Topological and Algebraic Properties of Topological Group Cohomology and LHS-type Spectral Sequences

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Prof. Dr. Christoph Wockel
Prof. Dr. Christoph Schweigert
Prof. Dr. Friedrich Wagemann

Dedicated to my family.

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Abstract

The aim of the current thesis is to investigate algebraic and topological properties encoded in cohomology classes of locally continuous group cohomology, and also in which cases LHS-style spectral sequences exist for this model. One of the three main results of the current thesis shows, for a large class of topological groups, that each cohomology class has a cocycle representative whose restriction to an open and dense subset is continuous. The second main result relates to lifting obstructions on principal bundles. If $P \to X$ is a principal G-bundle and we have a topological central extension K of G by some abelian group Z, one might ask when we can lift P to a K-bundle over X. We show that, under reasonable assumptions on the spaces involved, we can always have such a lift if K is topologically the product $Z \times G$. The third result relates to LHS spectral sequences. We prove a generalization to the locally continuous group cohomology of the classical result of Lyndon, Hochschild and Serre, in the case of finite quotients.

Abstract

Ziel der vorliegenden Arbeit ist es, die in Kohomologieklassen lokal-stetiger Gruppenkohomologie enkodierte algebraische und topologische Information zu untersuchen und zu verstehen, in welchen Fällen LHS-ähnliche Spektralsequenzen existieren. Eines unserer drei Hauptresultate zeigt, für eine grosse Klasse von topologischen Gruppen, dass jede Kohomologieklasse, einen Repräsentanten hat, der, eingeschränkt auf eine offene dichte Teilmenge, stetig ist. Das zweite Hauptresultat betrifft Obstruktion von Prinzipalbündel. Ist $P \to X$ ein G-Prinzipalbündel und K eine zentral Erweiterung von G durch eine abelsche Gruppe Z, dann ist es eine natürliche Frage, ob sich P zu einem K-Prinzipalbündel über X hocheben lässt. Wir zeigen, dass unter milden Bedingungen eine solche Hochhebung immer existiert, solange K topologisch das Produkt $Z \times G$ ist. Unser drittes Resultat betrifft LHS Spektralsequenzen. Wir zeigen eine Verallgemeinerugnen klassischer Resultate von Lyndon, Hoschschild and Serre zu lokal-stetigen Gruppenkohomologie unter der Annahme endlicher Quotienten.

Contents

Introduction

Topological groups are objects of interest in mathematics and its applications. Together with continuous actions they describe continuous symmetries, so for example they have applications to physics. Actually Lie groups (the smooth analogue of topological groups) play a central role in modern theoretical physics. This relates to a theorem by Noether which gives a one to one correspondence between symmetries and conserved quantities.

So it is of interest to have tools to help in the study of topological groups. Since they are groups we have in our disposal group cohomology but the latter is blind to the topological structure. On the other hand they are topological spaces so we could study cohomology theories that relate to them like sheaf cohomology, but those are blind to their algebraic structure. So it is obvious that one should look for some refinement of either or both which incorporates both data. We will refer to such a theory as *topological group cohomology.*

There have been many approaches for defining topological group cohomology in the past ([Hu52], [vE58], [HM62], [Seg70], [Moo76], [Cat77], [Del74], [Fla08], [KR12]), all with their advantages and shortcomings. The most common approach was to start with group cohomology and refine it to include also topological information. In abstract terms, group cohomology is defined as the right derived functors of the G-invariance functor on G-modules. Although it is possible to derive results using such an abstract definition, usually people prefer working with an explicit complex the so called *bar resolution*. The latter is defined as a complex whose nth degree consists of functions from G^n into A, for G a group and A a G-module[1] ([Wei94, Application 6.5.5]).

Now assume that G is a topological group and A a continuous G-module. We return to our question of what should be a cohomology theory for G. The most widely used approaches consisted of refining the bar complex by using functions with some topological properties. Let us see the reason why this actually makes sense. To this end we will discuss a bit about the interpretation of low degree cohomology, so that we get a bit of intuition of what properties our topological

[1] i.e. an abelian group with a left G-action which commutes with group multiplication in A

group cohomology should satisfy. The 0 degree cohomology is isomorphic to the G-invariant subgroup of the G-module A. The first degree cohomology classifies crossed homomorphisms modulo principal ones. Of particular importance is the second degree group cohomology $H^2_{\text{gr}}(G, A)$ since it related to the question of Schur about when projective representations lift to honest ones. So we recall ([Wei94, Theorem 6.6.3]) that $H^2_{\text{gr}}(G, A)$ is in one to one correspondence with equivalence classes of extensions

$$0 \to A \to E \to G \to 0. \tag{1}$$

In the topological world we would like also that $E \to G$ is a locally trivial bundle. Such extensions are of interest also in physics. In quantum mechanics the symmetries of a physical system are described as a representation $G \to \mathcal{U}(\mathbb{P}H)$ of G on a projective Hilbert space. Such a representation cannot in general be lifted to a representation on the Hilbert space, but due to a theorem of Wigner (appeared in [Wig59], and more recently reviewed in [TW87]) one can get a representation $\widetilde{G} \to \mathcal{U}(H)$ on the entire Hilbert space for some $\widetilde{G}$ which is a central extension of G by $U(1)$ such that $\widetilde{G} \to G$ is a principal $U(1)$-bundle. Now to see why it is reasonable to refine the bar complex by functions with "nice" topological properties we need to quickly recall the identification of $H^2_{\text{gr}}(G, A)$ with classes of extensions of the form (1). Starting with a group cocycle $f : G^2 \longrightarrow A$ one twists the group multiplication on the product $A \times G$ by

$$(a, g)(a', g') := (a + g.a' + f(g, g'), gg'),$$

call this group $E := A \rtimes_f G$. So in the topological world if one asks for E to have continuous multiplication it is obvious that f should have some topological regularity.

Unfortunately the obvious choice for f to be continuous makes the bundle $A \rtimes_f G$ topologically trivial. Nevertheless, if A is a vector space this is reasonable and people have extensively investigated (the first known to the author appearance of such theories was in [Hu52], later on the smooth case was discussed in more detail in [vE58], both continuous and smooth were investigated in detail in [HM62] and [BW00]) the cohomology of continuous cocycles which we denote by $H^n_{\text{c}}(G, A)$. For the current thesis the model that was of main importance was the so-called locally continuous group cohomology. This is defined as the subcomplex of the Bar complex which contains functions $f : G^n \longrightarrow A$ for which there exists an open neighborhood of $(e, \ldots, e)$ such that their restriction to it is continuous. In the following we denote this model by $H^n_{\text{lc}}(G, A)$. This was introduced in [Cat77], used in the study of projective representations in [TW87] and lately investigated extensively in [Nee02, FW12, WW15].

The aim of the current thesis was to shed some more light on certain properties of $H^n_{\mathrm{lc}}(G, A)$. Topics that were covered are the following:

- Locally continuous functions are not very well documented. We study basic properties of them and also investigate if there are reasonable topologies on the space of locally continuous functions between two topological spaces.
- In the case of discrete groups $H^n_{\mathrm{gr}}(G, A)$ can be computed with cocycles defined on any free G-set. Is there a corresponding result in the locally continuous model?
- Are there phenomena that further exhibit the interplay between algebraic and topological information in $H^n_{\mathrm{lc}}(G, A)$? To this end we investigate the following
  - Does the process of going to cohomology produce representatives with better regularity properties?
  - If $Z \to \widetilde{G} \to G$ is a central extension of G, then for a principal G-bundle $P \to X$ one could ask whether we can lift it to a principal $\widetilde{G}$-bundle. This was investigated in [Gro55] and more recently in [NWW13], and it was shown that it relates to the class in $H^2_{\mathrm{lc}}(G, Z)$ that classifies the extension. Here we will see that this obstruction vanishes if the bundle $\widetilde{G} \to G$ is topologically trivial.
- A powerful tool to compute group cohomology is the so called LHS spectral sequence[2] which relates the cohomology of G with that of N and G/N for any normal subgroup of G. Is there a corresponding result for locally continuous group cohomology and under which assumptions on N?

We take now some space to give a short outline of the thesis.

Chapter 1 The first chapter is meant as a reminder/quick introduction to the basic facts of homological algebra that will be used throughout. We introduce chain and double complexes in abelian categories. Then follows a short exposition of derived functors and with them we recall the definition of group cohomology. In the second section spectral sequences are introduced with a specialization towards bounded ones. After that the reader is reminded of how to construct a convergent spectral sequence from a bounded filtration on a chain complex ([Wei94, Section 5.4]). This construction of course gives two spectral sequences from each bounded double complex and both of them converge to the cohomology of the total complex. Finally, this result, with

[2]named after R. Lyndon, G. Hochschild and J.-P. Serre

the help of the language of delta functors, gives the well-known Grothendieck spectral sequence. Examples of the latter include the Leray-Serre spectral sequence for singular cohomology and the LHS spectral sequence for group cohomology.

Chapter 2 In this chapter, models of topological group cohomology are reviewed with an emphasis on the locally continuous one. There are number of reasons for us to be interested in this particular model. To name a couple, it has an easy explicit description and the very definition of locally continuous functions relates straightforwardly to the local triviality of bundles ([Nee04, Proposition 2.6]). Also, there is a locally smooth version of it for Lie groups, which has been known to have a close relation to Lie algebra cohomology ([Nee04]). In this chapter we start with some first basic facts about locally continuous functions. In particular, we introduce a possible topology on the set of locally continuous functions between two spaces, which makes the evaluation function locally continuous. The locally continuous model is introduced afterwards. More precisely, if X and Y are two pointed topological spaces, denote by $\mathrm{Map}_{\mathrm{lc}\bullet}(X, Y)$ pointed functions from X to Y such that f is continuous on a neighborhood of the base of X. Then $H^n_{\mathrm{lc}}(G, A)$ is defined as the cohomology of the complex $\mathrm{Map}_{\mathrm{lc}\bullet}(G^{*+1}, A)^G$ with differential the same as in the bar resolution in the group cohomology case. The locally smooth version for Lie groups is denoted by $H^n_{\mathrm{ls}}(G, A)$. Our first result refers to the fact that locally continuous group cohomology of a topological group G can be computed via cocycles defined on any free G-space, a well-known fact for group cohomology ([HS53], [Wei94]) but it was missing so far in the case of locally continuous group cohomology. The result reads

Proposition (Cohomology of free *G*-spaces.). *Let G be a topological group, A a continuous G-module and X a G-space. Assume that there exists a pointed locally continuous function $\psi : X \longrightarrow G$ that is G-equivariant. Then*

$$H^n_{\mathrm{lc}}(G, A) \cong H^n\left(\mathrm{Map}_{\mathrm{lc}\bullet}\left(X^{*+1}, A\right)^G\right).$$

This implies that locally continuous group cohomology of any subgroup K for which the projection $G \to K\backslash G$ has a local section, can be computed with cocycles defined on the total space.
After that we take a moment to review other models that were proposed for topological group cohomology in the past. There are two other refinements of the classical bar complex, the continuous group cohomology (in [HM62]

using continuous functions) and the measurable model ([Moo76] using measurable functions). Another proposal for topological group cohomology was made by Segal and Mitchison ([Seg70]) via a notion of "derivable functors", which are computed by a special kind of resolution called "soft", utilizing Segal's construction of universal bundles. Another way to define topological group cohomology is via sheaf cohomology of a simplicial construction of BG ([Del74]). All but the continuous model have been shown in the past to be isomorphic for a large class of topological groups, while recently ([WW15]) all the models were put into a unified framework via a Comparison Theorem.

Chapter 3 In the third chapter we talk about the interplay between topological and algebraic information contained in the classes of $H^n_{\text{lc}}(G, A)$. The first result shows that under nice assumptions on G each cohomology class is represented by a cocycle with some regularity properties which are much better than expected when seeing the definition of locally continuous functions.

Theorem. *Assume G is a topological group such that all G^n are paracompact for each n. If there exists a good, countable and locally finite cover $\mathcal{U}_\bullet$ on $BG_\bullet$ then any class on $H^n_{\text{lc}}(G, A)$ is represented by a cocycle continuous on an open and dense subset. Furthermore if G is a Lie group then every class in $H^n_{ls}(G, A)$ is represented by a cocycle smooth on an open and dense subset.*

Which gives as a Corollary the following.

Corollary. *If G is a finite-dimensional second countable Lie group then each class of $H^n_{\text{lc}}(G, A)$ is represented by a cocycle which is continuous on an open and dense subset. Furthermore each class in $H^n_{\text{ls}}(G, A)$ is represented by a cocycle smooth on an open and dense subset.*

The second result in this chapter discusses obstruction classes in low-degree Čech cohomology. It is well-known that a topological central group extension

$$0 \to Z \to K \to G \to 0$$

gives a short exact sequence of the corresponding shaves of continuous functions on any space X, and this gives an exact sequence of the low-degree Čech cohomology ([Gro55]). Now the above extension is classified by $H^2_{\text{lc}}(G, A)$. If G is connected there is a direct generalization from group cohomology (e.g. [Wei94, Theorem 6.6.3]) to locally continuous group cohomology (in the same way as in [Nee04, Section 2]), if G is not connected the relation of $H^2_{\text{lc}}(G, A)$ and extensions becomes more obscure but their equivalence still holds due to

[Seg70, Proposition 4.3] and [WW15, Teorem 4.5]. So identifying topological central extensions with $H^2_{\mathrm{lc}}(G, A)$ one can define a function

$$\mathrm{obs}\ :\ \check{H}^1(X, \underline{G}) \times H^2_{\mathrm{lc}}(G, Z) \longrightarrow \check{H}^2(X, \underline{Z}),$$

which in some sense describes the obstruction of lifting a principal G-bundle to a principal K-bundle. In [Gro55, Proposition 5.7.2] it is shown that the lift exists if and only if obs vanishes. In the current work we show that the kernel of obs contains the continuous group cohomology. For the following result we assume that all the underlying spaces are CW-complexes and are compactly generated. Explicitly, we show the following.

Theorem (Vanishing obstruction classes). *Let Z be either discrete or an Eilenberg-Maclane space for a discrete group. Let*

$$0 \to Z \to K \to G \to 0$$

be a central group extension and $P \to X$ a principal G-bundle. If the above topological group extension is represented by a globally continuous group cocycle then

$$\mathrm{obs}([P], [K \to G]) = 0 \in \check{H}^2(X, \underline{Z}).$$

To show the previous we show that obs can be fully characterized by

$$\check{H}^1(X, \underline{G}) \times H^2_{\mathrm{lc}}(G, Z) \to [X, BG] \times \left[BG, B^2Z\right] \to \left[X, B^2Z\right],$$

where the first arrow assigns to a Čech cocycle a classifying map for its associated G-bundle and to each central extension of G by Z the class of Bf where f is a classifying map for the extension $K \to G$ (and which can be chosen to be also a group homomorphism). Then we show that under the assumptions above, if f is null-homotopic then so is Bf.

Chapter 4 Our initial point of interest in locally continuous group cohomology was whether there are constructions of LHS spectral sequences. In this chapter we present results that were derived in that initial direction. We start by reviewing the classical result for discrete group cohomology of Lyndon/Hochschild-Serre. Then we recall similar results of Moore's for the measurable group cohomology and of Hochschild-Mostow's for the continuous group cohomology. The main result of this chapter is the following.

Theorem (LHS spectral sequences for finite quotients). *Assume that N is an open normal subgroup of an arbitrary topological group G. Assume that A is an arbitrary topological G-module. Then we have the following:*

1. *There is a spectral sequence*

$$E_2^{p,q} := H_{\mathrm{gr}}^p\left(G/N, H_{\mathrm{c}}^q(N,A)\right) \Rightarrow H_{\mathrm{c}}^{p+q}(G,A).$$

2. *If $\left|G/N\right| < \infty$, then there is a spectral sequence*

$$E_2^{p,q} := H_{\mathrm{gr}}^p\left(G/N, H_{\mathrm{lc}}^q(N,A)\right) \Rightarrow H_{\mathrm{lc}}^{p+q}(G,A).$$

3. *Assume further that G is a Lie group and A a smooth G-module. If $\left|G/N\right| < \infty$, then there is a spectral sequence*

$$E_2^{p,q} := H_{\mathrm{gr}}^p\left(G/N, H_{\mathrm{ls}}^q(N,A)\right) \Rightarrow H_{\mathrm{ls}}^{p+q}(G,A).$$

Examples to which this proposition can be applied include relating the cohomology of the identity component to that of the total space.
To prove this result we use a construction similar to the Grothendieck spectral sequence and try to deal with the extra problems introduced due to topology and continuity. More explicitly, we use a double complex and relate the two spectral sequences derived from it. The one will collapse because $H_{\mathrm{gr}}^p\left(G/N, \mathrm{Map}_{\mathrm{lc}}(G^{q+1},A)^n\right)$ vanishes for $p > 0$ due to the finiteness condition, and so the cohomology of the double complex is isomorphic to the cohomology of G. The other spectral sequence will give the required E_2-term.

In the appendix we present some technical results and definitions.

Conventions

We will use some conventions throughout. If X is either a pointed topological space or a pointed set then we will denote by $*_X$ the distinguished (or base) point of X. If $\{U_i\}_{i\in I}$ is a collection of subsets of some set X, then we will denote $U_{i_0,i_1,\ldots,i_n} := \bigcap_{n=0}^{n} U_{i_n}$. Also commonly we will use $pr_i : \prod_{i\in I} X_i \longrightarrow X_i$ for the projection to the i-th factor and maybe sometimes even drop the i from the subscript of pr_i if it is obvious to which factor we are projecting. Similarly we will use i_i for the "inclusions" into a coproduct. If x is an element of $\prod_{i\in I} X_i$, where of course we will be talking about some category whose objects will have some underlying set, like abelian groups, topological spaces and so on, we will denote by x_i or x^i the element $pr_i\,(x)$. If I is a set whose elements are tuples of other elements, we will never write $pr_{(i_0,\ldots,i_n)}$ but rather $pr_{i_0,\ldots,i_n}$ and similarly $x_{i_0,\ldots,i_n}$ for the elements. All those conventions are quite common but we felt they should be included to avoid possible confusion.

Chapter 1

General Background

In this first chapter we will recall some basic notions of general homological algebra and spectral sequences. Most of the material covered should be familiar to the reader. The purpose of the chapter is to fix notation. The references we are using are [Wei94], [KS06], [McC01].

1.1 Homological Algebra

1.1.1 Cochain Complexes in Abelian Categories

In this section we will always assume that $\mathcal{A}$ is a category with a 0 object unless explicitly stated otherwise.

Definition 1.1.1. We denote by $\mathbf{Ch}(\mathcal{A})$ the category with

Objects families $C^\bullet = \{C^n\}_{n\in\mathbb{Z}}$ of objects of $\mathcal{A}$ together with morphisms $d^n_{C^\bullet} : C^n \longrightarrow C^{n+1}$ such that $d^{n+1}_{C^\bullet} \circ d^n_{C^\bullet} = 0$ for all n,

Morphisms families $f^\bullet = \{f^n \in \mathcal{A}(C^n, D^n)\}_{n\in\mathbb{Z}}$, such that $f^{n+1} \circ d^n_{C^\bullet} = d^n_{D^\bullet} \circ f^n$ for all n.

The objects of $\mathbf{Ch}(\mathcal{A})$ will be called cochain complexes or simply complexes, while for each complex the morphisms $d^n : C^n \longrightarrow C^{n+1}$ will be called differentials. The object C^n will be called the n-th degree of the complex $C^\bullet$. If $\mathcal{A}$ is also abelian we will use common notations for images, kernels and cohomology. So for each n in $\mathbb{Z}$ we define the following functors $B^n, Z^n, H^n : \mathbf{Ch}(\mathcal{A}) \longrightarrow \mathcal{A}$, given on objects by $B^n(C^\bullet) = \operatorname{im}\left(d^{n-1}_{C^\bullet}\right)$, $Z^n(C^\bullet) = \ker\left(d^n_{C^\bullet}\right)$ and

$$H^n(C^\bullet) = \operatorname{coker}\left(B^n(C^\bullet) \to Z^n(C^\bullet)\right),$$

while if $f^\bullet : C^\bullet \longrightarrow D^\bullet$ is a morphism of cochain complexes then $B^n(f^\bullet)$, $Z^n(f^\bullet)$ and $H^n(f^\bullet)$ are the morphisms induced by the universal properties of $B^n(D^\bullet)$, $Z^n(D^\bullet)$ and $H^n(C^\bullet)$ respectively. Easily one sees that if $\mathcal{A}$ admits injective or projective limits indexed by some category $\mathcal{I}$ so does $\mathbf{Ch}(\mathcal{A})$. Explicitly let $\alpha : \mathcal{I} \longrightarrow \mathbf{Ch}(\mathcal{A})$ be a functor, then we find its limit degree-wise

$$(\mathrm{colim}\alpha)^n \cong \mathrm{colim}\alpha^n.$$

If $\mathcal{A}$ is abelian, one can easily check the fact that monomorphisms and epimorphisms in $\mathbf{Ch}(\mathcal{A})$ are normal, which together with the above comment shows that the category of cochain complexes of $\mathcal{A}$ is additive (resp. abelian) if $\mathcal{A}$ is additive (resp. abelian).

Remark 1.1.2. We will denote by $\mathbf{Ch}^*(\mathcal{A})$ with $*$ in $\{b, +, -\}$, the full subcategories of $\mathbf{Ch}(\mathcal{A})$ that consist of: a) for $* = b$ complexes $C^\bullet$ for which all but finitely many C^n vanish and call them bounded complexes, b) for $* = +$ complexes $C^\bullet$ for which there exists a in $\mathbb{Z}$ such that $C^n = 0$ for all $n < a$, and call those complexes bounded below[1] and c) for $* = -$ complexes $C^\bullet$ for which there exists an a in $\mathbb{Z}$ such that $C^n = 0$ for all $n > a$ and call them bounded above complexes.

Definition 1.1.3. In an abelian category a complex $C^\bullet$ is called acyclic if, $H^n(C^\bullet) = 0$ for all $n \neq 0$.

We assume now that $\mathcal{A}$ is abelian. Cochain complexes are rarely useful themselves, one is usually more interested in their cohomology. If $f : C^\bullet \longrightarrow D^\bullet$ is a cochain morphism such that $H^n(f)$ is an isomorphism for all n, then f is called a quasi-isomorphism and the two complexes are called quasi-isomorphic. A way to check if f is a quasi-isomorphism is the notion of homotopies.

Definition 1.1.4. Let f, g be two cochain morphisms between $C^\bullet$ and $D^\bullet$. A cochain homotopy (or simply homotopy) between them s is a collection of morphisms $\{s^n : C^n \longrightarrow D^{n-1}\}_{n\in\mathbb{Z}}$ such that $f^n - g^n = d^{n-1}_{D^\bullet} \circ s^n + s^{n+1} \circ d^n_{C^\bullet}$ for all n. If such a homotopy exists we call f and g homotopy equivalent or homotopic.
A morphism f will be called null-homotopic if f is homotopic to the zero morphism. If s is a homotopy between f and the 0 morphism then s is also referred to as a cochain contraction for f. If s is a cochain contraction of the identity $id_{C^\bullet}$, then s is called a cochain contraction of the complex $C^\bullet$.
A complex $C^\bullet$ is called split if we are given a family of morphisms $s^n : C^n \longrightarrow C^{n-1}$ such that $d^n = d^n \circ s^{n+1} \circ d^n$ for all n. It is called split exact if $id_{C^\bullet}$ is homotopic to the 0 morphism.

[1] if we are considering a subcategory of $\mathbf{Ch}^+(\mathcal{A})$ for which all complexes have a common such a we will write $\mathbf{Ch}^{\geq a}(\mathcal{A})$ to denote it

The following is obvious.

Lemma 1.1.5. *If two cochain morphisms f and g are homotopic then $H^n(f) = H^n(g)$ for all n.*

We will use the notion of double complexes later on, we recall their definition in order to fix some notation.

Definition 1.1.6. Let $\mathcal{A}$ be category with a 0 object. A double complex is an object of the category $\mathbf{Ch}(\mathbf{Ch}(\mathcal{A}))$.

Remark 1.1.7. Explicitly a double complex is a family $\{C^{p,q}\}_{p,q\in\mathbb{Z}}$, of objects of $\mathcal{A}$, and families of morphisms

$$\left\{d_h^{p,q} \in \mathcal{A}\left(C^{p,q}, C^{p+1,q}\right)\right\}_{p,q\in\mathbb{Z}}$$

and

$$\left\{d_v^{p,q} \in \mathcal{A}\left(C^{p,q}, C^{p,q+1}\right)\right\}_{p,q\in\mathbb{Z}},$$

satisfying

$$\begin{aligned} d_h^{p,q} \circ d_h^{p-1,q} = 0 \ , \ d_v^{p,q} \circ d_v^{p,q-1} = 0, \text{ and} \\ d_h^{p,q+1} \circ d_v^{p,q} = d_v^{p+1,q} \circ d_h^{p,q}, \end{aligned}$$

for all p and q in $\mathbb{Z}$. The morphisms d_h will be called horizontal differentials and the d_v vertical differentials. In case $\mathcal{A}$ is additive (in fact in most cases it will be abelian) we can perform the following total complex constructions. Assume that $\mathcal{A}$ admits countable products, then we denote by

$$\mathrm{tot}^{\prod} : \mathbf{Ch}(\mathbf{Ch}(\mathcal{A})) \longrightarrow \mathbf{Ch}(\mathcal{A}) \tag{1.1}$$

the functor, which assigns to a double complex C a complex $\mathrm{tot}^{\prod}(C)$ whose n-degree is $\mathrm{tot}^{\prod}(C)^n := \prod_{p+q=n} C^{p,q}$, and the differentials $D^n : \mathrm{tot}^{\prod}(C)^n \longrightarrow \mathrm{tot}^{\prod}(C)^{n+1}$ are determined by demanding

$$pr_{p,n+1-p} \circ D^n = d_h^{p-1,n-p+1} \circ pr_{p-1,n-p+1} + (-1)^p \, d_v^{p,n-p} \circ pr_{p,n-p}. \tag{1.2}$$

If the category admits countable coproducts we can define a similar functor $\mathrm{tot}^{\coprod}$. We will make use of a bounded version of them. We will denote by tot the functor

$$\mathrm{tot} : \mathbf{Ch}^+\left(\mathbf{Ch}^+(\mathcal{A})\right) \longrightarrow \mathbf{Ch}^+(\mathcal{A}), \tag{1.3}$$

given by $\operatorname{tot}\left(C^{\bullet,\bullet}\right)^n := \bigoplus_{p+q=n} C^{p,q}$ and differentials as in (1.2).

In some cases double complexes are defined in another way. That is instead of commuting, the differentials anticommute. Then the definition of the total complex is the same as above without the $(-1)^p$ factor in its total differential (1.2). Both definitions though are equivalent since we can switch from commuting to anticommuting differentials by simply multiplying the horizontal differentials by $(-1)^p$.

1.1.2 Derived Functors

Definition 1.1.8. A cohomological δ-functor $T : \mathcal{A} \longrightarrow \mathcal{B}$ between two abelian categories $\mathcal{A}$ and $\mathcal{B}$ is a collection of additive functors $\{T^n : \mathcal{A} \longrightarrow \mathcal{B}\}_{n\in\mathbb{Z}}$, together with morphisms

$$\left\{{}_T\delta^n_{A,B,C} : T^n(C) \longrightarrow T^{n+1}(A)\right\}_{n\in\mathbb{Z}},$$

for each short exact sequence $0 \to A \to B \to C \to 0$, natural in the sense that if we have a morphism of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] \arrow[d, equal] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \arrow[d, equal] \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}, \tag{1.4}$$

then the diagram

$$\begin{tikzcd} T^n(C) \arrow[r] \arrow[d] & T^{n+1}(A) \arrow[d] \\ T^n(C') \arrow[r] & T^{n+1}(A') \end{tikzcd} \tag{1.5}$$

commutes. A morphism of delta functors $\phi : S \longrightarrow T$ is a collection of natural transformations $\phi^n : T^n \longrightarrow S^n$, commuting with the morphisms δ.

A δ-functor T is called universal if, given another one S and a natural transformation $f : T^0 \longrightarrow S^0$, there exists unique morphism of δ-functor $\phi : T \longrightarrow S$ such that $\phi^0 = f$.

Definition 1.1.9. An object I in a category $\mathcal{C}$ is called injective if, given a morphism $f : A \longrightarrow I$ and a monomorphism $g : A \longrightarrow B$, there exists a morphism $\tilde{f} : B \longrightarrow I$ such that $f = \tilde{f} \circ g$. Diagrammatically we have the following

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "g"] \arrow[d, "f"'] & B \arrow[dl, dashed, "\exists \tilde{f}"] \\ & I & \end{tikzcd} \tag{1.6}$$

Definition 1.1.10. A (right) resolution of an object A of an abelian category $\mathcal{A}$ is an acyclic complex $R^\bullet$ of $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ together with a morphism $\epsilon : A \longrightarrow R^0$, such that the following sequence is exact

$$0 \longrightarrow A \xrightarrow{\epsilon} R^0 \longrightarrow R^1. \tag{1.7}$$

An injective resolution of A is a right resolution for which all R^i are injective objects of $\mathcal{A}$.

Remark 1.1.11. We will use sometimes the notation $A \xrightarrow{\epsilon} R^\bullet$ to encode the information of Definition 1.1.10.

We will give some technical lemmas which will ensure that our definition of derived functors later on is sound.

Lemma 1.1.12 ([KS06] Lemma 13.2.4). *Let f be a morphism between an exact complex $X^\bullet$ and a bounded below complex of injectives $I^\bullet$. Then f is homotopic to* 0.

Corollary 1.1.13. *Assume A, B are objects of an abelian category $\mathcal{A}$ and f a morphism between them. Let $A \xrightarrow{i_R} R^\bullet$ be a right resolution of A, let $I^\bullet$ be a complex of injectives in $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ and let l be a morphism $l : B \longrightarrow I^0$. Then there exists a cochain complex morphism F from $R^\bullet$ to $I^\bullet$, which is unique up to homotopy, such that $l \circ f = F^0 \circ i_R$.*

Corollary 1.1.14. *Exact bounded below complexes of injectives are split.*

Definition 1.1.15. A category $\mathcal{A}$ has enough injectives if for each object A there exists a monomorphism $A \to I$ into an injective object.

Injective objects of $\mathcal{A}$ and $\mathbf{Ch}(\mathcal{A})$ are related as follows.

Lemma 1.1.16. *An object in $\mathbf{Ch}(\mathcal{A})$ is injective if and only if it is a split exact complex of injectives. It follows that if $\mathcal{A}$ has enough injectives so does $\mathbf{Ch}(\mathcal{A})$.*

With the help of Corollary 1.1.13, we can now define right derived functors.

Definition 1.1.17. Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor between abelian categories. Assume that $\mathcal{A}$ has enough injectives. We define the functors $R^iF : \mathcal{A} \longrightarrow \mathcal{B}$, to be given on an object A of $\mathcal{A}$ by

$$R^iF(A) := H^i(F(I)) \tag{1.8}$$

where $A \to I^\bullet$ is an injective resolution of A. R^iF are called the right derived functors of F.

Remark 1.1.18. Definition 1.8 is sound since by Corollary 1.1.13 if $A \xrightarrow{\epsilon'} J^\bullet$ is another injective resolution of A, then $H^n(F(I^\bullet)) \cong H^n(F(J^\bullet))$ for all n. Actually those isomorphisms are natural in the following sense. Since the category $\mathcal{A}$ has enough injectives we can make a choice of an injective resolution $I(A)$ for each object A in $\mathcal{A}$ and by Corollary 1.1.13 we can also make a choice of a cochain morphism $I(A) \to I(B)$ for each morphism between two objects A and B, so we get a functor $I(-) : \mathcal{A} \longrightarrow \mathbf{Ch}^{\geq 0}(\mathcal{A})$. If $J(-)$ represents another such choice of injective resolutions the functors $H^n \circ F \circ I$ and $H^n \circ F \circ J$ are naturally isomorphic by Corollary 1.1.13 (see also [Wei94] Lemma 2.4.1).

Example 1.1.19. Let $\mathcal{A}$ be an abelian category and assume both $\mathcal{A}$ and its opposite category have enough injectives and A and B are objects of $\mathcal{A}$. The functors $\mathcal{A}(A,-) : \mathcal{A} \longrightarrow \mathbf{Ab}$ and $\mathcal{A}(-,B) : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathbf{Ab}$ are left exact. One can show[2] that their right derived functors are isomorphic on objects, those derived functors are usually called Ext functors and are denoted by

$$\mathrm{Ext}^n_{\mathcal{A}}(A,B) := R^i\mathcal{A}(A,-)(B) \cong R^i\mathcal{A}(-,B)(A). \tag{1.9}$$

Since we will use it later on to define explicitly group cohomology, we will take some space here to explain how one can compute $R^n(\mathcal{A}(-,B))(A)$. By definition $R^n\mathcal{A}(-,B)(A) = H^n(\mathcal{A}(I^*,B))$ where $A \to I^\bullet$ is an injective resolution of A in $\mathcal{A}^{\mathrm{op}}$. Note that such a resolution is equivalent to a sequence of morphisms $\partial_n : I^n \longrightarrow I^{n-1}$ for n bigger than 0, which is exact except in degree 0 where $\operatorname{coker}(\partial_1) \cong A$, and now both morphisms and colimits are considered in $\mathcal{A}$. Since the objects I^n are injective in $\mathcal{A}^{\mathrm{op}}$ they are projective in $\mathcal{A}$, i.e. for each epimorphism $B' \to B''$ and each morphism $I^n \to B''$ there is a morphism completing the following diagram

$$\begin{tikzcd} & I^n \arrow[dl, dashed] \arrow[d] & \\ B' \arrow[r] & B'' \arrow[r] & 0. \end{tikzcd} \tag{1.10}$$

[2]we will outline one of the possible proofs in the section on spectral sequences

Examples of projective objects are free modules in categories of modules over rings. So to compute $\operatorname{Ext}^*_{\mathcal{A}}(A, B)$ we can do either of the following. We can find an injective resolution $B \xrightarrow{e} I^\bullet$ of B in $\mathcal{A}$ and compute the cohomology of the complex $\mathcal{A}(A, I^\bullet)$. The other option is to find projective objects P_n and morphisms $\partial_n : P_n \longrightarrow P_{n-1}$ for all naturals n which satisfy the conditions written above and then compute the cohomology of the complex $\mathcal{A}(P_\bullet, B)$.

Remark 1.1.20. It is known that right derived functors form universal cohomological δ-functors (e.g. [Wei94] Theorem 2.7.4). In the next section we review the main example that is interesting for the current work.

1.1.3 Group Cohomology

Throughout this section G will be a group and **G-Mod** will denote the category with objects G-modules, i.e. abelian groups with a left G-action by group automorphisms, and morphisms G-equivariant group homomorphisms. As is common, if A is a G-module we will denote by A also its underlying abelian group and implicitly assume some action is given. Trivially **G-Mod** is abelian.

Definition 1.1.21. Define by $-^G : \mathbf{G\text{-}Mod} \longrightarrow \mathbf{Ab}$ the functor given on objects by $A^G := \left\{a \in A \middle| g.a = a,\, \forall g \in G\right\}$. It is trivially left exact. Its right derived functors are called the group cohomology of G

$$H^n_{\text{gr}}(G, A) := R^n\left(-^G\right)(A). \tag{1.11}$$

Theorem 1.1.22. *We consider $\mathbb{Z}$ as a trivial G-module, i.e. $g.z := z$ for all g and z.*

a) Let A be a G-module, then:

$$A^G \cong \mathbf{G\text{-}Mod}(\mathbb{Z}, A). \tag{1.12}$$

The following natural isomorphisms follow from the universality of delta functors

$$\begin{aligned} H^n_{\text{gr}}(G, A) = R^n\left(-^G\right)(A) &\cong R^n\left(\mathbf{G\text{-}Mod}(\mathbb{Z}, -)\right)(A) \\ &= \operatorname{Ext}^n_{\mathbf{G\text{-}Mod}}(\mathbb{Z}, A) \\ &= R^n\left(\mathbf{G\text{-}Mod}(-, A)\right)(\mathbb{Z}). \end{aligned}$$

b) Let X be a set with a left free action of G. Consider the following complex of G-modules

$$\left\{B_n^X := \mathrm{span}_{\mathbb{Z}}\left\{(x_0, \ldots, x_n) \,\middle|\, x_i \in X\right\}\right\}_{n\in\mathbb{N}}, \tag{1.13}$$

with differentials $d_n(x_0, \ldots, x_n) := \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n)$ *and each* B_n^X *is endowed with the G-module structure generated by*

$$g.(x_0, \ldots, x_n) := (gx_0, \ldots, gx_n).$$

This gives a resolution of $\mathbb{Z}$ in the category of G-modules. Since each B_n^X is a free abelian group, it is also a projective G-module. So we get an injective resolution $\mathbb{Z} \to B_\bullet^X$ in $\mathbf{G\text{-}Mod}^{\mathrm{op}}$ *(by Example 1.1.19).*

Example 1.1.23. a) The most usual example is the case that $X = G$, and then B_n^G is called the bar resolution.

b) Let G, $\widetilde{G}$ be two groups such that G is a normal subgroup of $\widetilde{G}$, then to compute $H_{\mathrm{gr}}^n(G, A)$ via some B_n^X as above, set $X = \tilde{G}$.

Let X be a G-space. We endow $\mathbf{Set}(X^n, A)$ with a G-module structure defined via

$$(g.f)(x_1, \ldots, x_n) := g.f\left(g^{-1}x_1, \ldots, g^{-1}x_n\right) \tag{1.14}$$

Corollary 1.1.24 (Explicit definition of group cohomology). *Let G be a group and A be a G-module. Denote by $C^\bullet$ the complex whose n-degree is 0 for $n < 0$ and* $\mathbf{Set}(G^{n+1}, A)^G$ *(homogeneous cochains) for $n \geq 0$, and its differentials are*

$$df(g_0, \ldots, g_n) := \sum_{i=0}^{n} (-1)^i f(g_0, \ldots, \hat{g}_i, \ldots, g_n). \tag{1.15}$$

Then

$$H_{\mathrm{gr}}^n(G, A) \cong H^n(C^\bullet). \tag{1.16}$$

Proof. Simply one notes that $\mathbf{Set}(G^{n+1}, A)^G \cong \mathbf{G\text{-}Mod}\left(B_n^G, A\right)$. □

Corollary 1.1.25. *Let N be a normal subgroup of G. N acts freely on G, and since* $\mathbf{N\text{-}Mod}\left(B_n^G, A\right) \cong \mathbf{Set}\left(G^{n+1}, A\right)^N$*, we have*

$$H^n_{\text{gr}}(N, A) \cong H^n\left(\mathbf{Set}\left(G^{\bullet+1}, A\right)^N\right) \tag{1.17}$$

1.2 Spectral Sequences

1.2.1 Basic Definitions

Spectral sequences are closely related to the notion of filtrations. So we quickly recall them here. A filtration of an object A in a category $\mathcal{A}$ is family $FA := \{F^n A\}_{n\in\mathbb{Z}}$ of objects of $\mathcal{A}$, each coming with a monomorphism $i_n : F^n A \longrightarrow A$ and another one $i_n^{n-1} : F^n A \longrightarrow F^{n-1}A$, s.t. $i_{n-1} \circ i_n^{n-1} = i_n$ for all n in $\mathbb{Z}$. We will call the filtration bounded if there exists integers n_b and n_a for which $F^{n_b}A = 0$ and $i_n = id_A$ for all $n \leq n_a$. A filtration of a complex $C^\bullet$ in $\mathbf{Ch}^{\geq 0}(\mathcal{A})$ will be called canonically bounded if for all degrees n of $C^\bullet$, $F^{n+1}C^n = 0$ and $i_m = id_{C^n}$ for all $m \leq 0$. In this section $\mathcal{A}$ will always denote an abelian category.

Definition 1.2.1.
1. A (cohomological) spectral sequence E in $\mathcal{A}$ consists of the following piece of data
   (a) a family $\{E_r^{i,j}\}$ of objects of $\mathcal{A}$, with i, j in $\mathbb{Z}$ and r in $\mathbb{N}$[3].
   (b) morphisms $d_r^{i,j} : E_r^{i,j} \longrightarrow E_r^{i+r,j-r+1}$ for all i, j in $\mathbb{Z}$ and r in $\mathbb{N}$, which we will call differentials and satisfy
   $$d_r^{i+r,j-r+1} \circ d_r^{i,j} = 0,$$
   (c) isomorphisms
   $$S^{i,j} : E_{r+1}^{i,j} \longrightarrow \ker\left(d_r^{i,j}\right)/\mathrm{im}\left(d_r^{i-r,j+r-1}\right).$$
2. A morphism of spectral sequences is a family of morphisms $f_r^{p,q} : E_r^{p,q} \longrightarrow E_r'^{p,q}$, such that $(d')_r^{p,q} \circ f_r^{p,q} = f_r^{p+r,q-r+1} \circ d_r^{p,q}$ and for each r, $f_{r+1}^{p,q}$ are the maps which are induced in cohomology from $f_r^{p,q}$.

There is another description of a spectral sequence which will come in handy later on. Assume that we have a spectral sequence E. For each a and r in $\mathbb{N}$ we can get a tower of subobjects

[3] if we are given such objects only of $r \geq a$ for some natural a, then we say that the spectral sequence starts at a

$$0 = B_a^{p,q} \subseteq B_{a+1}^{p,q} \subseteq \dots B_{a+r}^{p,q} \subseteq Z_{a+r}^{p,q} \subseteq \cdots \subseteq Z_{a+1}^{p,q} \subseteq Z_a^{p,q} = E_a^{p,q}, \tag{1.18}$$

which satisfy $E_n^{p,q} \cong Z_n^{p,q} \big/ B_n^{p,q}$ for all $n \geq a$ and

$$Z_n^{p,q} \big/ Z_{n+1}^{p,q} \cong B_{n+1}^{p+n,q-n+1} \big/ B_n^{p+n,q-n+1}. \tag{1.19}$$

More useful is that the converse also holds. So assume that for some a in $\mathbb{N}$ we are given a tower of subobjects as in (1.18), and also isomorphisms as in (1.19). Then defining $E_n^{p,q} := Z_n^{p,q} \big/ B_n^{p,q}$ and

$$\begin{aligned} d_r^{p,q} : E_r^{p,q} = Z_r^{p,q} \big/ B_r^{p,q} &\twoheadrightarrow Z_r^{p,q} \big/ Z_{r+1}^{p,q} \\ &\cong B_{r+1}^{p+r,q-r+1} \big/ B_r^{p+r,q-r+a} \\ &\hookrightarrow Z_r^{p+r,q-r+1} \big/ B_r^{p+r,q-r+1} = E_r^{p+r,q-r+1} \end{aligned}$$

give a spectral sequence starting at a.

Remark 1.2.2. We will never refer to the isomorphisms $S^{i,j}$ of Definition 1.2.1 explicitly. We will always implicitly assume a "canonical" identification of E_{r+1} with the cohomology of d_r.

Example 1.2.3. Let $C^\bullet$ be a complex in $\mathcal{A}$, and assume it has a filtration $FC^\bullet$. We will see later on that from that data one can derive a spectral sequence starting at degree 1 with $E_1^{p,q} = H^{p+q}\left(F^pC^\bullet \big/ F^{p+1}C^\bullet\right)$.

Definition 1.2.4.

1. A spectral sequence is called bounded, if for each r there are only finitely many terms $E_r^{p,q} \neq 0$. In that case for each pair (p,q) there exists a in $\mathbb{N}$ such that $E_{r+1}^{p,q} \cong E_r^{p,q}$ for all $r \geq a$. This stable value will be denoted by $E_\infty^{p,q}$.

2. Let $\{H^n\}_{n\in\mathbb{Z}}$ be a family of objects of $\mathcal{A}$. We say that a bounded spectral sequence E converges to $H^\bullet$, if for all n we are given a finite filtration

$$0 = F^sH^n \subseteq \cdots \subseteq F^tH^n = H^n$$

and isomorphisms $E_\infty^{p,q} \cong F^pH^{p+q} \big/ F^{p+1}H^{p+q}$. In that case we will write

$$E_a^{p,q} \Rightarrow H^{p+q}.$$

Remark 1.2.5. This is a good point for a comment on usual notation of spectral sequences and some ambiguities that occur. The usual jargon in results about spectral sequences reads usually roughly as follows "There is a spectral sequence starting at degree a converging to the cohomology H". Clearly this statement does not include enough information to define a spectral sequence. Knowing the E_a term does not give us the differentials d_a, and even if we had them, we could compute the E_{a+1}-term but at this stage again we have no information about the differentials. So this statement is a bit problematic at first glance. What people actually mean by statements of the previous type is the following, "There is a procedure, to start from the E_a term and iteratively build a spectral sequence, which in the end converges to H". The main point here is that there is a procedure. So if one needs differentials or higher degree terms he has to go through the proof of the above result and see how the differentials are constructed. Having made this remark we will also use the above abusive jargon, but the reader should always have in her/his mind that the actual definition of the spectral sequence exists in the proof of the result and not on its statement.

Remark 1.2.6. In general spectral sequences can be unbounded and their convergence becomes a more subtle issue. In the current work this generality is not necessary since we will be mainly interested about "first quadrant" spectral sequences, i.e. $E_r^{p,q} = 0$ for all r and $p, q < 0$. So in the following by spectral sequence we will always mean a bounded one and we will not mention it.

1.2.2 Spectral Sequence of filtration

The most usual short description of spectral sequences is "algorithms for computing cohomology". In this paragraph we will see how this can be made concrete. The introduction of spectral sequences was done by Jean Leray, when he tried to compute sheaf cohomology by using filtrations. Until today the most commonly used (and maybe only) way for one to construct a spectral sequence is via filtration of complexes. As we mentioned we will only consider bounded spectral sequences, so we will assume that the filtrations on the complexes will always be bounded. We have the following well-known theorem.

Theorem 1.2.7 ([Wei94] Theorem 5.5.1). *Let $\mathcal{A}$ be an abelian category. Consider a filtered complex $FC^\bullet$ in $\mathcal{A}$. Assume that the filtration is bounded, then there is a bounded convergent spectral sequence*

$$E_1^{p,q} := H^{p+q}\left(F^pC^\bullet / F^{p+1}C^\bullet\right) \Rightarrow H^{p+q}\left(C^\bullet\right). \tag{1.20}$$

Proof. As remarked earlier such a statement does not make sense by itself, but rather should be accompanied with the construction of the spectral sequence. We

will only mention the construction here for completeness. We will not repeat the proof that the construction actually gives the result since this can be found in many places, e.g. [Wei94], [McC01].
We define

$$\begin{aligned}
\eta^{p,n} &: F^pC^n \longrightarrow F^pC^n / F^{p+1}C^n =: E_0^{p,n} \\
A_r^{p,n} &:= F^pC^n \times_{C^{n+1}} F^{p+r}C^{n+1} \\
Z_r^{p,n} &:= \eta^{p,n}\left(A_r^{p,n}\right) \\
B_{r+1}^{p+r,n+1} &:= \eta^{p+r,n+1}\left(d\left(A_r^{p,n}\right)\right) \\
E_r^{p,n} &:= Z_r^{p,n} / B_r^{p,n}.
\end{aligned}$$

$Z_r^{p,n}$ and $B_r^{p,n}$ give a tower of submodules as in (1.18), one can show that

$$Z_r^{p,n} / Z_{r+1}^{p,n} \cong B_{r+1}^{p+r,n+1} / B_r^{p+r,n+1}.$$

It should be noted that this last isomorphism is induced by the differential on $C^\bullet$. The filtration of $H^{p+q}\left(C^\bullet\right)$, which provides the isomorphism with the E_∞-terms, is the one induced on cohomology from the one on $C^\bullet$. $\square$

Remark 1.2.8. The idea, behind the construction of the spectral sequence in the proof of Theorem (1.2.7), is basically that we step by step "approach" the cohomology of $C^\bullet$, by considering not exactly cocycles, but rather cochains that are iteratively included in smaller subsets $F^{p+r}C^\bullet$. We will not repeat the details here, but we should point out that this construction is as in the book of Weibel, even though his proofs use modules, each step in the proof can be carried out exactly in any abelian category in an element free way using the technical facts presented in Appendix A.

An interesting source of such spectral sequences is double complexes.

Corollary 1.2.9. *Let $C^{\bullet,\bullet}$ be a first quadrant double complex in $\mathcal{A}$. Then there are two spectral sequences converging to the cohomology of the total complex*

$$^{II}E_2^{p,q} := H_*^p H_\times^q \left(C^{*,\times}\right) \Rightarrow H^{p+q}\left(\mathrm{tot}\left(C^{\bullet,\bullet}\right)\right) \tag{1.21}$$

and

$$^{I}E_2^{p,q} := H_*^p H_\times^q \left(C^{\times,*}\right) \Rightarrow H^{p+q}\left(\mathrm{tot}\left(C^{\bullet,\bullet}\right)\right) \tag{1.22}$$

Proof. This follows from the Theorem (1.2.7), by simply noticing that a double complex gives rise to two filtrations. We will describe only one here, the other is exactly the same. We consider the filtration

$$F^p \left(\mathrm{tot}\left(C^{\bullet,\bullet}\right)\right)^n := \bigoplus_{i=p}^{n} C^{i,n-i}.$$

It is clearly an increasing filtration of $\mathrm{tot}\left(C^{\bullet,\bullet}\right)^n$, and is compatible with the total differential. We apply Theorem (1.2.7) to this filtration to get a spectral sequence as in (1.20). Let us compute the E_2-term. Clearly

$$F^p \left(\mathrm{tot} C\right)^n / F^{p+1} \left(\mathrm{tot} C\right)^n \cong C^{p,n-p}.$$

Then $E_1^{p,q} = H^{p+q}\left(C^{p,\bullet-p}\right) \cong H^q\left(C^{p,\bullet}\right)$. The d_1 differential is induced from the total differential. Clearly it is the map that the horizontal differential induces in cohomology[4]. □

Remark 1.2.10. We will give an intuitive more explicit description of the two spectral sequences in Corollary (1.2.9). For simplicity we consider that $\mathcal{A}$ is the category of modules over some ring R. We will give only a rough outline skipping many technical and bookkeeping details. Up to equivalence we can describe elements of $E_r^{p,q}$ by tuples $a \in C^{p,q} \bigoplus C^{p+1,q-1} \bigoplus \cdots \bigoplus C^{p+r-1,q-r+1}$, such that $d_v a^{p,q} = 0$, $d_h a^{p,q} = d_v a^{p+1,q-1}, \ldots\ d_h a^{p+r-2,q-r+2} = d_v a^{p+r-1,q-r+1}$. While the differential $d_r^{p,q}$ is induced on classes by the assignment $a \mapsto \left(d_h a^{p+r-1,q-r+1}, 0, \ldots, 0\right)$. The "up to equivalence" part gives the convergence. Many of the relations come from moding out classes coming from the total differentials and only make sure that things are well defined. It is only a matter of bookkeeping to write it explicitly so we will avoid it. There is an important relation though, which is that $a \backsim 0$ if $a^{p,q} = 0$. That means that even though the term $E_{n+2}^{0,n}$, has classes represented by cocycles of the total differential, it is only a quotient of the cohomology of the total complex by exactly $F^1 H^n\left(\mathrm{tot} C\right)$. Similarly for the other E_∞-terms we see the usual isomorphism about convergence as in part (2) of Definition (1.2.4).

Example 1.2.11. As an example use of Corollary 1.2.9, we will outline a possible proof of the fact that $R^n\left(\mathcal{A}\left(-,B\right)\right)\left(A\right) \cong R^n\left(\mathcal{A}\left(A,-\right)\right)\left(B\right)$. Pick an injective resolution $I^\bullet$ in $\mathcal{A}$ of B and an injective resolution $P^\bullet$ of A in $\mathcal{A}^{\mathrm{op}}$. We can get a first quadrant double complex $C^{p,q} = \mathcal{A}\left(P^p, I^q\right)$ with differentials the obvious ones. The spectral sequences coming from filtering either by columns or rows will collapse. To see that note that in the E_1-page for one of them we get $H^q\left(\mathcal{A}\left(P^p, I^*\right)\right) =$

[4] Note that this is a bit wrong since we consider commuting double complexes so the total differential is $\left(d_v, (-1)^p d_h\right)$, so more precisely the d_1 differential in the spectral sequence is $\left((-1)^p d_h\right)_*$

$R^q\left(\mathcal{A}\left(P^p,-\right)\right)(B)$. But since each P^p is a projective object in $\mathcal{A}$ it is clear that $\mathcal{A}\left(P^p,-\right)$ are exact functors so their derived functors vanish except in degree 0. So the E_2-page of the spectral sequence will be 0 except from when $q=0$ when we will have

$$E_2^{p,0}=H^p\left(\mathcal{A}\left(P^*,B\right)\right)=R^p\left(\mathcal{A}\left(-,B\right)\right)(A).$$

So we already know that the cohomology of the total complex is isomorphic to $R^q\left(\mathcal{A}\left(-,B\right)\right)(A)$. If we filter in the other way we will see that the cohomology of the entire complex is isomorphic to $R^q\left(\mathcal{A}\left(A,-\right)\right)(B)$.

An important application of Corollary (1.2.9), is the so called Grothendieck spectral sequence. This is quite well-known but it relates to our construction of LHS spectral sequences, so it is quickly recalled here. The following is part of [Wei94, Theorem 5.8.3].

Theorem 1.2.12 (Grothendieck Spectral Sequences). *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "G\circ F"'] & & \mathcal{D} \arrow[dl, "G"] \\ & \mathcal{A} & \end{tikzcd},$$

be left exact functors between abelian categories. Assume that $\mathcal{C}$ and $\mathcal{D}$ have enough injectives, and that $F(I)$ is G-acyclic if I is injective. Then for all objects A of $\mathcal{C}$ there exists a convergent spectral sequence

$$E_2^{p,q}=R^pGR^qF(A)\Rightarrow R^{p+q}(G\circ F)(A). \tag{1.23}$$

Chapter 2

Topological Group Cohomology

In Subsection 1.1.3 we saw the definition of group cohomology. Moving to topological groups one might ask what should be a cohomology theory for them. This is not straightforward since the category of topological abelian groups is not abelian. So there is no obvious abstract way to define cohomology for topological groups as is the case with discrete groups. One has to create models by hand and maybe in an ad hoc way. This was done by many people in the past (e.g. [HM62], [Moo76], [Cat77], [Seg70]). We will review some properties that such a cohomology theory should satisfy. Assume that G is a topological group and A a topological abelian group with a continuous left action of G by group automorphisms. The latter will be called continuous G-module. Assume that $H^n_{\text{tgc}}(G, A)$ are the cohomology groups of G with values in A. We would like first of all that $H^n_{\text{tgc}}(G, -)$ are δ-functors (in the sense of Definition A.1.15). Distinguished composable morphisms in the category of continuous G-modules will be considered sequences

$$0 \to A' \to A \to A'' \to 0$$

which are in any case short exact as abelian groups, and also satisfy some topological condition, e.g. it is a trivial or a locally trivial bundle. Another property that the groups H^n_{tgc} should satisfy is the interpretation of the low degree terms. We recall that in the discrete group theory case $H^0(G, A)$ is the G-invariant subspace of A, $H^1(G, A)$ are equivalence classes of crossed homomorphisms (or derivations) and $H^2(G, A)$ are equivalence classes of extensions $A \to E \to G$[1]. We would like to have similar interpretations in the topological case. Something useful for example would be if $H^2_{\text{tgc}}(G, A)$ classifies extensions of topological groups $A \to E \to G$ which are locally trivial bundles.
In the first section we will spend some time investigating one of them, the so called

[1]actually equivalence classes of such extensions for which conjugation of elements of A in E, is the same as the action of G.

locally continuous model. In the second section we will review some of the other models that were proposed by people over the years in an attempt to create such a cohomology theory and the classification theory of them that was discovered recently in [WW15].

2.1 Locally Continuous Model

The locally continuous model we will discuss in this section is defined as the cohomology of a generalized bar complex. But instead of using arbitrary functions we use the so called locally continuous. Before we introduce the model we take a moment to discuss those weird functions.

2.1.1 Locally Continuous Functions

Locally continuous functions are not very well documented and unfortunately have some very bad properties (e.g. they do not compose). We will mention a few properties of them before introducing locally continuous group cohomology. Let us fix some notation first. Let X, Y be two topological spaces. We denote by $\mathrm{Map}_{\mathrm{lc}}(X, Y)$ the set functions $f : X \longrightarrow Y$, whose restriction to some non-empty open subset U of X is continuous. In case the space X is pointed and the subsets U are also neighborhoods of the base point of X, we denote the corresponding set of functions by $\mathrm{Map}'_{\mathrm{lc}}(X, Y)$. Assuming that Y is also pointed, the subset of $\mathrm{Map}'_{\mathrm{lc}}(X, Y)$ containing pointed functions is denoted by $\mathrm{Map}_{\mathrm{lc}\bullet}(X, Y)$. Functions belonging to any of those sets will be called in general locally continuous functions, and if f is one, the open subset U such that $f\big|_U$ is continuous will be called its neighborhood of continuity.
Composing locally continuous functions is a bit tricky. In essence they compose if the image of the first intersects non-trivially the neighborhood of continuity of the second. We state a proposition to clarify that.

Proposition 2.1.1. *Let X, Y, Z be topological spaces, and $f : X \longrightarrow Y$, $g : Y \longrightarrow Z$ two functions of the underlying sets. Assume $f\big|_U$ and $g\big|_V$ are continuous, with U, V (possibly empty) open subsets of X and Y respectively. Then $g \circ f\big|_{U \cap f^{-1}(V)}$, is continuous.*

The proof is elementary and straightforward. Note that the result is not very interesting because empty functions are continuous. Let us see some interesting specific situations.

Corollary 2.1.2. *i) If g is continuous and f is locally continuous then $g \circ f$ is locally continuous.*

ii) If $g\big|_V$ is continuous for V open, f is continuous and $f^{-1}(V) \neq \varnothing$ then $g \circ f$ is locally continuous.

iii) If $g\big|_V$ is continuous for V open, $f\big|_U$ is continuous for U open and $V \cap f^{-1}(U) \neq \varnothing$ then $g \circ f$ is locally continuous.

Assuming further that X, Y, Z are pointed, we get the following.

iv) If $g\big|_V$ is continuous for V an open neighborhood of $*_Y$, $f\big|_U$ is continuous for U an open neighborhood of $*_X$ and $V \cap f^{-1}(U) \neq \varnothing$ then $g \circ f$ is an element of $\mathrm{Map}'_{\mathrm{lc}}(X, Z)$.

iiv) If g, f are locally continuous on open neighborhoods of the base points and pointed then $g \circ f$ is pointed and locally continuous on a neighborhood of the base point of X.

We see that we can form a category $\mathbf{LTop}_\bullet$, by considering objects pointed topological spaces and morphisms between X and Y the set $\mathrm{Map}_{\mathrm{lc}\bullet}(X, Y)$. It is usually of interest in a category $\mathcal{C}$ if the hom-functor of the category can be turned into an endofunctor, and it is even more interesting if this endofunctor becomes right adjoint to the product in the category. This is true for example for the category of sets, certain interesting subcategories of topological spaces (pointed and not), and many other examples. Stated in explicit terms, assume we are in a subcategory of topological spaces and continuous functions between them, we want morphisms $f : X \times Y \longrightarrow Z$, to be in one to one correspondence with morphisms $F : X \longrightarrow \mathrm{Hom}(Y, Z)$[2], via the assignment $F(x)(y) = f(x, y)$. For topological spaces this holds if Y is locally compact and one endows $\mathrm{Hom}(Y, Z)$ with the compact open topology. We will see in the following, that such a result will always fail for locally continuous functions (at least in some reasonable generality, because of course if X, Y, Z are discrete, such a result holds trivially).
An interesting tool concerning morphisms and mapping spaces is the evaluation map, i.e. $\mathrm{ev} : \mathrm{Map}_{\mathrm{lc}}(X, Y) \times X \longrightarrow Y,\ (f, x) \mapsto f(x)$. Clearly there is no topology on $\mathrm{Map}_{\mathrm{lc}}(X, Y)$ making ev continuous. To argue for that one simply notes that each function of the underlying sets $f : X \longrightarrow Y$, between two topological spaces can be written as $f = \mathrm{ev}|_F \circ (i_f \times id_X)$, where F is a subset of $\mathbf{Set}(X, Y)$, $\mathrm{ev}|_F : F \times X \longrightarrow Y$ defined as above, and $i_f \times id_X : \{f\} \times X \longrightarrow F \times X$ the obvious inclusion map. Clearly if $\mathrm{ev}|_F$ is continuous then f is continuous, and so F contains only continuous functions. In the following we will turn ev into a locally continuous function. Denote by $\mathrm{Map}_{\mathrm{lc}}(U; X, Y)$, the subset of $\mathrm{Map}_{\mathrm{lc}}(X, Y)$, having functions f such that $f\big|_U$ is continuous. Denote also $CO_{\mathrm{lc}}(K, O)$, the locally continuous functions such that $f(K) \subseteq U$.

[2] we will use the generic notation $\mathrm{Hom}(-, -)$, for subfunctors of $\mathbf{Set}(-, -)$ when we are dealing with topological spaces

Definition 2.1.3. We denote by τ^{co+lc} the topology on $\mathrm{Map}_{\mathrm{lc}}(X,Y)$, generated by the union of

$$\left\{\mathrm{Map}_{\mathrm{lc}}(U;X,Y) \,\middle|\, U \text{ open in } X\right\}$$

and

$$\left\{CO_{\mathrm{lc}}(K,V) \,\middle|\, K \text{ compact in } X \text{ and } V \text{ open in } Y\right\}.$$

So τ^{co+lc} is a refined version of the usual compact open topology. The interesting fact about it, is the following result.

Proposition 2.1.4. *Let X and Y be topological spaces with X being locally compact. For any open subset $U \subseteq X$, $\mathrm{ev}\big|_{\mathrm{Map}_{\mathrm{lc}}(U;X,Y)\times U}$ is continuous.*

Proof. Let (f,u) be a tuple in $\mathrm{Map}_{\mathrm{lc}}(U;X,Y)\times U$. Since X is locally compact and U is open in X, it follows that U is also locally compact. By assumption $f\big|_U$ is continuous, so for each neighborhood N of $f(u)$, there exists a compact neighborhood $K \subseteq U$ of u, such that $f\big|_U(K) \subseteq N$. Clearly f is in $CO_{\mathrm{lc}}(K,N)$. And finally we deduce

$$\mathrm{ev}\left(\left(CO(K,N)\cap \mathrm{Map}_{\mathrm{lc}}(U;X,Y)\right)\times K\right) \subseteq N.$$

And so $\mathrm{ev}\big|_{\mathrm{Map}_{\mathrm{lc}}(U;X,Y)\times U}$ is continuous. □

It is clear that $\mathrm{Map}_{\mathrm{lc}}(X,Y)$ with topology τ^{co+lc} is Hausdorff if Y is so.

Proposition 2.1.5. *The following injections*

$$\left(\mathrm{Map}_{\mathrm{lc}}(X,Y),\tau^{co+lc}\right) \hookrightarrow \left(\mathrm{Map}_{\mathrm{lc}}(X,Y),\tau^{co}\right) \hookrightarrow \left(\mathrm{Map}_{\mathrm{lc}}(X,Y),\tau^{pc}\right),$$

are continuous, where τ^{co} denotes the compact-open topology and τ^{pc} the topology of point-wise convergence. So if Y is Hausdorff the above three spaces are Hausdorff.

Proof. τ^{pc} is the topology induced by Y^X. A subbasis for Y^X is given by all sets of the form $CO(\{x\},N)$, which are all included in τ^{co}, and by definition $\tau^{co+lc} \supseteq \tau^{co}$, so the above functions are continuous. The Hausdorff property results from the fact that products of Hausdorff spaces are Hausdorff. □

Let Hom be any of $\mathrm{Map}_{\mathrm{lc}}$, $\mathrm{Map}'_{\mathrm{lc}}$ or $\mathrm{Map}_{\mathrm{lc}\bullet}$ and $\otimes$ either the product of topological spaces or the smash product of pointed topological spaces in the respective cases. We will close this subsection with some arguments that the sets $\mathrm{Hom}\,(X \otimes Y, Z)$ and $\mathrm{Hom}\,(X, \mathrm{Hom}\,(Y, Z))$ cannot be identified in general for arbitrary topological spaces (or pointed topological spaces in the appropriate cases) , for any topology on $\mathrm{Hom}\,(Y, Z)$. We denote by

$$\phi \;:\; \mathrm{Hom}\,(X \otimes Y, Z) \longrightarrow \mathbf{Set}\,(X, \mathbf{Set}\,(Y, Z))$$

the function defined by $\phi\,(f)\,(x)\,(y) := f\,(x \otimes y)$, and we denote by

$$\psi \;:\; \mathrm{Hom}\,(X, \mathrm{Hom}\,(Y, Z)) \longrightarrow \mathbf{Set}\,(X \otimes Y, Z)$$

the function defined by $\psi\,(F)\,(x \otimes y) := F\,(x)\,(y)$. Also denote by

$$\mathrm{for}_1 \;:\; \mathrm{Hom}\,(X \otimes Y, Z) \longrightarrow \mathbf{Set}\,(X \otimes Y, Z)$$

and

$$\mathrm{for}_2 \;:\; \mathrm{Hom}\,(X, \mathrm{Hom}\,(Y, Z)) \longrightarrow \mathbf{Set}\,(X, \mathbf{Set}\,(Y, Z))$$

the functions associating to some continuous function f its underlying function on sets. We have the following easy lemma which exhibits that ϕ and ψ cannot provide lifts which are inverse isomorphisms, which in turn means that the sets $\mathrm{Hom}\,(X \otimes Y, Z)$ and $\mathrm{Hom}\,(X, \mathrm{Hom}\,(Y, Z))$ cannot be identified in the usual way.

Lemma 2.1.6. *There is no topology on* $\mathrm{Hom}\,(\mathbb{R}, \mathbb{R})$*, such that* ϕ *factors through* for_2.

It is actually obvious that something stronger holds, namely that there is a subset $U \subseteq \mathrm{Hom}\,(\mathbb{R} \otimes \mathbb{R}, \mathbb{R})$ such that $\mathrm{for}_2^{-1}\,(\phi\,(U)) = \varnothing$. This is pretty straightforward since there are locally continuous functions which have restrictions to some subsets which are nowhere continuous.

Proof. An easy contradiction to the existence of such a lift is provided by the following function. Define $f : \mathbb{R} \otimes \mathbb{R} \longrightarrow \mathbb{R}$, given by $f\,(x \otimes y) := xy$ for $x^2 + y^2 \leq 1$, and $f\,(x \otimes y) := 1 - D\left(\frac{xy}{2}\right)$ otherwise, where D is the Dirichlet function. Clearly f is locally continuous but $\phi\,(f)\,(2)$ is nowhere continuous so not locally continuous and not an element of $\mathrm{Hom}\,(\mathbb{R}, \mathbb{R})$. $\square$

Lemma 2.1.6 is the main reason for restricting to finite quotients in our attempt to derive an LHS result for the locally continuous group cohomology in Subsection 4.2.

2.1.2 Locally Continuous Group Cohomology

Locally continuous group cohomology was first introduced by Cattaneo in [Cat77], while first used in applications (in physics in fact) by Tuynman and Wiegerinck in [TW87]. Recently, the locally smooth case was studied by Neeb (e.g. in [Nee02]), while (as we will see in the next paragraph) their relation to other cohomology theories for topological groups was studied by Wagemann and Wockel in [WW15]. We will give a short recap of it in this section.
First let us fix some notation. Assume G is a topological group, X and Y are G-spaces and F be a subset of $\mathbf{Set}\left(X,Y\right)$. Denote by

$$F^G := \left\{ f \in F \middle| \text{ such that } g.f\left(x\right) = f\left(g.x\right) \right\}, \tag{2.1}$$

in some sense the "invariant" functions of F[3]. Let A be a continuous G-module. We define the following families of abelian groups

$$C^n_{\mathrm{lc}}\left(G,A\right) := \mathrm{Map}'_{\mathrm{lc}}\left(G^{n+1},A\right)^G, \tag{2.2}$$

and

$$C^n_{\mathrm{lc}\bullet}\left(G,A\right) := \begin{cases} \mathrm{Map}'_{\mathrm{lc}}\left(G,A\right)^G & \text{for } n=0 \\ \mathrm{Map}_{\mathrm{lc}\bullet}\left(G^{n+1},A\right)^G & \text{for } n>0 \end{cases}. \tag{2.3}$$

Remark 2.1.7. If we have defined (2.3) as $\mathrm{Map}_{\mathrm{lc}\bullet}\left(G^{n+1},A\right)^G$, for all n, we would get cohomology groups isomorphic except in degree 0, where they would be trivial. But it is desirable for group cohomology theories to have $H^0\left(G,A\right) \cong A^G$. So we had to introduce this "non-homogeneity".

The assignment

$$d^n\left(f\right)\left(g_0,\ldots,g_{n+1}\right) := \sum_{i=0}^{n+1} \left(-1\right)^i f\left(g_0,\ldots,\widehat{g_i},\ldots,g_{n+1}\right), \tag{2.4}$$

makes both (2.2) and (2.3) complexes of abelian groups.
The most commonly used is the first one but the second has functions with nicer properties as is revealed by Corollary 2.1.2. Nevertheless they have the same cohomology.

Proposition 2.1.8. *The cohomology of the complexes (2.2) and (2.3) is isomorphic.*

[3] Note that it might be that the set F might not be a G-space via an action of the form $\left(g.f\right)\left(x\right) = gf\left(g^{-1}x\right)$, this is why the previous will not be the G-invariance functor applied to some G-set, and one needs a notation as above.

Proof. This follows from the dual Dold-Kan correspondence, since the complex of "normalized" cochains, i.e. $f(g_0, \ldots, g_n) = 0$ if for some i, $g_i = g_{i+1}$, injects to both and by the Dold-Kan correspondence those inclusions are quasi-isomorphisms. □

Definition 2.1.9. The locally continuous group cohomology of G with values in A, is defined for n in $\mathbb{N}$ as

$$H^n_{\mathrm{lc}}(G, A) := H^n\left(C^*_{\mathrm{lc}\bullet}(G, A)\right). \tag{2.5}$$

From now on we also denote by $\mathrm{Map}_{\Delta\mathrm{c}}(X^n, Y)$ the set of functions $f : X^n \longrightarrow Y$ such that there exits some open neighborhood U of the diagonal of X^n, such that $f\big|_U$ is continuous. We call those functions "diagonally continuous". In the previous definitions of locally continuous group cohomology (e.g. in [WW15]) people have used those functions instead. The cohomology is canonically isomorphic though. The easy way to see that is to simply notice that the assignments

$$\phi : \mathrm{Map}'_{\mathrm{lc}}(G^n, A) \longrightarrow \mathrm{Map}_{\Delta\mathrm{c}}\left(G^{n+1}, A\right)^G : f \mapsto F,$$

with $F(g_0, \ldots, g_n) := g_0.f\left(g_0^{-1}g_1, \ldots, g_{n-1}^{-1}g_n\right)$, and

$$\psi : \mathrm{Map}_{\Delta\mathrm{c}}\left(G^{n+1}, A\right)^G \longrightarrow \mathrm{Map}'_{\mathrm{lc}}(G^n, A) : F \mapsto f,$$

with $f(g_1, \ldots, g_n) := F(1, g_1, g_1g_2, \ldots, g_1 \ldots g_n)$, which are isomorphisms, imply actually that $\mathrm{Map}'_{\mathrm{lc}}(G^n, A) \cong \mathrm{Map}'_{\mathrm{lc}}(G^{n+1}, A)^G$. But we will get this isomorphism as a corollary of a somewhat stronger statement.

Lemma 2.1.10. *Let f be in $\mathrm{Map}'_{\mathrm{lc}}(X^n, Y)^G$. Then there exists an open neighborhood U of $\bigcup_{g\in G}(g.*_X, \ldots, g.*_X)$, such that $f\big|_U$ is continuous.*

Proof. By assumption $f\big|_V$ is continuous, where V is an open neighborhood of $(*_X, \ldots, *_X)$. We denote $V_g := g.V$ for g in G. Note that

$$f^{-1}(W) \bigcap g.\Omega = g.\left(f^{-1}\left(g^{-1}.W\right) \bigcap \Omega\right), \tag{2.6}$$

holds for all subsets $W \subseteq Y$ and $\Omega \subseteq X^n$.
Denote by $U := \bigcup_{g\in G} V_g$ and for all subsets W using (2.6) one gets

$$f|_U^{-1}(W) = \left(\bigcup_{g\in G} g.\left(f|_V^{-1}\left(g^{-1}.W\right)\right)\right) \bigcap U. \tag{2.7}$$

Now for any element $g \in G$ and any G-space X the assignment $x \mapsto g.x$ is a homeomorphism, and arbitrary unions of open subsets are open so U is open and furthermore it contains $\bigcup_{g\in G} (g.*_X, \dots, g.*_X)$. Also $f|_V$ is continuous, $g^{-1}.W$ are open and so the right hand side of (2.7) is an open subset of U. So $f|_U$ is continuous, and the claim is proven. □

We can define now the complex $\left\{ \mathrm{Map}_{\Delta \mathrm{c}} \left(G^{n+1}, A\right)^G \right\}_{n\in\mathbb{N}}$, with the same differential as in (2.4). The homology of this complex is the one that was used so far. But as a corollary of the above one gets:

Proposition 2.1.11. *By Lemma 2.1.10 we deduce that*

$$\mathrm{Map}'_{\mathrm{lc}} \left(G^{n+1}, A\right)^G \cong \mathrm{Map}_{\Delta \mathrm{c}} \left(G^{n+1}, A\right)^G . \tag{2.8}$$

Now in the case of group cohomology, the bar resolution can be constructed by any set on which G acts freely. We now show a similar version in the case of locally continuous group cohomology.

Proposition 2.1.12. *Let G be a topological group, A a continuous G-module and X a G-space. Assume that there exists a pointed locally continuous function $\psi : X \longrightarrow G$ that is G-equivariant. Then:*

$$H^n_{\mathrm{lc}} (G, A) \cong H^n \left(\mathrm{Map}_{\mathrm{lc}\bullet} \left(X^{*+1}, A\right)^G \right) . \tag{2.9}$$

Proof. Note that the assumption gives straight on that the action is free. To see that let us assume that for some $g \in G$ and $x \in X$ we have $g.x = x$. Then

$$\psi(x) = \psi(g.x) = g.\psi(x),$$

so $g = e$. Now let us proceed to prove the main claim. We define

$$\begin{aligned}
\phi &: G \longrightarrow X, \\
i &: \mathrm{Map}_{\mathrm{lc}\bullet} \left(X^{n+1}, A\right)^G \longrightarrow \mathrm{Map}_{\mathrm{lc}\bullet} \left(G^{n+1}, A\right)^G , \\
l &: \mathrm{Map}_{\mathrm{lc}\bullet} \left(G^{n+1}, A\right)^G \longrightarrow \mathrm{Map}_{\mathrm{lc}\bullet} \left(X^{n+1}, A\right)^G ,
\end{aligned}$$

where $\phi(g) := g.*_X$, $i(f) := f \circ \phi^n$ and $l(f) := f \circ \psi^n$. By assumptions they are well-defined and they commute with the differentials. Easily one gets $i \circ l = id_{\mathrm{Map}_{\mathrm{lc}\bullet}(G^{n+1},A)^G}$. So $i_* \circ l_* = id_*$ in cohomology. In the other direction we are not so lucky and have to find a homotopy. We define

$$s(f)(x_0, \dots, x_n) := \sum_{i=0}^{n} (-1)^i f\left(\phi\left(\psi\left(x_0\right)\right), \dots, \phi\left(\psi\left(x_i\right)\right), x_i, \dots, x_n\right) . \tag{2.10}$$

One computes

$$
\begin{aligned}
& d \circ s(f)(x_0, \ldots, x_n) \\
& \quad = \sum_{i=0}^{n} (-1)^i s(f)(x_0, \ldots, \hat{x}_i, \ldots, x_n) \\
& \quad = \sum_{i=0}^{n} (-1)^i \Bigg(\underbrace{\sum_{j=0}^{i-1} (-1)^j f(\phi(\psi(x_0)), \ldots, \phi(\psi(x_j)), x_j, \ldots, \hat{x}_i, \ldots, x_n)}_{B'} \\
& \qquad \underbrace{- \sum_{j=i+1}^{n} (-1)^j f\Big(\phi(\psi(x_0)), \ldots, \phi\widehat{(\psi(x_i)}), \ldots, \phi(\psi(x_j)), x_j, \ldots, x_n\Big)}_{A'} \Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
& s \circ d(f)(x_0, \ldots, x_n) \\
& = \sum_{j=0}^{n} (-1)^j df(\phi(\psi(x_0)), \ldots, \phi(\psi(x_j)), x_j, \ldots, x_n) \\
& = \sum_{j=0}^{n} (-1)^j \Bigg(\underbrace{\sum_{i=0}^{j-1} (-1)^i f\Big(\phi(\psi(x_0)), \ldots, \phi\widehat{(\psi(x_i)}), \ldots, \phi(\psi(x_j)), x_j, \ldots, x_n\Big)}_{A} \\
& \quad + (-1)^j f(\phi(\psi(x_0)), \ldots, \phi(\psi(x_{j-1})), x_j, \ldots, x_n) \\
& \quad + (-1)^{j+1} f(\phi(\psi(x_0)), \ldots, \phi(\psi(x_j)), x_{j+1}, \ldots, x_n) \\
& \quad \underbrace{- \sum_{i=j+1}^{n} (-1)^i f(\phi(\psi(x_0)), \ldots, \phi(\psi(x_j)), x_j, \ldots, \hat{x}_i, \ldots, x_n)}_{B} \Bigg).
\end{aligned}
$$

Notice that in $(d \circ s + s \circ d)(f)(x_0, \ldots, x_n)$, A will cancel A' and B will cancel B'. So finally

$$\begin{aligned}
&(d\circ s+s\circ d)(f)(x_0,\ldots,x_n)\\
&\quad=\sum_{j=0}^{n}(-1)^{2j}\Big(f\left(\phi\left(\psi\left(x_0\right)\right),\ldots,\phi\left(\psi\left(x_{j-1}\right)\right),x_j,\ldots,x_n\right)\\
&\qquad-f\left(\phi\left(\psi\left(x_0\right)\right),\ldots,\phi\left(\psi\left(x_j\right)\right),x_{j+1},\ldots,x_n\right)\Big)\\
&\quad=\sum_{j=0}^{n}f\left(\phi\left(\psi\left(x_0\right)\right),\ldots,\phi\left(\psi\left(x_{j-1}\right)\right),x_j,\ldots,x_n\right)\\
&\qquad-\sum_{j=1}^{n+1}f\left(\phi\left(\psi\left(x_0\right)\right),\ldots,\phi\left(\psi\left(x_{j-1}\right)\right),x_j,\ldots,x_n\right)\\
&\quad=f\left(x_0,\ldots,x_n\right)+\sum_{j=1}^{n}f\left(\phi\left(\psi\left(x_1\right)\right),\ldots,\phi\left(\psi\left(x_{j-1}\right)\right),x_j,\ldots,x_n\right)\\
&\qquad-\sum_{j=1}^{n}f\left(\phi\left(\psi\left(x_1\right)\right),\ldots,\phi\left(\psi\left(x_{j-1}\right)\right),x_j,\ldots,x_n\right)\\
&\qquad-f\left(\phi\left(\psi\left(x_0\right)\right),\ldots,\phi\left(\psi\left(x_n\right)\right)\right)\\
&\quad=id\left(f\right)\left(x_0,\ldots,x_n\right)-\left(l\circ i\right)\left(f\right)\left(x_0,\ldots,x_n\right)\\
&\quad=\left(id-l\circ i\right)\left(f\right)\left(x_0,\ldots,x_n\right).
\end{aligned}$$

Which means that

$$\left(l\circ i\right)_*=l_*\circ i_*=id_*. \tag{2.11}$$

Which shows the claim. □

We can adapt slightly the previous to get the following for diagonally continuous cochains.

Corollary 2.1.13. *With X, G and A defined as above, further assume that there exists a pointed continuous G-equivariant function $\psi : X \longrightarrow G$. Then it follows that*

$$H^n_{\mathrm{lc}}\left(G,A\right)\cong H^n\left(\mathrm{Map}_{\Delta\mathrm{c}}\left(X^{*+1},A\right)^G\right). \tag{2.12}$$

Proof. The proof of the previous Proposition can be carried over word by word. The only interesting check is the well-definiteness of l. If $f|_U$ is continuous on a neighborhood of the diagonal of G^{n+1}, then by Proposition 2.1.1, one easily sees that $f \circ \psi^{n+1}$ is continuous on $\left(\psi^{n+1}\right)^{-1}(U) \supseteq \Delta^{n+1}(X)$. $\square$

Corollary 2.1.14. *Let K be a closed subgroup of G, such that the projection $G \longrightarrow K\backslash G$ has a local section. Then*

$$H^n_{\mathrm{lc}}(K, A) \cong H^n\left(\mathrm{Map}_{\mathrm{lc}\bullet}\left(G^{*+1}, A\right)^K\right). \tag{2.13}$$

Proof. Since the bundle is locally trivial, take a local split and extend it non-continuously to $K\backslash G$. Call this function σ in $\mathrm{Map}_{\mathrm{lc}\bullet}(K\backslash G, G)$. Note that σ can be chosen to be pointed. The assignment $\psi(g) := g \cdot \sigma(p(g))^{-1}$, gives ψ in $\mathrm{Map}_{\mathrm{lc}\bullet}(G, K)^K$. $\square$

Remark 2.1.15. Note that the proof of Proposition 2.1.12 is a bit sloppy in one point. The 0 degree cochains are not pointed. But still $l(f)$ defined as in the proof above is locally continuous since ψ is. In general an easy criterion for locally continuous functions to compose is that the first functions is pointed. The last doesn't need to satisfy that criterion, so since in our case f will be post-composed with pointed locally continuous we get again locally continuous (but not necessarily pointed).

Remark 2.1.16. The above lemma is one more hint that the locally continuous cohomology is the correct one. The above is a known statement for group cohomology, but in the case of continuous group cohomology it holds only for very particular cases. For example Corollary 2.1.14 has been proven in the continuous setting only for vector space coefficients, G locally compact and G/K paracompact [HM62, Lemma 3.4].

Finally we give some usual identification between the so called homogeneous and inhomogeneous cochains which will be used in the Chapters 3 and 4.

Lemma 2.1.17. *Consider the complex*

$$C^n_{\mathrm{inhmg}} := \mathrm{Map}_{\mathrm{lc}}(G^n, A), \tag{2.14}$$

with differential

$$\begin{aligned} d^n(f)(g_1, \ldots, g_{n+1}) :=& g_1.f(g_2, \ldots, g_{n+1}) \\ &+ \sum_{i=1}^{n} (-1)^i f(g_1, \ldots g_i g_{i+1}, \ldots, g_{n+1}) \\ &+ (-1)^{n+1} f(g_1, \ldots, g_n). \end{aligned}$$

Then this complex is isomorphic to the complex (2.2).

Proof. Let us call the cochains in (2.2) homogeneous and in (2.14) inhomogeneous. If f is a homogeneous cochain, then

$$(g_1, \ldots, g_n) \mapsto f\left(e, g_1, g_1 g_2, \ldots, g_1 \ldots g_n\right)$$

defines an inhomogeneous cochain. While if F is an inhomogeneous cochain then the assignment

$$(g_0, \ldots, g_n) \mapsto g_0.F\left((g_0)^{-1} g_1, \ldots, (g_0)^{-1} g_n\right)$$

defines a homogeneous cochain. The two constructions are inverse to each other and give an identification of the two complexes. ☐

Remark 2.1.18. By Proposition 2.1.11 and Lemma 2.1.17, the complex defined in (2.3) agrees with the ones used in [Nee02], [WW15], [Cat77], [FW12] and other similar approaches.

Remark 2.1.19. In the next section we will talk about other topological group cohomology models. Most of them are defined in a similar fashion. One takes the bar complex and considers a subcomplex of functions with certain properties (measurable, continuous and so on). In most of those models the above lemma still works and we will use it in some places later on, using the jargon that we move from homogeneous to inhomogeneous cochains.

2.2 Review of other models

There were other approaches towards defining topological group cohomology. We will take a short time reviewing them.

1. **Continuous Group Cohomology**

   The most straightforward way to generalize group cohomology is obviously to consider a short of "continuous bar complex". Assume that G is a topological group and A a continuous G-module, then the continuous group cohomology of G with values in A is defined as

   $$H^n_{\mathrm{c}}(G, A) := H^n\left(\mathbf{Top}\left(G^{\bullet+1}, A\right)^G\right), \tag{2.15}$$

   where the differential of $\mathbf{Top}\left(G^{\bullet+1}, A\right)^G$ is defined as in (2.4). The proof of Proposition 2.1.12 can be adapted to the continuous case as well.

Proposition 2.2.1. *Let X be a continuous G-space, and assume that there exists a continuous G-equivariant function $\psi : X \longrightarrow G$. Then*

$$H_{\mathrm{c}}^{n}(G, A) \cong H^{n}\left(\mathbf{Top}\left(X^{\bullet+1}, A\right)^{G}\right). \tag{2.16}$$

The proof is the same as in Proposition 2.1.12 where we simply do not have to worry about local continuity. And also as in Corollary 2.1.14 we get that if $G \cong K \times G/K$ topologically for some closed subgroup K, then

$$H_{\mathrm{c}}^{n}(K, A) \cong H^{n}\left(\mathbf{Top}\left(G^{\bullet+1}, A\right)^{K}\right). \tag{2.17}$$

When this model appeared it was defined for the case that G is locally compact Hausdorff and A a Hausdorff topological (real) vector space. Since the category of topological G-modules is not abelian, one is faced with the problem of how to define cohomology in those categories. In [HM62] they tried to reproduce the setting of defining cohomology via a special kind of resolutions which they called strongly injective. They show that the modules $\mathbf{Top}(G^{q}, A)$ are strongly injective and so they use that explicit model when necessary.
This model however has quite a few drawbacks. We mentioned in the introduction to the chapter what properties a topological group cohomology theory should possess. Unfortunately continuous group cohomology lacks most of them. More precisely $H_{\mathrm{c}}^{2}(G, A)$ classifies group extensions

$$0 \to A \to E \to G \to 0$$

which are topologically trivial as bundles. It also behaves badly as a "δ-functor". To get a long exact sequence in cohomology from a short exact sequence of topological G-modules

$$0 \to A' \to A \to A'' \to 0$$

it is required that that sequence is topologically trivial. For example we do not get a long exact sequence for $\mathbb{Z} \to \mathbb{R} \to S^{1}$. The model has its uses though when one is primarily interested for vector space coefficients. For example for finite dimensional vector spaces it is obvious that there are long exact sequence in cohomology. But since people are interested also in the case that the coefficient module is not a vector space there were other approaches to define topological group cohomology.

2. **Measurable Group Cohomology**

In [Moo76] Moore introduces and studies extensively another model. The idea was to replace continuous with measurable functions. To quickly recall the setup, let X be a measurable space, and A a complete metrizable space. We define then

$$\mathrm{U}\left(X,A\right) := \left\{ f \;:\; X \longrightarrow A \middle| \, f \text{ is measurable} \right\} / \sim \tag{2.18}$$

where the equivalence relation is generated by identifying functions which are the same almost everywhere. The measurable group cohomology is defined for a locally compact Hausdorff group G with values in a Polish G-module A[4] as

$$H^n_\mu\left(G,A\right) := H^n\left(\mathrm{U}\left(G^{\bullet+1},A\right)^G\right) \tag{2.19}$$

with the usual differential. In [Moo76] it is shown that the groups $H^n_\mu\left(G,A\right)$ have all the desirable properties mentioned in the introduction.

3. **Segal Mitchison Model**

From a homotopy theoretic point, arbitrary topological spaces, are not very well behaved. The category of compactly generated spaces seems to behaving much better so people tried to focus more on them. While other usual subcategories of **Top**, like locally compact Hausdorff and so on, have a product isomorphic to the usual product in **Top**, the product in the category of compactly generated spaces has a topology which is finer than the usual product topology. So taking groups internal to the category of compactly generated spaces give objects that are not topological groups in the classical sense, since the multiplication needs only be continuous for the compactly generated topology on the product. We will call groups internal to compactly generated Hausdorff spaces, k-groups. Segal and Mitchison[5] introduced a group cohomology theory for those groups in [Seg70]. We will call this (not very creatively) the Segal-Mitchison model. To define it, let G be a k-group and A a locally contractible abelian k-group. We denote by EA the universal bundle of A and we define

[4] i.e. a separable, completely metrizable topological abelian group with a continuous left G-action

[5] In the paper only G. Segal appears as an author but in the very first line of the paper he explicitly says that it was joint work with G. J. Mitchison.

$$E_{(G)}(A) := \mathbf{Top}(G, EA) \tag{2.20}$$
$$B_{(G)}(A) := \mathbf{Top}(G, EA) \big/ A \tag{2.21}$$

which gives an exact complex

$$0 \longrightarrow A \longrightarrow E_{(G)}(A) \longrightarrow E_{(G)}B_{(G)}(A) \longrightarrow E_{(G)}B^2_{(G)}(A) \longrightarrow \cdots \tag{2.22}$$

and the Segal Mitchison cohomology of G with values in A is defined as

$$H^n_{\mathrm{SM}}(G, A) := H^n\left(\left(E_{(G)}B^{\bullet}_{(G)}(A)\right)^G\right). \tag{2.23}$$

Again in [Seg70] they proved that it has the correct low degree interpretation and that it behaves like a delta functor for short exact sequences of topological G-modules which are locally trivial bundles.

4. **Simplicial Group Cohomology**

   Another approach to define topological group cohomology is via cohomology of simplicial spaces. An approach to define the latter was given by Deligne in [Del74], Friedlander in [Fri82] and more recently by Joshua in [Jos02]. The upshot of the method is that we get to work with abelian categories so there are a lot of tools available in our disposal, the downside is that explicit constructions in those categories are prohibitively difficult. Let us quickly recall the setup.
   Denote by **kTop** the category of compactly generated Hausdorff spaces[6]. Let $X_\bullet : \Delta^{op} \longrightarrow \mathbf{kTop}$ be a simplicial k-space. To $X_\bullet$ we associate a small site $\mathbf{Top}(X_\bullet)$ as follows

   (a) Objects are open subset inclusions $U \hookrightarrow X_n$ for all n in $\mathbb{N}$,
   (b) morphisms between $U \hookrightarrow X_n$ and $V \hookrightarrow X_m$ are arrows $U \to V$ such that the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, hook] & V \arrow[d, hook] \\ X_n \arrow[r] & X_m \end{tikzcd} \tag{2.24}$$

   commutes, where $X_n \to X_m$ is a structure map of $X_\bullet$,

[6] we call them k-spaces

(c) coverings for each $U \hookrightarrow X_n$ are usual coverings $\{V_a \to U\}$, when we view U as an open subset of X_n.

We denote by $\mathbf{Ab}(X_\bullet)$, the category of abelian sheaves on $\mathbf{Top}(X_\bullet)$. In [Del74] and [Fri82] it is shown that $\mathbf{Ab}(X_\bullet)$ is an abelian category.

Remark 2.2.2. It can easily be seen that objects of $\mathbf{Ab}(X_\bullet)$ have the following explicit description. We denote by $d_p^i := X_\bullet(\delta_i^p)$ and $s_p^i := X_\bullet(\sigma_i^p)$. A sheaf $E^\bullet$ on a simplicial space $X_\bullet$ is uniquely determined by a family of sheaves $\{E^n\}_{n\in\mathbb{N}}$ on the spaces X_n together with morphisms of sheaves $D_i^p : d_p^{i\,*}E^{p-1} \longrightarrow E^p$ and $S_i^p : s_p^{i\,*}E^{p+1} \longrightarrow E^p$, satisfying a certain long list of compatibility equations ([WW15, Chapter 2]). Clearly each sheaf E^n is defined by $E^n(U \hookrightarrow X_n) := E^\bullet(U \hookrightarrow X_n)$. To see how the D_i^p and S_i^p arise note that if an open subset V of X_{p-1} contains $d_p^i(U)$, where U is an open subset of X_p then there exists a unique lift of d_p^i between $U \to V$, giving a morphism in $\mathbf{Top}(X_\bullet)$. We remember also that

$$d_p^{i\,*}\left(E^{p-1}\right)(U) = \varinjlim E^{p-1}(V)$$

where the limit runs over all open subsets V which contain $d_p^i(U)$. By the argument above we get morphisms

$$E^\bullet\left(\begin{tikzcd} U \arrow[r] \arrow[d, hook] & V \arrow[d, hook] \\ X_p \arrow[r] & X_{p-1} \end{tikzcd}\right) : E^\bullet\left(\begin{tikzcd} V \arrow[d, hook] \\ X_{p-1} \end{tikzcd}\right) \longrightarrow E^\bullet\left(\begin{tikzcd} U \arrow[d, hook] \\ X_p \end{tikzcd}\right). \tag{2.25}$$

Then $D_i^p(U)$ is simply the universal morphism out of $d_p^{i\,*}(E^{p-1})(U)$. The S_i^p have a similar description. The compatibility conditions can be derived by using the fact that $E^\bullet$ is a functor. On the other direction assume we are given sheaves E^n over each space X_n together with morphisms D_i^p and S_i^p satisfying certain compatibility conditions. We can define the sheaf on $\mathbf{Top}(X_\bullet)$ to be $E^\bullet(U \hookrightarrow X_n) := E^n(U \hookrightarrow X_n)$ on objects and the D_i^p's and S_i^p's can be used to define what $E^\bullet$ assigns to morphisms.

Definition 2.2.3. The sheaf cohomology of $X_\bullet$ is defined as the δ-functor

$$H^n_{\mathrm{Sh}}(X_\bullet, -) : \mathbf{Ab}(X_\bullet) \longrightarrow \mathbf{Ab} \tag{2.26}$$

given by

$$H^n_{\mathrm{Sh}}(X_\bullet, A^\bullet) := \mathrm{Ext}^n_{\mathbf{Ab}(X_\bullet)}(\mathbb{Z}, A^\bullet), \tag{2.27}$$

where $\mathbb{Z}$ denotes the constant abelian sheaf on $\mathbf{Top}(X_\bullet)$.

Assume that G is a k-group, $BG_\bullet$ denotes the simplicial k-space given by $BG_n := G^n$, for face maps $\delta^n_i : [n-1] \longrightarrow [n]$

$$\begin{aligned}
BG_\bullet(\delta^n_i)(g_1, \dots, g_n) &:= (g_1, \dots, g_i g_{i+1}, \dots, g_n), && 1 < i < n \\
BG_\bullet(\delta^n_i)(g_1, \dots, g_n) &:= (g_2, \dots, g_n), && i = 1 \\
BG_\bullet(\delta^n_i)(g_1, \dots, g_n) &:= (g_1, \dots, g_{n-1}), && i = n
\end{aligned}$$

and for degeneracy maps $\sigma^n_i : [n+1] \longrightarrow [n]$

$$BG_\bullet(\sigma^n_i)(g_1, \dots, g_n) := (g_1, \dots, g_i, e, g_{i+1}, \dots, g_n).$$

Let A be a continuous G-module. We consider the sheaf

$$A^\bullet_{\mathrm{glob,c}} : \mathbf{Top}(BG_\bullet)^{op} \longrightarrow \mathbf{Ab}, \tag{2.28}$$

given by

$$A^\bullet_{\mathrm{glob,c}}(U \hookrightarrow G^n) := \mathbf{Top}(U, A)$$

and for any morphism f of $\mathbf{Top}(BG_\bullet)$, $A^\bullet_{\mathrm{glob,c}}(f)$ is constructed from

$$\begin{aligned}
A^\bullet_{\mathrm{glob,c}}(BG_\bullet(\delta^n_i))(f) &:= f \circ BG_\bullet(\delta^n_i), && i > 0 \\
A^\bullet_{\mathrm{glob,c}}(BG_\bullet(\delta^n_i))(f) &:= L_{g_0} \circ f \circ BG_\bullet(\delta^n_i), && i = 0 \\
A^\bullet_{\mathrm{glob,c}}(BG_\bullet(\sigma^n_i))(f) &:= f \circ BG_\bullet(\sigma^n), && \forall i.
\end{aligned}$$

Definition 2.2.4. The (continuous) simplicial group cohomology of G with values in A is defined as

$$H^n_{\mathrm{simp,c}}(G, A) := H^n_{\mathrm{Sh}}\left(BG_\bullet, A^\bullet_{\mathrm{glob,c}}\right). \tag{2.29}$$

This model is shortly discussed in Section II of [WW15], for example it is proved that $H^n_{\text{simp,c}}(G, -)$ is a δ-functor with respect to short exact sequences $A' \to A \to A''$ which are locally trivial bundles ([WW15, Lemma II.11]).

Remark 2.2.5 (Čech Cohomology). Classical sheaf cohomology is usually computed via Čech cohomology. A generalization of what Čech cohomology should be for abelian sheaves on sites has been worked out in several places (e.g. [KS06, 18.7]). In the case of sheaves on simplicial spaces we can use the following explicit description following [WW15, III] or [Bry00]. A cover of a simplicial space $X_\bullet$ is determined by a simplicial set $I_\bullet$ and a family of coverings

$$\mathcal{U}_\bullet := \left\{ \mathcal{U}^p := \{U_i^p\}_{i \in I_p} \right\}_{p \in \mathbb{N}}$$

for each space X_p such that

$$X_\bullet\left(\delta_i^p\right)\left(U_j^p\right) \subseteq U^{p-1}_{I_\bullet\left(\delta_i^p\right)(j)} \tag{2.30}$$

and

$$X_\bullet\left(\sigma_i^{p-1}\right)\left(U_j^{p-1}\right) \subseteq U^p_{I_\bullet\left(\sigma_i^{p-1}\right)(j)}. \tag{2.31}$$

We will say that a cover $\mathcal{U}_\bullet$ is good (resp. countable/locally finite/finite) if each of $\mathcal{U}^p$ is good (resp. countable/locally finite/finite). For simplicity we will sloppily denote by d_p^i or s_p^i any of the simplicial maps for $X_\bullet$ or $I_\bullet$ i.e. we will write $d_p^i\left(U_j^p\right) \subseteq U^{p-1}_{d_p^i(j)}$ [7]. We consider now the bigraded object in **Top**

$$\check{C}^{p,q}\left(\mathcal{U}_\bullet, E^\bullet\right) := \prod_{i_0,\dots,i_q \in I_p} E^p\left(U_{i_0,\dots,i_q}\right), \tag{2.32}$$

which we turn into a double complex with the following maps. The vertical differential is the Čech differential, i.e.

$$\left(d_v^{p,q}(x)\right)_{i_0,\dots,i_{q+1}} := \sum_{j=0}^{q+1} (-1)^j x_{i_0,\dots,\widehat{i_j},\dots,i_{q+1}}. \tag{2.33}$$

[7] Examples of such covers can be obtained from the covers as defined in [WW15, Definition 3.1] by further assuming that each open set U_i^p is connected. In particular good such covers.

To describe the differential in the other direction first note that by the assumption (2.30)

$$d^i_{p+1}\left(U^{p+1}_{i_0,\dots,i_q}\right) \subseteq \bigcap_{k=0}^{q} d^i_{p+1}\left(U^{p+1}_{j_k}\right) \subseteq \bigcap_{k=0}^{q} U^p_{d^i_{p+1}(j_k)}, \tag{2.34}$$

which shows that we have the following composition of maps

$$\begin{aligned}
\left(d^{p,q}_i\right)_{i_0,\dots,i_q} : \prod_{j_0,\dots,j_q\in I_p} E^p\left(U_{j_0,\dots,j_q}\right) &\xrightarrow{pr_{d^i_{p+1}(i_0),\dots,d^i_{p+1}(i_q)}} E^p\left(U^p_{d^i_{p+1}(i_0),\dots,d^i_{p+1}(i_q)}\right)\\
&\longrightarrow \varinjlim_{V\supseteq d^i_{p+1}\left(U^{p+1}_{i_0,\dots,i_q}\right)} E^p(V)\\
&\xrightarrow{D^{p+1}_i\left(U^{p+1}_{i_0,\dots,i_q}\right)} E^{p+1}\left(U^{p+1}_{i_0,\dots,i_q}\right).
\end{aligned}$$

Then the horizontal differential of (2.32) is defined by requiring

$$pr_{i_0,\dots,i_q} \circ d^{p,q}_h = \sum_{i=0}^{p+1} (-1)^{i+q} \left(d^{p,q}_i\right)_{i_0,\dots,i_q}. \tag{2.35}$$

Note that the q in the exponent of (-1) is superfluous, it ensures that the differentials anticommute so the total differential is the sum of the vertical and the horizontal differential. We could skip it and introduce a $(-1)^p$ in the total differential as we mentioned in Remark 1.1.7. The Čech cohomology of $\mathcal{U}_\bullet$ with values in $E^\bullet$ is defined as

$$\check{H}^n\left(\mathcal{U}_\bullet, E^\bullet\right) := H^n\left(\operatorname{tot}\check{C}^{\bullet,\bullet}\left(\mathcal{U}_\bullet, E^\bullet\right)\right). \tag{2.36}$$

Some of the above models were know to be isomorphic but a general classification was missing for a long time. That was remedied in [WW15, Theorem IV.5]. We recall it here along with some important corollaries. The notion of δ-functors used in the theorem is given in Definition A.1.15 (see also [WW15, Definition VI.1]). We denote by **G-Mod** the category of continuous G-modules in the realm of k-spaces.

Theorem 2.2.6. *Let* $(H^n : \textbf{G-Mod} \longrightarrow \textbf{Ab})_{n\in\mathbb{N}}$ *be a δ-functor such that*

1. $H^0(A) \cong A^G$

2. *$H^n(A) \cong H^n_{\mathrm{c}}(G,A)$ for contractible A,*

then $(H^n)_{n\in\mathbb{N}}$ is equivalent to $(H^n_{\mathrm{SM}}(G,-))_{n\in\mathbb{N}}$ as a δ-functor. Moreover each morphism between δ-functors with properties 1 and 2, that is an isomorphism for $n=0$, is an isomorphism of δ-functors.

Also as a corollary one gets ([WW15, Corollaries IV.7, IV.8 and Remark IV.13]).

Corollary 2.2.7. 1. *If G^n is paracompact then $H^n_{\mathrm{SM}}(G,A) \cong H^n_{\mathrm{simp,c}}(G,A)$.*

2. *If furthermore we are given a god cover $\mathcal{U}_\bullet$ of $BG_\bullet$ then $H^n_{\mathrm{SM}}(G,A) \cong \check{H}^n\left(\mathcal{U}_\bullet, A^\bullet_{\mathrm{glob,c}}\right)$.*

3. *If the product topology on the set theoretic product G^n is compactly generated for all n, then $H^n_{\mathrm{lc}}(G,A) \cong H^n_{\mathrm{SM}}(G,A)$.*

4. *If furthermore G is locally compact Hausdorff and A a Polish G-module, then $H^n_{\mathrm{lc}}(G,A) \cong H^n_{\mu}(G,A)$.*

Remark 2.2.8 (Lie group cohomology). Lie groups in many cases are more important than topological groups[8], so people have tried to define cohomology theories for Lie groups as well. Most of the models that appeared in this section have their counterparts in the smooth world. So there is smooth group cohomology as a counterpart to the continuous one, also for the simplicial case one could take the sheaf of smooth instead of continuous functions and of course there is a locally smooth group cohomology. We will not review any of them here, this is done quite extensively elsewhere (e.g. [WW15], [Nee02], [HM62], [vE58], [SP11]). We will say a few words only for the locally smooth group cohomology.
We will use the conventions from [WW15], i.e. G is a group object in the category of manifolds modelled on locally convex spaces and A is a G-module in this category. We can again define locally continuous functions between smooth manifolds as in Subsection 2.1.1, i.e. a function $f : M \longrightarrow N$ of the underlying sets between two manifolds is called locally smooth if there is an open neighborhood U of M such that $f|_U$ is smooth. We will use the notation $\mathrm{Map}_{\mathrm{ls}}(M,N)$, $\mathrm{Map}'_{\mathrm{ls}}(M,N)$, $\mathrm{Map}_{\mathrm{ls}\bullet}(M,N)$ and $\mathrm{Map}_{\Delta\mathrm{s}}(M^n,N)$ for the "smooth counterparts" of the sets of locally continuous functions appearing in Subsection 2.1.1. Locally smooth group cohomology is defined obviously as

$$H^n_{\mathrm{ls}}(G,A) := H^n\left(\mathrm{Map}_{\Delta\mathrm{s}}\left(G^{\bullet+1},A\right)^G\right). \tag{2.37}$$

[8]Although by Hilbert's 5th problem for "sufficiently nice" spaces the distinction is unnecessary

By [WW15, Proposition I.7] locally smooth and locally continuous group cohomology groups are isomorphic if G is finite-dimensional and $A \cong \mathfrak{a}/\Gamma$ where $\mathfrak{a}$ is quasi-complete locally convex space on which G acts smoothly and Γ is a discrete submodule. All the results of Subsections 2.1.1 and 2.1.2 have equivalent ones in the locally smooth setting (except from the continuity of the evaluation map). In particular Proposition 2.1.1 and Corollary 2.1.2 hold by changing everywhere the word continuous with smooth, Lemma 2.1.10 and Propostion 2.1.11 still hold trivially for locally smooth functions. There is also an equivalent of Proposition 2.1.12 with G a Lie group A a smooth G-module and X a manifold with a smooth G-action assuming that ψ is locally smooth and of course equivalent of Corollaries 2.1.13 and 2.1.14 for the locally smooth cohomologies. We will make use of those in Chapter 4 to derive a spectral sequence also for Lie group cohomology. We will use the generic term "Lie group" to refer to a group object in some appropriate category of manifolds and smooth functions and then a smooth G-module will be an abelian Lie group with a smooth G-action (in the appropriate category). Note that the above comments hold in a wide variety of choices for a category of manifolds.

Chapter 3

Properties of topological group cohomology

As we argued before, a model of topological group cohomology should encapsulate in some way topological and algebraic information in a non-trivial way. We present two results in this chapter which exhibit exactly this non-trivial interplay for the locally continuous model. The first relates to regularity properties of cocycle representatives of classes in $H^n_{\mathrm{lc}}(G, A)$. To make this somewhat precise we mention that Theorem 3.1.2 tells us that if G and its products admit good, countable and locally finite covers then each class of $H^n_{\mathrm{lc}}(G, A)$ is represented by a cocycle continuous on an open and dense subset.
The second result exhibiting the interplay between topological and algebraic information relates to obstruction classes in low degree Čech cohomology. Roughly we recall ([Gro55], or a more recent exposition can be found for example in [NWW13]) that if $Z \to K \to G$ is a central topological group extension and $P \to X$ is a principal G-bundle, then a natural question that arises is if there is a lift $\hat{P} \to X$ to a principal K-bundle. It has been shown that the obstruction to the existence of such a lift is described by the kernel of a function $\check{\delta}^1_k : \check{H}^1(X, \underline{G}) \longrightarrow \check{H}^2(X, \underline{Z})$. If K and G are abelian this is simply the connecting homomorphism of Čech cohomology related to the short exact sequence

$$\underline{Z} \to \underline{K} \to \underline{G}$$

of sheaves of continuous functions. It is shown (e.g. in [Gro55]) that even when K and G are not abelian, there is a construction of such a $\check{\delta}^1_K$ which is almost identical to the usual connecting homomorphism of the abelian case. It is known that the above lifting exists if and only if $\check{\delta}^1_K([P])$ vanishes. In this chapter we show that this class always vanishes if $K \to G$ is topologically trivial. But such classes $[K]$ of $H^2_{\mathrm{lc}}(G, Z)$ are described by the purely topological requirement that there is a representing cocycle which is globally continuous.

3.1 Regularity properties of cocycles

We will use the isomorphism between locally continuous group cohomology and the Čech cohomology of the sheaf of continuous functions on $BG_\bullet$ (Corollary 2.2.7) to investigate regularity properties of representatives of classes of locally continuous group cocycles. We will use a third cohomology theory as a tool to relate the first two. To this end we introduce the sheaf of locally continuous functions. Let X be a pointed topological space. We denote by $A^X_{\text{loc,c}}$ the sheaf on X defined on open subsets by

$$U \mapsto \begin{cases} \operatorname{Map}'_{\text{lc}}(U,A) & \text{if} \quad *_X \in U \\ \mathbf{Set}(U,A) & \text{otherwise} \end{cases} \tag{3.1}$$

and on morphisms by the usual restriction of functions. It is more or less obvious that this sheaf is acyclic, but we give below a Lemma with a more refined statement. To further fix some notation, we will call a cochain x in $\check{C}^n\left(\mathcal{U}, A^X_{\text{loc,c}}\right)$ continuous on an open and dense subset if there exists an open and dense subset of X such that the restriction of each of the $x_{i_0,\ldots,i_n}$ to it is continuous. We will also give results in the smooth case, so if X is a smooth manifold and A an abelian Lie group then we will denote by $A^X_{\text{loc,s}}$ the sheaf as above with functions being locally smooth, and a cochain x is smooth on an open and dense subset if there is an open and dense subset of X such that the restriction of each of the $x_{i_0,\ldots,i_n}$ to it is smooth. The first technical Lemma is the following.

Lemma 3.1.1. *Assume $\mathcal{U}$ is a countable and locally finite cover of X. Then the Čech complex $\check{C}^n\left(\mathcal{U}, A^X_{\text{loc,c}}\right)$ is homotopy equivalent to the one with degree 0 equal to $\check{H}^0\left(\mathcal{U}, A^X_{\text{loc,c}}\right)$ and 0 everywhere else. Furthermore, the contracting homotopy can be chosen such that it carries cochains continuous on an open and dense subset to ones which are also continuous on an open and dense subset. Furthermore, if X is a smooth manifold and A an abelian Lie group the respective assertions hold for $\check{C}\left(\mathcal{U}, A^X_{\text{loc,s}}\right)$.*

Proof. Since the cover is countable we can, without loss of generality, assume that $\mathcal{U}$ is indexed by the naturals[1] such that $*_X \in U_0$. If the cover is actually finite we will assume that it is indexed by all the naturals up to $\#(I)$. We construct

[1] of course we always assume the AoC

now a sort of a partition of unity[2]. We consider the locally continuous functions $\{\lambda_i : X \longrightarrow \{0,1\}\}_{i\in\mathbb{N}}$ defined by

$$\lambda_0(x) := \begin{cases} 1 & \text{if} \quad x \in U_0 \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

and

$$\lambda_{n+1}(x) := \begin{cases} 1 & \text{if} \quad x \in U_{n+1} \setminus \left(\bigcup_{i=0}^{n} U_i \right) \\ 0 & \text{otherwise} \end{cases}. \tag{3.3}$$

It is obvious that $\sum_{i\in\mathbb{N}} \lambda_i = 1$. It is well known that the existence of such functions makes the Čech complex collapse. Let us quickly review how. If f is a function defined on a subset of X denote by $[f]_0$ the one extended by 0 outside this subset. We define now the contracting homotopy $h^q : \check{C}^q\left(\mathcal{U}, A^X_{\text{loc,c}}\right) \longrightarrow \check{C}^{q-1}\left(\mathcal{U}, A^X_{\text{loc,c}}\right)$ by

$$h^q\,(x)_{i_0,\ldots,i_{q-1}} := \sum_{i\in\mathbb{N}} \lambda_i \left[x_{i,i_0,\ldots,i_{q-1}}\right]_0. \tag{3.4}$$

It is well-defined and straightforward calculations show that

$$\check{\delta}^{q-1} \circ h^q + h^{q+1} \circ \check{\delta}^q = id_{\check{C}^q\left(\mathcal{U}, A^X_{\text{loc,c}}\right)}.$$

This shows the first part so let us proceed to the second part. To make things a bit more presentable we denote by $\widetilde{U}_n := U_n \setminus \left(\bigcup_{i=0}^{n-1} U_i \right)$. By the definition of the λ_i's we see immediately that

$$h^q(x)_{i_0,\ldots,i_{q-1}}\big|_{\widetilde{U}_n} = x_{n,i_0,\ldots,i_{q-1}}. \tag{3.5}$$

Now assume that x is a cochain continuous on an open and dense subset V, meaning that $h^q(x)_{i_0,\ldots,i_{q-1}}$ is continuous on $\widetilde{U}_n \cap V$ for all n. We recall that a function which is continuous on open subsets is continuous also on their union. This shows that the restriction of $h^q(x)_{i_0,\ldots,i_{q-1}}$ to

[2] the functions will not be continuous

$$\begin{aligned} U_V &:= \bigcup_{n\in\mathbb{N}} \operatorname{int}\left(\widetilde{U}_n \cap V\right) \\ &= \bigcup_{n\in\mathbb{N}} \left(\operatorname{int}\left(\widetilde{U}_n\right) \cap \operatorname{int}(V)\right) \\ &= \left(\bigcup_{n\in\mathbb{N}} \operatorname{int}\left(\widetilde{U}_n\right)\right) \cap V \end{aligned}$$

is continuous. The finite intersection of open and dense subsets is open and dense. By assumption V is open and dense and

$$U := \left(\bigcup_{n\in\mathbb{N}} \operatorname{int}\left(\widetilde{U}_n\right)\right) \tag{3.6}$$

is open. So if U is dense, so is U_V. Clearly $\left\{\widetilde{U}_n\right\}_{n\in\mathbb{N}}$ is a cover of X and each point of X is contained in exactly one $\widetilde{U}_n$. So all the points not in U are contained in

$$\bigcup_{n\in\mathbb{N}} \partial\left(\widetilde{U}_n\right).$$

We assume now that U is not dense. Then there exists a point w not in U and a neighborhood W of w (which without loss of generality can be chosen to be open) such that $W \cap U$ is empty. This implies the following

$$W \subseteq \bigcup_{n\in\mathbb{N}} \partial\left(\widetilde{U}_n\right) \subseteq \bigcup_{n\in\mathbb{N}} \partial\left(U_n\right).$$

Now since by assumption the cover is locally finite there is an open neighborhood Ω of w such that $\Omega \cap U_n$ is empty for all but finitely many U_n. Now the points in Ω can belong only on finitely many boundaries otherwise Ω will intersect non-trivially infinitely many U_n. This shows that there is a finite subset I of $\mathbb{N}$ such that

$$W \cap \Omega \subseteq \bigcup_{n\in I} \partial\left(U_n\right).$$

Since all U_n are open, each $\partial\left(\widetilde{U}_n\right)$ has empty interior and since the boundaries are closed they are nowhere dense. Now the finite union of nowhere dense subsets is nowhere dense. This means that

$$\varnothing = \operatorname{int}\left(\operatorname{closure}\left(\bigcup_{n\in I}\partial\left(U_n\right)\right)\right) = \operatorname{int}\left(\bigcup_{n\in\mathbb{N}}\operatorname{closure}\left(\partial\left(U_n\right)\right)\right).$$

And finally our assumption is contradicted since $W\cap\Omega$ lies within the interior of $\bigcup_{n\in I}\partial\left(U_n\right)$ and contains w. This means that all neighborhoods of w intersect U non-trivially. Since this holds for all points of X it follows that U is dense. And finally $h^q(x)$ is continuous on an open and dense subset, as asserted. The proof in the smooth case works word by word by replacing any instance of the word "continuous" with "smooth". This concludes the proof. □

Theorem 3.1.2. *Assume G is a topological group such that all G^n are paracompact for each n. If there exists a good, countable and locally finite cover $\mathcal{U}_\bullet$ on $BG_\bullet$ then any class in $H^n_{\mathrm{lc}}(G,A)$ is represented by a cocycle continuous on an open and dense subset. Furthermore, if G is a Lie group and there exists $\mathcal{U}_\bullet$ as above, then every class in $H^n_{ls}(G,A)$ is represented by a cocycle smooth on an open and dense subset.*

Proof. First we note that there is a canonical identification for each p and each cover $\mathcal{V}_\bullet$ of $BG_\bullet$ of the complex $\check{C}^{p,*}\left(\mathcal{V}_\bullet, A^\bullet_{\mathrm{loc,c}}\right)$ with the complex $\check{C}^*\left(\mathcal{V}_p, A^{G^p}_{\mathrm{loc,c}}\right)$. So by the existence of $\mathcal{U}_\bullet$ we can apply Lemma 3.1.1 to each G^p to get contracting homotopies of the complexes $\check{C}^{p,*}\left(\mathcal{V}_\bullet, A^\bullet_{\mathrm{loc,c}}\right)$ which in turn implies that we can use the staircase Lemma A.1.12. Very easily one notes that $\ker d_v^{*,0}$ is canonically isomorphic with $\mathrm{Map}'_{\mathrm{lc}}(G^*,A)$[3]. So $\check{H}^n\left(\mathcal{U}_\bullet, A^\bullet_{\mathrm{loc,c}}\right)$ is isomorphic to $H^n_{\mathrm{lc}}(G,A)$. To write the isomorphism, recall the notation from the Appendix A of the "total horizontal differential" $D_h^n : \operatorname{tot}(C)^n \longrightarrow \operatorname{tot}(C)^{n+1}$ and the "total column-wise homotopy" $M^n : \operatorname{tot}(C)^n \longrightarrow \operatorname{tot}(C)^{n-1}$. Then the isomorphism is provided at the level of cocycles by sending a cocycle x of $\operatorname{tot}\left(\check{C}^{*,*}\left(\mathcal{U}_\bullet, A^\bullet_{\mathrm{loc,c}}\right)\right)$ to

$$\sum_{i=0}^{n}\left(-D_h^{n-1}M^n\right)^{n-i}x^{i,n-i}.^{4}$

Now assume that each $x^{i,n-i}$ is continuous on an open and dense subset. When D_h^{n-1} and M^n are applied to an element of the product which has non-vanishing components on only one spot they become the normal horizontal differential $d_h^{p,q}$ and the usual homotopy $h^{p,q}$. Since their superscripts become redundant once we know to which $x^{p,q}$ we want to apply them we drop the bookkeeping subscripts of the morphisms.

By Lemma 3.1.1, to show that $(-d_h h)^{n-i}x^{i,n-i}$ is continuous on an open and dense

[3] note we are using here the inhomogeneous cocycles introduced in 2.14

[4] of course we introduce some sloppiness to ease notation by not writting the inclusion maps

subset it is enough to show that all horizontal differentials send cochains that are continuous on an open and dense subset to ones that are continuous on an open and dense subset. This follows from simply unwinding the definitions, but let us see explicitly why.
We resume the shorthand notation $d_p^i := BG_\bullet\left(\delta_i^p\right)$ from Remark 2.2.5. Note that for $i = 0, p$ it is simply a projection so in particular open. Otherwise it is group multiplication, which is also open. The pullback sheaf under an open map has a simpler description which in our case reads

$$\left(d_p^i\right)^{-1}\left(A^\bullet_{\text{loc,c}}\right)(U) \cong A^\bullet_{\text{loc,c}}\left(d_p^i(U)\right).$$

This means that the $\left(d_i^{p,q}\right)_{i_0,\dots,i_q}$ have a simpler description (since the D_i^p are simple precomposition with the d_p^i), i.e. if x is an element of $\prod\limits_{j_0,\dots,j_q \in I_p} A^\bullet_{\text{loc,c}}\left(U_{j_0,\dots,j_q}\right)$ then

$$\left(d_i^{p,q}\right)_{i_0,\dots,i_q}(x) = x_{d_{p+1}^i(i_0),\dots,d_{p+1}^i(i_q)} \circ d_{p+1}^i\big|_{U_{d_{p+1}^i(i_0),\dots,d_{p+1}^i(i_q)}}.$$

Now by assumption all $x_{j_0,\dots,j_q}$ are continuous on an open and dense subset. Denote this subset by V. Then Corollary 2.1.2 implies that $\left(d_i^{p,q}\right)_{i_0,\dots,i_q}(x)$ is continuous when restricted to $\left(d_{p+1}^i\right)^{-1}(V)$. The latter is open and dense since d_{p+1}^i is continuous and open. This shows that each $pr_{i_0,\dots,i_q} \circ d_h^{p,q}$, restricted to $\bigcap\limits_{i=0}^{p+1}\left(d_{p+1}^i\right)^{-1}(V)$, is continuous. The latter is a finite intersection of open and dense subsets so it is open and dense.
This shows that each term $(-d_h h)^{n-i}\, x^{i,n-i}$ is continuous on an open and dense subset if $x^{i,n-i}$ is so. The above sum is continuous on the intersection of the subsets that each term is continuous on. This intersection is a finite intersection of open and dense subsets. So the entire sum is continuous on an open and dense subset under our assumption.
So far we did not use the assumption on $\mathcal{U}_\bullet$ being a good cover. Under this assumption now by [WW15, Comparison Theorem and Proposition 3.4] the inclusion of continuous functions on locally continuous ones induces an isomorphism

$$\check{H}^n\left(\mathcal{U}_\bullet, A^\bullet_{\text{glob,c}}\right) \xrightarrow{i_*} \check{H}^n\left(\mathcal{U}_\bullet, A^\bullet_{\text{loc,c}}\right).$$

Thus each class of $\check{H}^n\left(\mathcal{U}_\bullet, A^\bullet_{\text{loc,c}}\right)$ is represented by a cocycle c for which all $c^{i,n-i}$ are continuous on an open and dense subset (namely the entire space). Since the staircase argument gives an isomorphism, it follows that each class of $H^n_{\text{lc}}(G, A)$ is represented by a cocycle continuous on an open and dense subset.

Finally, all the arguments carry over to the smooth case by simply changing the word continuous to smooth. □

Remark 3.1.3. Note that starting from a good, countable and locally finite cover $\mathcal{U}_\bullet$ of $BG_\bullet$, one can possibly in a straightforward but very tedious way describe the subset of G^n, for which the restriction of a cocycle under the maps $\check{H}^n\left(\mathcal{U}_\bullet, A^\bullet_{\text{glob,c}}\right) \to \check{H}^n\left(\mathcal{U}_\bullet, A^\bullet_{\text{loc,c}}\right) \to H^n_{\text{lc}}(G,A)$, is continuous. Or one could also describe the discontinuities. Both can be achieved by going through the proof. We will give a rough description for the discontinuities. It is obvious from the proof that each application of the homotopy we constructed creates discontinuities contained in $\bigcup_{n\in\mathbb{N}} \partial\left(\widetilde{U}^p_n\right)$ (recall Proof of Lemma 3.1.1). Then the discontinuities will be transferred by the face maps of $BG_\bullet$. In the end we will have a countable union of preimages of boundaries.

Remark 3.1.4. Lemma 3.1.1 and Theorem 3.1.2 show that the existence of nice covers on G ensure that the cohomology classes of $H^n_{\text{lc}}(G,A)$ can be represented by cocycles with good regularity properties. Lemma 3.1.1 can be applied for example to locally compact, Hausdorff and second countable spaces. The requirement for G to have good, countable and locally finite covers restricts a bit the class of the examples. Note that the assumption on the cover to be good has nothing to do with the constructed cocycle but only relates to whether or not classes in $\check{H}^n\left(\mathcal{U}_\bullet, A^\bullet_{\text{loc,c}}\right)$ have a representative with nice properties. If the cover is good, we can use [WW15, Remark 4.11] to obtain such ones from the globally continuous Čech model. In spite of this restriction, some of the most interesting topological groups have such covers as we note in the following Corollary.

Corollary 3.1.5. *If G is a finite-dimensional second countable Lie group, then each class of $H^n_{\text{lc}}(G,A)$ has a cocycle which is continuous on an open and dense subset.*
Similarly, each class in $H^n_{\text{ls}}(G,A)$ is represented by a cocycle smooth on an open and dense subset.

Proof. The result follows simply from the fact that a second countable finite-dimensional manifold M admits good, countable and locally finite refinements for each open cover. This can be found in many places (e.g. it is included in [Pet06, Proof of Theorem 89]), but let us outline the idea. Firstly there is a compact exhaustion $\{K_i\}_{i\in\mathbb{N}}$ of M since it is in particular second countable, locally compact and Hausdorff[5]. For each point g in G denote by $i_g := \min\left\{i \in \mathbb{N} \middle| g \in K_i\right\}$. Since M can be endowed with a Riemannian metric, for each point g in M we can pick

[5]this can be constructed for example as in [War83, Lemma 1.9]

a geodesic ball V_g around it which is contained in $\left(\operatorname{int}\left(K_{i_g+1}\right) \setminus K_{i_g-1}\right) \cap U_j$ where U_j is an open set belonging to the original cover. Since each K_i is compact we can pick for each i finitely many g_j such that the corresponding V_{g_j} cover K_i. Clearly this is a good, countable and locally finite refinement of the original cover.
To construct a good, countable and locally finite cover on $BG_\bullet$ take one on G pull it back with the face maps to G^2, refine it to one with such properties. Continuing this procedure for all n we get a good, countable and locally finite cover $\mathcal{U}_\bullet$ on $BG_\bullet$. This means that we can apply Theorem 3.1.2 to get the claim. □

3.2 Lifting Obstructions

We turn now our attention to the second advertised result of this chapter, the relation of properties of $H^2_{\mathrm{lc}}(G, Z)$ with lifting obstruction classes. The material in this section is closely related to (and mostly inspired by) the work in [NWW13], so we will also adopt some of their notation.
We will use the following assumptions throughout this section without further mentioning. All spaces[6] will be CW-complexes, X will be a paracompact Hausdorff space, G will be a k-group[7], Γ is a discrete abelian group, Z is a topological abelian group such that its identity component Z_0 is an Eilenberg-Maclane space $\mathrm{K}(\Gamma, 1)$ and is considered as a G-module with the trivial action. The latter means that $H^2_{\mathrm{lc}}(G, Z)$ classifies equivalence classes of central extensions of G by Z.

Remark 3.2.1. We should make a short comment on the last statement. Let us denote by $\operatorname{Ext}(G, Z)$ the equivalence classes of all extensions K of G by Z for which the projection $K \to G$ is a Z-principal bundle. There are two subtleties involved in relating the latter with classes in $H^2_{\mathrm{lc}}(G, Z)$. The first involves a subtlety that already appears in classical group cohomology. The second cohomology group classifies those classes of group extensions for which conjugation in K of elements z of Z, agrees with the action of G on Z. So if Z is not a trivial G-module then $H^2_{\mathrm{lc}}(G, Z)$ will classify no central extension. The second subtlety relates to G being connected or not and how to give the equivalence. If G is connected as we mentioned in the introduction the classical methods of creating extensions from group cocycles follow in exactly the same way using in essence [Bou98, I.2], as is done in [Nee04, Section 2] in the smooth case. If G is not connected it is not so straightforward how to associate group cocycles to extensions but still $H^2_{\mathrm{lc}}(G, Z)$ classifies topological extensions of G by Z (for which conjugation in the extension

[6] including the underlying topological space of topological groups
[7] we are also assuming that the product topology and the k-product topology on all G^n agree, examples of such behaviour include locally compact Hausdorff topological groups

agrees with the action of G on Z) as follows from [Seg70, Proposition 4.3] and the comparison theorem [WW15, Theorem 4.5].
In this section the results relate only to central extensions so we will put all the above under the rug. We have to note though that by [Gro55] the function obs can be defined on $\check{H}^1(X, \underline{G}) \times \operatorname{Ext}(G, Z)$ but for the current work this is not necessary.

To state the main theorem we will quickly recall some constructions from [NWW13] (or [Gro55]). We recall the construction of a function

$$\text{obs} : \check{H}^1(X, G) \times H^2_{\text{lc}}(G, Z) \longrightarrow \check{H}^2(X, Z). \tag{3.7}$$

From now one let $Z \hookrightarrow K \to G$ be a topological central group extension. This induces an exact sequence of the Čech cohomology groups of the sheaves of continuous functions [Gro55]

$$\check{H}^1(X, \underline{Z}) \longrightarrow \check{H}^1(X, \underline{K}) \longrightarrow \check{H}^1(X, \underline{G}) \xrightarrow{\delta_1^K} \check{H}^2(X, \underline{Z}).$$

The morphism δ_1^K is described in a similar way as in the case of the long exact sequence for abelian coefficients. We quickly recall it here. Since X is paracompact by [Gro55, Lemma 5.7.1] a class in $\check{H}^1(X, \underline{G})$ is represented by an open cover $\mathcal{U} := \{U_i\}_{i \in I}$ of X and a collection of continuous functions

$$c := \{c_{i,j} : U_i \cap U_j \longrightarrow G\}_{i,j \in I}$$

satisfying

$$c_{i,k} c_{k,j} c_{i,j}^{-1} = e$$

for all i, j, k in I. Furthermore, by the same Lemma the cover $\mathcal{U}$ and the collection c can be chosen such that there exist a continuous lift $\widetilde{c_{i,j}} : U_i \cap U_j \longrightarrow K$ for each of the $c_{i,j}$. Then

$$\delta_1^K([\{c_{i,j}\}]) := \left[\left\{\widetilde{c_{i,k}}\widetilde{c_{k,j}}\widetilde{c_{i,j}}^{-1}\right\}\right]$$

is well-defined, independent of the choice of the representative $\{c_{i,j}\}$ and the lifts [Gro55]. In the same paper it is shown that δ_1^K depends only on the class of the extension in $H^2_{\text{lc}}(G, Z)$. So if $P \to X$ is a principal G-bundle we define $\text{obs}([P], [K]) := \delta_1^K([P])$.
The aim of this section is to show that obs vanishes on classes in $H^2_{\text{lc}}(G, Z)$ which are in the image of the natural inclusion $H^2_{\text{c}}(G, Z) \to H^2_{\text{lc}}(G, Z)$. We make it precise in the following Theorem.

Theorem 3.2.2. *Let Z be either discrete or an Eilenberg-Maclane space for a discrete group. Let*

$$0 \to Z \to K \to G \to 0 \tag{3.8}$$

be a topological central group extension and $P \to X$ be a principal G-bundle. If (3.8) is represented by a globally continuous group cocycle then

$$\operatorname{obs}\left([P],[K]\right) = 0 \in \check{H}^2\left(X, \underline{Z}\right). \tag{3.9}$$

Theorem 3.2.2 will follow from some simple lemmas.
We recall the construction of a function

$$\Psi \; : \; H^2_{\mathrm{lc}}\left(G, Z\right) \longrightarrow \left[BG, B^2 Z\right], \tag{3.10}$$

outlined for example in [MS00, Section 7.1]. Let a topological group extension (3.8) represent a class in $H^2_{\mathrm{lc}}\left(G, Z\right)$. In [Seg70, Appendix A] it is shown that the geometric realization of K gives a universal bundle for K if the latter is locally contractible[8]. Since $Z \to K$ is injective the restriction of the free action of K on EK gives a free action of Z on EK. EK is contractible so by [Swi75, Theorem 11.35] EK is also a universal bundle for Z and BZ is homotopy equivalent to EK/Z. Also since Z is central in K, it is also central in EK, so EK/Z is a topological group. Clearly

$$G \cong \operatorname{coeq}\left(pr_K, \mu_K \circ \left(i_Z \times id_K\right)\right)$$

where $pr_K : Z \times K \longrightarrow K$ is the projection to K and $\mu_K \circ (i_Z \times id_K) : Z \times K \longrightarrow K$ is the group multiplication of K post-composed by the obvious inclusion into $K \times K$. The continuous function $f \; : \; G \longrightarrow EK/Z$ which factorizes $K \to EK \to EK/Z$ is easily seen to be a classifying map for K. The latter composition is a group homomorphism, the bundle projection $K \to G$ is a surjective group homomorphism and the diagram

$$\begin{tikzcd} K \arrow[r] \arrow[d] & EK \arrow[d] \\ G \arrow[r, "f"] & EK/Z \end{tikzcd}$$

[8] which is the case for us since we are assuming it is in particular a CW-complex

commutes, so f is a group homomorphism as well. So by [Swi75, 11.37] f gives rise to a continuous function

$$Bf \;:\; BG \longrightarrow B^2 Z \tag{3.11}$$

which is unique up to homotopy, such that the diagram

$$\begin{tikzcd} {[-;BG]} \arrow[r, "\psi", "\cong"'] \arrow[d, "Bf\circ"] & \check{H}^1\left(-,\underline{G}\right) \arrow[d, "f_*"] \\ {[-;B^2Z]} \arrow[r, "\phi", "\cong"'] & \check{H}^1\left(-,\underline{BZ}\right) \end{tikzcd} \tag{3.12}$$

commutes[9]. It is also obvious that the construction depends only on the class of the extension K so we finally define

$$\Psi\left([Z \to K \to G]\right) := [Bf]\,. \tag{3.13}$$

Lemma 3.2.3. *Left $f \,:\, G \longrightarrow BZ$ be a classifying map of a topological group extension (3.8), constructed as above. Denote by $\delta \,:\, \check{H}^1\left(BG,\underline{BZ}\right) \longrightarrow \check{H}^2\left(BG,\underline{Z}\right)$ the connecting homomorphism of the long exact sequence coming from the coefficient sequence $\underline{Z} \to \underline{EZ} \to \underline{BZ}$. Then*

$$\mathrm{obs}^K\left([EG]\right) = \delta\left(\left[Bf^*\left(EBZ\right)\right]\right). \tag{3.14}$$

Proof. First we note that since EZ is contractible δ is an isomorphism. Let $c_{i,j} \,:\, U_i \cap U_j \longrightarrow G$ be a cocycle representing the class of $[EG]$. Since f is a continuous group homomorphism $fc_{i,j} \,:\, U_i \cap U_j \longrightarrow BZ$ is a cocycle representing a class in $\check{H}^1\left(BG,BZ\right)$. We will write δ_1 for obs^K. As we discussed in the beginning of the section $\delta\left([\{fc_{i,j}\}]\right)$ is constructed by finding continuous lifts $\widetilde{fc}_{i,j} \,:\, U_i \cap U_j \longrightarrow EBZ$ and then

$$\delta\left([\{fc_{i,j}\}]\right) = \left[\left\{\widetilde{fc}_{i,k}\widetilde{fc}_{k,j}\widetilde{fc}_{i,j}^{-1}\right\}\right].$$

Also

[9] Recall $f_*\left([P]\right) = \left[\left(P \times BZ\right)/G\right]$ where G acts on $P \times BZ$ on the right as $\left(p,z\right).g = \left(p.g, f\left(g\right)^{-1} z\right)$.

$$\delta_1\left([\{c_{i,j}\}]\right) = \left[\left\{\tilde{\mathrm{c}}_{i,k}\tilde{\mathrm{c}}_{k,j}\tilde{\mathrm{c}}_{i,j}^{-1}\right\}\right]$$

where $\tilde{\mathrm{c}}_{i,j}$ are continuous lifts of the $c_{i,j}$'s to $K \cong G \times_{BZ} EZ$. The functions

$$a_{i,j} \,:\, U_i \cap U_j \longrightarrow G \times EZ,\; b \mapsto \left(c_{i,j}(b), \widetilde{fc}_{i,j}(b)\right)$$

restrict to K since

$$fc_{i,j} = p\widetilde{fc}_{i,j}.$$

Those restrictions clearly give lifts of the $c_{i,j}$'s. But

$$\tilde{\mathrm{c}}_{i,k}\tilde{\mathrm{c}}_{k,j}\tilde{\mathrm{c}}_{i,j}^{-1} = \left(e, \widetilde{fc}_{i,k}\widetilde{fc}_{k,j}\widetilde{fc}_{i,j}^{-1}\right)$$

so $\delta\left([\{fc_{i,j}\}]\right) = \delta_1\left([\{c_{i,j}\}]\right)$. Let $l \,:\, BG \longrightarrow B^2Z$ be a classifying map for the bundle constructed with cocycle $\{fc_{i,j}\}$. By [Swi75, Theorem 9.13] Bf is the unique up to homotopy map that makes the diagram (3.12) commute. Note now that $\phi\left([l]\right) = l^*\left(EBZ\right)$, which by definition is the bundle constructed from cocycle data $fc_{i,j}$. So

$$l^*\left(EBZ\right) \cong f_*\left(EG\right) = f_*\left(\psi\left(id_{BG}\right)\right).$$

So $[l] = [Bf]$ and $Bf^*\left(EBZ\right) \cong l^*\left(EBZ\right)$. Finally

$$\mathrm{obs}^K\left([EG]\right) = \delta_1\left([\{c_{i,j}\}]\right) = \delta\left([\{fc_{i,j}\}]\right) = \delta\left([l^*\left(EBZ\right)]\right) = \delta\left([Bf^*\left(EBZ\right)]\right).$$

□

Denote by $\phi_1 \,:\, \check{H}^1\left(X, \underline{G}\right) \longrightarrow [X, BG]$ the inverse natural isomorphism of taking pullbacks. Note also that EZ is contractible so the long exact sequence in cohomology gives natural isomorphisms $\check{H}^n\left(-, \underline{Z}\right) \to \check{H}^1\left(-, B^{n-1}Z\right)$, and composing those with the natural isomorphisms which are inverse to taking pullbacks, we get natural isomorphisms $\psi_n \,:\, \check{H}^n\left(-, \underline{Z}\right) \longrightarrow [-, B^nZ]$.

Lemma 3.2.4. *The diagram*

$$\begin{tikzcd}
\check{H}^1\left(X, \underline{G}\right) \times H^2_{\mathrm{lc}}\left(G, Z\right) \arrow[r, "\mathrm{obs}"] \arrow[d, "\phi_1 \times \Psi"] & \check{H}^2\left(X, \underline{Z}\right) \arrow[d, "\psi_2(X)", "\cong"'] \\
{[X, BG] \times [BG, B^2Z]} \arrow[r] & {[X, B^2Z]}
\end{tikzcd} \tag{3.15}$$

commutes, where the bottom arrow is composition.

Proof. Let $P \to X$ be a G-principal bundle. We choose a representative l of $\phi_1\Big([G \to P \to X]\Big)$. Going through the lower side of diagram (3.15)

$$\Psi\Big([Z \to K \to G]\Big) \circ \phi_1\Big([G \to P \to X]\Big) = [Bf] \circ [l] = [Bf \circ l]$$

We also note that

$$[P] = [l^* (EG)]$$

and by [NWW13, Lemma 2.2]

$$\delta_1 ([P]) = l^* \delta_1 ([EG]) .$$

Denote also by $\phi_2 := \psi_2 (X)$ and $\phi_2' := \psi_2 (BG)$. It follows by Lemma 3.2.3 that $\phi_2' (\delta_1 ([EG])) = [Bf]$. We compute the upper side of diagram (3.15)

$$\begin{aligned} \phi_2\Bigg(\text{obs}\Big([G \to P \to X], [Z \to K \to G]\Big)\Bigg) &= \phi_2\Big(l^* \delta_1\big([EG]\big)\Big) \\ &= \phi_2'\Bigg(\delta_1\Big([EG]\Big)\Bigg) \circ [l] \\ \text{(By Lemma 3.2.3)} \qquad &= [Bf \circ l] \end{aligned}$$

And the claim is proven. □

Lemma 3.2.5. *Assume f classifies a topologically trivial bundle. Then if Z is a discrete group Γ or an Eilenberg-Maclane space $\mathrm{K}(\Gamma, n)$ Bf is homotopic to* 0.

Proof. Since f classifies a trivial bundle it is homotopic to 0 (since $0^* (EZ) \cong G \times Z$). By [Bre97, Theorem II.11.12] the induced function $H^q_{\text{Sh}} (f^p)$ between $H^q_{\text{Sh}} (BZ^p, \underline{\Gamma})$ and $H^q_{\text{Sh}} (G^p, \underline{\Gamma})$ vanish. By [Fri82, Proposition 2.4] there are spectral sequences

$$E_1^{p,q} = H^q_{\text{Sh}} (BZ^p, \Gamma) \Rightarrow H^{p+q}_{\text{Sh}} \left(B (BZ)_\bullet , \underline{\Gamma}^\bullet_{\text{glob,c}}\right)$$

and

$$E_1^{p,q} = H^q_{\text{Sh}} (G^p, \Gamma) \Rightarrow H^{p+q}_{\text{Sh}} \left(BG_\bullet, \underline{\Gamma}^\bullet_{\text{glob,c}}\right)$$

and their constructions are natural both for the simplicial space and the sheaf. So $H^q_{\text{Sh}} (f^p)$ are maps of spectral sequences and the induced map on the cohomology of the simplicial spaces are $H^q_{\text{Sh}} (Bf)$. By [WW15, Cor IV.8] and [Seg70, Proposition 3.3] there are natural isomorphisms

$$H^n_{\mathrm{Sh}}\left(B\left(BZ\right)_\bullet, \underline{\Gamma}^\bullet_{\mathrm{glob,c}}\right) =: H^n_{\mathrm{simp,c}}\left(B^2Z, \underline{\Gamma}\right) \cong H^n_{\mathrm{Sh}}\left(B^2Z, \Gamma\right)$$

since Γ is discrete. So the induced morphism

$$H^n_{\mathrm{Sh}}\left(Bf\right) \,:\, H^n_{\mathrm{Sh}}\left(B^2Z, \Gamma\right) \longrightarrow H^n_{\mathrm{Sh}}\left(BG, \Gamma\right)$$

vanishes. Since our spaces are locally contractible sheaf cohomology is naturally isomorphic to singular cohomology. Also $H^n_{\mathrm{sing}}\left(-, \Gamma\right)$ is naturally isomorphic to $\left[-, \mathrm{K}\left(\Gamma, n\right)\right]$, so by the commutativity of

$$\begin{tikzcd}
H^n_{\mathrm{sing}}\left(B^2Z, \Gamma\right) \arrow[r, "\cong"] \arrow[d, "H^n_{\mathrm{sing}}(Bf)"] & \left[B^2Z, \mathrm{K}\left(\Gamma, n\right)\right] \arrow[d, "(\circ[Bf])"] \\
H^n_{\mathrm{sing}}\left(BG, \Gamma\right) \arrow[r, "\cong"] & \left[BG, \mathrm{K}\left(\Gamma, n\right)\right]
\end{tikzcd} \tag{3.16}$$

and the vanishing of $H^n_{\mathrm{sing}}\left(Bf\right)$ from above, we deduce that the morphism $\left(\circ\left[Bf\right]\right)$ vanishes.
Let now $Z = \Gamma$ then

$$\left[Bf\right] = \left(\circ\left[Bf\right]\right)\left(id_{\mathrm{K}(\Gamma,2)}\right) = 0.$$

If instead we have that $Z = \mathrm{K}\left(\Gamma, n\right)$ then

$$\left[Bf\right] = \left(\circ\left[Bf\right]\right)\left(id_{\mathrm{K}(\Gamma,n+2)}\right) = 0.$$

And the claim is proven. □

Finally Theorem 3.2.2 follows from Lemmas 3.2.4 and 3.2.5.

Chapter 4

The LHS Spectral Sequence

We already mentioned that $H^n_*(G,-)$ has usually the property of being a "delta-functor", i.e. to sent specific classes of short exact sequences

$$0 \to A' \to A \to A'' \to 0$$

to long exact sequences. A natural question is whether or not $H^n_*(-,A)$ has a similar property. To be more specific, let A be a G-module, if $N \hookrightarrow G \twoheadrightarrow G/N$ is a short exact sequence of groups[1], the N-invariant subgroup of A, which we denote by A^N, comes with a canonical G/N-module structure. Do the cohomology groups $H^n_*(G,A)$, $H^n_*(N,A)$ and $H^n_*\left(G/N, A^N\right)$ fit into a long exact sequence? The answer is negative in most cases. The relation between those groups, in the case of discrete groups was first investigated by Lyndon ([Lyn48]) and shortly after fully described by Hochschild and Serre ([HS53]). They found that the natural morphisms

$$H^n_{\mathrm{gr}}\left(G/N, A^N\right) \to H^n_{\mathrm{gr}}(G,A) \to H^n_{\mathrm{gr}}(N,A) \tag{4.1}$$

do not fit into a long exact sequence, but are part of the information (the edge homomorphisms) of a spectral sequence of the form

$$E_2^{p,q} := H^p_{\mathrm{gr}}\left(G/N, H^q_{\mathrm{gr}}(N,A)\right) \Rightarrow H^{p+q}_{\mathrm{gr}}(G,A)\,. \tag{4.2}$$

Because of the people that derived it, it is in the literature more commonly referred to as the LHS spectral sequence. If instead of groups we have topological groups, and instead of group cohomology we have some model for topological group cohomology, we will call such a result (not very creatively) as LHS result.
Note that (4.2) is a special case of a Grothendieck spectral sequence. In the first section we will recall the classic result in the case of discrete groups and also

[1] clearly in that case N is normal in G

recall some results derived by Hochschild/Mostow and Moore for the continuous and measurable cases respectively [HM62], [Moo76]. In the second section we will derive a LHS results for the locally continuous and locally smooth model, as well give results on the continuous and smooth case for arbitrary coefficients but with restrictions on G and G/N.
Since we will deal in this chapter with relations between cohomology groups of different groups let us take a moment to fix some notation. Let $\rho : G \longrightarrow K$ be a group homomorphism (in the topological cases also continuous). If A is a G'-module with an action μ, then $\mu \circ (\rho \times id_A)$ defines an action of G on A. It is easy to see that this assignment gives a well defined functor $\rho^* : \mathbf{K\text{-}Mod} \longrightarrow \mathbf{G\text{-}Mod}$ which is exact. If H is a subgroup of a group G then we will use i_H for the inclusion. In the latter cases if A is a G-module we will abusively also write A for $(i_H)^* (A)$.

4.1 Classical LHS results

The classical LHS result first appeared in the PhD thesis of Lyndon in [Lyn48] and was later discussed in full by Hochschild and Serre in [HS53]. Here we follow Weibel's exposition in [Wei94]. The main theorem is the following.

Theorem 4.1.1. *Let N be a normal subgroup of a group G and A be a G-module. Then there exists a convergent spectral sequence of the form*

$$E_2^{p,q} := H_{\mathrm{gr}}^p \left(G/N, H_{\mathrm{gr}}^q (N, A)\right) \Rightarrow H_{\mathrm{gr}}^{p+q} (G, A) \tag{4.3}$$

Proof. We denote by $p : G \longrightarrow G/N$ the obvious projection. The functor $-^G : \mathbf{G\text{-}Mod} \longrightarrow \mathbf{Ab}$ decomposes as

$$\begin{tikzcd} \mathbf{G\text{-}Mod} \arrow[rr, "-_G^N"] \arrow[dr, "-^G"'] & & \mathbf{G/N\text{-}Mod} \arrow[dl, "-^{G/N}"] \\ & \mathbf{Ab} & \end{tikzcd},$$

where A_G^N is the N-invariant subgroup of A, with G/N-module structure given by $gN.a := g.a$ (the latter is well defined since we consider the N-invariant subgroup). We denote it like that to distinguish it from the functor $-^N : \mathbf{N\text{-}Mod} \longrightarrow \mathbf{Ab}$. It is easy to see that $-_G^N$ is right adjoint to the exact functor $p^* : \mathbf{G/N\text{-}Mod} \longrightarrow \mathbf{G\text{-}Mod}$, so $-_G^N$ preserves injectives. So we can apply

Theorem 1.2.12 to get a spectral sequence

$$E_2^{p,q} := R^p\left(-^{G/N}\right)\left(R^q\left(-_G^N\right)(A)\right) \Rightarrow R^{p+q}\left(-^G\right)(A) = H_{\mathrm{gr}}^{p+q}(G,A) \tag{4.4}$$

Denote for $:$ **G/N-Mod** $\longrightarrow$ **Ab** the obvious forgetful functor. Now it is obvious that for $\circ -_G^N \cong -^N \circ (i_N)^*$, but since for and $(i_N)^*$ are exact it follows that

$$\text{for} \circ R^n\left(-_G^N\right)(A) \cong H_{\mathrm{gr}}^n(N,A). \tag{4.5}$$

And the claim is proven. □

Similar results do not follow so easily for topological group cohomology. To get them one has to take the construction of a Grothendieck spectral sequence and check which steps can still be carried out in the respective models.
We will first discuss the measurable model. Moore managed to get an LHS result in a quite decent generality for his setup. We recall [Moo76, Theorem 9].

Theorem 4.1.2. *Let G be a locally compact Hausdorff group, N a closed normal subgroup and A a Polish G-module. If $H_\mu^n(N,A)$ is Hausdorff for all n in $\mathbb{N}$, then there is a convergent spectral sequence of the form*

$$E_2^{p,q} := H_\mu^p\left(G/N, H_\mu^q(N,A)\right) \Rightarrow H_\mu^{p+q}(G,A). \tag{4.6}$$

Proof. We will not rewrite the proof here. We will simply give a quick sketch of a construction. Recall the definition of $\mathrm{U}(X,A)$ in (2.18). Let us start with the double complex

$$C^{p,q} \cong \mathrm{U}\left(\left(G/N\right)^{q+1}, \mathrm{U}\left(G^{p+1},A\right)^N\right)^{G/N} \tag{4.7}$$

with the obvious differentials. We recall that by Corollary 1.2.9 we get two spectral sequences by filtering by columns and by rows. The one of them will collapse because of [Moo76, Theorem 1] (the reason might be obscure now but we have a similar proof later on for the continuous case) and the cohomology of the total complex is identified with $H_\mu^n(G,A)$. By [Moo76, Proposition 4] one can deduce that

$$H^q\left(\mathrm{U}\left(\left(G/N\right)^{p+q}, \mathrm{U}\left(G^{*+1},A\right)^N\right)^{G/N}\right) \cong \mathrm{U}\left(\left(G/N\right)^{p+1}, H_\mu^q(N,A)\right)^{G/N},$$

and the result follows. Again the computation is roughly straightforward and we will present a similar one later on. □

Remark 4.1.3. The assumption in Theorem 4.1.2 that $H_{\mu}^{q}(N,A)$ is Hausdorff stems from one of the major problems in defining cohomology theories for topological modules. If the groups one considers are restricted to some subcategory of **Top**, it may happen (especially in the case of the Hausdorff condition) that a cokernel of a continuous linear map does not exist. To see that, let $i : A \longrightarrow B$ be a monomorphism between two topological modules. The quotient, in the category of say Hausdorff spaces, is $B\big/\overline{i(A)}$. This has the correct topological structure but the wrong algebraic structure. Usually people prefer to consider the correct algebraic quotient and endow the space with the topology induced by $p : B \longrightarrow B\big/i(A)$. The problem is that this space is in general not Hausdorff[2] and so it will not be a cokernel in our category of topological groups. The assumption Moore makes reflects this problem.

The continuous case is more problematic. A result appears in the proof of [HM62, Theorem 7.1] for vector space coefficients. We recall the notion of continuously injective objects and resolution from [HM62, Section 2].

Proposition 4.1.4. *Let N be a closed normal subgroup of a locally compact Hausdorff group G, A a finite-dimensional Hausdorff real vector space with a continuous left G-action. Assume further that A has a continuously injective G-module resolution that is also continuously injective as an N-resolution. Assumer further that $H_{\mathrm{c}}^{q}(G,A)$ is a finite-dimensional Hausdorff real vector space. Then there is a convergent spectral sequence*

$$E_2^{p,q} := H_{\mathrm{c}}^{p}\left(G\big/N, H_{\mathrm{c}}^{q}(N,A)\right) \Rightarrow H_{\mathrm{c}}^{p+q}(G,A)\,. \tag{4.8}$$

Proof. We will not rewrite the proof but give a short sketch of the construction though. Consider the continuously injective G-resolution $A \to X^{\bullet}$ from the assumption. The double complex $C^{p,q} = \mathbf{Top}\left(\left(G\big/N\right)^{q+1}, \left(X^{p}\right)^{N}\right)^{G/N}$ gives as usual two spectral sequences by Corollary 1.2.9. One collapses by the fact that $\left(X^{p}\right)^{N}$ are continuously injective as $G\big/N$-modules[3], and one gets

$$H^{p+q}\left(\mathrm{tot}\left(C^{\bullet,\bullet}\right)\right) \cong H^{p+q}\left(\left(\left(X^{p}\right)^{N}\right)^{G/N}\right) \cong H_{\mathrm{c}}^{p+q}(G,A)\,. \tag{4.9}$$

Note that we did not use any of the two technical assumptions we made to derive the above isomorphisms. Both assumptions are there to refine the E_2-term

[2] for instance if $i(A)$ is dense (but not closed) in B

[3] The reason for that is the same as in the discrete case. Simply note that the N-invariance functor on continuous G-modules is right adjoint to the forgetful functor and the latter clearly preserves strong exactness in the sense of [HM62, Section 2].

of the other spectral sequence. Since by assumption $A \to X^\bullet$ is strongly injective also as an N-resolution we get that

$$H^q_{\mathrm{c}}(N, A) \cong H^q\left((X^\bullet)^N\right). \tag{4.10}$$

Since continuous linear surjections to finite-dimensional spaces are split, the assumption on $H^q_{\mathrm{c}}(N, A)$ gives that

$$H^q\left(\mathbf{Top}\left(\left(G/N\right)^{q+1}, (X^\bullet)^N\right)^{G/N}\right) \cong \mathbf{Top}\left(\left(G/N\right)^{p+1}, H^q_{\mathrm{c}}(N, A)\right)^{G/N}. \tag{4.11}$$

So the claim follows. □

Remark 4.1.5. A slightly extended result of Proposition (4.1.4) was given in [BW00, Theorem IX.4.3]. In the next section we will give an LHS result for the continuous case for arbitrary coefficient modules but only if the sequence $N \hookrightarrow G \twoheadrightarrow G/N$ is topologically trivial.

4.2 LHS results for discrete and finite quotients

In this section we will discuss the locally continuous case and also some result in the continuous case for arbitrary coefficient modules. Unfortunately the nasty behaviour of locally continuous functions does not allow for a general result for that model. We will also give a result for the locally smooth model. We recall that a Lie group will be a group object in some category of manifolds. For the next theorem to be applicable the only assumption on this category is that it has finite products and coproducts and that if Γ is a discrete space then $\Gamma \times M \cong \coprod_{\gamma\in\Gamma} M_\gamma$, where M is manifold in this category and $M_\gamma \cong M$. Our main result is the following.

Theorem 4.2.1. *Assume that N is an open normal subgroup of an arbitrary topological group G. Assume that A is an arbitrary topological G-module. Then we have the following:*

1. *There is a spectral sequence*

$$E_2^{p,q} := H^p_{\mathrm{gr}}\left(G/N, H^q_{\mathrm{c}}(N, A)\right) \Rightarrow H^{p+q}_{\mathrm{c}}(G, A). \tag{4.12}$$

2. *If* $\left|G/N\right| < \infty$*, then there is a spectral sequence*

$$E_2^{p,q} := H_{\mathrm{gr}}^p\left(G/N, H_{\mathrm{lc}}^q(N, A)\right) \Rightarrow H_{\mathrm{lc}}^{p+q}(G, A). \tag{4.13}$$

3. *Assume further that* G *is a Lie group and* A *a smooth* G*-module. If again* $\left|G/N\right| < \infty$*, then there is a spectral sequence*

$$E_2^{p,q} := H_{\mathrm{gr}}^p\left(G/N, H_{\mathrm{ls}}^q(N, A)\right) \Rightarrow H_{\mathrm{ls}}^{p+q}(G, A). \tag{4.14}$$

We will break the proof down into two technical lemmas. We will use the following three double complexes

$$A_{\mathrm{c}}^{p,q} := \mathbf{Set}\left(\left(G/N\right)^{p+1}, \mathbf{Top}\left(G^{q+1}, A\right)^N\right)^{G/N} \tag{4.15}$$

$$A_{\mathrm{lc}}^{p,q} := \mathbf{Set}\left(\left(G/N\right)^{p+1}, \mathrm{Map}_{\Delta\mathrm{c}}\left(G^{q+1}, A\right)^N\right)^{G/N} \tag{4.16}$$

$$A_{\mathrm{ls}}^{p,q} := \mathbf{Set}\left(\left(G/N\right)^{p+1}, \mathrm{Map}_{\Delta\mathrm{s}}\left(G^{q+1}, A\right)^N\right)^{G/N} \tag{4.17}$$

with differentials given by the same formulas

$$d_v^{p,q}(f)(g_0N, \ldots, g_{p+1}N)\left(g_0', \ldots g_q'\right) := \sum_{i=0}^{p+1} (-1)^i f\left(g_0N, \ldots, \widehat{g_iN}, \ldots, g_{p+1}N\right)\left(g_0', \ldots, g_q'\right) \tag{4.18}$$

and

$$d_h^{p,q}(f)(g_0N, \ldots, g_pN)\left(g_0', \ldots g_{q+1}'\right) := \sum_{i=0}^{q+1} (-1)^i f(g_0N, \ldots, g_pN)\left(g_0', \ldots, \widehat{g_i'}, \ldots, g_{p+1}'\right) \tag{4.19}$$

in cases (4.15), (4.16) and (4.17) respectively.

Lemma 4.2.2. *Under the assumptions of Theorem 4.2.1 we have that*

$$H^p\left(A_{\mathrm{c}}^{\bullet,q}\right) \cong 0, \quad p > 0, \tag{4.20}$$

$$H^p\left(A_{\mathrm{lc}}^{\bullet,q}\right) \cong 0, \quad p > 0 \tag{4.21}$$

and

$$H^p\left(A_{\mathrm{ls}}^{\bullet,q}\right) \cong 0, \quad p > 0 \tag{4.22}$$

in the respective cases.

Proof. The proof is in principle the same in all cases. Let Hom denote $\mathrm{Map}_{\Delta \mathrm{c}}$ or $\mathrm{Map}_{\Delta \mathrm{s}}$ or **Top**. In all cases we claim that if

$$f \in \mathbf{Set}\left(\left(G/N\right)^{p+1}, \mathrm{Hom}\left(G^{q+1}, A\right)^N\right)^{G/N},$$

then

$$h(f)(g_0N, \ldots, g_{p-1}N)\left(g_0', \ldots, g_q'\right) := \\ (-1)^p f\left(g_0N, \ldots, g_{p-1}N, g_0'N\right)\left(g_0', \ldots, g_q'\right) \tag{4.23}$$

defines an element in $\mathbf{Set}\left(\left(G/N\right)^p, \mathrm{Hom}\left(G^{q+1}, A\right)^N\right)^{G/N}$. Clearly the only non-trivial check is that for all elements $(g_0N, \ldots, g_{p-1}N)$ of $\left(G/N\right)^p$, the function $h(f)(g_0N, \ldots, g_{p-1}N)$ is an element of $\mathrm{Hom}\left(G^{q+1}, A\right)$.
Let us not consider the invariant subspaces for now since the arguments work also without considering the equivariance of functions. Let f be an element of $\mathbf{Set}\left(\left(G/N\right)^{p+1}, \mathrm{Hom}\left(G^{q+1}, A\right)\right)$. Denote by $\mathcal{N}_e^o\left(G^{q+1}\right)$ the set of open neighborhoods of the diagonal of G^{q+1}. In the three cases we are considering there is always a function

$$l \,:\, \left(G/N\right)^{p+1} \longrightarrow \mathcal{N}_{\mathrm{diag}}^o\left(G^{q+1}\right)$$

such that $f\left(g_0N, \ldots, g_pN\right)|_{l(g_0N, \ldots, g_pN)}$, has the "global property for functions in $\mathrm{Hom}\left(G^{q+1}, A\right)$". To explain what we mean by that if Hom denote locally continuous functions then the previous restriction of f is continuous and the equivalent for smooth. Clearly if Hom $=$ **Top** then the function l can be chosen to be constant and assign to all tuples in $\left(G/N\right)^{p+1}$ the entire space G^{q+1}. Under the respective assumptions of Theorem 4.2.1 the space

$$V := \bigcap_{c \in \left(G/N\right)^{p+1}} l(c)$$

is open. This is so in the locally continuous and smooth cases due to the finiteness conditions and in the continuous case due to the fact that all $l(c) = G^{q+1}$ for all c so $V = G^{q+1}$. Now we denote by GHom the space of functions having the global property corresponding to the local property that the functions of Hom have[4]. Clearly $f\left(g_0 N, \ldots, g_p N\right)\Big|_V$ is always an element of $\mathrm{GHom}\,(V, A)$. So we have an element $\tilde{f}$ of $\mathbf{Set}\left(\left(G/N\right)^{p+1}, \mathrm{GHom}\,(V, A)\right)$ defined by

$$\tilde{f}\left(g_0 N, \ldots, g_p N\right) := f\left(g_0 N, \ldots, g_p N\right)\Big|_V.$$

But by the definition of the coproduct we have the following isomorphism

$$\mathbf{Set}\left(\left(G/N\right)^{p+1}, \mathrm{GHom}\,(V, A)\right) \cong \mathrm{GHom}\left(\coprod_{c \in \left(G/N\right)^{p+1}} V_c, A\right)$$

where $V_c = V$ for all c. Also we have assumed that in the underlying category of spaces we are working in, coproducts of some fixed space V indexed by some small set I are isomorphic to the product $V \times I$. So

$$\mathbf{Set}\left(\left(G/N\right)^{p+1}, \mathrm{GHom}\,(V, A)\right) \cong \mathrm{GHom}\left(\left(G/N\right)^{p+1} \times V, A\right). \tag{4.24}$$

Call $p : G \longrightarrow G/N$ the natural projection and $p_0 : G^{q+1} \longrightarrow G$ the projection to the first factor. We can define a function $\tilde{\Delta} : V \longrightarrow p\left(p_0\left(V\right)\right) \times V$ by $\tilde{\Delta}\left(v_0, \ldots, v_q\right) := \left(v_0 N, v_0, \ldots, v_q\right)$. Clearly $\tilde{\Delta}$ is an element of $\mathrm{GHom}\,(V, p\left(p_0\left(V\right)\right) \times V)$. We note that clearly equation (4.24) holds if we replace $\left(G/N\right)^{p+1}$ with any discrete space, in particular with $\left(G/N\right)^p$. Denote by $\tilde{h}$ the following composition

$$\begin{aligned}
\mathbf{Set}\left(\left(G/N\right)^{q+1}, \mathrm{GHom}\,(V, A)\right) &\xrightarrow{\cong} \mathrm{GHom}\left(\left(G/N\right)^{q+1} \times V, A\right) \\
\text{(restriction of functions)} \qquad &\longrightarrow \mathrm{GHom}\left(\left(G/N\right)^{q} \times p\left(p_0\left(V\right)\right) \times V, A\right) \\
&\xrightarrow{\circ\tilde{\Delta}} \mathrm{GHom}\left(\left(G/N\right)^{q} \times V, A\right) \\
&\xrightarrow{\cong} \mathbf{Set}\left(\left(G/N\right)^{q}, \mathrm{GHom}\,(V, A)\right).
\end{aligned}$$

[4]Clearly GHom = Hom if Hom = **Top**

Clearly now

$$h\left(f\right)\left(g_0 N,\ldots,g_{p-1}N\right)\big|_V = (-1)^p\,\tilde{h}\left(\tilde{f}\right)\left(g_0 N,\ldots,g_{p-1}N\right).$$

This implies that $h\left(f\right)\left(g_0 N,\ldots,g_{p-1}N\right)$ is an element of $\operatorname{Hom}\left(G^{q+1},A\right)$.
It is a trivial check to see that $h\left(f\right)$ has the correct equivariant properties, i.e. that for all elements of $\left(G/N\right)^{p+1}$ the function $h\left(f\right)\left(c\right)$ is N-equivariant and that $h\left(f\right)$ is G/N-equivariant.
So in either case we get a well defined group homomorphism

$$\begin{aligned} h^{p,q} \ : \ \mathbf{Set}\left(\left(G/N\right)^{p+1}, \operatorname{Hom}\left(G^{q+1},A\right)^N\right)^{G/N} &\longrightarrow \\ &\mathbf{Set}\left(\left(G/N\right)^{p}, \operatorname{Hom}\left(G^{q+1},A\right)^N\right)^{G/N}. \end{aligned}$$

But a straightforward check reveals that

$$d_v^{p-1,q}\circ h^{p,q} + h^{p+1,q}\circ d_v^{p,q} = id \text{ for } p>0. \tag{4.25}$$

This shows the claim. □

The previous result dealt with the convergence of (4.12), (4.13) and (4.14). We now turn to the E_2-term.

Lemma 4.2.3. *Assume that the bundle $N\hookrightarrow G\twoheadrightarrow G/N$ is topologically trivial. Then*

$$H^q\left(A_{\mathrm{c}}^{p,\bullet}\right)\cong \mathbf{Set}\left(\left(G/N\right)^{p+1}, H_{\mathrm{c}}^q\left(N,A\right)\right)^{G/N}, \tag{4.26}$$

$$H^q\left(A_{\mathrm{lc}}^{p,\bullet}\right)\cong \mathbf{Set}\left(\left(G/N\right)^{p+1}, H_{\mathrm{lc}}^q\left(N,A\right)\right)^{G/N} \tag{4.27}$$

and

$$H^q\left(A_{\mathrm{ls}}^{p,\bullet}\right)\cong \mathbf{Set}\left(\left(G/N\right)^{p+1}, H_{\mathrm{ls}}^q\left(N,A\right)\right)^{G/N}. \tag{4.28}$$

Proof. Recall the usage of the generic symbol Hom from the Proof of Lemma 4.2.2. Note that the functor

$$\mathbf{Set}\left((G/N)^p, -\right) \,:\, \mathbf{Ab} \longrightarrow \mathbf{Ab}$$

is exact. This implies that

$$H^q\left(\mathbf{Set}\left((G/N)^p, \mathrm{Hom}\left(G^{\bullet+1}, A\right)^N\right)\right) \cong \mathbf{Set}\left((G/N)^p, H^q\left(\mathrm{Hom}\left(G^{\bullet+1}, A\right)^N\right)\right).$$

Also by Lemma 2.1.17 we have the following isomorphisms

$$\mathbf{Set}\left((G/N)^p, \mathrm{Hom}\left(G^{\bullet+1}, A\right)^N\right) \cong \mathbf{Set}\left((G/N)^{p+1}, \mathrm{Hom}\left(G^{\bullet+1}, A\right)^N\right)^{G/N}$$

as double complexes. So there is an induced isomorphism of the spectral sequences they produce. So

$$\begin{aligned} H^q & \left(\mathbf{Set}\left((G/N)^{p+1}, \mathrm{Hom}\left(G^{\bullet+1}, A\right)^N\right)^{G/N}\right) \\ & \cong H^q\left(\mathbf{Set}\left((G/N)^p, \mathrm{Hom}\left(G^{q+1}, A\right)^N\right)\right) \\ & \cong \mathbf{Set}\left((G/N)^p, H^q\left(\mathrm{Hom}\left(G^{\bullet+1}, A\right)^N\right)\right) \\ & \cong \mathbf{Set}\left((G/N)^{p+1}, H^q\left(\mathrm{Hom}\left(G^{\bullet+1}, A\right)^N\right)\right)^{G/N}. \end{aligned}$$

Now the bundle $N \hookrightarrow G \twoheadrightarrow G/N$ is topologically trivial, so by Corollary (2.1.14) and the comment under Proposition 2.2.1 we get

$$H^q_{\mathrm{c}}(N, A) \cong H^q\left(\mathbf{Top}\left(G^{\bullet+1}, A\right)^N\right) \tag{4.29}$$

$$H^q_{\mathrm{lc}}(N, A) \cong H^q\left(\mathrm{Map}_{\Delta\mathrm{c}}\left(G^{\bullet+1}, A\right)^N\right), \tag{4.30}$$

and

$$H^q_{\mathrm{ls}}(N, A) \cong H^q\left(\mathrm{Map}_{\Delta\mathrm{s}}\left(G^{\bullet+1}, A\right)^N\right). \tag{4.31}$$

And the claim is proven. □

The proof of Theorem 4.2.1 is already laid out so let us finalize the

Proof (of Theorem 4.2.1). Again we will use the generic expression Hom for either **Top** or $\mathrm{Map}_{\Delta c}$ to deal with both cases at once. We also denote by $C^{p,q}$ the complexes $A_{\mathrm{c}}^{p,q}$ or $A_{\mathrm{lc}}^{p,q}$ or $A_{\mathrm{ls}}^{p,q}$. From $C^{p,q}$ we can get due to Corollary (1.2.9) two spectral sequences

$$ {}^{I}E_2^{p,q} = H_\times^p H_*^q \left(C^{*,\times}\right) \Rightarrow H^{p+q}\left(\mathrm{tot}\left(C^{\bullet,\bullet}\right)\right), \tag{4.32} $$

$$ {}^{II}E_2^{p,q} = H_\times^p H_*^q \left(C^{\times,*}\right) \Rightarrow H^{p+q}\left(\mathrm{tot}\left(C^{\bullet,\bullet}\right)\right). \tag{4.33} $$

By Lemma 4.2.2 we have that $H_*^q\left(C^{*,p}\right)$ vanishes for $q > 0$, and obviously for $q = 0$

$$ H_*^0\left(C^{*,p}\right) \cong \mathrm{Hom}\left(G^{p+1}, A\right)^G. \tag{4.34} $$

Also since ${}^{I}E$ collapses, we compute

$$ {}^{I}E_2^{p,q} \cong {}^{I}E_\infty^{p,q} \cong H^{p+q}\left(\mathrm{tot}\left(C^{\bullet,\bullet}\right)\right) \cong H_*^{p+q}\left(G, A\right), \tag{4.35} $$

where $*$ is either c or lc or ls. Also by Lemma 4.2.3 we get that

$$ H_\bullet^q\left(C^{p,\bullet}\right) \cong \mathbf{Set}\left(\left(G/N\right)^{p+1}, H^q\left(N, A\right)\right)^{G/N}. \tag{4.36} $$

Which means that in all cases

$$ {}^{II}E_2^{p,q} \cong H_{\mathrm{gr}}^p\left(G/N, H^q\left(N, A\right)\right). \tag{4.37} $$

Finally (4.35) and (4.37) proves Theorem 4.2.1. □

Let us give a slightly more general result of Theorem 4.2.1 for the continuous model.

Proposition 4.2.4. *Assume that N is a closed normal subgroup of a locally compact Hausdorff topological group G, such that topologically $G \cong N \times G/N$. Let A be an arbitrary topological G-module. Assume that*

$$ Z^q\left(\mathbf{Top}\left(G^{\bullet+1}, A\right)^N\right) \to H^q\left(\mathbf{Top}\left(G^{\bullet+1}, A\right)^N\right) $$

has a global topological section for all q. Then there exists a spectral sequence

$$ E_2^{p,q} := H_{\mathrm{c}}^p\left(G/N, H_{\mathrm{c}}^q\left(N, A\right)\right) \Rightarrow H_{\mathrm{c}}^{p+q}\left(G, A\right). \tag{4.38} $$

Proof. The proof is the same as in the previous theorem with a couple of small tweaks. We start with a double complex similar to $A_{\mathrm{c}}^{p,q}$,

$$C^{p,q} := \mathbf{Top}\left(\left(G/N\right)^{q+1}, \mathbf{Top}\left(G^{p+1}, A\right)^{N}\right)^{G/N} \tag{4.39}$$

with differentials as before. The proof of Lemma 4.2.2 works in a similar way. To see that note that the contracting homotopy h we constructed there, in the current case is the composition of the "exponential law" isomorphisms i.e.

$$\mathbf{Top}\left(X, \mathbf{Top}\left(Y, Z\right)\right) \cong \mathbf{Top}\left(X \times Y, Z\right)$$

and the continuous function

$$\left(g_0 N, \ldots, g_{q-1}N, g_0', \ldots g_p'\right) \mapsto \left(g_0 N, \ldots, g_{q-1}N, g_0' N, g_0', \ldots, g_p\right).$$

So one of the two spectral sequences associated to (4.39) collapses and so the cohomology of the total complex is identified with that of $H_{\mathrm{c}}^{p}\left(G, A\right)$. Clearly $\mathbf{Top}\left(\left(G/N\right)^{p}, -\right)$ is exact for topologically trivial bundles, and so by our assumption of the existence of a global section for $H^{q}\left(\mathbf{Top}\left(G^{\bullet+1}, A\right)^{N}\right)$, we get that

$$H^{q}\left(\mathbf{Top}\left((G/N)^{p+1}, \mathbf{Top}\left(G^{\bullet+1}, A\right)^{N}\right)^{G/N}\right)$$
$$\cong \mathbf{Top}\left((G/N)^{p+1}, H^{q}\left(\mathbf{Top}(G^{\bullet+1}, A)^{N}\right)\right)^{G/N}.$$

By Proposition (2.2.1) $H^{q}\left(\mathbf{Top}\left(G^{\bullet+1}, A\right)^{N}\right) \cong H_{\mathrm{c}}^{q}\left(N, A\right)$.
The result then follows exactly as in the Proof of Theorem 4.2.1. □

We close by mentioning some easy examples of Theorem 4.2.1. Note that the main examples are in the case of $N = G_0$, the connected identity component of G.

Example 4.2.5. Let S be a surface. We denote by $\mathrm{Homeo}^{+}\left(S, \partial S\right)$ the group of orientation preserving homeomorphisms which restrict to the identity on the boundary. Also, let $\mathrm{Homeo}_0\left(S, \partial S\right)$ denote the connected component of the identity of $\mathrm{Homeo}^{+}\left(S, \partial S\right)$. The mapping class group is defined as (e.g. in [FM12])

$$\mathrm{Mod}\left(S\right) := \mathrm{Homeo}^{+}\left(S, \partial S\right) / \mathrm{Homeo}_0\left(S, \partial S\right) \tag{4.40}$$

i.e. homotopy classes of orientation preserving homeomorphisms. Clearly theorem 4.2.1 is applicable and we get

$$E_2^{p,q} = H^p_{\mathrm{gr}}\left(\mathrm{Mod}\left(S\right), H^q_{\mathrm{c}}\left(\mathrm{Homeo}_0\left(S, \partial S\right), \mathbb{Z}\right)\right) \Rightarrow H^{p+q}_{\mathrm{c}}\left(\mathrm{Homeo}^+\left(S, \partial S\right), \mathbb{Z}\right)$$

For example when S is the annulus it is known that $\mathrm{Mod}\left(S\right) \cong \mathbb{Z}$. The cohomology groups of $\mathbb{Z}$ is computed in [Wei94] Example 6.14. So we get short exact sequences[5]

$$\begin{aligned} 0 \to H^p_{\mathrm{c}}\left(\mathrm{Homeo}_0\left(S, \partial S\right), \mathbb{Z}\right)_{\mathrm{Mod}(S)} \to H^p_{\mathrm{c}}\left(\mathrm{Homeo}^+\left(S, \partial S\right), \mathbb{Z}\right) \to \\ \to H^p_{\mathrm{c}}\left(\mathrm{Homeo}_0\left(S, \partial S\right), \mathbb{Z}\right)^{\mathrm{Mod}(S)} \to 0. \end{aligned}$$

In [FM12] it is shown that if S is either the twice or thrice punctured sphere $\mathrm{Mod}\left(S, \partial S\right)$ is finite (Proposition 2.3), so we get also spectral sequences for the locally continuous model by Theorem 4.2.1.

Remark 4.2.6. An LHS look-alike result for topological group cohomology can be derived for the simplicial group cohomology model we discussed in Subsection 2.2 Part 4. Note that $\mathbf{Ab}\left(BG_\bullet\right)$ is abelian and the section functor $\Gamma : \mathbf{Ab}\left(BG_\bullet\right) \longrightarrow \mathbf{Ab}$ factors as

$$\mathbf{Ab}\left(BG_\bullet\right) \xrightarrow{p_*} \mathbf{Ab}\left(B\left(G/N\right)_\bullet\right) \xrightarrow{\Gamma} \mathbf{Ab}\,.$$

So for any G, any normal closed subgroup N and any continuous G-module A, we get by (1.2.12) a spectral sequence

$$E_2^{p,q} := H^p_{\mathrm{Sh}}\left(B\left(G/N\right)_\bullet, R^q p_*\left(A^\bullet_{\mathrm{glob,c}}\right)\right) \Rightarrow H^{p+q}_{\mathrm{simp,c}}\left(G, A\right). \tag{4.41}$$

The identification of the stalks of $R^q p_*\left(A^\bullet_{\mathrm{glob,c}}\right)$ is extremely difficult. An attempt was made in [Jos02]. In [Jos02, Section 5] an LHS result for algebraic groups is presented, but as in part 2 of Theorem 4.2.1, a restriction has to be made to $\left|G/N\right| < \infty$.

Remark 4.2.7. As a closing remark to the chapter (and to some extent to the thesis), we will discuss a bit the reasons for the restriction to finite quotients. We saw hints of this reason throughout, but to try to make it a bit concrete, it relates to the fact that there is not a very good notion of what $H^n_{\mathrm{lc}}\left(G, A\right)$ is as a topological space. Actually something somehow more general happens. If we pick

[5] if A is G-module we denote, as is common, by A_G the quotient group of A by the submodule generated by elements of the form $g.a - a$.

a category $\mathcal{C}$, and we consider group objects over it, it is not straightforward how to define a cohomology theory of them, and even if we find a way to generalize the classical notions, usually the cohomology groups will have a not very easily accessible structure as objects in $\mathcal{C}$. In our discussion of locally continuous group cohomology for example we found a way to define a group cohomology theory for group objects in an appropriate category of spaces, but, as was outlined in the subsection on locally continuous functions, the sets $\mathrm{Map}_{\mathrm{lc}\bullet}(G^{n+1}, A)$ do not have a topological structure which is easy and useful to work with. This problems appear also in other setups. For example in the previous Remark 4.2.6, we mentioned that in the case of algebraic groups an LHS result is derived only in case G/N is finite. This allows the author there to identify the stalks of $R^q p_* \left(A^{\bullet}_{glob,c}\right)$ with the cohomology of N.
Going back to topological groups, there is another model of topological group cohomology, introduced in [Fla08], defined via sheaf cohomology. What he does is to view topological groups by the Yoneda embedding as sheaves and then consider the cohomology of the abelian ones. By the Comparison Theorem this is again isomorphic with the Segal-Mitchison cohomology [WW15, Remark 4.12]. In [Fla08, Corollary 6] he gives an LHS spectral sequence result for arbitrary G/N but he fails to identify the coefficient modules of the E_2-term with the cohomology of N. In applications he does so by assuming that the quotient is finite. But an "honest" LHS result should have the following properties, the E_2-term should involve in a clear way the cohomology of G/N and the cohomology of N, and the spectral sequence should converge to the cohomology of G. The problem is that usually by making the convergence part correct we create difficult to work with E_2-terms (as happened in [Jos02] or [Fla08]), or we could get easily accessible E_2-terms but then we do not know where they converge.
Note that we could change our exposition to fall in the first case as well. One can show that there is actually a spectral sequence, for all closed normal subgroups of G, which converges to the locally continuous group cohomology of G, and the E_2-term involves information about G/N and N but not in a straightforward way. The way is very similar to what we did, but now starting from the double complex $\mathrm{Map}_{\mathrm{lc}\bullet}\left(\left(G/N\right)^{p+1} \times G^{q+1}, A\right)^G$. Following similar steps as above we would obtain a spectral sequence

$$E_2^{p,q} := H^p_\times H^q_* \left(\mathrm{Map}_{\mathrm{lc}\bullet}\left(\left(G/N\right)^{\times+1} \times G^{*+1}, A\right)^G\right) \Rightarrow H^{p+q}_{\mathrm{lc}}(G, A).$$

We see that the E_2-term is definitely not easy to work with. The reason that the E_2-term has this unpleasant form relates to the fact that $\mathrm{Map}_{\mathrm{lc}}(X, -)$ fails to be right adjoint to the product in any reasonable category of topological spaces.

A setup, where the above problems are solved, is described in [Ram08], where he discusses bornological spaces. In his thesis he actually makes very explicit that the LHS result he gets is closely related to the fact that there is a nice bornology on the set of bounded functions which makes this bornological space in the usual sense adjoint to taking products.
We said in the introduction that a topological group cohomology should have classes which correctly incorporate topological and algebraic information about the group G. Having now such theories, we know further that they contain also interesting algebraic information about how their classes are related (being abelian groups). It seems that it will be of interesting further research if one could find a way to make the cohomology groups contain information about how their classes are related topologically.

Appendix A

Appendix

This appendix is a collection of various technical facts we used in various parts of the thesis. No authenticity is claimed, everything in this appendix has definitely appeared in one place or another in the past. We include it here so we can make the exposition a bit self-contained.
In the following $\mathcal{C}$ will be a category and c an object of it. We give the following definitions.

Definition A.1.8. 1. $\mathrm{Sub}\,(c)$ is the category with objects monomorphisms $i_s : s \longrightarrow c$, in $\mathcal{C}$, and morphisms the obvious commutative diagrams.

2. Assume that s and s' are subobjects of c. If their product exists in $\mathrm{Sub}\,(c)$, then it will be denoted as $s \cap s'$, and called their intersection. If their coproduct exist in $\mathrm{Sub}\,(c)$ it will be denoted as $s \cup s'$, and called their union.

Remark A.1.9. Usually we will be sloppy and identify the monomorphism with its source object, which will imply that we view some canonical monomorphism attach to it. An important thing to observe is that this notion is not refined enough. To give an example if $\mathcal{C}$ is topological spaces, then this definition would give as "subspaces" all continuous injections, while only embeddings actually make sense as "subspaces" in the classical sense. So people sometimes refine this notion to regular monomorphisms[1], or strong monomorphisms e.t.c. We will only use the language for abelian categories so there such confusion does not occur because monomorphisms are normal and all those notions coincide.

Intersections and unions do not always exists. In fact unions might not exist even if the category admits arbitrary limits. Let us see reasonable cases in which they do.

[1] equalizers of a pair of arrows

Proposition A.1.10. *Assume that $\mathcal{C}$ admits pullbacks, then the intersection of two subobjects s and s', always exists and $s \cap s' \cong s \times_c s'$. Assume that $\mathcal{C}$ is an abelian category, then their union exists and $s \cup s' \cong s \coprod_{s \times_c s'} s'$.*

Proof. The case of intersections is trivial. We will prove the second part. Let s and s' be subobjects of c' and c' a subobject of c. We need to show that the unique morphism $s \coprod_{s \times_c s'} s' \to c'$, is mono. Since such a morphism will factorize $m : s \coprod_{s \times_c s'} s' \longrightarrow c$, it is enough to show that m is mono. Consider a morphism $L \xrightarrow{l} s \coprod_{s \times_c s'} s'$ such that $ml = 0$. We remember that in abelian categories the pullback of $B \xrightarrow{g} C \xleftarrow{f} A$ exists and is $\ker(fpr_a - gpr_B)$, while the pushout of $B \xleftarrow{g} C \xrightarrow{f} A$ is $\operatorname{coker}(i_a f - i_B g)$. By [KS06, Lemma 8.3.2] we can complete the following diagram

$$\begin{tikzcd}
L'' \arrow[rr, two heads, "p''"] \arrow[dr, "\eta"] \arrow[ddr, "\eta"'] & & L' \arrow[r, two heads, "p'"] \arrow[d, "l'"] & L \arrow[d, "l"] \\
& s \times_c s' \arrow[r, "(i_s pr_s - i_{s'} pr_{s'})k"] \arrow[d, "\omega = id"] & s \oplus s' \arrow[r, two heads, "p"] \arrow[d, "\begin{pmatrix} id_s & 0 \\ 0 & -id_{s'} \end{pmatrix}"] & s \coprod_{s \times_c s'} s' \arrow[d, "m"] \\
& s \times_c s' \arrow[r, hook, "k"] & s \oplus s' \arrow[r, two heads, "fpr_s - gpr_{s'}"] & c
\end{tikzcd} \tag{A.1}$$

ω and m exists since

$$(fpr_s - gpr_{s'}) \begin{pmatrix} id_s & 0 \\ 0 & -id_{s'} \end{pmatrix} (i_s pr_s k - i_s pr_{s'} k) = fpr_s k - pr_{s'} k = 0.$$

By assumption $ml = 0$. $L' = L \times_{s \coprod_{s \times_c s'} s'} (s \bigoplus s')$. While L'' exists by [KS06, Lemma 8.3.2] , since $k = \ker(fpr_s - gpr_{s'})$, and $(fpr_s - gpr_{s'}) \begin{pmatrix} id_s & 0 \\ 0 & -id_{s'} \end{pmatrix} l' = mpl' = mlp' = 0$. And $l'p'' = (i_s pr_s k - i_{s'} pr_{s'} k)\, \eta$, but then

$$lp'p'' = pl'p'' = p\,(i_s pr_s k - i_{s'} pr_{s'} k)\, \eta = 0.$$

But $p'p''$ is an epimorphism so $l = 0$ and m is mono. □

Let $s \xhookrightarrow{i_s} c$ be a subobject of c, $s' \xhookrightarrow{i_s} c'$ be a subobject of c' and consider f in $\mathcal{C}(c, c')$. We will denote by $f(s) := \operatorname{im}(f i_s)$ if the latter one exists. While the "preimage" of a subobject is defined as $f^{-1}(s') := c \times_{c'} s'$ supposing the latter one exists. In abelian categories one can show that the usual identities hold, i.e.

$$\begin{aligned} f^{-1}(s \cap s') &\cong f^{-1}(s) \cap f^{-1}(s') \\ f^{-1}(s \cup s') &\cong f^{-1}(s) \cup f^{-1}(s') \\ f(s'' \cup s''') &\cong f(s'') \cup f(s''') . \end{aligned}$$

Now assume that $S \overset{i_S}{\hookrightarrow} M$ is a subobject in an abelian category $\mathcal{A}$. We will denote by $M/S := \operatorname{coker}(i_S)$. Usual results that concern subobjects and quotients that hold in some category of modules also hold in an arbitrary abelian category, using the above definitions of them. For example the proof of existence of spectral sequences in Weibel can be carried out step by step in arbitrary abelian categories, since he is only making use of the above definitions and results as well as the Noether isomorphism theorems. The latter ones also hold in arbitrary abelian categories. The result appears in many places (e.g. in [BP09]) but for completeness we repeat it here.

Proposition A.1.11 (Noether's Isomorphism Theorems). *1) Assume f is a morphism between A and A' then by definition $\ker(f) \to A$ and $\operatorname{im}(f) \to A'$ are monomorphisms and $A/\ker(f) \cong \operatorname{im}(f)$.*

2) Let S, T in $\operatorname{Sub}(M)$. Then $S \cap T$ and $S \cup T$ are in $\operatorname{Sub}(M)$ and also

$$(S \cup T)/S \cong T/(S \cap T) .$$

3) Assume that $T \hookrightarrow S \hookrightarrow M$. Then S/T in $\operatorname{Sub}(M/T)$. Moreover every object of $\operatorname{Sub}(M/T)$ is isomorphic to S'/T with $T \hookrightarrow S' \hookrightarrow M$. Also

$$(M/T)/(S/T) \cong M/S.$$

Proof. 1) Follows from definitions and basic properties of abelian categories.

2) By the definition of $S \cup T$, the diagram

$$\begin{tikzcd} S \times_M T \arrow[r, hook] \arrow[d, hook] & T \arrow[d, hook] \\ S \arrow[r, hook] & S \coprod_{S \times_M T} T \end{tikzcd} \tag{A.2}$$

Is pushout. So opposite arrows have isomorphic cokernels.

3) Let L' be a subobject of M/T. We have the following diagram

$$\begin{tikzcd}
 & T \arrow[d, hook] \\
L' \arrow[r, "l'"] \arrow[d, two heads, "p'"] & M \arrow[d, two heads, "p"] \\
L \arrow[r, hook, "l"] & M/T
\end{tikzcd} \tag{A.3}$$

Since the square is pullback, L' is a subobject of M. Also p and p' have isomorphic kernels, and $L' \cong L/\ker(p') \cong L/T$.
For the other part we have the following diagram

$$\begin{tikzcd}
T \arrow[r, equal] \arrow[d, hook, "i_T^S"] & T \arrow[d, hook, "i_T"] & \\
S \arrow[r, hook, "i_S"] \arrow[d, two heads, "p_T^S"] & M \arrow[r, two heads, "p_S"] \arrow[d, two heads, "p_T"] & M/S \arrow[d, shift right, "(p_T)_*"'] \\
S/T \arrow[r, hook] & M/T \arrow[r, two heads] \arrow[ur, "f"] & (M/T)\big/(S/T) . \arrow[u, shift right, "f_*"']
\end{tikzcd} \tag{A.4}$$

f exists since $p_S i_T = p_S i_s i_T^S = 0$. But then

$$f\,(i_S)_*\, p_T^S = f p_T i_S = p_S i_S = 0,$$

and p_T^s is an epimorphism so we get f_*. Trivially it is inverse to $(p_T)_*$. □

To write the isomorphism between Čech cohomology and locally continuous group cohomology we used the so-called staircase argument. Even though it is quite well-known we thought to present here a formal proof of it for the sake of completeness. One can carry out all the following arguments in any abelian category in an element free way using matrix notation for morphisms between direct products. But to save some space and make the proofs more readable we will only present it for some category of modules over a ring R. So the setup is the following, let C be a first quadrant double complex in the category $R-\mathbf{Mod}$. We use the convention that the differentials anticommute. We assume that there is a given homotopy for the vertical differential, i.e. we are given a collection of morphisms $h^{p,q} : C^{p,q} \longrightarrow C^{p,q-1}$ such that

$$d_v^{p,q-1} \circ h^{p,q} + h^{p,q+1} \circ d_v^{p,q} = id_{C^{p,q}},$$

for all p in $\mathbb{N}$ and $q > 0$. Before we present the Lemma, let us fix some notation. We denote by $M^n : \operatorname{tot}(C)^n \longrightarrow \operatorname{tot}(C)^{n-1}$ the unique morphism between products such that

$$pr_{p,n-1-p} \circ M^n \circ i_{p,n-p} = h^{p,n-p}$$

for all p less or equal than $n-1$. We also define the "total horizontal differential" $D_h^n : \operatorname{tot}(C)^n \longrightarrow \operatorname{tot}(C)^{n+1}$ to be the unique map between products such that

$$pr_{p+1,n-p} \circ D_h^n \circ i_{p,n-p} = d_h^{p,n-p}$$

for p greater than 0 and $pr_{0,n+1} \circ D_h^n = 0$.

We recall the following Lemma.

Lemma A.1.12 (Staircase argument). *Under the above assumptions the inclusion* $\ker d_v^{n,0} \hookrightarrow C^{n,0}$, *induces an isomorphism in cohomology*

$$H^n(\operatorname{tot} C) \cong H^n\left(\ker d_v^{\bullet,0}\right). \tag{A.5}$$

The inverse of the inclusion can be described on the cocycle level by sending a cocycle x of $\operatorname{tot} C$ *to the element*

$$i_{n,0}^{-1}\left(\sum_{i=0}^{n}\left(-D_h^{n-1}M^n\right)^{n-i}\left(i_{i,n-i}\left(x^{i,n-i}\right)\right)\right). \tag{A.6}$$

Remark A.1.13. We will make a couple of remarks before we give a proof. First of all Lemma A.1.12 is making an implicit claim, namely that if x is a cocycle of the total complex then formula (A.6) actually defines an element of $\ker d_v^{n,0}$. This might not be obvious at this stage[2], it is though a straightforward result included in the proof. Actually there is a much stronger statement, which usually goes by the name acyclic assembly lemma. Under the assumptions of the previous Lemma the inclusion of $\ker(d_v^{*,0})$ in the total complex is a homotopy equivalence. But it is not true that the formula we gave provides the homotopy inverse. One of the reasons is that if x is only a cochain of the total complex the formula will not give something in the required kernel. One needs to include an extra term (one

[2]in fact it is not obvious to the author without the proof

similar to the second sum in equation [BT82, Proposition 9.5]) to the formula A.6 to make it a homotopy inverse to the inclusion, but this term will always vanish on cocycles of the total complex so we can describe the isomorphism in cohomology without this term.

Proof. We give an outline of one of the possible tedious proofs. We define the following complexes for all natural numbers p

$$F^p\left(\operatorname{tot} C\right)^n = \begin{cases} \bigoplus_{i=p}^{n} C^{i,n-i} & \text{if } 0 \leq p \leq n \\ 0 & \text{otherwise} \end{cases} \tag{A.7}$$

and each has differential the restriction of the differential on the total complex. For $p \leq n$ we define morphisms

$$I_p^n \;:\; F^p\left(\operatorname{tot} C\right)^n \longrightarrow F^{p+1}\left(\operatorname{tot} C\right)^n \tag{A.8}$$

by

$$I_p^n\left(x^{p,n-p},\ldots,x^{n,0}\right) := \left(x^{p+1,n-p-1} - d_h^{p,n-p-1}h^{p,n-p}\left(x^{p,n-p}\right), x^{p+2,n-p-2},\ldots,x^{n,0}\right)$$

They commute with the differential for $n \geq p+2$, i.e.

$$I_p^{n+1}D^n = D^n I_p^n \tag{A.9}$$

for $n \geq p+1$, so they induce morphisms in cohomology for $n \geq p+2$. It is trivial that I_p^n is left inverse to the inclusion of $F^{p+1}\left(\operatorname{tot} C\right)^n$ in $F^p\left(\operatorname{tot} C\right)^n$. We define now morphisms

$$s_p^n \;:\; F^p\left(\operatorname{tot} C\right)^n \longrightarrow F^p\left(\operatorname{tot} C\right)^{n-1} \tag{A.10}$$

for $n \geq p+2$ by

$$s_p^n\left(x^{p,n-p},\ldots,x^{n,0}\right) := \left(h^{p,n-p}x^{p,n-p},0,\ldots,0\right) \tag{A.11}$$

and straightforward calculations show that for $n \geq p+2$ we get that

$$id_{F^p(\operatorname{tot} C)^n} - i \circ I_p^n = D^{n-1} \circ s_p^n + s_p^{n+1}D^n \tag{A.12}$$

and so the induced morphism in cohomology of I_p^n is right inverse as well to the inclusion and so they are both isomorphisms for $n \geq p+2$. This shows that we have a chain of isomorphisms

$$\tilde{I}^n := \left(I_{n-2}^n\right)_* \circ \left(I_{n-1}^n\right)_* \circ \cdots \circ \left(I_0^n\right)_* \;:\; H^n\left(\operatorname{tot} C\right) \longrightarrow H^n\left(F^{n-2}\left(\operatorname{tot} C\right)\right) \tag{A.13}$$

for all $n \geq 2$. Finally we argue that the morphism

$$l \,:\, F^{n-2}\left(\operatorname{tot} C\right) \longrightarrow C^{n,0} \tag{A.14}$$

given by

$$l\left(x\right) := x^{n,0} - d_h h\left(x^{n-1,1}\right) + d_h h d_h h\left(x^{n-2,2}\right)^{3} \tag{A.15}$$

induces an isomorphism in cohomology

$$l_* \,:\, H^n\left(F^{n-2}\left(\operatorname{tot} C\right)\right) \longrightarrow H^n\left(\ker d_v^{\bullet,0}\right). \tag{A.16}$$

The first part of this last claim is that l induces a morphism in cohomology. This follows from straightforward calculations since if x is a cocycle of $F^{n-2}\left(\operatorname{tot} C\right)^n$, then

$$d_h l\left(x\right) = d_h x^{n,0} \tag{A.17}$$

since d_h is a differential, and this last term vanishes because x is a cocycle. Also after some calculations, using the fact that d_v and d_h anticommute and the fact that h is a homotopy for d_v, one can show that $d_v l\left(x\right)$ also vanishes for x a cocycle. Also l sends images of D^{n-1} to images of $d_h^{n-1,0}|_{\ker d_v^{n-1,0}}$. To see that note that

$$lD\left(x\right) = d_h\left(x^{n-1,0} - hd_h x^{n-2,1} - hx_v x^{n-1,0} + hd_h hd_v x^{n-2,1} - d_h hx^{n-2,1}\right) \tag{A.18}$$

where the last term does not follow exactly from the computation of $lD(x)$ but it can be added since it is trivially 0, and the reason we added it is because the rest of the terms inside the parenthesis do not vanish under d_v, so they would not belong to $\ker d_v$. But a simple computation shows that d_v applied on them gives $d_v d_h hx^{n-2,1}$, which is then counterbalanced by the addition of the extra 0 term above. So finally, l does gives a morphism in cohomology as asserted. Furthermore, clearly

$$l \circ i\left(k\right) = k \tag{A.19}$$

and if x is a cocycle of $F^{n-2}\left(\operatorname{tot} C\right)^n$

$$\begin{aligned} il(x) &= \left(0, 0, x^{n,0} - d_h hx^{n-1,1} + d_h hd_h hx^{n-2,2}\right) \\ &= \left(x^{n-2,2}, x^{n-1,1}, x^{n,0}\right) \\ &\quad - \left(x^{n-2,2}, x^{n-1,1}, d_h hx^{n-1,1} - d_h hd_h hx^{n-2,2}\right). \end{aligned}$$

[3]from now we start dropping some tedious bookkeeping indices

But the last line is equal to

$$D\left(hx^{n-2,2}, hx^{n-1,1} - hd_h hx^{n-2,2}\right),$$

i.e. $il(x)$ is cohomologous to x for all cocycles of $F^{n-2}(\operatorname{tot} C)^n$. This finally shows that l_* is inverse to the inclusion $H^n(\ker d_v^{\bullet,0}) \to H^n(F^{n-2}(\operatorname{tot} C))$.
Finally, putting everything together we get an isomorphism

$$l_* \circ \tilde{I}^n \,:\, H^n(\operatorname{tot} C) \longrightarrow H^n\left(\ker d_v^{\bullet,0}\right). \tag{A.20}$$

By the construction above it is exactly described on the cocycle level by formula (A.6) □

Remark A.1.14. Note that the above construction is using one of the filtrations to obtain the usual spectral sequences from the double complex C. Of course the above isomorphism follows directly by a spectral sequence argument as we did in the proof of Theorem 4.2.1. It is obvious that the above isomorphism and the one coming from the spectral sequence argument are the same.

We used the notion of "generalized δ-functors" in a couple of places. Note that in Definition 1.1.8 the abelian group structure of the source category was only very "lightly" used. Let us recall a more general definition (e.g. [WW15, Definition VI.1]).

Definition A.1.15.

1. A category with short exact sequences, is a category $\mathcal{C}$ together with a collection of composable morphisms $A \to B \to C$ in $\mathcal{C}$ called short exact sequences. A morphism between short exact sequences $A \to B \to C$ and $A' \to B' \to C'$ are morphisms $A \to A'$, $B \to B'$ and $C \to C'$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[d] \\ A' \arrow[r] & B' \arrow[r] & C' \end{tikzcd} \tag{A.21}$$

commutes.

2. Let $\mathcal{C}$ be a category with short exact sequences and $\mathcal{A}$ an abelian category. A δ-functor from $\mathcal{C}$ to $\mathcal{A}$ is a collection of functors

$$\{T^n \,:\, \mathcal{C} \longrightarrow \mathcal{A}\}_{n\in\mathbb{Z}} \tag{A.22}$$

such that for each short exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of $\mathcal{C}$ there is a long exact sequence in $\mathcal{A}$

$$\cdots \longrightarrow T^n(A) \xrightarrow{T^n(f)} T^n(B) \xrightarrow{T^n(g)} T^n(C) \xrightarrow{\delta^n_{A,B,C}} T^{n+1}(A) \longrightarrow \cdots \quad \text{(A.23)}$$

and the morphisms $\delta^n_{A,B,C}$ are natural in the sense that for each morphism of short exact sequences as in (A.21), the diagram

$$\begin{tikzcd} T^n(C) \arrow[r, "\delta^n_{A,B,C}"] \arrow[d] & T^{n+1}(A) \arrow[d] \\ T^n(C') \arrow[r, "\delta^n_{A',B',C'}"] & T^{n+1}(A') \end{tikzcd}$$

commutes.

Remark A.1.16. Morphisms between δ-functors are defined similarly as in Definition 1.1.8.

Example A.1.17. The functors $H^\bullet_*(G,-)$ for $*$ in $\{\mathrm{gr}, \mathrm{lc}, \mathrm{c}, \mathrm{SM}\}$ are δ-functors. Note that the notion of delta functor depends on the choice of short exact sequences in the source category. For the group cohomology case of course the choice is short exact sequences of G-modules. For the case of continuous group cohomology the choice is short exact sequences of topological G-modules for which the underlying bundle is topologically trivial. For the other models the choice is short exact sequence of topological G-modules for which the underlying bundle has a local section.

Example A.1.18. A cohomological functor $H : \mathcal{D} \longrightarrow \mathcal{A}$ between a triangulated category $\mathcal{D}$ and an abelian category $\mathcal{A}$ defines a delta functor $H \circ T^n$, where T is the translation automorphism of $\mathcal{D}$. Of course as short exact sequences in $\mathcal{D}$ we consider $A \to B \to C$, such that there exists a distinguished triangle $A \to B \to C \to TA$.

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