

# Homogeneous almost hypercomplex and almost quaternionic pseudo-Hermitian manifolds with irreducible isotropy groups

Dissertation

zur Erlangung des Doktorgrades der Fakultät für Mathematik, Informatik und  
Naturwissenschaften der Universität Hamburg

vorgelegt im Fachbereich Mathematik von

Benedict Meinke

Hamburg 2015

Datum der Disputation: 18.12.2015

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# Acknowledgements

First of all I would like to thank my advisor Vicente Cortés for suggesting the topic and his patient mentoring, continuous encouragement and support, and helpful discussions during the last four years.

Special thanks go to Abdelghani Zeghib and the ENS de Lyon for helpful discussions and hospitality during my stay in Lyon in October 2013.

I would like to thank Marco Freibert for carefully reading my thesis and giving me valuable feedback. I also thank his wife Antonia Freibert for baking me birthday cookies and sending them to me from Denmark. Thanks also go to Christoph Wockel who helped me find references and to David Lindemann and Reiner Lauterbach for helpful discussions.

Furthermore, I would like to express my gratitude to my office mate Owen Vaughan and to our secretary Eva Kuhlmann for the friendly atmosphere in the department.

Finally, I want to thank my friends and my family for their support.



# Notation and conventions

We give here a collection of the notation used in this thesis.

- **Numbers and vector spaces:**

We denote by  $\mathbb{R}$  the set of real numbers, by  $\mathbb{C}$  the set of complex numbers, and by  $\mathbb{H}$  the set of quaternions. We will refer to all of these sets as fields, although  $\mathbb{H}$  is a skew-field. Let  $\mathbb{R}_+$  be the set of real positive numbers. We denote by  $i$  the imaginary unit of  $\mathbb{C}$ . Let  $\mathbf{i}$ ,  $\mathbf{j}$ , and  $\mathbf{k}$  be quaternions that satisfy the quaternionic relations, namely  $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$  and  $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$ . We will write  $\mathbb{F}$  if we do not want to specify any of the three fields  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ . For  $x \in \mathbb{F}$  we denote by  $\Re(x)$  and  $\Im(x)$  the real and the imaginary part of  $x$ , respectively. For the purely imaginary subspace  $\Im(\mathbb{H}) = \text{span}_{\mathbb{R}}\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$  of  $\mathbb{H}$  we will sometimes use the symbol  $\mathbb{I}$ . If we speak of a subfield  $\mathbb{F}' \subset \mathbb{F}$  we mean one of the fields above. Let  $\mathbb{F}^* := \mathbb{F} \setminus \{0\}$  be the multiplicative group of  $\mathbb{F}$ . For an element  $x \in \mathbb{F}$  we denote by  $\bar{x}$  the conjugate of  $x$  and by  $\|x\| := \sqrt{x\bar{x}}$  the norm of  $x$ .

Let  $\mathbb{N}$  be the set of natural numbers. The letter  $n$  will always denote a natural number. We will denote by  $\mathbb{F}^n$  the (right) vector space of  $n$ -tuples of elements of  $\mathbb{F}$ . Let  $e_1, \dots, e_n$  be the standard basis of  $\mathbb{F}^n$ . An  $\mathbb{F}$ -subspace of  $\mathbb{F}^n$  is a subspace over  $\mathbb{F}$ . Analogously, if  $\mathbb{F}' \subset \mathbb{F}$  is a subfield, then an  $\mathbb{F}'$ -subfield of  $\mathbb{F}^n$  is a subspace of  $\mathbb{F}^n$  over  $\mathbb{F}'$ . Let  $\langle \cdot, \cdot \rangle_0$  be the standard inner product on  $\mathbb{F}^n$ , i.e.  $\langle x, y \rangle_0 = \sum_{i=1}^n x_i \bar{y}_i$ . Let  $B^n(\mathbb{F}) = \{v \in \mathbb{F}^n \mid \langle v, v \rangle_0 < 1\}$  be the open ball of radius 1 and let  $S^n(\mathbb{F}) = \{v \in \mathbb{F}^n \mid \langle v, v \rangle_0 = 1\}$  be the sphere. For  $\mathbb{F} = \mathbb{C}$  we denote by  $\omega_0$  the Kähler form of  $\langle \cdot, \cdot \rangle_0$  defined by  $\omega_0(x + iy, x' + iy') := \langle x, y' \rangle_0 - \langle y, x' \rangle_0$  where  $x, x', y, y' \in \mathbb{R}^n$ .

Let  $k$  be a natural number. Then  $\mathbb{F}^{k,n}$  denotes the space  $\mathbb{F}^{n+k}$  endowed with the non-degenerate bilinear form  $\langle \cdot, \cdot \rangle_k$  defined by  $\langle x, y \rangle_k := -\sum_{i=1}^k x_i \bar{y}_i + \sum_{i=k+1}^{n+k} x_i \bar{y}_i$ . Let  $S_k^n = \{v \in \mathbb{F}^{k,n} \mid \langle v, v \rangle_k = 1\}$  be the pseudo-sphere. For  $\mathbb{F} = \mathbb{C}$  we define as above the Kähler form  $\omega_k$  of  $\langle \cdot, \cdot \rangle_k$  by  $\omega_k(x + iy, x' + iy') := \langle x, y' \rangle_k - \langle y, x' \rangle_k$ . We call a vector  $v \in \mathbb{F}^{k,n}$  spacelike, if  $\langle v, v \rangle_k > 0$ , lightlike, if  $\langle v, v \rangle_k = 0$ , and timelike, if  $\langle v, v \rangle_k < 0$ . For  $\mathbb{F}^{1,n}$  we will denote by  $e_0, e_1, \dots, e_n$  the standard basis where  $e_0$  is a timelike vector.

We call an  $\mathbb{F}$ -subspace  $\mathcal{V}$  of  $\mathbb{F}^{k,n}$  spacelike, if  $\langle \cdot, \cdot \rangle_k$  restricted to  $\mathcal{V} \times \mathcal{V}$  is positive definite, and timelike if the restriction is non-degenerate and  $\mathcal{V}$  contains timelike vectors. If  $\mathcal{V}$  is space- or timelike, then we denote by  $\mathcal{V}^\perp$  its orthogonal complement

in  $\mathbb{F}^{k,n}$  with respect to  $\langle \cdot, \cdot \rangle_k$ .

Let  $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$  and  $\mathcal{V}$  be a vector space over  $\mathbb{F}$ . Then  $\Lambda^k \mathcal{V}^*$  denotes the vector space of  $k$ -forms on  $\mathcal{V}$  with values in  $\mathbb{F}$ . If  $\alpha \in \Lambda^k \mathcal{V}^*$  and  $\mathcal{U} \subset \mathcal{V}$  is an  $\mathbb{F}$ -subspace, then we denote the restriction of  $\alpha$  to  $\mathcal{U} \times \dots \times \mathcal{U}$  by  $\alpha|_{\mathcal{U}}$ . For  $\alpha \in \Lambda^{k_1} \mathcal{V}^*$ ,  $\beta \in \Lambda^{k_2} \mathcal{V}^*$  we denote by  $\alpha \wedge \beta \in \Lambda^{k_1+k_2} \mathcal{V}^*$  the wedge-product. Let  $\alpha \in \Lambda^k \mathcal{V}^*$  and  $x \in \mathcal{V}$ , then we denote by  $\iota_x \alpha$  the interior product.

• **Lie groups and Lie algebras:**

We denote by  $G$ ,  $H$ ,  $K$ , and  $U$  Lie groups and the associated Lie algebras by  $\mathfrak{g}$ ,  $\mathfrak{h}$ ,  $\mathfrak{k}$ , and  $\mathfrak{u}$ , respectively. The connected components of the identity are denoted by  $G^0$ ,  $H^0$ ,  $K^0$ , and  $U^0$ .  $\mathcal{Z}(G)$  is the center of  $G$ .

For a Lie group  $G$  let  $\mu$  be the Haar measure of  $G$ . Let  $f : G \rightarrow \mathbb{C}$  be a Borel-measurable function. Then we define the essential supremum and essential infimum by

$$\begin{aligned} \text{ess sup } f &:= \inf \{a \in \mathbb{R} \mid \mu(\{g \in G \mid |f(g)| > a\}) = 0\}, \\ \text{ess inf } f &:= \sup \{b \in \mathbb{R} \mid \mu(\{g \in G \mid |f(g)| < b\}) = 0\}. \end{aligned}$$

Notice that  $\|f\|_\infty := \text{ess sup } f$  is a semi-norm on the vector space

$$\mathcal{L}^\infty(G) := \{f : G \rightarrow \mathbb{C} \mid f \text{ Borel-measurable, } \|f\|_\infty < \infty\}.$$

We define  $\mathcal{N} := \{f : G \rightarrow \mathbb{C} \mid f \text{ Borel-measurable, } \|f\|_\infty = 0\}$ . Then we obtain a Banach space by defining  $L^\infty(G) := \mathcal{L}^\infty(G)/\mathcal{N}$ .

Let  $M(n; \mathbb{F})$  be the set of  $n \times n$ -matrices with elements of  $\mathbb{F}$  in the entries and let  $\mathbb{1}_n$  be the  $n \times n$ -identity matrix. For a matrix  $A \in M(n; \mathbb{F})$  we denote by  $\overline{A}$  the conjugate matrix and by  $A^T$  its transpose.

We denote by  $\text{GL}(n; \mathbb{F}) := \{A \in M(n; \mathbb{F}) \mid A \text{ is invertible}\}$  the general linear group. If  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{R}$ , we denote by

$$\text{SL}(n; \mathbb{F}) := \{A \in \text{GL}(n; \mathbb{F}) \mid \det A = 1\}$$

the special linear group. Let

$$\text{U}(n; \mathbb{F}) := \{A \in \text{GL}(n; \mathbb{F}) \mid \langle Av, Aw \rangle_0 = \langle v, w \rangle_0 \quad \forall v, w \in \mathbb{F}^n\}$$

be the Lie group of unitary transformations with respect to  $\langle \cdot, \cdot \rangle_0$ . Equivalently,  $\text{U}(n; \mathbb{F})$  consists of the matrices  $A$  which satisfy  $A\overline{A}^T = \mathbb{1}_n$ . If we specify the field  $\mathbb{F}$ , we will use also the notation

$$\text{Sp}(n) := \text{U}(n; \mathbb{H}), \quad \text{U}(n) := \text{U}(n; \mathbb{C}), \quad \text{O}(n) := \text{U}(n; \mathbb{R});$$

$$\text{SU}(n) := \text{U}(n) \cap \text{SL}(n; \mathbb{C}), \quad \text{SO}(n) := \text{O}(n) \cap \text{SL}(n; \mathbb{R}).$$

Sometimes  $\text{Sp}(1)$  denotes also the Lie group which acts on  $\mathbb{H}^n$  by multiplication of quaternions of norm one from the right.



Analogously, we define for  $p, q \in \mathbb{N}$  the Lie group

$$U(p, q; \mathbb{F}) := \left\{ A \in \text{GL}(p+q; \mathbb{F}) \mid \langle Av, Aw \rangle_p = \langle v, w \rangle_p \quad \forall v, w \in \mathbb{F}^{p,q} \right\}.$$

Equivalently,  $U(p, q; \mathbb{F})$  consists of the matrices  $A$  which satisfy  $A\mathbb{1}_{p,q}\overline{A}^T = \mathbb{1}_{p,q}$  where

$$\mathbb{1}_{p,q} := \begin{pmatrix} -\mathbb{1}_p & 0 \\ 0 & \mathbb{1}_q \end{pmatrix}.$$

The maximal Lie subgroup of  $U(p, q; \mathbb{F})$  is  $U(p; \mathbb{F}) \times U(q; \mathbb{F})$ . If we specify the field  $\mathbb{F}$ , we will use the following notation

$$\text{Sp}(p, q) := U(p, q; \mathbb{H}), \quad U(p, q) := U(p, q; \mathbb{C}), \quad O(p, q) := U(p, q; \mathbb{R}),$$

$$\text{SU}(p, q) := U(p, q) \cap \text{SL}(p+q; \mathbb{C}), \quad \text{SO}(p, q) := O(p, q) \cap \text{SL}(p+q; \mathbb{R}).$$

Furthermore,  $\text{SO}^0(p, q)$  denotes the connected component of the identity of  $\text{SO}(p, q)$ .

The quaternionic Heisenberg group will be denoted by  $\text{Heis}_n(\mathbb{H})$  and consists of the matrices of the form

$$\begin{pmatrix} 1 & -\bar{t}^T & -\frac{1}{2}\|t\|^2 + s \\ 0 & \mathbb{1}_{n-1} & t \\ 0 & 0 & 1 \end{pmatrix}$$

where  $t \in \mathbb{H}^{n-1}$ ,  $\|t\|^2 = \langle t, t \rangle_0$ ,  $s \in \Im(\mathbb{H})$ .

- **Manifolds:**

We denote by  $M$  always a finite dimensional manifold. If  $M$  is connected, then  $\tilde{M}$  denotes its universal cover. If  $p \in M$ , then  $T_p M$  is the tangent space of  $M$  at  $p$ ,  $TM$  the tangent bundle and  $\text{End}(TM)$  the endomorphism bundle.  $\mathfrak{X}(M)$  is the Lie algebra of smooth vector fields on  $M$ . The pair  $(M, g)$  denotes a semi-Riemannian manifold and  $\nabla$  is the Levi-Civita connection. Sometimes we omit the semi-Riemannian metric  $g$  and just write  $M$  for a semi-Riemannian manifold. Let  $R$  and  $\text{Ric}$  be the Riemannian curvature tensor and the Ricci tensor, respectively. If  $N$  is a vector bundle over  $M$ , then  $\Gamma(N)$  denotes the set of smooth sections. We denote by  $\text{Iso}(M, g)$  the isometry group of  $(M, g)$ . If  $J$  is an almost complex structure on  $M$  and if  $g$  is Hermitian, then we define

$$\text{Iso}(M, g, J) := \left\{ f \in \text{Iso}(M, g) \mid df_p \circ J_p = J_{f(p)} \circ df_p \quad \forall p \in M \right\}.$$

If  $(I, J, K)$  is an almost hypercomplex structure on  $M$  and if  $g$  is Hermitian, then we define

$$\text{Iso}(M, g, (I, J, K)) := \text{Iso}(M, g, I) \cap \text{Iso}(M, g, J) \cap \text{Iso}(M, g, K).$$

Furthermore,  $\omega_I := g(I\cdot, \cdot)$ ,  $\omega_J := g(J\cdot, \cdot)$ , and  $\omega_K := g(K\cdot, \cdot)$  denote the Kähler forms.

If  $Q$  is an almost quaternionic structure on  $M$  and if  $g$  is Hermitian, then we define

$$\text{Iso}(M, g, Q) := \left\{ f \in \text{Iso}(M, g) \mid df_p Q_p (df_p)^{-1} = Q_{f(p)} \quad \forall p \in M \right\}.$$

Furthermore, we denote by  $\Omega$  the fundamental 4-form.

If a Lie group  $G$  acts on a manifold  $M$ , then we denote by  $G_p$  the stabilizer and by  $G \cdot p$  the orbit for a point  $p \in M$ .

The projective space  $\mathbb{F}P^n$  is the manifold which consists of all one-dimensional  $\mathbb{F}$ -subspaces of  $\mathbb{F}^{1,n}$  which we also call lines. The hyperbolic space  $H^n(\mathbb{F})$  is the manifold consisting of all timelike lines and its boundary will be denoted by  $\partial H^n(\mathbb{F})$  and consists of the lightlike lines. If  $M \subset H^n(\mathbb{F})$  is a totally geodesic submanifold, then  $I(M)$  and  $K(M)$  denotes the Lie subgroup of  $U(1, n; \mathbb{F})$  which preserves  $M$  or fixes  $M$  pointwise, respectively. The connected components of the identity of  $I(M)$  and  $K(M)$  are denoted by  $I^0(M)$  and  $K^0(M)$ , respectively.

If we consider  $\mathbb{F}^{1,n}$  as manifold, then we denote it by  $\text{Mink}_{n+1}(\mathbb{F})$ . The real, complex, and quaternionic de Sitter and Anti de Sitter spaces are denoted by

$$\begin{aligned} \text{dS}_{n+1}(\mathbb{R}) &= \text{SO}^0(1, n+1)/\text{SO}^0(1, n), & \text{AdS}_{n+1}(\mathbb{R}) &= \text{SO}^0(2, n)/\text{SO}^0(1, n), \\ \text{dS}_{n+1}(\mathbb{C}) &= \text{SU}(1, n+1)/\text{U}(1, n), & \text{AdS}_{n+1}(\mathbb{C}) &= \text{SU}(2, n)/\text{U}(1, n), \\ \text{dS}_{n+1}(\mathbb{H}) &= \text{Sp}(1, n+1)/\text{Sp}(1, n) \times \text{Sp}(1), & \text{AdS}_{n+1}(\mathbb{H}) &= \text{Sp}(2, n)/\text{Sp}(1) \times \text{Sp}(1, n). \end{aligned}$$

We mention four more manifolds, namely

$$\begin{aligned} \text{CdS}_{n+1} &= \text{SO}^0(1, n+2)/\text{SO}^0(1, n) \times \text{SO}(2), & \text{CAdS}_{n+1} &= \text{SO}^0(3, n)/\text{SO}(2) \times \text{SO}^0(1, n), \\ \text{HdS}_{n+1} &= \text{SO}^0(1, n+4)/\text{SO}^0(1, n) \times \text{SO}(4), & \text{HAdS}_{n+1} &= \text{SO}^0(5, n)/\text{SO}(4) \times \text{SO}^0(1, n). \end{aligned}$$

# Chapter 1

## Introduction

In this thesis we study pseudo-Riemannian almost hypercomplex respectively almost quaternionic homogeneous spaces with  $\mathbb{H}$ -irreducible isotropy groups.

An almost hypercomplex Hermitian manifold is a Riemannian manifold which admits three orthogonal almost complex structures which satisfy the quaternionic relations. All three almost complex structures together are also called an almost hypercomplex structure. Such a manifold is called hyper-Kähler if the three almost complex structures are parallel with respect to the Levi-Civita connection. An equivalent definition is that the holonomy group is contained in  $\mathrm{Sp}(n)$ .

An almost quaternionic Hermitian manifold is a Riemannian manifold  $M$  which admits a three-dimensional subbundle  $Q \subset \Gamma(\mathrm{End}(TM))$  which is locally generated by an almost hypercomplex structure. Such a manifold is called quaternionic Kähler if  $\dim M > 4$  and the almost quaternionic structure  $Q$  is parallel with respect to the Levi-Civita connection. This definition is equivalent to the holonomy group being contained in  $\mathrm{Sp}(n)\mathrm{Sp}(1)$ .

Hyper-Kähler and quaternionic Kähler manifolds have been intensively studied because they are Einstein manifolds, see also [Be] for a good overview. Furthermore hyper-Kähler manifolds are Ricci-flat. Specific types of hyper-Kähler respectively quaternionic Kähler manifolds studied in [Be] are those which are in addition homogeneous or symmetric spaces. The concept of hyper-Kähler and quaternionic Kähler geometry can be also applied to pseudo-Riemannian manifolds. Then the definitions are equivalent to the holonomy group being contained in  $\mathrm{Sp}(p, q)$  respectively  $\mathrm{Sp}(p, q)\mathrm{Sp}(1)$ .

In this thesis we focus on almost hypercomplex and almost quaternionic homogeneous manifolds with index 4 which have  $\mathbb{H}$ -irreducible isotropy groups, i.e. the isotropy representation has no non-trivial invariant quaternionic subspaces. This investigation is based on the work of Ahmed and Zeghib [AZ] who studied almost complex pseudo-Hermitian manifolds which are pseudo-Riemannian manifolds endowed with an orthogonal almost complex structure. Such a manifold is called pseudo-Kähler if the associated Kähler form is closed and the Nijenhuis-tensor vanishes. In [AZ] the authors focus on almost complex homogeneous spaces with index 2. The observation is that such manifolds are already pseudo-Kähler manifolds if the isotropy group acts  $\mathbb{C}$ -irreducibly, i.e. the isotropy represen-

tation has no non-trivial invariant complex subspaces. If furthermore the Lie algebra of the isotropy group acts  $\mathbb{C}$ -irreducibly, then these homogeneous spaces are locally symmetric.

**Theorem 1.1** ([AZ]). *Let  $(M, g, J)$  be a connected almost complex pseudo-Hermitian manifold with index 2 and  $\dim M = 2n + 2 \geq 8$ , such that there exists a connected Lie subgroup  $G \subset \text{Iso}(M, g, J)$  acting transitively on  $M$ . If the isotropy group  $H$  acts  $\mathbb{C}$ -irreducibly, then  $(M, g, J)$  is a pseudo-Kähler manifold. If furthermore  $\mathfrak{h}$  acts  $\mathbb{C}$ -irreducibly, then  $(M, g, J)$  is locally isometric to one of the following symmetric spaces:*

$$\text{Mink}_{n+1}(\mathbb{C}), \text{dS}_{n+1}(\mathbb{C}), \text{AdS}_{n+1}(\mathbb{C}), \text{CdS}_{n+1}, \text{CAdS}_{n+1}.$$

The crucial observation of Ahmed and Zeghib is that a real valued 3-form on  $\mathbb{C}^{1,n}$  vanishes for  $n \geq 3$  if it is  $\text{SO}^0(1, n)$ -invariant and that an antisymmetric bilinear form  $\mathbb{C}^{1,n} \times \mathbb{C}^{1,n} \rightarrow \mathbb{C}^{1,n}$  vanishes for  $n \geq 3$  if it is  $\text{SO}^0(1, n)$ -equivariant. Since for any homogeneous space a geometrical property that holds at one point also holds at every point, by considering the differential of the Kähler form and the Nijenhuis tensor these two facts imply, with the assumption of a  $\mathbb{C}$ -irreducible isotropy group, that the manifolds are pseudo-Kähler. More precisely, the authors of [AZ] proved that if the isotropy group is  $\mathbb{C}$ -irreducible, then the connected component of the Zariski closure of the linear isotropy group contains  $\text{SO}^0(1, n)$  up to conjugacy. This is due to the fact that all connected  $\mathbb{C}$ -irreducible Lie subgroups of  $\text{U}(1, n)$  contain  $\text{SO}^0(1, n)$  up to conjugacy. These Lie subgroups have been first classified by Di Scala and Leistner in [DSL]. Nevertheless the authors of [AZ] gave an independent proof. In this thesis we are going to study an analogous problem as in [AZ] but for the almost hypercomplex and almost quaternionic case. There are some analogues between (pseudo-)Kähler manifolds and (psuedo-)hyper-Kähler and quaternionic (pseudo-)Kähler manifolds. By the Hitchin Lemma, see Lemma 2.5, for an almost hypercomplex manifold it is enough to check that the three Kähler forms are closed to ensure that the manifolds are (pseudo-)hyper-Kähler. Hence, we will prove that the Zariski closure of an  $\mathbb{H}$ -irreducible Lie subgroup of  $\text{Sp}(1, n)$  contains  $\text{SO}^0(1, n)$  and that a real valued 3-form on  $\mathbb{H}^{1,n}$  with  $n \geq 3$  vanishes if it is  $\text{SO}^0(1, n)$ -invariant. The strategy is as follows. In Section 3.1 we will provide an elementary proof that such a 3-form vanishes. We also classify all connected  $\mathbb{H}$ -irreducible Lie subgroups of  $\text{Sp}(1, n)$  to ensure that the Zariski closure of an  $\mathbb{H}$ -irreducible Lie subgroup of  $\text{Sp}(1, n)$  contains  $\text{SO}^0(1, n)$  if  $n \geq 2$ . This classification is the first important result of this thesis.

**Theorem A.** *Let  $H \subset \text{Sp}(1, n)$  be a connected and  $\mathbb{H}$ -irreducible Lie subgroup. Then  $H$  is conjugated to one of the following groups:*

- $\text{SO}^0(1, n), \text{SO}^0(1, n) \cdot \text{U}(1), \text{SO}^0(1, n) \cdot \text{Sp}(1)$  if  $n \geq 2$ ,
- $\text{SU}(1, n), \text{U}(1, n)$ ,
- $\text{Sp}(1, n)$ ,
- $U^0 = \{A \in \text{Sp}(1, 1) | A\Phi = \Phi A\} \cong \text{Spin}^0(1, 3)$  with  $\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  if  $n = 1$ .

For this classification we will start in an analogous way as in [AZ], i.e. we will distinguish between amenable and non-amenable Lie subgroups. Then we will make use of a classification result by Chen and Greenberg in [CG], who classified all totally geodesic submanifolds of the hyperbolic space  $H^n(\mathbb{H})$  and the subgroups of  $\mathrm{Sp}(1, n)$  preserving them. By using these results we can shorten the analogous proofs of Ahmed and Zeghib for subgroups of  $\mathrm{U}(1, n)$ , see Propositions 3.1 and 3.4. With this results we will prove the following Theorem.

**Theorem B.** *Let  $(M, g, (I, J, K))$  be a connected almost hypercomplex pseudo-Hermitian manifold of index 4 and  $\dim M = 4n + 4 \geq 16$ , such that there exists a connected Lie group  $G \subset \mathrm{Iso}(M, g, (I, J, K))$  acting transitively on  $M$ . If the isotropy group  $H := G_p$ ,  $p \in M$ , acts  $\mathbb{H}$ -irreducibly, then  $(M, g, (I, J, K))$  is a pseudo-hyper-Kähler manifold. If furthermore  $\mathfrak{h}$  acts  $\mathbb{H}$ -irreducibly, then  $(M, g, (I, J, K))$  is locally isometric to  $\mathrm{Mink}_{n+1}(\mathbb{H})$ .*

Notice that similar to the case of Ahmed and Zeghib the  $\mathbb{H}$ -irreducibility of the Lie algebra of the isotropy group implies that the homogeneous space is locally symmetric.

There is an analogue of the Hitchin Lemma for the almost quaternionic case. Swann showed in [S] that an almost quaternionic (pseudo-)Hermitian manifold of dimension greater than four is quaternionic (pseudo-)Kähler if the fundamental 4-form is closed. As in the hyper-Kähler case we will make use of the fact that the Zariski closure of an  $\mathbb{H}$ -irreducible Lie subgroup of  $\mathrm{Sp}(1, n)$  contains  $\mathrm{SO}^0(1, n)$  up to conjugacy. This implies that the fundamental 4-form is invariant under  $\mathrm{SO}^0(1, n)$ . In Section 3.1 we will show that a real-valued 5-form on  $\mathbb{H}^{1, n}$  with  $n \geq 5$  vanishes if it is  $\mathrm{SO}^0(1, n)$ -invariant, see Lemma 3.11. This result will imply that the fundamental 4-form is closed. Our next important result is the following Theorem.

**Theorem C.** *Let  $(M, g, Q)$  be a connected almost quaternionic pseudo-Hermitian manifold of index 4 and  $\dim M = 4n + 4 \geq 24$ , such that there exists a connected Lie subgroup  $G \subset \mathrm{Iso}(M, g, Q)$  acting transitively on  $M$ . Let  $H := G_p$ ,  $p \in M$ , denote the isotropy group. If the intersection of the linear isotropy group with  $\mathrm{Sp}(1, n)$  acts  $\mathbb{H}$ -irreducibly, then  $(M, g, Q)$  is a quaternionic pseudo-Kähler manifold. If furthermore  $M$  is a reductive homogeneous space and  $\mathfrak{h} \cap \mathfrak{sp}(1, n)$  acts  $\mathbb{H}$ -irreducibly, then  $M$  is locally symmetric.*

Theorems A, B and C are the most important results of this thesis.

The structure of this thesis is as follows. In Chapter 2 we give an overview of the basic concepts. In particular we will introduce the notion of homogeneous and symmetric spaces in general, almost hypercomplex and almost quaternionic manifolds, amenable Lie groups, hyperbolic spaces  $H^n(\mathbb{F})$ , and the Zariski topology.

In Chapter 3 we will prove the algebraic results that are needed for the proof of the Theorems B and C. In particular we will investigate real-valued 3- and 5-forms on  $\mathbb{H}^{1, n}$  which are  $\mathrm{SO}^0(1, n)$ -invariant and show that they vanish for  $n \geq 3$ , respectively  $n \geq 5$  by using elementary methods. Furthermore we will classify all connected  $\mathbb{H}$ -irreducible Lie subgroups of  $\mathrm{Sp}(1, n)$  up to conjugacy by using the classification of the totally geodesic

submanifolds of  $H^n(\mathbb{H})$  and the Lie subgroups of  $\mathrm{Sp}(1, n)$  preserving them by Chen and Greenberg [CG].

In Chapter 4 we will finally prove Theorems B and C. For the proof of Theorem B we will first show with the above results that the Kähler forms are closed and then we will study the universal cover  $\tilde{M} = \tilde{G}/H^0$  by considering all possibilities for  $H^0$ . In all cases it will turn out that  $\tilde{M}$  is globally isomorphic to  $\mathrm{Mink}_{n+1}(\mathbb{H})$ . For the proof of Theorem C we will show in an analogous way as in the proof of Theorem B that the fundamental 4-form is closed. Finally in Chapter 5 we will point out some open problems that are left for future research.

## Chapter 2

# Basic Concepts

### 2.1 Homogeneous and symmetric spaces

In this section we give an overview of homogeneous and symmetric spaces. The reader can find more details for example in [Ba], [Be], [He], [KN2], [O], and [Wa].

**Definition 2.1.** Let  $G$  be a Lie group and  $M$  be a differentiable manifold. A **Lie group action** of  $G$  on  $M$  is a differentiable map  $G \times M \rightarrow M, (g, p) \mapsto g \cdot p$  such that

- (i) for all  $g \in G$  the map  $x \in M \mapsto g \cdot x \in M$  is a diffeomorphism,
- (ii)  $e \cdot p = p$  for all  $p \in M$ ,
- (iii)  $g \cdot (h \cdot p) = (gh) \cdot p$  for all  $g, h \in G$  and all  $p \in M$ .

In the following  $G$  will be always a Lie group that acts on a manifold  $M$ .

**Definition 2.2.** Let  $p \in M$ . The **stabilizer** of the point  $p \in M$  is by definition the subgroup  $G_p := \{g \in G \mid g \cdot p = p\}$ . The subset  $G \cdot p := \{g \cdot p \mid g \in G\}$  is called the **orbit** of  $p$ .

Notice that  $G_p$  is a closed subgroup of  $G$ , since the action is continuous. Hence,  $G_p$  is a Lie subgroup of  $G$ .

**Theorem 2.1** ([Wa, Theorem 3.58]). Let  $H$  be a closed subgroup of a Lie group  $G$ , and let  $G/H$  be the set  $\{gH \mid g \in G\}$  of left cosets modulo  $H$ . Let  $\pi : G \rightarrow G/H$  denote the natural projection  $\pi(g) = gH$ . Then  $G/H$  has a unique manifold structure such that

- (i)  $\pi : G \rightarrow G/H$  is smooth.
- (ii) There exist local smooth sections of  $G/H$  in  $G$ ; that is, if  $gH \in G/H$ , there is a neighbourhood  $W$  of  $gH$  and a  $C^\infty$ -map  $\tau : W \rightarrow G$  such that  $\pi \circ \tau = \text{Id}$ .

The dimension of  $G/H$  is  $\dim G - \dim H$ .

**Definition 2.3.** A Lie group action  $G \times M \rightarrow M$  is called **transitive**, if  $G \cdot p = M$ .

Let  $H \subset G$  be a closed subgroup. Then the Lie group action

$$G \times G/H \rightarrow G/H, (g_1, g_2H) \mapsto g_1g_2H$$

is called the **natural action** of  $G$  on  $G/H$ . This action is transitive.

Let  $G \times M \rightarrow M$  be a Lie group action,  $p \in M$ , and  $H := G_p$ . There is a natural map  $\varphi : G/H \rightarrow M, gH \mapsto g \cdot p$ . Notice that  $\varphi$  is well defined since

$$g_1H = g_2H \Rightarrow g_2^{-1}g_1H = H \Rightarrow g_2^{-1}g_1 \in H \Rightarrow g_1 \cdot p = g_2 \cdot p.$$

**Proposition 2.1** ([O, Chapter 11, Proposition 13]). *Let  $G \times M \rightarrow M$  be a transitive action and  $p \in M, H := G_p$ . Then the natural map  $\varphi : G/H \rightarrow M$  is a diffeomorphism.*

**Theorem 2.2** ([O, Chapter 9, Theorem 32]). *If  $(M, g)$  is a semi-Riemannian manifold, there is a unique way to make  $\text{Iso}(M, g)$  a manifold such that:*

- (i)  $\text{Iso}(M, g)$  is a Lie group.
- (ii) The natural action  $\text{Iso}(M, g) \times M \rightarrow M, (f, p) \mapsto f(p)$  is smooth.
- (iii) A homomorphism  $\beta : \mathbb{R} \rightarrow \text{Iso}(M, g)$  is smooth if the map  $\mathbb{R} \times M \rightarrow M$  sending  $(t, p)$  to  $\beta(t)p$  is smooth.

**Definition 2.4.** *Let  $(M, g)$  be a semi-Riemannian manifold and  $\text{Iso}(M, g)$  its isometry group. If  $\text{Iso}(M, g)$  acts transitively on  $M$ , then  $M$  is called a **homogeneous space**.*

If  $M$  is a homogeneous space, then any geometrical properties at one point of  $M$  hold at every point. By Proposition 2.1 every homogeneous space can be considered as a coset space  $G/H$  where  $G$  is a Lie subgroup of  $\text{Iso}(M, g)$  that acts transitively on  $M$ . The stabilizer  $H$  is also called the **isotropy group**. The representation

$$\rho : H \rightarrow \text{GL}(T_pM), h \mapsto dh_p$$

is called the **isotropy representation**. The image  $\rho(H)$  is called the **linear isotropy group**.

**Definition 2.5.** *A homogeneous space  $M = G/H$  is a reductive homogeneous space if there is a vector subspace  $\mathfrak{m} \subset \mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  and  $\text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$ .*

**Proposition 2.2** ([O, Chapter 11, Proposition 22]). *Let  $M = G/H$  be a reductive homogeneous space and  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ .*

- (i) The linear isotropy group  $\rho(H)$  acting on  $T_pM$  corresponds under  $d\pi$  to  $\text{Ad}(H)$  on  $\mathfrak{m}$ .
- (ii) Requiring  $d\pi : \mathfrak{m} \cong T_pM$  to be a linear isometry establishes a one-to-one correspondence between  $\text{Ad}(H)$ -invariant scalar products on  $\mathfrak{m}$  and  $G$ -invariant metrics on  $M$ .



**Definition 2.6.** A connected semi-Riemannian manifold  $(M, g)$  is called (semi-Riemannian) **symmetric space** if there exists for each  $p \in M$  an isometry  $s_p \in \text{Iso}(M, g)$  such that  $s_p(p) = p$  and  $d(s_p)_p = -\text{Id}_{T_p M}$ . The isometry  $s_p$  is called **global symmetry** of the point  $p$ . A semi-Riemannian manifold  $(M, g)$  is called **locally symmetric** if  $\nabla R = 0$ .

Recall that for a connected manifold an isometry is uniquely determined by its value and differential at one point. Hence, for a symmetric space  $(M, g)$  the isometry  $s_p$  is unique.

**Lemma 2.1** ([O, Chapter 8 and 9]). A symmetric space is locally symmetric, geodesically complete, and homogeneous.

**Lemma 2.2** ([O, Chapter 8, Corollary 21]). A geodesically complete, simply connected, locally symmetric semi-Riemannian manifold is a symmetric space.

**Definition 2.7.** Let  $G$  be a connected Lie group and  $H$  a closed subgroup of  $G$ . The pair  $(G, H)$  is called a **symmetric pair** if there exists an involutive smooth automorphism  $\sigma : G \rightarrow G$  such that  $H_\sigma^0 \subset H \subset H_\sigma$ , where  $H_\sigma$  is the set of fixed points of  $\sigma$  and  $H_\sigma^0$  the connected component of the identity of  $H_\sigma$ .

**Theorem 2.3** ([O, Chapter 11, Theorem 29]). Let  $(G, H)$  be a symmetric pair. Then any  $G$ -invariant metric tensor on  $M = G/H$  makes  $M$  a semi-Riemannian symmetric space such that  $s_p \circ \pi = \pi \circ \sigma$ , where  $\pi : G \rightarrow M$  is the projection.

If  $(M, g)$  is a symmetric space, then  $G = \text{Iso}^0(M, g)$  and  $H = \text{Iso}^0(M, g)_p$  form a symmetric pair such that  $M = G/H$ , see [O, Chapter 11].

**Lemma 2.3** ([O, Chapter 11, Lemma 30]). Let  $(G, H)$  be a symmetric pair. Then

- (i)  $\mathfrak{h} = \{X \in \mathfrak{g} \mid d\sigma(X) = X\}$ .
- (ii)  $\mathfrak{g}$  is the direct sum of  $\mathfrak{h}$  and the subspace  $\mathfrak{m} = \{X \in \mathfrak{g} \mid d\sigma(X) = -X\}$ .
- (iii)  $\text{Ad}(H)(\mathfrak{m}) \subset \mathfrak{m}$ .
- (iv)  $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ ,  $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$ ,  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ .

As a consequence any symmetric space is a reductive homogeneous space.

**Definition 2.8.** A **symmetric Lie algebra** is a triple  $(\mathfrak{g}, \mathfrak{h}, \tau)$  consisting of a Lie algebra  $\mathfrak{g}$ , a Lie subalgebra  $\mathfrak{h}$ , and an involutive automorphism  $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$  such that  $\mathfrak{h}$  consists of all elements of  $\mathfrak{g}$  which are fixed by  $\tau$ .

**Remark 2.1** ([KN2, Chapter XI]). If  $(G, H)$  is a symmetric pair and  $\sigma : G \rightarrow G$  is the involutive automorphism, then  $(\mathfrak{g}, \mathfrak{h}, d\sigma)$  is a symmetric Lie algebra. Conversely, if  $(\mathfrak{g}, \mathfrak{h}, \tau)$  is symmetric Lie algebra and if  $G$  is a simply connected Lie group with Lie algebra  $\mathfrak{g}$ , then  $\tau$  induces an involutive automorphism  $\sigma : G \rightarrow G$  such that for any closed subgroup  $H$  with  $G_\sigma^0 \subset H \subset G_\sigma$ ,  $(G, H)$  is a symmetric pair.

Let  $(\mathfrak{g}, \mathfrak{h}, \tau)$  be a symmetric Lie algebra. Since  $\tau$  is an involution its eigenvalues are 1 and

$-1$ , and  $\mathfrak{h}$  is the eigenspace of  $1$ . Let  $\mathfrak{m}$  be the eigenspace for  $-1$ . Then the decomposition  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  is called the **canonical decomposition** of  $(\mathfrak{g}, \mathfrak{h}, \tau)$ . Furthermore,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}.$$

These relations characterize a symmetric Lie algebra.

In the following we collect some facts about totally geodesic submanifolds in general and in the special case for a symmetric space.

**Definition 2.9.** A semi-Riemannian submanifold  $M$  of  $\overline{M}$  is called **totally geodesic** provided its second fundamental form vanishes:  $\Pi = 0$ .

**Proposition 2.3** ([O, Chapter 4, Proposition 13]). For a semi-Riemannian submanifold  $M \subset \overline{M}$  the following are equivalent.

- (i)  $M$  is totally geodesic in  $\overline{M}$ .
- (ii) Every geodesic of  $M$  is also a geodesic of  $\overline{M}$ .
- (iii) If  $v \in T_p \overline{M}$  is tangent to  $M$ , then the  $\overline{M}$ -geodesic  $\gamma_v$  lies initially in  $M$ .
- (iv) If  $\alpha$  is a curve in  $M$  and  $v \in T_{\alpha(0)} M$ , then parallel translation of  $v$  along  $\alpha$  is the same for  $M$  and for  $\overline{M}$ .

**Lemma 2.4** ([O, Chapter 4, Lemma 14]). Let  $M$  and  $N$  be complete, connected, totally geodesic semi-Riemannian submanifolds of  $\overline{M}$ . If there is a point  $p \in M \cap N$  at which  $T_p M = T_p N$ , then  $M = N$ .

**Proposition 2.4** ([He, Chapter IV.7, Proposition 7.1]). Let  $M$  be a semi-Riemannian manifold and  $S$  a totally geodesic submanifolds of  $M$ . If  $M$  is locally symmetric, then the same holds for  $S$ .

**Definition 2.10.** Let  $\mathfrak{g}$  be a real Lie algebra. A subspace  $\mathfrak{m}$  of  $\mathfrak{g}$  is called **Lie triple system** if  $X, Y, Z \in \mathfrak{m}$  implies  $[X, [Y, Z]] \in \mathfrak{m}$ .

**Theorem 2.4** ([He, Chapter IV.7, Theorem 7.2]). Let  $(M, g)$  be a semi-Riemannian symmetric space and  $(G, H)$  a symmetric pair such that  $M = G/H$ . Denote by  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$  the canonical decomposition and identify  $\mathfrak{m} \cong T_p M, p \in M$ . Let  $\mathfrak{s} \subset \mathfrak{m}$  be a Lie triple system. Then  $S := \exp(\mathfrak{s})$  is a totally geodesic submanifold of  $M$  satisfying  $T_p S \cong \mathfrak{s}$ . Furthermore,  $S$  is a symmetric space.

On the other hand, if  $S$  is a totally geodesic submanifold of  $M$  and  $p \in S$ , then the subspace  $\mathfrak{s} \cong T_p S$  of  $\mathfrak{m}$  is a Lie triple system.

**Remark 2.2.** In [He] Proposition 2.4 and Theorem 2.4 are stated for the Riemannian case, but the proofs work also in the pseudo-Riemannian case.

Finally, we cite the de Rham-Wu decomposition Theorem for simply connected semi-Riemannian manifolds.

**Theorem 2.5** ([Wu], [Ba, Satz 5.8]). *Let  $(M, g)$  be a geodesically complete, simply-connected semi-Riemannian manifold and  $TM = \mathcal{E}_1 \oplus \mathcal{E}_2$  a decomposition of the tangent bundle into two non-degenerate, orthogonal and parallel distributions. For  $p \in M$  let  $M_j(p)$  be the maximal connected integral submanifold of the distribution  $\mathcal{E}_j$  through the point  $p$ . Then  $(M, g)$  is isometric to the product of  $(M_1(p), g_1)$  and  $(M_2(p), g_2)$ , where  $g_j$  denotes the induced metric on  $M_j(p)$ :*

$$(M, g) \cong (M_1(p), g_1) \times (M_2(p), g_2).$$

**Remark 2.3.** *The integral submanifolds  $M_j(p)$  in Theorem 2.5 are given by*

$$M_j(p) = \left\{ q \in \tilde{M} \left| \begin{array}{l} \exists \text{ a piecewise smooth curve } \gamma : I \rightarrow M_j(p) \text{ connecting} \\ p \text{ and } q \text{ such that } \gamma'(t) \in (\mathcal{E}_j)_{\gamma(t)} \text{ for all } t \end{array} \right. \right\}.$$

## 2.2 Hyper-Kähler and quaternionic Kähler manifolds

In this section we give a very short overview of hyper-Kähler and quaternionic Kähler manifolds. The interested reader can find more details in [A], [Ba], [Be], and [I].

**Definition 2.11.** *Let  $\mathcal{V}$  be a finite dimensional real vector space. A **hypercomplex structure** on  $\mathcal{V}$  is a triple  $(I, J, K)$  of anti-commuting complex structures on  $\mathcal{V}$  such that  $IJ = K$ .*

**Remark 2.4.** *Notice that the dimension of a real vector space endowed with a hypercomplex structure is a multiple of 4.*

**Definition 2.12.** *Let  $M$  be a smooth  $n$ -dimensional manifold. An **almost hypercomplex structure** on  $M$  is a triple  $(I, J, K)$  consisting of three smooth sections  $I, J, K \in \Gamma(\text{End}(TM))$  such that  $(I_p, J_p, K_p)$  is a hypercomplex structure on  $T_p M$  for all  $p \in M$ . The pair  $(M, (I, J, K))$  is called an **almost hypercomplex manifold**.*

**Definition 2.13.** *Let  $(M, (I, J, K))$  be an almost hypercomplex manifold. A (pseudo-) Riemannian metric  $g$  on  $M$  is called **Hermitian** if  $I, J$ , and  $K$  are skew symmetric with respect to  $g$ . Then the triple  $(M, g, (I, J, K))$  is called an **almost hypercomplex (pseudo-) Hermitian manifold**.*

**Remark 2.5.** *Notice that an almost complex structure  $I$  on a (pseudo-) Riemannian manifold  $(M, g)$  is skew symmetric if and only if  $I$  is orthogonal, i.e.  $g(IX, IY) = g(X, Y)$  for all  $X, Y \in \mathfrak{X}(M)$ . In particular the three almost complex structures  $I, J, K$  of an almost hypercomplex (pseudo-) Hermitian manifold are orthogonal.*

**Definition 2.14.** *Let  $(M, g, (I, J, K))$  be an almost hypercomplex (pseudo-)Hermitian manifold. Then we define*

$$\text{Iso}(M, g, (I, J, K)) := \{ f \in \text{Iso}(M, g) \mid df \circ I = I \circ df, df \circ J = J \circ df, df \circ K = K \circ df \}.$$

**Definition 2.15.** Let  $(M, g, (I, J, K))$  be an almost hypercomplex (pseudo-) Hermitian manifold. Then each of the three almost complex structures defines a Kähler 2-form by

$$\omega_I(X, Y) := g(IX, Y), \quad \omega_J(X, Y) := g(JX, Y), \quad \omega_K(X, Y) := g(KX, Y)$$

with  $X, Y \in \mathfrak{X}(M)$ .

**Definition 2.16.** An almost hypercomplex (pseudo-) Hermitian manifold  $(M, g, (I, J, K))$  is called **(pseudo-)hyper-Kähler manifold** if  $\nabla I = 0$ ,  $\nabla J = 0$ , and  $\nabla K = 0$ , where  $\nabla$  denotes the Levi-Civita connection.

**Lemma 2.5** ([Hi, Lemma 6.8]). Let  $(M, g, (I, J, K))$  be an almost hypercomplex (pseudo-) Hermitian manifold. If the Kähler forms  $\omega_I$ ,  $\omega_J$ , and  $\omega_K$  are closed, then  $(M, g, (I, J, K))$  is a (pseudo-)hyper-Kähler manifold.

**Lemma 2.6** ([Be, Theorem 14.13]). A (pseudo-)hyper-Kähler manifold is Ricci-flat.

**Definition 2.17.** Let  $\mathcal{V}$  be a finite dimensional real vector space. A **quaternionic structure** on  $\mathcal{V}$  is a three-dimensional subspace  $Q \subset \text{End}(\mathcal{V})$ , which is spanned by a hypercomplex structure  $(I, J, K)$ . Then the hypercomplex structure  $(I, J, K)$  is called **subordinate** to the quaternionic structure  $Q$ .

**Definition 2.18.** Let  $M$  be a smooth  $n$ -dimensional manifold. An **almost quaternionic structure** on  $M$  is a three-dimensional subbundle  $Q \subset \text{End}(TM)$  such that for every  $p \in M$  there exists an open neighbourhood  $U$  of  $p$  with an almost hypercomplex structure  $(I, J, K)$  on  $U$  and  $Q|_U = \text{span}_{\mathbb{R}} \{I, J, K\}$ . The pair  $(M, Q)$  is called an **almost quaternionic manifold**.

**Remark 2.6.** The almost quaternionic structure defines pointwise a quaternionic structure on the tangent spaces.

Notice that in general the local almost hypercomplex structure can not be extended globally on  $M$ . An example is the quaternionic projective space  $\mathbb{H}P^n$ , which does not admit an almost complex structure for topological reasons.

**Definition 2.19.** Let  $(M, Q)$  be an almost quaternionic manifold. A (pseudo-) Riemannian metric  $g$  on  $M$  is called **Hermitian** if  $Q$  consists of skew symmetric endomorphisms with respect to  $g$ . Then the triple  $(M, g, Q)$  is called **almost quaternionic (pseudo-) Hermitian manifold**.

Let  $(M, g, Q)$  be an almost quaternionic (pseudo-) Hermitian manifold. Then we can define locally the Kähler forms  $\omega_I$ ,  $\omega_J$ , and  $\omega_K$ , which depend on  $I, J$  and  $K$ . The 4-form

$$\Omega := \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K \tag{2.1}$$

does not depend on the local almost hypercomplex structure and can be extended globally to  $M$ .

**Definition 2.20.** Let  $(M, g, Q)$  be an almost quaternionic (pseudo-) Hermitian manifold. The 4-form  $\Omega$  on  $M$  which is locally defined by equation (2.1) is called the **fundamental 4-form**.

**Definition 2.21.** An almost quaternionic (pseudo-) Hermitian manifold  $(M, g, Q)$  of dimension  $4n \geq 8$  is called **quaternionic (pseudo-) Kähler manifold** if the almost quaternionic structure  $Q$  is parallel with respect to the Levi-Civita connection  $\nabla$ , i.e.  $\nabla_X \Gamma(Q) \subset \Gamma(Q)$  for all  $X \in \mathfrak{X}(M)$ .

**Theorem 2.6** ([S, Theorem A.3]). Let  $(M, g, Q)$  be an almost quaternionic (pseudo-) Hermitian manifold of dimension  $4n \geq 12$ . If the fundamental 4-form  $\Omega$  is closed, then  $(M, g, Q)$  is a quaternionic (pseudo-) Kähler manifold.

**Definition 2.22.** Let  $(M, g, Q)$  be an almost quaternionic (pseudo-) Hermitian manifold. If  $X \in T_p M$ ,  $p \in M$ , then the 4-plane  $Q(X) := \text{span}_{\mathbb{R}} \{X, IX, JX, KX\}$  is called the **quaternionic 4-plane determined by  $X$** . A (non-degenerate) two dimensional subspace  $E$  of  $Q(X)$  is called **quaternionic plane**. The sectional curvature of a quaternionic plane  $E$  is called **quaternionic sectional curvature**.

An almost quaternionic (pseudo-) Hermitian manifold is called **quaternionic space form** if its quaternionic sectional curvatures are equal to a constant.

It is known that an almost quaternionic (pseudo-) Hermitian manifold  $(M, g, Q)$  is a quaternionic space form if and only if the Riemannian curvature has the form

$$\begin{aligned} R(X, Y)Z &= \frac{c}{4} (g(Y, Z)X - g(X, Z)Y + g(IY, Z)IX - g(IX, Z)IY \\ &\quad + 2g(X, IY)IZ + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ \\ &\quad + g(KY, Z)KX - g(KX, Z)KY + 2g(X, KY)KZ) \end{aligned}$$

for some constant  $c \in \mathbb{R}$ , see for instance [A], [I].

**Proposition 2.5** ([I, Section 5]). A quaternionic space form is locally symmetric.

## 2.3 Amenable groups

Amenable groups have been first introduced in 1929 by J. von Neumann in [N] for discrete groups but he named them *messbar*, which is the German word for measurable, because they are closely related to the Banach-Tarski paradoxon. In 1949 M. M. Day named these groups *amenable*, see [D]. V. Runde suggests in [R] that Day chose this translation as a pun. The reader can find an introduction to the historical origin of Amenability Theory in [P, Chapter 0]

In this section we will give a short overview of the theory of amenable groups from the Lie theoretical point of view, which will be needed in Section 3.2. The reader can find proofs of the following propositions and more details in [G], [P], [R], and [Z].

For motivation let us first consider the well known arithmetic mean of real numbers. Let  $n \in \mathbb{N}$  and  $a_1, \dots, a_n \in \mathbb{R}$ . Then

$$\min \{a_1, \dots, a_n\} \leq \frac{a_1 + \dots + a_n}{n} \leq \max \{a_1, \dots, a_n\}.$$

The term in the middle is the arithmetic mean of the real numbers  $a_1, \dots, a_n$ . If we set  $X := \{1, \dots, n\} \subset \mathbb{N}$ , then the arithmetic mean can be considered as a linear functional on the vector space of real-valued function on  $X$ . More precisely, if  $f : X \rightarrow \mathbb{R}$ , the arithmetic mean is defined as

$$M(f) := \frac{1}{n} \sum_{k=1}^n f(k).$$

In particular the inequality

$$\inf \{f(x) | x \in X\} \leq M(f) \leq \sup \{f(x) | x \in X\}$$

holds. We will now generalize the concept of a mean by replacing  $X$  with a Lie group. First, notice that the above inequality only makes sense for real valued bounded functions on a set  $X$ . In the above example all functions on  $X$  are bounded just because  $X$  is finite. One could think of defining a mean of a Lie group on the set of bounded functions but this does not reflect the topological structure of the Lie group. Actually, we will define the concept of a mean on the set of essentially bounded functions with respect to the Haar measure of the Lie group.

Let  $G$  be a Lie group and denote  $L^\infty(G)$  the Banach space of essentially bounded complex valued Borel measurable functions on  $G$ .

**Definition 2.23.** Let  $G$  be a Lie group. A continuous linear functional  $M : L^\infty(G) \rightarrow \mathbb{C}$  is called a **mean** on  $G$  if

- (1)  $M(\bar{f}) = \overline{M(f)}$  for all  $f \in L^\infty(G)$  and
- (2)  $\text{ess inf}(f) \leq M(f) \leq \text{ess sup}(f)$  for all  $f \in L^\infty(G)$  which are real valued almost everywhere.

**Remark 2.7.** Condition (1) in Definition 2.23 ensures that  $M(f)$  is a real number if  $f$  is real valued almost everywhere. Condition (2) is equivalent to

$$M(f) \geq 0 \text{ if } f \geq 0 \text{ almost everywhere, and } M(1) = 1.$$

*Proof:* Let  $f \in L^\infty(G)$  be real valued almost everywhere and non-negative. Then condition (2) implies  $0 \leq \text{ess inf}(f) \leq M(f) \leq \text{ess sup}(f)$ . Hence,  $M(f) \geq 0$ . Furthermore,

$$1 = \text{ess inf}(1) \leq M(1) \leq \text{ess sup}(1) = 1.$$

Hence,  $M(1) = 1$ .

Conversely, let  $f$  be real valued almost everywhere. We have  $\text{ess inf}(f) \leq f(x)$  for almost all  $x \in G$ . This implies  $g := f - \text{ess inf}(f) \geq 0$  almost everywhere. By assumption  $0 \leq M(g) = M(f) - \text{ess inf}(f)M(1)$ . Hence,  $M(f) \geq \text{ess inf}(f)$ . Analogously, one can show  $M(f) \leq \text{ess sup}(f)$ .  $\square$

Let  $f \in L^\infty(G)$ . For a fixed  $x \in G$  we define  $f_x : G \rightarrow \mathbb{C}$  by  $g \mapsto f(x^{-1}g)$ .

**Definition 2.24.** Let  $G$  be a Lie group. A mean  $M : L^\infty(G) \rightarrow \mathbb{C}$  on  $G$  is called **left-invariant** if  $M(f_x) = M(f)$  for all  $f \in L^\infty(G)$  and for all  $x \in G$ .

A Lie group  $G$  is called **amenable** if there exists a left-invariant mean on  $G$ .

**Example 2.1.** A compact Lie group  $G$  is amenable. If  $\mu$  is the normalized Haar measure on  $G$ , then a left-invariant mean on  $G$  is defined by

$$M(f) := \int_G f d\mu.$$

**Proposition 2.6** ([R, Example 1.1.5]). An abelian Lie group is amenable.

**Remark 2.8.** The proof of Proposition 2.6 is non-constructive.

**Proposition 2.7** ([G, Chapter 2 and 3]). Let  $G$  be a Lie group.

- (i) Let  $H$  be a Lie group and  $\pi : G \rightarrow H$  be a continuous surjective group homomorphism. If  $G$  is amenable, then  $H$  is amenable.
- (ii) If  $G$  is amenable and  $H \subset G$  is a closed subgroup, then  $H$  is amenable.
- (iii) If  $G$  is amenable and  $H \subset G$  is a connected Lie subgroup, then  $H$  is amenable.
- (iv) If  $G$  is amenable and  $N \subset G$  is a closed normal subgroup, then  $G/N$  is amenable.
- (v) If  $H$  is an amenable Lie group and  $G$  is also amenable, then  $G \ltimes H$  is amenable.
- (vi) Let  $N \subset G$  be a closed normal subgroup. If  $N$  and  $G/N$  are amenable, then  $G$  is amenable.

**Proposition 2.8** ([R, Example 1.2.11]). A solvable Lie group is amenable.

**Proposition 2.9** ([G, Cor. 3.3.3]). Let  $G$  be a connected Lie group and let  $G = L \cdot R$  be the Levi decomposition of  $G$ . Then  $G$  is amenable if and only if the semi-simple Levi factor  $L$  is compact.

Now we will consider a special example of an amenable group in detail, because it will play an important role in the proof of Proposition 3.1.

**Example 2.2.** We consider the action of the Lie group  $\mathrm{Sp}(1, n)$  on the right-vector space  $\mathbb{H}^{1, n}$ .  $\mathrm{Sp}(1, n)$  preserves the quadratic form

$$\gamma_0(q) := \langle q, q \rangle_1 = -|q_0|^2 + \sum_{i=1}^n |q_i|^2,$$

where  $q = (q_0, q_1, \dots, q_n)^T \in \mathbb{H}^{1, n}$ . The corresponding matrix to  $\gamma_0$  is

$$\Gamma_0 = \begin{pmatrix} -1 & 0 \\ 0 & \mathbb{1}_n \end{pmatrix},$$

i.e.  $\gamma_0(q) = \bar{q}^T \Gamma_0 q$ . We will now show that the subgroup preserving a lightlike line is amenable.

For simplification we will not use  $\gamma_0$  but an equivalent quadratic form, namely

$$\gamma_1(q) := \bar{q}_0 q_n + \bar{q}_n q_0 + \sum_{i=1}^{n-1} |q_i|^2.$$

The corresponding matrix of  $\gamma_1$  is

$$\Gamma_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{1}_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

i.e.  $\gamma_1(q) = \bar{q}^T \Gamma_1 q$ . Denote by

$$\mathfrak{T} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \mathbb{1}_{n-1} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Since  $\bar{\mathfrak{T}}^T \Gamma_1 \mathfrak{T} = \Gamma_0$ ,  $\gamma_0$  and  $\gamma_1$  are indeed equivalent. Hence,  $\mathrm{Sp}(1, n)$  is conjugated to the Lie group

$$G := \left\{ A \in \mathrm{GL}(n+1, \mathbb{H}) \mid \bar{A}^T \Gamma_1 A = \Gamma_1 \right\}.$$

It follows that a stabilizer in  $\mathrm{Sp}(1, n)$  of a lightlike line with respect to  $\gamma_0$  is conjugated to a stabilizer in  $G$  of a lightlike line with respect to  $\gamma_1$ .

The advantage of using  $\gamma_1$  is that the vector  $e_0$  becomes a lightlike vector. Let  $e_0 \mathbb{H}$  be the quaternionic line spanned by  $e_0$ . We denote the subgroup of  $G$  preserving  $e_0 \mathbb{H}$  by  $G_{e_0 \mathbb{H}}$ . Let  $\Phi \in G_{e_0 \mathbb{H}}$  and write

$$\Phi = \begin{pmatrix} a & v^T & b \\ u & A & x \\ c & w^T & d \end{pmatrix},$$

with  $a, b, c, d \in \mathbb{H}$ ,  $A \in M(n-1, \mathbb{H})$  and  $u, v, w, x \in \mathbb{H}^{n-1}$ . First of all, since  $\Phi e_0 \in e_0 \mathbb{H}$ , we have  $c = 0$  and  $u = 0$ . From  $\bar{\Phi}^T \Gamma_1 \Phi = \Gamma_1$  it follows that

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbb{1}_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \bar{a}w^T & \bar{a}d \\ \bar{w}a & \bar{v}w^T + \bar{A}^T A + \bar{w}v^T & \bar{v}d + \bar{A}^T x + \bar{w}b \\ \bar{d}a & \bar{b}w^T + \bar{x}^T A + \bar{d}v^T & \bar{b}d + \bar{x}^T x + \bar{d}b \end{pmatrix}.$$

This gives us the equations

$$\bar{a}d = 1, \tag{2.2}$$

$$\bar{a}w^T = 0, \tag{2.3}$$

$$\bar{v}w^T + \bar{A}^T A + \bar{w}v^T = \mathbb{1}_{n-1}, \tag{2.4}$$

$$\bar{v}d + \bar{A}^T x + \bar{w}b = 0, \tag{2.5}$$

$$\bar{b}d + \bar{x}^T x + \bar{d}b = 0. \tag{2.6}$$



Equation (2.2) implies  $d = \bar{a}^{-1}$  and in particular  $a \neq 0$ . From equation (2.3) it follows that  $w = 0$ . Now we conclude from equation (2.4) that  $A \in \mathrm{Sp}(n-1)$ . Equation (2.5) implies  $\bar{v} = -\bar{A}^T x \bar{a}$  which is equivalent to  $v^T = -a \bar{x}^T A$ .

Let  $t \in \mathbb{H}^{n-1}$  and  $\lambda \in \mathbb{H}$ , such that  $x = At$  and  $b = a\lambda$ . Now we can decompose

$$\Phi = \begin{pmatrix} a & -a\bar{t}^T & a\lambda \\ 0 & A & At \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\bar{t}^T & \lambda \\ 0 & \mathbb{1}_{n-1} & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Equation (2.6) implies

$$\bar{\lambda} \bar{a} \bar{a}^{-1} + a^{-1} a \lambda + \bar{t}^T \bar{A}^T A t = 0,$$

which is equivalent to  $\bar{\lambda} + \lambda = -\bar{t}^T t$ . Hence,  $\Re(\lambda) = -\frac{1}{2} \|t\|^2$ . Summarizing,

$$\Phi = \begin{pmatrix} a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & \bar{a}^{-1} \end{pmatrix} \begin{pmatrix} 1 & -\bar{t}^T & -\frac{1}{2} \|t\|^2 + s \\ 0 & \mathbb{1}_{n-1} & t \\ 0 & 0 & 1 \end{pmatrix}$$

with  $s \in \Im(\mathbb{H})$ ,  $t \in \mathbb{H}^{n-1}$ ,  $A \in \mathrm{Sp}(n-1)$ , and  $a \in \mathbb{H}^*$ . Finally, we see  $G_{e_0\mathbb{H}} = (\mathbb{H}^* \times \mathrm{Sp}(n-1)) \ltimes \mathrm{Heis}_{n-1}(\mathbb{H})$ .

$\mathrm{Sp}(n-1)$  is compact and it follows from Example 2.1 that  $\mathrm{Sp}(n-1)$  is amenable. The amenability of  $\mathbb{H}^* = \mathbb{R}_+ \times \mathrm{Sp}(1)$  follows from Example 2.1 and Propositions 2.6 and 2.7. The Heisenberg group  $\mathrm{Heis}_{n-1}(\mathbb{H})$  is solvable and hence amenable by Proposition 2.8. Finally Proposition 2.7 implies the amenability of  $G_{e_0\mathbb{H}}$ .

## 2.4 Hyperbolic spaces

In this section we give an overview of the hyperbolic spaces  $H^n(\mathbb{F})$ . The following results are taken from [CG]. Recall that  $\mathbb{F}$  denotes  $\mathbb{R}$ ,  $\mathbb{C}$ , or  $\mathbb{H}$ .

Denote by  $\mathcal{V} := \mathbb{F}^{1,n}$  the (right-)vector space  $\mathbb{F}^{n+1}$  endowed with the bilinear form  $\langle \cdot, \cdot \rangle_1$ . Let  $U(1, n; \mathbb{F})$  be the group of unitary transformations, i.e. the group of all  $\mathbb{F}$ -linear automorphisms  $A : \mathcal{V} \rightarrow \mathcal{V}$  such that  $\langle Av, Aw \rangle_1 = \langle v, w \rangle_1$  for all  $v, w \in \mathcal{V}$ . Analogously,  $U(n; \mathbb{F})$  denotes the group of  $\mathbb{F}$ -linear automorphisms with respect to  $\langle \cdot, \cdot \rangle_0$ .

We define the subsets

$$\mathcal{V}_- := \{v \in \mathcal{V} \mid \langle v, v \rangle_1 < 0\}, \quad \mathcal{V}_+ := \{v \in \mathcal{V} \mid \langle v, v \rangle_1 > 0\}, \quad \mathcal{V}_0 = \{v \in \mathcal{V} \mid \langle v, v \rangle_1 = 0\}.$$

Notice that  $\mathcal{V}_-$ ,  $\mathcal{V}_+$  and  $\mathcal{V}_0$  are invariant under the action of  $U(1, n; \mathbb{F})$ .

Denote by  $\mathbb{F}P^n$  the projective space of  $\mathcal{V} = \mathbb{F}^{1,n}$ , i.e. the set of all  $\mathbb{F}$ -lines in  $\mathcal{V}$ , and let  $\pi : \mathcal{V} \setminus \{0\} \rightarrow \mathbb{F}P^n$ ,  $v \mapsto v \cdot \mathbb{F} =: [v]$  be the projection. Then we define the hyperbolic space by  $H^n(\mathbb{F}) := \pi(\mathcal{V}_-)$ . The boundary  $\partial H^n(\mathbb{F})$  of  $H^n(\mathbb{F})$  in  $\mathbb{F}P^n$  is given by  $\pi(\mathcal{V}_0)$ .

Since  $g(v \cdot \lambda) = g(v) \cdot \lambda$  for  $g \in U(1, n; \mathbb{F})$ ,  $v \in \mathcal{V}$ , and  $\lambda \in \mathbb{F}$ , the action of  $U(1, n; \mathbb{F})$  on  $\mathcal{V}$  induces an action on  $\mathbb{F}P^n$ . Furthermore,  $U(1, n; \mathbb{F})$  acts on  $H^n(\mathbb{F})$  and  $\partial H^n(\mathbb{F})$ , since the sets  $\mathcal{V}_-$  and  $\mathcal{V}_0$  are  $U(1, n; \mathbb{F})$ -invariant.

**Proposition 2.10** ([CG, Prop. 2.2.1]).  $U(1, n; \mathbb{F})$  acts transitively on  $H^n(\mathbb{F})$  and doubly transitively on  $\partial H^n(\mathbb{F})$ , i.e. for every  $p_1, p_2, q_1, q_2 \in \partial H^n(\mathbb{F})$  there exists an element  $g \in U(1, n; \mathbb{F})$  such that  $g(p_1) = p_2$  and  $g(q_1) = q_2$ .

Since  $U(1, n; \mathbb{F})$  acts transitively on the hyperbolic space, we can  $H^n(\mathbb{F})$  identify with the coset space  $U(1, n; \mathbb{F})/U(1; \mathbb{F}) \times U(n; \mathbb{F})$ . In particular  $H^n(\mathbb{F})$  is a symmetric space.

Now we will discuss the ball model of the hyperbolic space. If  $v = (v_0, \dots, v_n)^T \in \mathcal{V}_-$ , the condition

$$\langle v, v \rangle_1 = -|v_0|^2 + \sum_{i=1}^n |v_i|^2 < 0$$

implies  $v_0 \neq 0$ . Hence, we can define the inhomogeneous coordinates by

$$p = (v_1 v_0^{-1}, \dots, v_n v_0^{-1})^T \in \mathbb{F}^n.$$

By definition we have  $\langle p, p \rangle_0 = \sum_{i=1}^n |p_i|^2 < 1$ . In this way we can identify  $H^n(\mathbb{F})$  with the open ball  $B^n(\mathbb{F}) = \{p \in \mathbb{F}^n \mid \langle p, p \rangle_0 < 1\}$ . We get a map  $P : \mathcal{V}_- \rightarrow B^n(\mathbb{F})$  by  $P(v) := (v_1 v_0^{-1}, \dots, v_n v_0^{-1})^T$ . The boundary  $\partial H^n(\mathbb{F})$  in this model is given by the sphere  $S^n(\mathbb{F}) = \{p \in \mathbb{F}^n \mid \langle p, p \rangle_0 = 1\}$ .

Let  $\mathbb{F}' \subset \mathbb{F}$  be one of the subfields  $\mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$  and  $1 \leq m \leq n$ . We have a natural inclusion  $\mathbb{F}'^{1,m} \subset \mathbb{F}^{1,n}$ , namely  $(v_0, \dots, v_m)^T \mapsto (v_0, \dots, v_m, 0, \dots, 0)$ . This induces an inclusion  $H^m(\mathbb{F}') \subset H^n(\mathbb{F})$ .

From now on we will use the ball model of the hyperbolic space. Denote by  $\overline{H^n(\mathbb{F})} = H^n(\mathbb{F}) \cup \partial H^n(\mathbb{F})$ .

**Proposition 2.11** ([CG, Prop. 2.4.3]). (i)  $H^1(\mathbb{R})$  is a geodesic in  $H^n(\mathbb{F})$ . Every geodesic is equivalent under  $U(1, n; \mathbb{F})$  to  $H^1(\mathbb{R})$ .

(ii) The geodesics at 0 are precisely the  $\mathbb{R}$ -lines through 0. These are all equivalent under the isotropy group  $U(1; \mathbb{F}) \times U(n; \mathbb{F})$ .

(iii) Let  $p, q \in \overline{H^n(\mathbb{F})}$ . Then there is a unique geodesic which connects  $p$  to  $q$ .

Since  $H^n(\mathbb{F})$  is a homogeneous space, every totally geodesic submanifold is equivalent under  $U(1, n; \mathbb{F})$  to a totally geodesic submanifold that contains 0. Proposition 2.11 (ii) implies that the totally geodesic submanifolds at 0 are intersections of real subspaces of  $\mathbb{F}^n$  with  $H^n(\mathbb{F}) = B^n(\mathbb{F})$ .

**Proposition 2.12** ([CG, Prop. 2.5.1]). Any totally geodesic submanifold of  $H^n(\mathbb{F})$  is equivalent under  $U(1, n; \mathbb{F})$  to one of the following:

(i)  $H^m(\mathbb{F}')$ , where  $\mathbb{F}' \subseteq \mathbb{F}$ ,  $\mathbb{F}' = \mathbb{R}, \mathbb{C}$ , or  $\mathbb{H}$ , and  $1 \leq m \leq n$ ;

(ii)  $H^1(\mathbb{I}) := e_1 \mathbb{I} \cap B^n(\mathbb{H})$ , where  $e_1 = (1, 0, \dots, 0)^T \in \mathbb{H}^n$  and  $\mathbb{I} = \text{span}_{\mathbb{R}} \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ . It occurs only if  $\mathbb{F} = \mathbb{H}$ .

These are all non-equivalent under  $U(1, n; \mathbb{F})$ .

Let  $X$  be a subset of  $\mathcal{V}$ . The span of  $X$ , denoted by  $\langle X \rangle$ , is the smallest  $\mathbb{F}$ -subspace containing  $X$ . If  $Y$  is a subset of  $\overline{H^n(\mathbb{F})}$  (as a subset of  $\mathbb{F}P^n$ ), the span  $\langle Y \rangle$  is defined by  $\langle Y \rangle = \pi(\langle \pi^{-1}(Y) \rangle \cap \mathcal{V}_-)$ .

**Lemma 2.7** ([CG, Lemma 3.1.1]). *Let  $Y$  be a subset of  $\partial H^n(\mathbb{F})$  which contains at least three elements and is pointwise fixed by some  $g \in \mathrm{U}(1, n; \mathbb{F})$ . Then there is a totally geodesic submanifold  $M \subset \langle Y \rangle$  of  $H^n(\mathbb{F})$  such that  $\langle M \rangle = \langle Y \rangle$  and  $M$  is pointwise fixed by  $g$ .*

The group  $\mathrm{U}(1, n; \mathbb{F})$  acts on  $\overline{H^n(\mathbb{F})}$ . Since  $\overline{H^n(\mathbb{F})}$  is a closed ball and  $g \in \mathrm{U}(1, n; \mathbb{F})$  is continuous,  $g$  has a fixed point in  $\overline{H^n(\mathbb{F})}$  by Brouwer's fixed point theorem.

**Definition 2.25.** *An element  $g \in \mathrm{U}(1, n; \mathbb{F})$  is called*

- *elliptic* if it has a fixed point in  $H^n(\mathbb{F})$ ,
- *parabolic* if it has exactly one fixed point in  $\overline{H^n(\mathbb{F})}$  and this lies on  $\partial H^n(\mathbb{F})$ ,
- *loxodromic* if it has exactly two fixed points in  $\overline{H^n(\mathbb{F})}$  and these belong to  $\partial H^n(\mathbb{F})$ ,
- *hyperbolic* if it is loxodromic and conjugated to an element different from  $\mathbb{1}_{n+1}$  of  $\mathrm{SO}^0(1, 1) \times \{\mathbb{1}_{n-1}\} \subset \mathrm{U}(1, n; \mathbb{F})$ .

Lemma 2.7 implies that an element  $g \in \mathrm{U}(1, n; \mathbb{F})$  with more than two fixed points in  $\partial H^n(\mathbb{F})$  fixes pointwise a totally geodesic submanifold of  $H^n(\mathbb{F})$ , i.e.  $g$  is elliptic. Hence, Definition 2.25 covers all possibilities. If two elements of  $\mathrm{U}(1, n; \mathbb{F})$  are conjugated, then they have the same (elliptic, parabolic, loxodromic) type.

**Remark 2.9.** *Notice that an element  $f \in \mathrm{U}(1, n; \mathbb{F})$  is elliptic if and only if  $f$  generates a precompact subgroup.*

Next we present some further properties of elliptic elements.

**Lemma 2.8** ([CG, Lemma 3.3.2]). *Let  $g \in \mathrm{U}(1, n; \mathbb{F})$  be elliptic, and let  $p, q \in \partial H^n(\mathbb{F})$ . If  $g$  fixes  $p$  and  $q$ , then it fixes every point on the unique geodesic connecting  $p$  and  $q$ .*

**Definition 2.26.** *Let  $g \in \mathrm{U}(1, n; \mathbb{F})$  be elliptic. An eigenvalue  $\lambda$  of  $g$  is called*

- (i) *of positive type* if there exists an eigenvector of  $\lambda$  in  $\mathcal{V}_-$ ,
- (ii) *of negative type* if there exists an eigenvector of  $\lambda$  in  $\mathcal{V}_+$ .

If  $\mathbb{F} = \mathbb{H}$ ,  $g \in \mathrm{U}(1, n; \mathbb{H}) = \mathrm{Sp}(1, n)$  and  $v \in \mathcal{V}$ ,  $\lambda \in \mathbb{H}$ , such that  $g(v) = v\lambda$ , then  $A(v\mu) = (v\mu)\mu^{-1}\lambda\mu$  for all  $\mu \in \mathbb{H} \setminus \{0\}$ , i.e.  $v\mu$  is an eigenvector of  $g$  with eigenvalue  $\mu^{-1}\lambda\mu$ . Thus the eigenvalues of  $g$  occur in similarity classes.

**Lemma 2.9** ([CG, Lemma 3.2.1]). *Let  $g \in \mathrm{U}(1, n; \mathbb{F})$  be elliptic. Every eigenvalue of  $g$  has positive or negative type. The eigenvalues fall into  $n$  similarity classes of positive type (which may not be different) and one similarity class of negative type (which may coincide with one of the positive classes).*

**Proposition 2.13** ([CG, Prop. 3.2.2]). *Let  $g \in \mathrm{U}(1, n; \mathbb{F})$  be elliptic, let  $\Lambda_0$  be its negative class of eigenvalues, and let  $\Lambda_1, \dots, \Lambda_n$  be its positive classes. Let  $F(g)$  denote the set of fixed points of  $g$  in  $H^n(\mathbb{F})$ .*

- (i) *If  $\Lambda_0 \neq \Lambda_i$  for all  $1 \leq i \leq n$ , then  $F(g)$  contains only one point.*
- (ii) *Suppose that  $\Lambda_0$  coincides with exactly  $m$  of the classes  $\Lambda_i$ ,  $1 \leq i \leq n$ . Then  $F(g)$  is a totally geodesic submanifold, which is equivalent to  $H^m(\mathbb{F})$  if  $\Lambda_0 \subset \mathbb{R}$ , and to  $H^m(\mathbb{C})$  if  $\Lambda_0 \not\subset \mathbb{R}$ .*

**Remark 2.10.** *One has to pay attention with the notation. The authors of [CG] denote by  $\mathbb{C}$  a subfield of  $\mathbb{F}$  which contains  $\mathbb{R}$  and is isomorphic to the field of complex numbers. Hence, in Proposition 2.13,  $\mathbb{C}$  could be for example  $\mathrm{span}_{\mathbb{R}}\{1, \mathbf{j}\}$ .*

Let  $M \subset H^n(\mathbb{F})$  be a totally geodesic submanifold and denote by  $I(M)$  the subgroup of  $\mathrm{U}(1, n; \mathbb{F})$  which leaves  $M$  invariant.

**Lemma 2.10** ([CG, Lemma 4.2.1]). *Let  $M = H^m(\mathbb{F}')$  be a totally geodesic submanifold of  $H^n(\mathbb{F})$ . Then the elements  $g \in I(M)$  are of the form*

$$g = \begin{pmatrix} A\lambda & 0 \\ 0 & B \end{pmatrix},$$

where  $A \in \mathrm{U}(1, m; \mathbb{F}')$ ,  $B \in \mathrm{U}(n-m; \mathbb{F})$ , and  $\lambda \in N^+(\mathbb{F}', \mathbb{F}) := \{\lambda \in \mathrm{U}(1; \mathbb{F}) \mid \lambda \mathbb{F}' \lambda^{-1} = \mathbb{F}'\}$ .

**Lemma 2.11** ([CG, Lemma 4.2.2]). *Let  $M = H^1(\mathbb{I}) \subset H^n(\mathbb{H})$ . Then the elements of  $g \in I(M)$  are of the form*

$$g = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}, \quad \text{where } A = \begin{pmatrix} a & -b \\ \varepsilon b & \varepsilon a \end{pmatrix} \in \mathrm{Sp}(1, 1),$$

$\varepsilon = \pm 1$ , and  $B \in \mathrm{Sp}(n-1)$ .

Let  $M$  be a totally geodesic submanifold of  $H^n(\mathbb{F})$  and let  $K(M)$  be the subgroup of  $I(M)$  which leaves  $M$  pointwise fixed. Then  $K(M)$  is a compact normal subgroup of  $I(M)$ . Let  $I^0(M)$  and  $K^0(M)$  be the connected components of the identity of  $I(M)$  and  $K(M)$ , respectively. We have the following Proposition.

**Proposition 2.14** ([CG, Proposition 4.2.1]). *Let  $M$  be a totally geodesic submanifold in  $H^n(\mathbb{F})$  and let  $I(M)$  be the stabilizer of  $M$  in  $\mathrm{U}(1, n; \mathbb{F})$ . Let  $K(M)$  be the subgroup of  $I(M)$  which leaves  $M$  pointwise fixed. Then there exists a Lie subgroup  $U(M) \subset I(M)$  such that  $I(M) = K(M)U(M)$  (almost semidirect product). The identity component  $U^0(M)$  is a simple Lie group when  $\dim M > 1$ , and  $I^0(M) = K^0(M)U^0(M)$  is an almost direct product.  $U^0(M)$  induces the connected isometry group of  $M$ .*

The following table covers all possibilities of  $I(M)$  for a totally geodesic submanifold  $M \subset H^n(\mathbb{F})$ .

Table 2.1: Decomposition of  $I(M)$ 

	$I(M) = K(M)U(M)$
$M = H^m(\mathbb{H}) \subset H^n(\mathbb{H})$	$K(M) = \{\pm \mathbb{1}_{m+1}\} \times \mathrm{Sp}(n-m), U(M) = \mathrm{Sp}(1, m) \times \{\mathbb{1}_{n-m}\},$ $K^0(M) = \{\mathbb{1}_{m+1}\} \times \mathrm{Sp}(n-m);$
$M = H^m(\mathbb{C}) \subset H^n(\mathbb{H})$	$K(M) = \mathrm{U}(1) \cdot \mathbb{1}_{m+1} \times \mathrm{Sp}(n-m), U(M) = \mathrm{SU}(1, m) \cdot \{\pm 1, \pm \mathbf{j}\} \times \{\mathbb{1}_{n-m}\},$ $U^0(M) = \mathrm{SU}(1, m) \times \{\mathbb{1}_{n-m}\};$
$M = H^m(\mathbb{R}) \subset H^n(\mathbb{H})$	$K(M) = \mathrm{Sp}(1) \cdot \mathbb{1}_{m+1} \times \mathrm{Sp}(n-m), U(M) = \mathrm{O}(1, m) \times \{\mathbb{1}_{n-m}\},$ $U^0(M) = \mathrm{SO}^0(1, m) \times \{\mathbb{1}_{n-m}\};$
$M = H^1(\mathbb{I}) \subset H^n(\mathbb{H})$	$K(M) = \{\pm \mathbb{1}_2\} \times \mathrm{Sp}(n-1), U(M) = U \times \{\mathbb{1}_{n-1}\},$ $K^0(M) = \{\mathbb{1}_2\} \times \mathrm{Sp}(n-1), U^0(M) = U^0 \times \{\mathbb{1}_{n-1}\}.$
$M = H^m(\mathbb{C}) \subset H^n(\mathbb{C})$	$K(M) = \mathrm{U}(1) \cdot \mathbb{1}_{m+1} \times \mathrm{U}(n-m), U(M) = \mathrm{SU}(1, m) \times \{\mathbb{1}_{n-m}\};$
$M = H^m(\mathbb{R}) \subset H^n(\mathbb{C})$	$K(M) = \mathrm{U}(1) \mathbb{1}_{m+1} \times \mathrm{U}(n-m), U(M) = \mathrm{O}(1, m) \times \{\mathbb{1}_{n-m}\},$ $U^0(M) = \mathrm{SO}^0(1, m) \times \{\mathbb{1}_{n-m}\};$
$M = H^m(\mathbb{R}) \subset H^n(\mathbb{R})$	$K(M) = \{\pm \mathbb{1}_{m+1}\} \times \mathrm{O}(n-m), U(M) = \mathrm{O}(1, m) \times \{\mathbb{1}_{n-m}\},$ $K^0(M) = \{\mathbb{1}_{m+1}\} \times \mathrm{O}(n-m), U^0(M) = \mathrm{SO}^0(1, m) \times \{\mathbb{1}_{n-m}\};$

In the case  $M = H^1(\mathbb{I})$  the Lie group  $U \subset \mathrm{Sp}(1, 1)$  is given by

$$U = \left\{ A \in \mathrm{Sp}(1, 1) \left| A = \begin{pmatrix} a & -b \\ \varepsilon b & \varepsilon a \end{pmatrix}, \varepsilon = \pm 1 \right. \right\}.$$

We will now investigate the identity component  $U^0$  in more detail. We see easily that  $U^0$  is contained in the subgroup of  $U$  consisting of the matrices with  $\varepsilon = 1$ . We denote this Lie group by  $S$ . Next we show that  $U^0 = S$ . It is sufficient to show that  $S$  is connected. The elements of  $S$  are precisely the elements of  $\mathrm{Sp}(1, 1)$  which commute with

$$\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence,  $S$  is an algebraic group. It is known that algebraic groups have finitely many connected components, see [Mi]. Recall that a Lie group with finitely many connected components is connected if and only if a maximal compact subgroup is connected. This follows for instance from [HN, Theorem 14.3.11]. So we have to check that a maximal compact subgroup of  $S$  is connected. A maximal compact subgroup of  $S$  is contained in a maximal compact subgroup of  $\mathrm{Sp}(1, 1)$ , which is conjugated to  $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ . Some computations show that the Lie algebra  $\mathfrak{s}$  is compatible with a Cartan decomposition of  $\mathfrak{sp}(1, 1)$ . Hence, a maximal compact subgroup of  $S$  is  $\{A \in \mathrm{Sp}(1, 1) | q \cdot \mathbb{1}_2, q \in \mathrm{Sp}(1)\} \cong \mathrm{Sp}(1)$ . Since  $\mathrm{Sp}(1)$  is connected, the Lie group  $S$  is connected and we conclude  $U^0 = S$ . Furthermore, since  $\mathrm{Sp}(1)$  is simply connected,  $U^0$  is simply connected. Its Lie algebra is

$$\mathfrak{u} = \{X \in \mathfrak{sp}(1, 1) | X\Phi = \Phi X\}.$$

A basis of  $\mathfrak{u}$  is for example

$$x = \frac{\mathbf{i}}{2}\Phi, \quad y = \frac{\mathbf{j}}{2}\Phi, \quad z = \frac{\mathbf{k}}{2}\Phi,$$

$$u = \frac{\mathbf{i}}{2}\mathbb{1}_2, \quad v = \frac{\mathbf{j}}{2}\mathbb{1}_2, \quad w = \frac{\mathbf{k}}{2}\mathbb{1}_2.$$

One can show that this Lie algebra is isomorphic to  $\mathfrak{so}(1, 3)$ , see Lemma A.1 for a proof. This implies  $U^0 \cong \text{Spin}^0(1, 3)$ .

**Lemma 2.12** ([CG, Lemma 4.3.1]). *Let  $p \in H^n(\mathbb{F})$  and let  $(g_n)_n$  be a sequence in  $U(1, n; \mathbb{F})$  such that  $\lim_{n \rightarrow \infty} g_n(p) = q \in \partial H^n(\mathbb{F})$ . Then  $\lim_{n \rightarrow \infty} g_n(p') = q$  for all  $p' \in H^n(\mathbb{F})$ .*

**Definition 2.27.** *Let  $G$  be a subgroup of  $U(1, n; \mathbb{F})$  and let  $p \in H^n(\mathbb{F})$ . The **limit set** of  $G$  is defined to be the set  $\mathcal{L}(G) := \overline{G \cdot p} \cap \partial H^n(\mathbb{F})$ . Lemma 2.12 implies that  $\mathcal{L}(G)$  does not depend on  $p$ .*

**Lemma 2.13** ([CG, Lemma 4.3.2]). *Let  $G$  be a subgroup of  $U(1, n; \mathbb{F})$ .*

- (a)  $\mathcal{L}(G)$  is invariant under  $G$ .
- (b) If  $G'$  is a subgroup of  $G$ , then  $\mathcal{L}(G') \subset \mathcal{L}(G)$ .
- (c) If  $G'$  is a subgroup of finite index,  $\mathcal{L}(G') = \mathcal{L}(G)$ .
- (d) If  $\overline{G}$  is the closure of  $G$  in  $U(1, n; \mathbb{F})$ , then  $\mathcal{L}(\overline{G}) = \mathcal{L}(G)$ .

**Lemma 2.14** ([CG, Lemma 4.3.4]). *Let  $N$  be a normal subgroup of  $G \subset U(1, n; \mathbb{F})$ . Then  $G$  leaves  $\mathcal{L}(N)$  invariant. Furthermore if  $\mathcal{L}(N) \neq \emptyset$  and the elements of  $G$  do not have a common fixed point in  $\partial H^n(\mathbb{F})$ , then  $\mathcal{L}(N) = \mathcal{L}(G)$ .*

**Theorem 2.7** ([CG, Theorem 4.4.1]). *Let  $G$  be a connected Lie subgroup of  $U(1, n; \mathbb{F})$ . Then one of the following is true.*

- (a) *The elements of  $G$  have a common fixed point in  $\overline{H^n(\mathbb{F})}$ .*
- (b) *There is a proper, totally geodesic submanifold  $M$  in  $H^n(\mathbb{F})$  such that  $\dim M > 1$ ,  $\mathcal{L}(G) = \partial M = \overline{M} \cap \partial H^n(\mathbb{F})$ , and  $G = K \cdot U^0(M)$ , where  $K \subset K^0(M)$  is a connected Lie subgroup.*
- (c)  $\mathbb{F} = \mathbb{C}$  and  $G = \text{SU}(1, n)$ .
- (d)  $G = U^0(1, n; \mathbb{F})$ .

**Remark 2.11.** *The condition  $\dim M > 1$  in case (b) is due to the following fact. If  $G$  preserves a proper totally geodesic submanifold  $M \subset H^n(\mathbb{F})$  such that  $\dim M = 1$  and  $\mathcal{L}(G) = \partial M$ , then  $\partial M$  consists of exactly two points. By Lemma 2.13,  $G$  preserves  $\partial M$ . Furthermore since  $G$  is connected, it fixes both points inside of it. Thus case (a) holds.*

**Theorem 2.8** ([CG, Theorem 4.4.2]). *Let  $G$  be a closed subgroup (not necessarily connected) of  $U(1, n; \mathbb{F})$ . Then one of the following is true.*

- (a)  $G$  is discrete.
- (b) *The elements of  $G$  have a common fixed point in  $\overline{H^n(\mathbb{F})}$ .*
- (c)  $G$  leaves invariant a proper, totally geodesic submanifold.

(d)  $\mathbb{F} = \mathbb{C}$  and  $\mathrm{SU}(1, n) \subset G$ .

(e)  $\mathrm{U}^0(1, n; \mathbb{F}) \subset G$ .

**Remark 2.12.** We will point out some details of the Theorems 2.7 and 2.8, since they are needed later. The proof of Theorem 2.8 makes use of Theorem 2.7. If  $G$  is not discrete,  $G^0$  has no common fixed point in  $\overline{H^n(\mathbb{F})}$ , and (b) of Theorem 2.7 holds for  $G^0$ , then Lemma 2.14 implies that  $G$  preserves  $\partial M = \mathcal{L}(G^0) = \mathcal{L}(G)$ . Notice that  $M$  is the union of all geodesics whose endpoints lie in  $\partial M$ . Since the elements of  $G$  map geodesics to geodesics and the geodesics are uniquely determined by their endpoints, see Proposition 2.11, it follows that  $G$  preserves  $M$ . This argument will be used later in the proof of Proposition 3.4. Furthermore, if (c) of Theorem 2.8 holds for  $G$  and the totally geodesic submanifold  $M$  preserved by  $G$  has dimension greater than one, then (b) of Theorem 2.7 holds for  $G^0$ .

Notice that in case (c) of Theorem 2.8 the totally geodesic submanifold  $M$  is allowed to be one dimensional. If we are additionally not in case (b), then  $G$  can interchange the two endpoints of  $M$ , since  $G$  is not assumed to be connected. In that case  $G^0$  fixes both points in  $\partial M$ , so (a) of Theorem 2.7 holds for  $G^0$ . Furthermore, notice that  $\partial M = \mathcal{L}(G^0)$ .

## 2.5 The Zariski topology

In this section we will recall the definitions of algebraic varieties and the Zariski topology. The reader can find more details in [OV1].

In the following  $\mathbb{F}$  will denote  $\mathbb{C}$  or  $\mathbb{R}$ . Let  $\mathbb{F}[X_1, \dots, X_n]$  be the polynomial algebra of  $X_1, \dots, X_n$  with coefficients in  $\mathbb{F}$ .

**Definition 2.28.** An **algebraic variety**  $V$  in  $\mathbb{F}^n$  is a subset of  $\mathbb{F}^n$  consisting of the common zeros of some subset  $S \subset \mathbb{F}[X_1, \dots, X_n]$ , i.e.

$$V = \{x = (x_1, \dots, x_n)^T \in \mathbb{F}^n \mid f(x_1, \dots, x_n) = 0 \text{ for all } f \in S\}.$$

**Remark 2.13.** Notice that algebraic varieties are closed with respect to the standard topology of  $\mathbb{F}^n$ , because they can be written as intersections of zeros of polynomials, which are closed sets.

**Example 2.3.** The known matrix Lie groups like  $\mathrm{SO}(p, q)$ ,  $\mathrm{SU}(p, q)$ ,  $\mathrm{Sp}(p, q)$  ect. are examples for algebraic varieties. For instance  $\mathrm{Sp}(p, q)$  consists of the matrices  $A \in \mathrm{GL}(p+q, \mathbb{H})$  which satisfy  $\overline{A}^T \mathbb{1}_{p,q} A = \mathbb{1}_{p,q}$ , i.e. the elements of  $\mathrm{Sp}(p, q)$  are the common zeros of  $(p+q)^2$  different polynomials in  $4(p+q)$  variables.

**Proposition 2.15** ([OV1, Chapter 2]). Any algebraic variety can be determined by a finite system of equations.

**Definition 2.29.** The topology on  $\mathbb{F}^n$  where the closed sets are precisely the algebraic varieties is called the **Zariski topology**. If  $M \subset \mathbb{F}^n$ , then  $M^{\mathrm{Zar}}$  denotes the closure of  $M$  with respect to the Zariski topology.

**Remark 2.14.** Let  $M \subset \mathbb{F}^n$  and  $S = \{f \in \mathbb{F}[X_1, \dots, X_n] \mid f(x) = 0 \text{ for all } x \in M\}$ . Then the Zariski closure  $M^{Zar}$  is given by

$$M^{Zar} = \{x \in \mathbb{F}^n \mid f(x) = 0 \text{ for all } f \in S\}.$$

**Lemma 2.15.** Let  $(M, g)$  be an  $n$ -dimensional  $G$ -homogeneous (pseudo-)Riemannian manifold with  $G \subset \text{Iso}(M, g)$ . Then the Zariski closure of the linear isotropy group  $H$  of a point  $p \in M$  preserves the (pseudo-)Riemannian metric  $g$ .

*Proof:* Let  $(q, n - q)$  be the signature of  $g$ ,  $p \in M$  and  $H := G_p$  the isotropy group. We consider  $H$  as a subgroup of  $O(q, n - q)$  and identify  $T_p M$  with  $\mathbb{R}^{q, n-q}$ .

Let  $S := \{f \in \mathbb{F}[X_1, \dots, X_n] \mid f(x) = 0 \text{ for all } x \in H\}$ . Then the Zariski closure  $H^{Zar}$  is given by

$$H^{Zar} = \{X \in M(n, \mathbb{R}) \mid f(X) = 0 \text{ for all } f \in S\}.$$

We denote by  $e_1, \dots, e_n$  the standard basis of  $\mathbb{R}^{q, n-q}$  and consider the polynomials

$$f_{i,j}(X) = g_p(Xe_i, Xe_j) - g_p(e_i, e_j)$$

with  $X \in M(n, \mathbb{R})$ . Since  $H$  is a subgroup of  $\text{Iso}(M, g)$ , we have  $f_{i,j} \in S$  for all  $1 \leq i, j \leq n$ . Hence,  $f_{i,j}(X) = 0$  for all  $X \in H^{Zar}$  and  $1 \leq i, j \leq n$ .  $\square$



## Chapter 3

# Algebraic results

In this chapter we will prove the algebraic results which will be needed for the proofs of the Theorems 4.1 and 4.2 in Chapter 4.

In Section 3.1 we start with studying real-valued 3- and 5-forms on  $\mathbb{H}^{n+1}$  which are  $\mathrm{SO}^0(1, n)$ -invariant. We present here elementary proofs without using any representation theory besides of the concept of irreducibility.

In Section 3.2 we will classify all connected  $\mathbb{H}$ -irreducible Lie subgroups of  $\mathrm{Sp}(1, n)$ .

### 3.1 $\mathrm{SO}^0(1, n)$ -invariant forms

**Lemma 3.1.** *Let  $\mathbb{F}$  denote  $\mathbb{R}$  or  $\mathbb{C}$  and let  $\alpha \in \Lambda^k(\mathbb{F}^{1, n})^*$  be  $\mathrm{SO}^0(1, n)$ -invariant. If  $n \geq k \geq 1$ , then  $\alpha = 0$ .*

*Proof:* We prove the Lemma by induction over  $k$ . Let  $k = 1, n \geq 1$  and  $\alpha : \mathbb{F}^{1, n} \rightarrow \mathbb{F}$  be an  $\mathbb{F}$ -linear and  $\mathrm{SO}^0(1, n)$ -invariant map. The kernel  $\ker \alpha \subseteq \mathbb{F}^{1, n}$  is an  $\mathrm{SO}^0(1, n)$ -invariant  $\mathbb{F}$ -subspace. Since  $\mathrm{SO}^0(1, n)$  acts  $\mathbb{F}$ -irreducibly on  $\mathbb{F}^{1, n}$ , we have  $\ker \alpha = \{0\}$  or  $\mathbb{F}^{1, n}$ . But  $\alpha$  can not be injective because  $n \geq 1$ . Hence,  $\ker \alpha = \mathbb{F}^{1, n}$ , i.e.  $\alpha = 0$ .

Assume now that the claim holds for some  $k \in \mathbb{N}$ . Let  $n \geq k + 1$  and  $\alpha \in \Lambda^{k+1}(\mathbb{F}^{1, n})^*$  be  $\mathrm{SO}^0(1, n)$ -invariant. Let  $x \in \mathbb{F}^{1, n}$  be spacelike, i.e.  $\langle x, x \rangle_1 > 0$ . We consider the decomposition  $\mathbb{F}^{1, n} = \mathbb{F}x \oplus (\mathbb{F}x)^\perp$ .

The form  $\iota_x \alpha$  vanishes if we insert an element of  $\mathbb{F}x$ , so we restrict  $\iota_x \alpha$  to  $(\mathbb{F}x)^\perp$ . Notice that  $\iota_x \alpha$  is  $\mathrm{SO}^0(1, n)_x$ -invariant. The action of  $\mathrm{SO}^0(1, n)_x$  on  $(\mathbb{F}x)^\perp$  is equivalent to the action of  $\mathrm{SO}^0(1, n-1)$  on  $\mathbb{F}^{1, n-1}$ . Since  $n-1 \geq k$ , we know by induction hypothesis that  $\iota_x \alpha|_{(\mathbb{F}x)^\perp} = 0$ . Hence,  $x \in \ker \alpha$ . Since  $\mathbb{F}^{1, n}$  is generated by spacelike vectors, we conclude  $\alpha = 0$ . This finishes the proof.  $\square$

**Lemma 3.2** ([AZ, Fact 2.1]). *Let  $n \geq 2$  and  $\alpha \in \Lambda^2 \mathcal{V}^*$ , where  $\mathcal{V}$  denotes  $\mathbb{C}^{1, n}$  considered as real vector space. If  $\alpha$  is  $\mathrm{SO}^0(1, n)$ -invariant, then  $\alpha \in \mathbb{R} \cdot \omega_1$ , where  $\omega_1$  is the Kähler form.*

**Remark 3.1.** *The proof of Lemma 3.2 works also for  $\mathrm{SO}(n)$ -invariant forms.*

**Lemma 3.3.** *Let  $n \geq 3$  and  $\alpha \in \Lambda^2 \mathcal{V}^*$ , where  $\mathcal{V}$  denotes  $\mathbb{C}^n$  considered as real vector spaces. If  $\alpha$  is  $\mathrm{SO}(n)$ -invariant, then  $\alpha \in \mathbb{R} \cdot \omega_0$ , where  $\omega_0$  is the Kähler form.*

**Lemma 3.4.** *Let  $n \geq 3$  and  $\alpha \in \mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}^*$ , with  $\mathcal{V} = \mathbb{C}^{1,n}$  considered as a real vector space. If  $\alpha$  is  $\mathrm{SO}^0(1, n)$ -invariant, then  $\alpha = 0$ .*

*Proof:* Let  $x \in \mathbb{R}^{1,n}$  be spacelike, i.e.  $\langle x, x \rangle_1 > 0$ . The proof consists of three Claims.

*Claim 1:*  $\alpha(\varepsilon_1 x, \varepsilon_2 x, \cdot)|_{(\mathbb{C}x)^\perp} = 0$ ,  $\alpha(\varepsilon_1 x, \cdot, \varepsilon_2 x)|_{(\mathbb{C}x)^\perp} = 0$  and  $\alpha(\cdot, \varepsilon_1 x, \varepsilon_2 x)|_{(\mathbb{C}x)^\perp} = 0$  for  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$ .

*Claim 2:*  $\alpha(\varepsilon_1 x, \varepsilon_2 x, \varepsilon_3 x) = 0$  for  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, i\}$ .

*Claim 3:*  $\alpha(\varepsilon x, \cdot, \cdot)|_{(\mathbb{C}x)^\perp} = 0$  for  $\varepsilon \in \{1, i\}$ .

The Claims imply that  $\alpha(u, \cdot, \cdot) = 0$  for all  $u \in \mathbb{C}x$ . Since the spacelike vectors of  $\mathbb{R}^{1,n}$  generate  $\mathbb{C}^{1,n}$ , this implies the Lemma. Thus we just have to prove the Claims.

*Proof of Claim 1:* The action of  $\mathrm{SO}^0(1, n)_x$  on  $(\mathbb{C}x)^\perp$  is equivalent to the action of  $\mathrm{SO}^0(1, n-1)$  on  $\mathbb{C}^{1, n-1} \cong \mathbb{R}^{1, n-1} \oplus i\mathbb{R}^{1, n-1}$ . Since  $n-1 \geq 2$ ,  $\mathrm{SO}^0(1, n)_x \cong \mathrm{SO}^0(1, n-1)$  acts  $\mathbb{R}$ -irreducibly on  $\mathbb{R}^n$  and  $i\mathbb{R}^n$ . First, we consider the  $\mathbb{R}$ -linear map

$$\varphi := \alpha(\varepsilon_1 x, \varepsilon_2 x, \cdot)|_{(\mathbb{C}x)^\perp} : (\mathbb{C}x)^\perp \cong \mathbb{C}^{1, n} = \mathbb{R}^{1, n} \oplus i\mathbb{R}^{1, n} \rightarrow \mathbb{R},$$

with  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$ . Notice that  $\varphi$  is  $\mathrm{SO}^0(1, n)_x$ -invariant. The same holds for  $\varphi_1 := \varphi|_{\mathbb{R}^{1, n-1}}$  and  $\varphi_2 := \varphi|_{i\mathbb{R}^{1, n-1}}$ .

The  $\mathrm{SO}^0(1, n)_x$ -invariance of  $\varphi_1$  and  $\varphi_2$  implies that the real subspaces  $\ker \varphi_1 \subset \mathbb{R}^{1, n-1}$  and  $\ker \varphi_2 \subset i\mathbb{R}^{1, n-1}$  are invariant under the action of  $\mathrm{SO}^0(1, n)_x$ . By the  $\mathbb{R}$ -irreducibility of  $\mathrm{SO}^0(1, n)_x$  the only possibilities for  $\ker \varphi_1$  are  $\{0\}$  and  $\mathbb{R}^{1, n-1}$ . Analogously, we have  $\ker \varphi_2 = \{0\}$  or  $\ker \varphi_2 = i\mathbb{R}^{1, n-1}$ . Since  $n \geq 3$ ,  $\varphi_1$  and  $\varphi_2$  are not injective, so we have  $\ker \varphi_1 = \mathbb{R}^{1, n-1}$  and  $\ker \varphi_2 = i\mathbb{R}^{1, n-1}$ . This implies  $\varphi = 0$ . A similar computation shows  $\alpha(\cdot, \varepsilon_1 x, \varepsilon_2 x)|_{(\mathbb{C}x)^\perp} = 0$  and  $\alpha(\varepsilon_1 x, \cdot, \varepsilon_2 x)|_{(\mathbb{C}x)^\perp} = 0$ .

*Proof of Claim 2:* Let  $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \{1, i\}$ . Let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}^{1, n}$  be a smooth curve, such that  $\gamma(t)$  is spacelike for all  $t \in \mathbb{R}$ , and denote by  $r(t) := \sqrt{\langle \gamma(t), \gamma(t) \rangle_1}$  the length function. Now we consider the equation

$$\alpha(\varepsilon_1 \gamma(t), \varepsilon_2 \gamma(t), \varepsilon_3 \gamma(t)) = r(t)^3 \alpha \left( \varepsilon_1 \frac{\gamma(t)}{r(t)}, \varepsilon_2 \frac{\gamma(t)}{r(t)}, \varepsilon_3 \frac{\gamma(t)}{r(t)} \right).$$

Since  $\frac{\gamma(t)}{r(t)}$  lies in  $\{p \in \mathbb{R}^{1, n} \mid \langle p, p \rangle_1 = 1\} = S_1^n$  and  $\mathrm{SO}^0(1, n)$  acts transitively on  $S_1^n$ , there exists for every  $t \in \mathbb{R}$  an element  $A(t) \in \mathrm{SO}^0(1, n)$  such that  $A(t) \frac{\gamma(t)}{r(t)} = \frac{x}{l}$ , with  $l := \sqrt{\langle x, x \rangle_1}$ . The  $\mathrm{SO}^0(1, n)$ -invariance of  $\alpha$  implies that

$$\alpha(\varepsilon_1 \gamma(t), \varepsilon_2 \gamma(t), \varepsilon_3 \gamma(t)) = \frac{r(t)^3}{l^3} \alpha(\varepsilon_1 x, \varepsilon_2 x, \varepsilon_3 x). \quad (3.1)$$

Differentiating equation (3.1) yields

$$\begin{aligned} \frac{3r(t)^2 r'(t)}{l^3} \alpha(\varepsilon_1 x, \varepsilon_2 x, \varepsilon_3 x) &= \alpha(\varepsilon_1 \gamma'(t), \varepsilon_2 \gamma(t), \varepsilon_3 \gamma(t)) + \alpha(\varepsilon_1 \gamma(t), \varepsilon_2 \gamma'(t), \varepsilon_3 \gamma(t)) \\ &\quad + \alpha(\varepsilon_1 \gamma(t), \varepsilon_2 \gamma(t), \varepsilon_3 \gamma'(t)). \end{aligned}$$

Taking a second derivative and evaluating at  $t = 0$  gives

$$\begin{aligned} \frac{6r(0)r'(0)^2 + 3r(0)^2r''(0)}{I^3} \alpha(\varepsilon_1 x, \varepsilon_2 x, \varepsilon_3 x) &= \alpha(\varepsilon_1 \gamma''(0), \varepsilon_2 \gamma(0), \varepsilon_3 \gamma(0)) + \alpha(\varepsilon_1 \gamma(0), \varepsilon_2 \gamma''(0), \varepsilon_3 \gamma(0)) \\ &\quad + \alpha(\varepsilon_1 \gamma(0), \varepsilon_2 \gamma(0), \varepsilon_3 \gamma''(0)) + 2\alpha(\varepsilon_1 \gamma'(0), \varepsilon_2 \gamma'(0), \varepsilon_3 \gamma(0)) \\ &\quad + 2\alpha(\varepsilon_1 \gamma'(0), \varepsilon_2 \gamma(0), \varepsilon_3 \gamma'(0)) + 2\alpha(\varepsilon_1 \gamma(0), \varepsilon_2 \gamma'(0), \varepsilon_3 \gamma'(0)) \end{aligned} \quad (3.2)$$

All terms on the right hand side of (3.2) are of the form  $\alpha(\varepsilon_1 v, \varepsilon_2 w, \varepsilon_3 w)$ ,  $\alpha(\varepsilon_1 v, \varepsilon_2 w, \varepsilon_3 v)$  or  $\alpha(\varepsilon_1 w, \varepsilon_2 w, \varepsilon_3 v)$  for  $v, w \in \mathbb{R}^{1,n}$ . It follows from Claim 1 that such terms vanish, if  $v, w$  are spacelike and  $v \perp w$ . So if we find a curve  $\gamma$ , such that  $\gamma(t)$  is spacelike for all  $t \in \mathbb{R}$ ,  $\gamma'(0)$  and  $\gamma''(0)$  are both spacelike and both orthogonal to  $\gamma(0)$  and in addition  $r''(0) \neq 0$ , then (3.2) implies  $\alpha(\varepsilon_1 x, \varepsilon_2 x, \varepsilon_3 x) = 0$ , since  $\gamma(0) \perp \gamma'(0)$  implies  $r'(0) = 0$ . For instance such a curve is given by

$$\gamma : \mathbb{R} \rightarrow \mathbb{R}^{1,n}, t \mapsto (t + \sqrt{2}) \cos(t) e_1 + ((t - \sqrt{2}) \sin(t) + 1) e_2,$$

with  $e_1 = (0, 1, 0, \dots, 0)^T, e_2 = (0, 0, 1, 0, \dots, 0)^T \in \mathbb{R}^{1,n}$ . This proves Claim 2.

*Proof of Claim 3:* We consider  $(\mathbb{C}x)^\perp \cong \mathbb{C}^{1,n-1} = \mathbb{R}^{1,n-1} \oplus i\mathbb{R}^{1,n-1}$ . Let  $b_1, \dots, b_n$  be an orthonormal basis of  $\mathbb{R}^{1,n-1}$  where  $b_1$  is a timelike vector.

First, we show that  $\alpha(\cdot, \varepsilon_1 b_j, \varepsilon_2 b_j)|_{(\mathbb{C}b_j)^\perp} = 0$  for all  $1 \leq j \leq n$  and  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$ . The  $\mathbb{R}$ -linear map

$$\varphi := \alpha(\cdot, \varepsilon_1 b_j, \varepsilon_2 b_j)|_{(\mathbb{C}b_j)^\perp} : (\mathbb{C}b_j)^\perp \cong \mathbb{C}^n \rightarrow \mathbb{R}$$

is invariant under

$$SO^0(1, n)_{b_j} \cong \begin{cases} SO(n) & \text{if } j = 1, \\ SO^0(1, n-1) & \text{if } j \geq 2. \end{cases}$$

As in the proof of Claim 1 it follows that  $\varphi = 0$ .

Let  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$  and  $\mathcal{U} := \text{span}_{\mathbb{C}} \{b_j, b_k\}^\perp \subset \mathbb{C}^{1,n}$  for  $j \neq k$ . We consider the  $\mathbb{R}$ -linear map

$$\psi := \alpha(\cdot, \varepsilon_1 b_j, \varepsilon_2 b_k)|_{\mathcal{U}} : \mathcal{U} \cong \mathbb{C}^{n-1} = \mathbb{R}^{n-1} \oplus i\mathbb{R}^{n-1} \rightarrow \mathbb{R}.$$

The map  $\psi$  is invariant under the action of

$$G := \{A \in SO^0(1, n) \mid Ab_j = b_j, Ab_k = b_k\}.$$

The same holds for  $\psi_1 := \psi|_{\mathbb{R}^{n-1}}$  and  $\psi_2 := \psi|_{i\mathbb{R}^{n-1}}$ . We have

$$G \cong \begin{cases} SO^0(1, n-2) & \text{if } b_j \text{ and } b_k \text{ are spacelike} \\ SO(n-1) & \text{if } b_j \text{ or } b_k \text{ is timelike.} \end{cases}$$

We have to distinguish the cases  $n \geq 4$  and  $n = 3$ .

Case  $n \geq 4$ : Since  $n-2 \geq 2$ ,  $G$  acts irreducibly on  $\mathbb{R}^{n-1}$  and  $i\mathbb{R}^{n-1}$  in both cases. Since  $\ker \psi_1 \subset \mathbb{R}^{n-1}$  and  $\ker \psi_2 \subset i\mathbb{R}^{n-1}$  are  $G$ -invariant subspaces and neither  $\psi_1$  nor  $\psi_2$  can be injective,  $\ker \psi_1 = \mathbb{R}^{n-1}$  and  $\ker \psi_2 = i\mathbb{R}^{n-1}$ , i.e.  $\psi = 0$ .

Case  $n = 3$ : If  $b_j$  or  $b_k$  is timelike then  $G \cong SO(2)$  acts irreducibly on  $\mathbb{R}^2$  and  $i\mathbb{R}^2$ . As above this implies  $\psi = 0$ .

Denote by  $v_{\pm} = (1, \pm 1)^T$ . If  $b_j$  and  $b_k$  are spacelike, then  $G \cong \mathrm{SO}^0(1, 1)$  acts reducibly on  $\mathbb{R}^{1,1} = \mathbb{R} \cdot v_+ \oplus \mathbb{R} \cdot v_-$ . We have

$$G \cong \mathrm{SO}^0(1, 1) = \left\{ \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

Since  $\psi_j$  is  $G$ -invariant it follows for  $A \in G$  that

$$\psi_j(v_{\pm}) = \psi_j(Av_{\pm}) = (\cosh(t) \pm \sinh(t))\psi_j(v_{\pm}) = e^{\pm t}\psi_j(v_{\pm})$$

for all  $t \in \mathbb{R}$ . Hence,  $\psi_j(v_{\pm}) = 0$ . This implies as above  $\psi = 0$ .

Summarizing, we have in both cases  $\psi = 0$ . Since  $\mathbb{C}x \subset \mathcal{U}$ ,  $\alpha(\varepsilon x, \varepsilon_1 b_j, \varepsilon_2 b_k) = 0$  for  $\varepsilon \in \{1, i\}$ . The multilinearity of  $\alpha$  implies Claim 3.  $\square$

**Remark 3.2.** *The proof of Lemma 3.4 works also with some simplifications if  $\alpha$  is  $\mathrm{SO}(n)$ -invariant.*

**Lemma 3.5.** *Let  $n \geq 4$  and  $\alpha \in \mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}^*$ , with  $\mathcal{V} = \mathbb{C}^n$  considered as a real vector space. If  $\alpha$  is  $\mathrm{SO}(n)$ -invariant, then  $\alpha = 0$ .*

**Lemma 3.6.** *Let  $n \geq 3$  and  $\alpha \in \Lambda^3(\mathbb{H}^{1,n})^*$ , where  $\mathbb{H}^{1,n} = \mathbb{C}^{1,n} \oplus \mathbb{C}^{1,n} \cdot \mathbf{j}$  is considered as real vector space. If  $\alpha$  is  $\mathrm{SO}^0(1, n)$ -invariant, then  $\alpha = 0$ .*

*Proof:* Let  $J : \mathbb{H}^{1,n} \rightarrow \mathbb{H}^{1,n}$  denote the right multiplication with  $\mathbf{j}$ . The value of  $\alpha$  is given by the values of  $\alpha|_{\mathbb{C}^{1,n}}, (J^*\alpha)|_{\mathbb{C}^{1,n}}, \alpha(J \cdot, \cdot, \cdot)|_{\mathbb{C}^{1,n}}$  and  $\alpha(\cdot, J \cdot, J \cdot)|_{\mathbb{C}^{1,n}}$ . They all are  $\mathrm{SO}^0(1, n)$ -invariant.

Let  $\mathcal{V}$  denote  $\mathbb{C}^{1,n}$  considered as real vector space. Then the above trilinear maps can be considered as elements of  $\mathcal{V}^* \otimes \mathcal{V}^* \otimes \mathcal{V}^*$ . By Lemma 3.4, they all vanish. Summarizing, we have  $\alpha = 0$ .  $\square$

**Lemma 3.7.** *Let  $n \geq 4$  and  $\alpha \in \Lambda^4 \mathcal{V}^*$ , where  $\mathcal{V}$  denotes  $\mathbb{C}^{1,n}$  considered as real vector space. If  $\alpha$  is  $\mathrm{SO}^0(1, n)$ -invariant, then  $\alpha \in \mathbb{R} \cdot (\omega_1 \wedge \omega_1)$ , where  $\omega_1$  denotes the Kähler form.*

*Proof:* The value of  $\alpha$  is given by the values of  $\alpha(\cdot, \cdot, \cdot, \cdot)|_{\mathbb{R}^{1,n}}, \alpha(i \cdot, \cdot, \cdot, \cdot)|_{\mathbb{R}^{1,n}}, \alpha(\cdot, i \cdot, \cdot, i \cdot)|_{\mathbb{R}^{1,n}}, \alpha(\cdot, i \cdot, i \cdot, i \cdot)|_{\mathbb{R}^{1,n}}$  and  $\alpha(i \cdot, i \cdot, i \cdot, i \cdot)|_{\mathbb{R}^{1,n}}$ . All these forms are  $\mathrm{SO}^0(1, n)$ -invariant.

By Lemma 3.1,  $\alpha(\cdot, \cdot, \cdot, \cdot)|_{\mathbb{R}^{1,n}} = 0$  and  $\alpha(i \cdot, i \cdot, i \cdot, i \cdot)|_{\mathbb{R}^{1,n}} = 0$ . Next we show  $\alpha(i \cdot, \cdot, \cdot, \cdot)|_{\mathbb{R}^{1,n}} = 0$  and  $\alpha(\cdot, i \cdot, i \cdot, i \cdot)|_{\mathbb{R}^{1,n}} = 0$ . For that it is sufficient to consider an element  $\beta \in (\mathbb{R}^{1,n})^* \otimes \Lambda^3(\mathbb{R}^{1,n})^*$  which is  $\mathrm{SO}^0(1, n)$ -invariant.

Let  $x_0 \in \mathbb{R}^{1,n}$  be a spacelike vector. Then  $\beta(x_0, x_0, \cdot, \cdot)$  vanishes if we insert an element of  $\mathbb{R}x_0$ , so we consider its restriction to  $(\mathbb{R}x_0)^{\perp} \cong \mathbb{R}^{1,n-1}$ . This restriction is invariant under the stabilizer  $\mathrm{SO}^0(1, n)_{x_0} \cong \mathrm{SO}^0(1, n-1)$ . Lemma 3.1 implies  $\beta(x_0, x_0, \cdot, \cdot)|_{(\mathbb{R}x_0)^{\perp}} = 0$ . Hence  $\iota_{x_0}\beta$  vanishes if we insert an element of  $\mathbb{R}x_0$ , so we consider the restriction of  $\iota_{x_0}\beta$  to  $(\mathbb{R}x_0)^{\perp} \cong \mathbb{R}^{1,n-1}$ . Again, this restriction is  $\mathrm{SO}^0(1, n)_{x_0} \cong \mathrm{SO}^0(1, n-1)$ -invariant and Lemma 3.1 implies its vanishing. Hence,  $x_0 \in \ker \beta$ . Since  $\mathbb{R}^{1,n}$  is generated by spacelike

vectors,  $\beta = 0$ . This implies  $\alpha(i\cdot, \cdot, \cdot, \cdot)|_{\mathbb{R}^{1,n}} = 0$  and  $\alpha(\cdot, i\cdot, i\cdot, i\cdot)|_{\mathbb{R}^{1,n}} = 0$ .

Next we show that there exists a real number  $\lambda$  such that

$$\alpha(x, iy, z, iu) = \lambda \det \begin{pmatrix} \langle x, y \rangle_1 & \langle y, z \rangle_1 \\ \langle x, u \rangle_1 & \langle z, u \rangle_1 \end{pmatrix},$$

for  $x, y, z, u \in \mathbb{R}^{1,n}$ .

Let again  $x_0 \in \mathbb{R}^{1,n}$  be a spacelike vector and consider the restriction of  $\alpha(x_0, ix_0, \cdot, i\cdot)$  to  $(\mathbb{R}x_0)^\perp \cong \mathbb{R}^{1,n-1}$ . There exists a real matrix  $A$  such that  $\alpha(x_0, ix_0, v, iw) = \langle v, Aw \rangle_1$  for all  $v, w \in (\mathbb{R}x_0)^\perp$ . Since  $\alpha$  is  $SO^0(1, n)$ -invariant,  $\alpha(x_0, ix_0, \cdot, i\cdot)|_{(\mathbb{R}x_0)^\perp}$  is  $SO^0(1, n)_{x_0} \cong SO^0(1, n-1)$ -invariant. We have

$$\begin{aligned} \alpha(x_0, ix_0, v, iw) &= \langle v, Aw \rangle_1 = \langle Bv, BAw \rangle_1, \\ \alpha(x_0, ix_0, v, iw) &= \alpha(x_0, ix_0, Bv, iBw) = \langle Bv, ABw \rangle_1, \end{aligned}$$

for all  $B \in SO^0(1, n)_{x_0}$  and all  $v, w \in \mathbb{R}^{1,n-1}$ . This implies  $[A, B] = 0$ . Since  $n-1 \geq 3$ ,  $SO^0(1, n)_{x_0}$  acts irreducibly and by Schur's lemma,  $A = \tilde{\lambda} \text{Id}$ . Since  $\alpha$  is real-valued,  $\tilde{\lambda}$  is a real number.

Hence,

$$\begin{aligned} \alpha(x_0, ix_0, v, iw)|_{(\mathbb{R}x_0)^\perp} &= \tilde{\lambda} \langle v, w \rangle_1 \\ &= \frac{\tilde{\lambda}}{\langle x_0, x_0 \rangle_1} \langle x_0, x_0 \rangle_1 \langle v, w \rangle_1. \end{aligned}$$

Now we set  $\lambda := \frac{\tilde{\lambda}}{\langle x_0, x_0 \rangle_1}$ . Since  $\alpha(x_0, ix_0, \cdot, i\cdot)$  vanishes if we insert an element of  $\mathbb{R}x_0$  it follows

$$\alpha(x_0, ix_0, z, iu) = \lambda(\langle x_0, x_0 \rangle_1 \langle z, u \rangle_1 - \langle x_0, z \rangle_1 \langle x_0, u \rangle_1),$$

for  $z, u \in \mathbb{R}^{1,n}$ .

Now we consider the restriction of  $\alpha(x_0, i\cdot, \cdot, i\cdot)$  to  $(\mathbb{R}x_0)^\perp$ . It is an  $\mathbb{R}$ -trilinear map, which is antisymmetric in the first and third entry and  $SO^0(1, n)_{x_0}$ -invariant. If we consider it as an  $SO^0(1, n-1)$ -invariant element of  $\mathcal{W}^* \otimes \mathcal{W}^* \otimes \mathcal{W}^*$ , where  $\mathcal{W}$  denotes  $\mathbb{C}^{1,n-1}$  as real vector space, then we can apply Lemma 3.4 since  $n-1 \geq 3$  and conclude its vanishing.

An elementary calculation shows

$$\alpha(x_0, iy, z, iu) = \lambda(\langle x_0, y \rangle_1 \langle z, u \rangle_1 - \langle x_0, u \rangle_1 \langle y, z \rangle_1), \quad (3.3)$$

for  $y, z, u \in \mathbb{R}^{1,n} \cong \mathbb{R}x_0 \oplus (\mathbb{R}x_0)^\perp$ .

Analogously there exists a real number  $\mu$  for any other spacelike vector  $x_1 \in \mathbb{R}^{1,n}$  such that

$$\alpha(x_1, iy, z, iu) = \mu(\langle x_1, y \rangle_1 \langle z, u \rangle_1 - \langle x_1, u \rangle_1 \langle y, z \rangle_1), \quad (3.4)$$

for all  $y, z, u \in \mathbb{R}^{1,n}$ .

If we set  $z = x_1$  in (3.3) and  $z = x_0$  in (3.4), we get

$$\begin{aligned} \alpha(x_0, iy, x_1, iu) &= \lambda(\langle x_0, y \rangle_1 \langle x_1, u \rangle_1 - \langle x_0, u \rangle_1 \langle y, x_1 \rangle_1), \\ \alpha(x_1, iy, x_0, iu) &= \mu(\langle x_1, y \rangle_1 \langle x_0, u \rangle_1 - \langle x_1, u \rangle_1 \langle y, x_0 \rangle_1) \\ &= -\mu(\langle x_1, u \rangle_1 \langle y, x_0 \rangle_1 - \langle x_1, y \rangle_1 \langle x_0, u \rangle_1) \end{aligned}$$

for all  $y, u \in \mathbb{R}^{1,n}$ . The relation  $\alpha(x_0, iy, x_1, iu) = -\alpha(x_1, iy, x_0, iu)$  implies  $\lambda = \mu$ . Since spacelike vectors generate  $\mathbb{R}^{1,n}$ , it follows that

$$\begin{aligned}\alpha(x, iy, z, iu) &= \lambda (\langle x, y \rangle_1 \langle z, u \rangle_1 - \langle x, u \rangle_1 \langle y, z \rangle_1) \\ &= \lambda \det \begin{pmatrix} \langle x, y \rangle_1 & \langle x, u \rangle_1 \\ \langle y, z \rangle_1 & \langle z, u \rangle_1 \end{pmatrix}\end{aligned}$$

for all  $x, y, z, u \in \mathbb{R}^{1,n}$ .

Now we consider complex vectors  $X, Y, Z, U \in \mathbb{C}^{1,n}$  and write  $X = X_0 + iX_1, Y = Y_0 + iY_1, Z = Z_0 + iZ_1$  and  $U = U_0 + iU_1$  with  $X_j, Y_j, Z_j, U_j \in \mathbb{R}^{1,n}, j = 0, 1$ . We get

$$\begin{aligned}\alpha(X, Y, Z, U) &= \alpha(X_0, Y_0, iZ_1, iU_1) + \alpha(X_0, iY_1, Z_0, iU_1) + \alpha(X_0, iY_1, iZ_1, U_0) \\ &\quad + \alpha(iX_1, Y_0, Z_0, iU_1) + \alpha(iX_1, Y_0, iZ_1, U_0) + \alpha(iX_1, iY_1, Z_0, U_0) \\ &= -\alpha(X_0, iZ_1, Y_0, iU_1) + \alpha(X_0, iY_1, Z_0, iU_1) - \alpha(X_0, iY_1, U_0, iZ_1) \\ &\quad - \alpha(Y_0, iX_1, Z_0, iU_1) + \alpha(Y_0, iX_1, U_0, iZ_1) - \alpha(Z_0, iX_1, U_0, iY_1) \\ &= \lambda (-\langle X_0, Z_1 \rangle_1 \langle Y_0, U_1 \rangle_1 + \langle X_0, U_1 \rangle_1 \langle Z_1, Y_0 \rangle_1 + \langle X_0, Y_1 \rangle_1 \langle Z_0, U_1 \rangle_1 \\ &\quad - \langle X_0, U_1 \rangle_1 \langle Y_1, Z_0 \rangle_1 - \langle X_0, Y_1 \rangle_1 \langle U_0, Z_1 \rangle_1 + \langle X_0, Z_1 \rangle_1 \langle Y_1, U_0 \rangle_1 \\ &\quad - \langle Y_0, X_1 \rangle_1 \langle Z_0, U_1 \rangle_1 + \langle Y_0, U_1 \rangle_1 \langle X_1, Z_0 \rangle_1 + \langle Y_0, X_1 \rangle_1 \langle U_0, Z_1 \rangle_1 \\ &\quad - \langle Y_0, Z_1 \rangle_1 \langle X_1, U_0 \rangle_1 - \langle Z_0, X_1 \rangle_1 \langle U_0, Y_1 \rangle_1 + \langle Z_0, Y_1 \rangle_1 \langle X_1, U_0 \rangle_1)\end{aligned}$$

Next we insert the vectors  $X, Y, Z$  and  $U$  in  $\omega_1 \wedge \omega_1$ .

$$\begin{aligned}\omega_1 \wedge \omega_1(X, Y, Z, U) &= 2(\omega_1(X, Y)\omega_1(Z, U) - \omega_1(X, Z)\omega_1(Y, U) + \omega_1(X, U)\omega_1(Y, Z)) \\ &= 2((\langle X_0, Y_1 \rangle_1 - \langle X_1, Y_0 \rangle_1)(\langle Z_0, U_1 \rangle_1 - \langle Z_1, U_0 \rangle_1) \\ &\quad - (\langle X_0, Z_1 \rangle_1 - \langle X_1, Z_0 \rangle_1)(\langle Y_0, U_1 \rangle_1 - \langle Y_1, U_0 \rangle_1) \\ &\quad + (\langle X_0, U_1 \rangle_1 - \langle X_1, U_0 \rangle_1)(\langle Y_0, Z_1 \rangle_1 - \langle Y_1, Z_0 \rangle_1)) \\ &= 2(\langle X_0, Y_1 \rangle_1 \langle Z_0, U_1 \rangle_1 - \langle X_0, Y_1 \rangle_1 \langle Z_1, U_0 \rangle_1 - \langle X_1, Y_0 \rangle_1 \langle Z_0, U_1 \rangle_1 \\ &\quad + \langle X_1, Y_0 \rangle_1 \langle Z_1, U_0 \rangle_1 - \langle X_0, Z_1 \rangle_1 \langle Y_0, U_1 \rangle_1 + \langle X_0, Z_1 \rangle_1 \langle Y_1, U_0 \rangle_1 \\ &\quad + \langle X_1, Z_0 \rangle_1 \langle Y_0, U_1 \rangle_1 - \langle X_1, Z_0 \rangle_1 \langle Y_1, U_0 \rangle_1 + \langle X_0, U_1 \rangle_1 \langle Y_0, Z_1 \rangle_1 \\ &\quad - \langle X_0, U_1 \rangle_1 \langle Y_1, Z_0 \rangle_1 - \langle X_1, U_0 \rangle_1 \langle Y_0, Z_1 \rangle_1 + \langle X_1, U_0 \rangle_1 \langle Y_1, Z_0 \rangle_1).\end{aligned}$$

Comparing both equations, we see that  $\alpha$  and  $\omega_1 \wedge \omega_1$  coincide up to a constant.  $\square$

**Lemma 3.8.** *Let  $n \geq 5$  and  $\alpha \in \Lambda^5 \mathcal{V}^*$ , where  $\mathcal{V}$  denotes  $\mathbb{C}^{1,n}$  considered as real vector space. If  $\alpha$  is  $\text{SO}^0(1, n)$ -invariant, then  $\alpha = 0$ .*

*Proof:* Let  $x \in \mathbb{R}^{1,n}$  be a spacelike vector and  $u \in \mathbb{C}x$ . We have  $\alpha(u, u, \cdot, \cdot, \cdot) = 0$ , so we consider  $\alpha(u, iu, \cdot, \cdot, \cdot)$ . It vanishes if we insert an element of  $\mathbb{C}x$ . Its restriction to  $(\mathbb{C}x)^\perp \cong \mathbb{C}^n$  is invariant under the stabilizer  $\text{SO}^0(1, n)_x \cong \text{SO}^0(1, n-1)$ . Hence, Lemma 3.4 gives us  $\alpha(u, iu, \cdot, \cdot, \cdot)|_{(\mathbb{C}x)^\perp} = 0$ , which implies that  $\iota_u \alpha$  vanishes if we insert an element of  $\mathbb{C}x$ .

Next we consider the restriction of  $\iota_u \alpha$  to  $(\mathbb{C}x)^\perp$ . It is  $\text{SO}^0(1, n)_x \cong \text{SO}^0(1, n-1)$ -invariant

and since  $n - 1 \geq 4$  we can apply Lemma 3.7, so there exists for every  $u \in \mathbb{C}x$  a real number  $\phi(u)$ , such that  $\iota_u \alpha|_{(\mathbb{C}x)^\perp} = \phi(u) \omega_1 \wedge \omega_1$ . This defines an  $\mathbb{R}$ -linear map  $\phi : \mathbb{C}x \rightarrow \mathbb{R}$ . Since the real dimension of  $\mathbb{C}x$  is 2, there exists a non-zero element  $u_0 \in \ker \phi$ . Hence,  $u_0 \in \ker \alpha$ . Since  $\ker \alpha$  is an  $\text{SO}^0(1, n)$ -invariant subspace of  $\mathbb{C}^{1, n}$  it has to be equal  $\{0\}$ ,  $\mathbb{C}^{1, n}$  or  $c \cdot \mathbb{R}^{1, n}$  for some complex number  $c \in \mathbb{C} \setminus \{0\}$ . We already found a non-zero vector  $u_0 \in \ker \alpha$ . Hence,  $\ker \alpha = c \cdot \mathbb{R}^{1, n}$  or  $\mathbb{C}^{1, n}$ .

Assume  $\ker \alpha = c \cdot \mathbb{R}^{1, n}$ . Then  $\alpha$  induces a 5-form  $\tilde{\alpha} \in \Lambda^5(\mathcal{V}/c \cdot \mathbb{R}^{1, n})^*$ , which is  $\text{SO}^0(1, n)$ -invariant. Since  $\mathcal{V}/c \cdot \mathbb{R}^{1, n} \cong \mathbb{R}^{1, n}$ , we can apply Lemma 3.1 and get  $\tilde{\alpha} = 0$ . Hence,  $\ker \alpha = \mathbb{C}^{1, n}$ , i.e.  $\alpha = 0$ .  $\square$

**Lemma 3.9.** *Let  $n \geq 5$  and  $\alpha \in \mathcal{V}^* \otimes \Lambda^4 \mathcal{V}^*$ , where  $\mathcal{V}$  denotes  $\mathbb{C}^{1, n}$  considered as real vector space. If  $\alpha$  is  $\text{SO}^0(1, n)$ -invariant, then  $\alpha = 0$ .*

*Proof:* The idea of the proof is to show that  $\mathbb{C}x \subset \mathcal{W} := \ker(v \mapsto \alpha(v, \cdot, \cdot, \cdot, \cdot))$  for every spacelike vector  $x \in \mathbb{R}^{1, n} \subset \mathbb{C}^{1, n} = \mathcal{V}$ .  $\mathbb{C}x \subset \mathcal{W}$  follows from several Claims. Let  $x \in \mathbb{R}^{1, n} \subset \mathbb{C}^{1, n}$  be a spacelike vector and  $u \in \mathbb{C}x$ .

*Claim 1:*  $(\iota_u \alpha)(x, ix, \cdot, \cdot) : \mathbb{C}^{1, n} \times \mathbb{C}^{1, n} \rightarrow \mathbb{R}$  vanishes.

*Claim 2:*  $(\iota_u \alpha)(\tilde{u}, \cdot, \cdot, \cdot)|_{(\mathbb{C}x)^\perp} = 0$  for all  $\tilde{u} \in \mathbb{C}x$ .

*Claim 3:*  $(\iota_u \alpha)(y, z, v, iw) = -(\iota_u \alpha)(y, z, iv, w)$  for all  $y, z, v, w \in (\mathbb{C}x)^\perp$ .

First we show that the Claims indeed imply  $u \in \mathcal{W}$ . Notice that Claims 1 and 2 imply that  $\iota_u \alpha$  vanishes if we insert an element of  $\mathbb{C}x$ . So we restrict  $\iota_u \alpha$  to  $(\mathbb{C}x)^\perp \cong \mathbb{C}^{1, n-1}$ . It is an  $\text{SO}^0(1, n)_x \cong \text{SO}^0(1, n-1)$ -invariant form. Since  $n - 1 \geq 4$  we can apply Lemma 3.7. Hence, there exists a real number  $\mu$ , such that  $\iota_u \alpha = \mu \cdot \omega_1 \wedge \omega_1$ . But  $\omega_1 \wedge \omega_1$  does not satisfy the equation in Claim 3. For example consider the vectors  $Y = i(e_0 + e_1)$ ,  $Z = e_1$ ,  $V = ie_0$  and  $W = i(e_0 + e_1)$ . Then we have  $\omega_1 \wedge \omega_1(Y, Z, V, iW) = 2$  but  $\omega_1 \wedge \omega_1(Y, Z, iV, W) = 0$ . Hence,  $\mu = 0$ . Summarizing we have shown  $\mathbb{C}x \subset \mathcal{W}$  for all spacelike vectors  $x \in \mathbb{R}^{1, n}$ . Since the spacelike vectors of  $\mathbb{R}^{1, n}$  generate  $\mathbb{C}^{1, n}$ , we have  $\mathcal{V} = \mathcal{W}$ , i.e.  $\alpha = 0$ . This finishes the proof of the Lemma and we have just to prove the Claims. From now on we identify  $(\mathbb{C}x)^\perp$  with  $\mathbb{C}^{1, n-1} \cong \mathbb{R}^{1, n-1} \oplus i\mathbb{R}^{1, n-1}$  and denote by  $b_1, \dots, b_n$  an orthonormal basis of  $\mathbb{R}^{1, n-1}$ , where  $b_1$  is timelike.

*Proof of Claim 1:* Denote by  $\tilde{\alpha} := (\iota_u \alpha)(x, ix, \cdot, \cdot)$ . We see that  $\tilde{\alpha}$  vanishes if we insert an element of  $\mathbb{C}x$ . Now we restrict  $\tilde{\alpha}$  to  $(\mathbb{C}x)^\perp \cong \mathbb{C}^{1, n-1}$  and let  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$ . Let  $\mathcal{U} := \text{span}_{\mathbb{C}}\{b_j, b_k\}^\perp \subset \mathbb{C}^{1, n}$  for  $j \neq k$ , which is isomorphic to  $\mathbb{C}^{n-1}$ . We consider

$$\beta := \alpha(\cdot, \cdot, \cdot, \varepsilon_1 b_j, \varepsilon_2 b_k)|_{\mathcal{U}} : \mathcal{U} \times \mathcal{U} \times \mathcal{U} \rightarrow \mathbb{R}.$$

$\beta$  is invariant under

$$G := \{A \in \text{SO}^0(1, n) | Ab_j = b_j, Ab_k = b_k\} \cong \begin{cases} \text{SO}^0(1, n-2) & \text{if } b_j \text{ and } b_k \text{ are spacelike} \\ \text{SO}(n-1) & \text{if } b_j \text{ or } b_k \text{ is timelike.} \end{cases}$$

Since  $n - 2 \geq 3$ ,  $n - 1 \geq 4$ , we can apply Lemma 3.4 and Lemma 3.5, respectively, i.e.  $\beta = 0$ . Since  $u, x, ix \in \mathcal{U}$ ,  $(\iota_u \alpha)(x, ix, \varepsilon_1 b_j, \varepsilon_2 b_k) = 0$ . Analogously,  $(\iota_u \alpha)(x, ix, b_j, ib_j) = 0$

for  $1 \leq j \leq n$ . This finishes the proof of Claim 1.

*Proof of Claim 2:* Let  $\tilde{u} \in \mathbb{C}x$ . We consider the restriction of the 3-form  $(\iota_u \alpha)(\tilde{u}, \cdot, \cdot, \cdot)$  to  $(\mathbb{C}x)^\perp \cong \mathbb{C}^{1, n-1}$ . It is  $\mathrm{SO}^0(1, n)_x \cong \mathrm{SO}^0(1, n-1)$ -invariant. Since  $n-1 \geq 4$ , we can apply Lemma 3.4, i.e.  $(\iota_u \alpha)(\tilde{u}, \cdot, \cdot, \cdot)|_{(\mathbb{C}x)^\perp} = 0$ . This finishes the proof of Claim 2.

*Proof of Claim 3:* Let  $\beta := \iota_u \alpha|_{(\mathbb{C}x)^\perp}$ . We identify  $\mathrm{SO}^0(1, n)_x$  with  $\mathrm{SO}^0(1, n-1)$  and  $(\mathbb{C}x)^\perp$  with  $\mathbb{C}^{1, n-1}$ .

The 2-form  $\beta(b_j, ib_j, \cdot, \cdot)$  vanishes if we insert an element of  $\mathbb{C}b_j \subset \mathbb{C}^{1, n-1}$ . Its restriction to  $(\mathbb{C}b_j)^\perp$  is invariant under

$$\mathrm{SO}^0(1, n-1)_{b_j} \cong \begin{cases} \mathrm{SO}^0(1, n-2) & \text{if } b_j \text{ is spacelike,} \\ \mathrm{SO}(n-1) & \text{if } b_j \text{ is timelike.} \end{cases}$$

Since  $n-2 \geq 3$  and  $n-1 \geq 4$  respectively, we can apply Lemma 3.2 and Lemma 3.3 respectively, i.e.  $\beta(b_j, ib_j, \cdot, \cdot) \in \mathbb{R} \cdot \omega_1$  or  $\beta(b_j, ib_j, \cdot, \cdot) \in \mathbb{R} \cdot \omega_0$ , respectively. Since  $\omega_1$  and  $\omega_0$  are of type  $(1, 1)$ , summarizing the above,

$$\beta(b_j, ib_j, iv, w) = -\beta(b_j, ib_j, v, iw)$$

holds for all  $v, w \in (\mathbb{C}x)^\perp$ . Let  $j \neq k$  and  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$ . Denote by  $\mathcal{U} := \mathrm{span}_{\mathbb{C}} \{b_j, b_k\} \subset \mathbb{C}^{1, n}$ . Next we show that the 2-form  $\beta(\varepsilon_1 b_j, \varepsilon_2 b_k, \cdot, \cdot)$  vanishes on  $\mathcal{U}$ . For that it is sufficient to consider the  $\mathbb{R}$ -linear map

$$\varphi := \alpha(\cdot, b_j, b_k, ib_j, ib_k)|_{\mathcal{U}^\perp} : \mathcal{U}^\perp \cong \mathbb{C}^{n-1} = \mathbb{R}^{n-1} \oplus i\mathbb{R}^{n-1}.$$

The map  $\varphi$  is invariant under

$$G := \{A \in \mathrm{SO}^0(1, n) | Ab_j = b_j, Ab_k = b_k\} \cong \begin{cases} \mathrm{SO}^0(1, n-2) & \text{if } b_j \text{ and } b_k \text{ are spacelike,} \\ \mathrm{SO}(n-1) & \text{if } b_j \text{ or } b_k \text{ is timelike.} \end{cases}$$

The same holds for  $\varphi_1 := \varphi|_{\mathbb{R}^{n-1}}$  and  $\varphi_2 := \varphi|_{i\mathbb{R}^{n-1}}$ . Since  $n-1 \geq 4$ ,  $G$  acts  $\mathbb{R}$ -irreducibly on  $\mathbb{R}^{n-1}$  and  $i\mathbb{R}^{n-1}$ . Furthermore  $\varphi_1$  and  $\varphi_2$  can not be injective, i.e.  $\ker \varphi_1 = \mathbb{R}^{n-1}$  and  $\varphi_2 := i\mathbb{R}^{n-1}$ . Since  $u \in \mathcal{U}^\perp$ ,  $\beta(b_j, b_k, ib_j, ib_k) = 0$ .

Now we consider  $\mathcal{U}$  as a subspace of  $(\mathbb{C}x)^\perp \cong \mathbb{C}^{1, n-1}$ . Then  $\mathcal{U}^\perp \subset \mathbb{C}^{1, n-1}$  and the 2-form  $\beta(\varepsilon_1 b_j, \varepsilon_2 b_k, \cdot, \cdot)|_{\mathcal{U}^\perp}$  is invariant under

$$H := \{A \in \mathrm{SO}^0(1, n-1) | Ab_j = b_j, Ab_k = b_k\} \cong \begin{cases} \mathrm{SO}^0(1, n-3) & \text{if } b_j \text{ and } b_k \text{ are spacelike,} \\ \mathrm{SO}(n-2) & \text{if } b_j \text{ or } b_k \text{ is timelike.} \end{cases}$$

Since  $n-3 \geq 2$  and  $n-2 \geq 3$  respectively, we can apply Lemma 3.2 and Lemma 3.3 respectively, i.e.  $\beta(\varepsilon_1 b_j, \varepsilon_2 b_k, \cdot, \cdot)|_{\mathcal{U}^\perp}$  is an element of  $\mathbb{R} \cdot \omega_1$  or  $\mathbb{R} \cdot \omega_0$  respectively.

Finally, we consider the  $\mathbb{R}$ -linear map  $\beta(\varepsilon_1 b_j, \varepsilon_2 b_k, v, \cdot)|_{\mathcal{U}^\perp}$  for some  $v \in \mathcal{U}$ . Again, it is  $H$ -invariant. Since  $H$  acts irreducibly on  $\mathbb{R}^{n-2}$  and  $i\mathbb{R}^{n-2}$  it follows that  $\beta(\varepsilon_1 b_j, \varepsilon_2 b_k, v, w) = 0$  for all  $v \in \mathcal{U}$  and all  $w \in \mathcal{U}^\perp$ .

Summarizing,

$$\beta(\varepsilon_1 b_j, \varepsilon_2 b_k, iv, w) = -\beta(\varepsilon_1 b_j, \varepsilon_2 b_k, v, iw)$$



holds for all  $v, w \in (\mathbb{C}x)^\perp$ . Since it is sufficient to check Claim 3 for vectors  $y, z$  belonging to our real basis  $b_1, \dots, b_n, ib_1, \dots, ib_n$  of  $(\mathbb{C}x)^\perp \cong \mathbb{C}^{1, n-1}$ , this proves Claim 3.  $\square$

**Remark 3.3.** Lemma 3.9 is of course a generalization of Lemma 3.8. We gave an independent proof of Lemma 3.8 to show up that this statement is much easier to prove.

**Lemma 3.10.** Let  $n \geq 5$  and  $\alpha \in \Lambda^2 \mathcal{V}^* \otimes \Lambda^3 \mathcal{V}^*$ , where  $\mathcal{V}$  denotes  $\mathbb{C}^{1, n}$  considered as real vector space. If  $\alpha$  is  $\text{SO}^0(1, n)$ -invariant, then  $\alpha = 0$ .

*Proof:* The Lemma follows from several Claims. Let  $x \in \mathbb{R}^{1, n}$  be spacelike and  $u \in \mathbb{C}x$ .

*Claim 1:*  $\alpha(u, iu, u, iu, \cdot) : \mathbb{C}^{1, n} \rightarrow \mathbb{R}$  vanishes.

*Claim 2:*  $\alpha(u, iu, \tilde{u}, \cdot, \cdot) : \mathbb{C}^{1, n} \times \mathbb{C}^{1, n} \rightarrow \mathbb{R}$  vanishes for all  $\tilde{u} \in \mathbb{C}x$ .

*Claim 3:*  $\alpha(u, iu, \cdot, \cdot, \cdot) : \mathbb{C}^{1, n} \times \mathbb{C}^{1, n} \times \mathbb{C}^{1, n} \rightarrow \mathbb{R}$  vanishes.

*Claim 4:*  $\alpha(u, \cdot, \tilde{u}, i\tilde{u}, \cdot) : \mathbb{C}^{1, n} \times \mathbb{C}^{1, n} \rightarrow \mathbb{R}$  vanishes for all  $\tilde{u} \in \mathbb{C}x$ .

*Claim 5:*  $\alpha(u, \cdot, \tilde{u}, \cdot, \cdot) : \mathbb{C}^{1, n} \times \mathbb{C}^{1, n} \times \mathbb{C}^{1, n} \rightarrow \mathbb{R}$  vanishes for all  $\tilde{u} \in \mathbb{C}x$ .

*Claim 6:*  $\iota_u \alpha|_{(\mathbb{C}x)^\perp} = 0$ .

The Claims 3 and 5 imply that  $\iota_u \alpha$  vanishes if we insert an element of  $\mathbb{C}x$ . Together with Claim 6 it follows that  $\iota_u \alpha = 0$  for all  $u \in \mathbb{C}x$ , i.e.  $\mathbb{C}x \subset \mathcal{W} := \ker(v \mapsto \alpha(v, \cdot, \cdot, \cdot, \cdot))$ . Since the spacelike vectors of  $\mathbb{R}^{1, n}$  generate  $\mathbb{C}^{1, n}$ ,  $\mathcal{W} = \mathbb{C}^{1, n}$ , i.e.  $\alpha = 0$ . Thus we just have to prove the Claims.

From now on we identify  $(\mathbb{C}x)^\perp$  with  $\mathbb{C}^{1, n-1} = \mathbb{R}^{1, n-1} \oplus i\mathbb{R}^{1, n-1}$  and denote by  $b_1, \dots, b_n$  an orthonormal basis of  $\mathbb{R}^n$  where  $b_1$  is timelike.

*Proof of Claim 1:* The  $\mathbb{R}$ -linear map  $\alpha(u, iu, u, iu, \cdot) : \mathbb{C}^{1, n} \rightarrow \mathbb{R}$  vanishes on  $\mathbb{C}x$ . So we consider the restriction  $\varphi := \alpha(u, iu, u, iu, \cdot)|_{(\mathbb{C}x)^\perp} : \mathbb{C}^{1, n-1} = \mathbb{R}^{1, n-1} \oplus i\mathbb{R}^{1, n-1} \rightarrow \mathbb{R}$ . The  $\mathbb{R}$ -linear map  $\varphi$  is  $\text{SO}^0(1, n)_x \cong \text{SO}^0(1, n-1)$ -invariant. The same holds for  $\varphi_1 := \varphi|_{\mathbb{R}^{1, n-1}}$  and  $\varphi_2 := \varphi|_{i\mathbb{R}^{1, n-1}}$ . Since  $n-1 \geq 4$ ,  $\text{SO}^0(1, n-1)$  acts  $\mathbb{R}$ -irreducibly on  $\mathbb{R}^{1, n-1}$  and  $i\mathbb{R}^{1, n-1}$ . Since  $n \geq 5$ , neither  $\varphi_1$  nor  $\varphi_2$  are injective, i.e.  $\ker \varphi_j \neq \{0\}$ . Furthermore,  $\ker \varphi_1 \subset \mathbb{R}^{1, n-1}$  and  $\ker \varphi_2 \subset i\mathbb{R}^{1, n-1}$  are  $\text{SO}^0(1, n-1)$ -invariant subspaces. Hence,  $\varphi_1 = 0$  and  $\varphi_2 = 0$ , i.e.  $\varphi = 0$ . This finishes the proof of Claim 1.

*Proof of Claim 2:* Let  $\tilde{u} \in \mathbb{C}x$ . It follows from Claim 1 that  $\alpha(u, iu, \tilde{u}, \cdot, \cdot)$  vanishes if we insert an element of  $\mathbb{C}x$ .

We consider  $(\mathbb{C}b_j)^\perp$  as a subspace of  $\mathbb{C}^{1, n}$ . Then  $\alpha(\cdot, \cdot, \cdot, b_j, ib_j)|_{(\mathbb{C}b_j)^\perp}$  is  $\text{SO}^0(1, n)_{b_j}$ -invariant and since  $n \geq 5$ , we can apply Lemma 3.4 if  $j > 1$  and Lemma 3.5 if  $j = 1$ . Since  $u, \tilde{u} \in (\mathbb{C}b_j)^\perp$ , we have  $\alpha(u, iu, \tilde{u}, b_j, ib_j) = 0$ .

Let  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$  and  $j \neq k$ . Denote by  $\mathcal{U} := \text{span}_{\mathbb{C}} \{b_j, b_k\}^\perp \subset \mathbb{C}^{1, n}$ . Then  $\alpha(\cdot, \cdot, \cdot, \varepsilon_1 b_j, \varepsilon_2 b_k)|_{\mathcal{U}}$  is invariant under

$$\{A \in \text{SO}^0(1, n) | Ab_j = b_j, Ab_k = b_k\} \cong \begin{cases} \text{SO}^0(1, n-2) & b_j \text{ and } b_k \text{ are spacelike,} \\ \text{SO}(n-1) & b_j \text{ or } b_k \text{ is timelike.} \end{cases}$$

Since  $n-2 \geq 3$ , we can apply Lemma 3.4 and Lemma 3.5, respectively. Since  $u, \tilde{u} \in \mathcal{U}$ ,

$$\alpha(u, iu, \tilde{u}, \varepsilon_1 b_j, \varepsilon_2 b_k) = 0.$$

The multilinearity of  $\alpha$  implies the Claim 2.

*Proof of Claim 3:* Claim 2 implies that  $\alpha(u, iu, \cdot, \cdot, \cdot)$  vanishes if we insert an element of  $\mathbb{C}x$ . Its restriction to  $(\mathbb{C}x)^\perp$  is  $\text{SO}^0(1, n)_x \cong \text{SO}^0(1, n-1)$ -invariant. Since  $n-1 \geq 4$ , we can apply Lemma 3.4, i.e.  $\alpha(u, iu, \cdot, \cdot, \cdot) = 0$ . This finishes the proof of Claim 3.

*Proof of Claim 4:* Let  $\tilde{u} \in \mathbb{C}x$ . It follows from Claim 1 that  $\alpha(u, \cdot, \tilde{u}, i\tilde{u}, \cdot)$  vanishes if we insert an element of  $\mathbb{C}x$ . So we have to show that it also vanishes on  $(\mathbb{C}x)^\perp$ .

Let  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$  and  $\mathcal{U} := \text{span}_{\mathbb{C}} \{b_j, b_k\}^\perp \subset \mathbb{C}^{1, n}$  with  $j \neq k$ . Then  $\alpha(\cdot, \varepsilon_1 b_j, \cdot, \cdot, \varepsilon_2 b_k)|_{\mathcal{U}}$  is invariant under

$$\{A \in \text{SO}^0(1, n) | Ab_j = b_j, Ab_k = b_k\} \cong \begin{cases} \text{SO}^0(1, n-2) & b_j \text{ and } b_k \text{ are spacelike,} \\ \text{SO}(n-1) & b_j \text{ or } b_k \text{ is timelike.} \end{cases}$$

Since  $n-2 \geq 3$ , we can apply Lemma 3.4 and Lemma 3.5, respectively. Since  $u, \tilde{u} \in \mathcal{U}$ ,

$$\alpha(u, \varepsilon_1 b_j, \tilde{u}, i\tilde{u}, \varepsilon_2 b_k) = 0.$$

Analogously,  $\alpha(u, \varepsilon_1 b_j, \tilde{u}, i\tilde{u}, \varepsilon_2 b_j) = 0$  for all  $1 \leq j \leq n$  by using Lemma 3.4 and Lemma 3.5, respectively. The multilinearity of  $\alpha$  implies Claim 4.

*Proof of Claim 5:* Let  $\tilde{u} \in \mathbb{C}x$ . The Claims 3 and 4 imply that  $\alpha(u, \cdot, \tilde{u}, \cdot, \cdot)$  vanishes if we insert an element of  $\mathbb{C}x$ . So we consider its restriction to  $(\mathbb{C}x)^\perp \cong \mathbb{C}^n$ . It is  $\text{SO}^0(1, n)_x \cong \text{SO}^0(1, n-1)$ -invariant. Since  $n-1 \geq 4$ , we can apply Lemma 3.4, which implies Claim 5.

*Proof of Claim 6:* Let  $y \in (\mathbb{C}x)^\perp$  be spacelike and  $v \in \mathbb{C}y$ . Let  $\mathcal{U} := \text{span}_{\mathbb{C}} \{x, y\}^\perp \cong \mathbb{C}^{1, n-2}$ . The 3-form  $(\iota_u \alpha)(v, \cdot, \cdot, \cdot)|_{\mathcal{U}}$  is invariant under

$$\{A \in \text{SO}^0(1, n) | Ax = x, Ay = y\} \cong \text{SO}^0(1, n-2).$$

Since  $n-2 \geq 3$ , we can apply Lemma 3.4, i.e.  $(\iota_u \alpha)(v, \cdot, \cdot, \cdot)|_{\mathcal{U}} = 0$ . Next we show that  $(\iota_u \alpha)(v, \cdot, \cdot, \cdot)|_{(\mathbb{C}x)^\perp}$  vanishes if we insert an element of  $\mathbb{C}y$ .

Let  $\tilde{v} \in \mathbb{C}y$  and  $\varphi := (\iota_u \alpha)(v, \tilde{v}, i\tilde{v}, \cdot)|_{\mathcal{U}} : \mathcal{U} \cong \mathbb{C}^{1, n-2} = \mathbb{R}^{1, n-2} \oplus i\mathbb{R}^{1, n-2} \rightarrow \mathbb{R}$ . The  $\mathbb{R}$ -linear map  $\varphi$  is  $\text{SO}^0(1, n-2)$ -invariant. The same holds for  $\varphi_1 := \varphi|_{\mathbb{R}^{1, n-2}}$  and  $\varphi_2 := \varphi|_{i\mathbb{R}^{1, n-2}}$ . Since  $\ker \varphi_1$  and  $\ker \varphi_2$  are  $\text{SO}^0(1, n-2)$ -invariant subspaces and neither  $\varphi_1$  nor  $\varphi_2$  can be injective, they both vanish. Hence,  $\varphi = 0$ . It follows that  $\beta := (\iota_u \alpha)(v, \tilde{v}, \cdot, \cdot)$  vanishes if we insert an element of  $\mathbb{C}y$ .

Next we show that the bilinear form  $\beta$  also vanishes on  $\mathcal{U} \cong \mathbb{C}^{1, n-2} \cong \mathbb{R}^{1, n-2} \oplus i\mathbb{R}^{1, n-2}$ . Let  $\varepsilon_1, \varepsilon_2 \in \{1, i\}$  and denote by  $d_1, \dots, d_{n-1}$  be an orthonormal basis of  $\mathbb{R}^{1, n-2}$  where  $d_1$  is timelike. Similarly as in the in the proof of Claim 2, it follows that  $(\iota_u \alpha)(v, \tilde{v}, \varepsilon_1 d_j, \varepsilon_2 d_k) = 0$  for all  $1 \leq j, k \leq n-1$ . The multilinearity of  $\alpha$  implies that  $(\iota_u \alpha)(v, \tilde{v}, \cdot, \cdot)$  vanishes on  $\mathcal{U}$ .

Summarizing the above, we have shown that  $(\iota_u \alpha)(v, \cdot, \cdot, \cdot)$  vanishes on  $(\mathbb{C}x)^\perp$  for every  $v \in \mathbb{C}y$ , where  $y \in (\mathbb{C}x)^\perp$  is spacelike. Since the spacelike vectors generate  $(\mathbb{C}x)^\perp$ , we have proven Claim 6.  $\square$

**Lemma 3.11.** *Let  $n \geq 5$  and  $\alpha \in \Lambda^5(\mathbb{H}^{1, n})^*$  where  $\mathbb{H}^{1, n}$  is considered as real vector space. If  $\alpha$  is  $\text{SO}^0(1, n)$ -invariant, then  $\alpha = 0$ .*

*Proof:* Let  $\mathbb{H}^{1,n} = \mathbb{C}^{1,n} \oplus \mathbb{C}^{1,n} \cdot \mathbf{j}$  and consider  $\mathcal{V} = \mathbb{C}^{1,n}$  as real vector space. Denote by  $J : \mathbb{H}^{1,n} \rightarrow \mathbb{H}^{1,n}$  the right multiplication with  $\mathbf{j}$ .

The value of  $\alpha$  is given by the values of  $\alpha|_{\mathcal{V}}, J^* \alpha|_{\mathcal{V}} \in \Lambda^5 \mathcal{V}^*, \alpha(J \cdot, \cdot, \cdot, \cdot, \cdot)|_{\mathcal{V}}, \alpha(\cdot, J \cdot, J \cdot, J \cdot, J \cdot)|_{\mathcal{V}} \in \mathcal{V}^* \otimes \Lambda^4 \mathcal{V}^*$  and  $\alpha(J \cdot, J \cdot, \cdot, \cdot, \cdot)|_{\mathcal{V}}, \alpha(\cdot, \cdot, J \cdot, J \cdot, J \cdot)|_{\mathcal{V}} \in \Lambda^2 \mathcal{V}^* \otimes \Lambda^3 \mathcal{V}^*$ . All these forms are  $\mathrm{SO}^0(1, n)$ -invariant. It follows from the Lemmas 3.8, 3.9 and 3.10 that they all vanish. Hence,  $\alpha = 0$ .  $\square$

**Lemma 3.12.** *Let  $n \geq 3$  and  $\beta : \mathbb{F}^{1,n} \times \mathbb{F}^{1,n} \rightarrow \mathbb{F}^{1,n}$  be an  $\mathbb{R}$ -bilinear form. If  $\beta$  is  $\mathrm{SO}^0(1, n)$ -equivariant, then  $\beta = 0$ .*

*Proof:* First we consider the case  $\mathbb{F} = \mathbb{R}$ . Let  $v, w \in \mathbb{R}^{1,n}$  be two linearly independent lightlike vectors and  $\mathcal{V} := \mathrm{span}_{\mathbb{R}} \{v, w\}$ . We define the group  $G := \{A \in \mathrm{SO}^0(1, n) | A(\mathcal{V}) = \mathcal{V}\}$ . Notice that the action of  $G$  on  $\mathcal{V}$  is equivalent to the action of  $\mathrm{SO}^0(1, 1)$  on  $\mathbb{R}^{1,1}$  and that  $\mathcal{V}^\perp$  is  $G$ -invariant. Hence, for every  $A \in G$  there exists some  $\lambda \in \mathbb{R}^*$  such that  $Av = \lambda v$ ,  $Aw = \frac{1}{\lambda} w$ . The  $\mathrm{SO}^0(1, n)$ -equivariance of  $\beta$  implies that the vector  $\beta(v, w)$  is fixed by  $G$ . Since  $G$  fixes no non-zero in  $\mathcal{V}$ , we obtain  $\beta(v, w) \in \mathcal{V}^\perp$ . The action of  $G$  on  $\mathcal{V}^\perp$  is equivalent to the action of  $\mathrm{SO}(n-1)$  on  $\mathbb{R}^{n-1}$ . Since  $n-1 \geq 2$ ,  $\mathrm{SO}(n-1)$  acts  $\mathbb{R}$ -irreducibly on  $\mathbb{R}^{n-1}$  and we obtain  $\beta(v, w) = 0$ . Next we show that  $\beta(v, v) = 0$ . The one-dimensional subspace  $\mathbb{R} \cdot \beta(v, v)$  is  $G$ -invariant. Since  $G$  acts  $\mathbb{R}$ -irreducibly on the at least two-dimensional subspace  $\mathcal{V}^\perp$ , it follows  $\beta(v, v) \in \mathcal{V}$ . Let  $A \in G$  be non-trivial. Then there exists a real number  $\lambda = e^t, t \in \mathbb{R}^*$ , such that  $Av = \lambda v$  and  $Aw = \frac{1}{\lambda} w$ . Since  $A\beta(v, v) = \lambda^2 \beta(v, v)$ ,  $\beta(v, v)$  is an eigenvector of  $A$ . Notice that  $A$  decomposes the subspace  $\mathcal{V}$  into the eigenspaces of  $\lambda$  and  $\frac{1}{\lambda}$ . Since  $\lambda^2$  is no eigenvalue of  $A$ , it follows  $\beta(v, v) = 0$ . Analogously, we obtain  $\beta(w, w) = 0$ . Since the lightlike vectors generate  $\mathbb{R}^{1,n}$ , we finally conclude  $\beta = 0$ .

If  $\mathbb{F} = \mathbb{C}$  then we restrict  $\beta$  pairwise to the subspaces  $\mathbb{R}^{1,n}$  and  $i\mathbb{R}^{1,n}$ , project it to one of the subspaces and apply the above. Analogously, for  $\mathbb{F} = \mathbb{H}$  we do the same for the subspaces  $\mathbb{R}^{1,n}, \mathbb{R}^{1,n}\mathbf{i}, \mathbb{R}^{1,n}\mathbf{j}$ , and  $\mathbb{R}^{1,n}\mathbf{k}$ . This finishes the proof.  $\square$

**Remark 3.4.** *Lemma 3.12 is not true for  $n = 2$ . If  $\mathbb{F} = \mathbb{R}$ , then the pseudo-Euclidean analogue of the cross-product in the  $\mathbb{R}^{1,2}$  is a counterexample. More precisely, the antisymmetric  $\mathbb{R}$ -bilinear map*

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \times \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} := \begin{pmatrix} -(x_2 y_3 - y_2 x_3) \\ -(x_1 y_3 - y_1 x_3) \\ x_1 y_2 - y_1 x_2 \end{pmatrix}$$

with  $(x_1, x_2, x_3)^T, (y_1, y_2, y_3)^T \in \mathbb{R}^3$  is  $\mathrm{SO}^0(1, 2)$ -equivariant.

**Remark 3.5.** *Recall that we can consider an  $\mathbb{R}$ -bilinear map  $\beta : \mathbb{F}^{1,n} \times \mathbb{F}^{1,n} \rightarrow \mathbb{F}^{1,n}$  as an element of  $(\mathbb{F}^{1,n})^* \otimes (\mathbb{F}^{1,n})^* \otimes \mathbb{F}^{1,n}$ . If  $\beta$  is  $\mathrm{SO}^0(1, n)$ -equivariant, then  $\beta$  is  $\mathrm{SO}^0(1, n)$ -invariant as an element of  $(\mathbb{F}^{1,n})^* \otimes (\mathbb{F}^{1,n})^* \otimes \mathbb{F}^{1,n}$ . Since the representations of  $\mathrm{SO}^0(1, n)$  on the vector spaces  $\mathbb{F}^{1,n}$  and  $(\mathbb{F}^{1,n})^*$  are equivalent, we are for  $\mathbb{F} = \mathbb{C}$  in the situation of Lemma 3.4. In other words for  $\mathbb{F} = \mathbb{C}$  the Lemmas 3.12 and 3.4 are equivalent. Analogously, for  $\mathbb{F} = \mathbb{H}$  Lemma 3.12 implies Lemma 3.6. Nevertheless we gave two different proofs. The*

advantage of the proof of Lemma 3.4 is that it can be easily modified to prove that an  $\mathbb{R}$ -trilinear map  $\alpha : \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \times \mathbb{C}^{n+1} \rightarrow \mathbb{R}$  vanishes, if  $n \geq 3$  and  $\alpha$  is  $\mathrm{SO}(n+1)$ -invariant, see Lemma 3.5. The proof of Lemma 3.12 does not work for such a map  $\alpha$ , since it makes use of the existence of lightlike vectors which do not exist in the Euclidean space.

### 3.2 Connected $\mathbb{H}$ -irreducible Lie subgroups of $\mathrm{Sp}(1, n)$

**Proposition 3.1.** *Let  $H \subset \mathrm{Sp}(1, n)$  be a non-precompact, connected Lie subgroup. Then*

- (i)  *$H$  is amenable if and only if  $H$  is contained in a stabilizer of a lightlike line.*
- (ii) *If  $H$  is non-amenable, then  $H$  is conjugated to one of the following groups*
  - *$K \cdot (\mathrm{SO}^0(1, m) \times \{\mathbb{1}_{n-m}\})$ , where  $2 \leq m \leq n$  and  $K \subset (\mathrm{Sp}(1) \cdot \mathbb{1}_{m+1}) \times \mathrm{Sp}(n-m)$  is a connected Lie subgroup, such that its projection on  $\mathrm{Sp}(1) \cdot \mathbb{1}_{m+1}$  is either  $\{\mathbb{1}_{m+1}\}$ ,  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$ , or  $\mathrm{Sp}(1) \cdot \mathbb{1}_{m+1}$ ,*
  - *$K \cdot (\mathrm{SU}(1, m) \times \{\mathbb{1}_{n-m}\})$ , where  $1 \leq m \leq n$  and  $K \subset (\mathrm{U}(1) \cdot \mathbb{1}_{m+1}) \times \mathrm{Sp}(n-m)$  is a connected Lie subgroup, such that its projection on  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$  is either  $\{\mathbb{1}_{m+1}\}$  or  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$ ,*
  - *$K \cdot (\mathrm{Sp}(1, m) \times \{\mathbb{1}_{n-m}\})$ , where  $1 \leq m \leq n$  and  $K \subset \{\mathbb{1}_{m+1}\} \times \mathrm{Sp}(n-m)$  is a connected Lie subgroup,*
  - *$K \cdot (U^0 \times \{\mathbb{1}_{n-1}\})$ , where  $U^0 = \{A \in \mathrm{Sp}(1, 1) | A\Phi = \Phi A\} \cong \mathrm{Spin}^0(1, 3)$  with  $\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $K \subset \{\mathbb{1}_2\} \times \mathrm{Sp}(n-1)$  is a connected Lie subgroup.*

*Proof:* Let  $H \subset \mathrm{Sp}(1, n)$  be a non-precompact, connected Lie subgroup. We denote by  $H^n(\mathbb{H}) = \mathrm{Sp}(1, n)/\mathrm{Sp}(1) \times \mathrm{Sp}(n)$  the quaternionic hyperbolic space. The Lie group  $\mathrm{Sp}(1, n)$  acts on  $H^n(\mathbb{H})$  and on its boundary  $\partial H^n(\mathbb{H})$ , which we identify with the set of lightlike lines of  $\mathbb{H}^{1, n}$ .

*Proof of (i):* Suppose  $H$  is contained in the stabilizer of a lightlike line. The stabilizers of a lightlike line are conjugated to

$$(\mathbb{H}^* \times \mathrm{Sp}(n-1)) \ltimes \mathrm{Heis}_{n-1}(\mathbb{H}),$$

which is amenable, see Example 2.2. Hence  $H$  is conjugated to a connected Lie subgroup of  $(\mathbb{H}^* \times \mathrm{Sp}(n-1)) \ltimes \mathrm{Heis}_{n-1}(\mathbb{H})$ . By Proposition 2.7 we know that connected Lie subgroups of amenable groups are amenable. Hence,  $H$  itself is amenable.

Now we prove the converse. Let  $H$  be amenable and denote by  $H = L \cdot R$  its Levi decomposition. By Proposition 2.9 the semi-simple Levi factor  $L$  is compact.

Now we apply Theorem 2.7 to  $H$ . We will show that (a) holds for  $H$ . We can exclude the case (c) just because  $\mathbb{F} = \mathbb{H} \neq \mathbb{C}$ . Suppose we are in case (d) which means  $H = \mathrm{Sp}(1, n)$ . Since  $\mathrm{Sp}(1, n)$  is a simple Lie group and non-compact, this is a contradiction to the amenability of  $H$ , so this is not possible. Suppose we are in case (b). Hence,

there exists a proper totally geodesic submanifold  $M \subset H^n(\mathbb{H})$  such that  $\dim(M) > 1$  and  $H = K \cdot U^0(M)$  where  $K \subset K^0(M)$ . By Proposition 2.14 the Lie group  $U^0(M)$  is simple. Furthermore, according to Table 2.1 the Lie group  $U^0(M)$  is non-precompact. This is again a contradiction to the amenability of  $H$ . It follows that we are in case (a). Hence, the elements of  $H$  have a common fixed point  $p$  in  $\overline{H^n(\mathbb{H})}$ . Suppose  $p \in H^n(\mathbb{H})$ . Then  $H$  is conjugated to a connected subgroup of  $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ . This implies that  $H$  is precompact which is a contradiction to the non-precompactness of  $H$ . It follows that the fixed point  $p$  lies in  $\partial H^n(\mathbb{H})$ . Equivalently,  $H$  preserves a lightlike line. This finishes the proof of (i).

*Proof of (ii):* By Proposition 2.9 the semi-simple Levi factor of  $H$  is non-compact. Now we apply Theorem 2.7 to  $H$  and consider the four cases (a), (b), (c), and (d). As before we exclude the case (c). Suppose we are in (a). Then the elements of  $H$  have a common fixed point  $p$  in  $\overline{H^n(\mathbb{H})}$ . Since  $H$  is non-amenable, (i) implies that  $p \in H^n(\mathbb{H})$ . But this is again a contradiction to the non-precompactness of  $H$ . Hence, we are not in case (a). If we are in case (b), then there exists a proper totally geodesic submanifold  $M \subset H^n(\mathbb{H})$  such that  $\dim M > 1$  and  $H = KU^0(M)$  where  $K \subset K^0(M)$  is a connected Lie subgroup. According to Proposition 2.12 there are four possibilities for  $M$ .

*Case 1:*  $M = H^m(\mathbb{R})$  for some  $2 \leq m \leq n$ . By Table 2.1 we obtain  $H = K \cdot (\mathrm{SO}^0(1, m) \times \{\mathbb{1}_{n-m}\})$  where  $K \subset \mathrm{Sp}(1) \cdot \mathbb{1}_{m+1} \times \mathrm{Sp}(n-m)$ . Let  $C$  be the projection of  $K$  to  $\mathrm{Sp}(1) \cdot \mathbb{1}_{m+1}$  and denote by  $C = L \cdot R$  its Levi decomposition where  $L$  is connected and semi-simple and  $R$  is connected and solvable. Recall that  $L \cap R$  is discrete. Since  $R$  is precompact,  $\overline{R}$  is compact. It is known that a compact solvable Lie group is a torus. Hence,  $R$  is abelian. A maximal torus of  $\mathrm{Sp}(1) \cdot \mathbb{1}_{m+1}$  is for example  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$  which is a one-dimensional Lie group. If  $R$  is non-trivial, then it follows  $R = \mathrm{U}(1) \cdot \mathbb{1}_{m+1}$ , since  $R$  is connected. If  $L$  is non-trivial, it is at least three-dimensional. Since  $L$  is connected, it follows  $L = \mathrm{Sp}(1) \cdot \mathbb{1}_{m+1}$ . Notice that  $L$  and  $R$  can not be both non-trivial, since  $L \cap R$  is discrete. This gives us the first case in the list of (ii).

*Case 2:*  $M = H^m(\mathbb{C})$  for some  $1 \leq m \leq n$ . By Table 2.1 we obtain  $H = K \cdot (\mathrm{SU}(1, m) \times \{\mathbb{1}_{n-m}\})$  where  $K \subset \mathrm{U}(1) \cdot \mathbb{1}_{m+1} \times \mathrm{Sp}(n-m)$  is a connected Lie subgroup. Let  $C$  be the projection of  $K$  on  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$  and denote by  $C = L \cdot R$  its Levi decomposition. Since  $\mathrm{U}(1)$  is one-dimensional, the Levi factor  $L$  is trivial. If the solvable radical  $R$  is non-trivial, then it follows  $R = \mathrm{U}(1) \cdot \mathbb{1}_{m+1}$ , since  $R$  is connected. This gives us the second case in the list of (ii).

*Case 3:*  $M = H^m(\mathbb{H})$  for some  $1 \leq m < n$ . By Table 2.1 we obtain  $H = K \cdot \mathrm{Sp}(1, m) \times \{\mathbb{1}_{n-m}\}$  where  $K \subset \{\mathbb{1}_{m+1}\} \times \mathrm{Sp}(n-m)$  is a connected Lie subgroup. This is the third case in the list of (ii) with  $m < n$ .

*Case 4:*  $M = H^1(\mathbb{I})$ . By Table 2.1 we obtain  $H = KU^0 \times \{\mathbb{1}_{n-1}\}$  where  $K \subset \{\mathbb{1}_2\} \times \mathrm{Sp}(n-1)$  is a connected Lie subgroup and  $U^0 = \{A \in \mathrm{Sp}(1, 1) | A\Phi = \Phi A\} \cong \mathrm{Spin}^0(1, 3)$ . This is the fourth case in the list of (ii).

Finally, by Theorem 2.7, there is the last case (d). Then we have  $H = \mathrm{Sp}(1, n)$  which is

the third case in the list of (ii) with  $m = n$ . This finishes the proof.  $\square$

**Remark 3.6.** *The methods in the proof of Proposition 3.1 can be applied to the connected, non-precompact Lie subgroups of  $\mathrm{U}(1, n)$  and  $\mathrm{O}(1, n)$ . Thus there exist complex and real versions of Proposition 3.1. The complex version is given in [AZ].*

**Proposition 3.2** ([AZ]). *Let  $H \subset \mathrm{U}(1, n)$  be a connected, non-precompact Lie subgroup. Then*

- (i)  *$H$  is amenable if and only if  $H$  is contained in the stabilizer of a lightlike line.*
- (ii) *If  $H$  is non-amenable, then  $H$  is conjugated to one of the following groups*
  - *$K \cdot (\mathrm{SO}^0(1, m) \times \{\mathbb{1}_{n-m}\})$  where  $2 \leq m \leq n$  and  $K \subset (\mathrm{U}(1) \cdot \mathbb{1}_{m+1}) \times \mathrm{U}(n-m)$  is a connected Lie subgroup, such that its projection on  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$  is either  $\{\mathbb{1}_{m+1}\}$  or  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$ ,*
  - *$K \cdot (\mathrm{SU}(1, m) \times \{\mathbb{1}_{n-m}\})$  where  $1 \leq m \leq n$  and  $K \subset \mathrm{U}(1) \cdot \mathbb{1}_{m+1} \times \mathrm{U}(n-m)$  is a connected Lie subgroup, such that its projection to  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$  is either  $\{\mathbb{1}_{m+1}\}$  or  $\mathrm{U}(1) \cdot \mathbb{1}_{m+1}$ .*

**Proposition 3.3.** *Let  $H \subset \mathrm{O}(1, n)$  be a connected, non-precompact Lie subgroup. Then*

- (i)  *$H$  is amenable if and only if  $H$  is contained in the stabilizer of a lightlike line.*
- (ii) *If  $H$  is non-amenable, then  $H$  is conjugated to  $K \cdot (\mathrm{SO}^0(1, m) \times \{\mathbb{1}_{n-m}\})$  where  $2 \leq m \leq n$  and  $K \subset \{\mathbb{1}_{m+1}\} \times \mathrm{SO}(n-m)$  is a connected Lie subgroup.*

**Corollary 3.1.** *Let  $H \subset \mathrm{Sp}(1, n)$  be a connected and  $\mathbb{H}$ -irreducible Lie subgroup. Then  $H$  is conjugated to one of the following groups:*

- $\mathrm{SO}^0(1, n)$ ,  $\mathrm{SO}^0(1, n) \cdot \mathrm{U}(1)$ ,  $\mathrm{SO}^0(1, n) \cdot \mathrm{Sp}(1)$  if  $n \geq 2$ ,
- $\mathrm{SU}(1, n)$ ,  $\mathrm{U}(1, n)$ ,
- $\mathrm{Sp}(1, n)$ ,
- $U^0 = \{A \in \mathrm{Sp}(1, 1) \mid A\Phi = \Phi A\} \cong \mathrm{Spin}^0(1, 3)$  with  $\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  if  $n = 1$ .

**Remark 3.7.** *Notice that only the Lie groups  $\mathrm{Sp}(1, n)$ ,  $\mathrm{SO}^0(1, n) \cdot \mathrm{Sp}(1)$  and  $U^0 \cong \mathrm{Spin}^0(1, 3)$  act  $\mathbb{C}$ -irreducibly on  $\mathbb{H}^{1, n}$ . The other Lie groups  $\mathrm{SO}^0(1, n)$ ,  $\mathrm{SO}^0(1, n) \cdot \mathrm{U}(1)$ ,  $\mathrm{SU}(1, n)$ , and  $\mathrm{U}(1, n)$  act  $\mathbb{C}$ -reducibly on  $\mathbb{H}^{1, n}$ .*

*Furthermore, the matrices in  $U^0$  have the form  $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ . Hence,  $U^0$  does not contain  $\mathrm{SO}^0(1, 1)$ .*

*Proof:* The idea is to apply Proposition 3.1. Notice that  $H$  is non-precompact since otherwise it would be contained in a maximal compact subgroup of  $\mathrm{Sp}(1, n)$ , which is conjugated to  $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ . In that case  $H$  would not act  $\mathbb{H}$ -irreducibly contradicting the assumption.

By Proposition 3.1 we have to consider the amenable and non-amenable case. If  $H$  is amenable, then it is contained in the stabilizer of a lightlike line. This is again a contradiction to the  $\mathbb{H}$ -irreducibility of  $H$ .

Therefore  $H$  is conjugated to one of the Lie subgroups of the list in Proposition 3.1. If  $H$  is conjugated to one of the first three Lie groups, then it can only act irreducibly if  $m = n$ , since otherwise  $\mathbb{H}^{1,m} \subset \mathbb{H}^{1,n}$  would be an  $\mathbb{H}$ -invariant subspace. This gives us the Lie groups  $\mathrm{SO}^0(1, n)$ ,  $\mathrm{SO}^0(1, n) \cdot \mathrm{U}(1)$ ,  $\mathrm{SO}^0(1, n) \cdot \mathrm{Sp}(1)$  if  $n \geq 2$ , and  $\mathrm{SU}(1, n)$ ,  $\mathrm{U}(1, n)$  and  $\mathrm{Sp}(1, n)$  for  $n \geq 1$ . They all act  $\mathbb{H}$ -irreducibly.

So we have to consider the fourth case of Proposition 3.1,  $H = K \cdot (U^0 \times \mathbb{1}_{n-1})$  where  $U^0 \subset \mathrm{Sp}(1, 1)$  is isomorphic to  $\mathrm{Spin}^0(1, 3)$  and  $K \subset \mathbb{1}_2 \times \mathrm{Sp}(n-1)$ . The Lie group  $H$  can act irreducibly only if  $n = 1$ . Thus we have to check that  $U^0 \subset \mathrm{Sp}(1, 1)$  indeed acts  $\mathbb{H}$ -irreducibly. Since  $U^0$  is connected, it acts  $\mathbb{H}$ -irreducibly if and only if its Lie algebra  $\mathfrak{u} \subset \mathfrak{sp}(1, 1)$  acts  $\mathbb{H}$ -irreducibly on  $\mathbb{H}^{1,1}$ . So we have to show, that there is no one-dimensional  $\mathfrak{u}$ -invariant  $\mathbb{H}$ -subspace of  $\mathbb{H}^{1,1}$ .

Recall that the matrices

$$\begin{aligned} x &= \frac{i}{2}\Phi, \quad y = \frac{j}{2}\Phi, \quad z = \frac{k}{2}\Phi, \\ u &= \frac{i}{2}\mathbb{1}_2, \quad v = \frac{j}{2}\mathbb{1}_2, \quad w = \frac{k}{2}\mathbb{1}_2 \end{aligned}$$

form a basis of  $\mathfrak{u}$ . The matrices can not be diagonalized simultaneously, e.g. we have for  $x$  and  $y$  the eigenspace decompositions

$$\mathbb{H}^{1,1} = \begin{pmatrix} i \\ -1 \end{pmatrix} \cdot \mathbb{H} \oplus \begin{pmatrix} i \\ 1 \end{pmatrix} \cdot \mathbb{H} \quad \text{and} \quad \mathbb{H}^{1,1} = \begin{pmatrix} j \\ -1 \end{pmatrix} \cdot \mathbb{H} \oplus \begin{pmatrix} j \\ 1 \end{pmatrix} \cdot \mathbb{H}.$$

Hence,  $\mathfrak{u}$  acts  $\mathbb{H}$ -irreducibly. □

**Proposition 3.4.** *Let  $n \geq 2$  and  $H \subset \mathrm{Sp}(1, n)$  be an  $\mathbb{H}$ -irreducible Lie subgroup. Then the Zariski closure  $H^{Zar}$  contains a subgroup which is conjugated to  $\mathrm{SO}^0(1, n)$ . In particular, the connected component of  $H^{Zar}$  is  $\mathbb{H}$ -irreducible.*

*Proof:* We will follow the arguments of [AZ].  $H^{Zar} \subset \mathrm{Sp}(1, n)$  is an algebraic group and hence it has finitely many connected components, see [Mi]. It is also a closed subgroup with respect to the standard topology of  $\mathrm{Sp}(1, n)$ .

The group  $H^{Zar}$  is not compact since otherwise  $H \subset H^{Zar}$  would be precompact and hence it would be conjugated to a Lie subgroup of the maximal compact subgroup  $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ . Then  $H$  would preserve a one-dimensional timelike subspace contradicting its  $\mathbb{H}$ -irreducibility. In particular,  $H^{Zar}$  is non-precompact.

Let  $L := (H^{Zar})^0$  denote the connected component of the identity. It is also non-precompact since otherwise  $H^{Zar}$  could be written as finite union of precompact sets of the form  $gL$  for  $g \in H^{Zar}$  and hence  $H^{Zar}$  would be precompact.

Now we apply Theorem 2.7 to  $L$  and consider the cases (a), (b), (c), and (d). First we exclude the case (c), since  $\mathbb{F} = \mathbb{H}$ .

Next we exclude the case (a) by supposing the opposite, i.e. suppose that  $L$  has a common

fixed point in  $\overline{H^n(\mathbb{H})}$ . Notice that such a fixed point can not lie in  $H^n(\mathbb{H})$ , since otherwise  $L$  would be contained in the stabilizer of this point which is a maximal compact subgroup of  $\mathrm{Sp}(1, n)$ . This would be a contradiction to the non-precompactness of  $L$ . Hence, all common fixed points of  $L$  lie in  $\partial H^n(\mathbb{H})$ . Let  $F \subset \partial H^n(\mathbb{H})$  be the set of fixed points of  $L$ . Notice that  $|F| \leq 2$ , since otherwise Lemma 2.7 implies that every element of  $L$  is elliptic and furthermore, by Lemma 2.8,  $L$  fixes every point on the geodesic connecting two elements of  $F$ , i.e.  $L$  would preserve timelike lines which again contradicts the non-precompactness of  $L$ . So  $F$  consists of one or two elements. Since  $L$  is a normal subgroup of  $H^{Zar}$ , the set  $F$  is preserved by  $H^{Zar}$ .

If  $F = \{p\}$ , then  $H^{Zar}$  fixes  $p$ . But this is a contradiction to the irreducibility of  $H^{Zar}$ . If  $F = \{p, q\}$ , then the corresponding lightlike lines in  $\mathbb{F}^{1, n}$  span a timelike subspace  $\mathcal{V}$  with  $\dim_{\mathbb{H}} \mathcal{V} = 2$ , which is preserved by  $H^{Zar}$ . Since  $n \geq 2$ ,  $\mathcal{V}$  is a proper  $H^{Zar}$ -invariant subspace which contradicts the  $\mathbb{H}$ -irreducibility of  $H^{Zar}$ .

Summarizing we are not in case (a). Hence, we are in case (b) or (d). If we are in case (d), then  $L = \mathrm{Sp}(1, n)$  and we are done. So we have to consider the case (b), i.e. there exists a proper totally geodesic submanifold  $M$  with  $\dim M > 1$ ,  $\mathcal{L}(L) = \partial M$  and  $L = KU^0(M)$ , where  $K \subset K^0(M)$  is a connected Lie subgroup. Notice that  $M$  is the union of all geodesics whose endpoints lie in  $\partial M$ .

Since  $H^{Zar}$  has finitely many connected components,  $L$  is a subgroup of finite index. Then Lemma 2.13 implies that  $\mathcal{L}(H^{Zar}) = \mathcal{L}(L) = \partial M$  and that  $H^{Zar}$  leaves  $\partial M$  invariant. Recall that  $H^{Zar}$  acts by isometries on  $H^n(\mathbb{H})$ . In particular it maps geodesics to geodesics. Since  $M$  is the union of all geodesics whose endpoints lie in  $\partial M$ , we conclude that  $H^{Zar}$  leaves  $M$  invariant. Hence,  $H^{Zar} \subset I(M)$ . Now we study the possibilities for the proper totally geodesic submanifold  $M$  case by case.

If  $M = H^1(\mathbb{I})$ , then Lemma 2.11 implies that there is a proper  $H^{Zar}$ -invariant subspace, since  $n \geq 2$ , which contradicts the  $\mathbb{H}$ -irreducibility of  $H^{Zar}$ . If  $M = H^m(\mathbb{H})$  with  $1 \leq m < n$ , then Lemma 2.10 implies that there is a proper  $H^{Zar}$ -invariant subspace which is again a contradiction. Hence, we have  $M = H^m(\mathbb{C})$  with  $1 \leq m \leq n$  or  $M = H^m(\mathbb{R})$  with  $2 \leq m \leq n$ . Since  $H^{Zar}$  is  $\mathbb{H}$ -irreducible, Lemma 2.10 implies in both cases  $m = n$ .

Recall that  $L = KU^0(M)$  where  $K \subset K^0(M)$  is a connected Lie subgroup. Consulting Table 2.1 we obtain  $\mathrm{SO}^0(1, n) \subset U^0(M)$ . This finishes the proof.  $\square$

**Proposition 3.5.** *Let  $n \geq 2$  and  $H \subset \mathrm{U}(1, n; \mathbb{F})$  be an  $\mathbb{F}$ -irreducible subgroup. If  $N \subset H$  is a normal subgroup which is closed in  $\mathrm{U}(1, n; \mathbb{F})$ , then one of the following is true.*

- (i)  $N$  is discrete.
- (ii)  $N = \mathrm{U}(1) \cdot \mathbb{1}_{n+1}$  if  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{H}$ , or  $N = \mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}$  if  $\mathbb{F} = \mathbb{H}$ .
- (iii)  $N$  contains  $\mathrm{SO}^0(1, n)$ .

*Proof:* We will apply Theorem 2.8 to  $N$  and discuss the cases (a), (b), (c), (d), and (e). Assume that  $N$  is not discrete. If (d) or (e) holds for  $N$ , then we have  $\mathrm{SO}^0(1, n) \subset N$ , so



we are in (iii).

Assume now that (b) holds for  $N$ . If  $N$  has no common fixed point in  $H^n(\mathbb{F})$ , then there is a common fixed point in  $\partial H^n(\mathbb{F})$ . Let  $F \subset \partial H^n(\mathbb{F})$  be the set of common fixed points of  $N$  on the boundary. Notice that  $F$  consists of either one or two elements. If  $F$  has exactly one element, then  $H$  fixes this point, since  $N$  is a normal subgroup of  $H$ . But this contradicts the  $\mathbb{F}$ -irreducibility of  $H$ . If  $F$  has exactly two elements, then  $F$  is preserved by  $H$ . It follows that  $H$  preserves the two dimensional  $\mathbb{F}$ -subspace spanned by the two lightlike lines. Since  $n \geq 2$ , this is again a contradiction to the  $\mathbb{F}$ -irreducibility of  $H$ . It follows that  $N$  has a common fixed point in  $H^n(\mathbb{F})$ . Hence, all elements of  $N$  are elliptic. Let  $g \in N$  and  $F(g)$  the set of fixed points of  $g$  in  $H^n(\mathbb{F})$ . By Proposition 2.13,  $F(g)$  is either a singleton or a totally geodesic submanifold. Hence,  $M := \bigcap_{g \in N} F(g)$  is a totally geodesic submanifold and the set of all common fixed points in  $H^n(\mathbb{F})$  of  $N$ . Since  $N$  is a normal subgroup,  $M$  is preserved by  $H$ , i.e.  $H \subset I(M)$ . The  $\mathbb{F}$ -irreducibility of  $H$  implies that  $M$  is not a singleton. Lemmas 2.10 and 2.11 imply that  $M$  is  $H^n(\mathbb{R})$ ,  $H^n(\mathbb{C})$ , or  $H^n(\mathbb{H})$ . Consulting Table 2.1 we get the possibilities for  $K(M)$ . If  $\mathbb{F} = \mathbb{R}$ , then  $K(M) = \{\pm \mathbb{1}_{n+1}\}$ . If  $\mathbb{F} = \mathbb{C}$ , then  $K(M) = \{\pm \mathbb{1}_{n+1}\}$  or  $\mathrm{U}(1) \cdot \mathbb{1}_{n+1}$ . If  $\mathbb{F} = \mathbb{H}$ , then  $K(M) = \{\pm \mathbb{1}_{n+1}\}$ ,  $\mathrm{U}(1) \cdot \mathbb{1}_{n+1}$ , or  $\mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}$ . By assumption  $N$  is not discrete. Since  $\mathrm{Sp}(1)$  has no two dimensional Lie subgroup, we obtain  $N = \mathrm{U}(1) \cdot \mathbb{1}_{n+1}$  or  $\mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}$ , so we are in case (ii).

Assume now that (c) holds for  $N$ , i.e. there is a proper totally geodesic submanifold  $M$  such that  $N \subset I(M)$ . By Remark 2.12, we know  $\mathcal{L}(N^0) = \partial M$ . If we apply Lemma 2.14 to  $N^0$  and  $N$ , then it follows  $\partial M = \mathcal{L}(N^0) = \mathcal{L}(N)$ . Applying Lemma 2.14 again to  $N$  and  $H$ , it follows  $\partial M = \mathcal{L}(N) = \mathcal{L}(H)$ . Hence,  $H$  preserves  $M$ . The Lemmas 2.10 and 2.11 imply that  $M$  is either  $H^n(\mathbb{R})$  or  $H^n(\mathbb{C})$ . Since  $N^0 = KU^0(M)$ , we obtain from Table 2.1 that  $\mathrm{SO}^0(1, n) \subset N$ . This finishes the proof.  $\square$

**Corollary 3.2.** *Let  $n \geq 2$  and  $H \subset \mathrm{U}(1, n; \mathbb{F})$  be a closed subgroup, which acts  $\mathbb{F}$ -irreducibly on  $\mathbb{F}^{1, n}$ . Then one of the following is true.*

- (i)  $H$  is discrete.
- (ii)  $H^0 = \mathrm{U}(1) \cdot \mathbb{1}_{n+1}$  if  $\mathbb{F} = \mathbb{C}$  or  $\mathbb{H}$ , or  $H^0 = \mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}$  if  $\mathbb{F} = \mathbb{H}$ .
- (iii)  $H^0$  contains  $\mathrm{SO}^0(1, n)$ .

*Proof:* This follows from Proposition 3.5 by setting  $N = H^0$ .  $\square$

**Corollary 3.3.** *Let  $n \geq 2$  and  $H \subset \mathrm{Sp}(1, n)\mathrm{Sp}(1)$  be a closed subgroup which acts  $\mathbb{H}$ -irreducibly on  $\mathbb{H}^{1, n}$ . Then one of the following is true.*

- (i)  $H \cap \mathrm{Sp}(1, n)$  is discrete.
- (ii)  $H \cap \mathrm{Sp}(1, n) = \mathrm{U}(1) \cdot \mathbb{1}_{n+1}$  or  $\mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}$ .
- (iii)  $H \cap \mathrm{Sp}(1, n)$  contains  $\mathrm{SO}^0(1, n)$ .

*Proof:* A subgroup  $H \subset \mathrm{Sp}(1, n)\mathrm{Sp}(1)$  is  $\mathbb{H}$ -irreducibly if and only if  $\mathrm{pr}_{\mathrm{Sp}(1, n)}(H)$  is  $\mathbb{H}$ -irreducible. Notice that  $H \cap \mathrm{Sp}(1, n)$  is a normal subgroup of  $\mathrm{pr}_{\mathrm{Sp}(1, n)}(H)$  which is closed in  $\mathrm{Sp}(1, n)$ . The Corollary follows now from Proposition 3.5.  $\square$

# Chapter 4

## Main results

### 4.1 Classification of isotropy $\mathbb{H}$ -irreducible almost hypercomplex pseudo-Hermitian manifolds of index 4

**Theorem 4.1.** *Let  $(M, g, (I, J, K))$  be a connected almost hypercomplex pseudo-Hermitian manifold of index 4 and  $\dim M = 4n + 4 \geq 16$ , such that there exists a connected Lie group  $G \subset \text{Iso}(M, g, (I, J, K))$  acting transitively on  $M$ . If the isotropy group  $H := G_p$ ,  $p \in M$ , acts  $\mathbb{H}$ -irreducibly, then  $(M, g, (I, J, K))$  is a pseudo-hyper-Kähler manifold. If furthermore  $\mathfrak{h}$  acts  $\mathbb{H}$ -irreducibly, then  $(M, g, (I, J, K))$  is locally isometric to  $\text{Mink}_{n+1}(\mathbb{H})$ .*

*Proof:* Let  $\rho : H \rightarrow \text{GL}(T_p M)$  be the isotropy representation. We identify  $H$  with its image  $\rho(H)$ . Since  $H$  preserves the metric  $g$  and the almost hypercomplex structure, we can consider  $H$  as a subgroup of  $\text{Sp}(1, n)$ .

First we show that  $M$  is a hyper-Kähler manifold. This follows from Lemma 2.5 by showing that the Kähler forms  $\omega_I$ ,  $\omega_J$ , and  $\omega_K$  are closed. Notice that since  $H$  preserves the hypercomplex structure and the metric  $g$ , it preserves  $d(\omega_I)_p$ ,  $d(\omega_J)_p$ , and  $d(\omega_K)_p$ , too. Since  $G$  acts transitively, it is sufficient to show that  $(d\omega_I)_p = 0$ ,  $(d\omega_J)_p = 0$ , and  $(d\omega_K)_p = 0$ . Lemma 2.15 implies that  $H^{\text{Zar}}$  also preserve the three 3-forms. By Proposition 3.4 we know that  $\text{SO}^0(1, n) \subset H^{\text{Zar}}$ . It follows that the three 3-forms are also preserved by  $\text{SO}^0(1, n)$ . If we identify  $T_p M$  with  $\mathbb{H}^{1, n}$ , we can consider the 3-forms as elements of  $\Lambda^3(\mathbb{H}^{1, n})^*$ . By assumption we have  $n \geq 3$ . Thus we can apply Lemma 3.6 and conclude  $d(\omega_I)_p = 0$ ,  $d(\omega_J)_p = 0$ , and  $d(\omega_K)_p = 0$ . Hence,  $M$  is a pseudo-hyper-Kähler manifold.

From now on we assume that  $\mathfrak{h}$  acts  $\mathbb{H}$ -irreducibly. In order to prove that  $M$  is locally isometric to  $\text{Mink}_{n+1}(\mathbb{H})$  we will investigate its universal cover  $\tilde{M} = \tilde{G}/H^0$  and show that it is globally isometric to  $\text{Mink}_{n+1}(\mathbb{H})$ . Since  $\mathfrak{h}$  acts  $\mathbb{H}$ -irreducibly, the same holds for  $H^0$ . According to Corollary 3.1,  $H^0$  is one of the following Lie groups

$$\text{SO}^0(1, n), \text{SO}^0(1, n) \cdot \text{U}(1), \text{SO}^0(1, n) \cdot \text{Sp}(1), \text{SU}(1, n), \text{U}(1, n), \text{Sp}(1, n).$$

Next we show that  $\tilde{M}$  is a reductive homogeneous space, i.e. we show that there exists an  $\text{Ad}(H^0)$ -invariant subspace  $\mathfrak{m} \subset \mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ .

If  $H^0$  is one of the semi-simple Lie groups above, then the action of  $\text{Ad}(H^0)$  on  $\mathfrak{g}$  is completely reducible. In that case  $\tilde{M}$  is a reductive homogeneous space. So we have to consider the cases where  $H^0$  is either  $\text{SO}^0(1, n) \cdot \text{U}(1)$  or  $\text{U}(1, n) = \text{SU}(1, n) \cdot \text{U}(1)$ .

Let  $\mathfrak{s}$  be either  $\mathfrak{so}(1, n)$  or  $\mathfrak{su}(1, n)$ . Then we have  $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{u}(1)$ . We consider the adjoint representation of  $\mathfrak{s}$  on  $\mathfrak{g}$ . Since  $\mathfrak{s}$  is simple,  $\mathfrak{s}$  acts completely reducibly on  $\mathfrak{g}$  and  $\mathfrak{s}$  is an irreducible  $\mathfrak{s}$ -invariant subspace. Furthermore, there exists an  $\mathfrak{s}$ -invariant complement  $\mathfrak{m}$  of  $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{u}(1)$  which is isomorphic to  $\mathfrak{g}/\mathfrak{h} \cong T_p \tilde{M} \cong \mathbb{H}^{1, n}$ . Hence, the  $\mathfrak{s}$ -module  $\mathfrak{g}$  decomposes into  $\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{s} \oplus \mathfrak{u}(1)$ . Notice that  $\mathfrak{m} \cong \mathbb{H}^{1, n}$  decomposes into four respectively two irreducible  $\mathfrak{s}$ -invariant subspaces which are equivalent to  $\mathbb{R}^{1, n}$  respectively  $\mathbb{C}^{1, n}$ . These three submodules  $\mathfrak{s}$ ,  $\mathfrak{u}(1)$ ,  $\mathbb{R}^{1, n}$  respectively  $\mathbb{C}^{1, n}$  are pairwise inequivalent. Since  $\mathfrak{s}$  and  $\mathfrak{u}(1)$  commute,  $\mathfrak{u}(1)$  preserve the isotypical  $\mathfrak{s}$ -submodules. It follows that the isotypical submodule  $\mathfrak{m}$  is  $\mathfrak{u}(1)$ -invariant and thus also  $\mathfrak{h}$ -invariant. Hence,  $\mathfrak{m}$  is invariant under  $\text{Ad}(H^0)$ . Thus we have shown that  $\tilde{M}$  is a reductive homogeneous space.

From now on we identify  $\mathfrak{m}$  with the tangent space  $T_p \tilde{M} \cong \mathbb{H}^{1, n}$ . Next we show that  $(\mathfrak{g}, \mathfrak{h}, \tau)$  is a symmetric Lie algebra where  $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$  is defined by  $X + Y \mapsto X - Y$  with  $X \in \mathfrak{h}$ ,  $Y \in \mathfrak{m}$ . By Remark 2.1, it is sufficient to show that  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ . For that we consider the restriction of the Lie bracket  $[\cdot, \cdot]$  to  $\mathfrak{m} \times \mathfrak{m}$  and denote its projection to  $\mathfrak{m}$  by  $\beta$ . We have to show that  $\beta$  vanishes. We know that  $\beta$  is  $\text{Ad}(H^0)$ -equivariant. Since  $n \geq 3$ , we know  $\text{SO}^0(1, n) \subset H^0$ . Thus we can apply Lemma 3.12 by identifying  $\mathfrak{m} \cong \mathbb{H}^{1, n}$  and obtain  $\beta = 0$ . So  $(\mathfrak{g}, \mathfrak{h}, \tau)$  is indeed a symmetric Lie algebra.

The above implies that  $\tilde{G}/H^0$  is a symmetric space, since  $\tilde{G}$  is simply connected and  $H^0$  connected, see Remark 2.1. Now we study the six possibilities for  $H^0$  case by case.

*Case 1:* Assume  $H^0 = \text{Sp}(1, n)$ . We will show that the quaternionic sectional curvature of  $\tilde{M}$  is constant. Let  $X \in T_p \tilde{M} \cong \mathbb{H}^{1, n}$  be a non-lightlike vector such that  $g_p(X, X) = \pm 1$ ,  $Q(X) := \text{span}_{\mathbb{R}} \{X, I_p X, J_p X, K_p X\}$ ,  $E \subset Q(X)$  be a two-dimensional subspace, and  $C_p(E)$  the quaternionic sectional curvature, see Definition 2.22. Let  $Y = Xq_1$ ,  $Z = Xq_2$  be an orthonormal basis of  $E$  where  $q_1, q_2 \in \mathbb{H} \cong \text{span}_{\mathbb{R}} \{\text{Id}_{T_p \tilde{M}}, I_p, J_p, K_p\}$ ,  $|q_1| = |q_2| = 1$ . Notice that since  $g_p(Y, Z) = 0$ ,  $q_1$  and  $q_2$  are orthogonal considered as elements of  $\mathbb{R}^4$  endowed with the standard Euclidean inner product. Since  $\text{Sp}(1, n)$  acts transitively on the pseudo-spheres there exists an element  $A \in \text{Sp}(1, n)$  such that  $AX = Xq_1^{-1}$ . Since  $H^0$  acts by isometries,  $A$  preserves the curvature. Hence, we obtain  $g_p(R_p(Xq_1, Xq_2)Xq_2, Xq_1) = g_p(R_p(X, Xq_1^{-1}q_2)Xq_1^{-1}q_2, X)$ . Recall that two quaternions are conjugated if and only if they have the same real part and the same norm, see [CG, Lemma 1.2.2]. Let  $\lambda \in \mathbb{C}$  be a complex number such that  $\lambda = \mu q_1^{-1} q_2 \mu^{-1}$  for some  $\mu \in \mathbb{H}$  with  $|\mu| = 1$ . Since  $\mu^{-1}$  has norm one,  $g_p$  is invariant under multiplication with  $\mu^{-1}$  from the right. Let  $B \in \text{Sp}(1, n)$  such that  $BX = X\mu$ . Then  $B$  preserves the curvature and by the invariance under  $\mu^{-1}$  we obtain  $g_p(R_p(X, Xq_1^{-1}q_2)Xq_1^{-1}q_2, X) = g_p(R_p(X, X\lambda)X\lambda, X)$ . Since  $X$  and  $X\lambda$  are orthonormal, it follows  $\lambda = \pm i$ . Hence,  $C_p(E) = C_p(F)$ , where  $F = \text{span}_{\mathbb{R}} \{X, I_p X\}$ , i.e. the quaternionic sectional curvature is independent of the two-dimensional subspace of  $Q(X)$ . Furthermore, we can find an element in  $\text{Sp}(1, n)$  which maps  $X$  to any other vector

$Y \in T_p M$  with  $g_p(X, X) = g_p(Y, Y)$ . This implies  $C_p(E) = C_p(F)$  for all two-dimensional subspaces  $E \subset Q(X)$  and  $F \subset Q(Y)$ .

Let  $c \in \mathbb{R}$  be the quaternionic sectional curvature for the two-dimensional subspaces of  $Q(X)$  with  $X$  spacelike. Then  $C_p - c$  is the restriction of a polynomial with infinitely many zeros. Hence, the quaternionic sectional curvature at  $p$  is constant independent whether  $X$  is a time- or spacelike vector. Since  $\tilde{M}$  is a homogeneous space, the quaternionic sectional curvature is constant for all points. By Theorem B.2 this implies that  $\tilde{M}$  is globally isometric to  $\text{Mink}_{n+1}(\mathbb{H})$ ,  $\text{dS}_{n+1}(\mathbb{H})$ , or to the universal cover of  $\text{AdS}_{n+1}(\mathbb{H})$  up to scale. Since  $\text{Mink}_{n+1}(\mathbb{H})$  is the only Ricci-flat manifold in the list, we have  $\tilde{M} \cong \text{Mink}_{n+1}(\mathbb{H})$ .

*Case 2:* Assume that  $H^0$  is  $\text{SO}^0(1, n) \cdot \text{U}(1)$ ,  $\text{SU}(1, n)$ ,  $\text{U}(1, n)$ , or  $\text{SO}^0(1, n)$ . Then  $H^0$  is  $\mathbb{C}$ -reducible.

We will decompose the simply connected manifold  $\tilde{M}$  into a product of two totally geodesic submanifolds by using Theorem 2.5, so we have to check the assumptions. Since  $\tilde{M}$  is a symmetric space, it is geodesically complete. In order to apply Theorem 2.5 we have to decompose the tangent bundle into two non-degenerate, orthogonal, and parallel distributions.

The tangent space  $T_p \tilde{M}$  decomposes into two  $H^0$ -invariant orthogonal  $I_p$ -complex subspaces of real index 2. We can identify these spaces with  $\mathbb{C}^{1,n}$  and  $\mathbb{C}^{1,n} \cdot j$ . Let  $q \in \tilde{M}$  and  $g \in \tilde{G}$  such that  $gp = q$ . Then we define two geometric distributions by  $(\mathcal{E}_1)_q := dg_p(\mathbb{C}^{1,n})$  and  $(\mathcal{E}_2)_q := dg_p(\mathbb{C}^{1,n} \cdot j)$ . Notice that the definition is independent of the choice of  $g$ . Both distributions are non-degenerate and orthogonal, so we have  $T\tilde{M} = \mathcal{E}_1 \oplus \mathcal{E}_2$ . Next we show that  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are both parallel, i.e. we have to show that  $\nabla_X Y \in \Gamma(\mathcal{E}_j)$  for all  $X \in \Gamma(TM)$  and  $Y \in \Gamma(\mathcal{E}_j)$ . We define a tensor field by

$$T : \Gamma(T\tilde{M}) \times \Gamma(\mathcal{E}_j) \rightarrow \Gamma(\mathcal{E}_j^\perp), (X, Y) \mapsto \text{pr}_{\mathcal{E}_j^\perp} \nabla_X Y.$$

If we restrict the first component of  $T$  to  $\Gamma(\mathcal{E}_j)$ , we obtain at  $p$  an  $\mathbb{R}$ -bilinear map  $\beta : \mathbb{C}^{1,n} \times \mathbb{C}^{1,n} \rightarrow \mathbb{C}^{1,n}$  which is  $H^0$ -equivariant. Since  $n \geq 3$  and  $\text{SO}^0(1, n) \subset H^0$ , we can apply Lemma 3.12 and get  $\beta = 0$ . Analogously, we obtain the same if we restrict the first component of  $T$  to  $\Gamma(\mathcal{E}_j^\perp)$ . Since  $\tilde{M}$  is a homogeneous space, this implies  $\nabla_X Y \in \Gamma(\mathcal{E}_j)$  for  $X \in \Gamma(T\tilde{M})$  and  $Y \in \Gamma(\mathcal{E}_j)$ .

Now Theorem 2.5 implies that  $(\tilde{M}, g)$  is isometric to the product of the maximal connected integral submanifolds of  $\mathcal{E}_1$  and  $\mathcal{E}_2$  through  $p$  which we denote by  $M_1(p)$  and  $M_2(p)$ . Next we show that these submanifolds are homogeneous. Let  $q \in M_1(p)$  and  $g \in \tilde{G}$  such that  $gp = q$ . We will show that  $gM_1(p) \subset M_1(p)$  by considering the associated foliation of the distribution  $\mathcal{E}_1$ . We have  $gM_1(p) = M_1(q)$ . Since  $g$  maps leaves to leaves, it maps the leaf which contains  $p$  and  $q$  into itself. It follows that  $M_1(q) = M_1(p)$ . Analogously, we obtain  $gM_2(p) \subset M_2(p)$  if  $gp \in M_2(p)$ .

Since  $\tilde{G}$  acts transitively on  $\tilde{M}$ , the group  $G_j := \left\{ g \in \tilde{G} \mid gM_j(p) \subset M_j(p) \right\}$  acts transitively on  $M_j(p)$ . Since  $M_j(p)$  is connected,  $G_j^0$  acts transitively on  $M_j(p)$ . Its isotropy group  $(G_j^0)_p$  coincides with  $H^0$  which acts  $\mathbb{C}$ -irreducibly on  $T_p M_j(p) \cong \mathbb{C}^{1,n}$ . If we restrict the almost complex structure  $I$  to  $M_j(p)$  and denote it by  $I_j$ , we can apply Theorem 1.1

to  $(M_j(p), g_j, I_j)$ . Hence,  $M_j(p)$  is globally isometric to  $\text{Mink}_{n+1}(\mathbb{C})$ ,  $\text{dS}_{n+1}(\mathbb{C})$ ,  $\text{CdS}_{n+1}$  or to the universal cover of  $\text{AdS}_{n+1}(\mathbb{C})$  or  $\mathbb{C}\text{AdS}_{n+1}$ .

Since  $\tilde{M}$  is a pseudo-hyper-Kähler manifold, it is Ricci-flat. Let  $\text{Ric}_j$  denote the Ricci-tensor of  $M_j(p)$ . Since  $g = g_1 + g_2$ , we obtain  $\text{Ric} = \text{Ric}_1 + \text{Ric}_2$ . This implies that  $M_1(p)$  and  $M_2(p)$  are also Ricci-flat. But in the list of Theorem 1.1 occurs only one Ricci-flat space, namely  $\text{Mink}_{n+1}(\mathbb{C})$ . Thus,  $M_j(p) \cong \text{Mink}_{n+1}(\mathbb{C})$ . This implies  $\tilde{M} \cong M_1(p) \times M_2(p) \cong \text{Mink}_{n+1}(\mathbb{C}) \times \text{Mink}_{n+1}(\mathbb{C}) \cong \text{Mink}_{n+1}(\mathbb{H})$ .

*Case 3:* Assume  $H^0 = \text{SO}^0(1, n) \cdot \text{Sp}(1)$ . The  $\text{SO}^0(1, n)$ -action decomposes the tangent space  $T_p \tilde{M} \cong \mathfrak{m}$  into four real  $\text{SO}^0(1, n)$ -invariant subspaces  $\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3$ , and  $\mathfrak{n}_4$  which we can identify with  $\mathbb{R}^{1, n}$ ,  $\mathbb{R}^{1, n} \cdot \mathbf{i}$ ,  $\mathbb{R}^{1, n} \cdot \mathbf{j}$ , and  $\mathbb{R}^{1, n} \cdot \mathbf{k}$ , respectively. Notice that these subspaces are not  $H^0$ -invariant.

Next we show that all four subspaces are Lie triple systems. Since  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ , it follows  $[\mathfrak{n}_j, \mathfrak{n}_j] \subset \mathfrak{h} = \mathfrak{so}(1, n) \oplus \mathfrak{sp}(1)$ . Recall that  $\mathfrak{sp}(1) = \mathbb{R} \cdot \mathbf{i} \oplus \mathbb{R} \cdot \mathbf{j} \oplus \mathbb{R} \cdot \mathbf{k}$ . If we consider the  $\mathbb{R}$ -bilinear map  $\alpha_j := \text{pr}_{\mathbb{R} \cdot \mathbf{i}}[\cdot, \cdot]_{|\mathfrak{n}_j \times \mathfrak{n}_j}$  and identify  $\mathfrak{n}_j$  with  $\mathbb{R}^{1, n}$ , we get a 2-form  $\alpha_j : \mathbb{R}^{n+1} \times \mathbb{R}^{n+1} \rightarrow \mathbb{R}$  which is  $\text{SO}^0(1, n)$ -invariant. Lemma 3.1 implies  $\alpha_j = 0$ . Analogously, we get the same if we project  $[\cdot, \cdot]_{|\mathfrak{n}_j \times \mathfrak{n}_j}$  to  $\mathbb{R} \cdot \mathbf{j}$  and  $\mathbb{R} \cdot \mathbf{k}$ , so we have  $[\mathfrak{n}_j, \mathfrak{n}_j] \subset \mathfrak{so}(1, n)$ . Since  $\mathfrak{so}(1, n)$  preserves  $\mathfrak{n}_j$ , it follows  $[[\mathfrak{n}_j, \mathfrak{n}_j], \mathfrak{n}_j] \subset \mathfrak{n}_j$ . This proves that  $\mathfrak{n}_j$  is a Lie triple system. If we identify the subspaces  $\mathfrak{n}_j$ ,  $j = 1, \dots, 4$ , with the orthogonal subspaces  $\mathbb{R}^{1, n}$ ,  $\mathbb{R}^{1, n} \cdot \mathbf{i}$ ,  $\mathbb{R}^{1, n} \cdot \mathbf{j}$ ,  $\mathbb{R}^{1, n} \cdot \mathbf{k}$  of  $T_p \tilde{M}$ , it follows that

$$N_1 := \exp_p(\mathbb{R}^{1, n}), \quad N_2 := \exp_p(\mathbb{R}^{1, n} \cdot \mathbf{i}), \quad N_3 := \exp_p(\mathbb{R}^{1, n} \cdot \mathbf{j}), \quad N_4 := \exp_p(\mathbb{R}^{1, n} \cdot \mathbf{k})$$

are connected totally geodesic submanifolds of  $\tilde{M}$  which are itself symmetric spaces. Let  $G_j$  be the connected component of the identity of the subgroup which preserves  $N_j$ . Then we have  $N_j = G_j / (G_j)_p$  with  $(G_j)_p = G_j \cap H^0 = \text{SO}^0(1, n)$ . Notice that  $N_j$  is a real Lorentz manifold. Similarly as in Case 1, it follows that the sectional curvature of  $N_j$  is constant. Notice that there exists an element  $h_j \in H^0$  such that  $d(h_j)_p(T_p N_1) = T_p N_j$ ,  $j = 2, 3, 4$ . Lemma 2.4 implies that the four totally geodesic submanifolds are pairwise isometric. Thus it is sufficient to study only one of the totally geodesic submanifolds, for instance  $N_1$ .

Let  $Y \in \Gamma(TN_1)$  and  $X \in \Gamma(TN_j)$  with  $j \neq 1$  and let  $\bar{X}, \bar{Y}$  be some arbitrary extensions of  $X, Y$  to  $\tilde{M}$ . Similarly as in Case 2, it follows that  $\nabla_{\bar{X}} \bar{Y}(p) \in T_p N_1$ . This implies

$$(R(Z, X)Y)(p) \in T_p N_1 \tag{4.1}$$

for every vector field  $Z$ .

Let  $X, Y \in \Gamma(TN_1)$  and  $e_1, \dots, e_{4n+4}$  be a local frame of  $T\tilde{M}$  such that  $e_{(n+1)(j-1)+1}, \dots, e_{(n+1)j}$  is a local frame of  $TN_j$ . Now we consider the Ricci-tensor of  $\tilde{M}$ . Since  $\tilde{M}$  is a

pseudo-Hyper-Kähler manifold, it is Ricci-flat, i.e.

$$\begin{aligned}
 0 &= \text{Ric}(X, Y) = \sum_{i=1}^{4n+4} g(R(e_i, X)Y, e_i) \\
 &= \sum_{i=1}^{n+1} g(R(e_i, X)Y, e_i) + \sum_{i=n+2}^{2n+2} g(R(e_i, X)Y, e_i) \\
 &\quad + \sum_{i=2n+3}^{3n+3} g(R(e_i, X)Y, e_i) + \sum_{i=3n+4}^{4n+4} g(R(e_i, X)Y, e_i).
 \end{aligned}$$

From equation 4.1 it follows that the last three terms vanish at  $p$ . Since  $N_1$  is a totally geodesic submanifold, this observation and the Gauss equation imply that at  $p$  the Ricci tensor of  $N_1$  vanishes. Since  $N_1$  is a homogeneous space, this implies that  $N_1$  is Ricci-flat. We already know that  $N_1$  is a Lorentz manifold of constant sectional curvature. Hence,  $N_1 \cong \text{Mink}_{n+1}(\mathbb{R})$ . It follows that  $N_j \cong \text{Mink}_{n+1}(\mathbb{R})$  for  $j = 2, 3, 4$ .

Next we will show that the Riemannian curvature  $R$  of  $\tilde{M}$  vanishes at  $p$ . For that we show  $g_p(R_p(x, y)z, w) = 0$  for all  $x, y, z, w \in T_p\tilde{M}$ . It is sufficient to check the claim for vectors tangent to one of the totally geodesic submanifolds.

If  $x, y, z \in T_pN_j$ , the claim is clear since  $N_j$  is flat and totally geodesic. Let  $z \in T_pN_j$  and let  $x \in T_pN_j$  and  $y \in T_pN_j^\perp$ . If  $w \in T_pN_j^\perp$ , then  $g_p(R_p(x, y)z, w) = 0$ , since  $R_p(x, y)z \in T_pN_j$  by equation 4.1. If  $w \in T_pN_j$ , then  $g_p(R_p(x, y)z, w) = g_p(R_p(z, w)x, y) = 0$ , since  $R(z, w)x \in T_pN_j$ .

Let  $z \in T_pN_j$  and  $x, y \in T_pN_j^\perp$ . If  $w \in T_pN_j^\perp$ , then  $g_p(R_p(x, y)z, w) = 0$ . If  $w \in T_pN_j$ , then  $g_p(R_p(x, y)z, w) = -g_p(R_p(y, z)x, w) - g_p(R_p(z, x)y, w) = 0$ , since  $R_p(y, z)x, R_p(z, x)y \in T_pN_j^\perp$ .

This finally proves the claim. Since  $\tilde{M}$  is a homogeneous space,  $g(R(X, Y)Z, W)$  vanishes completely on  $\tilde{M}$  for  $X, Y, Z, W \in \Gamma(T\tilde{M})$ . This implies that the Riemannian curvature tensor vanishes, i.e.  $\tilde{M}$  is flat and thus  $\tilde{M} \cong \text{Mink}_{n+1}(\mathbb{H})$ . This finishes the proof.  $\square$

**Remark 4.1.** *Theorem 4.1 implies that the homogeneous space  $M = G/H$  is locally symmetric, if  $\mathfrak{h}$  acts  $\mathbb{H}$ -irreducibly. Actually we know even more. Recall that the isotropy representation  $\rho : H \rightarrow \text{GL}(T_pM)$  acts on  $T_pM$  by  $\rho(h)(v) = dh_p(v)$ . By definition every element of  $H$  fixes the point  $p$ . It follows that if  $-\text{Id}_{T_pM} \in \rho(H)$ , then  $M$  is a globally symmetric space. This is the case if  $H^0$  is  $\text{SO}^0(1, n) \cdot \text{U}(1)$ ,  $\text{SO}^0(1, n) \cdot \text{Sp}(1)$ ,  $\text{U}(1, n)$ , or  $\text{Sp}(1, n)$ . If additionally  $n$  is an odd number, then this holds even for  $\text{SO}^0(1, n)$  and  $\text{SU}(1, n)$ .*

## 4.2 Homogeneous almost quaternionic pseudo-Hermitian manifolds of index 4 with $\mathbb{H}$ -irreducible isotropy group

**Theorem 4.2.** *Let  $(M, g, Q)$  be a connected almost quaternionic pseudo-Hermitian manifold of index 4 and  $\dim M = 4n + 4 \geq 24$ , such that there exists a connected Lie subgroup*

$G \subset \text{Iso}(M, g, Q)$  acting transitively on  $M$ . Let  $H := G_p$ ,  $p \in M$ , denote the isotropy group. If the intersection of the linear isotropy group with  $\text{Sp}(1, n)$  acts  $\mathbb{H}$ -irreducibly, then  $(M, g, Q)$  is a quaternionic pseudo-Kähler manifold. If furthermore  $M$  is a reductive homogeneous space and  $\mathfrak{h} \cap \mathfrak{sp}(1, n)$  acts  $\mathbb{H}$ -irreducibly, then  $M$  is locally symmetric.

*Proof:* Let  $\rho : H \rightarrow \text{GL}(T_p M)$  be the isotropy representation. We identify  $H$  with its image  $\rho(H)$ . Since  $H$  preserves the metric  $g$  and the almost quaternionic structure  $Q$ , we can consider  $H$  as a subgroup of  $\text{Sp}(1, n)\text{Sp}(1)$ .

First we show that  $M$  is a quaternionic pseudo-Kähler manifold. This follows from Theorem 2.6 by showing that the fundamental 4-form  $\Omega$  is closed. Since  $G$  acts transitively, it is sufficient to show that  $d\Omega_p = 0$ . Since  $H$  preserves the quaternionic structure  $Q$  and the metric  $g$ , it preserves  $d\Omega_p$ . By assumption  $H \cap \text{Sp}(1, n)$  acts  $\mathbb{H}$ -irreducibly. From Proposition 3.4 we know that  $\text{SO}^0(1, n) \subset (H \cap \text{Sp}(1, n))^{Zar}$ . It follows that  $d\Omega_p$  is  $\text{SO}^0(1, n)$ -invariant. If we identify  $T_p M$  with  $\mathbb{H}^{1, n}$ , we can consider  $d\Omega_p$  as an element of  $\Lambda^5(\mathbb{H}^{1, n})^*$ . Lemma 3.11 implies  $d\Omega_p = 0$ . This proves that  $M$  is a quaternionic pseudo-Kähler manifold.

Let us assume now that  $\mathfrak{h} \cap \mathfrak{sp}(1, n)$  acts  $\mathbb{H}$ -irreducibly and that  $M$  is a reductive homogeneous space. Then the same holds for the universal cover  $\tilde{M} \cong \tilde{G}/H^0$ , i.e. there is an  $\text{Ad}(H^0)$ -invariant subspace  $\mathfrak{m} \subset \mathfrak{g}$  such that  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ . The subspace  $\mathfrak{m}$  can be identified with  $T_p \tilde{M} \cong \mathbb{H}^{1, n}$ . If we restrict the Lie bracket  $[\cdot, \cdot]$  to  $\mathfrak{m} \times \mathfrak{m}$  and project it to  $\mathfrak{m}$ , we obtain an antisymmetric  $\mathbb{R}$ -bilinear map  $\beta : \mathbb{H}^{1, n} \times \mathbb{H}^{1, n} \rightarrow \mathbb{H}^{1, n}$  which is  $\text{Ad}(H^0)$ -equivariant. Since  $\text{SO}^0(1, n) \subset (H \cap \text{Sp}(1, n))^0 = H^0 \cap \text{Sp}(1, n)$ , Lemma 3.12 implies  $\beta = 0$ . Hence,  $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ . It follows that  $(\mathfrak{g}, \mathfrak{h}, \tau)$  is a symmetric Lie algebra where  $\tau : \mathfrak{g} \rightarrow \mathfrak{g}$  is defined by  $X + Y \mapsto X - Y$  for  $X \in \mathfrak{h}$ ,  $Y \in \mathfrak{m}$ . Hence,  $\tilde{M}$  is a symmetric space. This proves that  $M$  is locally symmetric.  $\square$

**Remark 4.2.** Theorem 4.2 says that  $\tilde{M}$  is a simply connected pseudo-Riemannian symmetric space of quaternionic Kähler type. Those spaces have been classified in [AC]. One step to complete the classification in Theorem 4.2 is to consult the list in [AC] and check which of them admit a pseudo-Riemannian metric of index 4. Such spaces are for example the quaternionic space forms  $\text{dS}_{n+1}(\mathbb{H}) = \text{Sp}(1, n+1)/\text{Sp}(1, n) \times \text{Sp}(1)$  and  $\text{AdS}_{n+1}(\mathbb{H}) = \text{Sp}(2, n)/\text{Sp}(1) \times \text{Sp}(1, n)$ . Two more symmetric spaces in the list are  $\mathbb{H}\text{dS}_{n+1} = \text{SO}^0(1, n+4)/\text{SO}^0(1, n) \times \text{SO}(4)$  and  $\mathbb{H}\text{AdS}_{n+1} = \text{SO}^0(5, n)/\text{SO}(4) \times \text{SO}^0(1, n)$ . The remaining symmetric spaces in [AC] need more investigation.



## Chapter 5

# Open problems

In this last chapter we point out some problems which are open and could be considered in future research.

In Theorem 4.1 we proved that an irreducible isotropy group implies that the considered manifolds are indeed pseudo-hyper-Kähler, if its dimension is  $4n + 4 \geq 16$ . This is due to Lemma 3.6 which ensures that the three Kähler forms are closed. By Remark 3.5 we know that Lemma 3.6 does not hold if  $n < 3$ . If  $M$  is a homogeneous manifold as in Theorem 4.1 but of dimension 8 or 12, then it is not clear that the  $\mathbb{H}$ -irreducibility of its isotropy group implies that  $M$  is a pseudo-Hyper-Kähler manifold. This yields the following problem.

**Problem 1.** *Let  $M$  be a manifold as in Theorem 4.1 but of dimension 8 or 12. Decide whether the  $\mathbb{H}$ -irreducibility of the isotropy group implies that  $M$  is pseudo-hyper-Kähler and if not find further assumptions.*

The additional assumption that  $\mathfrak{h}$  acts  $\mathbb{H}$ -irreducibly in Theorem 4.1 ensures that  $H^0$  is  $\mathbb{H}$ -irreducible. This is used to classify the universal cover  $\tilde{M} = \tilde{G}/H^0$ . If we do not assume that  $\mathfrak{h}$  acts  $\mathbb{H}$ -irreducibly, we know by Corollary 3.2 that  $H$  could be discrete or  $H^0 = \mathrm{U}(1) \cdot \mathbb{1}_{n+1}$  or  $\mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}$ . But we can exclude that  $H$  is discrete as follows.

Let  $M$  be as in Theorem 4.1. Assume that  $H$  is discrete. Then  $G \cong G/\{e\} \rightarrow G/H = M$  is a covering of  $M$ . In particular, we can identify  $\mathfrak{g}$  with  $T_p M \cong \mathbb{H}^{1,n}$ . Notice that  $H$  acts on  $\mathfrak{g}$  by conjugacy. Then the Lie bracket  $[\cdot, \cdot]$  defines at  $p$  an  $H^{Zar}$ -equivariant antisymmetric bilinear form. Since  $\mathrm{SO}^0(1, n) \subset H^{Zar}$ , Lemma 3.12 implies that  $[\cdot, \cdot]$  vanishes. In other words the connected Lie group  $G$  is abelian. Hence, the Lie subgroup  $H$  is abelian. But this is a contradiction, since by assumption  $H$  acts non-trivially on  $T_p M$ . This type of argument is due to [AZ].

Hence, we are left with the cases  $H^0 = \mathrm{U}(1) \cdot \mathbb{1}_{n+1}$  and  $H^0 = \mathrm{Sp}(1) \cdot \mathbb{1}_{n+1}$  for future investigation.

**Problem 2.** *Classify all manifolds  $M = G/H$  as in Theorem 4.1 with  $H^0 = \mathrm{U}(1)$  or  $H^0 = \mathrm{Sp}(1)$ .*

In Theorem 4.2 the assumption  $\dim M = 4n + 4 \geq 24$  is due to the fact that Lemma 3.11 holds for  $n \geq 5$ . The lower dimensional case is an open problem.

**Problem 3.** *Let  $M$  be a manifold as in Theorem 4.2 but of dimension 8, 12, 16, or 20. Decide whether the  $\mathbb{H}$ -irreducibility of the isotropy group implies that  $M$  is quaternionic pseudo-Kähler and if not find further assumptions.*

We assumed in Theorem 4.2 that  $H \cap \mathrm{Sp}(1, n)$  acts  $\mathbb{H}$ -irreducibly to ensure that  $H^{Zar}$  contains  $\mathrm{SO}^0(1, n)$ , since we only classified all connected  $\mathbb{H}$ -irreducible Lie subgroups of  $\mathrm{Sp}(1, n)$ .

**Problem 4.** *Classify all connected  $\mathbb{H}$ -irreducible Lie subgroups of  $\mathrm{Sp}(1, n)\mathrm{Sp}(1)$ .*

This classification could be used to replace the additional assumption in Theorem 4.2 that  $\mathfrak{h} \cap \mathfrak{sp}(1, n)$  acts  $\mathbb{H}$ -irreducibly by requiring that  $\mathfrak{h}$  acts  $\mathbb{H}$ -irreducibly.

## Appendix A

# Facts about Lie groups and Lie algebras

In this chapter we cite one general fact about connected Lie groups and prove furthermore that the Lie algebra  $\mathfrak{u}$  appearing in Section 2.4 is isomorphic to  $\mathfrak{so}(1, 3)$ .

The reader can find general information about Lie groups and Lie algebras for example in [Ba], [He], [HN], [KN1], [Kn], [O], [OV1], [OV2], and [Wa].

**Theorem A.1** ([OV2], Levi decomposition). *Let  $G$  be a connected Lie group. There exists a virtual connected Lie subgroup semisimple  $L \subset G$  such that  $G = L \cdot R$  and  $\dim(L \cap R) = 0$ , where  $R$  denotes the radical of  $G$ . The Lie group  $L$  is called Levi factor.*

**Lemma A.1.** *The matrices*

$$\begin{aligned} x &= \frac{i}{2}\Phi, \quad y = \frac{j}{2}\Phi, \quad z = \frac{k}{2}\Phi, \\ u &= \frac{i}{2}\mathbb{1}_2, \quad v = \frac{j}{2}\mathbb{1}_2, \quad w = \frac{k}{2}\mathbb{1}_2, \end{aligned}$$

with

$$\Phi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

form a basis of the Lie algebra  $\mathfrak{u} := \text{span}_{\mathbb{R}}\{x, y, z, u, v, w\}$  which is isomorphic to  $\mathfrak{so}(1, 3)$ .

*Proof:* We have the following commutators

$$\begin{aligned} [x, y] &= -w, & [y, z] &= -u, & [z, u] &= y, & [u, v] &= w, & [v, w] &= u, \\ [x, z] &= v, & [y, u] &= -z, & [z, v] &= -x, & [u, w] &= -v, \\ [x, u] &= 0, & [y, v] &= 0, & [z, w] &= 0, \\ [x, v] &= z, & [y, w] &= x, \\ [x, w] &= -y \end{aligned}$$

A basis of  $\mathfrak{so}(1, 3)$  is given by

$$\tilde{x} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{y} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tilde{z} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{u} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \tilde{v} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tilde{w} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We have the following commutators

$$\begin{aligned} [\tilde{x}, \tilde{y}] &= -\tilde{w}, & [\tilde{y}, \tilde{z}] &= -\tilde{u}, & [\tilde{z}, \tilde{u}] &= \tilde{y}, & [\tilde{u}, \tilde{v}] &= \tilde{w}, & [\tilde{v}, \tilde{w}] &= \tilde{u}, \\ [\tilde{x}, \tilde{z}] &= \tilde{v}, & [\tilde{y}, \tilde{u}] &= -\tilde{z}, & [\tilde{z}, \tilde{v}] &= -\tilde{x}, & [\tilde{u}, \tilde{w}] &= -\tilde{v}, \\ [\tilde{x}, \tilde{u}] &= 0, & [\tilde{y}, \tilde{v}] &= 0, & [\tilde{z}, \tilde{w}] &= 0, \\ [\tilde{x}, \tilde{v}] &= \tilde{z}, & [\tilde{y}, \tilde{w}] &= \tilde{x}, \\ [\tilde{x}, \tilde{w}] &= -\tilde{y} \end{aligned}$$

It follows that  $\mathfrak{u}$  and  $\mathfrak{so}(1, 3)$  are isomorphic. □

## Appendix B

# Spaces with constant quaternionic sectional curvature

It is a well know that simply connected, complete Riemannian manifolds with constant sectional curvature  $c$  are isometric to each other, see [KN1, Chapter VI, Theorem 7.10]. There is an analogous result for simply connected, complete Kähler manifolds of constant holomorphic sectional curvature  $c$ , see [KN2, Chapter IX, Theorem 7.9]. In this chapter we present the semi-Riemannian analogue for simply connected, complete quaternionic Kähler manifolds with constant quaternionic sectional curvature  $c$ , which can be obtained by some modifications of the known results.

Recall that manifolds with constant quaternionic sectional curvature  $c$  are called quaternionic space forms and that their Riemannian curvature tensor has the form

$$\begin{aligned} R(X, Y)Z = & \frac{c}{4} (g(Y, Z)X - g(X, Z)Y + g(IY, Z)IX - g(IX, Z)IY \\ & + 2g(X, IY)IZ + g(JY, Z)JX - g(JX, Z)JY + 2g(X, JY)JZ \\ & + g(KY, Z)KX - g(KX, Z)KY + 2g(X, KY)KZ). \end{aligned}$$

By Proposition 2.5, such manifolds are locally symmetric.

**Theorem B.1** ([KN1, Chapter VI, Theorem 7.8]). *Let  $M$  and  $M'$  be simply connected, complete differentiable manifolds with linear connections. Let  $T$ ,  $R$  and  $\nabla$  (respectively  $T'$ ,  $R'$  and  $\nabla'$ ) be the torsion, the curvature and the covariant differentiation of  $M$  (respectively  $M'$ ). Assume  $\nabla T = 0$ ,  $\nabla R = 0$ ,  $\nabla' T' = 0$  and  $\nabla' R' = 0$ . If  $F$  is a linear isomorphism of  $T_{x_0}M$  onto  $T_{y_0}M'$  and maps the tensors  $T_{x_0}$  and  $R_{x_0}$  at  $x_0$  into the tensors  $T'_{y_0}$  and  $R'_{y_0}$  at  $y_0$  respectively, then there is a unique affine isomorphism  $f$  of  $M$  onto  $M'$  such that  $f(x_0) = y_0$  and that the differential of  $f$  at  $x_0$  coincides with  $F$ .*

**Theorem B.2.** *Any two simply connected complete quaternionic (pseudo-)Kähler manifolds of constant quaternionic sectional curvature  $c$  are isometric to each other.*

*Proof:* We follow the arguments of [KN2]. Let  $M$  and  $M'$  be two simply connected complete quaternionic (pseudo-) Kähler manifolds and choose a point  $p \in M$  and  $p' \in M'$ . Then any linear isomorphism  $F : T_pM \rightarrow T_{p'}M'$  preserving both the metric and the almost

quaternionic structure necessarily maps the curvature tensor at  $p$  into the curvature tensor at  $p'$ . Then by Theorem B.1, there exists a unique affine isomorphism  $f$  of  $M$  such that  $f(p) = p'$  and that  $df_p = F$ . Let  $q$  be any point of  $M$  and  $\gamma$  a curve from  $p$  to  $q$ . We set  $q' = f(q)$  and  $\gamma' = f \circ \gamma$ . Since the parallel displacement along  $\gamma'$  corresponds to that along  $\gamma$  under  $f$  and since the metric tensors and the almost quaternionic structures of  $M$  and  $M'$  are parallel, the affine isomorphism  $f$  maps the metric tensor and the almost quaternionic structure of  $M$  into those of  $M'$ . Hence,  $M$  and  $M'$  are isometric.  $\square$

The following table presents some examples of quaternionic space forms with zero, positive, and negative quaternionic sectional curvature in the Riemannian case and in the pseudo-Riemannian case with index 4. We call such manifolds of quaternionic Lorentz type.

Table B.1: Quaternionic space forms

	Quaternionic Hermitian	Quaternionic Lorentz
<b>flat</b>	$\mathbb{H}^n$	$\text{Mink}_{n+1}(\mathbb{H}) = \mathbb{H}^{1,n}$
<b>positive</b>	$\mathbb{H}P^n = \text{Sp}(n+1)/\text{Sp}(1) \times \text{Sp}(n)$	$\text{dS}_{n+1}(\mathbb{H}) = \text{Sp}(1, n+1)/\text{Sp}(1, n) \times \text{Sp}(1)$
<b>negative</b>	$H^n(\mathbb{H}) = \text{Sp}(1, n)/\text{Sp}(1) \times \text{Sp}(1, n)$	$\text{AdS}_{n+1}(\mathbb{H}) = \text{Sp}(2, n)/\text{Sp}(1, n) \times \text{Sp}(1)$

# Zusammenfassung

In dieser Doktorarbeit betrachten wir homogene fast hyperkomplexe bzw. fast quaternionische pseudo-Hermitesche Mannigfaltigkeiten mit Index 4, die eine  $\mathbb{H}$ -irreduzible Isotropiegruppe besitzen. Wir zeigen, dass diese Mannigfaltigkeiten bereits pseudo-Hyperkähler bzw. quaternionisch pseudo-Kähler sind, falls die Dimension mindestens 16 bzw. 24 beträgt. Hierfür zeigen wir, dass gewisse Multilinearformen auf dem  $\mathbb{H}^{1,n}$  verschwinden, sofern sie invariant unter  $SO^0(1,n)$  sind. Weiter klassifizieren wir alle zusammenhängenden  $\mathbb{H}$ -irreduziblen Lie Untergruppen von  $Sp(1,n)$  bis auf Konjugation.

Zudem zeigen wir, dass die betrachteten homogenen Räume lokal symmetrisch sind, falls die Lie Algebra der Isotropiegruppe selbst  $\mathbb{H}$ -irreduzibel wirkt. Im hyperkähler Fall sind die Mannigfaltigkeiten sogar lokal isometrisch zum quaternionischen Minkowski-Raum  $\text{Mink}_{n+1}(\mathbb{H})$ .

# Abstract

In this thesis we consider homogeneous almost hypercomplex respectively almost quaternionic pseudo-Hermitian manifolds with index 4, which have an  $\mathbb{H}$ -irreducible isotropy group. We show that these manifolds are already pseudo-Hyper-Kähler respectively quaternionic pseudo-Kähler if their dimension is at least 16 respectively 24. For that we show that some multilinear forms on  $\mathbb{H}^{1,n}$  vanish if they are invariant under  $\mathrm{SO}^0(1, n)$ . We classify all connected  $\mathbb{H}$ -irreducible Lie subgroups of  $\mathrm{Sp}(1, n)$  up to conjugacy.

Furthermore we show that the considered homogeneous spaces are locally symmetric, if the Lie algebra of the isotropy group itself acts  $\mathbb{H}$ -irreducibly. In the hyper-Kähler case it turns out that the manifolds are locally isometric to the quaternionic Minkowski space  $\mathrm{Mink}_{n+1}(\mathbb{H})$ .



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