Tree-decompositions in finite and infinite graphs

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To Sarah

Contents

Introduction

In Chapters 1 and 2, we build tree-decompositions that display the global structure of infinite and finite graphs. These tree-decompositions of infinite graphs are an important tool to study infinite graphic matroids, which are the topic of Chapter 3.

Chapter 4 is independent of the others and contains results on harmonic functions on infinite graphs.

0.1 End-preserving spanning trees

In 1931, Freudenthal introduced a notion of *ends* for second countable Hausdorff spaces [63], and in particular for locally finite graphs[1] [64]. These ends are intended as 'points at infinity' that compactify the graph when it is locally finite (ie, locally compact). The compacification is similar to the familiar 1-point compactification of locally compact Hausdorff spaces but finer: the two-way infinite ladder, for example, has two such points at infinity, one at either 'end', see Figure 1.

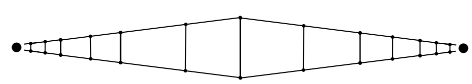

Figure 1: The two-way infinite ladder has two ends indicated at as the two thick points on the very left and the very right side.

Independently, in 1964, Halin [70] introduced a notion of *ends* for graphs, taking his cue directly from Carathéodory's *Primenden* of simply connected regions of the complex plane [33]. For locally finite graphs these two notions of ends agree.

For graphs that are not locally finite, Freudenthal's topological definition still makes sense, and gave rise to the notion of *topological ends* of arbitrary graphs [54]. In general, this no longer agrees with Halin's notion of ends, although it does for trees.

Halin [70] conjectured that the end structure of every connected graph can be displayed by the ends of a suitable spanning tree of that graph. He proved

[1]A *locally finite* graph is one in which all vertices have finite degree

this for countable graphs. Halin's conjecture was finally disproved in the 1990s by Seymour and Thomas [95], and independently by Thomassen [102].

In Chapter 1, we shall prove Halin's conjecture in amended form, based on the topological notion of ends rather than Halin's own graph-theoretical notion. We shall obtain it as a corollary of the following theorem, which proves a conjecture of Diestel [49] of 1992 (again, in amended form):

Theorem 1. *Every graph has a tree-decomposition* $(T, \mathcal{V})$ *of finite adhesion such that the ends of* T *define precisely the topological ends of* G.
See Section 3.3 for definitions.

We use Theorem 1 as a tool to show that the topological cycles of any graph together with its topological ends induce a matroid, see Section 0.3 below. The tree-decompositions constructed for the proof of Theorem 1 are based on earlier versions for finite graphs, which are a central technique in the following section.

0.2 Canonical tree-decompositions

One approach for understanding the global structure of mathematical objects such as graphs or groups is to decompose them into parts which cannot be further decomposed, and to analyse how those parts are arranged to make up the whole. Here we shall decompose a k-connected graph into the '$(k+1)$-connected pieces'; and the global structure will be tree-like. The idea is modelled on the well-known block-cutvertex tree, which for $k = 1$ displays the global structure of a connected graph 'up to 2-connectedness'. Extending this to $k = 2$, Tutte proved that every finite connected graph G has a tree-decomposition of adhesion 2 into '3-connected minors' [105]. Chapter 2 is about extending this result to higher connectivities.

One way to define k-indecomposable objects is the following: a $(k + 1)$-*block* in a graph is a maximal set of at least $k + 1$ vertices, no two of which can be separated in the ambient graph by removing at most k vertices. We prove that every finite graph has a (canonical) tree-decomposition of adhesion at most k such that any two different $(k + 1)$-blocks are contained in different parts of the decomposition [42]. Under weak but necessary conditions, these tree-decompositions can be combined into a single tree-decomposition that distinguishes all the $(k+1)$-blocks for all k simultaneously. We call $(k+1)$-blocks satisfying this necessary condition robust, see Section 2.1 for details.

Another notion of highly connected pieces in a graph is that of *tangles*. These were introduced by Robertson and Seymour in [94] and are a central notion in their theory of graph minors. With the same proof as that of the aforementioned theorem, one can construct a tree-decompositions that does not only distinguish all the (robust) blocks but also all the tangles. This implies and strengthens an important result of the Graph Minors Project of Robertson and Seymour [94]. An important feature of our tree-decompositions is that they are invariant under the group of automorphisms of the graph, whereas theirs is not. Our techniques also allow us to give another simpler proof of the original result of Robertson and Seymour, see Section 2.4.

Hundertmark [78] introduced *k-profiles*, which are a common generalisation of k-blocks and tangles of order k. Together with Lemanczyk [79], he used the proof of the decomposition theorem of [42] in order to construct a tree-decomposition that distinguishes all (robust) profiles.

We can further improve the above tree-decompositions so that they display all k-blocks that could possibly be isolated at all in a tree-decomposition, canonical or not. More precisely, we call a k-block *separable* if it appears as a part in some tree-decomposition of adhesion less than k of G. The results culminate in our proof of the following theorem, which was conjectured by Diestel [48] (see also [40]).

Theorem 2 (Carmesin, Gollin)**.** *For any fixed k, every finite graph G has a canonical tree-decomposition $\mathcal{T}$ of adhesion less than k that distinguishes efficiently every two distinct k-profiles, and which has the further property that every separable k-block is equal to the unique part of $\mathcal{T}$ in which it is contained.*

We can also extend the aforementioned theorem of Hundertmark and Lemanczyk in a similar way, see Section 2.6 for details.

The largest k for which G contains k-blocks is a graph invariant, called the *block number*. In Section 2.5, we investigate this further and relate it to other graph invariants such as the average degree.

0.3 Infinite matroids of graphs

In 2013, Bruhn, Diestel, Kriesell, Pendavingh and Wollan gave axiomatic foundations for infinite matroids with duality in terms of independent sets, bases, circuits, closure and rank [30]. This breakthrough opened the way to building a theory of infinite matroids, see for example [1, 2, 3, 4, 17, 18, 23, 28, 29, 32, 45].

A fundamental result of finite matroid theory is Whitney's theorem that any finite 3-connected graph G can be reconstructed just from the information of which edge sets form cycles [108]. The set of edge sets of G forming cycles is the set of circuits of a matroid, called the *cycle matroid* of G. Matroid duality extends planar duality of finite graphs in the sense that finite dual planar graphs have dual cycle matroids.

There are two natural cycle matroids associated with an infinite but locally finite graph G: the first is obtained as the limit of the cycle matroids of its finite subgraphs. The second is obtained as the limit of the cycle matroids of its finite contraction minors. Whilst the first limit can be understood as a direct limit, a limit matroid of the second type is represented by topological space which is the inverse limit of the corresponding contraction graphs. If G and G^* are locally finite dual planar graphs, then the subgraph limit of G is the dual of contraction limit of G^* [29]. Thus here matroid duality extends the planar duality of the underlying graphs by the duality between these two limits.

This also means that, unlike for finite graphs, there are non-isomorphic cycle matroids associated to the same infinite graph. This rises the question what an infinite graphic matroid is. In this section we offer two independent approaches towards a notion of 'infinite graphic matroids'.

0.3.1 Approach 1: Topological cycle matroids

The subgraph and contraction limit constructions give a matroid for any graph. However, unlike the subgraph limit construction, the contraction limit construction of graphs is limited in the sense that every connected component of such a limit matroid is countable (after deleting parallel edges), see Section 3.2 for details.

Nevertheless, there is yet another construction which for locally finite graphs agrees with the contraction limit construction: We consider the topological space consisting of the graph and its topological ends as in Section 0.1, and define *topological circles* to be homeomorphic images of the unit cycle. We prove the following:

Theorem 3. *The topological circles in any graph together with its topological ends form the circuits of a matroid.*

This topological construction gives genuinely new matroids for graphs of arbitrarily high cardinality. In turn we already need the full power of Halin's conjecture mentioned above to prove that these objects have a base.

0.3.2 Approach 2: Matroids with all finite minors graphic

A central result in finite matroid theory is Tutte's characterisation of the class of finite matroids which arise as cycle matroids of graphs by a finite list `Forb` of forbidden minors [104]. In this section we extend this characterisation to infinite matroids.

Graph-like spaces were introduced by Thomassen and Vela [103]. These are topological spaces whose topological circles very often form the set of circuits of a matroid, see Section 3.4 for examples. These matroids are graphic in the sense that all their finite minors are cycle matroids of graphs, that is, none of these minors is in `Forb`. Moreover, all these matroids have to be *tame*, see Section 3.4 for a definition and an explanation of why this is a property graphic matroids really should have. Tutte's characterisation extends as follows to infinite matroids:

Theorem 4 (Bowler, Carmesin, Christian). *A 3-connected matroid can be represented by a graph-like space if and only if it is tame and it has no finite minor in* `Forb`.

A remarkable consequence of this theorem is that every circuit in a 3-connected tame matroid with no finite minor in `Forb` is countable. To show this, we first introduce *pseudo-circles*, a more general notion of topological circles in graph-like spaces, which are allowed to be uncountable. Then we construct for each matroid in our class a graph-like space representing this matroid in the weak sense that its circuits are given by the pseudo-circles of the graph-like space. Working in this representation, we show that all the pseudo-circles are countable. Hence they are actual topological circles and the graph-like space represents the matroid in the strong sense of Theorem 4.

Since graphs together with the topological ends are examples of graph-like spaces, the second approach deals with a larger class of matroids than the first. Other examples captured by the second approach are 'Psi-matroids'. These are generic enough to provide lots of counterexamples [18]. In Section 3.2, we extend Theorem 3 to Psi-matroids by basically using the same proof. Having said this, it remains an open problem whether these two approaches lead to the same class of infinite matroids:

Open Question 0.3.1. *Is there a graph-like space inducing a 3-connected matroid which is not a minor of a Psi-matroid?*

Bowler showed that any such graph-like space cannot be compact [16].

0.4 Harmonic functions on infinite graphs

Harmonic functions on infinite graphs are discrete analogues of harmonic functions on Riemannian Manifolds. Many theorems in this area are about the relation between the discrete and the continuous setting. The discrete analogue of Brownian motions are random walks; and an infinite graph is *transient* if a random walk has a positive probability to escape to infinity.

Benjamini and Schramm [10] proved that every transient planar graph with bounded vertex degrees admits non-constant harmonic functions with finite Dirichlet energy; we will call such a function a *Dirichlet harmonic function* from now on. Combining this with results of He and Schramm [74] yields that the one-ended bounded degree planar graphs admitting a bounded harmonic function are precisely those that admit an accumulation-free circle packing in the unit disc; whilst the others have an accumulation-free circle packing in the complex plane. This nicely corresponds to the continuous setting: the unit disc admits non-constant bounded harmonic functions, whilst the complex plane does not.

We extend the Benjamini-Schramm-result to unbounded degree graphs by replacing the transience condition with a stronger one, which we call *roundabout-transience*.

Theorem 5 (Carmesin, Georgakopoulos)**.** *Every locally finite roundabout-transient plane graph admits a Dirichlet harmonic function.*
See Chapter 4 for definitions.

In Chapter 4, we shall explain a sense in which this theorem is best-possible. Furthermore Theorem 5 can be further applied to prove a conjecture of Georgakopoulos about Dirichlet harmonic function on non-amenable planar locally finite graphs.

0.5 Acknowledgements and basis of this thesis

This thesis is based on the ten papers [35, 42, 39, 40, 41, 44, 19, 37, 22, 43], some of which are joint work; see Appendix A for details. Additionally, it contains Section 2.4, which is just published here.

I am grateful to Nathan Bowler and to my supervisor Reinhard Diestel. I enjoy working with Nathan and the way he thinks about problems complements mine very well. I thank Reinhard for his very clear and extremely helpful advice. I am grateful that he took special care of the financial support for me and my family. Thirdly, I benefited from Reinhard's foresightedness; in particular for pushing the Infinite Matroids Project in Hamburg from its very beginning.

Chapter 1

All graphs have tree-decompositions displaying their topological ends

1.0.1 Introduction

In 1931, Freudenthal introduced a notion of *ends* for second countable Hausdorff spaces [63], and in particular for locally finite graphs [64]. Independently, in 1964, Halin [70] introduced a notion of *ends* for graphs, taking his cue directly from Carathéodory's *Primenden* of simply connected regions of the complex plane [33]. For locally finite graphs these two notions of ends agree.

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Halin [70] conjectured that the end structure of every connected graph can be displayed by the ends of a suitable spanning tree of that graph. He proved this for countable graphs. Halin's conjecture was finally disproved in the 1990s by Seymour and Thomas [95], and independently by Thomassen [102].

In this paper we shall prove Halin's conjecture in amended form, based on the topological notion of ends rather than Halin's own graph-theoretical notion. We shall obtain it as a corollary of the following theorem, which proves a conjecture of Diestel [49] of 1992 (again, in amended form):

Theorem 1.0.1. *Every graph has a tree-decomposition $(T, \mathcal{V})$ of finite adhesion such that the ends of T define precisely the topological ends of G.*
See Subsection 1.0.2 for definitions.

The tree-decompositions constructed for the proof of Theorem 1.0.1 have several further applications. In [36] we use them to answer the question to what extent the ends of a graph - now in Halin's sense - have a tree-like structure at all. In [37], we apply Theorem 1.0.1 to show that the topological cycles of any graph together with its topological ends induce a matroid.

This paper is organised as follows. In Subsection 1.0.2 we explain the problems of Diestel and Halin in detail, after having given some basic definitions. In Subsection 1.0.3 we continue with examples related to these problems. Subsection 1.0.4 only contains material that is relevant for Subsection 1.0.5 in which we prove that every graph has a nested set of separations distinguishing the vertex ends efficiently. In Subsection 1.0.6, we use this theorem to prove Theorem 1.0.1. Then we deduce Halin's amended conjecture.

1.0.2 Definitions

Throughout, notation and terminology for graphs are that of [52] unless defined differently. And G always denotes a graph.

A *vertex end* in a graph G is an equivalence class of rays (one-way infinite paths), where two rays are equivalent if they cannot be separated in G by removing finitely many vertices. Put another way, this equivalence relation is the transitive closure of the relation relating two rays if they intersect infinitely often.

Let X be a locally connected Hausdorff space. Given a subset $Y \subseteq X$, we write $\overline{Y}$ for the closure of Y, and $F(Y) := \overline{Y} \cap \overline{X \setminus Y}$ for its frontier. In order to define the topological ends of X, we consider infinite sequences $U_1 \supseteq U_2 \supseteq ...$ of non-empty connected open subsets of X such that each $F(U_i)$ is compact and $\bigcap_{i\geq 1} \overline{U}_i = \emptyset$. We say that two such sequences $U_1 \supseteq U_2 \supseteq ...$ and $U'_1 \supseteq U'_2 \supseteq ...$ are *equivalent* if for every i there is some j with $U_i \supseteq U'_j$. This relation is transitive and symmetric [63, Satz 2]. The equivalence classes of those sequences are the *topological ends* of X [54, 63, 77].

For the simplical complex of a graph G, Diestel and Kühn described the topological ends combinatorically: a vertex *dominates* a vertex end ω if for some (equivalently: every) ray R belonging to ω there is an infinite fan of v-R-paths that are vertex-disjoint except at v. In [54], they proved that the topological ends are given by the undominated vertex ends. Hence in this paper, we take this as our definition of *topological end of* G.

We denote the complement of a set X by $X^{\complement}$. For an edge set X, we denote by $V(X)$, the set of vertices incident with edges from X. For a vertex set W, we denote by s_W, the set of those edges with at least one endvertex in W.

For us, a *separation* is just an edge set. A *vertex-separation* in a graph G is an ordered pair (A, B) of vertex sets such that there is no edge of G with one endvertex in $A \setminus B$ and the other in $B \setminus A$. A separation X *induces* the vertex-separation $(V(X), V(X^{\complement}))$. Thus in general there may be several separations inducing the same vertex-separation. The *boundary* $\partial(X)$ of a separation X is the set of those vertices adjacent with an edge from X and one from $X^{\complement}$. The *order* of X is the size of $\partial(X)$. A separation X is *componental* if there is a component C of $G - \partial(X)$ such that $s_C = X$. Two separations X and Y are *nested* if one of the following 4 inclusions is true: $X \subseteq Y$, $X^{\complement} \subseteq Y$, $Y \subseteq X$ or $Y \subseteq X^{\complement}$. If there is a vertex in $\partial(Y) \setminus V(X)$, then it is incident with an edge from $Y \setminus X$ and an edge from $Y^{\complement} \setminus X$. Thus if additionally, X and Y are nested, then either $X^{\complement} \subseteq Y$ or $Y \subseteq X^{\complement}$. We shall refer to the four sets $\partial(Y) \setminus V(X)$, $\partial(Y) \setminus V(X^{\complement})$, $\partial(X) \setminus V(Y)$ or $\partial(X) \setminus V(Y^{\complement})$ as the *links of X and Y*.

A vertex end ω *lives* in a separation X of finite order if $V(X)$ contains one (equivalently: every) ray belonging to ω. Similarly, we define when a vertex end lives in a component. A separation X of finite order *distinguishes* two vertex ends ω and μ if one of them lives in X and the other in $X^{\complement}$. It distinguishes them *efficiently* if X has minimal order amongst all separations distinguishing ω and μ.

A *tree-decomposition* of G consists of a tree T together with a family of subgraphs $(P_t | t \in V(T))$ of G such that every vertex and edge of G is in at least one of these subgraphs, and such that if v is a vertex of both P_t and P_w, then it is a vertex of each P_u, where u lies on the v-w-path in T. Moreover, each edge of G is contained in precisely one P_t. We call the subgraphs P_t, the *parts* of the tree-decomposition. Sometimes, the "Moreover"-part is not part of the definition of tree-decomposition. However, both these two definitions give the same concept of tree-decomposition since any tree-decomposition without this additionally property can easily be changed to one with this property by deleting edges from the parts appropriately. The *adhesion* of a tree-decomposition is finite if adjacent parts intersect only finitely. Given a directed edge tu of T, the *separation corresponding to tu* consists of those edges contained in parts P_w, where w is in the component of $T - t$ containing u.

In [18, 73, 100], tree-decompositions of finite adhesion are used to study the structure of infinite graphs. In [49, Problem 4.3], Diestel wanted to know whether every graph G has a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion that somehow encodes the structure of the graph with its ends.

Let us be more precise: Given a vertex end ω, we take $O(\omega)$ to consist of those oriented edges tu of T such that ω lives in its corresponding separation. Note that $O(\omega)$ contains precisely one of tu and ut. Furthermore this orientation $O(\omega)$ of T points towards a node of T or to an end of T. We say that ω *lives* in the part for that node or that end, respectively.

A vertex end ω is *thin* if every set of vertex-disjoint rays belonging to ω is finite; otherwise ω is *thick*. Diestel asked whether every graph has a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion such that different thick vertex ends live in different parts and such that the ends of T *define precisely* the thin vertex ends. Here the ends of T *define precisely* a set W of vertex ends of G if in every end of T there lives a unique vertex end and it is in W and conversely every vertex end in W lives in some end of T.

Unfortunately, that is not true: In Example 1.0.3, we construct a graph such that each of its tree-decompositions of finite adhesion has a part in which two (thick) vertex ends live. Moreover, in Example 1.0.6, we construct a graph that does not have a tree-decomposition of finite adhesion such that the ends of its decomposition tree define precisely the thin vertex ends.

Hence the remaining open question is whether there is a natural subclass of the vertex ends (similar to the class of thin vertex ends) such that every graph has a tree-decomposition of finite adhesion such that the ends of its decomposition tree define precisely the vertex ends in that subclass. Theorem 1.0.1 above answers this question affirmatively.

It is impossible to construct a tree-decomposition as in Theorem 1.0.1 with the additional property that for any two topological ends ω and μ, there is a separation corresponding to an edge of the tree that separates ω and μ efficiently, see Example 1.0.7.

A recent development in the theory of infinite graphs seeks to extend theorems about finite graphs and their cycles to infinite graphs and the topological circles formed with their ends, see for example [15, 27, 55, 56, 67, 99], and [47] for a survey. We expect that Theorem 1.0.1 has further applications in this direction aside from the one mentioned in the Introduction.

A rooted spanning tree T of a graph G is *end-faithful* for a set Ψ of vertex ends if each vertex end $\omega \in \Psi$ is uniquely represented by T in the sense that T contains a unique ray belonging to ω and starting at the root. For example, every normal spanning tree is end-faithful for all vertex ends. Halin conjectured that every connected graph has an end-faithful tree for all vertex ends. At the end of Subsection 1.0.6, we show that Theorem 1.0.1 implies the following nontrivial weakening of this disproved conjecture:

Corollary 1.0.2. *Every connected graph has an end-faithful spanning tree for the topological ends.*

One might ask whether it is possible to construct an end-faithful spanning tree for the topological ends with the additional property that it does not include any ray to any other vertex end. However, this is not possible in general. Indeed, Seymour and Thomas constructed a graph G with no topological end that does not have a rayless spanning tree [95].

1.0.3 Example section

Throughout this section, we denote by T_2 the infinite rooted binary tree, whose nodes are the finite 0-1-sequences and whose ends are the infinite ones. In particular, its root is denoted by the empty sequence ϕ.

Example 1.0.3. In this example, we construct a graph G such that all its tree-decompositions of finite adhesion have a part in which two vertex ends live. We obtain G from T_2 by adding a single vertex v_ω for each of the continuum many ends ω of T_2, which we join completely to the unique ray belonging to ω starting at the root. Note that the vertex ends of G are the ends of T_2. For a finite path P of T_2 starting at ϕ, we denote by $A(P)$, the set of those vertex ends of G whose corresponding 0-1-sequence begins with the finite 0-1-sequence which is the last vertex of P.

Suppose for a contradiction that there is a tree-decomposition $(T, P_t | t \in V(T))$ of G of finite adhesion such that in each of its parts lives at most one vertex end.

Lemma 1.0.4. *For each $k \in \mathbb{N}$, there is a separation X_k corresponding to a directed edge $t_k u_k$ of T together with a finite path P_k of T of length k starting at ϕ satisfying the following.*

1. *uncountably many vertex ends of $A(P_k)$ live in X_k;*
2. *$X_{k+1} \subseteq X_k$;*
3. *$P_k \subseteq P_{k+1}$;*
4. *If $v_\omega \in \partial(X_k)$, then ω does not live in X_{k+1}.*

Proof. We start the construction with picking $P_0 = \phi$ and X_0 such that uncountably many vertex ends live in it. Assume that we already constructed for all $i \leq k$ separations X_i and P_i satisfying the above. Let Q_k and R_k be the two paths of T_2 starting at ϕ of length $k+1$ extending P_k. Then $A(P_k)$ is a disjoint union of $A(Q_k)$ and $A(R_k)$. For P_{k+1} we pick one of these two paths of length $k+1$ such that uncountably many vertex ends of $A(P_{k+1})$ live in X_k;

Let S_k be the component of $T - t_k$ containing u_k. Let F_k be the set of those directed edges of S_k directed away from u_k. Note that if some separation X corresponds to some $ab \in F_k$, then $X \subseteq X_k$. Actually, we will find $t_{k+1}u_{k+1}$ in F_k. We colour an edge of F_k *red* if uncountably many vertex ends of $A(P_{k+1})$ live in the separation corresponding to that edge.

Suppose for a contradiction that there is a constant c such that for each r, there are at most c red edges of F_k with distance r from $t_k u_k$ in T. Let W be the subforest of T consisting of the red edges. Note that W is a tree with at most c vertex ends. By construction, only countably many vertex ends of $A(P_{k+1})$ living in X_k can live in parts of nodes not belonging to W or ends not belonging to W. As W itself has only countably many nodes and ends, uncountably many vertex ends of $A(P_{k+1})$ have to live in the same part or some end τ.

The second is not possible since then uncountably of the v_ω would be eventually contained in the finite separators whose corresponding edges converge towards τ. Thus we get a contradiction to the assumption that no two vertex ends live in the same part P_t.

Hence there is some distance r such that there are at least $|\partial(X_k)| + 1$ red edges of F_k with distance r from $t_k u_k$ in T. Each vertex end ω with $v_\omega \in \partial(X_k)$ can live in at most one separation corresponding to one of these edges. Hence amongst these red edges we can pick $t_{k+1}u_{k+1}$ such that no such ω lives in its corresponding separation X_{k+1}. Clearly, X_{k+1} and P_{k+1} have the desired properties, completing the construction. □

Lemma 1.0.5. *Let X_k and P_k be as in Lemma 1.0.4. Then $P_k \subseteq V(X_k)$.*

Proof. By 1, uncountably many vertex ends of $A(P_k)$ live in X_k. Thus infinitely many of their corresponding vertices v_ω are in $V(X_k)$. Since only finitely many of these vertices can be in $\partial(X_k)$, one of these vertices has all its incident edges in X_k. Since P_k is in its neighbourhood, it must be that $P_k \subseteq V(X_k)$. $\square$

Having proved Lemma 1.0.4 and Lemma 1.0.5, it remains to derive a contradiction from the existence of the X_k and P_k. By construction $R = \bigcup_{k\in\mathbb{N}} P_k$ is ray. Let μ be its vertex end. By Lemma 1.0.5, $R \subseteq V(X_k)$ so that μ lives in each X_k. Hence $v_\mu \in V(X_k)$ for all k. Let e be any edge of G incident with v_μ. As each edge of G is in precisely one part P_t, the edge e is eventually not in X_k. Hence v_μ is eventually in $\partial(X_k)$, contradicting 4 of Lemma 1.0.4. Hence there is no tree-decomposition $(T, P_t|t \in V(T))$ of G of finite adhesion such that in each of its parts lives at most one vertex end.

Example 1.0.6. In this example, we construct a graph G that does not have a tree-decomposition $(T, P_t|t \in V(T))$ of finite adhesion such that the thin vertex ends of G define precisely the ends of T. Let Γ be the set of those ends of T_2 whose 0-1-sequences are eventually constant and let $\omega_1, \omega_2, \ldots$ be an enumeration of Γ. We represent each end ω of T_2 by the unique ray $R(\omega)$ starting at the root and belonging to ω.

For $n \in \mathbb{N}^*$, let H_n be the graph obtained by T_2 by deleting each ray $R(\omega_i)$ for each $i \leq n$. We obtain G from T_2 by adding for each natural number n the graph H_n where we join each vertex of T_2 with each of its clones in the graphs H_n. Note that a vertex in $R(\omega_n)$ has at most n clones.

It is clear from this construction that T_2 is a subtree of G whose ends are those of G. For every vertex end ω not in Γ, there are infinitely many vertex-disjoint rays in G belonging to ω, one in each H_n. For $\omega_n \in \Gamma$ and $v \in R(\omega_n)$, let $S_n(v)$ be the set of v and its clones. Each ray in G belonging to ω intersects the separators $S_n(v)$ eventually. Thus as $|S_n(v)| \leq n$, there are at most n vertex-disjoint rays belonging to ω_n. Hence the thin vertex ends of G are precisely those in Γ.

Suppose G has a tree-decomposition $(T, P_t|t \in V(T))$ of finite adhesion such that the thin vertex ends live in different ends of T. It remains to show that there is a vertex end of T in which no vertex end of Γ lives. For that, we shall recursively construct a sequence of separations $(A_n|n \in \mathbb{N}^*)$ that correspond to edges of T satisfying the following.

1. A_{n+1} is a proper subset of A_n;

2. infinitely many vertex ends of Γ live in A_n but none of $\{\omega_1, \ldots, \omega_n\}$.

We start the construction by picking an edge of T arbitrarily; one of the two separations corresponding to that edge satisfies 2 and we pick such a separation for A_1. Now assume that we already constructed $A_1, \ldots, A_n$ satisfying 1 and 2. By assumption, there are two distinct vertex ends α and β in Γ that live in A_n. If possible, we pick $\beta = \omega_{n+1}$. Since α and β live in different ends of T, there must be some separation A_{n+1} corresponding to an edge of T such that α lives in A_{n+1} but β does not.

We claim that A_{n+1} is a proper subset of A_n. Indeed, A_{n+1} and A_n are nested and as α lives in both of them, either $A_n \subseteq A_{n+1}$ or $A_{n+1} \subseteq A_n$. Since β witnesses that the first cannot happen, it must be that A_{n+1} is a proper subset of A_n.

Having seen that A_{n+1} satisfies 1, note that it also satisfies 2 since by construction one vertex end of Γ lives in A_{n+1}, which entails that infinitely many vertex ends of Γ live in A_{n+1} because for each finite separator S of G, each infinite component of $G - S$ contains infinitely many vertex ends from Γ.

Having constructed the sequence of separations $(A_n | n \in \mathbb{N}^*)$ as above, let e_n be the edge of T to which A_n corresponds. The set of the edges e_n lies on a ray of T but no vertex end in Γ lives in the end of that ray by 2, completing this example.

Example 1.0.7. In this example, we construct a graph G such that for any tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion that distinguishes the topological ends, there are two topological ends such that no separation corresponding to an edge of T distinguishes them efficiently.

Given two graphs G and H, by $G \times H$, we denote the graph with vertex set $V(G) \times V(H)$ where we join two vertices (g,h) and (g',h') by an edge if both $g = g'$ and $hh' \in E(G)$ or both $h = h'$ and $gg' \in E(G)$. Given a set of natural numbers X, by $\overline{X}$ we denote the graph with vertex set X where two vertices are adjacent if they have distance 1.

We start the construction with the graph $\overline{\mathbb{N}^*} \times \overline{\{1,2,3,4,5\}}$. Then for each $k \in \mathbb{N}^*$, we glue on the vertex set $R_k = \{1,...,k\} \times \{4,5\}$ the graph $H_k = \overline{\mathbb{N}^*} \times (\overline{\{1,...,k\}} \times \overline{\{4,5\}})$ by identifying $(l,i) \in R_k$ with $(1,l,i)$. Let ω_k be the vertex end whose subrays are eventually in H_k. Note that ω_k is undominated.

Similarly, we glue the graphs $H'_k = \overline{\mathbb{N}^*} \times (\overline{\{1,...,k\}} \times \overline{\{1,2\}})$ on the vertex sets $R'_k = \{1,...,k\} \times \{1,2\}$. By μ_k we denote the vertex end whose subrays are eventually in H'_k.

For $k < m$, the separator $S_k = (\{1,...,k\} \times \{4\}) + (k,5)$ separates ω_k from μ_m and every other separator separating ω_k from ω_m has strictly larger order. Note that $G - S_k$ has precisely two components, one containing $(1,1)$ and the other containing $(1,5)$. Thus every separation X with $\partial(X) = S_k$ has the property that precisely one of $(1,1)$ and $(1,5)$ is in $V(X)$.

Now let $(T, P_t | t \in V(T))$ be a tree-decomposition of finite adhesion that distinguishes the set of topological ends. Let P_t be a part containing $(1,1)$ and P_u be a part containing $(1,5)$. If X corresponds to an edge e of T and precisely one of $(1,1)$ and $(1,5)$ is in $V(X)$, then e lies on the finite t-u-path in T. Thus there are only finitely many such X so that there is some $k \in \mathbb{N}^*$ such that S_k is not the separator of any X corresponding to an edge of T. Thus there are two topological ends that are not distinguished efficiently by $(T, P_t | t \in V(T))$.

1.0.4 Separations and profiles

In this section, we define profiles and prove some intermediate lemmas that we will apply in Subsection 1.0.5.

Profiles

Profiles [39] are slightly more general objects than tangles which are a central concept in Graph Minor Theory. Readers familiar with tangles will not miss a lot if they just think of tangles instead of profiles. In fact, they can even skip the definition of robustness of a profile below as tangles are always robust.

For two separations X and Y, we denote by $L(X,Y)$ the intersection of $V(X) \cap V(Y)$ and $V(X^{\complement}) \cup V(Y^{\complement})$. Note that $\partial(X \cap Y) \subseteq L(X,Y)$ and there may be vertices in $L(X,Y)$ that only have neighbours in $X \setminus Y$ and $Y \setminus X$ so that they are not in $\partial(X \cap Y)$.

Remark 1.0.8. $|L(X,Y)| + |L(X^{\complement}, Y^{\complement})| = |\partial(X)| + |\partial(Y)|$. $\square$

Definition 1.0.9. A profile[1] P of order $k+1$ is a set of separations of order at most k that does not contain any singletons and that satisfies the following.

(P0) for each X with $\partial(X) \leq k$, either $X \in P$ or $X^{\complement} \in P$;

(P1) no two $X, Y \in P$ are disjoint;

(P2) if $X, Y \in P$ and $|L(X,Y)| \leq k$, then $X \cap Y \in P$;

(P3) if $X \in P$, then there is a componental separation $Y \subseteq X$ with $Y \in P$.

Note that (P1) implies that $\emptyset \notin P$. Under the presence of (P0) the axiom (P1) is equivalent to the following: if $X \in P$ and $X \subseteq Y$ with $\partial(Y) \leq k$, then $Y \in P$. So far profiles have only been defined for finite graph [39], and for them the definition given here is equivalent to one in [39]. Indeed, for finite graphs, there is an easy induction argument which proves (P3) from the other axioms. In infinite graphs, we get a different notion of profile if we do not require (P3) - for example if we leave out (P3), there is a profiles of order 3 on the infinite star.

If we replace '$L(X,Y)$' by '$\partial(X,Y)$', then this will define *tangles*; indeed, under the presence of (P1) it can be shown that the modified (P2) is equivalent to the axiom that no three small sides cover G. Thus every tangle of order $k+1$ induces a profile of order $k+1$, where a separation X of order at most k is in the induced profile if and only if the tangle says that it is the big side (formally, this means that X is not in the tangle). However, there are profiles of order $k+1$ that do not come from tangles, see [41, Section 6].

A separation X *distinguishes* two profiles P and Q if $X \in P$ and $X^{\complement} \in Q$ or vice versa: $X \in Q$ and $X^{\complement} \in P$. It distinguishes them *efficiently* if X has minimal order amongst all separations distinguishing P and Q. Given $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$, a profile P of order $k+1$ is *r-robust* if there does not exist a separation X of order at most r together with a separation Y of order $\ell \leq k$ such that $L(X,Y) < \ell$ and $L(X^{\complement}, Y) < \ell$ and $Y \in P$ but both $Y \setminus X$ and $Y \setminus X^{\complement}$ are not in P. Note that every profile of order $k+1$ is r-robust for every $r \leq k$.

[1] In [39], profiles were introduced using vertex-separations. However, it is straightforward to check that the definition given here gives the same concept of profiles.

The notion of a profile is closely related to the well-known notion of a haven, defined next. Two subgraph of an ambient graph *touch* if they share a vertex or there is an edge of the ambient graph connecting a vertex from the first subgraph with a vertex from the second one. A vertex *touches* if the subgraph just consisting of that vertex touches. A *haven* of order $k+1$ consists of a choice of a component of $G-S$ for each separator S of size at most k such that any two of these chosen components touch. Note that if a component C is a component of both $G-S$ and $G-T$ for separators of order at most k, then it is in the haven for S if and only if it is in haven for T. Hence we can just say that a component is in a haven without specifying a particular separator.

Given a profile P of order $k+1$, for each separator S of order at most k, there is a unique component C of $G-S$ such that $s_C \in P$ by (P1) and (P3). By (P1), the collection of these components is a haven of order $k+1$. We say that this *haven* is induced by P. A haven of order $k+1$ is *good* if for any two separators S and T of size at most k and the components C and D of $G-S$ and $G-T$ that are in the haven, the set $C \cap D$ is also in the haven as soon as there are at most k vertices in $S \cup T$ that touch both C and D.

Remark 1.0.10. *A haven is good if and only if it is induced by a profile.* □

In [36], we further explain the connections between vertex ends, havens and profiles.

Torsos

An $\mathcal{N}$*-block* is a maximal set of vertices no two of which are separated by a separation in $\mathcal{N}$. A separation $X \in \mathcal{N}$ *distinguishes* two $\mathcal{N}$-blocks B and D if there are vertices in $B \setminus \partial(X)$ and $D \setminus \partial(X)$. Note that if B and D are different $\mathcal{N}$-blocks, then there is some $X \in \mathcal{N}$ distinguishing them.

Until the end of this subsection, let us fix a nested set $\mathcal{N}$ of separations and an $\mathcal{N}$-block B. We obtain the *torso* $G_T[B]$ of B from $G[B]$ by adding those edges xy such that there is some $X \in \mathcal{N}$ with $x, y \in \partial(X)$. This definition is compatible with the usual definition of torso [52] in the context of tree-decompositions: if $\mathcal{N}$ is the set of separations corresponding to the edges of a tree-decomposition, then the vertex set of every maximal part is an $\mathcal{N}$-block and its torso is just the torso of that part.

Lemma 1.0.11. *Let C be a component of $G-B$ whose neighbourhood $N(C)$ is finite. Then there is some $X \in \mathcal{N}$ such that $N(C) \subseteq \partial(X)$.*

In particular, $N(C)$ is complete in $G_T[B]$.

Proof. Let $U \subseteq N(C)$ be maximal such that there is some $X \in \mathcal{N}$ separating a vertex of C from B with $U \subseteq \partial(X)$. Suppose for a contradiction there is some $y \in N(C) \setminus U$. Pick $X \in \mathcal{N}$ with $U \subseteq \partial(X)$. Then $\partial(X)$ contains a vertex of C. Pick such an X such that the distance from y to $\partial(X) \cap C$ is minimal. Let $z \in \partial(X) \cap C$ with minimal distance to y and let $Z \in \mathcal{N}$ be a separation separating z from B. Without loss of generality we may assume that $B \subseteq V(X)$ and $B \subseteq V(Z)$. Since z is in the link $\partial(X) \setminus V(Z)$ and X and Z are nested, the link $\partial(X) \setminus V(Z^{\complement})$ is empty. Thus $U \subseteq \partial(Z)$. By the minimality of the distance, it cannot be that $X^{\complement} \subseteq Z^{\complement}$. So $X \subseteq Z^{\complement}$ as this is the only left possibility for X and Z to be nested. Hence $B \subseteq \partial(Z) \cap \partial(X)$. Hence $y \in U$, which is the desired contradiction. Thus $U = N(C)$. □

Given a separation Y of G that is nested with $\mathcal{N}$, the separation Y_B *induced by* Y in the torso $G_T[B]$ is obtained from $Y \cap E(G[B])$ by adding those edges $xy \in E(G_T[B])$ such that there is some $X \in \mathcal{N}$ with $x, y \in \partial(X)$ and $V(X) \subseteq V(Y)$ or $V(X^{\complement}) \subseteq V(Y)$.

Remark 1.0.12. $\partial(Y_B) \subseteq \partial(Y)$. □

The vertex-separation (C, D) of G induced by Y *induces* the separation $(C \cap B, D \cap B)$ of $G_T[B]$. In general $(C \cap B, D \cap B)$ differs from the vertex-separation induced by Y_B.

Remark 1.0.13. *Let H be a haven of order $k+1$. Assume that for every vertex set $S \subseteq B$ of at most k vertices the unique component C_S of $G - S$ in H intersects B. Let H_B be the haven induced by H: for each $S \subseteq B$ of at most k vertices, H_B picks the unique component C^S of $G_T[B] - S$ that includes $C_S \cap B$. Then H_B is a haven of order $k+1$. Moreover, if H is good, then so is H_B.*

Proof. If C_S and D_S touch, then so do C^S and D^S by Lemma 1.0.11. Thus H_B is a haven of order $k+1$. The 'Moreover'-part is clear. □

Let P be a profile of order $k+1$ and H be its induced good haven, then under the circumstances of Remark 1.0.13 we define the profile P_B *induced* by P on $G_T[B]$ to be the profile induced by H_B. Note that P_B has order $k+1$.

Remark 1.0.14. *If P is r-robust, then so is P_B.* □

Lemma 1.0.15. *Let $r \in \mathbb{N} \cup \{\infty\}$, and $k \leq r$ be finite. Let $\mathcal{N}$ be a nested set of separations of order at most k. Let P and Q be two r-robust profiles distinguished efficiently by a separation Y of order $l \geq k+1$ that is nested with $\mathcal{N}$. Then there is a unique $\mathcal{N}$-block B containing $\partial(Y)$.*

Moreover, P_B and Q_B are well-defined and r-robust profiles of order at least $l+1$, which are distinguished efficiently by Y_B.

Proof. Since Y is nested with any $Z \in N$, no Z can separate two vertices in $\partial(Y)$ because then both links $\partial(Y) \setminus V(Z)$ and $\partial(Y) \setminus V(Z^{\complement})$ would be nonempty. Let B be the set of those vertices that are not separated by any $Z \in \mathcal{N}$ from $\partial(Y)$. Clearly, B is the unique $\mathcal{N}$-block containing $\partial(Y)$.

Let H be the haven induced by P. Let $S \subseteq B$ be so that there is a component C of $G - S$ that is in H. Suppose for a contradiction that C does not intersect B. Then by Lemma 1.0.11, the neighbourhood $N(C)$ of C is complete in $G_T[B]$ and $|N(C)| \leq k$.

Since $(V(Y) \cap B, V(Y^{\complement}) \cap B)$ is a vertex-separation of $G_T[B]$ either $N(C) \subseteq V(Y) \cap B$ or $N(C) \subseteq V(Y^{\complement}) \cap B$. By symmetry, we may assume that $Y \in P$. Then the second cannot happen since the component of $G - \partial(Y)$ that is in

H touches C. Hence s_C distinguishes P and Q, contradicting the efficiency of Y. Thus H_B is well-defined and a good haven of order $l+1$ by Remark 1.0.13. Thus P_B is an r-robust profile of order at least $l+1$. The same is true for Q_B whose corresponding havens we denote by J and J_B.

If P_B and Q_B are distinguished by a separation X of order less than l, then H_B and J_B will pick different components of $G_T[B] - \partial(X)$. Then in turn H and J will pick different components of $G - \partial(X)$, which is impossible by the efficiency of Y. Thus by Remark 1.0.12 it remains to show that Y_B distinguishes P_B and Q_B.

Let U and W be the components of $G_T[B] - \partial(Y)$ picked by H_B and J_B, respectively. Since $s_U \subseteq Y_B$ and $s_W \subseteq Y_B^{\complement}$, the separation Y_B distinguishes P_B and Q_B by (P1). $\square$

Given a set $\mathcal{P}$ of r-robust profiles of order at least $l+1$, in the circumstances of Lemma 1.0.15, we let $\mathcal{P}_B$ be the set of those $P \in \mathcal{P}$ distinguished efficiently from some other $Q \in \mathcal{P}$ by a separation Y nested with $\mathcal{N}$ with $|\partial(Y)| \geq k+1$ and $\partial(Y) \subseteq B$. By $\mathcal{P}(B)$ we denote the set of *induced profiles* P_B for $P \in \mathcal{P}_B$.

Extending separations of the torsos

We define an operation $Y \mapsto \hat{Y}$ that extends each separation Y of the torso $G_T[B]$ to a separation $\hat{Y}$ of G in such a way that $\hat{Y}$ is nested with every separation of $\mathcal{N}$.

For each $X \in \mathcal{N}$ at least one of $V(X)$ and $V(X^{\complement})$ includes B. We pick $X[B] \in \{X, X^{\complement}\}$ such that $B \subseteq V(X[B])$. Let $\mathcal{M} = \{X[B]^{\complement} \mid X \in \mathcal{N}\}$. We shall ensure that $X \subseteq \hat{Y}$ or $X \subseteq \hat{Y}^{\complement}$ for every $X \in \mathcal{M}$, which implies that $\hat{Y}$ is nested with every separation in $\mathcal{N}$.

Let (C, D) be the vertex-separation of the torso $G_T[B]$ induced by Y. An edge e of G is *forced at step* 1 (by Y) if one of its incident vertices is in $C \setminus D$. A separation $X \in \mathcal{M}$ is *forced at step* $2n+2$ if there is an edge $e \in X$ that is forced at step $2n+1$ and X is not forced at some step $2j+2$ with $j < n$. An edge e of G is *forced at step* $2n+1$ for $n > 0$ if there is some $X \in \mathcal{M}$ containing e that is forced at step $2n$ and e is not forced at some step $2j+1$ with $j < n$.

The separation $\hat{Y}$ consists of those edges that are forced at some step.

Remark 1.0.16. *If $Y \subseteq Z$, then $\hat{Y} \subseteq \hat{Z}$.* $\square$

Remark 1.0.17. *$X \subseteq \hat{Y}$ or $X \subseteq \hat{Y}^{\complement}$ for every $X \in \mathcal{M}$.*
In particular, $\hat{Y}$ is nested with every separation of $\mathcal{N}$.

Proof. If X intersects $\hat{Y}$, then $X \subseteq \hat{Y}$ by construction. $\square$

There are easy examples of nested separations Y and Z of the torso $G_T[B]$ such that $\hat{Y}$ and $\hat{Z}$ are not nested. These examples motivate the definition of $\tilde{\mathcal{L}}$ below.

Given a nested set $\mathcal{L}$ of separations of $G_T[B]$, the *extension* $\tilde{\mathcal{L}}$ of $\mathcal{L}$ (depending on a well-order $(Y_\alpha \mid \alpha \in \beta)$ of $\mathcal{L}$) is the set $\{\tilde{Y} \mid Y \in \mathcal{L}\}$, where $\tilde{Y}$ is defined as follows: For the smallest element Y_0 of the well-order, we just let $\tilde{Y}_0 = \hat{Y}_0$ and $\widetilde{Y_0^{\complement}} = (\hat{Y}_0)^{\complement}$.

Assume that we already defined $\tilde{Y}_\alpha$ and $\widetilde{Y_\alpha^{\complement}}$ for all $\alpha < \gamma$. Let $Z_\alpha \in \{Y_\alpha, Y_\alpha^{\complement}\}$ be such that $Z_\alpha \subseteq Y_\gamma$ or $Y_\gamma \subseteq Z_\alpha$. We let $\tilde{Y}_\gamma$ consist of those edges that are first forced by Y_γ or second contained in some $\tilde{Z}_\alpha$ with $Z_\alpha \subseteq Y_\gamma$ or third both contained in every $\tilde{Z}_\alpha$ with $Y_\gamma \subseteq Z_\alpha$ and not forced by $Y_\gamma^{\complement}$. We define $\widetilde{Y_\gamma^{\complement}}$ similarly with '$Y_\gamma^{\complement}$' in place of 'Y_γ' and '$Z_\alpha^{\complement}$' in place of 'Z_α'.

Lemma 1.0.25 below says that no edge is forced by both Y and $Y^{\complement}$. Using that and Remark 1.0.16, a transfinite induction over $(Y_\alpha \mid \alpha \in \beta)$ gives the following:

Remark 1.0.18. *1. If $Z_\alpha \subseteq Y_\gamma$, then $\tilde{Z}_\alpha \subseteq \tilde{Y}_\gamma$;*

2. If $Y_\gamma \subseteq Z_\alpha$, then $\tilde{Y}_\gamma \subseteq \tilde{Z}_\alpha$;

3. $\widetilde{Y_\gamma^{\complement}} = (\tilde{Y}_\gamma)^{\complement}$;

4. $\tilde{Y}_\gamma$ contains all edges forced by Y_γ;

5. $\widetilde{Y_\gamma^{\complement}}$ contains all edges forced by $Y_\gamma^{\complement}$;

□

Lemma 1.0.19. *Let $\mathcal{N}$ be a nested set of separations and let B and D be distinct $\mathcal{N}$-block. Let $\mathcal{L}_B$ and $\mathcal{L}_D$ be nested sets of separations of $G_T[B]$ and $G_T[D]$, respectively. Then $\tilde{\mathcal{L}}_B$ is a set of nested separations. If $X \in \mathcal{L}_B$ and $Y \in \mathcal{L}_D$, then $\tilde{X}$ and $\tilde{Y}$ are nested. Moreover, they are nested with every separation in $\mathcal{N}$.*

Proof. $\tilde{\mathcal{L}}_B$ is nested by 1 and 2 of Remark 1.0.18. It is easily proved by transfinite induction over the underlying well-order of $\mathcal{L}_B$ that for every $Z \in \mathcal{N}$ either $Z[B]^{\complement} \subseteq \tilde{X}$ or $\tilde{X} \subseteq Z[B]$. This implies the 'Moreover'-part.

There is some $Z \in \mathcal{N}$ distinguishing B and D. By exchanging the roles of B and D if necessary, we may assume that $Z[B] = Z$ and $Z[D] = Z^{\complement}$. Thus $\tilde{X} \subseteq Z$ or $\tilde{X}^{\complement} \subseteq Z$. And $\tilde{Y} \subseteq Z^{\complement}$ or $\tilde{Y}^{\complement} \subseteq Z^{\complement}$. Hence one of $\tilde{X}$ or $\tilde{X}^{\complement}$ is included in Z which in turn is included in one of $\tilde{Y}$ or $\tilde{Y}^{\complement}$. Thus $\tilde{X}$ and $\tilde{Y}$ are nested. □

Remark 1.0.20. *Let Y be a separation in a nested set $\mathcal{L}$ of $G_T[B]$. Then $\partial(\tilde{Y}) \subseteq \partial(Y)$.*

Proof. Let (C, D) be the vertex-separation induced by Y. If v is a vertex of B not in $C \cap D$, then all its incident edges are either all forced by Y at step 1 or else all forced by $Y^{\complement}$ at step 1, yielding that v cannot be in $\partial(\tilde{Y})$. If v is not in B then it is easily proved by induction on a well-order of $\mathcal{L}$ that all its incident edges are in $\tilde{Y}$ or else all of them are in $\tilde{Y}^{\complement}$. □

Remark 1.0.21. *Let B, P_B and Q_B as in Lemma 1.0.15. Let $\mathcal{L}$ be a nested set of separations in $G_T[B]$. If $X \in \mathcal{L}$ distinguishes P_B and Q_B in $G_T[B]$, then $\tilde{X}$ distinguishes P and Q.*

Proof. By construction there are different components F and K of $G - \partial(X)$ such that $s_F \in P$ and $s_K \in Q$. Clearly, every edge in s_F is forced by X, and every edge in s_K is forced by $X^{\complement}$. Thus $s_F \subseteq \tilde{X}$ and $s_K \subseteq \widetilde{X^{\complement}} = (\tilde{X})^{\complement}$. Hence $\tilde{X}$ distinguishes P and Q. □

Now we prepare to prove Lemma 1.0.25 below:

Remark 1.0.22. *Let $X \in \mathcal{M}$ that contains some edge e forced by Y. Then each endvertex v of e in $C \setminus D$ is in the boundary $\partial(X)$ of X.*

Proof. By assumption $v \in V(X^{\complement})$ and thus $v \in \partial(X)$. □

Remark 1.0.23. *Assume there is at least one edge forced by Y. Then no $X \in \mathcal{M}$ contains all edges of G which are forced by Y at steps 1.*

Proof. If X is not forced by Y at step 2, then this is clear. Otherwise there is a vertex $v \in \partial(X)$ that is in $C \setminus D$ by Remark 1.0.22. Thus there is an edge e incident with v contained in $X^{\complement}$. □

Remark 1.0.24.

1. *No edge is forced by both Y and $Y^{\complement}$ at step 1.*
2. *No $X \in \mathcal{M}$ contains edges forced by Y at step 1 and edges forced by $Y^{\complement}$ at step 1.*

Proof. 1 follows from the fact that (C, D) is a vertex-separation of the torso $G_T[B]$. To see 2, we have to additionally apply Remark 1.0.22 and the corresponding fact for $Y^{\complement}$. □

Lemma 1.0.25. *No edge is forced by both Y and $Y^{\complement}$.*

Proof. In this proof, we run step m for forcing by $Y^{\complement}$ in between step m and step $m+1$ for forcing by Y. Suppose for a contradiction, there is some step m such that just after step m there is an edge e that is forced by both Y and $Y^{\complement}$ or there is some $X \in \mathcal{M}$ containing edges forced by Y and edges forced by $Y^{\complement}$. Let k be minimal amongst all such m. Thus k must be odd. By 1 and 2 of Remark 1.0.24, $k \geq 3$.

Case 1: there is some $X \in \mathcal{M}$ containing an edge e_C forced by Y and an edge e_D forced by $Y^{\complement}$ just after step k. Then precisely one of e_C and e_D was forced at step k, say e_D (the case with e_C will be analogue). Let $Z \in \mathcal{M}$ be a separation forcing e_D, which exists as $k \geq 3$.

We shall show that X and Z are not nested by showing that all the four intersections $X \cap Z$, $X \cap Z^{\complement}$, $X^{\complement} \cap Z$ and $X^{\complement} \cap Z^{\complement}$ are nonempty: First $e_D \in X \cap Z$. Let f an edge forcing Z for $Y^{\complement}$. By minimality of k, first $f \in X^{\complement} \cap Z$. Second, the separation Z does not contain any edge forced by Y just before step k. Thus $e_C \in X \cap Z^{\complement}$. Furthermore, there is some edge forced by Y in $X^{\complement} \cap Z^{\complement}$ by Remark 1.0.23. Thus X and Z are not nested, which gives the desired contradiction in this case.

Case 2: there is some edge e that is forced by both Y and $Y^{\complement}$ just after step k. We shall only consider the case that e was first forced by Y and then by $Y^{\complement}$ (the other case will be analogue). As $k \geq 3$, there is a separation $Z \in \mathcal{M}$ forcing e for $Y^{\complement}$. Let f be an edge forcing Z for $Y^{\complement}$. If e is forced by Y at step 1, then at the step before k the separation Z will contain edges forced by Y and edges forced by $Y^{\complement}$, which is impossible by minimality of k. Thus there is a separation $X \in \mathcal{M}$ forcing e for Y. Let g be an edge forcing X for Y. By minimality of k, we have $g \in X \cap Z^{\complement}$ and $f \in X^{\complement} \cap Z$. Similar as in the last case we deduce that X and Z are not nested, which gives the desired contradiction. □

Miscellaneous

Lemma 1.0.26. *Let X and Y be two separations such that there is a component C of $G - \partial(X)$ with $s_C = X$ and C does not intersect $\partial(Y)$. Then X and Y are nested.*

Proof. By the definition of nestedness, it suffices to show that $X \subseteq Y$ or $X \subseteq Y^{\complement}$. For that, by symmetry, it suffices to show that if there is some edge $e_1 \in X \cap Y$, then any other edge e_2 of X must also be in Y. For that note that e_1 has an endvertex v in C and that there is a path P included in C from v to some endvertex of e_2. As no vertex of P is in $\partial(Y)$ and $e_1 \in Y$ it must be that $e_2 \in Y$, as desired. □

Lemma 1.0.27. *Let X, Y and Z be separations such that first X and Y are not nested and second $X \cap Y$ and Z are not nested. Then Z is not nested with X or Y.*

Proof. Recall that if A and Z are nested, then one of $A \subseteq Z$, $A \subseteq Z^{\complement}$, $A^{\complement} \subseteq Z$ or $A^{\complement} \subseteq Z^{\complement}$ is true. If one of $A \subseteq Z$ or $A \subseteq Z^{\complement}$ is false for $A = X \cap Y$, then it is also false for both $A = X$ or $A = Y$. If one of $A^{\complement} \subseteq Z$ or $A^{\complement} \subseteq Z^{\complement}$ is false for $A = X \cap Y$, then it is false for at least one of $A = X$ or $A = Y$. Suppose for a contradiction that $X \cap Y$ is not nested with Z but X and Y are. By exchanging the roles of X and Y if necessary, we may assume by the above that $X^{\complement} \subseteq Z$ and $Y^{\complement} \subseteq Z^{\complement}$. Then $X^{\complement} \subseteq Y$, contradicting the assumption that X and Y are not nested. □

A separation X is *tight* if $\partial(X) = \partial(s_C)$ for every component C of $G - \partial(X)$.

Lemma 1.0.28. *Let X be a separation of order k. Let Y be a tight separation such that $G - \partial(Y)$ has at least $k+1$ components. Then one of the links $\partial(Y) \setminus V(X)$ or $\partial(Y) \setminus V(X^{\complement})$ is empty.*

Proof. Suppose not for a contradiction, then there are $v \in \partial(Y) \setminus V(X)$ and $w \in \partial(Y) \setminus V(X^{\complement})$. Then v and w are in the neighbourhood of every component C of $G - \partial(Y)$. Thus there are $k + 1$ internally disjoint paths from v to w, contradiction that fact that $\partial(X)$ separates v from w. □

Given two vertices v and w, a separator S separates v and w *minimally* if each component of $G - S$ containing v or w has the whole of S in its neighbourhood.

Lemma 1.0.29 ([72, Statement 2.4]). *Given vertices v and w and $k \in \mathbb{N}$, there are only finitely many distinct separators of size at most k separating v from w minimally.*

1.0.5 Distinguishing the profiles

The aim in this section is to construct a nested set of separations of finite order that distinguishes any two vertex ends efficiently, which is needed in the proof of Theorem 1.0.1. A related result is proved in [42]. Actually, we shall prove the stronger statement that for each $r \in \mathbb{N} \cup \{\infty\}$ there is a nested set $\mathcal{N}$ of separations that distinguishes any two r-robust profiles efficiently.

Overview of the proof

We shall construct the set $\mathcal{N}$ as an ascending union of sets $\mathcal{N}_k$ one for each $k \in \mathbb{N}$, where $\mathcal{N}_k$ is a nested set of separations of order at most k distinguishing efficiently any two r-robust profiles of order $k+1$. Any two r-robust profiles of order $k+2$ that are not distinguished by $\mathcal{N}_k$ will live in the same $\mathcal{N}_k$-block. We obtain $\mathcal{N}_{k+1}$ from $\mathcal{N}_k$ by adding for each $\mathcal{N}_k$-block a nested set $\tilde{\mathcal{N}}_{k+1}(B)$ that distinguishes efficiently any two r-robust profiles of order $k+2$ living in B. Working in the torsos $G_T[B]$ will ensure that the sets $\tilde{\mathcal{N}}_{k+1}(B)$ for different blocks B will be nested with each other.

Summing up, we are left with the task of finding in these torso graphs $G_T[B]$ a nested set distinguishing efficiently all r-robust profiles of order $k+2$. Theorem 1.0.31 deals with this problem if $G_T[B]$ is "nice enough". In order to make all torso graphs nice enough, we add in an additional step in which we enlarge $\mathcal{N}_k$ a little bit so that for the larger nested set the new torso graphs are the old ones with the junk cut off. Lemma 1.0.30 will be the main lemma we use to enlarge $\mathcal{N}_k$.

Finishing the overview, we first state Lemma 1.0.30 and Theorem 1.0.31 and introduce the necessary definitions for that.

For any r-robust profile P and $k \in \mathbb{N}$, the restriction P_k of P to the set of separations of order at most k is an r-robust profile, whose order is the minimum of $k+1$ and the order of P. An *r-profile set* is a set of r-robust profiles such that if $P \in \mathcal{P}$ then for each $k \in \mathbb{N}$ the restriction P_k is in $\mathcal{P}$. Until the end of Subsection 1.0.5, let us a fix a graph G together numbers $k, r \in \mathbb{N} \cup \{\infty\}$ with $k \leq r$ and an r-profile set $\mathcal{P}$.

A set $\mathcal{N}$ of nested sets is *extendable* (for $\mathcal{P}$) if for any two distinct profiles in $\mathcal{P}$ of the same order, there is some separation X nested with $\mathcal{N}$ that distinguishes these two profiles efficiently.

By $R(k, r, \mathcal{P}, G)$ we denote the set of those separations whose order is finite and at most k that distinguish efficiently two profiles in $\mathcal{P}$ in the graph G. It may happen for some $X \in R(k, r, \mathcal{P}, G)$ that $G - \partial(X)$ has a component C such that $\partial(s_C)$ is a proper subset of $\partial(X)$. By $S(k, r, \mathcal{P}, G)$, we denote the set of all separations s_C for such components C of $G - \partial(X)$ for some $X \in R(k, r, \mathcal{P}, G)$. If it is clear from the context what G is, we shall just write $R(k, r, \mathcal{P})$ or $S(k, r, \mathcal{P})$, or even just $R(k, r)$ or $S(k, r)$.

Lemma 1.0.30. *If $R(k-1,r)=\emptyset$, then $S(k,r)$ is a nested extendable set of separations.*

A separation X *strongly disqualifies* a set Y if $|\partial(Y)|$ is strictly larger than both $|L(X,Y)|$ and $|L(X^{\complement},Y)|$. A set X *disqualifies* a set Y if it strongly disqualifies Y or $Y^{\complement}$. Note that every $X\in R(k,r)$ is tight if and only if $S(k,r)=\emptyset$.

Theorem 1.0.31. *Let $k\in\mathbb{N}$ and $r\in\mathbb{N}\cup\{\infty\}$ with $k\le r$. Assume that $S(k,r)=\emptyset$ and $R(k,r)=\emptyset$. Any set $\mathcal{N}$ of nested tight separations of order at most k that are not disqualified by any $X\in R(r,r)$ is extendable.*

In particular, any maximal such set distinguishes any two profiles of order $k+1$ in $\mathcal{P}$.

Proof of Lemma 1.0.30.

Lemma 1.0.32. *If X distinguishes two r-robust profiles P_1 and P_2 efficiently, then X is not disqualified by any separation Y with $\partial(Y)\le r$.*

Proof. We may assume that $X\in P_1$ and $X^{\complement}\in P_2$. Suppose for a contradiction that Y strongly disqualifies X. Then $|L(X,Y)|<|\partial(X)|$ and $|L(X,Y^{\complement})|<|\partial(X)|$. As neither $X\cap Y$ nor $X\cap Y^{\complement}$ is in P_2, these two sets cannot be in P_1 either since X distinguishes P_1 and P_2 efficiently. This contradicts the assumption that P_1 is r-robust. Similarly, one shows that Y cannot strongly disqualify $X^{\complement}$, and thus Y does not disqualify X. □

Lemma 1.0.33. *Let X and Y be two separations distinguishing profiles in $\mathcal{P}$ efficiently with $k=|\partial(X)|\le|\partial(Y)|$. Let C be a component of $G-\partial(X)$ such that $\partial(s_C)$ is a proper subset of $\partial(X)$.*

If $R(k-1,r)=\emptyset$, then C does not intersect $\partial(Y)$.

Proof. Let P and P' be two profiles in $\mathcal{P}$ distinguished efficiently by X, where $X\in P$.

Sublemma 1.0.34. *$G-\partial(X)$ has two components D and K different from C such that $s_D\in P$ and $s_K\in P'$.*

Proof. s_C can be in at most one of P and P'. By the efficiency of X it actually cannot be in precisely one of them. Thus s_C is in none of them. Hence the components D and K of $G-\partial(X)$ such that $s_D\in P$ and $s_K\in P'$, which exist by (P3), are different from C. □

Let Q and Q' be two profiles in $\mathcal{P}$ distinguished efficiently by Y, where $Y\in Q$. Since $|\partial(X)|\le|\partial(Y)|$, we have $X\in Q$ or $X^{\complement}\in Q$. By exchanging the roles of X and $X^{\complement}$ if necessary, we may assume that $X\in Q$. By Sublemma 1.0.34, we may assume that $s_C\subseteq X$ by replacing X by $X\cup s_C$ if necessary.

Sublemma 1.0.35. *Either $|L(X,Y)|\le|\partial(Y)|$ and $X\cap Y\in Q$ or else $|L(X,Y^{\complement})|\le|\partial(Y)|$ and $X\cap Y^{\complement}\in Q'$.*

Proof. **Case 1:** $X^{\complement} \in Q'$.

If $|L(X^{\complement}, Y^{\complement})| < |\partial(X)|$, then $X^{\complement} \cap Y^{\complement} \in Q'$ by (P2) so that $X^{\complement} \cap Y^{\complement}$ will distinguish Q and Q', which is impossible by the efficiency of Y. Thus $|L(X,Y)| \leq |\partial(Y)|$ by Remark 1.0.8, yielding that $X \cap Y \in Q$ by (P2), as desired.

Case 2: $X \in Q'$.

By Lemma 1.0.32, Y does not strongly disqualify $X^{\complement}$. Thus either $|L(Y^{\complement}, X^{\complement})| \geq |\partial(X)|$ or $|L(Y, X^{\complement})| \geq |\partial(X)|$. In the first case, $|L(Y^{\complement}, X)| \leq |\partial(Y)|$ by Remark 1.0.8. Then $Y^{\complement} \cap X \in Q'$ by (P2). Similarly in the second case, $|L(Y,X)| \leq |\partial(Y)|$. Then $Y \cap X \in Q$ by (P2), as desired. □

Sublemma 1.0.36. *One of C and D does not meet $\partial(Y)$.*

Proof. First we consider the case that $|L(X,Y)| \leq |\partial(Y)|$ and $X \cap Y \in Q$. By (P3), there is a component F of $G - \partial(Y \cap X)$ such that $s_F \in Q$. By the efficiency of Y, it must be that $\partial(s_F) = \partial(Y \cap X)$ as s_F distinguishes Q and Q'. Thus the union F' of F and the link $\partial(Y) \setminus V(X^{\complement})$ is connected.

Suppose for a contradiction that both C and D meet $\partial(Y)$, then they both meet $\partial(Y)$ in vertices of the link $\partial(Y) \setminus V(X^{\complement})$. Since C and D are components, they both must contain F', and hence are equal, which is the desired contradiction. Thus at most one of C and D can meet $\partial(Y)$.

By Sublemma 1.0.35 it remains to consider the case where $|L(X, Y^{\complement})| \leq |\partial(Y)|$ and $X \cap Y^{\complement} \in Q'$, which is dealt with analogous to the above case. □

Recall that $\partial(s_C) \subseteq \partial(s_D)$. By Sublemma 1.0.36, one of the links $\partial(s_C) \setminus V(Y)$ and $\partial(s_C) \setminus V(Y^{\complement})$ must be empty since otherwise there would a path joining these two links and avoiding $\partial(Y)$, which is impossible. By symmetry, we may assume that $\partial(s_C) \setminus V(Y)$ is empty. Thus $\partial(Y \setminus s_C) \subseteq \partial(Y)$. Since $R(k-1, r) = \emptyset$, and $s_C \notin P$, it must be that $s_C \notin Q$. Thus $Y \setminus s_C \in Q$ by (P2) so that $Y \setminus s_C$ distinguishes Q and Q'. By the efficiency of Y, it must be that $\partial(Y \setminus s_C) = \partial(Y)$. Hence $\partial(Y) \cap C$ is empty, as desired. □

Proof of Lemma 1.0.30. Let $X \in R(k,r)$ and $Y \in R(r,r)$ of order at least k. Let C be a component of $G - \partial(X)$ and D be a component of $G - \partial(Y)$. In order to see that $S(k,r)$ is a nested, it suffices to show that for any such C and D that the separations s_C and s_D are nested. This is true by Lemma 1.0.33 and Lemma 1.0.26. In order to see that $S(k,r)$ is an extendable, it suffices to show that for any such C and Y that the separations s_C and Y are nested. This is true by Lemma 1.0.33 and Lemma 1.0.26, as well. □

Proof of Theorem 1.0.31.

Before we prove Theorem 1.0.31, we need some intermediate lemmas. Throughout this subsection, we assume that $S(k,r)$ is empty. Let U be the set of those tight separations of order at most k that are not disqualified by any $X \in R(r,r)$. Note $R(k,r) \subseteq U$.

Lemma 1.0.37. *For any componental separation $X \in R(r,r)$, there are only finitely many $Y \in U$ not nested with X.*

Proof. First, we show that X is nested with every $Y \in U$ such that the link $\partial(X) \setminus V(Y)$ is empty. By Lemma 1.0.26, it suffices to show that $\partial(Y) \setminus V(X^{\complement})$ is empty. As X does not strongly disqualify $Y^{\complement}$, one of the links $\partial(Y) \setminus V(X)$ and $\partial(Y) \setminus V(X^{\complement})$ is empty. Hence we may assume that $\partial(Y) \setminus V(X)$ is empty. If Y is not nested with X, there must be a component of C of $G - \partial(Y)$ all of whose neighbours are in $\partial(X) \cap \partial(Y)$. As Y is tight, it must be that $\partial(Y) = \partial(X) \cap \partial(Y)$ so that $\partial(Y) \setminus V(X^{\complement})$ is empty. Hence X and Y are nested by Lemma 1.0.26.

Similarly one shows that X is nested with every $Y \in U$ such that the link $\partial(X) \setminus V(Y)$ is empty.

It remains to show that there are only finitely many $Y \in U$ not nested with X such that both links $\partial(X) \setminus V(Y)$ and $\partial(X) \setminus V(Y^{\complement})$ are nonempty. By Lemma 1.0.29, there are only finitely many triples (v, w, T) where $v, w \in \partial(X)$ and T is a separator of size at most k separating v and w minimally. Since each $\partial(Y)$ for some Y as above is such a separator T, it suffices to show that there are only finitely many $Z \in U$ with $\partial(Z) = \partial(Y)$. This is true as $G - \partial(Y)$ has at most $\partial(X) + 1$ components by Lemma 1.0.28. $\square$

Lemma 1.0.38. *Let $\mathcal{N}$ be a nested subset of U. For any two distinct profiles P and Q in $\mathcal{P}$ of the same order that are not distinguished by any separation of order less than k, there is some separation $X \in R(k,r) \subseteq U$ that is nested with $\mathcal{N}$ and distinguishes P and Q efficiently.*

Proof. First, we show that there is some $X \in U$ distinguishing P and Q efficiently that is nested with all but finitely many separations of $\mathcal{N}$. Since $S(k,r)$ is empty, $R(k,r)$ is a subset of U. Thus U contains some separation A distinguishing P and Q efficiently. By (P3), we can pick such an A that is componental. By Lemma 1.0.37, A is nested with all but finitely many separations of $\mathcal{N}$. Hence we can pick X distinguishing P and Q efficiently such that it is not nested with a minimal number of $Y \in \mathcal{N}$.

Suppose for a contradiction that there is some $Y \in \mathcal{N}$ that is not nested with X. We may assume that Y does not distinguish P and Q since otherwise Y would distinguish P and Q efficiently. Thus either both $Y \in P$ and $Y \in Q$ or both $Y^{\complement} \in P$ and $Y^{\complement} \in Q$. Since $Y^{\complement}$ is nested with $\mathcal{N}$, we may by symmetry assume that $Y \in P$ and $Y \in Q$.

Since X does not strongly disqualify $Y^{\complement}$ by the definition of U, either $|L(X, Y^{\complement})| \geq |\partial(Y)|$ or $|L(X^{\complement}, Y^{\complement})| \geq |\partial(Y)|$. By symmetry, we may assume that $|L(X, Y^{\complement})| \geq |\partial(Y)|$. By exchanging the roles of P and Q if necessary, we may assume that $X \in P$ and $X^{\complement} \in Q$. By Remark 1.0.8, $|L(X^{\complement}, Y)| \leq |\partial(X)|$. Note that $X^{\complement} \cap Y \notin P$ as $X^{\complement} \notin P$ by (P1) but $X^{\complement} \cap Y \in Q$ by (P2). Thus $X^{\complement} \cap Y$ distinguishes P and Q efficiently. Any separation in $\mathcal{N}$ not nested with $X^{\complement} \cap Y$ is by Lemma 1.0.27 not nested with X. As Y is nested with $X^{\complement} \cap Y$, the separation $X^{\complement} \cap Y$ violates the minimality of X. Hence X is nested with $\mathcal{N}$, completing the proof. $\square$

Proof of Theorem 1.0.31. By Lemma 1.0.38 any nested subset of U is extendable. □

Proof of the main result of this section.

In this subsection, we proof the following.

Theorem 1.0.39. *For any graph G and any $r \in \mathbb{N} \cup \{\infty\}$, there is a nested set of separation $\mathcal{N}$ that distinguishes efficiently any two r-robust profiles of the same order.*

First we need an intermediate lemma, for which we fix some notation. Let us fix some $r \in \mathbb{N} \cup \{\infty\}$, some finite $k \leq r$ and an r-profile set $\mathcal{P}$. Let $\mathcal{N}$ be a nested set of separations of order at most k that is extendable for $\mathcal{P}$ and that distinguishes efficiently any two profiles of $\mathcal{P}$ that can be distinguished by a separation of order at most k. For each $\mathcal{N}$-block B, let $\mathcal{P}(B)$ be defined as after Lemma 1.0.15. And let $\mathcal{N}_B$ be a set of nested separations of $G_T[B]$ that is extendable for $\mathcal{P}(B)$. We abbreviate $\mathcal{M} = \mathcal{N} \cup \bigcup \tilde{\mathcal{N}}_B$, where the union ranges over all $\mathcal{N}$-blocks B.

Lemma 1.0.40. *The set $\mathcal{M}$ is nested and extendable for $\mathcal{P}$.*

Proof. $\mathcal{M}$ is nested by Lemma 1.0.19.

It remains to show for every $l \geq k+1$ and any two profiles P and Q in $\mathcal{P}$ that are distinguished efficiently by a separation of order l that there is a separation nested with $\mathcal{M}$ that distinguishes P and Q efficiently. We may assume that P and Q both have order $l+1$ as $\mathcal{P}$ is an r-profile set. By Lemma 1.0.15 and since $\mathcal{N}$ is extendable, there is a unique $\mathcal{N}$-block B such that some separation Y of order l of $G_T[B]$ distinguishes P_B and Q_B.

As $\mathcal{N}_B$ is extendable, there is a separation Z of $G_T[B]$ nested with $\mathcal{N}_B$ that distinguishes P_B and Q_B efficiently. By Lemma 1.0.19, $\tilde{Z}$ is nested with $\mathcal{M}$, and it distinguishes P and Q by Remark 1.0.21 and it does so efficiently by Remark 1.0.20. □

Proof of Theorem 1.0.39. We shall construct the nested set $\mathcal{N}$ of Theorem 1.0.39 as a nested union of sets $\mathcal{N}_k$ one for each $k \in \mathbb{N} \cup \{-1\}$, where $\mathcal{N}_k$ is a nested extendable set of separations of order at most k that distinguishes any two r-robust profiles efficiently that are distinguished by a separation of order at most k. We start the construction with $\mathcal{N}_{-1} = \emptyset$. Assume that we already constructed $\mathcal{N}_k$ with the above properties. For an $\mathcal{N}_k$-block B, we define $\mathcal{P}(B)$ as indicated after Lemma 1.0.15.

Sublemma 1.0.41. *The set $R(k, r, \mathcal{P}(B), G_T[B])$ is empty.*

Proof. Suppose for a contradiction, two profiles P_B and Q_B in $\mathcal{P}(B)$ can be distinguished by a separation X of order at most k. Then $\tilde{X}$ has order at most $|\partial(X)|$ by Remark 1.0.20 and by Remark 1.0.21 it distinguishes the profiles P and Q which induce P_B and Q_B. So P and Q are distinguished by $\mathcal{N}_k$ by the induction hypothesis. This contradicts the assumption that P and Q are both in $\mathcal{P}(B)$. □

By Sublemma 1.0.41, we can apply Lemma 1.0.30 to $G_T[B]$ and $\mathcal{P}(B)$, yielding that the set $S(k+1, r, \mathcal{P}(B), G_T[B])$ is a nested extendable set of separations. For each $S(k+1, r, \mathcal{P}(B), G_T[B])$-block B', we define $\mathcal{P}(B')$ as indicated after Lemma 1.0.15.

Sublemma 1.0.42. *The set $S(k+1, r, \mathcal{P}(B'), G_T[B'])$ is empty.*

Proof. Suppose for a contradiction, there is some $X \in S(k+1, r, \mathcal{P}(B'), G_T[B'])$. Then there is some $Y \in R(k+1, r, \mathcal{P}(B'), G_T[B'])$ so that there is a component C of $G_T[B'] - \partial(Y)$ with $s_C = X$. By Remark 1.0.20, Remark 1.0.21 and the definition of $\mathcal{P}(B')$, the separation $\tilde{Y}$ distinguishes efficiently two profiles in $\mathcal{P}(B)$ so that $\tilde{Y} \in R(k+1, r, \mathcal{P}(B), G_T[B])$. By Remark 1.0.20, $\tilde{Y}$ has order precisely $k+1$ since $\tilde{\tilde{Y}}$ has order $k+1$ because it distinguishes two profiles that are not distinguished by $\mathcal{N}_k$. Hence $\tilde{X} \in S(k+1, r, \mathcal{P}(B), G_T[B])$ by Remark 1.0.20. Thus X is the empty, which is the desired contradiction. $\square$

By Zorn's Lemma we pick a maximal set $\mathcal{N}(B')$ of nested tight separations of order at most k in $G_T[B']$ that are not disqualified by any $X \in R(r, r, \mathcal{P}(B'), G_T[B'])$. By Theorem 1.0.31 the set $\mathcal{N}(B')$ is extendable and distinguishes any two r-robust profiles of order $k+2$ in $\mathcal{P}(B')$.

Let $\mathcal{N}_{k+1}(B)$ be the union of the sets $\tilde{\mathcal{N}}(B')$ together with $S(k+1, r, \mathcal{P}(B), G_T[B])$, where the union ranges over all $S(k+1, r, \mathcal{P}(B), G_T[B])$-blocks B'. By Lemma 1.0.40, $\mathcal{N}_{k+1}(B)$ is a nested and extendable set of separation of order at most $k+1$ in $G_T[B]$. Let $\mathcal{N}_{k+1}$ be the union of the sets $\tilde{\mathcal{N}}_{k+1}(B)$ together with $\mathcal{N}_k$, where the union ranges over all $\mathcal{N}_k$-blocks B. Applying Lemma 1.0.40 again, we get that $\mathcal{N}_{k+1}$ is a nested and extendable set of separation of order at most $k+1$ in G.

Sublemma 1.0.43. *$\mathcal{N}_{k+1}$ distinguishes efficiently any two r-robust profiles P and Q of G that are distinguished by a separation of order at most $k+1$.*

Proof. We may assume that P and Q both have order $k+2$. Let A distinguish P and Q efficiently. If A has order at most k, by the induction hypothesis, there is a separation $\hat{A}$ in $\mathcal{N}_k$ distinguishing P and Q efficiently. So $\hat{A}$ is in $\mathcal{N}_{k+1}$ by construction.

Otherwise there is a separation X distinguishing P and Q efficiently that is nested with $\mathcal{N}_k$ as $\mathcal{N}_k$ is extendable. By Lemma 1.0.15, there is an $\mathcal{N}_k$-blocks B such that P_B and Q_B are r-robust profiles in $G_T[B]$ of order $k+2$ in $\mathcal{P}(B)$, which are distinguished efficiently by X_B. Using the fact that $\mathcal{N}_{k+1}(B)$ is extendable and then applying Lemma 1.0.15 again, we find an $S(k+1, r, \mathcal{P}(B))$-block B' such that P_B and Q_B induce different r-robust profiles of order $k+2$ in $G_T[B']$, which are distinguished efficiently by some separation Z of order at most $k+1$. By construction, we find such a Z in $\mathcal{N}(B')$. Applying Remark 1.0.20 twice yields that the order of $\tilde{\tilde{Z}}$ is at most $k+1$. Thus $\tilde{\tilde{Z}}$ distinguishes P and Q efficiently by Remark 1.0.21. As $\tilde{\tilde{Z}}$ is in $\mathcal{N}_{k+1}$, this completes the proof. $\square$

Finally, the nested union $\mathcal{N}$ of the sets $\mathcal{N}_k$ is a nested set of separations that distinguishes efficiently any two r-robust profiles of the same order, as desired. $\square$

For a vertex end ω, let P_ω^k be the set of those separations of order at most k, in which ω lives. It is straightforward to show that P_ω^k is an ∞-robust profile of order $k+1$. Hence Theorem 1.0.39 has the following consequence.

Corollary 1.0.44. *For any graph G, there is a nested set $\mathcal{N}$ of separations that distinguishes any two vertex ends efficiently.* $\square$

1.0.6 A tree-decomposition distinguishing the topological ends

In this section, we prove Theorem 1.0.1 already mentioned in the Introduction. A key lemma in the proof of Theorem 1.0.1 is the following.

Lemma 1.0.45. *Let G be a graph with a finite nonempty set W of vertices. Then G has a star decomposition $(S, Q_s | s \in V(S))$ of finite adhesion such that each topological end lives in some Q_s where s is a leaf.*

Moreover, only the central part Q_c contains vertices of W, and for each leaf s, there lives an topological end in Q_s, and $Q_s \setminus Q_c$ is connected.

Proof that Lemma 1.0.45 implies Theorem 1.0.1. We shall recursively construct a sequence $\mathcal{T}^n = (T^n, P_t^n | t \in V(T^n))$ of tree-decomposition of G of finite adhesion as follows. We starting by picking a vertex v of G arbitrarily and we obtain $\mathcal{T}^1$ by applying Lemma 1.0.45 with $W = \{v\}$. Assume that we already constructed $\mathcal{T}^n$. For each leaf s of $\mathcal{T}^n$, we denote by W_s the set of those vertices in Q_s also contained in some other part of $\mathcal{T}^n$. Note that W_s is contained in the part adjacent to Q_s and thus is finite. By Lemma 1.0.45, we obtain a star decomposition $\mathcal{T}_s$ of $G[Q_s]$ such that no $w \in W_s$ is contained in a leaf part of $\mathcal{T}_s$ and such that each topological end living in Q_s lives in a leaf of $\mathcal{T}_s$. We obtain $\mathcal{T}^{n+1}$ from $\mathcal{T}^n$ by replacing each leaf part Q_s by $\mathcal{T}_s$, which is well-defined as the set W_s is contained in a unique part of $\mathcal{T}_s$.

By r, we denote the center of $\mathcal{T}_1$. For each $j < m < n$, the balls of radius j around r in T^m and T^n are the same. Thus we take T to be the tree whose nodes are those that are eventually a node of T^n. For each $t \in V(T)$, the parts P_t^n are the same for n larger than the distance between t and r, and we take P_t to be the limit of the P_t^n.

It is easily proved by induction that each vertex in W_s for s a leaf of T^n has distance at least $n-1$ from v in G. Thus for each $j < n$ the ball of radius j around v in G is included in the union over all parts P_t^n where t is in the ball of radius j around r in $\mathcal{T}_n$. Hence $(T, P_t | t \in V(T))$ is a tree-decomposition, and it has finite adhesion by construction.

It remains to show that the ends of T define precisely the topological ends of G, which is done in the following four sublemmas.

Sublemma 1.0.46. *Each topological end ω of G lives in an end of T.*

Proof. There is a unique leaf s of T^n such that ω lives in P^n_s. Let s_n be the predecessor of s in T^n. Then ω lives in the end of T to which $s_1 s_2 \dots$ belongs. □

Sublemma 1.0.47. *In each end τ of T, there lives a vertex end of G.*

Proof. For a directed paths P, we shall denote by $\overleftarrow{P}$ the directed path with the inverse ordering of that of P.

Let $s_1 s_2 ...$ be the ray in T starting at r that belongs to τ. By construction, the sets W_{s_i} are disjoint and finite. For each $w \in W_{s_i}$, we pick a path P_w from w to v. Since $W_{s_{i-1}}$ separates w from v, there is a first $w' \in W_{s_{i-1}}$ appearing on P_w. Now we apply the Infinity Lemma in the form of [52, Section 8] on the graph whose vertex set is the disjoint union of the sets W_{s_i}, and we put in all the edges ww'. Thus this graph has a ray $w_1 w_2 ...$ where $w_i \in W_{s_i}$. Then $K = v\overleftarrow{P_{w_1}}w_1\overleftarrow{P_{w_2}}w_2\overleftarrow{P_{w_3}}...$ is an infinite walk with the property that the distance between v and a vertex k on K is at least n if k appears after $\overleftarrow{P_{w_n}}$. In particular, K traverses each vertex only finitely many times. Thus K is a connected locally finite graph, and thus contains a ray R. Since R meets each of the sets W_{s_i}, the end to which R belongs lives in τ, as desired. □

Sublemma 1.0.48. *No two distinct vertex ends ω_1 and ω_2 of G live in the same end τ of T.*

Proof. Suppose for a contradiction, there are such ω_1, ω_2 and τ. Let U be a finite separator separating ω_1 from ω_2 and let n be the maximal distances between v and a vertex in U. Then there is a leaf s of T^{n+1} such that τ lives in Q_s. Let C_i be the component of $G - U$ in which ω_i lives. Since W_s separates U from $Q_s \setminus W_s$, it must be that the connected set $Q_s \setminus W_s$ is contained in a component of $G - U$. As ω_i lives in $Q_s \setminus W_s$ by assumption, it must be that $Q_s \setminus W_s \subseteq C_i$. Hence C_1 and C_2 intersect, which is the desired contradiction. □

Sublemma 1.0.49. *No vertex u dominates a vertex end ω living in some end of T.*

Proof. Suppose for a contradiction u does. Let n be the distance between u and v in G. Then there is a leaf s of T^{n+1} such that ω lives in Q_s. Thus the finite set W_s separates u from ω, contradicting the assumption that u dominates ω. □

Sublemma 1.0.46, Sublemma 1.0.47, Sublemma 1.0.48 and Sublemma 1.0.49 imply that the ends of T define precisely the topological ends of G, as desired. □

Remark 1.0.50. *Let $(T, \leq)$ be the tree order on T as in the proof of Theorem 1.0.1 where the root r is the smallest element. We remark that we constructed $(T, \leq)$ such that $(T, P_t | t \in V(T))$ has the following additional property: For each edge tu with $t \leq u$, the vertex set $\bigcup_{w \geq u} V(P_w) \setminus V(P_t)$ is connected.*

Moreover, we construct $(T, P_t | t \in V(T))$ such that if st and tu are edges of T with $s \leq t \leq u$, then $V(P_s) \cap V(P_t)$ and $V(P_t) \cap V(P_u)$ are disjoint.

In order to prove Lemma 1.0.45, we need the following.

Lemma 1.0.51. *Let G be a connected graph and $W \subseteq V(G)$ finite and nonempty. Then there is a set $\mathcal{X}$ of disjoint edge sets X of finite boundary such that every vertex end not dominated by some $w \in W$ lives in some $X \in \mathcal{X}$ and no edge e in any $X \in \mathcal{X}$ is incident with a vertex of W.*

Proof that Lemma 1.0.51 implies Lemma 1.0.45. We may assume that G in Lemma 1.0.45 is connected. Let $C = V(E \setminus \bigcup \mathcal{X}) \cup \bigcup_{X \in \mathcal{X}} \partial(X)$. For $X \in \mathcal{X}$ let $\mathcal{Q}_X$ consist of sets of the form $\partial(X) \cup Q$, where Q is a component of $G - \partial(X)$ with $Q \subseteq V(X)$. Let $\mathcal{Q}$ be the union over $\mathcal{X}$ of the sets $\mathcal{Q}_X$. Let $\mathcal{R}$ be the set of those H in $\mathcal{Q}$ such that some topological end lives in $V(H)$. Note that each topological end lives in some $R \in \mathcal{R}$ and that W does not intersect any such R. We obtain C' from C by adding the vertex sets of all $H \in \mathcal{Q} \setminus \mathcal{R}$. We consider $S = \mathcal{R} \cup \{C'\}$ as a star with center C'. It is straightforward to verify that $(S, s|s \in V(S))$ is a star decomposition with the desired properties. □

The rest of this section is devoted to the proof of Lemma 1.0.51. We shall need the following lemma.

Lemma 1.0.52. *Let G be a connected graph and $W \subseteq V(G)$ finite. There is a nested set N of nonempty separations of finite order such that every vertex end not dominated by some $w \in W$ lives in some $X \in N$ and no edge e in some $X \in N$ is incident with a vertex of W.*

Moreover, if $X, Y \in N$ are distinct with $X \subseteq Y$, then the order of Y is strictly larger than the order of X.

Proof. We obtain G_W from G by first deleting W and then adding a copy of K_ω, the complete graph on countably many vertices, which we join completely to the neighbourhood of W. Applying Corollary 1.0.44 to G_W, we obtain a nested set N' of separations of finite order such that any two vertex ends of G_W are distinguished efficiently by a separation in N'. Let τ be the vertex end to which the rays of the newly added copy of K_ω belong. Let N'' consist of those separations in N' that distinguish τ efficiently from some other vertex end. As the separations in N'' distinguish efficiency, no $X \in N''$ contains an edge incident with a vertex of the newly added copy of K_ω.

Given $k \in \mathbb{N}$, a *k-sequence* $(X_\alpha | \alpha \in \gamma)$ (for N'') is an ordinal indexed sequence of elements of N'' of order at most k such that if $\alpha < \beta$, then $X_\alpha \subseteq X_\beta$. We obtain N''' from N'' by adding $\bigcup_{\alpha \in \gamma} X_\alpha$ for all k-sequences (X_α) for all k. Clearly, $N'' \subseteq N'''$ and N''' is nested. Given $k \in \mathbb{N}$, the set N_k consists of those $X \in N'''$ of order at most k, and N'_k consists of the inclusion-wise maximal elements of N_k.

We let $N = \bigcup_{k \in \mathbb{N}} N'_k$. By construction, each $X \in N$ contains no edge incident with a vertex of the newly added copy of K_ω, and thus it can be considered as an edge set of G, whose boundary is the same as the boundary in G_W. We claim that N has all the properties stated in Lemma 1.0.52: By construction, each $X \in N$ is nonempty. Since $N \subseteq N'''$, the set N is nested. The "Moreover"-part is clear by construction. Thus it remains to show that each vertex end ω of G not dominated by some vertex in W lives in some element of N.

Let R be a ray belonging to ω. Since ω is not dominated by any vertex in W, for each $w \in W$ there is a finite vertex set S_w separating a subray R_w of R from w. Then $S = \bigcup_{w \in W} S_w \setminus W$ separates $R' = \bigcap_{w \in W} R_w$ from W in G but also in G_W. Let ω' be the vertex end of G_W to which R' belongs. Note that S witnesses that $\omega' \neq \tau$. Thus there is some $X \in N'''$ in which ω' lives. Let k be the order of X. By Zorn's lemma, N''' contains an inclusion-wise maximal element X' of order at most k including X. By construction X' is in N'_k and includes a subray of R'. Thus ω lives in X', which completes the proof. $\square$

Next we show how Lemma 1.0.52 implies Lemma 1.0.51. A good candidate for $\mathcal{X}$ in Lemma 1.0.51 might be the inclusion-wise maximal elements of N. However, there might be an infinite strictly increasing sequence of members in N, whose orders are also strictly increasing, so that we cannot expect that the union over the members of this sequence has finite order, and hence cannot be in N. Thus we have to make a more sophisticated choice for $\mathcal{X}$ than just taking the maximal members of N.

Lemma 1.0.52 implies Lemma 1.0.51. Let N be as in Lemma 1.0.52. Let $X \in N$ be such that there is another $Y \in N$ with $X \subseteq Y$, then the order of Y is strictly larger than the order of X. We denote the set of such Y of minimal order by $D(X)$. Let H be the digraph with vertex set N where we put in the directed edge XY if $Y \in D(X)$. A *connected component of* H, is a connected component of the underlying graph of H.

Sublemma 1.0.53. *Let $X', Y' \in N$. Then $X' \subseteq Y'$ if and only if there is a directed path from X' to Y'. Moreover, if $X, Y \in N$ are not joined by a directed path, then they are disjoint.*

Proof. Clearly, if there is a directed path from X' to Y', then $X' \subseteq Y'$. Conversely, let $X', Y' \in N$ with $X' \subseteq Y'$. Let (X_n) be a sequence of distinct separations in N such that $X' \subseteq X_1 \subseteq ... \subseteq X_n \subseteq Y'$. By Lemma 1.0.52, $n \leq |\partial(Y')| - |\partial(X')| + 1$. Thus there is a maximal such chain (Z_n), which satisfies $Z_1 = X'$ and $Z_n = Y'$ and $X_{i+1} \in D(X_i)$ for all i between 1 and $n-1$. Hence $Z_1 ... Z_n$ is a path from X' to Y'.

To see that "Moreover"-part, let $X, Y \in N$. As G is connected, there is an edge e incident with some vertex in W. Since e is not in $X \cup Y$ and X and Y are nested, X and Y must be disjoint if they are not joined by a directed path. $\square$

Sublemma 1.0.54. *Each vertex v of H has out-degree at most 1*

Proof. Suppose for a contradiction v has out-degree at least 2. Then there are distinct $X, Y \in D(v)$ so that neither $X \subseteq Y$ nor $Y \subseteq X$. Thus X and Y are disjoint by Sublemma 1.0.53. Since $v \subseteq X \cap Y$, this is the desired contradiction. $\square$

Sublemma 1.0.55. *Any undirected path P joining two vertices v and w contains a vertex u such that vPu and wPu are directed paths which are directed towards u.*

Proof. It suffices to show that P contains at most one vertex of out-degree 0 on P. If it contained two such vertices then between them would be a vertex of out-degree 2, which is impossible by Sublemma 1.0.54. $\square$

We define $\mathcal{X}$ as the union of sets $\mathcal{X}_C$, one for each component C of H. The $\mathcal{X}_C$ are defined as follows: If C has a vertex v_C of out-degree 0, then by Sublemma 1.0.55 C cannot contain a second such vertex and for any other vertex v in C, there is a directed path from v to v_C directed towards v_C. Hence v_C includes any other $v \in V(C)$. We let $\mathcal{X}_C = \{v_C\}$.

Otherwise, C includes a ray $X_1X_2\dots$ as C cannot contain a directed cycle by Sublemma 1.0.53. In this case, we take $\mathcal{X}_C$ to be the set consisting of the $Y_i = X_i \setminus X_{i-1}$ for each $i \in \mathbb{N}$, where $Y_1 = X_1$. Note that the order of Y_i is bounded by the sum of the orders of X_i and X_{i-1}, and thus finite.

Since no $Y \in N$ contains an edge incident with some $w \in W$, the same is true for any $Y \in \mathcal{X}$. Any two distinct $X, Y \in \mathcal{X}$ are disjoint: If X and Y are in the same $\mathcal{X}_C$, this is clear by construction. Otherwise it follows from the definition of Y_i and Sublemma 1.0.53. Thus it remains to prove the following:

Sublemma 1.0.56. *Each vertex end ω not dominated by some vertex of W lives in some $X \in \mathcal{X}$.*

Proof. By Lemma 1.0.52, there is some $Z \in N$ in which ω lives. Let C be the component of H containing Z. If $\mathcal{X}_C = \{v_C\}$, then $Z \subseteq v_C$. Otherwise let the X_i and the Y_i be as in the construction of $\mathcal{X}_C$. If $Z = X_j$ for some j. Then we pick j minimal such that ω lives in X_j. Since ω does not live in X_{j-1}, it must live in Y_j, as desired.

Thus we may assume that Z is not equal to any X_j. Let P be a path joining Z and $X_1 = Y_1$. By Sublemma 1.0.55, P contains a vertex u such that ZPu and X_1Pu are directed paths which are directed towards u. If $u = X_1$, then $Z \subseteq Y_1$, and we are done. Otherwise X_1Pu is a subpath of the ray $X_1X_2\dots$ since the out-degree is at most 1 so that $u = X_j$ for some j.

We pick P such that the j with $u = X_j$ is minimal and have the aim to prove that then $Z \subseteq Y_j$. Since $Z \subseteq X_j$, it remains to show that Z and X_{j-1} are disjoint. Suppose for a contradiction, there is a directed path Q joining Z and X_{j-1}. If Q is directed towards Z, then $Z = X_m$ for some m, contrary to our assumption. Thus Q is directed towards X_{j-1}. But then $ZQX_{j-1}PX_1$ has a smaller j-value, which contradicts the minimality of P. Hence there cannot be such a Q, and thus Z and X_{j-1} are disjoint by Sublemma 1.0.53. Having shown that $Z \subseteq Y_j$, we finish the proof by concluding that then ω also lives in Y_j. $\square$

$\square$

Finally we deduce Corollary 1.0.2.

Proof that Theorem 1.0.1 implies Corollary 1.0.2. By Theorem 1.0.1, G has a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion such that the ends of T define precisely the topological ends of T, and we choose this tree-decomposition as in Remark 1.0.50. In particular, we can pick a root r of T such that for each edge tu with $t \leq u$, the vertex set $\bigcup_{w \geq u} V(P_w) \setminus V(P_t)$ is connected.

Thus for each such edge tu, there is a finite connected subgraph S_u of $G[\bigcup_{w \geq u} V(P_w)]$ that contains $V(P_t) \cap V(P_u)$. Let Q_t be a maximal subforest of the union of the S_u, where the union ranges over all upper neighbours u of t. We recursively build a maximal subset U of $V(T)$ such that if $a, b \in U$, then Q_a and Q_b are vertex-disjoint. In this construction, we first add the nodes of T with smaller distance from the root. This ensures by the "Moreover"-part of Remark 1.0.50 that U contains infinitely many nodes of each ray of T.

Let S' be the union of those Q_t with $t \in U$. We obtain S by extending S' to a spanning tree of G, and rooting it at some $v \in V(S)$ arbitrarily. By the Star-Comb-Lemma [52, Section 8], each spanning tree of G contains for each topological end ω a ray belonging to ω.

Thus it remains to show that S does not contain two disjoint rays R_1 and R_2 that both belong to the same topological end ω of G. Suppose there are such R_1, R_2 and ω. Let $t_1 t_2 \ldots$ be the ray of T in which ω lives. Let n be so large that both R_1 and R_2 meet P_{t_n}. Then for each $m \geq n$, the set S_{t_m} contains a path joining R_1 and R_2. Thus the set $Q_{t_{m-1}}$ contains such a path. Since $Q_{t_{m-1}} \subseteq S$ for infinitely many m, the tree S contains a cycle, which is the desired contradiction. $\square$

Chapter 2

Canonical tree-decompositions

In Section 2.1, we show that every finite graph has a canonical tree-decomposition distinguishing all robust blocks efficiently. We improve these tree-decompositions in Section 2.2 so that they additionally distinguish all the maximal tangles. In Section 2.2 we investigate further properties of these tree-decompositions.

In Section 2.4, we use our techniques to give a simpler proof of the tangle-tree theorem of Robertson and Seymour. In Section 2.5, we study the block number.

2.1 Connectivity and tree structure in finite graphs

2.1.1 Introduction

Ever since graph connectivity began to be systematically studied, from about 1960 onwards, it has been an equally attractive and elusive quest to 'decompose a k-connected graph into its $(k+1)$-connected components'. The idea was modelled on the well-known block-cutvertex tree, which for $k = 1$ displays the global structure of a connected graph 'up to 2-connectedness'. For general k, the precise meaning of what those '$(k+1)$-connected components' should be varied, and came to be considered as part of the problem. But the aim was clear: it should allow for a decomposition of the graph into those 'components', so that their relative structure would display some more global structure of the graph.

While originally, perhaps, these 'components' were thought of as subgraphs, it soon became clear that, for larger k, they would have to be defined differently. For $k = 2$, Tutte [105] found a decomposition which, in modern terms,[1] would

[1] Readers not acquainted with the terminology of graph minor theory can skip the details of this example without loss. The main point is that those 'torsos' are not subgraphs, but subgraphs plus some additional edges reflecting the additional connectivity that the rest of the graph provides for their vertices.

be described as a tree-decomposition of adhesion 2 whose torsos are either 3-connected or cycles.

For general k, Robertson and Seymour [94] re-interpreted those '$(k+1)$-connected components' in a radically new (but interesting) way as 'tangles of order k'. They showed, as a cornerstone of their theory on graph minors, that every finite graph admits a tree-decomposition that separates all its maximal tangles, regardless of their order, in that they inhabit different parts of the decomposition. Note that this solves the modified problem for all k simultaneously, a feature we shall achieve also for the original problem.

More recently still, Dunwoody and Krön [61], taking their lead directly from Tutte (and from Dunwoody's earlier work on tree-structure induced by edge-cuts [60]), followed up Tutte's observation that his result for $k = 2$ can alternatively be described as a tree-like decomposition of a graph G into cycles and vertex sets that are '2-inseparable': such that no set of at most 2 vertices can separate any two vertices of that set in G. Note that such 'k-inseparable' sets of vertices, which were first studied by Mader [87], differ markedly from k-connected subgraphs, in that their connectivity resides not on the set itself but in the ambient graph. For example, joining $r > k$ isolated vertices pairwise by $k+1$ independent paths of length 2, all disjoint, makes this set into a 'k-block', a maximal k-inseparable set of vertices. This then plays an important structural (hub-like) role for the connectivity of the graph, but it is still independent.

External connectivity of a set of vertices in the ambient graph had been considered before in the context of tree-decompositions and tangles [53, 93]. But it was Dunwoody and Krön who realized that k-inseparability can serve to extend Tutte's result to $k > 2$: they showed that the k-blocks of a finite k-connected graph can, in principle, be separated canonically in a tree-like way [61]. We shall re-prove this in a simpler and stronger form, extend it to graphs of arbitrary connectivity, and cast the 'tree-like way' in the standard form of tree-decompositions. We show in particular that every finite graph has a canonical tree-decomposition of adhesion at most k such that distinct k-blocks are contained in different parts (Theorem 1); this appears to solve the original problem for fixed k in a strongest-possible way. For graphs whose k-blocks have size at least $3k/2$ for all k, a weak but necessary additional assumption, these decompositions can be combined into one unified tree-decomposition that distinguishes all the blocks of the graph, simultaneously for all k (Theorem 2).

Our paper is independent of the results stated in [61].[2] Our approach will be as follows. We first develop a more general theory of separation systems to deal with the following abstract problem. Let S be a set of separations in a graph, and let $\mathcal{I}$ be a collection of *S-inseparable* sets of vertices, sets which, for every separation $(A, B) \in S$, lie entirely in A or entirely in B. Under what condition does S have a *nested* subsystem N that still separates all the sets in $\mathcal{I}$? In a further step we show how such nested separation systems N can be captured by tree-decompositions.[3]

[2]The starting point for this paper was that, despite some effort, we were unable to verify some of the results claimed in [61].

[3]It is easy to see that tree-decompositions give rise to nested separation systems. The

The gain from having an abstract theory of how to extract nested subsystems from a given separation system is its flexibility. For example, we shall use it in [78] to prove that every finite graph has a canonical (in the sense above) tree-decomposition separating all its maximal tangles. This improves on the result of Robertson and Seymour [94] mentioned earlier, in that their decomposition is not canonical in our sense: it depends on an assumed vertex enumeration to break ties when choosing which of two crossing separations should be picked for the nested subsystem. The choices made by our decompositions will depend only on the structure of the graph. In particular, they will be invariant under its automorphisms, which thus act naturally also on the set of parts of the decomposition and on the associated decomposition tree.

To state our main results precisely, let us define their terms more formally. In addition to the terminology explained in [52] we say that a set X of vertices in a graph G is *k-inseparable* in G if $|X| > k$ and no set S of at most k vertices separates two vertices of $X \setminus S$ in G. A maximal k-inseparable set of vertices is a *k-block*,[4] or simply a *block*. The smallest k for which a block is a k-block is the *rank* of that block; the largest such k is its *order*.

The intersections $V_t \cap V_{t'}$ of 'adjacent' parts in a tree-decomposition $(T, \mathcal{V})$ of G (those for which tt' is an edge of T) are the *adhesion sets* of $(T, \mathcal{V})$; the maximum size of such a set is the *adhesion* of $(T, \mathcal{V})$. A tree-decomposition of adhesion at most k *distinguishes* two k-blocks b_1, b_2 of G if they are contained in different parts, V_{t_1} and V_{t_2} say. It does so *efficiently* if the t_1–t_2 path in the decomposition tree T has an edge tt' whose adhesion set (which will separate b_1 from b_2 in G) has size $\kappa(b_1, b_2)$, the minimum size of a b_1–b_2 separator in G. The tree-decomposition $(T, \mathcal{V})$ is $\mathrm{Aut}(G)$-*invariant* if the automorphisms of G act on the set of parts in a way that induces an action on the tree T.

Theorem 1. *Given any integer $k \geq 0$, every finite graph G has an $\mathrm{Aut}(G)$-invariant tree-decomposition of adhesion at most k that efficiently distinguishes all its k-blocks.*

Unlike in the original problem, the graph G in Theorem 1 is not required to be k-connected. This is a more substantial improvement than it might seem. It becomes possible only by an inductive approach which refines, for increasing $\ell = 0, 1, \ldots$, each part of a given tree-decomposition of G of adhesion at most ℓ by a finer tree-decomposition of adhesion at most $\ell+1$, until for $\ell = k$ the desired decomposition is achieved. The problem with this approach is that, in general, a graph G need not admit a unified tree-decomposition that distinguishes its ℓ-blocks for all $\ell \in \mathbb{N}$ simultaneously. Indeed, we shall see in Section 2.1.6 an example where G has two ℓ-blocks separated by a unique separation of order at most ℓ, as well as two $(\ell + 1)$-blocks separated by a unique separation of order

converse is less clear.

[4] Belonging to the same k-block is not an equivalence relation on $V(G)$, but almost: distinct k-blocks can be separated by k or fewer vertices. A long cycle has exactly one k-block for $k \in \{0, 1\}$ and no k-block for $k \geq 2$. A large grid has a unique k-block for $k \in \{0, 1\}$, five 2-blocks (each of the corner vertices with its neighbours, plus the set of non-corner vertices), and one 3-block (the set of its inner vertices). It has no k-block for $k \geq 4$.

at most $\ell + 1$, but where these two separations 'cross': we cannot adopt both for the same tree-decomposition of G. The reason why this inductive approach nonetheless works for a proof of Theorem 1 is that we aim for slightly less there: at stage ℓ we only separate those ℓ-blocks of G that contain a k-block for the fixed k given in the theorem, not all the ℓ-blocks of G.

However, there is a slight strengthening of the notion of a block that does make it possible to construct an overall tree-decomposition separating all the blocks of a graph at once. We shall call such blocks *robust*. Their precise definition is technical and will be given later; it essentially describes the exact way in which the offending block of the above counterexample lies in the graph.[5] In practice 'most' blocks of a graph will be robust, including all k-blocks that are complete or have size at least $3k/2$.

If all the blocks of a graph G are robust, how will they lie in the unified tree-decomposition of G that distinguishes them all? Some blocks (especially those of large order) will reside in a single part of this decomposition, while others (of smaller order) will inhabit a subtree consisting of several parts. Subtrees accommodating distinct k-blocks, however, will be disjoint. Hence for any fixed k we can contract them to single nodes, to reobtain the tree-decomposition from Theorem 1 in which the k-blocks (for this fixed k) inhabit distinct single parts. As k grows, we thus have a sequence $(T_k, \mathcal{V}_k)_{k\in\mathbb{N}}$ of tree-decompositions, each refining the previous, that gives rise to our overall tree-decomposition in the last step of the sequence.

Formally, let us write $(T_m, \mathcal{V}_m) \preccurlyeq (T_n, \mathcal{V}_n)$ for tree-decompositions $(T_m, \mathcal{V}_m)$ and $(T_n, \mathcal{V}_n)$ if the decomposition tree T_m of the first is a minor of the decomposition tree T_n of the second, and a part $V_t \in \mathcal{V}_m$ of the first decomposition is the union of those parts $V_{t'}$ of the second whose nodes t' were contracted to the node t of T_m.

Theorem 2. *For every finite graph G there is a sequence $(T_k, \mathcal{V}_k)_{k\in\mathbb{N}}$ of tree-decompositions such that, for all k,*

(i) $(T_k, \mathcal{V}_k)$ *has adhesion at most k and distinguishes all robust k-blocks;*

(ii) $(T_k, \mathcal{V}_k) \preccurlyeq (T_{k+1}, \mathcal{V}_{k+1})$*;*

(iii) $(T_k, \mathcal{V}_k)$ *is* $\mathrm{Aut}(G)$*-invariant.*

The decomposition $(T_k, \mathcal{V}_k)$ will in fact distinguish distinct robust k-blocks b_1, b_2 efficiently, by (i) for $k' = \kappa(b_1, b_2)$ and (ii). In Section 2.1.6 we shall prove Theorem 2 in a stronger form, which also describes how blocks of different rank or order are distinguished.

This paper is organized as follows. In Section 2.1.2 we collect together some properties of pairs of separations, either crossing or nested. In Section 2.1.3 we define a structure tree T associated canonically with a nested set of separations of a graph G. In Section 2.1.4 we construct a tree-decomposition of G modelled

[5]Thus we shall prove that our counterexample is essentially the only one: all graphs not containing it have a unified tree-decomposition distinguishing all their blocks.

on T, and study its parts. In Section 2.1.5 we find conditions under which, given a set S of separations and a collection $\mathcal{I}$ of S-inseparable set of vertices, there is a nested subsystem of S that still separates all the sets in $\mathcal{I}$. In Section 2.1.6, finally, we apply all this to the case of k-separations and k-blocks. We shall derive a central result, Theorem 2.1.24, which includes Theorems 1 and 2 as special cases.

2.1.2 Separations

Let $G = (V, E)$ be a finite graph. A *separation* of G is an ordered pair (A, B) such that $A, B \subseteq V$ and $G[A] \cup G[B] = G$. A separation (A, B) is *proper* if neither $A \setminus B$ nor $B \setminus A$ is empty. The *order* of a separation (A, B) is the cardinality of its *separator* $A \cap B$; the sets A, B are its *sides*. A separation of order k is a *k-separation*.

A separation (A, B) *separates* a set $I \subseteq V$ if I meets both $A \setminus B$ and $B \setminus A$. Two sets I_0, I_1 are *weakly separated* by a separation (A, B) if $I_i \subseteq A$ and $I_{1-i} \subseteq B$ for an $i \in \{0, 1\}$. They are *properly separated*, or simply *separated*, by (A, B) if in addition neither I_0 nor I_1 is contained in $A \cap B$.

Given a set S of separations, we call a set of vertices *S-inseparable* if no separation in S separates it. A maximal S-inseparable set of vertices is an *S-block*, or simply a *block* if S is fixed in the context.

Lemma 2.1.1. *Distinct S-blocks b_1, b_2 are separated by some $(A, B) \in S$.*

Proof. Since b_1 and b_2 are maximal S-inseparable sets, $b := b_1 \cup b_2$ can be separated by some $(A, B) \in S$. Then $b \setminus B \neq \emptyset \neq b \setminus A$, but being S-inseparable, b_1 and b_2 are each contained in A or B. Hence (A, B) separates b_1 from b_2. $\square$

A set of vertices is *small* with respect to S if it is contained in the separator of some separation in S. If S is given from the context, we simply call such a set *small*. Note that if two sets are weakly but not properly separated by some separation in S then at least one of them is small.

Let us look at how different separations of G can relate to each other. The set of all separations of G is partially ordered by

$$(A, B) \leq (C, D) \;:\Leftrightarrow\; A \subseteq C \text{ and } B \supseteq D. \tag{2.1}$$

Indeed, reflexivity, antisymmetry and transitivity follow easily from the corresponding properties of set inclusion on $\mathcal{P}(V)$. Note that changing the order in each pair reverses the relation:

$$(A, B) \leq (C, D) \;\Leftrightarrow\; (B, A) \geq (D, C). \tag{2.2}$$

Let (C, D) be any proper separation.

No proper separation (A, B) is $\leq$-comparable with both (C, D) and (D, C). In particular, $(C, D) \not\leq (D, C)$. (2.3)

Indeed, if $(A, B) \leq (C, D)$ and also $(A, B) \leq (D, C)$, then $A \subseteq C \subseteq B$ and hence $A \setminus B = \emptyset$, a contradiction. By (2.2), the other cases all reduce to this case by changing notation: just swap (A, B) with (B, A) or (C, D) or (D, C).

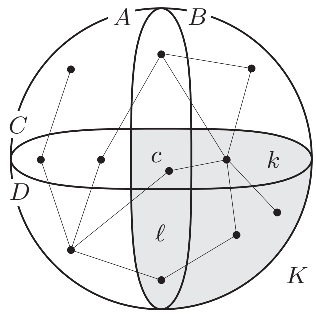


Figure 2.1: The cross-diagram $\{(A, B), (C, D)\}$ with centre c and a corner K and its links k, ℓ.

The way in which two separations relate to each other can be illustrated by a *cross-diagram* as in Figure 2.1. In view of such diagrams, we introduce the following terms for any set $\{(A, B), (C, D)\}$ of two separations, not necessarily distinct. The set $A \cap B \cap C \cap D$ is their *centre*, and $A \cap C$, $A \cap D$, $B \cap C$, $B \cap D$ are their *corners*. The corners $A \cap C$ and $B \cap D$ are *opposite*, as are the corners $A \cap D$ and $B \cap C$. Two corners that are not opposite are *adjacent*. The *link* between two adjacent corners is their intersection minus the centre. A corner minus its links and the centre is the *interior* of that corner; the rest – its two links and the centre – are its *boundary*. We shall write ∂K for the boundary of a corner K.

A corner forms a separation of G together with the union of the other three corners. We call these separations *corner separations*. For example, $(A \cap C, B \cup D)$ (in this order) is the corner separation for the corner $A \cap C$ in $\{(A, B), (C, D)\}$.

The four corner separations of a cross-diagram compare with the two separations forming it, and with the inverses of each other, in the obvious way:

Any two separations (A, B), (C, D) *satisfy* $(A \cap C, B \cup D) \le (A, B)$. (2.4)

If (I, J) *and* (K, L) *are distinct corner separations of the same cross-diagram, then* $(I, J) \le (L, K)$. (2.5)

Inspection of the cross-diagram for (A, B) and (C, D) shows that $(A, B) \le (C, D)$ if and only if the corner $A \cap D$ has an empty interior and empty links, i.e., the entire corner $A \cap D$ is contained in the centre:

$$(A, B) \le (C, D) \;\Leftrightarrow\; A \cap D \subseteq B \cap C. \tag{2.6}$$

Another consequence of $(A, B) \le (C, D)$ is that $A \cap B \subseteq C$ and $C \cap D \subseteq B$. So both separators live entirely on one side of the other separation.

A separation (A, B) is *tight* if every vertex of $A \cap B$ has a neighbour in $A \setminus B$ and another neighbour in $B \setminus A$. For tight separations, one can establish that $(A, B) \le (C, D)$ by checking only one of the two inclusions in (2.1):

If (A, B) *and* (C, D) *are separations such that* $A \subseteq C$ *and* (C, D) *is tight, then* $(A, B) \le (C, D)$. (2.7)

Indeed, suppose $D \nsubseteq B$. Then as $A \subseteq C$, there is a vertex $x \in (C \cap D) \setminus B$. As (C, D) is tight, x has a neighbour $y \in D \setminus C$, but since $x \in A \setminus B$ we see that $y \in A$. So $A \setminus C \neq \emptyset$, contradicting our assumption.

Let us call (A, B) and (C, D) *nested*, and write $(A, B) \| (C, D)$, if (A, B) is comparable with (C, D) or with (D, C) under $\leq$. By (2.2), this is a symmetrical relation. For example, we saw in (2.4) and (2.5) that the corner separations of a cross-diagram are nested with the two separations forming it, as well as with each other.

Separations (A, B) and (C, D) that are not nested are said to *cross*; we then write $(A, B) \nparallel (C, D)$.

Nestedness is invariant under 'flipping' a separation: if $(A, B) \| (C, D)$ then also $(A, B) \| (D, C)$, by definition of $\|$, but also $(B, A) \| (C, D)$ by (2.2). Thus although nestedness is defined on the separations of G, we may think of it as a symmetrical relation on the unordered pairs $\{A, B\}$ such that (A, B) is a separation.

By (2.6), nested separations have a simple description in terms of cross-diagrams:

Two separations are nested if and only if one of their four corners has an empty interior and empty links. (2.8)

In particular:

Neither of two nested separations separates the separator of the other. (2.9)

The converse of (2.9) fails only if there is a corner with a non-empty interior whose links are both empty.

Although nestedness is reflexive and symmetric, it is not in general transitive. However when transitivity fails, we can still say something:

Lemma 2.1.2. *If $(A, B) \| (C, D)$ and $(C, D) \| (E, F)$ but $(A, B) \nparallel (E, F)$, then (C, D) is nested with every corner separation of $\{(A, B), (E, F)\}$, and for one corner separation (I, J) we have either $(C, D) \leq (I, J)$ or $(D, C) \leq (I, J)$.*

Proof. Changing notation as necessary, we may assume that $(A, B) \leq (C, D)$, and that (C, D) is comparable with (E, F).[6] If $(C, D) \leq (E, F)$ we have $(A, B) \leq (E, F)$, contrary to our assumption. Hence $(C, D) \geq (E, F)$, or equivalently by (2.2), $(D, C) \leq (F, E)$. As also $(D, C) \leq (B, A)$, we thus have $D \subseteq F \cap B$ and $C \supseteq E \cup A$ and therfore

$$(D, C) \leq (F \cap B, E \cup A) \underset{(2.5)}{\leq} (L, K)$$

for each of the other three corner separations (K, L) of $\{(A, B), (E, F)\}$. $\square$

[6] Note that such change of notation will not affect the set of corner separations of the cross-diagram of (A, B) and (E, F), nor the nestedness (or not) of (C, D) with those corner separations.

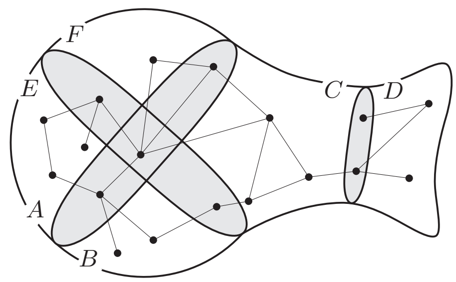


Figure 2.2: Separations as in Lemma 2.1.2

Figure 2.2 shows an example of three separations witnessing the non-transitivity of nestedness. Its main purpose, however, is to illustrate the use of Lemma 2.1.2. We shall often be considering which of two crossing separations, such as (A, B) and (E, F) in the example, we should adopt for a desired collection of nested separations already containing some separations such as (C, D). The lemma then tells us that we can opt to take neither, but instead choose a suitable corner separation.

Note that there are two ways in which three separations can be pairwise nested. One is that they or their inverses form a chain under $\leq$. But there is also another way, which will be important later; this is illustrated in Figure 2.3.

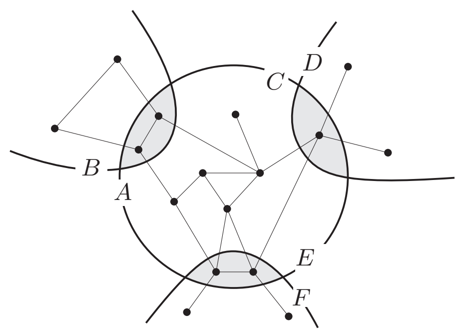


Figure 2.3: Three nested separations not coming from a $\leq$-chain

We need one more lemma.

Lemma 2.1.3. *Let N be a set of separations of G that are pairwise nested. Let (A, B) and (C, D) be two further separations, each nested with all the separations in N. Assume that (A, B) separates an N-block b, and that (C, D) separates an N-block $b' \neq b$. Then $(A, B) \| (C, D)$. Moreover, $A \cap B \subseteq b$ and $C \cap D \subseteq b'$.*

Proof. By Lemma 2.1.1, there is a separation $(E, F) \in N$ with $b \subseteq E$ and $b' \subseteq F$. Suppose $(A, B) \nparallel (C, D)$. By symmetry and Lemma 2.1.2 we may assume that

$$(E, F) \leq (A \cap C, B \cup D).$$

But then $b \subseteq E \subseteq A \cap C \subseteq A$, contradicting the fact that (A, B) separates b. Hence $(A, B) \| (C, D)$, as claimed.

If $A \cap B \not\subseteq b$, then there is a $(K, L) \in N$ which separates $b \cup (A \cap B)$. We may assume that $b \subseteq L$ and that $A \cap B \not\subseteq L$. The latter implies that $(K, L) \not\leq (A, B)$ and $(K, L) \not\leq (B, A)$. So $(K, L) \| (A, B)$ implies that either $(L, K) \leq (A, B)$ or $(L, K) \leq (B, A)$. Thus $b \subseteq L \subseteq A$ or $b \subseteq L \subseteq B$, a contradiction to the fact that (A, B) separates b. Similarly we obtain $C \cap D \subseteq b'$. □

2.1.3 Nested separation systems and tree structure

A set S of separations is *symmetric* if $(A, B) \in S$ implies $(B, A) \in S$, and *nested* if every two separations in S are nested. Any symmetric set of proper separations is a *separation system*. Throughout this section and the next, we consider a fixed nested separation system N of our graph G.

Our aim in this section will be to describe N by way of a *structure tree* $T = T(N)$, whose edges will correspond to the separations in N. Its nodes[7] will correspond to subgraphs of G. Every automorphism of G that leaves N invariant will also act on T. Although our notion of a separation system differs from that of Dunwoody and Krön [61, 60], the main ideas of how to describe a nested system by a structure tree can already be found there.

Our main task in the construction of T will be to define its nodes. They will be the equivalence classes of the following equivalence relation $\sim$ on N, induced by the ordering $\leq$ from (2.1):

$$(A, B) \sim (C, D) \ :\Leftrightarrow \begin{cases} (A, B) = (C, D) \text{ or} \\ (B, A) \text{ is a predecessor of } (C, D) \text{ in } (N, \leq). \end{cases} \tag{2.10}$$

(Recall that, in a partial order $(P, \leq)$, an element $x \in P$ is a *predecessor* of an element $z \in P$ if $x < z$ but there is no $y \in P$ with $x < y < z$.)

Before we prove that this is indeed an equivalence relation, it may help to look at an example: the set of vertices in the centre of Figure 2.3 will be the node of T represented by each of the equivalent nested separations (A, B), (C, D) and (E, F).

Lemma 2.1.4. *The relation $\sim$ is an equivalence relation on N.*

Proof. Reflexivity holds by definition, and symmetry follows from (2.2). To show transitivity assume that $(A, B) \sim (C, D)$ and $(C, D) \sim (E, F)$, and that all these separations are distinct. Thus,

(i) (B, A) is a predecessor of (C, D);

(ii) (D, C) is a predecessor of (E, F).

And by (2.2) also

(iii) (D, C) is a predecessor of (A, B);

(iv) (F, E) is a predecessor of (C, D).

[7] While our graphs G have *vertices*, structure trees will have *nodes*.

By (ii) and (iii), (A, B) is incomparable with (E, F). Hence, since N is nested, (B, A) is comparable with (E, F). If $(E, F) \leq (B, A)$ then by (i) and (ii), $(D, C) \leq (C, D)$, which contradicts (2.3) (recall that all separations in a separation system are required to be proper). Thus $(B, A) < (E, F)$, as desired.

Suppose there is a separation $(X, Y) \in N$ with $(B, A) < (X, Y) < (E, F)$. As N is nested, (X, Y) is comparable with either (C, D) or (D, C). By (i) and (ii), $(X, Y) \not< (C, D)$ and $(D, C) \not< (X, Y)$. Now if $(C, D) \leq (X, Y) < (E, F)$ then by (iv), (C, D) is comparable to both (E, F) and (F, E), contradicting (2.3). Finally, if $(D, C) \geq (X, Y) > (B, A)$, then by (iii), (D, C) is comparable to both (B, A) and (A, B), again contradicting (2.3). We have thus shown that (B, A) is a predecessor of (E, F), implying that $(A, B) \sim (E, F)$ as claimed. □

Note that, by (2.3), the definition of equivalence implies:

Distinct equivalent proper separations are incomparable under $\leq$. (2.11)

We can now define the nodes of $T = T(N)$ as planned, as the equivalence classes of $\sim$:

$$V(T) := \big\{[(A, B)] : (A, B) \in N\big\}.$$

Having defined the nodes of T, let us define its edges. For every separation $(A, B) \in N$ we shall have one edge, joining the nodes represented by (A, B) and (B, A), respectively. To facilitate notation later, we formally give T the abstract edge set

$$E(T) := \big\{\{(A, B), (B, A)\} \mid (A, B) \in N\big\}$$

and declare an edge e to be incident with a node $\mathcal{X} \in V(T)$ whenever $e \cap \mathcal{X} \neq \emptyset$ (so that the edge $\{(A, B), (B, A)\}$ of T joins its nodes $[(A, B)]$ and $[(B, A)]$). We have thus, so far, defined a multigraph T.

As $(A, B) \not\sim (B, A)$ by definition of $\sim$, our multigraph T has no loops. Whenever an edge e is incident with a node $\mathcal{X}$, the non-empty set $e \cap \mathcal{X}$ that witnesses this is a singleton set containing one separation. We denote this separation by $(e \cap \mathcal{X})$. Every separation $(A, B) \in N$ occurs as such an $(e \cap \mathcal{X})$, with $\mathcal{X} = [(A, B)]$ and $e = \{(A, B), (B, A)\}$. Thus,

Every node $\mathcal{X}$ of T is the set of all the separations $(e \cap \mathcal{X})$ such that e is incident with $\mathcal{X}$. In particular, $\mathcal{X}$ has degree $|\mathcal{X}|$ in T. (2.12)

Our next aim is to show that T is a tree.

Lemma 2.1.5. *Let $W = \mathcal{X}_1 e_1 \mathcal{X}_2 e_2 \mathcal{X}_3$ be a walk in T with $e_1 \neq e_2$. Then $(e_1 \cap \mathcal{X}_1)$ is a predecessor of $(e_2 \cap \mathcal{X}_2)$.*

Proof. Let $(e_1 \cap \mathcal{X}_1) = (A, B)$ and $(e_2 \cap \mathcal{X}_2) = (C, D)$. Then $(B, A) = (e_1 \cap \mathcal{X}_2)$ and $(B, A) \sim (C, D)$. Since $e_1 \neq e_2$ we have $(B, A) \neq (C, D)$. Thus, (A, B) is a predecessor of (C, D) by definition of $\sim$. □

And conversely:

Lemma 2.1.6. *Let $(E_0, F_0), \ldots, (E_k, F_k)$ be separations in N such that each (E_{i-1}, F_{i-1}) is a predecessor of (E_i, F_i) in $(N, \leq)$. Then $[(E_0, F_0)], \ldots, [(E_k, F_k)]$ are the nodes of a walk in T, in this order.*

Proof. By definition of $\sim$, we know that $(F_{i-1}, E_{i-1}) \sim (E_i, F_i)$. Hence for all $i = 1, \ldots, k$, the edge $\{(E_{i-1}, F_{i-1}), (F_{i-1}, E_{i-1})\}$ of T joins the node $[(E_{i-1}, F_{i-1})]$ to the node $[(E_i, F_i)] = [(F_{i-1}, E_{i-1})]$. □

Theorem 2.1.7. *The multigraph $T(N)$ is a tree.*

Proof. We have seen that T is loopless. Suppose that T contains a cycle $\mathcal{X}_1 e_1 \cdots \mathcal{X}_{k-1} e_{k-1} \mathcal{X}_k$, with $\mathcal{X}_1 = \mathcal{X}_k$ and $k > 2$. Applying Lemma 2.1.5 $(k-1)$ times yields

$$(A, B) := (e_1 \cap \mathcal{X}_1) < \ldots < (e_{k-1} \cap \mathcal{X}_{k-1}) < (e_1 \cap \mathcal{X}_k) = (A, B),$$

a contradiction. Thus, T is acyclic; in particular, it has no parallel edges.

It remains to show that T contains a path between any two given nodes $[(A, B)]$ and $[(C, D)]$. As N is nested, we know that (A, B) is comparable with either (C, D) or (D, C). Since $[(C, D)]$ and $[(D, C)]$ are adjacent, it suffices to construct a walk between $[(A, B)]$ and one of them. Swapping the names for C and D if necessary, we may thus assume that (A, B) is comparable with (C, D). Reversing the direction of our walk if necessary, we may further assume that $(A, B) < (C, D)$. Since our graph G is finite, there is a chain

$$(A, B) = (E_0, F_0) < \cdots < (E_k, F_k) = (C, D)$$

such that (E_{i-1}, F_{i-1}) is a predecessor of (E_i, F_i), for every $i = 1, \ldots, k$. By Lemma 2.1.6, T contains the desired path from $[(A, B)]$ to $[(C, D)]$. □

Corollary 2.1.8. *If N is invariant under a group $\Gamma \leq \mathrm{Aut}(G)$ of automorphisms of G, then Γ also acts on T as a group of automorphisms.*

Proof. Any automorphism α of G maps separations to separations, and preserves their partial ordering defined in (2.1). If both α and α^{-1} map separations from N to separations in N, then α also preserves the equivalence of separations under $\sim$. Hence Γ, as stated, acts on the nodes of T and preserves their adjacencies and non-adjacencies. □

2.1.4 From structure trees to tree-decompositions

Throughout this section, N continues to be an arbitrary nested separation system of our graph G. Our aim now is to show that G has a tree-decomposition, in the sense of Robertson and Seymour, with the structure tree $T = T(N)$ defined in Section 2.1.3 as its decomposition tree. The separations of G associated with the edges of this decomposition tree[8] will be precisely the separations in N identified by those edges in the original definition of T.

Recall that a *tree-decomposition* of G is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(G)$, one for every node of T, such that:

[8] as in the theory of tree-decompositions, see e.g. [52, Lemma 12.3.1]

(T1) $V(G) = \bigcup_{t \in T} V_t$;

(T2) for every edge $e \in G$ there exists a $t \in T$ such that both ends of e lie in V_t;

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the t_1–t_3 path in T.

To define our desired tree-decomposition $(T, \mathcal{V})$, we thus have to define the family $\mathcal{V} = (V_{\mathcal{X}})_{\mathcal{X} \in V(T)}$ of its parts: with every node $\mathcal{X}$ of T we have to associate a set $V_{\mathcal{X}}$ of vertices of G. We define these as follows:

$$V_{\mathcal{X}} := \bigcap \{ A \mid (A, B) \in \mathcal{X} \} \tag{2.13}$$

Example 2.1.9. Assume that G is connected, and consider as N the nested set of all proper 1-separations (A, B) and (B, A) such that $A \setminus B$ is connected in G. Then T is very similar to the block-cutvertex tree of G: its nodes will be the blocks in the usual sense (maximal 2-connected subgraphs or bridges) plus those cutvertices that lie in at least three blocks.

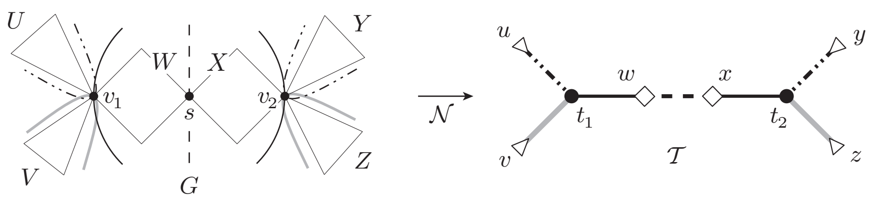


Figure 2.4: T has an edge for every separation in N. Its nodes correspond to the blocks and some of the cutvertices of G.

In Figure 2.4, this separation system N contains all the proper 1-separations of G. The separation (A, B) defined by the cutvertex s, with $A := U \cup V \cup W$ and $B := X \cup Y \cup Z$ say, defines the edge $\{(A, B), (B, A)\}$ of T joining its nodes $w = [(A, B)]$ and $x = [(B, A)]$.

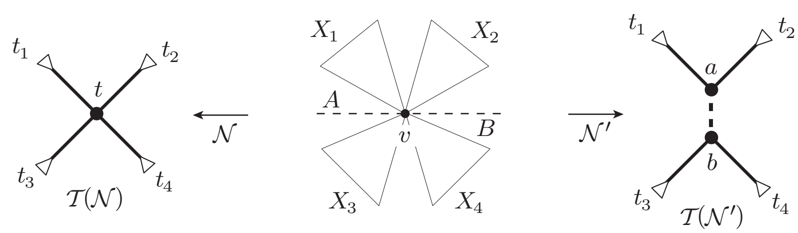


Figure 2.5: $T' = T(N')$ has distinct nodes a, b whose parts in the tree-decomposition $(T', \mathcal{V})$ coincide: $V_a = \{v\} = V_b$.

In Figure 2.5 we can add to N one of the two crossing 1-separations not in N (together with its inverse), to obtain a set N' of separations that is still nested. For example, let

$$N' := N \cup \{(A, B), (B, A)\}$$

with $A := X_1 \cup X_2$ and $B := X_3 \cup X_4$. This causes the central node $\mathcal{X}$ of T to split into two nodes $a = [(A, B)]$ and $b = [(B, A)]$ joined by the new edge $\{(A, B), (B, A)\}$. However the new nodes a, b still define the same part of the tree-decomposition of G as t did before: $V_a = V_b = V_t = \{v\}$.

Before we prove that $(T, \mathcal{V})$ is indeed a tree-decomposition, let us collect some information about its parts $V_{\mathcal{X}}$, the vertex sets defined in (2.13).

Lemma 2.1.10. *Every $V_{\mathcal{X}}$ is N-inseparable.*

Proof. Let us show that a given separation $(C, D) \in N$ does not separate V_t. Pick $(A, B) \in \mathcal{X}$. Since N is nested, and swapping the names of C and D if necessary, we may assume that (A, B) is $\leq$-comparable with (C, D). If $(A, B) \leq (C, D)$ then $V_t \subseteq A \subseteq C$, so (C, D) does not separate V_t. If $(C, D) < (A, B)$, there is a $\leq$-predecessor (E, F) of (A, B) with $(C, D) \leq (E, F)$. Then $(F, E) \sim (A, B)$ and hence $V_t \subseteq F \subseteq D$, so again (C, D) does not separate V_t. $\square$

The sets $V_{\mathcal{X}}$ will come in two types: they can be

- N-blocks (that is, maximal N-inseparable sets of vertices), or
- 'hubs' (defined below).

Nodes $\mathcal{X} \in T$ such that V_t is an N-block are *block nodes*. A node $\mathcal{X} \in T$ such that $V_t = A \cap B$ for some $(A, B) \in t$ is a *hub node* (and V_t a *hub*).

In Example 2.1.9, the N-blocks were the (usual) blocks of G; the hubs were singleton sets consisting of a cutvertex. Example 2.1.15 will show that $\mathcal{X}$ can be a hub node and a block node at the same time. Every hub is a subset of a block: by (2.9), hubs are N-inseparable, so they extend to maximal N-inseparable sets.

Hubs can contain each other properly (Example 2.1.15 below). But a hub V_t cannot be properly contained in a separator $A \cap B$ of any $(A, B) \in t$. Let us prove this without assuming that V_t is a hub:

Lemma 2.1.11. *Whenever $(A, B) \in t \in T$, we have $A \cap B \subseteq V_t$. In particular, if $V_t \subseteq A \cap B$, then $V_t = A \cap B$ is a hub with hub node t.*

Proof. Consider any vertex $v \in (A \cap B) \setminus V_t$. By definition of V_t, there exists a separation $(C, D) \in t$ such that $v \notin C$. This contradicts the fact that $B \subseteq C$ since $(A, B) \sim (C, D)$. $\square$

Lemma 2.1.12. *Every node of T is either a block node or a hub node.*

Proof. Suppose $\mathcal{X} \in T$ is not a hub node; we show that $\mathcal{X}$ is a block node. By Lemma 2.1.10, $V_{\mathcal{X}}$ is N-inseparable. We show that V_t is maximal in $V(G)$ with this property: that for every vertex $x \notin V_t$ the set $V_t \cup \{x\}$ is not N-inseparable.

By definition of V_t, any vertex $x \notin V_t$ lies in $B \setminus A$ for some $(A, B) \in t$. Since $\mathcal{X}$ is not a hub node, Lemma 2.1.11 implies that $V_t \not\subseteq A \cap B$. As $V_t \subseteq A$, this means that V_t has a vertex in $A \setminus B$. Hence (A, B) separates $V_t \cup \{x\}$, as desired. $\square$

Conversely, all the N-blocks of G will be parts of our tree-decomposition:

Lemma 2.1.13. *Every N-block is the set $V_{\mathcal{X}}$ for a node $\mathcal{X}$ of T.*

Proof. Consider an arbitrary N-block b.

Suppose first that b is small. Then there exists a separation $(A, B) \in N$ with $b \subseteq A \cap B$. As N is nested, $A \cap B$ is N-inseparable by (2.9), so in fact $b = A \cap B$ by the maximality of b. We show that $b = V_t$ for $t = [(A, B)]$. By Lemma 2.1.11, it suffices to show that $V_t \subseteq b = A \cap B$. As $V_t \subseteq A$ by definition of V_t, we only need to show that $V_t \subseteq B$. Suppose there is an $x \in V_t \setminus B$. As $x \notin A \cap B = b$, the maximality of b implies that there exists a separation $(E, F) \in N$ such that

$$F \not\supseteq b \subseteq E \text{ and } x \in F \setminus E \tag{$*$}$$

(compare the proof of Lemma 2.1.1). By $(*)$, all corners of the cross-diagram $\{(A, B), (E, F)\}$ other than $B \cap F$ contain vertices not in the centre. Hence by (2.8), the only way in which (A, B) and (E, F) can be nested is that $B \cap F$ does lie in the centre, i.e. that $(B, A) \leq (E, F)$. Since $(B, A) \neq (E, F)$, by $(*)$ and $b = A \cap B$, this means that (B, A) has a successor $(C, D) \leq (E, F)$. But then $(C, D) \sim (A, B)$ and $x \notin E \supseteq C \supseteq V_t$, a contradiction.

Suppose now that b is not small. We shall prove that $b = V_t$ for $t = t(b)$, where $t(b)$ is defined as the set of separations (A, B) that are minimal with $b \subseteq A$. Let us show first that $t(b)$ is indeed an equivalence class, i.e., that the separations in $t(b)$ are equivalent to each other but not to any other separation in N.

Given distinct $(A, B), (C, D) \in \mathcal{X}(b)$, let us show that $(A, B) \sim (C, D)$. Since both (A, B) and (C, D) are minimal as in the definition of $t(b)$, they are incomparable. But as elements of N they are nested, so (A, B) is comparable with (D, C). If $(A, B) \leq (D, C)$ then $b \subseteq A \cap C \subseteq D \cap C$, which contradicts our assumption that b is not small. Hence $(D, C) < (A, B)$. To show that (D, C) is a predecessor of (A, B), suppose there exists a separation $(E, F) \in N$ such that $(D, C) < (E, F) < (A, B)$. This contradicts the minimality either of (A, B), if $b \subseteq E$, or of (C, D), if $b \subseteq F$. Thus, $(C, D) \sim (A, B)$ as desired.

Conversely, we have to show that every $(E, F) \in N$ equivalent to some $(A, B) \in t(b)$ also lies in $t(b)$. As $(E, F) \sim (A, B)$, we may assume that $(F, E) < (A, B)$. Then $b \not\subseteq F$ by the minimality of (A, B) as an element of $t(b)$, so $b \subseteq E$. To show that (E, F) is minimal with this property, suppose that $b \subseteq X$ also for some $(X, Y) \in N$ with $(X, Y) < (E, F)$. Then (X, Y) is incomparable with (A, B): by (2.11) we cannot have $(A, B) \leq (X, Y) < (E, F)$, and we cannot have $(X, Y) < (A, B)$ by the minimality of (A, B) as an element of $t(b)$. But $(X, Y) \| (A, B)$, so (X, Y) must be comparable with (B, A). Yet if $(X, Y) \leq (B, A)$, then $b \subseteq X \cap A \subseteq B \cap A$, contradicting our assumption that b is not small, while $(B, A) < (X, Y) < (E, F)$ is impossible, since (B, A) is a predecessor of (E, F).

Hence $t(b)$ is indeed an equivalence class, i.e., $t(b) \in V(T)$. By definition of $t(b)$, we have $b \subseteq \bigcap \{ A \mid (A, B) \in t(b) \} = V_{t(b)}$. The converse inclusion follows from the maximality of b as an N-inseparable set. $\square$

We have seen so far that the parts V_t of our intended tree-decomposition associated with N are all the N-blocks of G, plus some hubs. The following proposition shows what has earned them their name:

Proposition 2.1.14. *A hub node $\mathcal{X}$ has degree at least 3 in T, unless it has the form $t = \{(A,B),(C,D)\}$ with $A \supsetneq D$ and $B = C$ (in which case it has degree 2).*

Proof. Let $(A,B) \in t$ be such that $V_t = A \cap B$. As $(A,B) \in t$ but $V_t \neq A$, we have $d(t) = |t| \geq 2$; cf. (2.12). Suppose that $d(t) = 2$, say $t = \{(A,B),(C,D)\}$. Then $B \subseteq C$ by definition of $\sim$, and $C \setminus B = (C \cap A) \setminus B = V_t \setminus B = \emptyset$ by definition of V_t and $V_t \subseteq A \cap B$. So $B = C$. As (A,B) and (C,D) are equivalent but not equal, this implies $D \subsetneq A$. $\square$

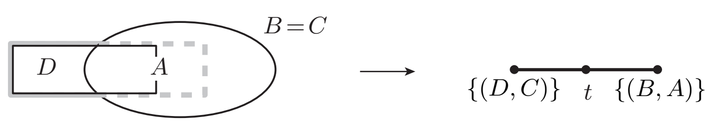


Figure 2.6: A hub node $t = \{(A,B),(C,D)\}$ of degree 2

Figure 2.6 shows that the exceptional situation from Proposition 2.1.14 can indeed occur. In the example, we have $N = \{(A,B),(B,A),(C,D),(D,C)\}$ with $B = C$ and $D \subsetneq A$. The structure tree T is a path between two block nodes $\{(D,C)\}$ and $\{(B,A)\}$ with a central hub node $\mathcal{X} = \{(A,B),(C,D)\}$, whose set $V_{\mathcal{X}} = A \cap B$ is not a block since it is properly contained in the N-inseparable set $B = C$.

Our last example answers some further questions about the possible relationships between blocks and hubs that will naturally come to mind:

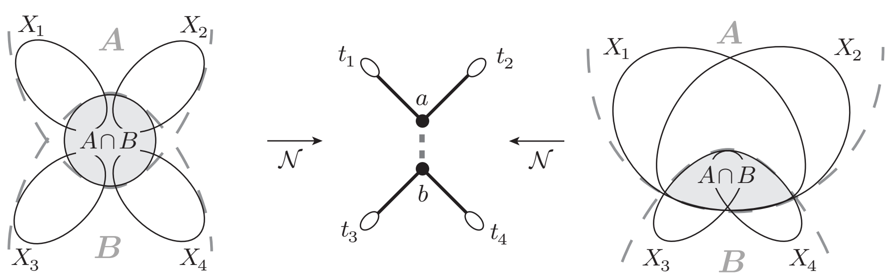


Figure 2.7: The two nested separation systems of Example 2.1.15, and their common structure tree

Example 2.1.15. Consider the vertex sets $X_1, \ldots, X_4$ shown on the left of Figure 2.7. Let A be a superset of $X_1 \cup X_2$ and B a superset of $X_3 \cup X_4$, so that $A \cap B \not\subseteq X_1 \cup \cdots \cup X_4$ and different X_i do not meet outside $A \cap B$. Let N consist of (A,B), (B,A), and $(X_1,Y_1), \ldots, (X_4,Y_4)$ and their inverses (Y_i,X_i), where $Y_i := (A \cap B) \cup \bigcup_{j \neq i} X_j$. The structure tree $T = T(N)$ has four block nodes $\mathcal{X}_1, \dots, \mathcal{X}_4$, with $t_i = [(X_i, Y_i)]$ and $V_{\mathcal{X}_i} = X_i$, and two central hub nodes

$$a = \{(A, B), (Y_1, X_1), (Y_2, X_2)\} \quad \text{and} \quad b = \{(B, A), (Y_3, X_3), (Y_4, X_4)\}$$

joined by the edge $\{(A, B), (B, A)\}$. The hubs corresponding to a and b coincide: they are $V_a = A \cap B = V_b$, which is also a block.

Let us now modify this example by enlarging X_1 and X_2 so that they meet outside $A \cap B$ and each contain $A \cap B$. Thus, $A = X_1 \cup X_2$. Let us also shrink B a little, down to $B = X_3 \cup X_4$ (Fig. 2.7, right). The structure tree T remains unchanged by these modifications, but the corresponding sets V_t have changed:

$$V_b = A \cap B \subsetneq X_1 \cap X_2 = X_1 \cap Y_1 = X_2 \cap Y_2 = V_a,$$

and neither of them is a block, because both are properly contained in X_1, which is also N-inseparable.

Our next lemma shows that deleting a separation from our nested system N corresponds to contracting an edge in the structure tree $T(N)$. For a separation (A, B) that belongs to different systems, we write $[(A, B)]_N$ to indicate in which system N we are taking the equivalence class.

Lemma 2.1.16. *Given $(A, B) \in N$, the tree $T' := T(N')$ for*

$$N' = N \setminus \{(A, B), (B, A)\}$$

arises from $T = T(N)$ by contracting the edge $e = \{(A, B), (B, A)\}$. The contracted node z of T' satisfies $z = x \cup y \setminus e$ and $V_z = V_x \cup V_y$, where $x = [(A, B)]_N$ and $y = [(B, A)]_N$, and $V(T') \setminus \{z\} = V(T) \setminus \{x, y\}$.[9]

Proof. To see that $V(T') \setminus \{z\} = V(T) \setminus \{x, y\}$ and $z = x \cup y \setminus e$, we have to show for all $(C, D) \in N'$ that $[(C, D)]_N = [(C, D)]_{N'}$ unless $[(C, D)]_N \in \{x, y\}$, in which case $[(C, D)]_{N'} = x \cup y \setminus e$. In other words, we have to show:

> *Two separations $(C, D), (E, F) \in N'$ are equivalent in N' if and only if they are equivalent in N or are both in $x \cup y \setminus e$.* $(*)$

Our further claim that $T' = T/e$, i.e. that the node-edge incidences in T' arise from those in T as defined for graph minors, will follow immediately from the definition of these incidences in T and T'.

Let us prove the backward implication of $(*)$ first. As $N' \subseteq N$, predecessors in $(N, \leq)$ are still predecessors in N', and hence $(C, D) \sim_N (E, F)$ implies $(C, D) \sim_{N'} (E, F)$. Moreover if $(C, D) \in x$ and $(E, F) \in y$ then, in N, (D, C) is a predecessor of (A, B) and (A, B) is a predecessor of (E, F). In N', then, (D, C) is a predecessor of (E, F), since by Lemma 2.1.6 and Theorem 2.1.7

[9]The last identity says more than that there exists a canonical bijection between $V(T') \setminus \{z\}$ and $V(T) \setminus \{x, y\}$: it says that the nodes of $T - \{x, y\}$ and $T' - z$ are the same also as sets of separations.

there is no separation $(A', B') \neq (A, B)$ in N that is both a successor of (D, C) and a predecessor of (E, F). Hence $(C, D) \sim_{N'} (E, F)$.

For the forward implication in $(*)$ note that if (D, C) is a predecessor of (E, F) in N' but not in N, then in N we have a sequence of predecessors $(D, C) < (A, B) < (E, F)$ or $(D, C) < (B, A) < (E, F)$. Then one of (C, D) and (E, F) lies in x and the other in y, as desired.

It remains to show that $V_z = V_x \cup V_y$. Consider the sets

$$x' := x \setminus \{(A, B)\} \quad \text{and} \quad y' := y \setminus \{(B, A)\}\,;$$

then $z = y' \cup x'$. Since all $(E, F) \in x'$ are equivalent to (A, B) but not equal to it, we have $(B, A) \leq (E, F)$ for all those separations. That is,

$$B \ \subseteq \bigcap_{(E,F)\,\in\,x'} E \ = V_{x'}. \tag{2.14}$$

By definition of V_x we have $V_x = V_{x'} \cap A$. Hence (2.14) yields $V_{x'} = V_x \cup (B \setminus A)$, and since $A \cap B \subseteq V_x$ by Lemma 2.1.11, we have $V_{x'} = V_x \cup B$. An analogous argument yields

$$V_{y'} = \bigcap_{(E,F)\,\in\,y'} E \ = \ V_y \cup A.$$

Hence,

$$\begin{aligned} V_z &= \bigcap_{(E,F)\,\in\,z} E \\ &= V_{x'} \cap V_{y'} \\ &= (V_x \cup B) \cap (V_y \cup A) \\ &= (V_x \cap V_y) \cup (V_x \cap A) \cup (V_y \cap B) \cup (B \cap A) \\ &= (V_x \cap V_y) \cup V_x \cup V_y \cup (B \cap A) \\ &= V_x \cup V_y. \end{aligned}$$

□

Every edge e of T separates T into two components. The vertex sets V_t for the nodes t in these components induce a corresponding separation of G, as in [52, Lemma 12.3.1]. This is the separation that defined e:

Lemma 2.1.17. *Given any separation $(A, B) \in N$, consider the corresponding edge $e = \{(A, B), (B, A)\}$ of $T = T(N)$. Let T_A denote the component of $T - e$ that contains the node $[(A, B)]$, and let T_B be the other component. Then $\bigcup_{t \in T_A} V_t = A$ and $\bigcup_{t \in T_B} V_t = B$.*

Proof. We apply induction on $|E(T)|$. If T consists of a single edge, the assertion is immediate from the definition of T. Assume now that $|E(T)| > 1$. In particular, there is an edge $e^* = xy \neq e$.

Consider $N' := N \setminus e^*$, and let $T' := T(N')$. Then $T' = T/e^*$, by Lemma 2.1.16. Let z be the node of T' contracted from e^*. Define T'_A as the component of $T' - e$

that contains the node $[(A,B)]$, and let T'_B be the other component. We may assume $e^* \in T_A$. Then

$$V(T_A) \setminus \{x,y\} = V(T'_A) \setminus \{z\} \text{ and } V(T_B) = V(T'_B).$$

As $V_z = V_x \cup V_y$ by Lemma 2.1.16, we can use the induction hypothesis to deduce that

$$\bigcup_{t \in T_A} V_t = \bigcup_{t \in T'_A} V_t = A \quad \text{and} \quad \bigcup_{t \in T_B} V_t = \bigcup_{t \in T'_B} V_t = B,$$

as claimed. $\square$

Let us summarize some of our findings from this section. Recall that N is an arbitrary nested separation system of an arbitrary finite graph G. Let $T := T(N)$ be the structure tree associated with N as in Section 2.1.3, and let $\mathcal{V} := (V_t)_{t \in T}$ be defined by (2.13). Let us call the separations of G that correspond as in [52, Lemma 12.3.1] to the edges of the decomposition tree of a tree-decomposition of G the separations *induced by* this tree-decomposition.

Theorem 2.1.18. *The pair $(T, \mathcal{V})$ is a tree-decomposition of G.*

(i) *Every N-block is a part of the decomposition.*

(ii) *Every part of the decomposition is either an N-block or a hub.*

(iii) *The separations of G induced by the decomposition are precisely those in N.*

(iv) *Every $N' \subseteq N$ satsfies $(T', \mathcal{V}') \preccurlyeq (T, \mathcal{V})$ for $T' = T(N')$ and $\mathcal{V}' = V(T')$.*[10]

Proof. Of the three axioms for a tree-decomposition, (T1) and (T2) follow from Lemma 2.1.13, because single vertices and edges form N-inseparable vertex sets, which extend to N-blocks. For the proof of (T3), let $e = \{(A,B),(B,A)\}$ be an edge at t_2 on the t_1–t_3 path in T. Since e separates t_1 from t_3 in T, Lemmas 2.1.17 and 2.1.11 imply that $V_{t_1} \cap V_{t_3} \subseteq A \cap B \subseteq V_{t_2}$.

Statement (i) is Lemma 2.1.13. Assertion (ii) is Lemma 2.1.12. Assertion (iii) follows from Lemma 2.1.17 and the definition of the edges of T. Statement (iv) follows by repeated application of Lemma 2.1.16. $\square$

2.1.5 Extracting nested separation systems

Our aim in this section will be to find inside a given separation system S a nested subsystem N that can still distinguish the elements of some given set $\mathcal{I}$ of S-inseparable sets of vertices. As we saw in Sections 2.1.3 and 2.1.4, such a nested subsystem will then define a tree-decomposition of G, and the sets from $\mathcal{I}$ will come to lie in different parts of that decomposition.

This cannot be done for all choices of S and $\mathcal{I}$. Indeed, consider the following example of where such a nested subsystem does not exist. Let G be the 3×3-grid, let S consist of the two 3-separations cutting along the horizontal and the

[10] See the Introduction for the definition of $(T', \mathcal{V}') \preccurlyeq (T, \mathcal{V})$.

vertical symmetry axis, and let $\mathcal{I}$ consist of the four corners of the resulting cross-diagram. Each of these is S-inseparable, and any two of them can be separated by a separation in S. But since the two separations in S cross, any nested subsystem contains at most one of them, and thus fails to separate some sets from $\mathcal{I}$.

However, we shall prove that the desired nested subsystem does exist if S and $\mathcal{I}$ satisfy the following condition. Given a separation system S and a set $\mathcal{I}$ of S-inseparable sets, let us say that S *separates* $\mathcal{I}$ *well* if the following holds for every pair of crossing – that is, not nested – separations $(A, B), (C, D) \in S$:

> *For all $I_1, I_2 \in \mathcal{I}$ with $I_1 \subseteq A \cap C$ and $I_2 \subseteq B \cap D$ there is an $(E, F) \in S$ such that $I_1 \subseteq E \subseteq A \cap C$ and $F \supseteq B \cup D$.*

Note that such a separation satisfies both $(E, F) \leq (A, B)$ and $(E, F) \leq (C, D)$.

In our grid example, S did not separate $\mathcal{I}$ well, but we can mend this by adding to S the four corner separations. And as soon as we do that, there is a nested subsystem that separates all four corners – for example, the set of the four corner separations.

More abstractly, the idea behind the notion of S separating $\mathcal{I}$ well is as follows. In the process of extracting N from S we may be faced with a pair of crossing separations (A, B) and (C, D) in S that both separate two given sets $I_1, I_2 \in \mathcal{I}$, and wonder which of them to pick for N. (Obviously we cannot choose both.) If S separates $\mathcal{I}$ well, however, we can avoid this dilemma by choosing (E, F) instead: this also separates I_1 from I_2, and since it is nested with both (A, B) and (C, D) it will not prevent us from choosing either of these later too, if desired.

Let us call a separation $(E, F) \in S$ *extremal* in S if for all $(C, D) \in S$ we have either $(E, F) \leq (C, D)$ or $(E, F) \leq (D, C)$. In particular, extremal separations are nested with all other separations in S. Being extremal implies being $\leq$-minimal in S; if S is nested, extremality and $\leq$-minimality are equivalent. If $(E, F) \in S$ is extremal, then E is an S-block; we call it an *extremal block* in S.

A separation system, even a nested one, typically contains many extremal separations. For example, given a tree-decomposition of G with decomposition tree T, the separations corresponding to the edges of T that are incident with a leaf of T are extremal in the (nested) set of all the separations of G corresponding to edges of T.[11]

Our next lemma shows that separating a set $\mathcal{I}$ of S-inseparable sets well is enough to guarantee the existence of an extremal separation among those that separate sets from $\mathcal{I}$. Call a separation $\mathcal{I}$-*relevant* if it weakly separates some two sets in $\mathcal{I}$. If all the separations in S are $\mathcal{I}$-relevant, we call S itself $\mathcal{I}$-*relevant.*

Lemma 2.1.19. *Let R be a separation system that is $\mathcal{I}$-relevant for some set $\mathcal{I}$ of R-inseparable sets. If R separates $\mathcal{I}$ well, then every $\leq$-minimal $(A, B) \in R$ is extremal in R. In particular, if $R \neq \emptyset$ then R contains an extremal separation.*

[11] More precisely, every such edge of T corresponds to an inverse pair of separations of which, usually, only one is extremal: the separation (A, B) for which A is the part V_t with t a leaf of T. The separation (B, A) will not be extremal, unless $T = K^2$.

Proof. Consider a $\leq$-minimal separation $(A, B) \in R$, and let $(C, D) \in R$ be given. If (A, B) and (C, D) are nested, then the minimality of (A, B) implies that $(A, B) \leq (C, D)$ or $(A, B) \leq (D, C)$, as desired. So let us assume that (A, B) and (C, D) cross.

As (A, B) and (C, D) are $\mathcal{I}$-relevant and the sets in $\mathcal{I}$ are R-inseparable, we can find opposite corners of the cross-diagram $\{(A, B), (C, D)\}$ that each contains a set from $\mathcal{I}$. Renaming (C, D) as (D, C) if necessary, we may assume that these sets lie in $A \cap C$ and $B \cap D$, say $I_1 \subseteq A \cap C$ and $I_2 \subseteq B \cap D$. As R separates $\mathcal{I}$ well, there exists $(E, F) \in R$ such that $I_1 \subseteq E \subseteq A \cap C$ and $F \supseteq B \cup D$, and hence $(E, F) \leq (A, B)$ as well as $(E, F) \leq (C, D)$. By the minimality of (A, B), this yields $(A, B) = (E, F) \leq (C, D)$ as desired. $\square$

Let us say that a set S of separations *distinguishes* two given S-inseparable sets I_1, I_2 (or *distinguishes* them *properly*) if it contains a separation that separates them. If it contains a separation that separates them weakly, it *weakly distinguishes* I_1 from I_2. We then also call I_1 and I_2 *(weakly) distinguishable* by S, or *(weakly) S-distinguishable*.

Here is our main result for this section:

Theorem 2.1.20. *Let S be any separation system that separates some set $\mathcal{I}$ of S-inseparable sets of vertices well. Then S has a nested $\mathcal{I}$-relevant subsystem $N(S, \mathcal{I}) \subseteq S$ that weakly distinguishes all weakly S-distinguishable sets in $\mathcal{I}$.*

Proof. If $\mathcal{I}$ has no two weakly distinguishable elements, let $N(S, \mathcal{I})$ be empty. Otherwise let $R \subseteq S$ be the subsystem of all $\mathcal{I}$-relevant separations in S. Then $R \neq \emptyset$, and R separates $\mathcal{I}$ well. Let $\mathcal{E} \subseteq R$ be the subset of those separations that are extremal in R, and put

$$\overline{\mathcal{E}} := \{(A, B) \mid (A, B) \text{ or } (B, A) \text{ is in } \mathcal{E}\}.$$

By Lemma 2.1.19 we have $\overline{\mathcal{E}} \neq \emptyset$, and by definition of extremality all separations in $\overline{\mathcal{E}}$ are nested with all separations in R. In particular, $\overline{\mathcal{E}}$ is nested.

Let

$$\mathcal{I}_{\mathcal{E}} := \{I \in \mathcal{I} \mid \exists (E, F) \in \mathcal{E} : I \subseteq E\}.$$

This is non-empty, since $\mathcal{E} \subseteq R$ is non-empty and $\mathcal{I}$-relevant. Let us prove that $\mathcal{E}$ weakly distinguishes all pairs of weakly distinguishable elements $I_1, I_2 \in \mathcal{I}$ with $I_1 \in \mathcal{I}_{\mathcal{E}}$. Pick $(A, B) \in R$ with $I_1 \subseteq A$ and $I_2 \subseteq B$. Since $I_1 \in \mathcal{I}_{\mathcal{E}}$, there is an $(E, F) \in \mathcal{E}$ such that $I_1 \subseteq E$. By the extremality of (E, F) we have either $(E, F) \leq (A, B)$, in which case $I_1 \subseteq E$ and $I_2 \subseteq B \subseteq F$, or we have $(E, F) \leq (B, A)$, in which case $I_1 \subseteq E \cap A \subseteq E \cap F$. In both cases I_1 and I_2 are weakly separated by (E, F).

As $\mathcal{I}' := \mathcal{I} \setminus \mathcal{I}_{\mathcal{E}}$ is a set of S-inseparable sets with fewer elements than $\mathcal{I}$, induction gives us a nested $\mathcal{I}'$-relevant subsystem $N(S, \mathcal{I}')$ of S that weakly distinguishes all weakly distinguishable elements of $\mathcal{I}'$. Then

$$N(S, \mathcal{I}) := \overline{\mathcal{E}} \cup N(S, \mathcal{I}')$$

is $\mathcal{I}$-relevant and weakly distinguishes all weakly distinguishable elements of $\mathcal{I}$. As $\mathcal{I}' \subseteq \mathcal{I}$, and thus $N(S, \mathcal{I}') \subseteq R$, the separations in $\overline{\mathcal{E}}$ are nested with those in $N(S, \mathcal{I}')$. Hence, $N(S, \mathcal{I})$ too is nested. □

An important feature of the proof of Theorem 2.1.20 is that the subset $N(S, \mathcal{I})$ it constructs is *canonical*, given S and $\mathcal{I}$: there are no choices made anywhere in the proof. We may thus think of N as a recursively defined operator that assigns to every pair $(S, \mathcal{I})$ as given in the theorem a certain nested subsystem $N(S, \mathcal{I})$ of S. This subsystem $N(S, \mathcal{I})$ is canonical also in the structural sense that it is invariant under any automorphisms of G that leave S and $\mathcal{I}$ invariant.

To make this more precise, we need some notation. Every automorphism α of G acts also on (the set of) its vertex sets $U \subseteq V(G)$, on the collections $\mathcal{X}$ of such vertex sets, on the separations (A, B) of G, and on the sets S of such separations. We write U^α, $\mathcal{X}^\alpha$, $(A, B)^\alpha$ and S^α and so on for their images under α.

Corollary 2.1.21. *Let S and $\mathcal{I}$ be as in Theorem 2.1.20, and let $N(S, \mathcal{I})$ be the nested subsystem of S constructed in the proof. Then for every automorphism α of G we have $N(S^\alpha, \mathcal{I}^\alpha) = N(S, \mathcal{I})^\alpha$. In particular, if S and $\mathcal{I}$ are invariant under the action of a group Γ of automorphisms of G, then so is $N(S, \mathcal{I})$.*

Proof. The proof of the first assertion is immediate from the construction of $N(S, \mathcal{I})$ from S and $\mathcal{I}$. The second assertion follows, as

$$N(S, \mathcal{I})^\alpha = N(S^\alpha, \mathcal{I}^\alpha) = N(S, \mathcal{I})$$

for every $\alpha \in \Gamma$. □

2.1.6 Separating the k-blocks of a graph

We now apply the theory developed in the previous sections to our original problem, of how to 'decompose a graph G into its $(k+1)$-connected components'. In the language of Section 2.1.5, we consider as S the set of all proper k-separations of G, and as $\mathcal{I}$ the set of its k-blocks. Our results from Section 2.1.5 rest on the assumption that the set R of $\mathcal{I}$-relevant separations in S separates $\mathcal{I}$ well (Lemma 2.1.19). So the first thing we have to ask is: given crossing k-separations (A, B) and (C, D) such that $A \cap C$ and $B \cap D$ contain k-blocks b_1 and b_2, respectively, is there a k-separation (E, F) such that $b_1 \subseteq E \subseteq A \cap C$?

If G is k-connected, there clearly is. Indeed, as the corners $A \cap C$ and $B \cap D$ each contain a k-block, they have order at least $k+1$, so their boundaries cannot have size less than k. But the sizes of these two corner boundaries sum to $|A \cap B| + |C \cap D| = 2k$, so they are both exactly k. We can thus take as (E, F) the corner separation $(A \cap C, B \cup D)$.

If G is not k-connected, we shall need another reason for these corner separations to have order at least k. This is a non-trivial problem. Our solution will be to assume inductively that those k-blocks that can be separated by a separation of order $\ell < k$ are already separated by such a separation selected earlier in the induction. Then the two corner separations considered above will have order at least k, since the k-blocks in the two corners are assumed not to have been separated earlier.

This approach differs only slightly from the more ambitious approach to build, inductively on ℓ, one nested set of separations which, for all ℓ at once, distinguishes every two ℓ-blocks by a separation of order at most ℓ. We shall construct an example showing that such a unified nested separation system need not exist. The subtle difference between our approach and this seemingly more natural generalization is that we use ℓ-separations for $\ell < k$ only with the aim to separate k-blocks; we do not aspire to separate all ℓ-blocks, including those that contain no k-block.

However we shall be able to prove that the above example is essentially the only one precluding the existence of a unified nested set of separations. Under a mild additional assumption saying that all blocks considered must be 'robust', we shall obtain one unified nested set of separations that distinguishes, for all ℓ simultaneously, all ℓ-blocks by a separation of order at most ℓ. All ℓ-blocks that have size at least $\frac{3}{2}\ell$ will be robust.

Once we have found our nested separation systems, we shall convert them into tree-decompositions as in Section 2.1.4. Both our separation systems and our tree-decompositions will be canonical in that they depend only on the structure of G. In particular, they will be invariant under the automorphism group $\mathrm{Aut}(G)$ of G.

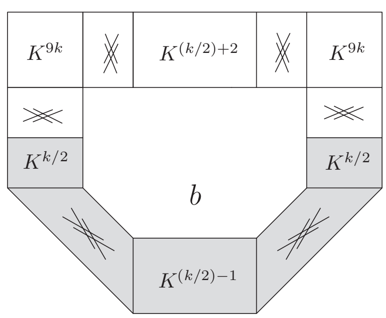


Figure 2.8: A horizontal k-separation needed to distinguish two k-blocks, crossed by a vertical $(k+1)$-separation needed to distinguish two $(k+1)$-blocks.

Let us now turn to our example showing that a graph need not have a 'unified' nested separation system N of separations of mixed order that distinguishes, for every ℓ, distinct ℓ-blocks by a separation in N of order at most ℓ. The graph depicted in Figure 2.8 arises from the disjoint union of a $K^{(k/2)-1}$, two $K^{k/2}$, a $K^{(k/2)+2}$ and two K^{9k}, by joining the $K^{(k/2)-1}$ completely to the two $K^{k/2}$, the $K^{(k/2)+2}$ completely to the two K^{9k}, the left $K^{k/2}$ completely to the left K^{9k}, and the right $K^{k/2}$ completely to the right K^{9k}. The horizontal k-separator consisting of the two $K^{k/2}$ defines the only separation of order at most k that distinguishes the two k-blocks consisting of the top five complete graphs versus the bottom three. On the other hand, the vertical $(k+1)$-separator consisting of the $K^{(k/2)-1}$ and the $K^{(k/2)+2}$ defines the only separation of order at most $(k+1)$ that distinguishes the two $(k+1)$-blocks consisting, respectively, of the left $K^{k/2}$ and K^{9k} and the $K^{(k/2)+2}$, and of the right $K^{k/2}$ and K^{9k} and the $K^{(k/2)+2}$. Hence any separation system that distinguishes all k-blocks as well as all $(k+1)$-blocks must contain both separations. Since the two separations cross, such a system cannot be nested.

In view of this example it may be surprising that we can find a separation system that distinguishes, for all $\ell \geq 0$ simultaneously, all *large* ℓ-blocks of G, those with at least $\lfloor \frac{3}{2}\ell \rfloor$ vertices. The example of Figure 2.8 shows that this value is best possible: here, all blocks are large except for the k-block b consisting of the two $K^{k/2}$ and the $K^{(k/2)-1}$, which has size $\frac{3}{2}k - 1$.

Indeed, we shall prove something considerably stronger: that the only obstruction to the existence of a unified tree-decomposition is a k-block that is not only not large but positioned exactly like b in Figure 2.8, inside the union of a k-separator and a larger separator crossing it.

Given integers k and K (where $k \leq K$ is the interesting case, but it is important formally to allow $k > K$), a k-inseparable set U is called *K-robust*[12] if for every k-separation (C, D) with $U \subseteq D$ and every separation (A, B) of order at most K such that $(A, B) \nparallel (C, D)$ and

$$|\partial(A \cap D)| < k > |\partial(B \cap D)|\,, \tag{2.15}$$

we have either $U \subseteq A$ or $U \subseteq B$. By $U \subseteq D$ and (2.15), the only way in which this can fail is that $|A \cap B| > k$ and U is contained in the union T of the boundaries of $A \cap D$ and $B \cap D$ (Fig. 2.9): exactly the situation of b in Figure 2.8.

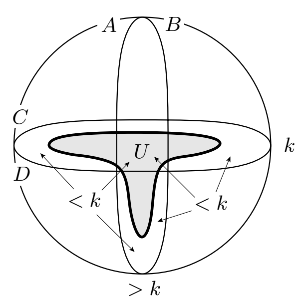


Figure 2.9: The shaded set U is k-inseparable but not K-robust.

It is obvious from the definition of robustness that

$$\textit{for } k \geq K\textit{, every } k\textit{-inseparable set is } K\textit{-robust.} \tag{2.16}$$

Let us call a k-inseparable set, in particular a k-block of G, *robust* if it is K-robust for every K (equivalently, for $K = |G|$). Our next lemma says that

[12]The parameter k is important here, too, but we suppress it for readability; it will always be stated explicitly in the context.

large k-blocks, those of size at least $\lfloor \frac{3}{2}k \rfloor$, are robust. But there are more kinds of robust sets than these: the vertex set of any K^{k+1} subgraph, for example, is a robust k-inseparable set.

Lemma 2.1.22. *Large k-blocks are robust.*

Proof. By the remark following the definition of 'K-robust', it suffices to show that the set $T = \partial(A \cap D) \cup \partial(B \cap D)$ in Figure 2.9 has size at most $\frac{3}{2}k - 1$, regardless of the order of (A, B). Let $\ell := |(A \cap B) \setminus C|$ be the size of the common link of the corners $A \cap D$ and $B \cap D$. By $|C \cap D| = k$ and (2.15) we have $2\ell \leq k - 2$, so $|T| = k + \ell \leq \frac{3}{2}k - 1$ as desired. □

For the remainder of this paper, a *block* of G is again a subset of $V(G)$ that is a k-block for some k. The smallest k for which a block b is a k-block is its *rank*; let us denote this by $r(b)$. A block b that is given without a specified k is called *K-robust* if it is K-robust as an $r(b)$-inseparable set. When we speak of a 'robust k-block' b, however, we mean the (stronger, see below) robustness as a k-inseparable set, not just as an $r(b)$-inseparable set.

It is not difficult to find examples of K-robust blocks that are k-blocks but are not K-robust as a k-block, only as an ℓ-block for some $\ell < k$. A k-inseparable set that is K-robust as a k'-inseparable set for $k' > k$, however, is also K-robust as a k-inseparable set. More generally:

Lemma 2.1.23. *Let k, k' and K be integers.*

(i) *Every k-inseparable set I containing a K-robust k'-inseparable set I' with $k \leq k'$ is K-robust.*

(ii) *Every block b that contains a K-robust block b' is K-robust.*

Proof. (i) Suppose that I is not K-robust, and let this be witnessed by a k-separation (C, D) crossed by a separation (A, B) of order $m \leq K$. Put $S := C \cap D$ and $L := (A \cap B) \setminus C$. Then $I \subseteq S \cup L$, as remarked after the definition of 'K-robust'.

Extend S into L to a k'-set S' that is properly contained in $S \cup L$ (which is large enough, since it contains $I' \subseteq I$), and put $C' := C \cup S'$. Then (C', D) is a k'-separation with separator S' and corners $D \cap A$ and $D \cap B$ with (A, B), whose boundaries by assumption have size less than $k \leq k'$. As I' is K-robust, it lies in one of these corners, say $I' \subseteq A \cap D$. Since

$$|I'| > k' \geq k > |\partial(A \cap D)|,$$

this implies that I' has a vertex in the interior of the corner $A \cap D$. As $I' \subseteq I$, this contradicts the fact that $I \subseteq S \cup L$.

(ii) The block b is an $r(b)$-inseparable set containing the K-robust $r(b')$-inseparable set b'. If $b = b'$ then $r(b) = r(b')$. If $b \supsetneq b'$, then b' is not maximal as an ℓ-inseparable set for any $\ell \leq r(b)$, giving $r(b') > r(b)$. Hence $r(b) \leq r(b')$ either way, so b is a K-robust block by (i). □

Let us call two blocks *distinguishable* if neither contains the other. It is not hard to show that distinguishable blocks b_1, b_2 can be separated in G by a separation of order $r \leq \min\{r(b_1), r(b_2)\}$. We denote the smallest such r by

$$\kappa(b_1, b_2) \leq \min\{r(b_1), r(b_2)\},$$

and say that b_1 and b_2 are *k-distinguishable* for a given integer k if $\kappa(b_1, b_2) \leq k$. Note that distinct k-blocks are k-distinguishable, but they might also be ℓ-distinguishable for some $\ell < k$.

A set S of separations *distinguishes* two k-blocks if it contains a separation of order at most k that separates them. It *distinguishes* two blocks b_1, b_2 given without a specified k if it contains a separation of order $r \leq \min\{r(b_1), r(b_2)\}$ that separates them.[13] If S contains a separation of order $\kappa(b_1, b_2)$ that separates two blocks or k-blocks b_1, b_2, we say that S distinguishes them *efficiently*.

Theorem 2.1.24. *For every finite graph G and every integer $k \geq 0$ there is a tight, nested, and* $\mathrm{Aut}(G)$*-invariant separation system N_k that distinguishes every two k-distinguishable k-robust blocks efficiently. In particular, N_k distinguishes every two k-blocks efficiently.*

Proof. Let us rename the integer k given in the theorem as K. Recursively for all integers $0 \leq k \leq K$ we shall construct a sequence of separation systems N_k with the following properties:

(i) N_k is tight, nested, and $\mathrm{Aut}(G)$-invariant;

(ii) $N_{k-1} \subseteq N_k$ (put $N_{-1} := \emptyset$);

(iii) every separation in $N_k \setminus N_{k-1}$ has order k;

(iv) N_k distinguishes every two K-robust k-blocks.

(v) every separation in $N_k \setminus N_{k-1}$ separates some K-robust k-blocks that are not distinguished by N_{k-1}.

We claim that N_K will satisfy the assertions of the theorem for $k = K$. Indeed, consider two K-distinguishable K-robust blocks b_1, b_2. Then

$$\kappa := \kappa(b_1, b_2) \leq \min\{K, r(b_1), r(b_2)\},$$

so b_1, b_2 are κ-inseparable and extend to distinct κ-blocks b_1', b_2'. These are again K-robust, by Lemma 2.1.23 (i). Hence by (iv), $N_\kappa \subseteq N_K$ distinguishes $b_1' \supseteq b_1$ from $b_2' \supseteq b_2$, and it does so efficiently by definition of κ.

It remains to construct the separation systems N_k.

Let $k \geq 0$ be given, and assume inductively that we already have separation systems $N_{k'}$ satisfying (i)–(v) for $k' = 0, \ldots, k-1$. (For $k = 0$ we have nothing

[13] Unlike in the definition just before Theorem 2.1.20, we no longer require that the blocks we wish to separate be S-inseparable for the entire set S.

but the definiton of $N_{-1} := \emptyset$, which has $V(G)$ as its unique N_{-1}-block.) Let us show the following:

$$\text{For all } 0 \leq \ell \leq k, \text{ any two } K\text{-robust } \ell\text{-blocks } b_1, b_2 \text{ that are not distinguished by } N_{\ell-1} \text{ satisfy } \kappa(b_1, b_2) = \ell. \tag{2.17}$$

This is trivial for $\ell = 0$; let $\ell > 0$. If $\kappa(b_1, b_2) < \ell$, then the $(\ell-1)$-blocks $b_1' \supseteq b_1$ and $b_2' \supseteq b_2$ are distinct. By Lemma 2.1.23 (i) they are again K-robust. Thus by hypothesis (iv) they are distinguished by $N_{\ell-1}$, and hence so are b_1 and b_2, contrary to assumption.

By hypothesis (iii), every k-block is N_{k-1}-inseparable, so it extends to some N_{k-1}-block; let $\mathcal{B}$ denote the set of those N_{k-1}-blocks that contain more than one K-robust k-block. For each $b \in \mathcal{B}$ let $\mathcal{I}_b$ be the set of all K-robust k-blocks contained in b. Let S_b denote the set of all those k-separations of G that separate some two elements of $\mathcal{I}_b$ and are nested with all the separations in N_{k-1}.

Clearly S_b is symmetric and the separations in S_b are proper (since they distinguish two k-blocks), so S_b is a separation system of G. By (2.17) for $\ell = k$, the separations in S_b are tight. Our aim is to apply Theorem 2.1.20 to extract from S_b a nested subsystem N_b that we can add to N_{k-1}.

Before we verify the premise of Theorem 2.1.20, let us prove that it will be useful: that the nested separation system $N_b \subseteq S_b$ it yields can distinguish[14] all the elements of $\mathcal{I}_b$. This will be the case only if S_b does so, so let us prove this first:

Claim 2.1.25. *$(*) S_b$ distinguishes every two elements of $\mathcal{I}_b$. For a proof of $(*)$ we have to find for any two k-blocks $I_1, I_2 \in \mathcal{I}_b$ a separation in S_b that separates them. Applying Lemma 2.1.1 with the set S of all separations of order at most k, we can find a separation $(A, B) \in S$ such that $I_1 \subseteq A$ and $I_2 \subseteq B$. Choose (A, B) so that it is nested with as many separations in N_{k-1} as possible. We prove that $(A, B) \in S_b$, by showing that (A, B) has order exactly k and is nested with every separation $(C, D) \in N_{k-1}$. Let $(C, D) \in N_{k-1}$ be given.*

Being elements of $\mathcal{I}_b$, the sets I_1 and I_2 cannot be separated by fewer than k vertices, by (2.17). Hence (A, B) has order exactly k. Since I_1 is k-inseparable it lies on one side of (C, D), say in C, so $I_1 \subseteq A \cap C$. As (C, D) does not separate I_1 from I_2, we then have $I_2 \subseteq B \cap C$.

Let $\ell < k$ be such that $(C, D) \in N_\ell \setminus N_{\ell-1}$. By hypothesis (v) for ℓ, there are K-robust ℓ-blocks $J_1 \subseteq C$ and $J_2 \subseteq D$ that are not distinguished by $N_{\ell-1}$. By (2.17),

$$\kappa(J_1, J_2) = \ell. \tag{2.18}$$

Let us show that we may assume the following:

$$\text{The corner separations of the corners } A \cap C \text{ and } B \cap C \text{ are nested with every separation } (C', D') \in N_{k-1} \text{ that } (A, B) \text{ is nested with.} \tag{2.19}$$

[14] As the elements of $\mathcal{I}_b$ are k-blocks, we have two notions of 'distinguish' that could apply: the definition given before Theorem 2.1.20, or that given before Theorem 2.1.24. However, as S_b consists of k-separations and all the elements of $\mathcal{I}_b$ are S_b-inseparable, the two notions coincide.

Since (C, D) and (C', D') are both elements of N_{k-1}, they are nested with each other. Thus,

$$(A, B) \| (C', D') \| (C, D).$$

Unless (A, B) is nested with (C, D) (in which case our proof of $(*)$ is complete), this implies by Lemma 2.1.2 that (C', D') is nested with all the corner separations of the cross-diagram for (A, B) and (C, D), especially with those of the corners $A \cap C$ and $B \cap C$ that contain I_1 and I_2. This proves (2.19).

Since the corner separations of $A \cap C$ and $B \cap C$ are nested with the separation $(C, D) \in N_{k-1}$ that (A, B) is not nested with (as we assume), (2.19) *and the choice of (A, B) imply that*

$$|\partial(A \cap C)| \geq k + 1 \quad \text{and} \quad |\partial(B \cap C)| \geq k + 1.$$

Since the sizes of the boundaries of two opposite corners sum to

$$|A \cap B| + |C \cap D| = k + \ell,$$

this means that the boundaries of the corners $A \cap D$ and $B \cap D$ have sizes $< \ell$. Since J_2 is K-robust as an ℓ-block, we thus have $J_2 \subseteq A \cap D$ or $J_2 \subseteq B \cap D$, say the former. But as $J_1 \subseteq C \subseteq B \cup C$, this contradicts (2.18), *completing the proof of $(*)$.*

Let us now verify the premise of Theorem 2.1.20:

Claim 2.1.26. *$(**)$ S_b separates $\mathcal{I}_b$ well. Consider a pair $(A, B), (C, D) \in S_b$ of crossing separations with sets $I_1, I_2 \in \mathcal{I}_b$ such that $I_1 \subseteq A \cap C$ and $I_2 \subseteq B \cap D$. We shall prove that $(A \cap C, B \cup D) \in S_b$.*

By (2.17) *and $I_1, I_2 \in \mathcal{I}_b$, the boundaries of the corners $A \cap C$ and $B \cap D$ have size at least k. Since their sizes sum to $|A \cap B| + |C \cap D| = 2k$, they each have size exactly k. Hence $(A \cap C, B \cup D)$ has order k and is nested with every separation $(C', D') \in N_{k-1}$ by Lemma 2.1.2, because $(A, B), (C, D) \in S_b$ implies that (A, B) and (C, D) are both nested with $(C', D') \in N_{k-1}$. This completes the proof of $(**)$.*

By $(*)$ and $(**)$, Theorem 2.1.20 implies that S_b has a nested $\mathcal{I}_b$-relevant subsystem $N_b := N(S_b, \mathcal{I}_b)$ that weakly distinguishes all the sets in $\mathcal{I}_b$. But these are k-inseparable and hence of size $> k$, so they cannot lie inside a k-separator. So N_b even distinguishes the sets in $\mathcal{I}_b$ properly. Let

$$N_{\mathcal{B}} := \bigcup_{b \in \mathcal{B}} N_b \quad \text{and} \quad N_k := N_{k-1} \cup N_{\mathcal{B}}.$$

Let us verify the inductive statements (i)–(v) *for k. We noted earlier that every S_b is tight, hence so is every N_b. The separations in each N_b are nested with each other and with N_{k-1}. Separations from different sets N_b are nested by Lemma 2.1.3. So the entire set N_k is nested. Since N_{k-1} is $\mathrm{Aut}(G)$-invariant, by hypothesis* (i), *so is $\mathcal{B}$. For every automorphism α and every $b \in \mathcal{B}$ we then have $\mathcal{I}_{b^\alpha} = (\mathcal{I}_b)^\alpha$ and $S_{b^\alpha} = (S_b)^\alpha$, so Corollary 2.1.21 yields $(N_b)^\alpha = N_{b^\alpha}$.*

Thus, $N_{\mathcal{B}}$ *is* Aut(G)*-invariant too, completing the proof of (i). Assertions* (ii) *and* (iii) *hold by definition of* N_k*. Assertion* (iv) *is easy too: if two* K*-robust* k*-blocks are not distinguished by* N_{k-1} *they will lie in the same* N_{k-1}*-block* b*, and hence be distinguished by* N_b*. Assertion* (v) *holds, because each* N_b *is* $\mathcal{I}_b$*-relevant.* □

Let us call two blocks b_1, b_2 of G *robust* if there exists a k for which they are robust k-blocks.[15] For $k = |G|$, Theorem 2.1.24 then yields our 'unified' nested separation system that separates all robust blocks by a separation of the lowest possible order:

Corollary 2.1.27. *For every finite graph* G *there is a tight, nested, and* Aut(G)*-invariant separation system* N *that distinguishes every two distinguishable robust blocks efficiently.* □

Let us now turn the separation systems N_k of Theorem 2.1.24 and its proof into tree-decompositions:

Theorem 2.1.28. *For every finite graph* G *and every integer* K *there is a sequence* $(T_k, \mathcal{V}_k)_{k \le K}$ *of tree-decompositions such that, for all* $k \le K$*,*

(i) *every* k*-inseparable set is contained in a unique part of* $(T_k, \mathcal{V}_k)$;

(ii) *distinct* K*-robust* k*-blocks lie in different parts of* $(T_k, \mathcal{V}_k)$;

(iii) $(T_k, \mathcal{V}_k)$ *has adhesion at most* k ;

(iv) *if* $k > 0$ *then* $(T_{k-1}, \mathcal{V}_{k-1}) \preccurlyeq (T_k, \mathcal{V}_k)$;

(v) Aut(G) *acts on* T_k *as a group of automorphisms.*

Proof. Consider the nested separation system N_K given by Theorem 2.1.24. As in the proof of that theorem, let N_k be the subsystem of N_K consisting of its separations of order at most k. By Theorem 2.1.24, N_K is Aut(G)-invariant, so this is also true for all N_k with $k < K$.

Let $(T_k, \mathcal{V}_k)$ be the tree-decomposition associated with N_k as in Section 2.1.4. Then (v) holds by Corollary 2.1.8, (iii) and (iv) by Theorem 2.1.18 (iii) and (iv). By (iii) and [52, Lemma 12.3.1], any k-inseparable set is contained in a unique part of $(T_k, \mathcal{V}_k)$, giving (i). By (iv) in the proof of Theorem 2.1.24, N_k distinguishes every two K-robust k-blocks, which implies (ii) by (i) and Theorem 2.1.18 (iii). □

From Theorem 2.1.28 we can finally deduce the two results announced in the Introduction, Theorems 1 and 2.

Theorem 1 follows by taking as K the integer k given in Theorem 1, and then considering the decomposition $(T_k, \mathcal{V}_k)$ for $k = K$. Indeed, consider two k-blocks b_1, b_2 that Theorem 1 claims are distinguished efficiently by $(T_k, \mathcal{V}_k)$. By Theorem 2.1.28 (ii), b_1 and b_2 lie in different parts of $(T_k, \mathcal{V}_k)$. Let $k' := \kappa(b_1, b_2) \le k$.

[15] By Lemma 2.1.23 (i), this is equivalent to saying that they are robust $r(b_i)$-blocks, that is, K-robust $r(b_i)$-blocks for $K = |G|$.

By Lemma 2.1.23 (i), the k'-blocks $b'_1 \supseteq b_1$ and $b'_2 \supseteq b_2$ are again K-robust. Hence by Theorem 2.1.28 (ii) for k', they lie in different parts of $(T_{k'}, \mathcal{V}_{k'})$. Consider an adhesion set of $(T_{k'}, \mathcal{V}_{k'})$ on the path in $T_{k'}$ between these parts. By Theorem 2.1.28 (iii), this set has size at most k', and by Theorem 2.1.28 (iv) it is also an adhesion set of $(T_k, \mathcal{V}_k)$ between the two parts of $(T_k, \mathcal{V}_k)$ that contain b_1 and b_2.

Theorem 2 follows from Theorem 2.1.28 for $K = |G|$; recall that robust k-blocks are K-robust for $K = |G|$.

2.1.7 Outlook

There are two types of question that arise from the context of this paper, but which we have not addressed.

The first of these concerns its algorithmic aspects. How hard is it

- to decide whether a given graph has a k-block;
- to find all the k-blocks in a given graph;
- to compute the canonical tree-decompositions whose existence we have shown?

Note that our definitions leave some leeway in answering the last question. For example, consider a graph G that consists of two disjoint complete graphs K, K' of order 10 joined by a long path P. For $k = 5$, this graph has only two k-blocks, K and K'. One tree-decomposition of G that is invariant under its automorphisms has as parts the graphs K, K' and all the K_2s along the path P, its decomposition tree again being a long path. This tree-decomposition is particularly nice also in that it also distinguishes the ℓ-blocks of G not only for $\ell = k$ but for all ℓ such that G has an ℓ-block, in particular, for $\ell = 1$.

However if we are only interested in k-blocks for $k = 5$, this decomposition can be seen as unnecessarily fine in that it has many parts containing no k-block. We might, in this case, prefer a tree-decomposition that has only two parts, and clearly there is such a tree-decomposition that is invariant under $\mathrm{Aut}(G)$, of adhesion 1 or 2 depending on the parity of $|P|$.

This tree-decomposition, however, is suboptimal in yet another respect: we might prefer decompositions in which any part that does contain a k-block contains nothing but this k-block. Our first decomposition satisfies this, but there is another that does too while having fewer parts: the path-decomposition into three parts whose middle part is P and whose leaf parts are K and K'.

We shall look at these possibilities and associated algorithms in more detail in [40]. However we shall not make an effort to optimize these algorithms from a complexity point of view, so the above three questions will be left open.

Since our tree-decompositions are canonical, another obvious question is whether they, or refinements, can be used to tackle the graph isomorphism problem. Are there natural classes of graphs for which we can

- describe the parts of our canonical tree-decompositions in more detail;

- use this to decide graph isomorphism for such classes in polynomial time?

Another broad question that we have not touched upon, not algorithmic, is the following. Denote by $\beta(G)$ the greatest integer k such that G has a k-block (or equivalently: has a k-inseparable set of vertices). This seems to be an interesting graph invariant; for example, in a network G one might think of the nodes of a $\beta(G)$-block as locations to place some particularly important servers that should still be able to communicate with each other when much of the network has failed.

From a mathematical point of view, it seems interesting to ask how β interacts with other graph invariants. For example, what average degree will force a graph to contain a k-block for given k? What can we say about the structure of graphs that contain no k-block but have large tree-width?

Some preliminary results in this direction are obtained in [41], but even for the questions we address we do not have optimal results.

Acknowledgement

Many ideas for this paper have grown out of an in-depth study of the treatise [61] by Dunwoody and Krön, which we have found both enjoyable and inspiring.

2.2 Canonical tree-decompositions of finite graphs I. Existence and algorithms

2.2.1 Introduction

Given an integer k, a *k-block* X in a graph G is a maximal set of at least k vertices no two of which can be separated in G by fewer than k other vertices; these may or may not lie in X. Thus, k-blocks can be thought of as highly connected pieces of a graph, but their connectivity is measured not in the subgraph they induce but in the ambient graph.

Extending results of Tutte [105] and of Dunwoody and Krön [61], three of us and Maya Stein showed that every finite graph G admits, for every integer k, a tree-decomposition $(T, \mathcal{V})$ of adhesion $< k$ that distinguishes all its k-blocks [42]. These decompositions are *canonical* in that the map $G \mapsto (T, \mathcal{V})$ commutes with graph isomorphisms. In particular, the decomposition $(T, \mathcal{V})$ constructed for G is invariant under the automorphisms of G.

Our next aim, then, was to find out more about the tree-decompositions whose existence we had just proved. What can we say about their parts? Will every part contain a k-block? Will those that do consist of just their k-block, or might they also contain some 'junk'? Such questions are not only natural; their answers will also have an impact on the extent to which our tree-decompositions can be used for an obvious potential application, to the graph isomorphism problem in complexity theory. See Grohe and Marx [68] for recent progress on this.

When we analysed our existence proof in view of these questions, we found that even within the strict limitations imposed by canonicity we can make choices that will have an impact on the answers. For example, we can obtain different decompositions (all canonical) if we seek to, alternatively, minimize the number of inessential parts, minimize the sizes of the parts, or just of the essential parts, or achieve a reasonable balance between these properties. (A part is called *essential* if it contains a k-block, and *inessential* otherwise.)

In this paper we describe a large family of algorithms[16] that each produce a canonical tree-decomposition for given G and k. Their parameters can be tuned to optimize this tree-decomposition in terms of criteria such as those above. In [40] we shall apply these results to specify algorithms from the family described here for which we can give sharp bounds on the number of inessential parts, or which under specified conditions ensure that some or all essential parts consist only of the corresponding k-block.

The existence theorems which our algorithms imply will extend our results from [42] in that the decompositions constructed will not only distinguish all the k-blocks of a graph, but also its tangles of order k. (Tangles were introduced by Robertson and Seymour [94] and can also be thought of as indicating highly connected parts of a graph.) In order to treat blocks and tangles in a unified way, we work with a common generalization called 'profiles'. These appear to be of interest in their own right, as a way of locating desirable local substructures in very general discrete structures. More about profiles, including further generalizations of our existence theorems to such general structures (including matroids), can be found in [78]. More on k-blocks, including different kinds of examples and their relationship to tangles, can be found in [41].

All graphs in this paper will be finite, undirected and simple. Any graph-theoretic terms not defined here are explained in [52]. Unless otherwise mentioned, $G = (V, E)$ will denote an arbitrary finite graph.

2.2.2 Separation systems

A pair (A, B) of subsets of V such that $A \cup B = V$ is called a *separation* of G if there is no edge $e = \{x, y\}$ in E with $x \in A \setminus B$ and $y \in B \setminus A$. If (A, B) is a separation such that neither $A \subseteq B$ nor $B \subseteq A$, then (A, B) is a *proper* separation of G. A separation that is not proper is called *improper*. The *order* $\mathrm{ord}(A, B)$ of a separation (A, B) is the cardinality of its *separator* $A \cap B$. A separation of order k is called a *k-separation*. By simple calculations we obtain:

Lemma 2.2.1. *For any two separations (A, B) and (C, D), the orders of the separations $(A \cap C, B \cup D)$ and $(B \cap D, A \cup C)$ sum to $|A \cap B| + |C \cap D|$.* □

[16] We should point out that our reason for thinking in terms of algorithms is not, at this stage, one of complexity considerations: these are interesting, but they are not our focus here. Describing a decomposition in terms of the algorithm that produces it is simply the most intuitive way to ensure that it will be canonical: as long as the instructions of how to obtain the decomposition refer only to invariants of the graph (rather than, say, to a vertex enumeration that has to be chosen arbitrarily at some point), the decomposition that this algorithm produces will also be an invariant.

We define a partial ordering on the set of separations of G by

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \wedge B \supseteq D. \tag{2.20}$$

A separation (A, B) is *nested* with (C, D), written as $(A, B) \| (C, D)$, if it is $\leq$-comparable with either (C, D) or (D, C). Since

$$(A, B) \leq (C, D) \Leftrightarrow (D, C) \leq (B, A), \tag{2.21}$$

the relation $\|$ is reflexive and symmetric.[17] Two separations that are not nested are said to *cross*.

A separation (A, B) is *nested* with a set S of separations, written as $(A, B) \| S$, if $(A, B) \| (C, D)$ for every $(C, D) \in S$. A set S of separations is *nested* with set S' of separations, written as $S \| S'$, if $(A, B) \| S'$ for every $(A, B) \in S$; then also $(C, D) \| S$ for every $(C, D) \in S'$.

A set of separations is called *nested* if every two of its elements are nested; it is called *symmetric* if whenever it contains a separation (A, B) it also contains (B, A). The minimal symmetric set containing a given set of separations is called its *symmetric closure*. A symmetric set of proper separations is called a *system* of separations, or *separation system*.[18]

A separation (A, B) *separates* a set $X \subseteq V$ if X meets both $A \setminus B$ and $B \setminus A$. Given a set S of separations, we say that X is *S-inseparable* if no separation in S separates X. An *S-block* of G is a maximal S-inseparable set of vertices.

Recall that a *tree-decomposition* of G is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V(G)$, one for every node of T, such that:

(T1) $V(G) = \bigcup_{t \in T} V_t$;

(T2) for every edge $e \in G$ there exists a $t \in T$ such that both ends of e lie in V_t;

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the t_1–t_3 path in T.

The sets V_t in such a tree-decomposition are its *parts*. Their intersections $V_t \cap V_{t'}$ for edges tt' of the *decomposition tree* T are the *adhesion sets* of $(T, \mathcal{V})$; their maximum size is the *adhesion* of $(T, \mathcal{V})$.

Deleting an oriented edge $e = t_1 t_2$ of T divides $T - e$ into two components $T_1 \ni t_1$ and $T_2 \ni t_2$. Then $(\bigcup_{t \in T_1} V_t, \bigcup_{t \in T_2} V_t)$ is a separation of G with separator $V_{t_1} \cap V_{t_2}$ [52, Lemma 12.3.1]; we say that our edge e *induces* this separation. A node $t \in T$ is a *hub node* if the corresponding part V_t is the separator of a separation induced by an edge of T at t. If t is a hub node, we call V_t a *hub*.

As is easy to check, the separations induced by (the edges of T in) a tree-decomposition $(T, \mathcal{V})$ are nested. Conversely, we proved in [42] that every nested separation system is induced by some tree-decomposition:

[17] But it is not in general transitive, compare [42, Lemma 2.2].

[18] Alert: Both conditions, that a set of separations is symmetric and the separations themselves are proper, are restrictions we shall often need to impose, and for which we therefore need a simple term. We hope that readers remember both these restrictions when they see the term 'system', as making them explicit each time would be cumbersome.

Theorem 2.2.2. [42, Theorem 4.8] *Every nested separation system N is induced by a tree-decomposition $(T, \mathcal{V})$ of G such that*

(i) *every N-block of G is a part of the decomposition;*

(ii) *every part of the decomposition is either an N-block of G or a hub.*

See [42] for how these tree-decompositions are constructed from N.

Let k be a positive integer. A set I of at least k vertices is called $(< k)$-*inseparable* if it is S-inseparable for the set S of all separations of order $< k$, that is, if for every separation (A, B) of order less than k we have either $I \subseteq A$ or $I \subseteq B$. A maximal $(< k)$-inseparable set of vertices is called a *k-block* of G.

Since a k-block is too large to be contained in the separator $A \cap B$ of a separation (A, B) of order $< k$, it thus 'chooses' one of the sides A or B, the one containing it. Compared with choosing one side of every separation of order $< k$ arbitrarily, always choosing the side that contains a certain k-block b makes these choices consistent in a sense.

Another way of making consistent choices for small-order separations, but one that cannot necessarily be defined by setting a 'target' in this way, are *tangles*, introduced by Robertson and Seymour [94]. Like k-blocks, tangles have been considered as a way of identifying the highly connected parts of a graph, and so we wish to treat them together with k-blocks in a unified way.

This can be done by axiomatically writing down some minimum requirements on what makes choices of sides in separations 'consistent': in a way just strong enough to prove our decomposition results,[19] but weak enough to encompass both blocks and tangles.

Given k and a k-block b, consider the following set $P_k(b)$ of separations:

$$P_k(b) := \{ (A, B) : |A \cap B| < k \wedge b \subseteq B \}. \tag{2.22}$$

It is easy to verify that $P = P_k(b)$ has the following properties:

(P1) for every $(A, B) \in P$ and every separation (C, D) with $(C, D) \leq (A, B)$ we have $(D, C) \notin P$;

(P2) for all $(A, B), (C, D) \in P$ we have $(B \cap D, A \cup C) \notin P$.

Similarly, it is immediate to check that every tangle P satisfies (P1) and (P2).

Let us call an arbitrary set P of separations a *profile* if it satisfies (P1) and (P2). Note that (P1) says something only about nested separations, while

[19] The notion of k-profiles we are about to introduce arose when we noticed that, in our proofs of the existence of canonical tree-decompositions distinguishing k-blocks [42], all we really used about the k-blocks was the information of which side of each $(< k)$-separation they lay in. Forgetting the rest, and working with just the sets of these choices rather than concrete vertex sets, makes the decomposition theory more abstract but also more powerful. The fact that it also applies to tangles bears witness to this, as does the fact that it also works for matroids [78].

(P2) is essentially about crossing separations. Separation systems that satisfy (P1) but not necessarily (P2) will be play a role too later.[20]

Note that (P2) is reminiscent of the property of ultrafilters that 'the intersection of large sets are large'. An important difference, however, is that rather than demanding outright that $(A \cup C, B \cap D) \in P$, the indirect phrasing of (P2) requires this only when $(A \cup C, B \cap D)$ has order $< k$; if not, (P2) asks nothing of either this separation or its inverse. On the other hand, (P1) has a consequence reminiscent of assuming that an ultrafilter is non-principal:

If (A, B) is an improper separation, with $A \subseteq B$ say, then (B, A) is not contained in any set of separations satisfying (P1). (2.23)

Indeed, if $A \subseteq B$ then $(A, B) \leq (B, A)$, which implies (2.23).

While axioms (P1) and (P2) reflect the consistency in the choices which b or the tangle makes from each separation (A, B) (in that it 'chooses' $B \supseteq b$ rather than A), it is only when, as in (2.22) or in the definition of a tangle of order k, such a choice is made for *every* separation of order $< k$ that such consistent choices signify something 'big' in G.[21]

To give such rich profiles a name, let us call a set P of separations satisfying (P1) and (P2) a *k-profile* if it satisfies

Every separation in P has order $< k$, and for every separation (A, B) of order $< k$ exactly one of (A, B) and (B, A) lies in P. (2.24)

So the set $P_k(b)$ in (2.22) is a k-profile; this is the k-profile *induced by* b, and we call it the k-profile *of* b. A k-profile induced by some k-block is a *k-block profile*.

Since a k-block is a maximal $(< k)$-inseparable set of vertices, there is for every pair of distinct k-blocks b, b' a separation (A, B) of order $< k$ such that $(A, B) \in P_k(b)$ and $(B, A) \in P_k(b')$ [42, Lemma 2.1]. Hence $P_k(b) \neq P_k(b')$. Thus, while every k-block induces a k-profile, conversely a k-profile P is induced by at most one k-block, which we then denote by $b(P)$. All k-block profiles P then satisfy $P = P_k(b(P))$, and we say that b and P *correspond*.

Not every k-profile is induced by a k-block. For example, there are tangles of order k that are not induced by a k-block, such as the unique tangle of any order $k \geq 5$ in a large grid (which has no k-block for $k \geq 5$; see [41, Example 3]). Conversely, there are k-block profiles that are not tangles; indeed, there are graphs that have interesting k-block profiles but have no non-trivial tangle at all [41, Examples 4–5 and Section 6]. The notion of a k-profile thus

[20] As a typcial example, consider the union of three large complete graphs X_1, X_2, X_3 identified in a common triangle. The three 3-separations whose left side is one of X_1, X_2, X_3 satisfy (P1) but not (P2), because the separation $(B \cap D, A \cup C)$ in (P2) happens to be one of the original three separations. The analogous system with four complete graphs does satisfy (P2).

[21] Readers familiar with the notion of *preferences*, or *havens* – a way of making consistent choices of components of $G - X$ for vertex sets X – will recognize this: it is because a preference or haven assigns a component of $G - X$ to *every* set X of $< k$ vertices for some k that the bramble formed by these components has order $\geq k$.

unifies the ways in which k-blocks and tangles of order k 'choose' one side of every separation of order $< k$, but neither of these two instances of k-profiles generalizes the other.

Let S be any set of separations of G. An S-block X of G is called *large* (*with respect to* S) if it is not contained in the separator of a separation in S. If all the separations in S have order $< k$, an obvious but typical reason for an S-block to be large is that it has k or more vertices. In analogy to (2.22) we define for a large S-block X

$$P_S(X) := \{(A,B) \in S \mid X \subseteq B\} \subseteq S. \tag{2.25}$$

Clearly, $P_S(X)$ is a profile; we call it the *S-profile of X*. As before, the S-profiles $P_S(X)$ and $P_S(X')$ of distinct large S-blocks X, X' are distinct; if S is symmetric, they are incomparable under set-inclusion.

Not every k-profile has this form. For example, a tangle θ of order $k \geq 5$ in a large grid is not the S-profile of a large S-block X for any set $S \supseteq \theta$ of separations, since X would be contained in a large θ-block but the grid has none.

Although profiles are, formally, sets of separations, our intuition behind them is that they signify some 'highly connected pieces' of our graph G. Our aim will be to separate all these pieces in a tree-like way, and we shall therefore have to speak about sets of separations that, initially, are quite distinct from the profiles they are supposed to 'separate'. To help readers keep their heads in this unavoidable confusion, we suggest that they think of the sets S of separations discussed below as (initially) quite independent of the profiles P discussed along with them, the aim being to explore the relationship between the two.

A separation (A,B) *distinguishes* two subsets of V if one lies in A, the other in B, and neither in $A \cap B$. A set S of separations *distinguishes* two sets of vertices if some separation in S does.

A separation (A,B) *distinguishes* two sets P, P' of separations if each of $P \setminus P'$ and $P' \setminus P$ contains exactly one of (A,B) and (B,A). Thus, a $(<k)$-separation (A,B) distinguishes two k-blocks if and only if it distinguishes their k-profiles. A set of separations S *distinguishes* P from P' if some separation in S distinguishes them, and S *distinguishes* a set $\mathcal{P}$ of sets of separations if it distinguishes every two elements of $\mathcal{P}$. If all the separations in S have order $< k$, it thus distinguishes two k-blocks if and only if it distinguishes their k-profiles.

An asymmetric set P of separations (one containing no inverse of any of its elements) *orients* a set S of separations if, for every $(A,B) \in S$, either $(A,B) \in P$ or $(B,A) \in P \cap S$; we then call $P \cap S$ an *orientation* of S. If, in addition, some set X of vertices lies in B for every $(A,B) \in P \cap S$, we say that P *orients S towards X*. If P is a profile then so is $P \cap S$; we call it the *S-profile of P*.[22] More generally, every profile that is an orientation of S will be called an *S-profile*.

A profile orienting a set S of separations need not orient it towards any non-empty set of vertices: consider, for example, our earlier tangle θ with $S = \theta$.

[22]This formalizes the idea that P, thought of as a big chunk of G, lies on exactly one side of every separation in S. For example, if the separations in S have order $< k$, then any k-profile will orient S.

However, a profile P orienting a nested separation system N orients it towards the union X of the separators of the $\leq$-maximal separations in $P \cap N$, which is non-empty if G is connected. Using (P1) one can show that X is N-inseparable. However, it can be 'small in terms of N', that is, contained in a separator of a separation in N. In that case it may extend to more than one N-block of G, and P need not orient N towards any of these. However if X does not lie in a separator of N, it extends to a unique N-block, towards which P orients N. We then say that P *lives in* this N-block.

Given a set S of separations of G and a set $\mathcal{P}$ of profiles orienting S, let us say that two profiles $P, P' \in \mathcal{P}$ *agree on* S if their S-profiles coincide, that is, if $P \cap S = P' \cap S$. This is an equivalence relation on $\mathcal{P}$, whose classes we call the S-*blocks* of $\mathcal{P}$. By definition, elements P, P' of the same S-block $\mathcal{Q}$ of $\mathcal{P}$ have the same S-profile $P \cap S = P' \cap S$, which we call the S-*profile of* $\mathcal{Q}$.

A set P of separations satisfying (P1) is a (P1)-*set* of separations. A separation (A, B) *splits* a (P1)-set P if both $P \cup \{(A, B)\}$ and $P \cup \{(B, A)\}$ satisfy (P1). (This implies that neither (A, B) nor (B, A) is in P.) For example, the S-profile corresponding to an S-block $\mathcal{Q}$ of a set $\mathcal{P}$ of profiles orienting a separation system S is split by every separation (A, B) that distinguishes some distinct profiles in $\mathcal{Q}$. By (2.23), every separation splitting a (P1)-set of separations must be proper.

We shall need the following lemma. A *(P1)-orientation* of a separation system S is an orientation of S that satisfies (P1).

Lemma 2.2.3. *Let N be a nested separation system.*

(i) *Every proper separation $(A, B) \notin N$ that is nested with N splits a unique* (P1)*-orientation O of N. This set O is given by*

$$O = \{(C, D) \in N \mid (C, D) \leq (A, B)\} \cup \{(C, D) \in N \mid (C, D) \leq (B, A)\}.$$

(ii) *If two separations not contained in but nested with N split distinct* (P1)*-orientations of N, they are nested with each other.*

Proof. (i) Since (A, B) is nested with N, for every separation $(C, D) \in N$ either (C, D) or (D, C) is smaller than one of (A, B) or (B, A) and thus contained in

$$O := \{(C, D) \in N \mid (C, D) \leq (A, B)\} \cup \{(C, D) \in N \mid (C, D) \leq (B, A)\}.$$

By definition, O contains only separations from N. As we have seen, every separation from N or its inverse lies in O. Once we know that O satisfies (P1) it will follow that for every separation it contains it will not contain its inverse, so O will be an orientation of N.

To check that O satisfies (P1), consider separations $(E, F) \leq (C, D)$ with $(C, D) \in O$. Our aim is to show that $(F, E) \notin O$. This is clearly the case if $(E, F) \notin N$, since $O \subseteq N$ and N is symmetric, so we assume that $(E, F) \in N$. By definition of O, either $(C, D) \leq (A, B)$ or $(C, D) \leq (B, A)$; we assume the former. Then by transitivity $(E, F) \leq (A, B)$, and hence $(E, F) \in O$ by definition of O. To show that $(F, E) \notin O$ we need to check that $(F, E) \not\leq (A, B)$

and $(F,E) \not\leq (B,A)$. If $(F,E) \leq (A,B)$ then $(B,A) \leq (E,F) \leq (A,B)$ and hence $B \subseteq A$, contradicting our assumption that (A,B) is proper. If $(F,E) \leq (B,A)$ then $(A,B) \leq (E,F) \leq (A,B)$ and hence $(E,F) = (A,B)$, contradicting our assumption that $(A,B) \notin N$.

So O is a (P1)-orientation of N. In particular, O never contains the inverse of a separation it contains. This implies by the definition of O that also $O \cup \{(A,B)\}$ and $O \cup \{(B,A)\}$ satisfy (P1). Hence (A,B) splits O, as desired.

It remains to show that O is unique. Suppose (A,B) also splits a (P1)-orientation $O' \neq O$ of N. Let $(C,D) \in N$ distinguish O from O', with $(C,D) \in O$ and $(D,C) \in O'$ say. By definition of O, either $(C,D) \leq (A,B)$ or $(C,D) \leq (B,A)$. In the first case $O' \cup \{(A,B)\}$ violates (P1), since $(B,A) \leq (D,C) \in O' \cup \{(A,B)\}$ but also $(A,B) \in O' \cup \{(A,B)\}$. In the second case, $O' \cup \{(B,A)\}$ violates (P1), since $(A,B) \leq (D,C) \in O' \cup \{(B,A)\}$ but also $(B,A) \in O' \cup \{(B,A)\}$.

(ii) Consider separations $(A,B), (A',B') \notin N$ that are both nested with N. Assume that (A,B) splits the (P1)-orientation O of N, and that (A',B') splits the (P1)-orientation $O' \neq O$ of N. From (2.23) we know that (A,B) and (A',B') must be proper separations, so they satisfy the premise of (i) with respect to O and O'. As $O \neq O'$, there is a separation $(C,D) \in N$ with $(C,D) \in O$ and $(D,C) \in O'$. By the descriptions of O and O' in (i), the separation (C,D) is smaller than (A,B) or (B,A), and (D,C) is smaller than (A',B') or (B',A'). The latter is equivalent to (C,D) being greater than (B',A') or (A',B'). Thus, (B',A') or (A',B') is smaller than (C,D) and hence than (A,B) or (B,A), so (A,B) and (A',B') are nested. $\square$

We remark that the (P1)-set O in Lemma 2.2.3 (i) is usually an N-profile; it is not hard to construct pathological cases in which O fails to satisfy (P2), but such cases are rare.

2.2.3 Tasks and strategies

In this section we describe a systematic approach to distinguishing some or all of the k-profiles of G by (the separations induced by) canonical tree-decompositions of adhesion less than k. Since the separations induced by a tree-decomposition are nested, our main task in finding such a tree-decomposition will be to select from the set S of all $(< k)$-separations of G a nested subset N that will still distinguish all the k-profiles under consideration.

We begin by formalizing the notion of such 'tasks'. We then show how to solve 'feasible' tasks in various ways, and give examples showing how different strategies – all canonical in that they commute with graph isomorphisms – can produce quite different solutions.

Consider a separation system S and a set $\mathcal{P}$ of profiles. Let us call the pair $(S, \mathcal{P})$ a *task* if every profile in $\mathcal{P}$ orients S and S distinguishes $\mathcal{P}$. Another task $(S', \mathcal{P}')$ is a *subtask* of the task $(S, \mathcal{P})$ if $S' \subseteq S$ and $\mathcal{P}' \subseteq \mathcal{P}$.

The two conditions in the definition of a task are obvious minimum requirements which S and $\mathcal{P}$ must satisfy before it makes sense to look for a nested subset $N \subseteq S$ that distinguishes $\mathcal{P}$. But to ensure that N exists, S must also be rich enough (in terms of $\mathcal{P}$): the more profiles we wish to separate in a nested way, the more separations will we need to have available. For example, if S consists of two crossing separations (A,B), (C,D) and their inverses, and $\mathcal{P}$ contains the four possible orientations of S (which are clearly profiles), then S distinguishes $\mathcal{P}$ but is not nested, while the two subsystems $\{(A,B),(B,A)\}$ and $\{(C,D),(D,C)\}$ of S are nested but no longer distinguish $\mathcal{P}$. But if we enrich S by adding two 'corner separations' $(A\cap C, B\cup D)$, $(A\cup C, B\cap D)$ and their inverses, then these together with (A,B) and (B,A), say, form a nested subsystem that does distinguish $\mathcal{P}$.

More generally, we shall prove in this section that we shall be able to find the desired N if S and $\mathcal{P}$ satisfy the following condition:

Whenever $(A,B),(C,D)\in S$ cross and there exist $P,P'\in\mathcal{P}$ such that $(A,B),(C,D)\in P$ and $(B,A),(D,C)\in P'$, there exists a separation $(E,F)\in P\cap S$ such that $(A\cup C, B\cap D)\le (E,F)$. (2.26)

Anticipating our results, let us call a task $(S,\mathcal{P})$ *feasible* if S and $\mathcal{P}$ satisfy (2.26).

Let us take a moment to analyse condition (2.26). Note first that, like the given separations (A,B) and (C,D), the new separation (E,F) will again distinguish P from P': by assumption we have $(E,F)\in P$, and by (2.21) we have $(F,E)\le(B\cap D, A\cup C)\le(B,A)$, so $(F,E)\in P'$ by (P1) and the fact that P' orients S.

Now the idea behind (2.26) is that in our search for N we may find ourselves facing a choice between two crossing separations $(A,B),(C,D)\in S$ that both distinguish two profiles $P,P'\in\mathcal{P}$, and wonder which of these we should pick for N. (Clearly we cannot take both.) If (2.26) holds, we have the option to choose neither and pick (E,F) instead: it will do the job of distinguishing P from P', and since it is nested with both (A,B) and (C,D), putting it in N entails no prejudice to any future inclusion of either (A,B) or (C,D) in N.

Separations in S that do not distinguish any profiles in $\mathcal{P}$ are not really needed for N, and so we may delete them.[23] So let us call a separation $\mathcal{P}$*-relevant* if it distinguishes some pair of profiles in $\mathcal{P}$, denote by R the set of all $\mathcal{P}$-relevant separations in S, and call $(R,\mathcal{P})$ the *reduction* of $(S,\mathcal{P})$. If $(S,\mathcal{P})=(R,\mathcal{P})$, we call this task *reduced*. Since all the separations (A,B), (C,D), (E,F) in (2.26) are $\mathcal{P}$-relevant, R inherits (2.26) from S (and vice versa):

$(R,\mathcal{P})$ is feasible if and only if $(S,\mathcal{P})$ is feasible. (2.27)

Consider a fixed feasible task $(S,\mathcal{P})$. Our aim is to construct N inductively, adding a few separations at each step. A potential danger when choosing a new separation to add to N is to pick one that crosses another separation that we might wish to include later. This can be avoided if we only ever add separations that are nested with all other separations in S that we might still want to include in N. So this will be our aim.

[23] But do not have to: the freedom to discard or keep such separations will be our source of diversity for the tree-decompositions sought – which, as pointed out earlier, we may wish to endow with other desired properties than the minimum requirement of distinguishing $\mathcal{P}$.

At first glance, this strategy might seem both wasteful and unrealistic: why should there even be a separation in S that we can choose at the start, one that is nested with all others? However, we cannot easily be more specific: since we want our nested subsystem N to be canonical, we are not allowed to break ties between crossing separations without appealing to an invariant of G as a criterion, and it would be hard to find such a criterion that applies to a large class of graphs without specifying this class in advance. But the strategy is also more realistic than it might seem. This is because the set of pairs of profiles we need to distinguish by separations still to be picked decreases as N grows. As a consequence, we shall need fewer separations in S to distinguish them. We may therefore be able to delete from S some separations that initially prevented the choice of a desired separation (A,B) for N by crossing it, because they are no longer needed to distinguish profiles in what remains of $\mathcal{P}$, thus freeing (A,B) for inclusion in N.

To get started, we thus have to look for separations (A,B) in S that are nested with all other separations in S. This will certainly be the case for (A,B) if, for every $(C,D) \in S$, we have either $(C,D) \le (A,B)$ or $(D,C) \le (A,B)$;[24] let us call such separations (A,B) *extremal* in S. By (P1),

Distinct extremal separations are $\le$-incomparable and cannot lie in the same profile. (2.28)

Extremal separations always exist in a feasible task $(S,\mathcal{P})$, as long as S contains no superfluous separations (which might cross useful ones):

Lemma 2.2.4. *If $(S,\mathcal{P})$ is reduced, then every $\le$-maximal element of S is extremal in S.*

Proof. Let (A,B) be a maximal separation in S, and let $(C,D) \in S$ be any other separation. If (A,B) is nested with (C,D) it is comparable with (C,D) or (D,C). Hence either $(C,D) \le (A,B)$ or $(D,C) \le (A,B)$ by the maximality of (A,B), as desired. We may thus assume that (A,B) and (C,D) cross.

Since $(S,\mathcal{P})$ is reduced, (A,B) and (C,D) each distinguish two profiles from $\mathcal{P}$. Pick $P \in \mathcal{P}$ containing (A,B). Since P orients S, it also contains (C,D) or (D,C); we assume it contains (C,D). Now pick $P' \in \mathcal{P}$ containing (D,C). If also $(B,A) \in P'$, then by (2.26) there exists an $(E,F) \in S \cap P$ such that $(A,B) \le (A \cup C, B \cap D) \le (E,F)$. Since (A,B) and (C,D) cross, the first of these inequalities is strict, which contradicts the maximality of (A,B). Hence $(A,B) \in P' \cap P$. Since $\mathcal{P}$ is reduced, there exists $P'' \in \mathcal{P}$ containing (B,A). Applying (2.26) to P'' and either P or P', we again find an $(E,F) > (A,B)$ that contradicts the maximality of (A,B). □

Note that the proof of Lemma 2.2.4 uses crucially that $(S,\mathcal{P})$ is feasible.

Lemma 2.2.5. *If $(S,\mathcal{P})$ is reduced, then for every extremal separation (A,B) in S there is a unique profile $P_{(A,B)} \in \mathcal{P}$ such that $(A,B) \in P_{(A,B)}$.*

[24] This implies that (A,B) is maximal in S, but only because we are assuming that all separations in S are proper: improper separations (C,D) can satisfy $(D,C) < (A,B) < (C,D)$.

Proof. As $(S, \mathcal{P})$ is reduced, there is a profile $P \in \mathcal{P}$ containing (A, B). Suppose there is another such profile $P' \in \mathcal{P}$. Then P and P' are distinguished by some $(C, D) \in S$. Since (A, B) is extremal, we may assume that $(C, D) \leq (A, B)$. The fact that (D, C) lies in one of P, P' contradicts (P1) for that profile. $\square$

By Lemma 2.2.5, (A, B) distinguishes $P_{(A,B)}$ from any other profile in $\mathcal{P}$.

Let us call a profile P orienting S *extremal* with respect to S if it contains an extremal separation of S. This will be the greatest, and hence the only maximal, separation in $P \cap S$.

As we have seen, an extremal profile is distinguished from every other profile in $\mathcal{P}$ by some separation (A, B) that is nested with all the other separations in S; this makes (A, B) a good choice for N. The fact that made (A, B) nested with all other separations in S was its maximality in S (Lemma 2.2.4). In the same way we may ask whether, given any profile $P \in \mathcal{P}$ (not necessarily extremal), the separations that are $\leq$-maximal in $P \cap S$ will be nested with every other separation in S: these are the separations 'closest to P', much as (A, B) was closest to $P_{(A,B)}$ (although there can now be many such separations).

Let us prove that the following profiles have this property:

Call a profile P orienting S *well separated* in S if the set of $\leq$-maximal separations in $P \cap S$ is nested.

Note that extremal profiles are well separated.

Lemma 2.2.6. *Given a profile P orienting a separation system S, the following assertions are equivalent:*

(i) *P is well separated in S.*

(ii) *Every maximal separation in $P \cap S$ is nested with all of S.*

(iii) *For every two crossing separations $(A, B), (C, D) \in P \cap S$ there exists a separation $(E, F) \in P \cap S$ such that $(A \cup C, B \cap D) \leq (E, F)$.*

Proof. The implication (ii)→(i) is trivial; we show (i)→(iii)→(ii).

(i)→(iii): Suppose that P is well separated, and consider two crossing separations $(A, B), (C, D) \in P \cap S$. Let $(A', B') \geq (A, B)$ be maximal in $P \cap S$. Suppose first that $(A', B') \| (C, D)$. This means that (A', B') is $\leq$-comparable with either (C, D) or (D, C). Since (A, B) is not nested with (C, D) we have $(A', B') \not\leq (C, D)$ and $(A', B') \not\leq (D, C)$, and since both (C, D) and (A', B') are in P, axiom (P1) yields $(D, C) \not\leq (A', B')$. Hence $(C, D) \leq (A', B')$, and thus $(A \cup C, B \cap D) \leq (A', B')$. This proves (iii) with $(E, F) := (A', B')$.

Suppose now that (A', B') crosses (C, D). Let $(C', D') \geq (C, D)$ be maximal in $P \cap S$. Since (A', B') and (C', D') are both maximal in $P \cap S$ they are nested, by assumption in (i). As in the last paragraph, now with (C', D') taking the role of (A', B'), and (A', B') taking the role of (C, D), we can show that $(A, B) \leq (A', B') \leq (C', D')$ and hence $(A \cup C, B \cap D) \leq (C', D')$.[25] This proves (iii) with $(E, F) := (C', D')$.

[25]In the argument we need that (C, D) and (A', B') cross. This is why we first treated the case that they don't (but in that case we used that (A, B) and (C, D) cross).

(iii)→(ii): Suppose some maximal (A,B) in $P \cap S$ crosses some $(C,D) \in S$. As P orients S, and by symmetry of nestedness, we may assume that $(C,D) \in P$. By (iii), there is an $(E,F) \in P \cap S$ such that $(A \cup C, B \cap D) \leq (E,F)$, so $(A,B) \leq (E,F)$ as well as $(C,D) \leq (E,F)$. But then $(E,F) = (A,B)$ by the maximality of (A,B), and hence $(A,B) \| (C,D)$, contradicting the choice of (A,B) and (C,D). □

Let us call a separation (A,B) *locally maximal* in a task $(S,\mathcal{P})$ if there exists a well-separated profile $P \in \mathcal{P}$ such that (A,B) is $\leq$-maximal in $P \cap S$. Lemma 2.2.6 shows that these separations are a good choice for inclusion in N:

Corollary 2.2.7. *Locally maximal separations in a task $(S,\mathcal{P})$, not necessarily feasible, are nested with all of S.*

We have seen three ways of starting the construction of our desired nested subsystem $N \subseteq S$ for a feasible task $(S,\mathcal{P})$ by choosing for N some separations from S that are nested with all other separations in S: we may choose either

- the set $\mathsf{ext}(S,\mathcal{P})$ of extremal separations in S and their inverses; or
- the set $\mathsf{loc}(S,\mathcal{P})$ of all locally maximal separations in $(S,\mathcal{P})$ and their inverses; or
- the set $\mathsf{all}(S,\mathcal{P})$ of all separations in S that are nested with every separation in S (which is a symmetric set).

Clearly,

$$\mathsf{ext}(S,\mathcal{P}) \subseteq \mathsf{all}(S,\mathcal{P}) \supseteq \mathsf{loc}(S,\mathcal{P}) \tag{2.29}$$

in general, and

$$\emptyset \neq \mathsf{ext}(S,\mathcal{P}) \subseteq \mathsf{loc}(S,\mathcal{P}) \subseteq \mathsf{all}(S,\mathcal{P}) \tag{2.30}$$

if $S \neq \emptyset$ and $(S,\mathcal{P})$ is reduced,[26] since in that case every maximal separation in S is extremal (Lemma 2.2.4) and every extremal separation (A,B) is locally maximal for $P_{(A,B)} \in \mathcal{P}$.

Example 2.2.8. Let G consist of three large complete graphs X_1, X_2, X_3 threaded on a long path P, as shown in Figure 2.10. Let S be the set of all proper 1-separations. Let $\mathcal{P} = \{P_1, P_2, P_3\}$, where P_i is the 2-profile induced by X_i. Then $\mathsf{all}(S,\mathcal{P}) = S$, while $\mathsf{loc}(S,\mathcal{P})$ contains only the separations in S with separators x_1, x_2, y_2 and x_3, and $\mathsf{ext}(S,\mathcal{P})$ only those with separator x_1 or x_3.

How shall we proceed now, having completed the first step of our algorithm by specifying some nested subsystem $N \in \{\mathsf{ext}(S,\mathcal{P}), \mathsf{loc}(S,\mathcal{P}), \mathsf{all}(S,\mathcal{P})\}$ of S? The idea is that N divides G into chunks, which we now want to cut up further by adding more separations of S to N. While it is tempting to think of those

[26] In fact, all we need for an extremal separation (A,B) to be locally maximal is that it lies in some $P \in \mathcal{P}$. But this need not be the case if $(S,\mathcal{P})$ is not reduced: although one of (A,B) and (B,A) must lie in every $P \in \mathcal{P}$ (because P orients S), it might happen that this is always (B,A).

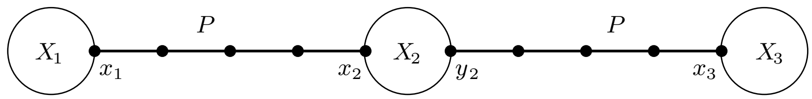


Figure 2.10: Different results for $\mathsf{ext}(S, \mathcal{P})$, $\mathsf{loc}(S, \mathcal{P})$ and $\mathsf{all}(S, \mathcal{P})$

'chunks' as the N-blocks of G, it turned out that this fails to capture some of the more subtle scenarios. Here is an example:

Example 2.2.9. Let G be the graph of Figure 2.11. Let N consist of the separations $(X_1, Y_1), \ldots, (X_4, Y_4)$ and their inverses (Y_i, X_i), where $Y_i := (A \cap B) \cup \bigcup_{j \neq i} X_j$, and let $S := N \cup \{(A, B), (B, A)\}$. Let $\mathcal{P}$ consist of the following six profiles: the orientations of S towards $X_1, \ldots, X_4$, respectively, and two further profiles P and P' which both orient N towards $A \cap B$ but of which P contains (A, B) while P' contains (B, A). Then N distinguishes all these profiles except P and P'. But these are distinguished by (A, B) and (B, A), so we wish to add these separations to N.

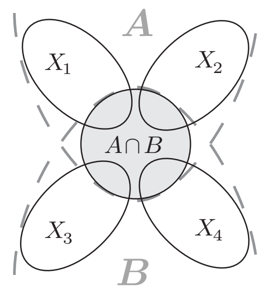


Figure 2.11: Two S-distinguishable profiles living in an S-inseparable N-block

The profiles P and P' live in the same N-block of G, the set $A \cap B$. But although S distinguishes P from P', it does not separate this N-block. We therefore cannot extend N to a separation system distinguishing $\mathcal{P}$ by adding only separations from S that separate an N-block of G.

The lesson to be learnt from Example 2.2.9 is that the 'chunks' into which N divides our graph G should not be thought of as the N-blocks of G. An alternative that the example suggests would be to think of them as the N-blocks of $\mathcal{P}$: the equivalence classes of $\mathcal{P}$ defined by how its profiles orient N. In the example, $\mathcal{P}$ has five N-blocks: the four singleton N-blocks consisting just of the profile P_i that orients N towards X_i, and another N-block $\mathcal{Q} = \{P, P'\}$. So the algorithm could now focus on the subtask $(R_{\mathcal{Q}}, \mathcal{Q})$ with $R_{\mathcal{Q}} = \{(A, B), (B, A)\}$ consisting of those separations from S that distinguish profiles in $\mathcal{Q}$.

More generally, we could continue our algorithm after finding N by iterating it with the subtasks $(R_{\mathcal{Q}}, \mathcal{Q})$ of $(S, \mathcal{P})$, where $\mathcal{Q}$ runs over the non-trivial N-blocks of $\mathcal{P}$ and $R_{\mathcal{Q}}$ is the set of $\mathcal{Q}$-relevant separations in S. This would indeed result in an overall algorithm that eventually produces a nested subsystem of S that distinguishes $\mathcal{P}$, solving our task $(S, \mathcal{P})$.

However, when we considered our three alternative ways of obtaining N, we also had a secondary aim in mind: rather than working with the reduction $(R, \mathcal{P})$ of $(S, \mathcal{P})$ straight away, we kept our options open to include more separations in N than distinguishing $\mathcal{P}$ requires, in order perhaps to produce a tree-decomposition into smaller parts.[27] In the same spirit, our secondary aim now as we look for ways to continue our algorithm from N is not to exclude any separation of $S \setminus N$ from possible inclusion into N without need, i.e., to subdivide $(S, \mathcal{P})$ into subtasks $(S_i, \mathcal{P}_i)$ if possible with $\bigcup_i S_i = S$.

In view of these two aims, the best way to think of the chunks left by N turned out to be neither as the (large) N-blocks of G, nor as the N-blocks of $\mathcal{P}$, but as something between the two: as the set $\mathcal{O}_N$ of all (P1)-orientations of N. Let us look at these in more detail.

Recall that since every $P \in \mathcal{P}$ orients N, it defines an N-profile $P \cap N$. Equivalent P, P' define the same N-profile $P \cap N = P' \cap N$, the N-profile of the N-block $\mathcal{Q}$ containing them. This is a (P1)-orientation of N. Conversely, given $O \in \mathcal{O}_N$, let us write $\mathcal{P}_O$ for the set of profiles $P \in \mathcal{P}$ with $P \cap N = O$. Note that $\mathcal{O}_N$ may also contain (P1)-orientations O of N, including N-profiles, that are not induced by any $P \in \mathcal{P}$, i.e., for which $\mathcal{P}_O = \emptyset$.

Similarly, every large N-block X of G defines an N-profile, the N-profile $P_N(X)$ of X. This is a (P1)-orientation of N. Again, $\mathcal{O}_N$ may also contain (P1)-orientations that are not of this form.[28]

Recall that a separation (A, B) *splits* $O \in \mathcal{O}_N$ if both $O \cup \{(A, B)\}$ and $O \cup \{(B, A)\}$ are again (P1)-orientations.[29] Let us write S_O for the set of separations in S that split O. These sets S_O extend our earlier sets $R_{\mathcal{Q}}$ in a way that encompasses all of $S \setminus N$, as intended:

Lemma 2.2.10. *Let N be a nested separation system that is oriented by every profile in $\mathcal{P}$ and nested with S.*[30]

(i) *$(\, S_O \mid O \in \mathcal{O}_N \,)$ is a partition of $S \setminus N$ (with $S_O = \emptyset$ allowed).*

(ii) *$(\, \mathcal{P}_O \mid O \in \mathcal{O}_N \,)$ is a partition of $\mathcal{P}$ (with $\mathcal{P}_O = \emptyset$ allowed).*

[27] In Example 2.2.8 with $\mathsf{ext}(S, \mathcal{P})$, where N consists of the proper 1-separations with separator x_1 or x_3, every N-block of $\mathcal{P}$ is trivial. But the middle N-block of G consists of X_2 and the entire path P, so we might cut it up further using the remaining 1-separations in S. If $\mathcal{P}$ consisted only of P_1 and P_3, then $\mathsf{ext}(S, \mathcal{P})$ would have produced the same N, and the middle N-block would not even have a profile from $\mathcal{P}$ living in it. But still, we might want to cut it up further.

[28] In Example 2.2.9, the set $A \cap B$ is a small S-block of G for the nested separation system S. The profiles P, P' are two (P1)-orientations of S orienting it towards $A \cap B$, but not towards any large S-block.

[29] In Example 2.2.9, the N-profile of $X = A \cap B$ could be split into the (P1)-orientations P and P' by adding the separations (A, B) and (B, A), although the large N-block X could not be separated by any separation in S. Thus, splitting the N-profile of a large N-block is more subtle than separating the N-block itself.

We remark that although all the (P1)-orientations considered in this example are in fact profiles, our aim to retain all the separations from $S \setminus N$ at this state requires that we do not restrict $\mathcal{O}_N$ to profiles: there may be separations in S (which we want to keep) that only split a (P1)-orientation of N that is not a profile, or separations that split an N-profile into two (P1)-separations that are not profiles.

[30] For better applicability of the lemma later, we do not require that $N \subseteq S$.

(iii) *The N-profile P of any N-block $\mathcal{Q}$ of $\mathcal{P}$ satisfies $\mathcal{P}_P = \mathcal{Q}$ and $S_P \supseteq R_{\mathcal{Q}}$.*

(iv) *The $(S_O, \mathcal{P}_O)$ are feasible tasks.*

Proof. (i) By Lemma 2.2.3, every separation $(A, B) \in S \backslash N$ splits a unique (P1)-orientation of N. Note that (A, B) is proper, since S is a separation system.

(ii) follows from the fact that every profile in $\mathcal{P}$ orients N and satisfies (P1).

(iii) The first assertion is immediate from the definition of an N-block of $\mathcal{P}$. For the second assertion let $(A, B) \in R_{\mathcal{Q}}$ be given, distinguishing $Q, Q' \in \mathcal{Q}$ say. By (i), we have $(A, B) \in S_O$ for some $O \in \mathcal{O}_N$. Since Q and Q' satisfy (P1), agree with P on N, and orient $\{(A, B), (B, A)\}$ differently, (A, B) splits the (P1)-orientation P of N. By the uniqueness of O this implies $P = O$. Hence, $(A, B) \in S_O = S_P$ as desired.

(iv) As S_O distinguishes $\mathcal{P}_O$, by (iii), we only have to show that $(S_O, \mathcal{P}_O)$ is feasible. As $(S, \mathcal{P})$ is feasible, there is a separation (E, F) in S for any two crossing separations $(A, B), (C, D) \in S_O$ distinguishing profiles $P, P' \in \mathcal{P}_O$ as in (2.26). Since (E, F) also distinguishes P from P', we have $(E, F) \in S_O$ by (iii). $\square$

We remark that the inclusion in Lemma 2.2.10 (iii) can be strict, since S_O may contain separations that do not distinguish any profiles in $\mathcal{P}$. Similarly, we can have $S_O \neq \emptyset$ for $O \in \mathcal{O}_N$ with $\mathcal{P}_O = \emptyset$.

The subtasks $(S_O, \mathcal{P}_O)$ will be 'easier' than the original task $(S, \mathcal{P})$, because we can reduce them further:

Example 2.2.11. The separations (X, Y) and (X', Y') in Figure 2.12 are $\mathcal{P}$-relevant (because they separate the profiles $P, P' \in \mathcal{P}$, say), so they will not be deleted when we reduce S (which is, in fact reduced already). They both belong to S_O for the middle (P1)-orientation O of N, but are no longer $\mathcal{P}_O$-relevant, where $\mathcal{P}_O = \{P_1, P_2, P_3\}$ as shown. We can therefore discard them when we reduce the subtask $(S_O, \mathcal{P}_O)$ before reapplying the algorithm to it, freeing (A, B) and (C, D) for adoption into N in the second step.

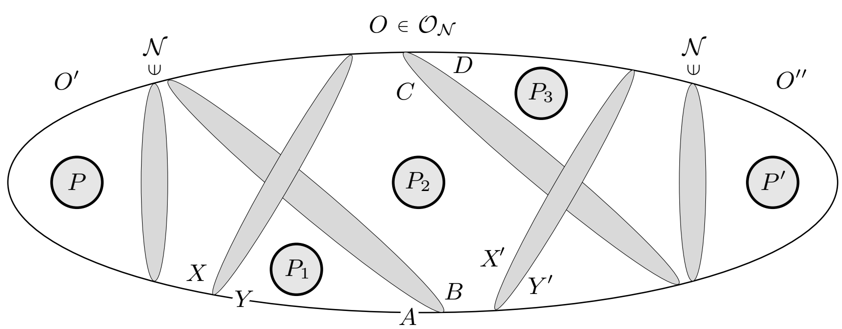


Figure 2.12: (X, Y) and (X', Y') are $\mathcal{P}$-relevant but no longer $\mathcal{P}_O$-relevant

More generally, reducing a subtask $(S', \mathcal{P}')$ will be the crucial step in getting our algorithm back afloat if it finds no separation in S' that is nested with all the others. Example 2.2.11 shows that this can indeed happen.[31] But after reducing $(S', \mathcal{P}')$ to $(R', \mathcal{P}')$, say, we know from (2.30) that each of $\mathsf{ext}, \mathsf{loc}, \mathsf{all}$ will find a separation in R' that is nested with all the others.

As notation for the double step of first reducing a task $(S, \mathcal{P})$, to $(R, \mathcal{P})$ say, and then applying ext, loc or all, let us define[32]

$$\mathsf{ext_r}(S,\mathcal{P}) := \mathsf{ext}(R,\mathcal{P}); \quad \mathsf{loc_r}(S,\mathcal{P}) := \mathsf{loc}(R,\mathcal{P}); \quad \mathsf{all_r}(S,\mathcal{P}) := \mathsf{all}(R,\mathcal{P}).$$

We shall view each of $\mathsf{ext}, \mathsf{loc}, \mathsf{all}, \mathsf{ext_r}, \mathsf{loc_r}, \mathsf{all_r}$ as a function that maps a given graph G and a feasible task $(S, \mathcal{P})$ in G to a nested subsystem N' of S'.

A *strategy* is a map $\sigma \colon \mathbb{N} \to \{\mathsf{ext}, \mathsf{loc}, \mathsf{all}, \mathsf{ext_r}, \mathsf{loc_r}, \mathsf{all_r}\}$ such that $\sigma(i) \in \{\mathsf{ext_r}, \mathsf{loc_r}, \mathsf{all_r}\}$ for infinitely many i. The idea is that, starting from some given task $(S, \mathcal{P})$, we apply $\sigma(i)$ at the ith step of the algorithm to the subtasks produced by the previous step, adding more and more separations to N. The requirement that for infinitely many i we have to reduce the subtasks first ensures that we cannot get stuck before N distinguishes all of $\mathcal{P}$.

Formally, we define a map $(\sigma, G, (S, \mathcal{P})) \mapsto N_\sigma(S, \mathcal{P})$ by which every strategy σ *determines* for every feasible task $(S, \mathcal{P})$ in a graph G some set $N_\sigma(S, \mathcal{P})$. We define this map recursively, as follows. Define σ^+ by setting $\sigma^+(i) := \sigma(i+1)$ for all $i \in \mathbb{N}$. Note that if σ is a strategy then so is σ^+. Let $s := |S|$, and let r_σ be the least integer r such that $\sigma(r) \in \{\mathsf{ext_r}, \mathsf{loc_r}, \mathsf{all_r}\}$. Our recursion is on s, and for fixed s on r_σ, for all G.

If $s = 0$, we let $N_\sigma(S, \mathcal{P}) := S = \emptyset$. Suppose now that $s \geq 1$; thus, $S \neq \emptyset$. Let $N := \sigma(0)(S, \mathcal{P})$. By Lemma 2.2.10 (iv), the subtasks $(S_O, \mathcal{P}_O)$ with $O \in \mathcal{O}_N$ are again feasible.

Assume first that $r_\sigma = 0$, i.e. that $\sigma(0) \in \{\mathsf{ext_r}, \mathsf{loc_r}, \mathsf{all_r}\}$, and let $(R, \mathcal{P})$ be the reduction of $(S, \mathcal{P})$. If $R \subsetneq S$ we let $N_\sigma(S, \mathcal{P}) := N_\sigma(R, \mathcal{P})$, which is already defined. If $R = S$ then $N \neq \emptyset$ by (2.30), and $|S_O| \leq |S \setminus N| < s$ for every $O \in \mathcal{O}_N$. Thus, $N_{\sigma^+}(S_O, \mathcal{P}_O)$ is already defined.

Assume now that $r_\sigma > 0$, i.e. that $\sigma(0) \in \{\mathsf{ext}, \mathsf{loc}, \mathsf{all}\}$. Then $r_{\sigma^+} < r_\sigma$, so again $N_{\sigma^+}(S_O, \mathcal{P}_O)$ is already defined. In either case we let

$$N_\sigma(S, \mathcal{P}) := N \cup \bigcup_{O \in \mathcal{O}_N} N_{\sigma^+}(S_O, \mathcal{P}_O)\,. \tag{2.31}$$

Theorem 2.2.12. *Every strategy σ determines for every feasible task $(S, \mathcal{P})$ in a graph G a nested subsystem N_σ of S that distinguishes all the profiles in $\mathcal{P}$.*

These sets N_σ are canonical in that, for each σ, the map $(G, S, \mathcal{P}) \mapsto N_\sigma$ commutes with all isomorphisms $G \mapsto G'$. In particular, if S and $\mathcal{P}$ are invariant under the automorphisms of G, then so is N_σ.

Proof. We apply induction along the recursion in the definition of $N_\sigma = N_\sigma(S, \mathcal{P})$. If $s = 0$, then $N_\sigma = S$ distinguishes all the profiles in $\mathcal{P}$, because $(S, \mathcal{P})$ is a task.

[31] More generally, if we apply $\mathsf{all}(S, \mathcal{P})$ in the first step to obtain N, say, then every subtask $(S_O, \mathcal{P}_O)$ with $O \in N_O$ will have this property: if a separation $(A, B) \in S_O$ was nested with all of S_O it would in fact be nested with all of S (and have been included in N), by Lemma 2.2.3 (ii).

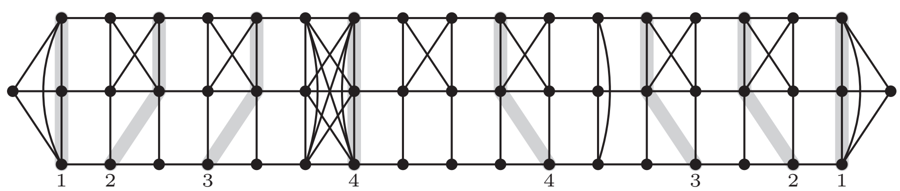
[32] For the remainder of this section, G and $(S, \mathcal{P})$ will no longer be fixed.

Suppose now that $s \geq 1$. Then N_σ is defined by (2.31). Both N and the sets $N_{\sigma^+}(S_O, \mathcal{P}_O)$ are subsets of S, hence so is N_σ. By definition, N is nested with all of S, in particular, with itself and the sets $N_{\sigma^+}(S_O, \mathcal{P}_O)$. These sets are themselves nested by induction, and nested with each other by Lemma 2.2.3 (ii). Thus, N_σ is a nested subset of S.

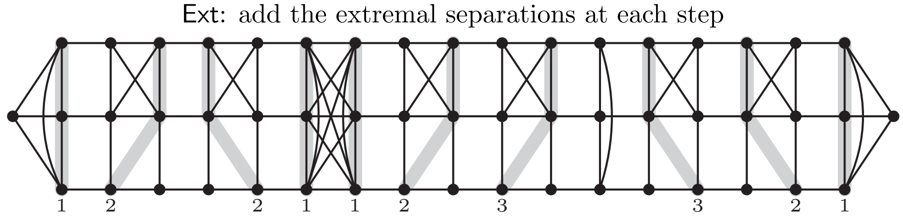
Any two profiles in the same N-block of $\mathcal{P}$ are, by induction, distinguished by $N_{\sigma^+}(S_O, \mathcal{P}_O)$ for their common (P1)-orientation O (cf. Lemma 2.2.10 (iii)). Profiles from different N-blocks of $\mathcal{P}$ are distinguished by N. Hence N_σ distinguishes $\mathcal{P}$.

Finally, the maps $(S, \mathcal{P}) \mapsto N_\sigma$ commute with all isomorphisms $G \mapsto G'$. Indeed, the maps $(S, \mathcal{P}) \mapsto N$ and hence $(S, \mathcal{P}) \mapsto \{\, (S_O, \mathcal{P}_O) \mid O \in \mathcal{O}_N \,\}$ do by definition of $\mathsf{ext}, \mathsf{loc}, \mathsf{all}, \mathsf{ext_r}, \mathsf{loc_r}, \mathsf{all_r}$, and the maps $(S_O, \mathcal{P}_O) \mapsto N_{\sigma^+}(S_O, \mathcal{P}_O)$ do by induction. $\square$

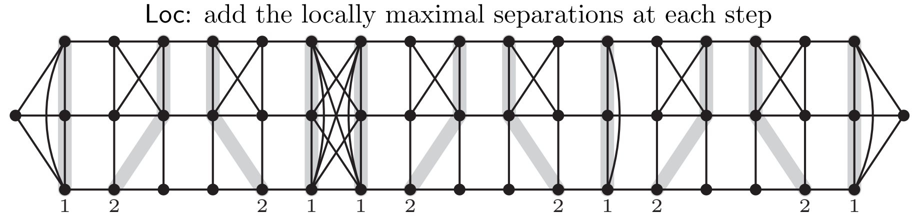
Let us complete this section with an example of how the use of different strategies can yield different nested separation systems. Unlike in the simpler Example 2.2.8, these will not extend each other, but will be incomparable under set inclusion. Let Ext, Loc and All denote the strategies given by setting $\mathsf{Ext}(i) = \mathsf{ext_r}$ and $\mathsf{Loc}(i) = \mathsf{loc_r}$ and $\mathsf{All}(i) = \mathsf{all_r}$, respectively, for all $i \in \mathbb{N}$.



Ext: add the extremal separations at each step



Loc: add the locally maximal separations at each step



All: add all possible separations at each step

Figure 2.13: Three different nested separation systems distinguishing the 4-blocks

Example 2.2.13. Let G be the 3-connected graph obtained from a (3×17)-grid by attaching two K^4s at its short ends, and some further edges as in Figure 2.13. Let S be the set of all its 3-separations, and $\mathcal{P}$ the set of all its 4-block profiles. It is not hard to show, and will follow from Lemma 2.2.14, that $(S,\mathcal{P})$ is a feasible task.

The grey bars in each of the three copies of the graph highlight the separators of the separations in $N_{\mathsf{Ext}}(S,\mathcal{P})$, in $N_{\mathsf{Loc}}(S,\mathcal{P})$, and in $N_{\mathsf{All}}(S,\mathcal{P})$. The step at which a separator was added is indicated by a number.

Note that the three nested separation systems obtained are not only $\subseteq$-incomparable. They are not even nested with each other: for every pair of N_{Ext}, N_{Loc} and N_{All} we can find a pair of crossing separations, one from either system.

2.2.4 Iterated strategies and tree-decompositions

Let us apply the results of Section 2.2.3 to our original problem of finding, for any set $\mathcal{P}$ of k-profiles in G, within the set S of all proper $(<k)$-separations of G a nested subset that distinguishes $\mathcal{P}$ (and hence gives rise to a tree-decomposition of adhesion $<k$ that does the same). This is easy now if G is $(k-1)$-connected:

Lemma 2.2.14. *If G is $(k-1)$-connected $(k\geq 1)$, then $(S,\mathcal{P})$ is a feasible task.*

Proof. The pair $(S,\mathcal{P})$ clearly is a task. So let us show that it is feasible (2.26). Let $(A,B)\nparallel(C,D)\in S$ and $P,P'\in\mathcal{P}$ be such that $(A,B),(C,D)\in P$ and $(B,A),(D,C)\in P'$. We first prove that $(E,F):=(A\cup C,B\cap D)$ has order at most $k-1$.

Suppose (E,F) has order greater than $k-1$. By Lemma 2.2.1 this implies that the separation $(X,Y):=(B\cup D,A\cap C)$ has order less than $k-1$, and hence is improper since G is $(k-1)$-connected. As both (A,B) and (C,D) are proper separations and hence $X\setminus Y=(B\cup D)\setminus(A\cap C)\neq\emptyset$, we then have $Y\subseteq X$. Then $(X,Y)\notin P'$, by (2.23). But by definition of (X,Y) and (P2) for P' we also have $(Y,X)\notin P'$. This contradicts (2.24).

We have shown that (E,F) has order at most $k-1$. By (2.24) and (P2) this implies that $(E,F)\in P$. To complete our proof of (2.26), it remains to show that $(E,F)\in S$, i.e. that (E,F) is proper. If it is improper then $F\subseteq E=V$, since $E\setminus F\supseteq A\setminus B\neq\emptyset$ and therefore $F\neq V$. Hence $(F,E)\in P$ by (P1) or (2.23), which contradicts (2.24) since also $(E,F)\in P$. $\square$

Coupled with Lemma 2.2.14, we can apply Theorem 2.2.12 as follows:

Corollary 2.2.15. *Every strategy σ determines for every $(k-1)$-connected graph G a canonical nested system of separations of order $k-1$ which distinguishes all the k-profiles of G.* $\square$

If G is not $(k-1)$-connected, the task $(S,\mathcal{P})$ consisting of the set S of all proper $(<k)$-separations of G and the given set $\mathcal{P}$ of k-profiles need not be feasible. Indeed, the separation $(A\cup C,B\cap D)$ in (2.26) might have order $\geq k$ even if both (A,B) and (C,D) have order $<k$. Then if $B\cap D$ induces a big complete graph, for example, there will be no (E,F) as required in (2.26).

However, if $\mathrm{ord}(A,B) = \mathrm{ord}(C,D) = k-1$ in this example, the separation $(B\cup D, A\cap C)$ will have some order $\ell < k-1$. This separation, too, distinguishes the profiles P, P' given in (2.26). Hence our dilemma of having to choose between (A,B) and (C,D) for inclusion in our nested subset N of S (which gave rise to (2.26) and the notion of feasibility) would not occur if we considered lower-order separations first: we would then have included $(B\cup D, A\cap C)$ in N, and would need neither (A,B) nor (C,D) to distinguish P from P'.

It turns out that this approach does indeed work in general. Given our set $\mathcal{P}$ of k-profiles, let us define for any $1 \le \ell \le k$ and $P \in \mathcal{P}$ the induced ℓ-profile

$$P_\ell := \{\, (A,B) \in P \mid \mathrm{ord}(A,B) < \ell \,\}, \quad \text{and set} \quad \mathcal{P}_\ell := \{\, P_\ell \mid P \in \mathcal{P} \,\}.$$

Note that distinct k-profiles P may induce the same ℓ-profile P_ℓ. Let $\kappa(P,P')$ denote the least order of any separation in G that distinguishes two profiles P, P'.

The idea now is to start with a nested set $N_1 \subseteq S$ of (<1)-separations that distinguishes $\mathcal{P}_1$, then to extend N_1 to a set $N_2 \subseteq S$ of (<2)-separations that distinguishes $\mathcal{P}_2$, and so on. The tasks $(S_O, \mathcal{P}_O)$ to be solved at step k, those left by the (P1)-orientations O of N_{k-1}, will then be feasible: since N_{k-1} already distinguishes $\mathcal{P}_{k-1}$, and hence distinguishes any $P, P' \in \mathcal{P}$ with $\kappa(P,P') < k-1$, any P, P' in a common $\mathcal{P}_O$ will satisfy $\kappa(P,P') = k-1$, and (2.26) will follow from Lemma 2.2.1 as in the proof of Lemma 2.2.14.

What is harder to show is that those $(S_O, \mathcal{P}_O)$ are indeed tasks: that S_O is rich enough to distinguish $\mathcal{P}_O$. This will be our next lemma. Let us say that a separation (A,B) of order ℓ that distinguishes two profiles P and P' does so *efficiently* if $\kappa(P,P') = \ell$. We say that (A,B) is $\mathcal{P}$-*essential* if it efficiently distinguishes some pair of profiles in $\mathcal{P}$. Note that for $\ell \le m$ we have $(\mathcal{P}_m)_\ell = \mathcal{P}_\ell$, and if (A,B) is $\mathcal{P}_\ell$-essential it is also $\mathcal{P}_m$-essential.

Lemma 2.2.16. *Let $\mathcal{P}$ be a set of k-profiles in G, let N be a nested system of $\mathcal{P}_{k-1}$-essential separations of G that distinguishes all the profiles in $\mathcal{P}_{k-1}$ efficiently, and let S be the set of all proper $(k-1)$-separations of G that are nested with N. Then for every* (P1)*-orientation O of N the pair $(S_O, \mathcal{P}_O)$ is a feasible task.*

Proof. As pointed out earlier, $(S_O, \mathcal{P}_O)$ will clearly be feasible once we know it is a task. Since all profiles in $\mathcal{P}_O$ are k-profiles and hence orient S_O, we only have to show that S_O distinguishes $\mathcal{P}_O$.

So consider distinct profiles $P_1, P_2 \in \mathcal{P}_O$. Being k-profiles, they are distinguished by a separation (A,B) of order at most $k-1$. Choose (A,B) nested with as many separations in N as possible; we shall prove that it is nested with all of N, giving $(A,B) \in S_O$ as desired. Note that $\mathrm{ord}(A,B) = k-1$, since N does not distinguish $P_1, P_2 \in \mathcal{P}_O$. As (A,B) distinguishes P_1 from P_2, we may assume $(B,A) \in P_1$ and $(A,B) \in P_2$.

Suppose (A,B) crosses a separation $(C,D) \in N$. Since every separation in N is $\mathcal{P}_{k-1}$-essential, by assumption, there are profiles $Q_1', Q_2' \in \mathcal{P}_{k-1}$ such that (C,D) distinguishes Q_1' from Q_2' efficiently. By definition of $\mathcal{P}_{k-1}$, this implies that there are distinct profiles $Q_1, Q_2 \in \mathcal{P}$ which (C,D) distinguishes efficiently. Then

$$m := \mathrm{ord}(C,D) < k-1 = \mathrm{ord}(A,B). \tag{2.32}$$

Hence (C,D) does not distinguish P_1 from P_2; we assume that $(D,C) \in P_1 \cap P_2 \cap Q_1$ and $(C,D) \in Q_2$. Since Q_2 is a k-profile it contains precisely one of (A,B) and (B,A), we assume $(A,B) \in Q_2$ (Figure 2.14).

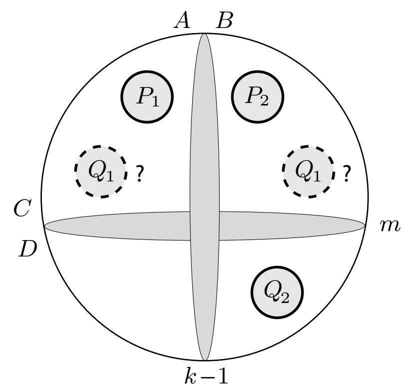


Figure 2.14: The known positions of P_1, P_2, Q_1 and Q_2

Now if $(X,Y) := (A \cup C, B \cap D)$ has order $< m$, then $(X,Y) \in Q_2$ by (P2), and $(Y,X) \in Q_1$ by (P1). Hence (X,Y) distinguishes Q_1 from Q_2 and has smaller order than (C,D), contradicting the fact that (C,D) distinguishes Q_1 and Q_2 efficiently. Thus (X,Y) has order at least m.

Hence by (2.32) and Lemma 2.2.1, the order of $(E,F) := (B \cup D, A \cap C)$ is at most $k-1$. Then $(E,F) \in P_1$ by (P2), and $(F,E) \in P_2$ by (P1). Thus (E,F) distinguishes P_1 from P_2. By [42, Lemma 2.2],[33] (E,F) is nested with every separation in N that (A,B) is nested with, and in addition (E,F) is also nested with (C,D). Hence, (E,F) is nested with more separations in N than (A,B) is, contradicting the choice of (A,B). $\square$

When we apply Lemma 2.2.16 inductively, we have to make sure that every N_ℓ we construct consists only of $\mathcal{P}_\ell$-essential separations. To ensure this, we have to reduce any task we tackle in the process of constructing N_ℓ. Given $k \geq 1$, a *k-strategy* is a k-tuple $(\sigma_1, \ldots, \sigma_k)$ of strategies σ_i each with range $\{\mathsf{ext_r}, \mathsf{loc_r}, \mathsf{all_r}\}$. The restriction in the range of k-strategies will reduce our freedom in shaping the decompositions, but Example 2.2.13 shows that considerable diversity remains.

Given G and $\mathcal{P}$, a k-strategy $\Sigma = (\sigma_1, \ldots, \sigma_k)$ *determines* the set $N_\Sigma = N_\Sigma(G, \mathcal{P})$ defined recursively as follows. For $k = 1$, let $N_\Sigma := N_{\sigma_1}(S, \mathcal{P})$, where S is the set of proper (< 1)-separations of G. Then for $k \geq 2$ let

$$N_\Sigma := N \cup \bigcup_{O \in \mathcal{O}_N} N_{\sigma_k}(S_O, \mathcal{P}_O)\,, \tag{2.33}$$

[33] Swap the names of (C,D) and (E,F) in the statement of the lemma in [42].

where $N = N_{\Sigma'}(G, \mathcal{P}_{k-1})$ for $\Sigma' = (\sigma_1, \ldots, \sigma_{k-1})$, and S is the set of proper $(k-1)$-separations of G that are nested with N. The pairs $(S_O, \mathcal{P}_O)$ are defined with reference to N, S and $\mathcal{P} = \mathcal{P}_k$ as before Lemma 2.2.10.

As before, the sets N_Σ will be *canonical* in that, for each Σ, the map $(G, \mathcal{P}) \mapsto N_\Sigma$ commutes with all isomorphisms $G \mapsto G'$. In particular, if $\mathcal{P}$ is invariant under the automorphisms of G, then so is N_Σ.

Theorem 2.2.17. *Every k-strategy Σ determines for every set $\mathcal{P}$ of k-profiles of a graph G a canonical nested system $N_\Sigma(G, \mathcal{P})$ of $\mathcal{P}$-essential separations of order $< k$ that distinguishes all the profiles in $\mathcal{P}$ efficiently.*

Proof. We show by induction on k that the recursive definition of $N_\Sigma(G, \mathcal{P})$ succeeds and that $N_\Sigma = N_\Sigma(G, \mathcal{P})$ has the desired properties. For $k = 1$ this follows from Corollary 2.2.15.

For $k \geq 2$ let N and S be defined as before the theorem. By the induction hypothesis, N is a nested system of $\mathcal{P}_{k-1}$-essential separations of G that distinguishes the profiles in $\mathcal{P}_{k-1}$ efficiently. For every (P1)-orientation O of N the pair $(S_O, \mathcal{P}_O)$ is a feasible task, by Lemma 2.2.16. By Theorem 2.2.12, then, σ_k determines a nested separation system $N_{\sigma_k}(S_O, \mathcal{P}_O) \subseteq S_O \subseteq S$ that distinguishes all the profiles in $\mathcal{P}_O$. By definition of S, all these $N_{\sigma_k}(S_O, \mathcal{P}_O)$ are nested with N, and they are nested with each other by Lemma 2.2.3. Hence N_Σ is well defined by (2.33) and forms a nested separation system.

Let us show that N_Σ has the desired properties. To show that N_Σ distinguishes the profiles in $\mathcal{P}$ efficiently, consider distinct $P, Q \in \mathcal{P}$. If $\kappa(P, Q) < k-1$, then $P_{k-1} \neq Q_{k-1}$ are distinct profiles in $\mathcal{P}_{k-1}$. So by the induction hypothesis there is a separation in $N \subseteq N_\Sigma$ that distinguishes P from Q efficiently. If $\kappa(P, Q) = k - 1$, we have $P_{k-1} = Q_{k-1}$. Then P and Q have the same N-profile O, and $P, Q \in \mathcal{P}_O$. Hence there is a separation in $N_{\sigma_k}(S_O, \mathcal{P}_O) \subseteq N_\Sigma$ that distinguishes P from Q; as it has order $k - 1$, it does so efficiently.

It remains to show that every separation $(A, B) \in N_\Sigma$ is $\mathcal{P}$-essential. If $(A, B) \in N$, this holds by the induction hypothesis and the definition of $\mathcal{P}_{k-1}$. So assume that $(A, B) \in N_\Sigma \setminus N$. Then there is a (P1)-orientation O of N such that $(A, B) \in N_{\sigma_k}(S_O, \mathcal{P}_O)$. Since $\sigma_k(i) \in \{\mathsf{ext_r}, \mathsf{loc_r}, \mathsf{all_r}\}$ for all $i \in \mathbb{N}$, we know that (A, B) distinguishes some $P, Q \in \mathcal{P}_O$. Then $\kappa(P, Q) = k - 1 = \mathrm{ord}(A, B)$, as otherwise N would distinguish P from Q by the induction hypothesis. Hence, (A, B) distinguishes P from Q efficiently, as desired. $\square$

It remains to translate our results from separation systems to tree-decompositions. Recall from Theorem 2.2.2 that every nested separation system N of G is induced by a tree-decomposition $(T, \mathcal{V})$: the separations of G that correspond to edges of T are precisely those in N. In [42] we showed that $(T, \mathcal{V})$ is uniquely determined by N.[34] Hence if N is determined by some k-strategy, as in Theorem 2.2.17, we may say that this k-strategy *defines* $(T, \mathcal{V})$ on G.

[34]We assume here that parts corresponding to different nodes of T are distinct. It is always possible to artificially enlarge the tree without changing the set of separations by duplicating a part.

If N comes from an application of Theorem 2.2.17, it will be canonical. In particular, if the set $\mathcal{P}$ of profiles considered is invariant under the automorphisms of G, then so is N, and hence so is T: the automorphisms of G will act on $V(T)$ as automorphisms of T. And many natural choices of $\mathcal{P}$ are invariant under the automorphisms of G: the set of all k-profiles for given k, the set of all k-block profiles, or the set of all tangles of order k to name some examples. All these can thus be distinguished in a single tree-decomposition:

Theorem 2.2.18. *Given $k \in \mathbb{N}$ and a graph G, every k-strategy defines a canonical tree-decomposition of adhesion $< k$ of G that distinguishes all its k-blocks and tangles of order k. In particular, such a decomposition exists.* □

Theorem 2.2.18 is not the end of this story, but rather a beginning. One can now build on the fact that these tree-decompositions are given constructively and study their details. For example, we may wish to find out more about the structure or size of their parts, or obtain bounds on the number of parts containing a k-block or accommodating a tangle of order k, compared with the total number of parts. The answers to such questions will depend on the k-strategy chosen. We shall pursue such questions in [40].

2.3 Canonical tree-decompositions of finite graphs II. Essential parts

2.3.1 Introduction

Given an integer k, a *k-block* X in a graph G is a maximal set of at least k vertices no two of which can be separated in G by fewer than k other vertices; these may or may not lie in X. Thus, k-blocks for large k can be thought of as highly connected pieces of a graph, but their connectivity is measured not in the subgraph they induce but in the ambient graph.

Another concept of highly connected pieces of a graph, formally quite different from k-blocks, is the notion of a tangle proposed by Robertson and Seymour [94]. Tangles are not defined directly in terms of vertices and edges, but indirectly by assigning to every low-order separation of the graph one of its two sides, the side in which 'the tangle' is assumed to sit. In order for this to make sense, the assignments of sides have to satisfy some consistency constraints, in line with our intuition that one tangle should not be able to sit in disjoint parts of the graph at once.

In a fundamental paper on graph connectivity and tree structure, Hundertmark [78] showed that high-order blocks and tangles have a common generalization, which he called 'profiles'. These also work for discrete structures other than graphs. We continue to work with profiles in this paper. All the reader needs to know about profiles is explained in [39], that is Section 2.2 of this thesis.

In [39] we described a family of algorithms which construct, for any finite graph G and $k \in \mathbb{N}$, a tree-decomposition of G that has two properties: it distinguishes all the k-blocks and tangles of order k in G, so that distinct blocks or tangles come to sit in distinct parts of the decomposition, and it is canonical in that the map assigning this decomposition to G commutes with graph isomorphisms.

In this follow-up to [39], we study these decompositions in more detail. Given k, let us call a part of such a decomposition *essential* if it contains a k-block or accommodates a tangle of order k. (Precise definitions will follow.) Since the aim of our tree-decompositions is to display how G can be cut up into its highly connected pieces, ideally every part of such a decomposition would be essential, and the essential parts containing a k-block would contain nothing else. (This makes no sense for tangles, since they cannot be captured by a set of vertices.)

Neither of these aims can always be attained. Our objective is to see when or to which extent they can. After providing in Section 2.3.2 some background on how tree-decompositions relate to oriented separation systems, we devote Section 2.3.3 to establishing upper bounds on the number of inessential parts in a canonical tree-decomposition of a graph that distinguishes all its k-profiles. These bounds depend in interesting ways on the algorithm chosen to find the decomposition. All the bounds we establish are sharp.

In Section 2.3.4 we investigate to what extent the decomposition parts containing a k-block can be required to contain nothing else. It turns out that there can be k-blocks that never occur as entire parts in a canonical tree-decomposition, due to a local obstruction in terms of the way in which these blocks are separated from the rest of G. We find a sufficient local condition on k-blocks X ensuring that any part containing X contains nothing else, but show that this condition is not in general necessary. It remains an open problem to either weaken our condition to one that is both necessary and sufficient, or to show that no such local condition exists. More generally, the following problem remains open: is there a canonical tree-decomposition of G in which all those k-blocks are parts that occur as a part in some canonical tree-decomposition of G? Finally, we establish some sufficient global conditions on G to ensure that any decomposition part containing a k-block contains nothing else; in particular, these conditions imply that there are no local obstructions to this as found earlier, for any k-block.

In order to read this paper with ease, the reader should be familiar with [39]; in particular with the terminology introduced in Section 2 there, the notions of a task and a strategy as defined in Section 3, and the notion of a k-strategy as defined in Section 4. The proofs in [39] need not be understood in detail, but Examples 1 and 4 make useful background.

Readers interested in k-blocks as such may refer to [41], where we relate the greatest number k such that G has a k-block to other graph invariants.

Throughout this paper, we consider a fixed finite graph $G = (V, E)$.

2.3.2 Orientations of decomposition trees

By [39, Theorem 2.2], every nested separation system N of our graph $G = (V, E)$ gives rise to a tree-decomposition $(T, \mathcal{V})$ that *induces* it, in that the (orientations of) edges of the decomposition tree T correspond to the separations in N. How exactly $(T, \mathcal{V})$ can be obtained from N is described in [42]. In this paper we shall be concerned with how profiles – in particular, blocks and tangles – correspond to nodes of T. This correspondence will be injective – distinct blocks or tangles will 'live in' distinct nodes of T – but it will not normally be onto: only some of the nodes of T will accommodate a block or tangle (all of some fixed order k).

However, there is a bijective correspondence between the nodes of T and a set of objects more general than profiles:[35] the (P1)-orientations of N. In this section we point out this correspondence. Let $\mathcal{V} = (V_t)_{t \in T}$.

As $(T, \mathcal{V})$ induces N, there is for every separation $(A, B) \in N$ an oriented edge $e = t_A t_B$ of T such that, if T_A denotes the component of $T - e$ that contains t_A and T_B denote the component containing t_B, we have

$$(A, B) = \Big(\bigcup_{t \in T_A} V_t \, , \, \bigcup_{t \in T_B} V_t \Big).$$

If $(T, \mathcal{V})$ was obtained from N as in [39, Theorem 2.2], then e is unique, and we say that it *represents* (A, B) *in* T.[36]

Every node $t \in T$ induces an orientation of the edges of T, towards it. This corresponds as above to an orientation $O(t)$ of N,

$$O(t) := \{(A, B) \in N \mid t \in T_B\},$$

from which we can reobtain the part V_t of $(T, \mathcal{V})$ as

$$V_t \;=\; \bigcap_{(A,B) \in O(t)} B. \tag{2.34}$$

We say that t *induces* the orientation $O(t)$ of N, and that the separations in $O(t)$ are *oriented towards* t.

Distinct nodes $t, t' \in T$ induce different orientations of N, since these orientations disagree on every separation that corresponds to an edge on the path tTt'. Also clearly, not all orientations of N are induced by a node of T. But it is interesting in our context to see which are:

Theorem 2.3.1. (i) *The orientations of N that are induced by nodes of T are precisely the* (P1)*-orientations of N.*

(ii) *An orientation of the set of all $(< k)$-separations of G orients the separations induced by any tree-decomposition of adhesion $< k$ towards a node of its decomposition tree if and only if it satisfies* (P1)*.*

Proof. (i) Let O be an orientation of N that is not induced by a node in T, and consider the corresponding orientation of (the edges of) T. Then there are edges e, e' of T that point in opposite directions. Indeed, follow the orientated

[35] Recall that profiles are sets of oriented separations satisfying two axioms, (P1) and (P2).

[36] In general if e is not unique, we can make it unique by contracting all but one of the edges of T inducing a given partition in N, merging the parts corresponding to the nodes of contracted edges.

edges of T to a sink t; this exists since T is finite. As t does not induce O, some oriented edge $e' = t't''$ has t lie in the component of $T - e'$ that contains t'. Then $T - e'$ contains a t'–t path. Its last edge e is oriented towards t, by the choice of t. The separations $(A, B), (C, D) \in O$ represented by e and e' then satisfy $(B, A) \leq (C, D)$, so O violates (P1).

For the converse implication suppose an orientation $O(t)$ induced by some $t \in T$ violates (P1). Then there are $(A, B), (C, D) \in O(t)$ with $(D, C) \leq (A, B)$. Let e be the oriented edge of T representing (A, B), and let f be the oriented edge representing (C, D).

Consider the subtrees T_A, T_B, T_C, T_D of T. Note that $T_B \cap T_D$ contains t, by definition of $O = O(t)$, and hence contains the component T_t of $T - e - f$ containing t.

If $f \in T_A$, then T_B is a connected subgraph of $T - f$ containing t, and hence contained in T_D. With $T_B \subseteq T_D$ we also have $B \subseteq D$. But now $(D, C) \leq (A, B)$ implies $B \subseteq D \subseteq A$, and so (A, B) is not a proper separation. But it is, since N is a separation system. Hence $f \in T_B$, and similarly $e \in T_D$.

Let us show that $t_B, t_D \in T_t$. Suppose f lies on the path in T from t to t_B. Then this path traverses f from t_D to t_C, since its initial segment from t to f lies in T_D (the component of $T - f$ containing t) and hence ends in t_D. But then $e \in T_C$, contrary to what we have shown. Thus $f \notin tTt_B$, and clearly also $e \notin tTt_B$. Therefore $t_B \in T_t$, and similarly $t_D \in T_t$.

Since $f \notin T_A$, we know that T_A is a connected subgraph of $T - f$ containing an end of e. Adding e to it we obtain a connected subgraph of $T - f$ that contains both ends of e and therefore meets T_t, and adding T_t too we obtain a connected subgraph of $T - f$ that contains both T_A and t_D. Therefore $T_A \subseteq T_D$, and thus $A \subseteq D$. Analogously, $C \subseteq B$. But now $(D, C) \leq (A, B)$ implies both $A \subseteq D \subseteq A$ and $C \subseteq B \subseteq C$, giving $(A, B) = (D, C)$. But then O contains both (C, D) and (D, C), which contradicts its definition as an orientation of N.

(ii) If a given orientation of the set S_k of all $(< k)$-separations of G satisfies (P1), then so does the orientation it induces on the nested separation system N induced by any tree-decomposition of adhesion $< k$. By (i), this orientation of N orients it towards a node of the decomposition tree.

Conversely, if an orientation of S_k violates (P1), then this is witnessed by separations $(A, B), (C, D) \in S_k$ with $(C, D) \leq (A, B)$ such that (A, B) is oriented towards B but (C, D) is oriented towards C. By [39, Theorem 2.2], $N = \{(A, B), (B, A), (C, D), (D, C)\}$ is induced by a tree-decomposition $(T, \mathcal{V})$. Since the orientation $\{(D, C), (A, B)\}$ which our given orientation of S_k induces on N violates (P1), we know from (i) that it does not orient N towards any node of T. $\square$

Theorem 2.3.1 (i) implies in particular that any profile P which orients N defines a unique node $t \in T$: the t that induces its N-profile $P \cap N = O(t)$. We say that P *inhabits* this node t and the corresponding part V_t. If P is a k-block profile, induced by the k-block X, say, then this is the case if and only if $X \subseteq V_t$.

Given a set $\mathcal{P}$ of profiles, we shall call a node t of T and the corresponding part V_t *essential (wrt. $\mathcal{P}$)* if there is a profile in $\mathcal{P}$ which inhabits t.

Nodes t such that $V_t \subseteq A \cap B$ for some $(A, B) \in N$ are called *hub nodes*; the node t itself is then a *hub*. Example 2 in [39] shows that distinct hub nodes $t, t' \in T$ may have the same hub $V_t = V_t'$. So the bijection established by Theorem 2.3.1 does not induce a similar correspondence between the (P1)-orientations of N and the parts of $(T, \mathcal{V})$ as a set, only as a family $\mathcal{V} = (V_t)_{t \in T}$. This is illustrated by [42, Figure 7].

Theorem 2.3.1 (ii) will not be needed in the rest of this paper. But it is interesting in its own right, in that it provides a converse to the following well-known fact in graph minor theory. Every haven [96], preference [93] or bramble [52] of order $\geq k$ in G orients the set S_k of all $(< k)$-separations of G (e.g., 'towards' that bramble). In particular, it orients the separations induced by any tree-decomposition of adhesion $< k$, and it orients these towards a node of that decomposition tree. But this fact has no converse: it is possible to orient S_k in such a way that the separations induced by any tree-decomposition of adhesion $< k$ are oriented towards a node t of the decomposition tree, but this orientation of S_k is not a haven or preference of order k; there is no bramble of order $\geq k$ 'living in' t.[37]

Theorem 2.3.1 (ii) shows that the (P1)-orientations of S_k, which are generalizations of havens or preferences of order k since these satisfy (P1), are the unique weakest-possible such generalization that still orients all tree-decompositions of adhesion $< k$ towards a node.

2.3.3 Bounding the number of inessential parts

Let $k \in \mathbb{N}$, and let $\mathcal{P}$ be a set of k-profiles of our graph G, both fixed throughout this section. Whenever we use the term 'essential' in this section, this will be with reference to this set $\mathcal{P}$.

Any canonical tree-decomposition distinguishing $\mathcal{P}$ has at least $|\mathcal{P}|$ essential parts, one for every profile in $\mathcal{P}$. Our aim in this section is to bound its number of inessential parts in terms of $|\mathcal{P}|$.

Variants of [39, Example 1] show that no such bounds exist if we ever use a strategy that has all, $\mathsf{all_r}$, ext or loc among its values, so we confine ourselves to strategies with values in $\{\mathsf{ext_r}, \mathsf{loc_r}\}$.

The definition of the parts of a tree-decomposition $(T, \mathcal{V})$ being somewhat complicated (see Section 2.3.2), rather than bounding the number $|\mathcal{V}| - |\mathcal{P}|$ of inessential parts of $(T, \mathcal{V})$ directly, we shall bound the number $|N|$ instead. Since $\frac{1}{2}|N|$ is the number of edges of T – as N contains 'oriented' separations, every edge of T appears twice – and $\frac{1}{2}|N| + 1$ its number of nodes, the number of inessential parts will then be $\frac{1}{2}|N| + 1 - |\mathcal{P}|$.

Our aim, then, will be to choose a strategy that minimizes $|N|$. Our strategies should therefore take values in $\{\mathsf{ext_r}, \mathsf{loc_r}\}$ only, i.e., we should reduce

[37]For example, identify three copies of K^5 in one vertex v, and orient every (< 2)-separation towards the side that contains two of these K^5. This is a (P1)-orientation of S_2 that is not a 2-haven or 2-preference and is not induced by a bramble of order ≥ 2, but which still orients the 1-separations of any tree-decomposition of adhesion 1 towards a node t (whose corresponding part could be either a K^5 or a K^1 hub).

our tasks before we tackle them, by deleting separations that do not distinguish any profiles in $\mathcal{P}$. Moreover, for a single reduced task $(S, \mathcal{P})$ we have $\mathsf{ext}(S, \mathcal{P}) \subseteq \mathsf{loc}(S, \mathcal{P})$ by [39, (11)], i.e., every separation chosen by ext is also chosen by loc. This suggests that the overall strategy Ext, which only uses $\mathsf{ext_r}$, should also return fewer separations than Loc, which only uses $\mathsf{loc_r}$ – perhaps substantially fewer, since if we select fewer separations at each step we also have more interim steps in which we reduce.

Surprisingly, this is not the case. Although our general bounds for Ext are indeed better than those for Loc (or the same, which is already a surprise), Example 2.3.5 below will show Loc yields better results than Ext for some graphs.

Let $|\mathcal{P}| =: p$. For single tasks $(S, \mathcal{P})$, we obtain the following bounds on $|N|$:

Lemma 2.3.2. *For every feasible task $(S, \mathcal{P})$ we have*

$$2(p-1) \leq |N_{\mathsf{Ext}}(S, \mathcal{P})| \leq 2p, \text{ and} \tag{2.35}$$

$$2(p-1) \leq |N_{\mathsf{Loc}}(S, \mathcal{P})| \leq 4(p-1). \tag{2.36}$$

Proof. The two lower bounds, which in fact hold for any strategy, follow from the fact that N gives rise to a tree-decomposition $(T, \mathcal{V})$ that induces it and distinguishes $\mathcal{P}$: this means that $|N| = 2\,(|T| - 1) \geq 2(p-1)$.

Let us now prove the upper bound in (2.35), by induction on p. Let $(R, \mathcal{P})$ be the reduction of $(S, \mathcal{P})$. If $p \leq 1$ then $R = \emptyset$, so the statement is trivial. Now assume that $p \geq 2$. Then $S \neq \emptyset$, since S distinguishes $\mathcal{P}$. Let $\mathcal{P}_{\mathcal{E}}$ be the set of profiles in $\mathcal{P}$ that are extremal in $(R, \mathcal{P})$. By [39, Lemma 3.1] we have $\mathcal{P}_{\mathcal{E}} \neq \emptyset$. Then

$$N_{\mathsf{Ext}}(S, \mathcal{P}) = \; N \;\cup \bigcup_{O \,\in\, \mathcal{O}_N} N_{\mathsf{Ext}}(S_O, \mathcal{P}_O)\,, \tag{2.37}$$

by definition of $N_{\mathsf{Ext}}(S, \mathcal{P})$, where $N = \mathsf{ext_r}(S, \mathcal{P})$. Every extremal $P \in \mathcal{P}$ is distinguished from all the other profiles in $\mathcal{P}$ by the separation (A, B) for which $P = P_{(A,B)}$, so P lies in a singleton class $\mathcal{P}_O = \{P\}$. Then $(S_O, \mathcal{P}_O)$ reduces to $(\emptyset, \mathcal{P}_O)$, giving $N_{\mathsf{Ext}}(S_O, \mathcal{P}_O) = \emptyset$ for these $O \in \mathcal{O}_N$. By the uniqueness of $P_{(A,B)}$ in [39, Lemma 3.2], no separation in N separates two non-extremal profiles from $\mathcal{P}$. So there is at most one other partition class $\mathcal{P}_O$ with $O \in \mathcal{O}_N$. If such a $\mathcal{P}_O$ exists it satisfies $\mathcal{P}_O = \mathcal{P} \setminus \mathcal{P}_{\mathcal{E}}$, and if it is non-empty the $O \in \mathcal{O}_N$ giving rise to it is unique. Therefore

$$N_{\mathsf{Ext}}(S, \mathcal{P}) = \mathsf{ext_r}(S, \mathcal{P}) \cup N_{\mathsf{Ext}}(S_O, \mathcal{P} \setminus \mathcal{P}_{\mathcal{E}})$$

for this O if $\mathcal{P} \setminus \mathcal{P}_{\mathcal{E}} \neq \emptyset$, and $N_{\mathsf{Ext}}(S, \mathcal{P}) = \mathsf{ext_r}(S, \mathcal{P})$ otherwise. In the first case we have

$$|N_{\mathsf{Ext}}(S_O, \mathcal{P} \setminus \mathcal{P}_{\mathcal{E}})| \leq \; 2\,|\mathcal{P} \setminus \mathcal{P}_{\mathcal{E}}|$$

by the induction hypothesis, and in both cases we have $|\mathsf{ext_r}(S, \mathcal{P})| \leq 2\,|\mathcal{P}_{\mathcal{E}}|$ by [39, Lemma 3.2 and (9)]. This completes the proof of (2.35).

For a proof of the upper bound in (2.36) let $(T, \mathcal{V})$ be a tree-decomposition of G that induces $N_{\mathsf{Loc}}(S, \mathcal{P})$ as in [39, Theorem 2.2]. Since $N_{\mathsf{Loc}}(S, \mathcal{P})$ contains only $\mathcal{P}$-relevant separations, all the leaves of T are essential. Furthermore, we shall prove the following:

For every edge $e = t_1t_2$ of T, either t_1 or t_2 is essential. (2.38)

Before we prove (2.38), let us show how it helps us establish the upper bound in (2.36). If (2.38) holds, then all the neighbours of an inessential node are essential. Let T' be obtained from T by deleting each inessential node and adding an edge from one of its neigbours to all its other neighbours. Let us show that

T' has p nodes and at least half as many edges as T. (2.39)

The first of these assertions holds by definition of T and p. For the second, note that for each inessential node we delete we lose exactly one edge. So to prove the second claim in (2.39) it suffices to show that T has at most $\|T\|/2$ inessential nodes. But this follows from (2.38) and the fact that the leaves of T are essential: every inessential node has at least two incident edges, and no edge is counted twice in this way (i.e., is incident with more than one inessential node).

By (2.39), T has at most $2(p-1)$ edges. Since $N_{\mathsf{Loc}}(S,\mathcal{P})$ is induced by $(T,\mathcal{V})$, this will establish the upper bound in (2.36).

So let us prove (2.38). Suppose T has an edge $e = t_1t_2$ with neither t_i essential. Let $(A,B) \in N_{\mathsf{Loc}}(S,\mathcal{P})$ be the separation which e induces. Let T_A denote the component of $T - e$ that contains t_1, and let T_B be the component containing t_2.

At the time (A,B) was chosen by Loc we had a nested separation system N and a (P1)-orientation O of N such that $(A,B) \in \mathsf{loc_r}(S_O,\mathcal{P}_O)$. (When $N = \emptyset$ at the start, we have $\mathsf{loc_r}(S_O,\mathcal{P}_O) = (S,\mathcal{P})$.) So there is a profile $P \in \mathcal{P}_O$ such that (A,B) or (B,A) is maximal in $(P \cap S_O, \leq)$, say (A,B). By the definition of a task, P orients S. By Lemma 2.3.1, therefore, P inhabits a unique node $t \in T$, making it essential. Then $(A,B) \in O(t)$, and hence $t \in T_B$. Since t_2 inessential by assumption, $t \neq t_2$.

The last edge e' on the t_2–t path in T induces a separation $(C,D) \in O(t) \subseteq P$, and $(A,B) \leq (C,D)$, or equivalently, $(D,C) \leq (B,A)$. Since (A,B) is $\mathcal{P}_O$-relevant there exists $P' \in \mathcal{P}_O$ with $(B,A) \in P'$. Then $(D,C) \in P'$ by (P1). But then (C,D) splits O, and thus lies in S_O. This contradicts the maximality of (A,B), completing the proof of (2.38) and hence of (2.36). □

It is easy to see that the upper bounds in Lemma 2.3.2 are tight. For example, if G consists of n disjoint large complete graphs threaded on a long path, then for $k = 3$ the canonical tree-decomposition produced by Loc will have n essential parts consisting of these complete graphs and $n-1$ inessential parts consisting of the paths between them. When n is even, this example also shows that the upper bound for Ext is best possible. In fact, the following example shows that the upper bound in Lemma 2.3.2 (i) is best possible for all canonical tree-decompositions (regardless of which strategy is used to produce it):

Example 2.3.3. Let G consist of an n-cycle C together with n large complete graphs $K_1,\dots,K_n$ each intersecting C in one edge and otherwise disjoint.

Then, for $n \geq 3$ and $k = 3$, any canonical tree-decomposition of G of adhesion $< k$ either has exactly one part or exactly the parts $C, K_1, \ldots, K_n$. This is because the 2-separations of G that induce 2-separations of C cannot be induced by a canonical tree-decomposition of G, since they cross their translates under suitable automorphisms of G.

It is more remarkable, perhaps, that the upper bound in Lemma 2.3.2 (i) is so low: that at most one part of the tree-decomposition is not inhabited by a profile from $\mathcal{P}$. For if G is $(k-1)$-connected and S is the set of all proper $(< k)$-separations, then every task $(S, \mathcal{P})$ is feasible [39, Lemma 4.1], and hence Lemma 2.3.2 gives the right overall bounds.

If G is not $(k-1)$-connected, the original task $(S, \mathcal{P})$ need not be feasible, and we have to use iterated strategies. Let Ext^k denote the k-strategy all whose entries are Ext, and let Loc^k denote the k-strategy which only uses Loc. Interestingly, having to iterate costs us a factor of 2 in the case of Ext, but it does not affect the upper bound for Loc. Hence for iterated strategies the two bounds coincide:

Theorem 2.3.4. *Let $\mathcal{P}$ be any set of k-profiles of G, and $p := |\mathcal{P}|$. Let $N_{\mathsf{Ext}^k}(\mathcal{P})$ and $N_{\mathsf{Loc}^k}(\mathcal{P})$ be obtained with respect to the set S of all proper $(< k)$-separations.*

(i) $2(p-1) \leq |N_{\mathsf{Ext}^k}(\mathcal{P})| \leq 4\,(p-1)$

(ii) $2(p-1) \leq |N_{\mathsf{Loc}^k}(\mathcal{P})| \leq 4\,(p-1)$

(iii) *If G is $(k-1)$-connected, then $|N_{\mathsf{Ext}^k}(\mathcal{P})| \leq 2p$.*

Proof. The lower bounds for N follow as in the proof of Lemma 2.3.2. Statement (iii) reduces to Lemma 2.3.2 (i), since $\mathsf{Ext}^k = \mathsf{Ext}$ now and the entire task $(S, \mathcal{P})$ is feasible [39, Lemma 4.1].

For the proof of the upper bounds in (i) and (ii), let us define a rooted tree (T, r) that represents the recursive definition of N_{Ext^k} and N_{Loc^k}, as follows. Let

$$V(T) := \{\emptyset\} \cup \bigcup_{1 \leq \ell \leq k} \mathcal{P}_\ell\,;$$

recall that $\mathcal{P}_\ell$ for $\ell \leq k$ is the set of all ℓ-profiles of G that extend to a k-profile in $\mathcal{P}$. We select $r = \emptyset$ as the root, and make it adjacent to every $P \in \mathcal{P}_1$. For $2 \leq \ell \leq k$ we join $P \in \mathcal{P}_\ell$ to the unique $P' \in \mathcal{P}_{\ell-1}$ which it induces (i.e., for which $P' \subseteq P$). This is clearly a tree, with levels $\{\emptyset\}, \mathcal{P}_1, \ldots, \mathcal{P}_k$. Let us call the vertices of T that are not in $\mathcal{P}_k$ its *internal* vertices.

The internal vertices of T correspond bijectively to the tasks which our iterated algorithm, either Ext^k or Loc^k, has to solve. Indeed, at the start the algorithm has to solve the task $(S', \mathcal{P}')$ with S' the set of proper 0-separations of G and $\mathcal{P}' = \mathcal{P}_1$ the set of 0-profiles that extend to a k-profile in $\mathcal{P}$. This task corresponds to r in that $\mathcal{P}'$ is the set of children of r. Later, for $\ell = 2, \ldots, k$ recursively, the algorithm at step ℓ receives as input some tasks $(S', \mathcal{P}')$, one for every $P \in \mathcal{P}_{\ell-1}$, in which $\mathcal{P}'$ is the set of ℓ-profiles in $\mathcal{P}_\ell$ extending P, and S' is the set of proper $(\ell - 1)$-separations of G that are nested with the set $N_{\ell-1}$ of nested $(< \ell - 1)$-separations distinguishing $\mathcal{P}_{\ell-1}$ which the algorithm has found so far. This task corresponds to $P \in V(T)$ in the same way, in that $\mathcal{P}'$ is the set of children of P.

Let $c(v)$ denote the number of children of an internal vertex v. Since Ext and Loc reduce every task before they solve it, the task $(S', \mathcal{P}')$ corresponding to a vertex v will add a separation to N only if $c(v) = |\mathcal{P}'| \geq 2$. Let (T', r') be obtained from (T, r) by suppressing any vertices with exactly one child; if r is suppressed, its first descendant with more than one child becomes the new root r'. The internal vertices of T' thus have degree at least 3, except that r' has degree at least 2. Let i denote the number of internal vertices of T'. Since the number of (non-root) leaves of T' is exactly p, we have at most $(p-1)$ internal vertices, that is, $i \leq p - 1$.

Now consider the construction of $N_{\mathsf{Ext}^k}(\mathcal{P})$. By (2.35) in Lemma 2.3.2, each internal vertex v of T' contributes at most $2c(v)$ separations. So there are at most twice as many separations in $N_{\mathsf{Ext}^k}(\mathcal{P})$ as there are edges in T':

$$|N_{\mathsf{Ext}^k}(\mathcal{P})| \leq 2\,\|T'\| = 2(p + i - 1) \leq 4(p-1).$$

During the construction of $N_{\mathsf{Loc}^k}(\mathcal{P})$, each internal vertex v of T' contributes at most $4(c(v)-1)$ separations, by (2.36) in Lemma 2.3.2. Writing I for the set of internal vertices of T', we thus obtain

$$|N_{\mathsf{Loc}^k}(\mathcal{P})| \leq 4 \sum_{v \in I} \big(c(v) - 1\big) = 4\big(\|T'\| -\ i\big) = 4\big(|T| - i - 1\big) = 4(p-1). \quad \square$$

It is easy to construct examples showing that all these bounds are sharp. Instead, let us give an example where Loc yields the best possible result of $2(p-1)$, while Ext does not:

Example 2.3.5. Consider the 3-connected graph with four 4-blocks shown in Figure 2.15. The grey bars indicate separators of chosen separations. Algorithm Ext chooses all these separations: first the two pairs of outer separations, then the two pairs of inner separations. On the other hand, Loc will choose the three pairs of 'straight' separations at the first step, and no further separations thereafter. Therefore Ext chooses one pair of separations more than Loc does.

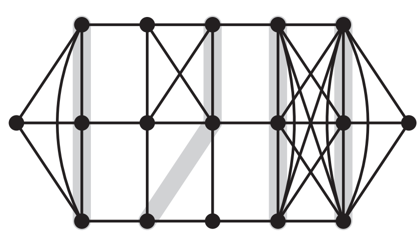


Figure 2.15: A graph where Loc chooses fewer separations than Ext.

2.3.4 Bounding the size of the parts

One of the first questions one may ask about canonical tree-decompositions is whether they can be chosen so as to witness the tree-width of the graph. Choosing the cycle C in Example 2.3.3 long, however, shows that this will not in general be the case: restricting ourselves to a set of separations that is invariant under all the automorphisms of G can result in arbitrarily large parts, and these need not even be essential.

However, if we restrict our attention from arbitrary k-profiles to (those induced by) k-blocks, we can try to make the essential parts small by reducing the junk they contain, the vertices contained in an essential part that do not belong to the k-block that made this part essential. Note that this aim conflicts with our earlier aim to reduce the number of inessential parts: since this junk is part of G, expunging it from the essential parts will mean that we have to have other parts to accommodate it.

In general, we shall not be able to reduce the junk in essential parts to zero unless we restrict the class of graphs under consideration. Our next example shows some graphs for which any tree-decomposition of adhesion at most k, canonical or not, has essential parts containing junk. The amount of junk in a part cannot even be bounded in terms the size of the k-block inhabiting it.

Example 2.3.6. Consider the 4-connected graph obtained by joining two adjacent vertices x, y to a K^5 as in Figure 2.16. This graph has a single 5-block K, the vertex set of the K^5. In any tree-decomposition of adhesion at most 4, the part containing K will contain x or y as well: since the 4-separations that separate x and y from K cross, at most one of them will be induced by the decomposition.

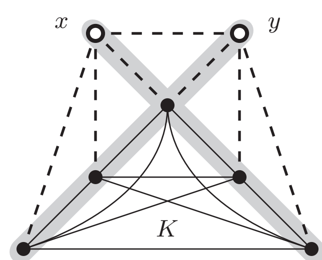


Figure 2.16: A K^5 with unavoidable junk attached

To increase the amount of junk in the part containing K, we can attach arbitrarily many pairs of adjacent vertices to the K^5 in the same way as we added x and y. This will not increase the size of the 5-block K, but the part containing K will also contain at least one vertex from each of those pairs.

The following theorem shows that the obstruction to obtaining essential parts without junk illustrated by the above example is, in a sense, the only such obstruction. Let us call a k-block X *well separated* in a separation system S of $(< k)$-separations if the k-profile $P_k(X) \cap S$ it induces in S is well separated, that is, if the maximal elements of $P_k(X) \cap S$ are nested with each other. (This fails in the example.) Recall that a separation (A, B) is *tight* if every vertex in $A \cap B$ has a neighbour in $A \setminus B$ and a neighbour in $B \setminus A$.

Theorem 2.3.7. *Let $1 \leq k \in \mathbb{N}$, and let S be a set of proper $(< k)$-separations that includes all the tight $(< k)$-separations. Then every graph G has a canonical*[38] *tree-decomposition all whose separations induced by tree-edges are in S such that*

(i) *distinct k-blocks lie in different parts;*

(ii) *parts containing a k-block that is well separated in S coincide with that k-block;*

(iii) *if the task $(S, \mathcal{P})$ with $\mathcal{P}$ the set of all k-block profiles is reduced and feasible, then every leaf part is a k-block.*[39]

Every such decomposition that satisfies (i)*, but not necessarily* (ii) *or* (iii)*, can be refined to such a tree-decomposition that also satisfies* (ii) *and* (iii)*.*

Proof. Let $\mathcal{P}$ be the set of all k-block profiles in G. Let $N \subseteq S$ be any nested separation system that distinguishes all its k-blocks and is canonical, i.e., invariant under the automorphisms of G. Such a set N exists by [39, Theorem 4.4]. (The separations provided by that theorem are tight, and hence in S, because they are $\mathcal{P}$-essential, i.e., distinguish two profiles in $\mathcal{P}$ efficiently.) Then the tree-decomposition $(T, \mathcal{V})$ that induces N by [39, Theorem 2.2] satisfies (i).

For (ii) we refine N by adding the locally maximal separations of $(S, \mathcal{P})$ and their inverses. These are nested with S by [39, Corollary 3.5]. Hence the refined separation system N' is again nested, and therefore induced by a tree-decomposition $(T', \mathcal{V}')$. This decomposition is again canonical, since the set of locally maximal separations is invariant under the automorphisms of G. Clearly, $(T', \mathcal{V}')$ still satisfies (i).

To show that $(T', \mathcal{V}')$ satisfies (ii), suppose it has a part that contains a well separated k-block X and a vertex v outside X. By the maximality of X as a $(< k)$-inseparable set, there is a separation $(A, B) \in S$ with $X \subseteq B$ and $v \in A \setminus B$. Clearly, $(A, B) \in P_k(X)$; choose (A, B) maximal in $P_k(X)$, the k-profile that X induces. Then $(A, B) \in N'$, by definition of N'. This contradicts our assumption that v lies in the same part of $(T', \mathcal{V}')$ as X.

To show that the decomposition $(T', \mathcal{V}')$ obtained for (ii) also satisfies (iii), consider a leaf part V_t. By the assumption in (iii), the separation $(A, B) \in N'$ that corresponds to the edge of T' at t and satisfies $B = V_t$ distinguishes two k-blocks. Let X be the k-block in V_t; it is unique, since N' distinguishes $\mathcal{P}$ but no separation in N' separates V_t. Let $P = P_k(X)$ be the k-profile that X induces. Let $(A', B') \geq (A, B)$ be maximal in S. By assumption in (iii), $B' \subseteq B$ too contains a k-block, which can only be X. Hence $(A', B') \in P$.

[38] Here, this means that the tree-decomposition will be invariant under the automorphisms of G if S is. This is the case, for example, is S consists of all the tight $(< k)$-separations.

[39] Recall that $(S, \mathcal{P})$ is feasible, for example, if G is $(k - 1)$-connected.

Since $(S, \mathcal{P})$ is reduced and feasible, by assumption in (iii), [39, Lemma 3.1] implies that (A', B') is extremal in S. Hence P is extremal, and therefore well separated. We thus have $V_t = X$ by (ii). $\square$

The idea behind allowing some flexibility for S in Theorem 2.3.7 is that this can make (ii) stronger by making more k-blocks well separated. For example, consider a tree-decomposition whose parts are all complete graphs K^5 and whose separations induced by tree edges all have order 3. The k-blocks for $k = 5$ are the K^5s, but none of these is well-separated in the set S of all proper $(< k)$-separations, since the natural 3-separations can be extended in many ways to pairwise crossing 4-separations that will be the locally maximal separations. However all the k-blocks are separated in the smaller set S' of all proper (< 4)-separations, which are precisely the tight $(< k)$-separations. So applying the theorem with this S' would exhibit that the essential parts of our decomposition are in fact k-blocks, a fact the theorem applied with S cannot see.

However, even with S the set of tight $(< k)$-separations, Theorem 2.3.7 (ii) can miss some parts in canonical tree-decompositions that are in fact k-blocks, because they are not well separated even in this restricted S:

Example 2.3.8. Let G consist of a large complete graph K to which three further large complete graphs are attached: K_1 and K_2 by separators S_1 and S_2, respectively, and K_{12} by the separator $S_1 \cap S_2$. If $|S_1| = |S_2| = k - 1$ and $S_1 \neq S_2$, the separations $(K_1 \cup K_{12}, K \cup K_2)$ and $(K_2 \cup K_{12}, K \cup K_1)$ are maximal in $P_k(K) \cap S$ for the k-block K and the set S of all tight $(< k)$-separations. They cross, since both have K_{12} on their 'small' side (Fig. 2.17).

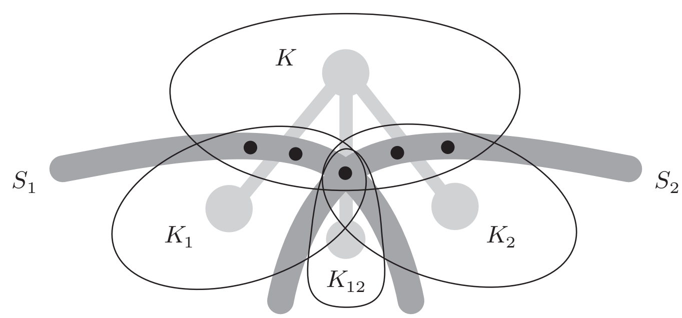


Figure 2.17: The 4-block K is a decomposition part but is not well-separated

So K is not well separated. But the (unique) canonical tree-decomposition of G that distinguishes its k-blocks still has K as a part: its parts are the four large complete graphs, the decomposition tree being a star with centre K.

We wonder whether the notion of being well separated can be weakened, or applied to a suitable set S of $(< k)$-separations, so as to give Theorem 2.3.7 (ii) a converse: so that every graph has a canonical tree-decomposition that distinguishes its k-blocks, whose separations induced by decomposition tree edges are in S, and in which every well separated k-block is a part, while conversely every k-block that occurs as a part in such a tree-decomposition is well separated in S.

Here is an attempt. Given a k-block X, let $S(X)$ denote the set of all tight separations (A, B) such that $X \subseteq B$ and $A \setminus B$ is a component of $G - X$. The separations in $S(X)$ are clearly nested.[40] Write S_k for the set of all $(< k)$-separations of G. Then the condition that

$$S(X) \subseteq S_k \tag{2.40}$$

is a weakening of X being well-separated in S_k. Indeed, if $(A, B) \in S(X)$ is not in S_k, i.e. has order $\geq k$, we can find two crossing separations both maximal in $P_k(X)$, as follows. Pick a vertex $a \in A \setminus B$. By the maximality of X as a $(< k)$-inseparable set, our vertex a can be separated from X by some $(C, D) \in S_k$, say with $a \in C \setminus D$ and $X \subseteq D$. Then $(C, D) \in P_k(X)$. Replace (C, D) with any maximal separation in $P_k(X)$ that is greater than it, and rename that separation as (C, D). Then still $a \in C \setminus D$ and $X \subseteq D$. As $A \setminus B$ is connected, and $A \cap B \subseteq X$ has size $\geq k$ although (A, B) is tight, it follows that $C \cap D$ contains a vertex $a' \in A \setminus B$. Like a, the vertex a' is separated from X by some maximal separation $(C', D') \in P_k(X)$. The separations (C, D) and (C', D') are easily seen to cross, so X is not well separated.

On the other hand, condition (2.40) holds for every k-block X that does occur as a part in a tree-decomposition of adhesion $< k$. Thus if (2.40) is still strong enough to imply that X is a part in some, or any, canonical such tree-decomposition, we shall have our desired converse of Theorem 2.3.7 (ii) with (2.40) replacing 'well separated'.

Given k, call a tree-decomposition of a graph *good* if it is canonical and distinguishes all the k-blocks of G.

Problem 2.3.9. *Let G be a graph and $k \in \mathbb{N}$.*

(i) *Given a k-block X satisfying* (2.40)*, is there always a good tree-decomposition in which X is a part?*

(ii) *Does G have a good tree-decomposition in which every k-block X that satisfies* (2.40) *is a part?*

A positive solution to Problem 2.3.9 (ii) would imply a positive solution also to the following problem, which can be stated with a minimum of technical overheads:

Problem 2.3.10. *Does every graph have a good tree-decomposition that includes among its parts all k-blocks that are a part in some good tree-decomposition?*

Are there any natural conditions ensuring that *every* essential part is a k-block? (In particular, such conditions will have to rule out Example 2.3.6.) We do not know the answer to this question. But we can offer the following:

[40] However, Example 2.3.8 with $X = K_1$ and $X' = K_2$ shows that for distinct k-blocks X, X' the sets $S(X)$ and $S(X')$ need not be nested: the separation $(A, B) \in S(X)$ with $A = K \cup K_2$ and $B = K_1 \cup K_{12}$ crosses the separation $(A', B') \in S(X')$ with $A' = K \cup K_1$ and $B = K_2 \cup K_{12}$.

Theorem 2.3.11. *Assume that G is $(k-1)$-connected, and that every pair x, y of adjacent vertices has one of the following properties:*

(i) *x and y have at least $k-3$ common neighbours;*

(ii) *x and y are joined by at least $\lfloor \frac{3}{2}(k-2) \rfloor$ independent paths other than xy;*

(iii) *x and y lie in a common k-block.*

Then G has a canonical tree-decomposition of adhesion $< k$ such that every part containing a k-block is a k-block. In particular distinct k-blocks are contained in different parts.

Proof. By Theorem 2.3.7 it suffices to show that every element P of the set $\mathcal{P}$ of k-block profiles is well separated in the set S of all the proper $(< k)$-separations.

We do this by applying [39, Lemma 3.4]. Given $P \in \mathcal{P}$, let $(A, B), (C, D)$ be crossing separations in $P \cap S$. If the separation $(A \cup C, B \cap D)$ has order $\leq k-1$, it is in $P \cap S$ by (P2), and we are done. If not, then the separation $(B \cup D, A \cap C)$ has order $< (k-1)$. Since G is $(k-1)$-connected, $(B \cup D, A \cap C)$ must be improper. This means that $A \cap C \subseteq B \cup D$, because $B \nsubseteq A$ as B contains a k-block. But since (A, B) and (C, D) cross, we cannot have $A \cap C \subseteq B \cap D$. By symmetry we may assume that there is a vertex $x \in (C \cap D) \setminus B$. As G is $(k-1)$-connected, (C, D) is tight, so x has a neighbour $y \in (A \cap B) \setminus D$. Let $e := xy$.

Suppose first that e satisfies (i). Since all common neighbours of x and y lie in $A \cap C$, this implies $k - 1 \leq |A \cap C| \leq k - 2$, a contradiction.

Now suppose that e satisfies (ii), and let $\mathcal{W}$ be a set of at least $\lfloor \frac{3}{2}(k-2) \rfloor$ independent x–y paths other than the edge xy. Let

$$X := (A \cap C) \setminus \{x, y\} \qquad Y := (A \cap B) \setminus C \qquad Z := (C \cap D) \setminus A\,.$$

Since $A \cap C \subseteq B \cup D$, we have

$$|X| + |Y| + |Z| \leq |A \cap B| - 1 + |C \cap D| - 1 = 2(k-2). \tag{2.41}$$

Every path in $\mathcal{W}$ that avoids X meets both Y and Z. As $|X| \leq (k-2) - 2$, this yields

$$|\mathcal{W}| \leq |X| + \tfrac{1}{2}\big(|Y| + |Z|\big) \leq |X| + (k-2) - \tfrac{1}{2}|X| \leq \tfrac{3}{2}(k-2) - 1,$$

a contradiction.

Finally assume that e satisfies (iii). Let X be a k-block containing x and y. As $x \notin B$ and $y \notin D$ we have $X \subseteq A \cap C$, contradicting $|A \cap C| \leq k-2$. $\square$

For $k = 2$, Theorem 2.3.11 (i) implies Tutte's theorem that every 2-connected graph has a tree-decomposition whose essential parts are precisely its 3-blocks. The decomposition obtained by any strategy starting with all is the decomposition provided by Tutte [105], in which the inessential parts have cycle torsos.

2.4 A short proof of the tangle-tree-theorem

2.4.1 Introduction

In this quick note, we give a simpler proof of the tangle-tree theorem of Robertson and Seymour [94], which says that every finite graph has a tree-decomposition that distinguishes all the maximal tangles.

Theorem 2.4.1. *Let N be any maximal set of separations each distinguishing some two tangles efficiently. Then any two distinct maximal tangles are distinguished efficiently by N.*

The tangle-tree theorem was extended to matroids by Geelen, Gerards and Whittle [65]. Although our proof is in terms of graphs, it immediately extends to matroids.

2.4.2 Preliminaries

The *order* $o(A, B)$ of a separation (A, B) is the size of the separator $A \cap B$. A separation (A, B) is *distinguishes* two profiles if (A, B) is small for precisely one of these profiles. It distinguishes the profiles P and Q *efficiency* if it distinguishes P and Q and every separation distinguishing them cannot have strictly smaller order. Given a tangle P, we write $(A, B) \in P$ if B is a big side in P.

Our proof relies on the following simple facts:

Lemma 2.4.2. $o(A, B) + o(C, D) = o(A \cap C, B \cup D) + o(B \cap D, A \cup C)$.

Lemma 2.4.3. *Let (A, B), (C, D) and (E, F) be separations such that (A, B) and (C, D) are not nested but (E, F) is nested with the other two separations. Then the corner separation $(A \cap C, B \cup D)$ is nested with (E, F).*

2.4.3 Proof

Proof of Theorem 2.4.1. Let N be any maximal set of separations each distinguishing some two tangles efficiently. Let (A, B) be a separation distinguishing two tangles P and Q of the same order efficiently. Amongst all such (A, B) we pick one which is not nested with a minimal number of separations of N. By the maximality of N, it suffices to show that (A, B) is nested with N.

Suppose for a contradiction, there is some (C, D) in N not nested with (A, B). Let k be the order of (A, B), and ℓ the order of (C, D). Let R and S be two tangles distinguished efficiently by (C, D) and without loss of generality $(C, D) \in R$.

Case 1: $k \geq \ell$. Then P and Q orient (C, D). If they orient it differently, then (C, D) is a candidate for (A, B) and thus (A, B) must be nested with N, which is the desired contradiction. Thus we may assume that (C, D) is in both P and Q. If one of the corner separations $(A \cap C, B \cup D)$ and $(B \cap C, A \cup D)$ has order at most k, then it would distinguish P and Q. But then this corner separation is better choice of (A, B) by Lemma 2.4.3, which is impossible.

Thus $(A \cap C, B \cup D)$ and $(B \cap C, A \cup D)$ have both order at least $k + 1$. Thus by Lemma 2.4.2, the two opposite corner separations $(A \cap D, B \cup C)$ and $(B \cap D, A \cup C)$ have both order at most $\ell - 1$. Hence these two corner separations cannot distinguish R and S. As both $(A \cap D, B \cup C)$ and $(B \cap D, A \cup C)$ are in S, they also must be in R. So R is not a tangle since the tree small sides $A \cap D$, $B \cap D$ and C cover the whole graph. This is the desired contradiction.

Case 2: $k < \ell$. Then R and S orient (A, B). They cannot orient it differently as (C, D) distinguishes them efficiently. Thus we may assume that (A, B) is in both R and S. If one of the corner separations $(A \cap C, B \cup D)$ and $(A \cap D, B \cup C)$ has order strictly less than ℓ, then it would distinguish R and S, which is impossible by the efficiency of (A, B). Thus these two corner separations have both order at least ℓ. Thus by Lemma 2.4.2, the two opposite corner separations $(B \cap C, A \cup D)$ and $(B \cap D, A \cup C)$ have both order at most k. By Lemma 2.4.3, these two corner separations cannot distinguish P and Q. Without loss of generality $(A, B) \in P$ so that these two corner separations are in Q. Hence they also both must be in P. So P is not a tangle since the tree small sides $B \cap C$, $B \cap D$ and A cover the whole graph. This is the desired contradiction.

□

2.5 k-Blocks: a connectivity invariant for graphs

2.5.1 Introduction

Given $k \in \mathbb{N}$, a set I of at least k vertices of a graph G is $(< k)$-*inseparable* if no set S of fewer than k vertices of G separates any two vertices of $I \setminus S$ in G. A maximal $(< k)$-inseparable set is a k-*block*. The degree of connectedness of such a set of vertices is thus measured in the ambient graph G, not only in the subgraph they induce. While the vertex set of a k-connected subgraph of G is clearly $(< k)$-inseparable in G, there can also be k-blocks that induce few or no edges.

The k-blocks of a graph were first studied by Mader [87]. They have recently received some attention because, unlike its k-connected subgraphs, they offer a meaningful notion of the 'k-connected pieces' into which the graph may be decomposed [42]. This notion is related to, but not the same as, the notion of a tangle in the sense of Robertson and Seymour [94]; see Section 2.5.6 and [78] for more on this relationship.

Although Mader [85] had already proved that graphs of average degree at least $4(k - 1)$ have k-connected subgraphs, and hence contain a k-block, he did not in [87] consider the analogous extremal problem for the weaker notion of a k-block directly.

Our aim in this paper is to study this problem: we ask what average or minimum degree conditions force a given finite graph to contain a k-block.

This question can, and perhaps should, be seen in a wider extremal context. Let $\beta(G)$ denote the *block number* of G, the greatest integer k such that G has a k-block (equivalently: has a $(< k)$-inseparable set of vertices). This β seems to be an interesting graph invariant[41], and one may ask how it interacts with other graph invariants, not just the average or minimum degree. Indeed, the examples we describe in Section 2.5.3 will show that containing a k-block for large k is compatible with having bounded minimum and average degree, even in all subgraphs.So k-blocks can occur in very sparse graphs, and one will need bounds on other graph invariants than δ and d to force k-blocks in such graphs.

There is an invariant dual to β: the least integer k such that a graph G has a *block-decomposition* of adhesion and width both at most k. Calling this k the *block-width* $\mathrm{bw}(G)$ of G, we can express the duality neatly as $\beta = \mathrm{bw}$.

All the graphs we consider are finite. Our paper is organized as follows. In Section 2.5.2 we introduce whatever terminology is not covered in [52], and give some background on tree-decompositions. In Section 2.5.3 we present examples of k-blocks, aiming to exhibit the diversity of the concept. In Section 2.5.4 we prove that graphs of minimum degree at least $2(k-1)$ have a k-block. If the graph G considered is $(k-1)$-connected, the minimum degree needed comes down to at most $\frac{3}{2}(k-1)$, and further to k if G contains no triangle. In Section 2.5.5 we show that graphs of average degree at least $3(k-1)$ contain a k-block. In Section 2.5.6 we clarify the relationship between k-blocks and tangles. In Section 2.5.7 we present a polynomial-time algorithm that decides whether a given graph has a k-block, and another that finds all the k-blocks in a graph. This latter algorithm gives rise to our duality theorem $\beta = \mathrm{bw}$.

2.5.2 Terminology and background

All graph-theoretic terms not defined within this paper are explained in [52]. Given a graph $G = (V, E)$, an ordered pair (A, B) of vertex sets such that $A \cup B = V$ is called a *separation* of G if there is no edge xy with $x \in A \setminus B$ and $y \in B \setminus A$. The sets A, B are the *sides* of this separation. A separation (A, B) such that neither $A \subseteq B$ nor $B \subseteq A$ is a *proper* separation. The *order* of a separation (A, B) is the cardinality of its *separator* $A \cap B$. A separation of order k is called a *k-separation*. A simple calculation yields the following:

Lemma 2.5.1. *Given any two separations (A, B) and (C, D) of G, the orders of the separations $(A \cap C, B \cup D)$ and $(B \cap D, A \cup C)$ sum to $|A \cap B| + |C \cap D|$.* □

Recall that a *tree-decomposition* of G is a pair $(T, \mathcal{V})$ of a tree T and a family $\mathcal{V} = (V_t)_{t \in T}$ of vertex sets $V_t \subseteq V$, one for every node of T, such that:

(T1) $V = \bigcup_{t \in T} V_t$;

(T2) for every edge $e \in G$ there exists a $t \in T$ such that both ends of e lie in V_t;

[41] For example, in a network G one might think of the nodes of a $\beta(G)$-block as locations to place some particularly important servers that should still be able to communicate with each other when much of the network has failed.

(T3) $V_{t_1} \cap V_{t_3} \subseteq V_{t_2}$ whenever t_2 lies on the t_1–t_3 path in T.

The sets V_t are the *parts* of $(T, \mathcal{V})$, its *width* is the number $\max_{t\in T} |V_t| - 1$, and the *tree-width* of G is the least width of any tree-decomposition of G.

The intersections $V_t \cap V_{t'}$ of adjacent parts in a tree-decomposition $(T, \mathcal{V})$ (those for which tt' is an edge of T) are its *adhesion sets*; the maximum size of such a set is the *adhesion* of $(T, \mathcal{V})$. The *interior* of a part V_t, denoted by V_t°, is the set of those vertices in V_t that lie in no adhesion set. By (T3), we have $V_t^\circ = V_t \setminus \bigcup_{t'\neq t} V_{t'}$.

Given an edge $e = t_1t_2$ of T, the two components $T_1 \ni t_1$ and $T_2 \ni t_2$ of $T - e$ define separations (A, B) and (B, A) of G with $A = \bigcup_{t\in T_1} V_t$ and $B = \bigcup_{t\in T_2} V_t$, whose separator is the adhesion set $V_{t_1} \cap V_{t_2}$ [52, Lemma 12.3.1]. We call these the separations *induced* by the tree-decomposition $(T, \mathcal{V})$. Note that the adhesion of a tree-decomposition is the maximum of the orders of the separations it induces.

A tree-decomposition *distinguishes* two k-blocks b_1, b_2 if it induces a separation that separates them. It does so *efficiently* if this separation can be chosen of order no larger than the minimum order of a b_1–b_2 separator in G. The tree-decomposition $(T, \mathcal{V})$ is $\mathrm{Aut}(G)$-*invariant* if the automorphisms of G act on the set of parts in a way that induces an action on the tree T. The following theorem was proved in [42]:

Theorem 2.5.2. *For every $k \in \mathbb{N}$, every graph G has an $\mathrm{Aut}(G)$-invariant tree-decomposition of adhesion at most k that efficiently distinguishes all its k-blocks.*

A tree-decomposition $(T, \mathcal{V})$ of a graph G is *lean* if for any nodes $t_1, t_2 \in T$, not necessarily distinct, and vertex sets $Z_1 \subseteq V_{t_1}$ and $Z_2 \subseteq V_{t_2}$ such that $|Z_1| = |Z_2| =: \ell$, either G contains ℓ disjoint Z_1–Z_2 paths or there exists an edge $tt' \in t_1Tt_2$ with $|V_t \cap V_{t'}| < \ell$. Since there is no such edge when $t_1 = t_2 =: t$, this implies in particular that, for every part V_t, any two subsets $Z_1, Z_2 \subseteq V_t$ of some equal size ℓ are linked in G by ℓ disjoint paths.

(However, the parts need not be $(< \ell)$-inseparable for any large ℓ; see Section 2.5.3.)

We call a tree-decomposition $(T, \mathcal{V})$ *k-lean* if none of its parts contains another, it has adhesion at most k, and for any nodes $t_1, t_2 \in T$, not necessarily distinct, and vertex sets $Z_1 \subseteq V_{t_1}$ and $Z_2 \subseteq V_{t_2}$ such that $|Z_1| = |Z_2| =: \ell \leq k + 1$, either G contains ℓ disjoint Z_1–Z_2 paths or there exists an edge $tt' \in t_1Tt_2$ with $|V_t \cap V_{t'}| < \ell$.

Thomas [101] proved that every graph G has a lean tree-decomposition whose width is no greater than the tree-width of G. By considering only separations of order at most k one can adapt the short proof of Thomas's theorem given in [8] to yield the following:

Theorem 2.5.3. *For every $k \in \mathbb{N}$, every graph has a k-lean tree-decomposition.*

2.5.3 Examples of k-blocks

In this section we discuss three different types of k-block.

Example 2.5.4. The vertex set of any k-connected subgraph is $(< k)$-inseparable, and hence contained in a k-block.

While a k-block as in Example 2.5.4 derives much or all of its inseparability from its own connectivity as a subgraph, the k-block in our next example will form an independent set. It will derive its inseparability from the ambient graph, a large grid to which it is attached.

Example 2.5.5. Let $k \geq 5$, and let H be a large $(m \times n)$-grid, with $m, n \geq k^2$ say.

Let G be obtained from H by adding a set $X = \{x_1, \ldots, x_k\}$ of new vertices, joining each x_i to at least k vertices on the grid boundary that form a (horizontal or vertical) path in Hso that every grid vertex obtains degree 4 in G (Figure 2.18). We claim that X is a k-block of G, and is its only k-block.

Any grid vertex can lie in a common k-block of G only with its neighbours, because these separate it from all the other vertices. As any k-block has at least $k \geq 5$ vertices but among the four G-neighbours of a grid vertex at least two are non-adjacent grid vertices, this implies that no k-block of G contains a grid vertex. On the other hand, every two vertices of X are linked by k independent paths in G, and hence cannot be separated by fewer than k vertices. Hence X is $(< k)$-inseparable, maximally so, and is thus the only k-block of G.

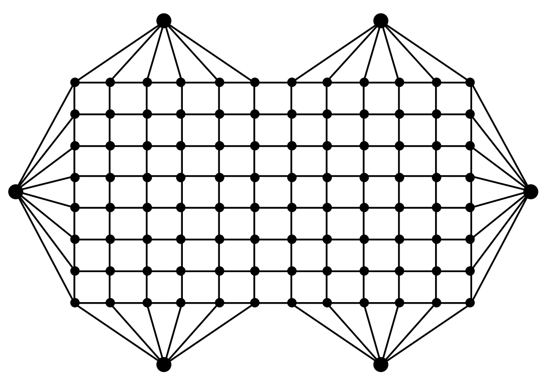


Figure 2.18: The six outer vertices form a 6-block

In the discussion of Example 2.5.5 we saw that none of the grid vertices lies in a k-block. In particular, the grid itself has no k-block when $k \geq 5$. Since every two inner vertices of the grid, those of degree 4, are joined in the grid by 4 independent paths, they form a (< 4)-inseparable set (which is clearly maximal):

Example 2.5.6. The inner vertices of any large grid H form a 4-block in H. However, H has no k-block for any $k \geq 5$.

The k-block defined in Example 2.5.5 gives rise to a tangle of large order (see Section 2.5.6), the same as the tangle specified by the grid H. This is in contrast to our last two examples, where the inseparability of the k-block will again lie in the ambient graph but in a way that need not give rise to a non-trivial tangle. (See Section 2.5.6 for when it does.) Instead, the paths supplying the required connectivity will live in many different components of the subgraph into which the k-block splits the original graph.

Example 2.5.7. Let X be a set of $n \geq k$isolated vertices. Join every two vertices of X by many (e.g., k) independent paths, making all these internally disjoint. Then X will be a k-block in the resulting graph.

Example 2.5.7 differs from Example 2.5.5 in that its graph has a tree-decomposition whose only part of order ≥ 3 is X. Unlike the grid in Example 2.5.5, the paths providing X with its external connectivity do not between them form a subgraph that is in any sense highly connected. We can generalize this as follows:

Example 2.5.8. Given $n \geq k$, consider a tree T in which every non-leaf node has $\binom{n}{k-1}$ successors. Replace each node t by a set V_t of n isolated vertices. Whenever t' is a successor of a node t in T, join $V_{t'}$ to a $(k-1)$-subset $S_{t'}$ of V_t by $(k-1)$ independent edges, so that these $S_{t'}$ are distinct sets for different successors t' of t. For every leaf t of T, add edges on V_t to make it complete. The k-blocks of the resulting graph G are all the sets V_t $(t \in T)$, but only the sets V_t with t a leaf of T induce any edges.

Interestingly, the k-blocks that we shall construct explicitly in our proofs will all be *connected*, i.e., induce connected subgraphs. Thus, our proof techniques seem to be insufficient to detect k-blocks that are disconnected or even independent, such as those in our examples. However, we do not know whether or not this affects the quality of our bounds or just their witnesses:

Problem 2.5.9. *Does every minimum or average degree bound that forces the existence of a k-block also force the existence of a connected $(< k)$-inseparable set?*

Even if the answer to this problem is positive, it will reflect only on how our invariant β relates to the invariants δ and d, and that for some graphs it may be more interesting to relate β to other invariants. The existence of a large k-block in Examples 2.5.5 and 2.5.7, for instance, will not follow from any theorem relating β to δ or d, since graphs of this type have a bounded average degreeindependent of k, even in all subgraphs. But they are key examples, which similar results about β and other graph invariants may be able to detect.

2.5.4 Minimum degree conditions forcing a k-block

Throughout this section, let $G = (V, E)$ be a fixed non-empty graph. We ask what minimum degree will force G to contain a k-block for a given integer $k > 0$.

Without any further assumptions on G we shall see that $\delta(G) \geq 2(k-1)$ will be enough. If we assume that G is $(k-1)$-connected – an interesting case, since for such G the parameter k is minimal such that looking for k-blocks can be non-trivial – we find that $\delta(G) > \frac{3}{2}k - \frac{5}{2}$ suffices. If G is $(k-1)$-connected but contains no triangle, even $\delta(G) \geq k$ will be enough. Note that this is best possible in the (weak) sense that the vertices in any k-block will have to have degree at least k, except in some very special cases that are easy to describe.

Conversely, we construct a $(k-1)$-connected graph of minimum degree $\lfloor \frac{3}{2}k - \frac{5}{2} \rfloor$ that has no k-block. So our second result above is sharp.

To enhance the readability of both the results and the proofs in this section, we give bounds on δ which force the existence of a $(k+1)$-block for any $k \geq 0$.

We shall often use the fact that a vertex of G together with k or more of its neighbours forms a $(<k+1)$-inseparable set as soon as these neighbours are pairwise not separated by k or fewer vertices. Let us state this as a lemma:

Lemma 2.5.10. *Let $v \in V$ and $N \subseteq N(v)$ with $|N| \geq k$. If no two vertices of N are separated in G by at most k vertices, then $N \cup \{v\}$ lies in a $(k+1)$-block.* $\square$

Here, then, is our first sufficient condition for the existence of a k-block. It is essentially due to Mader [86, Satz 7′], though with a different proof:

Theorem 2.5.11. *If $\delta(G) \geq 2k$, then G has a $(k+1)$-block. This $(k+1)$-block can be chosen to be connected in G and of size at least $\delta(G) + 1 - k$.*

Proof. If $k = 0$, then the assertion follows directly. So we assume $k > 0$. By Theorem 2.5.3, G has a k-lean tree-decomposition $(T, \mathcal{V})$, say with $\mathcal{V} = (V_t)_{t \in T}$. Pick a leaf t of T. (If T has only one node, we count it as a leaf.) Write $A_t := V_t \cap \bigcup_{t' \neq t} V_{t'}$ for the attachment set of V_t. As V_t is not contained in any other part of $(T, \mathcal{V})$,

We prove that V_t° extends to a $(k+1)$-block $B \subseteq V_t$ that is connected in G. Pick distinct vertices $v, v' \in V_t^\circ$. Let N be a set of k neighbours of v, and N' a set of k neighbours of v'. Note that $N \cup N' \subseteq V_t$. As our tree-decomposition is k-lean, there are $k+1$ disjoint paths in G between the $(k+1)$-sets $N \cup \{v\}$ and $N' \cup \{v'\}$. Hence v and v' cannot be separated in G by at most k other vertices.

We have thus shown that V_t° is $(<k+1)$-inseparable.In particular, A_t does not separate it, so V_t° is connected in G. Let B be a $(k+1)$-block containing V_t°. As A_t separates V_t° from $G \setminus V_t$, we have $B \subseteq V_t$. Every vertex of B in A_t sends an edge to V_t°, since otherwise the other vertices of A_t would separate it from V_t°. Hence B is connected. Since every vertex in V_t° has at least $\delta(G) - k$ neighbours in $V_t^\circ \subseteq B$, we have the desired bound of $|B| \geq \delta(G) + 1 - k$. $\square$

One might expect that our lower bound for the size of the $(k+1)$-block B found in the proof of Theorem 2.5.11 can be increased by proving that B must contain the adhesion set of the part V_t containing it. While we can indeed raise the bound a little (by at least 1, but we do not know how much at most), we show in Section 2.5.8 that B can lie entirely in the interior of V_t.

We also do not know whether the degree bound of $\delta(G) \geq 2k$ in Theorem 2.5.11 is sharp. The largest minimum degree known of a graph without a $(k+1)$-block is $\lfloor \frac{3}{2}k - 1 \rfloor$. This graph (Example 2.5.17 below) is k-connected, and we shall see that k-connected graphs of larger minimum degree do have $(k+1)$-blocks (Theorem 2.5.16). Whether or not graphs of minimum degree between $\frac{3}{2}k - 1$ and $2k$ and connectivity $< k$ must have $(k+1)$-blocks is unknown to us:

Problem 2.5.12. *Given $k \in \mathbb{N}$, determine the smallest value δ_k of δ such that every graph of minimum degree at least δ has a k-block.*

It is also conceivable that the smallest minimum degree that will force a *connected* $(k+1)$-block – or at least a connected $(< k+1)$-inseparable set, as found by our proof of Theorem 2.5.11 – is indeed $2k$ but possibly disconnected $(k+1)$-blocks can be forced by a smaller value of δ (compare Problem 2.5.9).

The degree bound of Theorem 2.5.11 can be reduced by imposing additional conditions on G. Our next aim is to derive a better bound on the assumption that G is k-connected, for which we need a few lemmas.

We say that a k-separation (A, B) is T-*shaped* (Fig. 2.19) if it is a proper separation and there exists another proper k-separation (C, D) such that $A \setminus B \subseteq C \cap D$ as well as $|A \cap C| \leq k$ and $|A \cap D| \leq k$. Obviously, (A, B) is T-shaped witnessed by (C, D) if and only if the two separations $(A \cap C, B \cup D)$ and $(A \cap D, B \cup C)$ have order at most k and are improper separations.

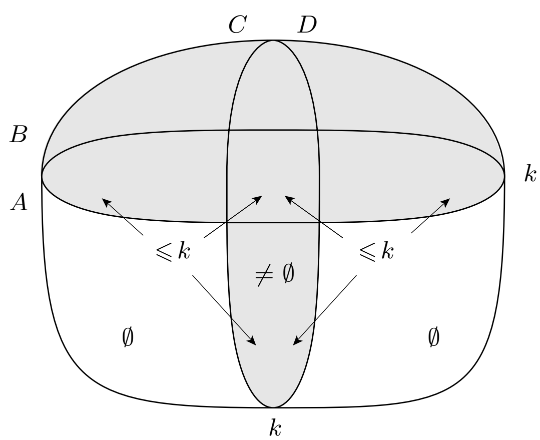


Figure 2.19: The separation (A, B) is T-shaped

Lemma 2.5.13. *If (A, B) is a T-shaped k-separation in G, then $|A| \leq \frac{3}{2}k$.*

Proof. Let (C, D) witness that (A, B) is T-shaped. Then

$$|A| \leq |A \cap B| + |(C \cap D) \setminus B| \leq k + \tfrac{1}{2}(2k - k) = \tfrac{3}{2}k. \qquad \square$$

When a k-separation (A, B) is T-shaped, no $(k+1)$-block of G can lie in A: with (C, D) as above, it would have to lie in either $A \cap C$ or $A \cap D$, but both these are too small to contain a $(k+1)$-block. Conversely, one may ask whether every proper k-separation (A, B) in a k-connected graph such that A contains no $(k+1)$-block must be T-shaped, or at least give rise to a T-shaped k-separation (A', B') with $A' \subseteq A$. This, however, is not true: some counterexamples are given in Section 2.5.8.

Interestingly, though, a global version of this does hold: a T-shaped k-separation must occur *somewhere* in every k-connected graph that has no $(k+1)$-block. More precisely, we have the following:

Lemma 2.5.14. *If G is k-connected, the following statements are equivalent:*

(i) *every proper k-separation of G separates two $(k+1)$-blocks;*

(ii) *no k-separation of G is T-shaped.*

Proof. We first assume (i) and show (ii). If (ii) fails, then G has a k-separation (A, B) that is T-shaped, witnessed by (C, D) say. We shall derive a contradiction to (i) by showing that A contains no $(k+1)$-block. If A contains a $(k+1)$-block, it lies in either $A \cap C$ or $A \cap D$, since no two of its vertices are separated by (C, D). By the definition of T-shaped, none of these two cases can occur, a contradiction.

Let us now assume (ii) and show (i). If (i) fails, there is a proper k-separation (A, B) such that A contains no $(k+1)$-block. Pick such an (A, B) with $|A|$ minimum. Since (A, B) is proper, there is a vertex $v \in A \setminus B$. Since G is k-connected, v has at least k neighbours, all of which lie in A. As A contains no $(k+1)$-block, Lemma 2.5.10 implies that there is a proper k-separation (C, D) that separates two of these neighbours. Then v must lie in $C \cap D$.

We first show that either $(A \cap C, B \cup D)$ has order at most k and $(A \cap C) \setminus (B \cup D) = \emptyset$ or $(B \cap D, A \cup C)$ has order at most k and $(B \cap D) \setminus (A \cup C) = \emptyset$. Let us assume that the first of these fails; then either $(A \cap C, B \cup D)$ has order $> k$ or $(A \cap C) \setminus (B \cup D) \neq \emptyset$. In fact, if the latter holds then so does the former: otherwise $(A \cap C, B \cup D)$ is a proper k-separation that contradicts the minimality of $|A|$ in the choice of (A, B). (We have $|A \cap C| < |A|$, since v has a neighbour in $A \setminus C$.)Thus, $(A \cap C, B \cup D)$ has order $> k$. As $|A \cap B| + |C \cap D| = 2k$, this implies by Lemma 2.5.1 that the order of $(B \cap D, A \cup C)$ is strictly less than k. As G is k-connected, this means that $(B \cap D, A \cup C)$ is not a proper separation, i.e., that $(B \cap D) \setminus (A \cup C) = \emptyset$ as claimed.

By symmetry, we also get the analogous statement for the two separations $(A \cap D, B \cup C)$ and $(B \cap C, A \cup D)$. But this means that one of the separations (A, B), (B, A), (C, D) and (D, C) is T-shaped,contradicting (ii). $\square$

Our next lemma says something about the size of the $(k+1)$-blocks we shall find.

Lemma 2.5.15. *If G is k-connected and $|A| > \frac{3}{2}k$ for every proper k-separation (A, B) of G, then either V is a $(k+1)$-block or G has two $(k+1)$-blocks of size at least* $\min\{\,|A| : (A, B)$ *is a proper k-separation*$\}$ *that are connected in G.*

Proof. By assumption and Lemma 2.5.13, G has no T-shaped k-separation, so by Lemma 2.5.14 every side of a proper k-separation contains a $(k+1)$-block.

By Theorem 2.5.3, G has a k-lean tree-decomposition $(T, \mathcal{V})$, with $\mathcal{V} = (V_t)_{t \in T}$ say. Unless V is a $(k+1)$-block, in which case we are done, this decomposition has at least two parts: since there exist two $(k+1)$-sets in V that are separated by some k-separation,the trivial tree-decomposition with just one part would not be k-lean.

So T has at least two leaves, and for every leaf t the separation $(A, B) := \big(V_t, \bigcup_{t' \neq t} V_{t'}\big)$ is a proper k-separation. It thus suffices to show that $A = V_t$ is a $(k+1)$-block;it will clearly be connected (as in the proof of Theorem 2.5.11).

As remarked at the start of the proof, there exists a $(k+1)$-block $X \subseteq A$. If $X \neq A$, then A has two vertices that are separated by a k-separation (C, D); we may assume that $X \subseteq C$, so $X \subseteq A \cap C$.

If $(A \cap C, B \cup D)$ has order $\leq k$, it is a proper separation (as $X \subseteq A \cap C$ has size $> k$);then its separator S has size exactly k, since G is k-connected. By the choice of (C, D) there is a vertex v in $(D \setminus C) \cap A$. The $k+1$ vertices of $S \cup \{v\} \subseteq A$ are thus separated in G by the k-set $C \cap D$ from $k+1$ vertices in $X \subseteq A \cap C$, which contradicts the leanness of $(T, \mathcal{V})$ for $V_t = A$.

So the order of $(A \cap C, B \cup D)$ is at least $k+1$. By Lemma 2.5.1, the order of $(B \cap D, A \cup C)$ must then be less than k, so by the k-connectedness of G there is no $(k+1)$-block in $B \cap D$.

The $(k+1)$-block X' which D contains (see earlier) thus lies in $D \cap A$. So A contains two $(k+1)$-blocks X and X', and hence two vertex sets of size $k+1$, that are separated by (C, D), which contradicts the k-leanness of $(T, \mathcal{V})$.

$\square$

Theorem 2.5.16. *If G is k-connected and $\delta(G) > \frac{3}{2}k - 1$,then either V is a $(k+1)$-block or G has at least two $(k+1)$-blocks. These can be chosen to be connected in G and of size at least $\delta(G) + 1$.*

Proof. For every proper k-separation (A, B) we have a vertex of degree $> \frac{3}{2}k - 1$ in $A \setminus B$, and hence $|A| \geq \delta(G) + 1 > \frac{3}{2}k$. The assertion now follows from Lemma 2.5.15. $\square$

To show that the degree bound in Theorem 2.5.16 is sharp, let us construct a k-connected graph H with $\delta(H) = \lfloor \frac{3}{2}k - 1 \rfloor$ that has no $(k+1)$-block.

Example 2.5.17. Let H_n be the ladder that is a union of $n \geq 2$ squares (formally: the cartesian product of a path of length n with a K^2).

For even k, let H be the lexicographic product of H_n and a complete graph $K = K^{k/2}$, i.e., the graph with vertex set $V(H_n) \times V(K)$ and edge set

$$\{\, (h_1, x)(h_2, y) \mid \text{either } h_1 = h_2 \text{ and } xy \in E(K) \text{ or } h_1 h_2 \in E(H_n) \,\},$$

see Figure 2.20. This graph H is k-connected and has minimum degree $\frac{3}{2}k - 1$. But it contains no $(k+1)$-block: among any $k+1$ vertices we can find two that are separated in H by a k-set of the form $V_{h_1} \cup V_{h_2}$, where $V_h := \{(h, x) \mid x \in K\}$.

If k is odd, let H' be the graph H constructed above for $k-1$, and let H be obtained from H' by adding a new vertex and joining it to every vertex of H'. Clearly, H is again k-connected and has minimum degree $\lfloor \frac{3}{2}k - 1 \rfloor$, and it has no $(k+1)$-block since H has no k-block.

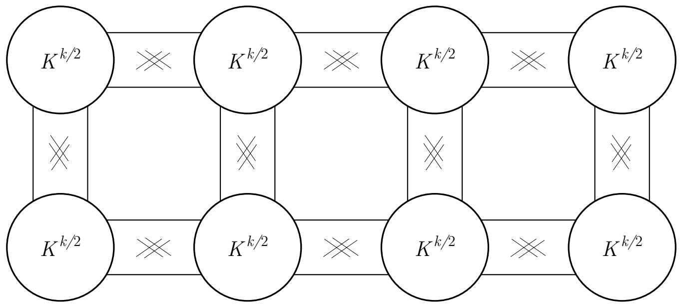


Figure 2.20: A k-connected graph without a $(k+1)$-block

Our next example shows that the connectivity bound in Theorem 2.5.16 is sharp: we construct for every odd k a $(k-1)$-connected graph H of minimum degree $\lfloor \frac{3}{2}k \rfloor$ whose largest $(k+1)$-blocks have size $k+1 < \delta(H)+1$.

Example 2.5.18. Let H_n be as in Example 2.5.17. Let H be obtained from H_n by replacing the degree-two vertices of H_n by complete graphs of order $(k+1)/2$ and its degree-three vertices by complete graphs of order $(k-1)/2$, joining vertices of different complete graphs whenever the corresponding vertices of H_n are adjacent. The minimum degree of this graph is $\lfloor \frac{3}{2}k \rfloor$, but it has only two $(k+1)$-blocks: the two K^{k+1}s at the extremes of the ladder.

We do not know whether the assumption of k-connectedness in Theorem 2.5.16 is necessary if we just want to force any $(k+1)$-block, not necessarily one of size $\geq \delta + 1$.

If, in addition to being k-connected, G contains no triangle, the minimum degree needed to force a $(k+1)$-block comes down to $k+1$, and the $(k+1)$-blocks we find are also larger:

Theorem 2.5.19. *If G is k-connected, $\delta(G) \geq k+1$, and G contains no triangle, then either V is a $(k+1)$-block or G has at least two $(k+1)$-blocks. These can be chosen to be connected in G and of size at least $2\delta(G)$.*

Proof. Since $2\delta(G) > \frac{3}{2}k$, it suffices by Lemma 2.5.15 to show that $|A| \geq 2\delta(G)$ for every proper k-separation (A, B) of G. Pick a vertex $v \in A \setminus B$. As $d(v) \geq k+1$, it has a neighbour w in $A \setminus B$. Since v and w have no common neighbour, we deduce that $|A| \geq d(v) + d(w) \geq 2\delta(G)$. $\square$

Any k-connected, k-regular, triangle-free graphshows that the degree bound in Theorem 2.5.19 is sharp, because of the following observation:

Proposition 2.5.20. *If G is k-connected and k-regular, then G has no $(k+1)$-block unless $G = K^{k+1}$ (which contains a triangle).*

Proof. Suppose G has a $(k+1)$-block X. Pick a vertex $x \in X$. The k neighbours of x in G do not separate it from any other vertex of X, so all the other vertices of X are adjacent to x. But then X consists of precisely x and its k neighbours, since $|X| \geq k+1$. As this is true for every $x \in X$, it follows that $G = K^{k+1}$. $\square$

If we strengthen our regularity assumption to transitivity (i.e., assume that for every two vertices u, v there is an automorphism mapping u to v), then G has no $(k+1)$-blocks, regardless of its degree:

Theorem 2.5.21. *If $\kappa(G) = k \geq 1$ and G is transitive, then G has no $(k+1)$-block unless $G = K^{k+1}$.*

Proof. Unless G is complete (so that $G = K^{k+1}$), it has a proper k-separation. Hence V is not a $(k+1)$-block. Let us show that G has no $(k+1)$-block at all.

If G has a $(k+1)$-block, it has at least two, since V is not a $(k+1)$-block but every vertex lies in a $(k+1)$-block, by transitivity. Hence any tree-decomposition that distinguishes all the $(k+1)$-blocks of G has at least two parts. By Theorem 2.5.2 there exists such a tree-decomposition $(T, \mathcal{V})$, which moreover has the property that every automorphism of G acts on the set of its parts. As $k \geq 1$, adjacent parts overlap in at least one vertex, so G has a vertex u that lies in at least two parts. But G also has a vertex v that lies in only one part (as long as no part of the decomposition contains another, which we may clearly assume): if t is a leaf of T and t' is its neighbour in T, then every vertex in $V_t \setminus V_{t'}$ lies in no other part than V_t (see Section 2.5.2). Hence no automorphism of G maps u to v, a contradiction to the transitivity of G. $\square$

Theorems 2.5.16 and 2.5.21 together imply a well-known theorem of Mader [84]and Watkins [106],which says that every transitive graph of connectivity k has minimum degree at most $\frac{3}{2}k - 1$.

2.5.5 Average degree conditions forcing a k-block

As before, let us consider a non-empty graph $G = (V, E)$ fixed throughout this section. We denote its average degree by $d(G)$. As in the previous section, we shall assume that $k \geq 0$ and consider $(k+1)$-blocks, to improve readability.

As remarked in the introduction, Mader [85] proved that if $d(G) \geq 4k$ then G has a $(k+1)$-connected subgraph. The vertex set of such a subgraph is $(< k+1)$-inseparable, and hence extends to a $(k+1)$-block of G. Our first aim will be to show that if we seek to force a $(k+1)$-block in G directly, an average degree of $d(G) \geq 3k$ will be enough.

In the proof of that theorem, we may assume that G is a minimal with this property, so its proper subgraphs will all have average degrees smaller than $3k$. The following lemma enables us to utilize this fact. Given a set $S \subseteq V$, write $E(S, V)$ for the set of edges of G that are incident with a vertex in S.

Lemma 2.5.22. *If $\lambda > 0$ is such that $d(G) \geq 2\lambda > d(H)$ for every proper subgraph $H \neq \emptyset$ of G, then $|E(S, V)| > \lambda |S|$ for every set $\emptyset \neq S \subsetneq V$.*

Proof. Suppose there is a set $\emptyset \neq S \subsetneq V$ such that $|E(S,V)| \leq \lambda |S|$. Then our assumptions imply

$$|E(G-S)| = |E| - |E(S,V)| \geq \lambda|V| - \lambda|S| = \lambda|V \setminus S|,$$

so the proper subgraph $G - S$ of G contradicts our assumptions. $\square$

Theorem 2.5.23. *If $d(G) \geq 3k$, then G has a $(k+1)$-block. This can be chosen to be connected in G and of size at least $\delta(G) + 1 - k$.*

Proof. If $k = 0$, then the assertion follows directly. So we assume $k > 0$. Replacing G with a subgraph if necessary, we may assume that $d(G) \geq 3k$ but $d(H) < 3k$ for every proper subgraph H of G. By Lemma 2.5.22, this implies that $|E(S,V)| > \frac{3}{2}k|S|$ whenever $\emptyset \neq S \subsetneq V$; in particular, $\delta(G) > \frac{3}{2}k$.

Let $(T, \mathcal{V})$ be a k-lean tree-decomposition of G, with $\mathcal{V} = (V_t)_{t \in T}$ say. Pick a leaf t of T. (If T has only one node, let t be this node.) Then $V_t^\circ \neq \emptyset$ by (T3), since V_t is not contained in any other part of $\mathcal{V}$.

If $|V_t^\circ| \leq k$ then, as also $|V_t \setminus V_t^\circ| \leq k$,

$$|E(V_t^\circ, V)| \leq \tfrac{1}{2}|V_t^\circ|^2 + k\,|V_t^\circ| \leq |V_t^\circ|\big(|V_t^\circ|/2 + k\big) \leq \tfrac{3}{2}k\,|V_t^\circ|,$$

which contradicts Lemma 2.5.22.So $|V_t^\circ| \geq k + 1 \geq 2$. The set V_t° extends to a $(k+1)$-block $B \subseteq V_t$ with the desired properties as in the proof of Theorem 2.5.11. $\square$

Since our graph of Example 2.5.17 contains no $(k+1)$-block, its average degree is a strict lower bound for the minimum average degree that forces a $(k+1)$-block. By choosing the ladder in the construction of that graph long enough, we can make its average degree exceed $2k-1-\epsilon$ for any $\epsilon > 0$. The minimum average degree that will force a $(k+1)$-block thus lies somewhere between $2k-1$ and $3k$.

Problem 2.5.24. *Given $k \in \mathbb{N}$, determine the smallest value d_k of d such that every graph of average degree at least d has a k-block.*

As we have seen, an average degree of $3k$ is sufficient to force a graph to contain a $(k+1)$-block. If we ask only that the graph should have a minor that contains a $(k+1)$-block, then a smaller average degree suffices:

Theorem 2.5.25. *If $d(G) \geq 2(k-1) > 0$, then G has a minor containing a $(k+1)$-block. This $(k+1)$-block can be chosen to be connected in the minor.*

Proof. Replacing G with a minor of itself if necessary, we may assume that $d(G) \geq 2(k-1)$ but $d(H) < 2(k-1)$ for every proper minor H of G. In particular, this holds for all subgraphs $\emptyset \neq H \subsetneq G$, so $\delta(G) \geq k$ by Lemma 2.5.22.

Let us show that any two adjacent vertices v and w have at least $k-1$ common neighbours. Otherwise, contracting the edge vw we lose one vertex and at most $k-1$ edges; as $|E|/|V| \geq k-1$ by assumption, this ratio (and hence the average degree) will not decrease, contradicting the minimality of G.

Let $(T, \mathcal{V})$ be a k-lean tree-decomposition of G, with $\mathcal{V} = (V_t)_{t \in T}$ say, and let t be a leaf of T. (If T has only one node, let t be this node.) We shall prove that V_t is $(< k+1)$-inseparable, and hence a $(k+1)$-block, in G.

As $(T, \mathcal{V})$ is k-lean, every vertex $a \in A_t := V_t \cap \bigcup_{t' \neq t} V_{t'}$ has a neighbour v in V_t°, as otherwise $X := A_t \setminus \{a\}$ would separate A_t from every set $X \cup \{v\}$ with $v \in V_t^\circ$, which contradicts k-leanness since $|X \cup \{v\}| = |A_t| \leq k$. As a and v have $k-1$ common neighbours in G, which must lie in V_t, we find that every vertex in A_t, and hence every vertex of V_t, has at least k neighbours in V_t.

As $V_t^\circ \neq \emptyset$ and hence $|V_t| \geq \delta(G)+1 \geq k+1$, it suffices to show that two vertices $u, v \in V_t$ can never be separated in G by $\leq k$ other vertices. But this follows from k-leanness: pick a set N_u of k neighbours of u in V_t and a set N_v of k neighbours of v in V_t to obtain two $(k+1)$-sets $N_u \cup \{u\}$ and $N_v \cup \{v\}$ that are joined in G by $k+1$ disjoint paths; hence u and v cannot be separated by $\leq k$ vertices. □

Recall that the graphs of Example 2.5.17 have average degrees of at least $2k-1-\epsilon$. So these graphs show that obtaining a $(k+1)$-block in G is indeed harder than obtaining a $(k+1)$-block in a minor of G, which these graphs must have by Theorem 2.5.25. (And they do: they even have $K^{3k/2}$-minors.)

2.5.6 Blocks and tangles

In this section we compare k-blocks with tangles, as introduced by Robertson and Seymour [94]. Our reason for doing so is that both notions have been advanced as possible approximations to the elusive "$(k+1)$-connected pieces" into which one might wish to decompose a k-connected graph, in analogy to its tree-like block-cutvertex decomposition (for $k = 1$), or to Tutte's tree-decomposition of 2-connected graphs into 3-connected torsos (for $k = 2$) [93, 42].

Let us say that a set θ of separations of order at most k of a graph $G = (V, E)$ is a *tangle of order k* of G if

(θ1) for every separation (A, B) of order $< k$ of G either (A, B) or (B, A) is in θ;

(θ2) for all $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \theta$ we have $G[A_1] \cup G[A_2] \cup G[A_3] \neq G$.

It is straightforward to verify that this notion of a tangle is consistent with the one given in [94].

Given a tangle θ, we think of the side A of a separation $(A, B) \in \theta$ as the *small side* of (A, B), and of B as its *large side*. (Thus, axiom (θ2) says that G is not the union of the subgraphs induced by at most three small sides.) If a set X of vertices lies in the large side of every separation in θ but not in the small side, we say that X *gives rise to* or *defines* the tangle θ.

If X is a $(< k)$-inseparable set of vertices, it clearly lies in exactly one of the two sides of any separation of order $< k$. Hence if we define θ as the set of those separations (A, B) of order $< k$ for which $X \subseteq B$, then θ satisfies (θ1), and V is not a union of at most two small sides of separations in θ. But it might be the union of three small sides, and indeed θ may fail to satisfy (θ2).So X might, or might not, define a tangle of order at most k.

An $(n\times n)$-grid minor of G, with $n \geq k$, also gives rise to a tangle of order k in G, but in a weaker sense: for every separation (A,B) of G of order less than k, exactly one side meets a branch set of every *cross* of the grid, a union of one column and one row. (Indeed, since crosses are connected and every two crosses meet, we cannot have one cross in $A \setminus B$ and another in $B \setminus A$.)

Since G can contain a large grid without containing a k-block (Example 2.5.6), it can thus have a large-order tangle but fail to have a k-block for any $k \geq 5$. Conversely, Examples 2.5.7 and 2.5.8 show that G can have k-blocks for arbitrarily large k without containing any tangle (other than those of order $\leq \kappa(G)$, in which the large side of every separation is all of V). For example, if G is a subdivided K^n with $n \geq k+1$,then its branch vertices form a k-block X, but when $n \leq \frac{3}{2}(k-1)$ the separations of order $< k$ whose large sides contain X do not form a tangle, since G is the union of three small sides of such separations (each with a separator consisting of two thirds of the branch vertices; compare [94, (4.4)]).

Any k-block of size $> \frac{3}{2}(k-1)$, however, does give rise to a tangle of order k:

Theorem 2.5.26. *Every $(< k)$-inseparable set of more than $\frac{3}{2}(k-1)$ vertices in $G = (V,E)$ defines a tangle of order k.*

Proof. Let X be a $(< k)$-inseparable set of more than $\frac{3}{2}(k-1)$ vertices, and consider the set θ of all separations (A,B) of order less than k with $X \subseteq B$. We show that θ is a tangle. As no two vertices of X can be separated by a separation in θ, it satisfies $(\theta 1)$. For a proof of $(\theta 2)$, it suffices to consider three arbitrary separations $(A_1,B_1),(A_2,B_2),(A_3,B_3)$ in θ and show that

$$E(A_1) \cup E(A_2) \cup E(A_3) \not\supseteq E, \qquad (*)$$

where $E(A_i)$ denotes the set of edges that A_i spans in G.

As $|X| > \frac{3}{2}(k-1)$, there is a vertex $v \in X$ that lies in at most one of the three sets $A_i \cap B_i$, say neither in $A_2 \cap B_2$ nor in $A_3 \cap B_3$. Let us choose v in A_1 if possible.Then, as $X \subseteq B_1$, there is another vertex $w \neq v$ in $X \setminus A_1$. As v and w lie in X, the set $(A_1 \cap B_1)\setminus\{v\}$ does not separate them. Hence there is an edge vu with $u \in B_1 \setminus A_1$. Since $v \notin A_2 \cup A_3$,the edge vu is neither in $E(A_2)$ nor in $E(A_3)$. But vu is not in $E(A_1)$ either, as $u \in B_1 \setminus A_1$, completing the proof of $(*)$. □

2.5.7 Finding k-blocks in polynomial time

We consider graphs $G = (V,E)$, with n vertices and m edges, say, and positive integers $k < n$.We shall present a simple algorithm that finds all the k-blocks of G in time polynomial in n, m and k. We start our algorithm with the following step, which we call *pre-processing.*

For two vertices x,y of G let $\kappa(x,y)$ denote the smallest size of a set of other vertices that separates x from y in G. We construct a graph H_k from G by adding, for every pair of non-adjacent vertices x,y, the edge xy if $\kappa(x,y) \geq k$, that is, if x and y cannot be separated by fewer than k other vertices. Moreover, we label every non-edge xy of H_k by some separation of order $\kappa(x,y) < k$ that separates x from y in G. This completes the pre-processing.

Note that all separations of order $< k$ of G are still separations of H_k,and that the k-blocks of G are the vertex sets of the maximal cliques of order $\geq k$ in H_k.

Lemma 2.5.27. *The pre-processing has running time $O(\min\{k, \sqrt{n}\} \cdot m \cdot n^2)$.*

Proof. We turn the problem of finding a minimal vertex separator between two vertices into one of finding a minimal edge cut between them. This is done in the usual way (see e.g. Even [62]) by constructing a unit-capacity network G' from G with $n' = 2\tilde{n}$ vertices and $m' = 2m + \tilde{n}$ directed edges, where $\tilde{n} = O(m)$ is the number of non-isolated vertices of G.

For every non-edge xy of G we start Dinitz's algorithm (DA) on G', which is designed to find an x–y separation of order $\kappa(x, y)$. If DA completes k iterations of its 'inner loop' (finding an augmenting path), then $\kappa(x, y) \geq k$; we then stop DA and let xy be an edge of H_k. Otherwise DA returns a separation (A, B) of order $< k$; we then keep xy as a non-edge of H_k and label it by (A, B). Since the inner loop has time complexity $O(m') = O(m)$ and DA has an overall time complexity of $O(\sqrt{n'} \cdot m') = O(\sqrt{n} \cdot m)$ (see e.g. [80]), this establishes the desired bound. $\square$

Now we describe the *main part of the algorithm.* We shall construct a rooted tree T, inductively by adding children to leaves of the tree constructed so far. We maintain two lists: a list $\mathcal{L}$ of some of the leaves of the current tree, and a list $\mathcal{B}$ of subsets of V. We shall change $\mathcal{L}$ by either deleting its last element or replacing it with two new elements that will be its children in our tree. Whenever we add an element t to $\mathcal{L}$ in this way, we assign it a set $X_t \subseteq V$. Think of the current list $\mathcal{L}$ as containing those t whose X_t we still plan to scan for k-blocks of G, and of $\mathcal{B}$ as the set of k-blocks found so far.

We start with a singleton list $\mathcal{L} = (r)$ and $\mathcal{B} = \emptyset$, putting $X_r = V$.

At a given step, stop with output $\mathcal{B}$ if $\mathcal{L}$ is empty; otherwise consider the last element t of $\mathcal{L}$. If $|X_t| < k$, delete t from $\mathcal{L}$ and do nothing further at this step.

Assume now that $|X_t| \geq k$. If X_t induces a complete subgraph in H_k, add X_t to $\mathcal{B}$, delete t from $\mathcal{L}$, and do nothing further at this step.

If not, find vertices $x, y \in X_t$ that are not adjacent in H_k. At pre-processing, we labeled the non-edge xy with a separation (A, B) of order $< k$ that separates x from y in G (and in H_k). Replace t in $\mathcal{L}$ by two new elements t' and t'', making them children of t in the tree under construction, and let $X_{t'} = X_t \cap A$ and $X_{t''} = X_t \cap B$. If $|X_t| > k$, do nothing further at this step. If $|X_t| = k$, then both $X_{t'}$ and $X_{t''}$ have size $< k$; we delete t' and t'' again from $\mathcal{L}$ and do nothing further in this step.

This completes the description of the main part of the algorithm. Let T be the tree with root r that the algorithm constructed: its nodes are those t that were in $\mathcal{L}$ at some point, and its edges were defined as nodes were added to $\mathcal{L}$.

Proposition 2.5.28. *The main part of the algorithm stops with output $\mathcal{B}$ the set of k-blocks of G.*

Proof. The algorithm clearly stops with $\mathcal{B}$ the set of vertex sets of the maximal cliques of H_k that have order $\geq k$. These are the k-blocks of G, by definition of H_k. $\square$

To analyse running time, we shall need a lemma that is easily proved by induction. A *leaf* in a rooted tree is a node that has no children, and a *branching node* is one that has at least two children.

Lemma 2.5.29. *Every rooted tree has more leaves than branching nodes.* $\square$

Lemma 2.5.30. *The main part of the algorithm stops after at most $4(n-k)$ steps. Its total running time is $O(\min\{m, n\} \cdot n^2)$.*

Proof. Each step takes $O(n^2)$ time, the main task being to check whether $H_k[X_t]$ is complete.It thus suffices to show that there are no more than $4(n-k)$ steps as long as $n \leq 2m$, which can be achieved by deleting isolated vertices.

At every step except the last (when $\mathcal{L} = \emptyset$) we considered the last element t of $\mathcal{L}$, which was subsequently deleted or replaced and thus never considered again. Every such t is a node of the tree T' obtained from T by deleting the children of nodes t with $|X_t| = k$. (Recall that such children t', t'' were deleted again immediately after they were created, so they do not give rise to a step of the algorithm.) Our aim, therefore, is to show that $|T'| \leq 4(n-k) - 1$.

By Lemma 2.5.29 it suffices to show that T' has at most $2(n-k)$ leaves. As $n \geq k+1$, this is the case if T' consists only of its root r. If not, then r is a branching node of T'. It thus suffices to show that below every branching node t of T' there are at most $2(|X_t| - k)$ leaves; for $t = r$ this will yield the desired result.

By definition of T', branching nodes t of T' satisfy $|X_t| \geq k+1$. So our assertion holds if the two children of t are leaves. Assuming inductively that the children t' and t'' of t satisfy the assertion (unless they are leaves), we find that, with $X_{t'} = X_t \cap A$ and $X_{t''} = X_t \cap B$ for some $(<k)$-separation (A, B) of G as in the description of the algorithm, the number of leaves below t is at most

$$2(|X_t \cap A| - k) + 2(|X_t \cap B| - k) \leq 2(|X_t| + (k-1) - 2k) \leq 2(|X_t| - k)$$

if neither t' nor t'' is a leaf, and at most

$$1 + 2(|X_t \cap B| - k) \leq 2(|X_t| - k)$$

if t' is a leaf but t'' is not (say), since $X_t \setminus B \neq \emptyset$ by the choice of (A, B). $\square$

Putting Lemmas 2.5.27 and 2.5.30 together, we obtain the following:

Theorem 2.5.31. *There is an $O(\min\{k, \sqrt{n}\} \cdot m \cdot n^2)$-time algorithm that finds, for any graph G with n vertices and m edges and any fixed $k < n$, all the k-blocks in G.* $\square$

Our algorithm can easily be adapted to find the k-blocks of G for all values of k at once. To do this, we run our pre-processing just once to construct the graph H_n, all whose non-edges xy are labeled by an x–y separation of minimum order and its value $\kappa(x, y)$. We can then use this information at the start of the proof of Lemma 2.5.30, when we check whether $H_k[X_t]$ is complete, leaving the running time of the main part of the algorithm at $O(n^3)$ as in Lemma 2.5.30. Running it separately once for each $k < n$, we obtain with Lemma 2.5.27:

Theorem 2.5.32. *There is an $O(\max\{m\sqrt{n}\,n^2, n^4\})$ algorithm that finds, for any graph G with n vertices and m edges, all the k-blocks of G (for all k).* □

Perhaps this running time can be improved if the trees T_k exhibiting the k-blocks are constructed simultaneously, e.g. by using separations of order ℓ for all T_k with $\ell < k$.

The mere decision problem of whether G has a k-block does not need our pre-processing, which makes the algorithm faster:

Theorem 2.5.33. *For fixed k, deciding whether a graph with n vertices and m edges has a k-block has time complexity $O(mn + n^2)$.*

Proof. Given k and a graph G, we shall find either a $(< k)$-inseparable set of vertices in G (which we know extends to a k-block) or a set S of at most $2(n - k) - 1$ separations of order $< k$ such that among any k vertices in G some two are separated by a separation in S (in which case G has no k-block).

Starting with $X = V(G)$, we pick a k-set of vertices in X and test whether any two vertices in this set are separated by a $(< k)$-separation (A, B) in G. If not, we have found a $(< k)$-inseparable set of vertices and stop with a yes-answer. Otherwise we iterate with $X = A$ and $X = B$.

Every separation found by the algorithm corresponds to a branching node of T. All these are nodes of T', of which there are at most $4(n - k) - 1$ (see the proof of Lemma 2.5.30). Testing whether a given pair of vertices is separated by some $(< k)$-separation of G takes at most k runs of the inner loop of Dinitz's algorithm (which takes $O(m+n)$ time), and we test at most $\binom{k}{2}$ pairs of vertices in X. □

Let us say that a set S of $(< k)$-separations in G *witnesses* that G has no k-block if among every k vertices of G some two are separated by a separation in S. Trivially, if G has no k-block then this is witnessed by some $O(n^2)$ separations. The proof of Theorem 2.5.33 shows that this bound can be made linear:

Corollary 2.5.34. *Whenever a graph of order n has no k-block, there is a set of at most $4(n - k) - 1$ separations witnessing this.* □

Let us call any tree T as in our main algorithm (at any stage), with each of its branching nodes t labelled by a separation $(A, B)_t$ of G that separates some two vertices of X_t, a *block-decomposition* of G. The sets X_t with t a leaf will be called its *leaf sets*.

The *adhesion* of a block-decomposition is the maximum order of the separations $(A, B)_t$. A block-decomposition is *k-complete* if it has adhesion $< k$ and every leaf set is $(< k)$-inseparable or has size $< k$. The *width* of a block-decomposition is the maximum order of a leaf set X_t. The *block-width* $\mathrm{bw}(G)$ of G is the least k such that G has a block-decomposition of adhesion and width both at most k.

Having block-width $< k$ can be viewed as dual to containing a k-block, much as having tree-width $< k-1$ is dual to containing a haven or bramble of order k, and having branch-width $< k$ is dual to containing a tangle of order k. Indeed, we have shown the following:

Theorem 2.5.35. *Let $\mathcal{D} = (T; (A, B)_t\,,\ t \in T)$ be a block-decomposition of a graph G, and let $k \in \mathbb{N}$.*

(i) *Every edge of G has both ends in some leaf set of T.*

(ii) *If $\mathcal{D}$ has adhesion $< k$, then any k-block of G is contained in a leaf set of T.*

(iii) *If $\mathcal{D}$ is k-complete, then every k-block of G is a leaf set, and all other leaf sets have size $< k$.* $\square$

Theorem 2.5.35 implies that G has a block-decomposition of adhesion and width both at most k if and only if G has no $(k+1)$-block. The least such k clearly equals the greatest k such that G has a k-block, its *block number* $\beta(G)$:

Corollary 2.5.36. *Every finite graph G satisfies $\beta(G) = \mathrm{bw}(G)$.* $\square$

By Theorem 2.5.33 and its proof, we obtain the following complexity bound:

Corollary 2.5.37. *Deciding whether a graph with n vertices and m edges has block-width $< k$, for k fixed, has time complexity $O(mn + n^2)$.* $\square$

For k variable, the proof of Theorem 2.5.33 yields a complexity of $O(k^3(m + n)(n - k))$. Alternatively, we can use pre-processing to obtain $O(\min\{k, \sqrt{n}\} \cdot m \cdot n^2)$ by Theorem 2.5.31.

The above duality between the block number and the block-width of a graph is formally reminiscent of the various known dualites for other width parameters, such as the tree-width, branch-width, path-width, rank-width, carving-width or clique-width of a graph. The 'width' to which these parameters refer, however, is usually that of a tree-like decomposition of the graph itself, which exhibits that it structurally resembles that tree. In our block-decompositions, on the other hand, the tree T merely indicates a recursion by which the graph can be decomposed into small sets: the separations used to achieve this, though of small order, will not in general be nested, and the structure of G will not in any intuitive sense be similar to that of T.

In [58], Diestel and Oum give a structural duality theorem for k-blocks in the sense of those traditional width parameters. The graph structure that is shown to witness the absence of a k-block is not a tree-structure, but one modelled on more general (though still tree-like) graphs. Whether or not a structural duality between k-blocks and tree-like decompositions exists remains an open problem. It has been formalized, and stated explicitly [58], with reference to a fundamental structural duality theorem between tangle-like 'dense objects' and tree-like decompositions, which implies all the traditional duality theorems for width parameters [57] but does not yield a duality theorem for k-blocks.

2.5.8 Further examples

In this section we discuss several examples dealing with certain situations of our results. In particular, we will describe one example that shows that the $(k+1)$-block found in Theorem 2.5.11 need not contain any vertex of the adhesion set that lies in the same part of the tree-decomposition, and we will describe two examples dealing with the notion of T-shaped and Lemma 2.5.14. All these examples are included only in this extended version of this paper.

Recall that in the proof of Theorem 2.5.11 we considered a k-lean tree tree-decomposition $(T, \mathcal{V})$ of a graph G with $\delta(G) \geq 2k$ and showed for each leaf t of T that V_t includes a $(k+1)$-block b. We now give an example where the adhesion set $V_t \cap V_{t'}$ lies completely outside b, where t' is the neighbour of t in T.

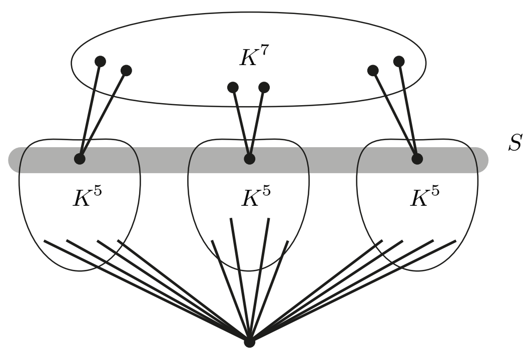


Figure 2.21: S lies outside the 4-block containing the K^7

Example 2.5.38. Let G be the graph in Figure 2.21 and let $(T, \mathcal{V})$ be the tree-decomposition with adhesion sets S and those 2-separators that contain one vertex in S and the lowest vertex. So T is a star with 4 leaves. It is not hard to show that $(T, \mathcal{V})$ is 3-lean. For every vertex x of the adhesion set S inside the upper part V_t, its two neighbours in V_t together with the bottom vertex separate it from any vertex in V_t° but its neighbours. Hence x does not lie in the 4-block b that contains V_t°. As no vertex of S lies in b, we conclude $V_t^\circ = b$.

Our next example shows that a local version of Lemma 2.5.14 as discussed just before the lemma is false. We considered there the question of whether every proper k-separation (A, B) in a k-connected graph such that A contains no $(k+1)$-block must be T-shaped, at least if A is minimal as above.

Example 2.5.39. Let $k = 6$, and let G be the complement of the disjoint union of three induced paths P_1, P_2, P_3 of length 2. Then each of three sets $V(P_i)$ is separated by the union of the other two. Hence any 7-block misses a vertex from each P_i and thus has at most 6 vertices. Hence, G has no 7-block.

But G is 6-connected, and its only proper 6-separations (A, B) have the form that either $A \setminus B$ consists of the ends of some P_i and $B \setminus A$ of its inner vertex, or vice versa. Let (A, B) be a 6-separation of the first kind. Obviously, A is minimal such that (A, B), for some B, is a proper 6-separation.

To show that (A, B) is not T-shaped, suppose it is, and let this be witnessed by another proper 6-separation (C, D). Then (C, D) is neither (A, B) nor (B, A).So the separators $A \cap B$ and $C \cap D$ meet in exactly one $V(P_i)$, say in $V(P_1)$. Then $C \cap D$ contains $V(P_2)$, say, while $A \cap B$ contains $V(P_3)$. By assumption, the ends of P_2 lie in $A \setminus B$. If the ends of P_3 lie in $C \setminus D$, say, we have $|A \cap C| = 7$. This contradicts the choice of (C, D), so (A, B) is not T-shaped.

So our envisaged local version of Lemma 2.5.14 is false. Since $|A| = 8 \leq \frac{3}{2}k$ in the above example, we could not simply use Lemma 2.5.13 to show that (A, B) is not T-shaped. In our next example A is larger, so that we can.

Example 2.5.40. Let G be the graph of Figure 2.22. It is 5-connected but has no 6-block. Let A be the vertex set that consists of the vertices of the upper three K^5s, and let B be the union of the vertex sets of the lower three complete graphs. Then (A, B) is a proper 5-separation, with A minimal. By Lemma 2.5.13, (A, B) is not T-shaped.

By Lemma 2.5.14, however, both these examples must have some T-shaped k-separation. In Example 2.5.39, the separation (B, A) is T-shaped. In Example 2.5.40, the separation (A', B') where A' consists of the two leftmost complete graphs and B' of the other four, is T-shaped.

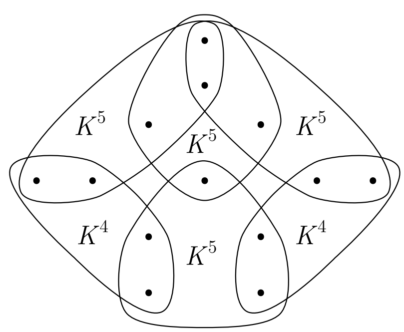


Figure 2.22: A 5-connected graph without a 6-block

2.5.9 Acknowledgements

We thank Jens Schmidt for pointing out reference [86] for Theorem 2.5.11, Matthias Kriesell for advice on the connectivity of triangle-free graphs, Paul Seymour for suggesting an algorithm of the kind indicated in Section 2.5.7, and Sang-il Oum for pointing out Corollary 2.5.34 and Theorem 2.5.35.

2.6 Canonical tree-decompositions of a graph that display its k-blocks

2.6.1 Introduction

Tangles in a graph G are orientations of the low order separations that consistently point towards some 'highly connected piece' of G. As a fundamental tool for their graph minors project, Robertson and Seymour [94] proved that every finite graph has a tree-decomposition that distinguishes every two maximal tangles.

More recently, k-*profiles* were introduced as a common generalisation of k-tangles and k-blocks [79]. Here, a k-*block* in a graph G is a maximal set of at least k vertices no two of which can be separated in G by removing less than k vertices. Carmesin, Diestel, Hamann and Hundertmark showed that every graph has a canonical tree-decomposition of adhesion less than k that distinguishes all its k-profiles [39].

In [40], these authors asked how one could improve the above tree-decompositions further so that they also display the structure of the k-blocks: it would be nice if we could compress any part containing a k-block so that it does not contain any 'junk'.

In this paper, we prove that this is possible simultaneously for all k-blocks that can be isolated at all in a tree-decomposition, canonical or not. More precisely, we call a k-block *separable* if it appears as a part in some tree-decomposition of adhesion less than k of G. We prove the following, which was conjectured by Diestel [48] (see also [40]).

Theorem 2.6.1. *Every finite graph G has a canonical tree-decomposition $\mathcal{T}$ of adhesion less than k that distinguishes efficiently every two distinct k-profiles, and which has the further property that every separable k-block is equal to the unique part of $\mathcal{T}$ in which it is contained.*

We also prove the following related result:

Theorem 2.6.2. *Every finite graph G has a canonical tree-decomposition $\mathcal{T}$ that distinguishes efficiently every two distinct maximal robust profiles, and which has the further property that every separable block inducing a maximal robust profile is equal to the unique part of $\mathcal{T}$ in which it is contained.*

See Subsection 2.6.2 for a definition of robust and [42] for an example showing that Theorem 2.6.2 fails if we leave out 'robust'. Theorem 2.6.2 without its description of the separable blocks is a result of Hundertmark and Lemanczyk [79], which implies the aforementioned theorem of Robertson and Seymour. In Subsection 2.6.4, we give an example showing that it is impossible to ensure that non-maximal robust separable blocks are also displayed by a tree-decomposition which distinguishes all the maximal robust profiles efficiently.

After recalling some preliminaries in Subsection 2.6.2, we develop the necessary tools in Subsection 2.6.3. Then we prove our main result in Subsection 2.6.4.

2.6.2 Preliminaries

Unless otherwise mentioned, G will always denote a finite, simple and undirected graph with vertex set $V(G)$ and edge set $E(G)$. Any graph-theoretic term and notation not defined here are explained in [51].

A vertex is called *central* in G if the greatest distance to any other vertex is minimal. It is well known that a finite tree T has either a unique central vertex or precisely two central adjacent vertices v and w. In the second case vw is called a *central edge*. For a vertex or edge to be central is obviously a property invariant under automorphisms of G.

Let us recall some notations from [39].

Separations

An ordered pair (A, B) of subsets of $V(G)$ is a *separation* of G if $A \cup B = V(G)$ and if there is no edge $e = vw \in E(G)$ with $v \in A \setminus B$ and $w \in B \setminus A$. The cardinality $|A \cap B|$ of the *separator* $A \cap B$ of a separation (A, B) is the *order* of (A, B) and a separation of order k is a *k-separation*.

A separation (A, B) is *proper* if neither $A \subseteq B$ nor $B \subseteq A$. Otherwise (A, B) is *improper*. A separation (A, B) is *tight* if every vertex in $A \cap B$ has a neighbour in $A \setminus B$ and a neighbour in $B \setminus A$.

The set of separations of G is partially ordered via

$$(A, B) \leq (C, D) :\Leftrightarrow A \subseteq C \ \wedge\ D \subseteq B.$$

For no two proper separations (A, B) and (C, D), the separation (A, B) is $\leq$-comparable with (C, D) and (D, C). In particular we obtain that (A, B) and (B, A) are not $\leq$-comparable.

A separation (A, B) is *nested* with a separation (C, D) if (A, B) is $\leq$-comparable with either (C, D) or (D, C). Since

$$(A, B) \leq (C, D) \ \Leftrightarrow\ (D, C) \leq (B, A),$$

being nested is symmetric and reflexive. Separations that are not nested are called *crossing*.

A separation (A, B) is *nested* with a set S of separations if (A, B) is nested with every $(C, D) \in S$. A set S of separations is *nested* with a set S' of separations if every $(A, B) \in S$ is nested with S' or equivalently every $(C, D) \in S'$ is nested with S.

A set N of separations is *nested* if its elements are pairwise nested. A set S of separations is *symmetric* if for every $(A, B) \in S$ it also contains its *inverse* separation (B, A). A symmetric set S of separations is also called a *separation system* or a *system of separations*, and if all its separations are proper, S is called a *proper separation system.* For a set S of separations the separation system *generated by* S is the separation system consisting of the separations in S and their inverses. A set S of separations is *canonical* if it is invariant under the automorphisms of G, i.e. for every $(A,B) \in S$ and for every $\varphi \in \mathrm{Aut}(G)$ we obtain $(\varphi[A], \varphi[B]) \in S$.

A separation (A,B) *separates* a vertex set $X \subseteq V(G)$ if X meets both $A \setminus B$ and $B \setminus A$. Given a set S of separations a vertex set $X \subseteq V(G)$ is *S-inseparable* if no separation $(A,B) \in S$ separates X. A maximal S-inseparable vertex set is an *S-block* of G.

For $k \in \mathbb{N}$ let $S_{<k}$ denote the set of separations of order less than k of G. The *$(<k)$-inseparable* sets are the $S_{<k}$-inseparable sets. So the *k-blocks* are exactly the $S_{<k}$-blocks of size at least k.

For two separations (A,B) and (C,D) not equal to $(V(G),V(G))$ consider a *cross-diagram* as in Figure 2.23. Every pair $(X,Y) \in \{A,B\} \times \{C,D\}$ denotes a *corner* of this cross-diagram, which we also denote by $\mathrm{cor}(X,Y)$. Let $\overline{X} \in \{A,B\} \setminus \{X\}$ and $\overline{Y} \in \{C,D\} \setminus \{Y\}$. In the diagram we consider the *center* $c{:=}A \cap B \cap C \cap D$ and for a corner $\mathrm{cor}(X,Y)$ as above the *interior* $\mathrm{int}(X,Y){:=}(X \cap Y) \setminus (\overline{X} \cup \overline{Y})$ and the *links* $\ell_X{:=}(X \cap Y \cap \overline{Y}) \setminus c$ and $\ell_Y{:=}(Y \cap X \cap \overline{X}) \setminus c$. The vertex set $X \cap Y$ is the disjoint union of $\mathrm{int}(X,Y)$ with ℓ_X, ℓ_Y and c and thus can be associated with the corner $\mathrm{cor}(X,Y)$.

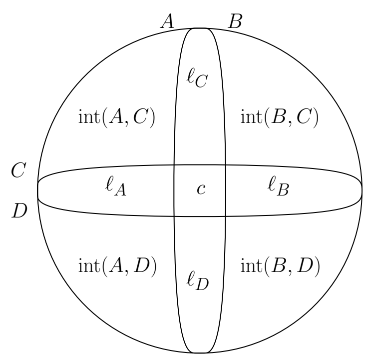


Figure 2.23: cross-diagram for (A,B) and (C,D)

Remark 2.6.3. *Two separations (A,B) and (C,D) are nested, if and only if for one of their corners $\mathrm{cor}(X,Y)$ the interior $\mathrm{int}(X,Y)$ and its links ℓ_X and ℓ_Y are empty.* □

For a corner $\mathrm{cor}(X,Y)$ there is a *corner separation* $(X \cap Y, \overline{X} \cup \overline{Y})$, which is again a separation of G.

Lemma 2.6.4. [42, Lemma 2.2] *For two crossing separations (A,B) and (C,D) any of its corner separation is nested with every separation that is nested with both (A,B) and (C,D).*

In particular a corner separation is nested with (A,B), (C,D) and all corner separations. A double counting argument yields:

Remark 2.6.5. *For any two separations (A,B) and (C,D), the orders of the separations $(A\cap C, B\cup D)$ and $(B\cap D, A\cup C)$ sum to $|A\cap B|+|C\cap D|$.* $\square$

Tree-decompositions

Recall that a *tree-decomposition* $\mathcal{T}$ of G is a pair $\big(T,(P_t)_{t\in V(T)}\big)$ of a tree T and a family of vertex sets $P_t\subseteq V(G)$ for every node $t\in V(T)$, such that

(T1) $V(G)=\bigcup_{t\in V(T)} P_t$;

(T2) for every edge $e\in E(G)$ there is a node $t\in V(T)$ such that both end vertices of e lie in P_t;

(T3) whenever t_2 lies on the t_1-t_3 path in T we obtain $P_{t_1}\cap P_{t_3}\subseteq P_{t_2}$.

The sets P_t are the *parts* of $\mathcal{T}$. For an edge $tt'\in E(T)$ the intersection $P_t\cap P_{t'}$ is the corresponding *adhesion set* and the maximum size of an adhesion set of $\mathcal{T}$ is the *adhesion* of $\mathcal{T}$. A node $t\in V(T)$ is a *hub node* if the corresponding part P_t is a subset of $P_{t'}$ for some neighbour t' of t. If t is a hub node, then P_t is a *hub*. A tree-decomposition $\mathcal{T}=\big(T,(P_t)_{t\in V(T)}\big)$ of G and a tree-decomposition $\mathcal{T}'=\big(T',(P'_t)_{t\in V(T')}\big)$ of G' are *isomorphic* if there is an isomorphism $\varphi: G\to G'$ and an isomorphism $\psi: T\to T'$ such that for every part P_t of $\mathcal{T}$ we obtain $\varphi[P_t]=P'_{\psi(t)}$. We say φ *induces* an isomorphism between $\mathcal{T}$ and $\mathcal{T}'$. A tree-decomposition $\mathcal{T}$ is *canonical* if it is invariant under the automorphisms of G, i.e. every automorphism of G induces an automorphism of $\mathcal{T}$.

Let $\big(T,(P_t)_{t\in V(T)}\big)$ be a tree-decomposition of G. For $t\in V(T)$ the *torso* H_t is the graph obtained from $G[P_t]$ by adding all edges joining two vertices in a common adhesion set $P_t\cap P_u$ for any $tu\in E(T)$. A separation (A,B) of $G[P_t]$ is a separation of H_t if and only if it does not separate any adhesion set $P_t\cap P_{t'}$ for $tt'\in E(T)$. A separation (A,B) of G with $A\cap B\subseteq P_t$ for some node $t\in V(T)$ that does not separate any adhesion set $P_t\cap P_{t'}$ for $tt'\in E(T)$ *induces* the separation $(A\cap P_t, B\cap P_t)$ of H_t.

Every oriented edge $\vec{e}=t_1t_2$ of T divides $T-e$ in two components T_1 and T_2 with $t_1\in V(T_1)$ and $t_2\in V(T_2)$. By [51, Lemma 12.3.1] $\vec{e}$ *induces* the separation $\big(\bigcup_{t\in V(T_1)} P_t, \bigcup_{t\in V(T_2)} P_t\big)$ of G such that the separator coincides with the adhesion set $P_{t_1}\cap P_{t_2}$. We say a separation is *induced* by $\mathcal{T}$ if it is induced by an oriented edge of T.

The set of separations induced by a tree-decomposition $\mathcal{T}$ (of adhesion less than k) is a nested system $N(\mathcal{T})$ of separations (of order less than k). We say $N(\mathcal{T})$ is *induced* by $\mathcal{T}$. Clearly if $\mathcal{T}$ is canonical, then so is $N(\mathcal{T})$.

Conversely, as proven in [42], every nested separation system N *induces* a tree-decomposition $\mathcal{T}(N)$:

Theorem 6. [42, Theorem 4.8] *Let N be a canonical nested separation system of G. Then there is a canonical*[42] *tree-decomposition $\mathcal{T}(N)$ of G such that*

(i) every N-block of G is a part of $\mathcal{T}(N)$;

(ii) every part of $\mathcal{T}(N)$ is either an N-block of G or a hub;

(iii) the separations of G induced by $\mathcal{T}(N)$ are precisely those in N;

(iv) every separation in N is induced by a unique oriented edge of $\mathcal{T}(N)$.

Profiles

Let S be a separation system. A subset $O \subseteq S$ is an *orientation* of S if for every $(A, B) \in S$ exactly one of (A, B) and (B, A) is an element of O. An orientation O of S is *consistent* if for every (A, B), $(C, D) \in S$ with $(A, B) \in O$ and $(C, D) \leq (A, B)$ we obtain $(C, D) \in O$ as well. A consistent orientation P of $S_{<k}$ is called a *k-profile* if it satisfies

(P) for all (A, B), $(C, D) \in P$ we have $(B \cap D, A \cup C) \notin P$.

In particular if the order $|(A \cup C) \cap (B \cap D)|$ of this corner separation is less than k, we have $(A \cup C, B \cap D) \in P$. Sometimes we omit the k and call P a *profile*.

It is easy to check that every k-block b *induces* a k-profile via

$$P_k(b) := \{(A, B) \in S_{<k} \mid b \subseteq B\}.$$

Also *tangles* of order k (or *k-tangles*), as introduced by Robertson and Seymour [94], are k-profiles. For more background on profiles, see [79].

For $r \in \mathbb{N}$, a k-profile P is *r-robust* if for any $(A, B) \in P$ and any $(C, D) \in S_{<r+1}$ one of $(A \cup C, B \cap D)$, $(A \cup D, B \cap C)$ either has order at least $k - 1$, or is in P. If P is r-robust for all $r \in \mathbb{N}$, then we call P *robust*.

A robust k-profile P is *maximal* if there does not exist a robust ℓ-profile Q with $P \subsetneq Q$ and $\ell > k$. Then P is just called a *maximal robust profile*.

Remark 2.6.6. *(i) Every k-profile is ℓ-robust for all $\ell < k$;*

(ii) if a k-block b contains a complete graph on k vertices, then the induced k-profile $P_k(b)$ is robust. □

The next lemma basically states that every k-profile induces a k-haven, as introduced by Seymour and Thomas [97].

Lemma 2.6.7. *Let $X \subseteq V(G)$ with $|X| < k$ and let Q be a k-profile. Then there exists a component C of $G - X$ such that $(V(G) \setminus C, C \cup X) \in Q$.*

Furthermore, $(V(G) \setminus C, C \cup N(C)) \in Q$ as well.

[42] In the original paper this theorem is stated without the canonicity since it holds in a greater generality. But it is clear from the proof that if N is canonical, then so is $\mathcal{T}(N)$.

Proof. Let $C_1, \dots, C_n$ denote the components of $G-X$ and for $i \in \{1, \dots, n\}$ let $(A_i, B_i){:=}(V(G) \setminus C_i, C_i \cup X)$. To reach a contradiction suppose that $(B_i, A_i) \in Q$ for all $i \in \{1, \dots, n\}$. Then (P) yields inductively for all $m \leq n$ that $\big(\bigcup_{i \leq m} B_i, \bigcap_{i \leq m} A_i\big) \in Q$, since their separators all equal X. Hence for $m = n$, we obtain $(V(G), X) \in Q$, contradicting the consistency of Q with $(X, V(G)) \leq (V(G), X)$. Thus there is a component C of $G - X$ such that $(A, B){:=}(V(G) \setminus C, C \cup X) \in Q$.

Now suppose $(C \cup N(C), V(G) \setminus C) \in Q$. Then (P) with (A, B) yields that $\big((V(G) \setminus C) \cup C \cup N(C), (C \cup X) \cap (V(G) \setminus C)\big) = (V(G), X) \in Q$, contradicting the consistency of Q again. $\square$

A k-profile Q *inhabits* a part P_t of a tree-decomposition $\big(T, (P_t)_{t \in V(T)}\big)$ if for every $(A, B) \in Q$ we obtain that $(B \setminus A) \cap P_t$ is not empty. Note that if for a node t every separation induced by an oriented edge ut of T has order less than k, then Q inhabits P_t if and only if all those separations are in Q.

Corollary 2.6.8. *Let $\big(T, (P_t)_{t \in V(T)}\big)$ be a tree-decomposition and let Q be a k-profile. If Q inhabits a part P_t, then $|P_t| \geq k$.*

Proof. Our aim is to show that if $|P_t| < k$, then any k-profile Q does not inhabit P_t. By Lemma 2.6.7 there is a component C of $G - P_t$ such that $(V(G) \setminus C, C \cup P_t) \in Q$. Since $(C \cup P_t) \setminus (V(G) \setminus C) = C$ and since $C \cap P_t$ is empty, we obtain that Q does not inhabit P_t. $\square$

A set $\mathcal{P}$ of profiles is *canonical* if for every $P \in \mathcal{P}$ and every automorphism φ of G the profile $\big\{\big(\varphi[A], \varphi[B]\big) \;\big|\; (A, B) \in P\big\}$ is also in $\mathcal{P}$.

Two profiles P and Q are *distinguishable* if there is a separation (A, B) with $(A, B) \in P$ and $(B, A) \in Q$. Such a separation *distinguishes* P and Q. It is said to distinguish P and Q *efficiently* if its order $|A \cap B|$ is minimal among all separations distinguishing P and Q. A set $\mathcal{P}$ of profiles is *distinguishable* if every two distinct $P, Q \in \mathcal{P}$ are distinguishable. A tree-decomposition $\mathcal{T}$ *distinguishes* two profiles P and Q (efficiently) if some (A, B) induced by $\mathcal{T}$ distinguishes them (efficiently).

For our main result of this paper, we will build on the following theorem.

Theorem 7. [79, Theorem 2.6][43] *Every graph G has a canonical tree-decomposition of adhesion less than k that distinguishes every two distinguishable $(k-1)$-robust ℓ-profiles of G for some values $\ell \leq k$ efficiently.*

Moreover, every separation induced by the tree-decomposition distinguishes some of those profiles efficiently.

2.6.3 Construction methods

Sticking tree-decompositions together

Given a tree-decomposi-tion $\mathcal{T}$ of G and for each torso H_t a tree-decomposition $\mathcal{T}^t$, our aim is to construct a new tree-decomposition $\overline{\mathcal{T}}$ of G by gluing together

[43] Since [79] is unpublished, see also [42, Theorem 6.3] for a version just concerning robust blocks or [36, Theorem 9.2] for a version also dealing with infinite graphs.

the tree-decompositions $\mathcal{T}^t$ of the torsos along $\mathcal{T}$ in a canonical way.

Example 2.6.9. First we shall give the construction of $\overline{\mathcal{T}}$ for a particular example: G is obtained from three edge-disjoint triangles intersecting in a single vertex by identifying two other vertices of distinct triangles. The tree-decomposition $\mathcal{T}$ of G and the tree-decompositions of the torsos are depicted in Figure 2.24 (a). In order to stick the tree-decompositions of the torsos together in a canonical way, we first have to refine them, see Figure 2.24 (b).

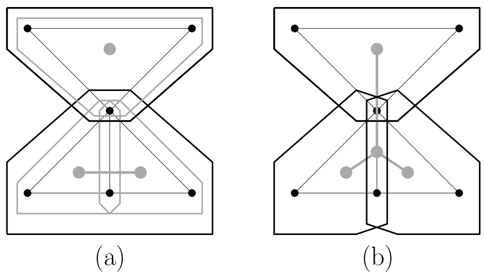


Figure 2.24: (a) shows the tree-decomposition $\mathcal{T}$ of G, drawn in black, and the tree-decompositions of the torsos, drawn in grey. (b) shows the canonically glued tree-decomposition $\overline{\mathcal{T}}$.

Before we can construct $\overline{\mathcal{T}}$, we need some preparation.

Construction 8. *Given a tree-decomposition* $\mathcal{T} = \big(T, (P_t)_{t\in V(T)}\big)$ *of* G*, we construct a new tree-decomposition* $\widetilde{\mathcal{T}} = \big(\widetilde{T}, (P_t)_{t\in V(\widetilde{T})}\big)$ *of* G *by contracting every edge* tu *of* T *where* $P_t = P_u$.[44] *In this tree-decomposition two adjacent nodes never have the same part. Let* $F \subseteq E(\widetilde{T})$ *be the set of edges* tu *where neither* $P_t \subseteq P_u$ *nor* $P_u \subseteq P_t$*. By subdividing every edge* $tu \in F$ *and assigning to the subdivided node* x *the part* $P_x{:=}P_t \cap P_u$*, we obtain a new tree-decomposition* $\widehat{\mathcal{T}} = \big(\widehat{T}, (P_t)_{t\in V(\widehat{T})}\big)$.

Remark 2.6.10. $\widehat{\mathcal{T}}$ *defined as in Construction 8 satisfies the following:*

(i) every separation induced by $\widehat{\mathcal{T}}$ *is also induced by* $\mathcal{T}$*;*

(ii) for every edge $tu \in E(T)$ *that induces a separation not induced by* $\widehat{\mathcal{T}}$ *we have* $P_t = P_u$*;*

(iii) for every edge $tu \in E(\widehat{T})$ *precisely one of* P_t *or* P_u *is a proper subset of the other;*

(iv) if $\mathcal{T}$ *distinguishes two profiles* Q_1 *and* Q_2 *efficiently, then so does* $\widehat{\mathcal{T}}$*;*

(v) if $\mathcal{T}$ *is canonical, then* $\widehat{\mathcal{T}}$ *is canonical as well.* □

[44] Here we understand the nodes of $\widehat{T}$ to be nodes of T, where a node obtained through the contraction of an edge tu to be identified with either t or u.

Lemma 2.6.11. *Let K be a complete subgraph of G and $\widehat{\mathcal{T}}$ as in Construction 8. Then there is a node t of $\widehat{T}$ with $V(K) \subseteq P_t$ such that P_t is fixed by every automorphism of G fixing K.*

Proof. As K is complete, there is a node $u \in V(\widehat{T})$ with $V(K) \subseteq P_u$.

Let W be the subforest of nodes w with $K \subseteq P_w$, which is connected as $\widehat{\mathcal{T}}$ is a tree-decomposition. Now W either has a central vertex t or a central edge tu such that P_u is a proper subset of P_t (cf Remark 2.6.10 (iii)). In both cases P_t is fixed by the automorphisms of G that fix K. □

Construction 9. *Let $\mathcal{T} = \big(T, (P_t)_{t \in V(T)}\big)$ be a tree-decomposition of G. For each $t \in V(T)$ let $\mathcal{T}^t = \big(T^t, (P^t_u)_{u \in V(T_t)}\big)$ be a tree-decomposition of the torso H_t. For each $\mathcal{T}^t$ let $\widehat{\mathcal{T}}^t$ be as in Construction 8. For $e = tu \in E(T)$ let A_e denote the adhesion set $P_t \cap P_u$. Since $H_t[A_e]$ is complete, we can apply Lemma 2.6.11: there is a node $\gamma(t,u)$ of $\widehat{T}^t$ with $A_e \subseteq P^t_{\gamma(t,u)}$ such that $P^t_{\gamma(t,u)}$ is fixed by every automorphism of H_t fixing K.*

We obtain a tree $\overline{T}$ from the disjoint union of the trees $\widehat{T}^t$ for all $t \in V(T)$ by adding the edges $\gamma(t,u)\gamma(u,t)$ for each $tu \in E(T)$. Let $\overline{P}_u$ be P^t_u for the unique $t \in V(T)$ with $u \in V(\widehat{T}^t)$. Then $\overline{\mathcal{T}} := \big(\overline{T}, (\overline{P}_t)_{t \in V(\overline{T})}\big)$ is a tree-decomposition of G.

Two torsos H_t and H_u of $\mathcal{T}$ are *similar*, if there is an automorphism of G that induces an isomorphism between H_t and H_u. The family $\big(\mathcal{T}^t\big)_{t \in V(T)}$ is *canonical* if all the $\mathcal{T}^t$ are canonical and for any two similar torsos H_t and H_u of $\mathcal{T}$ every automorphism of G that witnesses the similarity of H_t and H_u induces an isomorphism between $\mathcal{T}^t$ and $\mathcal{T}^u$.

Lemma 2.6.12. *The tree-decomposition $\overline{\mathcal{T}}$ as in Construction 9 satisfies the following:*

(i) for $t \in V(T)$ every node $u \in V(T^t)$ is also a node of $\overline{T}$ and $\overline{P}_u = P^t_u$;

(ii) every node $u \in V(\overline{T})$ that is not a node of any T^t is a hub node;

(iii) every separation (A, B) induced by $\overline{\mathcal{T}}$ is either induced by $\mathcal{T}$ or there is a node $t \in V(T)$ such that $(A \cap P_t, B \cap P_t)$ is induced by $\mathcal{T}^t$;

(iv) every separation induced by $\mathcal{T}$ is also induced by $\overline{\mathcal{T}}$;

(v) for every separation (C, D) induced by $\widehat{\mathcal{T}}^t$ there is a separation (A, B) induced by $\overline{\mathcal{T}}$ such that $A \cap B \subseteq P_t$ and $(A \cap P_t, B \cap P_t) = (C, D)$;

(vi) if $\mathcal{T}$ and the family of the $\mathcal{T}^t$ are canonical, then $\overline{\mathcal{T}}$ is canonical.

Proof. Whilst (i) is true by construction, the nodes added in the construction of $\widehat{\mathcal{T}}^t$ are hub nodes by definition, yielding (ii). Furthermore, (iii), (iv) and (v) follow by construction with Remark 2.6.10 (i) and the observation that for all $tu \in E(T)$ the adhesion sets $\overline{P}_{\gamma(t,u)} \cap \overline{P}_{\gamma(u,t)}$ and $P_t \cap P_u$ are equal. Finally, (iv) follows with Remark 2.6.10 (v) and Lemma 2.6.11 from the construction. □

Obtaining tree-decompositions from almost nested sets of separations

Theorem 6 gives a way how to transform a nested set of separations into a tree-decomposition. In this subsection, we extend this to sets of 'almost nested' separations.

For a separation (A, B) of G and $X \subseteq V(G)$, the pair $(A \cap X, B \cap X)$ is a separation of $G[X]$, which we call the *restriction* $(A, B)\restriction X$ of (A, B) to X. Note that $(A, B)\restriction X$ is proper if and only if (A, B) separates X. The *restriction* $S\restriction X$ to X of a set S of separations of G to X consists of the proper separations $(A, B)\restriction X$ with $(A, B) \in S$.

For a set S of separations of G let $\min_{\mathrm{ord}}(S)$ denote the set of those separations in S with minimal order. Note that if S is non-empty, then so is $\min_{\mathrm{ord}}(S)$, and that $\min_{\mathrm{ord}}$ commutes with graph isomorphisms.

A finite sequence $(\beta_0, \ldots, \beta_n)$ of vertex sets of G is called an *S-focusing sequence* if

(F1) $\beta_0 = V(G)$;

(F2) for all $i < n$, the separation system N_{β_i} generated by $\min_{\mathrm{ord}}(S\restriction\beta_i)$ is non-empty and is nested with the set $S\restriction\beta_i$;

(F3) β_{i+1} is an N_{β_i}-block of $G[\beta_i]$.

An S-focusing sequence $(\beta_0, \ldots, \beta_n)$ is *good* if

(F$*$) the separation system N_{β_n} generated by $\min_{\mathrm{ord}}(S\restriction\beta_n)$ is nested with the set $S\restriction\beta_n$.

Note that for an S-focusing sequence $(\beta_0, \ldots, \beta_n)$ we obtain $\beta_n \subseteq \beta_{n-1} \subseteq \ldots \subseteq \beta_0$. The set of all S-focusing sequences is partially ordered by extension, where $(V(G))$ is the smallest element. The subset $\mathcal{F}_S$ of all good S-focusing sequences is downwards closed in this partial order.

Lemma 2.6.13. *Let $(\beta_0, \ldots, \beta_n) \in \mathcal{F}_S$ and let $(A, B) \in S$. If $(A, B)\restriction\beta_n$ is proper, then $A \cap B \subseteq \beta_n$.*

Proof. By assumption $(A, B)\restriction\beta_n$ is proper, hence there are $a \in (\beta_n \cap A) \setminus B$ and $b \in (\beta_n \cap B) \setminus A$. Since $\beta_n \subseteq \beta_i$ for all $i \leq n$ the separations $(A, B)\restriction\beta_i$ are proper as well. Suppose for a contradiction there is a vertex $v \in (A \cap B) \setminus \beta_n$. Let $j < n$ be maximal with $v \in \beta_j$. Since β_{j+1} is an N_{β_j}-block of $G[\beta_j]$, there is a separation $(C, D) \in N_{\beta_j}$ with $v \in C \setminus D$ and $\{a, b\} \subseteq \beta_n \subseteq \beta_{j+1} \subseteq D$.

Now a, b and v witness that $(A, B)\restriction\beta_j$ and (C, D) are not nested: Indeed, a witnesses that D is not a subset of $B \cap \beta_j$. Similarly, b witnesses that D is not a subset of $A \cap \beta_j$. But v witnesses that neither $A \cap \beta_j$ nor $B \cap \beta_j$ is a subset of D. Thus we get a contradiction to the assumption that N_{β_j} is nested with the set $S\restriction\beta_j$. $\square$

A set S of separations of G is *almost nested* if all S-focusing sequences are good. In this case the maximal elements of $\mathcal{F}_S$ in the partial order are exactly the S-focusing sequences $(\beta_0, \ldots, \beta_n)$ with $N_{\beta_n} = \emptyset$, and hence $S\restriction\beta_n = \emptyset$.

Lemma 2.6.14. *Let S be an almost nested set of separations of G.*

(i) If $(\beta_0, \dots, \beta_n) \in \mathcal{F}_S$ is maximal, then β_n is an S-block.

(ii) If b is an S-block, there is a maximal $(\beta_0, \dots, \beta_n) \in \mathcal{F}_S$ with $\beta_n = b$.

Proof. Let $(\beta_0, \dots, \beta_n) \in \mathcal{F}_S$ be maximal. Then $S{\restriction}\beta_n$ is empty, i.e. no $(A, B) \in S$ induces a proper separation of $G[\beta_n]$. Hence β_n is S-inseparable. For every $v \in V(G) \setminus \beta_n$ there is an $i < n$ and a separation in N_{β_i} separating v from β_n. Hence β_n is an S-block.

Conversely given an S-block b, let $(\beta_0, \dots, \beta_n) \in \mathcal{F}_S$ be maximal with the property $b \subseteq \beta_n$, which exists since $(V(G)) \in \mathcal{F}_S$ and since $\mathcal{F}_S$ is finite. Since b is N_{β_n}-inseparable, there is some N_{β_n}-block β_{n+1} containing b. The choice of $(\beta_0, \dots, \beta_n)$ implies that $(\beta_0, \dots, \beta_{n+1}) \notin \mathcal{F}_S$ and hence $N_{\beta_n} = \emptyset$, i.e. $(\beta_0, \dots, \beta_n)$ is a maximal element of $\mathcal{F}_S$. Thus β_n is an S-block with $b \subseteq \beta_n$ and hence $b = \beta_n$. $\square$

Construction 10. *Let S be an almost nested set of separations of G. We recursively construct for every S-focusing sequence $(\beta_0, \dots, \beta_n)$ a tree-decomposition $\mathcal{T}^{\beta_n}$ of $G[\beta_n]$ so that the tree-decomposition $\mathcal{T}^{V(G)}{=:}\mathcal{T}(S)$ for the smallest S-focusing sequence $(V(G))$ is a tree-decomposition of G.*

For each maximal S-focusing sequence $(\beta_0, \dots, \beta_m)$ we take the trivial tree-decomposition of $G[\beta_m]$ with only a single part. Suppose that $\mathcal{T}^{\beta}$ has already been defined for every successor $(\beta_0, \dots, \beta_n, \beta)$ of $(\beta_0, \dots, \beta_n)$. To define $\mathcal{T}^{\beta_n}$ we start with the tree-decomposition $\mathcal{T}(N_{\beta_n})$ of $G[\beta_n]$ as given by Theorem 6. For each hub node h we take the trivial tree-decomposition of H_h and for each node t whose part is an N_{β_n}-block β, we take $\mathcal{T}^{\beta}$ given from the S-focusing sequence $(\beta_0, \dots, \beta_n, \beta)$. This is indeed a tree-decomposition of the torso H_t, which we will show in Theorem 11. Hence we can apply Construction 9 to $\mathcal{T}(N_{\beta_n})$ and the family of tree-decompositions of the torsos to get $\mathcal{T}^{\beta_n}$.

Given an S-focusing sequence $(\beta_0, \dots, \beta_n)$, any two separations in N_{β_n} have the same order ℓ. We call ℓ the *rank* of $(\beta_0, \dots, \beta_n)$. If N_{β_n} is empty, we set the rank to be ∞.

For an almost nested set S of separations of G two S-focusing sequences $(\beta_0, \dots, \beta_n)$ and $(\alpha_0, \dots \alpha_m)$ are *similar* if there is an automorphism ψ of G inducing an isomorphism between $G[\beta_n]$ and $G[\alpha_m]$. Similar S-focusing sequences clearly have the same rank. If S is canonical, then ψ induces an isomorphism between $\mathcal{T}(N_{\beta_n})$ and $\mathcal{T}(N_{\alpha_m})$ as obtained from Theorem 6.

Theorem 11. *The tree-decomposition $\mathcal{T}(S)$ as in Construction 10 is well-defined and satisfies the following:*

(i) every S-block of G is a part of $\mathcal{T}(S)$;

(ii) every part of $\mathcal{T}(S)$ is either an S-block of G or a hub;

(iii) for every separation (A, B) induced by $\mathcal{T}(S)$ there is a separation $(A', B') \in S$ such that $A \cap B = A' \cap B'$;

(iv) if S is canonical, then so is $\mathcal{T}(S)$.

Proof. We show inductively that for any S-focusing sequence $(\beta_0, \ldots, \beta_n)$ the tree-decomposition $\mathcal{T}^{\beta_n}$ has the following properties:

(a) every S-block included in β_n is a part of $\mathcal{T}^{\beta_n}$;

(b) every part of $\mathcal{T}^{\beta_n}$ is either an S-block or a hub;

(c) every separation (A, B) induced by $\mathcal{T}^{\beta_n}$ is proper;

(d) and for every separation (A, B) induced by $\mathcal{T}^{\beta_n}$ there is a separation $(A', B') \in S$ and an S-focusing sequence $(\beta_0, \ldots, \beta) \geq (\beta_0, \ldots, \beta_n)$ such that $(A', B') \upharpoonright \beta = (A, B)$.

Furthermore we show for canonical S by induction, that

(e) if $(\alpha_0, \ldots, \alpha_m)$ and $(\beta_0, \ldots, \beta_n)$ are similar, then $\mathcal{T}^{\alpha_m}$ and $\mathcal{T}^{\beta_n}$ are isomorphic;

(f) $\mathcal{T}^{\beta_n}$ is canonical.

The tree-decompositions for the maximal S-focusing sequences satisfy (a) and (b) by Lemma 2.6.14, and (c) and (d) since their trees do not have any edges. If for two S-blocks b_1 and b_2 there is an isomorphism between $G[b_1]$ and $G[b_2]$ induced by an automorphism of G, then clearly the tree-decompositions are isomorphic. Hence (e) and (f) hold for all S-focusing sequences of rank ∞.

Suppose for our induction hypothesis that for every S-focusing sequence $(\alpha_0, \ldots, \alpha_m)$ with rank greater than r the tree-decomposition $\mathcal{T}^{\alpha_m}$ of $G[\alpha_m]$ has the desired properties. Let $(\beta_0, \ldots, \beta_n)$ be an S-focusing sequence of rank r. Then for each successor $(\beta_0, \ldots, \beta_n, \beta)$ the tree-decomposition $\mathcal{T}^{\beta}$ is indeed a tree-decomposition of the corresponding torso: for a separation (A, B) induced by $\mathcal{T}^{\beta}$ consider (A', B') as given in (d). By (F$*$) we obtain that $(A', B') \upharpoonright \beta_n = (A, B)$ is nested with N_{β_n}, hence (A, B) does not separate any adhesion set in H_t. Hence $\mathcal{T}^{\beta_n}$ is indeed well-defined.

Lemma 2.6.12 (i), (ii) and (iii) and the induction hypothesis yield (a), (b) and (c) for $\mathcal{T}^{\beta_n}$. Also by Lemma 2.6.12 (iii) for a separation (A, B) induced by $\mathcal{T}^{\beta_n}$ either $(A, B) \in N_{\beta_n} \subseteq S \upharpoonright \beta_n$ or (A, B) induces a separation in $\mathcal{T}^{\beta}$ for an N_{β_n}-block β on the corresponding torso. In the first case $(\beta_0, \ldots, \beta_n)$ is the desired S-focusing sequence for (d) and in the second case the induction hypothesis yields $(A', B') \in S$ and the desired S-focusing sequence extending $(\beta_0, \ldots, \beta_n, \beta)$. Hence (d) holds for $\mathcal{T}^{\beta_n}$.

Suppose S is canonical. Let $(\alpha_0, \ldots, \alpha_m)$ be similar to $(\beta_0, \ldots, \beta_n)$. Then every automorphism of G that witnesses the similarity also witnesses that $\mathcal{T}(N_{\alpha_m})$ and $\mathcal{T}(N_{\beta_n})$ are isomorphic. Hence any torso of $\mathcal{T}(N_{\alpha_m})$ is similar to the corresponding torso of $\mathcal{T}(N_{\beta_n})$ and by induction hypothesis the tree-decompositions of the torsos are isomorphic. Therefore following Construction 9 yields (e). If two torsos H_t and H_u of $\mathcal{T}(N_{\beta_n})$ are similar, then either $V(H_t)$ and $V(H_u)$ are $N(\beta_n)$-blocks whose corresponding S-focusing sequences are similar and have rank greater than r, or they are hubs. If they are N_{β_n}-blocks, the chosen tree-decompositions are isomorphic by the induction hypothesis. If they are hubs, the chosen trivial tree-decompositions are isomorphic as witnessed by every automorphism of G witnessing the similarity of H_t and H_u. Hence this family of tree-decompositions of the torsos of $\mathcal{T}(N_{\beta_n})$ is canonical and with Lemma 2.6.12 (vi) we get (f).

Inductively the tree-decomposition $\mathcal{T}^{V(G)} = \mathcal{T}(S)$ of G satisfies (i), (ii) and (iv) by (a), (b) and (f). Finally, (iii) follows from (c), (d) and Lemma 2.6.13. $\square$

Extending a nested set of separations

In this subsection we give a condition for when we can extend a nested set of separations so that it distinguishes any two distinguishable profiles in a given set $\mathcal{P}$ efficiently.

Let N be a nested separation system of G and $\mathcal{T}(N) = \big(T, (P_t)_{t \in V(T)}\big)$ be the tree-decomposition of G as in Theorem 6. Recall that a separation (A, B) of G nested with N *induces* a separation $(A \cap P_t, B \cap P_t)$ of each torso H_t. An ℓ-profile $\widetilde{Q}$ of H_t is *induced* by a k-profile Q of G if for every $(A', B') \in \widetilde{Q}$ there is an $(A, B) \in Q$ which induces (A', B') on H_t.

Construction 12. *Let $t \in V(T)$ and let Q be a k-profile of G. We construct a profile $\widetilde{Q}^t$ of the torso H_t which is induced by Q.*

Case 1: *Q inhabits P_t.*
Let (A, B) be a proper separation of H_t of order less than k. By Lemma 2.6.7, there is a unique component C of $G - (A \cap B)$ with $(V(G) \setminus C, C \cup N(C)) \in Q$. As Q is consistent and inhabits P_t, the set $C \cap P_t$ is non-empty and either a subset of $A \setminus B$ or $B \setminus A$, but not both. If $(C \cap P_t) \subseteq (B \setminus A)$, then we let $(A, B) \in \widetilde{Q}^t$. Otherwise we let $(B, A) \in \widetilde{Q}^t$.

Case 2: *Q does not inhabit P_t and $(V(G) \setminus C, C \cup N(C)) \notin Q$ for all components C of $G - P_t$.*
Let (A, B) be a proper separation of H_t of order less than k. By Lemma 2.6.7, there is a unique component C of $G - (A \cap B)$ with $(V(G) \setminus C, C \cup N(C)) \in Q$. Since C is not a component of $G - P_t$, the set $C \cap P_t$ is non-empty by assumption, and we define $\widetilde{Q}^t$ as above.

Case 3: *Q does not inhabit P_t and there is a component C of $G - P_t$ such that $(V(G) \setminus C, C \cup N(C)) \in Q$.*
Let m denote the size of the neighbourhood of C. Let b be the m-block of H_t containing $N(C)$. For $\widetilde{Q}^t$ we take the m-profile induced by b.

The following is straightforward to check:

Remark 2.6.15. *The set $\widetilde{Q}^t$ as in Construction 12 is a profile of H_t induced by Q. Moreover, if Q is r-robust, then so is $\widetilde{Q}^t$.* $\square$

The next remark is a direct consequence of the relevant definitions.

Remark 2.6.16. *Let Q_1 and Q_2 be profiles of G.*

(i) If a separation (A, B) of G nested with N distinguishes Q_1 and Q_2 efficiently, then the induced separation $(A \cap P_t, B \cap P_t)$ of H_t distinguishes $\widetilde{Q}_1^t$ and $\widetilde{Q}_2^t$ efficiently for any part P_t where it is proper;

(ii) if a separation (A, B) of some torso H_t distinguishes $\widetilde{Q}_1^t$ and $\widetilde{Q}_2^t$, then any separation of G that induces (A, B) on H_t distinguishes Q_1 and Q_2. □

Lemma 2.6.17. *Let Q_1 and Q_2 be profiles of G which are not already distinguished efficiently by N. Let (A, B) distinguish them efficiently such that it is nested with N. Then there is a part P_t of $\mathcal{T}(N)$ such that the induced separation $(A \cap P_t, B \cap P_t)$ of the torso H_t is proper.*

Proof. Since (A, B) is nested with N, there is a part P_t such that $A \cap B \subseteq P_t$. Suppose that $(A \cap P_t, B \cap P_t)$ is not proper. Without loss of generality let $(B \setminus A) \cap P_t$ be empty and let $(A, B) \in Q_1$.

By Lemma 2.6.7 we obtain a component K of $G - (A \cap B)$ such that $(A, B) \leq (V(G) \setminus K, K \cup N(K)) \in Q_1$. By consistency of Q_2 the separation $(V(G) \setminus K, K \cup N(K))$ still distinguishes Q_1 and Q_2, and since (A, B) distinguishes Q_1 and Q_2 efficiently, the neighbourhood of K is $A \cap B$. Let u be the neighbour of t such that the by tu induced separation $(C_t, D_t) \in N$ satisfies $K \cup N(K) \subseteq D_t$. If $(B \setminus A) \cap P_u$ is empty, we obtain $(C_u, D_u) \in Q_1$ as before and by construction we obtain $(C_t, D_t) < (C_u, D_u)$.

Among all parts P_t containing $A \cap B$ such that $(B \setminus A) \cap P_t$ is empty, we choose a part P_x such that (C_x, D_x) is maximal with respect to the ordering of separations. Let y denote the neighbour of x such that xy induces (C_x, D_x). There is a vertex $v \in (C_x \cap D_x) \setminus (A \cap B)$, since otherwise (C_x, D_x) would distinguish Q_1 and Q_2 efficiently. Since we assumed that $(B \setminus A) \cap P_x$ is empty, we deduce that $v \in A \setminus B$. Therefore $(A \setminus B) \cap P_y$ is not empty. Hence if $(A \cap P_y, B \cap P_y)$ on H_y were improper, then $(B \setminus A) \cap P_y$ would be empty and (C_y, D_y) would contradict the maximality of (C_x, D_x). □

For a nested separation system N let $S_{<k}^N$ be the set of separations of order less than k of G nested with N.

Construction 13. *Let $N \subseteq S_{<r+1}$ be a nested separation system of G and let $\mathcal{P}$ be a set r-robust ℓ-profiles of G for some values $\ell \leq r + 1$, such that $S_{<r+1}^N$ distinguishes any two distinguishable profiles in $\mathcal{P}$ efficiently.*

Let $\mathcal{T}(N) = \big(T, (P_t)_{t \in V(T)}\big)$ be as in Theorem 6 and let $\mathcal{P}^t$ be the set of profiles $\widetilde{Q}^t$ of H_t for $Q \in \mathcal{P}$. Applying Theorem 7 to the graphs H_t and the maximal k of any k-profile in $\mathcal{P}^t$, we get a tree-decomposition $\mathcal{T}^t$ of H_t that distinguishes every two distinguishable profiles in $\mathcal{P}^t$ efficiently. Note that if $\mathcal{P}$ is canonical, then the family $(\mathcal{T}^t)_{t \in V(T)}$ is canonical as well. By applying Lemma 2.6.12 we obtain a tree-decomposition $\overline{\mathcal{T}}$ and the corresponding nested system $\overline{N}$ of separations of order at most r induced by $\overline{\mathcal{T}}$.

Theorem 14. *The nested separation system $\overline{N}$ as in Construction 13 satisfies the following.*

(i) $N \subseteq \overline{N}$*;*

(ii) $\overline{N}$ *distinguishes every two distinguishable profiles in* $\mathcal{P}$ *efficiently;*

(iii) if N *and* $\mathcal{P}$ *are canonical, then so is* $\overline{N}$*.*

Proof. Lemma 2.6.12 (iv) yields (i). For (ii), consider two distinguishable profiles $Q_1, Q_2 \in \mathcal{P}$ not already distinguished efficiently by N. By assumption, there is some $(A, B) \in S^N_{<r+1}$ distinguishing Q_1 and Q_2 efficiently.

By Lemma 2.6.17 and Remark 2.6.16 (i) there is a part P_t of $\mathcal{T}(N)$ such that $\widetilde{Q}^t_1$ and $\widetilde{Q}^t_2$ are distinguished efficiently by $(A \cap P_t, B \cap P_t)$. Hence Theorem 7, Remark 2.6.10 (iv), Lemma 2.6.12 (v) and Remark 2.6.16 (ii) yield a separation of order $|A \cap B|$ in $\overline{N}$ distinguishing Q_1 and Q_2, yielding (ii).

Finally, (iii) holds by construction. □

2.6.4 Proof of the main result

Given a k-block b and a component C of $G - b$, then $(C \cup N(C), V(G) \setminus C)$ is a separation. By $S_k(b)$ we denote the set of all those separations. Note that $S_k(b)$ is a nested set of separations, while for different (r-robust) k-blocks b, b' the union $S_k(b) \cup S_k(b')$ need not to be nested [40].

Lemma 2.6.18. *Let* b *be a* k*-block of* G*. Then* b *is separable if and only if every separation in* $S_k(b)$ *has order less than* k*.*

Proof. For the 'only if'-implication, let $\mathcal{T} = \big(T, (P_t)_{t \in V(T)}\big)$ be a tree-decomposition of adhesion less than k of G with $P_t = b$ for some $t \in V(T)$. Let C be a component of $G - b$. There is a separation (A, B) induced by $\mathcal{T}$ with $C \subseteq A \setminus B$ and $b \subseteq B$. Hence $N(C) \subseteq A \cap B$, and so has less than k vertices.

For the 'if'-implication, just consider the star-decomposition induced by $S_k(b)$, whose central part is b. This tree-decomposition has adhesion less than k if and only if all separations in $S_k(b)$ have order less than k. □

Remark 2.6.19. *Let* b *be a* k*-block of* G*. For all* $(A, B) \in S_k(b)$ *the separator* $A \cap B$ *is a subset of* b*.* □

Given some $r \in \mathbb{N}$ and a set $\mathcal{B}$ of distinguishable[45] r-robust k-blocks for some values $k \leq r + 1$, we define

$$S(\mathcal{B}) := \bigcup \big\{ S_k(b) \cap S_{<k} \;\big|\; b \text{ is a } k\text{-block in } \mathcal{B} \big\}.$$

Note that if the set of profiles induced by $\mathcal{B}$ is canonical, then so is $S(\mathcal{B})$.

Lemma 2.6.20. *Every separable* k*-block* $b \in \mathcal{B}$ *is an* $S(\mathcal{B})$*-block.*

[45] A set of blocks is *distinguishable* if the set of induced profiles is distinguishable.

Proof. Suppose for a contradiction there is a k'-block $b' \in \mathcal{B}$ and a separation $(A, B) \in S_{k'}(b') \cap S_{<k'} \subseteq S(\mathcal{B})$ separating b. Consider a separation (C, D) distinguishing b and b' efficiently with $b \subseteq C$ and $b' \subseteq D$. Since $|C \cap D| < k$, there is a vertex $v \in b \setminus (C \cap D)$. And since $(A \cap B) \subseteq b' \subseteq D$, the link ℓ_C is empty. Therefore we deduce that either $v \in A \setminus B$ or $v \in B \setminus A$. Let w denote a vertex of b such that (A, B) separates v and w. Both the corner separations $(A \cap C, B \cup D)$ and $(B \cap C, A \cup D)$ have order at most $|C \cap D| < k$. But one of them separates v from w, contradicting the $(< k)$-inseparability of b. Hence b is $S(\mathcal{B})$-inseparable.

Let X be an $S(\mathcal{B})$-inseparable set including b and let $v \in V(G) \setminus b$. Then there is some $(A, B) \in S_k(b)$ separating b from v. Lemma 2.6.18 implies that $(A, B) \in S_k(b) \cap S_{<k} \subseteq S(\mathcal{B})$ and thus v is not in X. Hence $X = b$. □

Lemma 2.6.21. *Let (A, B) and (C, D) be tight separations of G such that $A \setminus B$ is connected and the link ℓ_A is empty. Then (A, B) and (C, D) are nested.*

Proof. Since $A \setminus B$ is connected, either $\text{int}(A, C)$ or $\text{int}(A, D)$ is empty, say $\text{int}(A, C)$. Thus there cannot be a vertex in the link ℓ_C because it would have a neighbour in $A \setminus B$, which is impossible. Hence (A, B) and (C, D) are nested by Remark 2.6.3. □

Lemma 2.6.22. *Let $(A, B), (C, D) \in S(\mathcal{B})$ be crossing. Then the links ℓ_B and ℓ_D are empty.*

Moreover, the separation $(K \cup N(K), V(G) \setminus K)$ for every component K of $G[\text{int}(B, D)]$ is in $S(\mathcal{B})$ and its order is strictly less than the orders of both (A, B) and (C, D).

Proof. Let b_1 and b_2 be blocks in $\mathcal{B}$ such that $(A, B) \in S_{k_1}(b_1) \cap S_{<k_1}$ and $(C, D) \in S_{k_2}(b_2) \cap S_{<k_2}$. We may assume that the order k_2 of b_2 is at most the order k_1 of b_1. By Lemma 2.6.21, there are vertices $v_A \in \ell_A$ and $v_C \in \ell_C$. By Remark 2.6.19, $v_C \in b_1$. As (C, D) cannot separate b_1, the block b_1 is contained in $B \cap C$. In particular, the link ℓ_D is empty.

Let X be a component of $G - C \cap D$ that contains a vertex w of b_2. Note that X is unique as b_2 is a k_2-block. As ℓ_D is empty, X must be contained in $D \cap A$ or $D \cap B$. Since b_2 contains v_A, it must be contained in $D \cap A$. Indeed, otherwise the corner separation of $B \cap D$ would separated w from v_A. Hence ℓ_B is empty.

Let K be an arbitrary component of $G[\text{int}(B, D)]$. Let $E := K \cup N(K)$ and $F := V(G) \setminus K$. Since the center c is a subset of $b_1 \cap b_2$ and since $K \cap (b_1 \cup b_2)$ is empty, K is a component of both $G - b_1$ and $G - b_2$. Hence (E, F) is in both $S_{k_1}(b_1)$ and $S_{k_2}(b_2)$. And since $E \cap F \subseteq c$ and since ℓ_A and ℓ_C are not empty, we deduce that $|E \cap F| < \min\{|A \cap B|, |C \cap D|\}$. □

Lemma 2.6.23. *$S(\mathcal{B})$ is almost nested.*

Proof. We have to show that every $S(\mathcal{B})$-focusing sequence $(\beta_0, \ldots, \beta_n)$ is good, i.e. N_{β_n} is nested with $S(\mathcal{B}) \restriction \beta_n$. Let $(\beta_0, \ldots, \beta_n)$ be an $S(\mathcal{B})$-focusing sequence. Let $(A, B) \restriction \beta_n \in N_{\beta_n}$ and $(C, D) \restriction \beta_n \in S(\mathcal{B}) \restriction \beta_n$. If (A, B) and (C, D) are nested, then so are $(A, B) \restriction \beta_n$ and $(C, D) \restriction \beta_n$. Suppose (A, B) and (C, D) are crossing. By Lemma 2.6.22 ℓ_B and ℓ_D are empty. If $\text{int}(B, D) \cap \beta_n$ is empty, then by

Remark 2.6.3 $(A, B)\restriction\beta_n$ and $(C, D)\restriction\beta_n$ are nested. Hence by Lemma 2.6.22 it suffices to show that $(E \setminus F) \cap \beta_n$ is empty for every $(E, F) \in S(\mathcal{B})$ with $E \subseteq B \cap D$ whose order is strictly smaller than the order of (A, B).

Since $(A, B)\restriction\beta_n$ is proper, there is a $v \in \beta_n \setminus B \subseteq \beta_n \setminus E \subseteq (F \setminus E) \cap \beta_n$. Since $(A, B)\restriction\beta_n$ has minimal order among all separations in $S(\mathcal{B})\restriction\beta_n$, we deduce that $(E, F)\restriction\beta_n$ is improper and hence either $(F \setminus E) \cap \beta_n$ or $(E \setminus F) \cap \beta_n$ is empty. Now v witnesses that $(E \setminus F) \cap \beta_n$ is empty, as desired. □

Lemma 2.6.24. *Given $r \in \mathbb{N}$, let $\mathcal{P}$ be a set of r-robust distinguishable k-profiles for some values $k \leq r + 1$. Let N be a nested separation system such that for every $(C, D) \in N$, there is some ℓ-profile in $\mathcal{P}$ induced by an ℓ-block b with $(C \cap D) \subseteq b$. Then any two distinct $P, Q \in \mathcal{P}$ are distinguished efficiently by a separation nested with N.*

Proof. Let (A, B) distinguish $P, Q \in \mathcal{P}$ efficiently such that the number of separations in N nested with (A, B) is maximal. Without loss of generality let $(A, B) \in P$. Let $k{:=}|A \cap B|$. We prove that (A, B) is nested with N.

Suppose for a contradiction that there is some $(C, D) \in N$ not nested with (A, B). Let b be an $(\ell + 1)$-block such that $(C \cap D) \subseteq b$ whose induced profile $P_{\ell+1}(b)$ is in $\mathcal{P}$.

Case 1: $k \leq \ell$. Remark 2.6.19 implies that $C \cap D$ is $(\leq \ell)$-inseparable and hence one of the links ℓ_A or ℓ_B is empty. Without loss of generality let ℓ_B be empty. The orders of the corner separations $(A \cup D, B \cap C)$ and $(A \cup C, B \cap D)$ are less or equal than $|A \cap B|$. Hence they are oriented by P and Q. Applying Lemma 2.6.7 to $X{:=}A \cap B$ and P yields a component K of $G - X$ with $(V(G) \setminus K, K \cup N(K)) \in P$. In particular we get $K \subseteq B \setminus A$ by consistency. Since ℓ_B is empty and K is connected, we obtain $K \subseteq C \setminus D$ or $K \subseteq D \setminus C$. Therefore either $(A \cup D, B \cap C)$ or $(A \cup C, B \cap D)$ is in P by consistency to $(V(G) \setminus K, K \cup N(K))$, and not in Q by consistency to (B, A).

Hence there is a corner separation of (A, B) and (C, D) distinguishing P and Q efficiently. By Lemma 2.6.4 it is nested with every separation in N that is also nested with (A, B), as well as with (C, D). Hence it crosses strictly less separations of N than (A, B), contradicting the choice of (A, B). Thus (A, B) is nested with N.

Case 2: $k \geq \ell$. We prove this case by induction on k with Case 1 as the base case. By the efficiency of (A, B), the separation (C, D) does not distinguish P and Q. Thus we may assume that (C, D) is in both P and Q. If one of the corner separations $(A \cap D, B \cup C)$ or $(B \cap D, A \cup C)$ had order at most k, then it would violate the maximality of (A, B) by Lemma 2.6.4. Indeed, it would be nested with every separation in N that is also nested with (A, B), as well as with (C, D).

Hence we may assume that both these corner separations have order larger than k and therefore both links ℓ_A and ℓ_B are not empty. By Remark 2.6.5, the opposite corner separations $(A \cap C, B \cup D)$ and $(B \cap C, A \cup D)$ have order strictly less than $|C \cap D|$ and are in $P_{\ell+1}(b)$ since $C \cap D \subseteq b$. As b is r-robust, $(C, D) \in P_{\ell+1}(b)$. Hence (C, D) distinguishes P and $P_{\ell+1}(b)$.

By the induction hypothesis, there is a separation (E, F) of order at most ℓ distinguishing P and $P_{\ell+1}(b)$ efficiently that is nested with N. We may assume that $(E, F) \in P_{\ell+1}(b)$ and $(F, E) \in P$. Furthermore, (E, F) does not distinguish P and Q, since $|E \cap F| < |A \cap B|$. We claim that $(C, D) \leq (F, E)$. Indeed, since (C, D) and (F, E) are nested and P contains both of them, either $(C, D) \leq (F, E)$ or $(F, E) \leq (C, D)$. By consistency of $P_{\ell+1}(b)$, we can conclude that $(C, D) \leq (F, E)$.

If the order of $(E \cap B, F \cup A)$ is at most k, then it would distinguish P and Q efficiently. It would violate the maximality of (A, B) by Lemma 2.6.4 since it is nested with every separation in N that is also nested with (A, B), as well as with (C, D) itself as $(C, D) \geq (E, F) \geq (E \cap B, F \cup A)$. Thus we may assume that $(E \cap B, F \cup A)$ has order larger than k. Similarly we may assume that $(E \cap A, F \cup B)$ has order larger than k.

Again by Remark 2.6.5, the opposite corner separations $(F \cap A, E \cup B)$ and $(F \cap B, E \cup A)$ have order less than $|E \cap F|$. But by construction they separate ℓ_A and ℓ_B and hence b, contradicting the fact that b is $(\leq \ell)$-inseparable. $\square$

Theorem 15. *Let G be a finite graph, $r \in \mathbb{N}$ and let $\mathcal{P}$ be a canonical set of r-robust distinguishable ℓ-profiles for some values $\ell \leq r + 1$.*

Then G has a canonical tree-decomposition $\mathcal{T}$ that distinguishes efficiently every two distinct profiles in $\mathcal{P}$, and which has the further property that every separable block whose induced profile is in $\mathcal{P}$ is equal to the unique part of $\mathcal{T}$ in which it is contained.

Proof. Let $\mathcal{B}$ be the set of blocks whose induced profiles are in $\mathcal{P}$. We consider $S(\mathcal{B})$ as above. Lemma 2.6.23 and Construction 10 yield a canonical tree-decomposition $\mathcal{T}(S(\mathcal{B}))$ where by Lemma 2.6.20 and Theorem 11 (i) every separable $b \in \mathcal{B}$ is equal to the unique part in which it is contained.

Let N be the nested separation system induced by $\mathcal{T}(S(\mathcal{B})))$. With Lemma 2.6.24 we can apply Construction 13 to obtain $\overline{N}$, which by Theorem 14 (ii) distinguishes the profiles in $\mathcal{P}$ efficiently.

It is left to show that no separation $(A, B) \in \overline{N} \setminus N$ separates a separable k-block $b \in \mathcal{B}$. Suppose for a contradiction that $(A, B) \in \overline{N} \setminus N$ separates b. Let P_t be the part of $\mathcal{T}(S(\mathcal{B}))$ with $P_t = b$. Note that since the adhesion sets $P_t \cap P_u$ for any edge tu have size strictly smaller than k and since the only profile in $\mathcal{P}$ inhabiting P_t is $P_k(b)$, no profile in $\mathcal{P}$ induces an ℓ-profile for some $\ell \geq k + 1$ on the torso H_t. Then by Construction 13 and Lemma 2.6.12 (iii) the induced separation $(A \cap P_t, B \cap P_t)$ is a proper separation of H_t distinguishing two $(\leq k)$-profiles of H_t efficiently. But since H_t has no proper $(< k)$-separation, it has no two distinguishable $(\leq k)$-profiles.

Hence Theorem 6 yields a tree-decomposition $\mathcal{T}(\overline{N})$ with the desired properties. $\square$

Corollary 2.6.25. *Every finite graph G has a canonical tree-decomposition $\mathcal{T}$ that distinguishes efficiently every two distinct maximal robust profiles, and* *which has the further property that every separable block inducing a maximal robust profile is equal to the unique part of $\mathcal{T}$ in which it is contained.*

Proof. Since the set of maximal robust profiles is by definition distinguishable, we can apply Theorem 15. □

Corollary 2.6.26. *Every finite graph G has a canonical tree-decomposition $\mathcal{T}$ of adhesion less than k that distinguishes efficiently every two distinct k-profiles, and which has the further property that every separable k-block is equal to the unique part of $\mathcal{T}$ in which it is contained.*

Proof. By Remark 2.6.6 (i) any k-profile is $(k-1)$-robust. Since the set of all k-profiles is by definition distinguishable, we can apply Theorem 15. □

Theorem 15 fails if we do not require that $\mathcal{P}$ is distinguishable:

Example 2.6.27. Consider the graph obtained by two cliques K_1 and K_2 of size at least $k+1 \geq 7$ sharing $k-1$ vertices, together with a vertex v joined to two vertices of $K_1 - K_2$ and to two vertices of $K_2 - K_1$, see Figure 2.25.

Then $K_1 \cup K_2$ is a separable 5-block, as witnessed by the separation $(\{v\} \cup N(v), K_1 \cup K_2)$. But the two $(k+1)$-blocks K_1 and K_2 are only distinguished efficiently by $(K_1 \cup \{v\}, K_2 \cup \{v\})$. Since this separation crosses any separation separating v from $K_1 \cup K_2$, there is no tree-decomposition that distinguishes K_1 and K_2 efficiently such that there is a part equal to $K_1 \cup K_2$. Moreover, even the union of the parts inhabited by $P_5(K_1 \cup K_2)$ in any tree-decomposition that distinguishes K_1 and K_2 efficiently contains with v a vertex outside the block.

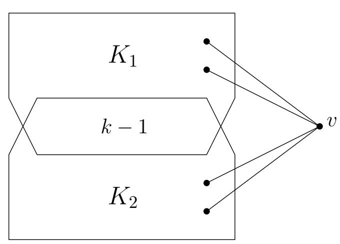


Figure 2.25: The graph of Example 2.6.27

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Chapter 3

Infinite graphic matroids

The proof of Theorem 3 is subdivided into three steps, the first of which is the proof of Theorem 1 above. The second gives a general recipe for how to build matroids from infinite tree-decompositions, see Section 3.1. In the third step, we use that recipe to build the topological cycle matroid of any graph from the tree-decomposition of Theorem 1, see Section 3.2.

In Section 3.4, we show that any infinite matroids that locally looks like a graph can be represented by a graph-like space.

3.1 Infinite trees of matroids

3.1.1 Introduction

In 2008, Bruhn et al [30] introduced several equivalent axiomatisations for infinite matroids, providing a foundation on which a theory of infinite matroids with duality can be built. We shall work with a slightly better behaved subclass of infinite matroids, called tame matroids. This class includes all finitary matroids and all the other motivating examples of infinite matroids but is easier to work with than the class of infinite matroids in general [1], [17], [18], [20], [21], [22]. In [18], we gave a construction by means of which finite matroids can be stuck together to get infinite tame matroids. The construction of [18] was restricted to the countable setting. In this paper, we extend it to the general setting.

A large collection of motivating examples of infinite matroids arises from locally finite graphs G. First of all, two well-established matroids associated to such a graph G are the *finite cycle matroid* $M_{FC}(G)$, whose circuits are the finite cycles in G, and the *topological cycle matroid* $M_{TC}(G)$, whose circuits are the edge sets of topological circles in the topological space $|G|$ obtained from G by adding the ends [29]. More generally, we say a tame matroid is a *G-matroid* if all of its circuits are edge sets of topological circles in $|G|$ and all of its cocircuits are bonds of G. Thus both $M_{FC}(G)$ and $M_{TC}(G)$ are G-matroids.

It turns out that any G-matroid M is determined by a set Ψ of ends of G, in that the circuits of M are the Ψ-*circuits*, that is, the edge sets of those topological circles that only use ends from Ψ [21]. Unfortunately, there are graphs G and sets Ψ of ends such that the set of Ψ-circuits is not the set of circuits of a matroid [18]. But this can only happen if Ψ is topologically unpleasant.

Theorem 3.1.1 ([18])**.** *Let Ψ be a Borel set of ends of a locally finite graph G. Then the Ψ-circuits of G are the circuits of a matroid.*

Ψ-circuits in graphs are a special case of a more general construction: A *tree of presentations* $\mathcal{T}$ over a field k consists of a tree $T_{\mathcal{T}}$ with a matroid presentaed over k at each node, where the ground sets of these matroids are only allowed to overlap if they are at adjacent nodes. Very roughly, the *circuits* of $\mathcal{T}$ are obtained by gluing together local circuits at the nodes of some subtree. Given a set Ψ of ends of T, the Ψ-circuits are those circuits of $\mathcal{T}$ for which the underlying subtree has all its ends in Ψ. The following theorem implies Theorem 3.1.1.

Theorem 3.1.2 ([18])**.** *Let k be a finite field and $\mathcal{T}$ be a tree of presentations*[1] *over k. If Ψ is a Borel set of ends of $T_{\mathcal{T}}$, then the Ψ-circuits are the circuits of a matroid, called the Ψ-matroid of $\mathcal{T}$.*

Ψ-matroids also appear naturally in the study of planar duality for infinite graphs [59] and in the reconstruction theorem of tame matroids from their canonical decompositions into 3-connected minors [21].

The Ψ-matroid construction of Theorem 3.1.2 can be thought of as giving, in a sense, a limit of the matroids induced from finite subtrees, but with the advantage that we are able to freely specify a great deal of information 'at infinity', namely the set Ψ. If we choose Ψ to be empty, this corresponds to taking the direct limit. On the other hand, taking Ψ to be the set of all ends corresponds to taking the inverse limit.

The purpose of this paper is to prove an extension of Theorem 3.1.2. We have proved this extension with an application in mind: it is used as a tool in the proof of an extension of Theorem 3.1.1 to arbitrary graphs [36]. We have tried to avoid the need for further generalisations by making the version in this paper as general as possible.

Next, let us discuss which of the restrictions from Theorem 3.1.2 we can weaken. First, in that theorem all of the matroids at the nodes are required to be finite. Allowing arbitrary infinite matroids at the nodes is unfortunately not possible - in fact, even for infinite stars of matroids in which the central node is infinite but all leaves are finite it is possible for our gluing construction to fail to give a matroid. But this is the only problem - that is, we are able to show that if the matroids at the nodes of the tree work well when placed at the centre of such stars, then they can also be glued together along arbitrary trees. The advantage of this approach is its great generality, but the disadvantage is that the class of

[1] In [18] we worked with 'trees of matroids' instead of 'trees of presentations'. We have made this change since the set of Ψ-circuits we get for such a tree may in general not only depend on the structure of the matroids attached to each node but also on their presentations.

stellar matroids, that is, those which fit well at the centre of stars, is not characterised in simpler terms. However, since in all existing applications the matroids involved can be easily seen to be stellar we do not see this as a great problem.

Second, in Theorem 3.1.2 the matroids at the nodes were required to be representable over a common field k. This continues to play a necessary role in our construction, because it is based on a gluing construction for finite matroids which in turn relies on representability over a common field. Because we now allow the matroids at the nodes to be infinite, we require them to be representable in the sense of [1], which introduced a notion of representability for infinitary matroids.

However, we are able to drop the requirement that k be finite. Thus we obtain the following more general result.

Theorem 3.1.3. *Let k be any field and let $\mathcal{T}$ be a stellar tree of presentations presented over k, and let Ψ be a Borel set of ends of $\mathcal{T}$. Then the Ψ-circuits are the circuits of a matroid.*

We not only extend Theorem 3.1.1, we also give a new and simpler proof of it, see Subsection 3.1.3.

The paper is organised as follows. After recalling some preliminaries in Subsection 3.1.2, in Subsection 3.1.3 we give a new proof of Theorem 3.1.1 which is simpler than the original one. However, to understand the rest of this paper, it is not necessary to read that section. In the proof of our main result, we will rely on the determinacy of certain games, and in Subsection 3.1.4 we prove a lemma that allows us to simplify winning strategies in these games. We then introduce presentations of infinite matroids over a field in Subsection 3.1.5, and the gluing construction along a tree in Subsection 3.1.6. The proof that this construction gives rise to matroids is given in Sections 3.1.7 and 3.1.8.

3.1.2 Preliminaries

Throughout, notation and terminology for (infinite) graphs are those of [52], and for matroids those of [92, 30]. We will rely on the following lemma from [52]:

Lemma 3.1.4 (König's Infinity Lemma [52])**.** *Let $V_0, V_1, \dots$ be an infinite sequence of disjoint non-empty finite sets, and let G be a graph on their union. Assume that every vertex v in V_n with $n \geq 1$ has a neighbour $f(v)$ in V_{n-1}. Then G includes a ray $v_0 v_1 \dots$ with $v_n \in V_n$ for all n.*

Note that this is equivalent to the usual formulation of König's Lemma, namely that every locally finite rayless tree is finite.

Given a graph G, the set of its ends is denoted by $\Omega(G)$. An end ω is in the *closure of* some edge set F if for each finite separator S, the unique component of $G \setminus S$ including a tail of each ray in ω contains a vertex incident with an edge of F in G.

A *walk* in a digraph is a sequence $w_1 ... w_n$ of vertices such that $w_i w_{i+1}$ is an edge for each $i < n$. For a walk $W = w_1 ... w_n$ and a vertex x in W, let i be minimal with $w_i = x$. Then we denote by Wx the walk $w_1 ... w_i$ and by xW the walk $w_i ... w_n$.

In this paper we define *tree decompositions* slightly differently than in [52]. Namely, we impose the additional requirement that each edge of the graph is contained in a unique part of the tree decomposition. Clearly, each tree decomposition in the sense of [52] can easily be transformed into such a tree decomposition of the same width. Throughout this paper, *even* means finite and a multiple of 2.

Having dealt with the graph theoretic preliminaries, we now define positional games. A *positional game* is played in a digraph D with a marked starting vertex a. The vertices of the digraph are called *positions* of the game. The game is played between two players between whom play alternates. At any point in the game, there is a *current position*, which initially is a. In each move, the player whose turn it is to play picks an out-neighbour x of the current position, and then the current position is updated to x. Thus a *play* in this game is encoded as a walk in D starting at an out-neighbour of a. If a player cannot move, they lose. If play continues forever, then the players between them generate an infinite walk starting at a neighbour of a. Then the first player wins if this walk is in the *set* Φ *of winning conditions*, which is part of the data of the positional game.

A *strategy for the first player* is a set σ of finite plays P all ending with a move of the first player such that the following is true for all $P \in \sigma$: Let m be a move of the second player such that Pm is a legal play. Then there is a unique move m' of the first player such that $Pmm' \in \sigma$. Furthermore, we require that σ is closed under *2-truncation*, that is, for every nontrivial $P \in \sigma$ there are some $P' \in \sigma$ and moves m and m' of the second player and the first player, respectively, such that $P'mm' = P$.

An infinite play *belongs to* a strategy σ for the first player if all its odd length finite initial plays are in σ. A strategy for the first player is *winning* if the first player wins in all infinite plays belonging to σ. Similarly, one defines *strategies* and *winning strategies* for the second player.

Finally, we summarise the matroid theoretic preliminaries. Given a matroid M, by $\mathcal{C}(M)$ we denote the set of circuits of M, and by $\mathcal{S}(M)$ we denote the set of *scrawls* of M, where a *scrawl* is just any (possibly empty) union of circuits.[2] The *orthogonality axioms*, introduced in [18], are as follows, where $\mathcal{C}$ and $\mathcal{D}$ are sets of subsets of a groundset E, and can be thought of as the sets of circuits and cocircuits of some matroid, respectively.

(O1) $|C \cap D| \neq 1$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(O2) For all partitions $E = P \dot\cup Q \dot\cup \{e\}$ either $P + e$ includes an element of $\mathcal{C}$ through e or $Q + e$ includes an element of $\mathcal{D}$ through e.

(O3) For every $C \in \mathcal{C}$, $e \in C$ and $X \subseteq E$, there is some $C_{min} \in \mathcal{C}$ with $e \in C_{min} \subseteq X \cup C$ such that $C_{min} \setminus X$ is minimal.

[2] Matroids can be axiomatised in terms of their scrawls [17].

(O3*) For every $D \in \mathcal{D}$, $e \in D$ and $X \subseteq E$, there is some $D_{min} \in \mathcal{D}$ with $e \in D_{min} \subseteq X \cup D$ such that $D_{min} \setminus X$ is minimal.

Theorem 3.1.5 ([18, Theorem 4.2])**.** *Let E be a countable set and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(E)$. Then there is a unique matroid M such that $\mathcal{C}(M) \subseteq \mathcal{C} \subseteq \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subseteq \mathcal{D} \subseteq \mathcal{S}(M^*)$ if and only if $\mathcal{C}$ and $\mathcal{D}$ satisfy (O1), (O2), (O3) and (O3*).*

We shall not be able to rely on this characterisation of matroids since we will be dealing with possibly uncountable groundsets E. So we will need an extra axiom. A set $I \subseteq E$ is *independent* if it does not include any nonempty element of $\mathcal{C}$. Given $X \subseteq E$, a *base of* X is a maximal independent subset of X. A *base* of $(\mathcal{C}, \mathcal{D})$ is a maximal independent subset of the ground set E.

(IM) Given an independent set I and a superset X, there exists a base of X including I.

The proof of Theorem 3.1.5 as in [18] also proves the following:

Corollary 3.1.6. *Let E be a set and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(E)$. Then there is a unique matroid M such that $\mathcal{C}(M) \subseteq \mathcal{C} \subseteq \mathcal{S}(M)$ and $\mathcal{C}(M^*) \subseteq \mathcal{D} \subseteq \mathcal{S}(M^*)$ if and only if $\mathcal{C}$ and $\mathcal{D}$ satisfy (O1), (O2), and (IM).*

We say that $(\mathcal{C}, \mathcal{D})$ is *tame* if the intersection of any set in $\mathcal{C}$ with any set in $\mathcal{D}$ is finite. In the proof of our main result we will be in the situation that we have a pair $(\mathcal{C}, \mathcal{D})$ of subsets of the powerset of some set E that satisfies (O1) and (O2) and is tame. We call such a pair an *orthogonality system*.

Theorem 3.1.7 ([18, Theorem 4.4])**.** *Any orthogonality system satisfies $(O3)$ and $(O3)^*$.*

Remark 3.1.8. *A set B is a base for an orthogonality system $(\mathcal{C}, \mathcal{D})$ if and only if for each $x \notin B$, there is some $o \in \mathcal{C}$ with $x \in o \subseteq B + x$ and for each $x \in B$, there is some $d \in \mathcal{D}$ with $x \in d \subseteq (E \setminus B) + x$.*

Given $X \subseteq E$, then the *restriction* $\mathcal{C}\restriction_X$ *of* $\mathcal{C}$ *to* X consists of those $o \in \mathcal{C}$ included in X. Similarly, the *contraction* $\mathcal{C}.X$ *of* $\mathcal{C}$ *to* X is the set of those $a \subseteq X$ such that there is some $o \in \mathcal{C}$ with $a = o \setminus X$. We let $(\mathcal{C}, \mathcal{D})\restriction_X = (\mathcal{C}\restriction_X, \mathcal{D}.X)$ and $(\mathcal{C}, \mathcal{D}).X = (\mathcal{C}.X, \mathcal{D}\restriction X)$. As usual we let $(\mathcal{C}, \mathcal{D}) \setminus X = (\mathcal{C}, \mathcal{D})\restriction_{(E \setminus X)}$ and $(\mathcal{C}, \mathcal{D})/X = (\mathcal{C}, \mathcal{D}).(E \setminus X)$.

Remark 3.1.9. *If $(\mathcal{C}, \mathcal{D})$ is an orthogonality system, then for any $X \subseteq E$ both $(\mathcal{C}, \mathcal{D})\restriction_X$ and $(\mathcal{C}, \mathcal{D}).X$ are orthogonality systems.* □

Corollary 3.1.10. *Let $(\mathcal{C}, \mathcal{D})$ be an orthogonality system such that for any two disjoint sets A and B the orthogonality system $(\mathcal{C}, \mathcal{D})/A \setminus B$ has a base. Then $(\mathcal{C}, \mathcal{D})$ satisfies (IM).*

Proof. Given I and X as in (IM), by assumption there is a base B of $(\mathcal{C}, \mathcal{D})/I \setminus (E \setminus X)$. It is straightforward to check that $I \cup B$ is a base of X with respect to $(\mathcal{C}, \mathcal{D})$. □

In orthogonality systems we already have a notion of connectedness: We say that two edges e and f are *in the same connected component* if there is some minimal nonempty $o \in \mathcal{C}$ containing both e and f.

Lemma 3.1.11. *Being in the same component is an equivalence relation.*

Moreover, e is in the same connected component as f if and only if there is some minimal nonempty $d \in \mathcal{D}$ containing both e and f.

Proof. Just as for finite matroids. □

A *connected component of* $(\mathcal{C}, \mathcal{D})$ is an equivalence class for the equivalence relation "being in the same component".

Lemma 3.1.12. *Given a connected component X of $(\mathcal{C}, \mathcal{D})$, then $(\mathcal{C}, \mathcal{D})\restriction_X = (\mathcal{C}, \mathcal{D}).X$.*

Proof. Just as for finite matroids. □

3.1.3 A simpler proof in a special case

The aim of this section is to give a simpler proof of a result from [18], Theorem 3.1.13 below, which implies Theorem 3.1.1. Given a locally finite graph G, a Ψ-*circuit of* G is the edge set of a topological cycle of G that only uses ends from Ψ. A $\Psi^{\complement}$-*bond of* G is the edge set of a bond of G that only has ends of $\Psi^{\complement}$ in its closure.

Theorem 3.1.13. *Let G be a locally finite graph and Ψ a Borel set of ends. Then there is a matroid $M_\Psi(G)$ whose circuits are the Ψ-circuits and whose cocircuits are the $\Psi^{\complement}$-bonds.*

Given a locally finite graph G with a tree decomposition $(T, P(t)|t \in V(T))$, the *torso* $\bar{P}(t)$ of a part $P(t)$ is the multigraph obtained from $P(t)$ by adding for each neighbour t' of t and any two $v, w \in V(P(t)) \cap V(P(t'))$ an edge (tt', v, w) between v and w. In this section, we shall only consider tree decompositions with each torso finite. A *precircuit* is a pair (S, o) where S is a connected subtree of T and o sends each $t \in V(S)$ to some $o(t) \subseteq E(\bar{P}(t))$ that has even degree at each vertex such that for any neighbour t' of t in T we have $o(t) \cap E(\bar{P}(t')) = \emptyset$ if $t' \notin V(S)$, and $o(t) \cap E(\bar{P}(t')) = o(t') \cap E(\bar{P}(t))$ otherwise. A *precocircuit* is the same as a precircuit except that here $o(t)$ is a cut of $\bar{P}(t)$ instead of a set that has even degree at each vertex. Given $\Psi \subseteq \Omega(T)$, a precircuit (S, o) is a Ψ-*precircuit* if all ends of S are in Ψ. Similarly, one defines a Ψ-precocircuit. We denote the set of underlying sets of Ψ-precircuits by $\mathcal{C}_\Psi(G)$, and the set of underlying sets of $\Psi^{\complement}$-precocircuits by $\mathcal{D}_\Psi(G)$.

The following follows from the fact that the finite circuits of any graph are the circuits of a matroid.

Remark 3.1.14. *The pair $(\mathcal{C}_\emptyset(G), \mathcal{D}_\emptyset(G))$ satisfies (O2).* □

It suffices to prove Theorem 3.1.13 for a graph G' obtained from a locally finite graph G by subdividing each edge. Indeed, then $M_\Psi(G)$ is a contraction minor of $M_\Psi(G')$. So from now on we fix a graph G' obtained from a connected locally finite graph G by subdividing each edge.[3] We abbreviate $\mathcal{C}_\Psi = \mathcal{C}_\Psi(G')$ and $\mathcal{D}_\Psi = \mathcal{D}_\Psi(G')$.

Lemma 3.1.15. *[Lemma 7.5, Lemma 7.6 and Lemma 7.7 from [18]] There is a tree-decomposition $(T, P(t)|t \in V(T))$ of G' with each part finite and a homeomorphism ι between $\Omega(G')$ and $\Omega(T)$ such that for each $\Psi \subseteq \Omega(G')$, the set of minimal nonempty elements in $\mathcal{C}_{\iota(\Psi)}$ is the set of Ψ-circuits of G' and set of minimal nonempty elements in $\mathcal{D}_{\iota(\Psi)}$ is the set of $\Psi^{\complement}$-bonds of G'.*

Furthermore, each $P(t)$ is connected and T is locally finite.

To simplify notation, we shall suppress the bijection ι from now on.

Lemma 3.1.16. *[Lemma 7.2 and Lemma 7.7 from [18]] The pair $(\mathcal{C}_\Psi, \mathcal{D}_\Psi)$ satisfies (O1) and tameness.*

Lemma 3.1.17. *The pair $(\mathcal{C}_\Psi, \mathcal{D}_\Psi)$ satisfies (O2) if and only if the pair consisting of the set of Ψ-circuits and the set of $\Psi^{\complement}$-bonds does.*

Proof. One implication is obvious, for the other assume that $(\mathcal{C}_\Psi, \mathcal{D}_\Psi)$ satisfies (O1), (O2) and tameness. Then by Theorem 3.1.7 $(\mathcal{C}_\Psi, \mathcal{D}_\Psi)$ satisfies (O3) and (O3*). Now let $E = P\dot\cup Q\dot\cup\{e\}$ be a partition. If there is some $o \in \mathcal{C}_\Psi$ with $e \in o \subseteq P + e$, we can pick it minimal by (O3), thus o is a Ψ-circuit. Otherwise by (O2), there is some $d \in \mathcal{D}_\Psi$ with $e \in d \subseteq Q + e$, and we conclude in the same way as above, which completes the proof. □

By Theorem 3.1.5, Lemma 3.1.16 and Lemma 3.1.17, the set of Ψ-circuits and the set of $\Psi^{\complement}$-bonds are the sets of circuits and cocircuits of a matroid if and only if $\mathcal{C}_\Psi$ and $\mathcal{D}_\Psi$ satisfy (O2). Thus to prove Theorem 3.1.13, it suffices to show that $(\mathcal{C}_\Psi, \mathcal{D}_\Psi)$ satisfies (O2). So let $E(G') = P\dot\cup Q\dot\cup\{e\}$ be a partition.

We consider T as a rooted tree rooted at the unique node t_e such that $e \in E(P(t_e))$. We consider the following positional game (D, a, Φ) played between two players called Sarah and Colin, where Sarah makes the first move. D is a directed graph whose underlying graph is bipartite with bipartition (D_1, D_2). The set D_1 is the union of sets $X(t)$, one for each $t \in V(T)$, where $X(t)$ is the set of those $F \subseteq E(\bar{P}(t)) \setminus Q$ that have even degree at each vertex. The set D_2 contains the singleton of the starting vertex a and includes the union of sets $Y(tt')$ one for each $tt' \in E(T)$, where $Y(tt')$ is the powerset of $E(\bar{P}(t)) \cap E(\bar{P}(t'))$ without the empty set. We have a directed edge from a to any $x \in X(t_e)$ containing e. For $tt' \in E(T)$ directed away from the root, we have an edge from $x \in X(t)$ to $y \in Y(tt')$ if $x \cap E(\bar{P}(t')) = y$. We have an edge from $y \in Y(tt')$ to $x \in X(t)$ if $x \cap E(\bar{P}(t)) = y$.

In order to complete the definition of the positional game, it remains to define Φ. Given a play Z, by $Z[n]$ we denote the unique node t with distance n

[3]Mathematically, it is not strictly necessary to work with G' instead of G, but this way we can cite a lemma from [18].

from t_e such that the $2n+1^{\text{st}}$ move of Z is in $X(t)$. For each infinite play Z, the sequence $(Z[n]|n \in \mathbb{N})$ is a ray of T which belongs to some end ω_Z. Let f be the map from the space of infinite positional plays to the space of ends of G defined via $f(Z) = \omega_Z$. It is straightforward to check that the map f is continuous. Let Φ be the inverse image of Ψ under f. As being Borel is preserved under inverse images of continuous maps, we get the following.

Remark 3.1.18. *If Ψ is Borel, then Φ is Borel and thus the positional game is determined.*

Lemma 3.1.19. *If Sarah has a winning strategy, there is some $o \in \mathcal{C}_\Psi$ with $e \in o \subseteq P + e$.* □

Given a winning strategy σ, then S_σ is the induced subforest of T whose nodes t are those such that some $F \in X(t)$ appears as a move in a play according to σ or are t_e. Note that S_σ is connected.

Lemma 3.1.20. *If Colin has a winning strategy σ, then S_σ has all ends in $\Psi^{\complement}$.*

Proof. Assume that S_σ has an end and let ω be an arbitrary end of S_σ. Then there is a ray $t_1 t_2 ...$ included in S_σ that belongs to ω with $t_1 = t_e$. For each $t_i t_{i+1}$, we pick some play P_i in σ with Colin's last move in $Y(t_i t_{i+1})$.

For each $t_i t_{i+1}$, we denote by Z_i the set of the initial plays of length i of some P_j with $j \geq i$. As T is locally finite and as each torso $\bar{P}(t)$ is finite, there are only finitely many possible plays ending with a move in $Y(t_i t_{i+1})$. Hence Z_i is finite.

We apply König's Infinity Lemma to the graph H whose vertex set is the union of the Z_i. We join $a_i \in Z_i$ with $a_{i+1} \in Z_{i+1}$ by an edge if a_{i+1} is an extension of a_i. By the Infinity Lemma, we get a ray $x_1 x_2 ...$ in H. By construction x_i is in σ. As σ is winning, the union of all these plays is in $\Phi^{\complement}$. Thus ω is in $\Psi^{\complement}$, which completes the proof. □

Proof of Theorem 3.1.13. As mentioned above, it suffices to show that $(\mathcal{C}_\Psi, \mathcal{D}_\Psi)$ satisfies (O2). By Remark 3.1.18, either Sarah or Colin has a winning strategy in the positional game above. By Lemma 3.1.19 we may assume that Colin has a winning strategy.

Let H be the graph obtained from G' by contracting all edges not in any $P(t)$ with $t \in V(S_\sigma)$. We obtain $\tilde{P}(t)$ from $P(t)$ by contracting all dummy edges (tt', v, w) with $t' \notin V(S_\sigma)$. By the "Furthermore"-part of Lemma 3.1.15, it is clear that H has a tree decomposition with tree S_σ whose torsos are the $\tilde{P}(t)$.

Suppose for a contradiction that there is some $o \in \mathcal{C}_\emptyset(H)$ with $e \in o \subseteq P + e$. Let $(S, \bar{o})$ be an $\emptyset$-precircuit with underlying set o. If Sarah always plays $\bar{o}(t)$ in the positional game above, Colin always challenges her at some $v \in V(S_\sigma)$ when he plays according to σ. As S is rayless, eventually Colin cannot challenge, which contradicts the fact that σ is winning.

Thus there cannot be such an o. Thus by Remark 3.1.14, there is some $d \in \mathcal{D}_\emptyset(H)$ with $e \in d \subseteq Q + e$. Then $d \in \mathcal{D}_\Psi(G')$ since all ends of S_σ are in $\Psi^{\complement}$ by Lemma 3.1.20. This completes the proof. □

3.1.4 Simplifying winning strategies

In this section we prove Lemma 3.1.21 which allows us to improve winning strategies in positional games. Our proof of Theorem 3.1.3 will rely on the determinacy of certain such games: this is why the set Ψ is required to be Borel.

Given a set σ of plays, by $\sigma(m)$ we denote the set of those moves that appear as m-th moves in plays of σ. Given two finite or infinite plays $P = p_1 \dots$ and $Q = q_1 \dots$ of the same length, then $P \sim_1 Q$ if the first player makes the same moves in both plays, that is, $p_i = q_i$ for all odd i. A winning strategy σ for the first player is *reduced* if there exists a total ordering $\leq$ of the set of positions with the following property: for any two plays $P = p_1 ... p_{2n+1}$ and $Q = q_1 ... q_{2n+1}$ in σ such that $p_1 ... p_{2n-1} \sim_1 q_1 ... q_{2n-1}$ and $p_1 ... p_{2n} q_{2n+1}$ is a legal play, we have $p_{2n+1} \leq q_{2n+1}$.

Lemma 3.1.21. *Let $\mathcal{G}$ be a positional game whose set Φ of winning conditions is closed under $\sim_1$. If the first player has a winning strategy σ, then the first player has a reduced winning strategy.*

Proof. First, we pick a well-order $\leq$ of the set of positions. Next we define a reduced winning strategy $\bar{\sigma}$ for the first player. The first player should play as follows. His first move should be the same as in σ. Whenever he has just made a move he should have in mind an auxiliary play according to σ which ends at the same position. Assume that the first player and the second player have already played $2n+1$ moves and let s be the current play, and s' the auxiliary play. Now assume that the second player's response is m. As σ is winning, there is a move t of the first player such that $s'mt \in \sigma$. Let X be the set of those pairs (m', u) such that $s'm'u \in \sigma$ and smu is a legal play. X is nonempty since $(m, t) \in X$. The first player picks $(m', u) \in X$ such that u is minimal with respect to $\leq$ and, subject to this, such that m' is minimal with respect to $\leq$. He plays u and imagines the auxiliary play $(smu)' = s'm'u$.

It is clear that this defines a strategy for the first player. Next, we show $\bar{\sigma}$ is winning. So let $(s_n | n \in \mathbb{N})$ be a sequence of plays according to $\bar{\sigma}$ with s_n of length $2n+1$, each extending the previous one. By construction, it is clear that s'_{n+1} extends s'_n. As $s'_n \in \sigma$, the union of the s'_n is in Φ. As Φ is closed under $\sim_1$, the union of the s_n is in Φ as well. Thus $\bar{\sigma}$ is winning.

By induction, it is straightforward to check that if $s, t \in \bar{\sigma}$ and $s \sim_1 t$, then $s' = t'$.

It remains to show that $\bar{\sigma}$ is reduced. So let $P = p_1 ... p_{2n+1}$ and $Q = q_1 ... q_{2n+1}$ in $\bar{\sigma}$ with $p_1 ... p_{2n-1} \sim_1 q_1 ... q_{2n-1}$ such that $p_1 ... p_{2n} q_{2n+1}$ is a legal play. Let $s = p_1 \dots p_{2n-1}$. Let $Q' = u_1 ... u_{2n+1}$. Then as noted above we have $s' = u_1 \dots u_{2n-1}$, so $s' u_{2n} u_{2n+1} \in \sigma$. So by the construction of p_{2n+1} we have $p_{2n+1} \leq u_{2n+1} = q_{2n+1}$. Thus $\bar{\sigma}$ is reduced, which completes the proof. $\square$

The following is a direct consequence of the definition of a reduced strategy.

Remark 3.1.22. *Let $\bar{\sigma}$ be a reduced winning strategy for the first player. Let $p_1 ... p_n \in \bar{\sigma}$ and $q_1 ... q_m \in \bar{\sigma}$ and assume there is some odd i such that $p_1 ... p_i \sim_1 q_1 ... q_i$. Then $p_1 ... p_i q_{i+1} ... q_m \in \bar{\sigma}$.* $\square$

3.1.5 Presentations

Fix a field k. For any set E and any element v of the vector space k^E, the *support* $\operatorname{supp}(v)$ is $\{e \in E | v(e) \neq 0\}$. To simplify our notation, we formally consider such a vector v to be a function with domain the support of v. This means that we can also consider v itself to be a member of other vector spaces k^F with $\operatorname{supp}(v) \subseteq F$. Note that addition and scalar multiplication of vectors are unambiguous with respect to this convention. If V is a subspace of k^E, we denote by $S(V)$ the set of supports of vectors in V.

For $v, w \in k^E$ we say that v and w are *orthogonal*, denoted $v \perp w$, if $\sum_{e \in E} v(e)w(e) = 0$. Here and throughout the paper such equalities are taken to include the claim that the sum on the left is well defined, in the sense that only finitely many summands are nonzero. That is, if $v \perp w$ then in particular $\operatorname{supp}(v) \cap \operatorname{supp}(w)$ is finite. If V and W are subspaces of k^E then we say they are *orthogonal*, denoted $V \perp W$, if $(\forall v \in V)(\forall w \in W) v \perp w$.

As in [18], we will need some extra linear structure over k to allow us to stick together matroids along sets of dummy edges of size more than 1. In fact, we will stick together presentations over k of the matroids in question to obtain a presentation of the resulting matroid. Therefore we must specify precisely what objects we will be taking as presentations of infinite matroids over k.

Definition 3.1.23. Let E be any set. A *presentation* Π on E consists of a pair (V, W) of orthogonal subspaces of k^E such that $S(V)$ and $S(W)$ satisfy (O2). Elements of V are called *vectors* of Π and elements of W are called *covectors*. We will sometimes denote the first element of Π by V_Π and the second by W_Π. We say that Π *presents* the matroid M if the circuits of M are the minimal nonempty elements of $S(V_\Pi)$ and the cocircuits of M are the minimal nonempty elements of $S(W_\Pi)$.

It is clear from the results of [1] that a tame matroid M is representable in the sense of that paper over k if and only if there is a presentation over k which presents M.

Note that for any presentation Π, the pair $(S(V_\Pi), S(W_\Pi))$ is an orthogonality system. Accordingly, we say a set is Π-*independent* when it is independent with respect to this orthogonality system.

Remark 3.1.24. *If E is a countable set then any presentation on E presents a matroid by Theorem 3.1.5.*

Definition 3.1.25. If V is a subspace of k^E then for X a subset of E we define the *restriction* $V\restriction_X$ of V to X to be $\{v \in V | \operatorname{supp}(v) \subseteq X\}$. We denote the restriction of V to $E \setminus Q$ by $V \backslash Q$, and say it is obtained from V by *removing* Q. Similarly, for X a subset of E we define the *contraction* $V.X$ of V to X to be $\{v\restriction_X | v \in V\}$. We denote the contraction of V to $E \setminus P$ by V/P, and say it is obtained from V by *contracting* P. We also define these terms for presentations as follows:

$$\begin{array}{rcl}
(V,W)\restriction_X & = & (V\restriction_X, W.X) \\
(V,W)\backslash Q & = & (V\backslash Q, W/Q) \\
(V,W).X & = & (V.X, W\restriction_X) \\
(V,W)/P & = & (V/P, W\backslash P)
\end{array}$$

All of these operations give rise to new presentations, called *minors* of the original presentation.

We will need some basic lemmas about presentations.

Lemma 3.1.26. *Let E be a finite set. Then a pair (V,W) of subspaces of E is a presentation on E if and only if V and W are orthogonal complements.*

Proof. For any subspace U of k^E, we will denote the orthogonal complement of U by $U^\perp$. Suppose first of all that $W = V^\perp$. We must show that $S(V)$ and $S(W)$ satisfy (O2), so suppose we have a partition $E = P\dot\cup Q\dot\cup\{e\}$. If there is no $v \in V$ with $e \in supp(v) \subseteq P+e$ then $\nVdash_e \notin V + k^P$, so $V^\perp \cap (k^P)^\perp \not\subseteq \langle \nVdash_e \rangle^\perp$. That is, there is some w which is in $V^\perp = W$ and is in $(k^P)^\perp$, so that $\mathrm{supp}(w) \subseteq Q+e$, but with $w(e) \neq 0$. Thus $e \in \mathrm{supp}(w) \subseteq Q + e$, as required.

Now suppose that (V,W) is a presentation, so that $S(V)$ and $S(W)$ satisfy (O2). Suppose for a contradiction that $W \neq V^\perp$, so that there is some $w \in V^\perp \setminus W$. As E is finite, we can choose such a w with minimal support. Since $w \neq 0$, we can pick some $e \in \mathrm{supp}(w)$. Let $P = E \setminus \mathrm{supp}(w)$ and $Q = \mathrm{supp}(w) - e$. Applying (O2) to the partition $E = P\dot\cup Q\dot\cup\{e\}$, we either get some $v \in V$ with $e \in \mathrm{supp}(v) \subseteq P + e$, so that $\mathrm{supp}(v) \cap \mathrm{supp}(w) = \{e\}$, contradicting our assumption that $w \in V^\perp$, or else we get some $w' \in W$ with $e \in \mathrm{supp}(w') \subseteq Q + e$. But in that case, letting $w'' = w - \frac{w(e)}{w'(e)}w'$ we have that $w'' \in V^\perp$ and $\mathrm{supp}(w'') \subsetneq \mathrm{supp}(w)$, so by minimality of the support of w we have $w'' \in W$. Thus $w = w'' + \frac{w(e)}{w'(e)}w' \in W$, which is again a contradiction. $\square$

Definition 3.1.27. Let $\Pi = (V,W)$ be a presentation on a set E and $x \in k^E$. Then $\Pi_x = (V_x, W^x)$ is the pair of orthogonal subspaces of k^{E+*} given by $V_x = V + \langle x - \nVdash_* \rangle$ and $W_x = \{w \in k^{E+*} | w\restriction_E \in W \text{ and } w(*) = \sum_{e\in E} w(e)x(e)\}$.

Remark 3.1.28. *If P and Q are disjoint subsets of E not meeting* $\mathrm{supp}(x)$ *then*

$$\Pi_x/P\backslash Q = (\Pi/P\backslash Q)_x\,.$$

If E is finite then it is clear (using the equivalent characterisation of presentations in Lemma 3.1.26) that Π_x is again a presentation. In fact this is more generally true:

Lemma 3.1.29. *Let $\Pi = (V,W)$ be a presentation on a set E and let $x \in k^E$ have finite support. Then Π_x is a presentation on $E + *$.*

Proof. It is clear that $V_x \perp W^x$, so we just have to prove (O2). Suppose we have some partition $E + * = P \dot\cup Q \dot\cup \{e\}$. Let F be the finite set $\mathrm{supp}(x) + e + *$. Now consider the presentation $\Pi' = (\Pi/(P \setminus F)\backslash(Q \setminus F))_x$ on the finite set F. By Remark 3.1.28, we have $\Pi' = \Pi_x/(P \setminus F)\backslash(Q \setminus F)$. We now apply (O2) in Π' to the partition $F = (P \cap F)\dot\cup(Q \cap F)\dot\cup\{e\}$. If we find a vector v of Π' with $e \in \mathrm{supp}(v) \subseteq (P \cap F) + e$ then we can extend v to a vector v' of Π_x which witnesses (O2) in that $e \in \mathrm{supp}(v) \subseteq P + e$. The case that there is a covector w of Π with $e \in \mathrm{supp}(w) \subseteq (Q \cap F) + e$ is dealt with similarly. $\square$

By Theorem 3.1.7, for any presentation (V, W) we must have that $S(V)$ satisfies (O3). We are now in a position to prove a more general (O3)-like principle.

Lemma 3.1.30. *Let Π be a presentation on a set E, v_0 a vector of Π, X a subset of E and F a finite subset of E disjoint from X. Then amongst the set $L_{v_0}^{F,X}(\Pi)$ of vectors v of Π such that $v\restriction_F = v_0\restriction_F$ and $\mathrm{supp}(v) \subseteq \mathrm{supp}(v_0) \cup X$, there is one with $\mathrm{supp}(v) \setminus X$ minimal.*

Proof. We put a preordering on $L_{v_0}^{F,X}(\Pi)$ by $v \le v'$ if $\mathrm{supp}(v) \setminus X \subseteq \mathrm{supp}(v') \setminus X$. The function $v \mapsto v - v_0\restriction_F + \mathbb{1}_*$ is an order-preserving bijection from $L_{v_0}^{F,X}(\Pi)$ to $L_{v_0 - v_0\restriction_F + \mathbb{1}_*}^{\{*\},X}(\Pi_{v_0\restriction_F})$. The latter collection has a minimal element by (O3) applied to the set of supports of vectors of the presentation $\Pi_{v_o\restriction_F}$. Hence the former collection also has a minimal element. $\square$

Remark 3.1.31. *Let $v \in L_{v_0}^{F,X}(\Pi)$ such that $\mathrm{supp}(v) \setminus X$ is minimal. Then the set $\mathrm{supp}(v) \setminus X$ is Π/X-independent.*

Corollary 3.1.32. *Let Π be a presentation on a set E, F a finite subset of E and P a subset of E disjoint from F. Then there is a Π-independent subset P' of P such that $(\Pi/P)\restriction_F = (\Pi/P')\restriction_F$.*

Proof. We successively apply Lemma 3.1.30 and Remark 3.1.31 to elements of a base of $(\Pi/P)\restriction_F$. $\square$

3.1.6 Trees of presentations

We can now mimic the construction of [18] to glue together trees of presentations.

Definition 3.1.33. A *tree of presentations* $\mathcal{T}$ consists of a tree T, together with functions $\overline{V}$ and $\overline{W}$ assigning to each node t of T a presentation $\Pi(t) = (\overline{V}(t), \overline{W}(t))$ on the ground set $E(t)$, such that for any two nodes t and t' of T, $E(t) \cap E(t')$ is finite and if $E(t) \cap E(t')$ is nonempty then tt' is an edge of T.

For any edge tt' of T we set $E(tt') = E(t) \cap E(t')$. We also define the *ground set* of $\mathcal{T}$ to be $E = E(\mathcal{T}) = \left(\bigcup_{t \in V(T)} E(t)\right) \setminus \left(\bigcup_{tt' \in E(T)} E(tt')\right)$.

We shall refer to the edges which appear in some $E(t)$ but not in E as *dummy edges* of $M(t)$: thus the set of such dummy edges is $\bigcup_{tt' \in E(T)} E(tt')$.

In sticking together such a tree of presentations, we shall make use of some additional information, namely a set Ψ of ends of T. We think of the ends in Ψ as being available to be used by the new vectors and those in the complement $\Psi^{\complement}$ of Ψ as being available to be used by the new covectors. More formally:

Definition 3.1.34. Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a tree of presentations. A *pre-vector* of $\mathcal{T}$ is a pair $(S, \overline{v})$, where S is a subtree of T and $\overline{v}$ is a function sending each node t of S to some $\overline{v}(t) \in \overline{V}(t)$, such that for each $t \in S$ we have $\overline{v}(t)\restriction_{E(tu)} = \overline{v}(u)\restriction_{E(tu)} \neq 0$ if $u \in S$, and $\overline{v}(t)\restriction_{E(tu)} = 0$ otherwise. The *underlying vector* $\overline{(S, \overline{v})}$ of $(S, \overline{v})$ is the element of $k^{E(\mathcal{T})}$ which at a given $e \in E(\mathcal{T})$ takes the value $\overline{v(t)(e)}$ if there is some $t \in S$ with $e \in E(t)$, and otherwise takes the value 0. The *support* $\mathrm{supp}(S, \overline{v})$ of a pre-vector is the support of the underlying vector.

Now let Ψ be a set of ends of T. A pre-vector $(S, \overline{v})$ is a Ψ*-pre-vector* if all ends of S are in Ψ. The space $V_\Psi(\mathcal{T})$ of Ψ*-vectors* is the subspace of k^E generated[4] by the underlying vectors of Ψ-pre-vectors.

A *pre-covector* of $\mathcal{T}$ is a pair $(S, \overline{w})$, where S is a subtree of T and $\overline{w}$ is a function sending each node t of S to some $\overline{w}(t) \in \overline{W}(t)$, such that for each $t \in S$ we have $\overline{w}(t)\restriction_{E(tu)} = -\overline{w}(u)\restriction_{E(tu)} \neq 0$ if $u \in S$, and $\overline{w}(t)\restriction_{E(tu)} = 0$ otherwise (note the change of sign from the definition of pre-vectors). Underlying covectors and supports are defined as above. A pre-covector $(S, \overline{w})$ is a Ψ*-pre-covector* if all ends of S are in Ψ. The space $W_\Psi(\mathcal{T})$ of $\Psi^{\complement}$*-covectors* is the subspace of k^E generated by the underlying covectors of $\Psi^{\complement}$-pre-covectors. Finally, $\Pi_\Psi(\mathcal{T})$ is the pair $(V_\Psi(\mathcal{T}), W_\Psi(\mathcal{T}))$. We may omit the subscripts from $V_\Psi(\mathcal{T})$, $W_\Psi(\mathcal{T})$ and $\Pi_\Psi(\mathcal{T})$ if the set of ends of T is empty.

Remark 3.1.35. *Let P and Q be sets which don't meet any of the sets $E(tu)$ with tu an edge of T. Then $\Pi_\Psi(\mathcal{T})/P\backslash Q = \Pi_\Psi(T, \overline{V}/P\backslash Q, \overline{W}\backslash P/Q)$, where $\overline{V}/P\backslash Q \colon t \mapsto \overline{V}(t)/P\backslash Q$ and $\overline{W}\backslash P/Q \colon t \mapsto \overline{W}(t)\backslash P/Q$.*

Our notation suggests that $V_\Psi(\mathcal{T})$ and $W_\Psi(\mathcal{T})$ should be orthogonal. This is often true, but as the following example shows some extra restriction is needed to ensure that intersections of supports of vectors with supports of covectors are finite.

Example 3.1.36. Let (V, W) be any presentation having some vector v of infinite support and some covector w of infinite support. Let $(e_i | i \in \mathbb{N})$ be an infinite sequence of distinct elements of $\mathrm{supp}(v) \setminus \mathrm{supp}(w)$ and $(f_i | i \in \mathbb{N})$ an infinite sequence of distinct elements of $\mathrm{supp}(w) \setminus \mathrm{supp}(v)$. We also introduce for each $i \in \mathbb{N}$ the presentation $\Pi_i = (V_i, W_i)$ on ground set $E_i = (e_i, f_i, g_i, h_i)$, where the g_i and h_i are all chosen distinct and outside E, and where $V_i = \{v \in k^{E_i} | v(f_i) = 0 \text{ and } v(g_i) = v(h_i)\}$ and $W_i = \{w \in k^{E_i} | w(e_i) = 0 \text{ and } w(g_i) = -w(h_i)\}$. Let $v_i \in V_i$ be the vector taking the value $v(e_i)$ at e_i, g_i and h_i and 0 at f_i. Let $w_i \in W_i$ be the covector taking the value 0 at e_i, $-w(f_i)$ at f_i and g_i and $w(f_i)$ at h_i.

Let T be the star with central node $*$ and whose leaves are the natural numbers. Then we get a tree of presentations $\mathcal{T} = (T, \overline{V}, \overline{W})$ by letting $\overline{V}(*) = V$ and $\overline{V}(i) = V_i$ for each $i \in \mathbb{N}$ and defining $\overline{W}$ similarly. We get a pre-vector

[4] under finite linear combinations

$(T, \overline{v})$ by letting $\overline{v}(*) = v$ and $\overline{v}(i) = v_i$ and a pre-covector $(T, \overline{w})$ by letting $\overline{w}(*) = w$ and $\overline{w}(i) = w_i$. Then the intersection of the supports of $(T, \overline{v})$ and $(T, \overline{w})$ includes $\bigcup_{i \in \mathbb{N}} \{g_i, h_i\}$, and so is infinite.

In order to avoid this sort of situation, we introduce the following restriction:

Definition 3.1.37. Let Π be a presentation on a set E, and let $\mathcal{F}$ be a set of disjoint subsets of E. We say that Π is *neat* with respect to $\mathcal{F}$ if for any $v \in V_\Pi$ and $w \in W_\Pi$ there are only finitely many $F \in \mathcal{F}$ meeting the supports of both v and w. We say that a tree $\mathcal{T} = (T, \overline{V}, \overline{W})$ of presentations is *neat* if for each node t of T the presentation $\Pi(t)$ is neat with respect to the set of sets $E(tu)$ with u adjacent to t in T.

Lemma 3.1.38. *Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a neat tree of presentations, and Ψ a set of ends of T. Then $V_\Psi(\mathcal{T}) \perp W_\Psi(\mathcal{T})$.*

Proof. It suffices to show that for any Ψ-pre-vector $(S, \overline{v})$ and any Ψ^{C}-pre-covector $(S', \overline{w})$ we have $\underline{(S, \overline{v})} \perp \underline{(S', \overline{w})}$. All ends of the tree $S \cap S'$ must be in $\Psi \cap \Psi^{\mathsf{C}} = \emptyset$: that is, $S \cap S'$ is rayless. Since $\mathcal{T}$ is neat, each vertex of $S \cap S'$ has finite degree in $S \cap S'$. Thus by König's Lemma the tree $S \cap S'$ is finite. The intersection of the supports of $(S, \overline{v})$ and $(S', \overline{w})$ is a subset of the finite set $\bigcup_{t \in S \cap S'} (\mathrm{supp}(\overline{v}(t)) \cap \mathrm{supp}(\overline{w}(t)))$ and so is finite.

For any edge tu of $S \cap S'$ and $e \in E(tu)$ we have $\overline{v}(t)(e)\overline{w}(t)(e) + \overline{v}(u)(e)\overline{w}(u)(e) = 0$, and so we have

$$\sum_{e \in E} \underline{(S, \overline{v})}(e)\underline{(S', \overline{w})}(e) = \sum_{t \in S \cap S'} \sum_{e \in E(t)} \overline{v}(t)(e)\overline{w}(t)(e) = 0\,.$$

□

However, our aim is to use the construction of $\Pi_\Psi(\mathcal{T})$ to produce matroids, so we are also interested in the question of when $\Pi_\Psi(\mathcal{T})$ presents a matroid, that is, the minimal nonempty Ψ-vectors and the minimal nonempty Ψ^{C}-covectors satisfy (O2) and (IM). It is not even clear that our construction will yield matroids when applied to the simplest sorts of trees, namely stars with all leaves finite. More precisely:

Definition 3.1.39. Let Π be a presentation on a set E and let $\mathcal{F}$ be a set of disjoint subsets of E. An $\mathcal{F}$*-star* of presentations around Π is a tree $(T, \overline{V}, \overline{W})$ of presentations where T is the star with central node $*$ and leaf set $\mathcal{F}$, $(\overline{V}(*), \overline{W}(*)) = \Pi$, and for each $F \in \mathcal{F}$ the set $E(F)$ is finite and $E(*F) = F$. We say that Π is *stellar* with respect to $\mathcal{F}$ if for any $\mathcal{F}$-star $\mathcal{T}$ of presentations around Π, the pair $\Pi_\emptyset(\mathcal{T})$ is a presentation and presents a matroid. We say that a tree $\mathcal{T} = (T, \overline{V}, \overline{W})$ of presentations is *stellar* if for each node t of T the presentation $\Pi(t)$ is stellar with respect to the set of sets $E(tu)$ with u adjacent to t in T.

Remark 3.1.40. *There are many examples of stellar presentations. For example, if* Π *is finitary*[5] *or* $\mathcal{F}$ *is finite then* Π *is stellar with respect to* $\mathcal{F}$*. If* Π' *is a minor of* Π *on the set* E' *and* Π *is stellar with respect to* $\mathcal{F}$ *then* Π' *is stellar with respect to* $\{F \cap E' | F \in \mathcal{F}\}$*. Furthermore, if* Π *is stellar with respect to* $\mathcal{F}$ *and* $\mathcal{F}'$ *is a set of disjoint sets such that each* $F' \in \mathcal{F}'$ *is a subset of some* $F \in \mathcal{F}$ *then* Π *is also stellar with respect to* $\mathcal{F}'$*. This fact, together with the construction given in Example 3.1.36, shows that if* Π *is stellar with respect to* $\mathcal{F}$ *then it is necessarily also neat with respect to* $\mathcal{F}$*.*

Our strategy, aiming at maximal generality, is to leave the question of precisely which presentations are stellar open but to reduce the question of when sticking together trees of presentations gives a presentation of a matroid to this problem. That is, we shall show that if $\mathcal{T}$ is a stellar tree of presentations and Ψ is a Borel set of ends then $\Pi_\Psi(\mathcal{T})$ is a presentation of a matroid. (O2) will be proved in Subsection 3.1.7 and (IM) in Subsection 3.1.8. We note, however, that the following question remains open:

Open Question 3.1.41. *If a presentation* Π *is neat with respect to some* $\mathcal{F}$*, must it also be stellar with respect to* $\mathcal{F}$*?*

We will rely on the following straightforward rearrangement of the definition of stellarity:

Lemma 3.1.42. *Let* $\Pi = (V, W)$ *be a presentation on a set* E *which is stellar with respect to* $\mathcal{F} \subseteq \mathcal{P}(E)$*, and let* $F_0 \in \mathcal{F}$ *and* $w_0 \in k^{F_0}$*. Let* Q *be a set disjoint from all* $F \in \mathcal{F}$*. For each* $F \in \mathcal{F} - F_0$ *let* $W(F)$ *be a subset of* k^F*. Suppose that for every* $v \in V$ *with* $v \not\perp w_0$ *and* $\mathrm{supp}(v) \cap Q = \emptyset$ *there is some* $F \in \mathcal{F} - F_0$ *and some* $w \in W(F)$ *such that* $w \not\perp v$*. Then there is some* $w \in W$ *such that* $w\restriction_{F_0} = w_0$*,* $\mathrm{supp}(w) \subseteq Q \cup \bigcup \mathcal{F}$*, and for each* $F \in \mathcal{F} - F_0$ *we have* $w\restriction_F \in \langle W(F)\rangle$*.*

Proof. Without loss of generality each $W(F)$ is a subspace of the corresponding space k^F. Let $V(F) = W(F)^\perp$, and $\Pi(F) = (V(F), W(F))$, which is a presentation by Lemma 3.1.26. Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be the $(\mathcal{F} - F_0)$-star of presentations around Π, where the presentation at the leaf F is $\Pi(F)$. Since Π is stellar, $\Pi_\emptyset(\mathcal{T})$ is a presentation. Now we consider the presentation $(\Pi_\emptyset(\mathcal{T})\backslash Q).F_0$, which by Lemma 3.1.26 consists of a pair (V_0, W_0) of complementary subspaces of k^{F_0}. What we have to prove is just that $w_0 \in W_0$.

Suppose not for a contradiction. Then there is some $v_0 \in V_0$ with $v_0 \not\perp w_0$. By definition this v_0 must arise as $\overline{v}(*)\restriction_{F_0}$ for some pre-vector $(S, \overline{v})$ of $\mathcal{T}$ whose support does not meet Q. Then $\overline{v}(*) \not\perp w_0$, so there is some $F \in \mathcal{F} - F_0$ and some $w \in W(F)$ such that $w \not\perp \overline{v}(*)\restriction_F = \overline{v}(F)$, contradicting the fact that $\overline{v}(F) \in V(F)$. $\square$

[5] A presentation (V, W) is *finitary* if every element in V is finite.

3.1.7 (O2) for trees of presentations

Our aim in this section is to show that, for any stellar tree $\mathcal{T} = (T, \overline{V}, \overline{W})$ of presentations and any Borel set Ψ of ends of T, the sets $S(V_\Psi(\mathcal{T}))$ and $S(W_\Psi(\mathcal{T}))$ satisfy (O2). Thus we begin by fixing such a $\mathcal{T}$ and Ψ. We also fix a partition $E(\mathcal{T}) = P \dot\cup Q \dot\cup \{e\}$. We shall consider the vertex t_0 of T with $e \in E(t_0)$ to be the root of T, and we consider T as a directed graph with the edges directed away from t_0. To prove (O2), it suffices to prove that there is either a Ψ-pre-vector $(S, \overline{v})$ with $e \in \mathrm{supp}(S, \overline{v}) \subseteq P + e$ or else a $\Psi^{\complement}$-pre-covector $(S, \overline{w})$ with $\in \mathrm{supp}(S, \overline{w}) \subseteq Q + e$. To this end, we recall two games, called the *circuit game* and *cocircuit game*, from [18]. To match the formalism of Subsection 3.1.4, we shall present these games as positional games.

To simplify notation in this section, we shall not distinguish between an end ω of a rooted tree T and the unique ray belonging to ω that starts at the root.

Definition 3.1.43. Let X be the set of pairs (t, v) with t a vertex of T and $v \in \overline{V}(t)$ such that $\mathrm{supp}(v) \cap Q = \emptyset$. Let Y be the set of pairs (tu, w) with tu an edge of T and $w \in k^{E(tu)}$.

The *circuit game* $\mathcal{G} = \mathcal{G}(T, \overline{V}, \overline{W}, \Psi, P, Q)$ is the positional game played on the digraph D with vertex set $X \sqcup Y \sqcup \{a\}$ and with edges given as follows:

- an edge from a to $(t_0, v) \in X$ when $e \in \mathrm{supp}(v)$.
- an edge from $(t, v) \in X$ to $(tu, w) \in Y$ when $v \not\perp w$.
- an edge from $(tu, w) \in Y$ to $(u, w) \in X$ when $v \not\perp w$.

Any infinite walk from an outneighbour of a in D induces an infinite walk from t_0 in T, which is an end of T. The set Φ of winning conditions of $\mathcal{G}$ is the set of infinite walks from outneighbours of a in D which induce walks to ends in Ψ. We call the two players of the circuit game Sarah and Colin, with Sarah playing first.

The *cocircuit game* is the game like the dual circuit game $\mathcal{G}(T, \overline{W}, \overline{V}, \Psi^{\complement}, Q, P)$ but with the roles of Sarah and Colin reversed.

It is not hard to see that this definition is just a reformulation of [18, Definition 8.1]. Using the arguments of that paper, we may now obtain the following results:

Lemma 3.1.44. *Either Sarah or Colin has a winning strategy in the circuit game.* $\square$

Lemma 3.1.45. *Colin has a winning strategy in the circuit game if and only if he has one in the cocircuit game.*

Proof. Just like the proof of [18, Lemma 8.5], but using Lemma 3.1.42 in place of [18, Sublemma 8.6] $\square$

From now on we shall assume that Sarah has a winning strategy σ in the circuit game: the argument if Colin has a winning strategy there is dual to the one which follows. Let S_σ be the subtree of T consisting of those vertices t for which there is some v such that Sarah might at some point play (t, v) when playing according to σ. We would like to mimic the argument of [18, Lemma 8.2] to construct a Ψ-precircuit from σ. In order to do this, we would need all ends of S_σ to be in Ψ. Although there is no reason to expect this to happen in general, it will happen if σ is reduced.

Lemma 3.1.46. *Let σ be a reduced winning strategy in the circuit game, and let S_σ be defined as above. Then all ends of S_σ are in Ψ.*

Proof. For any finite sequence s we denote the last element of s by $l(s)$. For any finite play s in $\mathcal{G}$, let $\hat{s}$ be the sequence of moves played by Sarah in s (that is, the sequence $(s_{2k+1}|0 \leq k \leq \text{length}(s)/2)$). Let $\tau = \{\hat{s}|s \in \sigma\}$.

First of all we will show that for any edge tu of T and any $s \in \tau$ with $\pi_1(l(s)) = t$ there are no more than $|E(tu)|$ extensions $s' \in \tau$ of s with $\pi_1(l(s')) = u$. Suppose for a contradiction that there are more than this. Then each such s' gives rise to a vector $\pi_2(l(s'))\restriction_{E(tu)}$ in $k^{E(tu)}$, and there must be some linear dependence of these vectors. So suppose that $\sum_{i=1}^n \lambda_i \pi_2(l(s^i))\restriction_{E(tu)} = 0$, where for each i λ_i is nonzero and s^i is an extension of s in τ with $\pi_1(l(s^i)) = u$. Let k be the length of s, and let j be such that $l(s^j)$ is maximal in the order $\leq$. Without loss of generality $j = n$. Let $s' = s'_1 ... s'_{2k+1} \in \sigma$ with $\widehat{s'} = s^n$. Then

$$\pi_2(s'_{2k}) \not\leq \pi_2(l(s^n))\restriction_{E(tu)} = -\frac{1}{\lambda_n} \sum_{i=1}^{n-1} \lambda_i \pi_2(l(s^i))\restriction_{E(tu)},$$

so there is some $i < n$ with $\pi_2(s'_{2k}) \not\leq \pi_2(l(s^i))\restriction_{E(tu)}$. But then $s'_1 s'_2 ... s'_{2k} l(s^i)$ is a legal play in $\mathcal{G}$ and $l(s^i) < s'_{2k+1}$, contradicting our assumption that σ is reduced. Thus there are at most $|E(tu)|$ (and in particular only finitely many) extensions $s' \in \tau$ of s with $\pi_1(l(s')) = u$.

Now let $\omega = (t_i|i \in \mathbb{N})$ be any end of S_σ. For each n, let τ_n be the set of those $s \in \tau$ with $\pi_1(l(s)) = t_n$. Then, repeatedly using what we have just shown, it follows by induction on n that each τ_n is finite. Let $f_n \colon \tau_{n+1} \to \tau_n$ be given by restriction. Then by König's Infinity Lemma we can find $s^n \in \sigma$ with $\widehat{s^n} \in \tau_n$ for each n such that $f_n(\widehat{s^{n+1}}) = \widehat{s^n}$ for each n. Let s be the infinite sequence $s^1_1 s^1_2 s^2_3 s^2_4 s^3_5 s^3_6 ...$. Then s is an infinite play according to σ by Remark 3.1.22, so since σ is winning we have $\omega \in \Psi$. $\square$

By Lemma 3.1.21, we may assume without loss of generality that Sarah's winning strategy σ is reduced, and so that all ends of S_σ are in Ψ. It follows, using the argument of [18, Lemma 8.2], that there is a Ψ-pre-vector $(S, \overline{v})$ with $e \in \text{supp}(S, \overline{v}) \subseteq P + e$. This completes our proof of (O2).

3.1.8 (IM) for trees of presentations

Our aim in this section is to show that gluing together stellar trees of presentations gives presentations which satisfy (IM), which is the only remaining part of the task of showing that this construction gives rise to matroids. To prove (IM), it suffices by Corollary 3.1.10 to show that we can construct a base. We will do this recursively, successively building the parts of the base at each node of the tree. When building the part of the base at a particular node, we will want to ignore the details of the branches of the tree which remain when this node is removed. To this end, we will replace each such branch by a finite matroid which retains just enough information for our argument. This will be done with the following Lemma:

Lemma 3.1.47. *Let $\Pi = (V, W)$ be a presentation on a set E, and let F be a finite subset of E. Then there are disjoint subsets P_F and Q_F of $E \setminus F$ such that $E \setminus (P_F \cup Q_F)$ is finite and $\Pi' \restriction_F = \Pi \restriction_F$ and $\Pi'.F = \Pi.F$, where $\Pi' = \Pi / P_F \setminus Q_F$.*

Proof. Let B_V be a (linear) basis of $V.F$ and B_W a (linear) basis of $W.F$. For each $v \in B_V$, choose some $\hat{v} \in V$ with $\hat{v}\restriction_F = v$. Similarly, for each $w \in B_W$ choose some $\hat{w} \in W$ with $\hat{w}\restriction_F = w$. Let $F' = F \cup \left[\left(\bigcup_{v \in B_V} \operatorname{supp}(\hat{v})\right) \cap \left(\bigcup_{w \in B_W} \operatorname{supp}(\hat{w})\right)\right]$, which is finite because it is the union of F with a finite union of sets of the form $\operatorname{supp}(\hat{v}) \cap \operatorname{supp}(\hat{w})$. Let $P_F = \bigcup_{v \in B_V} \operatorname{supp}(\hat{v}) \setminus F'$, and $Q_F = E \setminus (P_F \cup F')$. Thus P_F and Q_F are disjoint, and $E \setminus (P_F \cup Q_F) = F'$ is finite.

For each $v \in B_V$, we have $\operatorname{supp}(\hat{v}) \subseteq E \setminus Q_F$, so $v \in (V \setminus Q_F).F$. Thus $V.F \subseteq (V \setminus Q_F).F$. It is clear that the reverse inclusion $(V \setminus Q_F).F \subseteq V.F$ also holds, and so $(V \setminus Q_F).F = V.F$. Since by Lemma 3.1.26 any presentation on a finite set is determined by its set of vectors, we may deduce that $\Pi'.F = (\Pi \setminus Q_F).F = \Pi.F$. The proof that $\Pi'\restriction_F = \Pi\restriction_F$ is similar. $\square$

Using this, we can now obtain the lemma which will be applied at each node:

Lemma 3.1.48. *Let Π be a presentation on a set E which is stellar with respect to a set $\mathcal{F}$ of disjoint subsets of E. Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a tree of presentations, where T is a star with central node $*$ and leaf set $\mathcal{F}$, and $(\overline{V}(*), \overline{W}(*)) = \Pi$ and for each $F \in \mathcal{F}$ we have $E(*F) = F$. Let $E' = E \setminus \bigcup \mathcal{F}$. Let X and Y be disjoint subsets of $E(\mathcal{T})$ such that X is $S(V(\mathcal{T}))$-independent and Y is $S(W(\mathcal{T}))$-independent. Then there are disjoint subsets X' and Y' of $E(\mathcal{T})$ extending X and Y respectively such that:*

- *$E' \subseteq X' \cup Y'$*
- *X' is $S(V(\mathcal{T}))$-independent and Y' is $S(W(\mathcal{T}))$-independent.*
- *For any $e \in E' \setminus X'$ there is some $C \in S(V(\mathcal{T}))$ with $e \in C \subseteq X' + e$.*
- *For any $e \in E' \setminus Y'$ there is some $D \in S(W(\mathcal{T}))$ with $e \in D \subseteq Y' + e$.*
- *There do not exist leaves $F, F' \in \mathcal{F}$ such that there is a connected component of $\Pi(\mathcal{T})/X' \setminus Y'$ meeting both $E(F)$ and $E(F')$.*

Proof. For each $F \in \mathcal{F}$ we will denote the presentation $(\overline{V}(F), \overline{W}(F))$ by Π_F. By Remark 3.1.35 we may assume without loss of generality that X and Y are subsets of E'.

We begin by picking, for each $F \in \mathcal{F}$, sets P_F and Q_F as in Lemma 3.1.47 for the finite subset F of $E(F)$. Let $\tilde{V}(F) = \overline{V}(F)/P_F\backslash Q_F$ and $\tilde{W}(F) = \overline{W}(F)\backslash P_F/Q_F$. Taking $\tilde{V}(*) = V_\Pi$ and $\tilde{W}(*) = W_\Pi$ we get an $\mathcal{F}$-star $\tilde{\mathcal{T}} = (T, \tilde{V}, \tilde{W})$ of presentations around Π. By construction, X is $S(V(\tilde{\mathcal{T}}))$-independent and Y is $S(W(\tilde{\mathcal{T}}))$-independent. Since Π is stellar, we can choose a base B extending X and disjoint from Y for the matroid M presented by $\Pi(\tilde{\mathcal{T}})$: let B' be the base of the dual matroid M^* given by taking the complement of B. For each $F \in \mathcal{F}$, let X_F be an independent subset of $P_F \cup (B \cap E(F))$ such that $(\Pi_F/X_F)\restriction_F = (\Pi_F/(P_F \cup (B \cap E(F))))\restriction_F$ as in Corollary 3.1.32 and let Y_F be a coindependent subset of $Q_F \cup (B' \cap E(F))$ such that $(\Pi_F\backslash Y_F).F = (\Pi_F$ *backslash*$(Q_F \ \cup (B' \cap E(F)))).F$. Note that $(\Pi_F\backslash Y_F).F = (\Pi_F/X_F)\restriction_F$. Let $X' = (B \cap E') \cup \bigcup_{F \in \mathcal{F}} X_F$ and $Y' = (B' \cap E') \cup \bigcup_{F \in \mathcal{F}} Y_F$. It is clear that X' and Y' are disjoint, cover E', and respectively extend X and Y.

Now suppose for a contradiction that X' is $S(V(\mathcal{T}))$-dependent. Then there is some $\mathcal{T}$-prevector $(S, \hat{v})$ whose support C is nonempty and included in X'. The tree S cannot consist of just a single leaf of T by independence of of the sets X_F, so it must contain $*$. For each leaf F of T in S, we have $\hat{v}(*)\restriction_F \in \overline{V}(F)/X_F$, so by the definition of X_F we have $\hat{v}(*) \in (\overline{V}(F)/(P_F \cup (B \cap E(F))))\restriction_F$, that is, there is some vector $\hat{v}'(F)$ of $\tilde{V}(F)$ whose support is included in $(B \cap E(F)) \cup F$ and with $\hat{v}'(F)\restriction_F = \hat{v}(*)\restriction_F$. Letting $\hat{v}'(*) = \hat{v}(*)$, we obtain a $\tilde{\mathcal{T}}$-prevector $(S, \hat{v}')$ whose support is included in B, and so must be empty. So for each leaf F of T in S we have $\hat{v}'(F) \in k^F$ and so, by our choice of P_F and Q_F, $\hat{v}'(F) \in \overline{V}(F)$, so since $\hat{v}'(F) = \hat{v}(*)\restriction_F$ we have $\hat{v}(*)\restriction_F \in \overline{V}(F)$. Also, C cannot meet E', so since C is nonempty there is some leaf F of T in S for which the support of $\hat{v}(F)$ isn't a subset of F. Then $\hat{v}(F) - \hat{v}(*)\restriction_F$ is a vector in $\overline{V}(F)$ whose support is nonempty and included in X_F, contradicting the independence of X_F.

This shows that X' is $S(V(\mathcal{T}))$-independent, and a dual argument shows that Y' is $S(W(\mathcal{T}))$-independent.

Next we will show that for any $e \in E' \setminus X'$ there is some $C \in S(V(\mathcal{T}))$ with $e \in C \subseteq X' + e$. Since $e \in B'$, there is some circuit C_0 of M with $e \in C_0 \subseteq B + e$. Let $(S, \hat{v})$ be a $\tilde{\mathcal{T}}$-prevector with support C_0. Then for each leaf F of T in S, we have $\hat{v}(*)\restriction_F \in (\overline{V}(F)/(P_F \cup (B \cap E(F))))\restriction_F = (\overline{V}(F)/X_F)\restriction_F$, so that there is some $\hat{v}'(F) \in \overline{V}(F)$ with $\mathrm{supp}(\hat{v}'(F)) \subseteq X_F \cup F$ and $\hat{v}'(F)\restriction_F = \hat{v}(*)\restriction_F$. Letting $\hat{v}'(*) = \hat{v}(*)$, we get a $\mathcal{T}$-prevector $(S, \hat{v}')$ whose support is the desired C. A dual argument shows that for any $e \in E' \setminus Y'$ there is some $D \in S(W(\mathcal{T}))$ with $e \in D \subseteq Y' + e$.

It remains to prove the final condition of the Lemma. Suppose for a contradiction that this condition fails, and let $F \in \mathcal{F}$ such that there is a connected component of $\Pi(\mathcal{T})/X'\backslash Y'$ containing some edge e of $E(F)$ and some edge e' of $E(F')$ for some $F' \neq F$. Let C be a minimal nonempty element of $S(V(\mathcal{T})/X'\backslash Y')$ containing both e and e', and let $(S, \hat{v})$ be a $\mathcal{T}$-prevector whose support includes C but is a subset of $C \cup X'$. Both F and $*$ must be in S. Then the support of $\hat{v}(F)$ can't meet Y_F, so $\hat{v}(F)\restriction_F$ is a vector of $(\overline{V}(F)\backslash Y_F).F =$

$(\overline{V}(F)/X_F)\restriction_F$, so that there is some $v \in \overline{V}(F)$ with $\mathrm{supp}(v) \subseteq X_F \cup F$ and $v\restriction_F = \hat{v}(F)\restriction_F$. Then $(\{F\}, F \mapsto v(F) - v)$ is a $\mathcal{T}$-prevector whose support is a subset of $C \cup X'$ containing e but not e', contradicting the minimality of C. This completes the proof. $\square$

We now apply this lemma recursively to build the necessary bases. We will need a little notation for our recursive construction. For any tree T and directed edge st of T, let $T_{s\to t}$ be the subtree of T on the set of vertices u for which the unique path from s to u in T contains t. For $\mathcal{T} = (T, \bar{V}, \bar{W})$ a tree of presentations and st a directed edge of G, let $\mathcal{T}_{s\to t}$ be the tree of presentations $(T_{s\to t}, \bar{V}\restriction_{T_{s\to t}}, \bar{W}\restriction_{T_{s\to t}})$.

Theorem 3.1.49. *Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a stellar tree of presentations, and let Ψ be a Borel set of ends of $\mathcal{T}$. Then $\Pi_\Psi(\mathcal{T})$ presents a matroid.*

Proof. We have already shown that $\Pi_\Psi(\mathcal{T})$ is a presentation. Indeed, our results so far show that for each edge tt' of $\mathcal{T}$ the pair $\Pi_\Psi(\mathcal{T}_{t\to t'})$ is a presentation.

It remains to show that $S(V_\Psi(\mathcal{T}))$ satisfies (SM), for which by Remark 3.1.35 and Corollary 3.1.10 it is enough to show that there is some partition of $E(\mathcal{T})$ into a base X and a cobase Y, that is, Y is a subset of the $S(V_\Psi(\mathcal{T}))$-span of X and X is a subset of the $S(W_\Psi)(\mathcal{T}))$-span of Y. We build X and Y recursively. More precisely, we pick a root t_0 for T and order the vertices of T by the tree order $\leq$ with respect to this root. This is a well-founded order, and we construct subsets X_t and Y_t of $E(\mathcal{T})$ for each node t of T by recursion over $\leq$ such that:

1. X_t and Y_t are disjoint.
2. $X_t \subseteq X_{t'}$ and $Y_t \subseteq Y_{t'}$ for $t \leq t'$.
3. $X_{t'} \backslash X_t \subseteq E(\mathcal{T}_{t\to t'})$ and $Y_{t'} \backslash Y_t \subseteq E(\mathcal{T}_{t\to t'})$ for any edge tt' of T with $t \leq t'$.
4. $E(t) \cap E(\mathcal{T}) \subseteq X_t \cup Y_t$.
5. X_t is $S(V_\Psi(\mathcal{T}))$-independent and Y_t is $S(W_\Psi(\mathcal{T}))$-independent.
6. For any $e \in E(t) \cap E(\mathcal{T}) \setminus X_t$ there is some $C \in S(V(\mathcal{T}))$ with $e \in C \subseteq X_t + e$
7. For any $e \in E(t) \cap E(\mathcal{T}) \setminus Y_t$ there is some $D \in S(W(\mathcal{T}))$ with $e \in D \subseteq Y_t + e$
8. There is no edge tt' of T with $t \leq t'$ such that there is a connected component of $\Pi(\mathcal{T})/X_t \backslash Y_t$ meeting both $E(\mathcal{T}_{t\to t'})$ and $E(\mathcal{T}_{t'\to t})$.

If we can find such X_t and Y_t then the sets $X = \bigcup_{t\in V(T)} X_t$ and $Y = \bigcup_{t\in V(T)}(Y_t)$ will give the base and cobase we require: they will be disjoint by conditions 1, 2 and 3, will cover by condition 4 and will be respectively spanning and cospanning by conditions 6 and 7. It remains to show that this recursive construction can be carried out.

We construct X_{t_0} and Y_{t_0} by applying Lemma 3.1.48 to the star of presentations with central node $\Pi(t_0)$ and with a leaf for each neighbour t of t_0 in T labelled with the presentation $\Pi_\Psi(\mathcal{T}_{t_0\to t})$, and taking $X = Y = \emptyset$.

The construction of X_t and Y_t for $t \neq t_0$ is very similar. Let s be the predecessor of t in the tree order. Let E' be the set $E(t) \cap E(\mathcal{T})$ of real edges of $E(t)$. Let T' be the subtree $T_{t\to s} + t$ of T. Let $\Pi = (\Pi_\Psi(\mathcal{T}')/(X_s \cap E(\mathcal{T}_{t\to s})))\restriction_{E(t)\setminus E(st)}$. Note that by condition 8 applied at s we also have $\Pi = (\Pi_\Psi(\mathcal{T}')\backslash(Y_s \cap E(\mathcal{T}_{t\to s}))).(E(t) \setminus E(st))$. Then we build X' and Y' by applying Lemma 3.1.48 to the star of presentations with central node Π and with a leaf for each successor t' of t in T labelled with the presentation $\Pi_\Psi(\mathcal{T}_{t\to t'})$. We take the X and Y of the Lemma to be the intersections of X_s and Y_s with $E(\mathcal{T}_{s\to t})$ respectively. Finally, we let $X_t = X_s \cup X'$ and $Y_t = Y_s \cup Y'$. □

3.2 Topological cycle matroids of infinite graphs

3.2.1 Introduction

Many theorems about finite graphs and their cycles do not extend to infinite graphs and their finite cycles. However, many such theorems do extend to locally finite graphs together with their topological cycles, see for example [25, 55, 67, 99], and [47] for a survey. These *topological cycles* are homeomorphic images of the unit circle in the topological space obtained from the graph by adding certain points at infinity called ends.

Bruhn and Diestel gave an explanation why many of these theorems extended: the topological cycles of a locally finite graph form a matroid [29]. This matroidal point of view allowed for new proof techniques and abstracting the topological properties of the topological cycles often led to simpler proofs. For non-locally finite graphs various notions of end boundaries have been suggested [47], each of which gives rise to its own notion of topological cycles.

To compare these end boundaries we will not refer directly to topology but instead compare the matroids they induce. However for some of these notions, the matroids have finite circuits which are not finite cycles of the graph. A consequence of this is that there are non-isomorphic (even 3-connected) graphs inducing isomorphic matroids. For others we even do not always get a matroid.

Here we show that the topological end boundary, which had not been considered for this purpose before, lacks these defects. More precisely:

Theorem 3.2.1. *For any graph G, the topological cycles of G together with its topological ends form a matroid.*

Moreover, for non-isomorphic 3-connected graphs, these matroids are non-isomorphic.

Furthermore, all matroids that arise as cycle matroids for one of the other boundaries are minors of these cycle matroids. Theorem 3.2.1 implies for each of the other boundaries, a characterisations when the cycles induce a matroid.

The various notions of boundary and these characterisations are compared in Section 3.2.1.

The question whether the topological cycles in the graph G together with the boundary B induce a matroid is closely related to the question whether G has a spanning tree whose ends are equal to B. Indeed, any such spanning tree is an example of a base in the topological cycle matroid.

In our proof we use a result of [35] which ensures the existence of such spanning trees for the topological ends. We then combine this with the theory of trees of matroids [19].

Comparing the end boundaries

Bruhn and Diestel showed that the dual of the finite bond matroid of a graph G is given by the topological cycles of G together with its edge ends [29]. However, after deleting parallel edges, any component of such a matroid is countable.

Hence in order to construct matroids that are nontrivially uncountable, we have to consider topological cycles of different topological spaces. One such space is VTOP, which is obtained from the graph by adding the vertex ends. In Figure 3.1, we depicted a graph whose topological cycles in VTOP do not induce a matroid.

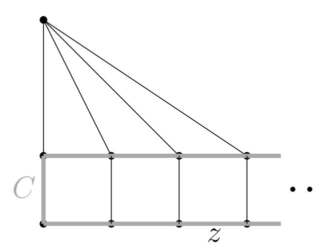

Figure 3.1: The dominated ladder is obtained from the one ended ladder by adding a vertex that is adjacent to every vertex on the upper side of the ladder. The topological cycles of VTOP of the dominated ladder do not induce a matroid as they violate the elimination axiom (C3): We cannot eliminate all the triangles from the grey cycle C.

The reason why this example works is that the topological cycle C goes through a dominated vertex end. Here a vertex v *dominates* a vertex end ω if there is an infinite v-fan to some ray belonging to ω.

One way to 'repair' VTOP is to identify each vertex ends with the vertices dominating it. The resulting space is called ITOP. A consequence of Theorem 3.2.1 is the following.

Corollary 3.2.2. *For any graph, the topological cycles of ITOP form a matroid.*

The matroids we get from Corollary 3.2.2 are more complicated than the ones for the edge ends in the sense that they are not always cofinitary. However, there are still non-isomorphic 3-connected graphs whose ITOP-matroids are isomorphic.

Another way to 'repair' VTOP is to delete the dominated vertex ends. Diestel and Kühn [54] showed that the remaining vertex ends are given by the topological ends, and in this case the topological cycles induce a matroid by Theorem 3.2.1.

In 1969, Higgs proved that the set of finite cycles and double rays of a graph G is the set of circuits of a matroid if and only if G does not have a subdivision of the Bean-graph [75]. Using Theorem 3.2.1, we get a result for the topological cycles of VTOP in the same spirit.

Corollary 3.2.3. *The topological cycles of VTOP induce a matroid if and only if G does not have a subdivision of the dominated ladder, which is depicted in Figure 3.1.*

Theorem 3.2.1 extends to 'Psi-Matroids': Given a set Ψ of topological ends, let C_Ψ be the set of those topological cycles that only use topological ends in Ψ. Let D_Ψ be the set of those bonds that have no topological end of Ψ in their closure. We prove the following strengthening of Theorem 3.2.1.

Theorem 3.2.4. *Let Ψ be a Borel set of topological ends. Then C_Ψ and D_Ψ are the sets of circuits and cocircuits of a matroid.*

We also can extend the main result of [18], see Subsection 3.3.3 for details.

This paper is organised as follows. After giving the necessary background in Section 3.3, we prove some intermediate results in Subsection 3.3.1. Then we prove Theorem 3.2.4 in Subsection 3.3.2. Finally, in Subsection 3.3.3 we deduce from it the other theorems mentioned in the Introduction.

3.3 Preliminaries

Throughout, notation and terminology for graphs are that of [52] unless defined differently. Instead of vertex ends, we shall just say end. And G always denotes a graph. We denote the complement of a set X by $X^{\complement}$. Throughout this paper, *even* always means finite and a multiple of 2. An edge set F in a graph is a *cut* if there is a partition of the set of vertices such that F is the set of edges with precisely one endvertex in each partition class. A vertex set *covers* a cut if every edge of the cut is incident with a vertex of that set. A cut is *finitely coverable* if there is a finite vertex set covering it. A *bond* is a minimal nonempty cut.

For us, a *separation* is just an edge set. The *boundary* $\partial(X)$ of a separation X is the set of those vertices adjacent with an edge from X and one from $X^{\complement}$. The *order* of X is the size of $\partial(X)$. Given a connected subgraph C of G, we denote the set of those edges with at least one endvertex in C by s_C. Given a separation X of finite order and a vertex end ω, then there is a unique component C of $G-\partial(X)$ in which ω lives. We say that ω *lives* in X if $s_C \subseteq X$.

A *tree-decomposition* of G consists of a tree T together with a family of subgraphs $(P_t | t \in V(T))$ of G such that every vertex and edge of G is in at least one of these subgraphs, and such that if v is a vertex of both P_t and P_w, then it is a vertex of each P_u, where u lies on the v-w-path in T. Moreover, each edge of G is contained in precisely one P_t. We call the subgraphs P_t, the *parts* of the tree-decomposition. Sometimes, the 'Moreover'-part is not part of the definition of tree-decomposition. However, both these two definitions give the same concept of tree-decomposition since any tree-decomposition without this additionally property can easily be changed to one with this property by deleting edges from the parts appropriately. Given a directed edge tu of T, the *separation corresponding to tu* is the set of those edges which are in parts P_w, where u lies on the unique t-w-path in T. The *adhesion* of a tree-decomposition is finite if any two adjacent parts intersect finitely. A key tool in our proof is the main result of [35], as follows.

Theorem 3.3.1. *Every graph G has a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion such that the ends of T are the undominated vertex ends of G.*

Remark 3.3.2. *([35, Remark 6.6]) Let $(T, \leq)$ be the tree order on T as in the proof of Theorem 3.3.1 where the root r is the smallest element. We remark that we constructed $(T, \leq)$ such that $(T, P_t | t \in V(T))$ has the following additional property: For each edge tu with $t \leq u$, the vertex set $\bigcup_{w \geq u} V(P_w) \setminus V(P_t)$ is connected.*

Moreover, we construct $(T, P_t | t \in V(T))$ such that if st and tu are edges of T with $s \leq t \leq u$, then $V(P_s) \cap V(P_t)$ and $V(P_t) \cap V(P_u)$ are disjoint.

Given a part P_t of a tree-decomposition, the *torso* H_t is the multigraph obtained from P_t by adding for each neighbour u of t in the tree a complete graph with vertex set $V(P_t) \cap V(P_u)$.

We denote the set of vertex ends of a graph G by $\Omega(G)$. A vertex v is in the *closure of* an edge set F if there is an infinite fan from v to $V(F)$. A vertex end ω is in the *closure of* an edge set F if every finite order separation X in which ω lives meets F. It is straightforward to show that a vertex end ω is in the closure of an edge set F if and only if every ray (equivalently: some ray) belonging to ω cannot be separated from F by removing finitely many vertices. A vertex end ω *lives in* a component C if it is in the closure of the edge set s_C. A *comb* is a subdivision of the graph obtain from the ray by attaching a leaf at each of its vertices. The set of these newly added vertices is the set of *teeth*. The Star-Comb-Lemma is the following.

Lemma 3.3.3. *(Diestel [50, Lemma 1.2]) Let U be an infinite set of vertices in a connected graph G. Then either there is a comb with all its teeth in U or a subdivision of the infinite star S with all leaves in U.*

Corollary 3.3.4. *Every infinite edge set has a vertex end or a vertex in its closure.* $\square$

Infinite matroids

An introduction to infinite matroids can be found in [30], whilst the axiomatisation of infinite matroids we work with here is the one introduced in [18]. Let $\mathcal{C}$ and $\mathcal{D}$ be sets of subsets of a groundset E, which can be thought of as the sets of circuits and cocircuits of some matroid, respectively.

(C1) The empty set is not in $\mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(O1) $|C \cap D| \neq 1$ for all $C \in \mathcal{C}$ and $D \in \mathcal{D}$.

(O2) For all partitions $E = P \dot\cup Q \dot\cup \{e\}$ either $P + e$ includes an element of $\mathcal{C}$ through e or $Q + e$ includes an element of $\mathcal{D}$ through e.

We follow the convention that if we put a * at an axiom A then this refers to the axiom obtained from A by replacing $\mathcal{C}$ by $\mathcal{D}$, for example (C1*) refers to the axiom that the empty set is not in $\mathcal{D}$. A set $I \subseteq E$ is *independent* if it does not include any nonempty element of $\mathcal{C}$. Given $X \subseteq E$, a *base of* X is a maximal independent subset Y of X.

(IM) Given an independent set I and a superset X, there exists a base of X including I.

The proof of [18, Theorem 4.2] also proves the following:

Theorem 3.3.5. *Let E be a some set and let $\mathcal{C}, \mathcal{D} \subseteq \mathcal{P}(E)$. Then there is a matroid M whose set of circuits is $\mathcal{C}$ and whose set of cocircuits is $\mathcal{D}$ if and only if $\mathcal{C}$ and $\mathcal{D}$ satisfy (C1), (C1*), (C2), (C2*), (O1), (O2), and (IM).*

Theorem 3.3.5 shows that the above axioms give an alternative axiomatisation of infinite matroids, which we use in this paper as a definition of infinite matroids. We call elements of $\mathcal{C}$ *circuits* and elements of $\mathcal{D}$ *cocircuits*. The *dual* of $(\mathcal{C}, \mathcal{D})$ is the matroid whose set of circuits is $\mathcal{D}$ and whose set of cocircuits is $\mathcal{C}$.

A matroid $(\mathcal{C}, \mathcal{D})$ is *finitary* if every element of $\mathcal{C}$ is finite, and it is *tame* if every element of $\mathcal{C}$ intersects any element of $\mathcal{D}$ only finitely. An example of a finitary matroid is the *finite-cycle matroids* of a graph G whose circuits are the edge sets of finite cycles of G and whose cocircuits are the bonds of G. We shall need the following lemma:

Lemma (Section 3.4, Lemma 3.4.9)**.** *Suppose that M is a matroid, and $\mathcal{C}$, $\mathcal{C}^*$ are collections of subsets of $E(M)$ such that $\mathcal{C}$ contains every circuit of M, $\mathcal{C}^*$ contains every cocircuit of M, and for every $o \in \mathcal{C}$, $b \in \mathcal{C}^*$, $|o \cap b| \neq 1$. Then the set of minimal nonempty elements of $\mathcal{C}$ is the set of circuits of M and the set of minimal nonempty elements of $\mathcal{C}^*$ is the set of cocircuits of M.*

Trees of presentations

In this subsection, we give a toy version of the definitions of [19], which are just enough to state the results of [19] we need in this paper. A tame matroid is *binary* if every circuit and cocircuit always intersect in an even number of edges.[6]

Roughly, a binary presentation of a tame matroid M is something like a pair of representations over $\mathbb{F}_2$, one of M and of the dual of M, formally:

Definition 3.3.6. Let E be any set. A *binary presentation* Π on E consists of a pair (V, W) of sets of subsets of E satisfying (02) and are orthogonal, that is, every $o \in V$ intersects any $d \in W$ evenly. We will sometimes denote the first element of Π by V_Π and the second by W_Π. We say that Π *presents* the matroid M if the circuits of M are the minimal nonempty elements of V_Π and the cocircuits of M are the minimal nonempty elements of W_Π.

Given a finitary binary matroid M, let $\overline{V_M}$ be the set of those finite edge sets meeting each cocircuit evenly, and let $\overline{W_M}$ be the set of those (finite or infinite) edge sets meeting each circuit evenly. Then $(\overline{V_M}, \overline{W_M})$ is called the *canonical presentation* of a M.

Definition 3.3.7. A *tree of binary presentations* $\mathcal{T}$ consists of a tree T, together with functions $\overline{V}$ and $\overline{W}$ assigning to each node t of T a binary presentation $\Pi(t) = (\overline{V}(t), \overline{W}(t))$ on the ground set $E(t)$, such that for any two nodes t and t' of T, if $E(t) \cap E(t')$ is nonempty then tt' is an edge of T.

For any edge tt' of T we set $E(tt') = E(t) \cap E(t')$. We also define the *ground set* of $\mathcal{T}$ to be $E = E(\mathcal{T}) = \left(\bigcup_{t \in V(T)} E(t)\right) \setminus \left(\bigcup_{tt' \in E(T)} E(tt')\right)$.

We shall refer to the edges which appear in some $E(t)$ but not in E as *dummy edges* of $M(t)$: thus the set of such dummy edges is $\bigcup_{tt' \in E(T)} E(tt')$.

A tree of binary presentations is a *tree of binary finitary presentations* if each presentation $\Pi(t)$ is a canonical presentation of some binary finitary matroid.

Definition 3.3.8. Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a tree of binary presentations. A *pre-vector* of $\mathcal{T}$ is a pair $(S, \overline{v})$, where S is a subtree of T and $\overline{v}$ is a function sending each node t of S to some $\overline{v}(t) \in \overline{V}(t)$, such that for each $t \in S$ we have $\overline{v}(t)\restriction_{E(tu)} = \overline{v}(u)\restriction_{E(tu)} \neq 0$ if $u \in S$, and $\overline{v}(t)\restriction_{E(tu)} = 0$ otherwise.

The *underlying vector* $\underline{(S, \overline{v})}$ of $(S, \overline{v})$ is the set of those edges in some $\overline{v}(t)$ for some $t \in V(T)$. Now let Ψ be a set of vertex ends of T. A pre-vector $(S, \overline{v})$ is a Ψ*-pre-vector* if all vertex ends of S are in Ψ. The space $V_\Psi(\mathcal{T})$ of Ψ*-vectors* consists of those sets that are a symmetric differences of finitely many underlying vectors of Ψ-pre-vectors.

pre-covectors are defined like pre-vectors with '$\overline{W}(t)$' in place of '$\overline{V}(t)$'. *underlying covectors* are defined similar to underlying vectors. A pre-covector $(S, \overline{w})$ is a Ψ*-pre-covector* if all vertex ends of S are in Ψ. The space $W_\Psi(\mathcal{T})$ of $\Psi^{\complement}$*-covectors* consists of those sets that are a symmetric differences of finitely many underlying covectors of $\Psi^{\complement}$-pre-covectors.

[6] In [17], it is shown that most of the equivalent characterisations of finite binary matroids extend to tame binary matroids.

Finally, $\Pi_\Psi(\mathcal{T})$ is the pair $(V_\Psi(\mathcal{T}), W_\Psi(\mathcal{T}))$.

The following is a consequence of the main result of [19], Theorem 8.3, and Lemma 6.8.

Theorem 3.3.9 ([19]). *Let $\mathcal{T} = (T, \overline{V}, \overline{W})$ be a tree of binary finitary presentations and Ψ a Borel set of vertex ends of T, then $\Pi_\Psi(\mathcal{T})$ presents a binary matroid. Moreover, the set of Ψ-vectors and $\Psi^{\complement}$-covectors satisfy (O1), (O2) and tameness.*

We shall also need the following related lemma, which is a combination of Lemma 6.6 and Lemma 6.8 from [19].

Lemma 3.3.10 ([19]). *Let $\mathcal{T} = (T, M)$ be a tree of binary finitary presentations and Ψ be any set of vertex ends of T. Any Ψ-vectors of $\mathcal{T}$ and any $\Psi^{\complement}$-covectors of $\mathcal{T}$ are orthogonal.*

3.3.1 Ends of graphs

The *simplicial topology of G* is obtained from the disjoint union of copies of the unit interval, one for each edge of G, by identifying two endpoints of these intervals if they correspond to the same vertex.

First we recall the definition of $|G|$ from [47], and then we give an equivalent one using inverse limits. Given a finite set of vertices S and a vertex end ω, by $C(S, \omega)$ we denote the component of $G - S$ in which ω lives. Let $\vec{\epsilon}$ be a function from the set of those edges with exactly one endvertex in $C(S, \omega)$ to $(0, 1)$. The set $C_{\vec{\epsilon}}(S, \omega)$ consists of all vertices of $C(S, \omega)$, all vertex ends living in $C(S, \omega)$, the set $e \times (0, 1)$ for each edge e with both endvertices in $C(S, \omega)$, together with for each edge f with exactly one endvertex $t(f)$ in $C(S, \omega)$, the set of those points on $f \times (0, 1)$ with distance less than $\vec{\epsilon}(f)$ from $t(f)$.

The point space of $|G|$ is the union of $\Omega(G)$, the vertex set $V(G)$ and a set $e \times (0, 1)$ for each edge e of G. A basis of this topology consists of the sets $C_{\vec{\epsilon}}(S, \omega)$ together with those sets O that are open considered as sets in the simplicial topology of G. Note that $|G|$ is Hausdorff.

Given a finite vertex set W of G, by $G^+[W]$ we denote the (multi-) graph obtained from G by contracting all edges not incident with a vertex of W. Thus the vertex set of $G^+[W]$ is W together with the set of components of $G - W$. We consider $G^+[W]$ as a topological space endowed with the simplicial topology. If $U \subseteq W$, then there is a continuous surjective map $f[W, U]$ from $G^+[W]$ to $G^+[U]$.

Theorem 3.3.11. *$|G|$ is the inverse limit of the topological spaces $G^+[W]$ with respect to the maps $f[W, U]$.*

Proof. For each vertex v of G, there is a point in the inverse limit which in the component for $G^+[W]$ takes the vertex whose branch set contains v. This is the point corresponding to the vertex v. Similarly, there are points in the inverse limit corresponding to interior points of edges. All other points in the inverse limit correspond to havens of order $< \infty$ of G. As explained in the appendix of [36], these are precisely the vertex ends of G. Thus $|G|$ and the inverse limit have the same point set. It is straightforward to check that they carry the same topology. □

In particular, $|G|$ has the following universal property: Suppose there is a topological space X and for each finite set W of vertices of G, a continuous function $f_W : X \to G^+[W]$ such that $f[W,U] \circ f_W = f_U$ for every $U \subseteq W$. Then there is a unique continuous function $f : X \to |G|$ such that $\pi_W \circ f = f_W$, where $\pi_W : |G| \to G^+[W]$ is the canonical projection.

A function f from S^1 to $|G|$ is *sparse* if $f^{-1}(v)$ never contains more than one point for each interior point v of an edge, and if there are two distinct points $x, y \in S^1$ with $f(x) = f(y)$, then there are two points z_1 and z_2 in different components of $S^1 - x - y$ both of whose f-values are different from $f(x)$ and not equal to interior points of edges.

Let f from S^1 to $|G|$ be a sparse continuous function. Then f meets an edge e in an interior point if and only if it traverses this edge precisely once. The set of those edges e is called the *edge set of* f, denoted by $E(f)$. If f is a topological cycle, we call $E(f)$ a *topological circuit*. An edge set F is *geometrically connected* if F meets every finitely coverable cut b with the property that two components of $G - b$ contain edges of F. Note that if the closure of an edge set F in $|G|$ is connected in $|G|$, then F is geometrically connected.

Lemma 3.3.12. *A nonempty edge set X is the set of edges of a sparse continuous function f from S^1 to $|G|$ if and only if it meets every finitely coverable cut evenly and is geometrically connected.*

Proof. For the 'only if'-implication, first note that the edge set of f is geometrically connected since connectedness is preserved under continuous images. Second, let F be a finitely coverable cut and let W be a finite vertex set covering it. If there is a sparse continuous function $f : S^1 \to |G|$, then $\pi_W \circ f : G^+[W] \to |G|$ is also continuous and its edge set Y meets F in $X \cap F$. Note that Lemma 3.3.12 is true with '$G^+[W]$' in place of '$|G|$'. So $X \cap F = Y \cap F$ is even, as F is a cut of $G^+[W]$.

The 'if'-implication is a consequence of Theorem 3.3.11: Suppose we have a geometrically connected set X meeting every finitely coverable cut evenly. Then for every finite vertex set W, the edge set $X \cap E(G^+[W])$ meets every cut of $G^+[W]$ evenly and is geometrically connected. Hence $X \cap E(G^+[W])$ is the edge set of a sparse continuous function f_W in $G^+[W]$. Each f_W is essentially given by a cyclic order on $E(f_W)$. As each vertex of W is incident with only finitely many vertices of X, the set $E(f_W)$ is finite. Thus we can use a standard compactness argument to ensure that $f_U = f[W,U] \circ f_W$ for every $U \subseteq W$. Then the limit of the f_W is continuous by the universal property of the limit and it is sparse by construction. □

The simplest example of a finitely coverable cut is the set of edges incident with a fixed vertex. Thus the edge set of a sparse continuous function has even degree at each vertex by Lemma 3.3.12. Thus we get the following.

Corollary 3.3.13. *Given a sparse continuous function f, then for every finite vertex set W only finitely many components of $G - W$ contain vertices incident with edges of $E(f)$.*

Proof. Let X be the set of those edges of $E(f)$ incident with vertices of W. Note that X is finite by Lemma 3.3.12. If two components of $G - W$ contain vertices incident with edges of $E(f)$, then s_D intersects X for every component D containing vertices incident with edges of $E(f)$ as $E(f)$ is geometrically connected by Lemma 3.3.12. Thus there are only finitely many such components D. $\square$

Having Lemma 3.3.12 and Corollary 3.3.13 in mind, the set F below can be sought of as the edge set of a topological cycle. Thus the following is an extension of the 'Jumping arc'-Lemma [52]:

Lemma 3.3.14. *Let F be an edge set meeting every finitely coverable cut evenly such that for every finite vertex set W only finitely many components of $G - W$ contain vertices of $V(F)$. Let b be a cut which does not intersect F evenly. Then there is an vertex end in the closure of both F and b.*

Given a finite vertex set W and a component D of $G - W$, we denote by $v(D)$ the vertex of $G^+[W]$ with branch set D.

Proof. First we show that for every finite vertex set W there is a component D of $G - W$ such that s_D contains infinitely many edges of both F and b. Suppose for a contradiction there is a vertex set W violating this. For a component D of $G - W$, let $X(D)$ be the set of those vertices in D incident with edges of b. Similarly, let $Y(D)$ be the set of those vertices in D incident with edges of F. Let U be the union of W with those $X(D)$ such that $Y(D)$ is infinite and those $Y(D)$ such that $Y(D)$ is finite.

By assumption $Y(D)$ is empty for all but finitely many D. Thus U is finite. Let G' be the graph obtained from $G^+[U]$ by deleting all vertices $v(K)$ for all components K of $G - U$ such that $Y(K)$ is empty.

Since $F \cap E(G')$ has even degree at each vertex of $G^+[U]$, the same is true for G'. On the other hand $b \cap E(G')$ is a cut by construction. Thus it intersects $F \cap E(G')$ evenly. As the intersection of b and F is included in $E(G')$ by construction, we get the desired contradiction.

Hence for every finite vertex set W there is a component D_W of $G - W$ such that s_{D_W} contains infinitely many edges of both F and b. By a standard compactness argument, we can pick the components D_W with the additional property that if $U \subseteq W$, then $f[U, W](v(D_W)) = v(D_U)$. Thus the components D_W define a haven of order $< \infty$ of G, which defines a vertex end ω as explained in the appendix of [36]. By construction the vertex end ω is in the closure of both F and b, completing the proof. $\square$

Lemma 3.3.15. *Let f be a sparse continuous function from S^1 to $|G|$ and let $x, y \in S^1$ such that $f(x)$ and $f(y)$ are distinct and not interior points of edges. Then for each connected component C of $S^1 - x - y$ there is an edge e_C of G such that $e_C \times (0, 1)$ is included in $f(C)$.*

Proof. We pick a finite vertex set W containing x and y. Clearly, the above lemma is true with '$G^+[W]$' in place of '$|G|$'. Thus for each connected component C of $S^1 - x - y$ there is an edge e_C of G such that $e_C \times (0,1)$ is included in $\pi_W(f(C))$. Hence $e_C \times (0,1)$ is included in $f(C)$. $\square$

3.3.2 Proof of Theorem 3.2.4

Given a connected graph G, we fix a tree-decomposition $(T, P_t | t \in V(T))$ as in Theorem 3.3.1 that has the additional properties of Remark 3.3.2. For an undominated vertex end ω of G, we denote the unique end of T in which it lives by $\iota_T(\omega)$. It is straightforward to check that ι_T is a homeomorphism from $\Omega(G)$ restricted to the undominated vertex ends to $\Omega(T)$.

For each $t \in V(T)$, let $M(t)$ be the finite-cycle matroid of the torso H_t. Let $\overline{V}(t) = V_{M(t)}$ and $\overline{W}(t) = W_{M(t)}$. Thus $\overline{V}(t)$ consists of those finite edge sets of H_t that have even degree at every vertex, and $\overline{W}(t)$ consists of the cuts of H_t.

Remark 3.3.16. *$\mathcal{T} = (T, \overline{V}, \overline{W})$ is a tree of binary finitary presentations.* $\square$

The aim of this section is to prove Theorem 3.2.4 from the Introduction. For that we have to show for each Borel set Ψ of undominated vertex ends of G that certain sets C_Ψ and D_Ψ are the sets of circuits and cocircuits of a matroid. By Theorem 3.3.9, we know that $\Pi_{\iota_T(\Psi)}(\mathcal{T})$ presents some matroid. In this section we prove that the circuits and cocircuits of that matroid are given by C_Ψ and D_Ψ.

To build this bridge from $\Pi_{\iota_T(\Psi)}(\mathcal{T})$ to the sets C_Ψ and D_Ψ, we start as follows. We have the two topological spaces $\Omega(G)$ and $\Omega(T)$, which each have their own Borel sets. The next lemma shows that these two systems of Borel sets are compatible:

Lemma 3.3.17. *The set of dominated vertex ends of G is Borel. In particular, for any set Ψ of undominated vertex ends, Ψ is Borel in $\Omega(G)$ if and only if $\iota_T(\Psi)$ is Borel in $\Omega(T)$.*

To prove this lemma, we need some intermediate lemmas. By $B_k(r)$ we denote the ball of radius k around a fixed vertex r.

Lemma 3.3.18. *The graph $G[B_k(r)]$ has a spanning tree Y_k of diameter at most $2k+1$.*

Proof. Proving this by induction over k, we may assume that $G[B_{k-1}(r)]$ has a spanning tree Y_{k-1} of diameter at most $2k-1$. Then Y_{k-1} together with all edges joining vertices in $B_k(r) \setminus B_{k-1}(r)$ to vertices in Y_{k-1} is a connected subgraph of $G[B_k(r)]$ with vertex set $B_k(r)$. Let Y_k be any of its spanning trees extending Y_{k-1}. Moreover, Y_k has diameter at most $2k+1$ by construction. $\square$

Lemma 3.3.19. *Let G be a graph with a fixed vertex r. The set Ω_k of those vertex ends dominated by some vertex in $B_k(r)$ is closed.*

Proof. In order to show that Ω_k is closed, we prove that its complement is open. For that it suffices to find for each ray R not dominated by some vertex in $B_k(r)$ some finite separator S_R disjoint from $B_k(r)$ that separates $B_k(r)$ from a tail of R.

Suppose for a contradiction that there is not such a finite separator S_R. Then we can recursively pick infinitely many $B_k(r)$-R-paths that are vertex-disjoint except possibly their starting vertices. Let U be the set of their starting vertices. The set U is infinite because otherwise some $u \in U$ would dominate R, which is impossible. By Lemma 3.3.18, $G[B_k(r)]$ has a rayless spanning tree Y_k. Applying the Star-Comb-Lemma [52, Lemma 8.2.2] to Y_k and U, we find a vertex v in $G[B_k(r)]$ together with an infinite fan whose endvertices are in U. Enlarging this fan by infinitely many of the previously chosen $B_k(r)$-R-paths, yields an infinite fan which witnesses that v dominates R, which is the desired contradiction. Thus there is such a finite set R_S for every ray R not dominated by some vertex in $B_k(r)$ and so Ω_k is closed. $\square$

Proof that Lemma 3.3.19 implies Lemma 3.3.17. By Lemma 3.3.19, the set of dominated vertex ends is a countable union of closed sets and thus Borel. $\square$

The next step in our proof of Theorem 3.2.4 is to give a more combinatorial description of the set C_Ψ defined in the Introduction. For a set A, we denote the set of minimal nonempty elements of A by A^{min}. Given a set Ψ of vertex ends of G, an edge set o is in $\mathcal{C}_\Psi$ if o meets every finitely coverable cut evenly and is geometrically connected. The next lemma implies that $C_\Psi = \mathcal{C}_\Psi^{min}$.

Lemma 3.3.20. *Given a Borel set Ψ of vertex ends of G, the following are equivalent for some nonempty edge set o.*

1. *$o \in \mathcal{C}_\Psi$;*
2. *o is the edge set of a sparse continuous function from S^1 to $|G|$ that only has vertex ends from Ψ in the closure;*
3. *o is the edge set of a sparse continuous function from S^1 to $|G| \setminus \Psi^\complement$.*

In particular, if o is minimal nonempty with one of these properties, then it is minimal nonempty with each of them. Furthermore o is minimal nonempty with one of these properties if and only if o is the edge set of a topological cycle in $|G| \setminus \Psi^\complement$.

Proof of Lemma 3.3.20. Clearly 2 and 3 are equivalent. And 1 and 2 are equivalent by Lemma 3.3.12. Thus 1,2 and 3 are equivalent.

To see the 'Furthermore'-part, first note that the edge set of a topological cycle in $|G| \setminus \Psi^\complement$ is a minimal nonempty edge set satisfying 3. To see the converse, let o be a minimal edge set which is the edge set of a sparse continuous function f from S^1 to $|G| \setminus \Psi^\complement$. Suppose for a contradiction that f is not injective. Then there are two distinct points $x, y \in S^1$ with $f(x) = f(y)$. By sparseness of f, there are two points z_1 and z_2 in different components of $S^1 - x - y$ whose f-values are different from $f(x)$. By Lemma 3.3.15 applied first to x and z_1 and second to x and z_2, for each of the two components C_1 and C_2 of $S^1 - x - y$ there is for each $i = 1, 2$ an edge e_i of G such that $e_i \times (0, 1)$ is included in $f(C_i)$.

We obtain the topological space K from $C_1 \cup \{x, y\} \subseteq S^1$ by identifying x and y. Note that K is homeomorphic to S^1. Moreover, the restriction $\bar{f}$ of f to $C_1 \cup \{x\}$ considered as a map from K to $|G|$ is continuous. However, the edge set of $\bar{f}$ is included in the edge set of f without e_2, violating the minimality of the edge set of f. Thus f is injective, and so o is the edge set of a topological cycle in $|G| \setminus \Psi^{\complement}$, completing the proof.

$\square$

Let $\mathcal{D}_\Psi$ be the set of cuts that do not have a vertex end of Ψ in their closure. Put another way, $d \in \mathcal{D}_\Psi$ if and only if d does not have a vertex end of Ψ in its closure and it meets every finite cycle evenly. Note that $D_\Psi = \mathcal{D}_\Psi^{min}$. The next step in our proof of Theorem 3.2.4 is to relate $\mathcal{C}_\Psi$ and $\mathcal{D}_\Psi$ to the sets of $\iota_T(\Psi)$-vectors of $\mathcal{T}$ and $\iota_T(\Psi)^{\complement}$-covectors of $\mathcal{T}$.

Lemma 3.3.21.

1. *The edge set of a finite cycle is an underlying vector of an $\emptyset$-pre-vector of $\mathcal{T}$;*
2. *Any finitely coverable bond is an underlying covector of an $\emptyset$-pre-covector of $\mathcal{T}$.*

Proof. In this proof we use the tree order $\leq$ on T as in Remark 3.3.2.

To see the second part, let d be a finitely coverable bond and let $V(G) = A \dot\cup B$ be a partition inducing d and let A' be a finite cover of d. Since G is connected, the partition is unique and both A and B are connected.

For $t \in V(T)$, let $x(t)$ be the set of crossing edges of the partition $V(P_t) = (A \cap V(P_t)) \dot\cup (B \cap V(P_t))$ in the torso H_t. Let S be the set of those nodes such that A and B both meet $V(P_t)$.

Our aim is to show that (S, x) is an $\emptyset$-pre-covector of $\mathcal{T}$, which then by construction has underlying set d. By construction, $x(t) \in \overline{W}(t)$. It remains to verify the followings sublemmas.

Sublemma 3.3.22. *S is connected. Moreover, for each $st \in E(S)$, $x(s)$ contains an edge of the torso H_t.*

Sublemma 3.3.23. *S is rayless.*

Proof of Sublemma 3.3.22. It suffices to show for each $st \in E(T)$ separating two vertices of S that $X = V(P_s) \cap V(P_t)$ contains vertices of both A and B. This follows from the fact that A and B are both connected and each has vertices in at least two components of $G - X$. $\square$

Proof of Sublemma 3.3.23. Suppose for a contradiction that S includes a ray $v_1 v_2 \ldots$. By taking a subray if necessary we may assume that $v_i < v_{i+1}$. As A' is finite, by the 'Moreover'-part of Remark 3.3.2 there is some m such that for all $w \geq v_m$ the part P_w does not contain vertices of A'. By Remark 3.3.2,

$X_i = \left(\bigcup_{w \geq v_{i+1}} V(P_w)\right) \setminus V(P_i)$ is connected. As $v_{m+2} \in S$, both A and B contain vertices of $P_{v_{m+2}} \subseteq X_m$. Thus X_m contains an edge of d, which is incident with a vertex of A'. This is a contradiction to the choice of m.

□

To see the first part, let o be the edge set of a finite cycle. We shall define for each node $t \in V(T)$ an edge set $x(t)$, which plays a similar role as in the last part. For that we need some preparation. Let $y(t) = o \cap E(P_t)$. Let $st \in E(T)$ with $s < t$. Let $Z(st)$ be the set of those vertices of $V(P_s) \cap V(P_t)$ incident with an odd number of edges of $y(t)$.

Sublemma 3.3.24. $|Z(st)|$ *is even.*

Proof. The set b of edges joining $V(P_s) \cap V(P_t)$ with $\left(\bigcup_{w \geq t} V(P_w)\right) \setminus V(P_s)$ is a cut. Thus o intersection b evenly. Since $b(st) \subseteq E(P_t)$ by Remark 3.3.2, the number $|Z(st)|$ has the same parity as $|o \cap b|$ and so is even. □

Thus there is a matching $M(st)$ of $Z(st)$ using only edges from $E(H_s) \cap E(H_t)$. We obtain $x(t)$ from $y(t)$ by adding all the sets $M(st)$ where s is a neighbour of t. Let S be the set of those nodes t where $x(t)$ is nonempty.

Our aim is to show that (S, x) is an $\emptyset$-pre-vector of $\mathcal{T}$, which then by construction has underlying set o. First note that S is finite as $y(t)$ is nonempty for only finitely many t. Thus it remains to verify the following sublemmas.

Sublemma 3.3.25. $x(t)$ *has even degree at each vertex of* H_t*.*

Sublemma 3.3.26. S *is connected. Moreover, for each* $st \in E(S)$*,* $x(s)$ *contains an edge of the torso* H_t*.*

Proof of Sublemma 3.3.25. By construction $x(t)$ has even degree at all vertices v in $V(H_t) \cap V(H_s)$, where $st \in E(T)$ with $s < t$. Hence if t is maximal in S, then $x(t)$ has even degree at all vertices of H_t. Otherwise the statement follows inductively from the statement for all the upper neighbours. Indeed, let $v \in V(H_t) \setminus V(H_s)$, where $st \in E(T)$ with $s < t$. Then the degree of v in $x(t)$ is congruent modulo 2 to the degree of v in o plus the sum of the degrees of v in $x(u)$, where the sum ranges over all upper neighbours u of t. □

Proof of Sublemma 3.3.26. It suffices to show for each $st \in E(T)$ separating two vertices of S that $M(st)$ is nonempty. Suppose for a contradiction that $M(st)$ is empty. Let T_s be the component of $T - t$ containing s. The symmetric difference D_s of all $x(u)$ with $u \in T_s$ contains only edges of o and has even degree at each vertex by Sublemma 3.3.25.

Moreover, T_s contains a vertex v of S. Either P_v contains an edge of o or it has a neighbour w such that $M(vw)$ is nonempty and P_w contains an edge of o. In the later case w is also in T_s. So in either case, D_s is nonempty.

Similarly, we define T_t and D_t, and deduce that D_t is nonempty. Since D_s and D_t are both nonempty, we deduce that o includes two edge disjoint cycles, which is the desired contradiction. □

□

Corollary 3.3.27. *Every $\Psi^{\complement}$-covector d of $\mathcal{T}$ is in $\mathcal{D}_\Psi$.*

Proof. First note that d has only vertex ends of $\Psi^{\complement}$ in its closure. Moreover d is a cut as it meets every finite cycle evenly by Lemma 3.3.21 and Lemma 3.3.10 as $\mathcal{T}$ is tree of binary finitary presentations. □

Let $\mathcal{F}_\Psi$ be the set of those edge sets o meeting every finitely coverable cut evenly such that for every finite vertex set W only finitely many components of $G - W$ contain vertices of $V(o)$. Note that $\mathcal{C}_\Psi \subseteq \mathcal{F}_\Psi$ by Lemma 3.3.12 and Corollary 3.3.13.

Lemma 3.3.28. *Any nonempty $o \in \mathcal{F}_\Psi$ includes a nonempty element of $\mathcal{C}_\Psi$. Hence, $\mathcal{F}_\Psi^{min} = \mathcal{C}_\Psi^{min}$.*

Proof. We say that edges e and f of o are in the *same geometric component* if o meets every finitely coverable cut d such that e and f are in different components of $G - d$. It is straightforward to check that being in the same geometric component is an equivalence relation. Pick some $e \in o$ and let u be its equivalence class. It suffices to show that u is in $\mathcal{C}_\Psi$, which is implies by the following two sublemmas.

Sublemma 3.3.29. *u is meets every finitely coverable cut evenly.*

Sublemma 3.3.30. *u is geometrically connected.*

Before proving these two sublemmas, we give a construction that is used in the proof of both these sublemmas. Let $x \in o$ and let b be a finitely coverable cut. For all $z \in b \cap (o \setminus u)$, there is a finitely coverable cut b_z such that x and z are in different components of $G - b_z$. Let $V(G) = A \dot\cup B$ be a partition inducing b, and let $V(G) = A_z \dot\cup B_z$ be a partition inducing b_z such that x has both its endvertices in A_z. Let d be the cut consisting of those edges with precisely one endvertex in the intersection of A and the finitely many A_z. Note that d is finitely coverable. By construction $d \cap u = d \cap o$. Moreover, $b \cap u = d \cap u$ since any $y \in u$ has both its endvertices in A_z.

Proof of Sublemma 3.3.29. Let b be a finitely coverable cut. Then $b \cap u = d \cap o$, and thus $b \cap u$ has even size. □

Proof of Sublemma 3.3.30. Let b be a finitely coverable cut such that there are edges x and y of u in different components of $G - b$. Thus there is a partition $V(G) = A \dot\cup B$ inducing b such that x has both endvertices in A and y has both endvertices in B. Then x and y are in different components of $G - d$. As x and y are in the same geometric component, d meets o. Thus b meets u, completing the proof. □

□

Lemma 3.3.31. *Every Ψ-vector o of $\mathcal{T}$ is in $\mathcal{F}_\Psi$.*

Proof. The set o meets every finitely coverable bond evenly by Lemma 3.3.21 and Lemma 3.3.10 as $\mathcal{T}$ is tree of binary finitary presentations. Since every finitely coverable cut is an edge-disjoint union of finitely many finitely coverable bonds, o meets each finitely coverable cut evenly.

The set o is a finite symmetric difference of sets o_i, which are underlying sets of Ψ-pre-vectors $(S_i, \overline{o}_i)$. Note that S_i is locally finite as each $\overline{o}_i$ is finite and for each $xy \in E(S_i)$, the set $\overline{o}_i(x)$ contains an edge of the torso of P_y. It suffices to show that there is no finite vertex set W together with an infinite set $\mathcal{A}$ of components of $G - W$ each containing a vertex of $V(o_i)$.

Suppose for a contradiction there is such a set W. By the 'Moreover'-part of Remark 3.3.2, there is a rayless subtree Q of T containing all nodes q such that its part P_q contains a vertex of W and the root r of T. For each $A \in \mathcal{A}$, there is an edge z_A in $o_i \cap s_A$. Let t_A be the unique node of T such that $z_A \in P_{t_A}$.

Next we define an edge e_A for each $A \in \mathcal{A}$. If $t_A \in Q$, we pick $e_A = z_A$. Otherwise, let q_A be the last node on the unique t_A-Q-path and u_A be the node before that. By Remark 3.3.2, P_{u_A} together with the parts above is connected. Thus all these parts are included in A. Thus the nodes u_A are distinct for different A. Moreover, q_A is on the path from t_A to some t_B for some other $b \in \mathcal{A}$. As S_i is connected and $t_A, t_B \in S_i$, it must be that $q_A \in S_i$. So u_A is in S_i, as well. Thus $\bar{o}_i(q_A)$ contains an edge of the torso of P_{u_A}. Pick such an edge for e_A. Summing up, we have picked for each $A \in \mathcal{A}$ an edge e_A in some $\overline{o}_i(q)$ with $q \in Q \cap S_i$ such that all these e_A are distinct.

Note that $S_i \cap Q$ is finite as S_i is locally finite and Q is rayless. Since each e_A is in some of the finite sets $\overline{o}_i(x)$ with $x \in S_i \cap Q$, we get the desired contradiction. $\square$

Theorem 3.3.32. *Let Ψ be a Borel set of vertex ends of an infinite connected graph G that are all undominated. Then there is a matroid M whose set of circuits is $\mathcal{C}_\Psi^{min}$ and whose set of cocircuits is $\mathcal{D}_\Psi^{min}$.*

Proof. By Lemma 3.3.17, $\iota_T(\Psi)$ is Borel. Thus we apply Theorem 3.3.9 to the tree of presentations $\mathcal{T}$, yielding that $\Pi_{\iota_T(\Psi)}(\mathcal{T})$ presents a matroid M. Note that $\mathcal{F}_\Psi$ and $\mathcal{D}_\Psi$ satisfy (01) by Lemma 3.3.14. Hence by Corollary 3.3.27 and Lemma 3.3.31, we can apply Lemma 3.4.9 to $\mathcal{F}_\Psi$ and $\mathcal{D}_\Psi$ and M. As $\mathcal{F}_\Psi^{min} = \mathcal{C}_\Psi^{min}$ by Lemma 3.3.28, we get the desired result. $\square$

Proof of Theorem 3.2.4. By considering distinct connected components separately, we may assume that G is connected. By Lemma 3.3.20, $\mathcal{C}_\Psi^{min}$ is the set of topological cycles in $|G| \setminus \Psi^{\complement}$. Thus Theorem 3.2.4 follows from Theorem 3.3.32. $\square$

3.3.3 Consequences of Theorem 3.2.4

First, we prove for any graph G that the set of topological circuits is the set of circuits of a matroid if and only if G does not have a subdivision of the dominated ladder H. This theorem was already mentioned in the Introduction, see Corollary 3.2.3. We start with a couple of preliminary lemmas.

Lemma 3.3.33. *Let ω be a dominated vertex end of a graph G such that there are two vertex-disjoint rays R and S belonging to ω. Then G has a subdivision of H.*

Proof. Let v be a vertex dominating ω. By taking subrays if necessary, we may assume that v lies on neither R nor S. As R and S belong to the same vertex end, there are infinitely many vertex-disjoint paths $P_1, P_2, \ldots$ from R to S. We may assume that no P_i contains v. Let r_i be the endvertex of P_i on R and s_i be the endvertex of P_i on S. By taking a subsequence of the P_i if necessary, we can ensure that the order in which the r_i appear on R is $r_1, r_2, \ldots$. Similarly, we may assume that the order in which the s_i appear on S is $s_1, s_2, \ldots$.

Let $Q_1, Q_2, \ldots$ be an infinite fan from v to $R \cup S$. So for one of R or S, say R, there is an infinite fan $Q'_1, Q'_2, \ldots$ from v to it that avoids the other ray. As each P_i and each Q'_j is finite, we can inductively construct infinite sets $I, J \subseteq \mathbb{N}$ such that for $i \in I$ and $j \in J$ the paths P_i and Q'_j are vertex-disjoint.

Indeed, just consider the bipartite graph with left hand side $(P_i | i \in \mathbb{N})$ and right hand side $(Q'_j | j \in \mathbb{N})$ and put an edge between two paths P_i and Q'_j if they share a vertex. Now we use that each vertex of this bipartite graph has only finitely many neighbours on the other side to construct an independent set of vertices that intersects both sides infinitely. Indeed, for each finite independent set, there are two vertices, one on the left and one on the right, such that the independent set together with these two vertices is still independent. So there is such an infinite independent set and I is its set of vertices on the left and J is its set of vertices on the right.

Finally, v together with R, S and $(P_i | i \in I)$ and $(Q'_j | j \in J)$ give rise to a subdivision of H, which completes the proof. □

Lemma 3.3.34. *Let o be a topological circuit that has the vertex end ω in its closure. Then there is a double ray both of whose tails belong to ω.*

This lemma already was proved in [21, Lemma 5.6] in a slightly more general context.

Proof of Corollary 3.2.3. If G has a subdivision of H, then as explained in the Introduction the topological set of topological circuits violates (C3).

Thus it remains to consider the case that G has no a subdivision of H. Now we apply Theorem 3.2.4 with Ψ the set of undominated vertex ends, which is Borel by Lemma 3.3.17.

It suffices to show that every topological circuit o of G is a Ψ-circuit. So let ω be a vertex end in the closure of o. Then by Lemma 3.3.34 there is a double ray both of whose tails belong to ω. If ω was not in Ψ, then G would have a subdivision of H by Lemma 3.3.33. Thus ω is in Ψ. As ω was arbitrary, this shows that every vertex end in the closure of o is in Ψ. □

Theorem 3.2.4 can also be used to extend a central result of [18] from countable graphs to graphs with a normal spanning tree as follows. Given a graph G with a normal spanning tree T, in [18] we constructed the Undomination graph

$U = U(G, T)$. This graph has the pleasant property that it has few enough edges to have no dominated vertex end but enough edges to have G as a minor. Moreover there is an inclusion $\tilde{u}$ from the set of vertex ends of G to the set of vertex ends of U. By Theorem 3.2.4, for every Borel set Ψ, the Ψ-circuits of $U(G, T)$ are the circuits of a matroid. Now we use the following theorem.

Theorem 3.3.35 ([18, Theorem 9.9])**.** *Assume that $(U, \tilde{u}(\Psi))$ induces a matroid M. Then (G, Ψ) induces the matroid M/C.*

We refer the reader to [18, Section 3] for a precise definition of when the pair (G, Ψ) consisting of a graph G and a vertex end set Ψ induces the matroid M. Very very roughly, this says that the set of certain 'topological circuits' which only use vertex ends from Ψ is the set of the circuits of M. However the topological space taken there is different from the one we take in this paper, so that the definition of topological circuit there does not match with the definition of topological circuit in this paper. For example, in this different notion a ray starting at a vertex v may also be a circuit if the vertex end it converges to is in Ψ and dominated by v. However these two notions of topological circuit are the same if no vertex is dominated by a vertex end. Thus combining Theorem 3.3.35 and Theorem 3.2.4, we get the following.

Corollary 3.3.36. *Let G be a graph with a normal spanning tree and $\Psi \subseteq \Omega(G)$ such that $\tilde{u}(\Psi)$ is Borel, then (G, Ψ) induces a matroid.*

For example, if we choose Ψ equal to the set of dominated vertex ends, then we get an interesting instance of this corollary: Like Theorem 3.2.4, this gives a recipe to associate a matroid (which we call $M_I(G)$) to every graph G that has a normal spanning tree which in general is neither finitary nor cofinitary. These two matroids need not be the same. For example, these two matroids differ for the graph obtained from the two side infinite ladder by adding a vertex so that it dominates precisely one of the two vertex ends.

In fact the circuits of the matroid $M_I(G)$ can be described topologically, namely they are the edge sets of topological cycles in the topological space ITOP, see [47] for a definition of ITOP. About ITOP, we shall only need the following fact, which is not difficult to prove: Given a graph G, we denote by G_I, the multigraph obtained from G by identifying any two vertices dominating the same vertex end. It is not difficult to show that G and G_I have the same topological cycles. Thus in order to study when the topological cycles of G induce a matroid, it is enough to study this question for the graphs G_I. In what follows, we show that the underlying simple graphs G_I' of G_I always has a normal spanning tree. This will imply the following:

Corollary 3.3.37. *The topological cycles of ITOP induce a matroid for every graph.*

Let H' be the graph obtained from the dominated ladder H by adding a clone of the infinite degree vertex of H. Note that G_I' has no subdivision of H'. Thus G_I' has a normal spanning tree due to the following criterion:

Theorem 3.3.38 (Halin [71])**.** *If G is connected and does not have a subdivision of the completes graph on countably many vertices, then G has a normal spanning tree.*

3.4 Matroids with all finite minors graphic

3.4.1 Introduction

There is a rich theory describing and employing the relationship between finite graphic matroids and finite graphs. In this paper, we will show how the foundations of this theory can be extended to infinite matroids [30]. A central result in the finite context is Tutte's characterisation by finitely many excluded minors of the class of matroids which can be represented by graphs [104].

Existing work with infinite graphic matroids has focused on a few possible constructions of matroids from infinite graphs, which generalise the construction of the cycle matroid of a finite graph. Most straightforwardly, for any infinite graph G we can consider the finite-cycle matroid, whose circuits are given by the finite cycles of G. We could also consider the algebraic-cycle matroid, whose circuits are given by finite cycles or double rays in G [76]. Alternatively, we can consider the topological cycle matroid, whose circuits are given by homeomorphic copies of the unit circle in the end-compactification of G [29]. Various ad-hoc extensions of these notions suggest themselves. For example, we could allow identification of ends with vertices in the definition of the topological cycle matroid [47].

Certain results about finite graphic matroids have been proved for these classes of infinite graphic matroids [26], [29], [31], [47], [99], and could also be proved about the ad-hoc extensions without too much trouble. But since all these notions fall far short of the natural boundary, namely the class of infinite matroids satisfying Tutte's excluded minor characterisation, in this paper we instead take the approach of isolating a notion of representation for which the representable matroids are precisely those satisfying Tutte's condition. Such matroids, and their representations, provide a natural context for the extension of results from finite to infinite graphic matroids.

That the existing approaches fall far short of providing representations of all graphic matroids is shown by examples like those depicted in Figure 3.2. Here the circuits of the matroids in question are again given by the (edge sets of) homeomorphic copies of the unit circle in the subspaces of the plane given in the pictures.

What these examples show is that infinite graphic matroids should, in general, be taken to be represented not by graphs but rather by graph-like topological spaces, in a sense akin to that of Thomassen and Vella [103]. This includes the existing approaches: the finite cycle matroid of a graph would be represented by its geometric realisation, the algebraic cycle matroid by a 1-point compactification and the topological cycle matroid by the end compactification.

We restrict our attention to tame matroids (those in which any intersection

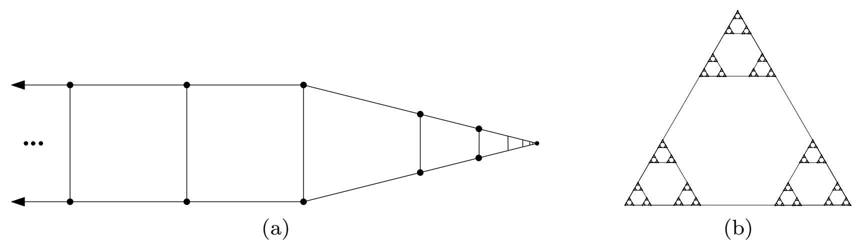

Figure 3.2: Subspaces of the plane inducing matroids

of a circuit with a cocircuit is finite) because this restriction has proved both natural and necessary in related representability problems [1], [17], [20]. We shall introduce a notion of representability of matroids over graph-like spaces for which we can prove the following:

Theorem 3.4.1. *A tame matroid satisfies Tutte's excluded minor characterisation if and only if it is representable over a graph-like space.*

We call matroids satisfying either of these equivalent conditions *graphic*.

At least for 3-connected matroids, the notion of representability is what you would hope: the circuits are given just as usual by homeomorphic copies of the unit circle. That this hope can be fulfilled is a little strange. After all, any circuit given in this way must be countable, and there is nothing in Tutte's excluded minor characterisation which appears to restrict the cardinality of circuits. We are saved by the following miraculous fact:

Theorem 3.4.2. *In any 3-connected tame matroid satisfying Tutte's excluded minor characterisation, all circuits are countable.*

In fact, in order to prove this we first introduce a notion of representability which doesn't entail any cardinality restrictions, then play the topological structure of the representing graph-like space off against the matroidal structure.

In an extended version of this work available at 'http://www.math.uni-hamburg.de/spag/dm/projects/matroids.html' and 'http://www.math.uni-hamburg.de/home/carmesin/', we show that the spaces in question are topologically well-behaved, and deduce essential desiderata, such as that the bases of the matroid correspond to minimal connected subspaces containing all vertices.

The structure of the paper is as follows. In Subsection 3.4.2, we recall some preliminary lemmas from the theory of infinite matroids. In Subsection 3.4.3 we introduce graph-like spaces and in Subsection 3.4.4 we introduce the subspaces which will play the role of topological circles. In Subsection 3.4.5 we introduce the notion of representation. In Subsection 3.4.6 we prove Theorem 3.4.1. In Subsection 3.4.7 we introduce a kind of forbidden substructure which we will make use of in our proof of Theorem 3.4.2 in Subsection 3.4.8. We conclude by discussing the notion of planarity for infinite matroids in Subsection 3.4.9.

3.4.2 Preliminaries

Throughout, notation and terminology for (infinite) graphs are those of [52], and for matroids those of [92, 30].

M always denotes a matroid and $E(M)$ (or just E), $\mathcal{I}(M)$ and $\mathcal{C}(M)$ denote its ground set and its sets of independent sets and circuits, respectively. For the remainder of this section we shall recall some basic facts about infinite matroids.

A set system $\mathcal{I} \subseteq \mathcal{P}(E)$ is the set of independent sets of a matroid if and only if it satisfies the following *independence axioms* [30].

(I1) $\varnothing \in \mathcal{I}(M)$.

(I2) $\mathcal{I}(M)$ is closed under taking subsets.

(I3) Whenever $I, I' \in \mathcal{I}(M)$ with I' maximal and I not maximal, there exists an $x \in I' \setminus I$ such that $I + x \in \mathcal{I}(M)$.

(IM) Whenever $I \subseteq X \subseteq E$ and $I \in \mathcal{I}(M)$, the set $\{I' \in \mathcal{I}(M) \mid I \subseteq I' \subseteq X\}$ has a maximal element.

A set system $\mathcal{C} \subseteq \mathcal{P}(E)$ is the set of circuits of a matroid if and only if it satisfies the following *circuit axioms* [30].

(C1) $\varnothing \notin \mathcal{C}$.

(C2) No element of $\mathcal{C}$ is a subset of another.

(C3) (Circuit elimination) Whenever $X \subseteq o \in \mathcal{C}(M)$ and $\{o_x \mid x \in X\} \subseteq \mathcal{C}(M)$ satisfies $x \in o_y \Leftrightarrow x = y$ for all $x, y \in X$, then for every $z \in o \setminus \left(\bigcup_{x \in X} o_x\right)$ there exists a $o' \in \mathcal{C}(M)$ such that $z \in o' \subseteq \left(o \cup \bigcup_{x \in X} o_x\right) \setminus X$.

(CM) $\mathcal{I}$ satisfies (IM), where $\mathcal{I}$ is the set of those subsets of E not including an element of $\mathcal{C}$.

For a base s of a matroid M, and $e \in E \setminus s$, there is a unique circuit o_e with $e \in o_e \subseteq s + e$. We call this circuit the *fundamental circuit* of e with respect to s. Similarly, for $f \in b$ we call the unique cocircuit b_f with $f \in b_f \subseteq (E \setminus s) + f$ the *fundamental cocircuit* of f with respect to s.

The following straightforward Lemmas can be proved as for finite matroids (see, for example, [17]).

Lemma 3.4.3. *Let M be a matroid and s be a base. Let o_e and b_f a fundamental circuit and a fundamental cocircuit with respect to s, then*

1. *$o_e \cap b_f$ is empty or $o_e \cap b_f = \{e, f\}$ and*
2. *$f \in o_e$ if and only if $e \in b_f$.*

Lemma 3.4.4. *For any circuit o containing two edges e and f, there is a cocircuit b such that $o \cap b = \{e, f\}$.*

Lemma 3.4.5. *Let I be some independent set in some matroid M. Then for each $e \in I$ there is a cocircuit b meeting I precisely in e*

Lemma 3.4.6. *Let M be a matroid with ground set $E = C\dot{\cup}X\dot{\cup}D$ and let o' be a circuit of $M' = M/C\backslash D$. Then there is an M-circuit o with $o' \subseteq o \subseteq o' \cup C$.*

Lemma 3.4.7. *Let M be a matroid, and let $w \subseteq E$. The following are equivalent:*

1. *w is a union of circuits of M.*
2. *w never meets a cocircuit of M just once.*

The basic theory of infinite binary matroids is introduced in [17]. One characterisation of such matroids given there is that every intersection of a circuit with a cocircuit is both finite and of even size.

Lemma 3.4.8. *Let M be a binary matroid and $X \subseteq E(M)$ with the property that it meets every circuit finitely and evenly. Then X is a disjoint union of cocircuits.*

Proof. By Zorn's Lemma, we can pick $Y \subseteq X$ maximal with the property that it is a disjoint union of cocircuits. As $Y \subseteq X$, the set Y meets every circuit finitely, and so meets every circuit evenly. By the choice of Y, the set $X \setminus Y$ does not include a circuit. But $X \setminus Y$ meets every circuit evenly, and so is empty by the dual of Lemma 3.4.7. This completes the proof. $\square$

Lemma 3.4.9. *Suppose that M is a matroid, and $\mathcal{C}$, $\mathcal{C}^*$ are collections of subsets of $E(M)$ such that $\mathcal{C}$ contains every circuit of M, $\mathcal{C}^*$ contains every cocircuit of M, and for every $o \in \mathcal{C}$, $b \in \mathcal{C}^*$, $|o \cap b| \neq 1$. Then the set of minimal nonempty elements of $\mathcal{C}$ is the set of circuits of M and the set of minimal nonempty elements of $\mathcal{C}^*$ is the set of cocircuits of M.*

Proof. The conditions imply that no element of $\mathcal{C}$ ever meets a cocircuit of M just once, so every element of $\mathcal{C}$ is a union of circuits of M by Lemma 3.4.7. Since every circuit of M is in $\mathcal{C}$, the minimal nonempty elements of $\mathcal{C}$ are precisely the circuits of M. The other claim is obtained by a dual argument. $\square$

A *switching sequence* for a base s in a matroid with ground set E is a finite sequence $(e_i|1 \leq i \leq n)$ whose terms are alternately in s and not in s and where for $i < n$ if $e_i \in s$ then $e_{i+1} \in b_{e_i}$ and if $e_i \notin s$ then $e_{i+1} \in o_{e_i}$.

Lemma 3.4.10. *Let M be a connected matroid with a base s, and e and f be edges of M. Then there is a switching sequence with first term e and last term f.*

Proof. Let e be any edge of M, and let X be the set of those $f \in E(M)$ for which there is such a switching sequence. Then $s \cap X$ is a base for X, since for any $f \in X \setminus s$ we have $o_f \subseteq X$. Similarly, $s \setminus X$ is a base for $E(M) \setminus X$, since for any $f \in E(M) \setminus X \setminus s$ and any $g \in o_f$ we have $f \in b_g$ by Lemma 3.4.3 and so $g \notin X$. Thus X and $E(M) \setminus X$ form a separation of M, and since M is connected this means that X must be the whole of E, completing the proof. $\square$

A k-separation of a matroid M is a partition (A, B) of the ground set of M such that each of A and B has size at least k and there are bases s_A and s_B of A and B and s of A such that $|s_A \cup s_B \setminus s| < k$. A 1-separation may also be called a *separation*. A matroid without l-separations for any $l < k$ is *k-connected*. A matroid is *connected* if it is 2-connected. Connected matroids can equivalently be characterised as those in which any 2 distinct edges lie on a common circuit [32].

3.4.3 Graph-like spaces

The key notion of this section is the following, which is based on a definition from [103]:

Definition 3.4.11. A *graph-like space* G is a topological space (also denoted G) together with a *vertex set* $V = V(G)$, an *edge set* $E = E(G)$ and for each $e \in E$ a continuous map $\iota_e^G \colon [0,1] \to G$ (the superscript may be omitted if G is clear from the context) such that:

- The underlying set of G is $V \sqcup [(0,1) \times E]$
- For any $x \in (0,1)$ and $e \in E$ we have $\iota_e(x) = (x, e)$.
- $\iota_e(0)$ and $\iota_e(1)$ are vertices (called the *endvertices* of e).
- $\iota_e\restriction_{(0,1)}$ is an open map.
- For any two distinct $v, v' \in V$, there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$.

The inner points of the edge e are the elements of $(0,1) \times \{e\}$.

Note that $V(G)$, considered as a subspace of G, is totally disconnected, and that G is Hausdorff.

Let e be an edge in a graph-like space with $\iota_e(0) \neq \iota_e(1)$. Then ι_e is a continuous injective map from a compact to a Hausdorff space and so it is a homeomorphism onto its image. The image is compact and so is closed, and therefore is the closure of $(0,1) \times \{e\}$ in G. So in this case ι_e is determined by the topology of G. The same is true if $\iota_e(0) = \iota_e(1)$: in this case we can lift ι_e to a continuous map from $S^1 = [0,1]/(0 = 1)$ to G, and argue as above that this map is a homeomorphism onto the closure of $(0,1) \times \{e\}$ in G. In this case, we say that e is a loop of G.

Next we shall define maps of graph-like spaces. Let G and G' be graph-like spaces. Two maps $\varphi_V : V(G) \to V(G')$ and $\varphi_E : E(G) \to (E(G') \times \{+, -\}) \sqcup V(G)$ induce a function φ sending points of G to points of G' as follows: a vertex v of G is mapped to $\varphi_V(v)$. Let e be an edge, and (r, e) one of its interior points. If $\varphi_E(e)$ is a vertex, then (r, e) is mapped to $\varphi_E(e)$. If $\varphi_E(e) = (f, +)$ for some $f \in E(G')$, then (r, e) is mapped to (r, f). Similarly, if $\varphi_E(e) = (f, -)$ for some $f \in E(G')$, then (r, e) is mapped to $(1-r, f)$. If a function arising in this way is continuous we call it a *map of graph-like spaces*. From this definition, it follows that if v is an endvertex of e, then $\varphi(v)$ is either an endvertex of or equal to the image of e.

Let us consider some examples of graph-like spaces. We shall write $[0, 1]$ for the unique graph-like space without loops having precisely one edge and two vertices. There are exactly seven maps of graph-like spaces from $[0, 1]$ to two copies of $[0, 1]$ glued together at a vertex: four of these have one of the copies of $[0, 1]$ as their image and the other three map the whole interval to a vertex. However, none of these maps is bijective nor has an inverse, even though the underlying topological spaces are homeomorphic.

Figures 3.2a and 3.2b from the introduction define graph-like spaces with vertices and edges as in the figures. In each case the topology is that induced by the embedding in the plane suggested by the figures. For a locally finite graph $G = (V, E)$, the topological space $|G|$ is a graph-like space with vertex set $V \cup \Omega(G)$ and edge set E (see [52] for the definition of $|G|$). Note that if G is finite, then $|G|$ is homeomorphic to the geometric realisation of G considered as a simplicial complex.

Lemma 3.4.12. *Let G be a graph-like space with only finitely many edges and finitely many vertices. Then G is homeomorphic to $|H|$ for some finite graph H.*

Proof. G is compact, since it is a union of finitely many compact subspaces. Let H be the graph with edge set $E(G)$ and vertex set $V(G)$, and in which v is an endpoint of e if and only if this is true in G. We now construct a map $\varphi\colon G \to |H|$ as follows: taking φ_V to be the identity and φ_E to be the function sending each edge e to $(e, +)$, we build φ as in the definition of a map of graph-like spaces.

It remains to show that the function φ is continuous: since it is a bijection from a compact to a Hausdorff space, it will then be a homeomorphism. We begin by noting that for any $e \in E(G)$, the restriction of φ to the image of ι_e^G is a homeomorphism, by the remarks following Definition 3.4.11. Now we need to show for any $x \in |H|$ that the inverse image of any open neighbourhood U of $\varphi(x)$ includes an open neighbourhood of x. If x is an interior point of an edge, this is clear. Otherwise, x is a vertex of $|H|$. Then there is an open neighbourhood $U' \subseteq U$ of x which only meets edges incident with x. For each such edge e, since the restriction of φ to the image of ι_e^G is a homeomorphism, there is an open set V_e of G with $V_e \cap \mathrm{Im}(\iota_e^G) = \varphi^{-1}(U') \cap \mathrm{Im}(\iota_e^G)$. Letting V be the intersection of the V_e, we obtain that V is an open neighbourhood of x included in $\varphi^{-1}(U)$, completing the proof that φ is continuous. $\square$

All the above examples of graph-like spaces will turn out to induce matroids. Before we can make this more explicit, we must first introduce the notions of topological circuits and bonds in a graph-like space. The discussion of topological circuits will be delayed until the next section, but we will introduce topological bonds now.

Definition 3.4.13. Given a pair of disjoint open subsets of a graph-like space G partitioning the vertices, we call the set of those edges having an endvertex in both sets *a topological cut* of G. A *topological bond* of G is a minimal nonempty topological cut of G.

Given a graph-like space G and a set of edges $R \subseteq E(G)$, we define the graph-like space $G\restriction_R$, the *restriction* of G to R, to have the same vertex set as G and edge set R. Then the ground set of $G\restriction_R$ is a subset of that of G, and we give it the subspace topology. Evidently, for any topological cut b of G, $b \cap R$ is a topological cut of $G\restriction_R$. The *deletion* of D from G, denoted by $G\backslash D$, is $G\restriction_{(E\backslash D)}$. We abbreviate $G\backslash\{e\}$ by $G - e$. The inclusion map g_D from $G\backslash D$ to G is a map of graph-like spaces.

Note that $G\restriction R$ has the same vertex set as G, even though only the vertices in the closure of $(0, 1) \times R$ play an important role in the new space. By analogy to the notation of [52], we also introduce a notation for the graph-like space whose edges are those in R but whose vertices are those in the closure of $(0, 1) \times R$. We will call this subspace the *standard subspace with edge set* R, and denote it $\overline{R}$.

Given a graph-like space G and $C \subseteq E(G)$, we define the *contraction* G/C of G onto C as follows:

Let $\equiv_C$ be the relation on the vertices of G defined by $u \equiv_C v$ if every topological cut with u and v in different parts meets C. It is easy to check that $\equiv_C$ is an equivalence relation. The vertex set of G/C is the set of $\equiv_C$-equivalence classes, and the edge set is $E(G) \setminus C$.

It remains to define the topology of G/C. We shall obtain this as the quotient topology derived from a function $f_C : G \to G/C$, to be defined next.

The function f_C sends each vertex to its $\equiv_C$-equivalence class and is bijective on the interior points of edges of $E \setminus C$. The two endpoints of an edge in C are in the same equivalence class, and we send all of its interior points to that equivalence class.

Taking this quotient topology ensures that G/C is a graph-like space, and makes f_C a map of graph-like spaces. In G/C, the endpoints of an edge are the equivalence classes of its endpoints in G. For any topological cut b of G with $b \cap C = \emptyset$, the two sides of b are closed under $\equiv_C$ by definition, and so b is also a topological cut in G/C.

We define $G.X := G/(E \setminus X)$ and $G/e := G/\{e\}$. It is straightforward to check for disjoint sets C and D that $(G\backslash D)/C$ and $(G/C)\backslash D$ are equal and the following diagram commutes.

$$\begin{tikzcd} G\backslash D \arrow[r, "g_D"] \arrow[d, "f_C"'] & G \arrow[d, "f_C"] \\ G/C\backslash D \arrow[r, "g_D"'] & G/C \end{tikzcd}$$

Contraction behaves especially well when applied to one side of a topological cut [22].

3.4.4 Pseudoarcs and Pseudocircles

When investigating a topological space, it is common to consider arcs in that space, that is, continuous injections from the unit interval to that space. We must consider maps from a slightly more general kind of domain. These domains, which we will call *pseudo-lines*, will be graph-like spaces built from total orders in the following way:

Definition 3.4.14. Let P be a totally ordered set. To construct the pseudo-line $L(P)$, we take as our vertex set V the set of initial segments of P, and as our edge set P itself. Next, we take a subbasis of the topology to consist of the sets of the type $S(p,r)^+$ or $S(p,r)^-$ defined below.

For every $p \in P$ and $r \in (0,1)$, let $S(p,r)^-$ contain precisely those vertices which do not contain p. Furthermore, let $S(p,r)^-$ contain all interior points of edges x with $x < p$ together with $(0,r) \times \{p\}$.

Similarly, let $S(p,r)^+$ contain precisely those vertices which contain p. Furthermore, let $S(p,r)^+$ contain all interior points of edges x with $x > p$ together with $(r,1) \times \{p\}$.

A *pseudo-path from v to w in a graph-like space G* is a map φ of graph-like spaces from a pseudo-line $L(P)$ to G with $\varphi(\emptyset) = v$ and $\varphi(P) = w$. The vertex v is called the *start-vertex* of the pseudo-path, and w is called the *end-vertex*.

A *pseudo-arc* is an injective pseudo-path. Any pseudo-arc is a homeomorphism onto its image since the domain is (as we shall soon show) compact, and the codomain is Hausdorff. Thus we will also refer to the images of pseudo-arcs as pseudo-arcs. In particular, a *pseudo-arc in* a graph-like space G is the image of such a map (in other words, it is a subspace of G which is also a pseudo-line).

Lemma 3.4.15. *The spaces $L(P)$ defined above are connected and compact.*

Proof. For the connectedness, let U be an open and closed set containing the start-vertex $\emptyset$. Since for any edge e the subspace topology of $\iota_e([0,1])$ is that of $[0,1]$, which is connected, the set $\iota_e([0,1])$ is either completely included in U or disjoint from U. Let $v = \{p \in P | S(p,1/2)^- \subseteq U\}$. Then the vertex v is in U since any neighbourhood of it meets U (even if $v = \emptyset$). So since U is open, it includes an open neighbourhood O of v. Since by our earlier remarks U includes all edges $p \in v$ and so also all vertices $w \subseteq v$, we may assume without loss of generality that either $v = P$ or else O has the form $S(p,r)^-$ for some $p \notin v$. In the second case we conclude that $p \in v$, which is impossible. Hence $v = P$. Since the closure of $\bigcup_{p \in P} \iota_p((0,1))$ is the whole of $L(P)$, the closed set U is the whole of $L(P)$. Hence $L(P)$ is connected, as desired.

It remains to show that $L(P)$ is compact. By Alexander's theorem, it suffices to check that any open cover by subbasic open elements has a finite subcover. Let $L(P) = \bigcup_{i \in I^+} S(p_i,r_i)^+ \cup \bigcup_{i \in I^-} S(p_i,r_i)^-$ be an open cover by subbasic open sets. Let $v = \{p \in P | \exists i \in I^- : p < p_i\}$.

First we consider the case where there is some $i \in I^+$ with $v \in S(p_i,r_i)^+$. Then $p_i \in v$, so there is some $j \in I^-$ such that $p_i < p_j$. This means that $S(p_i,r_i)^+$ and $S(p_j,r_j)^-$ cover $L(P)$.

Otherwise there is some $i \in I^-$ with $v \in S(p_i, r_i)^-$. Then $p_i \notin v$ and so p_i is maximal amongst the p_j with $j \in I^-$. Thus $v + p_i$ is contained in some $S(p_k, r_k)^+$ with $k \in I^+$. Then $S(p_i, r_i)^-$ and $S(p_k, r_k)^+$, together with some finite collection of sets from our cover covering the compact subspace $\iota_{p_i}([0,1])$, form a finite subcover, completing the proof. $\square$

Example 3.4.16. If $P = \omega_1$, then $L(P)$ is the *long line*, which is not homeomorphic to $[0,1]$.

Remark 3.4.17. *Any nontrivial pseudo-line is the closure of the set of interior points of its edges. Any nontrivial pseudo-arc in a graph-like space is the standard subspace corresponding to its set of edges.*

Remark 3.4.18. *Contracting a set of edges of a pseudo-line $L(P)$ corresponds to removing that set of edges from the associated poset P.*

Corollary 3.4.19. *Any contraction of a pseudo-line is a pseudo-line.* $\square$

Lemma 3.4.20. *Any nontrivial pseudo-line $L(P)$ with only countably many edges is homeomorphic to the unit interval.*

Proof. Let $\bar{\mathbb{Q}} = \mathbb{Q} \cap (0,1)$. Consider the lexicographic linear order on $P \times \bar{\mathbb{Q}}$. This is dense, countable and has neither a largest nor a smallest element. Since the theory of such linear orders is countably categorical, this order is isomorphic to the order of $\bar{\mathbb{Q}}$. Pick an isomorphism ϕ from $P \times \bar{\mathbb{Q}}$ to $\bar{\mathbb{Q}}$.

For any $x \in [0,1]$ such that there are $p \in P$ and $q, r \in \bar{\mathbb{Q}}$ with $\phi(p,q) < x < \phi(p,r)$ we set $f(x) = (p, \sup\{q \in \bar{\mathbb{Q}} | \phi(p,q) < x\})$ (in such cases, p is clearly uniquely determined). Otherwise we set $f(x) = \{p \in P | (\forall q \in \bar{\mathbb{Q}}) \phi(p,q) < x\}$. This gives an injection f from $[0,1]$ to $L(P)$. It is continuous by the definition of the topology on $L(P)$, and so is a homeomorphism since $[0,1]$ is compact and $L(P)$ is Hausdorff. $\square$

Lemma 3.4.21. *Let $s_1 <_L \ldots <_L s_n$ be finitely many edges of a pseudo-line L. Let $S = \bigcup_{i=1}^n \iota_{s_i}((0,1))$. Then $L \setminus S$ has $n+1$ components each of which is a pseudo-line. These are $S(s_1, 1/2)^- \setminus S$, and $S(s_{i+1}, 1/2)^- \cap S(s_i, 1/2)^+) \setminus S$ for $1 \leq i \leq n-1$ and $S(s_n, 1/2)^+ \setminus S$.*

Proof. The assertion follows by induction from the following. Let $e \in L$. Then $L - e$ has two components that are both pseudo-arcs. These are $S(e, 1/2)^- \setminus ((0,1) \times \{e\})$ and $S(e, 1/2)^+ \setminus ((0,1) \times \{e\})$. $\square$

We get a total order $\leqq$ on the set of points of the space $L(P)$ as follows: if v and w are vertices, we set $v \leqq w$ when $v \subseteq w$. If v is a vertex and (p,q) an interior point of an edge, we set $v \leqq (p,q)$ when $p \notin v$ and $(p,q) \leqq v$ when $p \in v$. Finally, we order the interior points of edges by the lexicographic order on $P \times (0,1)$.

Lemma 3.4.22. *Let X be a nonempty closed subset of a pseudo-line $L(P)$. Then X contains a $\leqq$-smallest and a $\leqq$-biggest element.*

Proof. First we show that X contains a $\leqq$-biggest element.

Let $v = \{p \in P | (\exists x \in X)(\exists r \in (0,1))(p,r) \leqq x\}$. If $v \in X$ then it is evidently the $\leqq$-biggest element of X. Otherwise, since X is closed, there must be some basic open set containing v but avoiding X. Without loss of generality this set is of the form $S(e,r)^+$. Then $e \in v$, and so there must be some $r' \in (0,1)$ with $(e,r') \in X$. Since X is closed there is a maximal such r'. Then (e,r') is the maximal element of X.

The proof that X contains a $\leqq_L$-smallest element is analogous. □

The *concatenation of two pseudo-lines L and M* is obtained from the disjoint union of L and M by identifying the end-vertex of L with the start-vertex of M.

Remark 3.4.23. *The concatenation of two pseudo-lines is a pseudo-line.* □

Remark 3.4.24. *Taking the concatenation of 2 pseudo-lines corresponds to taking the disjoint union of the two corresponding posets, where in the new ordering we take all elements of the second poset to be greater than all elements of the first.*

Let $P : L \to G$ and $Q : M \to G$ be two pseudo-arcs such that the end-vertex t_P of P is the start-vertex s_Q of Q. Then *their concatenation* is the function $f : (L \sqcup M)/(t_P = s_Q) \to G$ which restricted to L is just P and restricted to M is just Q. For a pseudo-arc $Q : M \to G$ and vertices x and y in the image of Q, we write xQy for the restriction of Q to those points of M that are both $\leqq_L$-bigger than $Q^{-1}(x)$ and $\leqq_L$-smaller than $Q^{-1}(y)$. Note that xQy is a pseudo-arc from x to y. If Q is a pseudo-arc from v to w and x and y are vertices in the image of Q, we abbreviate xQw by xQ and vQy by Qy.

Lemma 3.4.25. *Let $P : L \to G$ be a pseudo-arc from x to y and $Q : M \to G$ be a pseudo-arc from y to z. Then the concatenation of P and Q includes a pseudo-arc from x to z*

The corresponding Lemma about arcs needs the requirement that $x \neq z$. However, we avoid this requirement because there is a pseudo-line whose start- and end-vertex are equal, namely the trivial pseudo-line.

Proof. Let I be the intersection of the image of P with the image of Q, which is closed, being the intersection of two closed sets. Then $P^{-1}(I)$ is closed as P is continuous, and contains a $\leqq_L$-minimal element w by Lemma 3.4.22.

If w is not a vertex, then $P(w)$ is not a vertex and thus is contained in $\iota_e((0,1))$ for some edge e. Since P and Q both contain the whole of $\iota_e([0,1])$ if they contain some point from $\iota_e((0,1))$, the same is true for I. But then $\iota_e([0,1]) \subseteq I$, which contradicts the choice of w. Hence w is a vertex. Let $w' = P(w)$

Thus $w'Q$ is a pseudo-arc. By Remark 3.4.23, the concatenation of Pw' and $w'Q$ is the desired pseudo-arc since their images meet precisely in w'. □

A *pseudo-circle* is a graph-like space obtained by identifying the end-vertices of a nontrivial pseudo line.

We have the following relation between pseudo-lines and pseudo-circles. Every pseudo-circle C with one edge removed is a pseudo-line with endvertices the endvertices of the removed edge.

Conversely, let P and Q be pseudo-lines where P has endvertices s_P and t_P and Q has endvertices s_Q and t_Q. Then the graph-like space obtained from the disjoint union of P and Q by identifying s_P with t_Q and t_P with s_Q is a pseudo-circle or else is the trivial graph-like space.

So from Corollary 3.4.19 we obtain the following:

Corollary 3.4.26. *Any contraction of a pseudo-circle in which not all edges are contracted is a pseudo-circle.* □

Using Lemma 3.4.20 we get:

Corollary 3.4.27. *Any countable pseudo-circle is homeomorphic to S^1.* □

Definition 3.4.28. A *cyclic order on a set X* is a relation $R \subseteq X^3$, written $[a, b, c]_R$, that satisfies the following axioms:

1. Cyclicity: If $[a, b, c]_R$ then $[b, c, a]_R$.
2. Asymmetry: If $[a, b, c]_R$ then not $[c, b, a]_R$.
3. Transitivity: If $[a, b, c]_R$ and $[a, c, d]_R$ then $[a, b, d]_R$.
4. Totality: If a, b, and c are distinct, then either $[a, b, c]_R$ or $[c, b, a]_R$.

Remark 3.4.29. *The edge set of a pseudo-circle C has a canonical cyclic order R_C (up to choosing an orientation). Conversely, for any nonempty cyclic order there exists a pseudo-circle (unique up to isomorphism) such that its edge set has the same cyclic order.* □

We also get a cyclic order R'_C on the set of all points of a pseudo-circle C, corresponding to the order $\leqq$ on the set of points of a pseudo-line. Once more there are two canonical choices of cyclic order on C, one for each orientation of C; in fact, we shall take this as our definition of an orientation of C. For us, an orientation of a pseudo-circle C is a choice of one of the two canonical cyclic orders of the points of C.

Let $s \subseteq o$ and let $R \subseteq o^3$ be a cyclic order. The *cyclic order of s inherited from R* is R restricted to s^3. We say that e, g are *clockwise adjacent* in the cyclic order R if $[e, g, f]_R$ for any other f in o. In a finite cyclic order, for each e there is a unique g clockwise adjacent to e, which we denote by $n(e)$.

From Lemma 3.4.21 we obtain the following.

Corollary 3.4.30. *Let s be a finite nonempty set of edges of a pseudo-circle C. Let $S = \bigcup_{e \in s} \iota_e((0, 1))$. Then $L \setminus S$ has $|s|$ components each of which is a pseudo-line.*

For each such component there is a unique $e \in s$ such that the component contains precisely those edges f with $[e, f, n(e)]_{R_C}$, where $n(e)$ is taken with respect to the induced cyclic order on s. □

For a graph-like space G, we also use the term pseudo-circle to describe an injective map of graph-like spaces from a pseudo-circle to G, as well as the image of such a map. In particular, a *pseudo-circle in G* is the image of such a map (or, in other words, it is a subspace of G which is also a pseudo-circle). If G is a graph-like space and C is a pseudo-circle in G, the set of edges of C is called a *topological circuit* of G. Thus the pseudo-circles in G are precisely the standard subspaces of G corresponding to the topological circuits.

Lemma 3.4.31. *The intersection of a topological circuit with a topological cut is never only one edge.*

Proof. Suppose for a contradiction that there are a topological circuit o and a topological cut b that intersect in only one edge f. In the graph-like space $\overline{o}$, the set $b \cap o$ is a topological cut consisting of a single edge f. This contradicts the fact that removing any edge does not disconnect the pseudo-circle $\overline{o}$, which completes the proof. □

We can also show that the intersection of topological circuits with topological cuts is finite. In fact, we can prove something a little more general.

Lemma 3.4.32. *Let o be a set of edges in a graph-like space G such that $\overline{o}$ is compact. The the intersection of o with any topological cut b is finite.*

Proof. Let b be induced by the open sets U and U'. The sets $U \cap \overline{o}$ and $U' \cap \overline{o}$, together with all the sets $(0,1) \times \{e\}$ with $e \in o$, comprise an open cover of $\overline{o}$. So there is a finite subcover, which can only contain $(0,1) \times \{e\}$ for finitely many edges e. For any other edge f of o we must have $(0,1) \times \{f\} \subseteq U \cup U'$, and it must be a subset either of U or of V since it is connected: in particular, no such f can be in b. □

3.4.5 Graph-like spaces inducing matroids

In this section we will explain what it means for a graph-like space to induce a matroid and prove some fundamental facts about graph-like spaces inducing matroids which we will need in Subsection 3.4.6 and Subsection 3.4.8.

If for a graph-like space G there is a matroid M on $E(G)$ whose circuits are precisely the topological circuits of G and whose cocircuits are precisely the topological bonds of G, then we say that G *induces* M, and we may denote M by $M(G)$. Note that there can only be one such matroid since a matroid is uniquely defined by its set of circuits.

Example 3.4.33. For any finitely separable graph G the space $|G|$ induces the topological cycle matroid $M_C(G)$. The one-point compactification of a locally finite graph G induces the algebraic cycle matroid $M_A(G)$; if G is not locally finite and does not include a subdivision of the Bean graph, a similar construction can be used to construct a noncompact graph-like space that induces $M_A(G)$. Finally, the geometric realisation of G induces the finite cycle matroid $M_{FC}(G)$.

Lemma 3.4.34. *Let G be a graph-like space, and suppose G induces a matroid M. Then for any $C, D \subseteq E(M)$, the graph-like space $G/C\backslash D$ induces $M/C\backslash D$.*

Proof. Let $\mathcal{C}$ and $\mathcal{C}^*$ be respectively the collection of topological circuits and the collection of topological cuts of $G/C\backslash D$. We will show that every circuit of $M/C\backslash D$ is in $\mathcal{C}$, and that every cocircuit of $M/C\backslash D$ is in $\mathcal{C}^*$. Lemma 3.4.31 states that for every $o \in \mathcal{C}$, $b \in \mathcal{C}^*$, $|o \cap b| \neq 1$, so it will follow by Lemma 3.4.9 that the topological circuits of $G/C\backslash D$ are the circuits of $M/C\backslash D$ and that the minimal topological cuts (i.e. the topological bonds) of $G/C\backslash D$ are the cocircuits of $M/C\backslash D$, completing the proof.

Let o be a circuit of $M/C\backslash D$. By Lemma 3.4.6 there is a circuit o' of M such that $o \subseteq o' \subseteq o \cup C$. Since o' is a circuit of M, there is a pseudo-circle O in G with edge-set o'. Let $f_C : G \to G/C$ be as in the definition of the contraction G/C. Then $f_C\restriction_O$ is a map of graph-like spaces from O to a subspace of $G/C\backslash D$ that has edge-set o. If it describes a contraction of $O \cap C$, then Lemma 3.4.26 implies that o is a circuit of $G/C\backslash D$ as required. Otherwise, some vertex of $G/C\backslash D$ must contain two vertices p and q of O such that their deletion from the pseudo-circle O leaves two elements e and f of o in different components of $O - p - q$. Then by Lemma 3.4.4 there is a cocircuit b of $M/C\backslash D$ with $o \cap b = \{e, f\}$. Using the dual of Lemma 3.4.6, there is a cocircuit b' of M with $b \subseteq b' \subseteq b \cup D$, so that $o' \cap b' = \{e, f\}$. b' is a topological bond of G not meeting C and with p and q on opposite sides, contradicting the assumption that they are identified when we contract C.

Let b be a cocircuit of $M/C\backslash D$. It follows by the dual of Lemma 3.4.6 that there is a cocircuit b' of M (hence also a topological cut of G) such that $b \subseteq b' \subseteq b \cup D$. Let U, V be the disjoint open sets in G that partition $V(G)$ so that the set of edges with an end in each of U and V is b'. Let $f_C : G \mapsto G/C$ be the map of graph-like spaces describing the contraction of C from G. Since b' is disjoint from C, f_C does not identify any element of U with any element of V. Thus $f_C(U), f_C(V)$ are open sets in $G/C\backslash D$, and b is the set of edges with an end in each, showing that b is a topological cut of $G/C\backslash D$, as required.

$\square$

Proposition 3.4.35. *Let G be a graph-like space inducing a connected matroid M with a base s. Then for any edges e and f of M, and any endvertices v of e and w of f, there is a unique pseudo-arc from v to w that uses only edges in s.*

Proof. By Lemma 3.4.10, we can find a switching sequence $(e_i | 1 \leq i \leq n)$ for s with first term e and last term f. Pick a sequence $(v_i | 1 \leq i \leq n)$, with first term v and last term w, where for each i the vertex v_i is an endvertex of e_i. Then for any $i < n$ we can find a pseudo-arc from v_i to v_{i+1} using only edges of s: if $e_i \in s$ then we take an interval of the pseudo-arc $\overline{o_{e_{i+1}}} \setminus e_{i+1}$, and if $e_i \notin s$ then we take an interval of the pseudo-arc $\overline{o_{e_i}} \setminus e_i$. Repeatedly applying Lemma 3.4.25 we find the desired pseudo-arc from v to w.

To show uniqueness, we suppose for a contradiction that there are 2 distinct such pseudo-arcs R_1 and R_2. Then without loss of generality there is an edge e_0 in $R_1 \setminus R_2$.

Let $a \in R_1 \cap R_2$ be the $\leqq_{R_1}$-smallest point that is still $\leqq_{R_1}$-bigger than any point on e_0; such a point exists as the intersection of the two pseudo-arcs is closed. Similarly, let $b \in R_1 \cap R_2$ be the $\leqq_{R_1}$-biggest point that is still $\leqq_{R_1}$-smaller than any point on e_0. Then aR_1b and bR_2a are internally disjoint. Therefore aR_1bR_2a is a pseudo-circle all of whose edges are in s, a contradiction. □

Remark 3.4.36. *The proof of uniqueness above does not make use of the assumption that v and w are endvertices of edges.*

Let us call the pseudo-arc whose uniqueness is noted above vsw by analogy to the special case where s is a pseudo-arc. Next, we give a precise description of vsw.

Proposition 3.4.37. *The pseudo-arc vsw contains precisely those edges of s whose fundamental cocircuit with respect to s separates v from w. Its linear order is given by $e \leq f$ if and only if e lies on the same side as v of the fundamental cocircuit b_f of f.*

Proof. Let R be the pseudo-arc from v to w using edges in s only. Since R is connected, it must contain all edges whose fundamental cocircuit with respect to s separates v from w.

On the other hand let e be an edge on R. Let z_1 and z_2 be the endvertices of e, with $z_1 \leqq_R z_2$. Then by the above we can join v to z_1 by the pseudo-arc vRz_1 and w to z_2 by the pseudo-arc wRz_2. In G with the fundamental cocircuit of e removed, z_1 and z_2 lie on different sides, which we will call A_1 and A_2. Since $vRz_1 \subseteq A_1$ and $wRz_2 \subseteq A_2$, the fundamental cocircuit of e separates v from w, which completes the proof of the first part.

The second part is immediate from the definitions. □

3.4.6 Existence

Let G be a graph-like space inducing a matroid M. Then every finite minor of M is induced by a finite minor of G (finite in the sense that it only has finitely many edges) by Lemma 3.4.34. But this finite minor must consist simply of a graph, together with a (possibly infinite) collection of spurious vertices, by Lemma 3.4.12 applied to the closure of the set of edges. In particular, every finite minor of M is graphic. We also know that M has to be tame, by Lemma 3.4.32. The aim of this section is to prove that these conditions are also sufficient to show that M is induced by some graph-like space. More precisely, we wish to show:

Theorem 3.4.38. *Let M be a matroid. The following are equivalent.*

1. *There is a graph-like space G inducing M.*
2. *M is tame and every finite minor of M is the cycle matroid of some graph.*

The forward implication was proved above. The rest of this section will be devoted to proving the reverse implication. The strategy is as follows: we consider an extra structure that can be placed on certain matroids, with the following properties:

- There is such a structure on any matroid induced by a graph-like space (in particular, there is such a structure on any finite graphic matroid).
- Given such a structure on a matroid M, we can obtain a graph-like space inducing M.
- The structure is finitary.

Then we proceed as follows: given a tame matroid all of whose finite minors are graphic, we obtain a graph framework on each finite minor. Then the finitariness of the structure, together with the tameness of the matroid, allows us to show by a compactness argument that there is a graph framework on the whole matroid. From this graph framework, we build the graph-like space we need.

Graph frameworks

A *signing* for a tame matroid M is a choice of functions $c_o\colon o \to \{-1,1\}$ for each circuit o of M and $d_b\colon b \to \{-1,1\}$ for each cocircuit b of M such that for any circuit o and cocircuit b we have

$$\sum_{e\in o\cap b} c_o(e)d_b(e) = 0\,,$$

where the sums are evaluated over $\mathbb{Z}$. The sums are all finite since M is tame. A tame matroid is *signable* if it has a signing.

Signings for finite matroids were introduced in [107], where it was shown that a finite matroid is signable if and only if it is regular, i.e. representable over any field. This result was extended to tame infinite matroids, for a suitable infinitary notion of representability, in [17]. In [1] it is shown that the standard matroids associated to graphs are all signable. The construction for a graph G is as follows: we begin by choosing some orientation for each edge of G (equivalently, we choose some digraph whose underlying graph is G). We also choose a cyclic orientation of each circuit of the matroid and an orientation of each bond used as a cocircuit of the matroid. Then $c_o(e)$ is 1 if the orientation of e agrees with the orientation of o and -1 otherwise. Similarly, $d_b(e)$ is 1 if the orientation of e agrees with that of b and -1 otherwise. Then the terms $c_o(e)d_b(e)$ are independent of the orientation of e: such a term is 1 if o traverses b at e in a forward direction, and -1 if o traverses b at e in the reverse direction. Since o must traverse b the same number of times in each direction, all the sums in the definition evaluate to 0.

We therefore think of a signing, in a graphic context, as providing information about the cyclic orderings of the circuits and about the direction in which each edge in a given bond points relative to that bond. In order to reach the notion of a graph framework, we need to modify the notion of a signing in two ways. Firstly, we need to add some extra information specifying on which side of a bond b each edge not in b lies. Secondly, we need to add some conditions saying that these data induce well-behaved cyclic orderings on the circuits.

Recall that if s has a cyclic order R, then we say that $p, q \in s$ are *clockwise adjacent* in R if $[p, q, g]_R$ is in the cyclic order for all $g \in s - p - q$.

Definition 3.4.39. A *graph framework* on a matroid M consists of a signing of M and a map $\sigma_b : E \setminus b \to \{-1, 1\}$ for every cocircuit b, which we think of as telling us which side of the bond b each edge lies on, satisfying certain conditions. First, we require that these data induce a cyclic order R_o for each circuit o of M: For distinct elements e, f and g of M, we take $[e, f, g]_{R_o}$ if and only if both $e, f, g \in o$ and there exists a cocircuit b of M such that $b \cap o = \{e, f\}$ and $\sigma_b(g) = c_o(f)d_b(f)$. That is, we require that each such relation R_o satisfies the axioms for a cyclic order given in Definition 3.4.28. In particular, by asymmetry and totality, we require that this condition is independent from the choice of b: if o is a circuit with distinct elements e, f and g, and b and b' are cocircuits such that $o \cap b = o \cap b' = \{e, f\}$, then $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $\sigma_{b'}(g) = c_o(f)d_{b'}(f)$. Let o be a circuit, b be a cocircuit and s be a finite set with $b \cap o \subseteq s \subseteq o$. Then $s \subseteq o$ inherits a cyclic order $R_o\restriction_s$ from o. Our final conditions are as follows: for any two $p, q \in s$ clockwise adjacent in $R_o\restriction_s$ we require:

1. If $p, q \in b$, then $c_o(p)d_b(p) = -c_o(q)d_b(q)$.
2. If $p, q \notin b$, then $\sigma_b(p) = \sigma_b(q)$.
3. If $p \in b$ and $q \notin b$, then $c_o(p)d_b(p) = \sigma_b(q)$.
4. If $p \notin b$ and $q \in b$, then $c_o(q)d_b(q) = -\sigma_b(p)$.

Graph frameworks behave well with respect to the taking of minors. Let M be a matroid with a graph framework, and let $N = M/C\backslash D$ be a minor of M. For any circuit o of N we may choose by Lemma 3.4.6 a circuit o' of M with $o \subseteq o' \subseteq o \cup C$. This induces a function $c_{o'}\restriction_o : o \to \{-1, 1\}$. Similarly for any cocircuit b of N we may choose a cocircuit b' of N with $b \subseteq b' \subseteq b \cup D$, and this induces functions $d_{b'}\restriction_b : b \to \{-1, 1\}$ and $\sigma_{b'}\restriction_{E(N)\setminus b} : E(N) \setminus b \to \{-1, 1\}$. Then these choices comprise a graph framework on N, with R_o given by the restriction of $R_{o'}$ to o.

Next we show that every matroid induced by a graph-like space has a graph framework. Let M be a matroid induced by a graph-like space G. Fix for each topological bond of G a pair (U_b, V_b) of disjoint open sets in G inducing b, and fix an orientation $R'_{\overline{o}}$ of the pseudo-circle $\overline{o}$ inducing each topological circle o (recall from Subsection 3.4.4 that an orientation of a pseudo-circle is a choice of one of the two canonical cyclic orders of the set of points). For each topological circuit o, let the function $c_o : o \to \{-1, 1\}$ send e to 1 if $[\iota_e(0), \iota_e(0.5), \iota_e(1)]_{R'_{\overline{o}}}$, and to -1 otherwise. For each topological bond d_b, let the function $d_b : b \to \{-1, 1\}$ send e to 1 if $\iota_e(0) \in U_e$ and to -1 if $\iota_e(0) \in V_e$. Finally, for each topological bond d_b, let the function $\sigma_b : E \setminus b \to \{-1, 1\}$ send e to -1 if the end-vertices of e are both in U_b and to 1 if they are both in V_b.

Lemma 3.4.40. *The c_o, d_b and σ_b defined above give a graph framework on M.*

Proof. The key point will be that the cyclic ordering R_o we obtain on each circuit o will be that induced by the chosen orientation $R'_{\overline{o}}$. So let o be a topological circuit of G. First we show that for any distinct edges e, f and g in o and any topological bond b with $o \cap b = \{e, f\}$ we have $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R_{\overline{o}}}$. For any edge $e \in b$ we define $\iota_e^b \colon [0,1] \to G$ to be like ι_e but with the orientation changed to match b. That is, we set $\iota_e^b(r) = \iota_e(r)$ if $\iota_e(0) \in U_b$ and $\iota_e^b(r) = \iota_e(1-r)$ if $\iota_e(0) \in V_b$.

Since the pseudo-circle $\overline{o}$ with edge set o is compact, there can only be finitely many edges in o with both endpoints in U_b but some interior point not in U_b, so by adding the interiors of those edges to U_b if necessary we may assume without loss of generality that there are no such edges, and similarly we may assume that if an edge of o has both endpoints in V_b then all its interior points are also in V_b. Thus the two pseudo-arcs obtained by removing the interior points of e and f from $\overline{o}$ are both entirely contained in $U_b \cup V_b$. Since each of these two pseudo-arcs is connected and precisely one endvertex of e is in U_b, we must have that one of these pseudo-arcs, which we will call R^U is included in U_b. And the other, which we will call R^V, is included in V_b. The end-vertices of R^U must be $\iota_e^b(0)$ and $\iota_f^b(0)$, and those of R^V must be $\iota_e^b(1)$ and $\iota_f^b(1)$.

Suppose first of all that $\sigma_b(g) = 1$. Let R be the pseudo-arc $\iota_f^b(0) f \iota_f^b(1) R^V \iota_e^b(1)$. Then $c_o(f)d_b(f) = 1$ if and only if the ordering along R agrees with the orientation of $\overline{o}$, which happens if and only if $[\iota_f(0.5), \iota_g(0.5), \iota_e(0.5)]_{R'_{\overline{o}}}$, which is equivalent to $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R'_{\overline{o}}}$. The case that $\sigma_b(g) = -1$ is similar. This completes the proof that for any distinct edges e, f and g in o and any topological bond b with $o \cap b = \{e, f\}$ we have $\sigma_b(g) = c_o(f)d_b(f)$ if and only if $[\iota_e(0.5), \iota_f(0.5), \iota_g(0.5)]_{R'_{\overline{o}}}$.

In particular, the construction of Definition 3.4.39 really does induce cyclic orders on all the circuits. We now show that these cyclic orders satisfy (1)-(4). Let o, b, s, p and q be as in Definition 3.4.39. Without loss of generality $\overline{o}$ is the whole of G. We may also assume without loss of generality that all edges e are oriented so that $c_o(e) = 1$. Since $\overline{o}$ is compact we may as before assume that all interior points of edges not in s are in either U_b or V_b. Thus each of the pseudo-arcs obtained by removing the interior points of the edges in s, as in Corollary 3.4.30, is entirely included in U_b or V_b. Since they both lie on one of these pseudo-arcs, $\iota_p(1)$ and $\iota_q(0)$ are either both in U_b or both in V_b. We shall deal with the case that both are in V_b: the other is similar. In case (1), we get $d_b(p) = 1$ and $d_b(q) = -1$. In case (2), we get $\sigma_b(p) = \sigma_b(q) = 1$. In case (3), we get $d_b(p) = 1$ and $\sigma_b(q) = 1$. Finally in case (4) we get $\sigma_b(p) = 1$ and $d_b(q) = -1$. Since we are assuming that $c_o(p) = c_o(q) = 1$, in each case the desired equation is satisfied. This completes the proof. □

Since a graph framework is a finitary structure, we can lift it from finite minors to the whole matroid.

Lemma 3.4.41. *Let M be a tame matroid such that every finite minor is a cycle matroid of a finite graph. Then M has a graph framework.*

Proof. By Lemma 3.4.40 we get a graph framework on each finite minor of M. We will construct a graph framework for M from these graph frameworks by a compactness argument. Let $\mathcal{C}$ and $\mathcal{C}^*$ be the sets of circuits and of cocircuits of M. Let $H = \bigcup_{o\in\mathcal{C}} o \times \{o\} \sqcup \bigcup_{b\in\mathcal{C}^*} b \times \{b\} \sqcup \bigcup_{\tilde{b}\in\mathcal{C}^*} (E \setminus \tilde{b}) \times \{\tilde{b}\} \sqcup \bigcup_{o\in\mathcal{C}} o \times o^3$. Endow $X = \{-1, 1\}^H$ with the product topology. Any element in X encodes a choice of functions $c_o \colon e \mapsto x(o, e)$ for every circuit o, functions $d_b \colon e \mapsto x(b, e)$ and $\sigma_b \colon e \mapsto x(\tilde{b}, e)$ for every cocircuit $\tilde{b}$, and ternary relations $R_o = \{(e, f, g) \in o^3 | x(e, f, g) = 1\}$ for each circuit o.

To comprise a graph framework, these function have to satisfy several properties. These will be encoded by the following six types of closed sets.

For any circuit o and cocircuit b, let $C_{o,b} = \{x \in X | \sum_{e\in o\cap b} x(o, e)x(b, e) = 0\}$. Note that the functions c_o and d_b corresponding to any x in the intersection of all these closed sets will form a signing.

Secondly, for every circuit o, distinct edges $e, f, g \in o$ and cocircuit b such that $o\cap b = \{e, f\}$, let $C_{o,b,g} = \{x \in X|x(o, e, f, g) = x(\tilde{b}, g)x(o, f)x(b, f)\}$. So x is in the intersection of these closed sets if and only if the cyclic orders encoded by x are given as in Definition 3.4.39.

Thirdly any circuit o and distinct elements e, f, g of o we set $C_{o,e,f,g,\mathrm{Cyc}} = \{x \in X|x(o, e, f, g) = x(o, f, g, e)\}$. Note that for any x and o in the intersection of all these closed sets the relation R_o derived from x will satisfy the Cyclicity axiom. Similarly we get sets $C_{o,e,f,g,\mathrm{AT}}$ encoding the Asymmetry and Totality axioms and $C_{o,e,f,g,h,\mathrm{Trn}}$ encoding the Transitivity axiom.

Finally, for every circuit o, cocircuit b, finite set s with $o\cap b \subseteq s$, and $p, q \in s$ distinct, let $C_{b,o,s,p,q}$ denote the set of those x such that, if p and q are clockwise adjacent with respect to $R_o\restriction_s$, then the appropriate condition of (1)-(4) from Definition 3.4.39 is satisfied.

By construction, any x in the intersection of all those closed sets gives rise to a graph framework. As X has the finite intersection property, it remains to show that any finite intersection of those closed sets is nonempty. Given a finite family of those closed sets, let B and O be the set of all those cocircuits and circuits, respectively, that appear in the index of these sets. Let F be the set of those edges that either appear in the index of one of those sets or are contained in some set s or appear as the intersection of a circuit in O and a cocircuit in B. As the family is finite and M is tame, the sets B,O and F are finite.

By Lemma 4.6 from [17] we find a finite minor M' of M satisfying the following.

> For every M-circuit $o \in O$ and every M-cocircuit $b \in B$, there are M'-circuits o' and M'-cocircuits b' with $o' \cap F = o \cap F$ and $b' \cap F = b \cap F$ and $o' \cap b' = o \cap b$.

By Lemma 3.4.40 M' has a graph framework $((c'_o|o \in \mathcal{C}(M')), (d'_b|b \in \mathcal{C}^*(M')), (\sigma'_b|b \in \mathcal{C}^*(M')))$, giving cyclic orders $R'_{o'}$ on the circuits o'. Now by definition any x with $c_o\restriction_F = c'_o\restriction_F$ and $d_b\restriction_F = d'_b\restriction_F$ and $\sigma_b\restriction_F = \sigma'_b\restriction_F$ and $R_o\restriction_{o'} = R_{o'}$ for $o \in O$ and $b \in B$ will lie in the intersection of all the closed sets in the finite family, as required. This completes the proof. □

From graph frameworks to graph-like spaces

In this subsection, we prove the following lemma, which, together with Lemma 3.4.41, gives the reverse implication of Theorem 3.4.38.

Lemma 3.4.42. *Let M be a tame matroid with a graph framework $\mathcal{F}$. Then there exists a graph-like space $G = G(M, \mathcal{F})$ inducing M.*

We take our notation for the graph framework as in Definition 3.4.39.

We begin by defining G. The vertex set will be $V = \{-1, 1\}^{\mathcal{C}^*(M)}$, and of course the edge set will be $E(M)$. As in Definition 3.4.11, the underlying set of the topological space G will be $V \sqcup ((0, 1) \times E)$.

Next we give a subbasis for the topology of G. First of all, for any open subset U of $(0, 1)$ and any edge $e \in E(M)$ we take the set $U \times \{e\}$ to be open. The other sets in the subbasis will be denoted $U_b^i(\epsilon_b)$ where $i \in \{-1, 1\}$, $b \in \mathcal{C}^*(M)$ and $\epsilon_b : b \to (0, 1)$. Roughly, $U_b^1(\epsilon_b)$ should contain everything that is above b and $U_b^{-1}(\epsilon_b)$ should contain everything that is below b, so that removing the edges of b from G disconnects G. In other words, $G \setminus (\bigcup_{e \in b}(0, 1) \times \{e\})$ should be disconnected because the open sets $U_b^1(\epsilon_b)$ and $U_b^{-1}(\epsilon_b)$ should partition it (for every ϵ_b). Formally, we define $U_b^i(\epsilon_b)$ as follows.

$$\begin{aligned} U_b^i(\epsilon_b) \quad &= \quad \{v \in V | v(b) = i\} \cup \bigcup_{e \in E \setminus b, \sigma_b(e) = i} (0, 1) \times \{e\} \\ &\cup \bigcup_{e \in b, d_b(e) = i} (1 - \epsilon_b(e), 1) \times \{e\} \cup \bigcup_{e \in b, d_b(e) = -i} (0, \epsilon_b(e)) \times \{e\} \end{aligned}$$

To complete the definition of G, it remains to define the maps ι_e for every $e \in E(M)$. For each $r \in (0, 1)$, we must set $\iota_e(r) = (r, e)$. For $r \in \{0, 1\}$, we let:

$$\iota_e(0)(b) = \begin{cases} \sigma_b(e) & \text{if } e \notin b \\ -d_b(e) & \text{if } e \in b \end{cases} ; \iota_e(1)(b) = \begin{cases} \sigma_b(e) & \text{if } e \notin b \\ d_b(e) & \text{if } e \in b \end{cases} ;$$

Note that ι_e is continuous and $\iota_e\restriction_{(0,1)}$ is open. This completes the definition of G. Next, we check the following.

Lemma 3.4.43. *G is a graph-like space.*

Proof. The only nontrivial thing to check is that for any distinct $v, v' \in V$, there are disjoint open subsets U, U' of G partitioning $V(G)$ and with $v \in U$ and $v' \in U'$. Indeed, if $v \neq v'$, there is some $b \in \mathcal{C}^*$ such that $v(b) \neq v'(b)$, and then for any ϵ_b with $\epsilon_b(e) \leq 1/2$ for each $e \in E(M)$, the sets $U_b^1(\epsilon_b)$ and $U_b^{-1}(\epsilon_b)$ have all the necessary properties.

$\square$

Having proved that G is a graph-like space, it remains to show that G induces M. This will be shown in the next few lemmas.

Lemma 3.4.44. *Any circuit o of M is a topological circuit of G.*

The proof, though long, is simply a matter of unwinding the above definitions, and may be skipped.

Proof. By the symmetry of the construction of G, we may assume without loss of generality that $c_o(e) = 1$ for all $e \in o$. The graph framework of M induces a cyclic order R_o on o. From this cyclic order we get a corresponding pseudo-circle C with edge set o by Remark 3.4.29. We begin by defining a map f of graph-like spaces from C to G as follows. First we define $f(v)$ for a vertex v by specifying $f(v)(b)$ for each cocircuit b of M.

If $b \cap o = \emptyset$, then $(f(v))(b) = \sigma_b(e)$ for some $e \in o$. This is independent of the choice of e by condition (2) in the definition of graph frameworks. This ensures that $f^{-1}(U_b^i(\epsilon_b)) = C$ if $i = \sigma_b(e)$, and $f^{-1}(U_b^i(\epsilon_b)) = \emptyset$ if $i = -\sigma_b(e)$.

If $b \cap o =: s$ is nonempty, then s is finite as M is tame. The cyclic order of o induces a cyclic order on $s \cup \{v\}$: choose $p_{v,b}$ so that $p_{v,b}$ and v are clockwise adjacent in this cyclic order. We take $(f(v))(b) = d_b(p_{v,b})$.

Finally, we define the action of f on interior points of edges by $f(\iota_e^C(r)) = \iota_e^G(r)$ for $r \in (0,1)$. We may check from the definitions above that this formula also holds at $r = 0$ and $r = 1$. First we deal with the case that $r = 0$. We check the formula pointwise at each cocircuit b of M. In the case that $b \cap o = \emptyset$, we have $f(\iota_e^C(0))(b) = \sigma_b(e) = \iota_e^G(0)(b)$. Next we consider those b with $e \in b$. Let $s = o \cap b$, so that $p_{\iota_e^C(0),b}$ and e are clockwise adjacent in s. Thus $f(\iota_e^C(0))(b) = d_b(p_{\iota_e^C(0),b}) = -d_b(e) = \iota_e^G(0)(b)$ by condition (1) in the definition of graph frameworks and our assumption that $c_o(f) = 1$ for any $f \in o$. The other possibility is that $b \cap o$ is nonempty but $e \notin b$. In this case, let $s = b \cap o + e$, so that $p_{\iota_e^C(0),b}$ and e are clockwise adjacent in s. Thus $f(\iota_e^C(0))(b) = d_b(p_{\iota_e^C(0)}) = \sigma_b(e) = \iota_e^G(0)$ by condition (3) in the definition of graph frameworks and our assumption on c_o. The equality $f(\iota_e^C(1)) = \iota_e^G(1)$ may also be checked pointwise. The cases with $e \notin b$ are dealt with as before, but the case $e \in b$ needs a slightly different treatment: we note that in this case $p_{\iota_e^C(1),b} = e$, so that $f(\iota_e^C(1))(b) = d_b(e) = \iota_e^G(1)$.

It is clear by definition that f is injective on interior points of edges. To see that f is injective on vertices, let v and w be vertices of C such that $f(v) = f(w)$ and suppose for a contradiction that $v \neq w$. Since C is a pseudo-circle, there are two edges e and f in C such that v and w lie in different components of $C \backslash \{e, f\}$. By Lemma 3.4.4, there is a cocircuit b of M with $o \cap b = \{e, f\}$. Without loss of generality we have $e = p_{v,b}$. It follows that $f = p_{w,b}$. Since e and f are clockwise adjacent in the induced cyclic order on $\{e, f\}$, we have $f(v)(b) = d_b(e) = -d_b(f) = -f(w)(b)$ by condition (1) in the definition of graph frameworks and our assumption that $c_o(f) = 1$ for any $f \in o$. This is the desired contradiction. So f is injective.

To see that f is continuous, we consider the inverse images of subbasic open sets of G. It is clear that for any edge e and any open subset U of $(0,1)$, $f^{-1}(\{e\} \times U) = \{e\} \times U$ is open in C, so it remains to check that each set of the form $f^{-1}(U_b^i(\epsilon_b))$ is open in C. If $b \cap o = \emptyset$ then this set is either empty or the whole of C. So suppose that $b \cap o \neq \emptyset$, and let $x \in f^{-1}(U_b^i(\epsilon_b))$. If x is an interior point of an edge e then it is clear that some open neighborhood of x of the form $\{e\} \times U$ is included in $f^{-1}(U_b^i(\epsilon_b))$.

We are left with the case that x is a vertex and $s = b \cap o \neq \emptyset$. By Corollary 3.4.30, the component of $C \backslash s$ containing x is the pseudo-arc A consisting of all points y on C with $[a, y, b]_{R_C}$, together with a and b, for some vertices $a = \iota_p^C(1)$ and $b = \iota_q^C(0)$, where for any vertex v of A we have $p_{v,b} = p$ and where p and q are clockwise adjacent in the restriction of R_o to s. Since $f(x) \in U_b^i(\epsilon_b)$, we have $i = f(x)(b) = d_b(p)$ and so for any other vertex v of A we also have $f(v)(b) = d_b(p) = i$, so that $f(v) \in U_b^i(\epsilon_b)$. For any edge e of A, applying condition (3) in the definition of graph frameworks to p and e in the set $s + e$ gives $\sigma_b(e) = d_b(p) = i$, so that $f''(0,1) \times e = (0,1) \times e \subseteq U_b^i(\epsilon_b)$. By definition, we have $(1 - \epsilon_b(p), 1) \times \{p\} \subseteq U_b^i(\epsilon_b)$, and using condition (1) in the definition of graph frameworks we get $d_b(q) = -d_b(p) = -i$, so that $(0, \epsilon_b(q)) \times \{q\} \subseteq U_b^i(\epsilon_b)$. We have now shown that every point y of C with $[\iota_p^C(1 - \epsilon_b(p)), y, \iota_q^C(\epsilon_b(q))]_{R_C}$ is in $f^{-1}(U_b^i(\epsilon_b))$. But the set of such points is open in C, which completes the proof of the continuity of f.

We have shown that the map f is a map of graph-like spaces from the pseudo-circle C to G and that the edges in its image are exactly those in o, so that o is a topological circuit of G as required.

$\square$

It is clear that any cocircuit of M is a topological cut of G, as witnessed by the sets $U_b^{-1}(\frac{1}{2})$ and $U_b^1(\frac{1}{2})$. Combining this with Lemmas 3.4.44 and 3.4.31, we are in a position to apply Lemma 3.4.9 with $\mathcal{C}$ the set of topological circuits and $\mathcal{D}$ the set of topological cuts in G. The conclusion is Lemma 3.4.42, which together with Lemma 3.4.41 gives us Theorem 3.4.38.

3.4.7 A forbidden substructure

The next lemma gives a useful forbidden substructure for graph-like spaces inducing matroids.

Lemma 3.4.45. *Let G be a graph-like space, and let v be a vertex in it. Let $\{Q_n | n \in \mathbb{N}\}$ be a set of pseudo-arcs starting at v, and vertex-disjoint apart from that. Suppose also that the union of the edge sets of the Q_n is independent. Let y be a point in the closure of the set of their endvertices. Assume there is a nontrivial v-y-pseudo-arc P that is vertex-disjoint from all the $Q_n - v$.*

Then G does not induce a matroid.

Proof. First, we shall show that $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ does not include a pseudo-circle. Suppose for a contradiction that it includes a pseudo-circle K. Then K must include some edge e from P and some edge f from Q_m for some $m \in \mathbb{N}$. Going along K starting from f until we first hit the closed set P, we get two disjoint pseudo-arcs L_1 and L_2, one for each cyclic order of K. Formally, we consider the pseudo-arc $K - f$ endowed with the linear order $\leqq_{K-f}$. Let s be its start vertex and t be its endvertex. Let l_1 be the first point of $K - f$ in P, and let l_2 be the last point of $K - f$ in P. Then $L_1 = s(K - f)l_1$ and $L_2 = l_2(K - f)t$.

We shall show that each of these pseudo-arcs contains v. Since f and $P - v$ are in different components of $(P \cup Q_m) - v$, each L_i contains either v or some edge f'

in some Q_l with $l \in \mathbb{N} - m$. Note that fL_if' is included in $\bigcup_{n\in\mathbb{N}} Q_n$ and is an f-f'-pseudo-arc. By the independence of $\bigcup_{n\in\mathbb{N}} Q_n$ and Remark 3.4.36, it must be that $fL_if' = fQ_mvQ_lf'$. In particular, $v \in L_i$, as desired. This contradicts that L_1 and L_2 are disjoint. Thus $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ does not include a pseudo-circle.

Now suppose for a contradiction that G induces a matroid M. We pick $e \in P$ arbitrarily. Since $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ is M-independent as shown above, by Lemma 3.4.5 there must be a cocircuit meeting $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup P$ precisely in e. This cocircuit defines a topological cut of G with the two endvertices of e on different sides. This contradicts that $\left(\bigcup_{n\in\mathbb{N}} Q_n\right) \cup (P - e)$ is connected. □

Figure 3.3:

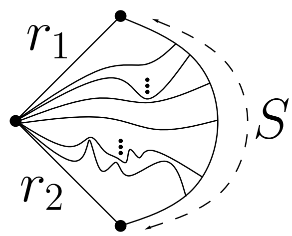

Figure 3.3: The situation of Lemma 3.4.46.

Lemma 3.4.46. *Let G be a graph-like space in which there is a pseudo-circle C with a vertex v of C that is indicent with two edges r_1 and r_2 of C. Let S be the pseudo-arc with edge set $E(C) - r_1 - r_2$. Assume there are infinitely many pseudo-arcs Q_n starting at v to points in S that are vertex-disjoint aside from v.*

If $\bigcup_{n\in\mathbb{N}} Q_n$ does not include a pseudo-circle, then G does not induce a matroid.

Proof. Without loss of generality, we may assume that the pseudo-arcs Q_n only meet S in their end-vertices. By Ramsey's theorem there is an infinite subset N of $\mathbb{N}$ such that the endpoints in S of the Q_n for $n \in N$ form a sequence that is either increasing or decreasing with respect to the linear order $\leqq_S$ of the pseudo-arc S. Let y be their limit point. Let P be the v-y-pseudo-arc included in C that avoids all the endpoints of those Q_n with $n \in N$. Note that P is nontrivial since it has to include either r_1 or r_2. Applying Lemma 3.4.45 now gives the desired result. □

Corollary 3.4.47. *Let G be a graph-like space, C a pseudo-circle of G, and r_1 and r_2 distinct edges of C. Let S_1 and S_2 be the two components of $C \setminus \{r_1, r_2\}$. If there is an infinite set W of edges of G each with one end-vertex in S_1 and the other in S_2 and with all of their end-vertices in S_2 distinct, then G does not induce a matroid.*

Proof. Let G' be the graph-like space obtained from G by contracting all edges of S_1. Then in G', there is a vertex v that is endvertex of all edges in W. On the other hand, the other endvertices are distinct for any two edges in W. Indeed, let b be the cocircuit meeting C in precisely r_1 and r_2. Then $W \subseteq b$ and no two endvertices in S_2 are identified.

The set $\overline{W}$ cannot include a pseudo-circle with at least 3 edges since then v would be an endvertex of at least 3 edges of that pseudo-circle, which is impossible. So by Lemma 3.4.46 with each of the Q_n given by a single edge of W, we obtain that G' does not induce a matroid. By Lemma 3.4.34, nor does G. □

3.4.8 Countability of circuits in the 3-connected case

Our aim in this section is to prove the following:

Theorem 3.4.48. *Any topological circuit in a graph-like space inducing a 3-connected matroid is countable.*

For the remainder of the section we fix such a graph-like space G, inducing a 3-connected matroid M, and we also fix a pseudo-circle C of G, whose edge set gives a circuit o of M.

We begin by taking a base s of M/o, and letting $G' = G/s$. Thus by Lemma 3.4.34 G' induces the matroid $M' = M/s$ in which o is a spanning circuit. For any $e \in o$, $o - e$ is a base of o and so $s \cup o - e$ is a base of M, which we shall denote s^e. We shall call the edges of $E(M') \setminus o$ which are not loops *bridges*. We denote the set of bridges by Br. The endpoints of each bridge lie on the pseudo-circle C' corresponding to o in G'. The edges of C' are the same as those of C, but the vertices are different: recall that the vertices of the contraction $G' = G/s$ were defined to be equivalence classes of vertices of G. Each of these can contain at most one vertex of C, since o is a circuit of M'. Thus each vertex of C' contains a unique vertex of C.

Lemma 3.4.49. *Let $g \in o$ and let f be a bridge with endpoints v' and w' in G'. Let v be the vertex of C contained in v', and w the vertex of C contained in w'. Let x be the endvertex of f in G contained in v', and y the endvertex of f in G in contained in w'. Then the fundamental circuit o_f of f with respect to the base s^g of M is given by concatenating 4 pseudo-arcs: the first, from x to y, consists of only f. The second, from y to w, contains only edges of s. The third, from w to v contains only edges of o - it is the interval of $C - g$ from w to v. The fourth, from v to x, contains only edges of s.*

Proof. $o_f \cap o$ must consist of the fundamental circuit of f with respect to the base $o - g$ of M' - that is, of the interval of $C' - g$ from w' to v'. So the pseudo-arc $v(C - g)w$, which is the closure of this set of edges, lies on the pseudo-circle $\bar{o}_f$. So $(\bar{o}_f - f) \setminus v(C - g)w$ consists of two pseudo-arcs joining v and w to x and y. These two pseudo-arcs use edges from s only. Since v and y lie in different connected components of $G\restriction_s$, we must have that the first goes from v to x, and the second goes from w to y. This completes the proof. □

Lemma 3.4.50. *For any distinct edges e and f of C, there is a bridge whose endvertices separate e from f in C.*

Proof. Since M is 3-connected, $\{e, f\}$ is not a bond of M, so we can pick some $g \notin \{e, f\}$ in the fundamental bond of f with respect to the base s^e. Then f lies in the fundamental circuit o_g of g, which is therefore not a subset of $s + g$. Thus g is a bridge, and since the fundamental circuit of g with respect to the base $o - e$ of M' contains f but not e the endpoints of g separate e from f. $\square$

Given that we are aiming to prove Theorem 3.4.48, we may as well assume that o has at least 2 elements, and by Lemma 3.4.50 we obtain that there is at least one bridge. We now fix a particular bridge e_0, and make use of the 3-connectedness of M to build a tree structure capturing the way the endpoints of the bridges divide up C'. We will call this tree the *partition tree*, and define it in terms of certain auxiliary sequences $(I_n \subseteq \mathrm{Br})$, $(J_n \subseteq V(C'))$ and (K_n) indexed by natural numbers, given recursively as follows:

We always construct J_n from I_n as the set of endvertices of elements of I_n, and K_n as the set of components of $C' \setminus J_n$. We take I_0 to be $\{e_0\}$, and I_{n+1} to be the set of bridges that have endvertices in different elements of K_n or at least one endvertex in J_n.

Then the nodes of the tree at depth n will be the elements of K_n, with p a child of q if and only if it is a subset of q.

Lemma 3.4.51. *Every bridge is in some I_n.*

Proof. Suppose not, for a contradiction, and let e be any bridge which is in no I_n. In particular, the endpoints of e both lie in the same component of $C - J_0$, so there is a pseudo-arc joining them in C that meets neither endvertex of e_0. Let f be any edge of this pseudo-arc. Let v_0' be any endvertex of e_0, and let v_0 be the unique vertex of C contained in v_0'.

For each n, let B_n be the element of K_n of which f is an edge, and let $B = \bigcap_{n \in \mathbb{N}} B_n$ and $A = C \setminus B$. Note that any 2 vertices in B are joined by a unique pseudo-arc in B, and that A has the same property. Since the two endvertices of e_0 (in G') avoid B_1, they are both in A. Since e is in no I_n, its two endvertices lie in B.

Let A_V be the set of endvertices v of edges of G such that the first point of $vs^f v_0$ on C is contained in a vertex in A. Let A_E be the set of edges of G that have both endvertices in A_V, and let $B_E = E(M) \setminus A_E$. Note that for any vertex $v \in A_V$, all edges of the unique v-C-path included in s_f lie in A_E. And for any $v \notin A_V$, all edges of the unique v-C-path included in s_f lie in B_E.

We shall show that (A_E, B_E) is a 2-separation of M, which will give the desired contradiction since we are assuming that M is 3-connected.

First, we show that $s^f \cap A_E$ is a base of A_E. It is clearly independent. Let g be any edge in $A_E \setminus s^f$. Suppose first of all that g is a bridge. We decompose the fundamental circuit of g as in Lemma 3.4.49, taking the notation from that lemma. Then since each of the endpoints x and y of g is in A_V, every edge of this fundamental circuit is in A_E, as required.

So suppose instead that g isn't a bridge, that is, g is a loop in M'. Let R_1 and R_2 be the pseudo-arcs from the endpoints x and y of g to v_0 which use only edges from s^f. Let z be the first point of R_1 to lie on R_2. Then zR_1v_0 and

zR_2v_0 must be identical, as both are pseudo-arcs from z to v_0 using only edges of s^f. Let k be the first point on this pseudo-arc that is in C. By assumption, $k \in A$. Also, xR_1zR_2y is a pseudo-arc from x to y using only edges from s^f, so must form (with g) the fundamental circuit of g with respect to s^f, so can meet C at most in a single vertex (since g is a loop in M'). Thus all edges in this fundamental circuit lie on either xR_1k or yR_2k, and so are in A_E, as required.

Next, we show that $(s_f \cap B_E) + f$ is a base of B_E. It is independent since A includes some edge as e_0 is a bridge. Let g be any edge in $B_E \setminus s^f - f$. If g isn't a bridge we can proceed as before, so we suppose it is a bridge. We decompose the fundamental circuit of g as in Lemma 3.4.49, taking the notation from that Lemma. At least one of v' and w' lies in B: without loss of generality it is v'. Suppose for a contradiction that w' is in A. Then either w' is in some J_n or it is an element of some K_n not containing f. In either case, $g \in I_{n+1}$ and so $v' \in J_{n+1}$, giving the desired contradiction since we are assuming $v' \in B$. Thus w' is also in B. Let R be the pseudo-arc from v to w in B. Then g is spanned by the pseudo-arc $xs^f vRws^f y$, which uses only edges of $s_f \cap B_E + f$. To see this we apply Lemma 3.4.49 with some edge not in B_1 in place of f of that lemma.

Since each of A_E and B_E has at least 2 elements, and the union of the bases for them given above only contains one more element than the base s^f of M, this gives a 2-separation of M, completing the proof. □

Lemma 3.4.52. *Every node of the Partition-tree has at most countably many children.*

Proof. Let $x \in K_n$ be a node of the Partition-tree. Then the closure $\bar{x}$ of the set of interior points of edges of x is a pseudo-arc. Let $\hat{x}$ be the set obtained from this pseudo-arc by removing its end-vertices. An x-bridge is a bridge with one endvertex in $\hat{x}$ and one in its complement. Thus every element of $J_{n+1} \cap x$ must be an endvertex of an x-bridge or of $\bar{x}$.

Let v_1 and v_2 be vertices of $\hat{x}$ with $v_1 \leqq_{\bar{x}} v_2$. Suppose for a contradiction that there are infinitely many elements of J_{n+1} between v_1 and v_2. Pick a corresponding set W of infinitely many x-bridges with different attachment points between v_1 and v_2. Since neither of v_1 and v_2 is an endpoint of $\bar{x}$, there are edges e_1 and e_2 in x such that all points of e_1 are $\leqq_{\bar{x}}$-smaller than v_1, and similarly all points of e_2 are $\leqq_{\bar{x}}$-bigger than v_2. Then by Corollary 3.4.47 with $r_1 = e_1$ and $r_2 = e_2$, G' does not induce a matroid, which gives the desired contradiction.

We have established that between any two elements of $J_{n+1} \cap \hat{x}$ there are only finitely many others. Hence $J_{n+1} \cap \hat{x}$ is finite or has the order type of $\mathbb{N}$, $-\mathbb{N}$ or $\mathbb{Z}$. In all these cases there are only countably many children of x, since these children are the connected components of $x \setminus (J_{n+1} \cap x)$. □

We now consider rays in the partition tree: a *ray* consists of a sequence $(k_n \in K_n | n \in \mathbb{N})$ such that for each n the node k_{n+1} is a child of k_n. Given such a ray, we call the set $\bigcap_{n \in \mathbb{N}} k_n$ its *partition class*.

Lemma 3.4.53. *The partition class of any ray includes at most one edge.*

Proof. Suppose for a contradiction that there is some ray (k_n) whose partition class includes 2 different edges e and f. Then by Lemma 3.4.50 there is a bridge g whose endvertices separate e from f in C. By Lemma 3.4.51, g lies in some I_n. But then e and f lie in different elements of K_n, so can't both lie in k_n, which is the desired contradiction. □

For any element k of K_n with $n \geq 1$, the *parent* $p(k)$ is the unique element of K_{n-1} including k.

An element k of K_n with $n \geq 2$ is *good* if no bridge in I_n has endvertices in two different components of $\overline{p(p(k)) \setminus k}$. Note that $\overline{p(p(k)) \setminus k}$ has at most two components. Note that if k is not good, there have to be two vertices in different components of not only $\overline{p(p(k)) \setminus k}$ but also $p(p(k)) \setminus k$.

Lemma 3.4.54. *Every node of the Partition-tree has at most one good child.*

Proof. Suppose for a contradiction that some $x \in K_n$ with $n \geq 1$ has two good children y_1 and y_2. Since they are different, there is an element i of J_{n+1} separating them, and a bridge e in I_{n+1} of which i is an endvertex. Since $i \notin J_n$, $e \notin I_n$ and so the other endvertex j of e must lie in $p(x) = p(p(y_1)) = p(p(y_2))$. Now the two endvertices of e have to be in different components of $\overline{p(p(y_1)) \setminus y_1}$ or $\overline{p(p(y_2)) \setminus y_2}$. Hence y_1 and y_2 cannot both be good at the same time, a contradiction. □

Lemma 3.4.55. *Let (k_n) be a ray whose partition class includes an edge. Then all but finitely many nodes on it are good.*

Proof. Let e be the edge in the partition class of this ray. Let f be any edge of $C \setminus k_0$.

Suppose for a contradiction that there is an infinite set N of natural numbers such that k_n is not good for any $n \in N$. Let N' be an infinite subset of N that does not contain 0, 1 or any pair of consecutive natural numbers. For each $n \in N'$, pick a bridge e_n in I_n with endvertices in both components of $\overline{p(p(k_n)) \setminus k_n}$, which is possible since k_n is not good. The endvertices of e_n are in J_n but not J_{n-2} and so we cannot find $m \neq n \in N'$ such that e_m and e_n share an endvertex. Applying Corollary 3.4.47 with $r_1 = e$, $r_2 = f$ and $W = \{e_n | n \in \mathbb{N}\}$ yields that G' does not induce a matroid, a contradiction. This completes the proof. □

Proof of Theorem 3.4.48. For each edge of C there is a unique ray whose partition class contains that edge. By Lemma 3.4.55, we can find a first node on that ray such that it and all successive nodes are good. This gives a map from the edges of C to the nodes of the partition tree. By Lemma 3.4.54 and Lemma 3.4.53, this map is injective. By Lemma 3.4.52 the partition tree has only countably many nodes. □

3.4.9 Planar graph-like spaces

A nice consequence of Theorem 3.4.48 is the following.

Corollary 3.4.56. *Let M be a tame 3-connected matroid such that all finite minors are planar. Then $E(M)$ is at most countable.*

Proof. Let e be some edge. By Lemma 3.4.10, there is a switching sequence from e to any other edge. Hence it suffices to show that there are only countably many different switching sequences starting at e. We show by induction that there are only countably many switching sequences of length n for each n. The case $n = 1$ is obvious. The first $n - 1$ elements of a switching sequence of length n form a switching sequence of length $n - 1$. On the other hand, there are only countably many ways to extend a given switching sequence of length $n - 1$ to one of length n since all circuits and cocircuits of M are countable by Theorem 3.4.48. Hence there are only countably many switching sequences of length n. This completes the proof. $\square$

This raises the question how to embed the graph-like space constructed from a tame matroid all of whose finite minors are planar in the plane. However, we shall construct such a matroid that does not seem to be embeddable in this sense the plane. Let N be the matroid whose circuits are the edge sets of topological circles in the topological space depicted in Figure 3.4. We omit the proof that this gives a matroid - it can be found in [34]. However, much of the complication of this matroid was introduced to make it 3-connected, and if we do not require 3-connectedness then it is easy to construct other simpler examples sharing the essential property of this matroid: it is tame and all finite minors are planar, but the topology of the graph-like space it induces has no countable basis of neighbourhoods for the vertex at the apex, so it cannot be embedded into the plane.

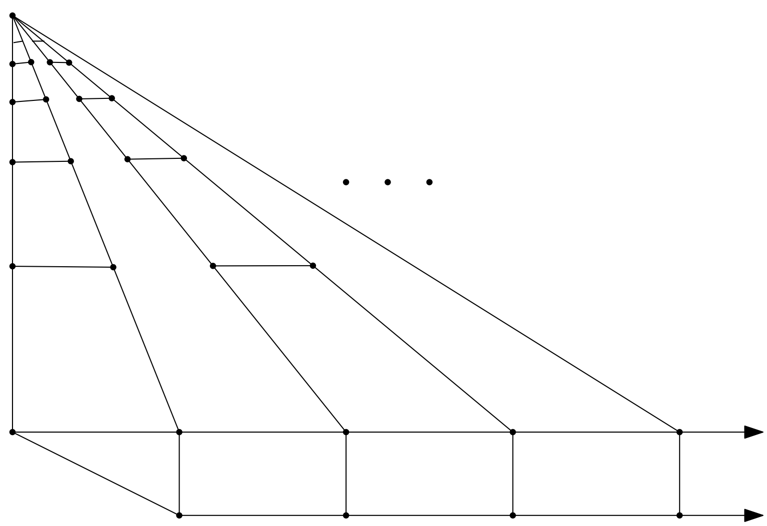

Figure 3.4: The matroid N.

Chapter 4

Every planar graph with the Liouville property is amenable

4.0.10 Introduction

A well-known result of Benjamini & Schramm states that every transient planar graph with bounded vertex degrees admits non-constant harmonic functions with finite Dirichlet energy; we will call such a function a *Dirichlet harmonic function* from now on. In particular, such a graph does not have the Liouville property. Two independent proofs of this theorem were given in [10, 11], one using circle packings and one using square tilings.

The bounded degree condition was essential in both these proofs, and is in fact necessary: consider for example a ray where the nth edge has been duplicated by 2^n parallel edges. Still, there are natural classes of unbounded degree graphs where such obstructions do not occur, and it is interesting to ask whether the above result remains true in them. Recently, planar graphs with unbounded degrees have been attracting a lot of interest, in particular due to research on coarse geometry [14], random walks [9, 66] and random planar graphs related to Liouville quantum gravity [5, 91, 6, 7, 9, 13, 46, 69, 81, 88]. Motivated by this, our main result extends the aforementioned result of Benjamini & Schramm to unbounded degree graphs by replacing the transience condition with a stronger one, which we call *roundabout-transience* and explain below

Theorem 4.0.57. *Let G be a locally finite roundabout-transient planar map. Then G admits a Dirichlet harmonic function.*

A *planar map G*, also called a *plane graph*, is a graph endowed with an embedding in the plane. The *roundabout graph G°* is obtained from G by replacing each vertex v with a cycle v° in such a way that the edges incident with v are incident with distinct vertices of v° (of degree 3), preserving their cyclic ordering; see also Subsection 4.0.13. We say that G is *roundabout-transient* if G° is transient[1]. In Subsection 4.0.13 we relate G° with circle packings of G.

Another way how one might try to strengthen the transience condition is to require that there is a flow f witnessing the transience which does not only have finite Dirichlet energy but finite norm in a different Hilbert space, where we give weights to the edges depending on the degrees of their endvertices. Following up, these ideas, we could show that Theorem 4.0.57 implies the following

Corollary 4.0.58. *Let G be a locally finite planar graph G such that there is a flow f of intensity 1 out of some vertex v such that $\sum_{vw\in E(G)}[deg(v)^2 + deg(w)^2]f(vw)^2$ is finite. Then G has a non-constant Dirichlet harmonic function.*

As shown in Subsection 4.0.17, the order of magnitude of the weights here is best-possible. Hence Corollary 4.0.58 is best-possible, which indicates a way in which Theorem 4.0.57 is tight.

Our work was partly motivated by a problem from [66], asking whether every simple planar graph with the Liouville property is (vertex-)*amenable*, by which we mean that for every $\epsilon > 0$ there is a finite set S of vertices of G such that less than $\epsilon|S|$ vertices outside S have a neighbour in S. As we show in Subsection 4.0.17,

Theorem 4.0.59. *Every locally finite non-amenable planar map is roundabout-transient.*

Combining this with Theorem 4.0.57 yields a positive answer to the aforementioned problem, and much more. This strengthens a result of Northschield [89], stating that every bounded degree non-amenable planar graph admits non-constant bounded harmonic functions, in two ways: it relaxes the bounded degree condition, and provides Dirichlet rather than bounded harmonic functions.

Benjamini [14] constructed a bounded degree non-amenable graph with the Liouville property. The last result shows that such a graph cannot be planar even if we drop the bounded degree assumption.

We think of Theorems 4.0.57 and 4.0.59 as indications that the notion of roundabout-transience is satisfied in many cases, and has strong implications. We expect it to find further applications, and propose some problems in Subsection 4.0.18.

We now give an overview of the proof of Theorem 4.0.57. As shown in [38], a graph admits Dirichlet harmonic functions if and only if it has two disjoint transient subgraphs T_1, T_2 such that the effective conductance between T_1 and T_2 is finite; see Theorem 4.0.61. To show that our graphs satisfy this condition, we start with a flow provided by T. Lyons' transience criterion (Theorem 4.0.60)— this flow lives in an auxiliary graph which for the purposes of this illustration can be thought of as a superimposition of G with its dual— we split that flow into four sub-flows using the square tiling techniques of [66], we use two subflows

[1]The authors coined this term in Warwick, where there are many roundabouts.

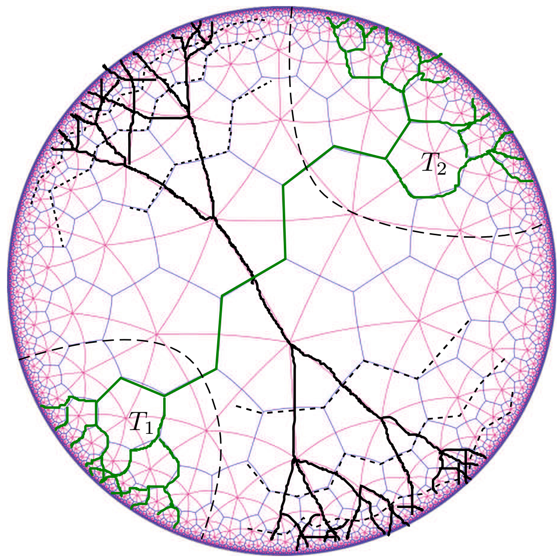

Figure 4.1: The two subgraphs T_1, T_2 delimited by the dashed curves are transient because of the green flow. The dual of the black flow (dashed) witnesses the fact that the effective conductance between T_1 and T_2 is finite because it has finite energy.

to obtain T_1, T_2, and we apply a duality argument to the other two subflows to show that the effective conductance between T_1 and T_2 is finite; see Figure 4.1.

The latter step can be thought of as an occurence of the idea that the effective resistance from the top to the bottom of a rectangle equals the effective conductance (or extremal length [109]) from left to right, with the aforementioned subflows showing finiteness of the top-to- bottom effective resistance. This idea was triggered by another result of Benjamini & Schramm [12], stating that every non-amenable graph contains a non-amenable tree.

4.0.11 Preliminaries

A *graph*, or *network*, G is a pair (V, E) where V is a set, called the set of *vertices* (or *nodes*) of G, and E is a set of pairs of elements of V, called the *edges*. In this paper all graphs are simple.

Given a vertex set X, by $E(X)$ we denote those edges with both endvertices in X. A locally finite graph G is *1-ended* if for every finite vertex S, the graph $G - S$ has only one infinite component.

Electrical network basics

All graphs in this paper are undirected. However, as we will want to describe flows of electrical current in our networks, we will need to be able to distinguish between the two possible orientations of an edge in order to be able to say in which direction current flows along that edge. A convenient solution is to introduce the set $\overrightarrow{E(G)}$ (or just $\vec{E}$) of *directed edges* of G to be the set of ordered pairs (x, y) such that $xy \in E$. Thus any edge $xy = yx \in E$ corresponds to two elements of $\vec{E}$, which we will denote by $\overrightarrow{xy}$ and $\overrightarrow{yx}$.

An *antisymmetric function* $i : \vec{E} \to \mathbb{R}$ satisfies $i(\overrightarrow{xy}) = -i(\overrightarrow{yx})$ for every edge $xy \in E$. All functions on $\vec{E}$ we will consider will have this property.

Given two dual plane graphs G and G^* and an orientation of the plane, there is a unique bijection * between the directed edges of G and G^* respecting this orientation. If we use * below we shall always assume that we picked some orientation - even if we do not say this explicitly. If F is an edge set, then F^* denotes image of F under *. The function * induces an operator on antisymmetric functions f on the directed edges of G. Given $f : \overrightarrow{E(G)} \to \mathbb{R}$, we denote the induced function from $\overrightarrow{E(G^*)}$ to $\mathbb{R}$ by f^*.

Given a function $i : \vec{E} \to \mathbb{R}$, we say that i satisfies *Kirchhoff's node law* at a vertex x if

$$\partial i(x) := \sum_{y \in N(x)} i(\overrightarrow{xy}) = 0 \tag{4.1}$$

holds, where $N(x)$ denotes the set of vertices sharing an edge with x (called the *neighbours* of x).

If i satisfies Kirchhoff's node law at everywhere except at one vertex o, then i is called a *flow from* o. By the *intensity* of i we will mean $\partial i(o)$. Usually we will assume that $\partial i(o) > 0$ when we use this term. Similarly, we define a Kirchhoff's node law at finite vertex sets and flow from a finite set $A \subset V(G)$.

Given $u : V \to \mathbb{R}$, the *induced antisymmetric function* ∂u is given by

$$\partial u(\overrightarrow{xy}) = u(x) - u(y) \tag{4.2}$$

If $i = \partial u$, we say that the pair i, u satisfies *Ohm's law*.

Suppose that a pair i, u as above satisfies Ohm's law, and i satisfies Kirchhoff's node law. Then, combining (4.1) with (4.2) we obtain $\sum_{y \in E(x)}(u(x) - u(y)) = 0$, and solving for $u(x)$ this can be rewritten as

$$u(x) = \frac{\sum_{y \in E(x)} u(y)}{d(x)}, \tag{4.3}$$

where the *degree* $d(x)$ of x is the number of edges incident with x.

If a function u satisfies the formula (4.3), then we say that u is *harmonic* at x. Note that the above implication can be reversed to yield that if u is harmonic at a vertex then it satisfies Kirchhoff's node law there. In other words, if the pair i, u satisfies Ohm's law, then u is harmonic at a vertex x if and only if i satisfies Kirchhoff's node law at x.

A function $u : V \to \mathbb{R}$ is *harmonic* if it is harmonic at every $x \in V$.

The (Dirichlet) *energy* of $i : \vec{E} \to \mathbb{R}$ is defined by

$$E(i) := \sum_{e \in \vec{E}} i^2(e).$$

Similarly, we define the energy of a function $u : V \to \mathbb{R}$ by
$E(u) := \sum_{xy \in E} (v(x) - v(y))^2$. We call u a *Dirichlet harmonic function* if u is harmonic and $E(u) < \infty$. We write $\mathcal{O}_{HD}$ for the class of graphs on which all Dirichlet harmonic functions are constant.

A *potential* on the network N is a function $u : V \to \mathbb{R}$. The *boundary* of the potential u is the set of vertices at which u is not harmonic.

A *walk* in G is a sequence of incident vertices and edges $x_0 e_{01} x_1 e_{12} x_2 \ldots x_k$ (where the x_j are vertices and the e_{jl} edges). A walk as above is *closed* if $x_k = x_0$. Kirchhoff's cycle law postulates that for every closed walk as above we have

$$\sum_{0 \leq n < k} i(\overrightarrow{x_n x_{n+1}}) = 0. \tag{4.4}$$

It is not hard to check that i satisfies Kirchhoff's cycle law if it does so for every *injective* closed walk, i.e. one for which the x_j are distinct for $0 \leq j < k$. Moreover, this is the case if and only if there is a potential u with $i = \partial u$.

Random walk basics

All random walks in this paper are simple and take place in discrete time, that is, if our the random walker is at a vertex x of our graph G at time n, then it is at each of the $d(x)$ neighbours of x with equal probability $1/d(x)$ at time $n+1$. The starting vertex of our random walk will always be deterministic, and usually denoted by o.

G is called *transient*, if the probability to visit any fixed vertex is strictly less than 1. We will make heavy use of T. Lyons classical characterisation of transience in terms of flows:

Theorem 4.0.60 ([83, 82])**.** *A locally finite graph G is transient if and only if for some (and hence for every) vertex $o \in V(G)$, G admits a flow from o with finite energy.*

If G is transient, then we can define a flow i out of any vertex o as follows. For every vertex $v \in V$, let $h(v)$ be the probability $p_v(o)$ that random walk from v will ever reach o. Thus $h(o) = 1$. Note that h is harmonic at every $v \neq o$. Let $i(\overrightarrow{xy}) := h(x) - h(y)$. By our discussion in Section 4.0.11, i is a flow out of o, and we call it the *random walk flow* out of o.

4.0.12 Known facts

HD facts

We shall use following characterisation of the locally finite graphs admitting Dirichlet harmonic functions:

Theorem 4.0.61 ([38])**.** *A locally finite graph G is not in $\mathcal{O}_{HD}$ if and only if there are transient vertex-disjoint subgraphs A and B such that there is a potential ρ of finite energy which is constant on A and B but takes different values on them.*

Corollary 4.0.62. *A locally finite graph G is not in $\mathcal{O}_{HD}$ if and only if there is a flow f and a potential ρ both of finite energy such that the supports of f and $\partial(\rho)$ intersect in precisely one edge.*

Proof. We may without loss of generality assume that the two graphs A and B of Theorem 4.0.61 are joined by an edge xy. Given two vertex-disjoint subgraphs A and B, there is a flow f of finite energy with $f(\overrightarrow{xy})$ nonzero and whose support is included in $(A \cup B) + xy$ if and only if A and B are transient. Thus Corollary 4.0.62 follows from Theorem 4.0.61. $\square$

Next, we give a new independent functional analytic proof of the 'if'-implication of Corollary 4.0.62. For that we need the following:

Lemma 4.0.63. *Let H be a Hilbert space space and V and W two orthogonal subspaces such that the orthogonal complement $V^\perp$ of V is not orthogonal to $W^\perp$. Then $V^\perp \cap W^\perp$ is nontrivial.*

Proof. Then $V^\perp + W^\perp = V \oplus W \oplus (V^\perp \cap W^\perp)$. By assumption, there are $v \in V^\perp$ and $w \in W^\perp$ with $< v|w > \neq 0$. Thus $v \in W \oplus (V^\perp \cap W^\perp)$ and $w \in V \oplus (V^\perp \cap W^\perp)$. Since V and W are orthogonal, the projection of v to $V \oplus (V^\perp \cap W^\perp)$ is contained in $(V^\perp \cap W^\perp)$. This projection is a nontrivial by assumption, completing the proof. $\square$

Proof of the 'if'-implication of Corollary 4.0.62. We consider the Hilbert space of antisymmetric functions on the edges with finite Dirichlet energy. Its scalar product is given by $\langle f \mid g \rangle = \sum_{e \in E(G)} f(e)g(e)$. Let C be its subspace generated by the characteristic functions of the finite cycles, and D be its subspace generated by the atomic bonds $b(v)$ given by the characteristic functions of the set of edges incident with a vertex v. Note that $f \in D^\perp$ and $\partial(\rho) \in C^\perp$. Thus by Lemma 4.0.63, there is some nontrivial $h \in C^\perp \cap D^\perp$, which is an antisymmetric function induced by a non-constant Dirichlet harmonic function. $\square$

A *cut* of a graph G is the set of edges between a set of vertices $U \subset V(G)$ and its complement $V(G) \setminus U$.

Corollary 4.0.64 ([98])**.** *Let G be a locally finite graph with a finite cut b such that $G - b$ has two transient components. Then G is not in $\mathcal{O}_{HD}$.*

Proof. Just apply Theorem 4.0.61 with any potential ρ which is constant on any component of $G - b$ and assigns different values on two transient components of $G - b$. $\square$

Theorem 4.0.65 ([38])**.** *Let H be a connected locally finite graph. Let G be a locally finite graph obtained from H by adding for each $n \in \mathbb{N}$ a path P_n of length 2^n such that P_n meets H and the other P_n only in its starting vertices. Then $G \in \mathcal{O}_{HD}$ if and only if $H \in \mathcal{O}_{HD}$.*

Given a locally finite graph G, an antisymmetric function f on $\overrightarrow{E(G)}$ *witnesses that a subgraph H of G is transient* if the restriction $\bar{f}$ of f to $\overrightarrow{E(H)}$ is a flow from some finite vertex set of finite energy. We can change $\bar{f}$ at finitely many edges to get a flow from a single vertex of finite energy. Thus $\bar{f}$ implies that H is transient by Theorem 4.0.60.

Recall that a *bond* of a graph is a minimal separating edge-set (i.e. a minimal nonempty cut).

Remark 4.0.66. *Let G and G^* be locally finite dual plane graphs. Let f be a flow of G of finite energy. Then one of the following is true.*

A) The function f^* satisfies Kirchhoff's cycle law;

B) there is a finite bond b of G such that f witnesses that the two components of $G - b$ are transient.

Proof. If f^* violates Kirchhoff's cycle law at a finite cycle C of G^*, then C considered as an edge set of G is a bond b and f witnesses that the two components of $G - b$ are transient. □

Electrical network facts

The following 'Monotone-Voltage Paths' lemma can be found in [82, Corollary 3.3]

Lemma 4.0.67. *Let G be a transient connected network and v the voltage function from the unit current flow i from a vertex o to ∞ with $v(\infty) = 0$. For every vertex x, there is a path from o to x along which v is monotone.*

4.0.13 Roundabout-transience

Given a locally finite plane graph G, informally the *roundabout graph* G° is obtained from G by replacing each vertex v by a roundabout of length equal to the degree of v so that every vertex gets degree 3. Formally, the vertex set of G° is the set of pairs (v, e) where e is an edge and v is an endvertex of e. The embedding of G gives us a cyclic order C_v of the set of edges incident with the vertex v. The edges of G° are of two types, for each edge $e = vw$ we have an edge joining (v, e) and (w, e). For any two edges e and f adjacent in the cyclic order C_v, we have an edge between (v, e) and (v, f).

Note that the roundabout graph G° is like G a plane graph.

In a slight abuse of notation, we shall suppress the inclusion map which maps the edge $e = vw$ to $\{(v, e), (w, e)\}$ in our notation, and we will just write things like $E(G) \subseteq E(G^\circ)$. The edges *going out off* a roundabout are those with precisely one endvertex in the roundabout. We say that a graph G is *roundabout-transient* if its roundabout graph G° is transient.

Remark 4.0.68. *Every cut of G is a cut of G°. Conversely, every cut b of G° with $b \subseteq E(G)$ is also a cut of G.* □

Remark 4.0.69. *We remark that the roundabout graph depends on the embedding of G. Thus roundabout-transience is a property of plane graphs and not of planar graphs. Indeed, let G be the graph obtained from T_2 by attaching 2^n leaves at each vertex at level n. It is straightforward to check that there is a non-roundabout-transient embedding of G in the plane as well as a roundabout-transient one. Still roundabout-transience implies transience, in the sense that if G admits a roundabout-transient embedding, then G is transient:*

Lemma 4.0.70. *If G° is transient, then so is G.*

Proof. Since G° is transient, it admits a flow f of finite energy from some vertex $o \in V(G^\circ)$ by Lyons' criterion, Theorem 4.0.60. We will show that f induces a flow of finite energy in G.

For a vertex $v \in V(G^\circ)$, let us denote by v° the set of vertices lying in the same roundabout as v. Note that f satisfies Kirchhoff's node law at every v° except o°. Therefore, the restriction f' of f to $\overrightarrow{E(G)}$ satisfies Kirchhoff's node law at every vertex of G except the vertex o°. In other words, f' is a flow from o°. Its energy is bounded from above by that of f, and so G is transient by Theorem 4.0.60. $\square$

In the following we will often use the notation $G^{*\circ}$, by which we mean that we apply first * and then $^\circ$. Thus $G^{*\circ}$ is the roundabout graph of the dual of G.

The *plane line graph* G^* of a plane graph G is the plane graph obtained from the roundabout graph G° by contracting all non-roundabout edges. Another way to define G^*, explaining the name we chose, is by letting the vertex set of G^* be the set of midpoints of edges of G and joining two such points with an arc whenever the corresponding edges are incident with a common vertex v of G and lie in the boundary of a common face of v. It is clear from this definition that

$$G^* = G^{**}. \tag{4.5}$$

A third equivalent definition of G^* can be given by considering a circle packing P of G, letting $V(G^*)$ be the set of intersection points of circles of P, and letting the arcs in P between these points be the edges of G^*. A fourth definition of G^* is as the dual of the bipartite graph G', with $V(G')$ consisting of the vertices and faces of G, and $E(G')$ joining each vertex of G to each of its incident faces.

Lemma 4.0.71. *Let G be a locally finite plane graph. Then G° is transient if and only if G^* is.*

Proof. This follows easily from Theorem 4.0.60: if G° has a flow f of finite energy from $o \in V(G^\circ)$, then f induces such a flow f' in G^* from the vertex corresponding to o by just restricting f to $E(G^*) \subset E(G^\circ)$.

Conversely, given a flow f' in G^* as above, we can construct a flow f on G° by letting $f(e) = f'(e)$ for every $e \in E(G^*)$ and letting $f(e)$ be the unique value that makes both endvertices of e satisfy Kirchhoff's node law, unless those vertices correspond to o in which case we let $f(e)$ be the unique value that makes exactly one endvertex of e satisfy Kirchhoff's node law. That such values always exist is an easy fact about Kirchhoff's node law. The energy $E(f)$ of f is finite because the contribution of each vertex to $E(f)$ is bounded above by a constant times the contribution of its corresponding vertex in G^* to $E(f')$. $\square$

Lemma 4.0.71, combined with the fact that $G^* = G^{**}$ (4.5), immediately yields

Corollary 4.0.72. *If G° is transient, then so is $G^{*\circ}$.*

Another way to state Corollary 4.0.72 is to say that G is roundabout-transient if and only if G^* is roundabout-transient.

4.0.14 Square tilings and the two crossing flows

In this section we use the theory of square tilings of transient planar graphs in order to find the special flows in our roundabout-transient G mentioned in the introduction. Square tilings in our sense were introduced in [10], and generalise a classical construction of Brooks et. al. [24] from finite plane graphs to infinite transient ones.

Let $\mathcal{C}$ denote the cylinder $(\mathbb{R}/\mathbb{Z}) \times \{0,1]$, or more generally, a cylinder $(\mathbb{R}/\mathbb{Z}) \times \{0,a]$ for some real $a > 0$ (which turns out to coincide with the effective resistance from a vertex o to infinity). A *square tiling* of a plane graph G is a mapping τ assigning to each edge e of G a square $\tau(e)$ contained in $\mathcal{C}$, where we allow $\tau(e)$ to be a 'trivial square' consisting of just a point (see Figure 4.2 for an example). A nice property of square tilings is that every vertex $x \in V$ can be associated with a horizontal line segment $\tau(x) \subset \mathcal{C}$ such that for every edge e incident with x, $\tau(e)$ is tangent to $\tau(x)$.

The construction of this τ is based on the random walk flow i out of a root vertex o (as defined in Section 4.0.11): the side length of the square $\tau(e)$ is chosen to be $|i(e)|$, and the placement of that square incide $\mathcal{C}$ is decided by a coordinate system where potentials of vertices induced by the flow i are used as coordinates. For example, the top circle of the cylinder $\mathcal{C}$ is the 'line segment' corresponding to o, because o has the highest potential. All other vertices and edges accumulate towards the base of $\mathcal{C}$, because their potentials (which equal the probability for random walk to return to o, normalised by the height of $\mathcal{C}$) converge to 0; see [66] for details.

We let $w(\tau(e))$ denote the width of the square $\tau(e)$. Our square tilings always have the following properties which we will use below:

1. Two of the sides of $\tau(e)$ are always parallel to the boundary circles of $\mathcal{C}$;

2. $w(\tau(e)) = |i(e)|$ for every $e \in \vec{E}$, where i denotes the random walk flow out of o;

3. the interiors of any two such squares $\tau(e), \tau(f)$ are disjoint;

4. every point of $\mathcal{C}$ lies in $\tau(e)$ for some $e \in E$;

5. every vertex x can be associated with a horizontal line segment $\tau(x) \subset \mathcal{C}$ so that for every edge e incident with x, $\tau(e)$ is tangent to $\tau(x)$, and every point of $\tau(x)$ is in $\tau(f)$ for some edge f incident with x, and
6. every face F can be associated with a vertical line segment $\tau(F) \subset \mathcal{C}$ so that for every edge e in the boundary of F, $\tau(e)$ is tangent to $\tau(F)$.

It was shown in [10] that a plane graph G admits a square tiling exactly when G is *uniquely absorbing*. We say that G is *uniquely absorbing*, if for every finite subgraph G_0 there is exactly one connected component D of $\mathbb{R}^2 \setminus G_0$ which is *absorbing*, that is, random walk on G visits $G \setminus D$ only finitely many times with positive probability (in particular, G is transient).

A *meridian* of $\mathcal{C}$ is a vertical line of the form $\{x\} \times \{0, 1\} \subset \mathcal{C}$ for some $x \in \mathbb{R}/\mathbb{Z}$. An important property of meridians that we will use below is that the net flow i crossing any meridian is zero; see [66, Lemma 6.6] for a more precise statement.

Lemma 4.0.73. *Let G and G^* be locally finite dual plane graphs. If G° is transient, then there are flows f and h of finite energy in the roundabout graphs G° and $G^{*\circ}$ respectively whose supports intersect in a single edge (of $E(G) = E(G^*)$).*

Here the graphs G° and $G^{*\circ}$ have precisely the edge set $E(G) = E(G^*)$ in common. In the proof below we think of G^* as being constructed from G° and $G^{*\circ}$ by contracting $E(G) = E(G^*)$. This way we can consider the roundabout of $v \in V(G)$ as a cycle of G^* .

Proof. We will first find appropriate auxiliary flows f', h' in G^* and use them to induce the desired flows f on G° and h on $G^{*\circ}$ by sending some flow along $E(G)$.

We distinguish two cases, according to whether G^* is uniquely absorbing.

If G^* is uniquely absorbing, then [10] provides a square tiling of G^* on a cylinder $\mathcal{C}$ as described above, with o being an arbitrary vertex of G^*.

Given a vertex $x \in V(G^*)$, we let $|x|$ denote the 'strip' of the cylinder $\mathcal{C}$ whose horizontal span coincides with that of the line segment $\tau(x)$ (as described in item 5). Then $\tau(x)$ separates $|x|$ into two rectangles, and we denote the bottom one (that is, the one not meeting $\tau(o)$) by $\lceil x \rceil$.

Next, we associate to this x a flow $\check{x}$ out of x that 'lives in $\lceil x \rceil$'. To define the flow $\check{x}$, for every $e \in \vec{E}(G^*)$ with $i(e) \geq 0$, where i is the random walk flow out of o, let $\check{x}(e) := w(\tau(e) \cap \lceil x \rceil)$ be the width of the rectangle $\tau(e) \cap \lceil x \rceil \subset \mathcal{C}$ corresponding to e. (Thus if $\tau(e)$ is contained in $\lceil x \rceil$, then $\check{x}(e) = i(e)$ by (2), and if $\lceil x \rceil$ dissects $\tau(e)$ then $\check{x}(e) < i(e)$.) Naturally, we extend $\check{x}$ to the remaining directed edges in the unique way that makes $\check{x}$ antisymmetric. By the aforementioned property of meridians proved in [66, Lemma 6.6], $\check{x}$ is indeed a flow out of x.

More generally, if M, M' are two meridians intersecting $\tau(x)$, we let $\lceil MxM' \rceil$ denote the rectangle of $\mathcal{C}$ bounded by M, x, M' and the bottom circle of $\mathcal{C}$, and define the flow out of x that *lives in* $\lceil MxM' \rceil$ similarly to $\check{x}$, except that we replace the rectangle $\lceil x \rceil$ with $\lceil MxM' \rceil$ in that definition.

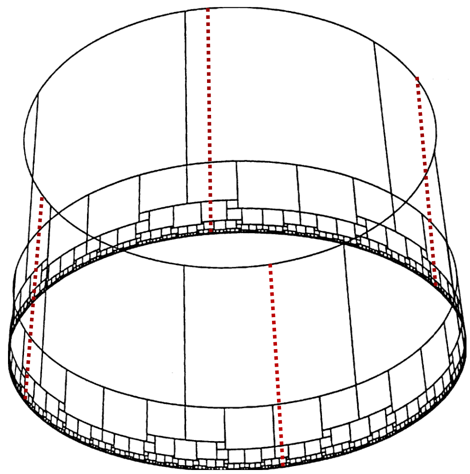

Figure 4.2: An example of a square tiling, with the four meridians M_i of Lemma 4.0.73 in dotted lines.

Our plan is to find four vertices $x_1, \ldots, x_4$ far enough from each other on $\mathcal{C}$ and flows f_i out of those vertices that live in appropriate disjoint rectangles, and combine these flows pairwise to obtain f', h'.

Now more precisely, we claim that we can choose four vertices $x_i, 1 \leq i \leq 4$ in G^*, a flow f_i out of each x_i, and a path P_i from x_i to o, so that these objects satisfy the following properties

1. $\mathrm{supp}(f_i) \cap \mathrm{supp}(f_j) = \emptyset$ for $i \neq j$; even stronger, no roundabout of G° meets both $\mathrm{supp}(f_i)$ and $\mathrm{supp}(f_j)$;

2. for every i and every edge e of P_i, no edge of the roundabout of G° containing e is in the support of any $f_j, 1 \leq j \leq 4$, and

3. the roundabout of G° containing the first edge of P_i does not contain x_j and does not contain any edge of P_j for $j \neq i$.

Before proving that such a choice is possible, let us first see how it helps us construct the desired flows f, h.

We claim that there is a tree T contained in G (we really mean G and not G^*) such that the set of leaves of T is $\{r_1, r_2, r_3, r_4\}$, where r_i denotes the roundabout of G° containing x_i, and such that no edge of G^* lying in a roundabout corresponding to a vertex in T is in the support of any f_i. Indeed, consider the subgraph H of G induced by the vertices of G whose roundabouts meet $\bigcup_{1 \leq i \leq 4} P_i$; that subgraph is connected since all P_i meet o, and its roundabouts avoid the supports of the f_i by (2). Letting T be a spanning tree of H, we can now use (3) to deduce that each r_i is a distinct leaf of T. If T has any further leaves, we can recursively prune them untill its set of leaves is $\{r_1, r_2, r_3, r_4\}$. Let T^* denote the subgraph of G^* spanned by the roundabouts in T.

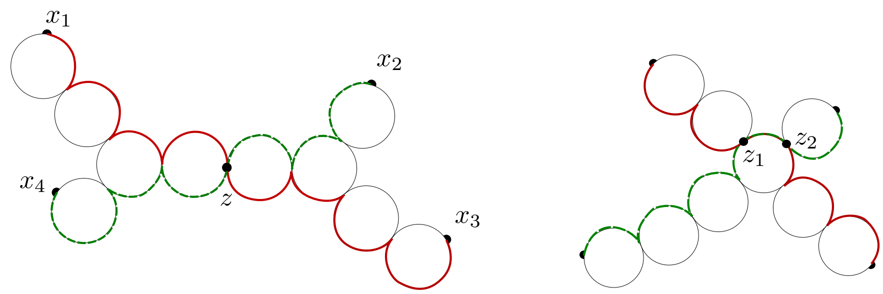

Figure 4.3: The 'tree' T^* and the paths P, Q.

There are two possible shapes for this T^* depending on whether T has a vertex of degree 4, as depicted in Figure 4.3. Assume without loss of generality that x_1, x_2, x_3, x_4 appear in that cyclic order along the outer face of T^*. Consider first the case where T has no vertex of degree 4. Easily, we can find an x_3-x_1 path P and an x_4-x_2 path Q such that $E(P) \cap E(Q) = \emptyset$, and there is a unique vertex $z \in T^*$ at which P, Q *cross*, that is, P contains two opposite edges of z and Q contains the other two (and so for every other vertex v in $P \cap Q$, we have no crossing at v). Figure 4.3 shows how to choose these paths P, Q.

In the other case, where T has a vertex of degree 4, we choose P, Q so that we have exactly two vertices z_1, z_2 meeting $P \cup Q$ in three edges, and all other vertices meet $P \cup Q$ in at most two edges; see the right side of Figure 4.3.

We can now construct the desired flow f' from a finite flow along P and an appropriate linear combination of f_1, f_3, where the coefficients, one positive and one negative, are tuned in such a way that Kirchhoff's node law (4.1) is satisfied at x_1 and x_3. Similarly, the flow h' can be constructed using a linear combination of f_2, f_4, and a finite flow along Q.

Note that f' induces a flow f on G° and h' induces a flow h on $G^{*\circ}$ by sending appropriate amounts of flow along the edges of G or G^* (as explained in the proof of Lemma 4.0.71). We claim that, in the case where T has no vertex of degree 4, the only edge in $\mathrm{supp}(f) \cap \mathrm{supp}(h)$ is the edge e_z of G corresponding to the vertex z of T^*, while in the case where T does have a vertex of degree 4, the only edge in $\mathrm{supp}(f) \cap \mathrm{supp}(h)$ is one of the two edges e_{z_1}, e_{z_2}. Indeed, the supports of the f_i meet no common roundabouts by (1), and as P, Q lie in T, the choice of T combined with (2) implies that no edge in $\mathrm{supp}(f_i)$ contributes to $\mathrm{supp}(f) \cap \mathrm{supp}(h)$. Thus the only possible intersections come from vertices of G^* in $P \cap Q$.

Now in the case where T has no vertex of degree 4, note that every vertex in $P \cap Q$ has all its 4 edges in $P \cup Q$. It is now straightforward to check using the definitions of the graphs $G^\circ, G^*, G^{*\circ}$ that z is the only vertex whose edge is in $\mathrm{supp}(f) \cap \mathrm{supp}(h)$, as z was the only vertex at which P and Q cross.

In the case where T does have a vertex of degree 4, similar arguments apply, and it is again straightforward to check that exactly one of e_{z_1}, e_{z_2} is in $\mathrm{supp}(f) \cap \mathrm{supp}(h)$ (which of the two depends on which of f', h' we use to induce a flow in G° and which in $G^{*\circ}$).

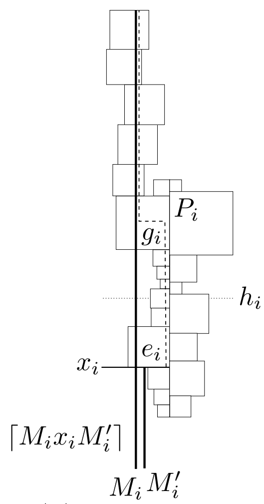

Figure 4.4: The choice of x_i, f_i and P_i.

Thus, in the uniquely absorbing case, it only remains to prove that we can indeed choose vertices x_i, flows f_i, and paths P_i with properties (1), (2) and (3) above.

For this, recall that the length of the circumference of $\mathcal{C}$ is 1, and let $M_i, 1 \leq 4$ denote the meridian of $\mathcal{C}$ whose width coordinate is $i/4 \pmod 1$. For each i, let $h_i \in (0, \frac{1}{16})$ be small enough that every roundabout of G° meeting M_i at a point whose height coordinate is less than h_i has width less than 1/8, where the width of a roundabout O is defined to be the maximum width of a line segment contained in $\tau[O]$; such a choice is possible because $\tau[O]$ is two squares wide at each horizontal level by (6) (where we use the fact that O bounds a face of G^*), and a square that starts close to the bottom of $\mathcal{C}$ cannot be very wide. In addition, we choose h_i even smaller, if needed, to ensure that if x is a vertex such that $\tau(x)$ meets M_i below height h_i, then $w(\tau(x)) < 1/8$; this is possible because there are are only finitely many edges e with $w(\tau(e))$ greater than any fixed constant since $\mathcal{C}$ has finite area, and $\tau(x)$ is at most three squares $\tau(e)$ wide by (5) and the fact that G^* is 4-regular.

Let $\lceil h_i M_i \rceil$ denote the subset of M_i with height coordinates ranging between zero and h_i, and $\lfloor h_i M_i \rfloor$ the subset of M_i with height coordinates ranging between h_i and 1.

For every $i \leq 4$, there is a lowermost edge e_i meeting $\lceil h_i M_i \rceil$ such that the roundabout O_i of G° containing e_i also contains an edge g_i meeting $\lfloor h_i M_i \rfloor$ (Figure 4.4); this is true because $\lfloor h_i M_i \rfloor$, being closed, only meets finitely many squares of positive area, and so there are finitely many roundabouts to choose from. There is at least one to choose from: a roundabout whose image contains the point of M_i at height h_i.

Let x_i denote the endvertex of e_i whose height coordinate is lower, and note that $\tau(x_i)$ meets M_i. Let M_i' be a meridian meeting $\tau(e_i)$ (and in particular

$\tau(x_i)$) close enough to M_i, but distinct from M_i, that the rectangle $\lceil M_i x_i M'_i \rceil$ bounded by M_i, x_i, M'_i and the bottom circle of $\mathcal{C}$, meets the τ image of no roundabout meeting $\lfloor h_i M_i \rfloor$; such a M'_i exists because, by the choice of e_i, O_i, no roundabout meeting $\lfloor h_i M_i \rfloor$ has an edge e meeting $\lceil M_i x_i M'_i \rceil$, or we would have chosen e instead of e_i. As we can choose M'_i as close to M_i as we wish, we may assume that $d(M_i, M'_i) < 1/16$, which will be useful later.

Let f_i be the flow out of x_i that lives in $\lceil M_i x_i M'_i \rceil$, as defined above. We claim that

If $e \in \mathrm{supp}(f_i)$, then $\tau(e)$ is contained in the open vertical strip of radius $1/8$ centered at M_i. (4.6)

Indeed, by the definition of f_i, if $e \in \mathrm{supp}(f_i)$, then $\tau(e)$ intersects the interior of $\lceil M_i x_i M'_i \rceil$. Then $\tau(e)$ cannot have a point at height higher that h_i, which we recall is less than $1/16$, because it would have to intersect the interior of $\tau(e_i)$ in that case, contradicting (3). Thus the height of $\tau(e)$ is at most $1/16$, and being a square, so is its width. Together with our assumption that $d(M_i, M'_i) < 1/16$, this proves our claim.

Note that (4.6), combined with the choice of the M_i, immediately implies that $\mathrm{supp}(f_i) \cap \mathrm{supp}(f_j) = \emptyset$ for $i \neq j$; in fact, it even implies the stronger statement of (1), because by (4.6) if edges e, f lie in a common roundabout then $\tau(e), \tau(f)$ must meet a common meridian.

It remains to construct the paths P_i: we let P_i start with the x_i-g_i path in O_i containing e_i, and continue with the g_i-o path consisting of all the edges whose τ-image meets M_i above $\tau(f_i)$. To make the later path well-defined, we would like M_i to meet no trivial squares $\tau(e)$ of zero width. This can easily be achieved: since G^* has only countably many edges, and every trivial square meets just one meridian, we can arrange for our 4 M_i to be among the uncountably many remaining ones by rotating $\mathcal{C}$ appropriately. The fact that the edges whose τ-image meets M_i above $\tau(g_i)$ form a g_i-o path now follows from (5) and the fact that $\tau(o)$ is the top circle of $\mathcal{C}$. In fact, by the above argument, we can even assume that M_i does not meet the boundary of any square $\tau(e)$, and so M_i uniquely determines that g_i-o path. Note that by construction,

every edge of P_i is in a roundabout O such that $\tau[O]$ meets M_i. (4.7)

To see that (2) is satisfied, recall that we chose h_i small enough that every roundabout of G° meeting M_i at a point whose height coordinate is less than h_i has width less than $1/8$, and P_i only uses roundabouts meeting M_i. Thus for $e \in E(P_i)$, $\tau(e)$ is contained in the vertical strip of radius $1/8$ centered at M_i. On the other hand, (4.6) says that the support of f_j is contained in the strip of radius $1/8$ centered at M_j, and so (2) follows from the fact that $d(M_i, M_j) \geq 1/4$.

Finally, we can prove (3) by a similar argument, now using the fact that $w(\tau(x_j)) < 1/8$ by the second part of our definition of h_j, and the fact that the roundabout containing the first edge e_i of P_i is contained in the strip of radius $1/8$ centered at M_i and every roundabout containing an edge of P_j meets M_j by (4.7).

Suppose now G^* is not uniquely absorbing. Then for some finite subgraph G_0 we have at least two absorbing components D_1, D_2 in $\mathbb{R}^2 \setminus G_0$. By elementary topological arguments, G_0 contains a cycle C such that both the interior I and the exterior O of C contain transient subgraphs of G^*.

If any of these subgraphs I, O is uniquely absorbing, then we can repeat the above arguments to that subgraph to obtain the two desired flows.

Hence it remains to consider the case where there is a cycle C_I in I and a cycle C_O in O that further separate each of I, O into two transient sides. In fact, we can iterate this argument as often as we like, to obtain many distinct transient subgraphs separated from any given cycle. Let us iterate it often enough to obtain four disjoint cycles $C_i, 1 \leq 4$, and inside each C_i a cycle D_i such that the interior of D_i is transient and no roundabout ot G° meets any two of these eight cycles.

We now apply Theorem 4.0.60 to each of the four interior sides of the D_i to obtain four transience currents f_i out of vertices x_i, such that the support of f_i is contained in D_i. We can then combine those flows pairwise in a way similar to the uniquely absorbing case to obtain the two desired flows f', h', and from them f, h: we can let o be an arbitrary vertex outside all C_i, and define T and the paths P, Q similarly. The fact that $|\operatorname{supp}(f) \cap \operatorname{supp}(h)| = 1$ follows from the same graph-theoretic arguments about the structure of G^*, for which we did not need the square tiling. □

4.0.15 Harmonic functions on plane graphs

In this section, we use Theorem 4.0.61 to prove a new existence criterion for Dirichlet harmonic functions, Theorem 4.0.76 below, which is used in the proof of Theorem 4.0.57. Before proving Theorem 4.0.76, we prove the following which we think is interesting in its own right, and which motivated the main result of this section.

Theorem 4.0.74. *Let G and G^* be locally finite 1-ended dual plane graphs. Then the following are equivalent:*

1. $G \notin \mathcal{O}_{HD}$;
2. $G^* \notin \mathcal{O}_{HD}$;
3. *there are flows f and h of finite energy of G and G^* respectively whose supports intersect in a single edge.*

Proof. By symmetry, it suffices to show that 1 is equivalent to 3. If $G \notin \mathcal{O}_{HD}$, then let f and ρ be as in Corollary 4.0.62. Then f and $\partial\rho$ witness 3.

For the converse suppose there are flows f and h as in 3. Then h^* satisfies Kirchhoff's cycle law by Remark 4.0.66 because 2 in that remark cannot be fulfilled as G is 1-ended and transient graphs are infinite. Thus h^* is induced by a potential ρ, which together with f witnesses that $G \notin \mathcal{O}_{HD}$ by Corollary 4.0.62. □

Example 4.0.75. We give a simple example that neither 2 nor 3 imply 1 in Theorem 4.0.74 if we leave out the assumption that G and G^* are 1-ended. Let H be the graph obtained from disjoint cycles C_n of length 2^n by gluing C_n and C_{n+1} together at a single edge for each n that are distinct for different n. We obtain the graph G from a triangle by gluing two copies of H at distinct edges of the triangle.

In the next theorem, we propose a strengthening of 3 which implies that $G \notin \mathcal{O}_{HD}$ - even if G has more than one end.

Theorem 4.0.76. *Let G and G^* be locally finite dual plane graphs such that there are flows f and h of finite energy in the roundabout graphs G° and $G^{*\circ}$ respectively whose supports intersect in a single edge. Then $G \notin \mathcal{O}_{HD}$.*

Proof. Let $h[G^*]$ be the restriction of h to $E(G^*)$, which is a flow of G^* as h satisfies Kirchhoff's node law at the set of vertices of each roundabout.

Case 1: $h[G^*]^*$ satisfies Kirchhoff's cycle lawin G. Then let ρ be a potential induced by $h[G^*]^*$, and let $f[G]$ be restriction of f to $E(G)$, which is a flow of G as f satisfies Kirchhoff's node law at the set of vertices of each roundabout. Then f and ρ witnesses that $G \notin \mathcal{O}_{HD}$ by Corollary 4.0.62.

Having dealt which case 1, the remaining case is by Remark 4.0.66:

Case 2: There is a finite bond b of G^* such that $h[G^*]$ witnesses that the two components D_1 and D_2 of $G^* - b$ are transient. The bond b considered as an edge set of G is the set of edges of a cycle C, see Figure 4.5.

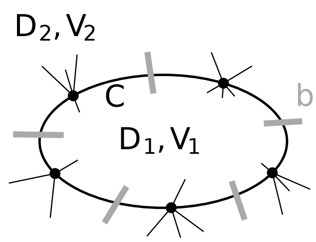

Figure 4.5: The cycle C, drawn thick, separates V_1 from V_2. In the dual, the bond b, drawn grey, separates D_1 from D_2.

Without loss of generality all edges of D_1 are contained in the interior of C, and the edges of D_2 in the exterior of C. Let V_1 be the set of vertices of G contained in the interior of C, and V_2 those vertices in the exterior. Since the set X of edges incident with a vertex of C contains a cut X' separating V_1 from V_2, by Corollary 4.0.64 it remains to show that $G[V_1]$ and $G[V_2]$ are both transient.

To see that $G[V_1]$ is transient, it suffices to show that $G[V_1]^\circ$ is transient by Lemma 4.0.70. Note that $G[V_1]$ and $G^*[D_1]$ are both locally finite and have locally finite duals. Moreover, the dual of $G[V_1]$ can be obtained from G^* by contracting all edges not in $G[V_1]$ (considered as edges of G^*). Thus the dual of $G[V_1]$ can be obtained from $G^*[D_1]$ by identifying finitely many vertices (and deleting finitely many loops). Since transience is invariant under changing finitely many edges or vertices, it remains to show that $G^*[D_1]^\circ$ is transient by Corollary 4.0.72. However, this is witnessed by $h[G^*]$. Summing up, the transience of $G[V_1]$ is inherited from $G^*[D_1]^\circ$ via $G[V_1]^\circ$.

Similarly, $G[V_2]$ is transient. Thus $G \notin \mathcal{O}_{HD}$ by Lemma 4.0.70 applied to X' and the $G[V_i]$.

□

4.0.16 Proof of the main result

Before proving Theorem 4.0.57, we need the following.

Lemma 4.0.77. *Let G be a locally finite plane roundabout-transient graph. Then there is a locally finite plane roundabout-transient supergraph H of G such that its dual H^* is locally finite, and $H \in \mathcal{O}_{HD}$ if and only if $G \in \mathcal{O}_{HD}$.*

Proof. To make sure that H^* is locally finite, we let G' be a supergraph of G obtained by 'triangulating' every infinite face of G in such a way that each vertex of G receives at most 2 new edges per incident face (any finite number would do in place of 2); this is easy to do. As $V(G)$ is countable, the set of newly added edges is countable. Take an enumeration of the set of newly added edges and subdivide the n-th edge 2^n-times. Call the resulting graph H. Note that H is locally finite and all its faces are finite. The roundabout graph of H has a subgraph which can be obtained from the roundabout graph of G by subdividing each edge at most twice.

Thus H is roundabout-transient. By Theorem 4.0.65 H is in $\mathcal{O}_{HD}$ if and only if G is in $\mathcal{O}_{HD}$, thus H has the desired properties. □

Proof of Theorem 4.0.57. By Lemma 4.0.77, we may assume without loss of generality that G^* is locally finite. Thus the theorem follows from combining Lemma 4.0.73 with Theorem 4.0.76. □

4.0.17 Applications

A vertex is in the *boundary* ∂X of some vertex set X if it is not in X but adjacent to a vertex in X. An infinite graph G is *non-amenable* if there is a constant $\gamma > 0$ such that the boundary ∂S has size at least $\gamma \cdot |S|$ for every finite vertex set S of G. The supremum of such values for γ is the *Cheeger-constant* $Ch(G)$ *of* G.

Lemma 4.0.78. *If a locally finite plane graph G is non-amenable, then so is its roundabout graph.*

Proof. Let X be a finite vertex set of the roundabout graph of G. Let $\overline{X}$ be the set of those vertices of G whose roundabouts meet X.

Sublemma 4.0.79. *Less than $6 \cdot |\overline{X}|$ vertices of X have all their neighbours in X.*

Proof. Let Y be the set of those vertices of X with all their neighbours in X. If $(v,e) \in Y$, then $(w,e) \in X$ where w is the other endvertex of e. Thus $|Y| \leq 2 \cdot |E(\overline{X})|$. As $(\overline{X}, E(\overline{X}))$ is plane, it has average degree less than 6. Thus $|E(\overline{X})| < 3 \cdot |\overline{X}|$, and thus $|Y| < 6 \cdot |\overline{X}|$. $\square$

If $|X| \geq 12 \cdot |\overline{X}|$, then at least $|X|/2$ vertices of X have a neighbour outside X. As the roundabout graph has maximal degree 3, the neighbourhood of X has then size at least $|X|/6$. Thus we may assume that $|X| < 12 \cdot |\overline{X}|$. Let $\overline{\overline{X}}$ be the set of those vertices of $\overline{X}$ whose whole roundabout is in X. Let $\epsilon = (|\overline{X}| - |\overline{\overline{X}}|)/|\overline{X}|$.

Sublemma 4.0.80. $|\partial X| > \frac{\epsilon}{12}|X|$

Proof. The roundabout of some $x \in \overline{X} \setminus \overline{\overline{X}}$ contains a vertex of ∂X, in formulas: $|\partial X| \geq |\overline{X} \setminus \overline{\overline{X}}| = \epsilon \cdot |\overline{X}|$. Thus the lemma follows from the assumption that $|X| < 12 \cdot |\overline{X}|$. $\square$

Sublemma 4.0.81. $|\partial X| \geq K(\epsilon) \cdot |X|$*, where* $K(\epsilon) = \frac{Ch(G) \cdot (1-\epsilon) - \epsilon}{12}$.

Proof. Each vertex in $\partial \overline{\overline{X}}$ is in $\overline{X} \setminus \overline{\overline{X}}$ or its roundabout contains a vertex of ∂X. Thus we estimate:

$$|\partial X| \geq |\partial \overline{\overline{X}}| - |\overline{X} \setminus \overline{\overline{X}}| \geq Ch(G) \cdot |\overline{\overline{X}}| - \epsilon |\overline{X}|$$

Note that $|\overline{\overline{X}}| = (1-\epsilon) \cdot |\overline{X}|$. Thus $|\partial X| \geq K(\epsilon) \cdot |X|$, where $K(\epsilon) = \frac{Ch(G)(1-\epsilon)-\epsilon}{12}$. $\square$

There is a positive constant δ - only depending on $Ch(G)$ - such that $K(\delta') \geq Ch(G)/24$ for all $\delta' \leq \delta$. Let γ be the minimum of $\frac{\delta}{12}$ and $\frac{Ch(G)}{24}$. Then $|\partial X| \geq \gamma \cdot |X|$ by Sublemma 4.0.80 and Sublemma 4.0.81. Hence the roundabout graph of G is non-amenable. $\square$

Proof of Theorem 4.0.59 (already mentioned in the Introduction). If G is non-amenable, then so is G° by Lemma 4.0.78. Every non-amenable locally finite graph is transient as it contains a subtree with positive Cheeger-constant by a result of Benjamini and Schramm [12]. $\square$

Corollary 4.0.82. *Every locally finite planar non-amenable graph G admits a non-constant Dirichlet harmonic function.*

Proof. Just combine Theorem 4.0.59 and Theorem 4.0.57. $\square$

Corollary 4.0.83. *Let G be a locally finite planar graph G such that there is a flow f of intensity 1 out of some vertex v such that $\sum_{v \in V(G)} deg(v) \left(\sum_{e|v \in e} |f(e)| \right)^2$ is finite. Then G has a non-constant Dirichlet harmonic function.*

Proof. For a vertex z of G°, we denote by $\vec{e_z}$ the unique directed edge not in any roundabout and pointing towards z.

By Theorem 4.0.57, it remains to extend f to a flow of G° from some vertex v' in the roundabout of v of finite energy by assigning values to the edges of the roundabout. At each roundabout C for a vertex $w \neq v$ of G, this is a finite Dirichlet-Problem: We want to find a function g_w assigning values to the directed edges of C such that at the vertex z it accumulates $-f(\vec{e_z})$. As f satisfies Kirchhoff's node law at w, the sum of the $f(\vec{e_z})$ is 0.

It is well-known that there is such a g_w and it is unique up to adding a multiple of the constant flow around C. Choosing g_w of minimal energy ensures for every $k \in C$ that $|g_w(k)| \leq \sum_{e|w\in e} |f(e)|$ since otherwise we could add a constant flow to g_w decreasing the energy. Pick a vertex v' in the roundabout for v. As above, there is a function g_v at the roundabout for v which at the vertex $z \neq v'$ accumulates $-f(\vec{e_z})$, and accumulates $1 - f(\vec{e_{v'}})$ at v'.

Then f together with the g_x defines a flow of G° from v' of intensity 1, whose energy is bounded by $\sum_{v\in V(G)} deg(v) \left(\sum_{e|v\in e} |f(e)|\right)^2$, and thus finite. $\square$

Given a locally finite graph G, for $e = vw$ we let $r(e) = deg(v)^2 + deg(w)^2$. The graph G is *super transient* if there is a flow from some vertex of intensity 1 such that its r-weighted energy is finite, that is, $\sum_{e\in E(G)} f(e)^2 r(e)$ is finite. Note that super transience implies transience. Note that G is super transient if and only if the graph $G[r]$ is transient, where we obtain $G[r]$ from G by subdividing each edge e $r(e)$-many times.

Corollary 4.0.84. *Every super transient planar locally finite graph G has a non-constant Dirichlet harmonic function.*

Proof. By Cauchy-Schwarz, $\left(\sum_{e|v\in e} |f(e)|\right)^2 \leq deg(v) \sum_{e|v\in e} f(e)^2$. Thus this follows from Corollary 4.0.83 $\square$

We can now re-prove the result of [10] that motivated our work:

Corollary 4.0.85 ([10])**.** *Every transient planar graph of bounded degree has a non-constant Dirichlet harmonic function.*

Proof. A transient bounded degree graph is super transient, so this follows from Corollary 4.0.84. $\square$

We remark that if we omit the assumption of planarity, then Corollary 4.0.84 and Corollary 4.0.83 become false as the example of the 3-dimensional grid $\mathbb{Z}^3$ shows. Indeed, it is in $\mathcal{O}_{HD}$ but transient and thus super transient as its degrees are uniformly bounded. The next example shows that Corollary 4.0.83 is best-possible.

Example 4.0.86. In this example, we show that the order of magnitude in Corollary 4.0.83 is best possible. More precisely, we construct a locally finite planar graph G without non-constant Dirichlet-harmonic functions but still with a flow f out of some vertex such that for every $\epsilon > 0$ the term $E_\epsilon(f) = \sum_{v\in V(G)} deg(v)^{(1-\epsilon)} \left(\sum_{e|v\in e} |f(e)|\right)^2$ is finite.

In this construction, we rely on the fact that the 2-dimensional grid $\mathbb{Z}^2$ has a subdivision T of the infinite binary tree T_2 such that edges at level n are subdivided at most 2^n-times. It is straightforward to construct this subdivision T recursively and we leave the details to the reader. We obtain G from $\mathbb{Z}^2$ by contracting for each edge e of T_2 all but one of its subdivision edges.

As the branch set of each vertex of G is finite, G and its dual are 1-ended. Moreover, the dual of G is obtained from $\mathbb{Z}^2$ by deleting edges. Thus by Theorem 4.0.74, $G \in \mathcal{O}_{HD}$.

Next we construct f. The subtree S of G consisting of those edges of T that are not contracted is isomorphic to T_2. Let f be the flow on T_2 which assigns edges at level n the value 2^{-n}. Thus f induces a flow on G with support S.

Next we estimate $E_\epsilon(f)$. A vertex v at level n of S has degree at most $8 \cdot 2^n$. Thus

$$E_\epsilon(f) \leq 1000 \cdot \sum_{n\in\mathbb{N}} 2^n \cdot 2^{n(1-\epsilon)} \cdot 2^{-2n} = 1000 \cdot \sum_{n\in\mathbb{N}} 2^{-\epsilon n}$$

Hence $E_\epsilon(f)$ is finite, completing this example.

4.0.18 Further remarks

As mentioned in the introduction, we expect our notion of roundabout-transience to find further applications. For example, we expect that the results of [90, Section 2] generalise from bounded-degree non-amenable planar maps to roundabout-transient ones.

A lot of this paper is motivated by [66], the main result of which states that the Poisson boundary of every bounded degree, uniquely absorbing, plane graph coincides with the boundary of the square tiling; this had been asked by Benjamini & Schramm [11]. We can now ask whether this generalises to graphs of unbounded degree using roundabout-transience:

Problem 1. *Does the Poisson boundary of every uniquely absorbing, roundabout-transient plane graph coincide with the boundary of its square tiling?*

A closely related result of [5] states that the Poisson boundary of every 1-ended triangulation of the plane coincides with the boundary of its circle packing. Again, we ask for a similar generalisation:

Problem 2. *Does the Poisson boundary of every 1-ended, roundabout-transient, triangulation of the plane coincide with the boundary of its circle packing?*

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Appendix A

We summarise the results of this thesis very briefly - first in English, then in German.

A.1 Summary

In Chapter 1, we show that every connected graph has a spanning tree that displays all its topological ends. This proves a 1964 conjecture of Halin in corrected form, and settles a problem of Diestel from 1992.

The techniques to build these spanning trees are based on earlier work on canonical tree-decompositions of finite graphs. These tree-decompositions are developed in Chapter 2 in order to decompose a finite graph into its highly connected pieces, which extends a central theorem of Tutte.

In Chapter 3, we study the relationship between (infinite) matroids and graphs. Perhaps unexpectedly, the trees of Chapter 1 appear here as an important tool. Another surprise of this Chapter is that any (3-connected tame) matroid with all finite minors graphic has only countable circuits, which allows for a topological representation of these matroids with the circuits being homeomorphic images of the unit circle.

In Chapter 4, we introduce the notion of *roundabout transience* of plane graphs, and show that every roundabout transient planar graph admits non-constant Dirichlet harmonic functions. A corollary of this is that every bounded degree planar transient graph admits non-constant Dirichlet harmonic functions, as originally shown by Bejamini and Schramm. However, our theorem is also non-trivial for unbounded degree graphs: for example non-amenable plane graphs are roundabout transient.

A.2 Zusammenfassung

In Kapitel 1 zeigen wir, dass jeder zusammenhängende Graph einen Spannbaum hat, der alle topologischen Enden darstellt. Dies beweist eine 50 Jahre alte Vermutung von Halin in korrigierter Form und löst ein Problem von Diestel von 1992.

Die Methoden, um diese Spannbäume zu konstruieren, basieren auf kanonischen Baumzerlegungen von endlichen Graphen. Diese Baumzerlegungen werden in Kapitel 2 entwickelt, um einen endlichen Graphen in seine hochzusammenhängenden Teile zu zerlegen. Dies erweitert einen zentralen Satz von Tutte.

In Kapitel 3 beschäftigen wir uns mit der Beziehung zwischen (unendlichen) Matroiden und Graphen. Überraschenderweise tauchen hier die Spannbäume aus Kapitel 1 als wichtiges Werkzeug auf. Eine andere Überraschung ist, dass alle (3-zusammenhängenden zahmen) Matroide, bei denen alle endlichen Minoren graphisch sind, nur abzählbare Kreise besitzen. Dies ermöglicht es diese Matroide durch topologische Räume darzustellen, wobei die Kreise des Matroids homeomorphe Bilder des Einheitskreises sind.

In Kapitel 4, führen wir den Begriff der *Kreisel-Transienz* von ebenen Graphen ein und zeigen, dass jeder Kreisel-transiente ebene Graph eine nicht-konstante Dirichlet harmonische Funktion hat. Dies impliziert, dass jeder ebene Graph mit beschränkten Graden eine nicht-konstante Dirichlet harmonische Funktion hat, wie erstmals von Benjamini und Schramm beweisen wurde. Darüber hinaus ist unser Satz ist auch im Falle unbeschränkter Grade interessant: zum Beispiel ist jeder nicht-mittelbare ebene Graph Kreisel-transient.

A.3 My contributions

Chapter 1 and Sections 2.4 and 3.2 are based on single authored papers. My coauthors and I contributed a fair share in all 8 joint papers on which this thesis is based. In particular, I highlight the following aspects:

- The formulation of Theorem 2.1.28., the main result of Chapter 2, is mine. Moreover I contributed lots of ideas towards its proof. In particular the strategy how to deal with separations of different sizes, which is the main difficulty.

- I first conjectured that every circuit in a 3-connected tame graphic matroid is countable and its proof in this thesis is mine. However Nathan contributed lots of ideas towards our first much more complicated proof.

- I invented the concept of roundabout transience and the overall proof strategy for Theorem 4.0.57., the main result of Chapter 4.

A.4 Papers on which this thesis is based

According to the Bibliography, this thesis is based on the following papers: 35, 42, 39, 40, 41, 44, 19, 37, 23 and 43. Here is a separate list of all these papers:

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den