^2 + g^{(\pm)} \overset{(3.76)}{=} &\pm \frac{4\sqrt{1\pm\tilde{r}^2}}{1 + c\sqrt{1 \pm \tilde{r}^2}}\left(ds \mp \frac{1}{2\rho_\pm} d^c_{J_1}\rho_\pm\right)^2 \\
&+ \sum_{\mu=1}^n \left(\rho_\pm^2\, d\zeta^\mu d\bar{\zeta}^\mu + \rho_\pm^{-2} d\eta_\mu d\bar{\eta}_\mu\right) \\
&\pm \rho_\pm^{-2}\Big|d\Big(\sum_{\mu=1}^n \zeta^\mu \eta_\mu\Big)\Big|^2 \mp 4\rho_\pm^{-2}\sqrt{1\pm\tilde{r}^2}\ |\partial_{J_1}\rho_\pm|^2
\end{aligned} \tag{4.60}$$

and the Killing vector field reads

$$Z_1^P = \pm \frac{1+c}{2}\frac{\partial}{\partial s} + 2i \sum_{\mu=1}^n \big(\eta_\mu \frac{\partial}{\partial \eta_\mu} - \bar{\eta}_\mu \frac{\partial}{\partial \bar{\eta}_\mu}\big). \tag{4.61}$$

Since $c \neq -1$, we can choose $M'_\pm = \{0\} \times M_\pm = \{s = 0\} \subset P$ as a Z_1^P-transversal submanifold. According to Theorem 4.1.2,

$$\begin{aligned} g'^c_\pm = \frac{1}{c+\sqrt{1\pm\tilde r^2}} \Bigg(&\mp \rho_\pm^{-2}\sqrt{1\pm\tilde r^2}\,(d\rho_\pm)^2 \mp \frac{1\pm\tilde r^2}{1+c\sqrt{1\pm\tilde r^2}} c\rho_\pm^{-2}(d^c_{J_1}\rho_\pm)^2 \\ &+ \sum_{\mu=1}^n (\rho_\pm^2\, d\zeta^\mu d\bar\zeta^\mu + \rho_\pm^{-2} d\eta_\mu d\bar\eta_\mu) \pm \rho_\pm^{-2}\big|d\big(\sum_{\mu=1}^n \zeta^\mu\eta_\mu\big)\big|^2 \\ &\mp \frac{1}{c+\sqrt{1\pm\tilde r^2}}\Big(4\big|\sum_{\mu=1}^n \eta_\mu d\zeta^\mu\big|^2 + \frac14\big(d\sqrt{1\pm\tilde r^2}\big)^2 + \frac14\big(d^c_{J_1}K_\pm\big)^2\Big)\Bigg) \end{aligned} \tag{4.62}$$

is a positive definite quaternionic Kähler metric of positive, respectively negative scalar curvature on $M'_\pm \approx M_\pm$.

Proposition 4.4.1 *(M'_-, g'_-) is incomplete for $c > 0$.*

***Proof*:** Consider the curve $\gamma : (0,1) \to M_-$,

$$\zeta^1 = \ldots = \zeta^n = 0, \quad \eta_1 = \frac{t}{2}, \quad \eta_2 = \ldots = \eta_n = 0, \quad 0 < t < 1,$$

which approaches the boundary $\{\tilde r^2 = 1\}$ of M_- for $t \to 1$. Its length ℓ_γ is given by

$$\begin{aligned} &\int_0^1 \sqrt{\frac14 g'_-\Big(\frac{\partial}{\partial\eta_1}+\frac{\partial}{\partial\bar\eta_1}, \frac{\partial}{\partial\eta_1}+\frac{\partial}{\partial\bar\eta_1}\Big)}\,\Bigg|_{\gamma(t)} dt \\ &= \int_0^1 \sqrt{\frac{1}{c+\sqrt{1-t^2}}\left(\frac{2\sqrt{1-t^2}}{1+\sqrt{1-t^2}}\Big(\frac{d}{dt}\sqrt{\frac{1+\sqrt{1-t^2}}{2}}\Big)^2 + \frac12\frac{1}{1+\sqrt{1-t^2}} + \frac14\frac{\big(\frac{d}{dt}\sqrt{1-t^2}\big)^2}{c+\sqrt{1-t^2}}\right)}\, dt \end{aligned}$$

Here, we used that $d^c(t) = d^c(\eta_1+\bar\eta_1) = 0$. Denote the three summands under the square root in the above integral by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. Then

$$\ell_\gamma = \int_0^1 \sqrt{\boldsymbol{a}+\mathbf{b}+\mathbf{c}}\, dt \le \int_0^1 (\sqrt{\mathbf{a}}+\sqrt{\mathbf{b}}+\sqrt{\mathbf{c}})\, dt,$$

where

$$\int_0^1 \sqrt{\mathbf{a}}\, dt \le \frac12\int_0^1 \frac{t}{\sqrt{1-t^2}}\frac{1}{1+\sqrt{1-t^2}}\, dt = -\frac12\log(1+\sqrt{1-t^2})\Big|_0^1$$

$$= \frac{1}{2}\log 2 < \infty,$$

$$\int_0^1 \sqrt{\mathbf{b}}\, dt \leq \frac{1}{\sqrt{2}} \int_0^1 \frac{1}{\sqrt{1-t^2}}\, dt = \frac{1}{\sqrt{2}} \arcsin t \Big|_0^1 = \frac{1}{\sqrt{2}} \frac{\pi}{2} < \infty,$$

$$\int_0^1 \sqrt{\mathbf{c}}\, dt = \frac{1}{2} \int_0^1 \frac{1}{c+\sqrt{1-t^2}} \frac{t}{\sqrt{1-t^2}}\, dt = -\frac{1}{2} \log(c+\sqrt{1-t^2}) \Big|_0^1$$
$$= \frac{1}{2} \log(1+\frac{1}{c}) \overset{(c>0)}{<} \infty.$$

This shows that ℓ_γ is finite and consequently that (M'_-, g'_-) is incomplete. □

Remark 4.4.2 Note that in the above proof the length of γ fulfills

$$\ell_\gamma \geq \frac{1}{2} \int_0^1 \frac{1}{c+\sqrt{1-t^2}} \frac{t}{\sqrt{1-t^2}}\, dt = -\frac{1}{2} \log(c+\sqrt{1-t^2}) \Big|_0^1.$$

For $c = 0$, the above integral diverges. In fact, we will see in the next subsection that for $c = 0$, (M'_-, g'_-) is isometric to $\mathbb{H}H^n$ and hence complete.

4.4.1 $c = 0$

The case $c = 0$ can be analyzed using the reverse construction of the HK/QK correspondence given in Theorem 4.2.1.

Proposition 4.4.3 *For $c = 0$, the quaternionic Kähler manifold (M'_+, g'_+) is isometric to the chart $\{q^0 \neq 0\}$ in*[12] *$(\mathbb{H}P^n)^o$ and (M'_-, g'_-) is isometric to $\mathbb{H}H^n$.*

***Proof*:** This proposition can be proven by applying Theorem 4.2.1 to $\hat{M}_+ = \mathbb{H}^{n+1}_{>0}$, respectively $\hat{M}_- = \mathbb{H}^{n,1}_{<0}$, endowed with the Killing vector field $\hat{X}$ induced by diagonal left-multiplication of $e^{it} \in S^1$ (scaled by a factor of two) with the choices M'_+ and M'_- as submanifolds of $P_\pm$. This has essentially already been done in Example 3.3.6 and Remarks 3.3.7 and 3.3.8. The result was summarized in Corollary 3.3.9 and Remark 3.3.10.

Note that M'_+ and M'_- can be identified with the manifolds $M_+ = \{(\zeta, \eta) \in \mathbb{C}^{2n}\}$ and $M_- = \{(\zeta, \eta) \in \mathbb{C}^{2n} \mid \tilde{r}^2 < 1,\ \|\zeta\|^2 < 1\}$. It is enough to check that the metric given in Eq. (4.62) agrees with Eqs. (3.68) and (3.67) for $c = 0$. □

[12] $(\mathbb{H}P^n)^o = \mathbb{H}P^n \backslash \{[q = z + jw]_{\mathbb{H}^*_{\text{right}}} \mid (z, w) \in \mathbb{C}^{2n+2}\backslash\{0\},\ \|z\|^2 = \|w\|^2,\ z \cdot w = 0\}$, see Eq. (3.56).

$$\begin{tikzcd}[column sep=small]
 & (\mathbb{H}^{n+2}_{>0})^o \arrow[dl, mapsto, "/\!\!/\!\!/_{\{z^{n+2}=0,\,|w_{n+2}|^2=1\}} S^1_{(q^{n+2})}"'] \arrow[dr, "\mathbb{H}^*"] & \\
\mathbb{H}^{n+1}_{>0} \arrow[d, mapsto, "/\!\!/\!\!/^{\mathrm{HK}}_{\{\lambda=1,\,\chi=0\}} S^1_{(\mathrm{diag.})}"'] & & (\mathbb{H}P^{n+1})^o \arrow[d, mapsto, "/\!\!/\!\!/^{\mathrm{QK}} S^1_{(\mathrm{diag.})}"] \\
T^*\mathbb{C}P^n \arrow[rr, mapsto, "f=\frac{1+\sqrt{1+\tilde{r}^2}}{2}", "\text{HK/QK corresp.}"'] & & (X(n))^o
\end{tikzcd}.$$

Figure 4.5: Analysis of the HK/QK correspondence for $T^*\mathbb{C}P^n$ with parameter $c = 1$ using the compatibility of the correspondence with quotient constructions. Here, $(\mathbb{H}^{n+2}_{>0})^o = \{\hat{q} = (q, q^{n+2}) \in \mathbb{H}^{n+2} \mid q \neq 0,\ q^{n+2} \neq 0\}$ and $(\mathbb{H}P^{n+1})^o = (\mathbb{H}^{n+2}_{>0})^o/\mathbb{H}^*$, $(X(n))^o = (\mathbb{H}P^{n+1})^o /\!\!/\!\!/ S^1_{(\mathrm{diag.})}$ denote the subsets of $\mathbb{H}P^{n+1}$, respectively $X(n) = SU(n+2)/S(U(2)\times U(n))$ corresponding to $(\mathbb{H}^{n+2}_{>0})^o$.

Remark 4.4.4 Patching the charts together in the case of $T^*\mathbb{C}P^n$, the above proposition gives the following HK/QK correspondences:

$$\begin{tikzcd}[column sep=small]
 & (\mathbb{H}^{n+1}_{>0})^o \arrow[dl, mapsto, "/\!\!/\!\!/_{\{\lambda=1,\,\chi=0\}} S^1_{(\mathrm{diag.})}"'] \arrow[dr, "\mathbb{H}^*"] & \\
T^*\mathbb{C}P^n \arrow[rr, mapsto, "f=\sqrt{1+\tilde{r}^2}/2", "\text{HK/QK cor.}"'] & & (\mathbb{H}P^n)^o
\end{tikzcd}, \quad
\begin{tikzcd}[column sep=small]
 & \mathbb{H}^{n,1}_{<0} \arrow[dl, mapsto, "/\!\!/\!\!/_{\{\lambda=-1,\,\chi=0\}} S^1_{(\mathrm{diag.})}"'] \arrow[dr, "\mathbb{H}^*"] & \\
\{\tilde{r}^2 < 1\} \subset T^*\mathbb{C}H^n \arrow[rr, mapsto, "f=-\sqrt{1-\tilde{r}^2}/2", "\text{HK/QK cor.}"'] & & \mathbb{H}H^n
\end{tikzcd}.$$

Here, the superscript ^{o} refers to the removal of the zero level set of the quaternionic Kähler, respectively homogeneous hyper-Kähler moment map associated with the S^1-action defined by diagonal left-multiplication on quaternionic vector space, e.g.

$$(\mathbb{H}^{n+1}_{>0})^o = \mathbb{H}^{n+1} \backslash \{q = z + jw \in \mathbb{H}^{n+1} \mid \|z\| = \|w\|,\ \langle z, \bar{w}\rangle = 0\}.$$

4.4.2 $c = 1$

The case $c = 1$ can be analyzed using the compatibility of the HK/QK correspondence with the hyper- and quaternionic Kähler quotient constructions (Theorem 4.3.1). The result of this analysis is summarized in Figures 4.5 and 4.6, and in Proposition 4.4.5.

$$\begin{tikzcd}
 & (\mathbb{H}^{n,2}_{<0})^o \arrow[dl, mapsto, "/\!/\!/_{\{z^{n+2}=0,\,|w_{n+2}|^2=1\}} S^1_{(q^{n+2})}"'] \arrow[dr, "\mathbb{H}^*"] & \\
\mathbb{H}^{n,1}_{<0} \arrow[d, mapsto, "/\!/\!/^{\mathrm{HK}}_{\{\lambda=-1,\,\chi=0\}} S^1_{(\mathrm{diag.})}"'] & & (\mathbb{H}H^{n,1})^o \arrow[d, mapsto, "/\!/\!/^{\mathrm{QK}} S^1_{(\mathrm{diag.})}"] \\
\{\tilde{r}^2 < 1\} \subset T^*\mathbb{C}H^n \arrow[rr, mapsto, "f=-\frac{1+\sqrt{1-\tilde{r}^2}}{2}", "\text{HK/QK corresp.}"'] & & (\tilde{X}(n))^o
\end{tikzcd}.$$

Figure 4.6: Analysis of the HK/QK correspondence for $T^*\mathbb{C}H^n$ with parameter $c = 1$ using the compatibility of the correspondence with quotient constructions. Here, $(\mathbb{H}^{n,2}_{<0})^o := \{\hat{q} = (q, q^{n+2}) \in \mathbb{H}^{n,2} \mid \langle q, q\rangle_{(n,1)} < 0,\ q^{n+2} \neq 0\}$ and $(\mathbb{H}H^{n,1})^o = (\mathbb{H}^{n,2}_{<0})^o/\mathbb{H}^*$, $(\tilde{X}(n))^o = (\mathbb{H}H^{n,1})^o /\!/\!/ S^1_{(\mathrm{diag.})}$ denote the subsets of $\mathbb{H}H^{n,1}$, respectively $\tilde{X}(n) = SU(n,2)/S(U(n)\times U(2))$ corresponding to $(\mathbb{H}^{n,2}_{<0})^o$.

We consider the chart $\widehat{M}_+ := \{q^{n+1} \neq 0\}$ in

$$(\mathbb{H}^{n+2}_{>0})^o := \{\hat{q} = \hat{z} + j\hat{w} = (q, q^{n+2}) \in \mathbb{H}^{n+2} \mid q \neq 0, q^{n+2} \neq 0\} \subset \mathbb{H}^{n+2}_{>0}$$

and

$$\widehat{M}_- := (\mathbb{H}^{n,2}_{<0})^o := \{\hat{q} = (q, q^{n+2}) \in \mathbb{H}^{n,2} \mid \langle q, q\rangle_{(n,1)} < 0, q^{n+2} \neq 0\} \subset \mathbb{H}^{n,2}_{<0},$$

where $q = (q^1, \ldots, q^{n+1})$ denotes a row vector consisting of the first $n+1$ standard quaternionic coordinates on $\widehat{M}_\pm$. We endow $\widehat{M}_\pm$ with the standard conical (pseudo-) hyper-Kähler structure $(\hat{g}, \hat{J}_1, \hat{J}_2, \hat{J}_3, \xi)$ (see Example 3.2.10), where in the case of $\widehat{M}_-$, $\hat{g}$ is chosen negative definite in the direction of q^{n+1} and q^{n+2}. We endow $\widehat{M}_\pm$ with the vector fields

$$\hat{X} = -2i\big(z^{n+2}\frac{\partial}{\partial z^{n+2}} - w_{n+2}\frac{\partial}{\partial w_{n+2}} - \bar{z}^{n+2}\frac{\partial}{\partial \bar{z}^{n+2}} + \bar{w}_{n+2}\frac{\partial}{\partial \bar{w}_{n+2}}\big) \tag{4.63}$$

and

$$\hat{Y} = 2i\sum_{a=1}^{n+2}\big(z^a\frac{\partial}{\partial z^a} - w_a\frac{\partial}{\partial w_a} - \bar{z}^a\frac{\partial}{\partial \bar{z}^a} + \bar{w}_a\frac{\partial}{\partial \bar{w}_a}\big). \tag{4.64}$$

Note that the submanifold $\widehat{M}_\pm \subset \mathbb{H}^{n+2}$ is chosen such that $\sigma\hat{g}\big|_{\mathrm{span}_{\mathbb{R}}(\hat{X},\hat{Y})}$ is positive definite, where $\sigma = +1$ for the case $\widehat{M}_+$ and $\sigma = -1$ for the case $\widehat{M}_-$.

The vector field $\hat{Y}$ induces a free S^1-action on $\widehat{M}_\pm$ that we denote by $G = S^1_{(\hat{Y})}$:

$$e^{it} \cdot (q, q^{n+2}) = (e^{it} q, e^{it} q^{n+2}).$$

The vector field $\hat{X}$ induces a free S^1-action on $\widehat{M}_\pm$ that we denote by $S^1_{(\hat{X})}$:

$$e^{it} \cdot (q, q^{n+2}) = (q, e^{-it} q^{n+2}).$$

$S^1_{(\hat{X})}$ and $S^1_{(\hat{Y})}$ commute and the action of $S^1_{(\hat{X})} \times S^1_{(\hat{Y})}$ on $\widehat{M}_\pm$ is free.

To use Theorem 4.3.1, we study the following diagram:

$$\begin{tikzcd}
 & & \widehat{M}_\pm \arrow[lld, mapsto, "/\!/\!/_{P_\pm} S^1_{(\hat{X})}"'] & & & \\
 & P_\pm \arrow[ur, hook] \arrow[ld, "S^1_{(\hat{X}|_P)}"] & & \widehat{M}_0^{(\pm)} \arrow[ul, hook'] \arrow[d, "S^1_{(\hat{Y}|_{\widehat{M}_0})}"] & & \\
M_\pm \arrow[dd, mapsto, bend right, "/\!/\!/^{\mathrm{HK}}_{\{\kappa^Y=0\}} S^1_{(Y)}"'] & & & \underline{\widehat{M}}_\pm & & M'_\pm \arrow[dd, mapsto, bend left, "/\!/\!/^{\mathrm{QK}} S^1_{(Y')}"] \\
 & & \underline{P}_\pm \arrow[ur, hook] \arrow[lld, "\underline{S}^1_{(\underline{\hat{X}}|_{\underline{P}_\pm})}"] & & & \\
\underline{M}_\pm \arrow[rrrrr, mapsto, "\text{HK/QK corresp.}"'] & & & & & \underline{M}'_\pm
\end{tikzcd} \quad . \tag{4.65}$$

The level set $P_\pm := \{\hat{\mu}^{\hat{X}} = (-\sigma, 0, 0)\} \subset \widehat{M}_\pm$ of the homogeneous hyper-Kähler moment map associated with $\hat{X}$ reads

$$\begin{aligned} P_\pm &= \{(\pm(|z^{n+2}|^2 - |w_{n+2}|^2), \mathrm{Re}(\mp 2iz^{n+2}w_{n+2}), \mathrm{Im}(\mp 2iz^{n+2}w_{n+2}) = (\mp 1, 0, 0)\} \\ &= \{z^{n+2} = 0, |w_{n+2}|^2 = 1\} \subset \widehat{M}_\pm \end{aligned} \tag{4.66}$$

(compare Example 3.4.6). We identify the hyper-Kähler quotient $\widehat{M}_\pm /\!/\!/_{P_\pm} S^1_{(\hat{X})}$ with the global section

$$M_\pm := \{(q, q^{n+2}) \in \widehat{M}_\pm \mid z^{n+2} = 0,\, w_{n+2} = 1\} \approx \begin{cases} \mathbb{H}^{n+1}_{>0} & (+) \\ \mathbb{H}^{n,1}_{<0} & (-) \end{cases} \tag{4.67}$$

in the S^1-bundle $P_\pm$. The vector field Y induced on $M_\pm$ by $\hat{Y}$ reads

$$Y = 2i\sum_{I=1}^{n+1}\big(z^I\frac{\partial}{\partial z^I} - w_I\frac{\partial}{\partial w_I} - \bar{z}^I\frac{\partial}{\partial \bar{z}^I} + \bar{w}_I\frac{\partial}{\partial \bar{w}_I}\big) \tag{4.68}$$

and the corresponding hyper-Kähler moment map κ^Y induced by the homogeneous hyper-Kähler moment map $\hat{\kappa}^{\hat{Y}}$ on $\widehat{M}_\pm$ associated with $\hat{Y}$ reads

$$\begin{aligned}\kappa^Y = \hat{\kappa}^{\hat{Y}}\Big|_{M_\pm\subset\widehat{M}_\pm} &= \big(\mp(|z^{n+2}|^2 - |w_{n+2}|^2) - \langle z,z\rangle + \langle w,w\rangle,\\ &\qquad \mathrm{Re}(2i(z^{n+2}w_{n+2} + \langle z,\bar{w}\rangle)),\\ &\qquad \mathrm{Im}(2i(z^{n+2}w_{n+2} + \langle z,\bar{w}\rangle))\big)\Big|_{\{z^{n+2}=0,\, w_{n+2}=1\}\subset\widehat{M}_\pm}\\ &= \big(\pm 1 - \langle z,z\rangle + \langle w,w\rangle,\, \mathrm{Re}(2i\langle z,\bar{w}\rangle),\, \mathrm{Im}(2i\langle z,\bar{w}\rangle)\big)\end{aligned} \tag{4.69}$$

(compare Example 3.4.7). Here and in the following, we use the notation from Example 3.4.7 with the index 0 replaced by $n+1$. The level set $\{\kappa^Y = 0\}\subset M_\pm$ then reads

$$\{\kappa^Y = 0\} = \{\lambda = \pm 1, \chi = 0\}\subset M_\pm,$$

where

$$\lambda = \langle z,z\rangle - \langle w,w\rangle = \pm(|z^{n+1}|^2 - |w_{n+1}|^2) + \sum_{\mu=1}^{n}(|z^\mu|^2 - |w_\mu|^2),$$

$$\chi = \langle z,\bar{w}\rangle = \pm z^{n+1}w_{n+1} + \sum_{\mu=1}^{n} z^\mu w_\mu.$$

In Example 3.4.7 and Remark 3.4.8, we showed that the resulting hyper-Kähler quotient $\underline{M}_\pm := M_\pm /\!\!/\!\!/_{\{\kappa^Y=0\}} S^1_{(Y)}$ is isomorphic to the chart $T^*(\{z^{n+1}\neq 0\})$ in $T^*\mathbb{C}P^n$, respectively to $\{\tilde{r}^2 < 1\}\subset T^*\mathbb{C}H^n$. In terms of the complex coordinates on $\underline{M}_\pm$ defined by $(\zeta^\mu = ((z^{n+1})^{-1}z^\mu, \eta_\mu = z^{n+1}w_\mu)_{\mu=1,\dots,n}$, $\underline{M}_\pm$ is given by the manifolds defined in Eqs. (4.48) and (4.49), and endowed with the hyper-Kähler structure defined by $\omega_+ = \omega_2 + i\omega_3 = \sum_{\mu=1}^n d\zeta^\mu\wedge d\eta_\mu$ and by the Kähler potential for the first Kähler form given in Eq. 4.51.

It remains to check that the Hamiltonian function $\underline{f}$ in Theorem 4.3.1 is the same as the one chosen in the beginning of this section when we directly applied the HK/QK correspondence to $T^*\mathbb{C}P^n$ and to $T^*\mathbb{C}H^n$. The function $\underline{f}$ on $\underline{M}_\pm$

is induced by

$$\begin{aligned}
\frac{\hat{g}(\xi,\xi)}{2}\Big|_{\{\kappa^Y=0\}\subset M_\pm\subset\widehat{M}_\pm} &= \frac{1}{2}\big(\pm|z^{n+2}|^2 \pm |w_{n+2}|^2 + \langle z,z\rangle + \langle w,w\rangle\big)\Big|_{\{\kappa^Y=0\}\subset M_\pm\subset\widehat{M}_\pm} \\
&= \frac{1}{2}(\pm 1 - \lambda + 2\langle z,z\rangle)\Big|_{\{\kappa^Y=0\}\subset\{z^{n+2}=0,\, w_{n+2}=1\}\subset\widehat{M}_\pm} \\
&= \frac{1}{2}(\pm 1 \mp 1 + 2{\rho_\pm}^2(\pm 1 + \|\zeta\|^2))\Big|_{\{\lambda=\pm 1,\, \chi=0,\, z^{n+2}=0,\, w_{n+2}=1\}} \\
&= \pm\frac{1+\sqrt{1\pm\tilde{r}^2}}{2},
\end{aligned} \tag{4.70}$$

which agrees with the Hamiltonian function chosen in Eq. (4.52) for the choice of parameter $c=1$ (see Eqs. (4.50) and (4.59) for the definition of $\tilde{r}$ and $\rho_\pm$).

To study the quaternionic Kähler (i.e. right) side of the diagram in Eq. (4.65), we choose the $\hat{J}_1\xi\big|_{P_\pm}$-transversal codimension one submanifold

$$M'_\pm = M_\pm = \{z^{n+2}=0, w_{n+2}=1\} \subset P_\pm \subset \widehat{M}_\pm \tag{4.71}$$

in the level set $P_\pm$. Together with the quaternionic (pseudo-)Kähler structure induced from $\widehat{M}_\pm$ via Theorem 3.2.6, $M'_\pm$ is isomorphic to the chart $\{q^{n+1}\neq 0\}$ in[13]

$$(\mathbb{H}P^{n+1})^o := (\mathbb{H}^{n+2}_{>0})^o/\mathbb{H}^*, \tag{4.72}$$

respectively to

$$(\mathbb{H}H^{n,1})^o := (\mathbb{H}^{n,2}_{<0})^o/\mathbb{H}^*. \tag{4.73}$$

The latter can be shown similarly to Example 3.2.10 (compare Example 2.1.15 and Remark 2.1.16).

For simplicity, we chose $M'_\pm$ equal to $M_\pm$. Note that technically, $M'_\pm$ does not fulfill the assumptions of Theorem 4.3.1, since it is not $S^1_{(\hat{Y})}$-invariant. Using the right-multiplication of $\mathbb{H}^*$ on $\widehat{M}_\pm$, we can however identify $M'_\pm$ with a $\hat{J}_1\xi\big|_{P_\pm}$-transversal codimension one submanifold in $P_\pm$ that is $S^1_{(\hat{Y})}$-invariant.

The S^1-action on $M'_\pm$ induced by $\hat{Y}$ is just the one given by diagonal left multiplication of $e^{it}\in S^1$ on quaternionic vector space. Let $Y'\in\mathfrak{X}(M'_\pm)$ denote the corresponding Killing vector field. The quaternionic Kähler quotient $\underline{M'_\pm} = M'_\pm /\!\!/\!\!/_{\{\kappa^Y=0\}} S^1_{(Y')}$ is then isomorphic to the chart $\{q^{n+1}\neq 0\}$ in

$$(X(n))^o := (\mathbb{H}P^{n+1})^o /\!\!/\!\!/ S^1_{(\mathrm{diag.})} \subset X(n), \tag{4.74}$$

[13]Note that although we use the same notation, the subset $(\mathbb{H}P^{n+1})^o$ of quaternionic projective space is different from the one defined in the last subsection.

respectively to

$$(\tilde{X}(n))^o := (\mathbb{H}H^{n,1})^o /\!\!/\!\!/ S^1_{(\text{diag.})} \subset \tilde{X}(n) \tag{4.75}$$

(see Example 2.2.10).

Further analysis of the diagram in Eq. (4.65) shows that the choices of S^1-bundle $\underline{P}_\pm$ with connection $\underline{\eta}$ and submanifold $\underline{M}'_\pm \subset \underline{P}_\pm$ in Theorem 4.3.1 agree with the choices made for $c = 1$ in the beginning of this section when we directly applied the HK/QK correspondence to $T^*\mathbb{C}P^n$ and to $\{\tilde{r}^2 < 1\} \subset T^*\mathbb{C}H^n$.

Applying Theorem 4.3.1 for $G = S^1_{(\hat{Y})}$ now gives the following result:

Proposition 4.4.5 *The manifolds given in Eqs.* (4.48) *and* (4.49)*, endowed with the metric given in Eq.* (4.62) *are, for* $c = 1$*, isometric to* $\{q^{n+1} \neq 0\} \subset (X(n))^o$ *and to* $(\tilde{X}(n))^o$*, respectively.*

The following final remark is in agreement with the fact that the quaternionic Kähler metric obtained from a direct application of the HK/QK correspondence to $\{\tilde{r} < 1\} \subset T^*\mathbb{C}H^n$ in the beginning of this section is incomplete at the boundary $\{\tilde{r} = 1\}$ for positive parameter $c > 0$ (see Proposition 4.4.1).

Remark 4.4.6 Note that $(\tilde{X}(n))^o$ is a proper subset of $\tilde{X}(n)$ and, hence, incomplete:
For $\hat{q} = \hat{z} + j\hat{w} = (0, \ldots, 0, \frac{1}{\sqrt{2}}, 0) + j(0, \ldots, \frac{1}{\sqrt{2}}, 0, 1) \in \mathbb{H}^{n,2}$, we have $\langle \hat{q}, \hat{q}\rangle_{(n,2)} = -1 < 0$, $\langle \hat{z}, \bar{\hat{w}}\rangle_{(n,2)} = 0$ and $\langle \hat{z}, \hat{z}\rangle_{(n,2)} = \langle \hat{w}, \hat{w}\rangle_{(n,2)}$, i.e. $\hat{q}$ is in the level set of the quaternionic Kähler moment map, but $\hat{q} \notin (\mathbb{H}^{n,2}_{<0})^o$ since $\langle q, q\rangle_{(n,1)} = 0$.

4.4.3 $c > 0$

Note that in the general case $c > 0$, the local geometry of $(M'_\pm, g'^c_\pm)$ can also be analyzed using the idea of Theorem 4.3.1 with local hyper- and quaternionic Kähler quotients. For this, one has to replace the vector fields $\hat{X}$, $\hat{Y}$ in the above subsection by

$$\hat{X}^c = -\frac{2i}{c}\big(z^{n+2}\frac{\partial}{\partial z^{n+2}} - w_{n+2}\frac{\partial}{\partial w_{n+2}} - \bar{z}^{n+2}\frac{\partial}{\partial \bar{z}^{n+2}} + \bar{w}_{n+2}\frac{\partial}{\partial \bar{w}_{n+2}}\big) \tag{4.76}$$

and

$$\hat{Y}^c = -\hat{X} + 2i\sum_{I=1}^{n+1}\big(z^I\frac{\partial}{\partial z^I} - w_I\frac{\partial}{\partial w_I} - \bar{z}^I\frac{\partial}{\partial \bar{z}^I} + \bar{w}_I\frac{\partial}{\partial \bar{w}_I}\big) \tag{4.77}$$

for $c > 0$. For c irrational, the integral curves of $\hat{Y}^c$ do not close. Nevertheless, we can consider local hyper-Kähler and quaternionic Kähler quotients by taking codimension one submanifolds in the level sets of the respective moment maps that are transversal to the Killing vector field. Let $\bar{Y}^c$ denote the vector field on $(\mathbb{H}P^{n+1})^o$, respectively $(\mathbb{H}H^{n,1})^o$ induced by $\hat{Y}^c$. Using a local version of Theorem 4.3.1 one can show that $(M'_\pm, g'^c_\pm)$ is locally isometric to the local quaternionic Kähler quotients $(\mathbb{H}P^{n+1})^o /\!/\!/ \bar{Y}^c$, respectively $(\mathbb{H}H^{n,1})^o /\!/\!/ \bar{Y}^c$.

Remark 4.4.7 Assume that $c \in \mathbb{R}^{>0}$ is rational. Let $\mathbf{p}, \mathbf{q} \in \mathbb{N}$ be coprime and such that $c = \frac{\mathbf{p}}{\mathbf{q}}$. The S^1-action induced by $\hat{X}^c$ remains unchanged and $\hat{Y}^c$ still induces the following free S^1-action on $\widehat{M}_\pm$:

$$e^{it} \cdot (q, q^{n+2}) = (e^{i\boldsymbol{p}t} q, e^{i\boldsymbol{q}t} q^{n+2}), \quad e^{it} \in S^1.$$

Note that for $c \neq 1$, the action of $S^1_{(\hat{X}^c)} \times S^1_{(\hat{Y}^c)}$ on $\widehat{M}_\pm$ is not free anymore. It has the following isotropy group at every point:

$$\mathbb{Z}_p = \{(e^{it}, e^{i\tilde{t}}) \in S^1_{(\hat{X})} \times S^1_{(\hat{Y})} \mid t = \frac{2\pi k}{\boldsymbol{p}}, \, \tilde{t} = \frac{2\pi k \mathbf{q}}{\boldsymbol{p}}, \, k = 0, \ldots, \boldsymbol{p} - 1\}.$$

[G1, GL] consider the quaternionic Kähler quotient of $\mathbb{H}P^{n+1}$ with respect to the S^1-action induced by $\bar{Y}^c$ for the case $c \geq 1$ (i.e. $\mathbf{p} \geq \mathbf{q}$) and show that for $c > 1$, $\mathbb{H}P^{n+1} /\!/\!/ S^1_{(\bar{Y}^c)}$ is a compact Riemannian orbifold whose smooth part is not locally symmetric for $c > 1$.

Remark 4.4.8 Note that for $c \to \infty$, $cg'_\pm$ converges to the original hyper-Kähler metric $g_\pm$ on $M_\pm$. According to [G1], the convergence is uniform in three derivatives.

Chapter 5

HK/QK correspondence for the c-map

In this chapter[1], we use the explicit formula for the metric given in Theorem 4.1.2 to show that the pseudo-hyper-Kähler structure on the cotangent bundle of a conical affine special Kähler manifold given by the rigid c-map is related to the quaternionic Kähler metric obtained from the supergravity c-map via the HK/QK correspondence. In fact, we get a one-parameter family of positive definite quaternionic Kähler metrics, which corresponds to one-loop corrections of the hypermultiplet moduli space in string theory compactifications on Calabi-Yau 3-folds (if the corresponding model is realized in string theory). As a corollary, this proves that the Ferrara-Sabharwal metric and its one-loop deformation are indeed quaternionic Kähler.

In Section 5.5, we derive the $Sp(1)$-connection one-form and the fundamental two-forms for the one-loop deformed Ferrara-Sabharwal metric with respect to the almost hypercomplex structure (J_1', J_2', J_3') obtained from the HK/QK correspondence. We also derive a holomorphic coordinate system for J_1', which as a corollary of the HK/QK correspondence is a globally defined compatible complex structure.

[1]Apart from Section 5.5, this chapter is identical to Section 4 of [ACDM] up to minor changes.

5.1 Conical affine and projective special Kähler geometry

First, we recall the definitions of conical affine and projective special Kähler manifolds [ACD, CM]:

Definition 5.1.1 *A **conical affine special Kähler manifold** (M, g_M, J, ∇, ξ) is a pseudo-Kähler manifold (M, g_M, J) endowed with a flat torsionfree connection ∇ and a vector field ξ such that*

i) $\nabla\omega_M = 0$, *where* $\omega_M := g_M(J\cdot\,,\cdot)$ *is the Kähler form,*

ii) $(\nabla_X J)Y = (\nabla_Y J)X$ *for all* $X, Y \in \mathfrak{X}(M)$,

iii) $\nabla\xi = D\xi = \mathrm{Id}$, *where* D *is the Levi-Civita connection,*

iv) g_M *is positive definite on* $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ *and negative definite on* $\mathcal{D}^\perp$.

Let (M, g_M, J, ∇, ξ) be a conical affine special Kähler manifold of complex dimension $n+1$. Then ξ and $J\xi$ are commuting holomorphic vector fields that are homothetic and Killing respectively [CM]. We assume that the holomorphic Killing vector field $J\xi$ induces a free S^1-action and that the holomorphic homothety ξ induces a free $\mathbb{R}^{>0}$-action on M. Then (M, g_M) is a metric cone over (S, g_S), where $S := \{p \in M \mid g_M(\xi|_p, \xi|_p) = 1\} \subset M$, $g_S := g_M\big|_S$; and $-g_S$ induces a Riemannian metric $g_{\bar{M}}$ on $\bar{M} := S/S^1_{J\xi}$. $(\bar{M}, -g_{\bar{M}})$ is obtained from (M, g, J) via a Kähler reduction with respect to $J\xi$ and, hence, $g_{\bar{M}}$ is a Kähler metric (see e.g. [CHM]). The corresponding Kähler form $\omega_{\bar{M}}$ is obtained from ω_M by symplectic reduction. This determines the complex structure $J_{\bar{M}}$.

Definition 5.1.2 *The Kähler manifold $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ is called a **projective special Kähler manifold**.*

More precisely, S is a Lorentzian Sasakian manifold and introducing the radial coordinate $r := \sqrt{g(\xi, \xi)}$, we can write the metric on M as [BC, MSY]

$$g_M = dr^2 + r^2\pi^* g_S, \quad g_S = g_M|_S = \tilde{\eta} \otimes \tilde{\eta}|_S - \bar{\pi}^* g_{\bar{M}}, \tag{5.1}$$

where

$$\tilde{\eta} := \frac{1}{r^2} g_M(J\xi, \cdot) = d^c \log r = i(\overline{\partial} - \partial)\log r \tag{5.2}$$

is the contact one-form when restricted to S and $\pi : M \to S = M/\mathbb{R}^{>0}_{\xi}$, $\bar{\pi} : S \to \bar{M} = S/S^1_{J\xi}$ are the canonical projection maps. From now on, we will drop π^* and $\bar{\pi}^*$ and identify, e.g., $g_{\bar{M}}$ with a (0,2) tensor field on M that has the distribution $\mathcal{D} = \mathrm{span}\{\xi, J\xi\}$ as its kernel and is invariant under ξ and $J\xi$.

Locally, there exist so-called **conical special holomorphic coordinates** $z = (z^I) = (z^0, \ldots, z^n) : U \xrightarrow{\sim} \tilde{U} \subset \mathbb{C}^{n+1}$ such that the geometric data on the domain $U \subset M$ is encoded in a holomorphic function $F : \tilde{U} \to \mathbb{C}$ that is homogeneous of degree 2 [ACD, CM]. Namely, we have [CM]

$$g_M\big|_U = \sum_{I,J} N_{IJ} dz^I d\bar{z}^J, \quad N_{IJ}(z,\bar{z}) := 2\mathrm{Im}\, F_{IJ}(z) := 2\mathrm{Im}\, \frac{\partial^2 F(z)}{\partial z^I \partial z^J}$$

$(I, J = 0, \ldots, n)$ and $\xi\big|_U = \sum z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I}$. The Kähler potential for $g_M\big|_U$ is given by $r^2\big|_U = g_M(\xi,\xi)\big|_U = \sum z^I N_{IJ} \bar{z}^J$.

The $\mathbb{C}^*$-invariant functions $X^\mu := \frac{z^\mu}{z^0}$, $\mu = 1, \ldots, n$, define a local holomorphic coordinate system on $\bar{M}$. The Kähler potential for $g_{\bar{M}}$ is $\mathcal{K} := -\log \sum_{I,J=0}^{n} X^I N_{IJ}(X) \bar{X}^J$, where $X := (X^0, \ldots, X^n)$ with $X^0 := 1$.

5.2 The rigid c-map

Now, we introduce the **rigid c-map**, which assigns to each affine special (pseudo-)Kähler manifold (M, g_M, J, ∇) and in particular to any conical affine special Kähler manifold (M, g_M, J, ∇, ξ) of real dimension $2n+2$ a (pseudo-) hyper-Kähler manifold $(N = T^*M, g_N, J_1, J_2, J_3)$ of dimension $4n+4$ [CFG, ACD].

From now on, we assume for simplicity that $(M \subset \mathbb{C}^{n+1}, g_M, J = J_{can}, \nabla, \xi)$ is a conical affine special Kähler manifold that is globally described by a homogeneous holomorphic function F of degree two defined on a $\mathbb{C}^*$-invariant domain M in standard holomorphic coordinates $z = (z^I) = (z^0, \ldots, z^n)$ induced from $\mathbb{C}^{n+1}$. Here, J_{can} denotes the standard complex structure induced from $\mathbb{C}^{n+1}$.

The real coordinates

$$(q^a)_{a=1,\ldots,2n+2} := (x^I, y_J)_{I,J=0,\ldots,n} := (\mathrm{Re}\, z^I, \mathrm{Re}\, F_J(z) := \mathrm{Re}\, \frac{\partial F(z)}{\partial z^J})$$

on M are ∇-affine and fulfill $\omega_M = -2 \sum dx^I \wedge dy_I$, where $\omega_M = g(J\cdot\,,\cdot)$ is the Kähler form on M [CM]. We consider the cotangent bundle $\pi_N : N := T^*M \to M$

and introduce real functions $(p_a) := (\tilde{\zeta}_I, \zeta^J)$ on N such that together with $(\pi_N^* q^a)$, they form a system of canonical coordinates.

Proposition 5.2.1 *In the above coordinates (z^I, p_a), the hyper-Kähler structure on $N = T^*M$ obtained from the rigid c-map is given by*

$$g_N = \sum dz^I N_{IJ} d\bar{z}^J + \sum A_I N^{IJ} \bar{A}_J, \tag{5.3}$$

$$\omega_1 = \frac{i}{2}\sum N_{IJ} dz^I \wedge d\bar{z}^J + \frac{i}{2}\sum N^{IJ} A_I \wedge \bar{A}_J, \tag{5.4}$$

$$\omega_2 = -\frac{i}{2}\sum (d\bar{z}^I \wedge \bar{A}_I - dz^I \wedge A_I), \tag{5.5}$$

$$\omega_3 = \frac{1}{2}\sum (dz^I \wedge A_I + d\bar{z}^I \wedge \bar{A}_I), \tag{5.6}$$

where $A_I := d\tilde{\zeta}_I + \sum_J F_{IJ}(z) d\zeta^J$ $(I = 0, \ldots, n)$ are complex-valued one-forms on N and $\omega_\alpha = g_N(J_\alpha \cdot, \cdot)$. (Here and in the following, we identify functions and one-forms on M with their pullbacks to N.)

***Proof*:** One can check by a direct calculation that the metric and Kähler forms, Eqs. (5.3)–(5.6) agree with the geometric data[2] for the rigid c-map given in Section 3 of [ACD] (see also Section 3 of [ACM]), up to a conventional sign in the definition of the Kähler forms $\omega_\alpha = g_N(J_\alpha \cdot\,, \cdot) = -g_N(\,\cdot, J_\alpha \cdot)$ in [ACD]. For instance, we can write ω_1 and ω_3 as

$$\begin{aligned} \omega_1 &= -2\sum dx^I \wedge dy_I + \frac{1}{2}\sum d\tilde{\zeta}_I \wedge d\zeta^I, \\ \omega_3 &= \sum dx^I \wedge d\tilde{\zeta}_I + \sum dy_I \wedge d\zeta^I = \sum dq^a \wedge dp_a. \end{aligned} \tag{5.7}$$

□

Remark 5.2.2 It follows from the intrinsic geometric description in [ACD] that the pseudo-hyper-Kähler structure is independent of the particular description of the special Kähler structure in terms of a holomorphic function F.

Remark 5.2.3 We introduce holomorphic functions w_I, $I = 0, \ldots, n$, on (N, J_1) that together with the holomorphic coordinates $z = (z^I)$ on (M, J) form a system of canonical holomorphic coordinates on $(N = T^*M, J_1)$. Then (w_I)

[2] Note that $J_2^* dz^I = i\sum_{J=0}^n N^{IJ}\bar{A}_J$, $J_2^* A_I = -i\sum_{J=0}^n N_{IJ} d\bar{z}^J$.

and $(\tilde{\zeta}_I, \zeta^J)$ are related by

$$\begin{aligned}\sum_I w_I dz^I + \bar{w}_I d\bar{z}^I &\stackrel{!}{=} \sum_I \tilde{\zeta}_I dx^I + \zeta^I dy_I \\ &= \sum_I \frac{\tilde{\zeta}_I}{2}(dz^I + d\bar{z}^I) + \frac{\zeta^I}{2}(\sum_J F_{IJ}(z)dz^J + \overline{F_{IJ}(z)}d\bar{z}^J),\end{aligned}$$

which is equivalent to

$$w_I = \frac{1}{2}(\tilde{\zeta}_I + \sum_J F_{IJ}(z)\zeta^J) \qquad (I = 0, \ldots, n). \tag{5.8}$$

With the identification (5.8), Eqs. (5.3)–(5.6) also agree, up to conventional factors, with the rigid c-map as given in Appendix B of [CFG] and throughout the physics literature.

5.3 The supergravity c-map

Let $(\bar{M}, g_{\bar{M}})$ be a projective special Kähler manifold of complex dimension n which is globally defined by a single holomorphic function F. The **supergravity c-map** [FS] associates with $(\bar{M}, g_{\bar{M}})$ a quaternionic Kähler manifold $(\bar{N}, g_{\bar{N}})$ of dimension $4n + 4$. Following the conventions of [CHM], we have $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ and

$$\begin{aligned} g_{\bar{N}} &= g_{\bar{M}} + g_G, \\ g_G &= \frac{1}{4\rho^2}d\rho^2 + \frac{1}{4\rho^2}(d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I))^2 + \frac{1}{2\rho}\sum \mathcal{I}_{IJ}(m)d\zeta^I d\zeta^J \\ &\quad + \frac{1}{2\rho}\sum \mathcal{I}^{IJ}(m)(d\tilde{\zeta}_I + \mathcal{R}_{IK}(m)d\zeta^K)(d\tilde{\zeta}_J + \mathcal{R}_{JL}(m)d\zeta^L), \end{aligned}$$

where $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)$, $I = 0, 1, \ldots, n$, are standard coordinates on $\mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$. The real-valued matrices $\mathcal{I}(m) := (\mathcal{I}_{IJ}(m))$ and $\mathcal{R}(m) := (\mathcal{R}_{IJ}(m))$ depend only on $m \in \bar{M}$ and $\mathcal{I}(m)$ is invertible with the inverse $\mathcal{I}^{-1}(m) =: (\mathcal{I}^{IJ}(m))$. More precisely,

$$\mathcal{N}_{IJ} := \mathcal{R}_{IJ} + i\mathcal{I}_{IJ} := \bar{F}_{IJ} + i\frac{\sum_K N_{IK}z^K \sum_L N_{JL}z^L}{\sum_{IJ} N_{IJ}z^I z^J}, \quad N_{IJ} := 2\mathrm{Im}F_{IJ}, \tag{5.9}$$

where F is the holomorphic prepotential with respect to some system of special holomorphic coordinates (z^I) on the underlying conical special Kähler manifold

$M \to \bar{M}$. Notice that the expressions are homogeneous of degree zero and, hence, well-defined functions on $\bar{M}$. It is shown in [CHM, Cor. 5] that the matrix $\mathcal{I}(m)$ is positive definite and hence invertible and that the metric $g_{\bar{N}}$ does not depend on the choice of special coordinates [CHM, Thm. 9]. It is also shown that $(\bar{N}, g_{\bar{N}})$ is complete if and only if $(\bar{M}, g_{\bar{M}})$ is complete [CHM, Thm. 5].

Using $(p_a)_{a=1,\dots,2n+2} := (\tilde{\zeta}_I, \zeta^J)_{IJ=0,\dots,n}$ and $(\hat{H}^{ab}) := \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix}$, we can combine the last two terms of g_G into $\frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b$, i.e. the quaternionic Kähler metric is given by

$$g_{FS} := g_{\bar{N}} = g_{\bar{M}} + \frac{1}{4\rho^2} d\rho^2 + \frac{1}{4\rho^2}(d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I))^2 + \frac{1}{2\rho}\sum dp_a \hat{H}^{ab} dp_b. \tag{5.10}$$

This metric is known as the **Ferrara-Sabharwal** metric.

5.4 HK/QK correspondence for the c-map

Again, we assume that $(M \subset \mathbb{C}^{n+1}, g_M, J = J_{can}, \nabla, \xi)$ is a conical affine special Kähler manifold that is globally described by a homogeneous holomorphic function F of degree two in standard holomorphic coordinates $z = (z^I) = (z^0, \dots, z^n)$ induced from $\mathbb{C}^{n+1}$. We want to apply the HK/QK correspondence to the hyper-Kähler manifold $(N = T^*M, g_N, J_1, J_2, J_3)$ of signature $(4, 4n)$ obtained from the rigid c-map (see Section 5.2). In [ACM], it was shown that the vector field $Z := 2(J\xi)^h = 2J_1\xi^h$ on N fulfills the assumptions of the HK/QK correspondence, i.e. it is a space-like ω_1-Hamiltonian Killing vector field with $\mathcal{L}_Z J_2 = -2J_3$. Here, $Y^h \in \mathfrak{X}(N)$ is defined for any vector field $Y \in \mathfrak{X}(M)$ by $Y^h(\pi_N^* q^a) = \pi_N^* Y(q^a)$ and $Y^h(p_a) = 0$ for all $a = 1, \dots, 2n+2$. (Y^h is the horizontal lift with respect to the flat connection ∇.)

Theorem 5.4.1 *Applying the HK/QK correspondence to (N, g_N, J_1, J_2, J_3) endowed with the ω_1-Hamiltonian Killing vector field Z gives (up to a constant conventional factor) the one-parameter family g^c_{FS} (5.11) of quaternionic pseudo-Kähler metrics, which includes the Ferrara-Sabharwal metric g_{FS} (5.10). The metric g^c_{FS} is positive definite and of negative scalar curvature on the domain $\{\rho > -2c\} \subset \bar{N}$ (which coincides with $\bar{N}$ if $c \geq 0$, see Section 5.3). If $c < 0$ the metric g^c_{FS} is of signature $(4n, 4)$ on the domain $\{-c < \rho < -2c\} \subset \bar{N}$. Furthermore, if $c > 0$ the metric g^c_{FS} is of signature $(4, 4n)$ on the domain*

$\bar{M} \times \{-c < \rho < 0\} \times \mathbb{R}^{2n+3} \subset \bar{M} \times \mathbb{R}^{<0} \times \mathbb{R}^{2n+3}$.

***Proof*:** We start from the hyper-Kähler structure on $N = T^*M$ given in Eqs. (5.3)–(5.6). As in Section 5.2, we identify functions and differential forms on M with their pullbacks to $\pi_N : N \to M$. We first compute the geometric data involved in the HK/QK correspondence, cf. Section 4.1. The moment map for $-\omega_1$ w.r.t. $Z = 2(J\xi)^h$ is given by $f := r^2 - c$, where $r := \|\xi\|_{g_M} = \sqrt{\sum z^I N_{IJ}\bar{z}^J}$ and $c \in \mathbb{R}$:

$$\omega_1(Z, \cdot) = -g_M(2\xi, \cdot) = -\sum(z^I N_{IJ} d\bar{z}^J + N_{IJ}\bar{z}^J dz^I) = -d(r^2) = -df,$$

since $\sum_I z^I \frac{\partial F_{IJ}(z)}{\partial z^K} = 0$. With $g_N(Z, Z) = 4g_M(\xi, \xi) = 4r^2$, we get

$$f_1 := f - \frac{1}{2} g_N(Z, Z) = -r^2 - c.$$

For the functions f and f_1 nowhere to vanish, we have to restrict N to $\{r^2 \neq |c|\} \subset N$. Using the contact one form $\tilde{\eta} := \frac{1}{r^2} g_M(J\xi, \cdot)$ on M (see Eq. (5.2)), we get

$$\beta := g_N(Z, \cdot) = 2g_M(J\xi, \cdot) = 2r^2\tilde{\eta}.$$

We consider the trivial S^1-principal bundle

$$P := N \times S^1, \quad S^1 = \{e^{is} \mid s \in \mathbb{R}\},$$

with the connection form

$$\eta = ds + \eta_N,$$

where η_N is the following one-form on N:

$$\eta_N := -\frac{1}{2} r^2 \tilde{\eta} + \eta_{can} = \frac{f_1 + c}{2}\tilde{\eta} + \eta_{can}, \quad \eta_{can} := \frac{1}{4}\sum(\tilde{\zeta}_I d\zeta^I - \zeta^I d\tilde{\zeta}_I).$$

Then

$$d\eta = d\eta_N = -\frac{1}{4} d\beta + d\eta_{can} = \omega_1 - \frac{1}{2} d\beta,$$

where we used that ω_1 can be written as

$$\omega_1 \overset{(5.7)}{=} \pi_N^* \omega_M + \frac{1}{2}\sum d\tilde{\zeta}_I \wedge d\zeta^I = \frac{1}{4} d\beta + d\eta_{can},$$

since $\pi_N^*\omega_M = \frac{1}{4}\pi_N^* dd^c(r^2)$ and $\pi_N^* d^c(r^2) = \pi_N^*(2r^2 d^c \log r) \overset{(5.2)}{=} \pi_N^*(2r^2\tilde{\eta}) = \beta$, see Section 5.1.

Now we compute the one-forms θ_a^P, $a = 0, 1, 2, 3$ on P, introduced in Eq. (3.47):

$$\begin{aligned}
\theta_0^P &= -\frac{1}{2}df = -rdr,\\
\theta_1^P &= \eta + \frac{1}{2}\beta = ds + \frac{1}{2}r^2\tilde{\eta} + \eta_{can} = ds + \frac{f+c}{2}\tilde{\eta} + \eta_{can},\\
\theta_2^P &= \frac{1}{2}\omega_3(Z,\cdot) = -\frac{i}{2}\sum(\bar{z}^I\bar{A}_I - z^I A_I) = -\mathrm{Im}\sum z^I A_I,\\
\theta_3^P &= -\frac{1}{2}\omega_2(Z,\cdot) = \frac{1}{2}\sum(z^I A_I + \bar{z}^I\bar{A}_I) = \mathrm{Re}\sum z^I A_I.
\end{aligned}$$

For the calculation of θ_2^P and θ_3^P, we used $Z = 2i\sum(z^I\frac{\partial}{\partial z^I} - \bar{z}^I\frac{\partial}{\partial \bar{z}^I})^h$ and Eqs. (5.5)–(5.6).

We compute the pseudo-Riemannian metric

$$g_P = \frac{2}{f_1}\eta^2 + \pi^* g_N \overset{(5.3)}{=} \frac{2}{f_1}(ds + \frac{c}{2}\tilde{\eta} + \eta_{can} + \frac{f_1}{2}\tilde{\eta})^2 + g_M + \sum A_I N^{IJ}\bar{A}_J$$

and the degenerate tensor field

$$\begin{aligned}
\tilde{g}_P &:= g_P - \frac{2}{f}\sum_{a=0}^{3}(\theta_a^P)^2\\
&= g_P - \frac{2}{f}\left(r^2dr^2 + (ds + \frac{c}{2}\tilde{\eta} + \eta_{can} + \frac{f}{2}\tilde{\eta})^2 + (\sum z^I A_I)(\sum \bar{z}^J\bar{A}_J)\right)\\
&= \left(\frac{2}{f_1} - \frac{2}{f}\right)(ds + \frac{c}{2}\tilde{\eta} + \eta_{can})^2 + \left(\frac{f_1}{2} - \frac{f}{2}\right)\tilde{\eta}^2 - \frac{2}{f}r^2dr^2 + g_M\\
&\qquad + \sum A_I N^{IJ}\bar{A}_J - \frac{2}{f}(\sum z^I A_I)(\sum \bar{z}^J\bar{A}_J).
\end{aligned}$$

As always, pullbacks from M and N to P are implied where necessary. Using $\frac{f_1}{2} - \frac{f}{2} = -r^2 = -(f+c)$, $\frac{2}{f_1} - \frac{2}{f} = -\frac{4}{f}\frac{f+c}{f+2c}$, $\frac{2}{f} = \frac{2}{r^2} + \frac{2c}{f(f+c)}$ and $g_M \overset{(5.1)}{=} dr^2 + r^2(\tilde{\eta}^2 - g_{\bar{M}})$, we get

$$\begin{aligned}
\tilde{g}_P = -r^2 g_{\bar{M}} - \frac{f+2c}{f}dr^2 &- \frac{4}{f}\frac{f+c}{f+2c}(ds + \frac{c}{2}\tilde{\eta} + \eta_{can})^2\\
&- \frac{2c}{f(f+c)}(\sum z^I A_I)(\sum \bar{z}^J\bar{A}_J)\\
&+ \sum A_I N^{IJ}\bar{A}_J - \frac{2}{r^2}(\sum z^I A_I)(\sum \bar{z}^J\bar{A}_J).
\end{aligned}$$

We claim that the last two terms can be combined into $-\frac{1}{2}\sum dp_a \hat{H}^{ab} dp_b$, which appeared in the Ferrara-Sabharwal metric (5.10). This will be proven in the lemma below, see Eq. (5.12).

We use the local coordinates

$$r = \sqrt{\sum z^I N_{IJ} \bar{z}^J},\ \phi := \arg z^0,\ X^\mu = \frac{z^\mu}{z^0}$$

on the conical affine special Kähler base M and choose the submanifold $N' = \{\phi = 0\} \subset P = N \times S^1$, which is transversal to

$$Z_1^P = (Z - \eta(Z)X_P) + f_1 X_P = Z + (r^2 + f_1)X_P = 2\partial_\phi - c\partial_s,$$

where $X_P = \partial_s$ is the fundamental vector field on P.

In these coordinates, we have

$$|z^0|^2 = r^2 e^{\mathcal{K}}$$

and, hence,

$$\begin{aligned}\tilde{\eta} &= \frac{1}{2} d^c \log r^2 = \frac{1}{2} d^c \log |z^0|^2 - \frac{1}{2} d^c \mathcal{K} = d\phi - \frac{1}{2} d^c \mathcal{K} \\ &= d\phi + \sum \frac{i N_{IJ}(X)}{2X^t N \bar{X}} (X^I d\bar{X}^J - \bar{X}^J dX^I)\end{aligned}$$

and

$$\begin{aligned}\sum(z^I A_I) \sum(\bar{z}^J \bar{A}_J) &= |z^0|^2 \sum(X^I A_I) \sum(\bar{X}^J \bar{A}_J) \\ &= r^2 e^{\mathcal{K}} |\sum(X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I)|^2,\end{aligned}$$

where $\mathcal{K} = -\log X^t N \bar{X}$, $X^t N \bar{X} := \sum X^I N_{IJ} \bar{X}^J$, is the Kähler potential for the projective special Kähler metric $g_{\bar{M}}$. Replacing the coordinates r and s by $\rho := f$ and $\tilde{\phi} := -4s$ and recalling that $\sigma = \operatorname{sgn} f$, we obtain the quaternionic Kähler metric $g' = \frac{1}{2|f|} \tilde{g}_P\big|_{N'}$ from the HK/QK correspondence (Theorem 4.1.2) such that $g_{FS}^c := -2\sigma g'$ is given by

$$\begin{aligned}g_{FS}^c = \frac{\rho + c}{\rho} g_{\bar{M}} &+ \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 + \frac{1}{4\rho^2} \frac{\rho + c}{\rho + 2c} (d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + c d^c \mathcal{K})^2 \\ &+ \frac{1}{2\rho} \sum dp_a \hat{H}^{ab} dp_b + \frac{2c}{\rho^2} e^{\mathcal{K}} \left|\sum(X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I)\right|^2. \quad (5.11)\end{aligned}$$

For $c = 0$, g^c_{FS} reduces to the Ferrara-Sabharwal metric (5.10).

Notice that the above metric g^c_{FS} obtained from the HK/QK correspondence is defined on a subset of $\bar{M} \times \mathbb{R}^* \times S^1 \times \mathbb{R}^{2n+2}$, where the $\mathbb{R}^*$-factor corresponds to the coordinate ρ (which may now take negative values) and the S^1-factor is parametrized by the coordinate $\tilde{\phi} = -4s$ considered modulo $8\pi\mathbb{Z}$. Replacing the above subset by its universal covering (that is replacing S^1 by $\mathbb{R}$) we obtain a subset of $\bar{M} \times \mathbb{R}^* \times \mathbb{R}^{2n+3}$. In particular, $g_{FS} = g^0_{FS}$ is defined on $\bar{N}$ as well as on the cyclic quotient $\bar{N}/\mathbb{Z} = \bar{M} \times \mathbb{R}^{>0} \times S^1 \times \mathbb{R}^{2n+2}$.

The pseudo-hyper-Kähler metric g_N has signature $(4, 4n)$ and Z is space-like. Hence, g' is negative definite if $f > 0$ and $f_1 < 0$, it has signature $(4, 4n)$ if $f_1 f > 0$ and it has signature $(8, 4(n-1))$ if $f < 0$ and $f_1 > 0$ (see Corollary 1 in [ACM]). Using $f = \rho$ and $f_1 = -\rho - 2c$, we get

$$\operatorname{sign} g' = \begin{cases} (0, 4n+4) & \text{for } \rho > \max\{0, -2c\} \\ (4, 4n) & \text{for } 0 < \rho < -2c,\ c < 0 \\ (4, 4n) & \text{for } -2c < \rho < 0,\ c > 0 \\ (8, 4(n-1)) & \text{for } \rho < \min\{0, -2c\}. \end{cases}$$

Taking into account that by definition $r^2 = g_M(\xi, \xi) > 0$, i.e. $\rho > -c$, we get

$$\operatorname{sign} g' = \begin{cases} (0, 4n+4) & \text{for } \rho > \max\{0,\ -2c\}\ (\Leftrightarrow r^2 > |c|) \\ (4, 4n) & \text{for } -c < \rho < \max\{0, -2c\}\ (\Leftrightarrow 0 < r^2 < |c|). \end{cases}$$

It remains to prove

Lemma 5.4.2

$$\sum dp_a \hat{H}^{ab} dp_b = -2 \sum A_I N^{IJ} \bar{A}_J + \frac{4}{r^2} (\sum z^I A_I)(\sum \bar{z}^J \bar{A}_J), \tag{5.12}$$

where, as in the last section, $(p_a) = (\tilde{\zeta}_I, \zeta^J)$ *and* $(\hat{H}^{ab}) = \begin{pmatrix} \mathcal{I}^{-1} & \mathcal{I}^{-1}\mathcal{R} \\ \mathcal{R}\mathcal{I}^{-1} & \mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} \end{pmatrix}$.

***Proof*:** Recall that $A_I = d\tilde{\zeta}_I + \sum_J F_{IJ} d\zeta^J$, $I = 0, \ldots, n$. We write $A = (A_I) = d\tilde{\zeta} + \boldsymbol{F} d\zeta$, where $d\tilde{\zeta} = (d\tilde{\zeta}_I)$, $d\zeta = (d\zeta^I)$ are form-valued column vectors and $\boldsymbol{F} := (F_{IJ})$.

First, we show that $\sum A_I N^{IJ} \bar{A}_J = \sum dp_a H^{ab} dp_b$ with

$$(H^{ab}) := \begin{pmatrix} N^{-1} & \frac{1}{2}N^{-1}R \\ \frac{1}{2}RN^{-1} & \frac{1}{4}(N + RN^{-1}R) \end{pmatrix},$$

where $R := 2\mathrm{Re}\,\boldsymbol{F}$:

$$\begin{aligned}\sum A_I N^{IJ} \bar{A}_J &= (d\tilde{\zeta}^t + d\zeta^t \boldsymbol{F})N^{-1}(d\tilde{\zeta} + \overline{\boldsymbol{F}}d\zeta) \\ &= (d\tilde{\zeta}^t + d\zeta^t \frac{1}{2}(R + iN))N^{-1}(d\tilde{\zeta} + \frac{1}{2}(R - iN)d\zeta) \\ &= d\tilde{\zeta}^t N^{-1} d\zeta + d\tilde{\zeta}^t \frac{1}{2} N^{-1}R\, d\zeta + d\zeta^t \frac{1}{2} RN^{-1} d\tilde{\zeta} + d\zeta^t \frac{1}{4}(N + RN^{-1}R)d\zeta.\end{aligned}$$

Now, we show that $(\sum z^I A_I)(\sum \bar{A}_J \bar{z}^J) = \sum dp_a \breve{H}^{ab} dp_b$ with

$$(\breve{H}^{ab}) := \frac{1}{2}\begin{pmatrix} z\bar{z}^t + \bar{z}z^t & z\bar{z}^t\overline{\boldsymbol{F}} + \bar{z}z^t\boldsymbol{F} \\ \overline{\boldsymbol{F}}\bar{z}z^t + \boldsymbol{F}z\bar{z}^t & \boldsymbol{F}z\bar{z}^t\overline{\boldsymbol{F}} + \overline{\boldsymbol{F}}\bar{z}z^t\boldsymbol{F} \end{pmatrix}:$$

$$\begin{aligned}(\sum z^I A_I)(\sum \bar{z}^J \bar{A}_J) &= (d\tilde{\zeta}^t z + d\zeta^t\, \boldsymbol{F}z)(\bar{z}^t d\tilde{\zeta} + \bar{z}^t\overline{\boldsymbol{F}}d\zeta) \\ &= d\tilde{\zeta}^t z\bar{z}^t d\tilde{\zeta} + d\tilde{\zeta}^t z\bar{z}^t\overline{\boldsymbol{F}}d\zeta + d\zeta^t\boldsymbol{F}z\bar{z}^t d\tilde{\zeta} + d\zeta^t\boldsymbol{F}z\bar{z}^t\overline{\boldsymbol{F}}d\zeta \\ &= d\tilde{\zeta}^t\frac{1}{2}(z\bar{z}^t + \bar{z}z^t)d\tilde{\zeta} + d\tilde{\zeta}^t\frac{1}{2}(z\bar{z}^t\overline{\boldsymbol{F}} + \bar{z}z^t\boldsymbol{F})d\zeta \\ &\quad + d\zeta^t\frac{1}{2}(\boldsymbol{F}z\bar{z}^t + \overline{\boldsymbol{F}}\bar{z}z^t)d\tilde{\zeta} + d\zeta^t\frac{1}{2}(\boldsymbol{F}z\bar{z}^t\overline{\boldsymbol{F}} + \overline{\boldsymbol{F}}\bar{z}z^t\boldsymbol{F})d\zeta.\end{aligned}$$

Hence, the right side of Eq. (5.12) is given by $\sum dp_a(-2H^{ab} + \frac{4}{r^2}\breve{H}^{ab})dp_b$.

To rewrite the left side of Eq. (5.12), we need to invert $\mathcal{I} = \mathrm{Im}\,\mathcal{N} = -\frac{1}{2}N + \frac{Nzz^tN}{2z^tNz} + \frac{N\bar{z}\bar{z}^tN}{2\bar{z}^tN\bar{z}}$. It is easy to check that the inverse of $\mathcal{I}$ is given by [MV]

$$\mathcal{I}^{-1} = -2N^{-1} + \frac{2}{z^tN\bar{z}}(z\bar{z}^t + \bar{z}z^t).$$

Using $\mathcal{R} = \mathrm{Re}\,\mathcal{N} = \frac{1}{2}R + \frac{iNzz^tN}{2z^tNz} - \frac{iN\bar{z}\bar{z}^tN}{2\bar{z}^tN\bar{z}}$, we obtain

$$\mathcal{I}^{-1}\mathcal{R} = -N^{-1}R + \frac{1}{z^tN\bar{z}}(z\bar{z}^t(R - iN) + \bar{z}z^t(R + iN)) = -N^{-1}R + \frac{2}{r^2}(z\bar{z}^t\overline{\boldsymbol{F}} + \bar{z}z^t\boldsymbol{F})$$

and hence

$$\mathcal{R}\mathcal{I}^{-1} = (\mathcal{I}^{-1}\mathcal{R})^t = -RN^{-1} + \frac{2}{r^2}(\overline{\boldsymbol{F}}\bar{z}z^t + \boldsymbol{F}z\bar{z}^t).$$

For the lower right block in $(\hat{H}^{ab})$, we calculate

$$\begin{aligned}\mathcal{R}\mathcal{I}^{-1}\mathcal{R} &= -\frac{1}{2}RN^{-1}R + \frac{1}{z^tN\bar{z}}(\overline{\boldsymbol{F}}\bar{z}z^t(R+iN) + \boldsymbol{F}z\bar{z}^t(R-iN)) \\ &\qquad + \frac{i}{z^tNz}(-\frac{1}{2}R+F)zz^tN - \frac{i}{\bar{z}^tN\bar{z}}(-\frac{1}{2}R+\overline{\boldsymbol{F}})\bar{z}\bar{z}^tN \\ &= -\frac{1}{2}RN^{-1}R + \frac{2}{r^2}(\overline{\boldsymbol{F}}\bar{z}z^t\boldsymbol{F} + \boldsymbol{F}z\bar{z}^t\overline{\boldsymbol{F}}) - \frac{Nzz^tN}{2z^tNz} - \frac{N\bar{z}\bar{z}^tN}{2\bar{z}^tN\bar{z}}\end{aligned}$$

and hence

$$\mathcal{I} + \mathcal{R}\mathcal{I}^{-1}\mathcal{R} = -\frac{1}{2}(N + RN^{-1}R) + \frac{2}{r^2}(\overline{\boldsymbol{F}}\bar{z}z^t\boldsymbol{F} + \boldsymbol{F}z\bar{z}^t\overline{\boldsymbol{F}}).$$

This shows that $(\hat{H}^{ab}) = -2(H^{ab}) + \frac{4}{r^2}(\breve{H}^{ab})$ and thus proves Eq. (5.12). □

This proves Theorem 5.4.1. □

Remark 5.4.3 Note that the quaternionic Kähler metric g^c_{FS} given in (5.11) agrees with the one-loop deformed Ferrara-Sabharwal metric first obtained in [RSV] (see also [APP], Eq. (2.93)).

5.5 The one-loop deformed Ferrara-Sabharwal metric

Definition 5.5.1 *For any $c \in \mathbb{R}$, the metric*

$$\begin{aligned}g^c_{FS} = \frac{\rho+c}{\rho}g_{\bar{M}} + \frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}d\rho^2 + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}(d\tilde{\phi} + \sum_{I=0}^{n}(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + cd^c\mathcal{K})^2 \\ + \frac{1}{2\rho}\sum_{a,\,b=1}^{2n+2} dp_a\hat{H}^{ab}dp_b + \frac{2c}{\rho^2}e^{\mathcal{K}}\left|\sum_{I=0}^{n}(X^I d\tilde{\zeta}_I + F_I(X)d\zeta^I)\right|^2 \quad (5.13)\end{aligned}$$

is defined[3] *on the domains*

$$\begin{aligned}N'_{(4n+4,\,0)} &:= \{\rho > -2c,\ \rho > 0\} \subset \bar{N}, \\ N'_{(4n,\,4)} &:= \{-c < \rho < -2c\} \subset \bar{N}, \\ N'_{(4,\,4n)} &:= \bar{M} \times \{-c < \rho < 0\} \times \mathbb{R}^{2n+3} \subset \bar{M} \times \mathbb{R}^{<0} \times \mathbb{R}^{2n+3} \quad (5.14)\end{aligned}$$

[3]The definition of $g_{\bar{M}}$ can be found in Section 5.1 and the definition of $\hat{H}^{ab}$ can be found in Section 5.3.

for any projective special Kähler manifold $\bar{M} \subset \mathbb{C}^n$ defined by a holomorphic function $F \in C^\infty(\bar{M})$, where $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$, $(X^\mu)_{\mu=1,\dots,n}$ are standard holomorphic coordinates on $\bar{M}$, $X^0 := 1$, the real coordinate ρ corresponds to the second factor and $(\tilde{\phi}, \tilde{\zeta}_I, \zeta^I)_{I=0,\dots,n}$ are standard real coordinates on $\mathbb{R}^{2n+3}$. The metric g^c_{FS} is called the ***one-loop deformed Ferrara-Sabharwal metric****.*

Proposition 5.5.2 *Let $\bar{M}$ be any pojective special Kähler manifold and g^c_{FS}, $g^{c'}_{FS}$ the one-loop deformed Ferrara-Sabharwal metric for positive deformation parameters $c, c' \in \mathbb{R}^{>0}$ defined on $\bar{N} = N'_{(4n+4,\,0)}$. Then $(\bar{N}, g^c_{FS})$ and $(\bar{N}, g^{c'}_{FS})$ are isometric.*

***Proof*:** Any $e^\lambda \in \mathbb{R}^{>0}$ acts diffeomorphically on $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ as follows:

$$\bar{N} \to \bar{N}, \quad (m, \rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)_{I=0,\dots,n} \mapsto (m, e^\lambda \rho, e^\lambda \tilde{\phi}, e^{\lambda/2}\tilde{\zeta}_I, e^{\lambda/2}\zeta^I)_{I=0,\dots,n}.$$

Under this action, $g^c_{FS} \mapsto g^{e^{-\lambda}c}_{FS}$. Choosing $e^\lambda = c/c'$, this shows that $(\bar{N}, g^c_{FS}) \approx (\bar{N}, g^{c'}_{FS})$. □

Remark 5.5.3 From Theorem 4.1.2 and the proof of Theorem 5.4.1, we obtain the following expressions for the components of the $Sp(1)$-connection one-form for g^c_{FS} with respect to the almost hyper-complex structure (J'_1, J'_2, J'_3) obtained from the HK/QK correspondence:

$$\begin{aligned}\bar{\theta}_1 &= -\frac{1}{4\rho}\Big(d\tilde{\phi} + (\rho + c)d^c\mathcal{K} - \sum_{I=0}^{n}(\tilde{\zeta}_I d\zeta^I - \zeta^I d\tilde{\zeta}_I)\Big)\\ \bar{\theta}_2 + i\bar{\theta}_3 &= i\frac{\sqrt{\rho + c}}{\rho} e^{\mathcal{K}/2}\sum_{I=0}^{n} X^I A_I. \end{aligned} \tag{5.15}$$

(Recall that $A_I = d\tilde{\zeta}_I + \sum_{J=0}^n F_{IJ}(X)d\zeta^J$, $I = 0, \dots, n$ and $\mathcal{K} = -\log\sum_{I,\,J=0}^n X^I N_{IJ}\bar{X}^J$.)

Remark 5.5.4 Due to the rescaling compared to g', the reduced scalar curvature of g^c_{FS} is

$$\nu = -\frac{1}{2\sigma}4\sigma = -2$$

(see Remark 4.1.7). Using $\bar{\omega}_\alpha = -(d\bar{\theta}_\alpha - 2\bar{\theta}_\beta \wedge \bar{\theta}_\gamma)$, the fundamental two-forms of g^c_{FS} with respect to (J'_1, J'_2, J'_3) are then found to be

$$\begin{aligned}
\bar\omega_1 &= -d\bar\theta_1 + i(\bar\theta_2 + i\bar\theta_3)\wedge(\bar\theta_2 - i\bar\theta_3)\\
&= \frac{1}{4\rho}\left(d\rho\wedge d^c\mathcal{K} + (\rho+c)\, dd^c\mathcal{K} - 2\sum_{I=0}^{n} d\tilde\zeta_I\wedge d\zeta^I\right) + \frac{1}{\rho}d\rho\wedge\bar\theta_1\\
&\quad + \frac{\rho+c}{\rho^2}e^{\mathcal{K}} i(\sum_I X^I A_I)\wedge(\sum_J \bar X^J \bar A_J)\\
&= \frac{\rho+c}{\rho}\frac{1}{4}dd^c\mathcal{K} + \frac{i}{2}\frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}\tau\wedge\bar\tau - \frac{i}{2}\frac{1}{\rho}\sum_{I,\,J=0}^{n} N^{IJ}A_I\wedge\bar A_J\\
&\quad + \frac{i}{2}\frac{2\rho+2c}{\rho^2}e^{\mathcal{K}}(\sum_I X^I A_I)\wedge(\sum_J \bar X^J\bar A_J), \qquad (5.16)\\
\bar\omega_2 + i\bar\omega_3 &= -d(\bar\theta_2 + i\bar\theta_3) + 2i\bar\theta_1\wedge(\bar\theta_2 + i\bar\theta_3) \qquad (5.17)\\
&= -i\frac{\sqrt{\rho+c}}{\rho}e^{\mathcal{K}/2}\sum_{\mu=1}^{n} dX^\mu\wedge A_\mu + \frac{\sqrt{\rho+c}}{2\rho^2}e^{\mathcal{K}/2}(\tau - 2i\rho\partial\mathcal{K})\wedge\sum_{I=0}^{n} X^I A_I,
\end{aligned}$$

where

$$\tau := d\tilde\phi + \sum_{I=0}^{n}(\zeta^I d\tilde\zeta_I - \tilde\zeta_I d\zeta^I) + c d^c\mathcal{K} + i\frac{\rho+2c}{\rho+c}d\rho \qquad (5.18)$$

and we used that $\sum_{I,\,J=0}^n iN^{IJ}A_I\wedge\bar A_J = \sum_{I=0}^n d\tilde\zeta_I\wedge d\zeta^I$ (see Eq. (7.95)).

Remark 5.5.5 As a direct corollary of the fact that the one-loop deformed Ferrara-Sabharwal metric g^c_{FS} is obtained from the HK/QK correspondence (Theorem 5.4.1), we have the result that J'_1 is a globally defined compatible integrable complex structure, see Remark 4.1.6. This was previously shown in [CLST]. Together with the expression

$$g^c_{FS} \overset{(5.12)}{=} \frac{\rho+c}{\rho}g_{\bar M} + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}|\tau|^2 - \frac{1}{\rho}\sum_{I,\,J=0}^{n} N^{IJ}A_I\,\bar A_J + \frac{2\rho+2c}{\rho^2}e^{\mathcal{K}}\Big|\sum_{I=0}^{n} X^I A_I\Big|^2 \qquad (5.19)$$

for the deformed Ferrara-Sabharwal metric, Eq. (5.16) shows that $(\tau, dX^\mu, A_I)^{\mu=1,\dots,n}_{I=0,\dots,n}$ is a coframe of holomorphic one-forms with respect to J'_1. This can be linealy combined into the coframe

$$\begin{aligned}
\Big(\tau + 2ic\partial\mathcal{K} - 2\sum_{I=0}^{n}\zeta^I A_I - \sum_{I,\,J,\,K=0}^{n}\zeta^I F_{IJK}(X)\zeta^J dX^K,&\\
dX^\mu,\ \frac{1}{2}(A_I - \sum_{J,\,K=0}^{n} F_{IJK}(X)\zeta^J dX^K)\Big)&
\end{aligned}$$

of closed holomorphic one-forms which corresponds to the J_1'-holomorphic coordinate system

$$(\chi,\, X^\mu,\, w_I = \frac{1}{2}(\tilde{\zeta}_I + \sum_{J=0}^{n} F_{IJ}(X)\zeta^J))^{\mu=1,\,\dots,\,n}_{I=0,\,\dots,\,n}, \tag{5.20}$$

where

$$\chi := \tilde{\phi} + i(\rho + c(\mathcal{K} + \log(\rho + c))) - \sum_{I=0}^{n} \zeta^I \tilde{\zeta}_I - \sum_{I,\,J=0}^{n} \zeta^I F_{IJ}(X)\zeta^J. \tag{5.21}$$

Chapter 6

Completeness of the one-loop deformed Ferrara-Sabharwal metric

In this chapter, we discuss the completeness of the one-loop deformed Ferrara-Sabharwal metric g^c_{FS} (see Definition 5.5.1) on the domain $N'_{(4n+4,\,0)}$ (where it is positive definite) for positive deformation parameter $c \in \mathbb{R}^{\geq 0}$. For $c < 0$, $(N'_{(4n+4,\,0)}, g^c_{FS})$ is incomplete [ACDM, Rem. 9].

In the first section, we recall the notion of completeness for Riemannian manifolds. In Section 6.2, we introduce projective special real geometry and the supergravity r-map. The latter assigns a complete projective special Kähler manifold to each complete projective special real manifold. In Section 6.3, we derive a sufficient condition for the completeness of $(N'_{(4n+4,\,0)}, g^c_{FS})$ for $c \in \mathbb{R}^{\geq 0}$. Recall that we construct $(N'_{(4n+4,\,0)}, g^c_{FS})$ from a projective special Kähler manifold. We prove the completeness of $(N'_{(4n+4,\,0)}, g^c_{FS})$ in the case that the projective special Kähler manifold is obtained from a complete projective special real manifold via the supergravity r-map and in the case of $\mathbb{C}H^n$. We also show progress in the case of a general complete special Kähler manifold.

As a corollary, we obtain deformations by complete quaternionic Kähler metrics of all known homogeneous quaternionic Kähler manifolds of negative scalar curvature (including symmetric spaces), except for quaternionic hyperbolic space. In the case of $\tilde{X}(n+1) = \frac{SU(n+1,\,2)}{S[U(n+1)\times U(2)]}$ we give a simple and explicit expression for the deformed metric.

In this chapter, we only discuss positive definite quaternionic Kähler metrics.

6.1 Completeness in Riemannian geometry

Since in this chapter, we restrict ourselves to positive definite Riemannian manifolds, we use the following definition for completeness:

Definition 6.1.1 *We call a Riemannian manifold (M, g)* ***complete*** *if every inextendible smooth curve in (M, g) has infinite length.*

For Riemannian manifolds, many otherwise different notions of completeness are equivalent:

Theorem 6.1.2 (Hopf-Rinow, see [O, Ch. 5, Th. 21])
For a Riemannian manifold (M, g), the following conditions are equivalent:

1. *(M, g) is complete.*
2. *(M, d_g) is complete as a metric space.*
3. *(M, g) is geodesically complete.*
4. *Any closed and bounded subset of M is compact.*

We will later prove the completeness of the one-loop deformed Ferrara-Sabharwal metric based on the completeness of the undeformed metric using the following obvious criterion:

Proposition 6.1.3 *Let (M, g) be a complete Riemannian manifold. If $\tilde{g}$ is a Riemannian metric on M such that $\tilde{g} \geq g$, then $(M, \tilde{g})$ is complete.*

6.2 Projective special real geometry and the supergravity r-map

Definition 6.2.1 *Let h be a homogeneous cubic polynomial in n variables with real coefficients and let $U \subset \mathbb{R}^n \setminus \{0\}$ be an $\mathbb{R}^{>0}$-invariant domain such that $h|_U > 0$ and such that $g_{\mathcal{H}} := -\partial^2 h\big|_{\mathcal{H}}$ is a Riemannian metric on the hypersurface $\mathcal{H} := \{x \in U \mid h(x) = 1\} \subset U$. Then $(\mathcal{H}, g_{\mathcal{H}})$ is called a* ***projective special real*** *(PSR) manifold.*

Define $\bar{M} := \mathbb{R}^n + iU \subset \mathbb{C}^n$. We endow $\bar{M}$ with the standard complex structure $J_{\bar{M}}$ and use holomorphic coordinates $(X^\mu = y^\mu + ix^\mu)_{\mu=1,\dots,n} \in \mathbb{R}^n + iU$. We define a Kähler metric

$$\begin{aligned} g_{\bar{M}} &= 2\sum_{\mu,\,\nu=1}^{n} g_{\mu\bar{\nu}} dX^\mu d\bar{X}^\nu := \sum_{\mu,\,\nu=1}^{n} \frac{\partial^2 \mathcal{K}}{\partial X^\mu \partial \bar{X}^\nu} dX^\mu d\bar{X}^\nu \\ &= \frac{1}{2}\sum_{\mu,\,\nu=1}^{n} \frac{\partial^2 \mathcal{K}}{\partial X^\mu \partial \bar{X}^\nu}(dX^\mu \otimes d\bar{X}^\nu + d\bar{X}^\nu \otimes dX^\mu) \end{aligned}$$

on $\bar{M}$ with Kähler potential

$$\mathcal{K}(X, \bar{X}) := -\log 8h(x) = -\log h\left(i(\bar{X} - X)\right). \tag{6.1}$$

Definition 6.2.2 *The correspondence $(\mathcal{H}, g_{\mathcal{H}}) \mapsto (\bar{M}, g_{\bar{M}})$ is called the **supergravity r-map**.*

Remark 6.2.3 With $\frac{\partial}{\partial X^\mu} = \frac{1}{2}\left(\frac{\partial}{\partial y^\mu} - i\frac{\partial}{\partial x^\mu}\right)$, we have

$$\begin{aligned} 2g_{\bar{M}}\left(\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial \bar{X}^\nu}\right) &= 2g_{\mu\bar{\nu}} = \frac{\partial^2 \mathcal{K}(X,\bar{X})}{\partial X^\mu \partial \bar{X}^\nu} =: \mathcal{K}_{\mu\bar{\nu}} \\ &= -\frac{1}{4}\frac{\partial^2 \log h(x)}{\partial x^\mu \partial x^\nu} = -\frac{h_{\mu\nu}(x)}{4h(x)} + \frac{h_\mu(x) h_\nu(x)}{4h^2(x)}, \end{aligned} \tag{6.2}$$

where $h_\mu(x) := \frac{\partial h(x)}{\partial x^\mu}$, $h_{\mu\nu}(x) := \frac{\partial^2 h(x)}{\partial x^\mu \partial x^\nu}$, etc., for $\mu, \nu = 1, \dots, n$.

The inverse of $(\mathcal{K}_{\mu\bar{\nu}})_{\mu,\,\nu=1,\dots,n}$,

$$\mathcal{K}_{\mu\bar{\nu}} = -\frac{\partial^2}{\partial X^\mu \partial \bar{X}^\nu}\log h(x) = -\frac{h_{\mu\nu}(x)}{4h(x)} + \frac{h_\mu(x) h_\nu(x)}{4h^2(x)}, \tag{6.3}$$

is given by $(\mathcal{K}^{\bar{\nu}\lambda})_{\nu,\,\lambda=1,\dots,n}$,

$$\mathcal{K}^{\bar{\nu}\lambda} = -4h(x)h^{\nu\lambda}(x) + 2x^\nu x^\lambda. \tag{6.4}$$

This can be shown using the fact that h is a homogeneous polynomial of degree three:

$$\begin{aligned} &\sum_{\mu=1}^{n} h_\mu(x) x^\mu = 3h(x), \quad \sum_{\nu=1}^{n} h_{\mu\nu}(x) x^\nu = 2h_\mu(x), \\ &\sum_{\rho=1}^{n} h_{\mu\nu\rho}(x) x^\rho = h_{\mu\nu}, \quad h_{\mu\nu\rho\sigma} = 0. \end{aligned} \tag{6.5}$$

Remark 6.2.4 Note that any manifold $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ in the image of the supergravity r-map is a projective special Kähler manifold (see Section 5.1). The corresponding conical affine special Kähler manifold is the trivial $\mathbb{C}^*$-bundle

$$M := \{z = z^0 \cdot (1, X) \in \mathbb{C}^{n+1} \mid z^0 \in \mathbb{C}^*,\, X \in \bar{M} = \mathbb{R}^n + iU\} \to \bar{M}$$

endowed with the standard complex structure J and the metric g_M defined by the holomorphic function

$$F : M \to \mathbb{C}, \quad F(z^0, \dots, z^n) = \frac{h(z^1, \cdots, z^n)}{z^0}.$$

Note that in general, the flat connection[1] ∇ on M is not the standard one induced from $\mathbb{C}^{n+1} \approx \mathbb{R}^{2n+2}$. The homothetic vector field ξ is given by $\xi = \sum_{I=0}^{n}(z^I \frac{\partial}{\partial z^I} + \bar{z}^I \frac{\partial}{\partial \bar{z}^I})$. To check that $g_{\bar{M}}$ is the corresponding projective special Kähler metric, one uses the fact that

$$8|z^0|^2 h(x) = \sum_{I,\,J=0}^{n} z^I N_{IJ}(z, \bar{z}) \bar{z}^J, \tag{6.6}$$

where as above, $x = (\operatorname{Im} X^1, \dots, \operatorname{Im} X^n) = (\operatorname{Im} \frac{z^1}{z^0}, \dots, \operatorname{Im} \frac{z^n}{z^0}) \in U$ (see [CHM]).

Definition 6.2.5 *A Kähler manifold $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ in the image of the supergravity r-map is called a* ***projective very special Kähler manifold****.*

Due to the following result, projective special real geometry constitutes a powerful tool for the construction of complete projective special Kähler manifolds. Since an analogous result exists for the supergravity c-map, the latter define complete quaternionic Kähler manifolds.

Theorem 6.2.6 [CHM]
The supergravity r-map preserves completeness, i.e. it assigns a complete projective special Kähler manifold to each complete projective special real manifold.

Remark 6.2.7 In low dimensions, it is possible to classify all complete projective special real manifolds up to linear isomorphisms of the ambient space. In the case of curves, there are exactly two examples [CHM]. In the case of surfaces, there exist precisely five discrete examples and a one-parameter family [CDL].

[1] ∇ is defined by $x^I = \operatorname{Re} z^I$ and $y_I = \operatorname{Re} F_I(z)$ being flat, $I = 0, \dots, n$ (see [ACD]).

Due to the following result, the question of completeness for a projective special real manifold $(\mathcal{H}, g_{\mathcal{H}})$ reduces to a simple topological question for the hypersurface $\mathcal{H} \subset \mathbb{R}^n$:

Theorem 6.2.8 [CNS, Thm. 2.6.]
Let $(\mathcal{H}, g_{\mathcal{H}})$ be a projective special real manifold of dimension $n-1$. If $\mathcal{H} \subset \mathbb{R}^n$ is closed, then $(\mathcal{H}, g_{\mathcal{H}})$ is complete.

6.3 Completeness of the one-loop deformed Ferrara-Sabharwal metric

Definition 6.3.1 *The **q-map** is the composition of the supergravity r- and c-map. It assigns a $(4n+4)$-dimensional quaternionic Kähler manifold to each $(n-1)$-dimensional projective special real manifold.*

Remark 6.3.2 Except for quaternionic hyperbolic space $\mathbb{H}H^{n+1}$, all Wolf spaces of non-compact type and all known homogeneous, non-symmetric quaternionic Kähler manifolds (called normal quaternionic Kähler manifolds or Alekseevsky spaces) are in the image of the supergravity c-map. While the series $\tilde{X}(n+1) = Gr_{0,2}(\mathbb{C}^{n+1,2})$ of non-compact Wolf spaces can be obtained via the supergravity c-map from the projective special Kähler manifold $\mathbb{C}H^n$ (with holomorphic prepotential $F = \frac{i}{2}((z^0)^2 - \sum_{\mu=1}^{n}(z^\mu)^2))$, which is not in the image of the supergravity r-map, all the other manifolds mentioned above are in the image of the q-map.

Below, we prove the completeness of the one-loop deformation of the Ferrara-Sabharwal metric with positive deformation parameter $c \in \mathbb{R}^{\geq 0}$ for all manifolds in the image of the q-map and for $\tilde{X}(n+1) = Gr_{0,2}(\mathbb{C}^{n+1,2})$. We also show progress in the case of a general special Kähler manifold.

Due to the following result, both the supergravity c-map and the q-map preserve completeness:

Theorem 6.3.3 [CHM]
The supergravity c-map assigns a complete quaternionic Kähler manifold of dimension $4n+4$ to each complete projective special Kähler manifold of dimension $2n$.

Let $(\bar{M} \subset \mathbb{C}^n, g_{\bar{M}}, J_{\bar{M}})$ be a projective special Kähler manifold which is globally defined by a single holomorphic function F on $M := \{z = z^0 \cdot (1, X) \mid z^0 \in \mathbb{C}^*,\ X \in \bar{M}\} \subset \mathbb{C}^{n+1}$. F is homogeneous of degree two in the standard holomorphic coordinates $(z^I)_{I=0,\ldots,n}$ on M. $g_{\bar{M}}$ has a Kähler potential $\mathcal{K} = -\log X^t N \bar{X} = -\log \sum_{I,J=0}^n X^I N_{IJ} \bar{X}^J$ in holomorphic coordinates $(X^\mu = \frac{z^\mu}{z^0})_{\mu=1,\ldots,n}$ on $\bar{M}$, where $X^0 := 1$ and

$$N = 2(\operatorname{Im} F_{IJ}(z))_{I,J=0,\ldots,n} = 2\big(\operatorname{Im} \frac{\partial^2 F(z)}{\partial z^I \partial z^J}\big)_{I,J=0,\ldots,n},$$

which is homogeneous of degree zero and hence defines a matrix-valued function on $\bar{M}$ (see Section 5.1). Note that

$$g_{\bar{M}} = -\frac{(dX)^t N (d\bar{X})}{X^t N \bar{X}} + (\partial\mathcal{K})(\bar{\partial}\mathcal{K}) = -\frac{(dX)^t N (d\bar{X})}{X^t N \bar{X}} + \frac{1}{4}(d\mathcal{K})^2 + \frac{1}{4}(d^c\mathcal{K})^2. \quad (6.7)$$

The first term in equation (6.7) has complex Lorentzian signature for special Kähler manifolds in the image of the supergravity r-map. For the flat conical affine special Kähler manifold $\mathbb{C}^{1,n}$ with prepotential $F = \frac{i}{2}((z^0)^2 - \sum_{\mu=1}^n (z^\mu)^2)$, it is positive definite.

We consider the one-loop deformed Ferrara-Sabharwal metric (see Eq. (5.13))

$$\begin{aligned} g^c_{FS} = {} & \frac{\rho + c}{\rho} g_{\bar{M}} + \frac{1}{4\rho^2}\frac{\rho + 2c}{\rho + c} d\rho^2 + \frac{1}{4\rho^2}\frac{\rho + c}{\rho + 2c}\Big(d\tilde{\phi} + \sum_{I=0}^n (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + c d^c\mathcal{K}\Big)^2 \\ & + \frac{1}{2\rho}\sum_{a,b=1}^{2n+2} dp_a \hat{H}^{ab} dp_b + \frac{2c}{\rho^2} e^{\mathcal{K}} \Big| \sum_{I=0}^n (X^I d\tilde{\zeta}_I + F_I(X) d\zeta^I) \Big|^2 \end{aligned} \quad (6.8)$$

for $c \in \mathbb{R}^{\geq 0}$ defined on $N'_{(4n+4,0)} = \bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ endowed with global coordinates

$$(X^\mu, \rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)^{\mu=1,\ldots,n}_{I=0,\ldots,n}.$$

Lemma 6.3.4 *Let $\epsilon > 0$. If $g_{\bar{M}} \geq \frac{k}{4}(d^c\mathcal{K})^2$ for some $k \in \mathbb{R}^{>0}$, then*

$$g^c_{FS} \geq \frac{1}{2}\frac{k\epsilon}{k\epsilon + c} g^0_{FS}$$

on $\{\rho > \epsilon\} \subset \bar{N}$.

***Proof*:** Note that $\sum_{a,b=1}^{2n+2} dp_a \hat{H}^{ab} dp_b \geq 0$ [MV]. We have

$$\begin{aligned}
&\frac{1}{4\rho^2}\frac{\rho+2c}{\rho+c}d\rho^2 + \frac{1}{2\rho}dp_a\hat{H}^{ab}dp_b + \frac{2c}{\rho^2}e^{\mathcal{K}}\left|\sum(X^I d\tilde{\zeta}_I + F_I(X)d\zeta^I)\right|^2 \\
\geq\ & \frac{1}{2}\frac{k\epsilon}{k\epsilon+c}\left(\frac{1}{4\rho^2}d\rho^2 + \frac{1}{2\rho}dp_a\hat{H}^{ab}dp_b\right), \qquad (6.9)
\end{aligned}$$

since $\frac{1}{2}\frac{k\epsilon}{k\epsilon+c} \leq \frac{1}{2} < 1 \leq \frac{\rho+2c}{\rho+c}$. Now with $\theta_0 := d\tilde{\phi} + \sum(\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I)$, we have

$$\begin{aligned}
&\frac{\rho+c}{\rho}g_{\bar{M}} + \frac{1}{4\rho^2}\frac{\rho+c}{\rho+2c}(\theta_0 + c d^c\mathcal{K})^2 \\
\stackrel{(6.7)}{=}\ & \underbrace{1}_{>\frac{1}{2}\frac{k\epsilon}{k\epsilon+c}}\, g_{\bar{M}} + \frac{c}{\rho}\underbrace{g_{\bar{M}}}_{\geq \frac{k}{4}(d^c\mathcal{K})^2} \\
&+ \frac{1}{4\rho^2}\underbrace{\frac{\rho+c}{\rho+2c}}_{\frac{1}{2}\leq\cdots\leq 1}\left(\underbrace{\frac{c}{k\epsilon+c}(\theta_0 + (k\epsilon+c)d^c\mathcal{K})^2}_{\geq 0} + \frac{k\epsilon}{k\epsilon+c}(\theta_0)^2 - kc\epsilon(d^c\mathcal{K})^2\right) \\
\geq\ & \frac{1}{2}\frac{k\epsilon}{k\epsilon+c}\left(g_{\bar{M}} + \frac{1}{4\rho^2}(\theta_0)^2\right) + \frac{ck}{4\rho^2}\underbrace{(\rho-\epsilon)}_{>0}(d^c\mathcal{K})^2 \\
\geq\ & \frac{1}{2}\frac{k\epsilon}{k\epsilon+c}\left(g_{\bar{M}} + \frac{1}{4\rho^2}(\theta_0)^2\right). \qquad (6.10)
\end{aligned}$$

Combining the inequalities (6.9) and (6.10), we have shown that

$$g^c_{FS} \stackrel{\rho>\epsilon}{\geq} \frac{1}{2}\frac{k\epsilon}{k\epsilon+c}g^0_{FS}.$$

□

Proposition 6.3.5 *If $(\bar{M}, g_{\bar{M}})$ is complete and $g_{\bar{M}} \geq \frac{k}{4}(d^c\mathcal{K})^2$, for some $k \in \mathbb{R}^{>0}$, then $(\bar{N}, g^c_{FS})$ is complete for every $c \in \mathbb{R}^{\geq 0}$.*

***Proof*:** $(\bar{N}, g^0_{FS})$ is complete by Theorem 6.3.3. Since every curve on $(\bar{N}, g^c_{FS})$ approaching $\rho = 0$ has infinite length, we can restrict to $\{\rho > \epsilon\} \subset \bar{N}$ for some $\epsilon > 0$. According to the above Lemma,

$$g^c_{FS} \geq \frac{1}{2}\frac{k\epsilon}{k\epsilon+c}g^0_{FS}.$$

Since $(\bar{N}, g^0_{FS})$ is complete, this shows that $(\bar{N}, g^c_{FS})$ is complete as well for $c \in \mathbb{R}^{\geq 0}$. □

6.3.1 Complex hyperbolic space

For the projective special Kähler manifold $\mathbb{C}H^n$ with quadratic holomorphic prepotential $F = \frac{i}{2}((z^0)^2 - \sum_{\mu=1}^n (z^\mu)^2)$, we have $-\frac{(dX)^t N (d\bar{X})}{X^t N \bar{X}} > 0$. Equation (6.7) then shows that $g_{\bar{M}} \geq \frac{1}{4}(d^c\mathcal{K})^2$, i.e. the assumption of Lemma 6.3.4 is fulfilled for $k = 1$. We know from the literature that $(\bar{N}, g^0_{FS})$ is isometric to the series of Wolf spaces

$$\tilde{X}(n+1) = \frac{SU(n+1,2)}{S[U(n+1) \times U(2)]} \tag{6.11}$$

of non-compact type.

Corollary 6.3.6 *For any $n \in \mathbb{N}_0$ and $c \in \mathbb{R}^{\geq 0}$, the deformed Ferrara-Sabharwal metric*[2]

$$\begin{aligned} g^c_{FS} = {} & \frac{\rho + c}{\rho} \frac{1}{1 - \|X\|^2} \Big(\sum_{\mu=1}^n dX^\mu d\bar{X}^\mu + \frac{1}{1 - \|X\|^2} \big| \sum_{\mu=1}^n \bar{X}^\mu dX^\mu \big|^2 \Big) \\ & + \frac{1}{4\rho^2} \frac{\rho + 2c}{\rho + c} d\rho^2 - \frac{2}{\rho} (dw_0 d\bar{w}_0 - \sum_{\mu=1}^n dw_\mu d\bar{w}_\mu) \\ & + \frac{\rho + c}{\rho^2} \frac{4}{1 - \|X\|^2} \big| dw_0 + \sum_{\mu=1}^n X^\mu dw_\mu \big|^2 \\ & + \frac{1}{4\rho^2} \frac{\rho + c}{\rho + 2c} \Big(d\tilde{\phi} - 4 \operatorname{Im} \big(\bar{w}_0 dw_0 - \sum_{\mu=1}^n \bar{w}_\mu dw_\mu \big) + \frac{2c}{1 - \|X\|^2} \operatorname{Im} \sum_{\mu=1}^n \bar{X}^\mu dX^\mu \Big)^2 \end{aligned} \tag{6.12}$$

on

$$\bar{N} = \{(X, \rho, \tilde{\phi}, w) \in \mathbb{C}^n \times \mathbb{R}^{>0} \times \mathbb{R} \times \mathbb{C}^{n+1} \mid \|X\|^2 < 1\}$$

defined by the holomorphic function $F = \frac{i}{2}((z^0)^2 - \sum_{\mu=1}^n (z^\mu)^2)$ on $\{z \in \mathbb{C}^{1,n} \mid \langle z, z \rangle > 0\}$ is a complete quaternionic Kähler metric. For $c = 0$, $(\bar{N}, g^c_{FS})$ is isometric to the symmetric space $\tilde{X}(n+1) = \frac{SU(n+1,2)}{S[U(n+1)\times U(2)]}$.

□

6.3.2 Manifolds in the image of the supergravity r-map

For quaternionic Kähler manifolds in the image of the q-map, we have $F(z) = \frac{h(z^1, \ldots, z^n)}{z^0}$ for a homogeneous cubic polynomial h, $\bar{M} = \mathbb{R}^n + iU$, where

[2] Note that $w_0 = \frac{1}{2}(\tilde{\zeta}_0 + i\zeta^0)$, $w_\mu = \frac{1}{2}(\tilde{\zeta}_\mu - i\zeta^\mu)$, $\mu = 1, \ldots, n$, see Eq. (5.20).

$U \subset \mathbb{R}^n$ is an $\mathbb{R}^{>0}$-invariant domain such that $h|_U > 0$ and such that $-\partial^2 h|_U$ has Lorentzian signature. The (positive definite) projective special Kähler metric is given by

$$g_{\bar{M}} = \frac{1}{4h(x)} \sum_{\mu,\nu=1}^{n} \left(-h_{\mu\nu}(x) + \frac{h_\mu(x) h_\nu(x)}{h(x)} \right) (dx^\mu dx^\nu + dy^\mu dy^\nu), \tag{6.13}$$

where $y + ix \in \bar{M}$. The Kähler potential is $\mathcal{K} = -\log 8h(x)$ and $d^c\mathcal{K} = -\frac{1}{h(x)} \sum_{\mu=1}^{n} h_\mu(x) dy^\mu$.

Lemma 6.3.7

$$g_{\bar{M}} \geq \frac{1}{12}(d^c\mathcal{K})^2.$$

***Proof*:** First, we show that

$$\tilde{g} := -\sum_{\mu,\nu=1}^{n} \frac{h_{\mu\nu}(x)}{h(x)} dy^\mu dy^\nu \geq -\frac{2}{3}(d^c\mathcal{K})^2. \tag{6.14}$$

Considering $\tilde{g}$ as a family of pseudo-Riemannian metrics on $\mathbb{R}^n$ depending on a parameter $x \in U$, the left hand side is positive definite on the orthogonal complement $Y^{\perp_{\tilde{g}}}$ of $Y := \sum_{\mu=1}^{n} x^\mu \partial_{y^\mu}$, while the right hand side is zero, since $\tilde{g}(Y, \cdot) = 2d^c\mathcal{K}$. In the direction of Y, we have $\tilde{g}(Y,Y) = -6 = -\frac{2}{3}(d^c\mathcal{K})^2(Y,Y)$.

Equation (6.14) implies

$$\begin{aligned} g_{\bar{M}} &\geq \frac{1}{4h(x)} \sum_{\mu,\nu=1}^{n} \left(-h_{\mu\nu}(x) + \frac{h_\mu(x) h_\nu(x)}{h(x)} \right) dy^\mu dy^n \\ &\geq -\frac{1}{6}(d^c\mathcal{K})^2 + \frac{1}{4}(d^c\mathcal{K})^2 = \frac{1}{12}(d^c\mathcal{K})^2. \end{aligned}$$

□

This shows that the assumption of Lemma 6.3.4 is fulfilled with $k = 1/3$ for projective special Kähler manifolds in the image of the supergravity r-map and proves the following corollary:

Corollary 6.3.8 *Let $(\mathcal{H}, g_{\mathcal{H}})$ be a complete projective special real manifold of dimension $n-1$ and g^c_{FS}, $c \in \mathbb{R}^{\geq 0}$, the one-loop deformed Ferrara-Sabharwal metric on $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$ defined by the projective special Kähler manifold $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ obtained from $(\mathcal{H}, g_{\mathcal{H}})$ via the supergravity r-map. Then $(\bar{N}, g^c_{FS})$ is a complete quaternionic Kähler manifold. $(\bar{N}, g^0_{FS})$ is the complete quaternionic Kähler manifold obtained from $(\mathcal{H}, g_{\mathcal{H}})$ via the q-map.*

□

Remark 6.3.9 For the case $n = 1$ ($h = x^3$), $(\bar{N}, g^0_{FS})$ is isometric to the symmetric space $G_2^*/SO(4)$. In this case we checked using computer algebra software that the squared pointwise norm of the Riemann tensor with respect to the metric is

$$\sum_{i,j,k,l,\tilde{i},\tilde{j},\tilde{k},\tilde{l}=1}^{8} R_{ijkl} g^{i\tilde{i}} g^{j\tilde{j}} g^{k\tilde{k}} g^{l\tilde{l}} R_{\tilde{i}\tilde{j}\tilde{k}\tilde{l}}$$
$$= \frac{128 \begin{pmatrix} 528c^7 + 2112c^6\rho + 3664c^5\rho^2 + 3568c^4\rho^3 \\ \quad + 2110c^3\rho^4 + 764c^2\rho^5 + 161c\rho^6 + 17\rho^7 \end{pmatrix}}{3(c+\rho)(2c+\rho)^6}.$$

For $c > 0$, this function is non-constant, which shows that $(\bar{N}, g^c_{FS})$ is not locally homogeneous for $c > 0$.

6.3.3 General projective special Kähler manifolds

Let $(\bar{M} \subset \mathbb{C}^n, g_{\bar{M}}, J_{\bar{M}})$ be a projective special Kähler manifold which is globally defined by a single holomorphic function F on $M := \{z = z^0 \cdot (1, X) \mid z^0 \in \mathbb{C}^*,\ X \in \bar{M}\} \subset \mathbb{C}^{n+1}$. F is homogeneous of degree two in the standard holomorphic coordinates $(z^I)_{I=0,\dots,n}$ on M. $g_{\bar{M}}$ has a Kähler potential $\mathcal{K} = -\log X^t N \bar{X} = -\log \sum_{I,J=0}^n X^I N_{IJ} \bar{X}^J$ in holomorphic coordinates $(X^\mu = \frac{z^\mu}{z^0})_{\mu=1,\dots,n}$ on $\bar{M}$, where $X^0 := 1$ and

$$N = 2(\operatorname{Im} F_{IJ}(z))_{I,J=0,\dots,n} = 2\Big(\operatorname{Im} \frac{\partial^2 F(z)}{\partial z^I \partial z^J}\Big)_{I,J=0,\dots,n},$$

which is homogeneous of degree zero and hence defines a matrix-valued function on $\bar{M}$ (see Section 5.1). Note that by assumption $X^t N \bar{X} > 0$. The metric on $\bar{M}$ can be written as

$$g_{\bar{M}} = b + \frac{1}{4}(d\mathcal{K})^2 + \frac{1}{4}(d^c\mathcal{K})^2, \tag{6.15}$$

where

$$b := -\frac{(dX)^t N (d\bar{X})}{X^t N \bar{X}} = -\frac{1}{X^t N \bar{X}} \sum_{\mu,\nu=1}^n N_{\mu\nu} dX^\mu d\bar{X}^\nu \tag{6.16}$$

is a non-degenerate pseudo-Riemannian metric on $\bar{M}$ (see [C–G] for the fact that for the matrix-valued function $n := (n_{\mu\nu})_{\mu,\nu=1,\dots,n} := (N_{\mu\nu})_{\mu,\nu=1,\dots,n}$, $n(p)$

is invertible at every point $p \in \bar{M}$). Define

$$\Delta := N_{00} - \sum_{\mu,\nu=1}^{n} N_{0\mu} n^{\mu\nu} N_{\nu 0}, \tag{6.17}$$

where $n^{-1} =: (n^{\mu\nu})_{\mu,\nu=1,\dots,n}$ is the pointwise inverse of n.

Proposition 6.3.10 *Let $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ be a complete projective special Kähler manifold. Assume that $d\mathcal{K} \in \Omega^1(\bar{M})$ is non-vanishing and that $\frac{\Delta}{X^t N \bar{X}} > 1$ or $\frac{\Delta}{X^t N \bar{X}} < -\epsilon$ for some $\epsilon > 0$. Then for $c \in \mathbb{R}^{\geq 0}$, the one-loop deformed Ferrara-Sabharwal metric g^c_{FS} (see Eq. (5.13)) is a complete quaternionic Kähler metric on $\bar{N} = \bar{M} \times \mathbb{R}^{>0} \times \mathbb{R}^{2n+3}$.*

***Proof*:** Define the non-vanishing vector field

$$Y := b^{-1}(d\mathcal{K}) = 4 \sum_{I=0}^{n} \sum_{\mu,\nu=1}^{n} \mathrm{Re}(X^I N_{I\mu} n^{\mu\nu} \frac{\partial}{\partial X^\nu}) \in \mathfrak{X}(\bar{M}).$$

Then

$$b(Y,Y) = d\mathcal{K}(Y) = -\frac{4}{X^t N \bar{X}} \sum_{I=0}^{n} \sum_{\mu,\nu=1}^{n} X^I N_{I\mu} n^{\mu\nu} N_{\nu J} \bar{X}^J = 4\big(\frac{\Delta}{X^t N \bar{X}} - 1\big).$$

Note that $d^c\mathcal{K}(J_{\bar{M}}Y) = d\mathcal{K}(Y)$ and $d^c\mathcal{K}(Y) = d\mathcal{K}(J_{\bar{M}}Y) = 0$. Since $g_{\bar{M}}$ is positive definite,

$$g_{\bar{M}}(Y,Y) = g_{\bar{M}}(J_{\bar{M}}Y, J_{\bar{M}}Y) = \frac{d\mathcal{K}(Y)}{4}(d\mathcal{K}(Y)+4) = 4\frac{\Delta}{X^t N \bar{X}}\big(\frac{\Delta}{X^t N \bar{X}} - 1\big) > 0,$$

which implies that either $\frac{\Delta}{X^t N \bar{X}} > 1$ or $\frac{\Delta}{X^t N \bar{X}} < 0$. We can split the tangent bundle of $\bar{M}$ as

$$T\bar{M} = \mathbb{R}Y \oplus^\perp \mathbb{R}J_{\bar{M}}Y \oplus^\perp H,$$

where $H := \{Y, J_{\bar{M}}Y\}^\perp = \ker d\mathcal{K} \cap \ker d^c\mathcal{K} \subset T\bar{M}$.

We want to show that $g_{\bar{M}} \geq \frac{k}{4}(d^c\mathcal{K})^2$ for some $k \in \mathbb{R}^{>0}$. Since the left side of the inequality is positive definite and the right side vanishes on $\mathbb{R}Y$ and on H, we just need to check that

$$4\frac{\Delta}{X^t N \bar{X}}\big(\frac{\Delta}{X^t N \bar{X}} - 1\big) = g_{\bar{M}}(J_{\bar{M}}Y, J_{\bar{M}}Y) \geq \frac{k}{4}(d^c\mathcal{K}(J_{\bar{M}}Y))^2 = 4k\big(\frac{\Delta}{X^t N \bar{X}} - 1\big)^2.$$

If $\frac{\Delta}{X^t N \bar{X}} > 1$, then this inequality is fulfilled for $k = 1$. If $\frac{\Delta}{X^t N \bar{X}} < 0$, it is equivalent to $k < 1$,

$$-\frac{\Delta}{X^t N \bar{X}} \geq \frac{k}{1-k}.$$

This can be fulfilled for some $k > 0$ if and only if $-\frac{\Delta}{X^t N \bar{X}}$ is bounded from above by a positive number. The proof is then finished by applying Proposition 6.3.5. $\square$

Remark 6.3.11 Note that for projective special Kähler manifolds in the image of the supergravity r-map, we have $\frac{\Delta}{X^t N \bar{X}} = -\frac{1}{2}$. For complex hyperbolic space, $\frac{\Delta}{X^t N \bar{X}} = \frac{1}{1-\|X\|^2} > 1$.

Chapter 7

Curvature of the q-map

In Section 7.1, we give explicit local realizations of the complex vector bundles E and H in Salamon's *E-H formalism* for quaternionic Kähler manifolds. Using local frames in E and H, we derive the formulas (7.48)-(7.51) for the E- and H-part of the Levi-Civita connection and the formulas (7.55)-(7.57) for the E-part R_E of the Riemann curvature tensor. These formulas are known from the *quaternionic vielbein formalism* used in the physics literature. In [FS], they were used to calculate the Levi-Civita connection and Riemann curvature for all manifolds in the image of the supergravity c-map. We also derive the formula (7.58), which expresses R_E in terms of a quartic tensor field Ω on E.

In Section 7.2, we recall the expression for the curvature of manifolds in the image of the supergravity r-map from [CDL] and express it in terms of a *unitary coframe*. In Section 7.3 we then calculate expressions for the Levi-Civita connection of all manifolds in the image of the one-loop deformed q-map and for the Riemann tensor of all manifolds in the image of the undeformed q-map. We also derive the quartic tensor field $\Omega \in \Gamma(S^4E^*)$ that determines the curvature tensor of the manifolds in the image of the q-map.

In the last section, we construct a series of complete quaternionic Kähler manifolds via the q-map. Using the results from Section 7.3, we calculate a curvature invariant (the pointwise norm of the Riemann tensor, sometimes called the *Kretschmann scalar*) for all members of the constructed series and show that it is a non-constant function. This shows that the constructed series consists of complete quaternionic Kähler manifolds that are not locally homogeneous.

Note that in this chapter, we will only discuss positive definite quaternionic Kähler manifolds.

7.1 *E*-*H* formalism

In this section, we will locally give an explicit realization of the complex vector bundles E and H over a quaternionic Kähler manifold that are used to identify the complexified tangent bundle of the quaternionic Kähler manifold with the tensor product $H \otimes_{\mathbb{C}} E$ in the so-called *E-H formalism* introduced in [Sa1]. In particular, we will prove the following proposition throughout the main text:

Proposition 7.1.1 *Let (M, g, Q) be a quaternionic Kähler manifold. Every choice of a local section*[1] $(J_1, J_2, J_3) \in \Gamma(U, S)$ *defines an isomorphism*

$$f : H \otimes_{\mathbb{C}} E \to T^{\mathbb{C}}U,\ h \otimes e \mapsto he \tag{7.1}$$

of complex vector bundles over $U \subset M$, where

$$H = \mathbb{R}\,\mathrm{Id}_{\mathrm{TU}} \oplus Q|_U,\ i_H = R_{J_1}, \tag{7.2}$$

$$E = T^{1,0}_{J_1}U = (\mathrm{Id} - iJ_1)T^{\mathbb{C}}U,\ i_E = J_1 = i, \tag{7.3}$$

are complex vector bundles endowed with quaternionic structure maps

$$j_H = -R_{J_2}, \quad j_E = J_2 \circ \rho, \tag{7.4}$$

(ρ is the standard real structure on $T^{\mathbb{C}}U$ given by complex conjugation) and with non-degenerate two-forms

$$\omega_H \in \Gamma(\Lambda^2 H^*),\ \omega_H(\mathrm{Id}, -J_2) = 1, \quad \omega_E = \frac{1}{2}(\omega_2 + i\omega_3) \in \Gamma(\Lambda^2 E^*) \tag{7.5}$$

that fulfill $j_H^* = \overline{\omega_H}$, $j_E^* = \overline{\omega_E}$. Under the identification of $T^{\mathbb{C}}U$ and $H \otimes_{\mathbb{C}} E$ given by f, ρ corresponds to $j_H \otimes j_E$ and the complexified metric $g_{\mathbb{C}}$ corresponds to $\omega_H \otimes \omega_E$. The action of J_i on $T^{\mathbb{C}}U$ corresponds to $L_{J_i} \otimes \mathrm{Id}_E$ and ω_H is invariant under L_{J_i}, $i = 1, 2, 3$.

We will then express the decomposition of the Riemann curvature tensor of a quaternionic Kähler manifold (into the sum of a multiple of the curvature tensor of quaternionic projective space and the quaternionic Weyl tensor) in terms of (local) frames in H and E. This proves formulas for the Riemann curvature tensor in the so-called *quaternionic vielbein formalism* used in the

[1] As in Chapter 3.6, S denotes the principal $SO(3)$-bundle of oriented orthonormal frames in Q.

physics literature. These formulas will be used in Section 7.3 to calculate the curvature tensor for all manifolds in the image of the q-map.

Vector bundles H and E, complex structures i_H and i_E

Let (M, g, Q) be a positive definite $4n$-dimensional quaternionic Kähler manifold and let $(J_1, J_2, J_3) \in \Gamma(U, S)$ be a frame in Q defined over an open subset $U \subset M$ such that $J_1 J_2 = J_3$ and $J_i^2 = -\mathrm{Id}$, $i = 1, 2, 3$. We define

$$E := T^{1,0}_{J_1} U \subset (TU)^{\mathbb{C}}, \tag{7.6}$$

$$H := \mathbb{R}\,\mathrm{Id}_{TU} \oplus Q|_U = \mathrm{span}\{\mathrm{Id}, J_1, J_2, J_3\} \subset \mathrm{End}(TU). \tag{7.7}$$

Note that E is only defined locally over U and depends on the choice of J_1. It is a complex vector bundle of $\mathrm{rank}_{\mathbb{C}} E = 2n$ with complex structure $i_E := J_1 = i$. H is a globally defined real vector bundle. To turn it into a complex vector bundle, we restrict it to U and choose a complex structure $i_H := R_{J_1}$ defined by right-multiplication with J_1. By abuse of notation, we will from now on denote the complex vector bundle $(H|_U, i_H)$ by H ($\mathrm{rank}_{\mathbb{C}} H = 2$).

Quaternionic structure map j_H, non-degenerate two-form ω_H and complex frame (h_1, h_2)

$$j_H := -R_{J_2} : H \to H,\ h \mapsto -hJ_2 \tag{7.8}$$

defines an i_H-antilinear (i.e. $i_H j_H = -j_H i_H$) structure map on H satisfying $(j_H)^2 = -\mathrm{Id}_H$. i_H and j_H commute with the natural action of $a+b\,i+c\,j+d\,k \in Sp(1)$ on H given by left-multiplication of $a\,\mathrm{Id}+b\,J_1+c\,J_2+d\,J_3$.

We choose the following complex frame for H:

$$(h_1 := \mathrm{Id},\ h_2 := -J_2). \tag{7.9}$$

We have $j_H(h_1) = h_2$. Let (h^1, h^2) be the (complex) dual frame[2] of H^*, i.e. $h^1, h^2 : H \to \mathbb{C}$ are $\mathbb{R}$-linear maps such that $h^\alpha((a + i_H b)h_\beta) = (a + ib)\delta^\alpha_\beta$ for

[2] In terms of the dual frame $(\eta^0, \eta^1, \eta^2, \eta^3)$ of the real frame $(\eta_0 := h_1, \eta_1 := i_H h_1, \eta_2 := h_2, \eta_3 := i_H h_2)$ of H, we have $h^1 = \eta^0 + i\eta^1$, $h^2 = \eta^2 + i\eta^3$.

$a, b \in \mathbb{R}$. Now, we define a non-degenerate two-form on H:

$$\omega_H := h^1 \wedge h^2 = \frac{1}{2} \sum_{\alpha,\,\beta=1}^{2} \epsilon_{\alpha\beta}\, h^\alpha \wedge h^\beta \in \Gamma(\Lambda^2 H^*), \tag{7.10}$$

where the real-valued 2×2 matrix $(\epsilon_{\alpha\beta})_{\alpha,\,\beta=1,\,2}$ is defined by $\epsilon_{12} = -\epsilon_{21} = 1$ and $\epsilon_{11} = \epsilon_{22} = 0$. ω_H can equivalently be characterized by being non-degenerate and fulfilling $\omega_H(h_1, h_2) = \omega_H(\mathrm{Id}\,,\, -J_2) = 1$. Since j_H is i_H-antilinear and $j_H(h_1) = h_2$, we have $j_H^* h^1 = -\overline{h^2}$, $j_H^* h^2 = \overline{h^1}$ and hence $j_H^* \omega_H = \overline{\omega_H}$. Since $L_{J_1}, L_{J_2}, L_{J_3}$ are i_H-linear and fulfill

$$L_{J_1} h_1 = i_H h_1,\ L_{J_1} h_2 = -i_H h_2,\ L_{J_2} h_1 = -h_2,\ L_{J_2} h_2 = h_1,$$

we have

$$L^*_{J_1} h^1 = i h^1,\ L^*_{J_1} h^2 = -i h^2,\ L^*_{J_2} h^1 = h^2,\ L^*_{J_2} h^2 = -h^1.$$

In combination with $J_3 = J_1 J_2$, this can be used to show that ω_H is $Sp(1)$-invariant. In terms of the frame (h_1, h_2), the action of (L_{J_α}) on H is given by

$$L_{J_1}^{(h_\alpha)} = \begin{pmatrix} i_H & 0 \\ 0 & -i_H \end{pmatrix},\ L_{J_2}^{(h_\alpha)} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ L_{J_3}^{(h_\alpha)} = \begin{pmatrix} 0 & i_H \\ i_H & 0 \end{pmatrix}. \tag{7.11}$$

The almost hyper-complex structure (J_1, J_2, J_3) defines the following symmetric forms on H:

$$\omega_H(L_{J_i}\,\cdot\,,\cdot) = \begin{cases} 2 i h^1 h^2 & (i=1) \\ h^1 h^1 + h^2 h^2 & (i=2) \\ i(h^2 h^2 - h^1 h^1) & (i=3). \end{cases} \tag{7.12}$$

Quaternionic structure map j_E, non-degenerate two-form ω_E and compact symplectic group $Sp(E_x)$

On $E = T^{1,0}_{J_1} U$, we define an i_E-antilinear structure map that squares to $-\mathrm{Id}_E$ by

$$j_E := \rho \circ J_2 = J_2 \circ \rho : E \to E,\ e \mapsto \overline{J_2 e} = J_2 \overline{e}, \tag{7.13}$$

where ρ denotes the standard real structure on $T^{\mathbb{C}}U$ given by complex conjugation. We define the non-degenerate two-form

$$\omega_E := \frac{1}{2}(\omega_2 + i\omega_3) \in \Gamma(\Lambda^2 E^*), \tag{7.14}$$

where $\omega_i = g(J_i \cdot, \cdot)|_U$, $i = 1, 2, 3$. Using the fact that $J_2J_3 = -J_3J_2$ and that g is J_2-invariant, one shows that $j_E^*\omega_2 = \overline{\omega_2}$ and $j_E^*\omega_3 = -\overline{\omega_3}$, which implies $j_E^*\omega_E = \overline{\omega_E}$. The fiber over a point $x \in U$ of the subbundle $Sp_{\mathbb{C}}(E) \subset \mathrm{End}(E)$ consists of all invertible endomorphisms of E_x that leave $\omega_E|_x$ invariant. We denote the compact symplectic group which consists of elements in $Sp_{\mathbb{C}}(E_x)$ commuting with $j_E|_x$ by $Sp(E_x)$, i.e. the corresponding subbundle of $\mathrm{End}(E)$ is

$$Sp(E) := Sp_{\mathbb{C}}(E)^{j_E} = \{A \in Sp_{\mathbb{C}}(E) \mid j_E A j_E^{-1} = A\}. \tag{7.15}$$

Isomorphism f between $T^{\mathbb{C}}U$ and $H \otimes_{\mathbb{C}} E$

Now, we want to identify $T^{\mathbb{C}}U$ with $H \otimes_{\mathbb{C}} E$ via the following isomorphism:

$$f : H \otimes_{\mathbb{C}} E \to T^{\mathbb{C}}U,\ h \otimes e \mapsto he. \tag{7.16}$$

Since i_H is defined via right-multiplication of J_1 and i_E via left-multiplication of J_1, f is $\mathbb{C}$-linear, i.e. $f \circ (i_H \otimes \mathrm{Id}_E) = f \circ (\mathrm{Id}_H \otimes i_E) = i \circ f$. The standard real structure $\rho : v \mapsto \overline{v}$ on $T^{\mathbb{C}}U$ is recovered via $\rho \circ f = f \circ (j_H \otimes j_E)$, since $j_H = -R_{J_2}$ and $j_E = J_2 \circ \rho$. Using a frame in E, one can show that

$$f^* g_{\mathbb{C}}|_U = \omega_H \otimes \omega_E, \tag{7.17}$$

where $g_{\mathbb{C}}$ is the complex bilinear extension of g (see Eq. (7.40) below).

Decomposition of the curvature tensor R, quaternionic Weyl tensor W

Now, we state the well-known decomposition of the Riemann curvature tensor of a quaternionic Kähler manifold:

Theorem 7.1.2 *The curvature tensor R of a quaternionic Kähler manifold admits the decomposition*

$$R = \nu R_{\mathbb{H}P^n} + W, \tag{7.18}$$

where $R_{\mathbb{H}P^n}$ is the curvature tensor of the standard metric[3] *of $\mathbb{H}P^n$, $\nu = \frac{\mathrm{scal}}{4n(n+2)}$ is the reduced scalar curvature and all traces of $W \in \Gamma(TM \otimes T^*M^{\otimes 3})$ are zero. Given a local section $(J_1, J_2, J_3) \in \Gamma(U, S)$, $R_{\mathbb{H}P^n}$ is given by*

$$\begin{aligned} R_{\mathbb{H}P^n}(X,Y)Z = \frac{1}{4}[g(Y,Z)X - g(X,Z)Y] - \frac{1}{2}\sum_{i=1}^{3}\omega_i(X,Y)J_iZ \\ + \frac{1}{4}\sum_{i=1}^{3}[\omega_i(Y,Z)J_iX - \omega_i(X,Z)J_iY] \end{aligned} \tag{7.19}$$

and in terms of the identification of $T^{\mathbb{C}}U$ with $H \otimes_{\mathbb{C}} E$ given in Proposition 7.1.1, W is an $(\mathrm{Id}_H \otimes \mathfrak{sp}(E))$-valued 2-form whose complex bilinear extension fulfills

$$W(he, h'e')(h''e'') = -\omega_H(h, h')\, h''\omega_E^{-1}(\Omega(e, e', e'', \cdot)) \tag{7.20}$$

$(h, h', h'' \in \Gamma(H),\ e, e', e'' \in \Gamma(E))$, where $\Omega \in \Gamma(S^4E^*)$ such that $j_E^*\Omega = \overline{\Omega}$.

***Proof*:** This theorem was proven in [A1, Sa1]. See also [Besse, ACDGV]. □

Remark 7.1.3

1. *W is called the **quaternionic Weyl tensor**. Since R and $R_{\mathbb{H}P^n}$ fulfill the Bianchi identity, W does as well:*

$$W(X,Y)Z + W(Y,Z)X + W(Z,X)Y = 0. \tag{7.21}$$

2. *Note, that we use the following convention to identify E with E^*:*

$$E \overset{\approx}{\to} E^*,\ v \mapsto \omega_E(v, \cdot). \tag{7.22}$$

We denote the inverse of the above map by ω_E^{-1}:

$$E^* \to E,\ \alpha \mapsto \omega_E^{-1}(\alpha). \tag{7.23}$$

3. *The condition $j_E^*\Omega = \overline{\Omega}$ ensures that in equation (7.20), W is the $\mathbb{C}$-linear extension of a real tensor field:*

$$W(\overline{he}, \overline{h'e'})(\overline{h''e''}) = W(j_Hhj_Ee, j_Hh'j_Ee')(j_Hh''j_Ee'') = \overline{W(he, h'e')(h''e'')}. \tag{7.24}$$

[3] Here, the metric on quaternionic projective space is normalized such that its reduced scalar curvature is equal to one.

E-H splitting of the curvature tensor of quaternionic projective space

Now, we give a slight refinement of the splitting of the curvature tensor of a quaternionic Kähler manifold given in Theorem 7.1.2 by splitting the curvature tensor of $\mathbb{H}P^n$ into an H- and an E-part (this was done for example in [KSW]):

Proposition 7.1.4

$$R_{\mathbb{H}P^n} = R^H_{\mathbb{H}P^n} + R^E_{\mathbb{H}P^n}, \tag{7.25}$$

where for $h, h', h'' \in \Gamma(H)$ and $e, e', e'' \in \Gamma(E)$,

$$R^H_{\mathbb{H}P^n}(he, h'e')(h''e'') := -\frac{1}{2}\omega_E(e,e')(\omega_H(h,h'')h' + \omega_H(h',h'')h)e'', \tag{7.26}$$

$$R^E_{\mathbb{H}P^n}(he, h'e')(h''e'') := -\frac{1}{2}\omega_H(h,h')h''(\omega_E(e,e'')e' + \omega_E(e',e'')e). \tag{7.27}$$

***Proof*:** Using the equation

$$\sum_{i=1}^{3} \omega_H(J_i h, h')J_i h'' = \omega_H(h,h'')h' + \omega_H(h',h'')h, \tag{7.28}$$

which can be checked by direct computation, one finds

$$\begin{aligned} -\frac{1}{2}\sum_{i=1}^{3}\omega_i(he,h'e')J_i h''e'' &\overset{(7.17)}{=} -\frac{1}{2}\omega_E(e,e')\sum_{i=1}^{3}\omega_H(J_i h, h')J_i h''e'' \\ &= -\frac{1}{2}\omega_E(e,e')(\omega_H(h,h'')h' + \omega_H(h',h'')h)e'' \\ &= R^H_{\mathbb{H}P^n}(he,h'e')h''e''. \end{aligned} \tag{7.29}$$

Equation (7.28) and the Bianchi-type identity

$$\omega_H(h,h')h'' + \omega_H(h',h'')h + \omega_H(h'',h)h' = 0 \tag{7.30}$$

imply

$$\begin{aligned} \frac{1}{4}\sum_{I=0}^{3}\omega_I(h'e',h''e'')J_I he &\overset{(7.17)}{=} \frac{1}{4}\omega_E(e',e'')[\omega_H(h',h'')h + \sum_{i=1}^{3}\omega_H(J_i h', h'')J_i h]e \\ &= \frac{1}{4}\omega_E(e',e'')[\omega_H(h',h'')h + \omega_H(h',h)h'' + \omega_H(h'',h)h']e \\ &= -\frac{1}{2}\omega_E(e',e'')\omega_H(h,h')h''e, \end{aligned} \tag{7.31}$$

and hence

$$\frac{1}{4}\sum_{I=0}^{3}[\omega_I(h'e',h''e'')J_I he - \omega_I(he,h''e'')J_I h'e'] = R^E_{\mathbb{H}P^n}(he,h'e')h''e''. \qquad (7.32)$$

Here, we used the notation $\omega_0 := g$ and $J_0 := \mathrm{Id}$. Equation (7.19) then shows $R_{\mathbb{H}P^n} = R^H_{\mathbb{H}P^n} + R^E_{\mathbb{H}P^n}$. ☐

Frame $(E_\Gamma) = (E_a, E_{\tilde{a}}) = (\beta_a, \alpha_a)$ in E and quaternionic vielbein $(f^{\alpha\Gamma})$ (coframe in $T^{\mathbb{C}}U$)

To make contact with formulas used in the physics literature, we will now express the objects defined above on E and $T^{\mathbb{C}}U$ in terms of a frame (E_Λ) of E and a corresponding frame $(f_{\alpha\Lambda}) = (h_\alpha E_\Lambda)$ of $T^{\mathbb{C}}U$. The coframe $(f^{\alpha\Lambda})$ dual to $(f_{\alpha\Lambda})$ is called a **quaternionic vielbein** in the physics literature.

Let $e_1, ..., e_n \in \Gamma(U, TM)$ such that $g(e_a, e_b) = \delta_{ab}$, $a, b = 1, \ldots, n$. Then $(e_a, J_1e_a, J_2e_a, J_3e_a)_{a=1,\ldots,n}$ is a local oriented orthonormal frame with respect to g that is adapted to the almost hypercomplex structure (J_1, J_2, J_3). Then we define the following complex frame of E:

$$(E_a := \beta_a := \frac{1}{2}(e_a - iJ_1e_a),\ E_{a+n} := \alpha_a := \frac{1}{2}(J_2e_a - iJ_3e_a))_{a=1,\ldots,n}. \qquad (7.33)$$

From now on, we will write $\tilde{a}$ for the index $a + n$, $a = 1, \ldots, n$. We have $j_E(\beta_a) = \alpha_a$, i.e. $j_E(E_a) = E_{\tilde{a}}$.

$$(E^a := \beta^a := e^a - iJ_1^*e^a,\ E^{\tilde{a}} := \alpha^a := -J_2^*e^a - iJ_3^*e^a)_{a=1,\ldots,n} \qquad (7.34)$$

is the dual frame of $E^* = \Omega^{1,0}_{J_1}U$, where $e^a := g(e_a, \cdot) \in \Omega^1U$.

We define $f_{\alpha\Gamma} := f(h_\alpha \otimes E_\Gamma) = h_\alpha E_\Gamma \in T^{\mathbb{C}}U$ for $\alpha = 1, 2$ and for $\Gamma = 1, \ldots, 2n = 1, \ldots n, \tilde{1}, \ldots, \tilde{n}$:

$$(f_{\alpha\Gamma})_{\alpha=1,2;\,\Gamma=1,\ldots,2n} = \begin{pmatrix} f_{1a} & f_{1\tilde{a}} \\ f_{2a} & f_{2\tilde{a}} \end{pmatrix}_{a=1,\ldots,n} = \begin{pmatrix} \beta_a & \alpha_a \\ -\bar{\alpha}_a & \bar{\beta}_a \end{pmatrix}_{a=1,\ldots,n}. \qquad (7.35)$$

$(f_{\alpha\Gamma})_{\alpha=1,2;\,\Gamma=1,\ldots,2n}$ constitutes a frame in $T^{\mathbb{C}}U$. The corresponding coframe is given by

$$(f^{\alpha\Gamma})_{\alpha=1,2;\,\Gamma=1,\dots,2n} = \begin{pmatrix} f^{1a} & f^{1\tilde{a}} \\ f^{2a} & f^{2\tilde{a}} \end{pmatrix}_{a=1,\dots,n} := \begin{pmatrix} \beta^a & \alpha^a \\ -\bar{\alpha}^a & \bar{\beta}^a \end{pmatrix}_{a=1,\dots,n}, \tag{7.36}$$

where $\beta^a = 2\overline{g(\beta_a,\cdot)}$, $\alpha^a = 2\overline{g(\alpha_a,\cdot)}$ are defined as in equation (7.34). Then $(f^*(f^{\alpha\Gamma}))(h_\beta \otimes E_\Delta) = f^{\alpha\Gamma}(f_{\beta\Delta}) = \delta^\alpha_\beta \delta^\Gamma_\Delta$, i.e. $(f^{-1})^*(h^\alpha \otimes E^\Gamma) = f^{\alpha\Gamma}$, where we naturally identify $(H \otimes E)^*$ with $H^* \otimes E^*$.

Formulas for Levi-Civita connection and curvature tensor in quaternionic vielbein formalism

Using the fact that the metric can be written as

$$g|_U = \sum_{a=1}^{n} (\beta^a \bar{\beta}^a + \alpha^a \bar{\alpha}^a) = \frac{1}{2}\sum_{a=1}^{n} (\beta^a \otimes \bar{\beta}^a + \alpha^a \otimes \bar{\alpha}^a + \bar{\beta}^a \otimes \beta^a + \bar{\alpha}^a \otimes \alpha^a) \tag{7.37}$$

and that $J_2^*\beta^a = J_2^* e^a - iJ_3^* e^a = -\bar{\alpha}^a$, one obtains

$$\omega_2 = g(J_2\,\cdot\,,\cdot)|_U = \frac{1}{2}\sum_{a=1}^{n} (\beta^a \wedge \alpha^a + \bar{\beta}^a \wedge \bar{\alpha}^a). \tag{7.38}$$

The $(2,0)_{J_1}$-form $\omega_E = \frac{1}{2}(\omega_2 + i\omega_3)$ can thus be written as

$$\omega_E = \frac{1}{2}\sum_{a=1}^{n} \beta^a \wedge \alpha^a = \frac{1}{4}\sum_{\Gamma,\Delta=1}^{2n} C_{\Gamma\Delta} E^\Gamma \wedge E^\Delta, \tag{7.39}$$

where $(C_{\Gamma\Delta})_{\Gamma,\Delta=1,\dots,2n}$ is defined by $C_{a\tilde{b}} = -C_{\tilde{a}b} = \delta_{ab}$, $C_{ab} = C_{\tilde{a}\tilde{b}} = 0$ $(a, b = 1, \dots, n)$.

In terms of the coframe $(f^{\alpha\Gamma})$, the metric can be written as

$$g|_U \overset{(7.37)}{\underset{(7.36)}{=}} \sum_{\alpha,\beta=1}^{2} \sum_{\Gamma,\Delta=1}^{2n} \frac{1}{2}\epsilon_{\alpha\beta} C_{\Gamma\Delta} f^{\alpha\Gamma} f^{\beta\Delta}. \tag{7.40}$$

We define the real-valued $2n \times 2n$ matrix

$$J := (J^\Gamma{}_\Delta)_{\Gamma,\Delta=1,\dots,2n} = \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{7.41}$$

Then for $v = \sum_{\Gamma=1}^{2n} v^\Gamma E_\Gamma \in E$, $j_E v = \sum_{\Gamma,\,\Delta=1}^{2n} J^\Gamma{}_\Delta \overline{v^\Delta} E_\Gamma$, i.e. with respect to the frame (E_Γ), j_E is represented by $J \circ {}^-$. In the definition of J and in the following, the splitting of matrices into block form corresponds to the splitting

$$E = \operatorname{span}\{E_a\}_{a=1,\dots,n} \oplus \operatorname{span}\{E_{\tilde{a}}\}_{a=1,\dots,n} \tag{7.42}$$

of E into two totally isotropic subspaces with respect to ω_E. With respect to the frame (E_Γ), the two-form ω_E is represented by $\frac{1}{2}C$, where as above

$$C = (C_{\Gamma\Delta})_{\Gamma,\,\Delta=1,\dots,2n} = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}. \tag{7.43}$$

Remark 7.1.5 In terms of the quaternionic frame $(f_{\alpha\Gamma})$, we have

$$\begin{aligned} \omega_E\Big(\sum_{\Gamma=1}^{2n} v^\Gamma E_\Gamma, \cdot\Big) &= \frac{1}{2} \sum_{\Gamma,\,\Delta=1}^{2n} v^\Gamma C_{\Gamma\Delta} E^\Delta, \\ \omega_E^{-1}\Big(\sum_{\Delta=1}^{2n} \alpha_\Delta E^\Delta\Big) &= 2 \sum_{\Delta,\,\Gamma=1}^{2n} \alpha_\Delta C^{\Delta\Gamma} E_\Gamma, \end{aligned}$$

where $(C^{\Gamma\Delta}) = (C_{\Gamma\Delta})^{-1} = -(C_{\Gamma\Delta})$. The quaternionic Weyl tensor is thus given by

$$W(f_{\alpha\Gamma}, f_{\beta\Delta}) f_{\gamma\Xi} \overset{(7.20)}{=} -2\epsilon_{\alpha\beta} \sum_{\Lambda'=1}^{2n} \Omega_{\Gamma\Delta\Xi\Lambda'} C^{\Lambda'\Lambda} f_{\gamma\Lambda}, \tag{7.44}$$

where $\Omega_{\Gamma\Delta\Xi\Lambda} = \Omega(E_\Gamma, E_\Delta, E_\Xi, E_\Lambda) \in C^\infty(U, \mathbb{C})$.

The Lie algebra of $Sp(E_x)$ consists of all endomorphisms $B \in \operatorname{End}(E_x)$ such that the matrix $\tilde{B} \in \operatorname{Mat}(2n, \mathbb{C})$ representing B with respect to the basis $(E_\Gamma|_x)_{\Gamma=1,\dots,2n}$ of E_x fulfills $\tilde{B}^t C + C\tilde{B} = 0$ and $J\tilde{B} = \overline{\tilde{B}} J$, i.e.

$$\mathfrak{sp}(E_x) = \{B \in \operatorname{End}(E_x) \mid \tilde{B} = \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix} \in \operatorname{Mat}(2n, \mathbb{C}),\ q^\dagger := \overline{q}^t = -q,\ t^t = t\} \tag{7.45}$$

for $x \in U$.

Since the Levi-Civita connection ∇ preserves Q, i.e. $\nabla_X \Gamma(Q) \subset \Gamma(Q)$, $X \in \Gamma(TU)$, the connection one-form A, $A(X) \in \Gamma(\mathfrak{so}(TU)) \subset \Gamma(\operatorname{End}(TU))$, normalizes Q, i.e. $[A(X), Q] \subset Q$. Thus $A(X)$ can be written as a linear combination of the inner automorphisms $\{J_\alpha\}$ of Q and a Q-linear part:

$$A(X) = \sum_{i=1}^{3} A^i(X)J_i + A_0(X), \quad [A_0(X), Q] = 0. \tag{7.46}$$

Then $\nabla_X J_i = [A(X), J_i] = \sum_{l=1}^{3} A^l(X)[J_l, J_i] = -2A^j(X)J_k + 2A^k(X)J_j$ for any cyclic permutation (i, j, k) of $(1, 2, 3)$. Comparing with Eq. (2.1), we see that $A^i(X) = -\bar{\theta}_i(X)$, where $\bar{\theta}_i$, $i = 1, 2, 3$, are the components of the local $Sp(1)$-connection one-form with respect to the frame (J_1, J_2, J_3). In terms of the splitting $T^{\mathbb{C}}U = H \otimes E$, we thus have

$$\begin{aligned} A(X) &= \sum_{i=1}^{3} A^i(X)J_i + A_0(X) \\ &\mathrel{\hat{=}} A_H(X) \otimes \mathrm{Id}_E + \mathrm{Id}_H \otimes A_E(X) \\ &= -\sum_{\alpha=1}^{3} \bar{\theta}_\alpha(X) L_{J_\alpha} \otimes \mathrm{Id}_E + \mathrm{Id}_H \otimes A_E(X), \end{aligned} \tag{7.47}$$

where $A_E(X) \in \Gamma(\mathfrak{sp}(E))$. We denote the matrix representing $A_E(X)$ in terms of the frame (E_Γ) by $\Theta(X) := \tilde{A}_E(X) := ({\Theta^\Gamma}_\Delta(X))_{\Gamma,\Delta=1,\ldots,2n}$. With respect to the frame $(f_{\alpha\Gamma})$, we thus have

$$f^{\alpha\Gamma}(\nabla_X f_{\beta\Delta}) = {p^\alpha}_\beta(X){\delta^\Gamma}_\Delta + {\delta^\alpha}_\beta {\Theta^\Gamma}_\Delta(X),$$

where, since ∇ is metric, $J\Theta(X) = \overline{\Theta(X)}J$ and $\Theta(X)^t C + C\Theta(X) = 0$, and

$$p = ({p^\alpha}_\beta) = \begin{pmatrix} {p^1}_1 & {p^1}_2 \\ {p^2}_1 & {p^2}_2 \end{pmatrix} \overset{(7.11)}{\underset{(7.47)}{=}} \begin{pmatrix} -i\bar{\theta}_1 & -\bar{\theta}_2 - i\bar{\theta}_3 \\ \bar{\theta}_2 - i\bar{\theta}_3 & i\bar{\theta}_1 \end{pmatrix}. \tag{7.48}$$

We write the $\mathfrak{sp}(E)$-part of local Levi-Civita connection one-form with respect to the frame (E_Γ) as

$$({\Theta^\Gamma}_\Delta) = \begin{pmatrix} q & t \\ -\bar{t} & \bar{q} \end{pmatrix}, \tag{7.49}$$

where q, t are complex 1-form-valued $n \times n$ matrices that are anti-Hermitian, respectively symmetric: $q^\dagger := \bar{q}^t = -q$ and $t^t = t$. Since ∇ is torsion-free, q and t are determined by the equation $0 = df^{\Gamma 1} + \sum_{\beta=1}^{2} {p^1}_\beta \wedge f^{\Gamma\beta} + \sum_{\Delta=1}^{2n} {\Theta^\Gamma}_\Delta \wedge f^{\Delta 1}$,

$\Gamma = 1, \ldots, 2n$, which is equivalent to

$$0 = d\beta^a + p^1{}_1 \wedge \beta^a - p^1{}_2 \wedge \bar{\alpha}^a + \sum_{b=1}^{n} (q^a{}_b \wedge \beta^b + t^a{}_b \wedge \alpha^b) \tag{7.50}$$

$$0 = d\alpha^a + p^1{}_1 \wedge \alpha^a + p^1{}_2 \wedge \bar{\beta}^a + \sum_{b=1}^{n} (-\bar{t}^a{}_b \wedge \beta^b + \bar{q}^a{}_b \wedge \alpha^b), \tag{7.51}$$

$a = 1, \ldots, n$.

The calculation of the curvature tensor $R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ leads to

$$f^{\alpha\Gamma}(R(X, Y) f_{\beta\Delta}) = \tilde{R}_H{}^{\alpha}{}_{\beta}(X, Y)\delta^{\Gamma}{}_{\Delta} + \delta^{\alpha}{}_{\beta} \tilde{R}_E{}^{\Gamma}{}_{\Delta}(X, Y), \tag{7.52}$$

where

$$\begin{aligned} \tilde{R}_H &= dp + p \wedge p \\ &= \begin{pmatrix} -id\bar{\theta}_1 + 2i\bar{\theta}_2 \wedge \bar{\theta}_3 & -(d\bar{\theta}_2 + id\bar{\theta}_3) + 2i\bar{\theta}_1 \wedge (\bar{\theta}_2 + i\bar{\theta}_3) \\ (d\bar{\theta}_2 - id\bar{\theta}_3) + 2i\bar{\theta}_1 \wedge (\bar{\theta}_2 - i\bar{\theta}_3) & id\bar{\theta}_1 - 2i\bar{\theta}_2 \wedge \bar{\theta}_3 \end{pmatrix} \\ &\overset{(2.7)}{=} \frac{\nu}{2} \begin{pmatrix} -i\omega_1 & -\omega_2 - i\omega_3 \\ \omega_2 - i\omega_3 & i\omega_1 \end{pmatrix} \end{aligned} \tag{7.53}$$

and

$$\tilde{R}_E = d\Theta + \Theta \wedge \Theta. \tag{7.54}$$

We write the E-part of the curvature tensor with respect to the frame (E_Γ) as

$$\tilde{R}_E = \begin{pmatrix} r & s \\ -\bar{s} & \bar{r} \end{pmatrix}, \tag{7.55}$$

where r, s are complex two-form valued $n \times n$ matrices that fulfill $r^\dagger = -r$, $s^t = s$. In components, we then have

$$r^a{}_b = dq^a{}_b + \sum_{c=1}^{n} (q^a{}_c \wedge q^c{}_b - t^a{}_c \wedge \bar{t}^c{}_b) \tag{7.56}$$

$$s^a{}_b = dt^a{}_b + \sum_{c=1}^{n} (q^a{}_c \wedge t^c{}_b + t^a{}_c \wedge \bar{q}^c{}_b), \tag{7.57}$$

$a, b = 1, \ldots, n$.

To express the E-part R_E of the curvature tensor in terms of the quartic symmetric tensor field Ω in E, we combine Theorem 7.1.2 and Proposition 7.1.4:

Corollary 7.1.6 *The E-part of the curvature tensor of a quaternionic Kähler manifold with respect to the frame (E_Γ) is given by*

$$\tilde{R}_E{}^\Lambda{}_\Xi = \sum_{\alpha,\beta=1}^{2} \sum_{\Delta=1}^{2n} \frac{\nu}{4} \epsilon_{\alpha\beta} C_{\Xi\Delta} f^{\alpha\Lambda} \wedge f^{\beta\Delta} + \sum_{\alpha,\beta=1}^{2} \sum_{\Lambda',\Gamma,\Delta=1}^{2n} C^{\Lambda\Lambda'} \Omega_{\Lambda'\Xi\Gamma\Delta} \epsilon_{\alpha\beta} f^{\alpha\Gamma} \wedge f^{\beta\Delta}. \tag{7.58}$$

***Proof*:** Since $\omega_H(h_\alpha, h_\beta) = \epsilon_{\alpha\beta}$ and $\omega_E(E_\Gamma, E_\Delta) = \frac{1}{2} C_{\Gamma\Delta}$, we have

$$f^{\delta\Lambda}(R^E_{\mathbb{H}P^n}(f_{\alpha\Gamma}, f_{\beta\Delta}) f_{\gamma\Xi}) \overset{(7.27)}{=} -\frac{1}{4} \epsilon_{\alpha\beta} \delta^\delta_\gamma (C_{\Gamma\Xi} \delta^\Lambda_\Delta + C_{\Delta\Xi} \delta^\Lambda_\Gamma). \tag{7.59}$$

The definition of $\Omega \in \Gamma(S^4 E^*)$ in Eq. (7.20) implies

$$f^{\delta\Lambda}(W(f_{\alpha\Gamma}, f_{\beta\Delta}) f_{\gamma\Xi}) = -2 \delta^\delta_\gamma \epsilon_{\alpha\beta} \sum_{\Lambda'=1}^{2n} \Omega_{\Gamma\Delta\Xi\Lambda'} C^{\Lambda'\Lambda}. \tag{7.60}$$

Due to the decomposition $R = \nu R_{\mathbb{H}P^n} + W$ in Theorem 7.1.2, the E-part of the curvature tensor with respect to the frame (E_Γ) is given as a linear combination of the terms in Eqs. (7.59) and (7.60):

$$\tilde{R}_E{}^\Lambda{}_\Xi(f_{\alpha\Gamma}, f_{\beta\Delta}) = \frac{\nu}{4} \epsilon_{\alpha\beta} C_{\Xi\Delta} \delta^\Lambda_\Gamma - \frac{\nu}{4} \epsilon_{\beta\alpha} C_{\Xi\Gamma} \delta^\Lambda_\Delta - 2\epsilon_{\alpha\beta} \sum_{\Lambda'=1}^{2n} \Omega_{\Gamma\Delta\Xi\Lambda'} C^{\Lambda'\Lambda}. \tag{7.61}$$

The above equation is equivalent to Eq. (7.58). □

7.2 Curvature of the supergravity r-map

In this section, we recall expressions for the Levi-Civita connection and the Riemann curvature tensor for projective special Kähler manifolds in the image of the supergravity r-map that we stated in [CDL]. We then derive formulas for the local connection one-form and the curvature tensor in terms of a *unitary coframe*.

Note that in this section, we will sometimes leave out summation symbols and employ Einstein's summation convention, i.e. every index that appears twice within one expression is summed over.

Recall that a projective special Kähler manifold $(\bar{M} = \mathbb{R}^n + iU \subset \mathbb{C}^n, g_{\bar{M}}, J_{\bar{M}})$ in the image of the supergravity r-map is defined by a real homogeneous cubic polynomial h in n variables and an $\mathbb{R}^{>0}$-invariant domain $U \subset \mathbb{R}^n \backslash \{0\}$ (see Section 6.2). The Kähler potential of $g_{\bar{M}}$ is

$$\mathcal{K}(X, \bar{X}) = -\log 8h(x) = -\log h\big(i(\bar{X} - X)\big),$$

where $(X^\mu = y^\mu + ix^\mu)_{(\mu=1,\dots,n)}$ are standard holomorphic coordinates on $\bar{M} = \mathbb{R}^n + iU \subset \mathbb{C}^n$. The metric then reads $g_{\bar{M}} = \mathcal{K}_{\mu\bar{\nu}} dX^\mu d\bar{X}^\nu$, where

$$\mathcal{K}_{\mu\bar{\nu}} = -\frac{\partial^2}{\partial X^\mu \partial \bar{X}^\nu} \log h(x) = -\frac{h_{\mu\nu}(x)}{4h(x)} + \frac{h_\mu(x) h_\nu(x)}{4h^2(x)}. \tag{7.62}$$

For all Kähler manifolds, the only non-vanishing Christoffel symbols are (see e.g. [Mo], Section 12.2, or [KN])

$$dX^\rho(\nabla_{\partial_{X^\sigma}} \partial_{X^\mu}) =: \Gamma^\rho_{\sigma\mu} = g^{\rho\bar{\kappa}} \partial_{X^\sigma} g_{\mu\bar{\kappa}} \tag{7.63}$$

and their complex conjugates. For manifolds in the image of the supergravity r-map, we have

$$\Gamma^\rho_{\sigma\mu} = -\frac{i}{2h}\left(h h^{\rho\kappa} h_{\kappa\mu\sigma} - h_\sigma \delta^\rho_\mu - h_\mu \delta^\rho_\sigma + \frac{1}{2} x^\rho h_{\mu\sigma} \right). \tag{7.64}$$

For the Riemann tensor

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \qquad (X, Y, Z \in \mathfrak{X}(\bar{M}))$$

in local coordinates, we have (see e.g. [Mo], Section 12.2, or [KN])

$$dX^\rho \left(R(\partial_{X^\mu}, \partial_{\bar{X}^\nu}) \partial_{X^\sigma} \right) =: R^\rho{}_{\sigma\mu\bar{\nu}} = -\partial_{\bar{X}^\nu} \Gamma^\rho_{\sigma\mu}. \tag{7.65}$$

The other non-vanishing components $R^{\bar{\rho}}{}_{\bar{\sigma}\mu\bar{\nu}}$, $R^\rho{}_{\sigma\bar{\mu}\nu}$ and $R^{\bar{\rho}}{}_{\bar{\sigma}\bar{\mu}\nu}$ of the curvature tensor can be obtained from this via symmetry and complex conjugation. The curvature tensor is given by [CDL, Theorem 3]

$$\begin{aligned} R^\rho{}_{\sigma\mu\bar{\nu}} = -\frac{i}{2} \partial_{x^\nu} \Gamma^\rho_{\sigma\mu} = -\frac{1}{4h^2} \Bigg[&\frac{1}{2} x^\rho (h h_{\mu\sigma\nu} - h_{\mu\sigma} h_\nu) + h_\mu h_\nu \delta^\rho_\sigma + h_\sigma h_\nu \delta^\rho_\mu \\ &- h \left(h_{\sigma\nu} \delta^\rho_\mu + h_{\mu\nu} \delta^\rho_\sigma - \frac{1}{2} h_{\mu\sigma} \delta^\rho_\nu \right) - h^2 h^{\rho\alpha} h_{\nu\alpha\beta} h^{\beta\gamma} h_{\gamma\mu\sigma} \Bigg] \end{aligned}$$

$$= -\delta^\rho_\sigma \mathcal{K}_{\mu\bar{\nu}} - \delta^\rho_\mu \mathcal{K}_{\sigma\bar{\nu}} + e^{2\mathcal{K}} \mathcal{K}^{\rho\bar{\alpha}} h_{\alpha\nu\beta} \mathcal{K}^{\beta\bar{\gamma}} h_{\gamma\mu\sigma}, \tag{7.66}$$

where

$$\mathcal{K}^{\bar{\nu}\lambda} = -4h(x)h^{\nu\lambda}(x) + 2x^\nu x^\lambda. \tag{7.67}$$

Unitary coframe $(\sigma^a = e^a - iJ^*_{\bar{M}} e^a = \sum_{\mu=1}^n e^a_\mu dX^\mu)_{a=1,\dots,n}$

Let $(e^a_\mu)_{a,\mu=1,\dots,n}$ be a real $n \times n$ matrix-valued function on some open subset in $\bar{M}$ such that $\sum_{a=1}^n e^a_\mu \bar{e}^a_\nu = \sum_{a=1}^n e^a_\mu e^a_\nu = \mathcal{K}_{\mu\bar{\nu}}$. Then the holomorphic one-forms

$$\sigma^a := \sum_{\mu=1}^n e^a_\mu dX^\mu \tag{7.68}$$

constitute a *unitary coframe* $(\sigma^a)_{a=1,\dots,n}$, i.e. the metric can locally be written as

$$g_{\bar{M}} = \sum_{a=1}^n \sigma^a \bar{\sigma}^a = \frac{1}{2} \sum_{a=1}^n (\sigma^a \otimes \bar{\sigma}^a + \bar{\sigma}^a \otimes \sigma^a). \tag{7.69}$$

Let $(\sigma_a := \sum_{\mu=1}^n e^\mu_a \frac{\partial}{\partial X^\mu})_{a=1,\dots,n}$ denote the corresponding local frame in $T^{1,0}\bar{M}$ dual to $(\sigma^a)_{a=1,\dots,n}$, i.e. $(e^\mu_a) = (e^a_\mu)^{-1}$. Then $\sigma^a = 2g_{\bar{M}}(\bar{\sigma}_a, \cdot)$.

For the Levi-Civita connection ∇, we denote the coefficients of the local connection one-form associated to the coframe (σ^a) by ${\omega^a}_b$, i.e. $\nabla_\cdot \sigma^a = \sum_{b=1}^n {\omega^a}_b(\cdot)\sigma^b$. Since the connection is metric, the complex one-form valued matrix $({\omega^a}_b)_{a,b=1,\dots,n}$ is anti-Hermitian and since the connection is torsion-free, it fulfills $d\sigma^a + \sum_{b=1}^n {\omega^a}_b \wedge \sigma^b = 0$, $a = 1, \dots, n$. A formula for the local connection one-form of any Kähler manifold with respect to a unitary coframe is given by

$${\omega^a}_b = \sum_{\mu=1}^n (e^a_\mu \bar{\partial} e^\mu_b - \bar{e}^b_\mu \partial \bar{e}^\mu_a). \tag{7.70}$$

In terms of the local connection one-form, the curvature tensor of a Kähler manifold is given by

$$R(X,Y)\sigma_c = \sum_{d=1}^n (d{\omega^d}_c + \sum_{c'=1}^n {\omega^d}_{c'} \wedge {\omega^{c'}}_c)(X,Y)\sigma_d =: \sum_{d=1}^n {\tilde{R}^d}_c(X,Y)\sigma_d. \tag{7.71}$$

Proposition 7.2.1 *In terms of the unitary coframe $(\sigma^a)_{a=1,\dots,n}$, the Riemann curvature tensor of a projective special Kähler manifold in the image of the su-*

pergravity r-map reads

$$\tilde{R}^a{}_b = -\delta^a_b \sum_{c=1}^{n} \sigma^c \wedge \bar{\sigma}^c - \sigma^a \wedge \bar{\sigma}^b + e^{2\mathcal{K}} \sum_{c,\, e,\, d=1}^{n} \tilde{h}_{adc}\tilde{h}_{ceb}\sigma^e \wedge \bar{\sigma}^d, \tag{7.72}$$

where $\tilde{h}_{abc} := \sum_{\mu,\,\nu,\,\sigma=1}^{n} e^\mu_a e^\nu_b e^\sigma_c h_{\mu\nu\sigma}$ *for* $a, b, c = 1, \ldots, n$.

***Proof*:** Using $\mathcal{K}_{\mu\bar{\nu}} = e^c_\mu \bar{e}^c_\nu$, $\mathcal{K}^{\mu\bar{\nu}} = e^\mu_c \bar{e}^\nu_c$ and the fact that $(e^a_\mu)_{a,\,\mu=1,\ldots,n} = (e^\nu_b)^{-1}_{\nu,\,b=1,\ldots,n}$, we find

$$\begin{aligned}
\tilde{R}^a{}_b(\sigma_e, \bar{\sigma}_d) &= \sigma^a(R(\sigma_e, \bar{\sigma}_d)\sigma_b) \\
&= e^a_\rho \, dX^\rho(R(\partial_{X^\mu}, \partial_{\bar{X}^\nu})\partial_{X^\sigma}) \, e^\mu_e \bar{e}^\nu_d e^\sigma_b \\
&\overset{(7.66)}{=} e^a_\rho \left(-\delta^\rho_\sigma \mathcal{K}_{\mu\bar{\nu}} - \delta^\rho_\mu \mathcal{K}_{\sigma\bar{\nu}} + e^{2\mathcal{K}}\mathcal{K}^{\rho\bar{\alpha}} h_{\alpha\nu\beta}\mathcal{K}^{\beta\bar{\gamma}} h_{\gamma\mu\sigma}\right) e^\mu_e \bar{e}^\nu_d e^\sigma_b \\
&= -\delta^a_b \delta_{de} - \delta^a_e \delta_{bd} + e^{2\mathcal{K}} \tilde{h}_{adc}\tilde{h}_{ceb}.
\end{aligned}$$

□

Recall from Remark 6.2.4 that a projective special Kähler manifold $\bar{M} = \mathbb{R}^n + iU$ in the image of the supergravity r-map is defined by the holomorphic *prepotential*

$$F : M \to \mathbb{C}, \quad F(z^0, \ldots, z^n) = \frac{h(z^1, \cdots, z^n)}{z^0}, \tag{7.73}$$

where M is the trivial $\mathbb{C}^*$-bundle

$$M := \{z = z^0 \cdot (1, X) \in \mathbb{C}^{n+1} \mid z^0 \in \mathbb{C}^*,\, X \in \bar{M} = \mathbb{R}^n + iU\} \to \bar{M}. \tag{7.74}$$

Recall that the complex $(n+1) \times (n+1)$ matrix-valued function

$$\boldsymbol{F}(z) := (F_{IJ}(z))_{I,\,J=0,\ldots,n} := \Big(\frac{\partial^2 F(z)}{\partial z^I \partial z^J}\Big)_{I,\,J=0,\ldots,n} \tag{7.75}$$

is homogeneous of degree zero and thus defines a function $\boldsymbol{F}(X)$ on $\bar{M}$. The same holds true for $N(z, \bar{z}) := (N_{IJ}(z, \bar{z}))_{I,\,J=0,\ldots,n} := 2\,\mathrm{Im}\,\boldsymbol{F}(z)$. The matrix N is invertible at every point in $\bar{M}$ and we denote the components of its inverse by N^{IJ}. Recall from Remark 6.2.4 that in terms of N, the Kähler potential can be written as

$$\mathcal{K} = -\log XN\bar{X} =: -\log(X^I N_{IJ}(X, \bar{X})\bar{X}^J), \tag{7.76}$$

where $X^0 := 1$. More generally, for every function $f_M(z)$ on M, we define a function $f_{\bar{M}}(X)$ on $\bar{M}$ by $f_{\bar{M}}(X) := f_M(1, X^1, \ldots, X^n)$. Like this, the function

$$F_{IJK}(z) := \frac{\partial^3 F(z)}{\partial z^I \partial z^J \partial z^K} \qquad (I, J, K = 0, \ldots, n) \tag{7.77}$$

defines a function $F_{IJK}(X)$ on $\bar{M}$.

Proposition 7.2.2 *The local connection one-form for the Levi-Civita connection with respect to the unitary coframe $(\sigma^a)_{a=1,\ldots,n}$ can be written as*

$$\omega^a{}_b = e^{-\mathcal{K}}\big((\bar{\partial} P^a_I) N^{IJ} \bar{P}^b_J - P^a_I N^{IJ} (\partial \bar{P}^b_J)\big) \tag{7.78}$$
$$= \delta^a_b \partial \mathcal{K} + e^{-\mathcal{K}} d(P^a_I N^{IJ}) \bar{P}^b_J + i e^{-\mathcal{K}} P^a_I N^{IK} \, \overline{dF_{KL}(X)} \, N^{LJ} \bar{P}^b_J, \tag{7.79}$$

where P^a_I are the components of the complex $n \times (n+1)$ matrix-valued function

$$(P^a_I)_{a=1,\ldots,n,\ I=0,\ldots,n} = (P^a_0, P^a_\mu)_{a,\mu=1,\ldots,n} := (-\sum_{\nu=1}^{n} e^a_\nu X^\nu, e^a_\mu)_{a,\mu=1,\ldots,n}. \tag{7.80}$$

Before we prove the above proposition, we state a few formulas for the matrix-valued functions $\boldsymbol{F}$, N, (P^a_I), and for $F_{IJK}(X)$ that can be easily checked or looked up in the physics literature on special geometry:

Remark 7.2.3

$$0 = \sum_{I=0}^{n} X^K F_{IJK}(X) \qquad (I, J = 0, \ldots, n), \tag{7.81}$$
$$0 = \sum_{I=0}^{n} P^a_I X^I \qquad (a = 1, \ldots, n), \tag{7.82}$$
$$\partial \mathcal{K} = -e^{\mathcal{K}} \sum_{I,J=0}^{n} \bar{X}^I N_{IJ} dX^J, \tag{7.83}$$
$$\mathcal{K}_{\mu\bar{\nu}} = -e^{\mathcal{K}}(N_{\mu\nu} - e^{\mathcal{K}} \sum_{K,L=0}^{n} N_{\mu K} \bar{X}^K X^L N_{L\nu}) \qquad (\mu, \nu = 1, \ldots, n), \tag{7.84}$$
$$\sum_{a=1}^{n} P^a_I \bar{P}^a_J = -e^{\mathcal{K}}(N_{IJ} - e^{\mathcal{K}} \sum_{K,L=0}^{n} N_{IK} \bar{X}^K X^L N_{LJ}) \qquad (I, J = 0, \ldots, n), \tag{7.85}$$
$$-e^{\mathcal{K}} \delta^{ab} = \sum_{I,J=0}^{n} P^a_I N^{IJ} \bar{P}^b_J \qquad (a, b = 1, \ldots, n). \tag{7.86}$$

Proof (of Proposition 7.2.2):

Multiplication of Eq. (7.86) by $-e^{-\mathcal{K}}e^{\mu}_{a}$ gives

$$e^{\mu}_{b} = e^{-\mathcal{K}}(X^{\mu}N^{0J} - N^{\mu J})\bar{P}^{b}_{J}.$$

This equation shows that

$$-e^{\mu}_{b}\bar{\partial}e^{a}_{\mu} = e^{-\mathcal{K}}((\bar{\partial}e^{a}_{\mu})N^{\mu J} - X^{\mu}(\bar{\partial}e^{a}_{\mu})N^{0J})\bar{P}^{b}_{J} = e^{-\mathcal{K}}(\bar{\partial}P^{a}_{I})N^{IJ}\bar{P}^{b}_{J}.$$

Using the above equation one then finds

$$\omega^{a}{}_{b} \overset{(7.70)}{=} -e^{\mu}_{b}\bar{\partial}e^{a}_{\mu} + \bar{e}^{\mu}_{a}\partial\bar{e}^{b}_{\mu} = e^{-\mathcal{K}}\big((\bar{\partial}P^{a}_{I})N^{IJ}\bar{P}^{b}_{J} - P^{a}_{I}N^{IJ}(\partial\bar{P}^{b}_{J})\big).$$

Adding $0 \overset{(7.86)}{=} \delta^{a}_{b}\partial\mathcal{K} + e^{-\mathcal{K}}\partial(P^{a}_{I}N^{IJ}\bar{P}^{b}_{J})$ to the above equation gives

$$\begin{aligned}\omega^{a}{}_{b} &= \delta^{a}_{b}\partial\mathcal{K} + e^{-\mathcal{K}}\big(d(P^{a}_{I}N^{IJ})\bar{P}^{b}_{J} - P^{a}_{I}(\bar{\partial}N^{IJ})\bar{P}^{b}_{J}\big)\\ &= \delta^{a}_{b}\partial\mathcal{K} + e^{-\mathcal{K}}d(P^{a}_{I}N^{IJ})\bar{P}^{b}_{J} + ie^{-\mathcal{K}}P^{a}_{I}N^{IK}\,\overline{dF_{KL}(X)}\,N^{LJ}\bar{P}^{b}_{J}.\end{aligned}$$

□

7.3 Curvature of the q-map

In this section, we will use the formulas (7.48)-(7.51) for the E- and H-part of the Levi-Civita connection and the formulas (7.55)-(7.57) for the E-part R_E of the Riemann curvature tensor derived in Section 7.1 to calculate the Levi-Civita connection of all manifolds in the image of the one-loop deformed q-map and the Riemann tensor of all manifolds in the image of the undeformed q-map. We also derive the quartic tensor field $\Omega \in \Gamma(S^4E^*)$ that determines the curvature tensor of the manifolds in the image of the q-map.

Again, we will sometimes leave out summation symbols and employ Einstein's summation convention.

As in the last section, let $(\bar{M} = \mathbb{R}^n + iU \subset \mathbb{C}^n, g_{\bar{M}}, J_{\bar{M}})$ be a projective special Kähler manifold in the image of the supergravity r-map. Let h denote the corresponding homogeneous cubic polynomial in n real variables. Let $(\sigma^1, \ldots, \sigma^n)$ be a unitary coframe on the projective special Kähler manifold $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$, i.e. $\sigma^a =: \sum_{\mu=1}^{n} e^{a}_{\mu}\, dX^{\mu}$, $a = 1, \ldots, n$, are locally defined $J_{\bar{M}}$-holomorphic one-forms such that $g_{\bar{M}} = \sum_{a=1}^{n} \sigma^a\bar{\sigma}^a = \frac{1}{2}\sum_{a=1}^{n}(\sigma^a \otimes \bar{\sigma}^a + \bar{\sigma}^a \otimes \sigma^a)$. Here, $(X^{\mu})_{\mu=1,\ldots,n}$ again denotes standard holomorphic coordinates on $\bar{M} \subset \mathbb{C}^n$. Since for Kähler manifolds in the image of the supergravity r-map $\mathcal{K}_{\mu\bar{\nu}}$ is real, we can choose e^a_μ to be real (see Eq. (7.62)). Recall how $\bar{M}$ is realized as a $\mathbb{C}^*$-quotient of a conical affine special Kähler manifold M defined by a holomorphic prepotential $F : M \to \mathbb{C}$ (see Eq. (7.73) and below). This in particular defines the matrix-valued functions $N = (N_{IJ})_{I,\,J=0,\dots,n}$, $\boldsymbol{F} = (F_{IJ})_{I,\,J=0,\dots,n}$, etc. on $\bar{M}$.

Starting from the projective *very* special Kähler manifold $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$, we now consider the one-loop deformed supergravity c-map. For $c \in \mathbb{R}$, let

$$N' := N'_{(4n+4,\,0)} \subset \bar{M} \times \mathbb{R}^{2n+4} \subset \mathbb{R}^{4n+4} \tag{7.87}$$

denote the domain where the one-loop deformed Ferrara-Sabharwal metric g^c_{FS} is positive definite (see Definition 5.5.1). As in Definition 5.5.1, we use standard real coordinates $(\rho, \tilde{\phi}, \tilde{\zeta}_I, \zeta^I)_{I=0,\dots,n}$ on the $\mathbb{R}^{2n+4}$ factor of N' and complex coordinates $(X^\mu)_{\mu=1,\dots,n}$ on $\bar{M}$. Note that on N', $\rho > 0$ and $\rho + 2c > 0$. We define the following complex-valued one-forms on N':

$$\begin{aligned}
\beta^0 &:= i e^{\mathcal{K}/2} \frac{\sqrt{\rho + 2c}}{\rho} \sum_{I=0}^{n} X^I A_I, \\
\beta^a &:= \sqrt{\frac{\rho + c}{\rho}} \sum_{I=0}^{n} P^a_I dX^I = \sqrt{\frac{\rho + c}{\rho}}\, \sigma^a, \\
\alpha^0 &:= -\frac{1}{2\rho} \sqrt{\frac{\rho + 2c}{\rho + c}} \left(d\rho - i \frac{\rho + c}{\rho + 2c} (d\tilde{\phi} + \sum_{I=0}^{n} (\zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I) + c d^c \mathcal{K}) \right), \\
\alpha^a &:= \frac{i}{\sqrt{\rho}} e^{-\mathcal{K}/2} \sum_{I,J=0}^{n} \overline{P}^a_I N^{IJ} A_J
\end{aligned} \tag{7.88}$$

$(a = 1, \dots, n)$, where $(P^a_I)_{I=0,\dots,n} = (P^a_0, P^a_\mu)_{\mu=1,\dots,n} = (-\sum_{\nu=1}^n X^\nu e^a_\nu, e^a_\mu)_{\mu=1,\dots,n}$ and $A_I = d\tilde{\zeta}_I + \sum_{J=0}^n F_{IJ}(X) d\zeta^J$, $I = 0, \dots, n$. Here, we trivially extend functions and one-forms from $\bar{M}$ to N', using the same notation (i.e. leaving out pullbacks). Let $Q = \mathrm{span}_{\mathbb{R}}\{J'_1, J'_2, J'_3\}$ denote the (trivial) quaternionic structure on N' obtained from the HK/QK correspondence (see Remarks 5.5.3 and 5.5.4).

Lemma 7.3.1 *The coframe*

$$(f^{\alpha\Gamma})_{\alpha=1,\,2;\,\Gamma=1,\dots,2n+2} = \begin{pmatrix} f^{1A} & f^{1\tilde{A}} \\ f^{2A} & f^{2\tilde{A}} \end{pmatrix}_{A=0,\dots,n} := \begin{pmatrix} \beta^A & \alpha^A \\ -\bar{\alpha}^A & \bar{\beta}^A \end{pmatrix}_{A=0,\dots,n} \tag{7.89}$$

in $(T^*N')^{\mathbb{C}}$ defines a unitary coframe for the one-loop deformed Ferrara-Sabharwal *metric, i.e. the metric reads*

$$g^c_{FS} = \sum_{A=0}^{n}(\beta^A\bar{\beta}^A + \alpha^A\bar{\alpha}^A), \tag{7.90}$$

and α^A, β^A are J_1-holomorphic and fulfill

$$\alpha^A = -J_2^*\bar{\beta}^A \qquad (A = 0, \ldots, n). \tag{7.91}$$

Proof: Note that $\tau = -2i\rho\sqrt{\frac{\rho+2c}{\rho+c}}\alpha^0$, where τ is given by Eq. (5.18). Furthermore,

$$\frac{1}{4}dd^c\mathcal{K} = \frac{i}{2}\partial\bar{\partial}\mathcal{K} = \frac{i}{2}\sigma^a \wedge \bar{\sigma}^a = \frac{i}{2}\frac{\rho}{\rho+c}\beta^a \wedge \bar{\beta}^a,$$

$\beta^0 \wedge \bar{\beta}^0 = e^{\mathcal{K}}\frac{\rho+2c}{\rho^2}(X^I A_I) \wedge (\bar{X}^J\bar{A}_J)$ and

$$\alpha^a \wedge \bar{\alpha}^a = \frac{1}{\rho}e^{-\mathcal{K}}N^{IK}\bar{P}^a_K P^a_L N^{LJ} A_I \wedge \bar{A}_J \overset{(7.85)}{=} -\frac{1}{\rho}(N^{IJ} - e^{\mathcal{K}}X^I\bar{X}^J)A_I \wedge \bar{A}_J.$$

Together with Eq. (5.16), this shows that the first fundamental two-form is given by

$$\bar{\omega}_1 = \frac{i}{2}\sum_{A=0}^{n}(\beta^A \wedge \bar{\beta}^A + \alpha^A \wedge \bar{\alpha}^A). \tag{7.92}$$

Note that

$$\begin{aligned}
\alpha^a \wedge \beta^a &= ie^{-\mathcal{K}/2}\frac{\sqrt{\rho+c}}{\rho}\,P^a_I\bar{P}^a_K N^{KJ}A_J \wedge dX^I \\
&\overset{(7.85)}{=} ie^{\mathcal{K}/2}\frac{\sqrt{\rho+c}}{\rho}\,(-A_I \wedge dX^I + e^{\mathcal{K}}\bar{X}^M N_{MI}X^J A_J \wedge dX^I) \\
&\overset{(7.83)}{=} +ie^{\mathcal{K}/2}\frac{\sqrt{\rho+c}}{\rho}\,(dX^I \wedge A_I + \partial\mathcal{K} \wedge X^I A_I).
\end{aligned} \tag{7.93}$$

Together with Eq. (5.17) and the definitions of β^0 and α^0, this shows that

$$\bar{\omega}_2 + i\bar{\omega}_3 = \sum_{A=0}^{n}\beta^A \wedge \alpha^A. \tag{7.94}$$

The statements of the lemma follow immediately from Eqs. (7.92) and (7.94). □

Before we proceed, we state a few more formulas that can be proven using the formulas in Remark 7.2.3 and the definitions of β^A, α^A. These formulas will be used in later proofs.

Remark 7.3.2 We have

$$N^{IJ}A_I \wedge \bar{A}_J = N^{IJ}(F_{IK} - \bar{F}_{IK})d\zeta^K \wedge d\tilde{\zeta}_J = id\zeta^J \wedge d\tilde{\zeta}_J, \tag{7.95}$$

$$dX^K \underset{(7.83)}{\overset{(7.85)}{=}} -X^K\partial\mathcal{K} - e^{-\mathcal{K}}\sqrt{\frac{\rho}{\rho+c}}\, N^{KJ}\bar{P}^a_J\beta^a, \tag{7.96}$$

$$\frac{i}{\sqrt{\rho}}e^{-\mathcal{K}/2}A_J \overset{(7.85)}{=} \sqrt{\frac{\rho}{\rho+2c}}\, N_{JL}\bar{X}^L\beta^0 - e^{-\mathcal{K}}P^b_J\alpha^b, \tag{7.97}$$

$$\begin{aligned}\frac{i}{\sqrt{\rho}}e^{-\mathcal{K}/2}d\zeta^L &= 2\frac{i}{\sqrt{\rho}}e^{-\mathcal{K}/2}N^{LM}\mathrm{Im}\,A_M = -2iN^{LM}\mathrm{Re}(\frac{i}{\sqrt{\rho}}e^{-\mathcal{K}/2}A_M)\\ &\overset{(7.97)}{=} -2i\sqrt{\frac{\rho}{\rho+2c}}\,\mathrm{Re}(\bar{X}^L\beta^0) + 2ie^{-\mathcal{K}}N^{LM}\mathrm{Re}(P^b_M\alpha^b),\end{aligned} \tag{7.98}$$

$$\begin{aligned}d(\bar{P}^a_I N^{IJ})N_{JL}\bar{X}^L &\overset{(7.82)}{=} -\bar{P}^a_I N^{IJ}d(N_{JL}\bar{X}^L)\\ &\overset{(7.81)}{=} -\sqrt{\frac{\rho}{\rho+c}}\,\bar{\beta}^a + i\bar{P}^a_I N^{IJ}dF_{JL}(X)\bar{X}^L,\end{aligned} \tag{7.99}$$

$$e^{-\mathcal{K}}\bar{P}^a_I N^{IJ}\bar{P}^b_M N^{ML}F_{JKL}(X)dX^K \underset{(7.81)}{\overset{(7.96)}{=}} -e^{-2\mathcal{K}}\sqrt{\frac{\rho}{\rho+c}}\,\widetilde{F}^{abc}(X)\beta^c, \tag{7.100}$$

where for $a, b, c = 1, \ldots, n$,

$$\tilde{F}^{abc}(X) := \sum_{I,J,K,L,M,N=0}^{n} \bar{P}^a_L N^{LI}\bar{P}^b_M N^{MJ}\bar{P}^c_N N^{NK}F_{IJK}(X). \tag{7.101}$$

Proposition 7.3.3

$$d\beta^0 = \frac{1}{2}\left((1+\frac{2c}{\rho+2c})\sqrt{\frac{\rho+c}{\rho+2c}}\,(\alpha^0+\bar{\alpha}^0) - id^c\mathcal{K}\right)\wedge\beta^0 + \sqrt{\frac{\rho+2c}{\rho+c}}\,\sum_{b=1}^{n}\alpha^b\wedge\beta^b,$$

$$d\beta^a = \frac{c}{2\sqrt{\rho+c}\sqrt{\rho+2c}}\,(\alpha^0+\bar{\alpha}^0)\wedge\beta^a - \sum_{b=1}^{n}{\omega^a}_b\wedge\beta^b,$$

$$\begin{aligned}d\alpha^0 &= \frac{1}{\sqrt{\rho+c}\sqrt{\rho+2c}}\left(-(\rho+c)+\frac{1}{2}\frac{c\rho}{\rho+2c}\right)\alpha^0\wedge\bar{\alpha}^0 + \frac{\rho}{\rho+2c}\sqrt{\frac{\rho+c}{\rho+2c}}\,\beta^0\wedge\bar{\beta}^0\\ &\quad - \sqrt{\frac{\rho+c}{\rho+2c}}\,\sum_{b=1}^{n}\alpha^b\wedge\bar{\alpha}^b - \frac{c}{\sqrt{\rho+c}\sqrt{\rho+2c}}\sum_{b=1}^{n}\beta^b\wedge\bar{\beta}^b,\end{aligned}$$

$$\begin{aligned}d\alpha^a &= \frac{1}{2}(\sqrt{\frac{\rho+c}{\rho+2c}}\,(\alpha^0+\bar{\alpha}^0) - id^c\mathcal{K})\wedge\alpha^a + \frac{\rho}{\sqrt{\rho+c}\sqrt{\rho+2c}}\,\beta^0\wedge\bar{\beta}^a\\ &\quad - \sum_{b=1}^{n}\overline{{\omega^a}_b}\wedge\alpha^b - ie^{\mathcal{K}}\sqrt{\frac{\rho}{\rho+c}}\,\sum_{b,c=1}^{n}\tilde{h}_{abc}\bar{\alpha}^b\wedge\beta^c,\end{aligned}$$

where $\tilde{h}_{abc} = \sum_{\mu,\nu,\sigma=1}^{n} e^\mu_a e^\nu_b e^\sigma_c h_{\mu\nu\sigma}$ *for* $a, b, c = 1, \ldots, n$ *and* $({\omega^a}_b)_{a,b=1,\ldots,n}$ *is the* *(pullback to N' of the) connection one-form of the Levi-Civita connection on $\bar{M}$ with respect to the given choice of coframe on $\bar{M}$, i.e. $(\omega^a{}_b)$ is anti-Hermitian and fulfills $d\sigma^a + \sum_{b=1}^n \omega^a{}_b \wedge \sigma^b = 0$, $a = 1, \dots, n$.*

***Proof*:** For $\beta^0 = ie^{\mathcal{K}/2}\frac{\sqrt{\rho+2c}}{\rho} X^I A_I$, we have

$$\begin{aligned}
d\beta^0 &\underset{(7.81)}{=}^4 \left(-\frac{1}{2\rho}\big(1+\frac{2c}{\rho+2c}\big)d\rho + \frac{1}{2}d\mathcal{K}\right)\wedge\beta^0 + ie^{\mathcal{K}/2}\frac{\sqrt{\rho+2c}}{\rho}\,dX^I\wedge A_I \\
&= \frac{1}{2}\big(1+\frac{2c}{\rho+2c}\big)\sqrt{\frac{\rho+c}{\rho+2c}}\,(\alpha^0+\bar{\alpha}^0)\wedge\beta^0 + (\partial\mathcal{K} - \frac{i}{2}d^c\mathcal{K})\wedge\beta^0 \\
&\qquad\qquad + ie^{\mathcal{K}/2}\frac{\sqrt{\rho+2c}}{\rho}\,dX^I\wedge A_I \\
&= \frac{1}{2}\left(\big(1+\frac{2c}{\rho+2c}\big)\sqrt{\frac{\rho+c}{\rho+2c}}\,(\alpha^0+\bar{\alpha}^0) - id^c\mathcal{K}\right)\wedge\beta^0 + \sqrt{\frac{\rho+2c}{\rho+c}}\,\alpha^b\wedge\beta^b,
\end{aligned}$$

since

$$\alpha^a\wedge\beta^a \overset{(7.93)}{=} +ie^{\mathcal{K}/2}\frac{\sqrt{\rho+c}}{\rho}\,dX^I\wedge A_I + \sqrt{\frac{\rho+c}{\rho+2c}}\,\partial\mathcal{K}\wedge\beta^0.$$

For $\beta^a = \sqrt{\frac{\rho+c}{\rho}}\,\sigma^a$, we have $d\sigma^a = -\omega^a{}_b\wedge\sigma^b$ by the definition of $(\omega^a{}_b)$, so

$$\begin{aligned}
d\beta^a &=^5 -\frac{c}{2\rho}\frac{1}{\rho+c}d\rho\wedge\beta^a - \omega^a{}_b\wedge\beta^b \\
&= \frac{c}{2\sqrt{\rho+c}\sqrt{\rho+2c}}\,(\alpha^0+\bar{\alpha}^0)\wedge\beta^a - \omega^a{}_b\wedge\beta^b.
\end{aligned}$$

For $\alpha^0 = -\frac{1}{2\rho}\sqrt{\frac{\rho+2c}{\rho+c}}\,\left(d\rho - i\frac{\rho+c}{\rho+2c}(d\tilde{\phi} + \zeta^I d\tilde{\zeta}_I - \tilde{\zeta}_I d\zeta^I + cd^c\mathcal{K})\right)$, we find

$$\begin{aligned}
d\alpha^0 &=^6 i\left(-\frac{1}{\rho} + \frac{1}{2}\frac{c}{(\rho+c)(\rho+2c)}\right)d\rho\wedge\operatorname{Im}\alpha^0 + i\frac{1}{2\rho}\sqrt{\frac{\rho+c}{\rho+2c}}\,(2d\zeta^I\wedge d\tilde{\zeta}_I + cdd^c\mathcal{K}) \\
&= \frac{2i}{\sqrt{\rho+c}\sqrt{\rho+2c}}\left(\rho+c-\frac{1}{2}\frac{c\rho}{\rho+2c}\right)\underbrace{\operatorname{Re}\alpha^0\wedge\operatorname{Im}\alpha^0}_{\frac{i}{2}\alpha^0\wedge\bar{\alpha}^0} \\
&\quad -\frac{1}{\rho}\sqrt{\frac{\rho+c}{\rho+2c}}\left(\rho\,\alpha^b\wedge\bar{\alpha}^b - \frac{\rho^2}{\rho+2c}\beta^0\wedge\bar{\beta}^0\right) - \frac{c}{\sqrt{\rho+c}\sqrt{\rho+2c}}\beta^b\wedge\bar{\beta}^b,
\end{aligned}$$

[4] $\frac{\partial}{\partial\rho}\left(\frac{\sqrt{\rho+2c}}{\rho}\right) = -\frac{1}{2\rho}\big(1+\frac{2c}{\rho+2c}\big)\frac{\sqrt{\rho+2c}}{\rho}$

[5] $\frac{\partial}{\partial\rho}\sqrt{\frac{\rho+c}{\rho}} = -\frac{c}{2\rho}\frac{1}{\rho+c}\sqrt{\frac{\rho+c}{\rho}}$

since

$$\begin{aligned}
\alpha^b \wedge \bar{\alpha}^b &= \frac{1}{\rho} e^{-\mathcal{K}} P^b_K \bar{P}^b_I N^{IJ} N^{KL} A_J \wedge \bar{A}_L \\
&\overset{(7.85)}{=} -\frac{1}{\rho} N^{JL} A_J \wedge \bar{A}_L + \frac{1}{\rho} e^{\mathcal{K}} X^J A_J \wedge \bar{X}^L \bar{A}_L \\
&\overset{(7.95)}{=} -\frac{i}{\rho} d\zeta^L \wedge \tilde{\zeta}_L + \frac{\rho}{\rho + 2c} \beta^0 \wedge \bar{\beta}^0
\end{aligned}$$

and $dd^c\mathcal{K} = 4\omega_{\bar{M}} = 2i\sigma^a \wedge \bar{\sigma}^a = \frac{2i\rho}{\rho+c} \beta^a \wedge \bar{\beta}^a$.
Finally, $\alpha^a = \frac{i}{\sqrt{\rho}} e^{-\mathcal{K}/2} \bar{P}^a_I N^{IJ} A_J$ fulfills

$$\begin{aligned}
d\alpha^a &= (-\frac{1}{2\rho} d\rho - \frac{1}{2} d\mathcal{K}) \wedge \alpha^a + \frac{i}{\sqrt{\rho}} e^{-\mathcal{K}/2} d(\bar{P}^a_I N^{IJ}) \wedge A_J \\
&\quad + \frac{i}{\sqrt{\rho}} e^{-\mathcal{K}/2} \bar{P}^a_I N^{IJ} dF_{JL}(X) \wedge d\zeta^L \\
&\overset{(7.97)}{=} \frac{1}{2}\sqrt{\frac{\rho + c}{\rho + 2c}} (\alpha^0 + \bar{\alpha}^0) \wedge \alpha^a - (\frac{i}{2} d^c\mathcal{K} + \bar{\partial}\mathcal{K}) \wedge \alpha^a \\
&\quad + \sqrt{\frac{\rho}{\rho + 2c}} d(\bar{P}^a_I N^{IJ}) N_{JL} \bar{X}^L \wedge \beta^0 - e^{-\mathcal{K}} d(\bar{P}^a_I N^{IJ}) P^b_J \wedge \alpha^b \\
&\quad + \frac{i}{\sqrt{\rho}} e^{-\mathcal{K}/2} \bar{P}^a_I N^{IJ} dF_{JL}(X) \wedge d\zeta^L \\
&\underset{(7.98)}{\overset{(7.99)}{=}} \frac{1}{2}\sqrt{\frac{\rho + c}{\rho + 2c}} (\alpha^0 + \bar{\alpha}^0) \wedge \alpha^a - (\frac{i}{2} d^c\mathcal{K} + \bar{\partial}\mathcal{K}) \wedge \alpha^a \\
&\quad \cancel{+ i\sqrt{\frac{\rho}{\rho + 2c}} \bar{P}^a_I N^{IJ} dF_{JL}(X) \bar{X}^L \wedge \beta^0} - \frac{\rho}{\sqrt{\rho + c}\sqrt{\rho + 2c}} \bar{\beta}^a \wedge \beta^0 \\
&\quad - e^{-\mathcal{K}} d(\bar{P}^a_I N^{IJ}) P^b_J \wedge \alpha^b \\
&\quad \cancel{-2i\sqrt{\frac{\rho}{\rho + 2c}} \bar{P}^a_I N^{IJ} dF_{JL}(X) \wedge \mathrm{Re}(\bar{X}^L \beta^0)} \\
&\quad + 2ie^{-\mathcal{K}} \bar{P}^a_I N^{IJ} dF_{JL}(X) N^{LM} \wedge \mathrm{Re}(P^b_M \alpha^b) \\
&\underset{(7.100)}{\overset{(7.81),\,(7.79)}{=}} \frac{1}{2}(\sqrt{\frac{\rho + c}{\rho + 2c}} (\alpha^0 + \bar{\alpha}^0) - id^c\mathcal{K}) \wedge \alpha^a + \frac{\rho}{\sqrt{\rho + c}\sqrt{\rho + 2c}} \beta^0 \wedge \bar{\beta}^a \\
&\quad - \overline{\omega^a_{\ b}} \wedge \alpha^b + ie^{-2\mathcal{K}} \sqrt{\frac{\rho}{\rho + c}} \widetilde{F}^{abc}(X) \bar{\alpha}^b \wedge \beta^c.
\end{aligned}$$

Finally, note that

$$\tilde{F}^{abc}(X) = -e^{3\mathcal{K}} \tilde{h}_{abc}(\mathrm{Im}\, X). \tag{7.102}$$

□

[6] $\frac{\partial}{\partial \rho}\left(-i\frac{\sigma_1}{2|\rho|}\sqrt{\frac{\rho+c}{|\rho+2c|}}\right) = -i\frac{\sigma_1}{2|\rho|}\sqrt{\frac{\rho+c}{|\rho+2c|}}\left(-\frac{1}{\rho} + \frac{1}{2}\frac{c}{(\rho+c)(\rho+2c)}\right)$

Recall that $\bar{\theta}_1 = -\frac{1}{2}\sqrt{\frac{\rho+2c}{\rho+c}}\,\mathrm{Im}\,\alpha^0 - \frac{1}{4}d^c\mathcal{K}$ and $\bar{\theta}_2 + i\bar{\theta}_3 = \sqrt{\frac{\rho+c}{\rho+2c}}\,\beta^0$ (see Remark 5.5.3). The H-part of the Levi-Civita connection is given by (see Eq. (7.48))

$$p = \begin{pmatrix} p^1{}_1 & p^1{}_2 \\ p^2{}_1 & p^2{}_2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4}(-id^c\mathcal{K} + \sqrt{\frac{\rho+2c}{\rho+c}}\,(\bar{\alpha}^0 - \alpha^0)) & -\sqrt{\frac{\rho+c}{\rho+2c}}\,\beta^0 \\ \sqrt{\frac{\rho+c}{\rho+2c}}\,\bar{\beta}^0 & \frac{1}{4}(-id^c\mathcal{K} + \sqrt{\frac{\rho+2c}{\rho+c}}\,(\bar{\alpha}^0 - \alpha^0)) \end{pmatrix}.$$

Corollary 7.3.4 *The E-part of the Levi-Civita connection with respect to the frame (E_Γ) for the one-loop deformed q-map is given by* $(\Theta^\Gamma{}_\Delta) = \begin{pmatrix} q^A{}_B & t^A{}_{\tilde{B}} \\ -\bar{t}^{\tilde{A}}{}_B & \bar{q}^{\tilde{A}}{}_{\tilde{B}} \end{pmatrix}$ *with* $(q^A{}_B) = q$,

$$q = \begin{pmatrix} \frac{i}{4}d^c\mathcal{K} + \frac{1}{4}\frac{1}{\sqrt{\rho+c}\sqrt{\rho+2c}}\left(3\rho + \frac{4c^2}{\rho+2c}\right)(\bar{\alpha}^0 - \alpha^0) & -\sqrt{\frac{\rho+c}{\rho+2c}}\,\alpha^b \\ \sqrt{\frac{\rho+c}{\rho+2c}}\,\bar{\alpha}^a & \omega^a{}_b + \frac{1}{4}(-id^c\mathcal{K} + \frac{\rho}{\sqrt{\rho+c}\sqrt{\rho+2c}}\,(\bar{\alpha}^0 - \alpha^0))\delta^a{}_b \end{pmatrix}$$

and

$$t = (t^A{}_{\tilde{B}}) = \begin{pmatrix} \frac{2c}{\rho+2c}\sqrt{\frac{\rho+c}{\rho+2c}}\,\beta^0 & \frac{c}{\sqrt{\rho+c}\sqrt{\rho+2c}}\,\beta^b \\ \frac{c}{\sqrt{\rho+c}\sqrt{\rho+2c}}\,\beta^a & ie^{\mathcal{K}}\sqrt{\frac{\rho}{\rho+c}}\,\tilde{h}_{abc}\,\alpha^c \end{pmatrix}.$$

***Proof*:** q is anti-Hermitian and t is symmetric. A straightforward calculation shows that the equations given in Proposition 7.3.3 agree with equations (7.50) and (7.51), when q and t are given as above. □

From now on, we restrict ourselves to the undeformed q-map, i.e. we set $c = 0$.

Proposition 7.3.5 *The E-part of the curvature two-form with respect to the frame (E_Γ) is given for any quaternionic Kähler manifold in the image of the* *q-map by* $(R_E{}^{\Gamma}{}_{\Delta}) = \begin{pmatrix} r^A{}_B & s^A{}_{\tilde{B}} \\ -\bar{s}^{\tilde{A}}{}_B & \bar{r}^{\tilde{A}}{}_{\tilde{B}} \end{pmatrix}$ *with* $(r^A{}_B) = r$,

$$r = \begin{pmatrix} \begin{array}{l} \frac{1}{2}\big(\alpha^0 \wedge \bar{\alpha}^0 - \beta^0 \wedge \bar{\beta}^0 \\ \quad + \sum_{C=0}^{n} \alpha^C \wedge \bar{\alpha}^C - \beta^C \wedge \bar{\beta}^C\big) \end{array} & \alpha^b \wedge \bar{\alpha}^0 + \bar{\beta}^b \wedge \beta^0 + ie^{\mathcal{K}} \tilde{h}_{bcd} \bar{\alpha}^c \wedge \beta^d \\ \begin{array}{l} \alpha^0 \wedge \bar{\alpha}^a + \bar{\beta}^0 \wedge \beta^a \\ \quad + ie^{\mathcal{K}} \tilde{h}_{acd} \alpha^c \wedge \bar{\beta}^d \end{array} & \begin{array}{l} \frac{1}{2}\delta^a{}_b \sum_{C=0}^{n} (\alpha^C \wedge \bar{\alpha}^C - \beta^C \wedge \bar{\beta}^C) \\ \quad - (\beta^a \wedge \bar{\beta}^b + \bar{\alpha}^a \wedge \alpha^b) \\ \quad - e^{2\mathcal{K}} \tilde{h}_{adc} \tilde{h}_{ceb} (\alpha^d \wedge \bar{\alpha}^e + \bar{\beta}^d \wedge \beta^e) \end{array} \end{pmatrix}$$

and $(s^A{}_{\tilde{B}}) = s$,

$$s = \begin{pmatrix} 0 & 0 \\ 0 & ie^{\mathcal{K}} \tilde{h}_{abc} (\beta^0 \wedge \bar{\beta}^c + \bar{\alpha}^0 \wedge \alpha^c) + e^{2\mathcal{K}} \tilde{h}_{abf} \tilde{h}_{fde} \bar{\alpha}^d \wedge \beta^e - 2S_{abcd} \alpha^c \wedge \bar{\beta}^d \end{pmatrix},$$

where

$$\begin{aligned} S_{abcd} := -\frac{1}{2} e^{2\mathcal{K}} \Big(& (\tilde{h}_{bcf} \tilde{h}_{fad} - 4\tilde{h}_{bc} \tilde{h}_{ad}) + (\tilde{h}_{acf} \tilde{h}_{fbd} - 4\tilde{h}_{ac} \tilde{h}_{bd}) + (\tilde{h}_{abf} \tilde{h}_{fcd} - 4\tilde{h}_{ab} \tilde{h}_{cd}) \\ & + 4\tilde{h}_a \tilde{h}_{bcd} + 4\tilde{h}_b \tilde{h}_{cda} + 4\tilde{h}_c \tilde{h}_{dab} + 4\tilde{h}_d \tilde{h}_{abc} \Big). \end{aligned}$$

***Proof*:** First, we calculate dq:

$$\begin{aligned} dq^0{}_0 &= \frac{i}{4} dd^c \mathcal{K} + \frac{3}{4}(d\bar{\alpha}^0 - d\alpha^0) \overset{(d\bar{\alpha}^0 = -d\alpha^0)}{=} -\frac{1}{2}\partial\bar{\partial}\mathcal{K} - \frac{3}{2} d\alpha^0 \\ &= -\frac{1}{2}\beta^c \wedge \bar{\beta}^c + \frac{3}{2}(\alpha^0 \wedge \bar{\alpha}^0 - \beta^0 \wedge \bar{\beta}^0 + \alpha^c \wedge \bar{\alpha}^c), \end{aligned}$$

$$\begin{aligned} dq^0{}_b &= -\overline{dq^b{}_0} = -d\alpha^b \\ &= -\frac{1}{2}(\alpha^0 + \bar{\alpha}^0 - id^c\mathcal{K}) \wedge \alpha^b - \beta^0 \wedge \bar{\beta}^b + \bar{\omega}^b{}_c \wedge \alpha^c + ie^{\mathcal{K}} \tilde{h}_{abc} \bar{\alpha}^a \wedge \beta^c, \end{aligned}$$

$$\begin{aligned} dq^a{}_b &= d\omega^a{}_b + \frac{1}{2}\delta^a{}_b(\partial\bar{\partial}\mathcal{K} - d\alpha^0) \\ &= d\omega^a{}_b + \frac{1}{2}\delta^a{}_b(\beta^c \wedge \bar{\beta}^c + \alpha^0 \wedge \bar{\alpha}^0 - \beta^0 \wedge \bar{\beta}^0 + \alpha^c \wedge \bar{\alpha}^c). \end{aligned}$$

Then with

$$(q^A{}_C \wedge q^C{}_B) = \begin{pmatrix} -\alpha^c \wedge \bar\alpha^c & -\frac{1}{2}(id^c\mathcal{K} + \bar\alpha^0 - \alpha^0) \wedge \alpha^b - \bar\omega^b{}_c \wedge \alpha^c \\ -\frac{1}{2}(id^c\mathcal{K} + \bar\alpha^0 - \alpha^0) \wedge \bar\alpha^a + \omega^a{}_c \wedge \bar\alpha^c & \omega^a{}_c \wedge \omega^c{}_b - \bar\alpha^a \wedge \alpha^b \end{pmatrix}$$

and

$$(t^A{}_C \wedge \bar t^C{}_B) = \begin{pmatrix} 0 & 0 \\ 0 & e^{2\mathcal{K}} \tilde h_{adc} \tilde h_{ceb} \alpha^d \wedge \bar\alpha^e \end{pmatrix},$$

we obtain

$$\begin{aligned} r &\overset{(7.56)}{=} dq + q \wedge q - t \wedge \bar t \\ &= \begin{pmatrix} \begin{matrix} \frac{1}{2}(\alpha^c \wedge \bar\alpha^c - \beta^c \wedge \bar\beta^c) \\ + \frac{3}{2}(\alpha^0 \wedge \bar\alpha^0 - \beta^0 \wedge \bar\beta^0) \end{matrix} & \alpha^b \wedge \bar\alpha^0 + \bar\beta^b \wedge \beta^0 + ie^{\mathcal{K}} \tilde h_{bcd} \bar\alpha^c \wedge \beta^d \\ \\ \begin{matrix} \alpha^0 \wedge \bar\alpha^a + \bar\beta^0 \wedge \beta^a \\ + ie^{\mathcal{K}} \tilde h_{acd} \alpha^c \wedge \bar\beta^d \end{matrix} & \begin{matrix} d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b - \bar\alpha^a \wedge \alpha^b - e^{2\mathcal{K}} \tilde h_{adc} \tilde h_{ceb} \alpha^d \wedge \bar\alpha^e \\ + \frac{1}{2}\delta^a{}_b(\beta^c \wedge \bar\beta^c + \alpha^c \wedge \bar\alpha^c + \alpha^0 \wedge \bar\alpha^0 - \beta^0 \wedge \bar\beta^0) \end{matrix} \end{pmatrix}. \end{aligned}$$

This can be brought into the form stated above using (see Eq. (7.72))

$$d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = -\delta^a{}_b \beta^c \wedge \bar\beta^c - \beta^a \wedge \bar\beta^b + e^{2\mathcal{K}} \tilde h_{adc} \tilde h_{ceb} \beta^e \wedge \bar\beta^d.$$

Since $e^a{}_\mu$ is real, we have $e^a{}_\mu de^\mu{}_b = \bar e^a{}_\mu d\bar e^\mu{}_b \overset{(7.70)}{=} \bar\omega^a{}_b + \bar m^a{}_b$, where $m^a{}_b := e^a{}_\mu \partial e^\mu{}_b + \bar e^b{}_\mu \partial \bar e^\mu{}_a$. Using $d(\tilde h_{abc}) = \tilde h_{dbc} e^d{}_\mu de^\mu{}_a + \tilde h_{adc} e^d{}_\mu de^\mu{}_b + \tilde h_{abd} e^d{}_\mu de^\mu{}_c$, we calculate

$$\begin{aligned} dt^a{}_d = ie^{\mathcal{K}} \tilde h_{abc}(x)\big(d\mathcal{K} \wedge \alpha^c + \frac{1}{2}(\alpha^0 + \bar\alpha^0 - id^c\mathcal{K}) \wedge \alpha^c + \beta^0 \wedge \bar\beta^c \\ - \bar\omega^c{}_d \wedge \alpha^d - ie^{\mathcal{K}} \tilde h_{cde}(x) \bar\alpha^d \wedge \beta^e\big) \\ + ie^{\mathcal{K}}\big(\tilde h_{dbc}(\bar\omega^d{}_a + \bar m^d{}_a) + \tilde h_{adc}(\bar\omega^d{}_b + \bar m^d{}_b) + \tilde h_{abd}(\bar\omega^d{}_c + \bar m^d{}_c)\big) \wedge \alpha^c. \end{aligned}$$

With

$$q^A{}_C \wedge t^C{}_B \overset{\omega^a{}_c = -\bar\omega^c{}_a}{=} \begin{pmatrix} 0 & 0 \\ 0 & -ie^{\mathcal{K}} \tilde h_{cbe} \bar\omega^c{}_a \wedge \alpha^e + \frac{i}{4} e^{\mathcal{K}} \tilde h_{abe}(-id^c\mathcal{K} + \bar\alpha^0 - \alpha^0) \wedge \alpha^e \end{pmatrix}$$

and

$$t^A{}_C \wedge \bar{q}^C{}_B = \begin{pmatrix} 0 & 0 \\ 0 & ie^{\mathcal{K}}\tilde{h}_{ace}\alpha^e \wedge \bar{\omega}^c{}_b + \frac{i}{4}e^{\mathcal{K}}\tilde{h}_{abe}(-id^c\mathcal{K} + \bar{\alpha}^0 - \alpha^0) \wedge \alpha^e \end{pmatrix},$$

we obtain

$$s \overset{(7.57)}{=} dt + q \wedge t + t \wedge \bar{q} = \begin{pmatrix} 0 & 0 \\ 0 & s^a{}_b \end{pmatrix},$$

where

$$\begin{aligned} s^a{}_b &= ie^{\mathcal{K}}\tilde{h}_{abc}\left((d\mathcal{K} + \bar{\alpha}^0 - id^c\mathcal{K}) \wedge \alpha^c + \beta^0 \wedge \bar{\beta}^c - ie^{\mathcal{K}}\tilde{h}_{cde}\bar{\alpha}^d \wedge \beta^e\right) \\ &\qquad + ie^{\mathcal{K}}(\tilde{h}_{dbc}\bar{m}^d{}_a + \tilde{h}_{adc}\bar{m}^d{}_b + \tilde{h}_{abd}\bar{m}^d{}_c) \wedge \alpha^c \\ &= 8e^{2\mathcal{K}}\tilde{h}_{abc}\tilde{h}_d\bar{\beta}^d \wedge \alpha^c + ie^{\mathcal{K}}\tilde{h}_{abc}(\beta^0 \wedge \bar{\beta}^c + \bar{\alpha}^0 \wedge \alpha^c) + e^{2\mathcal{K}}\tilde{h}_{abc}\tilde{h}_{cde}\bar{\alpha}^d \wedge \beta^e \\ &\qquad + ie^{\mathcal{K}}(\tilde{h}_{dbc}\bar{m}^d{}_a + \tilde{h}_{adc}\bar{m}^d{}_b + \tilde{h}_{abd}\bar{m}^d{}_c) \wedge \alpha^c. \end{aligned}$$

In the last equality, we used

$$d\mathcal{K} - id^c\mathcal{K} = 2\bar{\partial}\mathcal{K} = -\frac{i}{h(x)}h_\mu(x)e^\mu{}_d\bar{\beta}^d = -8ie^{\mathcal{K}}\tilde{h}_d\bar{\beta}^d.$$

Now, using

$$m^a{}_b = -e^a{}_\rho e^\mu{}_b e^\sigma{}_c \Gamma^\rho{}_{\mu\sigma}\beta^c$$

and (see (7.64) and (6.4))

$$\Gamma^\rho_{\sigma\mu} = -\frac{i}{2h}\left(hh^{\rho\kappa}h_{\kappa\mu\sigma} - h_\sigma\delta^\rho_\mu - h_\mu\delta^\rho_\sigma + \frac{1}{2}x^\rho h_{\mu\sigma}\right),$$

$$e^\nu{}_a e^\lambda{}_a = K^{\bar{\nu}\lambda} = -4h(x)h^{\nu\lambda}(x) + 2x^\nu x^\lambda,$$

we find

$$\tilde{h}_{dbc}\bar{m}^d{}_a = ie^{\mathcal{K}}(\tilde{h}_{bcf}\tilde{h}_{fda} - 4\tilde{h}_{bc}\tilde{h}_{da} + 4\tilde{h}_a\tilde{h}_{bcd} + 4\tilde{h}_d\tilde{h}_{abc})\bar{\beta}^d.$$

Hence,

$$\begin{aligned} s^a{}_b = ie^{\mathcal{K}}\tilde{h}_{abc}(\beta^0 \wedge \bar{\beta}^c + \bar{\alpha}^0 \wedge \alpha^c) + e^{2\mathcal{K}}\tilde{h}_{abf}\tilde{h}_{fde}\bar{\alpha}^d \wedge \beta^e \\ - e^{2\mathcal{K}}\Big((\tilde{h}_{bcf}\tilde{h}_{fad} - 4\tilde{h}_{bc}\tilde{h}_{ad}) + (\tilde{h}_{acf}\tilde{h}_{fbd} - 4\tilde{h}_{ac}\tilde{h}_{bd}) \\ + (\tilde{h}_{abf}\tilde{h}_{fcd} - 4\tilde{h}_{ab}\tilde{h}_{cd})\Big)\bar{\beta}^d \wedge \alpha^c \\ - 4e^{2\mathcal{K}}(\tilde{h}_a\tilde{h}_{bcd} + \tilde{h}_b\tilde{h}_{cda} + \tilde{h}_c\tilde{h}_{dab} + \tilde{h}_d\tilde{h}_{abc})\bar{\beta}^d \wedge \alpha^c. \end{aligned}$$

□

Remark 7.3.6 Note that the vanishing of the symmetric quartic tensor field

$$\begin{aligned}
&S_{abcd}\,\sigma^a \otimes \sigma^b \otimes \sigma^c \otimes \sigma^d \\
&= -\frac{1}{2}\frac{1}{4^3 h^2}\begin{pmatrix} 3h_{\tau(\mu\nu}\mathcal{K}^{\tau\tau'}h_{\sigma\rho)\tau'}\dots \\ \dots - 12h_{(\mu\nu}h_{\sigma\rho)} + 16h_{(\mu}h_{\nu\sigma\rho)} \end{pmatrix} dX^\mu \otimes dX^\nu \otimes dX^\sigma \otimes dX^\rho \\
&= -\frac{1}{2}\frac{1}{4^3 h^2}\begin{pmatrix} -12h_{\tau(\mu\nu}h^{\tau\tau'}h_{\sigma\rho)\tau'}\dots \\ \dots - 6h_{(\mu\nu}h_{\sigma\rho)} + 16h_{(\mu}h_{\nu\sigma\rho)} \end{pmatrix} dX^\mu \otimes dX^\nu \otimes dX^\sigma \otimes dX^\rho
\end{aligned}$$

on the projective special Kähler manifold $(\bar{M}, g_{\bar{M}}, J_{\bar{M}})$ is a necessary and sufficient condition for $(\bar{M}, g_{\bar{M}})$ to be symmetric [CV].

In the following theorem, we use the notation from Section 7.1.

Theorem 7.3.7 *The $Sp(E)$-curvature of manifolds in the image of the q-map can be written as*

$$\tilde{R}^{\Gamma}{}_{\Gamma'} = -\frac{1}{2}\epsilon_{\alpha\beta}C_{\Gamma'\Gamma''}f^{\alpha\Gamma} \wedge f^{\beta\Gamma''} + C^{\Gamma\Gamma^0}\Omega_{\Gamma^0\Gamma'\Gamma''\Gamma'''}\epsilon_{\alpha\beta}f^{\alpha\Gamma''} \wedge f^{\beta\Gamma'''}, \tag{7.103}$$

where the non-vanishing components of the symmetric quartic tensor field Ω are given by

$$\Omega_{00\tilde{0}\tilde{0}} = \frac{1}{2}, \quad \Omega_{0b\tilde{0}\tilde{d}} = \frac{1}{4}\delta_{bd}, \quad \Omega_{ab\tilde{c}\tilde{d}} = \frac{1}{4}(\delta_{ac}\delta_{bd} + \delta_{ad}\delta_{bc}) - \frac{1}{2}e^{2\mathcal{K}}\tilde{h}_{abf}\tilde{h}_{fcd},$$

$$\Omega_{\tilde{0}bcd} = \Omega_{0\tilde{b}\tilde{c}\tilde{d}} = -\frac{i}{2}e^{\mathcal{K}}\tilde{h}_{bcd}, \quad \Omega_{abcd} = \Omega_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = S_{abcd}$$

(and symmetrization thereof).

***Proof*:** First, note that

$$(\epsilon_{\alpha\beta}C_{\Gamma'\Gamma''}f^{\Gamma\alpha} \wedge f^{\Gamma''\beta})^{\Gamma=A,\tilde{A}}_{\Gamma'=B,\tilde{B}} = \begin{pmatrix} \beta^A \wedge \bar{\beta}^B + \bar{\alpha}^A \wedge \alpha^B & \beta^A \wedge \bar{\alpha}^{\tilde{B}} - \bar{\alpha}^A \wedge \beta^{\tilde{B}} \\ \alpha^{\tilde{A}} \wedge \bar{\beta}^B - \bar{\beta}^{\tilde{A}} \wedge \alpha^B & \alpha^{\tilde{A}} \wedge \bar{\alpha}^{\tilde{B}} + \bar{\beta}^{\tilde{A}} \wedge \beta^{\tilde{B}} \end{pmatrix}.$$

Define

$$U^{\Gamma\Gamma'} := \epsilon_{\alpha\beta}f^{\Gamma\alpha} \wedge f^{\Gamma'\beta} = \begin{pmatrix} -\beta^A \wedge \bar{\alpha}^B + \bar{\alpha}^A \wedge \beta^B & \beta^A \wedge \bar{\beta}^{\tilde{B}} + \bar{\alpha}^A \wedge \alpha^{\tilde{B}} \\ -\alpha^{\tilde{A}} \wedge \bar{\alpha}^B - \bar{\beta}^{\tilde{A}} \wedge \beta^B & \alpha^{\tilde{A}} \wedge \bar{\beta}^{\tilde{B}} - \bar{\beta}^{\tilde{A}} \wedge \alpha^{\tilde{B}} \end{pmatrix}$$

and

$$\mathring{R}_{\Gamma^0\Gamma'} := C_{\Gamma^0\Gamma}(\tilde{R}^{\Gamma}{}_{\Gamma'} + \frac{1}{2}\epsilon_{\alpha\beta}C_{\Gamma'\Gamma''}f^{\Gamma\alpha} \wedge f^{\Gamma''\beta}).$$

Then

$$\mathring{R}_{\tilde{A}B} = -\big(r^A{}_B + \frac{1}{2}(\beta^A \wedge \bar{\beta}^B + \bar{\alpha}^A \wedge \alpha^B)\big)$$

$$= \begin{pmatrix} \begin{array}{l} \frac{1}{2}(\beta^0 \wedge \bar{\beta}^0 + \bar{\alpha}^0 \wedge \alpha^0) \\ \quad + \frac{1}{2}(\beta^C \wedge \bar{\beta}^C + \bar{\alpha}^C \wedge \alpha^C) \end{array} & \frac{1}{2}(\beta^0 \wedge \bar{\beta}^b + \bar{\alpha}^0 \wedge \alpha^b) + ie^{\mathcal{K}}\tilde{h}_{bde}\beta^e \wedge \bar{\alpha}^d \\ \begin{array}{c} \frac{1}{2}(\beta^a \wedge \bar{\beta}^0 + \bar{\alpha}^a \wedge \alpha^0) \\ -ie^{\mathcal{K}}\tilde{h}_{ade}\alpha^d \wedge \bar{\beta}^e \end{array} & \begin{array}{l} \frac{1}{2}\delta_{ab}(\beta^C \wedge \bar{\beta}^C + \bar{\alpha}^C \wedge \alpha^C) \\ \quad + \frac{1}{2}(\beta^a \wedge \bar{\beta}^b + \bar{\alpha}^a \wedge \alpha^b) \\ \quad + e^{2\mathcal{K}}\tilde{h}_{adc}\tilde{h}_{ceb}(\alpha^d \wedge \bar{\alpha}^e + \bar{\beta}^d \wedge \beta^e) \end{array} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{4}(U^{0\tilde{0}} + U^{\tilde{0}0} + U^{C\tilde{C}} + U^{\tilde{C}C}) & \frac{1}{4}(U^{0\tilde{b}} + U^{\tilde{b}0}) - \frac{i}{2}e^{\mathcal{K}}\tilde{h}_{bde}U^{de} \\ \frac{1}{4}(U^{a\tilde{0}} + U^{\tilde{0}a}) - \frac{i}{2}e^{\mathcal{K}}\tilde{h}_{ade}U^{\tilde{d}\tilde{e}} & \begin{array}{l} \frac{1}{4}\delta_{ab}(U^{C\tilde{C}} + U^{\tilde{C}C}) + \frac{1}{4}(U^{a\tilde{b}} + U^{\tilde{b}a}) \\ \quad - \frac{e^{2\mathcal{K}}}{2}\tilde{h}_{adf}\tilde{h}_{feb}(U^{d\tilde{e}} + U^{\tilde{d}e}) \end{array} \end{pmatrix},$$

$$\mathring{R}_{\tilde{A}\tilde{B}} = -\big(s^A{}_B + \frac{1}{2}(\beta^A \wedge \bar{\alpha}^B - \bar{\alpha}^A \wedge \beta^B)\big)$$

$$= \begin{pmatrix} -\beta^0 \wedge \bar{\alpha}^0 & -\frac{1}{2}(\beta^0 \wedge \bar{\alpha}^b - \bar{\alpha}^0 \wedge \beta^b) \\ -\frac{1}{2}(\beta^a \wedge \bar{\alpha}^0 - \bar{\alpha}^a \wedge \beta^0) & \begin{array}{l} -\frac{1}{2}(\beta^a \wedge \bar{\alpha}^b - \bar{\alpha}^a \wedge \beta^b) + e^{2\mathcal{K}}\tilde{h}_{abf}\tilde{h}_{fde}\beta^d \wedge \bar{\alpha}^e \\ - ie^{\mathcal{K}}\tilde{h}_{abc}(\beta^0 \wedge \bar{\beta}^c + \bar{\alpha}^0 \wedge \alpha^c) + 2S_{abcd}\alpha^c \wedge \bar{\beta}^d \end{array} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}U^{00} & \frac{1}{4}(U^{0b} + U^{b0}) \\ \frac{1}{4}(U^{a0} + U^{0a}) & \begin{array}{l} \frac{1}{4}(U^{ab} + U^{ba}) - \frac{e^{2\mathcal{K}}}{2}\tilde{h}_{abf}\tilde{h}_{fde}U^{de} \\ \quad - \frac{i}{2}\tilde{h}_{abc}(U^{0\tilde{c}} + U^{\tilde{c}0}) + S_{abcd}U^{\tilde{c}\tilde{d}} \end{array} \end{pmatrix},$$

and $\mathring{R}_{AB} = \overline{\mathring{R}_{\tilde{A}\tilde{B}}}$, $\mathring{R}_{A\tilde{B}} = -\overline{\mathring{R}_{\tilde{A}B}}$.

Eq. (7.58) is equivalent to $\mathring{R}_{\Gamma\Gamma'} = \Omega_{\Gamma\Gamma'\Gamma''\Gamma'''}U^{\Gamma''\Gamma'''}$. Now $\mathring{R}_{\tilde{0}0} = \Omega_{\tilde{0}0\Gamma''\Gamma'''}U^{\Gamma''\Gamma'''}$

implies $\Omega_{\tilde{0}0CD} = \Omega_{\tilde{0}0\tilde{C}\tilde{D}} = 0$, $\Omega_{\tilde{0}00\tilde{d}} = \Omega_{\tilde{0}0c\tilde{0}} = 0$ and

$$\Omega_{\tilde{0}00\tilde{0}} = \frac{1}{2}, \quad \Omega_{\tilde{0}0c\tilde{d}} = \frac{1}{4}\delta_{cd}.$$

$\mathring{R}_{\tilde{a}0} = \Omega_{\tilde{a}0\Gamma''\Gamma'''}U^{\Gamma''\Gamma'''}$ implies $\Omega_{\tilde{a}0CD} = 0$, $\Omega_{\tilde{a}00\tilde{d}} = 0$, $\Omega_{\tilde{a}0c\tilde{d}} = 0$ and

$$\Omega_{\tilde{a}0\tilde{c}\tilde{d}} = -\frac{i}{2}e^{\mathcal{K}}\tilde{h}_{acd}.$$

$\mathring{R}_{\tilde{a}b} = \Omega_{\tilde{a}b\Gamma''\Gamma'''}U^{\Gamma''\Gamma'''}$ implies $\Omega_{\tilde{a}b\tilde{C}\tilde{D}} = 0$, $\Omega_{\tilde{a}bcd} = 0$, $\Omega_{\tilde{a}b\tilde{0}d} = 0$ and

$$\Omega_{\tilde{a}b\tilde{c}d} = \frac{1}{4}(\delta_{ab}\delta_{cd} + \delta_{bc}\delta_{ad}) - \frac{e^{2\mathcal{K}}}{2}\tilde{h}_{acf}\tilde{h}_{fdb}.$$

$\mathring{R}_{\tilde{0}b} = \Omega_{\tilde{0}b\Gamma''\Gamma'''}U^{\Gamma''\Gamma'''}$ implies $\Omega_{\tilde{0}b\tilde{0}\tilde{0}} = 0$, $\Omega_{\tilde{0}b\tilde{0}d} = 0$ and

$$\Omega_{\tilde{0}bcd} = -\frac{i}{2}e^{\mathcal{K}}\tilde{h}_{bcd}.$$

It remains to determine the components of the form Ω_{ABCD} and $\Omega_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}}$:
$\mathring{R}_{\tilde{A}\tilde{B}} = \Omega_{\tilde{A}\tilde{B}\Gamma''\Gamma'''}U^{\Gamma''\Gamma'''}$ implies $\Omega_{\tilde{0}\tilde{B}\tilde{C}\tilde{D}} = 0$ and

$$\Omega_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} = S_{abcd}.$$

Using $U^{AB} = \overline{U^{\tilde{A}\tilde{B}}}$ and $U^{A\tilde{B}} = -\overline{U^{\tilde{A}B}}$, we find that $\mathring{R}_{AB} = \overline{\mathring{R}_{\tilde{A}\tilde{B}}} = \Omega_{AB\Gamma''\Gamma'''}U^{\Gamma''\Gamma'''}$ implies $\Omega_{0BCD} = 0$ and

$$\Omega_{abcd} = S_{abcd}.$$

□

7.4 Example: A series of inhomogeneous complete quaternionic Kähler manifolds

In this section, we show that the members of a certain series of complete quaternionic Kähler manifolds constructed from the q-map are not locally homogeneous. This is done by calculating the pointwise norm of the Riemann tensor and showing that it is a non-constant function on the quaternionic Kähler manifold. We leave out the details of the calculation and just show some intermediate steps and the final result. Note that some simplifications of formulas were done using computer algebra software.

We will again leave out summation symbols and employ Einstein's summation convention.

For $n \in \mathbb{N}$, we consider the following series of projective special real manifolds:

$$\mathcal{H} = \{h = 1,\, x > 0\} \subset \mathbb{R}^n,\; h := x(x^2 - \sum_{i=1}^{n-1} (y_i)^2). \tag{7.104}$$

The projective special real manifold $(\mathcal{H}, g_{\mathcal{H}})$ is a closed subset of $\mathbb{R}^n$ and thus complete according to Theorem 6.2.8 (which was proven in [CNS]). Due to Theorems 6.2.6 and 6.3.3 (which were proven in [CHM]), the corresponding projective special Kähler manifold obtained from the supergravity r-map and the quaternionic Kähler manifold obtained from the q-map are complete as well.

The scalar curvature of the corresponding projective special Kähler manifold $\bar{M}$ in the image of the supergravity r-map can be calculated to be (see Theorem 3 in[7] [CDL] for the general formula)

$$\begin{aligned} scal_{\bar{M}} &= -2n^2 + n - 2h h_{\alpha\beta\gamma} h^{\alpha\alpha'} h^{\beta\beta'} h^{\gamma\gamma'} h_{\alpha'\beta'\gamma'} \\ &= -2n(n+1) + \frac{1}{32h^2} h_{\alpha\beta\gamma} \mathcal{K}^{\alpha\alpha'} \mathcal{K}^{\beta\beta'} \mathcal{K}^{\gamma\gamma'} h_{\alpha'\beta'\gamma'} \\ &= -n \cdot (2n-1) + 3h \cdot \frac{n-2}{h - 4x^3} + \frac{36x^3h^2}{(h-4x^3)^3}. \end{aligned} \tag{7.105}$$

We find the following expression for the squared pointwise norm of the quartic tensor field $B_{\mu\nu\sigma\rho} := h_{\mu\nu\kappa} \mathcal{K}^{\kappa\kappa'} h_{\kappa'\rho\sigma}$ on $\bar{M}$:

$$\begin{aligned} B_{\mu\nu\sigma\rho} \mathcal{K}^{\mu\mu'} \mathcal{K}^{\nu\nu'} \mathcal{K}^{\sigma\sigma'} \mathcal{K}^{\rho\rho'} B_{\mu'\nu'\sigma'\rho'} &= \frac{4096h^4}{(h-4x^3)^6} \cdot \Big(h^6(n-1)(n+3) \\ &- 4h^5(n+3)(5n-7)x^3 + 4h^4(n(41n+98) - 159)x^6 \\ &- 64h^3(n(11n+43) - 75)x^9 + 128h^2(n(13n+73) - 78)x^{12} \\ &- 2048h(n(n+7) - 3)x^{15} + 1024n(8+n)x^{18} \Big). \end{aligned} \tag{7.106}$$

The squared pointwise norm of the Riemann tensor of the projective special Kähler manifold is (see Theorem 3 in [CDL] for the general formula for the

[7]Note that compared to [CDL] we scaled the projective special Kähler metric $g_{\bar{M}}$ by a factor of $\frac{1}{2}$, which leads to a scaling of the scalar curvature $scal_{\bar{M}}$ by a factor of 2.

Riemann tensor)

$$\begin{aligned}
\|R_{\bar{M}}\|^2 &= 16R_{\bar{\mu}\nu\sigma\bar{\rho}}\mathcal{K}^{\mu\mu'}\mathcal{K}^{\nu\nu'}\mathcal{K}^{\sigma\sigma'}\mathcal{K}^{\rho\rho'}R_{\mu'\bar{\nu}'\bar{\sigma}'\rho'} \\
&= -32\, scal_{\bar{M}} - 32n(n+1) + \frac{1}{4^4h^4}B_{\rho\sigma\mu\nu}\mathcal{K}^{\rho\rho'}\mathcal{K}^{\sigma\sigma'}\mathcal{K}^{\mu\mu'}\mathcal{K}^{\nu\nu'}B_{\rho'\sigma'\mu'\nu'} \\
&= \frac{16}{(h-4x^3)^6}\Big(h^6(n(3n-8)+9) - 4h^5(n(17n-46)+57)x^3 \\
&\quad + 4h^4(n(161n-382)+537)x^6 - 64h^3(n(51n-97)+99)x^9 \\
&\quad + 128h^2(n(73n-107)+78)x^{12} - 2048h(n(7n-8)+3)x^{15} \\
&\quad + 1024n(9n-8)x^{18}\Big). \qquad (7.107)
\end{aligned}$$

For the squared pointwise norm of the tensor field

$$\begin{aligned}
S_{\mu\nu\sigma\rho} = -\frac{1}{2}e^{2\mathcal{K}}\Big((\tilde{h}_{bcf}\tilde{h}_{fad} - 4\tilde{h}_{bc}\tilde{h}_{ad}) &+ (\tilde{h}_{acf}\tilde{h}_{fbd} - 4\tilde{h}_{ac}\tilde{h}_{bd}) \\
&+ (\tilde{h}_{abf}\tilde{h}_{fcd} - 4\tilde{h}_{ab}\tilde{h}_{cd}) \\
&+ 4\tilde{h}_a\tilde{h}_{bcd} + 4\tilde{h}_b\tilde{h}_{cda} + 4\tilde{h}_c\tilde{h}_{dab} + 4\tilde{h}_d\tilde{h}_{abc}\Big)
\end{aligned}$$

on $\bar{M}$ (see the definition of S in Proposition 7.3.5), we find:

$$\begin{aligned}
&S_{\mu\nu\sigma\rho}\mathcal{K}^{\mu\mu'}\mathcal{K}^{\nu\nu'}\mathcal{K}^{\sigma\sigma'}\mathcal{K}^{\rho\rho'}S_{\mu'\nu'\sigma'\rho'} \qquad (7.108) \\
&= \frac{3x^6}{(h-4x^3)^6}\Big(h^4(n(n+16)+207) - 16h^3(n-2)(n+9)x^3 \\
&\qquad + 96h^2\left(n^2+n-6\right)x^6 - 256h(n-2)nx^9 + 256(n-2)nx^{12}\Big).
\end{aligned}$$

The squared pointwise norm of the quaternionic Weyl tensor is

$$\begin{aligned}
\frac{1}{64}\|\mathcal{W}\|^2 &= \Omega_{\Gamma\Gamma'\Gamma''\Gamma'''}C^{\Gamma\Delta}C^{\Gamma'\Delta'}C^{\Gamma''\Delta''}C^{\Gamma'''\Delta'''}\Omega_{\Delta\Delta'\Delta''\Delta'''} \\
&= 2\Omega_{ABCD}\Omega_{\tilde{A}\tilde{B}\tilde{C}\tilde{D}} - 8\Omega_{ABC\tilde{D}}\Omega_{\tilde{A}\tilde{B}\tilde{C}D} + 6\Omega_{AB\tilde{C}\tilde{D}}\Omega_{\tilde{A}\tilde{B}CD} \\
&= 2\Omega_{abcd}\Omega_{\tilde{a}\tilde{b}\tilde{c}\tilde{d}} - 8\Omega_{abc\tilde{0}}\Omega_{\tilde{a}\tilde{b}\tilde{c}0} + 6(\Omega_{00\tilde{0}\tilde{0}})^2 + 24\Omega_{0b\tilde{0}\tilde{d}}\Omega_{\tilde{0}\tilde{b}0d} + 6\Omega_{ab\tilde{c}\tilde{d}}\Omega_{\tilde{a}\tilde{b}cd} \\
&= 2S_{abcd}S_{abcd} + 2n(n+1) + scal_{\bar{M}} + \frac{3}{2}(n+1) \\
&\qquad + 6\left(\frac{1}{4^3}\|R_{\bar{M}}\|^2 + \frac{1}{4}scal_{\bar{M}} + \frac{n^2+n}{8}\right)
\end{aligned}$$

$$
\begin{aligned}
&= 2S_{abcd}S_{abcd} + \frac{1}{4}(11n+6)(n+1) + \frac{3}{32}\|R_{\bar{M}}\|^2 + \frac{5}{2}scal_{\bar{M}} \\
&= \frac{3}{2\left(h-4x^3\right)^6}\Big(h^6 n(n+1) - 4h^5(n+1)(5n-2)x^3 + 8h^4(n(21n+37)+112)x^6 \\
&\qquad - 256h^3(n(3n+10)-11)x^9 + 256h^2(n(8n+33)-20)x^{12} \\
&\qquad - 1024h(n(3n+11)+2)x^{15} + 2048(n+1)(n+2)x^{18}\Big) \\
&\quad + \frac{3n}{4}(n+1). \qquad (7.109)
\end{aligned}
$$

By evaluating the above function in different points, one can check that it is non-constant for $n > 1$. This gives the following proposition:

Proposition 7.4.1 *For $n > 1$, the series of manifolds obtained from the complete projective special real manifolds in Eq.* (7.104) *via the q-map consists of complete quaternionic Kähler manifolds that are not locally homogeneous.*

Remark 7.4.2 The curvature tensor of the quaternionic Kähler manifolds discussed above splits as $\mathcal{R} = \nu\mathcal{R}_{\mathbb{H}P^{n+1}} + \mathcal{W}$. Note that in our conventions for quaternionic Kähler manifolds obtained via the supergravity c-map from an $2n$-dimensional projective special Kähler manifold manifold, the reduced scalar curvature is $\nu = -2$. The squared pointwise norm of $\mathcal{R}_{\mathbb{H}P^{n+1}}$ is

$$
\|\mathcal{R}_{\mathbb{H}P^{n+1}}\|^2 = 20n^2 + 44n + 24 = 20(n+1)^2 + 4(n+1). \qquad (7.110)
$$

Using computer algebra software, we have calculated the squared pointwise norm $\|\mathcal{R}\|^2$ of the Riemann tensor for $n = 2$ and $n = 3$ and have checked that $\|\mathcal{R}\|^2 - 4\|\mathcal{R}_{\mathbb{H}P^{n+1}}\|^2$ agrees with the squared pointwise norm of the Weyl tensor in Eq. (7.109).

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D.V. Alekseevsky, V. Cortés, M. Dyckmanns and T. Mohaupt,
Quaternionic Kähler metrics associated with special Kähler manifolds,
J. Geom. Phys. **92** (2015), 271–287.

This collaboration was part of my doctoral project. The formulation of the HK/QK correspondence in Section 4.1 is (as opposed to its proof) taken from this publication. The same is true for the account of the Swann bundle construction in Section 3.6. The application of the HK/QK correspondence to the c-map in Sections 5.1–5.4 is part of this publication. The results on the Kähler/Kähler correspondence from this publication did not enter this doctoral thesis.

V. Cortés, M. Dyckmanns and D. Lindemann,
Classification of complete projective special real surfaces,
Proc. London Math. Soc. **109** (2014), no. 2, 423–445.

My contribution to this collaboration is mainly based on my M.Sc. thesis written in Hamburg under the supervision of Vicente Cortés. In this doctoral thesis, the classification of projective special real surfaces is only mentioned in a short remark in Chapter 6. Chapter 7 makes some use of the curvature results for projective very special Kähler manifolds from this publication.

M. Dyckmanns,
A twistor sphere of generalized Kähler potentials on hyperkähler manifolds,
arXiv:1111.3893 (hep-th).

This preprint contains the results of my M.A. thesis written in Stony Brook under the supervision of Martin Roček and is essentially unrelated to this doctoral thesis.