Shape optimization based on a fictitious domain method with curves

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Thorben Vehling

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und Prof. Dr. Thomas Slawig

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Prof. Dr. Bernd Siebert
Leiter des Fachbereichs Mathematik

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Introduction

In this work we investigate shape optimization problems for the two-dimensional Poisson equation with homogeneous Dirichlet boundary conditions.
In general the aim of shape optimization problems is as follows: Find the optimal shape Ω such that a given functional $\bar{J}(\bar{u}, \Omega)$ is minimized while the state variable $\bar{u}$ solves a given state equation $\bar{E}(\bar{u}, \Omega) = 0$, i.e.

$$(\bar{P}_\Omega) \qquad \begin{cases} \min \ \bar{J}(\bar{u}, \Omega) \\ \text{s.t.} \ \ \Omega \in \mathcal{O}_{\text{ad}}, \\ \bar{E}(\bar{u}, \Omega) = 0 \quad \text{in } \Omega, \end{cases}$$

In shape optimization, two approaches exist to characterize admissible domains, namely domain variation approaches and boundary variation approaches. The domain variation approach is further classified by the perturbation of identity method [MS76, BLUU09] and the velocity field method [SZ92]. In the boundary variation ap-

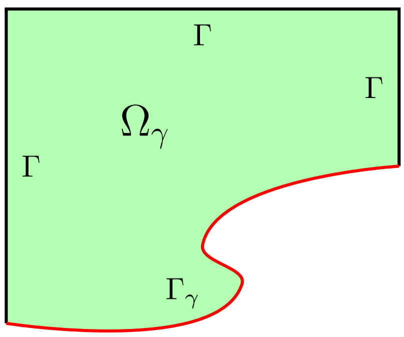


Figure 0.1.: A domain Ω_γ characterized through the boundary parts Γ and Γ_γ

proach, admissible domain are characterized via a boundary parametrization. Problems of this kind are considered in [KP98, KP01, GKM00, Sla00, Sla03], where Γ_γ is parametrized as a graph of a function γ.
In the present work we extend this approach to problems, where Γ_γ is parametrized by smooth curves. In detail: given a physical domain Ω_γ (see Figure 0.1), find Γ_γ such that a given functional $J(u, \gamma)$ is minimized while the state variable u solves the elliptic PDE

$$\begin{aligned} -\Delta u &= f_\gamma \quad \text{in } \Omega_\gamma, \\ u &= 0 \quad \text{on } \partial\Omega_\gamma := \overline{\Gamma} \cup \overline{\Gamma}_\gamma, \end{aligned}$$

see Figure 0.1.

In this setting the set of admissible domains $\mathcal{O}_{\mathrm{ad}}$ is defined through a set of admissible curves S_{ad}, i.e.

$$\mathcal{O}_{\mathrm{ad}} := \{\Omega_\gamma := \Omega(\gamma) \subset \mathbb{R}^2 : \partial\Omega_\gamma = \overline{\Gamma} \cup \overline{\Gamma}_\gamma, \gamma \in S_{\mathrm{ad}}\}, \tag{0.0.1}$$

where the set S_{ad} is specified below.
The model problem can then be written in the form of an optimal control problem

$$(P_\gamma) \qquad \begin{cases} \min\ J(u,\gamma) \\ \text{s.t.}\ \ \gamma \in S_{\mathrm{ad}}, \\ E(u,\gamma) = 0 \quad \text{in } \Omega_\gamma, \end{cases}$$

where the curve γ is acting as the control variable. The objective functional J and the state operator E are specified below.
We use a fictitious domain method (also referred to as embedding domain method) [GPP94] to reformulate the state equation as an equivalent problem defined on a simply shaped domain $\hat{\Omega}$ (called fictitious domain) in which the original domain Ω_γ is embedded (see Figure 1.1a).
This is done by extending the involved functions to functions defined on $\hat{\Omega}$ and using a boundary Lagrange multiplier technique to incorporate the boundary conditions on the boundary part Γ_γ which now is located inside the fictitious domain $\hat{\Omega}$.
This leads to the equivalent problem formulation

$$(\hat{P}_\gamma) \qquad \begin{cases} \min\ \hat{J}((\hat{u},\mathcal{G}),\gamma) \\ \text{s.t.}\ \ \gamma \in S_{\mathrm{ad}}, \\ \hat{E}((\hat{u},\mathcal{G}),\gamma) = 0 \quad \text{in } \hat{\Omega}, \end{cases}$$

which has computational advantages since the state equation has to be solved on the fixed domain, $\hat{\Omega}$, which does not change during the optimization process.

This work is structured as follows. In the first chapter we define the geometry of admissible domains by introducing a set of admissible curves. Then we discuss a family of elliptic partial differential equations, which are stated on the admissible domains with homogeneous Dirichlet boundary conditions and serve as our state equation in the shape optimization problem. In Chapter 2, we introduce the functional analytical framework in order to use the fictitious domain method in our setting as described above. Chapter 3 considers a class of shape optimization problems, on which we apply the fictitious domain method. Chapter 4 introduces the discretization via mixed finite elements methods, approximation of the integral representations of the derivatives and finite dimensional descent methods. In Chapter 5, we present numerical experiments.

1. Characterization of admissible domains

At the beginning we describe the geometry of admissible domains $\Omega_\gamma \in \mathcal{O}_{\text{ad}}$. Then we introduce the state equation which is later used in our shape optimization problem.

1.1. Admissible domains $\Omega_\gamma \in \mathcal{O}_{\text{ad}}$

We use admissible domains of the shape depicted in Figure 1.1a: The boundary $\partial\Omega_\gamma$ of an admissible domain $\Omega_\gamma \in \mathcal{O}_{\text{ad}}$ is composed of the fixed part

$$\Gamma := \{0\}\times]\alpha_1, 1[\cup[0,1] \times \{1\} \cup \{1\}\times]\beta_1, 1[,$$

and a variable part

$$\Gamma_\gamma := \{\gamma(t) = (\gamma_1(t), \gamma_2(t)) \in \mathbb{R}^2 : t \in I \subset \mathbb{R}\},$$

where $\alpha_1,\ \beta_1 \in [0,1]$ and $\gamma : I \to \mathbb{R}^2$ is a smooth curve with $\gamma(0) = (0, \alpha_1)$, $\gamma(1) = (1, \beta_1)$. We obtain $\partial\Omega_\gamma = \overline{\Gamma} \cup \overline{\Gamma}_\gamma$ and $\{(0,\alpha_1),(1,\beta_1)\} = \overline{\Gamma} \cap \overline{\Gamma}_\gamma$.
We assume that all admissible domains Ω_γ are embedded in the hold-all domain

$$\hat{\Omega} := (0,1) \times (0,1).$$

We denote by Ω_γ^c the complement of Ω_γ with respect to $\hat{\Omega}$, i.e. $\Omega_\gamma^c := \hat{\Omega} \setminus \overline{\Omega}_\gamma$. With the fixed fictitious boundary part

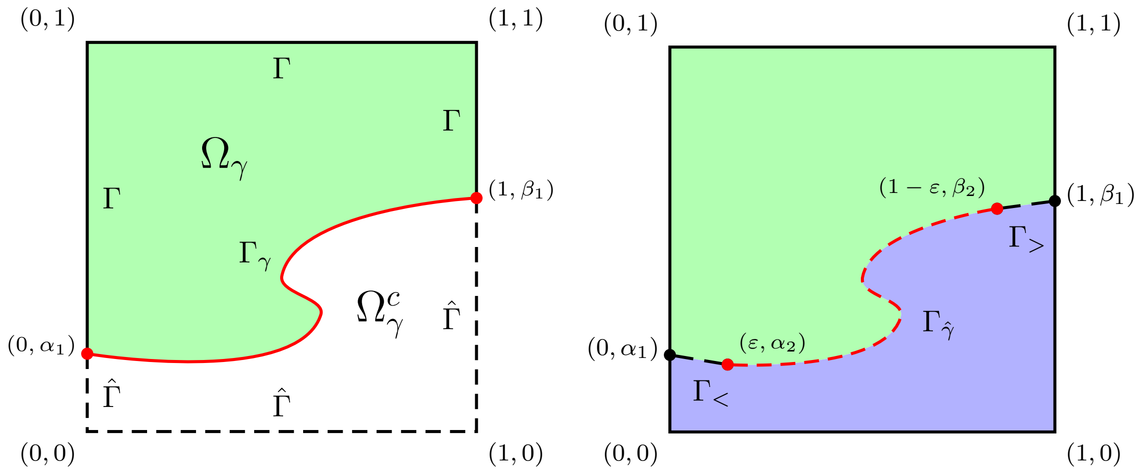
$$\hat{\Gamma} := \{0\}\times]0, \alpha_1[\cup[0,1] \times \{0\} \cup \{1\}\times]0, \beta_1[$$

we have $\partial\Omega_\gamma^c = \overline{\hat{\Gamma}} \cup \overline{\Gamma}_\gamma$ and $\{(0,\alpha_1),(1,\beta_1)\} = \overline{\hat{\Gamma}} \cap \overline{\Gamma}_\gamma$.

We impose several requirements on the set of admissible curves γ:

Assumption 1.1.1. *The curve γ characterizing the variable boundary part Γ_γ*

1. *connects the two endpoints $(0,\alpha_1)$ and $(1,\beta_1)$,*
2. *is contained in $\hat{\Omega}$,*
3. *satisfies $\gamma(t) \neq \gamma(s)$ for $t \neq s$,*
4. *behaves in such way that Ω_γ and Ω_γ^c are Lipschitz-domains,*



(a) Involved domains Ω_γ, Ω_γ^c and corresponding boundary parts Γ_γ, Γ and $\hat{\Gamma}$

(b) Composition of the boundary part Γ_γ of fixed segments $\Gamma_<$, $\Gamma_>$ and a variable part $\Gamma_{\hat{\gamma}}$.

Figure 1.1.: Fictitious domain setting, $\hat{\Omega} = (0,1) \times (0,1)$

5. *is sufficiently smooth to guarantee H^2-regularity of the solution of the state equation on Ω_γ.*

The aim of the next section is to discuss some sufficient conditions to construct a set S_{ad} of admissible curves which fulfill Assumption 1.1.1.

1.2. Admissible curves $\gamma \in S_{\text{ad}}$

Admissible curves $\gamma \in S_{\text{ad}}$ shall be parametrized over the interval $I := (0,1)$. To ensure Assumptions 1.1.1.2 and 1.1.1.4 we split the interval I for $0 < \varepsilon < 0.5$ into open intervals

$$I_< = (0,\varepsilon), \quad \hat{I} = (\varepsilon, 1-\varepsilon), \quad I_> = (1-\varepsilon, 1).$$

The boundary part Γ_γ is then composed of fixed boundary segments $\Gamma_<$, $\Gamma_>$, and a variable boundary part $\Gamma_{\hat{\gamma}}$ (see Figure 1.1b), i.e.

$$\overline{\Gamma}_\gamma = \overline{\Gamma}_< \cup \overline{\Gamma}_{\hat{\gamma}} \cup \overline{\Gamma}_>.$$

Let $\mathcal{S}(\hat{I}) := H^3(\hat{I})^2$. We define the equivalence relation

$$\gamma \sim \delta \text{ in } \mathcal{S}(\hat{I}) \;:\Longleftrightarrow\; \gamma(\hat{I}) = \delta(\hat{I}),$$

and the related quotient space $S(\hat{I}) := \mathcal{S}(\hat{I})/\sim$. In this way all curves in $H^3(\hat{I})^2$ that have the same image (and therefore shape an identical boundary part) are represented by one equivalence class.

For α_1, β_1 from above and α_2, $\beta_2 \in [2\varepsilon, 1-2\varepsilon]$ we define

$$\begin{aligned}
\hat{S}_{\text{ad}} := \Big\{\hat{\gamma} = (\hat{\gamma}_1, \hat{\gamma}_2)^T \in S(\hat{I}) : \|\hat{\gamma}\|_{H^3(\hat{I})^2} \leq c_s, \quad & (1.2.1)\\
\hat{\gamma}(\varepsilon) = (\varepsilon, \alpha_2)^T, \quad & (1.2.2)\\
\hat{\gamma}(1-\varepsilon) = (1-\varepsilon, \beta_2)^T, \quad & (1.2.3)\\
(\alpha_2 - \alpha_1)\,\dot{\hat{\gamma}}_1(\varepsilon) - \varepsilon\dot{\hat{\gamma}}_2(\varepsilon) = 0, \quad & (1.2.4)\\
(\beta_2 - \beta_1)\,\dot{\hat{\gamma}}_1(1-\varepsilon) + \varepsilon\dot{\hat{\gamma}}_2(1-\varepsilon) = 0, \quad & (1.2.5)\\
\ddot{\hat{\gamma}}(\varepsilon) = \ddot{\hat{\gamma}}(1-\varepsilon) = (0,0)^T, \quad & (1.2.6)\\
\|\hat{\gamma}(t_2) - \hat{\gamma}(t_1)\|_2 \geq c_r|t_2 - t_1| \text{ for all } t_1, t_2 \in \hat{I}, t_1 \neq t_2, \quad & (1.2.7)\\
\operatorname{dist}(\Gamma_{\hat{\gamma}}, \partial\hat{\Omega}) \geq \varepsilon\Big\}. \quad & (1.2.8)
\end{aligned}$$

Finally we can compose the set of admissible curves

Definition 1.2.1.

$$S_{ad} := \left\{ \gamma = (\gamma_1, \gamma_2)^T : \gamma(t) = \begin{cases} \left(t, \frac{1}{\varepsilon}\left(\alpha_2 - \alpha_1\right)t + \alpha_1\right)^T & t \in I_< \\ \hat{\gamma}(t) & t \in \hat{I} \\ \left(t, \frac{1}{\varepsilon}\left(\beta_2 - \beta_1\right)(1-t) + \beta_1\right)^T & t \in I_> \end{cases}, \hat{\gamma} \in \hat{S}_{ad} \right\}.$$

Remark 1.2.2. (i) *Let $\gamma \in S_{ad}$. Then γ fulfills the Assumption 1.1.1.*

(ii) *We could achieve a less technical definition for the set of admissible curves without splitting the parameter interval I and directly define*

$$\begin{aligned}
S'_{ad} := \Big\{\gamma = (\gamma_1, \gamma_2)^T \in S(I) : \|\gamma\|_{H^3(I)^2} \leq c_s,\\
\gamma(0) = (0, \alpha_1)^T,\\
\gamma(1) = (1, \beta_1)^T,\\
(\alpha_2 - \alpha_1)\dot{\gamma}_1(0) - \varepsilon\dot{\gamma}_2(0) = 0,\\
(\beta_2 - \beta_1)\dot{\gamma}_1(1) + \varepsilon\dot{\gamma}_2(1) = 0,\\
\|\gamma(t_2) - \gamma(t_1)\|_2 \geq c_r|t_2 - t_1| \textit{ for all } t_1, t_2 \in I, t_1 \neq t_2\Big\}.
\end{aligned}$$

However, in this situation it is difficult to formulate a condition analogously to condition (1.2.8) *to ensure Assumption 1.1.1.2.*

1.3. The state equation

For a function $f \in L^2(\hat{\Omega})$ and $\gamma \in S_{\text{ad}}$ we set $f_\gamma := f|_{\Omega_\gamma}$ which defines a function in $L^2(\Omega_\gamma)$.

We now consider the Poisson equation with homogeneous Dirichlet boundary values

$$-\Delta u = f_\gamma \quad \text{in } \Omega_\gamma, \tag{1.3.1}$$
$$u = 0 \quad \text{on } \partial\Omega_\gamma, \tag{1.3.2}$$

and define the spaces

$$U(\Omega_\gamma) := H_0^1(\Omega_\gamma),$$
$$Z(\Omega_\gamma) := H_0^1(\Omega_\gamma)^* = H^{-1}(\Omega_\gamma).$$

The weak formulation of (1.3.1)-(1.3.2) is given by the following problem: Find $u \in U(\Omega_\gamma)$ such that

$$\langle v, E(u,\gamma)\rangle_{Z(\Omega_\gamma)^*, Z(\Omega_\gamma)} := \int_{\Omega_\gamma} \nabla u(x)^T \nabla v(x)\, dx - \int_{\Omega_\gamma} f_\gamma(x) v(x)\, dx = 0 \quad \forall v \in U(\Omega_\gamma) \tag{1.3.3}$$

This formulation defines the state operator

$$E : \{(u,\gamma) : u \in U(\Omega_\gamma), \gamma \in S_{\text{ad}}\} \to \{z : z \in Z(\Omega_\gamma), \gamma \in S_{\text{ad}}\}.$$

Theorem 1.3.1 (Existence and uniqueness)**.** *The Poisson equation $E(u,\gamma) = 0$ for all $\gamma \in S_{ad}$ and $f_\gamma \in L^2(\Omega_\gamma)$ admits a unique weak solution $u \in U(\Omega_\gamma)$ which satisfies*

$$\|u\|_{H^1(\Omega_\gamma)} \le C \|f_\gamma\|_{L^2(\Omega_\gamma)}.$$

Here, the positive constant C depends on γ but not on f_γ.

For later reference we note:

Remark 1.3.2. *For $\gamma \in S_{ad}$ the following two assertions are equivalent:*

1. *$u \in U(\Omega_\gamma)$ satisfies $E(u,\gamma) = 0$.*
2. *$u \in U(\Omega_\gamma)$ minimizes the energy functional*

$$\mathcal{Q}(v) := \frac{1}{2}(\nabla v, \nabla v)_{L^2(\Omega_\gamma)^2} - (f_\gamma, v)_{L^2(\Omega_\gamma)}.$$

Theorem 1.3.3 (Regularity and uniform boundedness)**.** *The solution $u_\gamma := u(\gamma)$ of the Poisson equation $E(u,\gamma) = 0$ satisfies $u_\gamma \in H^2(\Omega_\gamma)$ for all $\gamma \in S_{ad}$, and the set*

$$\left\{\|u_\gamma\|_{H^2(\Omega_\gamma)}\right\}_{\gamma \in S_{ad}}$$

is uniformly bounded.

Proof. Since $\|f_\gamma\|_{L^2(\Omega_\gamma)} \le \|f\|_{L^2(\hat{\Omega})}$ the claim follows from regularity theory of weak solutions (see e.g. [Eva98]) together with our definition of S_{ad}. The latter implies that the domain Ω_γ is locally convex at $(0, \alpha_1)$, $(1, \beta_1)$, $(0, 1)$, and $(1, 1)$, and is of class $C^{1,1}$ uniformly in γ, otherwise. $\square$

2. The fictitious domain method

In this chapter we transform the Poisson problem given on Ω_γ to an equivalent problem posed on the fictitious domain $\hat{\Omega}$. From Definition 1.2.1 we have

$$\Omega_\gamma \subset \hat{\Omega} \text{ for all } \gamma \in S_{\text{ad}}.$$

Definition 2.0.4. *For $\gamma \in S_{ad}$ and $v \in L^2(\Omega_\gamma)$ we denote by*

$$\widetilde{v} := \begin{cases} v & \text{in } \Omega_\gamma, \\ 0 & \text{in } \Omega_\gamma^c, \end{cases}$$

the extension of v by zero onto $\hat{\Omega}$.

2.1. Interpretation of the boundary condition as an additional constraint

As already mentioned we have to impose the boundary condition $u|_{\Gamma_\gamma} = 0$ on the boundary part Γ_γ as a constraint, because Γ_γ no longer is part of the boundary $\partial\hat{\Omega}$. As an analogue of Remark 1.3.2 we study a constrained minimization problem on $\hat{\Omega}$:

$$(\hat{V}_\gamma) \qquad \begin{cases} \hat{\mathcal{Q}}(\hat{u}) = \min\limits_{\hat{v} \in H_0^1(\hat{\Omega})} \hat{\mathcal{Q}}(\hat{v}) := \dfrac{1}{2}(\nabla\hat{v}, \nabla\hat{v})_{L^2(\hat{\Omega})^2} - (\widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})} \\ \text{s.t. } \ \hat{v} = 0 \quad \text{on } \Gamma_\gamma. \end{cases}$$

In the following we introduce the function spaces for a proper treatment of the equality constraint in a Lagrange multiplier framework. The following results are collected from [KP98, Sla98, Sla00, KP01] and they are direct consequences of properties of the trace operator, see e.g. [Eva98], combined with our assumptions on S_{ad}.

Lemma 2.1.1. *Let Γ_0 be an open subset of $\partial\Omega_\gamma$, $\partial\Omega_\gamma^c$ or $\partial\hat{\Omega}$. Then, the space*

$$H^{1/2}(\Gamma_0) := \left\{ g \in L^2(\Gamma_0) : \int_{\Gamma_0}\int_{\Gamma_0} \frac{|g(x) - g(y)|^2}{|x-y|^2}\, dS(x)\, dS(y) < \infty \right\}$$

is a Hilbert space with inner product

$$(g,h)_{H^{1/2}(\Gamma_0)} = (g,h)_{L^2(\Gamma_0)} + \int_{\Gamma_0}\int_{\Gamma_0} \frac{(g(x)-g(y))(h(x)-h(y))}{|x-y|^2}\, dS(x)\, dS(y)$$

and the norm

$$\|g\|_{H^{1/2}(\Gamma_0)} = \sqrt{\|g\|^2_{L^2(\Gamma_0)} + \int_{\Gamma_0}\int_{\Gamma_0} \frac{|g(x)-g(y)|^2}{|x-y|^2}\, dS(x)\, dS(y)}.$$

Lemma 2.1.2. *For every $\gamma \in S_{ad}$ the Lions-Magenes space*

$$\begin{aligned} H_\gamma &:= H^{1/2}_{00}(\Gamma_\gamma) \\ &:= \Big\{ h \in H^{1/2}(\Gamma_\gamma) : \textit{there exists } \widetilde{h} \in H^{1/2}(\partial\Omega_\gamma) \textit{ with } \widetilde{h}|_{\Gamma_\gamma} = h, \widetilde{h}|_{\Gamma} = 0 \Big\} \end{aligned}$$

is a Hilbert space with the inner product

$$(g,h)_{H_\gamma} := (\widetilde{g}, \widetilde{h})_{H^{1/2}(\partial\Omega_\gamma)},$$

and the norm

$$\|g\|_{H_\gamma} := \|\widetilde{g}\|_{H^{1/2}(\partial\Omega_\gamma)},$$

where $\widetilde{g}$, $\widetilde{h} \in H^{1/2}(\partial\Omega_\gamma)$ satisfy $\widetilde{g}|_{\Gamma_\gamma} = g$, $\widetilde{h}|_{\Gamma_\gamma} = h$, $\widetilde{g}|_\Gamma = \widetilde{h}|_\Gamma = 0$.

Lemma 2.1.3. *Let $\gamma \in S_{ad}$. The trace operator*

$$\tau_\gamma u := u|_{\Gamma_\gamma}$$

is a linear continuous mapping from $H^1(\Omega_\gamma)$ onto $H^{1/2}(\Gamma_\gamma)$.

Lemma 2.1.4. *Let $\gamma \in S_{ad}$.*

(i) The trace operator

$$\tau_\gamma \hat{u} := \hat{u}|_{\Gamma_\gamma}$$

is a linear continuous mapping from $H^1(\hat{\Omega})$ onto $H^{1/2}(\Gamma_\gamma)$ and from $H^1_0(\hat{\Omega})$ onto H_γ.

(ii) τ_γ is surjective.

(iii) The family of trace operators $\{\tau_\gamma\}_{\gamma \in S_{ad}}$ is uniformly bounded in $\mathcal{L}(H^1_0(\hat{\Omega}), H_\gamma)$, i.e. for all $\hat{u} \in H^1_0(\hat{\Omega})$ there holds

$$\|\tau_\gamma \hat{u}\|_{H_\gamma} \le C \|\hat{u}\|_{H^1(\hat{\Omega})},$$

where C is independent of $\gamma \in S_{ad}$.

Theorem 2.1.5. *For $\hat{u} \in H^1_0(\hat{\Omega})$ there holds:*

$$\hat{u} = 0 \textit{ on } \Gamma_\gamma \quad \Longleftrightarrow \quad \langle h, \tau_\gamma \hat{u} \rangle_{H^*_\gamma, H_\gamma} = 0 \quad \textit{for all } h \in H^*_\gamma.$$

In a next step we use a pullback mapping to define our problems on spaces which are independent of γ.

Lemma 2.1.6. *The space*

$$H^{1/2}(I) := \left\{ g \in L^2(I) : \int_I \int_I \frac{|g(s)-g(t)|^2}{|s-t|^2}\, ds\, dt < \infty \right\}$$

is a Hilbert space with inner product

$$(g,h)_{H^{1/2}(I)} = (g,h)_{L^2(I)} + \int_I \int_I \frac{(g(s)-g(t))(h(s)-h(t))}{|s-t|^2}\, ds\, dt$$

and the norm

$$\|g\|_{H^{1/2}(I)} = \sqrt{\|g\|^2_{L^2(I)} + \int_I \int_I \frac{|g(s)-g(t)|^2}{|s-t|^2}\, ds\, dt.}$$

Lemma 2.1.7. *The space*

$$H_I := \left\{ g \in H^{1/2}(I) : \int_I \frac{|g(t)|^2}{t(1-t)}\, dt < \infty \right\}$$

is a Hilbert space with the inner product

$$(g,h)_{H_I} := (g,h)_{H^{1/2}(I)} + \int_I \frac{g(t)h(t)}{t(1-t)}\, dt,$$

and the norm

$$\|g\|_{H_I} := \left(\|g\|^2_{H^{1/2}(I)} + \int_I \frac{|g(t)|^2}{t(1-t)}\, dt \right)^{1/2}.$$

Lemma 2.1.8. *The space H_I is dense in $L^2(I)$.*

Since $\gamma \in S_{\mathrm{ad}}$ is bijective, our assumptions on S_{ad} directly imply the following results:

Lemma 2.1.9. *For every $\gamma \in S_{ad}$ the mapping $\mathcal{I}_\gamma$ with $\mathcal{I}_\gamma h := h \circ \gamma$ is an isomorphism from $L^2(\Gamma_\gamma)$ onto $L^2(I)$, from $H^{1/2}(\Gamma_\gamma)$ onto $H^{1/2}(I)$ and from H_γ onto H_I. The continuity of $\mathcal{I}_\gamma$ and of its inverse are uniform on S_{ad}. Here, uniformness on S_{ad} means that for all $g \in H_\gamma$ there holds*

$$\|\mathcal{I}_\gamma g\|_{H_I} \leq C \|g\|_{H_\gamma},$$

where C is independent of $\gamma \in S_{ad}$.

Lemma 2.1.10. *Let $\gamma \in S_{ad}$.*

1. *The trace operator*

   $$\mathcal{T}_\gamma := \mathcal{I}_\gamma \circ \tau_\gamma$$

   is a linear continuous mapping from $H^1(\Omega)$ onto $H^{1/2}(I)$ and from $H^1_0(\hat{\Omega})$ to H_I.

2. *$\mathcal{T}_\gamma$ is surjective.*

3. *The family of trace operators $\{\mathcal{T}_\gamma\}_{\gamma \in S_{ad}}$ is uniformly bounded in $\mathcal{L}(H_0^1(\hat{\Omega}), H_I)$, i.e. for all $\hat{u} \in H_0^1(\hat{\Omega})$ there holds*

$$\|\mathcal{T}_\gamma \hat{u}\|_{H_I} \leq C\|\hat{u}\|_{H^1(\hat{\Omega})},$$

where C is independent of $\gamma \in S_{ad}$.

Theorem 2.1.11. *For $\hat{u} \in H_0^1(\hat{\Omega})$ there holds:*

$$\hat{u} = 0 \text{ on } \Gamma_\gamma \quad \Longleftrightarrow \quad \langle \mathcal{H}, \mathcal{T}_\gamma \hat{u} \rangle_{H_I^*, H_I} = 0 \quad \text{for all } \mathcal{H} \in H_I^*.$$

2.2. Existence and uniqueness result

We are now in the position to reformulate problem $(\hat{V}_\gamma)$:

Theorem 2.2.1. *Let $\gamma \in S_{ad}$. The problem finding $\hat{u}_\gamma := \hat{u}(\gamma) \in H_0^1(\hat{\Omega})$ such that*

$$(\hat{V}) \quad \begin{cases} \hat{\mathcal{Q}}(\hat{u}_\gamma) = \min\limits_{\hat{v} \in H_0^1(\hat{\Omega})} \hat{\mathcal{Q}}(\hat{v}) = \dfrac{1}{2}(\nabla \hat{v}, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - (\widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})} \\ s.t. \;\; \langle \mathcal{H}, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} = 0 \quad \text{for all } \mathcal{H} \in H_I^*, \end{cases}$$

has a unique solution.
Furthermore there exists a unique $\mathcal{G}_\gamma := \mathcal{G}(\gamma) \in H_I^*$ such that $(\hat{u}_\gamma, \mathcal{G}_\gamma)$ is the unique saddle point of the Lagrangian $\hat{\mathcal{P}} : H_0^1(\hat{\Omega}) \times H_I^* \to \mathbb{R}$,

$$\hat{\mathcal{P}}(\hat{u}, \mathcal{G}) := \hat{\mathcal{Q}}(\hat{u}) - \langle \mathcal{G}, \mathcal{T}_\gamma \hat{u} \rangle_{H_I^*, H_I},$$

over the set $H_0^1(\hat{\Omega}) \times H_I^*$. The pair $(\hat{u}_\gamma, \mathcal{G}_\gamma) = (\hat{u}(\gamma), \mathcal{G}(\gamma))$ is the unique solution of

$$\begin{cases} (\nabla \hat{u}_\gamma, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} = (\widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})}, & \forall \hat{v} \in H_0^1(\hat{\Omega}), \\ -\langle \mathcal{H}, \mathcal{T}_\gamma \hat{u}_\gamma \rangle_{H_I^*, H_I} = 0, & \forall \mathcal{H} \in H_I^*. \end{cases} \tag{2.2.1}$$

We call this system *fictitious domain formulation* of the Poisson problem. We introduce the bounded linear operator $\hat{\mathcal{A}} : H_0^1(\hat{\Omega}) \to H_0^1(\hat{\Omega})^*$ with

$$\langle \hat{\mathcal{A}} \hat{u}, \hat{v} \rangle_{H_0^1(\hat{\Omega})^*, H_0^1(\hat{\Omega})} := (\nabla \hat{u}, \nabla \hat{v})_{L^2(\hat{\Omega})^2} = \int_{\hat{\Omega}} \nabla \hat{u}(x)^T \nabla \hat{v}(x)\, dx,$$

and the bounded linear functional $\hat{\mathcal{F}}_\gamma : H_0^1(\hat{\Omega}) \to \mathbb{R}$,

$$\langle \hat{\mathcal{F}}_\gamma, \hat{v} \rangle_{H_0^1(\hat{\Omega})^*, H_0^1(\hat{\Omega})} := (\widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})} = \int_{\Omega_\gamma} f_\gamma(x) \hat{v}(x)\, dx.$$

This allows us to rewrite (2.2.1) in the form

$$\begin{pmatrix} \hat{\mathcal{A}} & -\mathcal{T}_\gamma^* \\ -\mathcal{T}_\gamma & 0 \end{pmatrix} \begin{pmatrix} \hat{u}_\gamma \\ \mathcal{G}_\gamma \end{pmatrix} = \begin{pmatrix} \hat{\mathcal{F}}_\gamma \\ 0 \end{pmatrix},$$

and accordingly

$$\hat{E}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma) := \begin{pmatrix} \hat{\mathcal{A}}\hat{u}_\gamma - \mathcal{T}_\gamma^* \mathcal{G}_\gamma - \hat{\mathcal{F}}_\gamma \\ -\mathcal{T}_\gamma \hat{u}_\gamma \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{in } \left(H_0^1(\hat{\Omega}) \times H_I^*\right)^*. \tag{2.2.2}$$

This defines the fictitious domain state operator

$$\hat{E} : \left(H_0^1(\hat{\Omega}) \times H_I^*\right) \times S_{\text{ad}} \to \left(H_0^1(\hat{\Omega}) \times H_I^*\right)^*.$$

2.3. Equivalence of the problems on Ω_γ and $\hat{\Omega}$

Theorem 2.3.1. *Let $\gamma \in S_{ad}$ and $f \in L^2(\hat{\Omega})$.*

1. *Let $(\hat{u}_\gamma, \mathcal{G}_\gamma) \in H_0^1(\hat{\Omega}) \times H_I^*$ denote the solution of the fictitious domain formulation of the Poisson problem* (2.2.1) *on $\hat{\Omega}$. Then $u_\gamma := \hat{u}_\gamma|_{\Omega_\gamma} \in H^2(\Omega_\gamma) \cap H_0^1(\Omega_\gamma)$ is the unique solution of the weak formulation of the Poisson problem* (1.3.3) *on Ω_γ. On Ω_γ^c we have $\hat{u}_\gamma = 0$ and the Lagrange multiplier $\mathcal{G}_\gamma$ satisfies*

$$\langle \mathcal{G}_\gamma, h \rangle_{H_I^*, H_I} = \left(\frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \|\dot{\gamma}(\cdot)\|_2, h \right)_{L^2(I)} \qquad \forall h \in H_I. \tag{2.3.1}$$

   Here, $n_\gamma(t) := \frac{1}{\sqrt{\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2}} \begin{pmatrix} \dot{\gamma}_2(t) \\ -\dot{\gamma}_1(t) \end{pmatrix}$, $t \in (0,1)$, denotes the unit outer normal vector along the boundary part Γ_γ.

2. *Conversely, let $u_\gamma \in H^2(\Omega_\gamma) \cap H_0^1(\Omega_\gamma)$ be the solution of* (1.3.3)*. Then* (2.3.1) *uniquely defines an element $\mathcal{G}_\gamma \in H_I^*$ and $(\hat{u}_\gamma, \mathcal{G}_\gamma)$ with*

$$\hat{u}_\gamma := \begin{cases} u_\gamma & \textit{in } \Omega_\gamma, \\ 0 & \textit{in } \Omega_\gamma^c, \end{cases}$$

   is the unique solution of (2.2.1)*.*

Proof. (1) Let $(\hat{u}_\gamma, \mathcal{G}_\gamma)$ be the solution of (2.2.1). For $u_\gamma := \hat{u}_\gamma|_{\Omega_\gamma}$ we have $\mathcal{T}_\gamma u_\gamma = \mathcal{T}_\gamma \hat{u}_\gamma = 0$ and therefore $u_\gamma|_{\partial\Omega_\gamma} = 0$. Let $\widetilde{v}$ denote the extension by zero of $v \in H_0^1(\Omega_\gamma)$ onto $\hat{\Omega}$ which is in $H_0^1(\hat{\Omega})$. From (2.2.1) we get

$$(\nabla \hat{u}_\gamma, \nabla \widetilde{v})_{L^2(\hat{\Omega})^2} = (\widetilde{f}_\gamma, \widetilde{v})_{L^2(\hat{\Omega})} + \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \widetilde{v} \rangle_{H_I^*, H_I} \quad \forall v \in H_0^1(\Omega_\gamma).$$

Since $\widetilde{v}|_{\partial\Omega_\gamma} = 0$ and $\widetilde{v}|_{\Omega_\gamma^c} = 0$ we obtain

$$(\nabla \hat{u}_\gamma, \nabla \widetilde{v})_{L^2(\hat{\Omega})^2} = (\nabla u_\gamma, \nabla v)_{L^2(\Omega_\gamma)^2} = (f_\gamma, v)_{L^2(\Omega_\gamma)} \quad \forall v \in H_0^1(\Omega_\gamma),$$

so that u_γ is the unique solution of (1.3.3).
Consider analogously $u_\gamma^c := \hat{u}_\gamma|_{\Omega_\gamma^c}$, any $v^c \in H_0^1(\Omega_\gamma^c)$ and denote by $\bar{v}^c$ its extension by zero onto $\hat{\Omega}$. Then we get $\bar{v}^c \in H_0^1(\hat{\Omega})$ and with $\mathcal{T}_\gamma u_\gamma^c = \mathcal{T}_\gamma \hat{u}_\gamma = 0$ also $u_\gamma^c \in H_0^1(\Omega_\gamma^c)$. By definition we have $\widetilde{f}_\gamma|_{\Omega_\gamma^c} = 0$. Hence we get from (2.2.1)

$$\left(\nabla u_\gamma^c, \nabla v\right)_{L^2(\Omega_\gamma^c)^2} = 0 \quad \forall v \in H_0^1(\Omega_\gamma^c),$$

and thus $u_\gamma^c = 0$ on Ω_γ^c.
The first equation of the fictitious domain problem (2.2.1) now reads

$$(\nabla u_\gamma, \nabla \hat{v})_{L^2(\Omega_\gamma)^2} - (f_\gamma, \hat{v})_{L^2(\Omega_\gamma)} = \langle \mathcal{G}, \mathcal{T}_\gamma \hat{v}\rangle_{H_I^*, H_I} \quad \forall \hat{v} \in H_0^1(\hat{\Omega}).$$

Since $u_\gamma \in H^2(\Omega_\gamma)$ we can apply integration by parts and obtain

$$\int_{\Omega_\gamma} (-\Delta u_\gamma - f_\gamma)\, \hat{v}\, dx + \int_{\partial\Omega_\gamma} \frac{\partial u_\gamma}{\partial n_\gamma} \hat{v}\, dS(x) = \langle \mathcal{G}, \mathcal{T}_\gamma \hat{v}\rangle_{H_I^*, H_I} \quad \forall \hat{v} \in H_0^1(\hat{\Omega}). \tag{2.3.2}$$

We now choose $v \in H_0^1(\Omega_\gamma)$ and test this equation with its extension $\widetilde{v} \in H_0^1(\hat{\Omega})$ which fulfills $\widetilde{v}|_{\partial\Omega_\gamma} = 0$. Then we get

$$(-\Delta u_\gamma - f_\gamma, v)_{L^2(\Omega_\gamma)} = 0 \quad \forall v \in H_0^1(\Omega_\gamma),$$

i.e.

$$-\Delta u_\gamma - f_\gamma = 0 \quad \text{in } H^{-1}(\Omega_\gamma).$$

Since all terms are in $L^2(\Omega_\gamma)$ this equation is also valid in this space. Now we test (2.3.2) with an arbitrary $\hat{v} \in H_0^1(\hat{\Omega})$. Then we have $\mathcal{T}_\gamma \hat{v} \in H_I$, $\hat{v}|_\Gamma = 0$, and (2.3.2) implies

$$\begin{aligned}
\langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{v}\rangle_{H_I^*, H_I} &= \int_{\partial\Omega_\gamma} \frac{\partial u_\gamma}{\partial n_\gamma} \hat{v}\, dS(x) = \int_{\Gamma_\gamma} \frac{\partial u_\gamma}{\partial n_\gamma} \hat{v}\, dS(x) \\
&= \int_I \frac{\partial u_\gamma(\gamma(t))}{\partial n_\gamma} \hat{v}(\gamma(t)) \|\dot{\gamma}(t)\|_2\, dt = \left(\frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \|\dot{\gamma}(\cdot)\|_2, \mathcal{T}_\gamma \hat{v} \right)_{L^2(I)},
\end{aligned}$$

which is (2.3.1).

(2) From (2.3.1) it follows that $\mathcal{G}_\gamma \in H_I^*$:

$$
\begin{aligned}
\langle \mathcal{G}_\gamma, h \rangle_{H_I^*, H_I} &\leq \left| \left(\frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \|\dot\gamma(\cdot)\|_2, h \right)_{L^2(I)} \right| \\
&\leq \|\gamma\|_{W^{1,\infty}(I)^2} \left\| \frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \right\|_{L^2(I)} \|h\|_{L^2(I)} \\
&\leq \|\gamma\|_{W^{1,\infty}(I)^2} \left\| \frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \right\|_{L^2(I)} \|h\|_{H_I}.
\end{aligned}
$$

This means that $\mathcal{G}_\gamma$ is bounded since $\gamma \in S_{\text{ad}}$ is bounded in $W^{1,\infty}(I)^2$ by definition, $\|h\|_{L^2(I)} \leq \|h\|_{H_I}$ and

$$
\frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \in L^2(I).
$$

Obviously $\hat u_\gamma$ satisfies $\mathcal{T}_\gamma \hat u_\gamma = 0$, so it remains to show:

$$
(\nabla \hat u_\gamma, \nabla \hat v)_{L^2(\hat\Omega)^2} - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat v \rangle_{H_I^*, H_I} - (\widetilde{f}_\gamma, \hat v)_{L^2(\hat\Omega)} = 0 \quad \forall \hat v \in H_0^1(\hat\Omega). \tag{2.3.3}
$$

With the definition of $\hat u_\gamma$ and again by integration by parts we obtain for the left-hand side

$$
\begin{aligned}
&(\nabla \hat u_\gamma, \nabla \hat v)_{L^2(\hat\Omega)^2} - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat v \rangle_{H_I^*, H_I} - (\widetilde{f}_\gamma, \hat v)_{L^2(\hat\Omega)} \\
&\quad = (\nabla u_\gamma, \nabla \hat v)_{L^2(\Omega_\gamma)^2} - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat v \rangle_{H_I^*, H_I} - (f_\gamma, \hat v)_{L^2(\Omega_\gamma)} \\
&\quad = \int_{\Omega_\gamma} (-\Delta u_\gamma - f_\gamma)\, \hat v \, dx + \int_{\partial\Omega_\gamma} \frac{\partial u_\gamma}{\partial n_\gamma} \hat v \, dS(x) - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat v \rangle_{H_I^*, H_I}.
\end{aligned}
$$

The first term vanishes since

$$
-\Delta u_\gamma - f_\gamma = 0 \quad \text{in } L^2(\Omega_\gamma).
$$

Furthermore we have with $\hat v|_\Gamma = 0$

$$
\int_{\partial\Omega_\gamma} \frac{\partial u_\gamma}{\partial n_\gamma} \hat v \, dS(x) = \int_{\Gamma_\gamma} \frac{\partial u_\gamma}{\partial n_\gamma} \hat v \, dS(x) = \left(\frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \|\dot\gamma(\cdot)\|_2, \hat v(\gamma(\cdot)) \right)_{L^2(I)},
$$

and we get (2.3.3) from (2.3.1). □

Remark 2.3.2.

1. *Since the right-hand side of equation* (2.3.1) *is defined for all* $h \in H_I$ *and by Lemma 2.1.8 the space* H_I *is dense in* $L^2(I)$ *we may uniquely extend the functional* $\mathcal{G}_\gamma \in H_I^*$ *onto* $L^2(I)$ *by the definition*

$$
\langle \mathcal{G}_\gamma, h \rangle_{L^2(I)^*, L^2(I)} := \left(\frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \|\dot\gamma(\cdot)\|_2, h \right)_{L^2(I)} \quad \forall h \in L^2(I).
$$

2. *Identifying $L^2(I)$ with its dual space, the Lagrange multiplier satisfies*

$$\mathcal{G}_\gamma = \frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma}\|\dot\gamma(\cdot)\|_2 \quad \text{in } L^2(I).$$

2.4. Continuous dependence result

Lemma 2.4.1. *Let $\gamma \in S_{ad}$.*

1. *The trace operator*

$$\tau_{1,\gamma} u := \frac{\partial u}{\partial n_\gamma}$$

is a linear continuous mapping from $H^2(\Omega_\gamma)$ onto $L^2(\Gamma_\gamma)$.

2. *The family of trace operators $\{\tau_{1,\gamma}\}_{\gamma\in S_{ad}}$ is uniformly bounded in $\mathcal{L}(H^2(\hat\Omega), L^2(\Gamma_\gamma))$, i.e. for all $u \in H^2(\hat\Omega)$ there holds*

$$\|\tau_{1,\gamma} u\|_{L^2(\Gamma_\gamma)} \le C\|u\|_{H^2(\hat\Omega)},$$

where C is independent of $\gamma \in S_{ad}$.

Theorem 2.4.2. *The family of solutions $\{(\hat u_\gamma, \mathcal{G}_\gamma)\}_{\gamma\in S_{ad}} \subset H_0^1(\hat\Omega) \times H_I^*$ of (2.2.1) satisfies*

$$\|(\hat u_\gamma, \mathcal{G}_\gamma)\|_{H_0^1(\hat\Omega)\times H_I^*} \le C_1.$$

Moreover, the Lagrange multiplier satisfies

$$\|\mathcal{G}_\gamma\|_{L^2(I)} \le C_2$$

for all $\gamma \in S_{ad}$. Both constants C_1, C_2 are independent of $\gamma \in S_{ad}$.

Proof. Since $\left\{\|u_\gamma\|_{H^2(\Omega)}\right\}_{\gamma\in S_{\text{ad}}}$ is uniformly bounded by Theorem 1.3.3 the same holds for $\left\{\|u_\gamma\|_{H^1(\Omega)}\right\}_{\gamma\in S_{\text{ad}}}$. By Theorem 2.3.1 we know that the corresponding solutions of the fictitious domain problems $\hat u_\gamma$ are in $H_0^1(\hat\Omega)$ and vanish on Ω_γ^c. This implies

$$\|\hat u_\gamma\|_{H^1(\hat\Omega)} = \|u_\gamma\|_{H^1(\Omega_\gamma)} \quad \text{for all } \gamma \in S_{\text{ad}}.$$

Therefore $\left\{\|\hat u_\gamma\|_{H_0^1(\Omega)}\right\}_{\gamma\in S_{\text{ad}}}$ is uniformly bounded.
To estimate the H_I^* norm of $\mathcal{G}_\gamma$ we use (2.3.1) and compute

$$\langle \mathcal{G}_\gamma, h\rangle_{H_I^*,H_I} = (\mathcal{G}_\gamma, h)_{L^2(I)} \le \|\mathcal{G}_\gamma\|_{L^2(I)}\|h\|_{L^2(I)} \le \|\mathcal{G}_\gamma\|_{L^2(I)}\|h\|_{H_I},$$

which implies

$$\|\mathcal{G}_\gamma\|_{H_I^*} \le \|\mathcal{G}_\gamma\|_{L^2(I)}.$$

The boundedness of $\{\|\mathcal{G}_\gamma\|_{H_I^*}\}_{\gamma\in S_{\text{ad}}}$ is now a consequence of the fact that

$$\begin{aligned}
\|\mathcal{G}_\gamma\|_{L^2(I)} &= \left\| \frac{\partial u_\gamma(\gamma(\cdot))}{\partial n_\gamma} \|\dot\gamma(\cdot)\|_2 \right\|_{L^2(I)} \\
&\leq \|\gamma\|_{W^{1,\infty}(I)^2} \left\| \mathcal{I}_\gamma \left(\frac{\partial u_\gamma}{\partial n_\gamma} \right) \right\|_{L^2(I)} \\
&\leq \|\gamma\|_{W^{1,\infty}(I)^2} \|\mathcal{I}_\gamma\|_{\mathcal{L}(L^2(\Gamma_\gamma),L^2(I))} \left\| \frac{\partial u_\gamma}{\partial n_\gamma} \right\|_{L^2(\Gamma_\gamma)} \\
&\leq C \|u_\gamma\|_{H^2(\Omega_\gamma)},
\end{aligned}$$

where we used Lemma 2.1.9, Lemma 2.4.1, and the boundedness of $\left\{\|u_\gamma\|_{H^2(\Omega)}\right\}_{\gamma\in S_{\text{ad}}}$. □

In the next chapter we apply the fictitious domain method to shape optimization problems. In order to prove the existence of a solution we need a continuous dependence result in the following way: For a given sequence $\{\gamma_k\}_{k\in\mathbb{N}}$ that converges to a curve γ in $W^{1,\infty}(I)^2$ for $k\to\infty$, we obtain convergence of the state variables $\hat u_{\gamma_k}$ to $\hat u_\gamma$ in $H_0^1(\hat\Omega)$ and weak*-convergence of $\mathcal{G}_{\gamma_k}$ to $\mathcal{G}_\gamma$ in H_I^* for $k\to\infty$. The difficulty to prove this result lies in stating appropriate $W^{1,\infty}$-transformations that map Ω_γ onto Ω_{γ_k} for all k. On the other hand, to prove the derivative formulas of the reduced objective functional, we need the following convergence result which is a special case of the above mentioned conclusion: For a given curve $\gamma\in S_{\text{ad}}$ and an admissible direction $\bar\gamma$ specified below, we obtain convergence of the state variables $\hat u_{\gamma+\delta\bar\gamma}$ to $\hat u_\gamma$ in $H_0^1(\hat\Omega)$ and weak*-convergence of $\mathcal{G}_{\gamma+\delta\bar\gamma}$ to $\mathcal{G}_\gamma$ in H_I^* for $\delta\to 0$, if the sequence $\{\gamma+\delta\bar\gamma\}$ converges to γ in $W^{1,\infty}(I)^2$ for $\delta\to 0$.
In the following, we will prove the latter case by constructing appropriate domain transformations. In the general case, we are not able to state transformations analogously to the special case.

We calculate shape variations of the reduced objective function in normal direction. Unfortunately, normal variations of a domain do not preserve its regularity (cf. [BS98]), i.e. for $\gamma\in S_{\text{ad}}$ we obtain $\Gamma_\gamma\in C^{1,1}$, but for a smooth and scalar function $\bar g: I\to\mathbb{R}$ the deformed boundary $\Gamma_{\gamma+\bar g n_\gamma} = \{\gamma(t)+\bar g(t) n_\gamma(t) : t\in I\}$ is in general not of class $C^{1,1}$.
Since we require H^2-regularity of the solution of the state equation also on deformations $\Omega_{\gamma+\bar g n_\gamma}$, we define a set of even more regular curves.

Definition 2.4.3. *With S_{ad} from Definition 1.2.1 we set*

$$S_{ad}^+ := S_{ad} \cap \left\{\gamma\in H^4(I)^2 : \|\gamma\|_{H^4(I)^2} \leq c_p\right\}.$$

Now we claim $\gamma \in S_{ad}^+$ and characterize admissible directions $\bar{\gamma} = \bar{g} n_\gamma$ through

$$\begin{aligned} S'_\gamma := \{\bar{\gamma} \in I \to \mathbb{R}^2 : & \textit{ there exists } \bar{g} \in W^{1,\infty}(I) \textit{ with } \bar{\gamma} = \bar{g} n_\gamma \textit{ and} \\ & \textit{there exists } \delta_0 > 0 : \gamma \pm \delta\bar{\gamma} \in S_{ad}, \forall \delta \in [0, \delta_0]\}. \end{aligned}$$

We clarify that $\gamma \in S_{\mathrm{ad}}^+$ leads to $n_\gamma \in H^3(I)^2$ and for $\gamma \in S_{\mathrm{ad}}$ we obtain in general only $n_\gamma \in H^2(I)^2$. The requirement $\gamma \pm \delta_k \bar{\gamma} \in S_{\mathrm{ad}}$ implicitly implies that the function $\bar{g}$ has to be more smooth than $W^{1,\infty}(I)$. Furthermore, every $\bar{\gamma} \in S'_\gamma$ satisfies $\bar{\gamma}|_{I_<} = \bar{\gamma}|_{I_>} = 0$.

Definition 2.4.4. *For $\gamma \in S_{ad}^+$ let $\bar{\gamma} = \bar{g} n_\gamma \in S'_\gamma$ be an admissible direction. Then we define*

$$\begin{aligned} I_- &:= \{t \in I : \bar{g}(t) < 0\}, \\ I_+ &:= \{t \in I : \bar{g}(t) > 0\}. \end{aligned}$$

For $p \in [-1, 2]$ we define the open sets

$$\begin{aligned} \Gamma_{\gamma + p\delta\bar{\gamma}} &:= \{\gamma(t) + p\delta\bar{\gamma}(t) : t \in I\}, \\ \Gamma^+_{\gamma + p\delta\bar{\gamma}} &:= \{\gamma(t) + p\delta\bar{\gamma}(t) : t \in I_+\}, \\ \Gamma^-_{\gamma + p\delta\bar{\gamma}} &:= \{\gamma(t) + p\delta\bar{\gamma}(t) : t \in I_-\}, \end{aligned}$$

and for $-1 \le p < q \le 2$ we define

$$D^{\delta,\pm}_{p,q} := I_\pm \times (p\delta, q\delta),$$

and denote by $\Delta^{\delta,\pm}_{p,q}$ the domain that is bounded by $\Gamma^\pm_{\gamma+p\delta\bar{\gamma}}$ and $\Gamma^\pm_{\gamma+q\delta\bar{\gamma}}$, i.e.

$$\begin{aligned} \Delta^{\delta,+}_{p,q} &:= \bar{\Omega}_{\gamma+q\delta\bar{\gamma}} \setminus \left(\bar{\Omega}_{\gamma+p\delta\bar{\gamma}} \cap \bar{\Omega}_{\gamma+q\delta\bar{\gamma}}\right), \\ \Delta^{\delta,-}_{p,q} &:= \bar{\Omega}_{\gamma+p\delta\bar{\gamma}} \setminus \left(\bar{\Omega}_{\gamma+p\delta\bar{\gamma}} \cap \bar{\Omega}_{\gamma+q\delta\bar{\gamma}}\right). \end{aligned}$$

Finally we define

$$\begin{aligned} \Omega^\delta_{p,q} &:= \Omega_{\gamma+p\delta\bar{\gamma}} \cap \Omega_{\gamma+q\delta\bar{\gamma}}, \\ \Omega^{\delta,c}_{p,q} &:= \Omega^c_{\gamma+p\delta\bar{\gamma}} \cap \Omega^c_{\gamma+q\delta\bar{\gamma}}. \end{aligned}$$

We introduce an appropriate coordinate transformation in the following:

Lemma 2.4.5. *For $\gamma \in S_{ad}^+$ let $\bar{\gamma} = \bar{g} n_\gamma \in S'_\gamma$ be an admissible direction. For all $-1 \le p < q \le 2$ and $\delta > 0$ small enough, the coordinate transformation*

$$\begin{aligned} \Phi^{\delta,-}_{p,q} : D^{\delta,-}_{p,q} &\to \Delta^{\delta,-}_{p,q}, \\ (t, \xi) &\mapsto (x(t,\xi), y(t,\xi)) := \gamma(t) + \xi \bar{g}(t) n_\gamma(t), \end{aligned}$$

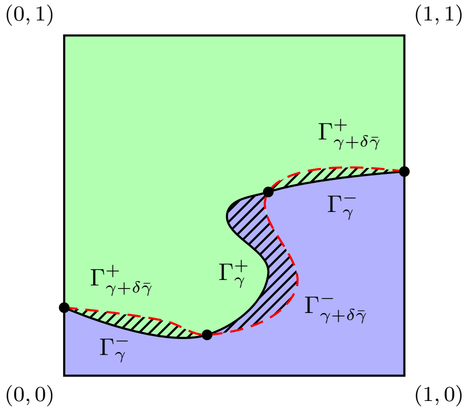


Figure 2.1.: Ω_γ and a shape variation $\Omega_{\gamma+\delta\bar{\gamma}}$

satisfies

$$\det\left(\left(\Phi^{\delta,-}_{p,q}\right)'(t,\xi)\right) = -\left(1+\xi\bar{g}(t)\kappa_\gamma(t)\right)\bar{g}(t)\|\dot{\gamma}(t)\|_2,$$

with the signed curvature $\kappa_\gamma(t) := \frac{\det(\dot{\gamma},\ddot{\gamma})}{\|\dot{\gamma}(t)\|_2^3}$, *and we can estimate*

$$\begin{aligned}\left|\det\left(\left(\Phi^{\delta,-}_{p,q}\right)'(t,\xi)\right)\right| &\leq C_1,\\ \left|\det\left(\left(\Phi^{\delta,-}_{p,q}\right)'(t,\xi)\right)\right| &\geq C_2|\bar{g}(t)|.\end{aligned}$$

Similar estimates hold for

$$\begin{aligned}\Phi^{\delta,+}_{p,q} : D^{\delta,+}_{p,q} &\to \Delta^{\delta,+}_{p,q},\\ (t,\xi) &\mapsto (x(t,\xi),y(t,\xi)) := \gamma(t)+\xi\bar{g}(t)n_\gamma(t).\end{aligned}$$

Proof. Let $\gamma = (\gamma_1,\gamma_2)^T$, $n_\gamma = (n_1,n_2)^T$ and $\Phi := \Phi^{\delta,-}_{p,q}$. We have

$$\begin{aligned}\Phi'(t,\xi) &= \begin{pmatrix}\frac{\partial}{\partial t}\left(\gamma_1(t)+\xi\bar{g}(t)n_1(t)\right) & \frac{\partial}{\partial \xi}\left(\gamma_1(t)+\xi\bar{g}(t)n_1(t)\right)\\ \frac{\partial}{\partial t}\left(\gamma_2(t)+\xi\bar{g}(t)n_2(t)\right) & \frac{\partial}{\partial \xi}\left(\gamma_2(t)+\xi\bar{g}(t)n_2(t)\right)\end{pmatrix}\\ &= \begin{pmatrix}\dot{\gamma}_1(t)+\xi\left(\bar{g}'(t)n_1(t)+\bar{g}(t)n_1'(t)\right) & \bar{g}(t)n_1(t)\\ \dot{\gamma}_2(t)+\xi\left(\bar{g}'(t)n_2(t)+\bar{g}(t)n_2'(t)\right) & \bar{g}(t)n_2(t)\end{pmatrix},\end{aligned}$$

and calculate

$$\begin{aligned}\det\Phi'(t,\xi) &= \left(\dot{\gamma}_1(t)+\xi\left(\bar{g}'(t)n_1(t)+\bar{g}(t)n_1'(t)\right)\right)\bar{g}(t)n_2(t)\\ &\quad - \left(\dot{\gamma}_2(t)+\xi\left(\bar{g}'(t)n_2(t)+\bar{g}(t)n_2'(t)\right)\right)\bar{g}(t)n_1(t)\\ &= \bar{g}(t)\left(\dot{\gamma}_1(t)n_2(t)-\dot{\gamma}_2(t)n_1(t)\right)\\ &\quad + \xi\bar{g}^2(t)\left(n_1'(t)n_2(t)-n_2'(t)n_1(t)\right).\end{aligned}$$

From

$$
\begin{aligned}
\det(\dot\gamma, n_\gamma) &= \dot\gamma_1(t) n_2(t) - \dot\gamma_2(t) n_1(t)\\
&= -\frac{\dot\gamma_1^2(t) + \dot\gamma_2^2(t)}{\sqrt{\dot\gamma_1^2(t) + \dot\gamma_2^2(t)}} = -\sqrt{\dot\gamma_1^2(t) + \dot\gamma_2^2(t)}\\
&= -\|\dot\gamma(t)\|_2,
\end{aligned}
$$

and

$$
\begin{aligned}
\det(n'_\gamma, n_\gamma) &= (n'_1(t) n_2(t) - n'_2(t) n_1(t))\\
&= \frac{\ddot\gamma_2(t)\|\dot\gamma(t)\|_2 - \dot\gamma_2(t)\frac{d}{dt}\|\dot\gamma(t)\|_2}{\|\dot\gamma(t)\|_2^2} \cdot \frac{-\dot\gamma_1(t)}{\|\dot\gamma(t)\|_2}\\
&\quad + \frac{\ddot\gamma_1(t)\|\dot\gamma(t)\|_2 - \dot\gamma_1(t)\frac{d}{dt}\|\dot\gamma(t)\|_2}{\|\dot\gamma(t)\|_2^2} \cdot \frac{\dot\gamma_2(t)}{\|\dot\gamma(t)\|_2}\\
&= -\frac{\dot\gamma_1(t)\ddot\gamma_2(t) - \dot\gamma_2(t)\ddot\gamma_1(t)}{\|\dot\gamma(t)\|_2^2}\\
&= -\frac{\det(\dot\gamma, \ddot\gamma)}{\|\dot\gamma(t)\|_2^2} = -\kappa_\gamma(t)\|\dot\gamma(t)\|_2
\end{aligned}
$$

we finally obtain

$$
\begin{aligned}
\det \Phi'(t,\xi) &= \bar g(t)\left(\dot\gamma_1(t) n_2(t) - \dot\gamma_2(t) n_1(t)\right)\\
&\quad + \xi \bar g^2(t)\left(n'_1(t) n_2(t) - n'_2(t) n_1(t)\right)\\
&= -\bar g(t)\|\dot\gamma(t)\|_2 - \xi g^2(t)\kappa_\gamma(t)\|\dot\gamma(t)\|_2\\
&= -\left(1 + \xi \bar g(t)\kappa_\gamma(t)\right)\bar g(t)\|\dot\gamma(t)\|_2.
\end{aligned}
$$

With the help of Definition 1.2.1 and Definition 2.4.3 and the embedding $H^4(I)^2 \hookrightarrow W^{2,\infty}(I)^2$ we can bound the $W^{2,\infty}(I)^2$-norm of γ from above by c_p and the 2-norm of $\dot\gamma$ by c_r from below. Therefore, we obtain

$$
\begin{aligned}
\kappa_\gamma(t) &= \frac{\det(\dot\gamma(t), \ddot\gamma(t))}{\|\dot\gamma(t)\|_2^2}\\
&\le \frac{|\dot\gamma_1(t)||\ddot\gamma_2(t)| + |\dot\gamma_2(t)||\ddot\gamma_1(t)|}{\|\dot\gamma(t)\|_2^2} \le C\frac{c_p^2}{c_r^2} =: \kappa_{\max}.
\end{aligned}
$$

Eventually, we estimate

$$
\begin{aligned}
|\det \Phi'(t,\xi)| &\le (1 + \delta\|\bar g\|_\infty \kappa_{\max})\,\|\bar g\|_\infty \sqrt{\|\dot\gamma_1\|_\infty^2 + \|\dot\gamma_2\|_\infty^2} \le C_1,\\
|\det \Phi'(t,\xi)| &\ge (1 - \delta\|\bar g\|_\infty \kappa_{\max})\,|\bar g(t)| c_r \ge C_2 |\bar g(t)|.
\end{aligned}
$$

□

Lemma 2.4.6. *For $\gamma \in S_{ad}^+$ let $\bar\gamma = \bar g n_\gamma \in S'_\gamma$ be an admissible direction. Then we* *obtain*

$$\lim_{\delta\to 0}\gamma + \delta\bar{\gamma} = \gamma \quad \text{in } W^{1,\infty}(I)^2. \tag{2.4.1}$$

Moreover, there exist transformations $\psi_\delta : \Omega_\gamma \to \Omega_{\gamma+\delta\bar{\gamma}}$*,* $\psi_\delta^c : \Omega_\gamma^c \to \Omega_{\gamma+\delta\bar{\gamma}}^c$*,* $\delta \le \delta_1 \le \delta_0$*, such that*

$$\lim_{\delta\to 0} T_\delta = \mathrm{id}_{\hat{\Omega}} \ \ \text{in } W^{1,\infty}(\hat{\Omega})^2,$$

$$T_\delta : \hat{\Omega} \to \hat{\Omega}, \quad T_\delta(x,y) = \begin{cases} \psi_\delta(x,y), & (x,y) \in \Omega_\gamma \\ \psi_\delta^c(x,y), & (x,y) \in \Omega_\gamma^c \end{cases}, \quad T_\delta(\Gamma_\gamma) = \Gamma_{\gamma+\delta\bar{\gamma}}.$$

Proof. For the first part we have

$$\begin{aligned} \|\gamma + \delta\bar{\gamma} - \gamma\|_{W^{1,\infty}(I)^2} &= \|\delta\bar{g}n_\gamma\|_{W^{1,\infty}(I)^2} \\ &\le \delta\|\bar{g}\|_{W^{1,\infty}(I)} \left\| \frac{1}{\sqrt{\dot{\gamma}_1(t)^2 + \dot{\gamma}_2(t)^2}} \begin{pmatrix} \dot{\gamma}_2(t) \\ -\dot{\gamma}_1(t) \end{pmatrix} \right\|_{W^{1,\infty}(I)^2} \\ &\le C\delta\|\bar{g}\|_{W^{1,\infty}(I)}\|\gamma\|_{W^{2,\infty}(I)^2} \to 0 \text{ for } \delta \to 0. \end{aligned}$$

For the second part we construct a decomposition of the domains Ω_γ, Ω_γ^c, $\Omega_{\gamma+\delta\bar{\gamma}}$, and $\Omega_{\gamma+\delta\bar{\gamma}}^c$ into disjoint domains which are transformed separately. The idea is to let T_δ behave mainly like the identity mapping and let the stretching/compression happen locally around Γ_γ and $\Gamma_{\gamma+\delta\bar{\gamma}}$.
We can choose δ_1 in such way that for all $p \in [-1, 2]$ and $\delta \le \delta_1$

$$\mathrm{dist}(\Gamma_{\gamma+p\delta\bar{\gamma}}, \partial\hat{\Omega}) > 0,$$

and such that we achieve for all $-1 \le p < q \le 2$ one-to-one mappings $\Phi_{p,q}^{\delta,\pm}$, $\delta \le \delta_1$ with

$$\begin{aligned} &\Phi_{p,q}^{\delta,\pm} : I_\pm \times (p\delta, q\delta) \to \Delta_{p,q}^{\delta,\pm}, \\ &\qquad\qquad (t,\xi) \mapsto (x(t,\xi), y(t,\xi)) := \gamma(t) + \xi\bar{g}(t)n_\gamma(t), \end{aligned}$$

and

$$\begin{aligned} &\left(\Phi_{p,q}^{\delta,\pm}\right)^{-1} : \Delta_{p,q}^{\delta,\pm} \to I_\pm \times (p\delta, q\delta), \\ &\qquad (x,y) \mapsto (t(x,y), \xi(x,y)) := \left(\gamma^{-1}(P_\gamma(x,y)), \left\|(x,y)^T - P_\gamma(x,y)\right\|_2\right), \end{aligned}$$

where $P_\gamma(x,y)$ is the orthogonal projection of (x,y) onto Γ_γ. This leads to the following decompositions:

$$
\begin{aligned}
\Omega_\gamma &= \Omega^\delta_{-1,2} \cup \Delta^{\delta,+}_{-1,0} \cup \Delta^{\delta,-}_{0,2}\,,\\
\Omega^c_\gamma &= \Omega^{\delta,c}_{-1,2} \cup \Delta^{\delta,+}_{0,2} \cup \Delta^{\delta,-}_{-1,0},\\
\Omega_{\gamma+\delta\bar\gamma} &= \Omega^\delta_{-1,2} \cup \Delta^{\delta,+}_{-1,1} \cup \Delta^{\delta,-}_{1,2}\,,\\
\Omega^c_{\gamma+\delta\bar\gamma} &= \Omega^{\delta,c}_{-1,2} \cup \Delta^{\delta,+}_{1,2} \cup \Delta^{\delta,-}_{-1,1}.
\end{aligned}
$$

Now, for $(x,y)\in\Omega^\delta_{-1,2}$ and $(x,y)\in\Omega^{\delta,c}_{-1,2}$ the transformations ψ_δ and ψ^c_δ are defined as the identity mapping. Furthermore, the transformation ψ_δ maps the domain $\Delta^{\delta,+}_{-1,0}$ into the domain $\Delta^{\delta,+}_{-1,1}$ (stretching) and $\Delta^{\delta,-}_{0,2}$ into $\Delta^{\delta,-}_{1,2}$ (compression). The transformation ψ^c_δ maps $\Delta^{\delta,+}_{0,2}$ into $\Delta^{\delta,+}_{1,2}$ (compression) and $\Delta^{\delta,-}_{-1,0}$ into $\Delta^{\delta,-}_{-1,1}$ (stretching).
We describe the stretching/compression process: With the definition of the mapping $\Phi^{\delta,\pm}_{p,q}$ we have for $-1\le p<q\le 2$

$$
\Delta^{\delta,\pm}_{p,q} = \{\Phi^{\delta,\pm}_{p,q}(t,\xi) : (t,\xi)\in I_\pm \times (p\delta, q\delta)\}.
$$

For $-1\le r<s\le 2$ a transformation $\Psi^{\delta,\pm}_{p,q,r,s} : \Delta^{\delta,\pm}_{p,q} \to \Delta^{\delta,\pm}_{r,s}$ can be described by the concatenations $\Psi^{\delta,\pm}_{p,q,r,s} = \Phi^{\delta,\pm}_{r,s} \circ \phi^\delta_{p,q,r,s} \circ \big(\Phi^{\delta,\pm}_{p,q}\big)^{-1}$, where $\phi^\delta_{p,q,r,s}$ is a smooth mapping from $\mathbb{R}\times(p\delta,q\delta)$ onto $\mathbb{R}\times(r\delta,s\delta)$.
Here, the mapping $\Psi^{\delta,\pm}_{p,q,r,s}$ leads to a stretching for $q-p<s-r$ and to a compression for $q-p>s-r$. We are now able to state a formula for the transformation T_δ:

$$
T_\delta(x,y) = \begin{cases}
\psi_\delta(x,y), \quad (x,y)\in\Omega_\gamma, \quad \psi_\delta(x,y) = \begin{cases} (x,y), & (x,y)\in\Omega^\delta_{-1,2},\\ \Psi^{\delta,+}_{-1,0,-1,1}(x,y), & (x,y)\in\Delta^{\delta,+}_{-1,0},\\ \Psi^{\delta,-}_{0,2,1,2}(x,y), & (x,y)\in\Delta^{\delta,-}_{0,2}\,, \end{cases}\\
\psi^c_\delta(x,y), \quad (x,y)\in\Omega^c_\gamma, \quad \psi^c_\delta(x,y) = \begin{cases} (x,y), & (x,y)\in\Omega^{\delta,c}_{-1,2},\\ \Psi^{\delta,+}_{0,2,1,2}(x,y), & (x,y)\in\Delta^{\delta,+}_{0,2}\,,\\ \Psi^{\delta,-}_{-1,0,-1,1}(x,y), & (x,y)\in\Delta^{\delta,-}_{-1,0}. \end{cases}
\end{cases}
$$

Due to this construction, we finally obtain

$$
\begin{aligned}
\|T_\delta - \mathrm{id}_{\hat\Omega}\|_{W^{1,\infty}(\hat\Omega)^2} &= \|\psi_\delta(x,y) - \mathrm{id}_{\Omega_\gamma}\|_{W^{1,\infty}(\Omega_\gamma)^2} + \|\psi^c_\delta(x,y) - \mathrm{id}_{\Omega^c_\gamma}\|_{W^{1,\infty}(\Omega^c_\gamma)^2}\\
&= \|\Psi^{\delta,+}_{-1,0,-1,1} - \mathrm{id}\|_{W^{1,\infty}(\Delta^{\delta,+}_{-1,0})^2} + \|\Psi^{\delta,-}_{0,2,1,2} - \mathrm{id}\|_{W^{1,\infty}(\Delta^{\delta,-}_{0,2})^2}\\
&\quad + \|\Psi^{\delta,+}_{0,2,1,2} - \mathrm{id}\|_{W^{1,\infty}(\Delta^{\delta,+}_{0,2})^2} + \|\Psi^{\delta,-}_{-1,0,-1,1} - \mathrm{id}\|_{W^{1,\infty}(\Delta^{\delta,-}_{-1,0})^2}\\
&\le C\delta\|\bar g\|_{W^{1,\infty}(I)}\|\gamma\|_{W^{2,\infty}(I)^2} \to 0 \text{ for } \delta\to 0.
\end{aligned}
$$

□

Lemma 2.4.7. *Let the assumptions of the previous lemma hold. Then*

$$\begin{aligned} &\lim_{\delta\to 0} \widetilde{f}_{\gamma+\delta\bar{\gamma}} = \widetilde{f}_{\gamma} \qquad \text{in } L^2(\hat{\Omega}),\\ &\lim_{\delta\to 0} \mathcal{T}_{\gamma+\delta\bar{\gamma}} = \mathcal{T}_{\gamma} \qquad \text{in } \mathcal{L}(H_0^1(\hat{\Omega}), H_I). \end{aligned}$$

Proof. For the first statement we have with $f \in L^\infty(\hat{\Omega})$,

$$\widetilde{f}_{\gamma} := \begin{cases} f & \text{in } \Omega_\gamma, \\ 0 & \text{in } \Omega_\gamma^c, \end{cases} \qquad\qquad \widetilde{f}_{\gamma+\delta\bar{\gamma}} := \begin{cases} f & \text{in } \Omega_{\gamma+\delta\bar{\gamma}}, \\ 0 & \text{in } \Omega_{\gamma+\delta\bar{\gamma}}^c, \end{cases}$$

and with the coordinate transformation from Lemma 2.4.5 we obtain

$$\begin{aligned} \|\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_{\gamma}\|^2_{L^2(\hat{\Omega})} &= \|f\chi_{\Delta_{0,1}^{\delta,+}}\|^2_{L^2(\hat{\Omega})} + \|f\chi_{\Delta_{0,1}^{\delta,-}}\|^2_{L^2(\hat{\Omega})} \\ &= \|f\|^2_{L^2(\Delta_{0,1}^{\delta,+})} + \|f\|^2_{L^2(\Delta_{0,1}^{\delta,-})} \\ &\le \|f\|^2_{L^\infty(\hat{\Omega})} \left(\int_{\Delta_{0,1}^{\delta,+}} 1\,dx + \int_{\Delta_{0,1}^{\delta,-}} 1\,dx \right) \\ &\le \|f\|^2_{L^\infty(\hat{\Omega})} \Bigg(\int_{I_+} \int_0^\delta \left| \det\left((\Phi_{0,1}^{\delta,+})'(t,\xi) \right) \right| \,dt\,d\xi \\ &\qquad + \int_{I_-} \int_0^\delta \left| \det\left((\Phi_{0,1}^{\delta,-})'(t,\xi) \right) \right| \,dt\,d\xi \Bigg) \\ &\le C\delta \to 0 \quad \text{ for } \quad \delta \to 0. \end{aligned}$$

Now we show

$$\|\mathcal{T}_{\gamma+\delta\bar{\gamma}} - \mathcal{T}_\gamma\|_{\mathcal{L}(H_0^1(\hat{\Omega}),H_I)} := \sup_{\|\hat{u}\|_{H_0^1(\hat{\Omega})}=1} \|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u} - \mathcal{T}_\gamma\hat{u}\|_{H_I} \to 0 \quad \text{for } \delta \to 0.$$

We have

$$\|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u} - \mathcal{T}_\gamma\hat{u}\|^2_{H_I} = \|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u} - \mathcal{T}_\gamma\hat{u}\|^2_{H^{1/2}(I)} + \int_I \frac{|\hat{u}(\gamma(t)+\delta\bar{\gamma}(t)) - \hat{u}(\gamma(t))|^2}{t(1-t)}\,dt.$$

With ε from the Definition of S_{ad}:

$$\begin{aligned}
&\int_I \frac{|\hat{u}(\gamma(t)+\delta\bar{\gamma}(t))-\hat{u}(\gamma(t))|^2}{t(1-t)}\,dt \\
&\quad= \int_I \frac{|\hat{u}(\gamma(t)+\delta\bar{\gamma}(t))-\hat{u}(\gamma(t))|^2}{t}\,dt + \int_I \frac{|\hat{u}(\gamma(t)+\delta\bar{\gamma}(t))-\hat{u}(\gamma(t))|^2}{1-t}\,dt \\
&\quad= \int_0^{\varepsilon} \frac{|\hat{u}(\gamma(t)+\delta\bar{\gamma}(t))-\hat{u}(\gamma(t))|^2}{t}\,dt + \int_{\varepsilon}^1 \frac{|\hat{u}(\gamma(t)+\delta\bar{\gamma}(t))-\hat{u}(\gamma(t))|^2}{t}\,dt \\
&\qquad + \int_0^{1-\varepsilon} \frac{|\hat{u}(\gamma(t)+\delta\bar{\gamma}(t))-\hat{u}(\gamma(t))|^2}{1-t}\,dt + \int_{1-\varepsilon}^1 \frac{|\hat{u}(\gamma(t)+\delta\bar{\gamma}(t))-\hat{u}(\gamma(t))|^2}{1-t}\,dt.
\end{aligned}$$

The second and third integral are bounded from above by $\frac{1}{\varepsilon}\|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}-\mathcal{T}_\gamma\hat{u}\|^2_{L^2(I)}$ and thus also by $\frac{1}{\varepsilon}\|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}-\mathcal{T}_\gamma\hat{u}\|^2_{H^{1/2}(I)}$. The other two integrals vanish. This implies

$$\|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}-\mathcal{T}_\gamma\hat{u}\|_{H_I} \le \left(1+\frac{2}{\varepsilon}\right)^{1/2} \|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}-\mathcal{T}_\gamma\hat{u}\|_{H^{1/2}(I)}. \tag{2.4.2}$$

Now let $\{\hat{v}_n\}_{n\in\mathbb{N}} \subset C^2(\bar{\hat{\Omega}}) \cap H_0^1(\hat{\Omega})$ with

$$\lim_{n\to\infty} \hat{v}_n = \hat{u} \quad \text{in } H_0^1(\hat{\Omega}).$$

Since the family $\{\mathcal{T}_\gamma\}_{\gamma\in S_{\text{ad}}}$ is uniformly bounded in $\mathcal{L}(H_0^1, H_I)$, we have for every $n\in\mathbb{N}$

$$\begin{aligned}
&\|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}-\mathcal{T}_\gamma\hat{u}\|_{H^{1/2}(I)} \\
&\quad\le \|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}-\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{v}_n\|_{H^{1/2}(I)} + \|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{v}_n-\mathcal{T}_\gamma\hat{v}_n\|_{H^{1/2}(I)} + \|\mathcal{T}_\gamma\hat{v}_n-\mathcal{T}_\gamma\hat{u}\|_{H^{1/2}(I)} \\
&\qquad\le \|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\|_{\mathcal{L}(H_0^1(\hat{\Omega}),H_I)}\|\hat{u}-\hat{v}_n\|_{H^1(\hat{\Omega})} \\
&\qquad\quad + \|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{v}_n-\mathcal{T}_\gamma\hat{v}_n\|_{H^{1/2}(I)} + \|\mathcal{T}_\gamma\|_{\mathcal{L}(H_0^1(\hat{\Omega}),H_I)}\|\hat{v}_n-\hat{u}\|_{H^1(\hat{\Omega})} \\
&\qquad\le C\|\hat{v}_n-\hat{u}\|_{H^1(\hat{\Omega})} + \|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{v}_n-\mathcal{T}_\gamma\hat{v}_n\|_{H^{1/2}(I)}.
\end{aligned} \tag{2.4.3}$$

Let T_δ be the transformation from Lemma 2.4.6. Then we have $T_\delta(\Gamma_\gamma) = \Gamma_{\gamma+\delta\bar{\gamma}}$. Define $\hat{w}_\delta := \hat{v}_n \circ T_\delta$, then we have $\hat{w}_\delta(\gamma(t)) = \hat{v}_n(\gamma(t)+\delta\bar{\gamma}(t))$ for all $t\in I$. This implies

$$(\mathcal{T}_\gamma\hat{w}_\delta)(t) = (\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{v}_n)(t) \quad \text{for all } t\in I,$$

and

$$\|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{v}_n-\mathcal{T}_\gamma\hat{v}_n\|_{H^{1/2}(I)} = \|\mathcal{T}_\gamma(\hat{w}_\delta-\hat{v}_n)\|_{H^{1/2}(I)} \le \|\mathcal{T}_\gamma\|_{\mathcal{L}(H_0^1(\hat{\Omega}),H_I)}\|\hat{w}_\delta-\hat{v}_n\|_{H^1(\hat{\Omega})}. \tag{2.4.4}$$

We now show that $\|\hat{w}_\delta-\hat{v}_n\|_{H^1(\hat{\Omega})} = \|\hat{v}_n\circ T_\delta-\hat{v}_n\|_{H^1(\hat{\Omega})} \to 0$ for $\|T_\delta - \mathrm{id}\,\|_{W^{1,\infty}(\hat{\Omega})^2} \to 0$.

In the next steps we drop the index n and obtain

$$\begin{aligned}
\|\hat{v}\circ T_\delta-\hat{v}\|^2_{H^1(\hat{\Omega})} &= \|\hat{v}\circ T_\delta-\hat{v}\|^2_{L^2(\hat{\Omega})}+\|\nabla(\hat{v}\circ T_\delta-\hat{v})\|^2_{L^2(\hat{\Omega})^2}\\
&= \|\hat{v}\circ T_\delta-\hat{v}\|^2_{L^2(\hat{\Omega})}+\|\left(T'_\delta\right)^T(\nabla\hat{v})\circ T_\delta-\nabla\hat{v}\|^2_{L^2(\hat{\Omega})^2}\\
&= \|\hat{v}\circ T_\delta-\hat{v}\|^2_{L^2(\hat{\Omega})}\\
&\quad+\|\left(T'_\delta\right)^T(\nabla\hat{v})\circ T_\delta-\left(\mathrm{id}'_{\hat{\Omega}}\right)^T(\nabla\hat{v})\circ\mathrm{id}_{\hat{\Omega}}\|^2_{L^2(\hat{\Omega})^2}\\
&\le \|\hat{v}\circ T_\delta-\hat{v}\|^2_{L^2(\hat{\Omega})}+\|\left(T'_\delta-\mathrm{id}'_{\hat{\Omega}}\right)^T(\nabla\hat{v})\circ T_\delta\|^2_{L^2(\hat{\Omega})^2}\\
&\qquad+\|\left(\mathrm{id}'_{\hat{\Omega}}\right)^T((\nabla\hat{v})\circ T_\delta-(\nabla\hat{v})\circ\mathrm{id}_{\hat{\Omega}})\|^2_{L^2(\hat{\Omega})^2}\\
&\le \int_{\hat{\Omega}}|\hat{v}(T_\delta(x,y))-\hat{v}(x,y)|^2\,d(x,y)\\
&\qquad+\|T_\delta-\mathrm{id}_{\hat{\Omega}}\|^2_{W^{1,\infty}(\hat{\Omega})^2}\|(\nabla\hat{v})\circ T_\delta\|^2_{L^2(\hat{\Omega})^2}\\
&\qquad+\int_{\hat{\Omega}}|\hat{v}_x(T_\delta(x,y))-\hat{v}_x(x,y)|^2\,d(x,y)\\
&\qquad+\int_{\hat{\Omega}}|\hat{v}_y(T_\delta(x,y))-\hat{v}_y(x,y)|^2\,d(x,y)\\
&=: A+B+C+D.
\end{aligned}$$

We use the mean value theorem for the terms under the integrals and obtain with $c\in(0,1)$:

$$\begin{aligned}
A &= \int_{\hat{\Omega}}|\hat{v}(T_\delta(x,y))-\hat{v}(x,y)|^2\,d(x,y)\\
&= \int_{\hat{\Omega}}|\nabla\hat{v}\left((\mathrm{id}_{\hat{\Omega}}+c(T_\delta-\mathrm{id}_{\hat{\Omega}}))(x,y)\right)^T(T_\delta-\mathrm{id}_{\hat{\Omega}})(x,y)|^2\,d(x,y)\\
&\le \|T_\delta-\mathrm{id}_{\hat{\Omega}}\|^2_{L^\infty(\hat{\Omega})^2}(\|\hat{v}_x\|^2_{L^\infty(\hat{\Omega})}+\|\hat{v}_y\|^2_{L^\infty(\hat{\Omega})}).
\end{aligned}$$

For B we have

$$\|T_\delta-\mathrm{id}_{\hat{\Omega}}\|^2_{W^{1,\infty}(\hat{\Omega})^2}\|(\nabla\hat{v})\circ T_\delta\|^2_{L^2(\hat{\Omega})^2}\le\|T_\delta-\mathrm{id}_{\hat{\Omega}}\|^2_{W^{1,\infty}(\hat{\Omega})^2}(\|\hat{v}_x\|^2_{L^\infty(\hat{\Omega})}+\|\hat{v}_y\|^2_{L^\infty(\hat{\Omega})}).$$

For C and D we obtain the estimates

$$\begin{aligned}
C &\le \|T_\delta-\mathrm{id}_{\hat{\Omega}}\|^2_{L^\infty(\hat{\Omega})^2}(\|\hat{v}_{xx}\|^2_{L^\infty(\hat{\Omega})}+\|\hat{v}_{xy}\|^2_{L^\infty(\hat{\Omega})}),\\
D &\le \|T_\delta-\mathrm{id}_{\hat{\Omega}}\|^2_{L^\infty(\hat{\Omega})^2}(\|\hat{v}_{yx}\|^2_{L^\infty(\hat{\Omega})}+\|\hat{v}_{yy}\|^2_{L^\infty(\hat{\Omega})}).
\end{aligned}$$

Altogether

$$\|\hat{v}_n\circ T_\delta-\hat{v}_n\|_{H^1(\hat{\Omega})}\le\|T_\delta-\mathrm{id}_{\hat{\Omega}}\|^2_{W^{1,\infty}(\hat{\Omega})^2}\|\hat{v}_n\|^2_{W^{2,\infty}(\hat{\Omega})}.$$

Using this estimate in (2.4.4) we get

$$\|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{v}_n - \mathcal{T}_\gamma \hat{v}_n\|_{H^{1/2}(I)} \le \|\mathcal{T}_\gamma\|_{\mathcal{L}(H_0^1(\hat{\Omega}),H_I)} \|T_\delta - \mathrm{id}_{\hat{\Omega}}\|_{W^{1,\infty}(\hat{\Omega})^2} \|\hat{v}_n\|_{W^{2,\infty}(\hat{\Omega})}.$$

In (2.4.3) we obtain

$$\|\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u} - \mathcal{T}_\gamma \hat{u}\|_{H^{1/2}(I)} \le C\|\hat{v}_n - \hat{u}\|_{H^1(\hat{\Omega})} + \|\mathcal{T}_\gamma\|_{\mathcal{L}(H_0^1(\hat{\Omega}),H_I)} \|T_\delta - \mathrm{id}_{\hat{\Omega}}\|_{W^{1,\infty}(\hat{\Omega})^2} \|\hat{v}_n\|_{W^{2,\infty}(\hat{\Omega})}.$$

For $\varepsilon \ge 0$ there exist $n_0 \in \mathbb{N}$ and $\delta_0 > 0$ such that

$$C\|\hat{v}_n - \hat{u}\|_{H^1(\hat{\Omega})} \le \frac{\varepsilon}{2} \quad \text{for all } n \ge n_0,$$

and since $\|T_\delta - \mathrm{id}_{\hat{\Omega}}\|_{W^{1,\infty}(\hat{\Omega})^2}$ tends to zero for $\delta \to 0$ also

$$\|\mathcal{T}_\gamma\|_{\mathcal{L}(H_0^1(\hat{\Omega}),H_I)} \|T_\delta - \mathrm{id}_{\hat{\Omega}}\|_{W^{1,\infty}(\hat{\Omega})^2} \|\hat{v}_n\|_{W^{2,\infty}(\hat{\Omega})} \le \frac{\varepsilon}{2} \quad \text{for all } n \ge n_0, \delta \le \delta_0.$$

Passing to the limit $n \to \infty$ and then $\delta \to 0$ gives strong convergence of $\mathcal{T}_{\gamma+\delta\bar{\gamma}}$ to $\mathcal{T}_\gamma$ in $\mathcal{L}(H_0^1(\hat{\Omega}), H^{1/2}(I))$ and because of (2.4.2) also in $\mathcal{L}(H_0^1(\hat{\Omega}), H_I)$. □

Theorem 2.4.8. *For $\gamma \in S_{ad}^+$ let $\bar{\gamma} = \bar{g}n_\gamma \in S'_\gamma$ be an admissible direction. Then the solutions of the problem* (2.2.1) *for $\gamma + \delta\bar{\gamma}$, γ satisfy*

$$\lim_{\delta\to 0} \hat{u}_{\gamma+\delta\bar{\gamma}} = \hat{u}_\gamma \quad \text{in } H_0^1(\hat{\Omega}),$$
$$w^* - \lim_{\delta\to 0} \mathcal{G}_{\gamma+\delta\bar{\gamma}} = \mathcal{G}_\gamma \quad \text{in } H_I^*.$$

Proof. Theorem 2.4.2 implies the existence of a subsequence of $\{\hat{u}_{\gamma+\delta_k\bar{\gamma}}\}$, denoted by $\{\hat{u}_{\gamma^k}\}$, which converges weakly in $H_0^1(\hat{\Omega})$ to an element $\hat{u} \in H_0^1(\hat{\Omega})$. We show that $\mathcal{T}_\gamma \hat{u} = 0$. Therefore, we estimate with $\mathcal{T}_\gamma = \mathcal{I}_\gamma \circ \tau_\gamma$

$$\begin{aligned}\|\mathcal{T}_\gamma \hat{u}\|_{L^2(I)} &\le \|\mathcal{T}_\gamma(\hat{u} - \hat{u}_{\gamma^k})\|_{L^2(I)} + \|\left(\mathcal{T}_\gamma - \mathcal{T}_{\gamma^k}\right)\hat{u}_{\gamma^k}\|_{H_I} \\ &\le \|\mathcal{I}_\gamma\|_{\mathcal{L}(L^2(\Gamma_\gamma),L^2(I))} \|\tau_\gamma(\hat{u} - \hat{u}_{\gamma^k})\|_{L^2(\Gamma_\gamma)} + \|\mathcal{T}_\gamma - \mathcal{T}_{\gamma^k}\|_{\mathcal{L}(H_0^1(\hat{\Omega}),H_I)} \|\hat{u}_{\gamma^k}\|_{H_0^1(\hat{\Omega})}.\end{aligned}$$

Both terms on the right hand side of the above inequality tend to zero for $k \to \infty$. For the first term we note that $\hat{u}_{\gamma^k}$ converges weakly in $H^1(\hat{\Omega})$ to $\hat{u}$ and therefore the traces $\tau_\gamma \hat{u}_{\gamma^k}$ converge strongly to the trace $\tau_\gamma \hat{u}$ in $L^2(\Gamma_\gamma)$. For the second term we use the fact that the sequence $\{\hat{u}_{\gamma^k}\}$ is weakly convergent and therefore bounded. The convergence follows with Lemma 2.4.7.
Next observe that for every $\hat{v} \in H_0^1(\hat{\Omega})$

$$\begin{aligned}\lim_{k\to\infty} \langle \mathcal{G}_{\gamma^k}, \mathcal{T}_{\gamma^k}\hat{v}\rangle_{H_I^*,H_I} &= \lim_{k\to\infty} \left\{ (\nabla \hat{u}_{\gamma^k}, \nabla \hat{v})_{L^2(\hat{\Omega})} - (\widetilde{f}_{\gamma^k}, \hat{v})_{L^2(\hat{\Omega})} \right\} \\ &= (\nabla \hat{u}, \nabla \hat{v})_{L^2(\hat{\Omega})} - (\widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})}.\end{aligned}$$

For arbitrary $\varphi \in H_I$ choose $\hat{v} \in H_0^1(\hat{\Omega})$ such that $\varphi = \mathcal{T}_\gamma \hat{v}$ and calculate

$$\langle \mathcal{G}_{\gamma^k}, \varphi \rangle_{H_I^*, H_I} = \langle \mathcal{G}_{\gamma^k}, \mathcal{T}_\gamma \hat{v} - \mathcal{T}_{\gamma^k} \hat{v} \rangle_{H_I^*, H_I} + \langle \mathcal{G}_{\gamma^k}, \mathcal{T}_{\gamma^k} \hat{v} \rangle_{H_I^*, H_I}. \tag{2.4.5}$$

The first term on the right tends to zero for $k \to \infty$ since $\{\mathcal{G}_{\gamma^k}\}$ is bounded (cf. Theorem 2.4.2) and $\mathcal{T}_{\gamma^k} \to \mathcal{T}_\gamma$ strongly (cf. Lemma 2.4.7). For the second term we obtain:

$$\lim_{k \to \infty} \langle \mathcal{G}_{\gamma^k}, \varphi \rangle_{H_I^*, H_I} = (\nabla \hat{u}, \nabla \hat{v})_{L^2(\hat{\Omega})} - (\widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})} =: \langle \mathcal{G}, \varphi \rangle_{H_I^*, H_I}.$$

It follows that $(\hat{u}, \mathcal{G})$ is the unique solution $(\hat{u}_\gamma, \mathcal{G}_\gamma)$ of (2.2.1) and that the original sequence satisfies $\hat{u}_{\gamma+\delta\bar{\gamma}} \rightharpoonup u_\gamma$ in $H_0^1(\hat{\Omega})$ and $\mathcal{G}_{\gamma+\delta\bar{\gamma}} \overset{*}{\rightharpoonup} \mathcal{G}_\gamma$ in H_I^* for $\delta \to 0$. To show strong convergence of $\{\hat{u}_{\gamma+\delta\bar{\gamma}}\}$ to $\hat{u}_\gamma$ in $H_0^1(\hat{\Omega})$ we argue as follows: We have

$$\begin{aligned} (\nabla \hat{u}_{\gamma+\delta\bar{\gamma}}, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - (\widetilde{f}_{\gamma+\delta\bar{\gamma}}, \hat{v})_{L^2(\hat{\Omega})} - \langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}} \hat{v} \rangle_{H_I^*, H_I} &= 0 \quad \text{and} \\ (\nabla \hat{u}_\gamma, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - (\widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})} - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} &= 0 \quad \text{for all } \hat{v} \in H_0^1(\hat{\Omega}). \end{aligned}$$

Inserting $\hat{v} = \hat{u}_{\gamma+\delta\bar{\gamma}}$ in the first and $\hat{v} = \hat{u}_\gamma$ in the second equation we obtain the relationships

$$|\hat{u}_{\gamma+\delta\bar{\gamma}}|^2_{H^1(\hat{\Omega})} = (\widetilde{f}_{\gamma+\delta\bar{\gamma}}, \hat{u}_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} \quad \text{and} \quad |\hat{u}_\gamma|^2_{H^1(\hat{\Omega})} = (\widetilde{f}_\gamma, \hat{u}_\gamma)_{L^2(\hat{\Omega})}.$$

Subtracting these identities yields

$$\begin{aligned} \Big| |\hat{u}_{\gamma+\delta\bar{\gamma}}|^2_{H^1(\hat{\Omega})} - |\hat{u}_\gamma|^2_{H^1(\hat{\Omega})} \Big| &= \Big| (\widetilde{f}_{\gamma+\delta\bar{\gamma}}, \hat{u}_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} - (\widetilde{f}_\gamma, \hat{u}_\gamma)_{L^2(\hat{\Omega})} \Big| \\ = \Big| (\widetilde{f}_{\gamma+\delta\bar{\gamma}}, \hat{u}_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} &- (\widetilde{f}_\gamma, \hat{u}_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} + (\widetilde{f}_\gamma, \hat{u}_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} - (\widetilde{f}_\gamma, \hat{u}_\gamma)_{L^2(\hat{\Omega})} \Big| \\ &= \Big| (\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{u}_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} + (\widetilde{f}_\gamma, \hat{u}_{\gamma+\delta\bar{\gamma}} - \hat{u}_\gamma)_{L^2(\hat{\Omega})} \Big| \\ \le \|\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma\|_{L^2(\hat{\Omega})} \|\hat{u}_{\gamma+\delta\bar{\gamma}}\|_{L^2(\hat{\Omega})} &+ \|\widetilde{f}_\gamma\|_{L^2(\hat{\Omega})} \|\hat{u}_{\gamma+\delta\bar{\gamma}} - \hat{u}_\gamma\|_{L^2(\hat{\Omega})} \to 0 \text{ for } \delta \to 0, \end{aligned}$$

and hence $\lim_{\delta \to 0} |\hat{u}_{\gamma+\delta\bar{\gamma}}|_{H^1(\hat{\Omega})} = |\hat{u}_\gamma|_{H^1(\hat{\Omega})}$. Since $|\cdot|_{H^1(\hat{\Omega})}$ and $\|\cdot\|_{H^1(\hat{\Omega})}$ are equivalent norms on $H_0^1(\hat{\Omega})$ we also have $\lim_{\delta \to 0} \|\hat{u}_{\gamma+\delta\bar{\gamma}}\|_{H^1(\hat{\Omega})} = \|\hat{u}_\gamma\|_{H^1(\hat{\Omega})}$. Together with the weak convergence of $\{\hat{u}_{\gamma+\delta\bar{\gamma}}\}$ to $\hat{u}_\gamma$ this implies strong converges

$$\hat{u}_{\gamma+\delta\bar{\gamma}} \to \hat{u}_\gamma \text{ in } H_0^1(\hat{\Omega}).$$

□

Furthermore, we assume that the following generalized version of Theorem 2.4.8 holds.

Assumption 2.4.9. *Let $\{\gamma_k\} \subset S_{ad}^+$ be a sequence which converges to γ in $W^{1,\infty}(I)^2$.*

Then the solutions of the problem (2.2.1) for γ^k, γ satisfy

$$\begin{aligned} \lim_{k\to\infty} \hat{u}_{\gamma^k} &= \hat{u}_\gamma \quad \textit{in } H_0^1(\hat{\Omega}), \\ w^* - \lim_{k\to\infty} \mathcal{G}_{\gamma^k} &= \mathcal{G}_\gamma \quad \textit{in } H_I^*. \end{aligned}$$

3. Applying the fictitious domain method to shape optimization

3.1. Shape optimization problem formulation

We consider the optimal control problem

$$(P_\gamma) \quad \begin{cases} \min\ J(u,\gamma) = \dfrac{1}{2}\displaystyle\int_{\Omega_T} (\widetilde{u}(x) - u_d(x))^2\,dx \\ \text{s.t.}\ \ \gamma \in S_{\text{ad}}, \\ \qquad E(u,\gamma) = 0 \quad \text{in } \Omega_\gamma, \end{cases}$$

where $\Omega_T \subset \hat{\Omega}$ is an observation domain and $u_d \in L^2(\Omega_T)$ denotes a given desired state in the tracking-type objective functional $J : \{(u,\gamma) : u \in U(\Omega_\gamma), \gamma \in S_{\text{ad}}\} \to \mathbb{R}$. The set of admissible curves S_{ad} is defined in Definition 1.2.1, and the operator E is defined in (1.3). We use the extension $\widetilde{u}$ of u, introduced in Definition 2.0.4, since $\Omega_\gamma^c \cap \Omega_T \neq \emptyset$ is possible in general.
As a consequence of Theorem 2.3.1, with the definition $\hat{J}((\hat{u},\mathcal{G}),\gamma) := J(\hat{u}|_{\Omega_\gamma},\gamma)$ and the operator $\hat{E}$ defined in (2.2.2) the optimization problem (P_γ) is equivalent to the shape optimization problem

$$(\hat{P}_\gamma) \quad \begin{cases} \min\ \hat{J}((\hat{u},\mathcal{G}),\gamma) = \dfrac{1}{2}\displaystyle\int_{\Omega_T} (\hat{u}(x) - u_d(x))^2\,dx \\ \text{s.t.}\ \ \gamma \in S_{\text{ad}}, \\ \qquad \hat{E}((\hat{u},\mathcal{G}),\gamma) = 0 \quad \text{in } \hat{\Omega}. \end{cases}$$

Since for given $\gamma \in S_{\text{ad}}$ the pair $(\hat{u}_\gamma,\mathcal{G}_\gamma) \in U := H_0^1(\hat{\Omega}) \times H_I^*$ is uniquely defined, the reduced control problem

$$(\hat{p}_\gamma) \quad \begin{cases} \min\ \hat{\jmath}(\gamma) := \hat{J}((\hat{u}_\gamma,\mathcal{G}_\gamma),\gamma) \\ \text{s.t.}\ \ \gamma \in S_{\text{ad}}, \end{cases} \tag{3.1.1}$$

is equivalent to $(\hat{P}_\gamma)$.

Lemma 3.1.1. *The objective functional J and the state operator E are Fréchet-*

differentiable with respect to u. The derivatives $J_u(u,\gamma)$ and $E_u(u,\gamma)$ satisfy

$$
\begin{aligned}
J_u(u,\gamma)v &= (u-u_d, v)_{L^2(\Omega_T\cap\Omega_\gamma)} \quad \textit{for all } v\in H_0^1(\Omega_\gamma),\\
E_u(u,\gamma)v &= (\nabla v, \nabla\cdot)_{L^2(\Omega_\gamma)^2} \quad \textit{for all } v\in H_0^1(\Omega_\gamma).
\end{aligned}
$$

Proof. For every $v\in H_0^1(\Omega_\gamma)$ there holds

$$
\begin{aligned}
d_uJ(u,\gamma;v) &= \lim_{\delta\to 0}\frac{1}{\delta}\Big\{J(u+\delta v,\gamma)-J(u,\gamma)\Big\}\\
&= \lim_{\delta\to 0}\frac{1}{2\delta}\Big\{(\widetilde{u}+\delta\widetilde{v}-u_d,\widetilde{u}+\delta\widetilde{v}-u_d)_{L^2(\Omega_T)}-(\widetilde{u}-u_d,\widetilde{u}-u_d)_{L^2(\Omega_T)}\Big\}\\
&= \lim_{\delta\to 0}\frac{1}{2\delta}\Big\{\delta(\widetilde{u}-u_d,\widetilde{v})_{L^2(\Omega_T)}+\delta(\widetilde{v},\widetilde{u}-u_d)_{L^2(\Omega_T)}+\delta^2(\widetilde{v},\widetilde{v})_{L^2(\Omega_T)}\Big\}\\
&= (\widetilde{u}-u_d,\widetilde{v})_{L^2(\Omega_T)} = (u-u_d,v)_{L^2(\Omega_T\cap\Omega_\gamma)}.
\end{aligned}
$$

We show that the linear functional

$$
J_u(u,\gamma): v\mapsto (u-u_d,v)_{L^2(\Omega_T\cap\Omega_\gamma)}, \quad v\in H_0^1(\Omega_\gamma),
$$

is bounded: For all $v\in L^2(\Omega_\gamma)$ we have

$$
|J_u(u,\gamma)v| = |(u-u_d,v)_{L^2(\Omega_T\cap\Omega_\gamma)}| \le \|u-u_d\|_{L^2(\Omega_T\cap\Omega_\gamma)}\|v\|_{L^2(\Omega_\gamma)}.
$$

Since u and u_d are fixed the claim for J follows.
Furthermore, for every $v\in H_0^1(\Omega_\gamma)$ there holds

$$
d_uE(u,\gamma;v) = \lim_{\delta\to 0}\frac{1}{\delta}\Big\{E(u+\delta v,\gamma)-E(u,\gamma)\Big\} = (\nabla v,\nabla\cdot)_{L^2(\Omega_\gamma)^2}.
$$

Clearly, the functional $E_u(u,\gamma): v\mapsto(\nabla v,\nabla\cdot)_{L^2(\Omega_\gamma)^2}$, $v\in H_0^1(\Omega_\gamma)$, is linear and bounded. $\square$

Lemma 3.1.2. *The objective functional $\hat{J}$ and fictitious domain state operator $\hat{E}$ are Fréchet-differentiable with respect to $(\hat{u},\mathcal{G})$. The derivatives $\hat{J}_{(\hat{u},\mathcal{G})}((\hat{u},\mathcal{G}),\gamma)$ and $\hat{E}_{(\hat{u},\mathcal{G})}((\hat{u},\mathcal{G}),\gamma)$ satisfy*

$$
\begin{aligned}
\hat{J}_{(\hat{u},\mathcal{G})}((\hat{u},\mathcal{G}),\gamma)(\hat{v},\mathcal{H}) &= \begin{pmatrix}(\hat{u}-u_d,\hat{v})_{L^2(\Omega_T)}\\ 0\end{pmatrix} \quad \textit{for all } \hat{v}\in H_0^1(\hat{\Omega}),\mathcal{H}\in H_I^*,\\
\hat{E}_{(\hat{u},\mathcal{G})}((\hat{u},\mathcal{G}),\gamma)(\hat{v},\mathcal{H}) &= \begin{pmatrix}\hat{\mathcal{A}} & -\mathcal{T}_\gamma^*\\ -\mathcal{T}_\gamma & 0\end{pmatrix}\begin{pmatrix}\hat{v}\\ \mathcal{H}\end{pmatrix} \quad \textit{for all } \hat{v}\in H_0^1(\hat{\Omega}),\mathcal{H}\in H_I^*.
\end{aligned}
$$

3.2. Existence result for the shape optimization problem

Theorem 3.2.1. *Under the Assumption 2.4.9 the domain optimization problem* $(\hat{P}_\gamma)$ *admits at least one solution* $(\hat{u}^*, \mathcal{G}^*) \in U$, $\gamma^* \in S_{ad}$ *with* $\hat{E}((\hat{u}^*, \mathcal{G}^*), \gamma^*) = 0$.

Proof. We denote by

$$F_{\text{ad}} := \{((\hat{u}, \mathcal{G}), \gamma) \in (H_0^1(\hat{\Omega}) \times H_I^*) \times S_{\text{ad}} : \hat{E}((\hat{u}, \mathcal{G}), \gamma) = 0\}$$

the feasible set. Since $\hat{J} \geq 0$ and F_{ad} is nonempty,

$$\hat{J}^* := \inf_{((\hat{u},\mathcal{G}),\gamma) \in F_{\text{ad}}} \hat{J}((\hat{u}, \mathcal{G}), \gamma) \geq 0,$$

so that a minimizing sequence $\{((\hat{u}^k, \mathcal{G}^k), \gamma^k)\} \subset F_{\text{ad}}$ exists with

$$\lim_{k\to\infty} \hat{J}((\hat{u}^k, \mathcal{G}^k), \gamma^k) = \hat{J}^*.$$

The sequence $\{\gamma^k\}$ is bounded in $H^3(I)^2$ since S_{ad} is bounded in $H^3(I)^2$ by definition. Since $H^3(I)^2$ is reflexive, there exists a weakly convergent subsequence $\{\gamma^{k_i}\} \subset \{\gamma^k\}$ and some $\gamma^* \in H^3(I)^2$ with $\gamma^{k_i} \rightharpoonup \gamma^*$ in $H^3(I)^2$ as $i \to \infty$. We show that $\gamma^* \in S_{\text{ad}}$, i.e. $\hat{\gamma}^* \in \hat{S}_{\text{ad}}$:
The fact that $\|\hat{\gamma}^{k_i}\|_{H^3(\hat{I})^2} \leq c_s$ for all i and the weakly lower semicontinuity of the norm implies $\|\hat{\gamma}^*\|_{H^3(\hat{I})^2} \leq c_s$. From the compact embedding $H^3(I)^2 \hookrightarrow\hookrightarrow C^2(\bar{I})^2$ we obtain strong convergence

$$\lim_{i\to\infty} \gamma^{k_i} = \gamma^* \text{ in } C^2(\bar{I})^2 \tag{3.2.1}$$

for a further subsequence.
Since $\hat{\gamma}^{k_i}(\varepsilon) = (\varepsilon, \alpha_2)^T$, $\hat{\gamma}^{k_i}(1-\varepsilon) = (1-\varepsilon, \beta_2)^T$, $(\alpha_2 - \alpha_1)\,\dot{\hat{\gamma}}_1^{k_i}(\varepsilon) - \varepsilon\dot{\hat{\gamma}}_2^{k_i}(\varepsilon) = 0$, $(\beta_2 - \beta_1)\,\dot{\hat{\gamma}}_1^{k_i}(1-\varepsilon) + \varepsilon\dot{\hat{\gamma}}_2^{k_i}(1-\varepsilon) = 0$, $\ddot{\hat{\gamma}}^{k_i}(\varepsilon) = \ddot{\hat{\gamma}}^{k_i}(1-\varepsilon) = (0,0)^T$, $\|\hat{\gamma}^{k_i}(t_2) - \hat{\gamma}^{k_i}(t_1)\|_2 \geq c_r|t_2 - t_1|$ for all $t_1, t_2 \in \hat{I}$, $t_1 \neq t_2$ and $\text{dist}(\Gamma_{\hat{\gamma}^{k_i}}, \partial\hat{\Omega}) \geq \varepsilon$ the same properties hold for γ^*. Therefore we have $\gamma^* \in S_{\text{ad}}$. The convergence in (3.2.1) in particular implies

$$\lim_{i\to\infty} \gamma^{k_i} = \gamma^* \text{ in } W^{1,\infty}(I)^2,$$

since $C^2(\bar{I})^2 \hookrightarrow\hookrightarrow C^{0,1}(\bar{I})^2 \equiv W^{1,\infty}(I)^2$, and

$$\lim_{i\to\infty} \widetilde{f}_{\gamma^{k_i}} = \widetilde{f}_{\gamma^*} \text{ in } L^2(\hat{\Omega}).$$

With Assumption 2.4.9 we then obtain $\hat{u}^{k_i} = \hat{u}_{\gamma^{k_i}} \to \hat{u}_{\gamma^*} = \hat{u}^*$ in $H_0^1(\hat{\Omega})$ and $\mathcal{G}^{k_i} = \mathcal{G}_{\gamma^{k_i}} \overset{*}{\rightharpoonup} \mathcal{G}_{\gamma^*} = \mathcal{G}^*$ in H_I^*. By the continuity of $\hat{E}$ we have that

$$0 = \hat{E}((\hat{u}^{k_i}, \mathcal{G}^{k_i}), \gamma^{k_i}) \to \hat{E}((\hat{u}^*, \mathcal{G}^*), \gamma^*),$$

and thus $((\hat{u}^*, \mathcal{G}^*), \gamma^*) \in F_{\text{ad}}$. Furthermore, by the continuity of $\hat{J}$ we obtain

$$\hat{J}((\hat{u}^{k_i}, \mathcal{G}^{k_i}), \gamma^{k_i}) \to \hat{J}((\hat{u}^*, \mathcal{G}^*), \gamma^*),$$

and therefore

$$\hat{J}^* = \lim_{i\to\infty} \hat{J}((\hat{u}^{k_i}, \mathcal{G}^{k_i}), \gamma^{k_i}) = \hat{J}((\hat{u}^*, \mathcal{G}^*), \gamma^*).$$

Hence, $((\hat{u}^*, \mathcal{G}^*), \gamma^*)$ is an optimal solution. $\square$

3.3. The adjoint equations

3.3.1. The adjoint equations associated with $(\hat{P}_\gamma)$ and (P_γ)

In this subsection we introduce the adjoint equation associated with the shape optimization problem $(\hat{P}_\gamma)$. We define the spaces

$$\begin{aligned} U = U_1 \times U_2 &:= H_0^1(\hat{\Omega}) \times H_I^*, \\ Z = Z_1 \times Z_2 &:= H^{-1}(\hat{\Omega}) \times H_I, \end{aligned}$$

and the Lagrange functional $\hat{L} : U \times S_{\text{ad}} \times Z^* \to \mathbb{R}$,

$$\hat{L}((\hat{u}, \mathcal{G}), \gamma, (\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma)) := \hat{J}((\hat{u}, \mathcal{G}), \gamma) - \langle(\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma), \hat{E}((\hat{u}, \mathcal{G}), \gamma)\rangle_{Z^*,Z}.$$

Inserting $(\hat{u}, \mathcal{G}) = (\hat{u}_\gamma, \mathcal{G}_\gamma)$ for arbitrary $(\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma) \in Z^*$ gives

$$\begin{aligned} \hat{L}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma, (\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma)) &= \hat{J}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma) - \langle(\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma), \hat{E}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma)\rangle_{Z^*,Z} \\ &= \hat{J}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma) = \hat{\jmath}(\gamma). \end{aligned}$$

Now we obtain the adjoint equation if we choose a special $(\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma) = (\hat{\lambda}^\Sigma_\gamma, \mathcal{M}^\Sigma_\gamma)$ such that

$$\hat{L}_{(\hat{u},\mathcal{G})}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma, (\hat{\lambda}^\Sigma_\gamma, \mathcal{M}^\Sigma_\gamma)) = 0.$$

The differentiability properties of $\hat{J}$ and $\hat{E}$ induce according differentiability properties of $\hat{L}$. We have

$$\begin{aligned} &\langle \hat{L}_{(\hat{u},\mathcal{G})}((\hat{u}, \mathcal{G}), \gamma, (\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma)), (\hat{v}, \mathcal{H})\rangle_{U^*,U} \\ &\quad = \langle \hat{J}_{(\hat{u},\mathcal{G})}((\hat{u}, \mathcal{G}), \gamma), (\hat{v}, \mathcal{H})\rangle_{U^*,U} - \langle(\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma), \hat{E}_{(\hat{u},\mathcal{G})}((\hat{u}, \mathcal{G}), \gamma)(\hat{v}, \mathcal{H})\rangle_{Z^*,Z} \\ &\quad = \langle \hat{J}_{(\hat{u},\mathcal{G})}((\hat{u}, \mathcal{G}), \gamma) - \hat{E}_{(\hat{u},\mathcal{G})}((\hat{u}, \mathcal{G}), \gamma)^*(\hat{\lambda}^\Sigma, \mathcal{M}^\Sigma), (\hat{v}, \mathcal{H})\rangle_{U^*,U}. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{L}_{(\hat{u},\mathcal{G})}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma, (\hat{\lambda}^\Sigma_\gamma, \mathcal{M}^\Sigma_\gamma)) &= \hat{J}_{(\hat{u},\mathcal{G})}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma) \\ &\quad - \hat{E}_{(\hat{u},\mathcal{G})}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma)^*(\hat{\lambda}^\Sigma_\gamma, \mathcal{M}^\Sigma_\gamma) = 0. \end{aligned}$$

This leads to the adjoint equation associated with the shape optimization problem $(\hat{P}_\gamma)$: For $\gamma \in S_{\mathrm{ad}}$ and $\hat{u}_\gamma$ (the solution of the state equation $\hat{E}((\hat{u}, \mathcal{G}), \gamma) = 0$) find $(\hat{\lambda}^\Sigma_\gamma, \mathcal{M}^\Sigma_\gamma) := (\hat{\lambda}^\Sigma(\gamma), \mathcal{M}^\Sigma(\gamma))$ such that

$$\begin{cases} (\nabla \hat{\lambda}^\Sigma_\gamma, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - \langle \mathcal{M}^\Sigma_\gamma, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} = (\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_T)}, & \forall \hat{v} \in H^1_0(\hat{\Omega}), \\ -\langle \mathcal{H}, \mathcal{T}_\gamma \hat{\lambda}^\Sigma_\gamma \rangle_{H_I^*, H_I} = 0, & \forall \mathcal{H} \in H_I^*, \end{cases} \tag{3.3.1}$$

or equivalently

$$\hat{E}_{(\hat{u}, \mathcal{G})}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma)^* (\hat{\lambda}^\Sigma_\gamma, \mathcal{M}^\Sigma_\gamma) = \hat{J}_{(\hat{u}, \mathcal{G})}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma),$$

which is

$$\begin{pmatrix} \hat{\mathcal{A}} & -\mathcal{T}^*_\gamma \\ -\mathcal{T}_\gamma & 0 \end{pmatrix} \begin{pmatrix} \hat{\lambda}^\Sigma_\gamma \\ \mathcal{M}^\Sigma_\gamma \end{pmatrix} = \begin{pmatrix} (\hat{u}_\gamma - u_d, \cdot)_{L^2(\Omega_T)} \\ 0 \end{pmatrix}.$$

The main difference of this fictitious domain formulation for $(\hat{\lambda}_\gamma, \mathcal{M}_\gamma)$ compared to (2.2.1) is that we expect $\hat{\lambda}^\Sigma_\gamma|_{\Omega^c_\gamma} \neq 0$, since the difference $\hat{u}_\gamma - \hat{u}_d$ in the right-hand side of the first equation of (3.3.1) may be non-zero in $\Omega_T \cap \Omega^c_\gamma$.
Now we derive the adjoint equation associated with the shape optimization problem (P_γ). Therefore, let us define the Lagrange function $L : U(\Omega_\gamma) \times S_{\mathrm{ad}} \times Z(\Omega_\gamma)^* \to \mathbb{R}$,

$$L(u, \gamma, \lambda) := J(u, \gamma) - \langle \lambda, E(u, \gamma) \rangle_{Z(\Omega_\gamma)^*, Z(\Omega_\gamma)} \ .$$

We insert $u = u_\gamma$ for arbitrary $\lambda \in Z(\Omega_\gamma)^*$and obtain the adjoint equation from setting the derivative of L with respect to the state variable u equal to zero, i.e. choose $\lambda = \lambda_\gamma$ such that

$$L_u(u_\gamma, \gamma, \lambda_\gamma) = 0.$$

Then we obtain: For $\gamma \in S_{\mathrm{ad}}$ and u_γ (the solution of the state equation $E(u, \gamma) = 0$) find $\lambda_\gamma := \lambda(\gamma) \in Z(\Omega_\gamma)^*$ such that

$$(\nabla \lambda_\gamma, \nabla v)_{L^2(\Omega_\gamma)^2} = (u_\gamma - u_d, v)_{L^2(\Omega_T \cap \Omega_\gamma)}, \quad \forall v \in U(\Omega_\gamma). \tag{3.3.2}$$

As a preparation of two following two theorems we state an equation on the complementary domain Ω^c_γ: Find $\lambda^c_\gamma := \lambda^c(\gamma) \in Z(\Omega^c_\gamma)^*$ such that

$$(\nabla \lambda^c_\gamma, \nabla v^c)_{L^2(\Omega^c_\gamma)^2} = (-u_d, v^c)_{L^2(\Omega_T \cap \Omega^c_\gamma)}, \quad \forall v^c \in U(\Omega^c_\gamma). \tag{3.3.3}$$

Theorem 3.3.1. *Let for $\gamma \in S_{ad}$ denote u_γ the solution of the Poisson equation $E(u, \gamma) = 0$.*

1. *The adjoint equation* (3.3.2) *has a unique solution $\lambda_\gamma \in H^2(\Omega_\gamma) \cap H^1_0(\Omega_\gamma)$.*
2. *The equation* (3.3.3) *has a unique solution $\lambda^c_\gamma \in H^2(\Omega^c_\gamma) \cap H^1_0(\Omega^c_\gamma)$.*
3. *The sets $\left\{\|\lambda_\gamma\|_{H^2(\Omega_\gamma)}\right\}_{\gamma \in S_{ad}}$ and $\left\{\|\lambda^c_\gamma\|_{H^2(\Omega^c_\gamma)}\right\}_{\gamma \in S_{ad}}$ are uniformly bounded.*

Proof. Apply the same techniques as in the proofs of Theorem 1.3.1 and Theorem 1.3.3. $\square$

Theorem 3.3.2. *Let for $\gamma \in S_{ad}$ denote u_γ the solution of the Poisson equation $E(u,\gamma) = 0$ and $(\hat{u}_\gamma, \mathcal{G}_\gamma)$ the solution of the fictitious domain formulation of the Poisson problem $\hat{E}((\hat{u}, \mathcal{G}), \gamma) = 0$.*

1. *Let $(\hat{\lambda}_\gamma^\Sigma, \mathcal{M}_\gamma^\Sigma) \in \Big(H^{-1}(\hat{\Omega}) \times H_I\Big)^*$ denote the solution of the adjoint equation (3.3.1) on $\hat{\Omega}$. Then $\lambda_\gamma := \hat{\lambda}_\gamma^\Sigma|_{\Omega_\gamma} \in H^2(\Omega_\gamma) \cap H_0^1(\Omega_\gamma)$ is the unique solution of the adjoint equation (3.3.2) on Ω_γ. Furthermore, $\lambda_\gamma^c := \hat{\lambda}_\gamma^\Sigma|_{\Omega_\gamma^c} \in H^2(\Omega_\gamma^c) \cap H_0^1(\Omega_\gamma^c)$ is the unique solution of (3.3.3) on Ω_γ^c and the Lagrange multiplier $\mathcal{M}_\gamma^\Sigma$ satisfies*
$$\langle \mathcal{M}_\gamma^\Sigma, h \rangle_{H_I^*, H_I} = \left(\left[\frac{\partial \hat{\lambda}_\gamma^\Sigma(\gamma(\cdot))}{\partial n_\gamma} \right] \|\dot{\gamma}(\cdot)\|_2, h \right)_{L^2(I)} \qquad \forall h \in H_I, \tag{3.3.4}$$
where $\left[\frac{\partial \hat{\lambda}_\gamma^\Sigma}{\partial n_\gamma}\right] = \left(\frac{\partial \lambda_\gamma}{\partial n_\gamma} - \frac{\partial \lambda_\gamma^c}{\partial n_\gamma}\right)$ is the jump of the normal derivative at Γ_γ.

2. *Conversely, let $\lambda_\gamma \in H^2(\Omega_\gamma) \cap H_0^1(\Omega_\gamma)$ be the solution of (3.3.2) and let $\lambda_\gamma^c \in H^2(\Omega_\gamma^c) \cap H_0^1(\Omega_\gamma^c)$ be the solution of (3.3.3). Then (3.3.4) uniquely defines an element $\mathcal{M}_\gamma^\Sigma \in H_I^*$ and $(\hat{\lambda}_\gamma^\Sigma, \mathcal{M}_\gamma^\Sigma)$ with*
$$\hat{\lambda}_\gamma^\Sigma := \begin{cases} \lambda_\gamma & \text{in } \Omega_\gamma, \\ \lambda_\gamma^c & \text{in } \Omega_\gamma^c, \end{cases}$$
is the unique solution of (3.3.1) in $H_0^1(\hat{\Omega}) \times H_I^*$.

Proof. The proof is mainly carried out analogously to the proof of Theorem 2.3.1. However, we recapitulate and extend a short part of the proof to demonstrate where the jump of the normal derivative $\left[\frac{\partial \hat{\lambda}_\gamma^\Sigma}{\partial n_\gamma}\right]$ pops up.
The first equation of (3.3.1) reads
$$(\nabla \lambda_\gamma, \nabla \hat{v})_{L^2(\Omega_\gamma)^2} + \big(\nabla \lambda_\gamma^c, \nabla \hat{v}\big)_{L^2(\Omega_\gamma^c)^2} - (\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_T \cap \Omega_\gamma)} \\ - (\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_T \cap \Omega_\gamma^c)} = \langle \mathcal{M}_\gamma^\Sigma, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} \quad \forall \hat{v} \in H_0^1(\hat{\Omega}).$$

Since λ and $\lambda^c \in H^2(\Omega_\gamma)$ we can apply integration by parts and obtain
$$- \int_{\Omega_\gamma} \Delta \lambda_\gamma \hat{v}\, dx - \int_{\Omega_\gamma^c} \Delta \lambda_\gamma^c \hat{v}\, dx - (\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_T \cap \Omega_\gamma)} - (\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_T \cap \Omega_\gamma^c)} \\ + \int_{\partial \Omega_\gamma} \frac{\partial \lambda_\gamma}{\partial n_\gamma} \hat{v}\, dS(x) + \int_{\partial \Omega_\gamma^c} \frac{\partial \lambda_\gamma^c}{\partial n_\gamma^c} \hat{v}\, dS(x) = \langle \mathcal{M}_\gamma^\Sigma, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} \quad \forall \hat{v} \in H_0^1(\hat{\Omega}). \tag{3.3.5}$$

We now choose $v \in H_0^1(\Omega_\gamma)$ and test the previous equation with its extension $\widetilde{v} \in H_0^1(\hat{\Omega})$ which fulfills $\widetilde{v}|_{\partial\Omega_\gamma} = 0$ and $\widetilde{v}|_{\partial\Omega_\gamma^c} = 0$. Then we get

$$-\Delta\lambda_\gamma = (u_\gamma - u_d, \cdot)_{L^2(\Omega_T \cap \Omega_\gamma)} \quad \text{in } L^2(\Omega_\gamma).$$

We now choose $v^c \in H_0^1(\Omega_\gamma^c)$ and test equation (3.3.5) with its extension $\bar{v}^c \in H_0^1(\hat{\Omega})$ which fulfills $\bar{v}^c|_{\partial\Omega_\gamma} = 0$ and $\bar{v}^c|_{\partial\Omega_\gamma^c} = 0$. Then we get

$$-\Delta\lambda_\gamma^c = (-u_d, \cdot)_{L^2(\Omega_T \cap \Omega_\gamma^c)} \quad \text{in } L^2(\Omega_\gamma^c).$$

Now we test equation (3.3.5) with an arbitrary $\hat{v} \in H_0^1(\hat{\Omega})$. Then we have $\mathcal{T}_\gamma \hat{v} \in H_I$, $\hat{v}|_\Gamma = 0$, and with $n_\gamma = -n_\gamma^c$ we get

$$\begin{aligned}
\langle \mathcal{M}_\gamma^\Sigma, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} &= \int_{\partial\Omega_\gamma} \frac{\partial\lambda_\gamma}{\partial n_\gamma} \hat{v}\, dS(x) + \int_{\partial\Omega_\gamma^c} \frac{\partial\lambda_\gamma^c}{\partial n_\gamma^c} \hat{v}\, dS(x) \\
&= \int_{\Gamma_\gamma} \frac{\partial\lambda_\gamma}{\partial n_\gamma} \hat{v}\, dS(x) - \int_{\Gamma_\gamma} \frac{\partial\lambda_\gamma^c}{\partial n_\gamma} \hat{v}\, dS(x) \\
&= \int_I \left(\frac{\partial\lambda_\gamma(\gamma(t))}{\partial n_\gamma} - \frac{\partial\lambda_\gamma^c(\gamma(t))}{\partial n_\gamma} \right) \hat{v}(\gamma(t)) \|\dot{\gamma}(t)\|_2\, dt \\
&= \left(\left(\frac{\partial\lambda_\gamma(\gamma(\cdot))}{\partial n_\gamma} - \frac{\partial\lambda_\gamma^c(\gamma(\cdot))}{\partial n_\gamma} \right) \|\dot{\gamma}(\cdot)\|_2, \mathcal{T}_\gamma \hat{v} \right)_{L^2(I)} \\
&= \left(\left[\frac{\partial\lambda_\gamma^\Sigma(\gamma(\cdot))}{\partial n_\gamma} \right] \|\dot{\gamma}(\cdot)\|_2, \mathcal{T}_\gamma \hat{v} \right)_{L^2(I)},
\end{aligned}$$

which is (3.3.4). □

Remark 3.3.3.

1. *Analogously to the extension of $\mathcal{G}_\gamma$ in Remark 2.3.2 we extend $\mathcal{M}_\gamma^\Sigma \in H_I^*$ onto $L^2(I)$ by the definition*

$$\langle \mathcal{M}_\gamma^\Sigma, h \rangle_{L^2(I)^*, L^2(I)} := \left(\left[\frac{\partial\hat{\lambda}_\gamma^\Sigma(\gamma(\cdot))}{\partial n_\gamma} \right] \|\dot{\gamma}(\cdot)\|_2, h \right)_{L^2(I)} \qquad \forall h \in L^2(I).$$

2. *Identifying $L^2(I)$ with its dual space the (adjoint) Lagrange multiplier satisfies*

$$\mathcal{M}_\gamma^\Sigma = \left[\frac{\partial\hat{\lambda}_\gamma^\Sigma(\gamma(\cdot))}{\partial n_\gamma} \right] \|\dot{\gamma}(\cdot)\|_2 \quad \text{in } L^2(I).$$

Theorem 3.3.4. *For $\gamma \in S_{ad}^+$ let $\bar{\gamma} = \bar{g} n_\gamma \in S_\gamma'$ be an admissible direction.*

Then the solutions of (3.3.1) *for* $\gamma + \delta\bar{\gamma}$, γ *satisfy*

$$\lim_{\delta\to 0} \hat{\lambda}^{\Sigma}_{\gamma+\delta\bar{\gamma}} = \hat{\lambda}^{\Sigma}_{\gamma} \quad \text{in } H^1_0(\hat{\Omega}),$$
$$w^* - \lim_{\delta\to 0} \mathcal{M}^{\Sigma}_{\gamma+\delta\bar{\gamma}} = \mathcal{M}^{\Sigma}_{\gamma} \quad \text{in } H^*_I.$$

Proof. Apply the same techniques as in the proof of Theorem 2.4.8. □

Assumption 3.3.5. *Let* $\{\gamma_k\} \subset S^+_{ad}$ *be a sequence which converges to* γ *in* $W^{1,\infty}(I)^2$. *Then the solutions of the problem* (3.3.1) *for* γ^k, γ *satisfy*

$$\lim_{k\to\infty} \hat{\lambda}^{\Sigma}_{\gamma^k} = \hat{\lambda}^{\Sigma}_{\gamma} \quad \text{in } H^1_0(\hat{\Omega}),$$
$$w^* - \lim_{k\to\infty} \mathcal{M}^{\Sigma}_{\gamma^k} = \mathcal{M}^{\Sigma}_{\gamma} \quad \text{in } H^*_I.$$

3.3.2. Fictitious domain formulation of the adjoint equation associated with (P_γ)

Since the adjoint equation (3.3.2) is the Poisson equation (1.3) in weak form with a different but sufficiently regular inhomogeneity, we can apply the theory of Section 1.3.
We can also derive a fictitious domain formulation of the adjoint equation (3.3.2) by introducing the additional (adjoint) Lagrange multiplier $\mathcal{M} \in H^*_I$, like in Chapter 2. We thus obtain: For $\gamma \in S_{\text{ad}}$ and $\hat{u}_\gamma$ (the solution of the state equation $\hat{E}((\hat{u}, \mathcal{G}), \gamma) = 0$) find $(\hat{\lambda}_\gamma, \mathcal{M}_\gamma) := (\hat{\lambda}(\gamma), \mathcal{M}(\gamma))$ such that

$$\begin{cases} (\nabla\hat{\lambda}_\gamma, \nabla\hat{v})_{L^2(\hat{\Omega})^2} - \langle \mathcal{M}_\gamma, \mathcal{T}_\gamma \hat{v}\rangle_{H^*_I, H_I} = (\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_T\cap\Omega_\gamma)}, & \forall \hat{v} \in H^1_0(\hat{\Omega}), \\ -\langle \mathcal{H}, \mathcal{T}_\gamma \hat{\lambda}_\gamma\rangle_{H^*_I, H_I} = 0, & \forall \mathcal{H} \in H^*_I, \end{cases} \tag{3.3.6}$$

or equivalently

$$\begin{pmatrix} \hat{\mathcal{A}} & -\mathcal{T}^*_\gamma \\ -\mathcal{T}_\gamma & 0 \end{pmatrix} \begin{pmatrix} \hat{\lambda}_\gamma \\ \mathcal{M}_\gamma \end{pmatrix} = \begin{pmatrix} (\hat{u}_\gamma - u_d, \cdot)_{L^2(\Omega_T\cap\Omega_\gamma)} \\ 0 \end{pmatrix}.$$

Furthermore, we obtain a fictitious domain formulation of (3.3.3): Find $(\hat{\lambda}^c_\gamma, \mathcal{M}^c_\gamma) := (\hat{\lambda}^c(\gamma), \mathcal{M}^c(\gamma))$ such that

$$\begin{cases} (\nabla\hat{\lambda}^c_\gamma, \nabla\hat{v})_{L^2(\hat{\Omega})^2} - \langle \mathcal{M}^c_\gamma, \mathcal{T}_\gamma \hat{v}\rangle_{H^*_I, H_I} = (-u_d, \hat{v})_{L^2(\Omega_T\cap\Omega^c_\gamma)}, & \forall \hat{v} \in H^1_0(\hat{\Omega}), \\ -\langle \mathcal{H}, \mathcal{T}_\gamma \hat{\lambda}^c_\gamma\rangle_{H^*_I, H_I} = 0, & \forall \mathcal{H} \in H^*_I, \end{cases} \tag{3.3.7}$$

or equivalently

$$\begin{pmatrix} \hat{\mathcal{A}} & -\mathcal{T}^*_\gamma \\ -\mathcal{T}_\gamma & 0 \end{pmatrix} \begin{pmatrix} \hat{\lambda}^c_\gamma \\ \mathcal{M}^c_\gamma \end{pmatrix} = \begin{pmatrix} (\hat{u}_\gamma - u_d, \cdot)_{L^2(\Omega_T\cap\Omega^c_\gamma)} \\ 0 \end{pmatrix}.$$

Lemma 3.3.6. *We obtain*

$$\hat{\lambda}_\gamma = \begin{cases} \lambda_\gamma & \text{in } \Omega_\gamma, \\ 0 & \text{in } \Omega_\gamma^c, \end{cases} \qquad \langle \mathcal{M}_\gamma, h\rangle_{H_I^*, H_I} = \left(\frac{\partial \lambda_\gamma(\gamma(\cdot))}{\partial n_\gamma} \|\dot{\gamma}(\cdot)\|_2, h \right)_{L^2(I)} \qquad \forall h \in H_I,$$

$$\hat{\lambda}_\gamma^c = \begin{cases} 0 & \text{in } \Omega_\gamma, \\ \lambda_\gamma^c & \text{in } \Omega_\gamma^c, \end{cases} \qquad \langle \mathcal{M}_\gamma^c, h\rangle_{H_I^*, H_I} = \left(\frac{\partial \lambda_\gamma^c(\gamma(\cdot))}{\partial n_\gamma^c} \|\dot{\gamma}(\cdot)\|_2, h \right)_{L^2(I)} \qquad \forall h \in H_I.$$

Moreover, we obtain

$$\hat{\lambda}_\gamma^\Sigma = \hat{\lambda}_\gamma + \hat{\lambda}_\gamma^c, \qquad \text{and} \qquad \mathcal{M}_\gamma^\Sigma = \mathcal{M}_\gamma + \mathcal{M}_\gamma^c.$$

Remark 3.3.7.

1. *The authors in [KP98, Sla00] studied the case, where the observation domain Ω_T is restricted by*

$$\Omega_T \subset\subset \Omega_\gamma \subset \hat{\Omega} \quad \text{for all } \gamma \in S_{ad}, \text{ i.e. } \operatorname{dist}(\Gamma_\gamma, \Omega_T) > 0 \text{ for all } \gamma \in S_{ad}. \tag{3.3.8}$$

   In this case, we obtain $\Omega_T \cap \Omega_\gamma^c = \emptyset$ and therefore $\hat{\lambda}_\gamma^c = 0$, $\mathcal{M}_\gamma^c = 0$. Furthermore, the fictitious domain formulation (3.3.6) *coincides with the adjoint equation* (3.3.1)*, i.e. $\hat{\lambda}_\gamma^\Sigma = \hat{\lambda}_\gamma$ and $\mathcal{M}_\gamma^\Sigma = \mathcal{M}_\gamma$. We note that* (3.3.8) *would require further restrictions on S_{ad}.*

2. *If it is possible from the application point of view, we recommend to track the L^2-error of $\hat{u}$ with respect to a desired state $\hat{u}_d \in L^2(\Omega_T)$ in the entire fictitious domain $\hat{\Omega}$, i.e. $\Omega_T = \hat{\Omega}$.*
   The choice of the observation domain Ω_T has a huge influence on the optimization process. If one chooses Ω_T with $|\Omega_T|$ small compared to $|\hat{\Omega}|$, or $\Omega_T \cap \Gamma_\gamma = \emptyset$ for some $\gamma \in S_{ad}$, it is an easy exercise to construct examples where the sequence $(\hat{u}^k(\gamma^k))$ from an optimization process tends to u_d in $L^2(\Omega_T)$ as k tends to ∞, but the sequence (γ^k) converges to an 'optimal shape', which significantly deviates from the expected aim.

3.4. First derivative of the reduced objective function

In this section we prove that the reduced objective function $\hat{\jmath}$,

$$\begin{aligned} \hat{\jmath} : S_{\text{ad}}^+ &\to \mathbb{R}, \\ \gamma &\mapsto \hat{J}((\hat{u}(\gamma), \mathcal{G}(\gamma)), \gamma), \end{aligned}$$

from our problem (3.1.1) is Fréchet-differentiable with respect to γ. This result is obtained even though the state equation operator $\hat{E}$ is not continuously Fréchet-differentiable. This Fréchet-differentiability is a necessary condition if one derives the derivative representation of $\hat{\jmath}'(\gamma)$ via the *adjoint approach*, cf. [HPUU08, Section 1.6].

In Theorem 3.4.1 we show that the reduced objective function $\hat{\jmath}$ is directional differentiable with respect to $\gamma \in S^+_{\text{ad}}$ in all admissible directions $\bar{\gamma} \in S'_\gamma$. In addition we obtain a suitable integral representation of the directional derivative $d\hat{\jmath}(\gamma, \bar{\gamma})$. In Theorem 3.4.2 we obtain the Gâteaux- and Fréchet-differentiability of the reduced objective function $\hat{\jmath}$.

The main key in the proof of the next theorem is to replace occurring (normal) derivatives of the state $\hat{u}$ by the Lagrange multiplier $\mathcal{G}$ using the relation (2.3.1). It turns out that occurring (normal) derivatives of the adjoint state $\hat{\lambda}^\Sigma = \hat{\lambda} + \hat{\lambda}^c$ cannot be replaced by the adjoint Lagrange multiplier $\mathcal{M}^\Sigma$. But since derivatives with respect to $\hat{\lambda}^c$ cancel out in the computation, we can replace the remaining derivatives with respect to $\hat{\lambda}$ by the Lagrange multiplier $\mathcal{M}$. We obtain the following result:

Theorem 3.4.1. *The reduced objective function $\hat{\jmath}$ has a directional derivative with respect to $\gamma \in S^+_{ad}$ in all admissible directions $\bar{\gamma} \in S'_\gamma$ which is given by*

$$d\hat{\jmath}(\gamma, \bar{\gamma}) = \int_I \mathcal{M}_\gamma(t)\mathcal{G}_\gamma(t) \frac{\bar{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\, dt.$$

Proof. The proof is structured as follows: In part 1 we show that the right-sided directional derivative of $\hat{\jmath}$ with respect to γ in an admissible direction $\bar{\gamma} \in S'_\gamma$ is given by

$$\begin{aligned} d^+\hat{\jmath}(\gamma, \bar{\gamma}) = \lim_{\delta \to 0^+} \frac{1}{2\delta} \Big\{ & (\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}} \rangle_{H_I^*, H_I} \\ & + \langle \mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}} \hat{u}_\gamma \rangle_{H_I^*, H_I} + (\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_\gamma)_{L^2(\hat{\Omega})} + \langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}} \hat{\lambda}^\Sigma_\gamma \rangle_{H_I^*, H_I} \\ & - \langle \mathcal{M}^\Sigma_\gamma, \mathcal{T}_\gamma \hat{u}_{\gamma+\delta\bar{\gamma}} \rangle_{H_I^*, H_I} \Big\} =: \lim_{\delta \to 0^+} \left(B_1(\delta) + \cdots + B_6(\delta) \right). \end{aligned}$$

In part 2 we prove the following right-sided limits

$$\begin{aligned} \lim_{\delta \to 0^+} B_4(\delta) &= \lim_{\delta \to 0^+} \frac{1}{2\delta} (\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_\gamma)_{L^2(\hat{\Omega})} = 0, \\ \lim_{\delta \to 0^+} B_1(\delta) &= \lim_{\delta \to 0^+} \frac{1}{2\delta} (\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} = 0. \end{aligned}$$

In part 3 we obtain

$$\begin{aligned}
\lim_{\delta\to 0^+} B_2(\delta) &= \lim_{\delta\to 0^+} -\frac{1}{2\delta}\langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{\lambda}^{\Sigma}_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I} \\
&= \frac{1}{2}\int_{I_+} \mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt + \frac{1}{2}\int_{I_-} \mathcal{G}_\gamma(t)\nabla\lambda^c_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt \\
\lim_{\delta\to 0^+} B_6(\delta) &= \lim_{\delta\to 0^+} -\frac{1}{2\delta}\langle \mathcal{M}^{\Sigma}_\gamma, \mathcal{T}_\gamma \hat{u}_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I} = \frac{1}{2}\int_{I_+} \mathcal{M}^{\Sigma}_\gamma(t)\nabla u_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt.
\end{aligned}$$

In part 4 we find

$$\begin{aligned}
\lim_{\delta\to 0^+} B_5(\delta) &= \lim_{\delta\to 0^+} \frac{1}{2\delta}\langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}} \hat{\lambda}^{\Sigma}_{\gamma}\rangle_{H_I^*,H_I} \\
&= \frac{1}{2}\int_{I_+} \mathcal{G}_\gamma(t)\nabla\lambda^c_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt + \frac{1}{2}\int_{I_-} \mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt, \\
\lim_{\delta\to 0^+} B_3(\delta) &= \lim_{\delta\to 0^+} \frac{1}{2\delta}\langle \mathcal{M}^{\Sigma}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}} \hat{u}_{\gamma}\rangle_{H_I^*,H_I} = \frac{1}{2}\int_{I_-} \mathcal{M}^{\Sigma}_\gamma(t)\nabla u_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt.
\end{aligned}$$

In part 5 we conclude that

$$d^+\hat{\jmath}(\gamma,\bar{\gamma}) = \int_I \mathcal{M}_\gamma(t)\mathcal{G}_\gamma(t)\frac{\bar{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\,dt.$$

Finally, in part 6 we find that the left- and right-sided directional derivatives of $\hat{\jmath}$ coincide, i.e.

$$d^-\hat{\jmath}(\gamma,\bar{\gamma}) = d^+\hat{\jmath}(\gamma,\bar{\gamma}) = d\hat{\jmath}(\gamma,\bar{\gamma}) = \int_I \mathcal{M}_\gamma(t)\mathcal{G}_\gamma(t)\frac{\bar{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\,dt.$$

Part 1: The right-sided directional derivative of $\hat{\jmath}$ with respect to γ in an admissible direction $\bar{\gamma} \in S'_\gamma$ is given by

$$\begin{aligned}
d^+\hat{\jmath}(\gamma,\bar{\gamma}) &= \lim_{\delta\to 0^+} \frac{1}{\delta}\left\{\hat{\jmath}(\gamma+\delta\bar{\gamma}) - \hat{\jmath}(\gamma)\right\} \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta}\left\{\frac{1}{2}\|\hat{u}_{\gamma+\delta\bar{\gamma}} - u_d\|^2_{L^2(\Omega_T)} - \frac{1}{2}\|\hat{u}_\gamma - u_d\|^2_{L^2(\Omega_T)}\right\}.
\end{aligned}$$

Using the identity $(a-c,a-c)-(b-c,b-c) = (a,a)-2(a,c)-(b,b)+2(b,c) =$

$(a-b, a-c)+(a-b, b-c)$ we obtain

$$\begin{aligned}
d^+\hat{j}(\gamma,\bar{\gamma}) &= \lim_{\delta\to 0^+}\frac{1}{\delta}\left\{\frac{1}{2}\|\hat{u}_{\gamma+\delta\bar{\gamma}}-u_d\|^2_{L^2(\Omega_T)}-\frac{1}{2}\|\hat{u}_\gamma-u_d\|^2_{L^2(\Omega_T)}\right\}\\
&= \lim_{\delta\to 0^+}\frac{1}{2\delta}\left\{(\hat{u}_{\gamma+\delta\bar{\gamma}}-\hat{u}_\gamma,\hat{u}_{\gamma+\delta\bar{\gamma}}-u_d)_{L^2(\Omega_T)}+(\hat{u}_{\gamma+\delta\bar{\gamma}}-\hat{u}_\gamma,\hat{u}_\gamma-u_d)_{L^2(\Omega_T)}\right\}\\
&=: \lim_{\delta\to 0^+}(A_1(\delta)+A_2(\delta)).
\end{aligned}$$

For $A_1(\delta)$ we can use the first equation of the adjoint system (3.3.1) for γ and $\gamma+\delta\bar{\gamma}$ with the test function $\hat{v}:=\hat{u}_{\gamma+\delta\bar{\gamma}}-\hat{u}_\gamma\in H^1_0(\hat{\Omega})$ and obtain

$$\begin{aligned}
A_1(\delta) &= \frac{1}{2\delta}(\hat{u}_{\gamma+\delta\bar{\gamma}}-\hat{u}_\gamma,\hat{u}_{\gamma+\delta\bar{\gamma}}-u_d)_{L^2(\Omega_T)}\\
&= \frac{1}{2\delta}\Big\{(\nabla\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}},\nabla\left(\hat{u}_{\gamma+\delta\bar{\gamma}}-\hat{u}_\gamma\right))_{L^2(\hat{\Omega})^2}-\langle\mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}},\mathcal{T}_{\gamma+\delta\bar{\gamma}}(\hat{u}_{\gamma+\delta\bar{\gamma}}-\hat{u}_\gamma)\rangle_{H_I^*,H_I}\Big\}.
\end{aligned}$$

Since $\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}_{\gamma+\delta\bar{\gamma}}=0$ we obtain

$$A_1(\delta)=\frac{1}{2\delta}\Big\{(\nabla(\hat{u}_{\gamma+\delta\bar{\gamma}}-\hat{u}_\gamma),\nabla\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})^2}+\langle\mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}},\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}_\gamma\rangle_{H_I^*,H_I}\Big\}. \tag{3.4.1}$$

For the first term in (3.4.1) we use the state equation (2.2.1) for γ and $\gamma+\delta\bar{\gamma}$, respectively, in both cases with the test function $\hat{v}=\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}}\in\hat{H}^1_0(\hat{\Omega})$. We obtain

$$\begin{aligned}
A_1(\delta) &= \frac{1}{2\delta}\Big\{(\nabla(\hat{u}_{\gamma+\delta\bar{\gamma}}-\hat{u}_\gamma),\nabla\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})^2}+\langle\mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}},\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}_\gamma\rangle_{H_I^*,H_I}\Big\}\\
&= \frac{1}{2\delta}\Big\{(\widetilde{f}_{\gamma+\delta\bar{\gamma}},\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})}+\langle\mathcal{G}_{\gamma+\delta\bar{\gamma}},\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I}-(\widetilde{f}_\gamma,\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})}\\
&\qquad -\langle\mathcal{G}_\gamma,\mathcal{T}_\gamma\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I}+\langle\mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}},\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}_\gamma\rangle_{H_I^*,H_I}\Big\}.
\end{aligned}$$

Finally, with $\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}}=0$ we have

$$\begin{aligned}
A_1(\delta) = \frac{1}{2\delta}\Big\{&(\widetilde{f}_{\gamma+\delta\bar{\gamma}}-\widetilde{f}_\gamma,\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})}\\
&-\langle\mathcal{G}_\gamma,\mathcal{T}_\gamma\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I}+\langle\mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}},\mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}_\gamma\rangle_{H_I^*,H_I}\Big\}.
\end{aligned}$$

In a similar way we obtain for $A_2(\delta)$ the expression

$$
\begin{aligned}
A_2(\delta) &= \frac{1}{2\delta}(\hat{u}_{\gamma+\delta\bar{\gamma}} - \hat{u}_\gamma, \hat{u}_\gamma - u_d)_{L^2(\Omega_T)} \\
&= \frac{1}{2\delta}\Big\{(\nabla\hat{\lambda}^\Sigma_\gamma, \nabla(\hat{u}_{\gamma+\delta\bar{\gamma}} - \hat{u}_\gamma))_{L^2(\hat{\Omega})^2} - \langle \mathcal{M}^\Sigma_\gamma, \mathcal{T}_\gamma(\hat{u}_{\gamma+\delta\bar{\gamma}} - \hat{u}_\gamma)\rangle_{H_I^*,H_I}\Big\} \\
&= \frac{1}{2\delta}\Big\{(\widetilde{f}_{\gamma+\delta\bar{\gamma}}, \hat{\lambda}^\Sigma_\gamma)_{L^2(\hat{\Omega})} + \langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{\lambda}^\Sigma_\gamma\rangle_{H_I^*,H_I} - (\widetilde{f}_\gamma, \hat{\lambda}^\Sigma_\gamma)_{L^2(\hat{\Omega})} \\
&\qquad - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma\hat{\lambda}^\Sigma_\gamma\rangle_{H_I^*,H_I} - \langle \mathcal{M}^\Sigma_\gamma, \mathcal{T}_\gamma\hat{u}_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I}\Big\}. \\
&= \frac{1}{2\delta}\Big\{(\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_\gamma)_{L^2(\hat{\Omega})} + \langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{\lambda}^\Sigma_\gamma\rangle_{H_I^*,H_I} \\
&\qquad - \langle \mathcal{M}^\Sigma_\gamma, \mathcal{T}_\gamma\hat{u}_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I}\Big\}.
\end{aligned}
$$

Altogether, using the expressions for $A_1(\delta)$ and $A_2(\delta)$, we obtain

$$
\begin{aligned}
d^+\hat{\jmath}(\gamma,\bar{\gamma}) = \lim_{\delta\to 0^+}\frac{1}{2\delta}\Big\{&(\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I} \\
&+ \langle \mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}_\gamma\rangle_{H_I^*,H_I} + (\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_\gamma)_{L^2(\hat{\Omega})} + \langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{\lambda}^\Sigma_\gamma\rangle_{H_I^*,H_I} \\
&- \langle \mathcal{M}^\Sigma_\gamma, \mathcal{T}_\gamma\hat{u}_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I}\Big\} =: \lim_{\delta\to 0^+}\left(B_1(\delta) + \cdots + B_6(\delta)\right).
\end{aligned}
$$

Part 2: In this part we show the following right-sided limit

$$
\lim_{\delta\to 0^+} B_4(\delta) = \lim_{\delta\to 0^+}\frac{1}{2\delta}(\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_\gamma)_{L^2(\hat{\Omega})} = 0.
$$

We have $f \in L^\infty(\hat{\Omega})$,

$$
\widetilde{f}_\gamma := \begin{cases} f & \text{in } \Omega_\gamma, \\ 0 & \text{in } \Omega^c_\gamma, \end{cases} \qquad\qquad \widetilde{f}_{\gamma+\delta\bar{\gamma}} := \begin{cases} f & \text{in } \Omega_{\gamma+\delta\bar{\gamma}}, \\ 0 & \text{in } \Omega^c_{\gamma+\delta\bar{\gamma}}, \end{cases}
$$

and $\hat{\lambda}^\Sigma_\gamma$ is the solution of the adjoint equation (3.3.1) with $\hat{\lambda}^\Sigma_\gamma = \lambda_\gamma$ in Ω_γ and $\hat{\lambda}^\Sigma_\gamma = \lambda^c_\gamma$ in Ω^c_γ. We thus obtain[1]

$$
\begin{aligned}
B_4(\delta) &= \frac{1}{2\delta}(\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_\gamma)_{L^2(\hat{\Omega})} \\
&= -\frac{1}{2\delta}\int_{\Delta^-_\delta} f(x)\lambda_\gamma(x)\,dx + \frac{1}{2\delta}\int_{\Delta^+_\delta} f(x)\lambda^c_\gamma(x)\,dx =: B_{4,1}(\delta) + B_{4,2}(\delta).
\end{aligned}
$$

[1]In this proof we have a slight change in the notation:

$$
\Delta^\pm_\delta := \Delta^{\delta,\pm}_{0,1}, \quad D^\pm_\delta := D^{\delta,\pm}_{0,1}, \quad \Phi_\pm := \Phi^{\delta,\pm}_{0,1}
$$

We will see that both terms, $B_{4,1}(\delta)$ and $B_{4,2}(\delta)$, tend to zero. We start with $B_{4,1}(\delta)$, apply the coordinate transformation of Lemma 2.4.5 and get

$$\begin{aligned}
|B_{4,1}(\delta)| &= \left| \frac{1}{2\delta} \int_{\Delta_\delta^-} f(x)\lambda_\gamma(x)\, d(x) \right| \leq \frac{\|f\|_{L^\infty(\hat{\Omega})}}{2\delta} \int_{\Phi_-(D_\delta^-)} |\lambda_\gamma(x)|\, dx \\
&= \frac{C_1}{\delta} \int_{I_-} \int_0^\delta |\lambda_\gamma(\gamma(t) + \xi \bar{g}(t) n_\gamma(t))| \left|\det \Phi'_-(t,\xi)\right|\, d\xi\, dt.
\end{aligned}$$

Now we use the estimates of Lemma 2.4.5 and the fact that $\mathcal{T}_\gamma \hat{\lambda}_\gamma^\Sigma = \mathcal{T}_\gamma \lambda_\gamma = 0$, i.e. $\lambda_\gamma(\gamma(t)) = 0$ for a.e $t \in I$. Therefore we obtain

$$\begin{aligned}
|B_{4,1}(\delta)| &\leq \frac{C_1}{\delta} \int_{I_-} \int_0^\delta |\lambda_\gamma(\gamma(t) + \xi \bar{g}(t) n_\gamma(t))| |\det \Phi'_-(t,\xi)|\, d\xi\, dt \\
&\leq \frac{C_1}{\delta} \max_{(t,\xi)\in D_\delta^-} \left|\det \Phi'_-(t,\xi)\right| \int_{I_-} \int_0^\delta |\lambda_\gamma(\gamma(t) + \xi \bar{g}(t) n_\gamma(t))|\, d\xi\, dt \\
&\leq \frac{C_2}{\delta} \int_{I_-} \int_0^\delta |\lambda_\gamma(\gamma(t) + \xi \bar{g}(t) n_\gamma(t)) - \lambda_\gamma(\gamma(t))|\, d\xi\, dt.
\end{aligned}$$

Using the identity

$$\lambda_\gamma(\gamma(t) + \xi \bar{g}(t) n_\gamma(t)) - \lambda_\gamma(\gamma(t)) = \int_0^\xi \nabla \lambda_\gamma(\gamma(t) + \eta \bar{g}(t) n_\gamma(t))^T (\bar{g}(t) n_\gamma(t))\, d\eta \quad (3.4.2)$$

which is valid[2] for $\lambda_\gamma \in H_0^1(\Omega_\gamma) \cap H^2(\Omega_\gamma)$, we obtain

$$\begin{aligned}
|B_{4,1}(\delta)| &\leq \frac{C_2}{\delta} \int_{I_-} \int_0^\delta \left| \int_0^\xi \nabla \lambda_\gamma(\gamma(t) + \eta \bar{g}(t) n_\gamma(t))^T (\bar{g}(t) n_\gamma(t))\, d\eta \right| d\xi\, dt \\
&\leq \frac{C_2}{\delta} \int_{I_-} \left(\int_0^\delta \int_0^\xi \|\nabla \lambda_\gamma(\gamma(t) + \eta \bar{g}(t) n_\gamma(t))\|_2\, d\eta\, d\xi \right) \|\bar{g}(t) n_\gamma(t)\|_2\, dt \\
&\leq \frac{C_2}{\delta} \int_{I_-} \left(\int_0^\delta \int_0^\xi \|\nabla \lambda_\gamma(\gamma(t) + \eta \bar{g}(t) n_\gamma(t))\|_2\, d\eta\, d\xi \right) |\bar{g}(t)|\, dt.
\end{aligned}$$

[2]For $t \in I_-$ the function $G(\xi) := \lambda_\gamma(\gamma(t) + \xi \bar{g}(t) n_\gamma(t))$ is differentiable on the interval $(0, \delta)$ with

$$G'(\xi) = \nabla \lambda_\gamma(\gamma(t) + \xi \bar{g}(t) n_\gamma(t))^T (\bar{g}(t) n_\gamma(t))$$

since $\lambda_\gamma \in H_0^1(\Omega_\gamma) \cap H^2(\Omega_\gamma)$. We obtain due to the fundamental theorem of calculus

$$G(\xi) - G(0) = \int_0^\xi G'(\eta)\, d\eta.$$

Using the identity

$$\int_a^b \int_a^\xi F(\eta)\, d\eta\, d\xi = \int_a^b F(\eta)(b-\eta)\, d\eta, \tag{3.4.3}$$

which is valid[3] for all $F \in L^1(a,b)$, we obtain

$$\begin{aligned}
|B_{4,1}(\delta)| &\le \frac{C_2}{\delta} \int_{I_-} \left(\int_0^\delta \int_0^\xi \left\| \nabla \lambda_\gamma(\gamma(t) + \eta \bar{g}(t) n_\gamma(t)) \right\|_2 \, d\eta\, d\xi \right) |\bar{g}(t)|\, dt \\
&\le \frac{C_2}{\delta} \int_{I_-} \left(\int_0^\delta \left\| \nabla \lambda_\gamma(\gamma(t) + \eta \bar{g}(t) n_\gamma(t)) \right\|_2 (\delta - \eta)\, d\eta \right) |\bar{g}(t)|\, dt.
\end{aligned}$$

The Hölder's inequality and Lemma 2.4.5 lead to

$$\begin{aligned}
|B_{4,1}(\delta)| &\le \frac{C_2}{\delta} \left(\int_{I_-} \int_0^\delta \left\| \nabla \lambda_\gamma(\gamma(t) + \eta \bar{g}(t) n_\gamma(t)) \right\|_2^2 |\bar{g}(t)|\, d\eta\, dt \right)^{1/2} \\
&\qquad \left(\int_{I_-} \int_0^\delta |\delta - \eta|^2 |\bar{g}(t)|\, d\eta\, dt \right)^{1/2} \\
&\le \frac{C_2}{\delta} \left(\int_{I_-} \int_0^\delta \left\| \nabla \lambda_\gamma(\gamma(t) + \eta \bar{g}(t) n_\gamma(t)) \right\|_2^2 |\det \Phi_-'(t,\eta)|\, d\eta\, dt \right)^{1/2} \\
&\qquad \left| \int_{I_-} \left[\frac{|\delta - \eta|^3 |\bar{g}(t)|}{3} \right]_{\eta=0}^{\delta} dt \right|^{1/2} \\
&\le \frac{C_2}{\delta} \left(\int_{\Phi_-(D_\delta^-)} \left\| \nabla \lambda_\gamma(x) \right\|_2^2 \, dx \right)^{1/2} \frac{\delta^{3/2}}{3} \left| \int_{I_-} |\bar{g}(t)|\, dt \right|^{1/2} \\
&\le C_3 \delta^{1/2} \|\lambda_\gamma\|_{H^1(\Delta_\delta^-)} \|\bar{g}\|_{L^1(I_-)}^{1/2} \\
&\le C_3 \delta^{1/2} \|\lambda_\gamma\|_{H^1(\Omega_\gamma)} \|\bar{g}\|_{L^1(I)}^{1/2} \\
&= C_4 \delta^{1/2} \longrightarrow 0 \quad \text{for } \delta \to 0^+.
\end{aligned}$$

For the second term $B_{4,2}(\delta)$ we use a similar argumentation as for $B_{4,1}(\delta)$. We just replace Δ_δ^- by Δ_δ^+, λ_γ by λ_γ^c, Φ_- by Φ_+, D_δ^- by D_δ^+ and I_- by I_+ and obtain

[3]For $F \in L^1(a,b)$ there exists an absolutely continuous function f, i.e. $f'(x) = F(x)$ a.e., with $f(x) - f(a) = \int_a^x F(\eta)\, d\eta$. Using this identity and integration by parts we obtain

$$\begin{aligned}
\int_a^b F(\eta)(b-\eta)\, d\eta &= f(\eta)(b-\eta)\big|_{\eta=a}^b - \int_a^b f(\eta)(-1)\, d\eta = f(a)(b-a) + \int_a^b f(\xi)\, d\xi \\
&= \int_a^b f(\xi) - f(a)\, d\xi = \int_a^b \int_a^\xi F(\eta)\, d\eta\, d\xi.
\end{aligned}$$

$$|B_{4,2}(\delta)| \leq C_5 \delta^{1/2} \|\lambda_\gamma^c\|_{H^1(\Delta_\delta^+)} \|\bar{g}\|_{L^1(I_+)}^{1/2} \leq C_6 \delta^{1/2} \longrightarrow 0 \quad \text{for } \delta \to 0^+.$$

Furthermore, the same techniques from above lead to

$$\lim_{\delta \to 0^+} B_1(\delta) = \lim_{\delta \to 0^+} \frac{1}{2\delta} (\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} = 0.$$

Part 3: In the next step we show that

$$\begin{aligned}
\lim_{\delta \to 0^+} B_2(\delta) &= \lim_{\delta \to 0^+} -\frac{1}{2\delta} \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}} \rangle_{H_I^*, H_I} \\
&= \frac{1}{2} \int_{I_+} \mathcal{G}_\gamma(t) \nabla \lambda_\gamma(\gamma(t))^T \bar{\gamma}(t)\, dt + \frac{1}{2} \int_{I_-} \mathcal{G}_\gamma(t) \nabla \lambda_\gamma^c(\gamma(t))^T \bar{\gamma}(t)\, dt \\
&\qquad =: \overline{B}_{2,1} + \overline{B}_{2,2}.
\end{aligned}$$

Let $\delta > 0$. We have $\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}} = \lambda_{\gamma+\delta\bar{\gamma}}$ on $\Omega_{\gamma+\delta\bar{\gamma}}$ and $\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}} = \lambda^c_{\gamma+\delta\bar{\gamma}}$ on $\Omega^c_{\gamma+\delta\bar{\gamma}}$ and obtain

$$\begin{aligned}
B_2(\delta) &= -\frac{1}{2\delta} \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}} \rangle_{H_I^*, H_I} = -\frac{1}{2\delta} \int_I \mathcal{G}_\gamma(t) \hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}}(\gamma(t))\, dt \\
&= -\frac{1}{2\delta} \int_{I_+} \mathcal{G}_\gamma(t) \lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t))\, dt - \frac{1}{2\delta} \int_{I_-} \mathcal{G}_\gamma(t) \lambda^c_{\gamma+\delta\bar{\gamma}}(\gamma(t))\, dt =: B_{2,1}(\delta) + B_{2,2}(\delta).
\end{aligned}$$

Now we prove

$$\lim_{\delta \to 0^+} (\overline{B}_{2,i} - B_{2,i}(\delta)) = 0, \quad i = 1, 2. \tag{3.4.4}$$

In the case $i = 1$ we obtain with $\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \delta\bar{\gamma}(t)) = 0$ a.e. in I_+ and the differentiability of $G(\xi) := \lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \xi\bar{\gamma}(t))$ for $t \in I_+$ in $(0, \delta)$ the expression

$$\begin{aligned}
B_{2,1}(\delta) &= -\frac{1}{2\delta} \int_{I_+} \mathcal{G}_\gamma(t) \lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t))\, dt \\
&= \frac{1}{2\delta} \int_{I_+} \mathcal{G}_\gamma(t) \left(\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \delta\bar{\gamma}(t)) - \lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t)) \right)\, dt \\
&= \frac{1}{2\delta} \int_{I_+} \mathcal{G}_\gamma(t) \left(\int_0^\delta \nabla \lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \xi\bar{\gamma}(t))^T \bar{\gamma}(t)\, d\xi \right)\, dt.
\end{aligned}$$

Now, the identity

$$\frac{1}{2} \int_{I_+} \mathcal{G}_\gamma(t) \nabla \lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t))^T \bar{\gamma}(t)\, dt = \frac{1}{2\delta} \int_{I_+} \mathcal{G}_\gamma(t) \left(\int_0^\delta \nabla \lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t))^T \bar{\gamma}(t)\, d\xi \right)\, dt,$$

which follows from the equation $1 = \frac{1}{\delta}\int_0^\delta d\xi$, leads to

$$
\begin{aligned}
\overline{B}_{2,1} - B_{2,1}(\delta) &= \frac{1}{2}\int_{I_+} \mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt \\
&\quad - \frac{1}{2\delta}\int_{I_+} \mathcal{G}_\gamma(t)\left(\int_0^\delta \nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t)+\xi\bar{\gamma}(t))^T\bar{\gamma}(t)\,d\xi\right)dt \\
&= -\frac{1}{2}\int_{I_+} \mathcal{G}_\gamma(t)\left(\nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t)) - \nabla\lambda_\gamma(\gamma(t))\right)^T\bar{\gamma}(t)\,dt \\
&\quad - \frac{1}{2\delta}\int_{I_+} \mathcal{G}_\gamma(t)\bigg(\int_0^\delta \big(\nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t)+\xi\bar{\gamma}(t)) \\
&\qquad\qquad\qquad - \nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t))\big)^T\bar{\gamma}(t)\,d\xi\bigg)\,dt \\
&=: B_{2,1,1}(\delta) + B_{2,1,2}(\delta).
\end{aligned}
$$

We show that both terms, $B_{2,1,1}(\delta)$ and $B_{2,1,2}(\delta)$, tend to zero for $\delta \to 0^+$. The first one can be estimated as

$$
\begin{aligned}
|B_{2,1,1}(\delta)| &= \frac{1}{2}\left|\int_{I_+} \mathcal{G}_\gamma(t)\left(\nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t)) - \nabla\lambda_\gamma(\gamma(t))\right)^T\bar{\gamma}(t)\,dt\right| \\
&\le \frac{1}{2}\|\bar{g}\|_{L^\infty(I_+)}\int_{I_+} |\mathcal{G}_\gamma(t)|\,\|\nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t)) - \nabla\lambda_\gamma(\gamma(t))\|_2\,dt \\
&\le \frac{1}{2}\|\bar{g}\|_{L^\infty(I)}\,\|\mathcal{G}_\gamma\|_{L^2(I_+)} \\
&\qquad \left(\int_{I_+} \|\nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t)) - \nabla\lambda_\gamma(\gamma(t))\|_2^2\,dt\right)^{1/2} \\
&\le \frac{1}{2}\|\bar{g}\|_{L^\infty(I)}\,\|\mathcal{G}_\gamma\|_{L^2(I)}\,\|\nabla\lambda_{\gamma+\delta\bar{\gamma}} - \nabla\lambda_\gamma\|_{L^2(\Gamma_\gamma^+)^2}
\end{aligned}
$$

with $\Gamma_\gamma^+ := \{(\gamma_1(t),\gamma_2(t)) : t \in I_+\}$. Because the first two terms are bounded we just have to show that the last one tends to zero. The family $\{\|\lambda_{\gamma+\delta\bar{\gamma}}\|_{H^2(\Omega_{\gamma+\delta\bar{\gamma}})}\}$ and thus also $\{\|\nabla\lambda_{\gamma+\delta\bar{\gamma}}\|_{H^1(\Omega_{\gamma+\delta\bar{\gamma}})^2}\}$ is bounded for $\delta \in [0,\delta_0]$ with some δ_0 sufficiently small. We consider the set $\Omega'_\gamma := \Omega_\gamma \cap \Omega_{\gamma+\delta_0\bar{\gamma}}$ which is clearly contained in all $\Omega_{\gamma+\delta\bar{\gamma}}$, $\delta \in [0,\delta_0]$. Thus the family $\{\nabla\lambda_{\gamma+\delta\bar{\gamma}}, \delta \in [0,\delta_0]\}$ is bounded in $H^1(\Omega'_\gamma)$. This implies that for any sequence $\{\delta_i\} \subset [0,\delta_0]$ with $\lim\delta_i = 0$ there exists a sub-sequence $\{\delta_{i'}\}$ such that we have weak convergence

$$\nabla\lambda_{\gamma+\delta_{i'}\bar{\gamma}} \rightharpoonup w \text{ in } H^1(\Omega'_\gamma)^2.$$

Due to the compact embedding $H^1(\Omega)^k \hookrightarrow\hookrightarrow L^2(\Omega)^k$ we obtain strong convergence

$$\nabla\lambda_{\gamma+\delta_{i'}\bar{\gamma}} \to w \text{ in } L^2(\Omega'_\gamma)^2.$$

Because we already know that $\hat{\lambda}^{\Sigma}_{\gamma+\delta\bar{\gamma}} \to \hat{\lambda}^{\Sigma}_{\gamma}$ in $H^1(\hat{\Omega})$ (see Theorem 3.3.4) which implies $\nabla\hat{\lambda}^{\Sigma}_{\gamma+\delta\bar{\gamma}} \to \nabla\hat{\lambda}^{\Sigma}_{\gamma}$ in $L^2(\hat{\Omega})^2$ for $\delta \to 0^+$ we obtain $w = \nabla\hat{\lambda}^{\Sigma}_{\gamma}|_{\Omega'_\gamma} = \nabla\lambda_\gamma|_{\Omega'_\gamma}$. Thus the weak convergence of $\{\nabla\lambda_{\gamma+\delta\bar{\gamma}}\}$ in $H^1(\Omega'_\gamma)$ is valid for the whole sequence and not only for a sub-sequence. We consider the traces of the functions of any sequence on the set $\Gamma^+_\gamma = \Gamma_\gamma \cap \partial\Omega'_\gamma = \{(\gamma_1(t), \gamma_2(t)) : t \in I_+\}$. The weak convergence

$$\nabla\lambda_{\gamma+\delta\bar{\gamma}} \rightharpoonup \nabla\lambda_\gamma \text{ in } H^1(\Omega'_\gamma)^2$$

for $\delta \to 0^+$ implies strong convergence

$$\nabla\lambda_{\gamma+\delta\bar{\gamma}}|_{\Gamma^+_\gamma} \to \nabla\lambda_\gamma|_{\Gamma^+_\gamma} \text{ in } L^2(\Gamma^+_\gamma)^2,$$

which implies that $B_{2,1,1}(\delta)$ tends to zero for $\delta \to 0^+$.
For the second term $B_{2,1,2}(\delta)$ we use analogously to equation (3.4.2) the identity

$$\nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \xi\bar{\gamma}(t)) - \nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t)) = \int_0^\xi \bar{\gamma}(t)^T H\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \eta\bar{\gamma}(t))\bar{\gamma}(t)\, d\eta,$$

where $H\lambda_{\gamma+\delta\bar{\gamma}} = \nabla^2\lambda_{\gamma+\delta\bar{\gamma}}$ is the Hessian matrix of $\lambda_{\gamma+\delta\bar{\gamma}}$. We obtain

$$\begin{aligned} B_{2,1,2}(\delta) &= -\frac{1}{2\delta}\int_{I_+} \mathcal{G}_\gamma(t)\left(\int_0^\delta \left(\nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \xi\bar{\gamma}(t)) - \nabla\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t))\right)^T \bar{\gamma}(t)\, d\xi\right) dt \\ &= -\frac{1}{2\delta}\int_{I_+} \mathcal{G}_\gamma(t)\left(\int_0^\delta \int_0^\xi \bar{\gamma}(t)^T H\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \eta\bar{\gamma}(t))\bar{\gamma}(t)\, d\eta d\xi\right) dt. \end{aligned}$$

Using for the inner two integrals the equality (3.4.3) we obtain

$$B_{2,1,2}(\delta) = -\frac{1}{2\delta}\int_{I_+} \mathcal{G}_\gamma(t)\left(\int_0^\delta \bar{\gamma}(t)^T H\lambda_{\gamma+\delta\bar{\gamma}}(\gamma(t) + \eta\bar{\gamma}(t))\bar{\gamma}(t)\,(\delta - \eta)\, d\eta\right) dt.$$

The inner integral of the previous equation is estimated as follows[4]:

$$
\begin{aligned}
&\left| \int_0^\delta \bar\gamma(t)^T H\lambda_{\gamma+\delta\bar\gamma}(\gamma(t)+\eta\bar\gamma(t))\bar\gamma(t)\,(\delta-\eta)\,d\eta \right| \\
&\le \int_0^\delta \|\bar\gamma(t)\|_2 \|H\lambda_{\gamma+\delta\bar\gamma}(\gamma(t)+\eta\bar\gamma(t))\|_F \|\bar\gamma(t)\|_2\,|\delta-\eta|\,d\eta \\
&\le \int_0^\delta \|H\lambda_{\gamma+\delta\bar\gamma}(\gamma(t)+\eta\bar\gamma(t))\|_F |\bar g(t)|^{1/2}\,|\delta-\eta|\,|\bar g(t)|^{3/2} d\eta \\
&\le \left(\int_0^\delta \|H\lambda_{\gamma+\delta\bar\gamma}(\gamma(t)+\eta\bar\gamma(t))\|_F^2 |\bar g(t)|\,d\eta \right)^{1/2} \left(\int_0^\delta |\delta-\eta|^2\,|\bar g(t)|^3\,d\eta \right)^{1/2} \\
&\le C_1 \left(\int_0^\delta \|H\lambda_{\gamma+\delta\bar\gamma}(\gamma(t)+\eta\bar\gamma(t))\|_F^2 |\det\Phi_+'(t,\eta)|\,d\eta \right)^{1/2} (\delta|\bar g(t)|)^{3/2}.
\end{aligned}
$$

The last estimation is done using the inequality $|\bar g(t)| \le |\det \Phi_+'(t,\eta)|$ from Lemma 2.4.5. We thus obtain

$$
\begin{aligned}
|B_{2,1,2}(\delta)| &\le \frac{1}{2\delta} \int_{I_+} |\mathcal{G}_\gamma(t)| \left| \int_0^\delta \bar\gamma(t)^T H\lambda_{\gamma+\delta\bar\gamma}(\gamma(t)+\eta\bar\gamma(t))\bar\gamma(t)\,(\delta-\eta)\,d\eta \right| \,dt \\
&\le C_2 \delta^{1/2} \int_{I_+} |\mathcal{G}_\gamma(t)|\,|\bar g(t)|^{3/2} \\
&\qquad\qquad \left(\int_0^\delta \|H\lambda_{\gamma+\delta\bar\gamma}(\gamma(t)+\eta\bar\gamma(t))\|_F^2 |\det\Phi_+'(t,\eta)|\,d\eta \right)^{1/2} dt,
\end{aligned}
$$

and furthermore, using Hölder's inequality, we have

$$
\begin{aligned}
|B_{2,1,2}(\delta)| &\le C_2 \delta^{1/2} \|\bar g\|_{L^\infty(I_+)}^{3/2} \left(\int_{I_+} |\mathcal{G}_\gamma(t)|^2 \,dt \right)^{1/2} \\
&\qquad \left(\int_{I_+} \int_0^\delta \|H\lambda_{\gamma+\delta\bar\gamma}(\gamma(t)+\eta\bar\gamma(t))\|_F^2\,|\det\Phi_+'(t,\eta)|\,d\eta\,dt \right)^{1/2} \\
&\le C_2 \delta^{1/2} \|\bar g\|_{L^\infty(I_+)}^{3/2} \left\|\mathcal{G}_\gamma\right\|_{L^2(I_+)} \left\|\lambda_{\gamma+\delta\bar\gamma}\right\|_{H^2(\Phi_+(I_+\times(0,\delta)))} \\
&\le C_2 \delta^{1/2} \|\bar g\|_{L^\infty(I)}^{3/2} \left\|\mathcal{G}_\gamma\right\|_{L^2(I)} \left\|\lambda_{\gamma+\delta\bar\gamma}\right\|_{H^2(\Omega_{\gamma+\delta\bar\gamma})}.
\end{aligned}
$$

---

[4] Here, we use the Frobenius norm

$$\|A\|_F = \sqrt{\sum_{i=1}^2 \sum_{j=1}^2 |a_{i,j}|^2}$$

for (2×2)-matrices and the inequality

$$\|Ax\|_2 \le \|A\|_F \|x\|_2.$$

The boundedness of $\{\|\lambda_\gamma\|_{H^2(\Omega_\gamma)}\}_{\gamma\in S_{\text{ad}}}$ (cf. Theorem 3.3.1) and $\{\|\mathcal{G}_\gamma\|_{L^2(I)}\}_{\gamma\in S_{\text{ad}}}$ (cf. Theorem 2.4.2) now imply $B_{2,1,2}(\delta) \to 0$ for $\delta \to 0^+$.
To prove equation (3.4.4) for $i = 2$, we use a similar argumentation as above. We just replace I_+ by I_-, $\lambda_{\gamma+\delta\bar{\gamma}}$ by $\lambda^c_{\gamma+\delta\bar{\gamma}}$, Γ^+_γ by $\Gamma^-_\gamma := \{(\gamma_1(t), \gamma_2(t)) : t \in I_-\}$ and Φ_+ by Φ_-.
If we switch the role of $\mathcal{G}_\gamma$ and $\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}}$ with $\mathcal{M}^\Sigma_\gamma$ and $\hat{u}_{\gamma+\delta\bar{\gamma}}$, respectively, and use the fact that $\hat{u}_{\gamma+\delta\bar{\gamma}} = u^c_{\gamma+\delta\bar{\gamma}} = 0$ on $\Omega^c_{\gamma+\delta\bar{\gamma}}$, we analogously obtain

$$\lim_{\delta\to 0^+} B_6(\delta) = \lim_{\delta\to 0^+} -\frac{1}{2\delta}\langle \mathcal{M}^\Sigma_\gamma, \mathcal{T}_\gamma \hat{u}_{\gamma+\delta\bar{\gamma}}\rangle_{H_I^*,H_I} = \frac{1}{2}\int_{I_+} \mathcal{M}^\Sigma_\gamma(t)\nabla u_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt. \tag{3.4.5}$$

Part 4: In this step we prove that

$$\begin{aligned}
\lim_{\delta\to 0^+} B_5(\delta) &= \lim_{\delta\to 0^+} \frac{1}{2\delta}\langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{\lambda}^\Sigma_\gamma\rangle_{H_I^*,H_I} \\
&= \frac{1}{2}\int_{I_+} \mathcal{G}_\gamma(t)\nabla\lambda^c_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt + \frac{1}{2}\int_{I_-} \mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt \\
&=: \overline{B}_{5,1} + \overline{B}_{5,2}.
\end{aligned}$$

Again, let $\delta > 0$. We use $\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}} = \lambda_{\gamma+\delta\bar{\gamma}}$ on $\Omega_{\gamma+\delta\bar{\gamma}}$ and $\hat{\lambda}^\Sigma_{\gamma+\delta\bar{\gamma}} = \lambda^c_{\gamma+\delta\bar{\gamma}}$ on $\Omega^c_{\gamma+\delta\bar{\gamma}}$ and obtain

$$\begin{aligned}
B_5(\delta) &= \frac{1}{2\delta}\langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{\lambda}^\Sigma_\gamma\rangle_{H_I^*,H_I} = \frac{1}{2\delta}\int_I \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\hat{\lambda}^\Sigma_\gamma(\gamma(t)+\delta\bar{\gamma}(t))\,dt \\
&= \frac{1}{2\delta}\int_{I_+} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\lambda^c_\gamma(\gamma(t)+\delta\bar{\gamma}(t))\,dt + \frac{1}{2\delta}\int_{I_-} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\lambda_\gamma(\gamma(t)+\delta\bar{\gamma}(t))\,dt \\
&=: B_{5,1}(\delta) + B_{5,2}(\delta).
\end{aligned}$$

We proceed in an analogous way as in the last part and show that

$$\lim_{\delta\to 0^+}(\overline{B}_{5,i} - B_{5,i}(\delta)) = 0, \quad i = 1, 2. \tag{3.4.6}$$

To show (3.4.6) for $i = 2$, we use $\lambda_\gamma(\gamma(t)) = 0$ a.e. in I_- and the differentiability of $G(\xi) := \lambda_\gamma(\gamma(t) + \xi\bar{\gamma}(t))$ for $t \in I_-$ in $(0, \delta)$, and obtain

$$\begin{aligned}
B_{5,2}(\delta) &= \frac{1}{2\delta}\int_{I_-} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\lambda_\gamma(\gamma(t)+\delta\bar{\gamma}(t))\,dt \\
&= \frac{1}{2\delta}\int_{I_-} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\left(\lambda_\gamma(\gamma(t)+\delta\bar{\gamma}(t)) - \lambda_\gamma(\gamma(t))\right)\,dt \\
&= \frac{1}{2\delta}\int_{I_-} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\left(\int_0^\delta \nabla\lambda_\gamma(\gamma(t)+\xi\bar{\gamma}(t))^T\bar{\gamma}(t)\,d\xi\right)\,dt.
\end{aligned}$$

Using the equation $1 = \frac{1}{\delta}\int_0^\delta d\xi$, we obtain for the difference

$$\begin{aligned}\overline{B}_{5,2} - B_{5,2}(\delta) = &\frac{1}{2}\int_{I_-} \mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt\\ &- \frac{1}{2}\int_{I_-} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt\\ &+ \frac{1}{2\delta}\int_{I_-} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\left(\int_0^\delta \nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\,d\xi\right)dt\\ &- \frac{1}{2\delta}\int_{I_-} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\left(\int_0^\delta \nabla\lambda_\gamma(\gamma(t)+\xi\bar{\gamma}(t))^T\bar{\gamma}(t)\,d\xi\right)dt,\end{aligned}$$

where the difference of the second and the third term equals to zero. We then obtain

$$\begin{aligned}\overline{B}_{5,2} - B_{5,2}(\delta) = &-\frac{1}{2}\int_{I_-}\Big(\mathcal{G}_{\gamma+\delta\bar{\gamma}}(t) - \mathcal{G}_\gamma(t)\Big)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\,dt\\ &- \frac{1}{2\delta}\int_{I_-} \mathcal{G}_{\gamma+\delta\bar{\gamma}}(t)\left(\int_0^\delta \left(\nabla\lambda_\gamma(\gamma(t)+\xi\bar{\gamma}(t)) - \nabla\lambda_\gamma(\gamma(t))\right)^T\bar{\gamma}(t)\,d\xi\right)dt\\ &=: B_{5,2,1}(\delta) + B_{5,2,2}(\delta).\end{aligned}$$

We again show that both terms, $B_{5,2,1}(\delta)$ and $B_{5,2,2}(\delta)$, tend to zero for $\delta \to 0^+$. We may express $B_{5,2,1}(\delta)$ as

$$\begin{aligned}B_{5,2,1}(\delta) &= (\mathcal{G}_{\gamma+\delta\bar{\gamma}}, h(\cdot))_{L^2(I)} - (\mathcal{G}_\gamma, h(\cdot))_{L^2(I)}\\ &= (\mathcal{G}_{\gamma+\delta\bar{\gamma}} - \mathcal{G}_\gamma, h(\cdot))_{L^2(I)},\end{aligned}$$

where

$$h(t) := \begin{cases} -\frac{1}{2}\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t), & t \in I_-,\\ 0, & t \in I_+,\end{cases}$$

is a $L^2(I)$-function.
Since H_I is dense in $L^2(I)$ by Lemma 2.1.8 there exists a sequence $\{\mathcal{H}_k\}_k$ in H_I such that

$$\lim_{k\to\infty} \mathcal{H}_k = h \quad \text{in } L^2(I).$$

For fixed k we may hence estimate

$$\begin{aligned}|B_{5,2,1}(\delta)| &\le |(\mathcal{G}_{\gamma+\delta\bar{\gamma}} - \mathcal{G}_\gamma, h)_{L^2(I)}|\\ &\le |(\mathcal{G}_{\gamma+\delta\bar{\gamma}}, (h-\mathcal{H}_k))_{L^2(I)}| + |(\mathcal{G}_\gamma, (\mathcal{H}_k - h))_{L^2(I)}| + |\langle \mathcal{G}_{\gamma+\delta\bar{\gamma}} - \mathcal{G}_\gamma, \mathcal{H}_k\rangle_{H_I^*,H_I}|.\end{aligned}$$

The first two terms on the right-hand side tend to zero since $\mathcal{H}_k \to h$ in $L^2(I)$ and the family $\{\mathcal{G}_\gamma\}_{\gamma\in S_{\mathrm{ad}}}$ is uniformly bounded in $L^2(I)$. The third term tends to zero since the Lagrange multipliers are weak-* convergent in H_I^*.
For the term $B_{5,2,2}(\delta)$ we use a similar argumentation as for $B_{2,1,2}(\delta)$. We just replace

$\lambda_{\gamma+\delta\bar{\gamma}}$ by λ_γ, I_+ by I_-, $\mathcal{G}_\gamma$ by $\mathcal{G}_{\gamma+\delta\bar{\gamma}}$ and Φ_+ by Φ_- and obtain

$$\begin{aligned}|B_{5,2,2}(\delta)| &\leq C\delta^{1/2}\|\bar{g}\|^{3/2}_{L^\infty(I_-)}\, \|\mathcal{G}_{\gamma+\delta\bar{\gamma}}\|_{L^2(I_-)}\, \|\lambda_\gamma\|_{H^2(\Phi_-(I_-\times(0,\delta)))} \\ &\leq C\delta^{1/2}\|\bar{g}\|^{3/2}_{L^\infty(I)}\, \|\mathcal{G}_{\gamma+\delta\bar{\gamma}}\|_{L^2(I)}\, \|\lambda_\gamma\|_{H^2(\Omega_\gamma)} \longrightarrow 0 \quad \text{for } \delta \to 0^+.\end{aligned}$$

In the same manner we prove equation (3.4.6) for $i = 1$. Eventually, if we switch the role of $\mathcal{G}_{\gamma+\delta\bar{\gamma}}$ and $\hat{\lambda}^\Sigma_\gamma$ with $\mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}}$ and $\hat{u}_\gamma$, respectively, and use the fact that $\hat{u}_\gamma = u^c_\gamma = 0$ on Ω^c_γ, we obtain

$$\lim_{\delta\to 0^+} B_3(\delta) = \lim_{\delta\to 0^+} \frac{1}{2\delta}\langle \mathcal{M}^\Sigma_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}_\gamma\rangle_{H^*_I,H_I} = \frac{1}{2}\int_{I_-} \mathcal{M}^\Sigma_\gamma(t)\nabla u_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt.$$

Part 5: Composing the results of parts 1–4, we obtain

$$\begin{aligned}d^+\hat{j}(\gamma,\bar{\gamma}) &= \lim_{\delta\to 0^+}\frac{1}{2\delta}\Big\{(\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}_{\gamma+\delta\bar{\gamma}})_{L^2(\hat{\Omega})} - \langle \mathcal{G}_\gamma, \mathcal{T}_\gamma\hat{\lambda}_{\gamma+\delta\bar{\gamma}}\rangle_{H^*_I,H_I} \\ &\qquad + \langle \mathcal{M}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{u}_\gamma\rangle_{H^*_I,H_I} + (\widetilde{f}_{\gamma+\delta\bar{\gamma}} - \widetilde{f}_\gamma, \hat{\lambda}_\gamma)_{L^2(\hat{\Omega})} \\ &\qquad + \langle \mathcal{G}_{\gamma+\delta\bar{\gamma}}, \mathcal{T}_{\gamma+\delta\bar{\gamma}}\hat{\lambda}_\gamma\rangle_{H^*_I,H_I} - \langle \mathcal{M}_\gamma, \mathcal{T}_\gamma\hat{u}_{\gamma+\delta\bar{\gamma}}\rangle_{H^*_I,H_I}\Big\} \\ &= \frac{1}{2}\int_{I_+}\mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt + \frac{1}{2}\int_{I_-}\mathcal{G}_\gamma(t)\nabla\lambda^c_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt \\ &\quad + \frac{1}{2}\int_{I_-}\mathcal{M}^\Sigma_\gamma(t)\nabla u_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt + \frac{1}{2}\int_{I_+}\mathcal{G}_\gamma(t)\nabla\lambda^c_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt \\ &\quad + \frac{1}{2}\int_{I_-}\mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt + \frac{1}{2}\int_{I_+}\mathcal{M}^\Sigma_\gamma(t)\nabla u_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt \\ &= \frac{1}{2}\int_I \mathcal{G}_\gamma(t)\left(\nabla\lambda_\gamma(\gamma(t)) + \nabla\lambda^c_\gamma(\gamma(t))\right)^T\bar{\gamma}(t)\, dt \\ &\quad + \frac{1}{2}\int_I \mathcal{M}^\Sigma_\gamma(t)\nabla u_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt.\end{aligned}$$

We have

$$
\begin{aligned}
\nabla\lambda_\gamma(\gamma(\cdot))^T\bar{\gamma}(\cdot) &= \begin{cases} d^-\lambda_\gamma(\gamma(\cdot),\bar{\gamma}(\cdot)) & \text{on } I_+ \\ d^+\lambda_\gamma(\gamma(\cdot),\bar{\gamma}(\cdot)) & \text{on } I_-. \end{cases} \\
&= \begin{cases} -d^+\lambda_\gamma(\gamma(\cdot),-\bar{\gamma}(\cdot)) & \text{on } I_+ \\ d^+\lambda_\gamma(\gamma(\cdot),\bar{\gamma}(\cdot)) & \text{on } I_-. \end{cases} \\
&= \begin{cases} -\underbrace{(\bar{g}(\cdot))}_{>0} d^+\lambda_\gamma(\gamma(\cdot),-n_\gamma(\cdot)) & \text{on } I_+ \\ \underbrace{(-\bar{g}(\cdot))}_{>0} d^+\lambda_\gamma(\gamma(\cdot),-n_\gamma(\cdot)) & \text{on } I_-. \end{cases} \\
&= -\bar{g}(\cdot)d^+\lambda_\gamma(\gamma(\cdot),-n_\gamma(\cdot)) \quad \text{on } I \\
&= \bar{g}(\cdot)d^-\lambda_\gamma(\gamma(\cdot),n_\gamma(\cdot)) \\
&= \bar{g}(\cdot)d^-\lambda_\gamma(\gamma(\cdot),n_\gamma(\cdot)) \underbrace{\|\dot{\gamma}(\cdot)\|_2 \frac{n_\gamma(\cdot)^T n_\gamma(\cdot)}{\|\dot{\gamma}(\cdot)\|_2}}_{=1} \\
&= d^-\lambda_\gamma(\gamma(\cdot),n_\gamma(\cdot))\|\dot{\gamma}(\cdot)\|_2 \frac{\bar{g}(\cdot)n_\gamma(\cdot)^T n_\gamma(\cdot)}{\|\dot{\gamma}(\cdot)\|_2} \\
&= \frac{\partial\lambda_\gamma(\gamma(\cdot))}{\partial n_\gamma}\|\dot{\gamma}(\cdot)\|_2 \frac{\bar{\gamma}(\cdot)^T n_\gamma(\cdot)}{\|\dot{\gamma}(\cdot)\|_2} \\
&= \mathcal{M}_\gamma(\cdot)\frac{\bar{\gamma}(\cdot)^T n_\gamma(\cdot)}{\|\dot{\gamma}(\cdot)\|_2},
\end{aligned}
$$

$$
\begin{aligned}
\nabla\lambda^c_\gamma(\gamma(\cdot))^T\bar{\gamma}(\cdot) &= \begin{cases} d^+\lambda^c_\gamma(\gamma(\cdot),\bar{\gamma}(\cdot)) & \text{on } I_+ \\ d^-\lambda^c_\gamma(\gamma(\cdot),\bar{\gamma}(\cdot)) & \text{on } I_-. \end{cases} \\
&= \bar{g}(\cdot)d^+\lambda^c_\gamma(\gamma(\cdot),n_\gamma(\cdot)) \quad \text{on } I \\
&= -d^-\lambda^c_\gamma(\gamma(\cdot),n^c_\gamma(\cdot))\|\dot{\gamma}(\cdot)\|_2 \frac{\bar{g}(\cdot)n_\gamma(\cdot)^T n_\gamma(\cdot)}{\|\dot{\gamma}(\cdot)\|_2} \\
&= -\mathcal{M}^c_\gamma(\cdot)\frac{\bar{\gamma}(\cdot)^T n_\gamma(\cdot)}{\|\dot{\gamma}(\cdot)\|_2},
\end{aligned}
$$

and analogously

$$\nabla u_\gamma(\gamma(\cdot))^T\bar{\gamma}(\cdot) = \mathcal{G}_\gamma(\cdot)\frac{\bar{\gamma}(\cdot)^T n_\gamma(\cdot)}{\|\dot{\gamma}(\cdot)\|_2}.$$

Therefore, using the expressions from above and the identity $\mathcal{M}^\Sigma = \mathcal{M} + \mathcal{M}^c$, we obtain

$$\begin{aligned}
d^+\hat{\jmath}(\gamma,\bar\gamma) &= \frac{1}{2}\int_I \mathcal{G}_\gamma(t)\left(\nabla\lambda_\gamma(\gamma(t)) + \nabla\lambda_\gamma^c(\gamma(t))\right)^T \bar\gamma(t)\,dt \\
&\quad + \frac{1}{2}\int_I \mathcal{M}_\gamma^\Sigma(t)\nabla u_\gamma(\gamma(t))^T\bar\gamma(t)\,dt \\
&= \frac{1}{2}\int_I \mathcal{G}_\gamma(t)\left(\mathcal{M}_\gamma(t) - \mathcal{M}_\gamma^c(t)\right)\frac{\bar\gamma(t)^T n_\gamma(t)}{\|\dot\gamma(t)\|_2}\,dt \\
&\quad + \frac{1}{2}\int_I \left(\mathcal{M}_\gamma(t) + \mathcal{M}_\gamma^c(t)\right)\mathcal{G}_\gamma(t)\frac{\bar\gamma(t)^T n_\gamma(t)}{\|\dot\gamma(t)\|_2}\,dt \\
&= \int_I \mathcal{G}_\gamma(t)\mathcal{M}_\gamma(t)\frac{\bar\gamma(t)^T n_\gamma(t)}{\|\dot\gamma(t)\|_2}\,dt.
\end{aligned}$$

Part 6: Eventually, we show that the left- and right-sided directional derivatives of $\hat\jmath$ coincide, i.e. $d^-\hat\jmath(\gamma,\bar\gamma) = d^+\hat\jmath(\gamma,\bar\gamma)$. This statement is not obvious, since the right-sided limits in parts 3 and 4 exist, but they are in general different from the left-sided limits. For example, in part 3 we found

$$\lim_{\delta\to 0^+} B_2(\delta) = \lim_{\delta\to 0^+} -\frac{1}{2\delta}\langle \mathcal{G}_\gamma, \mathcal{T}_\gamma\hat\lambda^\Sigma_{\gamma+\delta\bar\gamma}\rangle_{H_I^*,H_I} = \frac{1}{2}\int_{I_+} \mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar\gamma(t)\,dt,$$

but we get with the same techniques

$$\lim_{\delta\to 0^-} B_2(\delta) = \lim_{\delta\to 0^-} -\frac{1}{2\delta}\langle \mathcal{G}_\gamma, \mathcal{T}_\gamma\hat\lambda^\Sigma_{\gamma+\delta\bar\gamma}\rangle_{H_I^*,H_I} = \frac{1}{2}\int_{I_-} \mathcal{G}_\gamma(t)\nabla\lambda_\gamma(\gamma(t))^T\bar\gamma(t)\,dt.$$

However, it turns out that

$$\begin{aligned}
\lim_{\delta\to 0^-} B_2(\delta) &= \lim_{\delta\to 0^+} B_5(\delta), &\qquad \lim_{\delta\to 0^-} B_5(\delta) &= \lim_{\delta\to 0^+} B_2(\delta)\\
\lim_{\delta\to 0^-} B_6(\delta) &= \lim_{\delta\to 0^+} B_3(\delta), &\qquad \lim_{\delta\to 0^-} B_3(\delta) &= \lim_{\delta\to 0^+} B_6(\delta)
\end{aligned}$$

as well as $\lim_{\delta\to 0^-} B_1(\delta) = \lim_{\delta\to 0^-} B_4(\delta) = 0$. Therefore we obtain as in part 5

$$d^-\hat\jmath(\gamma,\bar\gamma) = \int_I \mathcal{G}_\gamma(t)\mathcal{M}_\gamma(t)\frac{\bar\gamma(t)^T n_\gamma(t)}{\|\dot\gamma(t)\|_2}\,dt,$$

i.e. $d^-\hat\jmath(\gamma,\bar\gamma) = d^+\hat\jmath(\gamma,\bar\gamma) = d\hat\jmath(\gamma,\bar\gamma)$. □

Theorem 3.4.2. *Under Assumption 2.4.9 and Assumption 3.3.5 the reduced objective functional $\hat\jmath$ is Fréchet-differentiable with respect to $\gamma \in S_{ad}^+$. The derivative $\hat\jmath'(\gamma)$ satisfies*

$$\langle \hat\jmath'(\gamma), \bar\gamma\rangle_{S^*,S} = \int_I \mathcal{M}_\gamma(t)\mathcal{G}_\gamma(t)\frac{\bar\gamma(t)^T n_\gamma(t)}{\|\dot\gamma(t)\|_2}\,dt \quad \textit{for all } \bar\gamma \in S'_\gamma. \tag{3.4.7}$$

Proof. The mapping

$$\begin{aligned} \hat{\jmath}'(\gamma) : S'_\gamma &\to \mathbb{R}, \\ \bar{\gamma} \mapsto d\hat{\jmath}(\gamma, \bar{\gamma}) &= \int_I \mathcal{M}_\gamma(t)\mathcal{G}_\gamma(t) \frac{\bar{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\, dt \end{aligned}$$

is clearly linear on S'_γ. We now show that it is also bounded on S'_γ:

$$\begin{aligned} |d\hat{\jmath}(\gamma, \bar{\gamma})| &= \left| \int_I \mathcal{G}_\gamma(t)\mathcal{M}_\gamma(t) \frac{\bar{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\, dt \right| \\ &\le \frac{\|\bar{\gamma}\|_{L^\infty(I)^2}}{c_r} \int_I |\mathcal{G}_\gamma(t)\mathcal{M}_\gamma(t)|\, dt \\ &\le \frac{\|\bar{\gamma}\|_{L^\infty(I)^2}}{c_r} \|\mathcal{G}_\gamma\|_{L^2(I)} \|\mathcal{M}_\gamma\|_{L^2(I)} \le C\|\bar{\gamma}\|_{L^\infty(I)^2} \end{aligned}$$

and C is independent of $\bar{\gamma}$. Therefore we have shown that $\hat{\jmath}'(\gamma)$ is a bounded linear functional on S'_γ which implies that it is the Gateaux-derivative of $\gamma \mapsto \hat{\jmath}(\gamma)$ at $\gamma \in S^+_{\text{ad}}$.
To prove that $\hat{\jmath}'(\gamma)$ is the Fréchet-derivative of $\gamma \mapsto \hat{\jmath}(\gamma)$ we have to show that the mapping

$$\gamma \mapsto \hat{\jmath}'(\gamma) \tag{3.4.8}$$

is a continuous operator from S^+_{ad} to $\mathcal{L}(S'_\gamma, \mathbb{R})$.
The family $\{\mathcal{G}_\gamma\}_{\gamma \in S_{\text{ad}}}$ is uniformly bounded in $L^2(I)$ (cf. Theorem 2.4.2). Hence for $\gamma_n \to \gamma$ in S_{ad} there exists a weakly convergent subsequence, i.e.

$$(\mathcal{G}_{\gamma_n}, h)_{L^2(I)} \to (g, h)_{L^2(I)} \quad \text{for all } h \in L^2(I)$$

for $n \to \infty$ with some $g \in L^2(I)$. On the other hand with Assumption 2.4.9 $\mathcal{G}_{\gamma_n}$ converges weak-* in H_I^* to $\mathcal{G}_\gamma$ which means

$$\langle \mathcal{G}_{\gamma_n}, h\rangle_{H_I^*, H_I} = (\mathcal{G}_{\gamma_n}, h)_{L^2(I)} \to \langle \mathcal{G}_\gamma, h\rangle_{H_I^*, H_I} = (\mathcal{G}_\gamma, h)_{L^2(I)} \quad \text{for all } h \in H_I$$

for $n \to \infty$. Because H_I is dense in $L^2(I)$ we have $g = \mathcal{G}_\gamma$. This implies that $\mathcal{G}_{\gamma_n}$ converges to $\mathcal{G}_\gamma$ in $L^2(I)$ and this shows that the mapping $\gamma \mapsto \mathcal{G}_\gamma$ is continuous. Using Assumption 3.3.5 the same arguments hold for the mapping $\gamma \mapsto \mathcal{M}_\gamma$. Furthermore, since Γ_γ is of class $C^{1,1}$ uniformly in γ, we have continuity of the mapping $\gamma \mapsto n_\gamma$. Altogether, we obtain continuity of $\gamma \mapsto \mathcal{G}_\gamma(\cdot)\mathcal{M}_\gamma(\cdot)\frac{\bar{\gamma}(\cdot)^T n_\gamma(\cdot)}{\|\dot{\gamma}(\cdot)\|_2}$ and thus the continuity of (3.4.8). □

Algorithm 3.4.3 (Compute the first derivative $\hat{\jmath}'(\gamma)$)**.**

1. *For $\gamma \in S^+_{ad}$ compute the solution $(\hat{u}_\gamma, \mathcal{G}_\gamma) = (\hat{u}(\gamma), \mathcal{G}(\gamma))$ of the fictitious domain formulation of the state equation* (2.2.1)*.*
2. *Compute the solution $(\hat{\lambda}_\gamma, \mathcal{M}_\gamma) = (\hat{\lambda}(\gamma), \mathcal{M}(\gamma))$ of the fictitious domain formu-*

lation of the adjoint equation (3.3.6).

3. *Obtain the first derivative* $\hat{\jmath}'(\gamma)$ *of the reduced objective functional* $\hat{\jmath}$ *via the integral representation* (3.4.7).

3.5. Approximation of the second derivative of the reduced objective function

In this section we discuss possibilities to derive a formula for the second derivative of the reduced objective function $\hat{\jmath}$. According to the theory of optimal control problems, second derivatives are established if $\hat{J}$ and $\hat{E}$ are twice continuously Fréchet-differentiable. In [HPUU08, Section 1.6.5] this is carried out using the *Lagrange function based approach*. As mentioned in the last section, in our setting the state equation operator $\hat{E}$ is not continuously differentiable with respect to γ. Therefore, a reliable way to calculate a second derivative $\hat{\jmath}''(\gamma)$ cannot be expected from the established theory. If we formally differentiate the integral representation (3.4.7) once again with respect to γ, we find that the second derivative of $\hat{\jmath}$ is directly coupled with the derivatives of the Lagrange multipliers $\mathcal{G}(\gamma)$ and $\mathcal{M}(\gamma)$. Unfortunately, we obtain that for an admissible direction $\tilde{\gamma} \in S'_\gamma$ the directional derivative $d\hat{\jmath}'(\gamma, \tilde{\gamma})$ does not exist in general, since the directional derivatives $(d\hat{u}(\gamma, \tilde{\gamma}), d\mathcal{G}(\gamma, \tilde{\gamma}))$ and $(d\hat{\lambda}(\gamma, \tilde{\gamma}), d\mathcal{M}(\gamma, \tilde{\gamma}))$ do not exist.
However, we can provide an appropriate approximation of the second derivative of $\hat{\jmath}$. More precisely, for an admissible direction $\tilde{\gamma} \in S'_\gamma$ we approximate operator-vector-products $\hat{\jmath}''(\gamma)\tilde{\gamma}$ by the symmetric directional derivative

$$d^{\pm}\hat{\jmath}'(\gamma, \tilde{\gamma}) := \lim_{\delta \to 0^+} \frac{1}{2\delta} \left\{ \hat{\jmath}'(\gamma + \delta\tilde{\gamma}) - \hat{\jmath}'(\gamma - \delta\tilde{\gamma}) \right\} \tag{3.5.1}$$

of the first derivative $\hat{\jmath}'(\gamma)$:
In Lemma 3.5.1 we show the existence of the one-sided directional derivatives

$$\begin{aligned} (\delta^+_{\tilde{\gamma}} \hat{u}_\gamma, \delta^+_{\tilde{\gamma}} \mathcal{G}_\gamma) &:= (d^+ \hat{u}(\gamma, \tilde{\gamma}), d^+ \mathcal{G}(\gamma, \tilde{\gamma})) \quad \text{and} \\ (\delta^-_{\tilde{\gamma}} \hat{u}_\gamma, \delta^-_{\tilde{\gamma}} \mathcal{G}_\gamma) &:= (d^- \hat{u}(\gamma, \tilde{\gamma}), d^- \mathcal{G}(\gamma, \tilde{\gamma})). \end{aligned}$$

In Lemma 3.5.2 we conclude in a similar way that

$$\begin{aligned} (\delta^+_{\tilde{\gamma}} \hat{\lambda}_\gamma, \delta^+_{\tilde{\gamma}} \mathcal{M}_\gamma) &:= (d^+ \hat{\lambda}(\gamma, \tilde{\gamma}), d^+ \mathcal{M}(\gamma, \tilde{\gamma})) \quad \text{and} \\ (\delta^-_{\tilde{\gamma}} \hat{\lambda}_\gamma, \delta^-_{\tilde{\gamma}} \mathcal{M}_\gamma) &:= (d^- \hat{\lambda}(\gamma, \tilde{\gamma}), d^- \mathcal{M}(\gamma, \tilde{\gamma})) \end{aligned}$$

exist. Finally, in Theorem 3.5.3 we prove that $\hat{\jmath}'$ possesses right- and left-sided directional derivatives $d^+\hat{\jmath}'(\gamma, \tilde{\gamma})$ and $d^-\hat{\jmath}'(\gamma, \tilde{\gamma})$.
We note that if right- and left-sided derivatives exist, the symmetric derivative also exists and is the arithmetic mean of the latter two quantities.

Lemma 3.5.1. *For $\gamma \in S_{ad}^{+}$ let $(\hat{u}_\gamma, \mathcal{G}_\gamma) = (\hat{u}(\gamma), \mathcal{G}(\gamma)) \in H_0^1(\hat{\Omega}) \times H_I^*$ denote the solution of the Poisson problem $\hat{E}((\hat{u}, \mathcal{G}), \gamma) = 0$. Then the mapping $\gamma \mapsto (\hat{u}_\gamma, \mathcal{G}_\gamma)$ has a symmetrical directional derivative with respect to γ in all admissible directions $\tilde{\gamma} \in S'_\gamma$. This directional derivative, denoted by*

$$(\delta_{\tilde{\gamma}}^{\pm}\hat{u}_\gamma, \delta_{\tilde{\gamma}}^{\pm}\mathcal{G}_\gamma) := (d^{\pm}\hat{u}(\gamma, \tilde{\gamma}), d^{\pm}\mathcal{G}(\gamma, \tilde{\gamma})),$$

is the arithmetic mean of the right- and left-sided directional derivatives $(\delta_{\tilde{\gamma}}^{+}\hat{u}_\gamma, \delta_{\tilde{\gamma}}^{+}\mathcal{G}_\gamma)$ and $(\delta_{\tilde{\gamma}}^{-}\hat{u}_\gamma, \delta_{\tilde{\gamma}}^{-}\mathcal{G}_\gamma)$ and can be computed by solving the following state sensitivity equation

$$\begin{cases} (\nabla \delta_{\tilde{\gamma}}^{\pm}\hat{u}_\gamma, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - \langle \delta_{\tilde{\gamma}}^{\pm}\mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{v}\rangle_{H_I^*, H_I} \\ \qquad = \displaystyle\int_I \mathcal{G}_\gamma(t) \nabla \hat{v}(\gamma(t))^T \tilde{\gamma}(t)\, dt + l_1^{\pm}(\hat{v}, f, \gamma, \tilde{\gamma}), \qquad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}), \\ -\langle \mathcal{H}, \mathcal{T}_\gamma \delta_{\tilde{\gamma}}^{\pm}\hat{u}_\gamma\rangle_{H_I^*, H_I} = \displaystyle\frac{1}{2}\int_I \mathcal{H}(t)\mathcal{G}_\gamma(t) \frac{\tilde{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\, dt, \quad \forall \mathcal{H} \in H_I^* \cap L^2(I)^*. \end{cases} \tag{3.5.2}$$

Here, $l_1^{\pm}(\hat{v}, f, \gamma, \tilde{\gamma}) := \frac{1}{2}\left(l^{+}(\hat{v}, f, \gamma, \tilde{\gamma}) + l^{-}(\hat{v}, f, \gamma, \tilde{\gamma})\right)$ and

$$l_1^{+}(\hat{v}, f, \gamma, \tilde{\gamma}) := \lim_{\delta \to 0^+} \frac{1}{\delta}(\widetilde{f}_{\gamma+\delta\tilde{\gamma}} - \widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})} \quad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}), \tag{3.5.3}$$

$$l_1^{-}(\hat{v}, f, \gamma, \tilde{\gamma}) := \lim_{\delta \to 0^-} \frac{1}{\delta}(\widetilde{f}_{\gamma+\delta\tilde{\gamma}} - \widetilde{f}_\gamma, \hat{v})_{L^2(\hat{\Omega})} \quad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}). \tag{3.5.4}$$

Proof. Let us define the *reduced state operator*

$$\hat{e}(\gamma) := \hat{E}((\hat{u}_\gamma, \mathcal{G}_\gamma), \gamma) = \begin{pmatrix} \hat{\mathcal{A}}\hat{u}_\gamma - \mathcal{T}_\gamma^* \mathcal{G}_\gamma - \hat{\mathcal{F}}_\gamma \\ -\mathcal{T}_\gamma \hat{u}_\gamma \end{pmatrix}, \quad \gamma \in S_{\mathrm{ad}}.$$

For a given $\delta \neq 0$ we investigate the following difference quotient of $\hat{e}(\gamma)$

$$\begin{aligned} D_\delta \hat{e}(\gamma; \tilde{\gamma}) &:= \frac{1}{\delta}\left\{\hat{e}(\gamma + \delta\tilde{\gamma}) - \hat{e}(\gamma)\right\} \\ &= \frac{1}{\delta}\left\{ \begin{pmatrix} \hat{\mathcal{A}}\hat{u}_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}_{\gamma+\delta\tilde{\gamma}}^* \mathcal{G}_{\gamma+\delta\tilde{\gamma}} - \hat{\mathcal{F}}_{\gamma+\delta\tilde{\gamma}} \\ -\mathcal{T}_{\gamma+\delta\tilde{\gamma}} \hat{u}_{\gamma+\delta\tilde{\gamma}} \end{pmatrix} - \begin{pmatrix} \hat{\mathcal{A}}\hat{u}_\gamma - \mathcal{T}_\gamma^* \mathcal{G}_\gamma - \hat{\mathcal{F}}_\gamma \\ -\mathcal{T}_\gamma \hat{u}_\gamma \end{pmatrix} \right\} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Our aim is to rearrange the terms in the foregoing equation to obtain a linear system for $D_\delta \hat{u}(\gamma; \tilde{\gamma}) := \frac{1}{\delta}\left\{\hat{u}_{\gamma+\delta\tilde{\gamma}} - \hat{u}_\gamma\right\}$ and $D_\delta \mathcal{G}(\gamma; \tilde{\gamma}) := \frac{1}{\delta}\left\{\mathcal{G}_{\gamma+\delta\tilde{\gamma}} - \mathcal{G}_\gamma\right\}$. In a first step we obtain

$$D_\delta \hat{e}(\gamma;\tilde{\gamma}) = \frac{1}{\delta}\begin{pmatrix} \hat{\mathcal{A}}\left(\hat{u}_{\gamma+\delta\tilde{\gamma}} - \hat{u}_\gamma\right) - \mathcal{T}_\gamma^*\left(\mathcal{G}_{\gamma+\delta\tilde{\gamma}} - \mathcal{G}_\gamma\right) \\ -\mathcal{T}_\gamma\left(\hat{u}_{\gamma+\delta\tilde{\gamma}} - \hat{u}_\gamma\right) \end{pmatrix} \\ - \frac{1}{\delta}\begin{pmatrix} \left(\mathcal{T}^*_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}^*_\gamma\right)\mathcal{G}_{\gamma+\delta\tilde{\gamma}} + \left(\hat{\mathcal{F}}_{\gamma+\delta\tilde{\gamma}} - \hat{\mathcal{F}}_\gamma\right) \\ \left(\mathcal{T}_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}_\gamma\right)\hat{u}_{\gamma+\delta\tilde{\gamma}} \end{pmatrix}$$

which can be written as

$$\begin{pmatrix} \hat{\mathcal{A}} & -\mathcal{T}^*_\gamma \\ -\mathcal{T}_\gamma & 0 \end{pmatrix}\begin{pmatrix} D_\delta \hat{u}(\gamma;\tilde{\gamma}) \\ D_\delta \mathcal{G}(\gamma;\tilde{\gamma}) \end{pmatrix} = \begin{pmatrix} R_1(\delta) + R_2(\delta) \\ R_3(\delta) \end{pmatrix}. \tag{3.5.5}$$

Here, we used the definitions

$$R_1(\delta) := \frac{1}{\delta}\left(\mathcal{T}^*_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}^*_\gamma\right)\mathcal{G}_{\gamma+\delta\tilde{\gamma}}, \quad R_2(\delta) := \frac{1}{\delta}\left(\hat{\mathcal{F}}_{\gamma+\delta\tilde{\gamma}} - \hat{\mathcal{F}}_\gamma\right),$$

$$R_3(\delta) := \frac{1}{\delta}\left(\mathcal{T}_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}_\gamma\right)\hat{u}_{\gamma+\delta\tilde{\gamma}}.$$

We test the linear system (3.5.5) with test functions $(\hat{v}, \mathcal{H})$. If δ tends to zero from above we obtain on the right-hand-side of (3.5.5) limits similar to the ones in the proof of Theorem 3.4.1. In order to be able to apply the same techniques as therein we assume more regularity on the test functions, i.e. $(\hat{v}, \mathcal{H}) \in H^1_0(\hat{\Omega}) \cap H^2(\hat{\Omega}) \times H^*_I \cap L^2(I)^*$. We obtain for the right-sided limits

$$\begin{aligned} \lim_{\delta\to 0^+} \langle R_1(\delta), \hat{v}\rangle_{H^1_0(\hat{\Omega})^*, H^1_0(\hat{\Omega})} &= \lim_{\delta\to 0^+} \frac{1}{\delta}\langle\left(\mathcal{T}^*_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}^*_\gamma\right)\mathcal{G}_{\gamma+\delta\tilde{\gamma}}, \hat{v}\rangle_{H^1_0(\hat{\Omega})^*, H^1_0(\hat{\Omega})} \\ &= \lim_{\delta\to 0^+} \frac{1}{\delta}\langle \mathcal{G}_{\gamma+\delta\tilde{\gamma}}, \left(\mathcal{T}_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}_\gamma\right)\hat{v}\rangle_{H^*_I, H_I} \\ &= \lim_{\delta\to 0^+} \frac{1}{\delta}\int_I \mathcal{G}_{\gamma+\delta\tilde{\gamma}}(t)\left(\hat{v}(\gamma(t) + \delta\tilde{\gamma}(t)) - \hat{v}(\gamma(t)\right)\, dt \\ &= \int_I \mathcal{G}_\gamma(t)\nabla\hat{v}(\gamma(t))^T\tilde{\gamma}(t)\, dt, \end{aligned}$$

$$\begin{aligned} \lim_{\delta\to 0^+} \langle \mathcal{H}, R_3(\delta)\rangle_{H^*_I, H_I} &= \lim_{\delta\to 0^+} \frac{1}{\delta}\langle \mathcal{H}, \left(\mathcal{T}_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}_\gamma\right)\hat{u}_{\gamma+\delta\tilde{\gamma}}\rangle_{H^*_I, H_I} \\ &= -\lim_{\delta\to 0^+} \frac{1}{\delta}\langle \mathcal{H}, \mathcal{T}_\gamma \hat{u}_{\gamma+\delta\tilde{\gamma}}\rangle_{H^*_I, H_I} \\ &= \int_{I_+} \mathcal{H}(t)\nabla u_\gamma(\gamma(t))^T\bar{\gamma}(t)\, dt, \end{aligned}$$

and with the definition (3.5.3)

$$\begin{aligned}
\lim_{\delta\to 0^+} \langle R_2(\delta), \hat{v}\rangle_{H_0^1(\hat{\Omega})^*, H_0^1(\hat{\Omega})} &= \lim_{\delta\to 0^+} \frac{1}{\delta}\langle \Big(\hat{\mathcal{F}}_{\gamma+\delta\tilde{\gamma}} - \hat{\mathcal{F}}_{\gamma}\Big), \hat{v}\rangle_{H_0^1(\hat{\Omega})^*, H_0^1(\hat{\Omega})} \\
&= \lim_{\delta\to 0^+} \frac{1}{\delta}(\widetilde{f}_{\gamma+\delta\tilde{\gamma}} - \widetilde{f}_{\gamma}, \hat{v})_{L^2(\hat{\Omega})} \\
&= l_1^+(\hat{v}, f, \gamma, \tilde{\gamma}).
\end{aligned}$$

We note that the limit $l_1^+(\hat{v}, f, \gamma, \tilde{\gamma})$ exist, but we do not have an explicit formula. Now, to compute the right-sided directional derivatives

$$(\delta_{\tilde{\gamma}}^+ \hat{u}_\gamma, \delta_{\tilde{\gamma}}^+ \mathcal{G}_\gamma) := \lim_{\delta\to 0^+} (D_\delta \hat{u}(\gamma;\tilde{\gamma}), D_\delta \mathcal{G}(\gamma;\tilde{\gamma})),$$

we have to solve (3.5.5) for $\delta \to 0^+$, which is

$$\begin{cases}
(\nabla \delta_{\tilde{\gamma}}^+ \hat{u}_\gamma, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - \langle \delta_{\tilde{\gamma}}^+ \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{v}\rangle_{H_I^*, H_I} \\
\qquad = \displaystyle\int_I \mathcal{G}_\gamma(t) \nabla \hat{v}(\gamma(t))^T \tilde{\gamma}(t)\, dt + l_1^+(\hat{v}, f, \gamma, \tilde{\gamma}), \qquad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}), \\
-\langle \mathcal{H}, \mathcal{T}_\gamma \delta_{\tilde{\gamma}}^+ \hat{u}_\gamma\rangle_{H_I^*, H_I} = \displaystyle\int_{I_+} \mathcal{H}(t)\mathcal{G}_\gamma(t) \frac{\tilde{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\, dt, \quad \forall \mathcal{H} \in H_I^* \cap L^2(I)^*.
\end{cases}$$

Analogously, to compute the left-sided directional derivatives

$$(\delta_{\tilde{\gamma}}^- \hat{u}_\gamma, \delta_{\tilde{\gamma}}^- \mathcal{G}_\gamma) := \lim_{\delta\to 0^-} (D_\delta \hat{u}(\gamma;\tilde{\gamma}), D_\delta \mathcal{G}(\gamma;\tilde{\gamma})),$$

we have to solve (3.5.5) for $\delta \to 0^-$, which is

$$\begin{cases}
(\nabla \delta_{\tilde{\gamma}}^- \hat{u}_\gamma, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - \langle \delta_{\tilde{\gamma}}^- \mathcal{G}_\gamma, \mathcal{T}_\gamma \hat{v}\rangle_{H_I^*, H_I} \\
\qquad = \displaystyle\int_I \mathcal{G}_\gamma(t) \nabla \hat{v}(\gamma(t))^T \tilde{\gamma}(t)\, dt + l_1^-(\hat{v}, f, \gamma, \tilde{\gamma}), \qquad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}), \\
-\langle \mathcal{H}, \mathcal{T}_\gamma \delta_{\tilde{\gamma}}^- \hat{u}_\gamma\rangle_{H_I^*, H_I} = \displaystyle\int_{I_-} \mathcal{H}(t)\mathcal{G}_\gamma(t) \frac{\tilde{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\, dt, \quad \forall \mathcal{H} \in H_I^* \cap L^2(I)^*.
\end{cases}$$

We call attention to the subtle differences in the right-hand sides of the resulting systems for $(\delta_{\tilde{\gamma}}^+ \hat{u}_\gamma, \delta_{\tilde{\gamma}}^+ \mathcal{G}_\gamma)$ and $(\delta_{\tilde{\gamma}}^- \hat{u}_\gamma, \delta_{\tilde{\gamma}}^- \mathcal{G}_\gamma)$.
Finally, for the symmetrical directional derivative

$$(\delta_{\tilde{\gamma}}^\pm \hat{u}_\gamma, \delta_{\tilde{\gamma}}^\pm \mathcal{G}_\gamma) := \lim_{\delta\to 0^+} \frac{1}{2\delta} \left(\{\hat{u}(\gamma+\delta\tilde{\gamma}) - \hat{u}(\gamma-\delta\tilde{\gamma})\}, \{\mathcal{G}(\gamma+\delta\tilde{\gamma}) - \mathcal{G}(\gamma-\delta\tilde{\gamma})\}\right)$$

we obtain

$$(\delta_{\tilde{\gamma}}^\pm \hat{u}_\gamma, \delta_{\tilde{\gamma}}^\pm \mathcal{G}_\gamma) = \frac{1}{2}\left((\delta_{\tilde{\gamma}}^+ \hat{u}_\gamma, \delta_{\tilde{\gamma}}^+ \mathcal{G}_\gamma) + (\delta_{\tilde{\gamma}}^- \hat{u}_\gamma, \delta_{\tilde{\gamma}}^- \mathcal{G}_\gamma)\right)$$

which can be computed by (3.5.2). $\square$

Lemma 3.5.2. *Let the assumptions of the previous lemma hold. Let* $(\delta^{\pm}_{\tilde{\gamma}}\hat{u}_\gamma, \delta^{\pm}_{\tilde{\gamma}}\mathcal{G}_\gamma)$ *be the symmetrical directional derivative of* $\gamma \mapsto (\hat{u}_\gamma, \mathcal{G}_\gamma)$ *and* $(\hat{\lambda}_\gamma, \mathcal{M}_\gamma) := (\hat{\lambda}(\gamma), \mathcal{M}(\gamma))$ *denote the solution of the fictitious domain formulation of the adjoint equation* (3.3.6)*. Then the mapping* $\gamma \mapsto (\hat{\lambda}_\gamma, \mathcal{M}_\gamma)$ *has a symmetrical directional derivative with respect to* γ *in all admissible directions* $\tilde{\gamma} \in S'_\gamma$*. This directional derivative, denoted by*

$$(\delta^{\pm}_{\tilde{\gamma}}\hat{\lambda}_\gamma, \delta^{\pm}_{\tilde{\gamma}}\mathcal{M}_\gamma) := (d^{\pm}\hat{\lambda}(\gamma, \tilde{\gamma}), d^{\pm}\mathcal{M}(\gamma, \tilde{\gamma})),$$

is the arithmetic mean of the right- and left-sided directional derivatives $(\delta^{+}_{\tilde{\gamma}}\hat{\lambda}_\gamma, \delta^{+}_{\tilde{\gamma}}\mathcal{M}_\gamma)$ *and* $(\delta^{-}_{\tilde{\gamma}}\hat{\lambda}_\gamma, \delta^{-}_{\tilde{\gamma}}\mathcal{M}_\gamma)$ *and can be computed by solving the adjoint sensitivity equation*

$$\begin{cases} (\nabla\delta^{\pm}_{\tilde{\gamma}}\hat{\lambda}_\gamma, \nabla\hat{v})_{L^2(\hat{\Omega})^2} - \langle \delta^{\pm}_{\tilde{\gamma}}\mathcal{M}_\gamma, \mathcal{T}_\gamma\hat{v}\rangle_{H_I^*,H_I} = \displaystyle\int_I \mathcal{M}_\gamma(t)\nabla\hat{v}(\gamma(t))^T\tilde{\gamma}(t)\,dt \\ \qquad + \displaystyle\int_{\Omega_T} \delta^{\pm}_{\tilde{\gamma}}\hat{u}_\gamma(x)\hat{v}(x)\,dx - l_2^{\pm}(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T), \qquad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}), \\ -\langle \mathcal{H}, \mathcal{T}_\gamma\delta^{\pm}_{\tilde{\gamma}}\hat{\lambda}_\gamma\rangle_{H_I^*,H_I} = \dfrac{1}{2}\displaystyle\int_I \mathcal{H}(t)\mathcal{M}_\gamma(t)\frac{\tilde{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\,dt, \qquad \forall \mathcal{H} \in H_I^* \cap L^2(I)^*. \end{cases} \tag{3.5.6}$$

Here, $l_2^{\pm}(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T) := \frac{1}{2}\left(l_2^{+}(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T) + l_2^{-}(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T)\right)$ *and*

$$l_2^{+}(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T) := \lim_{\delta\to 0^+} \frac{1}{\delta}(\widetilde{u_d|_{\Omega_T\cap\Omega_{\gamma+\delta\tilde{\gamma}}}} - \widetilde{u_d|_{\Omega_T\cap\Omega_\gamma}}, \hat{v})_{L^2(\Omega_T)} \quad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}),$$

$$l_2^{-}(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T) := \lim_{\delta\to 0^-} \frac{1}{\delta}(\widetilde{u_d|_{\Omega_T\cap\Omega_{\gamma+\delta\tilde{\gamma}}}} - \widetilde{u_d|_{\Omega_T\cap\Omega_\gamma}}, \hat{v})_{L^2(\Omega_T)} \quad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}).$$

Proof. The proof of this lemma is almost identical to the proof of the previous lemma. Let us define the *reduced adjoint state operator*

$$\hat{w}(\gamma) := \begin{pmatrix} \hat{\mathcal{A}}\hat{\lambda}_\gamma - \mathcal{T}_\gamma^*\mathcal{M}_\gamma - (\hat{u}_\gamma - u_d, \cdot)_{L^2(\Omega_T\cap\Omega_\gamma)} \\ -\mathcal{T}_\gamma\hat{\lambda}_\gamma \end{pmatrix}, \quad \gamma \in S_{\mathrm{ad}}.$$

For $D_\delta\hat{\lambda}(\gamma;\tilde{\gamma}) := \frac{1}{\delta}\left\{\hat{\lambda}_{\gamma+\delta\tilde{\gamma}} - \hat{\lambda}_\gamma\right\}$ and $D_\delta\mathcal{M}(\gamma;\tilde{\gamma}) := \frac{1}{\delta}\left\{\mathcal{M}_{\gamma+\delta\tilde{\gamma}} - \mathcal{M}_\gamma\right\}$, $\delta \neq 0$, we obtain

$$\begin{pmatrix} \hat{\mathcal{A}} & -\mathcal{T}_\gamma^* \\ -\mathcal{T}_\gamma & 0 \end{pmatrix} \begin{pmatrix} D_\delta\hat{\lambda}(\gamma;\tilde{\gamma}) \\ D_\delta\mathcal{M}(\gamma;\tilde{\gamma}) \end{pmatrix} = \begin{pmatrix} R_4(\delta) + R_5(\delta) \\ R_6(\delta) \end{pmatrix}, \tag{3.5.7}$$

where

$$R_4(\delta) := \frac{1}{\delta}\left(\mathcal{T}^*_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}^*_\gamma\right)\mathcal{M}_{\gamma+\delta\tilde{\gamma}}, \quad R_6(\delta) := \frac{1}{\delta}\left(\mathcal{T}_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}_\gamma\right)\hat{\lambda}_{\gamma+\delta\tilde{\gamma}},$$

$$R_5(\delta) := \frac{1}{\delta}\left((\hat{u}_{\gamma+\delta\tilde{\gamma}} - u_d, \cdot)_{L^2(\Omega_T\cap\Omega_{\gamma+\delta\tilde{\gamma}})} - (\hat{u}_\gamma - u_d, \cdot)_{L^2(\Omega_T\cap\Omega_\gamma)}\right).$$

The limits of $R_4(\delta)$ and $R_6(\delta)$ for $\delta \to 0^+$ are obtained as in the proof of the previous lemma with $(\hat{u}, \mathcal{G})$ interchanged by $(\hat{\lambda}, \mathcal{M})$, i.e.

$$\begin{aligned}
\lim_{\delta \to 0^+} \langle R_4(\delta), \hat{v} \rangle_{H_0^1(\hat{\Omega})^*, H_0^1(\hat{\Omega})} &= \lim_{\delta \to 0^+} \frac{1}{\delta} \langle \left(\mathcal{T}^*_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}^*_\gamma \right) \mathcal{M}_{\gamma+\delta\tilde{\gamma}}, \hat{v} \rangle_{H_0^1(\hat{\Omega})^*, H_0^1(\hat{\Omega})} \\
&= \int_I \mathcal{M}_\gamma(t) \nabla \hat{v}(\gamma(t))^T \tilde{\gamma}(t)\, dt,
\end{aligned}$$

$$\begin{aligned}
\lim_{\delta \to 0^+} \langle \mathcal{H}, R_6(\delta) \rangle_{H_I^*, H_I} &= \lim_{\delta \to 0^+} \frac{1}{\delta} \langle \mathcal{H}, (\mathcal{T}_{\gamma+\delta\tilde{\gamma}} - \mathcal{T}_\gamma)\, \hat{\lambda}_{\gamma+\delta\tilde{\gamma}} \rangle_{H_I^*, H_I} \\
&= \int_{I_+} \mathcal{H}(t) \nabla \lambda_\gamma(\gamma(t))^T \tilde{\gamma}(t)\, dt.
\end{aligned}$$

Furthermore we find

$$\begin{aligned}
&\lim_{\delta \to 0^+} \langle R_5(\delta), \hat{v} \rangle_{H_0^1(\hat{\Omega})^*, H_0^1(\hat{\Omega})} \\
&\quad = \lim_{\delta \to 0^+} \frac{1}{\delta} \left((\hat{u}_{\gamma+\delta\tilde{\gamma}} - u_d, \hat{v})_{L^2(\Omega_T \cap \Omega_{\gamma+\delta\tilde{\gamma}})} - (\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_T \cap \Omega_\gamma)} \right) \\
&\quad = \lim_{\delta \to 0^+} \frac{1}{\delta} (\hat{u}_{\gamma+\delta\tilde{\gamma}} - \hat{u}_\gamma, \hat{v})_{L^2(\Omega_T)} - \lim_{\delta \to 0^+} \frac{1}{\delta} (\widetilde{u_d|_{\Omega_T \cap \Omega_{\gamma+\delta\tilde{\gamma}}}} - \widetilde{u_d|_{\Omega_T \cap \Omega_\gamma}}, \hat{v})_{L^2(\Omega_T)} \\
&\quad = (\delta^+_{\tilde{\gamma}} \hat{u}_\gamma, \hat{v})_{L^2(\Omega_T)} - l_2^+(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T).
\end{aligned}$$

The equation (3.5.7) for $\delta \to 0^+$ leads to

$$\begin{cases}
(\nabla \delta^+_{\tilde{\gamma}} \hat{\lambda}_\gamma, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - \langle \delta^+_{\tilde{\gamma}} \mathcal{M}_\gamma, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} = \displaystyle\int_I \mathcal{M}_\gamma(t) \nabla \hat{v}(\gamma(t))^T \tilde{\gamma}(t)\, dt \\
\qquad + \displaystyle\int_{\Omega_T} \delta^+_{\tilde{\gamma}} \hat{u}_\gamma(x) \hat{v}(x)\, dx - l_2^+(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T) \qquad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}), \\
-\langle \mathcal{H}, \mathcal{T}_\gamma \delta^+_{\tilde{\gamma}} \hat{\lambda}_\gamma \rangle_{H_I^*, H_I} = \displaystyle\int_{I_+} \mathcal{H}(t) \mathcal{M}_\gamma(t) \frac{\tilde{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\, dt, \qquad \forall \mathcal{H} \in H_I^* \cap L^2(I)^*.
\end{cases}$$

For the left-sided directional derivatives we have

$$\begin{cases}
(\nabla \delta^-_{\tilde{\gamma}} \hat{\lambda}_\gamma, \nabla \hat{v})_{L^2(\hat{\Omega})^2} - \langle \delta^-_{\tilde{\gamma}} \mathcal{M}_\gamma, \mathcal{T}_\gamma \hat{v} \rangle_{H_I^*, H_I} = \displaystyle\int_I \mathcal{M}_\gamma(t) \nabla \hat{v}(\gamma(t))^T \tilde{\gamma}(t)\, dt \\
\qquad + \displaystyle\int_{\Omega_T} \delta^-_{\tilde{\gamma}} \hat{u}_\gamma(x) \hat{v}(x)\, dx - l_2^-(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T), \qquad \forall \hat{v} \in H_0^1(\hat{\Omega}) \cap H^2(\hat{\Omega}), \\
-\langle \mathcal{H}, \mathcal{T}_\gamma \delta^-_{\tilde{\gamma}} \hat{\lambda}_\gamma \rangle_{H_I^*, H_I} = \displaystyle\int_{I_-} \mathcal{H}(t) \mathcal{M}_\gamma(t) \frac{\tilde{\gamma}(t)^T n_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\, dt, \qquad \forall \mathcal{H} \in H_I^* \cap L^2(I)^*.
\end{cases}$$

After all,

$$(\delta_{\tilde{\gamma}}^{\pm}\hat{\lambda}_{\gamma}, \delta_{\tilde{\gamma}}^{\pm}\mathcal{M}_{\gamma}) = \frac{1}{2}\Big((\delta_{\tilde{\gamma}}^{+}\hat{\lambda}_{\gamma}, \delta_{\tilde{\gamma}}^{+}\mathcal{M}_{\gamma}) + (\delta_{\tilde{\gamma}}^{-}\hat{\lambda}_{\gamma}, \delta_{\tilde{\gamma}}^{-}\mathcal{M}_{\gamma})\Big)$$

can be computed by (3.5.6). □

We are now able to prove the main result of this section.

Theorem 3.5.3. *Let the assumptions of the previous lemma hold. Then the first derivative of the reduced objective function $\hat{\jmath}$ has a symmetrical directional derivative with respect to γ in all admissible directions $\tilde{\gamma} \in S'_{\gamma}$, $\tilde{\gamma} \neq 0$, which is given by*

$$\begin{aligned}\langle d^{\pm}\hat{\jmath}'(\gamma,\tilde{\gamma}),\bar{\gamma}\rangle_{S^*,S} = &\int_I (\delta_{\tilde{\gamma}}^{\pm}\mathcal{G}_{\gamma}(t)\mathcal{M}_{\gamma}(t) + \mathcal{G}_{\gamma}(t)\delta_{\tilde{\gamma}}^{\pm}\mathcal{M}_{\gamma}(t))\frac{\bar{\gamma}(t)^T\vec{n}_{\gamma}(t)}{\|\dot{\gamma}(t)\|_2}\,dt \\ &+ \int_I \mathcal{G}_{\gamma}(t)\mathcal{M}_{\gamma}(t)\frac{\bar{\gamma}(t)^T\left(\vec{n}_{\tilde{\gamma}}(t) - 2\vec{n}_{\gamma}(t)\vec{n}_{\gamma}(t)^T\vec{n}_{\tilde{\gamma}}(t)\right)\|\dot{\tilde{\gamma}}(t)\|_2}{\|\dot{\gamma}(t)\|_2^2}\,dt. \quad (3.5.8)\end{aligned}$$

Proof. For an admissible direction $\bar{\gamma}$ we define the operator

$$\begin{aligned}\mathcal{N} : S_{\mathrm{ad}}^{+} &\to L^{\infty}(I), \\ \gamma &\mapsto \mathcal{N}_{\gamma} := \mathcal{N}(\gamma) := \frac{\bar{\gamma}(\cdot)^T\vec{n}_{\gamma}(\cdot)}{\|\dot{\gamma}(\cdot)\|_2}.\end{aligned}$$

With the derivative

$$\begin{aligned}\frac{d}{d\varepsilon}\|\dot{\gamma} + \varepsilon\dot{\tilde{\gamma}}\|_2^2 &= \frac{d}{d\varepsilon}\left((\dot{\gamma}_1 + \varepsilon\dot{\tilde{\gamma}}_1)^2 + (\dot{\gamma}_2 + \varepsilon\dot{\tilde{\gamma}}_2)^2\right) \\ &= 2\left((\dot{\gamma}_1 + \varepsilon\dot{\tilde{\gamma}}_1)\dot{\tilde{\gamma}}_1 + (\dot{\gamma}_2 + \varepsilon\dot{\tilde{\gamma}}_2)\dot{\tilde{\gamma}}_2\right), \quad \forall\tilde{\gamma} \in S'_{\gamma},\end{aligned}$$

the directional derivative of $\mathcal{N}$ with respect to γ in an admissible direction $\tilde{\gamma} \in S'_{\gamma}$ is given by

$$d\mathcal{N}(\gamma,\tilde{\gamma}) = \frac{d}{d\varepsilon}\mathcal{N}(\gamma+\varepsilon\tilde{\gamma})\bigg|_{\varepsilon=0} = \frac{d}{d\varepsilon}\frac{\bar{\gamma}(\cdot)^T\vec{n}_{\gamma+\varepsilon\tilde{\gamma}}(\cdot)}{\|\dot{\gamma}(\cdot)+\varepsilon\dot{\tilde{\gamma}}(\cdot)\|_2}\bigg|_{\varepsilon=0} \tag{3.5.9}$$

$$= \frac{d}{d\varepsilon}\frac{\bar{\gamma}^T\begin{pmatrix}\dot{\gamma}_2+\varepsilon\dot{\tilde{\gamma}}_2\\ -\dot{\gamma}_1-\varepsilon\dot{\tilde{\gamma}}_1\end{pmatrix}}{\|\dot{\gamma}+\varepsilon\dot{\tilde{\gamma}}\|_2^2}\Bigg|_{\varepsilon=0} = \frac{\bar{\gamma}^T\begin{pmatrix}\dot{\tilde{\gamma}}_2\\ -\dot{\tilde{\gamma}}_1\end{pmatrix}\|\dot{\gamma}+\varepsilon\dot{\tilde{\gamma}}\|_2^2}{\|\dot{\gamma}+\varepsilon\dot{\tilde{\gamma}}\|_2^4}\Bigg|_{\varepsilon=0} \tag{3.5.10}$$

$$-2\,\frac{\bar{\gamma}^T\begin{pmatrix}\dot{\gamma}_2+\varepsilon\dot{\tilde{\gamma}}_2\\ -\dot{\gamma}_1-\varepsilon\dot{\tilde{\gamma}}_1\end{pmatrix}\left((\dot{\gamma}_1+\varepsilon\dot{\tilde{\gamma}}_1)\dot{\tilde{\gamma}}_1+(\dot{\gamma}_2+\varepsilon\dot{\tilde{\gamma}}_2)\dot{\tilde{\gamma}}_2\right)}{\|\dot{\gamma}+\varepsilon\dot{\tilde{\gamma}}\|_2^4}\Bigg|_{\varepsilon=0} \tag{3.5.11}$$

$$= \frac{\bar{\gamma}^T\begin{pmatrix}\dot{\tilde{\gamma}}_2\\ -\dot{\tilde{\gamma}}_1\end{pmatrix}\|\dot{\gamma}\|_2^2-2\bar{\gamma}^T\begin{pmatrix}\dot{\gamma}_2\\ -\dot{\gamma}_1\end{pmatrix}\left(\dot{\gamma}_1\dot{\tilde{\gamma}}_1+\dot{\gamma}_2\dot{\tilde{\gamma}}_2\right)}{\|\dot{\gamma}\|_2^4}, \tag{3.5.12}$$

and therefore

$$d\mathcal{N}(\gamma,\tilde{\gamma}) = \frac{\bar{\gamma}^T\vec{n}_{\tilde{\gamma}}\|\dot{\tilde{\gamma}}\|_2-2\bar{\gamma}^T\vec{n}_{\gamma}\vec{n}_{\gamma}^T\vec{n}_{\tilde{\gamma}}\|\dot{\tilde{\gamma}}\|_2}{\|\dot{\gamma}\|_2^2} = \frac{\bar{\gamma}^T\left(\vec{n}_{\tilde{\gamma}}-2\vec{n}_{\gamma}\vec{n}_{\gamma}^T\vec{n}_{\tilde{\gamma}}\right)\|\dot{\tilde{\gamma}}\|_2}{\|\dot{\gamma}\|_2^2}. \tag{3.5.13}$$

Now, we calculate the right-sided directional derivative of $\hat{\jmath}'$ with respect to γ in an admissible direction $\tilde{\gamma}\in S'_\gamma$ using the integral representation (3.4.7). We obtain

$$\begin{aligned}
\langle d^+\hat{\jmath}'(\gamma,\tilde{\gamma}),\bar{\gamma}\rangle_{S^*,S} &= \lim_{\delta\to 0^+}\frac{1}{\delta}\left\{\langle\hat{\jmath}'(\gamma+\delta\tilde{\gamma}),\bar{\gamma}\rangle_{S^*,S}-\langle\hat{\jmath}'(\gamma),\bar{\gamma}\rangle_{S^*,S}\right\}\\
&= \lim_{\delta\to 0^+}\frac{1}{\delta}\bigg\{\int_I \mathcal{G}_{\gamma+\delta\tilde{\gamma}}(t)\mathcal{M}_{\gamma+\delta\tilde{\gamma}}(t)\mathcal{N}_{\gamma+\delta\tilde{\gamma}}(t)\,dt\\
&\qquad -\int_I \mathcal{G}_{\gamma}(t)\mathcal{M}_{\gamma}(t)\mathcal{N}_{\gamma}(t)\,dt\bigg\}\\
&= \lim_{\delta\to 0^+}\frac{1}{\delta}\bigg\{\int_I \left(\mathcal{G}_{\gamma+\delta\tilde{\gamma}}(t)-\mathcal{G}_{\gamma}(t)\right)\mathcal{M}_{\gamma+\delta\tilde{\gamma}}(t)\mathcal{N}_{\gamma+\delta\tilde{\gamma}}(t)\,dt\\
&\qquad +\int_I \mathcal{G}_{\gamma}(t)\left(\mathcal{M}_{\gamma+\delta\tilde{\gamma}}(t)-\mathcal{M}_{\gamma}(t)\right)\mathcal{N}_{\gamma+\delta\tilde{\gamma}}(t)\,dt\\
&\qquad +\int_I \mathcal{G}_{\gamma}(t)\mathcal{M}_{\gamma}(t)\left(\mathcal{N}_{\gamma+\delta\tilde{\gamma}}(t)-\mathcal{N}_{\gamma}(t)\right)\,dt\bigg\}.
\end{aligned}$$

The equation (3.5.13) and the right-sided directional differentiability of $\mathcal{G}_\gamma$ and $\mathcal{M}_\gamma$

from Lemma 3.5.1 and Lemma 3.5.2 lead to

$$\begin{aligned}\langle d^+\hat{\jmath}'(\gamma,\tilde{\gamma}),\bar{\gamma}\rangle_{S^*,S} = &\int_I \left(\delta^+_{\tilde{\gamma}}\mathcal{G}_\gamma(t)\mathcal{M}_\gamma(t)+\mathcal{G}_\gamma(t)\delta^+_{\tilde{\gamma}}\mathcal{M}_\gamma(t)\right)\frac{\bar{\gamma}(t)^T\vec{n}_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\,dt\\ &+\int_I \mathcal{G}_\gamma(t)\mathcal{M}_\gamma(t)\frac{\bar{\gamma}(t)^T\left(\vec{n}_{\tilde{\gamma}}(t)-2\vec{n}_\gamma(t)\vec{n}_\gamma(t)^T\vec{n}_{\tilde{\gamma}}(t)\right)\|\dot{\tilde{\gamma}}(t)\|_2}{\|\dot{\gamma}(t)\|_2^2}\,dt.\end{aligned}$$

Clearly, for the left-sided directional derivative of $\hat{\jmath}'$ with respect to γ in an admissible direction $\tilde{\gamma}\in S'_\gamma$ we obtain

$$\begin{aligned}\langle d^-\hat{\jmath}'(\gamma,\tilde{\gamma}),\bar{\gamma}\rangle_{S^*,S} = &\int_I \left(\delta^-_{\tilde{\gamma}}\mathcal{G}_\gamma(t)\mathcal{M}_\gamma(t)+\mathcal{G}_\gamma(t)\delta^-_{\tilde{\gamma}}\mathcal{M}_\gamma(t)\right)\frac{\bar{\gamma}(t)^T\vec{n}_\gamma(t)}{\|\dot{\gamma}(t)\|_2}\,dt\\ &+\int_I \mathcal{G}_\gamma(t)\mathcal{M}_\gamma(t)\frac{\bar{\gamma}(t)^T\left(\vec{n}_{\tilde{\gamma}}(t)-2\vec{n}_\gamma(t)\vec{n}_\gamma(t)^T\vec{n}_{\tilde{\gamma}}(t)\right)\|\dot{\tilde{\gamma}}(t)\|_2}{\|\dot{\gamma}(t)\|_2^2}\,dt,\end{aligned}$$

respectively. As a consequence the symmetrical directional derivative of $\hat{\jmath}'$ with respect to γ in all admissible directions $\tilde{\gamma}\in S'_\gamma$ is given by 3.5.8. □

We summarize our results in the following algorithm:

Algorithm 3.5.4 (Compute the approximation of the second derivative $\hat{\jmath}''(\gamma)\tilde{\gamma}$)**.**

1. *For $\gamma\in S^+_{ad}$ compute the solution $(\hat{u}_\gamma,\mathcal{G}_\gamma)$ of the state equation and the solution $(\hat{\lambda}_\gamma,\mathcal{M}_\gamma)$ of the adjoint equation (as in the algorithm of the first derivative).*
2. *Compute the solution $(\delta^\pm_{\tilde{\gamma}}\hat{u},\delta^\pm_{\tilde{\gamma}}\mathcal{G})$ of the state sensitivity equation* (3.5.2)*.*
3. *Compute the solution $(\delta^\pm_{\tilde{\gamma}}\hat{\lambda},\delta^\pm_{\tilde{\gamma}}\mathcal{M})$ of the adjoint state sensitivity equation* (3.5.6)*.*
4. *Obtain the symmetric directional derivative $d^\pm\hat{\jmath}'(\gamma,\tilde{\gamma})$ as an approximation of operator-vector-products $\hat{\jmath}''(\gamma)\tilde{\gamma}$ via the integral representation* (3.5.8)*.*

3.6. Descent methods in a Hilbert space setting

In this section we present iterative optimization methods to solve the shape optimization problem (3.1.1) using the differentiability results from the previous sections. In detail, we discuss the gradient method, the BFGS quasi-Newton method and an inexact Newton-like method.
We assume that the optimal solution $\gamma^*\in S_{\text{ad}}$ of (3.1.1) is an interior point of S_{ad}, i.e. $\gamma^*\in\text{int}(S_{\text{ad}})$. Moreover, we suppose that we start with a strictly feasible interior point $\gamma^0\in\text{int}(S_{\text{ad}})$ and maintain strict feasibility during the optimization process, i.e. all iterates γ^k satisfy $\gamma^k\in\text{int}(S_{\text{ad}})$.
With this assumptions we can consider the unconstrained optimization problem

$$\min_{\gamma\in S(I)}\hat{\jmath}(\gamma)=\hat{J}((\hat{u}_\gamma,\mathcal{G}_\gamma),\gamma) \tag{3.6.1}$$

instead of (3.1.1), and we avoid the difficulty[5] to implement a projection operator $P_{S_{\mathrm{ad}}} : S(I) \to S_{\mathrm{ad}}$. We recall that $(\hat{u}_\gamma, \mathcal{G}_\gamma)$ is the solution of (2.2.1).
The first-order optimality conditions for a local minimum $\gamma^* \in S(I)$ of (3.6.1) reads as follows: $\gamma^* \in S(I)$ satisfies

$$\hat{\jmath}'(\gamma^*) = 0, \tag{3.6.2}$$

cf. [HPUU08, Section 2.2.1].

Definition 3.6.1. *An element $s \in S'_\gamma \subset S(I)$ is called descent direction of $\hat{\jmath}$ at γ, if $\hat{\jmath}(\gamma + \delta s)$ is decreasing at $\delta = 0$, i.e.*

$$\frac{d}{d\delta}\hat{\jmath}(\gamma + \delta s)\bigg|_{\delta=0} = \langle \hat{\jmath}'(\gamma), s\rangle_{S^*,S} < 0.$$

The steepest descent directions of $\hat{\jmath}$ at γ are defined by $s = \delta d_{sd}$, $\delta > 0$, where d_{sd} solves

$$\min_{\|d\|_S=1} \langle \hat{\jmath}'(\gamma), d\rangle_{S^*,S}.$$

We state the following general descent method:

Algorithm 3.6.2 (General descent method)**.**

0. Choose an initial curve $\gamma^0 \in S(I)$

For $k = 0, 1, 2, \dots$ (until convergence...):

1. If $\hat{\jmath}'(\gamma^k) = 0$, STOP.

2. Choose a descent direction $s^k \in S'_{\gamma^k} \subset S(I)$ such that $\langle \hat{\jmath}'(\gamma^k), s^k\rangle_{S^*,S} < 0$.

3. Choose a step size $\sigma_k > 0$ such that $\hat{\jmath}(\gamma^k + \sigma_k s^k) < \hat{\jmath}(\gamma^k)$.

4. Set $\gamma^{k+1} := \gamma^k + \sigma_k s^k$.

Since S is a Hilbert space, we can choose $S^* = S$ and $\nabla\hat{\jmath}\left(\gamma^k\right)$ is the Riesz-representative of the Fréchet-derivative $\hat{\jmath}'(\gamma^k)$, i.e

$$\langle \hat{\jmath}'(\gamma^k), s^k\rangle_{S^*,S} = (\nabla\hat{\jmath}(\gamma^k), s^k)_S, \quad \text{for all } s^k \in S'_{\gamma^k}. \tag{3.6.3}$$

If $H_k \in \mathcal{L}(S)$ is a symmetric and positive definite operator and $\nabla\hat{\jmath}(\gamma^k) \neq 0$, then

$$s^k = -H_k^{-1}\nabla\hat{\jmath}(\gamma^k) \tag{3.6.4}$$

is a descent direction. Clearly, we directly obtain

$$\langle \hat{\jmath}'(\gamma^k), s^k\rangle_{S^*,S} = -(\nabla\hat{\jmath}(\gamma^k), H_k^{-1}\nabla\hat{\jmath}(\gamma^k))_S < 0.$$

[5]We note that S_{ad} is non-convex, and projection points in non-convex sets may be non-unique.

Typical choices of $H_k \in \mathcal{L}(S)$ are given below. First, we will provide a commonly used step rule, see [HPUU08, 2.2.1.1].

Armijo rule:

Given a descent direction s^k of $\hat{\jmath}$ at γ^k, and a parameter $\rho \in (0,1)$, choose the maximum $\sigma_k \in \{1, 1/2, 1/4, \dots\}$ such that

$$\hat{\jmath}(\gamma^k + \sigma_k s^k) - \hat{\jmath}(\gamma^k) \le \rho\sigma_k \langle \hat{\jmath}'(\gamma^k), s^k\rangle_{S^*,S}.$$

Gradient method/steepest descent:

Choose $H_k = \mathrm{id}_S$ in (3.6.4) and obtain

$$s^k = -\nabla\hat{\jmath}(\gamma^k). \tag{3.6.5}$$

This is a steepest descent direction of $\hat{\jmath}$ at γ^k since

$$\begin{aligned}\min_{\|s^k\|_S=1} \langle \hat{\jmath}'(\gamma^k), s^k\rangle_{S^*,S} &= \min_{\|s^k\|_S=1} (\nabla\hat{\jmath}(\gamma^k), s^k)_S \\ &\ge -\|\nabla\hat{\jmath}(\gamma^k)\|_S \|s^k\|_S \\ &= \left(\nabla\hat{\jmath}(\gamma^k), -\frac{\nabla\hat{\jmath}(\gamma^k)}{\|\nabla\hat{\jmath}(\gamma^k)\|_S}\right)_S.\end{aligned}$$

BFGS quasi-Newton method:

Choose $H_0 = \mathrm{id}_S$ and use the BFGS operator update formula

$$H_{k+1} := H_k + \frac{(w^k, \,\cdot\,)_S}{(w^k, v^k)_S} w^k - \frac{(H_k v^k, \,\cdot\,)_S}{(H_k v^k, v^k)_S} H_k v^k, \tag{3.6.6}$$

at which

$$\begin{aligned}v^k &:= \gamma^{k+1} - \gamma^k, \\ w^k &:= \nabla\hat{\jmath}(\gamma^{k+1}) - \nabla\hat{\jmath}(\gamma^k).\end{aligned}$$

The operators H_{k+1} are clearly symmetric. They are also positive definite if the necessary condition $(w^k, v^k)_S > 0$ holds.

Inexact Newton-like method:

If we assume that $\hat{\jmath}(\gamma)$ is twice continuously Fréchet-differentiable with respect to γ, than $G(\gamma) := \nabla\hat{\jmath}(\gamma)$ would be continuously Fréchet-differentiable with respect to γ. The classical Newton method for (3.6.1) is obtained if Newton's method is applied to $G(\gamma) = \nabla\hat{\jmath}(\gamma) = 0$. Actually, this procedure corresponds to the scenario, in which we would choose $H_k = G'(\gamma^k)$: Thus, (3.6.4) turns to

$$G'(\gamma^k)s^k = -G(\gamma^k),$$

which yields the classical Newton direction. If we identify $G'(\gamma^k)s^k \in S$ with $\hat{\jmath}''(\gamma^k)s^k \in S^*$ than the previous equation leads to

$$\langle \hat{\jmath}''(\gamma^k)s^k, \cdot\rangle_{S^*,S} = -(\nabla\hat{\jmath}(\gamma^k), \cdot)_S. \tag{3.6.7}$$

Since $\hat{\jmath}(\gamma)$ is not twice continuously Fréchet-differentiable with respect to γ in our setting, we can still try to solve equation (3.6.7) if we replace $\hat{\jmath}''(\gamma^k)s^k$ by the symmetric derivative $d^{\pm}\hat{\jmath}'(\gamma^k, s^k)$. Thus, we obtain

$$\langle d^{\pm}\hat{\jmath}'(\gamma^k, s^k), \cdot\rangle_{S^*,S} = -(\nabla\hat{\jmath}(\gamma^k), \cdot)_S. \tag{3.6.8}$$

If we transfer the idea of inexact Newton methods, [DES82], we solve

$$\langle d^{\pm}\hat{\jmath}'(\gamma^k, s^k), \cdot\rangle_{S^*,S} = -(\nabla\hat{\jmath}(\gamma^k), \cdot)_S + (r^k, \cdot)_S, \text{ where } \frac{\|r^k\|_S}{\|\nabla\hat{\jmath}(\gamma^k)\|_S} \le \eta_k, \tag{3.6.9}$$

instead of (3.6.8). Here, $\{\eta_k\}$, with $\eta_k = \eta_k(\gamma^k) \ge 0$, is a sequence to control the level of accuracy. Obviously, $\eta_k = 0$ gives (3.6.8). Now we can apply iterative solvers that excepts operator-vector-products to solve (3.6.9) and obtain an inexact Newton-like (descent) direction.

4. Discretization of the shape optimization problem

In this chapter we follow the “first optimize, then discretize“ approach to solve the unconstrained optimization problem (3.6.1) numerically. Therefore, we discretize everything related to the state $(\hat{u}, \mathcal{G})$, the control γ, and to functionals, integrals and dualities, see Section 4.1. Then, in Section 4.3 we present finite-dimensional counterparts to the descent methods discussed in Section 3.6.
This approach differs from the “first discretize, then optimize“ approach from a structural point of view, where all quantities in problem (3.6.1) are discretized a-priori, and one solves the resulting finite dimensional optimization problem. For more details we refer to [HPUU08, Chapter 3].
We recall that the underlying objective function $\hat{J}$ in problem (3.6.1) tracks the L^2-error of $\hat{u}$ with respect to a desired state u_d in an observation domain $\Omega_T \subset \hat{\Omega}$. For simplicity we use $\Omega_T := \hat{\Omega}$ in the following.

4.1. Mixed finite element discretization

Our numerical realization is based on a mixed finite element method. We introduce two independent meshes, namely for the fictitious domain $\hat{\Omega}$ and for the interior boundary Γ_γ, see [KP01, Section 3].

4.1.1. Equidistant mesh on the fictitious domain $\hat{\Omega}$

On the fictitious domain $\hat{\Omega}$ we use a structured, uniform mesh. For $N \in \mathbb{N}$, $N \geq 3$, and $h^N := \frac{1}{N-1}$, we set

$$\overline{\hat{\Omega}} = \bigcup_{i=1}^{(N-1)^2} \bar{Q}_i,$$

where Q_i, $i = 1, \ldots, (N-1)^2$, are quadrilateral elements with N^2 equidistant mesh points

$$(x_i, y_j) = \left((i-1)h^N, (j-1)h^N\right), \quad i, j = 1, \ldots, N.$$

We approximate functions $\hat{u} \in H_0^1(\hat{\Omega})$ by continuous and piecewise bilinear functions $\hat{u}^N$. Therefore, we introduce

$$\hat{V}^N := \{\hat{v}^N \in C^0(\bar{\hat{\Omega}}) : \hat{v}^N|_{Q_i} \in \mathbb{P}_1(Q_i) \text{ for all } 1 \leq i \leq (N-1)^2\},$$

where $\mathbb{P}_1$ denotes the space of polynomials, in two variables, of degree less than or equal to one, and

$$\hat{U}^N := \{\hat{v}^N \in \hat{V}^N : \hat{v}^N|_{\partial\hat{\Omega}} = 0\} =: \langle \phi_1, \dots, \phi_{(N-2)^2} \rangle \subseteq H_0^1(\hat{\Omega}).$$

The resulting ansatz for $\hat{u}^N \in U^N$ then takes the form

$$\hat{u}^N(x, y) = \sum_{k=1}^{(N-2)^2} \hat{u}_k^N \Phi_k^N(x, y).$$

4.1.2. Equidistant partition of the interval I

In this part, we set for $M \in \mathbb{N}$, $M \geq 2$, and $h^M := \frac{1}{M-1}$,

$$I = \bigcup_{i=1}^{M-1} \bar{I}_i,$$

where $I_i = (t_i, t_{i+1})$, $i = 1, \dots, M-1$, are subintervals with M grid points

$$t_i = (i-1)h^M, \quad i = 1, \dots, M.$$

We approximate function $\mathcal{G} \in H_I^*$ by piecewise constant functions $\mathcal{G}^M$, and define

$$H_I^M := \operatorname{span}\{\chi_k^M\}_{k=1}^{M-1},$$

where $\chi_k^M|_{I_i} = \delta_{ki},\ k, i = 1, \dots, M-1$ denote the characteristic functions of our partition. This leads for $\mathcal{G}^M \in H_I^M$ to an ansatz

$$\mathcal{G}^M(t) = \sum_{i=1}^{M-1} g_i^M \chi_i^M(t).$$

Finally, we approximate curves $\gamma \in S(I)$ by piecewise linear curves γ^M. Here, we introduce

$$\begin{aligned} S_I^M = \{\gamma^M = (\gamma_1^M, \gamma_2^M)^T \in C^0(I)^2 : \gamma_j^M|_{I_i} \in \mathbb{P}_1(I_i) \text{ for all } j = 1, 2,\ 1 \leq i \leq M-1\} \\ =: \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} B_1^M, \dots, \begin{pmatrix} 1 \\ 0 \end{pmatrix} B_M^M, \begin{pmatrix} 0 \\ 1 \end{pmatrix} B_1^M, \dots, \begin{pmatrix} 0 \\ 1 \end{pmatrix} B_M^M \right\}, \end{aligned}$$

where B_k^M are piecewise linear hat functions with $B_k^M(t_i) = \delta_{ki}$, $k, i = 1, \dots, M$. This leads to an ansatz for $\gamma^M \in S_I^M$ of the form

$$\gamma^M(t) = \sum_{i=1}^{M} \vec{\gamma}_i^M B_i^M(t), \qquad \vec{\gamma}_i^M = \begin{pmatrix} \gamma_{1,i}^M \\ \gamma_{2,i}^M \end{pmatrix},\ i = 1, \dots, M. \tag{4.1.1}$$

Moreover, we obtain an approximation of

- the admissible domain Ω_γ, denoted by $\Omega_{\gamma^M} \subset \hat{\Omega}$,
- the curved boundary Γ_γ, denoted by

$$\Gamma_{\gamma^M} = \{\gamma^M(t), t \in (0,1)\},$$

- the trace operator $\mathcal{T}_\gamma$, denoted by

$$\mathcal{T}_{\gamma^M} : \hat{v}^N \longmapsto \hat{v}^N(\gamma^M(\cdot)),$$

- and the extension $\widetilde{f}_\gamma$, denoted by

$$\widetilde{f}_{\gamma^M} := \begin{cases} f_{\gamma^M} & \text{in } \Omega_{\gamma^M}, \\ 0 & \text{in } \Omega^c_{\gamma^M}. \end{cases}$$

4.1.3. The discrete state equation

In the following we replace the continuous quantities in the state equation (2.2.1) by their discrete counterparts introduced in Subsections 4.1.1 and 4.1.2. Then we obtain: For $\gamma^M \in S_I^M$, find $(\hat{u}^N_{\gamma^M}, \mathcal{G}^M_{\gamma^M}) = (\hat{u}^N(\gamma^M), \mathcal{G}^M(\gamma^M)) \in \hat{U}^N \times H_I^M$ with

$$\begin{cases} (\nabla \hat{u}^N_{\gamma^M}, \nabla \hat{v}^N)_{L^2(\hat{\Omega})^2} - \langle \mathcal{G}^M_{\gamma^M}, \mathcal{T}_{\gamma^M} \hat{v}^N \rangle_{H_I^*, H_I} = (\widetilde{f}_{\gamma^M}, \hat{v}^N)_{L^2(\hat{\Omega})}, & \forall \hat{v}^N \in \hat{U}^N, \\ -\langle \mathcal{H}^M, \mathcal{T}_{\gamma^M} \hat{u}^N_{\gamma^M} \rangle_{H_I^*, H_I} = 0, & \forall \mathcal{H}^M \in H_I^M. \end{cases} \tag{4.1.2}$$

Matrix notation allows us to rewrite (4.1.2) in the from: For $\gamma^M \in S_I^M$, find $(\underline{\hat{u}}^N_{\gamma^M}, \underline{g}^M_{\gamma^M}) = (\underline{\hat{u}}^N(\gamma^M), \underline{g}^M(\gamma^M)) \in \mathbb{R}^{(N-2)^2} \times \mathbb{R}^{M-1}$ with

$$\begin{pmatrix} \hat{A}^N & -(T^{N,M}_{\gamma^M})^T \\ -T^{N,M}_{\gamma^M} & 0 \end{pmatrix} \begin{pmatrix} \underline{\hat{u}}^N_{\gamma^M} \\ \underline{g}^M_{\gamma^M} \end{pmatrix} = \begin{pmatrix} \hat{F}^{N,M}_{\gamma^M} \\ 0 \end{pmatrix}. \tag{4.1.3}$$

Here, $\underline{\hat{u}}^N_{\gamma^M} = (\hat{u}^N_1, \ldots, \hat{u}^N_{(N-2)^2})^T$ and $\underline{g}^M_{\gamma^M} = (g^M_1, \ldots, g^M_{M-1})^T$ are the node vectors of $(\hat{u}^N_{\gamma^M}, \mathcal{G}^M_{\gamma^M}) \in \hat{U}^N \times H_I^M$, the matrix $\hat{A}^N := (\hat{a}^N_{ij})$ with

$$\hat{a}^N_{ij} := \int_{\hat{\Omega}} \nabla \Phi^N_i(x,y)^T \nabla \Phi^N_j(x,y)\, d(x,y), \quad i,j = 1, \ldots, (N-2)^2,$$

is the finite element stiffness matrix. Moreover, $T^{N,M}_{\gamma^M} = (t^{N,M}_{ij})$ with

$$t^{N,M}_{ij} := \langle \chi^M_i, \mathcal{T}_{\gamma^M} \Phi^N_j \rangle_{H_I^*, H_I}, \quad i = 1, \ldots, M-1,\ j = 1, \ldots, (N-2)^2,$$

is called the trace matrix and the vector $\hat{F}^{N,M}_{\gamma^M} = (\hat{f}^{N,M}_i)$ is given by

$$\hat{f}^{N,M}_i := \int_{\hat{\Omega}} \widetilde{f}_{\gamma^M}(x,y)\Phi^N_i(x,y)\, d(x,y), \quad i = 1, \dots, (N-2)^2.$$

Theorem 4.1.1. *Let $\gamma^M \in S^M_I$ of the form* (4.1.1) *satisfies the two conditions*

$$c_i := \sqrt{(\gamma^M_{1,i+1} - \gamma^M_{1,i})^2 + (\gamma^M_{2,i+1} - \gamma^M_{2,i})^2} > 2\sqrt{2}h^N, \quad i = 1, \dots, M-1, \tag{4.1.4}$$

$$\begin{aligned} &\langle \chi^M_i, \mathcal{T}_{\gamma^M}\Phi^N_j \rangle_{H^*_I, H_I} \neq 0, \langle \chi^M_k, \mathcal{T}_{\gamma^M}\Phi^N_j \rangle_{H^*_I, H_I} \neq 0, k \neq i \\ &\qquad \Longrightarrow k = i-1 \vee k = i+1. \end{aligned} \tag{4.1.5}$$

Then the discrete saddle point problem (4.1.3) *is uniquely solvable.*

Proof. The finite element stiffness matrix $\hat{A}^N$ is symmetric and positive definite. Therefore, (4.1.3) is uniquely solvable if and only if $\ker \left(T^{N,M}_{\gamma^M}\right)^T = \{0\}$. We have

$$\begin{aligned} &\ker \left(T^{N,M}_{\gamma^M}\right)^T = \{0\} \Longleftrightarrow \left(T^{N,M}_{\gamma^M}\right)^T \underline{g}^M = 0 \text{ implies } \underline{g}^M = 0 \\ &\quad \Longleftrightarrow \sum_{i=1}^{M-1} g^M_i \langle \chi^M_i, \mathcal{T}_{\gamma^M}\Phi^N_j \rangle_{H^*_I, H_I} = 0 \text{ for } j = 1, \dots, (N-2)^2, \text{ implies } \underline{g}^M = 0. \end{aligned}$$

Since $c_1 > 2\sqrt{2}h^N$ there exists an index $k \in \{1, \dots, (N-2)^2\}$ such that

$$\begin{aligned} &\langle \chi^M_1, \mathcal{T}_{\gamma^M}\Phi^N_k \rangle_{H^*_I, H_I} \neq 0 && \text{and} \\ &\langle \chi^M_i, \mathcal{T}_{\gamma^M}\Phi^N_k \rangle_{H^*_I, H_I} = 0 && \text{for } i = 2, 3, \dots, M-1. \end{aligned}$$

We obtain

$$\sum_{i=1}^{M-1} g^M_i \langle \chi^M_i, \mathcal{T}_{\gamma^M}\Phi^N_k \rangle_{H^*_I, H_I} = g^M_1 \langle \chi^M_1, \mathcal{T}_{\gamma^M}\Phi^N_k \rangle_{H^*_I, H_I} = 0 \quad \Longrightarrow \quad g^M_1 = 0.$$

Proceeding by induction assume that $g^M_1 = \dots = g^M_l = 0$ for some $l \in \{1, \dots, M-1\}$. Then there exists an index $k \in \{1, \dots, (N-2)^2\}$ such that

$$\begin{aligned} &\langle \chi^M_l, \mathcal{T}_{\gamma^M}\Phi^N_k \rangle_{H^*_I, H_I} \neq 0, \\ &\langle \chi^M_{l+1}, \mathcal{T}_{\gamma^M}\Phi^N_k \rangle_{H^*_I, H_I} \neq 0 && \text{and} \\ &\langle \chi^M_i, \mathcal{T}_{\gamma^M}\Phi^N_k \rangle_{H^*_I, H_I} = 0 && \text{for all } i \in \{1, \dots, M-1\} \setminus \{l, l+1\}. \end{aligned}$$

Now we obtain with the induction hypothesis

$$\begin{aligned}\sum_{i=1}^{M-1} g_i^M \langle \chi_i^M, \mathcal{T}_{\gamma^M}\Phi_k^N\rangle_{H_I^*,H_I} &= g_l^M \langle \chi_l^M, \mathcal{T}_{\gamma^M}\Phi_k^N\rangle_{H_I^*,H_I} + g_{l+1}^M \langle \chi_{l+1}^M, \mathcal{T}_{\gamma^M}\Phi_k^N\rangle_{H_I^*,H_I} \\ &= g_{l+1}^M \langle \chi_{l+1}^M, \mathcal{T}_{\gamma^M}\Phi_k^N\rangle_{H_I^*,H_I} = 0 \quad \Longrightarrow \quad g_{l+1}^M = 0.\end{aligned}$$

It follows that $g_1^M = \cdots = g_{M-1}^M = 0$ and therefore $\underline{g}^M = 0$. □

In [GG95] the authors prove error estimates for a slightly different setting. The result is mainly obtained by verifying a uniform inf-sup condition (also known as the Ladyženskaja-Babuška-Brezzi (LBB) condition) assuming that the mesh on Γ_{γ^M} is sufficiently large compared to the size of the grid on $\hat{\Omega}$. Due to our numerical computations we suggest an even higher ratio $N : M$ to avoid oscillations of the discrete Lagrange multiplier $\mathcal{G}^M$. Full details are given in Chapter 5.

4.1.4. The discrete adjoint equation

In this subsection we state the discrete counterpart of the adjoint equation (3.3.6). First of all, we rewrite the right hand side of the first equation of (3.3.6). Using $\Omega_T := \hat{\Omega}$, the notation

$$\hat{u} \mapsto \mathcal{R}_\gamma(\hat{u}) := \widetilde{\hat{u}|_{\Omega_\gamma}} = \begin{cases} \hat{u} & \text{in } \Omega_\gamma, \\ 0 & \text{in } \Omega_\gamma^c, \end{cases}$$

and the fact that $\mathcal{R}_\gamma(\hat{u}_\gamma) = \hat{u}_\gamma = 0$ we obtain

$$(\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_T \cap \Omega_\gamma)} = (\hat{u}_\gamma - u_d, \hat{v})_{L^2(\Omega_\gamma)} = (\hat{u}_\gamma - \mathcal{R}_\gamma(u_d), \hat{v})_{L^2(\hat{\Omega})}. \tag{4.1.6}$$

Let

$$u_d^{\bar{N}} := I^{\bar{N}} u_d = \sum_{i=1}^{(\bar{N}-2)^2} u_{d,i}^{\bar{N}} \Phi_i^{\bar{N}}(x,y) \tag{4.1.7}$$

be an approximation of u_d on a fine grid ($\bar{N} \geq N$), where $I^{\bar{N}} : L^2(\hat{\Omega}) \to \hat{U}^{\bar{N}}$ denotes a continuous interpolation operator.

In view of equation (4.1.6), we implement the following approximation $\mathcal{R}_{\gamma^M}^{\bar{N},M}(u_d^{\bar{N}})$ of $\mathcal{R}_\gamma(u_d)$: Set all coefficients $u_{d,i}^{\bar{N}}$ in (4.1.7) to zero, if the corresponding mesh points are contained in $\Omega_{\gamma^M}^c$. This modified coefficients are denoted by $u_{d,i}^{\bar{N}}|_{\gamma^M}$ and we obtain

$$\mathcal{R}_{\gamma^M}^{\bar{N},M}(u_d^{\bar{N}}) = \sum_{i=1}^{(\bar{N}-2)^2} u_{d,i}^{\bar{N}}|_{\gamma^M} \Phi_i^{\bar{N}}(x,y).$$

Now, let

$$\hat{\lambda}^N(x,y) = \sum_{i=1}^{(N-2)^2} \hat{\lambda}_i^N \Phi_i^N(x,y), \quad \mathcal{M}^M(t) = \sum_{i=1}^{M-1} \mu_i^M \chi_i^M(t),$$

and $\hat{u}^N_{\gamma^M} = \sum_{i=1}^{(N-2)^2} \hat{u}_i^N \Phi_i^N$ be the solution of the discrete state equation (4.1.2). The discrete form of the adjoint system (3.3.6) then reads: Find $(\hat{\lambda}^N_{\gamma^M}, \mathcal{M}^M_{\gamma^M}) = (\hat{\lambda}^N(\gamma^M), \mathcal{M}^M(\gamma^M)) \in \hat{U}^N \times H_I^M$

$$\begin{cases} (\nabla\hat{\lambda}^N_{\gamma^M}, \nabla\hat{v}^N)_{L^2(\hat{\Omega})^2} - \langle \mathcal{M}^M_{\gamma^M}, \mathcal{T}_{\gamma^M}\hat{v}^N\rangle_{H_I^*,H_I} \\ \qquad\qquad = (\hat{u}^N_{\gamma^M} - \mathcal{R}^{\bar{N},M}_{\gamma^M}(u_d^{\bar{N}}), \hat{v}^N)_{L^2(\hat{\Omega})}, \qquad \forall \hat{v}^N \in \hat{U}^N, \\ -\langle \mathcal{H}^M, \mathcal{T}_{\gamma^M}\hat{\lambda}^N_{\gamma^M}\rangle_{H_I^*,H_I} = 0, \qquad\qquad \forall \mathcal{H}^M \in H_I^M. \end{cases} \tag{4.1.8}$$

Note that we have different ansatz functions for $\hat{u}^N_{\gamma^M} \in U^N$ and $u_d^{\bar{N}} \in U^{\bar{N}}$. Let $\bar{N} - 1 = (N-1)2^l$ with $l > 0$. For the ansatz functions on the resulting hierarchical grids we can determine coefficients $r_{ij}^{N,\bar{N}}$, $i = 1,\dots,(N-2)^2$, $j = 1,\dots,(\bar{N}-2)^2$, such that

$$\Phi_i^N(x,y) = \sum_{j=1}^{(\bar{N}-2)^2} r_{ij}^{N,\bar{N}} \Phi_j^{\bar{N}}(x,y), \quad i = 1,\dots,(N-2)^2.$$

or equivalently

$$\underline{\Phi}^N = R^{N,\bar{N}} \underline{\Phi}^{\bar{N}} \tag{4.1.9}$$

with $\underline{\Phi}^N := (\Phi_1^N(x,y),\dots,\hat{\Phi}^N_{(N-2)^2}(x,y))^T$, $\underline{\Phi}^{\bar{N}} := (\Phi_1^{\bar{N}}(x,y),\dots,\hat{\Phi}^{\bar{N}}_{(\bar{N}-2)^2}(x,y))^T$ and the matrix $R^{N,\bar{N}} := \left(r_{ij}^{N,\bar{N}}\right)$.

If we test the first equation of (4.1.8) with $\hat{v}^N = \Phi_i^N$, we obtain for the right hand side

$$\begin{aligned} (\hat{u}^N_{\gamma^M} - \mathcal{R}^{\bar{N},M}_{\gamma^M}(u_d^{\bar{N}}), \Phi_i^N)_{L^2(\hat{\Omega})} &= (\sum_{k=1}^{(N-2)^2} \hat{u}_k^N \Phi_k^N - \sum_{l=1}^{(\bar{N}-2)^2} u^{\bar{N}}_{d,l}|_{\gamma^M} \Phi_l^{\bar{N}}, \Phi_i^N)_{L^2(\hat{\Omega})} \\ &= \sum_{k=1}^{(N-2)^2} \hat{u}_k^N (\Phi_k^N, \Phi_i^N)_{L^2(\hat{\Omega})} - \sum_{j=1}^{(\bar{N}-2)^2} r_{ij}^{N,\bar{N}} \sum_{l=1}^{(\bar{N}-2)^2} u^{\bar{N}}_{d,l}|_{\gamma^M} (\Phi_l^{\bar{N}}, \Phi_j^{\bar{N}})_{L^2(\hat{\Omega})}. \end{aligned}$$

Together with the finite element mass matrices $\hat{B}^N := (\hat{b}^N_{ij})$,

$$\hat{b}^N_{ij} := \int_{\hat{\Omega}} \Phi_i^N(x,y)\Phi_j^N(x,y)\, d(x,y), \quad i,j = 1,\dots,(N-2)^2,$$

and $\hat{B}^{\bar{N}} := (\hat{b}_{ij}^{\bar{N}})$,

$$\hat{b}_{ij}^{\bar{N}} := \int_{\hat{\Omega}} \Phi_i^{\bar{N}}(x,y)\Phi_j^{\bar{N}}(x,y)\,d(x,y), \quad i,j = 1,\dots,(\bar{N}-2)^2,$$

the equations (4.1.8) lead to the system: Find $(\underline{\hat{\lambda}}_{\gamma^M}^N, \underline{\mu}_{\gamma^M}^M) = (\underline{\hat{\lambda}}^N(\gamma^M), \underline{\mu}^M(\gamma^M)) \in \mathbb{R}^{(N-2)^2} \times \mathbb{R}^{M-1}$ with

$$\begin{pmatrix} \hat{A}^N & -(T_{\gamma^M}^{N,M})^T \\ -T_{\gamma^M}^{N,M} & 0 \end{pmatrix} \begin{pmatrix} \underline{\hat{\lambda}}_{\gamma^M}^N \\ \underline{\mu}_{\gamma^M}^M \end{pmatrix} = \begin{pmatrix} \hat{B}^N \underline{\hat{u}}_{\gamma^M}^N - R^{N,\bar{N}}(\hat{B}^{\bar{N}} \underline{u}_d^{\bar{N}}|_{\gamma^M}) \\ 0 \end{pmatrix}, \tag{4.1.10}$$

where $\underline{\hat{\lambda}}_{\gamma^M}^N = (\hat{\lambda}_1^N, \dots, \hat{\lambda}_{(N-2)^2}^N)^T$, and $\underline{\mu}^M := (\mu_1^M, \dots, \mu_{M-1}^M)^T$ are the node vectors of $(\hat{\lambda}_{\gamma^M}^N, \mathcal{M}_{\gamma^M}^M) \in \hat{U}^N \times H_I^M$ and $\underline{u}_d^{\bar{N}}|_{\gamma^M} := (u_{d,1}^{\bar{N}}|_{\gamma^M}, \dots, u_{d,(\bar{N}-2)^2}^{\bar{N}}|_{\gamma^M})^T$ is the node vector of $\mathcal{R}_{\gamma^M}^{\bar{N},M}(u_d^{\bar{N}})$. Clearly, (4.1.10) is uniquely solvable by Theorem 4.1.1.

4.2. Approximation with respect to the reduced objective functional $\hat{\jmath}$

4.2.1. An approximation $\hat{\jmath}^{(N,M)}$ of the reduced objective functional $\hat{\jmath}$

We set $S^M := H^1(I)^2$. For given $\gamma^M \in S_I^M \subset S^M$, let $(\hat{u}_{\gamma^M}^N, \mathcal{G}_{\gamma^M}^M) \in \hat{U}^N \times H_I^M$ be the solution of the discrete state equation (4.1.2), and $u_d^{\bar{N}}$ as in (4.1.7). Then, the function

$$\begin{aligned} \hat{\jmath}^{(N,M)} &: S_I^M \to \mathbb{R}, \\ &\gamma^M \mapsto \frac{1}{2}\int_{\hat{\Omega}} (\hat{u}_{\gamma^M}^N(x,y) - u_d^{\bar{N}}(x,y))^2\,d(x,y), \end{aligned}$$

denote the discrete approximation of the reduced objective functional $\hat{\jmath}$.
Using the relation (4.1.9) given in the previous section, we can evaluate $\hat{\jmath}^{(N,M)}(\gamma^M)$ via

$$\begin{aligned} \hat{\jmath}^{(N,M)}(\gamma^M) &= \frac{1}{2}\|\hat{u}_{\gamma^M}^N - \hat{u}_d^N\|_{L^2(\hat{\Omega})}^2 \\ &= \frac{1}{2}\|(\underline{\hat{u}}_{\gamma^M}^N)^T \underline{\Phi}^N - (\underline{\hat{u}}_d^{\bar{N}})^T \underline{\Phi}^{\bar{N}}\|_{L^2(\hat{\Omega})}^2 \\ &= \frac{1}{2}\|(\underline{\hat{u}}_{\gamma^M}^N)^T R^{N,\bar{N}} \underline{\Phi}^{\bar{N}} - (\underline{\hat{u}}_d^{\bar{N}})^T \underline{\Phi}^{\bar{N}}\|_{L^2(\hat{\Omega})}^2. \end{aligned}$$

With the definition $P^{N,\bar{N}} := \left(R^{N,\bar{N}}\right)^T$ we have

$$\begin{aligned}\hat{\jmath}^{(N,M)}(\gamma^M) &= \frac{1}{2}\|(P^{N,\bar{N}}\underline{\hat{u}}^N_{\gamma^M})^T\underline{\Phi}^{\bar{N}} - (\underline{\hat{u}}^{\bar{N}}_d)^T\underline{\Phi}^{\bar{N}}\|^2_{L^2(\hat{\Omega})} \\ &= \frac{1}{2}\|(P^{N,\bar{N}}\underline{\hat{u}}^N_{\gamma^M} - \underline{\hat{u}}^{\bar{N}}_d)^T\underline{\Phi}^{\bar{N}}\|^2_{L^2(\hat{\Omega})} \\ &= \frac{1}{2}(P^{N,\bar{N}}\underline{\hat{u}}^N_{\gamma^M} - \underline{\hat{u}}^{\bar{N}}_d)^T\hat{B}^{\bar{N}}(P^{N,\bar{N}}\underline{\hat{u}}^N_{\gamma^M} - \underline{\hat{u}}^{\bar{N}}_d).\end{aligned}$$

4.2.2. An approximation $\vec{n}_{\gamma^M}$ of the outer unit normal vector $\vec{n}_\gamma$

For a discretized curve $\gamma^M = \sum_{i=1}^M \vec{\gamma}_i^M B_i^M \in S_I^M$ let $\vec{n}_{\gamma^M} = (n_{1,\gamma^M}, n_{2,\gamma^M})^T$ denote the approximation of the outer unit normal vector $\vec{n}_\gamma$ along the boundary part Γ_γ. In the first place we define unit vectors that are orthogonal to the line connecting the points $\vec{\gamma}_i^M$ and $\vec{\gamma}_{i+1}^M$,

$$\vec{n}_i^{\text{aux}} := \frac{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\left(\vec{\gamma}_{i+1}^M - \vec{\gamma}_i^M\right)}{\left\|\vec{\gamma}_{i+1}^M - \vec{\gamma}_i^M\right\|_2}, \quad i = 1, \ldots, M-1.$$

Now, we define the approximation $\vec{n}_{\gamma^M}$ of the normal vector $\vec{n}_{\gamma^M}$ at t_i, $i = 2, \ldots, M-1$, i.e. in boundary points $\vec{\gamma}_i^M$ of Γ_{γ^M}:

$$\vec{n}_{\gamma^M}(t_i) := \begin{pmatrix} n_{1,i}^M \\ n_{2,i}^M \end{pmatrix} := \frac{\vec{n}_{i-1}^{\text{aux}} + \vec{n}_i^{\text{aux}}}{\left\|\vec{n}_{i-1}^{\text{aux}} + \vec{n}_i^{\text{aux}}\right\|_2}.$$

Moreover, we collect the components $n_{1,i}^M$, $n_{2,i}^M$, in vectors

$$\eta_1^M(\gamma^M) := \left(n_{1,2}^M, \ldots, n_{1,M-1}^M\right)^T \in \mathbb{R}^{M-2}, \tag{4.2.1}$$

$$\eta_2^M(\gamma^M) := \left(n_{2,2}^M, \ldots, n_{2,M-1}^M\right)^T \in \mathbb{R}^{M-2}. \tag{4.2.2}$$

We approximate the set of admissible directions S'_γ by $S'_{\gamma^M} := \operatorname{span}(\varphi^2_{\gamma^M}, \ldots, \varphi^{M-1}_{\gamma^M})$, where

$$\varphi^i_{\gamma^M}(t) := \vec{n}_{\gamma^M}(t_i)B_i^M(t), \quad i = 2, \ldots, M-1.$$

An admissible direction $\bar{\gamma}^M \in S'_{\gamma^M} \subset S_I^M$ can be written as

$$\bar{\gamma}^M(t) = \sum_{j=2}^{M-1} \bar{\gamma}_j^M \varphi^j_{\gamma^M}(t). \tag{4.2.3}$$

4.2.3. An approximation $\hat{\jmath}^{(N,M)\prime}(\gamma^M)$ of the first derivative $\hat{\jmath}'(\gamma)$ of the reduced objective functional $\hat{\jmath}$

For given $\gamma^M \in S_I^M \subset S^M$, let $(\hat{u}^N_{\gamma^M}, \mathcal{G}^M_{\gamma^M}) \in \hat{U}^N \times H_I^M$ be the solution of the discrete state equation (4.1.2), and $(\hat{\lambda}^N_{\gamma^M}, \mathcal{M}^M_{\gamma^M}) \in \hat{U}^N \times H_I^M$ be the solution of the discrete adjoint equation (4.1.8). Then, the function $\hat{\jmath}^{(N,M)\prime}(\gamma^M) : S'_{\gamma^M} \to \mathbb{R}$, with

$$\langle \hat{\jmath}^{(N,M)\prime}(\gamma^M), \bar{\gamma}^M \rangle_{(S^M)^*,S^M} = \int_I \mathcal{G}^M_{\gamma^M}(t)\mathcal{M}^M_{\gamma^M}(t) \frac{\bar{\gamma}^M(t)^T \vec{n}_{\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2}\, dt, \quad \forall \bar{\gamma}^M \in S'_{\gamma^M},$$

denotes the discrete approximation of the Fréchet-derivative $\hat{\jmath}'(\gamma)$.
In the first place we evaluate $\hat{\jmath}^{(N,M)\prime}(\gamma^M)$ at the basis functions of S_I^M, therefore we define the vectors

$$\mathcal{J}_1^{(N,M)}(\gamma^M) := \begin{pmatrix} \langle \hat{\jmath}^{(N,M)\prime}(\gamma^M), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_2^M \rangle_{(S^M)^*,S^M} \\ \vdots \\ \langle \hat{\jmath}^{(N,M)\prime}(\gamma^M), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_{M-1}^M \rangle_{(S^M)^*,S^M} \end{pmatrix} \in \mathbb{R}^{M-2},$$

$$\mathcal{J}_2^{(N,M)}(\gamma^M) := \begin{pmatrix} \langle \hat{\jmath}^{(N,M)\prime}(\gamma^M), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_2^M \rangle_{(S^M)^*,S^M} \\ \vdots \\ \langle \hat{\jmath}^{(N,M)\prime}(\gamma^M), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_{M-1}^M \rangle_{(S^M)^*,S^M} \end{pmatrix} \in \mathbb{R}^{M-2}.$$

Then we obtain the vector

$$\mathcal{J}^{(N,M)}_{\gamma^M} := (\langle \hat{\jmath}^{(N,M)\prime}(\gamma^M), \varphi^2_{\gamma^M} \rangle_{(S^M)^*,S^M}, \dots, \langle \hat{\jmath}^{(N,M)\prime}(\gamma^M), \varphi^{M-1}_{\gamma^M} \rangle_{(S^M)^*,S^M})^T \quad (4.2.4)$$

as the sum of two Hadamard products[1]

$$\mathcal{J}^{(N,M)}_{\gamma^M} = \mathcal{J}_1^{(N,M)}(\gamma^M) \circ \eta_1^M(\gamma^M) + \mathcal{J}_2^{(N,M)}(\gamma^M) \circ \eta_2^M(\gamma^M).$$

4.2.4. An approximation $\nabla\hat{\jmath}^{(N,M)}(\gamma^M)$ of the gradient $\nabla\hat{\jmath}(\gamma)$ of the reduced objective functional $\hat{\jmath}$

The equation (3.6.3) in the finite-dimensional setting reads

$$\langle \hat{\jmath}^{(N,M)\prime}(\gamma^M), s^M \rangle_{(S^M)^*,S^M} = (\nabla\hat{\jmath}^{(N,M)}(\gamma^M), s^M)_{S^M}, \quad \text{for all } s^M \in S'_{\gamma^M}. \quad (4.2.5)$$

[1] Let vectors A, $B \in \mathbb{R}^n$, and vectors $\vec{x}_i = (x_{1,i}, x_{2,i})^T \in \mathbb{R}^2$, $i = 1, \dots, n$ be given. Then, for $Z \in \mathbb{R}^n$ with $Z_i := (\vec{x}_i)^T \begin{pmatrix} A_i \\ B_i \end{pmatrix} = x_{1,i}A_i + x_{2,i}B_i$, $i = 1, \dots, n$, we obtain $Z = A \circ \xi_1 + B \circ \xi_2$, where $\xi_1 := (x_{1,1}, \dots, x_{1,n})^T$ and $\xi_2 := (x_{2,1}, \dots, x_{2,n})^T \in \mathbb{R}^n$. Here, $\cdot \circ \cdot$ denotes the entrywise product (Hadamard product) for matrices, i.e.: $(A \circ B)_{ij} = (A)_{ij} \cdot (B)_{ij}$.

Using the basis $\{\varphi^2_{\gamma^M}, \ldots, \varphi^{M-1}_{\gamma^M}\}$ of S'_{γ^M}, the ansatz $\nabla \hat{j}^{(N,M)}(\gamma^M) = \sum_{i=2}^{M-1} d_i^M \varphi^i_{\gamma^M}$ leads to the following system of equations:

$$\sum_{i=2}^{M-1} d_i^M (\varphi^i_{\gamma^M}, \varphi^j_{\gamma^M})_{S^M} = \langle \hat{j}^{(N,M)\prime}(\gamma^M), \varphi^j_{\gamma^M} \rangle_{(S^M)^*, S^M}, \quad j = 2, \ldots, M-1. \tag{4.2.6}$$

With the definitions (4.2.4) and

$$\widetilde{D}^M_{\gamma^M} := \Big((\varphi^i_{\gamma^M}, \varphi^j_{\gamma^M})_{S^M} \Big)_{i,j=2}^{M-1}$$

we immediately obtain the following linear system to compute the node vector $\underline{d}^M_{\gamma^M} := \underline{d}^M(\gamma^M) = (d_2^M, \ldots, d_{M-1}^M)^T$ of $\nabla \hat{j}^{(N,M)}(\gamma^M)$:

$$\widetilde{D}^M_{\gamma^M} \underline{d}^M_{\gamma^M} = \mathcal{J}^{(N,M)}_{\gamma^M}. \tag{4.2.7}$$

For the (i,j)-th entry of the matrix $\widetilde{D}^M_{\gamma^M}$ we obtain

$$\begin{aligned}
\Big(\widetilde{D}^M_{\gamma^M}\Big)_{ij} &= (\varphi^i_{\gamma^M}, \varphi^j_{\gamma^M})_{S^M} \\
&= (\vec{n}_{\gamma^M}(t_i) B_i^M, \vec{n}_{\gamma^M}(t_j) B_j^M)_{S^M} \\
&= \big(\vec{n}_{\gamma^M}(t_i)\big)^T \begin{pmatrix} (\binom{1}{0} B_i^M, \binom{1}{0} B_j^M)_{S^M} & (\binom{1}{0} B_i^M, \binom{0}{1} B_j^M)_{S^M} \\ (\binom{0}{1} B_i^M, \binom{1}{0} B_j^M)_{S^M} & (\binom{0}{1} B_i^M, \binom{0}{1} B_j^M)_{S^M} \end{pmatrix} \vec{n}_{\gamma^M}(t_j) \\
&= \big(\vec{n}_{\gamma^M}(t_i)\big)^T \begin{pmatrix} \alpha_{ij}^M + \beta_{ij}^M & 0 \\ 0 & \alpha_{ij}^M + \beta_{ij}^M \end{pmatrix} \vec{n}_{\gamma^M}(t_j).
\end{aligned}$$

Here, $\mathcal{A}^M := (\alpha_{ij}^M)$ with

$$\alpha_{ij}^M := \Big(\dot{B}_i^M, \dot{B}_j^M\Big)_{L^2(I)}, \quad i,j = 2, \ldots, M-1,$$

is the finite element stiffness matrix, and $\mathcal{B}^M := (\beta_{ij}^M)$ with

$$\beta_{ij}^M := \big(B_i^M, B_j^M\big)_{L^2(I)}, \quad i,j = 2, \ldots, M-1,$$

is the finite element mass matrix. Therefore, we obtain[2]

$$\widetilde{D}^M_{\gamma^M} = \left(\mathcal{A}^M + \mathcal{B}^M\right) \circ \left(\eta_1^M(\gamma^M)(\eta_1^M(\gamma^M))^T + \eta_2^M(\gamma^M)(\eta_2^M(\gamma^M))^T\right).$$

4.3. Finite-dimensional descent methods

In this section we investigate finite-dimensional approximations of Algorithm 3.6.2.

Algorithm 4.3.1.

0. Choose an initial curve $(\gamma^M)^0 \in S_I^M$, *and a symmetric, positive definite operator* $H_0^M \in \mathcal{L}(S^M)$

For $k = 0, 1, 2, \dots$ *(until convergence...):*

1. If $\hat{\jmath}^{(N,M)\prime}((\gamma^M)^k) = 0$, *STOP.*

2. Compute a descent direction $(s^M)^k \in S'_{(\gamma^M)^k} \subset S_I^M$ *from*

$$H_k^M (s^M)^k = -\nabla \hat{\jmath}^{(N,M)}((\gamma^M)^k). \tag{4.3.1}$$

3. Choose a step size $\sigma_k > 0$ *such that*

$$\hat{\jmath}^{(N,M)}((\gamma^M)^k + \sigma_k (s^M)^k) < \hat{\jmath}^{(N,M)}((\gamma^M)^k).$$

4. Set $(\gamma^M)^{k+1} := (\gamma^M)^k + \sigma_k (s^M)^k$.

5. Choose a symmetric, positive definite operator $H_{k+1}^M \in \mathcal{L}(S^M)$.

4.3.1. Finite-dimensional gradient method

In Algorithm 4.3.1 we choose $H_k^M = \mathrm{id}_{S^M}$ for all k. To obtain the steepest descent direction $(s^M)^k$ in (4.3.1), we compute the gradient $\nabla \hat{\jmath}^{(N,M)}((\gamma^M)^k)$ and set

$$(s^M)^k = -\nabla \hat{\jmath}^{(N,M)}((\gamma^M)^k). \tag{4.3.2}$$

Thus, using the ansatz $(s^M)^k = \sum_{i=2}^{M-1} (s_i^M)^k \varphi^i_{(\gamma^M)^k}$ and equation (4.2.7), we obtain the following relation to compute the node vector $\underline{s}^M_{(\gamma^M)^k} = ((s_2^M)^k, \dots, (s_{M-1}^M)^k)$ of

[2] Let matrices A, B, C, $D \in \mathbb{R}^{n\times n}$, and vectors $\vec{x}_i = (x_{1,i}, x_{2,i})^T \in \mathbb{R}^2$, $i = 1, \dots, n$ be given. Then, for $Z \in \mathbb{R}^{n\times n}$ with

$$Z_{ij} := (\vec{x}_i)^T \begin{pmatrix} A_{ij} & B_{ij} \\ C_{ij} & D_{ij} \end{pmatrix} \vec{x}_j = x_{1,i}A_{ij}x_{1,j} + x_{1,i}B_{ij}x_{2,j} + x_{2,i}C_{ij}x_{1,j} + x_{2,i}D_{ij}x_{2,j}$$

we obtain $Z = A \circ \left(\xi_1(\xi_1)^T\right) + B \circ \left(\xi_1(\xi_2)^T\right) + C \circ \left(\xi_2(\xi_1)^T\right) + D \circ \left(\xi_2(\xi_2)^T\right)$, where $\xi_1 := (x_{1,1}, \dots, x_{1,n})^T$ and $\xi_2 := (x_{2,1}, \dots, x_{2,n})^T \in \mathbb{R}^n$.

the finite-dimensional steepest descent direction $(s^M)^k$. We have

$$\underline{s}^M_{(\gamma^M)^k} = -\underline{d}^M_{(\gamma^M)^k},$$

or equivalently,

$$\widetilde{D}^M_{(\gamma^M)^k}\underline{s}^M_{(\gamma^M)^k} = -\mathcal{J}^{(N,M)}_{(\gamma^M)^k}. \tag{4.3.3}$$

4.3.2. Finite-dimensional BFGS quasi-Newton method

In Algorithm 4.3.1 we choose $H_0^M = \mathrm{id}_{S^M}$ and use the discrete version of the BFGS operator update (3.6.6), i.e.

$$H^M_{k+1} := H^M_k + \frac{((w^M)^k, \cdot\,)_{S^M}}{((w^M)^k, (v^M)^k)_{S^M}}(w^M)^k - \frac{(H^M_k(v^M)^k, \cdot\,)_{S^M}}{(H^M_k(v^M)^k, (v^M)^k)_{S^M}}H^M_k(v^M)^k, \tag{4.3.4}$$

at which

$$\begin{aligned}(v^M)^k &:= (\gamma^M)^{k+1} - (\gamma^M)^k,\\ (w^M)^k &:= \nabla\hat{\jmath}^{(N,M)}((\gamma^M)^{k+1}) - \nabla\hat{\jmath}^{(N,M)}((\gamma^M)^k).\end{aligned}$$

At the kth iteration, we use the basis $\{\varphi^2_{(\gamma^M)^k}, \dots, \varphi^{M-1}_{(\gamma^M)^k}\}$ of $S'_{(\gamma^M)^k}$ and the ansatz $(s^M)^k = \sum_{i=2}^{M-1}(s_i^M)^k\varphi^i_{(\gamma^M)^k}$. Then, with the relation (4.2.5) the equation (4.3.1) in Algorithm 4.3.1 leads to the following system of equations:

$$\begin{aligned}&\sum_{i=2}^{M-1}(s_i^M)^k(H^M_k\varphi^i_{(\gamma^M)^k}, \varphi^j_{(\gamma^M)^k})_{S^M}\\ &\qquad = -\langle \hat{\jmath}^{(N,M)\prime}((\gamma^M)^k), \varphi^j_{(\gamma^M)^k}\rangle_{(S^M)^*, S^M}, \quad j = 2, \dots, M-1.\end{aligned} \tag{4.3.5}$$

With (4.2.4) and the definition

$$\widetilde{H}^M_{(\gamma^M)^k} := \Big((H^M_k\varphi^i_{(\gamma^M)^k}, \varphi^j_{(\gamma^M)^k})_{S^M}\Big)_{i,j=2}^{M-1} \tag{4.3.6}$$

we obtain for (4.3.5) the linear system

$$\widetilde{H}^M_{(\gamma^M)^k}\underline{s}^M_{(\gamma^M)^k} = -\mathcal{J}^{(N,M)}_{(\gamma^M)^k} \tag{4.3.7}$$

to compute the node vector $\underline{s}^M_{(\gamma^M)^k} := ((s_2^M)^k, \dots, (s^M_{M-1})^k)^T$ of the finite-dimensional quasi-Newton direction $(s^M)^k$.
For $k = 0$ and $H_0^M = \mathrm{id}_{S^M}$ we obtain $\widetilde{H}^M_{(\gamma^M)^0} = \widetilde{D}^M_{(\gamma^M)^0}$. In the following we describe the update-process of the matrix $\widetilde{H}^M_{(\gamma^M)^k}$.

First of all the update for the operator H_k^M leads to an update for the matrices

$$\bar{H}_k^{1,1} := \Big((H_k^M \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_i^M, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_j^M)_{S^M}\Big)_{i,j=2}^{M-1}, \quad \bar{H}_k^{1,2} := \Big((H_k^M \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_i^M, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_j^M)_{S^M}\Big)_{i,j=2}^{M-1},$$
$$\bar{H}_k^{2,1} := \Big((H_k^M \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_i^M, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_j^M)_{S^M}\Big)_{i,j=2}^{M-1}, \quad \bar{H}_k^{2,2} := \Big((H_k^M \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_i^M, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_j^M)_{S^M}\Big)_{i,j=2}^{M-1},$$

which are independent of $(\gamma^M)^k$. Indeed, the (i,j)-th entry of the matrix $\widetilde{H}^M_{(\gamma^M)^{k+1}}$ is given by

$$\begin{aligned}
\Big(\widetilde{H}^M_{(\gamma^M)^{k+1}}\Big)_{ij} &= (H_{k+1}^M \varphi^i_{(\gamma^M)^{k+1}}, \varphi^j_{(\gamma^M)^{k+1}})_{S^M} \\
&= \big(\vec{n}_{(\gamma^M)^{k+1}}(t_i)\big)^T \begin{pmatrix} \big(\bar{H}_{k+1}^{1,1}\big)_{ij} & \big(\bar{H}_{k+1}^{1,2}\big)_{ij} \\ \big(\bar{H}_{k+1}^{2,1}\big)_{ij} & \big(\bar{H}_{k+1}^{2,2}\big)_{ij} \end{pmatrix} \vec{n}_{(\gamma^M)^{k+1}}(t_j),
\end{aligned}$$

i.e.

$$\begin{aligned}
\widetilde{H}^M_{(\gamma^M)^{k+1}} = \bar{H}_{k+1}^{1,1} &\circ \big(\eta_1^M((\gamma^M)^{k+1})(\eta_1^M((\gamma^M)^{k+1}))^T\big) \\
+ \bar{H}_{k+1}^{1,2} &\circ \big(\eta_1^M((\gamma^M)^{k+1})(\eta_2^M((\gamma^M)^{k+1}))^T\big) \\
+ \bar{H}_{k+1}^{2,1} &\circ \big(\eta_2^M((\gamma^M)^{k+1})(\eta_1^M((\gamma^M)^{k+1}))^T\big) \\
+ \bar{H}_{k+1}^{2,2} &\circ \big(\eta_2^M((\gamma^M)^{k+1})(\eta_2^M((\gamma^M)^{k+1}))^T\big).
\end{aligned}$$

The choice of $H_0^M = \mathrm{id}_{S^M}$ leads to $\bar{H}_0^{1,1} = H_0^{2,2} = \mathcal{A}^M + \mathcal{B}^M$ and $\bar{H}_k^{1,2} = \bar{H}_k^{2,1} = 0 \in \mathbb{R}^{(M-2)\times(M-2)}$. With the update formula (4.3.4) we derive update formulas for the matrices $\bar{H}_k^{1,1}$, $\bar{H}_k^{1,2}$, $\bar{H}_k^{2,1}$ and $\bar{H}_k^{2,2}$. For the (i,j)-th entry of the matrix $\bar{H}_{k+1}^{1,1}$ we obtain

$$\begin{aligned}
\big(\bar{H}_{k+1}^{1,1}\big)_{ij} &= (H_{k+1}^M \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_i^M, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_j^M)_{S^M} \\
&= (H_k^M \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_i^M, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_j^M)_{S^M} + \frac{((w^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_i^M)_{S^M}((w^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_j^M)_{S^M}}{((w^M)^k, (v^M)^k)_{S^M}} \\
&\quad - \frac{(H_k^M (v^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_i^M)_{S^M}(H_k^M (v^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_j^M)_{S^M}}{(H_k^M (v^M)^k, (v^M)^k)_{S^M}}.
\end{aligned}$$

With the definitions

$$\begin{aligned}
(\omega_1^M)^k &:= \Big(((w^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_2^M)_{S^M}, \ldots, ((w^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_{M-1}^M)_{S^M}\Big)^T, \\
(\omega_2^M)^k &:= \Big(((w^M)^k, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_2^M)_{S^M}, \ldots, ((w^M)^k, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_{M-1}^M)_{S^M}\Big)^T, \\
(\nu_1^M)^k &:= \Big((H_k^M (v^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_2^M)_{S^M}, \ldots, (H_k^M (v^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_{M-1}^M)_{S^M}\Big)^T, \\
(\nu_2^M)^k &:= \Big((H_k^M (v^M)^k, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_2^M)_{S^M}, \ldots, (H_k^M (v^M)^k, \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_{M-1}^M)_{S^M}\Big)^T,
\end{aligned}$$

this leads to

$$\bar{H}^{1,1}_{k+1} = \bar{H}^{1,1}_k + \frac{(\omega^M_1)^k \left((\omega^M_1)^k\right)^T}{((w^M)^k, (v^M)^k)_{S^M}} - \frac{(\nu^M_1)^k \left((\nu^M_1)^k\right)^T}{(H^M_k (v^M)^k, (v^M)^k)_{S^M}},$$

and analogously

$$\begin{aligned}
\bar{H}^{1,2}_{k+1} &= \bar{H}^{1,2}_k + \frac{(\omega^M_1)^k \left((\omega^M_2)^k\right)^T}{((w^M)^k, (v^M)^k)_{S^M}} - \frac{(\nu^M_1)^k \left((\nu^M_2)^k\right)^T}{(H^M_k (v^M)^k, (v^M)^k)_{S^M}},\\
\bar{H}^{2,1}_{k+1} &= \bar{H}^{2,1}_k + \frac{(\omega^M_2)^k \left((\omega^M_1)^k\right)^T}{((w^M)^k, (v^M)^k)_{S^M}} - \frac{(\nu^M_2)^k \left((\nu^M_1)^k\right)^T}{(H^M_k (v^M)^k, (v^M)^k)_{S^M}},\\
\bar{H}^{2,2}_{k+1} &= \bar{H}^{2,2}_k + \frac{(\omega^M_2)^k \left((\omega^M_2)^k\right)^T}{((w^M)^k, (v^M)^k)_{S^M}} - \frac{(\nu^M_2)^k \left((\nu^M_2)^k\right)^T}{(H^M_k (v^M)^k, (v^M)^k)_{S^M}}.
\end{aligned}$$

In the following, we provide formulas for the vectors $(\omega^M_1)^k$, $(\omega^M_2)^k$, $(\nu^M_1)^k$, and $(\nu^M_2)^k$, as well as for the values $((w^M)^k, (v^M)^k)_{S^M}$ and $(H^M_k (v^M)^k, (v^M)^k)_{S^M}$. We have

$$\begin{aligned}
(\omega^M_1)^k &= \left(((w^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_2)_{S^M}, \dots, ((w^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_{M-1})_{S^M}\right)^T\\
&= \left((\nabla \hat{j}^{(N,M)}((\gamma^M)^{k+1}), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_2)_{S^M}, \dots, (\nabla \hat{j}^{(N,M)}((\gamma^M)^{k+1}), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_{M-1})_{S^M}\right)^T\\
&\quad - \left((\nabla \hat{j}^{(N,M)}((\gamma^M)^{k}), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_2)_{S^M}, \dots, (\nabla \hat{j}^{(N,M)}((\gamma^M)^{k}), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_{M-1})_{S^M}\right)^T\\
&= \mathcal{J}^{(N,M)}_1((\gamma^M)^{k+1}) - \mathcal{J}^{(N,M)}_1((\gamma^M)^k),\\
(\omega^M_2)^k &= \mathcal{J}^{(N,M)}_2((\gamma^M)^{k+1}) - \mathcal{J}^{(N,M)}_2((\gamma^M)^k).
\end{aligned}$$

With $(v^M)^k = (\gamma^M)^{k+1} - (\gamma^M)^k = \sigma_k (s^M)^k = \sigma_k \sum_{i=2}^{M-1} (s^M_i)^k \varphi^i_{(\gamma^M)^k}$ we obtain

$$\begin{aligned}
(\nu^M_1)^k &= \left((H^M_k (v^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_2)_{S^M}, \dots, (H^M_k (v^M)^k, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_{M-1})_{S^M}\right)^T\\
&= \sigma_k \sum_{i=2}^{M-1} (s^M_i)^k \left((H_k \varphi^i_{(\gamma^M)^k}, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_2)_{S^M}, \dots, (H_k \varphi^i_{(\gamma^M)^k}, \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B^M_{M-1})_{S^M}\right)^T\\
&= \sigma_k \left(\left((\underline{s}^M)^k \circ \eta^M_1((\gamma^M)^k)\right)^T \bar{H}^{1,1}_k + \left((\underline{s}^M)^k \circ \eta^M_2((\gamma^M)^k)\right)^T \bar{H}^{2,1}_k\right)^T,\\
(\nu^M_2)^k &= \sigma_k \left(\left((\underline{s}^M)^k \circ \eta^M_1((\gamma^M)^k)\right)^T \bar{H}^{1,2}_k + \left((\underline{s}^M)^k \circ \eta^M_2((\gamma^M)^k)\right)^T \bar{H}^{2,2}_k\right)^T.
\end{aligned}$$

For the expression $((w^M)^k, (v^M)^k)_{S^M}$ we get

$$\begin{aligned}
((w^M)^k, (v^M)^k)_{S^M} &= (\nabla \hat{j}^{(N,M)}((\gamma^M)^{k+1}) - \nabla \hat{j}^{(N,M)}((\gamma^M)^k), (\gamma^M)^{k+1} - (\gamma^M)^k)_{S^M}\\
&= \sigma_k \sum_{i=2}^{M-1} (s^M_i)^k (\nabla \hat{j}^{(N,M)}((\gamma^M)^{k+1}) - \nabla \hat{j}^{(N,M)}((\gamma^M)^k), \varphi^i_{(\gamma^M)^k})_{S^M}\\
&= \sigma_k \left((\underline{s}^M)^k\right)^T \left((\omega^M_1)^k \circ \eta^M_1((\gamma^M)^k) + (\omega^M_2)^k \circ \eta^M_2((\gamma^M)^k)\right),
\end{aligned}$$

and finally we obtain

$$
\begin{aligned}
(H_k^M(v^M)^k, (v^M)^k)_{S^M} &= \sigma_k^2 (H_k^M (s^M)^k, (s^M)^k)_{S^M} \\
&= \sigma_k^2 \sum_{i=2}^{M-1} \sum_{j=2}^{M-1} (s_i^M)^k (H_k^M \varphi^i_{(\gamma^M)^k}, \varphi^j_{(\gamma^M)^k})_{S^M} (s_j^M)^k = \sigma_k^2 \left((\underline{s}^M)^k \right)^T \widetilde{H}^M_{(\gamma^M)^k} (\underline{s}^M)^k.
\end{aligned}
$$

4.3.3. Finite-dimensional inexact Newton-like method

In this subsection we outline the procedure to compute the node vector of a descent direction candidate based on the symmetric directional derivative $d^{\pm}\hat{\jmath}'(\gamma,\tilde{\gamma})$, compare (3.6.8). Therefore we write down discrete versions of the state sensitivity equation (3.5.2) and the adjoint state sensitivity equation (3.5.6). Then, we formulate a discrete approximation of $d^{\pm}\hat{\jmath}'(\gamma,\tilde{\gamma})$ given in Theorem 3.5.3. This is carried out with two simplifications: On the one hand we neglect the terms $l_1^{\pm}(\hat{v}, f, \gamma, \tilde{\gamma})$ in (3.5.2) and $l_2^{\pm}(\hat{v}, u_d, \gamma, \tilde{\gamma}, \Omega_T)$ in (3.5.6) in our approximation, on the other hand we use the same ansatz functions, namely $(\hat{v}^N, \mathcal{H}^M) \in \hat{U}^N \times H_I^M$, for the test functions as in the discrete state equation from Subsection 4.1.3.
The discrete versions of the state sensitivity equation (3.5.2) and the adjoint state sensitivity equation (3.5.6) read as follows: For $\gamma^M \in S_I^M$, $\Omega_T = \hat{\Omega}$, let $(\hat{u}^N_{\gamma^M}, \mathcal{G}^M_{\gamma^M}) \in \hat{U}^N \times H_I^M$ be the solution of (4.1.2) and $(\hat{\lambda}^N_{\gamma^M}, \mathcal{M}^M_{\gamma^M}) \in \hat{U}^N \times H_I^M$ be the solution of (4.1.2). Then, for given $\tilde{\gamma}^M \in S'_{\gamma^M}$ find $(\delta^{\pm}_{\tilde{\gamma}^M} \hat{u}^N_{\gamma^M}, \delta^{\pm}_{\tilde{\gamma}^M} \mathcal{G}^M_{\gamma^M}) \in \hat{U}^N \times H_I^M$ such that

$$
\begin{cases}
(\nabla \delta^{\pm}_{\tilde{\gamma}^M} \hat{u}^N_{\gamma^M}, \nabla \hat{v}^N)_{L^2(\hat{\Omega})^2} - \langle \delta^{\pm}_{\tilde{\gamma}^M} \mathcal{G}^M_{\gamma^M}, \mathcal{T}_{\gamma^M} \hat{v}^N \rangle_{H_I^*, H_I} & \\
\qquad = \displaystyle\int_I \mathcal{G}^M_{\gamma^M}(t) \nabla \hat{v}^N (\gamma^M(t))^T \tilde{\gamma}^M(t)\, dt, & \forall \hat{v}^N \in \hat{U}^N, \\
- \langle \mathcal{H}^M, \mathcal{T}_{\gamma^M} \delta^{\pm}_{\tilde{\gamma}^M} \hat{u}^N_{\gamma^M} \rangle_{H_I^*, H_I} = \dfrac{1}{2} \displaystyle\int_I \mathcal{H}^M(t) \mathcal{G}^M_{\gamma^M}(t) \frac{\tilde{\gamma}^M(t)^T n_{\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2}\, dt, & \forall \mathcal{H}^M \in H_I^M.
\end{cases}
$$

As well, find $(\delta^{\pm}_{\tilde{\gamma}^M} \hat{\lambda}^N_{\gamma^M}, \delta^{\pm}_{\tilde{\gamma}^M} \mathcal{M}^M_{\gamma^M}) \in \hat{U}^N \times H_I^M$ such that

$$
\begin{cases}
(\nabla \delta^{\pm}_{\tilde{\gamma}^M} \hat{\lambda}^N_{\gamma^M}, \nabla \hat{v}^N)_{L^2(\hat{\Omega})^2} - \langle \delta^{\pm}_{\tilde{\gamma}^M} \mathcal{M}^M_{\gamma^M}, \mathcal{T}_{\gamma^M} \hat{v}^N \rangle_{H_I^*, H_I} & \\
\qquad = \displaystyle\int_I \mathcal{M}^M_{\gamma^M}(t) \nabla \hat{v}^N (\gamma^M(t))^T \tilde{\gamma}^M(t)\, dt + \int_{\hat{\Omega}} \delta^{\pm}_{\tilde{\gamma}^M} \hat{u}^N_{\gamma^M}(x) \hat{v}^N(x)\, dx, & \forall \hat{v}^N \in \hat{U}^N, \\
- \langle \mathcal{H}^M, \mathcal{T}_{\gamma^M} \delta^{\pm}_{\tilde{\gamma}^M} \hat{\lambda}^N_{\gamma^M} \rangle_{H_I^*, H_I} = \dfrac{1}{2} \displaystyle\int_I \mathcal{H}^M(t) \mathcal{M}^M_{\gamma^M}(t) \frac{\tilde{\gamma}^M(t)^T n_{\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2}\, dt, & \forall \mathcal{H}^M \in H_I^M.
\end{cases}
$$

Moreover, let the function $d^{\pm}\hat{\jmath}^{(N,M)\prime}(\gamma^M, \tilde{\gamma}^M) : S'_{\gamma^M} \to \mathbb{R}$, with

$$
\begin{aligned}
&\langle d^{\pm}\hat{\jmath}^{(N,M)\prime}(\gamma^M, \tilde{\gamma}^M), \bar{\gamma}^M \rangle_{(S^M)^*, S^M} \\
&\qquad = \int_I (\delta^{\pm}_{\tilde{\gamma}^M} \mathcal{G}^M_{\gamma^M}(t) \mathcal{M}^M_{\gamma^M}(t) + \mathcal{G}^M_{\gamma^M}(t) \delta^{\pm}_{\tilde{\gamma}^M} \mathcal{M}^M_{\gamma^M}(t)) \frac{\bar{\gamma}^M(t)^T \vec{n}_{\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2} \, dt \\
&\quad + \int_I \mathcal{G}^M_{\gamma^M}(t) \mathcal{M}^M_{\gamma^M}(t) \frac{\bar{\gamma}^M(t)^T \left(\vec{n}_{\tilde{\gamma}^M}(t) - 2\vec{n}_{\gamma^M}(t) \vec{n}_{\gamma^M}(t)^T \vec{n}_{\tilde{\gamma}^M}(t)\right) \|\dot{\tilde{\gamma}}^M(t)\|_2}{\|\dot{\gamma}^M(t)\|_2^2} \, dt, \\
&\hspace{28em} \forall \bar{\gamma}^M \in S'_{\gamma^M},
\end{aligned}
$$

denote the discrete approximation of the symmetric directional derivative $d^{\pm}\hat{\jmath}'(\gamma, \tilde{\gamma})$ (which itself is an approximation of operator-vector-products $\hat{\jmath}''(\gamma)\tilde{\gamma}$).
In view of (3.6.8), the equation (4.3.1) in Algorithm 4.3.1 leads to the following system of equations:

$$
\begin{aligned}
&\langle d^{\pm}\hat{\jmath}^{(N,M)\prime}((\gamma^M)^k, (s^M)^k), \varphi^j_{(\gamma^M)^k} \rangle_{(S^M)^*, S^M} \\
&\qquad\qquad = -\langle \hat{\jmath}^{(N,M)\prime}((\gamma^M)^k), \varphi^j_{(\gamma^M)^k} \rangle_{(S^M)^*, S^M}, \quad j = 2, \dots, M-1.
\end{aligned}
\tag{4.3.8}
$$

Now, we define the vectors

$$
\begin{aligned}
d\mathcal{J}_1^{(N,M)}(\gamma^M)(\tilde{\gamma}^M) &:= \begin{pmatrix} \langle d^{\pm}\hat{\jmath}^{(N,M)\prime}(\gamma^M, \tilde{\gamma}^M), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_2^M \rangle_{(S^M)^*, S^M} \\ \vdots \\ \langle d^{\pm}\hat{\jmath}^{(N,M)\prime}(\gamma^M, \tilde{\gamma}^M), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_{M-1}^M \rangle_{(S^M)^*, S^M} \end{pmatrix} \in \mathbb{R}^{M-2}, \\
d\mathcal{J}_2^{(N,M)}(\gamma^M)(\tilde{\gamma}^M) &:= \begin{pmatrix} \langle d^{\pm}\hat{\jmath}^{(N,M)\prime}(\gamma^M, \tilde{\gamma}^M), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_2^M \rangle_{(S^M)^*, S^M} \\ \vdots \\ \langle d^{\pm}\hat{\jmath}^{(N,M)\prime}(\gamma^M, \tilde{\gamma}^M), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_{M-1}^M \rangle_{(S^M)^*, S^M} \end{pmatrix} \in \mathbb{R}^{M-2},
\end{aligned}
$$

similar to the case of the first derivative, cf Subsection 4.2.3. Then, as in (4.2.4) we obtain the vector

$$
\begin{aligned}
d\mathcal{J}_{\gamma^M}^{(N,M)}(\tilde{\gamma}^M) &:= \begin{pmatrix} \langle d^{\pm}\hat{\jmath}^{(N,M)\prime}(\gamma^M, \tilde{\gamma}^M), \varphi^2_{\gamma^M} \rangle_{(S^M)^*, S^M} \\ \vdots \\ \langle d^{\pm}\hat{\jmath}^{(N,M)\prime}(\gamma^M, \tilde{\gamma}^M), \varphi^{M-1}_{\gamma^M} \rangle_{(S^M)^*, S^M} \end{pmatrix} \in \mathbb{R}^{M-2} \\
&= d\mathcal{J}_1^{(N,M)}(\gamma^M)(\tilde{\gamma}^M) \circ \eta_1^M(\gamma^M) + d\mathcal{J}_2^{(N,M)}(\gamma^M)(\tilde{\gamma}^M) \circ \eta_2^M(\gamma^M).
\end{aligned}
$$

Finally, with the definition of the mapping

$$
\widetilde{K}^M_{\gamma^M} : \mathbb{R}^{M-2} \to S'_{\gamma^M} \to \mathbb{R}^{M-2} \tag{4.3.9}
$$

$$
\underline{\tilde{\gamma}}^M \mapsto \tilde{\gamma}^M \mapsto d\mathcal{J}_{\gamma^M}^{(N,M)}(\tilde{\gamma}^M), \tag{4.3.10}
$$

the system of equations (4.3.8) can be rewritten as

$$\widetilde{K}^M_{(\gamma^M)^k}(\underline{s}^M_{(\gamma^M)^k}) = -\mathcal{J}^{(N,M)}_{(\gamma^M)^k}. \tag{4.3.11}$$

Iterative methods which can handle operator-vector-products $\widetilde{K}^M_{(\gamma^M)^k}(\underline{\tilde{\gamma}}^M)$ are applied to solve (4.3.11) numerically. We summarize that at the kth iterate $(\gamma^M)^k$ we have to solve the discrete state and adjoint equation once. Then, for each operator-vector-product $\widetilde{K}^M_{(\gamma^M)^k}(\underline{\tilde{\gamma}}^M)$ we have to solve the discrete state sensitivity and adjoint state sensitivity equation once.

5. Numerical experiments

5.1. Discrete fictitious domain formulation

We study the stability of the linear system (4.1.3) by numerical experiments. Given a fixed discretization parameter N, we compute the numerical solutions $(\hat{u}^N_{\gamma^M}, \mathcal{G}^M_{\gamma^M}) \in \hat{V}^N \times H^M$ for different values of M. With increasing values of M we improve the approximation γ^M of γ, but we observe escalating oscillations of the Lagrange multiplier $\mathcal{G}^M_{\gamma^M}$, see Figure 5.1, cf. [KP01].

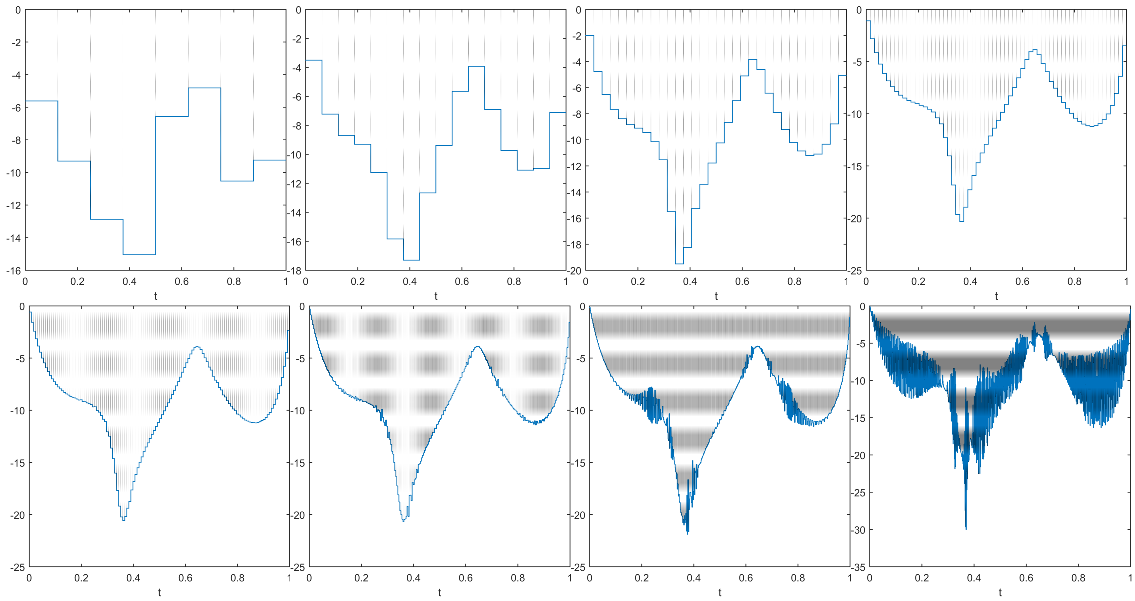


Figure 5.1.: The Lagrange multiplier $\mathcal{G}^M_{\gamma^M}$ for a fixed parameter N and different values of M.

As presented in Figure 5.2, the oscillations of $\mathcal{G}^M_{\gamma^M}$ do not only depend on the ratio of N and M, but also on the intersection configuration of γ^M with the underlying mesh for $\hat{u}^N_{\gamma^M}$.

In general, for $\gamma \in S_{\mathrm{ad}}$ the exact solution $(\hat{u}_\gamma, \mathcal{G}_\gamma)$ is not known, although we can easily construct an exact solution in the case where Γ_γ is the graph of a given function, or γ is a closed curve. According to [HPUU08] we compute alternatively a reference solution $(\hat{u}^{\bar{N}}_d, \mathcal{G}^{\bar{M}}_d) := (\hat{u}^{\bar{N}}_{\gamma^{\bar{M}}}, \mathcal{G}^{\bar{M}}_{\gamma^{\bar{M}}})$ on a fine grid where no oscillations of $\mathcal{G}_d$ occur ($\bar{N} = 2^{12} + 1 = 4097$ and $\bar{M} = 2^8 + 1 = 257$). Then we evaluate the errors

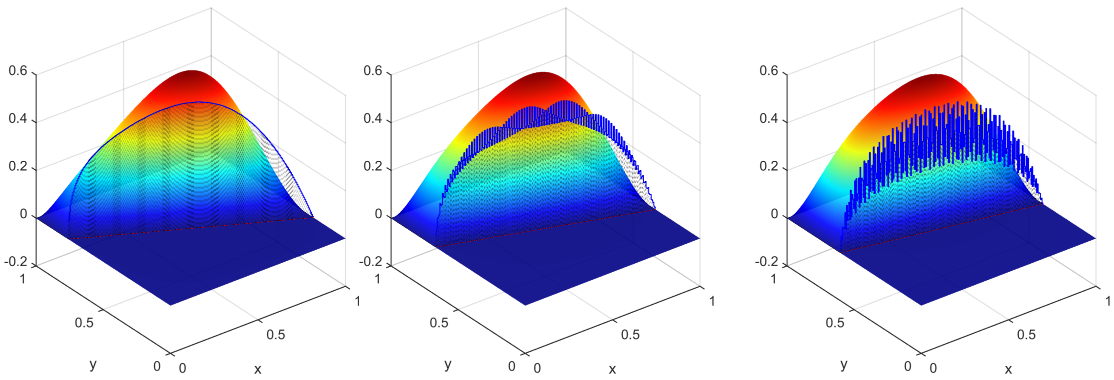
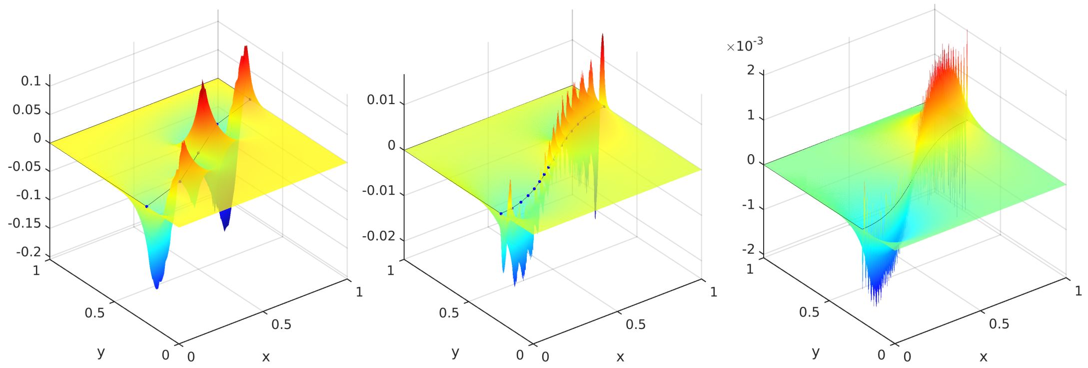


Figure 5.2.: Numerical solutions $(\hat{u}^N_{\gamma^M}, \mathcal{G}^M_{\gamma^M})$ where γ^M intersects the underlying mesh in different angels.



(a) $N = 65$, $M = 5$ (b) $N = 257$, $M = 17$ (c) $N = 1025$, $M = 65$

Figure 5.3.: Error functions $\hat{u}^{\bar{N}}_d - \hat{u}^N_{\gamma^M}$ on different discretization levels.

$\|\hat{u}^{\bar{N}}_d - \hat{u}^N_{\gamma^M}\|$ and $\|\mathcal{G}^{\bar{M}}_d - \mathcal{G}^M_{\gamma^M}\|$ for different norms. The error $\hat{u}^{\bar{N}}_d - \hat{u}^N_{\gamma^M}$ is maximal in the neighborhood of Γ_{γ^M}, see Figure 5.3.

In the following we introduce the *Experimental Order of Convergence* (EOC). For a function w and its numerical approximation w_h we assume an error estimation of the form $\|w_h - w\| \leq Ch^\alpha$ in an appropriate norm $\|\cdot\|$ with a convergence rate α. Now we can estimate the convergence rate α via

$$\alpha \approx \mathrm{EOC}_w(h_1, h_2) := \frac{\log(\|w_{h_1} - w\|) - \log(\|w_{h_2} - w\|)}{log(h_1) - log(h_2)}, \tag{5.1.1}$$

where w_{h_1} and w_{h_2} denote two numerical solutions for different discretization parameters h_1 and h_2. We use the abbreviations $E_{w_{L^2}}$ for the error in the L^2-norm, $E_{w_{\mathrm{sup}}}$ for the error in the L^∞-norm and $E_{w_{\mathrm{sem}}}$ for the error in the H^1-seminorm.

In Table 5.1 and Table 5.2 we present the EOCs of the state variables $\hat{u}^N_{\gamma^M}$ and $\mathcal{G}^M_{\gamma^M}$ for two different examples of γ^M.

N	M	$E_{u_{L^2}}$	EOC	$E_{u_{\sup}}$	EOC	$E_{u_{\text{sem}}}$	EOC	$E_{\mathcal{G}_{L^2}}$	EOC	$E_{\mathcal{G}_{\sup}}$	EOC
257	17	0.002191	-	0.008606	-	0.190050	-	0.4966	-	3.1315	-
513	33	0.001031	1.087	0.003072	1.486	0.128437	0.565	0.2615	0.925	1.9227	0.704
1025	65	0.000443	1.219	0.001326	1.212	0.083734	0.617	0.1330	0.975	1.0420	0.884
2049	129	0.000148	1.584	0.000569	1.222	0.048260	0.795	0.0616	1.111	0.4235	1.299
	∅		1.297		1.307		0.659		1.004		0.962
129	17	0.004571	-	0.013572	-	0.273752	-	0.5096	-	3.2372	-
257	33	0.002218	1.043	0.006560	1.049	0.188423	0.539	0.2702	0.915	2.0029	0.693
513	65	0.001035	1.099	0.003143	1.061	0.128101	0.557	0.1423	0.925	1.0971	0.868
1025	129	0.000444	1.222	0.001399	1.168	0.083671	0.614	0.0769	0.889	0.4424	1.310
	∅		1.122		1.093		0.570		0.910		0.957
65	17	0.009383	-	0.026943	-	0.395202	-	0.5398	-	3.4111	-
129	33	0.004598	1.029	0.013459	1.001	0.272786	0.535	0.2852	0.921	2.1064	0.695
257	65	0.002222	1.049	0.006597	1.029	0.188227	0.535	0.1506	0.921	1.1602	0.860
513	129	0.001036	1.101	0.003125	1.078	0.128065	0.556	0.0832	0.856	0.5039	1.203
	∅		1.060		1.036		0.542		0.899		0.920

Table 5.1.: Errors and EOCs for $\hat{u}^N_{\gamma^M}$ and $\mathcal{G}^M_{\gamma^M}$ (first Example).

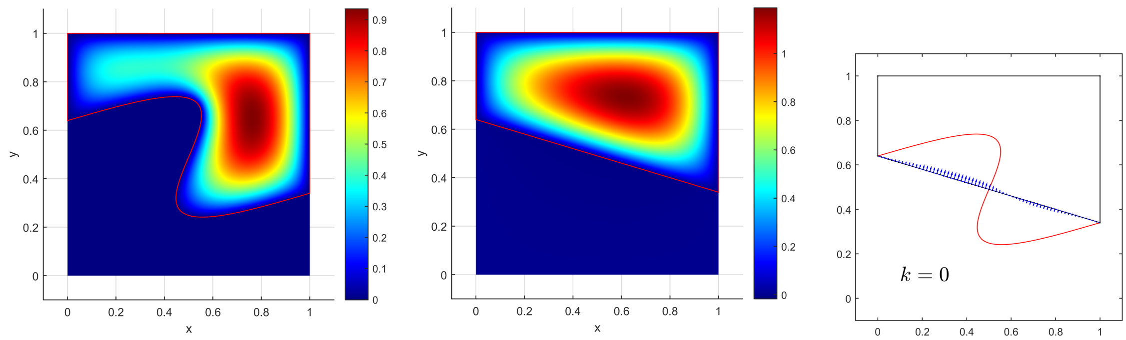
N	M	$E_{u_{L^2}}$	EOC	$E_{u_{\sup}}$	EOC	$E_{u_{\text{sem}}}$	EOC	$E_{\mathcal{G}_{L^2}}$	EOC	$E_{\mathcal{G}_{\sup}}$	EOC
257	17	0.030079	-	0.179247	-	0.979726	-	3.0322	-	18.1031	-
513	33	0.007580	1.989	0.072414	1.308	0.469020	1.063	1.7365	0.804	12.5769	0.525
1025	65	0.001902	1.995	0.024563	1.560	0.214587	1.128	0.8847	0.973	7.0653	0.832
2049	129	0.000410	2.213	0.006368	1.948	0.090080	1.252	0.4087	1.114	2.8637	1.303
	∅		2.066		1.605		1.148		0.964		0.887
129	17	0.030286	-	0.176336	-	0.962182	-	3.0521	-	18.4418	-
257	33	0.007841	1.950	0.070859	1.315	0.455037	1.080	1.7513	0.801	12.9625	0.509
513	65	0.002102	1.899	0.023602	1.586	0.218911	1.056	0.8893	0.978	7.0590	0.877
1025	129	0.000550	1.934	0.005682	2.055	0.110024	0.993	0.4110	1.114	2.9266	1.270
	∅		1.927		1.652		1.043		0.964		0.885
65	17	0.031141	-	0.168331	-	0.954532	-	3.1496	-	19.8796	-
129	33	0.008515	1.871	0.067720	1.314	0.469079	1.025	1.7697	0.832	13.2528	0.585
257	65	0.002707	1.653	0.021456	1.658	0.247158	0.924	0.9127	0.955	7.6771	0.788
513	129	0.000956	1.502	0.004863	2.141	0.148968	0.730	0.4341	1.072	3.2172	1.255
	∅		1.675		1.704		0.893		0.953		0.876

Table 5.2.: Errors and EOCs for $\hat{u}^N_{\gamma^M}$ and $\mathcal{G}^M_{\gamma^M}$ (second Example).

5.2. Discrete shape optimization examples

5.2.1. Comparison of different descent directions

The crucial step in Algorithm 4.3.1 is the choice of the operator $H^M_k \in \mathcal{L}(S^M)$ and the computation of the resulting descent direction $(s^M)^k \in S'_{(\gamma^M)^k}$. In the following example we compare and discuss the descent directions presented in Subsections 4.3.1-4.3.3.



(a) Desired state $\hat{u}_d^{\bar{N}}$. (b) Initial state $\hat{u}_{\gamma^0}^N$ (c) Ω_{γ_d} and Ω_{γ^0}

Figure 5.4.: Desired state $\hat{u}_d^{\bar{N}}$ (left) and initial state $\hat{u}_{\gamma^0}^N$ (right) from Example 5.2.1.

Example 5.2.1. *Let*

$$\gamma(t) = \left(t + \frac{1}{4}\sin(2\pi t), 0.64 - \frac{3}{10}t + \frac{1}{6}\sin(2\pi t)\right)^T$$

and $f(x, y) = 40$. For $\bar{N} = 2^{12} + 1 = 4097$ and $\bar{M} = 2^8 + 1 = 257$ we compute an approximation $\gamma_d := \gamma_d^{\bar{M}}$ of γ and a reference solution $(\hat{u}_d^{\bar{N}}, \mathcal{G}_d^{\bar{M}}) := (\hat{u}_{\gamma_d^{\bar{M}}}^{\bar{N}}, \mathcal{G}_{\gamma_d^{\bar{M}}}^{\bar{M}})$, which serve as the desired shape and the desired state in our optimization problem. The optimization process is performed with $N = 2^{10} + 1 = 1025$ and $M = 2^6 + 1 = 65$. As an initial guess γ^0 we use

$$\gamma^0(t) = \sum_{i=1}^{M} \vec{\gamma}_i^M B_i^M(t), \qquad \vec{\gamma}_i^M = (\gamma(1) - \gamma(0))t_i + \gamma(0),\ i = 1, \ldots, M.$$

which is just the straight line between the points $\gamma(0)$ and $\gamma(1)$. With γ^0 we compute the initial solution $(\hat{u}_{\gamma^0}^N, \mathcal{G}_{\gamma^0}^M)$. The desired state $\hat{u}_d^{\bar{N}}$ and the initial state $\hat{u}_{\gamma^0}^N$ as well as the desired shape Ω_{γ_d} and the initial shape Ω_{γ^0} are visualized in Figure 5.4.
In the first test we study the Algorithm 4.3.1 using the steepest descent direction computed via (4.3.3) (see Figure 5.5). Starting with the initial curve γ^0 the algorithm produces a sequence of curves $\{\gamma^k\}$ that converges to the desired shape γ_d.
In each iteration step we have to solve the state and the adjoint state equation. Since $N \approx 2^4 M$ we obtain non-oscillating Lagrange multipliers $\mathcal{G}_{\gamma^k}^M$, $\mathcal{M}_{\gamma^k}^M$ in each iteration. These functions can be seen in the left column of Figure 5.5 for selected iteration numbers k ($\mathcal{G}_{\gamma^k}^M$ in blue, $\mathcal{M}_{\gamma^k}^M$ in red). With $\mathcal{G}_{\gamma^k}^M$ and $\mathcal{M}_{\gamma^k}^M$ we compute the derivative $\hat{\jmath}^{(N,M)\prime}(\gamma^k)$ and the gradient $\nabla\hat{\jmath}^{(N,M)}(\gamma^k)$. As illustrated in the middle column of Figure 5.5 we plot a piecewise linear function interpolating the points

$$\left(t_j, \left(\mathcal{J}_{\gamma^k}^{(N,M)}\right)_j\right), \quad j = 2, \ldots, M-1, \tag{5.2.1}$$

(blue curve) and a piecewise linear function interpolating the points

$$\Big((t_j, (\underline{d}^M_{\gamma^k})_j \Big), \quad j = 2, \dots, M-1, \tag{5.2.2}$$

(red curve) to visualize the derivative $\hat{\jmath}^{(N,M)\prime}(\gamma^k)$ *and the gradient* $\nabla \hat{\jmath}^{(N,M)}(\gamma^k)$*, respectively. We observe that the points in* (5.2.1) *are sensitively dependent from the Lagrange multipliers. Instabilities in the Lagrange multipliers are immediately carried over to the coefficients in* $\mathcal{J}^{(N,M)}_{\gamma^k}$*. Nevertheless, occurring roughness in* (5.2.1) *is smoothed out in* (5.2.2)*.*
Eventually, in the right column of Figure 5.5 the desired shape γ_d *(red curve) is shown together with the k-th iterate* γ^k *(blue curve). According to the descent direction, the arrows indicate the direction of the movement from* γ^k *to the next iterate* γ^{k+1}*.*
It turns out that the steepest descent direction is very inefficient. We obtain a small decrease of the values $\hat{\jmath}^{(N,M)}(\gamma^k)$ *in each step and a slow convergence of* γ^k *to* γ_d*. After* 8015 *iterations, the optimization process terminates due to the stopping condition*

$$\|\gamma^{8014} - \gamma^{8015}\|_\infty \leq \sqrt{10^{-12}}(1 + \|\gamma^{8015}\|_\infty).$$

Next, we discuss the application of the BFGS quasi-Newton descent direction, which we obtain by the update formula (4.3.4)*. The BFGS quasi-Newton method approximates the Newton descent direction, but it avoids to evaluate the second derivative directly. Since the BFGS method performs well in practice, it is commonly used in the optimization community. In our example, using the BFGS descent direction, we observe a major improvement with respect to the convergence rate in comparison to the steepest decent method. We emphasize that the matrix* $\widetilde{H}^M_{(\gamma^M)^k}$ *in* (4.3.6) *can be evaluated exactly without large numerical effort. The algorithm stops after* 116 *iterations with*

$$\|\gamma^{115} - \gamma^{116}\|_\infty \leq \sqrt{10^{-12}}(1 + \|\gamma^{116}\|_\infty).$$

The BFGS method stops at a function value of $\hat{\jmath}^{(N,M)}(\gamma^{116}) \approx 2.41 \cdot 10^{-7}$*.*
Regarding the iteration number and the decrease of the function value we can do better, if we use an inexact Newton-like method, which we will describe in the following. In this method we obtain the descent direction by solving the system (3.6.9) *iteratively, and therefore only approximately. In our computation the iteration stops if the relative residual achieves*

$$\frac{\|r^k\|_S}{\|\nabla \hat{\jmath}(\gamma^k)\|_S} \leq 0.1. \tag{5.2.3}$$

Naturally, we need a sufficiently good initial guess γ^0 *to obtain a convergent inexact Newton-like method. In our example we cannot compute the inexact Newton-like descent direction for the given starting curve* γ^0*. Hence, we have to perform six BFGS quasi-Newton steps until the algorithm switches to inexact Newton-like steps.*

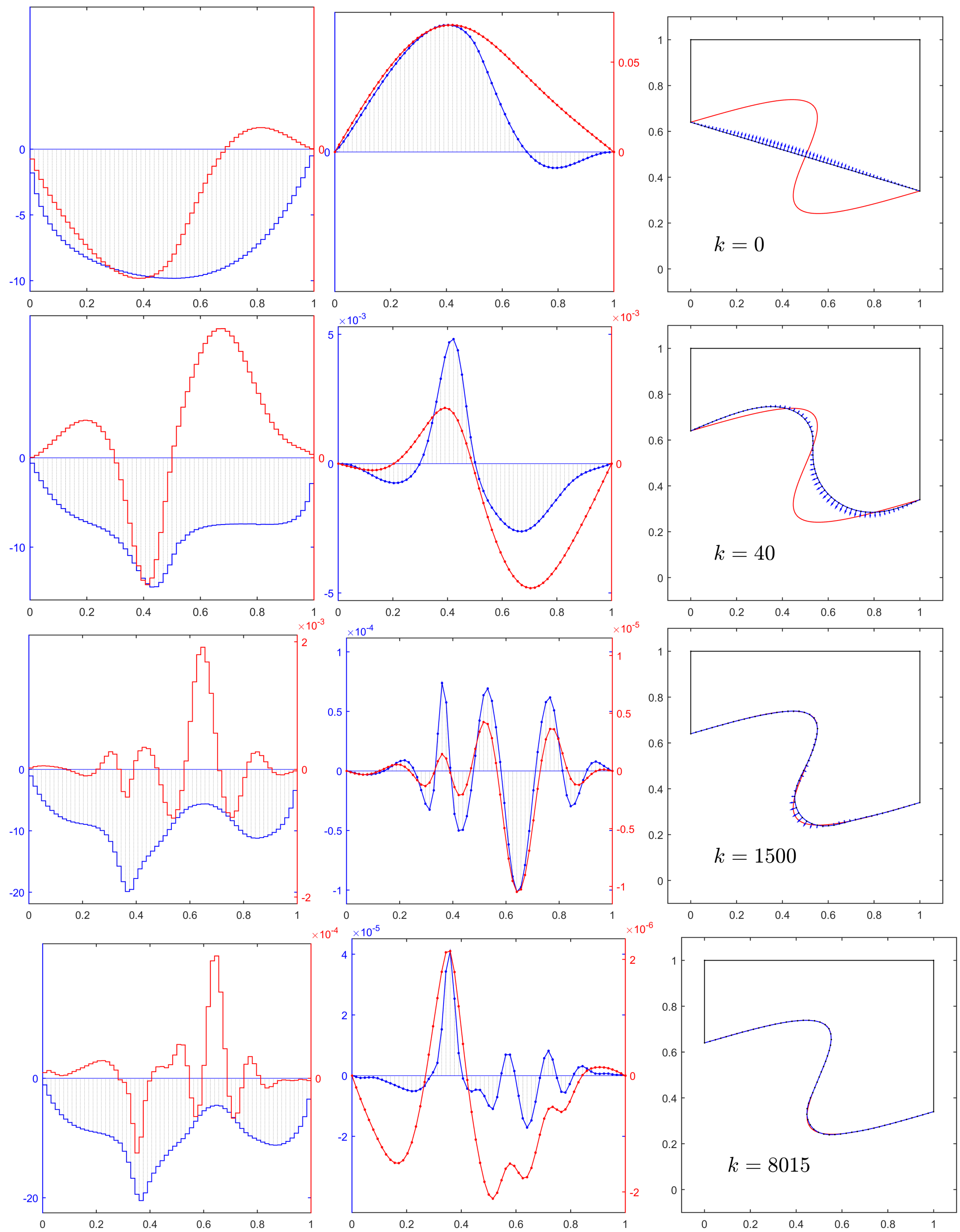


Figure 5.5.: Numerical results for Example 5.2.1 (**Gradient method**). Left column: Lagrange multipliers $\mathcal{G}_{\gamma^k}$ (blue) and $\mathcal{M}_{\gamma^k}$ (red). Middle column: Visualization of the Fréchet-derivative (blue) and the gradient (red). Right column: shape of the k-th iterate γ^k (blue) and the desired shape γ_d (red).

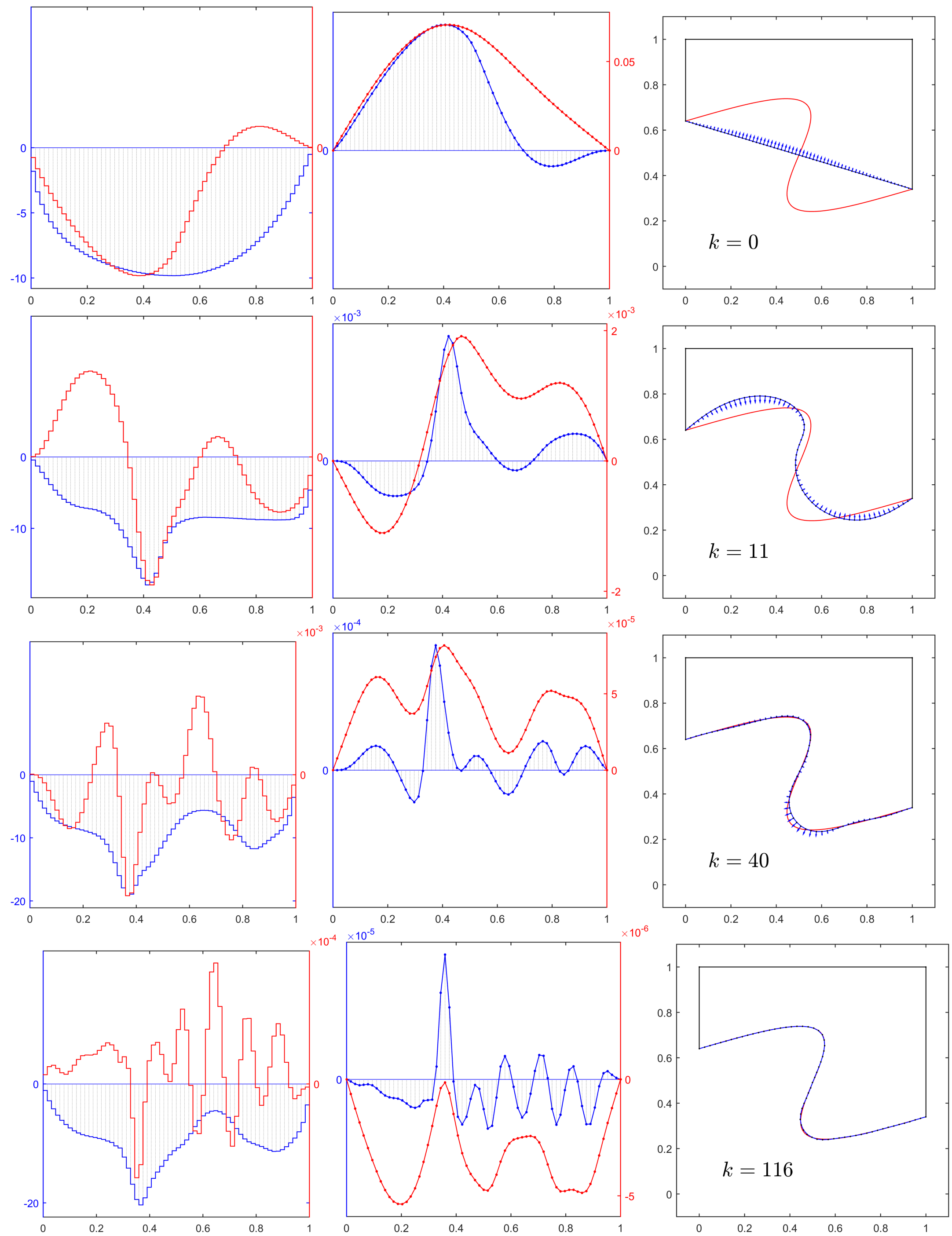


Figure 5.6.: Numerical results for Example 5.2.1 (**BFGS quasi-Newton method**). Left column: Lagrange multipliers $\mathcal{G}_{\gamma^k}$ (blue) and $\mathcal{M}_{\gamma^k}$ (red). Middle column: Visualization of the Fréchet-derivative (blue) and the gradient (red). Right column: shape of the k-th iterate γ^k (blue) and the desired shape γ_d (red).

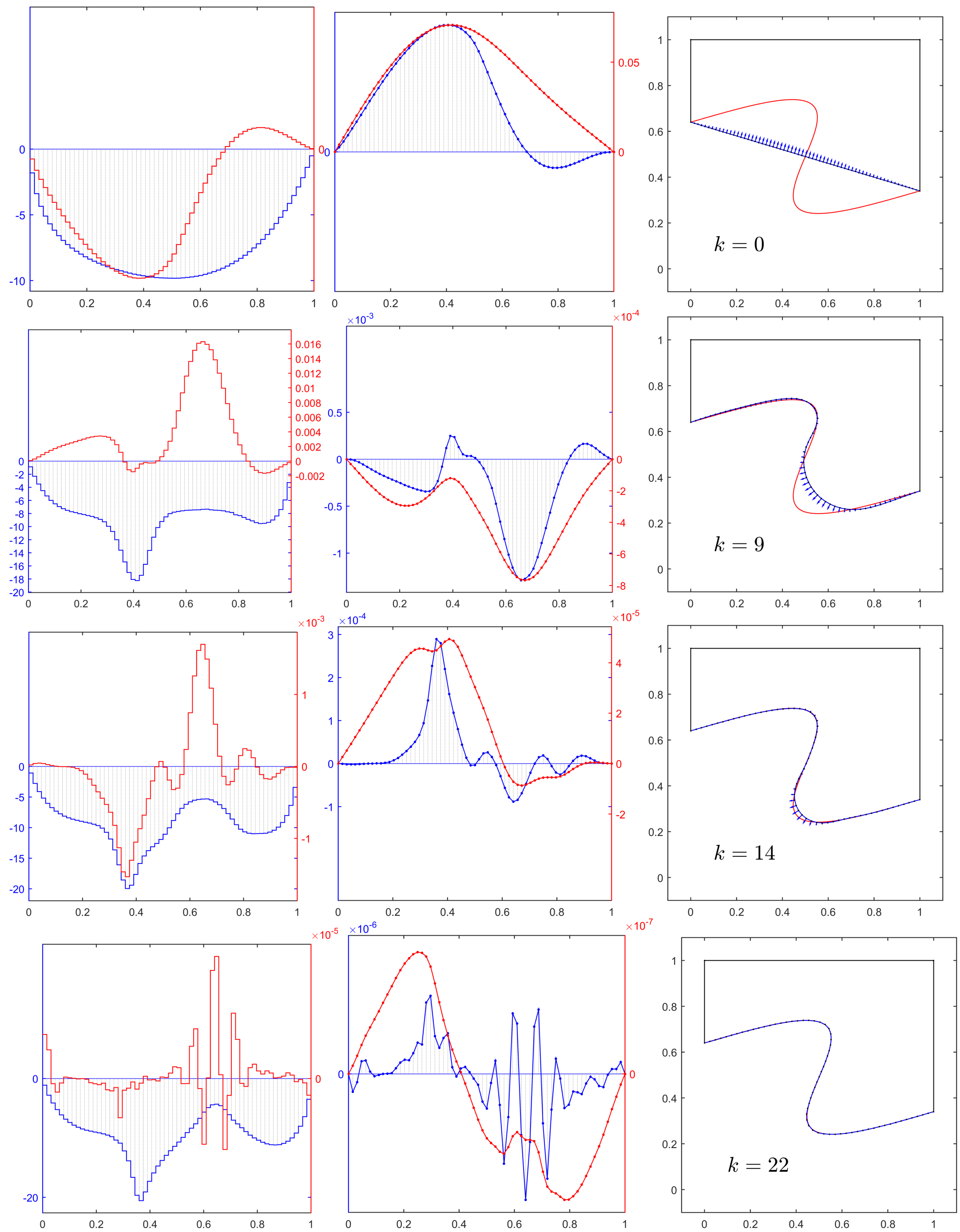


Figure 5.7.: Numerical results for Example 5.2.1 (**Inexact Newton-like method**). Left column: Lagrange multipliers $\mathcal{G}_{\gamma^k}$ (blue) and $\mathcal{M}_{\gamma^k}$ (red). Middle column: Visualization of the Fréchet-derivative (blue) and the gradient (red). Right column: shape of the k-th iterate γ^k (blue) and the desired shape γ_d (red).

At a total number of 22 iterations the optimization process terminates at $\hat{\jmath}^{(N,M)}(\gamma^{22}) \approx 1.28 \cdot 10^{-7}$ *with*

$$\|\gamma^{21} - \gamma^{22}\|_\infty \leq \sqrt{10^{-12}}(1 + \|\gamma^{22}\|_\infty).$$

In Figure 5.8, the convergence history of all three methods can be seen.

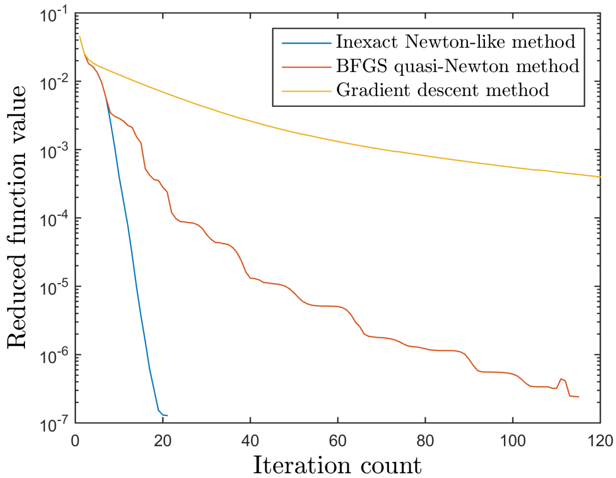


Figure 5.8.: Convergence history for the Gradient descent method, the BFGS quasi-Newton method and the inexact Newton-like method performed in Example 5.2.1

5.2.2. Convergence behavior of the control variable γ

In the following example we analyze the convergence behavior of the control variable γ.

Example 5.2.2. *Let*

$$\gamma(t) = (\gamma_1(t), \gamma_2(t))^T,$$

where γ_1 *is the one-dimensional cubic spline for the data*

$$X = \left(0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right), \quad Y \approx (0.00, 0.27, 0.46, 0.35, 0.40, 0.75, 0.68, 1.00),$$

and γ_2 *is the one-dimensional cubic spline for the data*

$$X = \left(0, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, 1\right), \quad Y \approx (0.69, 0.73, 0.67, 0.39, 0.06, 0.23, 0.49, 0.64),$$

and $f(x, y) = 46$.
As in Example 5.2.1, we compute a desired shape $\gamma_d := \gamma_d^{\bar{M}}$ *and a desired state* $(\hat{u}_d^{\bar{N}}, \mathcal{G}_d^{\bar{M}}) := (\hat{u}_{\gamma_d^{\bar{M}}}^{\bar{N}}, \mathcal{G}_{\gamma_d^{\bar{M}}}^{\bar{M}})$ *for* $\bar{N} = 2^{12} + 1 = 4097$ *and* $\bar{M} = 2^8 + 1 = 257$.

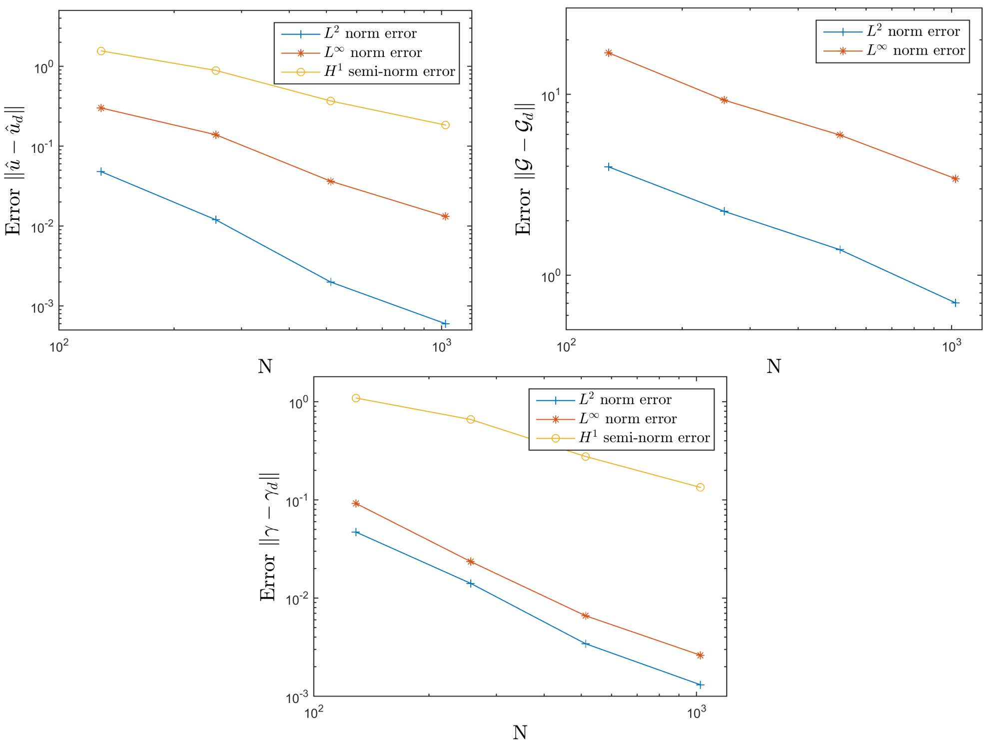


Figure 5.9.: Convergence histories for the inexact Newton-like method performed in Example 5.2.2 for different discretization levels (M_l, N_l).

The optimization process is performed with the inexact Newton-like method for different discretization parameters N_l, M_l:

$$\textit{Level } l: \quad N_l = 2^{6+l} + 1, M_l = 2^{2+l} + 1, l = 1, 2, 3, 4. \tag{5.2.4}$$

As an initial guess γ^0 in all discretization levels we use again the straight line between the points $\gamma(0)$ and $\gamma(1)$. On all discretization levels only inexact Newton-like steps are performed. No BFGS quasi-Newton steps were necessary to obtain suitable descent in the beginning of the optimization process. The convergence histories for the state and control are given in Figure 5.9. The convergence behavior of the inexact Newton-like method for level 4 is presented in Table 5.3. The evolution of the shapes for level 4 are visualized in Figure 5.10 and Figure 5.11.

k	$\hat{\jmath}^{(N,M)}(\gamma^k)$	$\lVert\nabla\hat{\jmath}^{(N,M)}(\gamma^k)\rVert_S$	$\lVert s^k\rVert_\infty$	cgIt	cgRelRes	#S
0	2.1416128117E-02	3.0755351744E-02	7.781E-02	6	8.95E-02	27
1	1.7732189968E-02	2.0003957380E-02	2.789E-02	2	7.49E-02	38
2	1.6306948087E-02	9.5475986532E-03	2.979E-02	4	9.18E-02	57
3	1.5444428148E-02	7.8407829933E-03	3.113E-02	3	8.54E-02	72
4	1.4773879911E-02	4.9484075880E-03	2.967E-02	4	8.69E-02	91
5	1.4216280172E-02	4.3811871065E-03	3.300E-02	4	7.95E-02	110
6	1.3619236803E-02	4.0104701774E-03	3.545E-02	4	8.91E-02	129
7	1.2915677921E-02	4.2285894581E-03	3.755E-02	4	8.80E-02	148
8	1.2035071531E-02	4.9210464706E-03	4.281E-02	4	8.03E-02	167
9	1.0837916318E-02	5.9947471664E-03	4.157E-02	4	7.05E-02	186
10	9.3261122301E-03	7.3238034484E-03	4.192E-02	4	7.27E-02	205
11	7.5589990926E-03	7.9718500540E-03	4.262E-02	4	7.79E-02	224
12	5.5808323592E-03	8.9373701530E-03	4.093E-02	3	9.80E-02	239
13	3.7161643106E-03	8.6474820934E-03	3.825E-02	4	7.34E-02	258
14	2.0589778655E-03	7.3866393202E-03	4.089E-02	4	7.18E-02	277
15	8.8084920620E-04	4.9707172447E-03	3.095E-02	4	8.25E-02	296
16	3.1672775257E-04	2.7260369279E-03	2.222E-02	5	7.09E-02	319
17	9.9448247441E-05	1.3578309042E-03	1.663E-02	5	9.33E-02	342
18	2.8609233634E-05	5.6663371914E-04	1.176E-02	8	7.88E-02	377
19	6.1057836691E-06	2.1059779195E-04	7.206E-03	11	9.16E-02	424
20	3.0829407738E-06	3.8114505727E-04	1.114E-03	4	9.21E-02	443
21	1.2316970066E-06	3.9103851689E-05	4.566E-03	20	9.62E-02	526
22	2.5686115786E-07	7.2813779299E-05	2.888E-04	4	9.01E-02	545
23	2.4052865614E-07	6.9376401137E-06	1.547E-03	34	9.54E-02	684
24	1.7890994530E-07	8.1124518436E-06	5.551E-05	7	9.35E-02	755
25	1.7890994530E-07	8.1124518436E-06	5.551E-05	7	9.35E-02	830
26	1.7890994530E-07	8.1124518436E-06				

Table 5.3.: The convergence behavior of the inexact Newton-like method from Example 5.2.2.

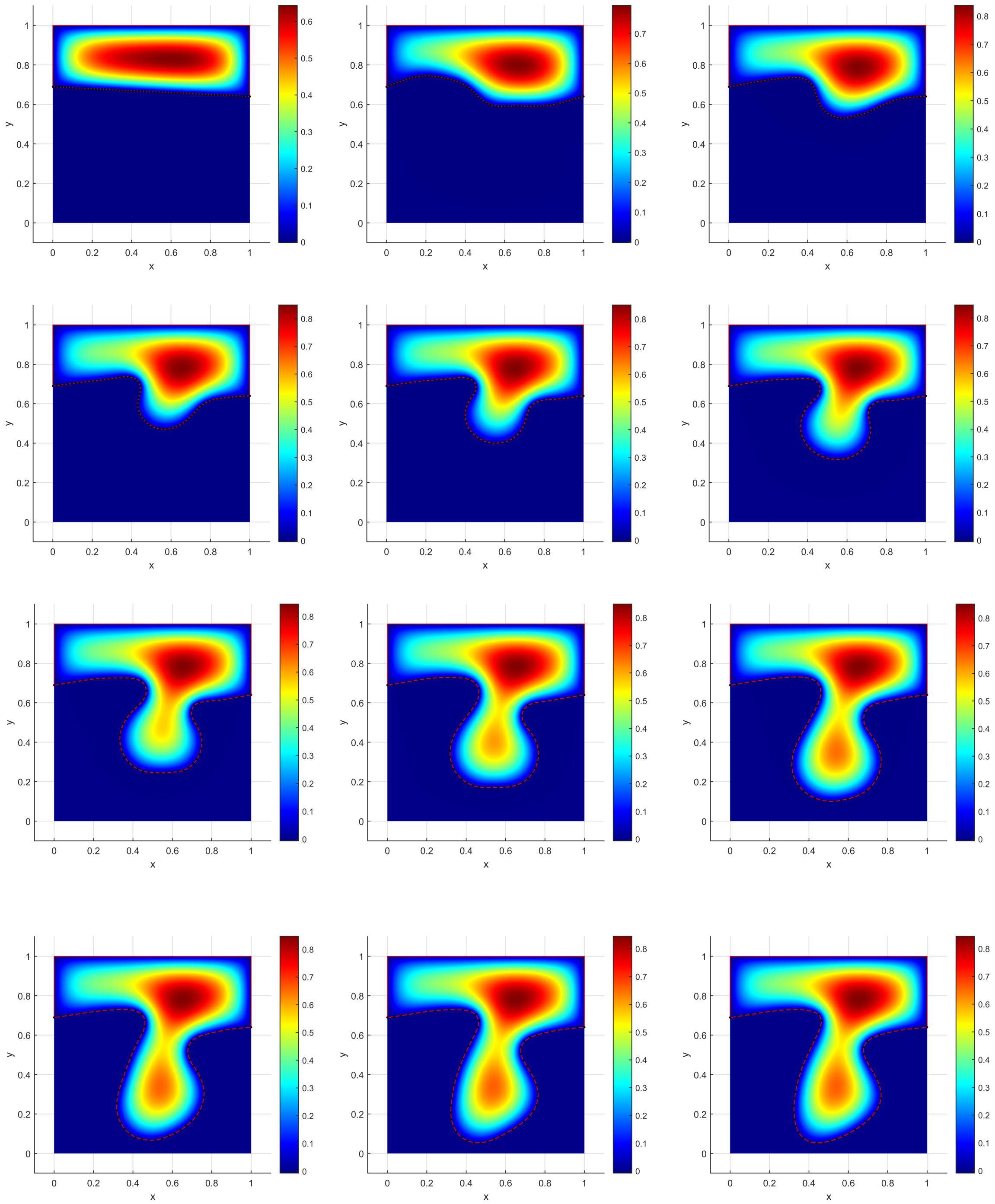


Figure 5.10.: Numerical results for Example 5.2.2. The snapshots are taken at iteration numbers $k = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 21, 26$.

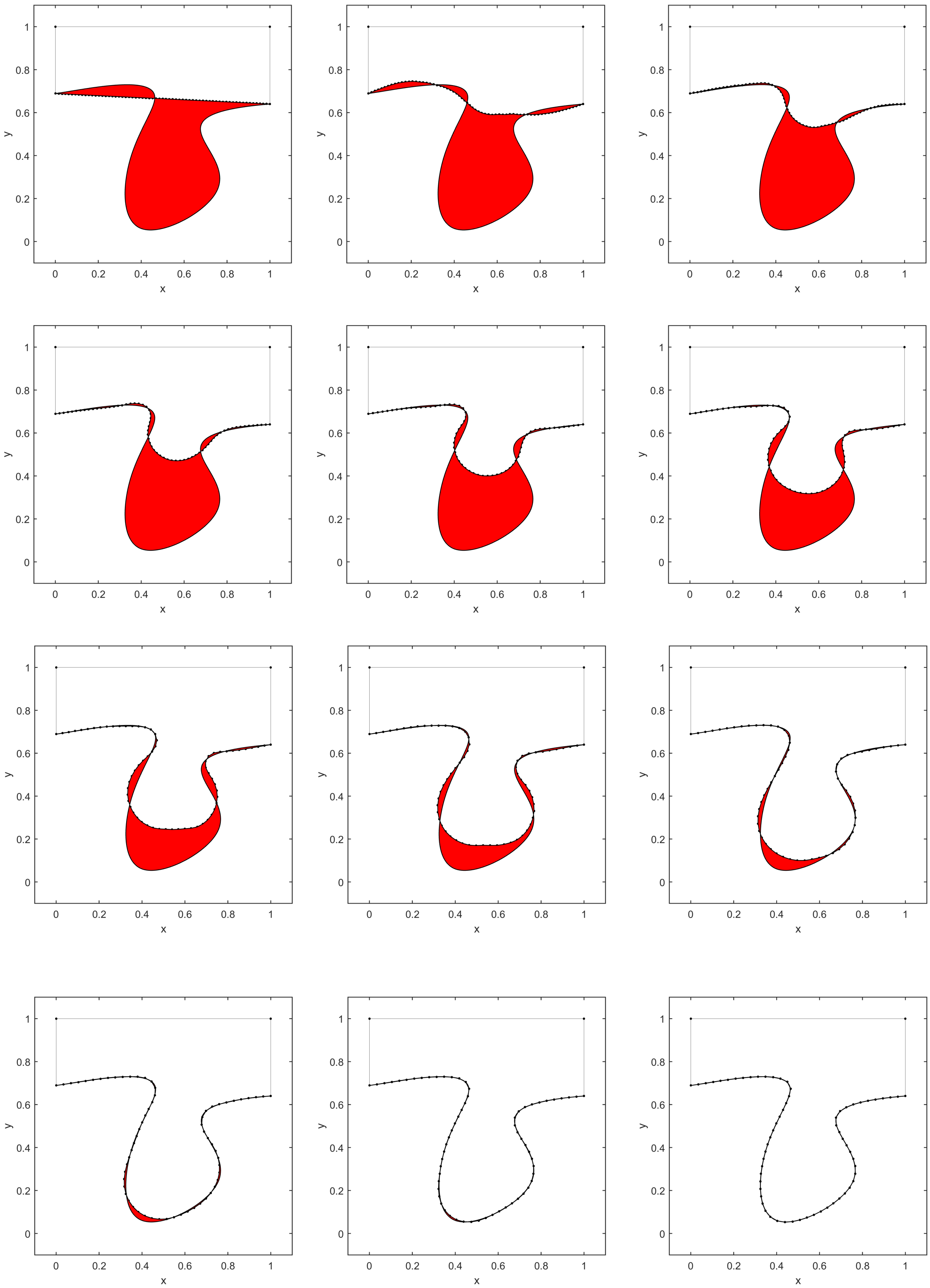


Figure 5.11.: Numerical results for Example 5.2.2. The snapshots are taken at iteration numbers $k = 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 21, 26$.

A. Discretization: detailed calculations

A.1. Knots and elements on $\hat{\Omega}$ and I

For the numerical realization the elements Q_i and the mesh points (x_i, y_i) are numbered as you see in Figure A.1. Create an index matrix with all mesh point numbers

$$
i\_all^N := \begin{pmatrix} 1 & N+1 & \cdots & (N-1)N+1 \\ 2 & N+2 & \cdots & (N-1)N+2 \\ \vdots & \vdots & \ddots & \vdots \\ N & 2N & \cdots & N^2 \end{pmatrix}
$$

and an element matrix with the mesh point numbers for each element

$$
ETN := \left(\begin{array}{cccc}
1 & N+1 & N+2 & 2 \\
2 & N+2 & N+3 & 3 \\
\vdots & \vdots & \vdots & \vdots \\
N-1 & 2N-1 & 2N & N \\
\hdashline
N+1 & 2N+1 & 2N+2 & N+2 \\
\vdots & \vdots & \vdots & \vdots \\
2N-1 & 3N-1 & 3N & 2N \\
\hdashline
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\hdashline
(N-2)N+1 & (N-1)N+1 & (N-1)N+2 & (N-2)N+2 \\
\vdots & \vdots & \vdots & \vdots \\
(N-1)N-1 & N^2-1 & N^2 & (N-1)N
\end{array}\right)
$$

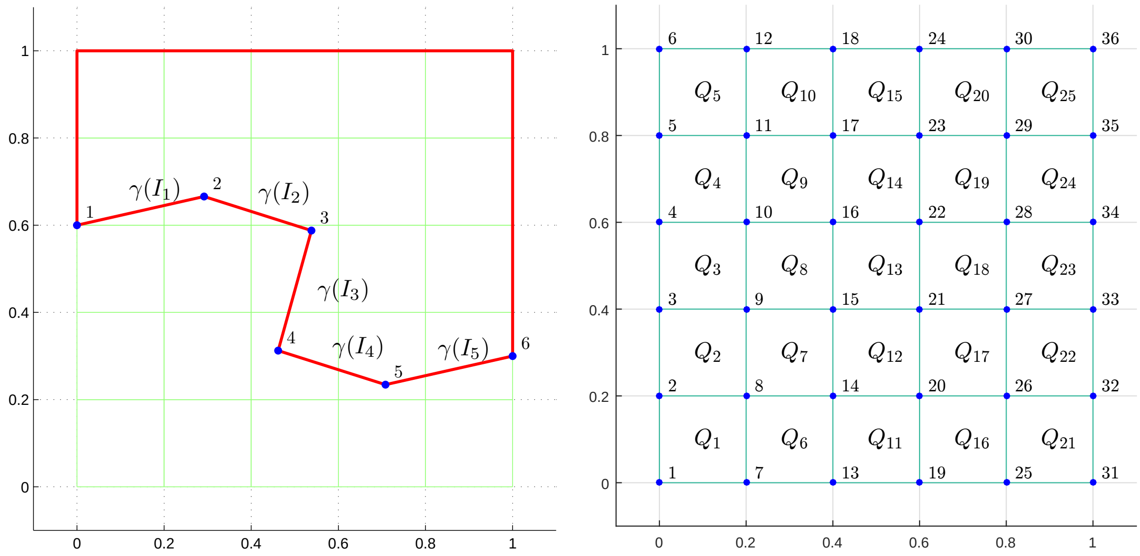


Figure A.1.: Numbering of Knots and Elements

A.2. Assembly of the finite element stiffness matrix $\hat{A}^N$

The goal of this section is to assemble the finite element stiffness matrix $\hat{A}^N := (\hat{a}_{ij}^N)$ with

$$\hat{a}_{ij}^N := \int_{\hat{\Omega}} \nabla\Phi_i^N(x,y)^T \nabla\Phi_j^N(x,y)\, d(x,y), \quad i,j = 1,\dots,N^2.$$

In the first place we define a square reference element $\hat{Q}$ with coordinates

$$\begin{aligned} (\hat{x}_1,\hat{y}_1) &:= (-1,-1), \quad & (\hat{x}_2,\hat{y}_2) &:= (\ \ 1,-1), \\ (\hat{x}_3,\hat{y}_3) &:= (\ \ 1,\ \ 1), \quad & (\hat{x}_4,\hat{y}_4) &:= (-1,\ \ 1), \end{aligned}$$

and basis shape functions on $\hat{Q}$:

$$\begin{aligned}
\hat{\phi}_1(\hat{x},\hat{y}) &:= \frac{1}{4}(\hat{x}-1)(\hat{y}-1), \ \nabla\hat{\phi}_1(\hat{x},\hat{y}) = \frac{1}{4}\begin{pmatrix}\hat{y}-1\\ \hat{x}-1\end{pmatrix}, \\
\hat{\phi}_2(\hat{x},\hat{y}) &:= -\frac{1}{4}(\hat{x}+1)(\hat{y}-1), \ \nabla\hat{\phi}_2(\hat{x},\hat{y}) = -\frac{1}{4}\begin{pmatrix}\hat{y}-1\\ \hat{x}+1\end{pmatrix}, \\
\hat{\phi}_3(\hat{x},\hat{y}) &:= \frac{1}{4}(\hat{x}+1)(\hat{y}+1), \ \nabla\hat{\phi}_3(\hat{x},\hat{y}) = \frac{1}{4}\begin{pmatrix}\hat{y}+1\\ \hat{x}+1\end{pmatrix}, \\
\hat{\phi}_4(\hat{x},\hat{y}) &:= -\frac{1}{4}(\hat{x}-1)(\hat{y}+1), \ \nabla\hat{\phi}_4(\hat{x},\hat{y}) = -\frac{1}{4}\begin{pmatrix}\hat{y}+1\\ \hat{x}-1\end{pmatrix}.
\end{aligned}$$

Let $Q_j \subset \hat{\Omega}$ be a Quadrilateral element with coordinates: (x_1^j, y_1^j), (x_2^j, y_2^j), (x_3^j, y_3^j), (x_4^j, y_4^j), then we define

$$F_j : \hat{Q} \to Q_j, \quad (\hat{x}, \hat{y}) \mapsto \sum_{k=1}^{4} \begin{pmatrix} x_k^j \\ y_k^j \end{pmatrix} \hat{\phi}_k(\hat{x}, \hat{y})$$

with

$$\begin{aligned}
(F_j)_1(\hat{x}, \hat{y}) &= \frac{1}{4}\Big(x_1^j(\hat{x}-1)(\hat{y}-1) - x_2^j(\hat{x}+1)(\hat{y}-1) \\
&\qquad + x_3^j(\hat{x}+1)(\hat{y}+1) - x_4^j(\hat{x}-1)(\hat{y}+1)\Big) \\
{}^1 &= \frac{1}{4}\left(-2x_1^j(\hat{x}-1) + 2x_2^j(\hat{x}+1)\right) \\
&= \frac{x_2^j - x_1^j}{2}\hat{x} + \frac{x_2^j + x_1^j}{2}
\end{aligned}$$

and

$$(F_j)_2(\hat{x}, \hat{y}) = \frac{y_3^j - y_2^j}{2}\hat{y} + \frac{y_3^j + y_2^j}{2}$$

For the inverse mapping

$$F_j^{-1} : Q \to \hat{Q}, \quad (x, y) \mapsto F_j^{-1}(x, y)$$

of F_j we obtain

$$\begin{aligned}
(F_j^{-1})_1(x, y) &= \frac{1}{x_2^j - x_1^j}\left(2x - x_1^j - x_2^j\right), \\
(F_j^{-1})_2(x, y) &= \frac{1}{y_3^j - y_2^j}\left(2y - y_2^j - y_3^j\right).
\end{aligned}$$

With the help of F_j^{-1} and the basis shape functions on $\hat{Q}$ we define basis functions ϕ_k^j,

$$\phi_k^j(x, y) := \hat{\phi}_k(F_j^{-1}(x, y)), \quad k = 1, 2, 3, 4,$$

on the quadrilateral element Q_j. We obtain

$$\begin{aligned}
\hat{\phi}_k(\hat{x}, \hat{y}) = \phi_k^j(F_j(\hat{x}, \hat{y})) &\Rightarrow \nabla\hat{\phi}_k(\hat{x}, \hat{y}) = \left(F_j'(\hat{x}, \hat{y})\right)^T \nabla\phi_k^j(F_j(\hat{x}, \hat{y})) \\
&\Rightarrow \nabla\phi_k^j(F_j(\hat{x}, \hat{y})) = \left(F_j'(\hat{x}, \hat{y})\right)^{-T} \nabla\hat{\phi}_k(\hat{x}, \hat{y}).
\end{aligned}$$

[1] $x_1^j = x_4^j$, $x_2^j = x_3^j$

Furthermore we calculate

$$F_j'(\hat{x},\hat{y}) = \sum_{k=1}^{4} \begin{pmatrix} x_k^j \\ y_k^j \end{pmatrix} \hat{\phi}_k'(\hat{x},\hat{y}) = \begin{pmatrix} \frac{x_2^j - x_1^j}{2} & 0 \\ 0 & \frac{y_3^j - y_2^j}{2} \end{pmatrix}$$

and with $h^N = x_2^j - x_1^j = y_3^j - y_2^j$ we obtain

$$F_j'(\hat{x},\hat{y}) = \frac{h^N}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad F_j'(\hat{x},\hat{y})^{-1} = \frac{2}{h^N} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$\det(F_j'(\hat{x},\hat{y})) = \frac{(h^N)^2}{4}, \quad F_j'(\hat{x},\hat{y})^{-1} F_j'(\hat{x},\hat{y})^{-T} = \frac{4}{(h^N)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.2.1}$$

Now we are able to calculate the local stiffness matrix $\hat{A}_{\text{loc}}^N := \big((\hat{a}_{\text{loc}}^N)_{kl}\big)$ with

$$(\hat{a}_{\text{loc}}^N)_{kl} := \int_{Q_j} \nabla \phi_k^j(x,y)^T \nabla \phi_l^j(x,y)\, d(x,y), \quad k,l = 1,2,3,4,$$

by transforming the integrals to the reference element $\hat{Q}$ and using the upper results. As we will see, the outcome is independent of Q_j and j:

$$\begin{aligned}
&\int_{Q_j = F_j(\hat{Q})} \nabla \phi_k^j(x,y)^T \nabla \phi_l^j(x,y)\, d(x,y) \\
&\quad = \int_{\hat{Q}} \nabla \phi_k^j(F_j(\hat{x},\hat{y}))^T \nabla \phi_l^j(F_j(\hat{x},\hat{y})) \left|\det(F_j'(\hat{x},\hat{y}))\right| d(\hat{x},\hat{y}) \\
&\quad = \int_{\hat{Q}} \Big(\big(F_j'(\hat{x},\hat{y})\big)^{-T} \nabla \hat{\phi}_k(\hat{x},\hat{y}) \Big)^T \Big(\big(F_j'(\hat{x},\hat{y})\big)^{-T} \nabla \hat{\phi}_l(\hat{x},\hat{y}) \Big) \left|\det(F_j'(\hat{x},\hat{y}))\right| d(\hat{x},\hat{y}) \\
&\quad = \int_{\hat{Q}} \nabla \hat{\phi}_k(\hat{x},\hat{y})^T \frac{4}{(h^N)^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \nabla \hat{\phi}_l(\hat{x},\hat{y}) \frac{(h^N)^2}{4}\, d(\hat{x},\hat{y}) \\
&\quad = \int_{\hat{Q}} \nabla \hat{\phi}_k(\hat{x},\hat{y})^T \nabla \hat{\phi}_l(\hat{x},\hat{y})\, d(\hat{x},\hat{y})
\end{aligned}$$

For $k = l = 1$ we obtain

$$\begin{aligned}
\int_{\hat{Q}} \nabla\hat{\phi}_1(\hat{x},\hat{y})^T \nabla\hat{\phi}_1(\hat{x},\hat{y})\, d(\hat{x},\hat{y}) &= \frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} \begin{pmatrix} \hat{y}-1 \\ \hat{x}-1 \end{pmatrix}^T \begin{pmatrix} \hat{y}-1 \\ \hat{x}-1 \end{pmatrix} d\hat{x}\, d\hat{y} \\
&= \frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} (\hat{y}-1)^2 + (\hat{x}-1)^2 \, d\hat{x}\, d\hat{y} \\
&= \frac{1}{16}\cdot 2(1-(-1)) \int_{-1}^{1} (\hat{x}-1)^2 \, d\hat{x} \\
&= \frac{1}{4}\left[\frac{1}{3}(\hat{x}-1)^3\right]_{-1}^{1} = \frac{2}{3},
\end{aligned}$$

for $k = 1$, $l = 2$ we have

$$\begin{aligned}
\int_{\hat{Q}} \nabla\hat{\phi}_1(\hat{x},\hat{y})^T \nabla\hat{\phi}_2(\hat{x},\hat{y})\, d(\hat{x},\hat{y}) &= -\frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} \begin{pmatrix} \hat{y}-1 \\ \hat{x}-1 \end{pmatrix}^T \begin{pmatrix} \hat{y}-1 \\ \hat{x}+1 \end{pmatrix} d\hat{x}\, d\hat{y} \\
&= -\frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} (\hat{y}-1)^2 + (\hat{x}^2-1) \, d\hat{x}\, d\hat{y} \\
&= -\frac{1}{16}\left(2\left(0-\left(-\frac{8}{3}\right)\right) + 2\left(-\frac{2}{3}-\frac{2}{3}\right)\right) = -\frac{1}{6},
\end{aligned}$$

and for $k = 1$, $l = 3$ we find

$$\begin{aligned}
\int_{\hat{Q}} \nabla\hat{\phi}_1(\hat{x},\hat{y})^T \nabla\hat{\phi}_3(\hat{x},\hat{y})\, d(\hat{x},\hat{y}) &= \frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} \begin{pmatrix} \hat{y}-1 \\ \hat{x}-1 \end{pmatrix}^T \begin{pmatrix} \hat{y}+1 \\ \hat{x}+1 \end{pmatrix} d\hat{x}\, d\hat{y} \\
&= \frac{1}{16}\int_{-1}^{1}\int_{-1}^{1} (\hat{y}^2-1) + (\hat{x}^2-1) \, d\hat{x}\, d\hat{y} \\
&= \frac{1}{16}\left(2\left(-\frac{2}{3}-\frac{2}{3}\right) + 2\left(-\frac{2}{3}-\frac{2}{3}\right)\right) = -\frac{1}{3}.
\end{aligned}$$

This leads to the following algorithm to assemble the stiffness matrix $\hat{A}^N$

Algorithm A.2.1.

function ASSEMBLESTIFFNESSMATRIX(N, ETN)

$\quad \hat{A}^N_{\text{loc}} \leftarrow \frac{1}{6}\begin{pmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{pmatrix}$

$\quad \hat{A}^N \leftarrow sparse(N^2, N^2)$

$\quad$ ***for*** $i \leftarrow 1, 4$ ***do***

$\quad\quad$ ***for*** $j \leftarrow 1, 4$ ***do***

$\quad\quad\quad \hat{A}^N \leftarrow \hat{A}^N + sparse(ETN(:,i), ETN(:,j), \hat{A}^N_{\text{loc}}(i,j), N^2, N^2)$

$\quad\quad$ ***end for***

    end for
    return $\hat{A}^N$
end function

A.3. Assembly of the finite element mass matrix $\hat{B}^N$

To assemble the global mass matrix, we proceed as in the last section. We calculate the local mass matrix $\hat{B}^N_{\text{loc}} := \left((\hat{b}^N_{\text{loc}})_{kl}\right)$ with

$$(\hat{b}^N_{\text{loc}})_{kl} := \int_{Q_j} \phi^j_k(x,y)\phi^j_l(x,y)\,d(x,y), \quad k,l = 1,2,3,4,$$

and transform these integrals to the reference element $\hat{Q}$. We obtain

$$\begin{aligned}
&\int_{Q_j=F_j(\hat{Q})} \phi^j_k(x,y)\phi^j_l(x,y)\,d(x,y) \\
&\qquad = \int_{\hat{Q}} \phi^j_k(F_j(\hat{x},\hat{y}))\phi^j_l(F_j(\hat{x},\hat{y}))\left|\det(F'_j(\hat{x},\hat{y}))\right|\,d(\hat{x},\hat{y}) \\
&\qquad = \frac{(h^N)^2}{4}\int_{\hat{Q}} \hat{\phi}_k(\hat{x},\hat{y})\hat{\phi}_l(\hat{x},\hat{y})\,d(\hat{x},\hat{y})
\end{aligned}$$

For $k = l = 1$ we have

$$\begin{aligned}
\frac{(h^N)^2}{4}\int_{\hat{Q}} \hat{\phi}_1(\hat{x},\hat{y})\hat{\phi}_1(\hat{x},\hat{y})\,d(\hat{x},\hat{y}) &= \frac{(h^N)^2}{4}\cdot\frac{1}{16}\int_{-1}^{1}\int_{-1}^{1}(\hat{x}-1)^2(\hat{y}-1)^2\,d\hat{x}\,d\hat{y} \\
&= \frac{(h^N)^2}{4}\cdot\frac{1}{16}\int_{-1}^{1}(\hat{x}-1)^2\left[\frac{1}{3}(\hat{y}-1)^3\right]_{\hat{y}=-1}^{1}\,d\hat{x} \\
&= \frac{(h^N)^2}{4}\cdot\frac{1}{16}\cdot\frac{8}{3}\left[\frac{1}{3}(\hat{x}-1)^3\right]_{\hat{x}=-1}^{1} \\
&= \frac{(h^N)^2}{4}\cdot\frac{1}{16}\cdot\frac{8}{3}\cdot\frac{8}{3} = \frac{(h^N)^2}{36}\cdot 4,
\end{aligned}$$

for $k = 1$, $l = 2$ we obtain

$$\begin{aligned}
\frac{(h^N)^2}{4} \int_{\hat{Q}} \hat{\phi}_1(\hat{x},\hat{y})\hat{\phi}_2(\hat{x},\hat{y})\, d(\hat{x},\hat{y}) &= \frac{(h^N)^2}{4} \cdot \left(-\frac{1}{16}\right) \int_{-1}^{1}\int_{-1}^{1} (\hat{x}-1)^2(\hat{y}^2-1)\, d\hat{x}\, d\hat{y} \\
&= \frac{(h^N)^2}{4} \cdot \left(-\frac{1}{16}\right) \int_{-1}^{1} (\hat{x}-1)^2 \left[\frac{1}{3}\hat{y}^3 - \hat{y}\right]_{\hat{y}=-1}^{1} d\hat{x} \\
&= \frac{(h^N)^2}{4} \cdot \left(-\frac{1}{16}\right) \cdot \left(-\frac{4}{3}\right) \left[\frac{1}{3}(\hat{x}-1)^3\right]_{\hat{x}=-1}^{1} \\
&= \frac{(h^N)^2}{4} \cdot \left(-\frac{1}{16}\right) \cdot \left(-\frac{4}{3}\right) \cdot \frac{8}{3} = \frac{(h^N)^2}{36} \cdot 2,
\end{aligned}$$

and for $k = 1$, $l = 3$ we have

$$\begin{aligned}
\frac{(h^N)^2}{4} \int_{\hat{Q}} \hat{\phi}_1(\hat{x},\hat{y})\hat{\phi}_3(\hat{x},\hat{y})\, d(\hat{x},\hat{y}) &= \frac{(h^N)^2}{4} \cdot \frac{1}{16} \int_{-1}^{1}\int_{-1}^{1} (\hat{x}^2-1)(\hat{y}^2-1)\, d\hat{x}\, d\hat{y} \\
&= \frac{(h^N)^2}{4} \cdot \frac{1}{16} \left(-\frac{4}{3}\right) \int_{-1}^{1} (\hat{x}^2-1)\, d\hat{x} \\
&= \frac{(h^N)^2}{4} \cdot \frac{1}{16} \left(-\frac{4}{3}\right)\left(-\frac{4}{3}\right) = \frac{(h^N)^2}{36}.
\end{aligned}$$

This leads to

$$\hat{B}^N_{\text{loc}} = \frac{(h^N)^2}{36} \begin{pmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{pmatrix}$$

and an algorithm analogously to Algorithm A.2.1.

A.4. Assembly of the trace matrix $T^{N,M}_{\gamma^M}$

In this section we assemble the trace matrix $T^{N,M}_{\gamma^M} = \left(t^{N,M}_{ij}\right)$ with

$$\begin{aligned}
t^{N,M}_{ij} &:= \langle \chi^M_i, \mathcal{T}_{\gamma^M} \Phi^N_j \rangle_{H^*_I, H_I} \\
&= \int_I \chi^M_i(t) \Phi^N_j(\gamma^M(t))\, dt, \quad i = 1, \dots, M-1,\ j = 1, \dots, N^2.
\end{aligned}$$

Therefore, we consider a subdivision of the parameter interval $I = \bigcup_{i=1} \bar{J}_i$ with intervals $J_i := (\tau_i, \tau_{i+1})$ where

$$\begin{aligned}
\text{either} \quad & \exists j : \tau_i = t_j, \text{ i.e. } \gamma^M(\tau_i) = \gamma^M(t_j) = \vec{\gamma}^M_j \\
\text{or} \quad & \gamma^M(\tau_i) \quad \text{is an intersection point of } \gamma^M \text{ with the mesh } (x_j, y_j).
\end{aligned}$$

We determine all intersection points and store all τ_i in a vector

$$\tau^{N,M} = \begin{bmatrix} \tau_1 & \cdots & \tau_{n\_isp} \end{bmatrix}.$$

To assemble the trace matrix correctly we also store index vectors i_isp^N, $i\_isp^M \in \mathbb{R}^{1 \times n\_isp-1}$ with the corresponding element numbers for elements Q_k and I_l respectively. For the example in Figure A.2 we obtain

$$\begin{array}{rl} \tau^{N,M} = & [\,0.0 \quad 0.14 \quad 0.20 \quad 0.29 \quad 0.37 \quad 0.40 \quad 0.54 \quad 0.60 \quad 0.71 \quad 0.80 \quad 0.86 \quad 1.0\,], \\ i\_isp^N = & [\ \ 4 \qquad 9 \qquad 9 \qquad 14 \qquad 13 \qquad 13 \qquad 12 \qquad 12 \qquad 17 \qquad 17 \qquad 22 \qquad], \\ i\_isp^M = & [\ \ 1 \qquad 1 \qquad 2 \qquad 2 \qquad 2 \qquad 3 \qquad 3 \qquad 4 \qquad 4 \qquad 5 \qquad 5 \qquad]. \end{array}$$

For $i = 1, \ldots, n\_isp-1$ let $k = i\_isp^M(i)$, $l = i\_isp^N(i)$. For $j = 1, 2, 3, 4$ we calculate

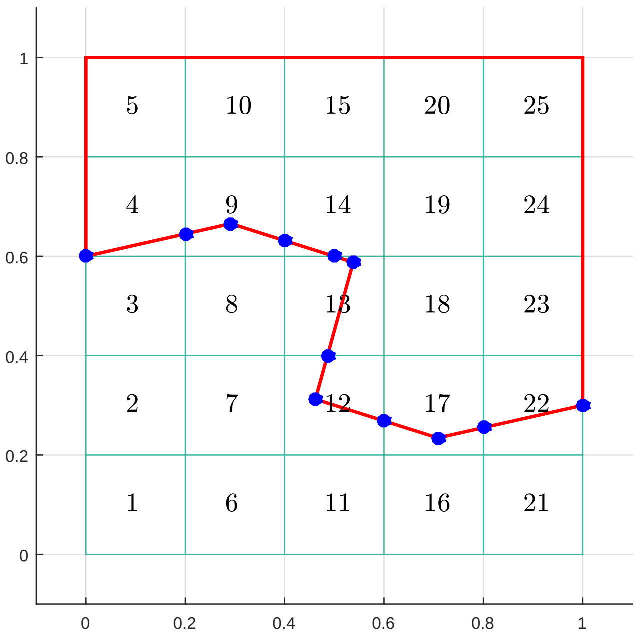


Figure A.2.: Intersection Points of γ^M and the mesh and vertices $\vec{\gamma}_j^M$.

$$int\_isp(j,i) := \int_{J_i} \chi_k^M(t)\phi_j^l(\gamma^M(t))\,dt = \int_{\tau_i}^{\tau_{i+1}} \phi_j^l(\gamma^M(t))\,dt.$$

By transforming these integrals to the interval $(-1, 1)$, substituting the basis functions ϕ_j^l with the basis shape functions $\hat{\phi}_j$ and using Gauss-Legendre quadrature with

ng Gauss nodes s_n and weights w_n, we obtain

$$\begin{aligned}
int_isp(j,i) &= \int_{\tau_i}^{\tau_{i+1}} \phi_j^l(\gamma^M(t))\,dt \\
&= \frac{\tau_{i+1}-\tau_i}{2}\int_{-1}^{1} \phi_j^l\left(\gamma^M\left(\frac{\tau_{i+1}-\tau_i}{2}s+\frac{\tau_{i+1}+\tau_i}{2}\right)\right)ds \\
&= \frac{\tau_{i+1}-\tau_i}{2}\int_{-1}^{1} \hat{\phi}_j\left(F_l^{-1}\left(\gamma^M\left(\frac{\tau_{i+1}-\tau_i}{2}s+\frac{\tau_{i+1}+\tau_i}{2}\right)\right)\right)ds \\
&= \frac{\tau_{i+1}-\tau_i}{2}\sum_{n=1}^{ng} w_n\hat{\phi}_j\left(F_l^{-1}\left(\gamma^M\left(\frac{\tau_{i+1}-\tau_i}{2}s_n+\frac{\tau_{i+1}+\tau_i}{2}\right)\right)\right).
\end{aligned}$$

We are now in the position to assemble the trace matrix via

Algorithm A.4.1.

1: ***function*** ASSEMBLETRACEMATRIX($\tau^{N,M}$, i_isp^N, i_isp^M)
2: $\quad int\_isp \leftarrow$ CALCINTEGRALSTRACEMATRIX$(\tau^{N,M}, i_isp^N, i_isp^M)$
3: $\quad T_{\gamma^M}^{N,M} \leftarrow sparse(M-1, N^2)$
4: $\quad$ ***for*** $i \leftarrow 1,4$ ***do***
5: $\qquad T_{\gamma^M}^{N,M} \leftarrow T_{\gamma^M}^{N,M} + sparse(i\_isp^M, ETN(i\_isp^N, i), int\_isp(i,:), M-1, N^2)$
6: $\quad$ ***end for***
7: $\quad$ ***return*** $T_{\gamma^M}^{N,M}$
8: ***end function***

A.5. Assembly of the vector $\hat{F}_{\gamma^M}^{N,M}$

Here, we assemble the vector $\hat{F}_{\gamma^M}^{N,M} = (\hat{f}_i^{N,M})$, $i = 1,\ldots,N^2$ with

$$\hat{f}_i^{N,M} := \int_{\hat{\Omega}} \widetilde{f}_{\gamma^M}(x,y)\Phi_i^N(x,y)\,d(x,y), \quad i = 1,\ldots,N^2,$$

which is one part of the right hand side of the discrete state equation. Therefore, we calculate on each element Q_i, $i = 1,\ldots,(N-1)^2$, and for $j = 1,2,3,4$

$$\int_{Q_i} \widetilde{f}_{\gamma^M}(x,y)\phi_j^i(x,y)\,d(x,y).$$

- Let $i\_omg^{N,M}$ be an index vector of element numbers i such that Q_i are completely contained in Ω_{γ^M} (cf. the red colored elements in Figure A.3). Then we obtain

$$\int_{Q_i} \widetilde{f}_{\gamma^M}(x,y)\phi_j^i(x,y)\,d(x,y) = \int_{Q_i} f(x,y)\phi_j^i(x,y)\,d(x,y). \tag{A.5.1}$$

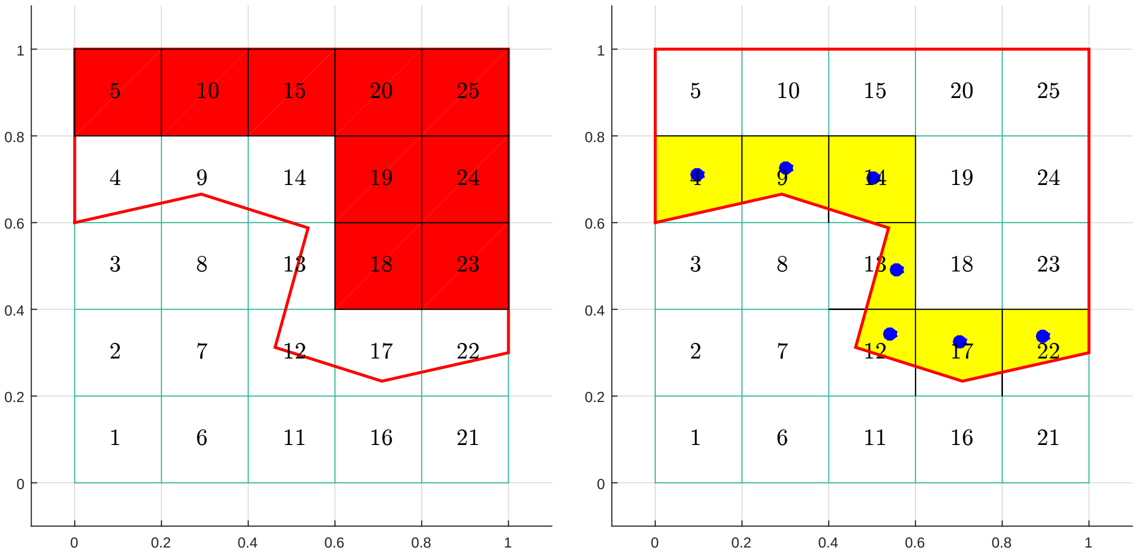


Figure A.3.: Red elements Q_i, which are completely contained in Ω_{γ^M} and Elements Q_i with yellow intersection areas $Q_i \cap \Omega_{\gamma^M}$ and its blue marked centroids.

- Let $i\_caa^{N,M}$ be an index vector of element numbers i such that Q_i are devided by γ^M (cf. the elements with yellow intersection areas $Q_i \cap \Omega_{\gamma^M}$ in Figure A.3). Then we obtain

$$\int_{Q_i} \widetilde{f}_{\gamma^M}(x,y)\phi_j^i(x,y)\, d(x,y) = \int_{Q_i \cap \Omega_{\gamma^M}} f(x,y)\phi_j^i(x,y)\, d(x,y). \tag{A.5.2}$$

- All integrals on the remainder elements equal zero.

First of all we approximate the integrals $\int_{Q_i} f(x,y)\phi_j^i(x,y)\, d(x,y)$ on all elements Q_i, $i = 1, \dots, (N-1)^2$. This is obtained by Gauss-Legendre quadrature with ng Gauss nodes $(\hat{x}_n, \hat{y}_n)$ and weights w_n, i.e.

$$\begin{aligned}
\int_{Q_i = F_i(\hat{Q})} f(x,y)\phi_j^i(x,y)\, d(x,y) &= \int_{\hat{Q}} f(F_i(\hat{x},\hat{y}))\phi_j^i(F_i(\hat{x},\hat{y}))\, |\det(F_i'(\hat{x},\hat{y}))|\, d(\hat{x},\hat{y}) \\
&= \frac{(h^N)^2}{4} \int_{\hat{Q}} f(F_i(\hat{x},\hat{y}))\hat{\phi}_j(\hat{x},\hat{y})\, d(\hat{x},\hat{y}) \\
&\approx \frac{(h^N)^2}{4} \sum_{n=1}^{ng} w_n f(F_i(\hat{x}_n,\hat{y}_n))\hat{\phi}_j(\hat{x}_n,\hat{y}_n) \\
&=: int_rhs(i,j)
\end{aligned}$$

The integrals in the matrix int_rhs are independent of γ^M and have to be evaluated only once before the optimization process starts. The integrals in (A.5.1) are then accessed by $int\_rhs(i\_omg^{N,M}, j)$.
For an approximation of the integrals in (A.5.2) we calculate the areas $A_{caa}(i)$ and

the centroids $(x_{caa}(i), y_{caa}(i))$ of the polygonal intersection areas $Q_i \cap \Omega_{\gamma^M}$ and define

$$\int_{Q_i \cap \Omega_{\gamma^M}} f(x,y)\phi_j^i(x,y)\, d(x,y) \approx A_{caa}(i) f(x_{caa}(i), y_{caa}(i))\phi_j^i(x_{caa}(i), y_{caa}(i)) \\ =: int\_caa(i,j)$$

Finally we can assemble the vector $\hat{F}^{N,M}_{\gamma^M}$ via

Algorithm A.5.1.

function $\textsc{assembleRHS\_StateEquation}(\tau^{N,M}, i\_isp^N, i\_isp^M)$
  $i\_caa^{N,M}, A_{caa}, x_{caa}, y_{caa} \leftarrow \textsc{calcCentroidAndArea}()$
  $int\_caa \leftarrow \textsc{calcIntegralsCentroidArea}()$
  $i\_omg^{N,M} \leftarrow \textsc{assembleKnots\_omg}()$
  $\hat{F}^{N,M}_{\gamma^M} \leftarrow zeros(N^2, 1)$
  for $i \leftarrow 1, 4$ ***do***
    $\hat{F}^{N,M}_{\gamma^M} \leftarrow \hat{F}^{N,M}_{\gamma^M} + sparse(ETN(i\_caa^{N,M}, i), 1, int\_caa(:, i), N^2, 1)$
      $+ sparse(ETN(i\_omg^{N,M}, i), 1, int\_rhs(i\_omg^{N,M}, i), N^2, 1)$
  end for
  return $\hat{F}^{N,M}_{\gamma^M}$
end function

A.6. Assembly of the finite element stiffness matrix $\mathcal{A}^M$ and the finite element mass matrix $\mathcal{B}^M$

The goal of this section is to assemble the finite element stiffness matrix $\mathcal{A}^M := (\alpha_{ij}^M)$ with

$$\alpha_{ij}^M := \left(\dot{B}_i^M, \dot{B}_j^M\right)_{L^2(I)}, \quad i,j = 2, \ldots, M-1.$$

Therefore we define a reference element $\hat{I} := (-1, 1)$ and basis shape functions on $\hat{I}$:

$$\begin{aligned} \hat{b}_1(\hat{t}) &= -\frac{1}{2}\left(\hat{t} - 1\right), & \dot{\hat{b}}_1(\hat{t}) &= -\frac{1}{2}, \\ \hat{b}_2(\hat{t}) &= \frac{1}{2}\left(\hat{t} + 1\right), & \dot{\hat{b}}_2(\hat{t}) &= \frac{1}{2}. \end{aligned}$$

For the j-th subinterval $I_j = (t_j, t_{j+1})$ we define

$$\begin{aligned} G_j &: \hat{I} \to I_j, & \hat{t} &\mapsto t_j \hat{b}_1(\hat{t}) + t_{j+1}\hat{b}_2(\hat{t}) = \frac{t_{j+1} - t_j}{2}\hat{t} + \frac{t_{j+1} + t_j}{2}, \\ G_j^{-1} &: I_j \to \hat{I}, & t &\mapsto G_j^{-1}(t) = \frac{1}{t_{j+1} - t_j}\left(2t - t_j - t_{j+1}\right). \end{aligned}$$

With the help of G_j^{-1} and the basis shape functions on $\hat{I}$ we define basis functions b_k^j,

$$b_k^j(t) := \hat{b}_k(G_j^{-1}(t)), \quad k = 1, 2,$$

on the subinterval $I_j = (t_j, t_{j+1})$. We obtain

$$\hat{b}_k(\hat{t}) = b_k^j(G_j(\hat{t})) \quad \Longrightarrow \quad \dot{\hat{b}}_k(\hat{t}) = \dot{G}_j(\hat{t})\dot{b}_k^j(G_j(\hat{t})).$$

We calculate the local stiffness matrix $\mathcal{A}_{\text{loc}}^M := \big((\alpha_{\text{loc}}^M)_{kl}\big)$ with

$$\begin{aligned}
(\alpha_{\text{loc}}^M)_{kl} &:= \int_{I_j=G_j(\hat{I})} \dot{b}_k^j(t)\dot{b}_l^j(t)\,dt \\
&= \int_{\hat{I}} \dot{b}_k^j(G_j(\hat{t}))\dot{b}_l^j(G_j(\hat{t}))\dot{G}_j(\hat{t})\,d\hat{t} \\
&= \int_{\hat{I}} \frac{\dot{\hat{b}}_k(\hat{t})\dot{\hat{b}}_l(\hat{t})}{\dot{G}_j(\hat{t})}\,d\hat{t} \\
&= \frac{2}{t_{j+1}-t_j}\int_{\hat{I}} \dot{\hat{b}}_k(\hat{t})\dot{\hat{b}}_l(\hat{t})\,d\hat{t}.
\end{aligned}$$

For $k = l = 1$ we obtain

$$\frac{2}{t_{j+1}-t_j}\int_{\hat{I}} \dot{\hat{b}}_1(\hat{t})\dot{\hat{b}}_1(\hat{t})\,d\hat{t} = \frac{2}{t_{j+1}-t_j}\frac{1}{4}\int_{-1}^{1} d\hat{t} = \frac{2}{t_{j+1}-t_j}\frac{1}{2},$$

and for $k = 1$, $l = 2$ we have

$$\frac{2}{t_{j+1}-t_j}\int_{\hat{I}} \dot{\hat{b}}_1(\hat{t})\dot{\hat{b}}_2(\hat{t})\,d\hat{t} = \frac{2}{t_{j+1}-t_j}\left(-\frac{1}{4}\right)\int_{-1}^{1} d\hat{t} = \frac{2}{t_{j+1}-t_j}\left(-\frac{1}{2}\right).$$

This leads to the following algorithm to assemble the stiffness matrix $\mathcal{A}^M$

Algorithm A.6.1.

1: ***function*** ASSEMBLESTIFFNESSMATRIXM*(M)*
2: $\quad h^M \leftarrow \frac{1}{M-1}$
3: $\quad ETM \leftarrow [(1 : NG-1)', (2 : NG)']$
4: $\quad \mathcal{A}_{\text{loc}}^M \leftarrow \frac{1}{h^M}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$
5: $\quad \mathcal{A}^M \leftarrow sparse(M, M)$
6: $\quad$ ***for*** $i \leftarrow 1, 2$ ***do***
7: $\quad\quad$ ***for*** $j \leftarrow 1, 2$ ***do***
8: $\quad\quad\quad \mathcal{A}^M \leftarrow \mathcal{A}^M + sparse(ETM(:, i), ETM(:, j), \mathcal{A}_{\text{loc}}^M(i, j), M, M)$
9: $\quad\quad$ ***end for***
10: $\quad$ ***end for***

$\mathcal{A}^M \leftarrow \mathcal{A}^M(2 : end - 1, 2 : end - 1)$
return $\mathcal{A}^M$
end function

To assemble the global mass matrix $\mathcal{B}^M$ we calculate the local mass matrix $\mathcal{B}^M_{\text{loc}} := \big((\beta^M_{\text{loc}})_{kl}\big)$ with

$$\begin{aligned}(\beta^M_{\text{loc}})_{kl} &:= \int_{I_j = G_j(\hat{I})} b^j_k(t) b^j_l(t)\, dt \\ &= \int_{\hat{I}} b^j_k(G_j(\hat{t})) b^j_l(G_j(\hat{t})) \dot{G}_j(\hat{t})\, d\hat{t} \\ &= \frac{t_{j+1} - t_j}{2} \int_{\hat{I}} \hat{b}_k(\hat{t}) \hat{b}_l(\hat{t})\, d\hat{t}.\end{aligned}$$

For $k = l = 1$ we obtain

$$\begin{aligned}\frac{t_{j+1} - t_j}{2} \int_{\hat{I}} \hat{b}_1(\hat{t}) \hat{b}_1(\hat{t})\, d\hat{t} &= \frac{t_{j+1} - t_j}{2} \frac{1}{4} \int_{-1}^{1} \left(\hat{t} - 1\right)^2\, d\hat{t} \\ &= \frac{t_{j+1} - t_j}{2} \frac{1}{4} \left[\frac{1}{3} \left(\hat{t} - 1\right)^3\right]_{-1}^{1} \\ &= \frac{t_{j+1} - t_j}{2} \frac{2}{3}.\end{aligned}$$

For $k = 1$, $l = 2$ we have

$$\begin{aligned}\frac{t_{j+1} - t_j}{2} \int_{\hat{I}} \hat{b}_1(\hat{t}) \hat{b}_2(\hat{t})\, d\hat{t} &= \frac{t_{j+1} - t_j}{2} \left(-\frac{1}{4}\right) \int_{-1}^{1} \left(\hat{t} - 1\right) \left(\hat{t} + 1\right)\, d\hat{t} \\ &= \frac{t_{j+1} - t_j}{2} \left(-\frac{1}{4}\right) \left[\frac{1}{3}\hat{t}^3 - \hat{t}\right]_{-1}^{1} \\ &= \frac{t_{j+1} - t_j}{2} \left(-\frac{1}{4}\right) \left(-\frac{2}{3} - \frac{2}{3}\right) = \frac{t_{j+1} - t_j}{2} \frac{1}{3}.\end{aligned}$$

This leads to

$$\mathcal{B}^M_{\text{loc}} = \frac{h^M}{6} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and an algorithm analogously to Algorithm A.6.1.

A.7. Details on evaluating the shape derivative vectors $\mathcal{J}_1^{(N,M)}(\gamma^M)$ and $\mathcal{J}_2^{(N,M)}(\gamma^M)$

The aim of this section is to assemble the vectors $\mathcal{J}_1^{(N,M)}(\gamma^M)$ and $\mathcal{J}_2^{(N,M)}(\gamma^M)$. For the ith entry of $\mathcal{J}_1^{(N,M)}(\gamma^M)$ we obtain

$$\begin{aligned}\left(\mathcal{J}_1^{(N,M)}(\gamma^M)\right)_i &= \langle \hat{j}^{(N,M)\prime}(\gamma^M), \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_i^M \rangle_{(S^M)^*,S^M} \\ &= \int_I \mathcal{G}_{\gamma^M}^M(t)\mathcal{M}_{\gamma^M}^M(t) \frac{\left(\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right) B_i^M(t)\right)^T \vec{n}_{\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2}\,dt \\ &= \int_I \mathcal{G}_{\gamma^M}^M(t)\mathcal{M}_{\gamma^M}^M(t) B_i^M(t)\frac{n_{1,\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2}\,dt.\end{aligned}$$

and

$$\begin{aligned}\left(\mathcal{J}_2^{(N,M)}(\gamma^M)\right)_i &= \langle \hat{j}^{(N,M)\prime}(\gamma^M), \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right) B_i^M \rangle_{(S^M)^*,S^M} \\ &= \int_I \mathcal{G}_{\gamma^M}^M(t)\mathcal{M}_{\gamma^M}^M(t) B_i^M(t)\frac{n_{2,\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2}\,dt.\end{aligned}$$

To assemble these integrals we calculate contributions on each subinterval (t_i, t_{i+1}) using the basis functions b_1^i, b_2^i. First we note that for $t \in (t_i, t_{i+1})$ we have

$$\begin{aligned}\dot{\gamma}^M(t) &= \vec{\gamma}_i^M \dot{B}_i^M(t) + \vec{\gamma}_{i+1}^M \dot{B}_{i+1}^M(t) = \vec{\gamma}_i^M \dot{b}_1^i(t) + \vec{\gamma}_{i+1}^M \dot{b}_2^i(t) \\ &= (M-1)\left(\vec{\gamma}_{i+1}^M - \vec{\gamma}_i^M\right), \\ \|\dot{\gamma}^M(t)\|_2 &= (M-1)\|\vec{\gamma}_{i+1}^M - \vec{\gamma}_i^M\|_2 \\ &= (M-1)\sqrt{(\gamma_{1,i+1}^M - \gamma_{1,i}^M)^2 + (\gamma_{2,i+1}^M - \gamma_{2,i}^M)^2} =: c_i(M-1)\end{aligned}$$

Now we obtain for b_j^i, $j = 1, 2$, on the subinterval (t_i, t_{i+1})

$$\begin{aligned}&\int_{t_i}^{t_{i+1}} \mathcal{G}_{\gamma^M}^M(t)\mathcal{M}_{\gamma^M}^M(t) b_j^i(t) \frac{n_{1,\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2}\,dt \\ &\quad = \frac{g_i^M \mu_i^M}{c_i^M(M-1)} \int_{t_i}^{t_{i+1}} b_j^i(t) n_{1,\gamma^M}(t)\,dt \\ &\quad = \frac{g_i^M \mu_i^M}{c_i^M(M-1)} \frac{t_{i+1}-t_i}{2} \int_{-1}^{1} b_j^i(G_i(\hat{t})) n_{1,\gamma^M}(G_i(\hat{t}))\,d\hat{t} \\ &\quad = \frac{g_i^M \mu_i^M}{c_i^M(M-1)} \frac{t_{i+1}-t_i}{2} \int_{-1}^{1} \hat{b}_j(\hat{t}) n_{1,\gamma^M}(G_i(\hat{t}))\,d\hat{t} \\ &\quad \approx \frac{g_i^M \mu_i^M}{c_i^M(M-1)} \frac{t_{i+1}-t_i}{2} \sum_{n=1}^{ng} \alpha_n \hat{b}_j(\hat{t}_n) n_{1,\gamma^M}(G_i(\hat{t}_n)) =: int\_lmp1(i,j),\end{aligned}$$

and equivalently

$$\int_{t_i}^{t_{i+1}} \mathcal{G}_{\gamma^M}^M(t)\mathcal{M}_{\gamma^M}^M(t)b_j^i(t)\frac{n_{2,\gamma^M}(t)}{\|\dot{\gamma}^M(t)\|_2}\,dt$$
$$\approx \frac{g_i^M\mu_i^M}{c_i^M(M-1)}\frac{t_{i+1}-t_i}{2}\sum_{n=1}^{ng}\alpha_n\hat{b}_j(\hat{t}_n)n_{2,\gamma^M}(G_i(\hat{t}_n)) =: int\_lmp2(i,j).$$

Finally we can assemble the vectors $\mathcal{J}_1^{(N,M)}(\gamma^M)$ and $\mathcal{J}_2^{(N,M)}(\gamma^M)$ via

Algorithm A.7.1.

1: ***function*** ASSEMBLESHAPEDERIVATIVE(γ^M,$\mathcal{G}^M$, $\mathcal{M}^M$)
2: $int\_lmp1, int\_lmp2 \leftarrow$ CALCINTEGRALSSHAPEDERIVATIVE(γ^M,$\mathcal{G}^M$, $\mathcal{M}^M$)
3: $\mathcal{J}_1^{(N,M)}(\gamma^M) \leftarrow zeros(M,1)$
4: $\mathcal{J}_2^{(N,M)}(\gamma^M) \leftarrow zeros(M,1)$
5: ***for*** $i \leftarrow 1,2$ ***do***
6: $\mathcal{J}_1^{(N,M)}(\gamma^M) \leftarrow \mathcal{J}_1^{(N,M)}(\gamma^M) + sparse(ETM(:,i),1,int\_lmp1(:,i),M,1)$
7: $\mathcal{J}_2^{(N,M)}(\gamma^M) \leftarrow \mathcal{J}_2^{(N,M)}(\gamma^M) + sparse(ETM(:,i),1,int\_lmp2(:,i),M,1)$
8: ***end for***
9: ***return*** $\mathcal{J}_1^{(N,M)}(\gamma^M)$, $\mathcal{J}_2^{(N,M)}(\gamma^M)$
10: ***end function***

A.8. Solving the discrete state system

In our optimization process we have to repeatedly solve linear systems with the matrix

$$\begin{pmatrix} \hat{A}^N & -(T_{\gamma^M}^{N,M})^T \\ -T_{\gamma^M}^{N,M} & 0 \end{pmatrix} \tag{A.8.1}$$

for different γ^M.
We describe the solution of (4.1.3) via the well-known Schur complement reduction, cf. [BGL05]. Since $\hat{A}^N$ is nonsingular, by

$$\begin{pmatrix} \hat{A}^N & -(T_{\gamma^M}^{N,M})^T \\ -T_{\gamma^M}^{N,M} & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -T_{\gamma^M}^{N,M}(\hat{A}^N)^{-1} & I \end{pmatrix}\begin{pmatrix} \hat{A}^N & 0 \\ 0 & S \end{pmatrix}\begin{pmatrix} I & -(\hat{A}^N)^{-1}(T_{\gamma^M}^{N,M})^T \\ 0 & I \end{pmatrix},$$

the *Schur complement* $S := -T_{\gamma^M}^{N,M}(\hat{A}^N)^{-1}(T_{\gamma^M}^{N,M})^T$ is also nonsingular.
If we multiply the first equation of (4.1.3) from left by $-T_{\gamma^M}^{N,M}(\hat{A}^N)^{-1}$ we obtain

$$-T_{\gamma^M}^{N,M}\underline{\hat{u}}_{\gamma^M}^N + T_{\gamma^M}^{N,M}(\hat{A}^N)^{-1}(T_{\gamma^M}^{N,M})^T\underline{g}_{\gamma^M}^M = -T_{\gamma^M}^{N,M}(\hat{A}^N)^{-1}\hat{F}_{\gamma^M}^{N,M}.$$

Using the second equation of (4.1.3), which is $-T_{\gamma^M}^{N,M}\,\hat{\underline{u}}_{\gamma^M}^{N} = 0$, we reach the following algorithm:

1. Compute $\underline{g}_{\gamma^M}^{M}$ from
$$T_{\gamma^M}^{N,M}(\hat{A}^N)^{-1}(T_{\gamma^M}^{N,M})^T\underline{g}_{\gamma^M}^{M} = -T_{\gamma^M}^{N,M}(\hat{A}^N)^{-1}\hat{F}_{\gamma^M}^{N,M}. \tag{A.8.2}$$

2. Obtain $\underline{u}_{\gamma^M}^{N}$ by solving
$$\hat{A}^N\underline{u}_{\gamma^M}^{N} = \hat{F}_{\gamma^M}^{N,M} + (T_{\gamma^M}^{N,M})^T\underline{g}_{\gamma^M}^{M}.$$

We note that building or factorizing the Schur complement S is too expensive, but we can solve (A.8.2) by an iterative method, where S is only needed in the form of matrix-vector products

$$\underline{p}^M = -S\underline{g}^M = T_{\gamma^M}^{N,M}(\hat{A}^N)^{-1}(T_{\gamma^M}^{N,M})^T\underline{g}^M.$$

Every action of S on $\underline{g}^M$ requires a matrix-vector product with $(T_{\gamma^M}^{N,M})^T$, the solution of a linear system with the matrix $\hat{A}^N$, and a matrix-vector product with $T_{\gamma^M}^{N,M}$.
To solve the linear systems with coefficient matrix $\hat{A}^N$ efficiently, we apply a preconditioned conjugate gradient (PCG) method [Saa03, HS52] and the commonly used incomplete Cholesky factorization with threshold dropping (ICT) [Man80] of $\hat{A}^N$ as a preconditioner. This procedure is realized with the MATLAB built-in functions `pcg` and `ichol`. Depending on the available memory it is worthwhile to save the incomplete Cholesky factor $\hat{L}^N$ for a drop tolerance value as low as possible. This increases the density of $\hat{L}^N$, but leads to significantly less PCG iterations in our computations. Nevertheless, the crucial point is to construct a preconditioner to solve (A.8.2) efficiently by an iterative method.
However, if enough memory is available, a sparse direct solver can be applied to solve linear systems with (A.8.1). We use the MATLAB built-in backslash operator.

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Abstract

In the present thesis we deal with a class of shape optimization problems for the two-dimensional Poisson equation with homogeneous Dirichlet boundary conditions. We consider tracking-type objective functionals and characterize admissible domains through admissible parametrizations of the boundary. These parametrizations serve as the control variable in the optimization process. A fictitious domain method is used to embed admissible domains into a larger, geometrically simpler reference domain, on which the analysis and the computations are then performed. Shape optimization problems of this kind were considered in [KP98, Sla00].

We generalize the problem setting from [KP98, Sla00] in the following way. On the one hand, the extension to a wider class of admissible domains is established, this means that the variable part of the boundary of admissible domains is not given as the graph of an admissible function, but rather as the image of an admissible curve. One the other hand, a more general class of objective functionals is considered. In particular, we are able to discard previously necessary restrictions to the observation domain, in which we track the L^2-error of the state with respect to a desired state. This substantially enhances the sensitivity of the objective functional with respect to boundary variations.

To study the shape optimization problem in the generalized setting, we provide an extended functional-analytic framework. The following results are then transferred to and proven within this framework: the existence of a solution of the corresponding shape optimization problem, Fréchet-differentiability of the reduced objective functional and a resulting integral representation for the first derivative. As an approximation of the second derivative of the reduced objective functional, we furthermore show the existence of the symmetrical directional derivative of its first derivative, for which an elegant integral representation is also established.

The differentiability results are then used to solve the shape optimization problems by iterative descent methods in an appropriate Hilbert space setting. On the one hand we discuss the steepest descent method and the BFGS quasi-Newton method. On the other hand, we are able to present an inexact Newton-like method using the results of the approximation of the second derivative.

Moreover, a mixed finite element discretization and finite-dimensional descent methods corresponding to the continuous case are provided to solve the shape optimization problem numerically. Eventually, the functionality and reliability of the developed methods are presented in numerical experiments.

Zusammenfasssung

Die vorliegende Arbeit behandelt eine Klasse von Gebietsoptimierungsproblemen für die zweidimensionale Poisson-Gleichung mit homogenen Dirichlet Randbedingungen. Dabei werden tracking-type Zielfunktionale betrachtet, und zulässige Gebiete werden durch zulässige Parametrisierungen des Gebietsrandes charakterisiert. Solche Parametrisierungen dienen als Kontrollvariable im Optimierungsprozess. Ein Einbettungsverfahren für die Zustandsgleichung wird angewendet, um zulässige Gebiete in ein größeres, geometrisch vereinfachtes Referenzgebiet zu integrieren, in welchem dann die Analysis und die erwünschten Berechnungen durchgeführt werden. Gebietsoptimierungsprobleme dieser Art wurden bereits in [KP98, Sla00] diskutiert.
Die Rahmenbedingungen der Probleme aus [KP98, Sla00] werden in dieser Dissertation auf folgende Weise verallgemeinert: Einerseits wird eine Erweiterung durch eine größere Klasse an zulässigen Gebieten etabliert. Hier ist der variable Teil des Randes zulässiger Gebiete nicht mehr durch den Graph einer zulässigen Funktion gegeben, sondern durch das Bild einer zulässigen Kurve. Andererseits wird eine größere Klasse von Zielfunktionalen betrachtet, genauer: auf die bisher notwendigen Einschränkungen hinsichtlich des Beobachtungsgebietes, in dem der L^2-Fehler des bestehenden Zustandes im Vergleich zum gewünschten Zustand gemessen wird, kann nun verzichtet werden. Dadurch wird die Sensitivität des Zielfunktionales in Bezug auf Randvariationen erheblich erhöht.
Zur Untersuchung der Gebietsoptimierungsprobleme in dem nun verallgemeinerten Setting wird zunächst ein erweiterter funktionalanalytischer Rahmen bereitgestellt. Darin werden dann die folgenden bekannten Resultate übertragen und bewiesen: die Existenz von Lösungen des korrespondierenden Gebietsoptimierungsproblems, Fréchet-Differenzierbarkeit des reduzierten Zielfunktionales, sowie eine daraus resultierende Integraldarstellung der ersten Ableitung. Als Approximation der zweiten Ableitung des reduzierten Zielfunktionales wird darüber hinaus die Existenz der symmetrischen Richtungsableitung ihrer ersten Ableitung gezeigt. Auch hierfür wird eine elegante Integraldarstellung hergeleitet.
Die Differenzierbarkeitsaussagen finden Anwendung bei der Lösung der Gebietsoptimierungsprobleme durch iterative Abstiegsverfahren in einer geeigneten Hilbert-Raum Konfiguration. Zunächst werden das Gradientenverfahren und das BFGS Quasi-Newton-Verfahren diskutiert, mit Hilfe der Ergebnisse zur Approximation der zweiten Ableitung wird darüber hinaus auch ein inexaktes, Newton-ähnliches Verfahren präsentiert.
Es werden eine gemischte Finite-Elemente-Diskretisierung und zum kontinuierlichen Fall korrespondierende, endlich-dimensionale Abstiegsverfahren bereitgestellt, um das Gebietsoptimierungsproblem auch numerisch zu behandeln. Abschließend wird die Funktionalität und Zuverlässigkeit der entwickelten Verfahren anhand numerischer Beispiele vorgestellt.

Lebenslauf

entfällt aus datenschutzrechtlichen Gründen