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Symmetries of 3d-TQFTs and the Brauer-Picard Group

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Introduction

For a finite tensor category \mathcal{C} the *Brauer-Picard group* $\text{BrPic}(\mathcal{C})$ is defined as the group of equivalence classes of invertible \mathcal{C} -bimodule categories, where the group structure is given by the relative Deligne tensor product. We can actually assign to \mathcal{C} the Brauer-Picard 3-group $\underline{\text{BrPic}}(\mathcal{C})$ of invertible \mathcal{C} -bimodule categories, \mathcal{C} -bimodule functors and natural transformations. Truncating the Brauer-Picard 3-group by identifying equivalent bimodule functors gives us the 2-group $\underline{\text{BrPic}}(\mathcal{C})$ and we can further truncate this 2-group to the group $\text{BrPic}(\mathcal{C})$ by identifying equivalent bimodule categories. This group is an important invariant of the tensor category \mathcal{C} and appears at several essential places in representation theory and mathematical physics, as explained below. In both these fields it is crucial to understand in detail the structure of the group $\text{BrPic}(\mathcal{C})$.

An important structural insight is the following result proven in Thm. 1.1 [ENO10] for \mathcal{C} a fusion category and in Thm. 4.1 [DN12] for \mathcal{C} a finite tensor category (not necessarily semisimple): There is a group isomorphism from the Brauer-Picard group to the group of equivalence classes of *braided autoequivalences* of the *Drinfeld center* $Z(\mathcal{C})$:

$$\text{BrPic}(\mathcal{C}) \cong \text{Aut}_{br}(Z(\mathcal{C})) \tag{1}$$

Instead of working with \mathcal{C} -bimodule categories and the rather difficult relative Deligne tensor product, we can hence work with autoequivalences where the group multiplication is the composition of monoidal functors. It turns out, that the group $\text{Aut}_{br}(Z(\mathcal{C}))$ is still complicated, because it incorporates, for e.g. $\mathcal{C} = H\text{-mod}$ the category of finite dimensional modules over a finite dimensional Hopf algebra H , the control over the group of Hopf automorphisms $\text{Aut}_{Hopf}(H)$ and 2-cocycles on H (see Definition 1.2.12). In the case $\mathcal{C} = \text{Rep}(G)$ of finite dimensional complex representations of a finite group G (respectively $\mathcal{C} = \text{Vect}_G$ which has the same Drinfeld center) computing $\text{Aut}_{br}(Z(\text{Vect}_G))$ and hence the Brauer-Picard group is already an interesting and non-trivial task.

Let us give two motivations for why the Brauer-Picard group is important.

Symmetries of Dijkgraaf-Witten Theories

A topological quantum field theory (TQFT) is a symmetric monoidal functor from the category of n -dimensional cobordisms to the category of vector spaces. In other

words, it assigns to a closed $(n-1)$ -manifold Σ a vector space $F(\Sigma)$, to an n -dimensional cobordisms class $[\Sigma_1, M, \Sigma_2]$ between two closed $(n-1)$ -manifolds Σ_1, Σ_2 a linear map $F(M) : F(\Sigma_1) \rightarrow F(\Sigma_2)$ and comes with a collection of isomorphisms $F(\emptyset) \simeq \mathbb{C}$, $F(M \amalg N) \simeq F(M) \otimes F(N)$ such that certain axioms are satisfied (see [Ati89] for Atiyah's Axioms). Such a TQFT produces topological invariants of n -manifolds and has turned out to be a particularly successful approach in low dimensional topology ($n = 1, 2, 3$) e.g. with the study of quantum invariants (see [RT91], [TV92] and [Tur10]).

Motivated by copy-and-paste procedures along submanifolds of higher codimension, the so-called extended TQFTs were introduced (see [La93]). In these theories, we assign values not just to n -manifolds and codimension one submanifolds, but also to submanifolds of higher codimension. In the case we go down to points, the TQFT is called fully extended. In other words, these are symmetric monoidal n -functors from the weak n -category of n -dimensional cobordisms to some symmetric monoidal weak n -category as target. For a detailed exposition and classification of fully extended TQFT we refer to [Lu09], where Lurie formulates the classification of n -dimensional *framed* fully extended TQFTs in Thm. 2.4.6 and the classification of TQFTs for more general tangential structures in Thm. 2.4.18 Section 3 of [Lu09] gives a detailed sketch of the proof of these Theorems. Here we want to consider theories that assign values to closed oriented 1-manifolds, compact oriented 2-manifolds with boundary and diffeomorphism classes of compact oriented 3-manifolds with corners. We call such theories oriented $(3, 2, 1)$ -extended TQFTs.

In general, the study of symmetries is a crucial aspect in the understanding of quantum field theories. It is certainly desirable to have a solid conceptual grasp of symmetries and symmetry groups. We are interested in symmetries of a certain class of $(3, 2, 1)$ -extended TQFTs, namely Dijkgraaf-Witten theories. These theories have a mathematically rigorous formulation as gauge theories with a finite structure group G and a topological Lagrangian represented by a 3-cocycle $\omega \in Z^3(G, \mathbb{C}^\times)$ (see [DW90] for the original construction). The groupoid of gauge fields and gauge transformations is given by the groupoid of principal G -bundles. A Dijkgraaf-Witten theory then assigns to an oriented 1-dimensional closed manifold Σ the linearization (quantization) of the groupoid of gauge fields. On 2-dimensional manifolds with boundary, the theory uses a so called pull-push construction to define linear functors between two linearizations of groupoids. On 3-dimensional manifolds with corners, the construction involves the choice of a natural isomorphism between a left and right adjoint. The property that the left and right adjoints are natural equivalent is called *ambidexterity*. For more details on the construction we refer to [FQ93], [Mo13] and to Chapter 2 of this thesis. See also [FHLT10] and [LH14] for more details on ambidexterity.

Symmetries of such theories can be viewed in two different ways. On the one hand, it is known that an oriented $(3, 2, 1)$ -extended TQFT F is uniquely determined by the anomaly free modular tensor category it assigns to the circle $\mathcal{B} = F(S^1)$. We refer to [BDSV15] for the precise statement and proof. From this point of view, it is natural to define the symmetry group of F by the group of braided autoequivalence

$\text{Aut}_{br}(\mathcal{B})$. As described above, the untwisted Dijkgraaf-Witten theory assigns to the circle the anomaly free modular tensor category $Z(\text{Rep}(G))$, which is equivalent to $DG\text{-mod}$, the representation category of the Drinfeld double DG . The symmetry group of an untwisted Dijkgraaf-Witten theory based on a finite group G is thus $\text{Aut}_{br}(DG\text{-mod})$. Using only this approach to symmetries however, has its drawbacks. In particular, from this point of view we do not know if a symmetry acts on other field theoretic quantities of the theory, such as boundary conditions and defects. Moreover, in case such an action exists, we do not know if this action is unique. Motivated by the study of symmetries in 2-dimensional conformal field theories (see [FFRS04], [FFRS07]), one should consider a different angle, namely defining a symmetry to be an invertible topological codimension one defect. A general approach to the study of defects in 3d-TQFTs was presented in [FSV13]. Given two oriented $(3, 2, 1)$ -extended TQFT theories F and F' with $F(S^1) = \mathcal{B}$ and $F'(S^1) = \mathcal{B}'$ for two anomaly free modular tensor categories \mathcal{B} and \mathcal{B}' living in two separated three dimensional regions, topological surface defects between these two theories exists if and only if \mathcal{B} and \mathcal{B}' are in the same Witt class (see [DMNO13]). This means that there exists a fusion category \mathcal{C} and a braided equivalence $\mathcal{B} \boxtimes \mathcal{B}'^{rev} \simeq Z(\mathcal{C})$, where \mathcal{B}'^{rev} is the category \mathcal{B}' but with opposite braiding. In case such an equivalence exist, defects between these two theories are described by the bicategory of \mathcal{C} -module categories, bimodule functors and natural transformations. If we are in the case $\mathcal{B} = \mathcal{B}' = Z(\mathcal{C})$ for a fusion category \mathcal{C} , we have a distinguished braided equivalence $Z(\mathcal{C}) \boxtimes Z(\mathcal{C})^{rev} \simeq Z(\mathcal{C} \boxtimes \mathcal{C}^{rev})$ and thus surface defects are described by $\mathcal{C} \boxtimes \mathcal{C}^{rev}$ -module categories or equivalently \mathcal{C} -bimodule categories. Symmetries correspond to invertible \mathcal{C} -bimodule categories and the symmetry group is the Brauer-Picard group $\text{BrPic}(\mathcal{C})$. We refer to [FSV13] and [FS15] for more details. In the untwisted Dijkgraaf-Witten case we have $\mathcal{C} = \text{Vect}_G$ for a finite group G and are thus interested in the Brauer-Picard group $\text{BrPic}(\text{Vect}_G)$.

It is desirable to have a firm conceptual understanding as well as a computational grip on this symmetry group. For this reason it is desirable to construct characteristic generators or generating subgroups yielding a convenient decomposition of the Brauer-Picard group. Additionally, we would like to have a physical interpretation of these generators in the Lagrangian setting described above.

Group Extensions of Fusion Categories

Another motivation for the Brauer-Picard group is given by the study of fusion categories. These form an interesting and rather accessible class of tensor categories and play an important role in areas like representation theory, noncommutative algebra and mathematical physics (see [ENO05],[ENO10],[EGNO15] and references therein). A full classification of fusion categories, which would include e.g. the classification of finite groups as a special case, is not available and seems to be too hard to archive.

An interesting approach to produce new examples of fusion categories is given by group extensions by finite groups. In fact, there is a large class of fusion categories that are per definition Morita equivalent to iterated group extensions by finite

groups. These are called weakly group-theoretical fusion categories. Several important classification results of fusion categories in particular dimensions are based on these iterated extensions (see [ENO08]).

Given a fusion category \mathcal{C} and a finite group Γ , a Γ -extension of \mathcal{C} is a (faithful) Γ -graded fusion category $\mathcal{D} = \bigoplus_{g \in \Gamma} \mathcal{D}_g$ such that $\mathcal{D}_1 = \mathcal{C}$. We would like to know under which conditions group extensions exists; this leads to certain obstructions which depend on $\text{BrPic}(\mathcal{C})$. If the obstructions vanish, we want to know all Γ -extensions of \mathcal{C} . The data classifying Γ -extensions of \mathcal{C} also depends on $\text{BrPic}(\mathcal{C})$. More precisely:

For a Γ -graded fusion category $\mathcal{D} = \bigoplus_{g \in \Gamma} \mathcal{D}_g$, each component \mathcal{D}_g is a full abelian subcategory of \mathcal{D} , and \mathcal{D}_1 is even a full tensor subcategory of \mathcal{D} . Moreover, each component \mathcal{D}_g is an invertible \mathcal{D}_1 -bimodule category and the tensor product of \mathcal{D} restricted to $\mathcal{D}_g \times \mathcal{D}_h$ induces a \mathcal{D}_1 -bimodule equivalence $M_{g,h} : \mathcal{D}_g \boxtimes_{\mathcal{D}_1} \mathcal{D}_h \xrightarrow{\sim} \mathcal{D}_{gh}$, where $\boxtimes_{\mathcal{D}_1}$ is the relative Deligne product over \mathcal{D}_1 . Further, for the family $M = \{M_{g,h}\}_{g,h \in \Gamma}$ there exists a natural equivalence of \mathcal{D}_1 -bimodule functors:

$$\alpha_{g,h,k} : M_{g,hk} \circ (\text{id}_{\mathcal{D}_g} \boxtimes_{\mathcal{D}_1} M_{h,k}) \xrightarrow{\sim} M_{gh,k} \circ (M_{g,h} \boxtimes_{\mathcal{D}_1} \text{id}_{\mathcal{D}_k})$$

fulfilling the pentagon axiom. It has been shown in [ENO10] that this data determines a Γ -extension of a fusion category \mathcal{C} . In other words, Γ -extensions of \mathcal{C} are in bijection with triples (c, M, α) , where $c : \Gamma \rightarrow \text{BrPic}(\mathcal{C})$ is a group homomorphism, M is a family of \mathcal{C} -bimodule functors as above and α is a family of natural transformations of \mathcal{C} -bimodule functors as above. In order for that to exist, c and M have to fulfill certain conditions, namely we have two obstructions $O_3(c) \in H^3(\Gamma, \text{Inv}(Z(\mathcal{C})))$ and $O_4(c, M) \in H^4(\Gamma, k^\times)$ that need to vanish, where $\text{Inv}(Z(\mathcal{C}))$ is the group of isomorphism classes of invertible objects of $Z(\mathcal{C})$. The H^3 -obstruction arises as a condition when for a chosen c and M we require a quasi-tensor category structure on $\mathcal{D} = \bigoplus_{g \in \Gamma} \mathcal{D}_g$, where by a quasi-tensor structure on \mathcal{D} we mean a functor $\otimes : \mathcal{D} \times \mathcal{D} \rightarrow \mathcal{D}$ together with a natural equivalence $\otimes \circ (\otimes \times \text{id}_{\mathcal{D}}) \simeq \otimes \circ (\text{id}_{\mathcal{D}} \times \otimes)$ where the pentagon axiom has not to be satisfied. The H^4 -obstruction arises when we additionally require the quasi-tensor structure to be a monoidal structure, hence when the pentagon axiom is satisfied.

We see that in order to have an explicit handle on the classification data and the obstructions, we need to understand the structure of the Brauer-Picard group. A decomposition into subgroups that are easy to control would therefore be very useful.

Outline

The goal of this thesis is to achieve a decomposition of $\text{BrPic}(\text{Rep}(G))$ into convenient subgroups analogous to a Bruhat decomposition for groups with a so-called Tits system (BN -pair). In this sense, our approach can be seen as the beginning of a structure theory of the Brauer-Picard group.

For any finite tensor category \mathcal{C} there is a group homomorphism:

$$\text{Ind}_{\mathcal{C}} : \text{Aut}_{\text{mon}}(\mathcal{C}) \rightarrow \text{BrPic}(\mathcal{C}) \stackrel{(1)}{\cong} \text{Aut}_{\text{br}}(Z(\mathcal{C}))$$

given by assigning to a monoidal automorphism $\Psi \in \text{Aut}_{\text{mon}}(\mathcal{C})$ the invertible \mathcal{C} -bimodule category ${}_{\Psi}\mathcal{C}$, where the left \mathcal{C} -module structure is given by precomposing with Ψ ; then we use the isomorphism (1) mentioned at the beginning of the introduction (see also Proposition 2.2.3 of this thesis). The image of this map gives us a natural subgroup of the Brauer-Picard group. If we can choose another category \mathcal{C}' and a braided equivalence $F : Z(\mathcal{C}') \xrightarrow{\sim} Z(\mathcal{C})$, then we get a different induction and a new subgroup of $\text{Aut}_{\text{br}}(Z(\mathcal{C}))$:

$$\text{Ind}_{\mathcal{C}', F} : \text{Aut}_{\text{mon}}(\mathcal{C}') \rightarrow \text{BrPic}(\mathcal{C}') \cong \text{Aut}_{\text{br}}(Z(\mathcal{C}')) \xrightarrow{F} \text{Aut}_{\text{br}}(Z(\mathcal{C}))$$

Consider a finite dimensional Hopf algebra H and let $\mathcal{C} = H\text{-mod}$ be the category of finite dimensional H -modules. Then $Z(H\text{-mod}) = DH\text{-mod} = H^* \bowtie H\text{-mod}$ and we have a canonical choice $\mathcal{C}' = H^*\text{-mod}$ and a canonical isomorphism of Hopf algebras $DH \xrightarrow{\sim} D(H^*)$ (see Thm. 3 in [Rad93]), that gives us a canonical braided equivalence $D(H^*)\text{-mod} \cong DH\text{-mod}$. Hence, we have two canonical subgroups of $\text{Aut}_{\text{br}}(DH\text{-mod})$, namely $\text{im}(\text{Ind}_{H\text{-mod}})$ and $\text{im}(\text{Ind}_{H^*\text{-mod}})$.

Now we introduce an additional set $\mathcal{R} \subset \text{Aut}_{\text{br}}(DH\text{-mod})$. For each decomposition of H into a Radford biproduct $H = A \bowtie K$ (see Sect. 10.6 [Mon93]), where A a Hopf algebra and K a Hopf algebra in the category $Z(A\text{-mod})$, for a choice of a Hopf algebra L in $Z(A\text{-mod})$ and a non-degenerate Hopf algebra pairing $\langle \cdot, \cdot \rangle : K \otimes L \rightarrow k$ in the category $Z(A\text{-mod})$, we can construct a canonical braided equivalence by Thm 3.20 in [BLS15]:

$$\Omega^{\langle \cdot, \cdot \rangle} : Z(A \bowtie K\text{-mod}) \xrightarrow{\sim} Z(A \bowtie L\text{-mod})$$

In the special case of $L := K$, the functor $\Omega^{\langle \cdot, \cdot \rangle}$ is a braided autoequivalence of $Z(A \bowtie K\text{-mod}) = DH\text{-mod}$. In this case, we identify $\langle \cdot, \cdot \rangle$ canonically with an isomorphism of Hopf algebras in $Z(A\text{-mod})$ that we denote by $\delta : K \xrightarrow{\sim} K^*$. We call the triple (A, K, δ) a partial dualization datum and $r_{A, K, \delta} := \Omega^{\langle \cdot, \cdot \rangle} \in \text{Aut}_{\text{br}}(DH\text{-mod})$ a partial dualization of H on K .

In the case of a group algebra $H = kG$ of a finite group G , we obtain for each decomposition of G as a semi-direct product $G = Q \ltimes N$ a decomposition of kG as a Radford biproduct $kG = kQ \bowtie kN$, where N is a normal subgroup of G . kN is a Hopf algebra in $Z(kQ\text{-mod})$, where kQ acts on kN by conjugation and where the kQ -coaction on kN is trivial. The existence of a Hopf isomorphism $\delta : kN \xrightarrow{\sim} k^N$ in the category $Z(kQ\text{-mod})$ forces N to be abelian and kN to be a self-dual kQ -module. Thus, for each partial dualization datum (Q, N, δ) , we obtain an element $r_{Q, N, \delta} \in \text{Aut}_{\text{br}}(DG\text{-mod})$. We denote the set of partial dualizations by \mathcal{R} .

Conjecture. The subgroups $\text{im}(\text{Ind}_{H\text{-mod}})$, $\text{im}(\text{Ind}_{H^*\text{-mod}})$ together with partial dualizations \mathcal{R} generate the group $\text{Aut}_{\text{br}}(DH\text{-mod})$. Further, $\text{Aut}_{\text{br}}(DH\text{-mod})$ decomposes into an (ordered) product of $\text{im}(\text{Ind}_{H\text{-mod}})$, $\text{im}(\text{Ind}_{H^*\text{-mod}})$ and \mathcal{R} .

Such a decomposition would give us effective control over the Brauer-Picard group $\text{BrPic}(\mathcal{C})$ through explicit and natural generators that have additionally an interesting field theoretic interpretation:

For this let us consider the case of a group algebra $H = kG$ with G a finite group.

As explained above, the (untwisted) Dijkgraaf-Witten theory is a topological gauge theory with principal G -bundles on a manifold M as classical fields and a Lagrangian ω (here trivial $\omega = 1$). Based on this gauge theoretic view, it is natural to expect automorphisms of G to be a symmetry of the classical and the quantized theory. Indeed, $\mathcal{V} = \text{Out}(G)$ is a subgroup of $\text{Aut}_{br}(DG\text{-mod})$ and since it already exists at the classical level, we call this a classical symmetry (see also Proposition 2.3.1 (i) and Proposition 6.2.1). It is both, a subgroup of $\text{im}(\text{Ind}_{\text{Vect}_G})$ and a subgroup of $\text{im}(\text{Ind}_{\text{Rep}(G)})$. More symmetries can be obtained by the following idea: equivalence classes of gauge fields are principal G -bundles and thus are in bijection with homotopy classes of maps from M to BG , the classifying space of G . One may view a Dijkgraaf-Witten theory based on (G, ω) as a σ -model with target space BG . Then the 3-cocycle ω can be viewed as a background field on the target space, and the choice of ω corresponds to the choice of a 2-gerbe. Even for trivial ω we obtain a non-trivial symmetry group of this 2-gerbe and hence an additional subgroup of automorphisms of the theory. These symmetries are again classical symmetries, the so-called *background field symmetries* $H^2(G, k^\times)$. Our subgroup $\text{im}(\text{Ind}_{\text{Vect}_G}) = \mathcal{B} \rtimes \mathcal{V}$ where $\mathcal{B} \cong H^2(G, k^\times)$ is therefore the semidirect product of the two classical symmetry groups from above (see also Proposition 2.3.1 (ii) and Proposition 6.3.2). An interesting implication of our result is that in order to obtain the full automorphism group one considers a second σ -model (the 'dual σ -model') associated to $\mathcal{C}' = \text{Rep}(G)$ which however has the same quantum field theory. This dual σ -model induces another subgroup of background field symmetries \mathcal{E} which is a subgroup of $\text{im}(\text{Ind}_{\text{Rep}(G)})$ (see Proposition 2.3.1 (iii) and Proposition 6.4.1). However, the group $\text{im}(\text{Ind}_{\text{Rep}(G)})$ is *not* a semidirect product of \mathcal{E} and \mathcal{V} in general.

The elements in \mathcal{R} have the field theoretic interpretation of so-called *partial dualities* (electric-magnetic-dualities). In so-called quantum double models (see e.g. [BCKA13] and [KaW07]), irreducible representations of DG have the interpretation of quasi-particle charges (anyon charges). These are parametrized by pairs $([g], \chi)$, where $[g]$ is a conjugacy class in G and χ an irreducible representation of the centralizer $\text{Cent}(g)$ (see Preliminaries of this thesis). The irreducibles of the form $([g], 1)$ are called *magnetic* and the ones of the form $([1], \chi)$ are called *electric*. Partial dualizations \mathcal{R} are symmetries of the quantized theory that exchange magnetic and electric charges (see equation (2.22), Proposition 6.5.1 and Section 7.1). These are only present at the quantum level, hence we call them quantum symmetries.

For the lazy Brauer-Picard group, which incorporates the abelian case as a special case, it is enough to dualize on direct abelian factors of G . A partial dualization is then induced by a Hopf automorphism of DG (Proposition 6.5.1). For the general Brauer-Picard group, we need to consider semi-direct abelian factors of G . Partial dualizations are induced by algebra isomorphisms that are not necessarily Hopf (see Section 7.1).

In gauge theories with abelian structure group $G = A$, we can dualize on A by $r_{1,A,\delta} \in \mathcal{R}$. The groups $\text{im}(\text{Ind}_{\text{Vect}_A})$ and $\text{im}(\text{Ind}_{\text{Rep}(A)})$ are conjugate subgroups by the partial dualization $r_{1,A,\delta}$. In the non-abelian case, it turns out $\text{im}(\text{Ind}_{\text{Vect}_G})$ and $\text{im}(\text{Ind}_{\text{Rep}(G)})$ are non-conjugate subgroups; they are usually not even isomorphic (see Chapter 7).

Let us now outline the structure of the thesis:

In Chapter 1, we give some preliminaries. In particular, we recall the Drinfeld double DG for a finite group G and its irreducible representations. Further, we give some basic facts about Hopf-Bigalois extensions, lazy Bigalois objects, lazy cohomology and its relation to monoidal autoequivalences.

In Chapter 2, we define $(3, 2, 1)$ -extended Dijkgraaf-Witten theories and investigate symmetries of these theories in the abelian case. In particular, we present the field theoretic construction of untwisted $(3, 2, 1)$ -extended Dijkgraaf Witten theories from [Mo13] and parts of the gauge theoretic construction that incorporates defects and boundary conditions. In particular, we consider a cylinder $S^1 \times [-1, 1]$ decorated with an invertible defect d at $S^1 \times \{0\}$. A TQFT based on an anomaly free modular tensor category \mathcal{C} assigns to this decorated cylinder the so-called transmission functor $F_d : \mathcal{C} \rightarrow \mathcal{C}$. We show that one can naturally associate to F_d a monoidal structure such that it becomes a braided autoequivalence of \mathcal{C} . Further, we show in Proposition 2.3.3 that in the abelian Dijkgraaf-Witten case $G = A$ the transmission functor is equivalent to (1) from above. For this, we use a decomposition of $\text{BrPic}(\text{Vect}_A)$ into natural generators, which is a special case of one of the main results, namely Theorem 6.6.1. Moreover, we give a field theoretic interpretation of these generators. This part of the thesis is mainly based on the publication [FPSV15].

In Chapter 3, we recall the classification of Galois algebras in [Mov93] and [Dav01]. Based on this we give an explicit formula for lazy 2-cocycles of k^G . This chapter is based on Sect. 3 of [LP15a].

In Chapter 4, we provide in Theorem 4.2.1 one of the main results of this thesis, namely a decomposition of the group of Hopf algebra automorphisms $\text{Aut}_{\text{Hopf}}(DG)$ into natural subgroups. As described in Proposition 4.1.1, with respect to the decomposition of the Drinfeld double as $k^G \rtimes kG$, these subgroups can be seen as upper triangular matrices E , lower triangular matrices B , block diagonal matrices $V \cong \text{Aut}(G)$ and $V_c \cong \text{Aut}_c(G)$ and so-called *reflections* on direct abelian factors of G . Our results use the approach [ABM12] Corollary 3.3 and on the work of Keilberg [Keil15]. He has determined a product decomposition (exact factorization) of $\text{Aut}_{\text{Hopf}}(DG)$ whenever G does not contain abelian direct factors. In [KS14] Keilberg and Schauenburg determined $\text{Aut}_{\text{Hopf}}(DG)$ in the general case, hence when G is allowed to have abelian direct factors using an approach that differs from ours. This chapter is based on Sect. 4 of [LP15a].

In Chapter 5, we address the problem of decomposing the Hopf cohomology of the Drinfeld double $H^2(DG^*)$ and in particular the group of lazy 2-cocycles $H_L^2(DG^*)$. For Hopf algebra tensor products such as $kG \otimes k^G$ and Doi twists of these, the Kac-Schauenburg sequence implies an easy decomposition of the second cohomology $H^2(kG \otimes k^G)$ into $H^2(G, k^\times)$, $H^2(k^G)$ and bialgebra pairings (see e.g. [Schau02]). However, the dual Drinfeld double DG^* is a Drinfeld twist of a tensor product. Thus, we can not apply known results. In Lemma 5.0.10 and Lemma 5.0.11 we provide

partial results that are needed to prove the full decomposition of $H^2(DG^*)$ and are in addition necessary for the decomposition of $\text{Aut}_{br}(DG\text{-mod})$. This chapter is based on Sect. 5 of [LP15a].

In Chapter 6, we define the group of lazy braided automorphisms $\text{Aut}_{br,L}(DG\text{-mod})$ (or equivalently $\text{BrPic}_L(\text{Vect}_G)$) and give some general facts and properties. Elements of this group are essentially determined by pairs (ϕ, σ) where $\phi \in \text{Aut}_{Hopf}(DG)$ and $\sigma \in H^2(DG^*)$. We then construct certain explicit elements in $\text{Aut}_{br,L}(DG\text{-mod})$ which are the natural candidates for generators. Combining the decompositions of the group of Hopf automorphisms $\text{Aut}_{Hopf}(DG)$ from Chapter 4 and the decomposition of the second cohomology group $H^2(DG^*)$ from Chapter 5, we give a decomposition of $\text{Aut}_{br,L}(DG\text{-mod})$ in Theorem 6.6.1 which is the main result of this thesis. This chapter is based on Sect. 4 and Sect. 5 of [LP15b].

In Chapter 7, we give the group $\text{Aut}_{br,L}(DG\text{-mod}) = \text{BrPic}_L(\text{Vect}_G)$ for certain G . Further, we compare the result of this thesis to the *full* Brauer-Picard group obtained in [NR14] in certain examples for G and thus provide evidence that the conjectured decomposition into $\text{im}(\text{Ind}_{\text{Vect}_G})$, $\text{im}(\text{Ind}_{\text{Rep}(G)})$ and partial dualizations \mathcal{R} is also true for $\text{Aut}_{br}(DG\text{-mod}) = \text{BrPic}(\text{Vect}_G)$. This chapter is based on Sect. 6 of [LP15b].

The main results of this thesis are the decomposition of $\text{Aut}_{br,L}(DG\text{-mod})$ in Theorem 6.6.1, the decomposition of $\text{Aut}_{Hopf}(DG)$ in Theorem 4.2.1 and the Proposition 2.3.3.

This thesis is based on the following three publications/preprints:

- Chapter 2: [FPSV15] J. Fuchs, J. Priel, C. Schweigert, A. Valentino, *On the Brauer groups of symmetries of abelian Dijkgraaf-Witten theories*, Commun. Math. Phys. 339, (2015), 385405, arXiv:1404.6646v3 [hep-th]
- Chapter 4 and 5: [LP15a] S. Lentner, J. Priel, *On monoidal autoequivalences of the category of Yetter-Drinfeld modules over a group: The lazy case*, Preprint (2015), arXiv:1511.03871 [math.QA]
- Chapter 6: [LP15b] S. Lentner, J. Priel, *A decomposition of the Brauer-Picard group of the representation category of a finite group*, Preprint (2015), arXiv:1506.07832v2 [math.QA]

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Chapter 1

Preliminaries

We assume some familiarity with standard notions and properties about Hopf algebras and representation theory that can be found in e.g. [Kass94]. Let us fix notation: We will denote a Hopf algebra over a field k by H , the multiplication on H by $\mu_H : H \otimes_k H \rightarrow H$, the comultiplication by $\Delta_H : H \rightarrow H \otimes_k H$, the antipode by $S_H : H \rightarrow H$, the unit by $\eta_H : k \rightarrow H$ and the counit by $\epsilon_H : H \rightarrow k$. We assume in this thesis that the field k is algebraically closed and of characteristic zero. Moreover, we denote a right H -coaction of an H -comodule M by $\delta_R : M \rightarrow M \otimes_k H$ and similarly left coactions by δ_L . Also, we use the Sweedler notation: $\Delta_H(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H$, $\delta_R(m) = m_{(0)} \otimes m_{(1)}$ and $\delta_L(m) = m_{(-1)} \otimes m_{(0)}$. For a finite group G , we denote by \hat{G} the group of 1-dimensional characters of G . We use the following notation for conjugation $g^t = t^{-1}gt$ and ${}^t g = tgt^{-1}$ for $g, t \in G$, whenever it is convenient.

We denote categories by \mathcal{C}, \mathcal{D} etc. and, as it is customary, use for objects in \mathcal{C}, \mathcal{D} the notation: $X \in \mathcal{C}, Y \in \mathcal{D}$.

1.1 The Drinfeld Double

Given a finite dimensional Hopf algebra H , one can construct another Hopf algebra DH , called the Drinfeld double of H . As a coalgebra DH is the tensor product $H^{op*} \otimes H$. We denote elements of DH by $(f \times h)$ for $f \in H^{op*}$ and $h \in H$. The algebra structure on DH is given by:

$$(f \times h)(f' \times h') = f * f'(S^{-1}(h_{(3)})(\cdot)h_{(1)}) \times h_{(2)}h'$$

where $f * f'$ denotes the convolution product of $f, f' \in H^{op*}$. For more details on the Drinfeld double we refer to Chapter 9 of [Kass94]. Let us spell out the Hopf algebra structure of the Drinfeld double for the following special case:

Definition 1.1.1. Let G be a finite group.

(i) Let us define the group algebra kG . As a k -vector space kG is spanned by elements of G . This vector space has a Hopf algebra structure as follows:

The multiplication on the basis of kG is just the multiplication in G , the comultiplication on kG , again on the basis is given by:

$$\Delta_{kG}(g) = g \otimes g$$

for $g \in G$. kG has an antipode defined by $S_{kG} : kG \rightarrow kG; g \mapsto g^{-1}$, a unit $\eta_{kG} : k \rightarrow kG; 1 \mapsto 1_G$ and counit $\epsilon_{kG} : kG \rightarrow k; g \mapsto 1$.

(ii) Let $k^G := \text{Hom}(kG, k)$ be the dual vector space of kG . k^G has a basis $\{e_g\}_{g \in G}$ where e_g is defined by $e_g(h) = \delta_{g,h}$ for $g, h \in G$. This vector space has a Hopf algebra structure as follows: The multiplication is $e_g * e_h = \delta_{g,h} e_g$. The comultiplication is

$$\Delta_{k^G}(e_g) = \sum_{g_1 g_2 = g} e_{g_1} \otimes e_{g_2}$$

k^G has an antipode defined by $S_{k^G} : k^G \rightarrow k^G; e_g \mapsto e_{g^{-1}}$, a unit $\eta_{k^G} : k \rightarrow k^G; 1 \mapsto \sum_{g \in G} e_g$ and a counit $\epsilon_{k^G} : k^G \rightarrow k; e_g \mapsto \delta_{g,1}$.

(iii) Then let DG be $k^G \otimes kG$ as a vector space. Denote the basis of DG by $\{e_x \times y\}_{x,y \in G}$. The multiplication is then

$$(e_g \times h)(e_{g'} \times h') = e_g(hg'h^{-1})(e_g \times hh')$$

and the comultiplication is

$$\Delta_{DG}(e_g \times h) = \sum_{g_1 g_2 = g} (e_{g_1} \times h) \otimes (e_{g_2} \times h)$$

Further, DG has an antipode $S(e_x \times y) = e_{y^{-1}x^{-1}y} \times y^{-1}$, a unit $1_{DG} = \sum_{x \in G} (e_x \times 1_G)$ and a counit $\epsilon(e_x \times y) = \delta_{x,1_G}$.

Later we also use the Hopf algebra DG^* , which is the dual Hopf algebra of DG . For this reason let us also spell out the multiplication and comultiplication of DG^* :

$$(x \times e_y)(x' \times e_{y'}) = (xx' \times e_y * e_{y'}) \quad \Delta_{DG^*}(x \times e_y) = \sum_{y_1 y_2 = y} (x \times e_{y_1}) \otimes (y_1^{-1} x y_1 \times e_{y_2})$$

In the case the group $G = A$ is abelian $DA \simeq k(\hat{A} \times A)$ and $DA^* \simeq k(A \times \hat{A})$ are isomorphic as Hopf algebras. In general there is no Hopf isomorphism from DG to DG^* , since the number of irreducible representations of DG (see below) differs from the number of irreducibles in $DG^*\text{-mod} \simeq \text{Rep}(G) \boxtimes \text{Vect}_G$, where \boxtimes is the Deligne tensor product.

$DG\text{-mod}$ is equivalent, as a braided monoidal category, to the category of kG -Yetter-Drinfeld-modules and the Drinfeld center $Z(\text{Vect}_G)$ of the category of G -graded vector spaces.

We recall that $DG\text{-mod}$ is a semisimple braided tensor category as follows:

- The simple objects of $DG\text{-mod}$ are induced modules $\mathcal{O}_g^\rho := kG \otimes_{k\text{Cent}_G(g)} V$, where $[g] \subset G$ is a conjugacy class and $\rho : \text{Cent}_G(g) \rightarrow \text{GL}(V)$ an isomorphism class of an irreducible representation of the centralizer of a representative $g \in [g]$. We have the following left DG -action on \mathcal{O}_g^ρ :

$$(e_h \times t).(y \otimes v) := e_h((ty)g(ty)^{-1})(ty \otimes v)$$

More explicitly: \mathcal{O}_g^ρ is a G -graded vector space consisting of $|[g]|$ copies of V :

$$\mathcal{O}_g^\rho := \bigoplus_{\gamma \in [g]} V_\gamma, \quad V_\gamma := V$$

Then the action of an element $(e_h \times 1) \in DG$ is given by projecting to the homogeneous component V_h . Choose a set of coset representatives $\{s_i \in G\}$ of $G/\text{Cent}_G(g) \simeq [g]$. Then for a homogeneous component V_γ with $\gamma \in [g]$ there is a unique representative $s_i \in G$ that corresponds under the conjugation action to the element $\gamma \in [g]$, thus $s_i g s_i^{-1} = \gamma$. For an $h \in G$, there is a unique representative s_j such that $h s_i \in s_j \text{Cent}_G(G)$. The action of an element $(1 \times h) \in DG$ is then given by

$$V_\gamma \rightarrow V_{h\gamma h^{-1}}; v \mapsto (1 \times h).v := h.v := \rho(s_j^{-1} h s_i)(v) \quad (1.1)$$

this is indeed well-defined, since $s_j g s_j^{-1} = h \gamma h^{-1}$.

- The monoidal structure on $DG\text{-mod}$ is given by the tensor product of DG -modules, i.e. with the diagonal action on the tensor product.
- The braiding $\{c_{M,N} : M \otimes N \xrightarrow{\sim} N \otimes M \mid M, N \in DG\text{-mod}\}$ on $DG\text{-mod}$ is defined by the universal R -matrix

$$R = \sum_{g \in G} (e_g \times 1) \otimes (1 \times g) = R_1 \otimes R_2 \in DG \otimes DG$$

$$c_{M,N}(m \otimes n) = \tau(R.(m \otimes n)) = R_2.n \otimes R_1.m \quad (1.2)$$

where $\tau : M \otimes N \rightarrow N \otimes M; m \otimes n \mapsto n \otimes m$ is the twist.

In the above convention we leave out the sum for $R = R_1 \otimes R_2 \in DG \otimes DG$. This should not be confused with the Sweedler notation for the coproduct or coaction.

Definition 1.1.2.

- (i) Let $\underline{\text{Aut}}_{\text{mon}}(DG\text{-mod})$ be the functor category of monoidal autoequivalences of $DG\text{-mod}$ and natural monoidal isomorphisms and $\text{Aut}_{\text{mon}}(DG\text{-mod})$ be the group of isomorphism classes of monoidal autoequivalences of $DG\text{-mod}$.
- (ii) Let $\underline{\text{Aut}}_{\text{br}}(DG\text{-mod})$ be the functor category of braided autoequivalences of $DG\text{-mod}$ and natural monoidal isomorphisms and $\text{Aut}_{\text{br}}(DG\text{-mod})$ be the group of isomorphism classes of braided monoidal autoequivalences of $DG\text{-mod}$.

1.2 Hopf-Galois-Extensions

In order to study monoidal automorphisms of DG -mod we will make use of the theory of Hopf-Galois extensions. For this, our main reference is [Schau91], [Schau96], [Schau02] and [BC04]. The motivation for this approach lies mainly in the relationship between Galois extensions and monoidal functors as formulated e.g. in [Schau96] and also stated in Proposition 1.2.5. Namely, for two Hopf algebras L, H , the category of monoidal functors between the category of H -comodules and the category of L -comodules is equivalent to the category of L - H -Bigalois objects. For this reason, we are led to the study of DG^* -Bigalois objects. Since DG is finite dimensional we can use the fact that a Bigalois object over a finite dimensional Hopf algebra can essentially be described by an automorphism of H and a 2-cocycle on H . We will see in a later section, that it is possible to handle the automorphism group of DG . On the other hand, the large set of 2-cocycles is hard to control and we need to introduce a special class of 2-cocycles, called lazy [Bich04] (sometimes invariant [BN14]), which have a better behavior in a certain sense. Those still give us a large class of interesting Bigalois objects. Let us introduce some basic notions and properties of Hopf-Galois extensions first.

Recall that a k -algebra A is called a right H -comodule algebra if A has a right H -coaction $\delta_R : A \rightarrow A \otimes_k H$ and δ_R is an algebra map. In this case we call the subalgebra $A^{coH} = \{a \in A \mid \delta_R(a) = a \otimes 1_H\}$ the coinvariants of H on A . Left comodule algebras and their coinvariants are defined similarly and we use analogous notation. Now we are ready to give the definition of a Hopf-Galois extension.

Definition 1.2.1. Let H be a bialgebra and A a right H -comodule algebra. Then A is called a *right H -Galois extension of $B := A^{coH}$* if the Galois map

$$\begin{aligned} \beta_A : A \otimes_B A &\xrightarrow{id_A \otimes \delta_R} A \otimes_B A \otimes_k H \xrightarrow{\mu_A \otimes id_H} A \otimes_k H \\ x \otimes y &\longmapsto x \otimes y_{(0)} \otimes y_{(1)} \longmapsto xy_{(0)} \otimes y_{(1)} \end{aligned}$$

is a bijection. A morphism of right H -Galois extensions A, A' is an H -colinear algebra morphism. Left H -Galois extensions and their morphisms are defined similarly. Denote by $\underline{\text{Gal}}_B(H)$ the category of right H -Galois extensions of B and by $\text{Gal}_B(H)$ the set of equivalence classes of right H -Galois extensions of B . We define a right H -Galois object A to be an H -Galois extension of $B = A^{coH} = k$ and we write $\text{Gal}(H) = \text{Gal}_k(H)$.

Let us consider some immediate examples and properties of Hopf-Galois extensions.

Lemma 1.2.2. (Example 2.1.2 and Lemma 4.4.1 [Schau91])

A bialgebra H is a Hopf algebra if and only if H is an H -Galois object.

Proof. First, let us remark that $H^{coH} = k$, because $h_{(1)} \otimes h_{(2)} = h \otimes 1$ for all $h \in H$ implies $h = \epsilon(h) \in k$ for all $h \in H$.

For the bialgebra H we want to show that the identity morphism id_H is convolution invertible if and only if the Galois map

$$\begin{aligned}\beta_H : H \otimes_k H &\rightarrow H \otimes_k H \\ h \otimes h' &\mapsto hh'_{(1)} \otimes h'_{(2)}\end{aligned}$$

is invertible. This follows from a more general statement: Let C be a k -coalgebra, A a k -algebra, $\text{End}_A^C(A \otimes_k C)$ the k -vector space of left A -module and right C -comodule endomorphisms of $A \otimes_k C$. Then the k -linear map

$$\begin{aligned}\text{Hom}_k(C, A) &\rightarrow \text{End}_A^C(A \otimes_k C) \\ f &\mapsto (a \otimes c \mapsto af(c_{(1)}) \otimes c_{(2)})\end{aligned}$$

is an anti-isomorphism of k -algebras, where the algebra structure on $\text{End}_A^C(A \otimes_k C)$ is the composition of maps and the algebra structure on $\text{Hom}_k(C, A)$ is convolution of maps. See the Lemma in [Mon93] on page 91. We take $C = A = H$ and notice that id_H is mapped to β_H under the above anti-isomorphism. It follows that id_H is convolution invertible if and only if β_H is bijective. \square

From now on H will always denote a Hopf algebra. In the following lemma we want to give another example of Hopf-Galois extensions and also show how field extensions of Galois type are related to Hopf-Galois extensions.

Lemma 1.2.3. (Section 8.1.2 [Mon93])

Let A/k be a finite field extension of k . Further, let G be a finite group acting on A , $H = (kG)^*$ the dual of the group algebra and $F = A^G$ the fixed points of A . Then A/F is a field extension of Galois type with Galois group G if and only if A is an H -Galois extension of F .

Proof. Assume that A/F is a Galois field extension with Galois group $G = \text{Aut}(A/F)$. Then we have $|G| = [A : F]$, where $[A : F]$ is the degree of the extension (dimension of A as an F -vector space). We want to show that the Galois map

$$\begin{aligned}\beta_A : A \otimes_F A &\rightarrow A \otimes_k k^G \\ a \otimes a' &\mapsto \sum_{f \in \text{Aut}(A/F)} af(a') \otimes e_f\end{aligned}$$

is bijective, where $e_f \in k^G$ defined by $e_f(f') = \delta_{f,f'}$ for $f' \in G = \text{Aut}(A/F)$. Let $\text{Aut}(A/F) = \{f_1, \dots, f_n\}$ and let (b_1, \dots, b_n) be an F -basis of A . Take a general element $z = \sum_{i,j=1}^n \alpha_{ij} b_i \otimes b_j \in A \otimes_F A$ with $\alpha_{ij} \in F$. If z is in the kernel of β_A , then:

$$\sum_{i,j,l=1}^n \alpha_{ij} b_i f_l(b_j) \otimes e_{f_l} = 0$$

and the linear independence of the e_{f_l} implies that $\sum_{i,j=1}^n \alpha_{ij} b_i f_l(b_j) = 0$ for all $l = 1, \dots, n$. Then the theorem of linear independence of field automorphisms

(Dedekind's lemma) implies that the f_i are linear independent over A , therefore we have $\sum_{i=1}^n \alpha_{ij} b_i = 0$ for all $j = 1, \dots, n$. Since b_i is an F -basis, this means that $z = 0$, hence β_A is injective. Since A and k^G are finite dimensional and since the domain and codomain of β_A have equal F -dimensions, β_A is also bijective.

For the converse assume that $\beta_A : A \otimes_F A \rightarrow A \otimes_k k^G$ is bijective, then we have $[A : F]^2 = \dim_F(A \otimes_k k^G)$. Also we have the following equation:

$$[A : F][F : k] \dim_k(k^G) = \dim_k(A \otimes_k k^G) = \dim_F(A \otimes_k k^G)[F : k]$$

Combining the last two identities we have $[A : F] = \dim_k(k^G) = |G|$, which proves that the extension A/F is of Galois type. \square

Lemma 1.2.4. Let H be a Hopf algebra and $f : A \rightarrow A'$ be a morphism of H -Galois extensions of B such that A' is faithfully flat over B , then f is bijective.

Proof. This is essentially the same proof as in [Bich10] Proposition 1.6. but without assuming $k = B$.

First we note that A acts on A' through f . Then the following diagram commutes:

$$\begin{array}{ccc} A' \otimes_B A & \xrightarrow{\text{id} \otimes f} & A' \otimes_B A' \xrightarrow{\beta_{A'}} A' \otimes_k H \\ \simeq \downarrow & & \simeq \downarrow \\ A' \otimes_A A \otimes_B A & \xrightarrow{\text{id} \otimes \beta_A} & A' \otimes_A A \otimes_k H \end{array}$$

Hence, the map $\text{id} \otimes_B f$ is bijective and since A' is faithfully flat over B , the map f is also bijective. \square

We now want to show how Hopf-Galois extensions of H correspond to certain monoidal functors from H -comodules to k -vector spaces. For this, recall the *cotensor product*: Let H, L be Hopf algebras, (A, δ_R) a right L -comodule and (B, δ_L) a left L -comodule. The cotensor product of A and B is the following vector space over k :

$$A \square_L B := \left\{ \sum a \otimes b \in A \otimes_k B \mid \sum \delta_R(a) \otimes b = \sum a \otimes \delta_L(b) \right\}$$

Also recall that given a Hopf algebra H a *fiber functor* $H\text{-comod} \rightarrow \text{Vect}_k$ is a k -linear, monoidal, exact and faithful functor that preserves colimits. We denote by $\text{Fun}_{fib}(H\text{-comod}, \text{Vect}_k)$ the set of monoidal equivalence classes of fiber functors. Given a right H -comodule Z we can define a k -linear functor $H\text{-comod} \rightarrow \text{Vect}_k; M \mapsto Z \square_H M$. In the following proposition we will see that given an algebra structure on Z that is compatible with the coaction we can also define a monoidal structure on this functor, which however is only *weak* monoidal (lax monoidal) in general, where 'weak' means that the morphisms in the natural family of the monoidal structure do not have to be isomorphisms. This functor is monoidal if and only if the Galois map is bijective. Let us state this more precisely:

Proposition 1.2.5. (Theorem 1.2 [Ulb89])

Let $\underline{\text{Gal}}(H)$ be the category where objects are given by H -Galois objects and morphisms by morphisms of H -Galois objects. Further, let $\underline{\text{Fun}}_{fib}(H\text{-comod}, \text{Vect}_k)$ be the category of fiber functors and monoidal natural transformations. Then the following is an equivalence of categories:

$$\begin{aligned} \underline{\text{Gal}}(H) &\xrightarrow{\sim} \underline{\text{Fun}}_{fib}(H\text{-comod}, \text{Vect}_k) \\ A &\mapsto (A \square_H \bullet, J^A) \end{aligned} \quad (1.3)$$

where the monoidal structure J^A on $A \square_H \bullet : H\text{-comod} \rightarrow \text{Vect}_k$ is given by

$$\begin{aligned} J_{V,W}^A : (A \square_H V) \otimes_k (A \square_H W) &\xrightarrow{\sim} A \square_H (V \otimes_k W) \\ \left(\sum x_i \otimes v_i \right) \otimes \left(\sum y_j \otimes w_j \right) &\mapsto \sum x_i y_j \otimes v_i \otimes w_j \end{aligned} \quad (1.4)$$

The last proposition was generalized to the case of Hopf-Galois extensions of general coinvariants $B = A^{coH}$ in [Schau91].

We are now interested in functors $H\text{-comod} \rightarrow L\text{-comod}$. Then we could again consider cotensoring with a right H -comodule Z , but for M a left H -comodule the k -vector space $Z \square_H M$ does not have a left L -comodule structure in general. Therefore we need to endow Z with an L - H -bicomodule structure. This motivates for us the following definition of Bigalois objects:

Definition 1.2.6. Let L, H be two Hopf algebras. An L - H -Bigalois object is a k -algebra A with the structure of a right H -Galois object and a left L -Galois object such that A is an L - H -bicomodule algebra. A morphism of L - H -Bigalois objects is a L - H -colinear algebra map. Denote by $\underline{\text{Bigoal}}(L, H)$ the category of L - H -Bigalois objects and by $\underline{\text{Bigoal}}(H)$ the category of H - H -Bigalois objects. Further, denote by $\text{Bigoal}(L, H)$ the set of isomorphism classes of L - H -Bigalois objects and by $\text{Bigoal}(H)$ the set of isomorphism classes of H - H -Bigalois objects.

An important result in the theory of Hopf-Galois extensions is the following Theorem of Schauenburg, which is the main result of [Schau96]:

Theorem 1.2.7. (Theorem 3.5 [Schau96])

Let H be a Hopf algebra and A a right H -Galois object. Define the map

$$\gamma : H \rightarrow A \otimes_k A; h \mapsto \beta_A^{-1}(1 \otimes h) =: \gamma(h)_1 \otimes \gamma(h)_2$$

where β_A is the Galois map. Let $L(H, A)$ be a Hopf algebra defined as follows: As an algebra we have $L(H, A) = (A \otimes_k A)^{coH}$, where $A \otimes_k A$ is equipped with the codiagonal H -comodule structure. Further, $L(H, A)$ has the following comultiplication, counit and antipode:

$$\begin{aligned} \Delta \left(\sum x \otimes y \right) &= \sum x_{(0)} \otimes \gamma(x_{(1)}) \otimes y \\ \epsilon \left(\sum x \otimes y \right) &= \sum xy \in A^{coH} = k \\ S \left(\sum x \otimes y \right) &= \sum y_{(0)} \otimes \gamma(y_{(1)})_1 x \gamma(y_{(1)})_2 \end{aligned}$$

Then, A is an $L(H, A)$ - H -Bialgebra object. In particular, there is an H -colinear algebra map (left $L(H, A)$ -coaction on A):

$$\delta : A \rightarrow L(H, A) \otimes A; a \mapsto a_{(0)} \otimes \gamma(a_{(1)})$$

If L' is another Hopf algebra, such that A is an L' - H -Bialgebra object with the left L' -coaction δ' , then there is a unique isomorphism of Hopf algebras $f : L(H, A) \xrightarrow{\sim} L'$ such that $\delta' = (f \otimes \text{id}_A) \circ \delta$.

We can interpret this statement as follows: Given an H -Galois objects, it is always an L - H -Bialgebra object and the Hopf algebra L is unique up to isomorphism.

Theorem 1.2.8. (Theorem 4.3 [Schau96])

Let \mathcal{H} be the category where the objects are given by Hopf algebras and the morphisms between two Hopf algebras L, H are given by isomorphism classes of L - H -Bialgebra objects, thus elements in $\text{Bialg}(L, H)$. The composition of morphisms is the cotensor product. Then \mathcal{H} is a groupoid. \mathcal{H} is called the Harrison groupoid.

Proof. The full proof can be found in [Schau96] page 16. We mention that the inverse of an L - H -Bialgebra object A is given by $A^{-1} = (H \otimes A)^{\text{co}H}$ where $H \otimes A$ is endowed with the codiagonal H -comodule structure. If H has a bijective antipode S then $A^{-1} \simeq A^{\text{op}}$ where the left H -comodule structure on A^{op} is: $a \mapsto S^{-1}(a_{(1)}) \otimes a_{(0)}$. The isomorphism class of the Hopf algebra H with the natural H - H -Bialgebra object structure is the identity morphism. □

Example 1.2.9. For a Hopf algebra automorphism $\phi \in \text{Aut}_{\text{Hopf}}(H)$ we obtain an H - H -Bialgebra object ${}_{\phi}H$ where the left H -coaction is the coproduct post-composed with ϕ . This yields a group homomorphism $\text{Aut}_{\text{Hopf}}(H) \rightarrow \text{Bialg}(H)$ which in general is neither surjective nor injective.

We now finally come to the classification of monoidal autoequivalences by Hopf-Bialgebra objects.

Proposition 1.2.10. (Section 5 [Schau96])

Let H, L be Hopf algebras, then we have a bijection of isomorphism classes of L - H -Bialgebra objects and monoidal equivalence classes of monoidal functors:

$$\begin{aligned} \text{Bialg}(L, H) &\simeq \text{Equiv}_{\text{mon}}(H\text{-comod}, L\text{-comod}) \\ A &\mapsto (A \square_H \bullet, J^A) \end{aligned}$$

where the monoidal structure J^A of the functor $A \square_H \bullet$ is given by

$$\begin{aligned} J_{V,W}^A : (A \square_H V) \otimes_k (A \square_H W) &\xrightarrow{\sim} A \square_H (V \otimes_k W) \\ \left(\sum x_i \otimes v_i \right) \otimes \left(\sum y_j \otimes w_j \right) &\mapsto \sum x_i y_j \otimes v_i \otimes w_j \end{aligned} \tag{1.5}$$

for $W, V \in H\text{-comod}$. Moreover, we have a group isomorphism

$$\text{Bigal}(H) \simeq \text{Aut}_{\text{mon}}(H\text{-comod})$$

and if H is finite dimensional we have a group isomorphism

$$\text{Bigal}(H^*) \simeq \text{Aut}_{\text{mon}}(H\text{-mod})$$

Example 1.2.11. In Example 1.2.9, we obtained for each $\phi \in \text{Aut}_{\text{Hopf}}(H) \cong \text{Aut}_{\text{Hopf}}(H^*)$ an H^* -Bigalois object ${}_{\phi}H^*$ isomorphic to H^* as an algebra but with left comodule structure post-composed by ϕ . Under the isomorphism above, this corresponds to the monoidal autoequivalence $(F_{\phi}, J^{\text{triv}})$ mapping an H -module M to the H -module ${}_{\phi}M$ given by pre-composing the module structure with ϕ (and a trivial monoidal structure).

There is a large class of H -Galois extensions that are isomorphic to H as H -comodules. The algebra structure of such extensions is then parametrized by twisting the algebra structure on H with a 2-cocycle. In the case H is finite-dimensional or pointed, all H -Galois extensions arise in this way.

Definition 1.2.12. (Lemma 1.6. [BC04])

- (i) Denote by $\text{Reg}^1(H)$ the group of convolution invertible, k -linear maps $\eta : H \rightarrow k$ such that $\eta(1) = 1$.
- (ii) Let $\text{Reg}^2(H)$ be the group of convolution invertible, k -linear maps $\sigma : H \otimes H \rightarrow k$ such that $\sigma(1, h) = \epsilon(h) = \sigma(h, 1)$
- (iii) A left 2-cocycle on H is a map $\sigma \in \text{Reg}^2(H)$ such that for all $a, b \in H$

$$\sigma(a_{(1)}, b_{(1)})\sigma(a_{(2)}b_{(2)}, c) = \sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) \quad (1.6)$$

We denote the set of 2-cocycles on H by $Z^2(H)$.

- (iv) We define a map $d : \text{Reg}^1(H) \rightarrow \text{Reg}^2(H)$ by

$$d\eta(a, b) = \eta(a_{(1)})\eta(b_{(1)})\eta^{-1}(a_{(2)}b_{(2)})$$

for all $a, b \in H$. We have $d\eta \in Z^2(H)$ and call 2-cocycles of this form exact. For other properties of d see [BC04] Lemma 1.6.

- (v) Two 2-cocycles σ, σ' are called cohomologous if there is an $\eta \in \text{Reg}^1(H)$ such that

$$\sigma'(a, b) = \eta(a_{(1)})\eta(b_{(1)})\sigma(a_{(2)}, b_{(2)})\eta^{-1}(a_{(3)}b_{(3)}) \quad \forall a, b \in H$$

We write $\sigma \sim \sigma'$ for two cohomologous 2-cocycles. \sim is an equivalence relation on $Z^2(H)$. Denote by $H^2(H) := Z^2(H)/\sim$ the set of 2-cohomology classes on H .

Proposition 1.2.13. ([Schau91])

An H -Galois object A is called *cleft* if one of the following equivalent conditions is satisfied:

- (i) There exists an H -comodule isomorphism $H \simeq A$.
- (ii) There exists an H -colinear convolution invertible map $H \rightarrow A$
- (iii) There exists a 2-cocycle σ such that $A \simeq {}_\sigma H$ as H -comodule algebras, where ${}_\sigma H$ is has the H -comodule structure of H and the following twisted algebra structure

$$a \cdot {}_\sigma b = \sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}$$

Proposition 1.2.14. ([Mon93], [Schau91])

- (i) If H is finite dimensional or pointed (which means that all simple H -comodules are 1-dimensional), then every H -Galois object is cleft.
- (ii) If H is finite dimensional or pointed, the map $\sigma \mapsto {}_\sigma H$ induces an bijection of sets $\mathbb{H}^2(H) \cong \text{Gal}(H)$.
- (iii) If H is finite dimensional or pointed, the unique Hopf algebra $L(H, A)$ from Proposition 1.2.7 is given by the Doi twist $L(H, A) := {}_\sigma H_{\sigma^{-1}}$ which is H as a coalgebra and has the following twisted algebra structure:

$$a \cdot b := \sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\sigma^{-1}(a_{(3)}, b_{(3)})$$

It is important to note that $\mathbb{Z}^2(H)$ as well as $\mathbb{H}^2(H)$ are not groups, because the convolution of two 2-cocycles is not a 2-cocycle in general. The convolution product $\sigma * \tau$ for $\sigma \in \mathbb{Z}^2(H)$ and $\tau \in \mathbb{Z}^2(L)$ is a 2-cocycle in $\mathbb{Z}^2(H)$ if $L \simeq {}_\sigma H_{\sigma^{-1}}$.

Corollary 1.2.15. Let H be a finite dimensional Hopf algebra. We have the following map from 2-cocycles to fiber functors:

$$\begin{aligned} \mathbb{Z}^2(H^*) &\rightarrow \underline{\text{Fun}}_{fib}(H\text{-mod}, \text{Vect}_k) & J_{V,W}^\sigma &: V \otimes_k W \xrightarrow{\simeq} V \otimes_k W \\ \sigma &\mapsto (\text{Forget}, J^\sigma) & v \otimes w &\mapsto \sigma_1.v \otimes \sigma_2.w \end{aligned}$$

Here we have identified $\sigma : H^* \otimes H^* \rightarrow k$ with an element $\sigma = \sigma_1 \otimes \sigma_2 \in H \otimes H$. This map induces a bijection of sets $\mathbb{H}^2(H^*) \simeq \text{Fun}_{fib}(H\text{-mod}, \text{Vect}_k)$.

Proof. Since H is finite dimensional, we already know that $\mathbb{H}^2(H^*) \simeq \text{Gal}(H^*) \simeq \text{Fun}_{fib}(H^*\text{-comod}, \text{Vect}_k)$ by Proposition 1.2.5 and Proposition 1.2.14. So we just need to check that the composition of these two bijections composed with the canonical bijection $\text{Fun}_{fib}(H^*\text{-comod}, \text{Vect}_k) \simeq \text{Fun}_{fib}(H\text{-mod}, \text{Vect}_k)$ is indeed the function given in the statement.

First, for every right H^* -comodule M we have ${}_\sigma H^* \square_{H^*} M \simeq M$ as k -vector spaces, where we assign a general element $\sum h \otimes m \in H^* \square_{H^*} M$ to $\sum \epsilon(h)m \in M$. The inverse is given by mapping $m \in M$ to $\delta_R(m) = m_{(0)} \otimes m_{(1)}$ which is indeed in $H^* \square_{H^*} M$ because of $(\Delta_H \otimes \text{id}) \circ \delta_R = (\text{id} \otimes \delta_R) \circ \delta_R$. We check that these are indeed inverses of each other: For $m \in M$ the composition of the two maps is $\epsilon(m_{(0)})m_{(1)} = m$. On the other hand $\sum h \otimes m$ is mapped to $\sum \epsilon(h)m_{(0)} \otimes m_{(1)} = \sum h \otimes m$. So the functor $H^* \square_{H^*} \bullet$ is indeed the forgetful functor. Now we only need to check that the

monoidal structure is the multiplication with $\sigma \in H \times H$. Let us pick a basis (h_i) of H and a corresponding dual basis (h^i) of H^* . Let $M, N \in H\text{-mod}$ and $m \in M$, $n \in N$. Then $\sigma.m \otimes n = \sum_{i,j} \sigma(h^i, h^j) h_i.m \otimes h_j.n$ is indeed the same map as the composition

$$M \otimes_k N \simeq H^* \square_{H^*} M \otimes_k H^* \square_{H^*} N \simeq H^* \square_{H^*} (M \otimes_k N) \simeq M \otimes_k N$$

which is $m \otimes n \rightarrow \sigma(m_{(-1)}, n_{(-1)}) m_{(0)} \otimes n_{(0)}$ where the right and left H^* -coactions are induced by the right and left H -actions: $\delta_L(m) = \sum_i h^i \otimes h_i.m$ and similarly for N . □

In general it is very hard to control the sets $Z^2(H)$, $H^2(H)$ and even more the subset of Galois objects with $L(A, H) \cong H$. For this reason, we consider below H -Galois objects that have an additional property. An implication of that property is that they can be described by certain cohomology groups.

1.3 Lazy Bigalois Objects and Lazy Cohomology

Definition 1.3.1.

- (i) An H - H -Bigalois object A is called *bicleft* if and only if $A \cong H$ as H -bicomodules. The group $\text{Bigal}_{bicleft}(H)$ of bicleft H - H -Bigalois objects is a normal subgroup of $\text{Bigal}(H)$.
- (ii) A right H -Galois object A is called *lazy* if there exists a unique left H -Galois structure such that A is bicleft. Hence it is a Galois object where there is a canonical isomorphism $L(H, A) \cong H$.
- (iii) An H - H -Bigalois object A is called *lazy* if it is lazy as a right H -Galois object. Denote the group of lazy H - H -Bigalois objects by $\text{Bigal}_{lazy}(H)$.

The terminology 'lazy' is motivated in terms of 2-cocycle twists on a Hopf algebra H , as explained further below.

Example 1.3.2. If H is cocommutative (e.g. $H = kG$), then all H -Galois objects and H - H -Bigalois objects are lazy (see Lemma 4.7 [Schau96]).

Since we are mainly interested in the cleft case, hence where all H -Galois objects are of the form ${}_\sigma H$ for a 2-cocycle σ , we discuss what additional property on σ corresponds to the lazy property of the Galois object ${}_\sigma H$.

Definition 1.3.3.

- (i) An $\eta \in \text{Reg}^1(H)$ is called *lazy* if it has the additional property $\eta * \text{id} = \text{id} * \eta$, where $*$ denotes the convolution product. Hence, if for all $h \in H$ we have

$$\eta(h_{(1)})h_{(2)} = h_{(1)}\eta(h_{(2)})$$

Denote by $\text{Reg}_L^1(H)$ the subgroup of such lazy regular maps.

(ii) A $\sigma \in \text{Reg}^2(H)$ is called *lazy* if it (convolution) commutes with the multiplication on H : $\sigma * \mu_H = \mu_H * \sigma$. Hence, if for all $a, b \in H$ we have

$$\sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)} = a_{(1)}b_{(1)}\sigma(a_{(2)}, b_{(2)})$$

Denote the subgroup of such lazy regular maps by $\text{Reg}_L^2(H)$. Accordingly, a 2-cocycle $\sigma \in Z^2(H)$ is called *lazy* if $\sigma \in \text{Reg}_L^2(H)$. Denote by $Z_L^2(H)$ the subgroup of lazy 2-cocycles. Note: The map d in Definition 1.2.12 maps $\text{Reg}_L^1(H)$ to the center of $Z_L^2(H)$.

(iii) An $\eta \in \text{Reg}^1(H)$ is called *almost lazy* if $d\eta$ is lazy. Denote by $\text{Reg}_{aL}^1(H)$ the group of almost lazy regular maps.

(iv) The second lazy cohomology group is then defined by

$$H_L^2(H) := Z_L^2(H)/d(\text{Reg}_L^1(H))$$

The set of almost lazy cohomology classes is defined as the set of cosets

$$H_{aL}^2(H) := Z_L^2(H)/d(\text{Reg}_{aL}^1(H))$$

Note that if we consider for a Hopf algebra H the Doi twist ${}_\sigma H_{\sigma^{-1}}$ for a lazy 2-cocycle $\sigma \in Z^2(H)$ as defined above, H and ${}_\sigma H_{\sigma^{-1}}$ are equal Hopf algebras (isomorphic via id_H). This motivates the terminology of a 'lazy' cocycle (see also [BC04]).

Proposition 1.3.4. (Proposition 3.6, Proposition 3.7 [BC04])

An H -Galois object A is bicleft if and only if there exists a lazy $\sigma \in Z_L^2(H)$ such that ${}_\sigma H_{\sigma^{-1}} \simeq A$ as H -bicomodule algebras. Further, the group of bicleft H -Bigalois objects is a normal subgroup of $\text{Bigal}(H)$ and

$$H_L^2(H) \simeq \text{Bigal}_{\text{bicleft}}(H)$$

Example 1.3.5.

(i) For $H = kG$ with G a finite group the 2-cocycle condition reduces to

$$\sigma(ab, c)\sigma(a, b) = \sigma(b, c)\sigma(a, bc) \quad \forall a, b, c \in G$$

The lazy condition is automatically fulfilled and d is the differential corresponding to the bar complex, hence

$$H^2(kG) = H_L^2(kG) = H^2(G)$$

(ii) For $H = k^G$ with G a finite group, a left 2-cocycle α is lazy if and only if for all $x, y, g \in G$ holds:

$$\alpha(e_x, e_y) = \alpha(e_{gxg^{-1}}, e_{gyg^{-1}})$$

An $\eta \in \text{Reg}^1(k^G)$ is lazy if and only if $\eta(e_x) = \eta(e_{gxg^{-1}})$ for all $g, x \in G$.

Similar formulas hold for lazy 2-cocycles on DG (see Lemma 5.0.1).

Definition 1.3.6.

(i) For a Hopf algebra H denote by $\text{Int}(H)$ the subgroup of *internal* Hopf automorphisms. These are $\phi \in \text{Aut}_{\text{Hopf}}(H)$ of the form $\phi(h) = xhx^{-1}$ for some invertible $x \in H$ such that for all $h \in H$:

$$(x \otimes x)\Delta(x^{-1})\Delta(h) = \Delta(h)(x \otimes x)\Delta(x^{-1}) \quad (1.7)$$

Note: For an invertible element $x \in H$ the conjugation $\phi(h) = xhx^{-1}$ is an algebra automorphism. It is a coalgebra automorphism if and only if (1.7) holds.

(ii) Denote by $\text{Inn}(H) \subset \text{Int}(H)$ the subgroup of *inner* Hopf automorphisms, hence $\phi \in \text{Aut}_{\text{Hopf}}(H)$ of the form $\phi(h) = xhx^{-1}$ for some group-like $x \in H$.

(iii) Let $\text{Out}_{\text{Hopf}}(H) := \text{Aut}_{\text{Hopf}}(H)/\text{Inn}(H)$ the subgroup of outer Hopf automorphisms.

Example 1.3.7. Let $H = DG$ for a finite group G . Then group-like elements are $G(DG) = \hat{G} \times G$, where \hat{G} is the group of 1-dimensional characters of G . We have $\text{Inn}(DG) \cong \text{Inn}(G)$. More precisely, each inner automorphism $\phi \in \text{Inn}(DG)$ is of the form $\phi(e_g \times h) = e_{tgt^{-1}} \times tgt^{-1}$ for some $t \in G$.

We now discuss how the previously defined subgroups interact:

Lemma 1.3.8. (Lemma 1.15 [BC04])

Let H be a finite dimensional Hopf algebra. $\text{Aut}_{\text{Hopf}}(H)$ acts on $Z_L^2(H^*)$ by

$$\phi.\sigma = (\phi \otimes \phi)(\sigma) \in H \otimes H \quad \phi \in \text{Aut}_{\text{Hopf}}(H) \quad \sigma \in Z^2(H^*)$$

where we identify a 2-cocycle on H^* with an element $\sigma = \sigma_1 \otimes \sigma_2 \in H \otimes H$. Then for all $\phi \in \text{Aut}_{\text{Hopf}}(H)$ we have:

- (i) If $\omega, \sigma \in Z^2(H^*)$, then $\phi.(\sigma * \omega) = (\phi.\sigma) * (\phi.\omega)$.
- (ii) If $\gamma \in \text{Reg}^1(H^*)$, then $\phi.d\gamma = d(\gamma \circ \phi)$.
- (iii) $\text{Inn}(H)$ acts trivially on $\text{Reg}_L^2(H^*)$.
- (iv) This action induces an action of $\text{Out}_{\text{Hopf}}(H)$ on lazy cohomology $H_L^2(H^*)$

Let us now come to the main statements of this section:

To every pair $(\phi, \sigma) \in \text{Aut}_{\text{Hopf}}(H) \times Z_L^2(H^*)$ we can assign an H^* -Bigalois object $\phi^*(\sigma H^*)$ that is ${}_{\sigma}H^*$ as an algebra, where the right H^* -coaction is given by the comultiplication in H^* and where the left H^* -coaction is given by the comultiplication in H^* post-composed with $\phi^* : H^* \rightarrow H^*$ the dual of ϕ .

We denote the monoidal functor corresponding to the Bigalois object $\phi^*(\sigma H^*)$ under the equivalence in Proposition 1.2.10 by $(F_\phi, J^\sigma) \in \underline{\text{Aut}}_{\text{mon}}(H\text{-mod})$. Then $F_\phi : H\text{-mod} \rightarrow H\text{-mod}$ assigns a left H -module (M, ρ_L) to the left H -module $(M, \rho_L \circ (\phi \otimes_k \text{id}))$ and the monoidal structure J^σ is given by

$$J_{M,N}^\sigma : {}_{\phi}M \otimes_k {}_{\phi}N \xrightarrow{\sim} {}_{\phi}(M \otimes_k N)$$

$$m \otimes n \mapsto \sigma_1.m \otimes \sigma_2.m$$

where we view the 2-cocycle as an element $\sigma = \sigma_1 \otimes \sigma_2 \in H \otimes_k H$. This assignment gives us a map to the group $\text{Aut}_{\text{mon}}(H\text{-mod})$:

$$\text{Aut}_{\text{Hopf}}(H) \times Z_L^2(H^*) \rightarrow \text{Aut}_{\text{mon}}(H\text{-mod})$$

where $\text{Aut}_{\text{Hopf}}(H)$ acts on $Z_L^2(H^*)$ as defined in Proposition 1.3.8.

Lemma 1.3.9. Let $\phi \in \text{Aut}_{\text{Hopf}}(H)$ be a Hopf automorphism and let $\sigma \in Z_L^2(H^*)$ a lazy 2-cocycle, then the following two statements are equivalent:

- The functor (F_ϕ, J^σ) is monoidal equivalent to $(\text{id}, J^{\text{triv}})$
- $\phi \in \text{Int}(H)$, hence ϕ has the form $\phi = x \cdot \text{id}_H \cdot x^{-1}$ for some invertible element $x \in H$ and σ is of the form

$$\sigma = \Delta(x)(x^{-1} \otimes x^{-1}) \quad (1.8)$$

Proof. Let η be the monoidal equivalence $(\text{id}, J^{\text{triv}}) \sim (F_\phi, J^\sigma)$. In particular, there is an H -module isomorphism for the regular H -module $\eta_H : H \xrightarrow{\sim} F_\phi(H) =: {}_\phi H$ such that $\eta_H \circ f = f \circ \eta_H$ for all H -module homomorphisms $f : H \rightarrow H$. Note that η_H is determined by what it does on 1_H hence by an invertible element $x := \eta(1) \in H$. Further, every H -module morphism $f : H \rightarrow H$ is determined by an $h := f(1) \in H$ and then the naturality property of η implies $\phi(h) = xhx^{-1}$. Since ϕ is a Hopf automorphism there are additional conditions on x . Since $x \in H$ is invertible, ϕ is an algebra automorphism. ϕ is a coalgebra automorphism if and only if

$$\Delta(x^{-1})\Delta(h)\Delta(x) = (x^{-1} \otimes x^{-1})\Delta(h)(x \otimes x)$$

which is equivalent to (1.7). Further, by definition there has to be an H -module isomorphism $\eta_{H \otimes H} : H \otimes H \xrightarrow{\sim} {}_\phi(H \otimes H)$ such that for all H -module morphisms $r : H \rightarrow H \otimes H$ the following diagram commutes:

$$\begin{array}{ccc} H & \xrightarrow{\eta_H} & {}_\phi H \\ r \downarrow & & r \downarrow \\ H \otimes H & \xrightarrow{\eta_{H \otimes H}} & {}_\phi(H \otimes H) \end{array}$$

Again, r is determined by $y := r(1) \in H \otimes H$ then the diagram above implies $\eta_{H \otimes H}(y) = x \cdot y$ for all $y \in H \otimes H$. Now we use that η is monoidal, hence in particular the diagram

$$\begin{array}{ccc} H \otimes H & \xrightarrow{\eta_H \otimes \eta_H} & {}_\phi H \otimes {}_\phi H \\ \downarrow J_{H \otimes H}^{\text{triv}} & & \downarrow J_{H \otimes H}^\sigma \\ H \otimes H & \xrightarrow{\eta_{H \otimes H}} & {}_\phi(H \otimes H) \end{array}$$

commutes which implies that $J_{H \otimes H}^\sigma(xh \otimes xh') = \Delta(x)(h \otimes h')$ for all $h, h' \in H$ which is equivalent to (1.8).

On the other hand let $(\phi, \sigma) \in \text{Aut}_{\text{Hopf}}(H) \times Z_L^2(H^*)$ such that $\phi(h) = xhx^{-1}$ for some invertible $x \in H$ such that equations (1.7) and (1.8) hold. It can be checked by direct calculation that ϕ is a well-defined algebra automorphism because x is invertible and coalgebra morphism because of (1.7). We claim that the following family of morphisms $\eta = \{\eta_M : M \rightarrow {}_\phi M; \eta_M(m) = x.m\}$ is a monoidal equivalence between $(\text{id}, J^{\text{triv}})$ and (F_ϕ, J^σ) . The fact that η is a natural equivalence follows from the construction. The fact that η is monoidal follows from (1.8). \square

Proposition 1.3.10. The following map is a group homomorphism

$$\text{Aut}_{\text{Hopf}}(H) \times Z_L^2(H^*) \rightarrow \text{Aut}_{\text{mon}}(H\text{-mod}); (\phi, \sigma) \mapsto (F_\phi, J^\sigma) \quad (1.9)$$

and induces a group homomorphism $\text{Out}_{\text{Hopf}}(H) \times H_L^2(H^*) \rightarrow \text{Aut}_{\text{mon}}(H\text{-mod})$.

Proof. We first check that (1.9) is indeed a homomorphism. The composition in the semi-direct product $(\phi, \sigma)(\phi', \sigma') = (\phi' \circ \phi, (\phi.\sigma')\sigma)$ is mapped to the functor $(F_{\phi' \circ \phi}, J^{(\phi.\sigma')\sigma})$. On the other hand the composition (F_ϕ, J^σ) with $(F_{\phi'}, J^{\sigma'})$ gives $F_{\phi'} \circ F_\phi = F_{\phi' \circ \phi}$ with the monoidal structure

$$F_{\phi'}(J_{M,N}^\sigma) \circ J_{M_\phi, N_\phi}^{\sigma'}(m \otimes n) = \sigma.(\phi.\sigma').(m \otimes n) = J_{M,N}^{\sigma*(\phi.\sigma')}(m \otimes n)$$

Let us now show that the map factorizes as indicated: The kernel of

$$\text{Aut}_{\text{Hopf}}(H) \times Z_L^2(H^*) \rightarrow \text{Out}_{\text{Hopf}}(H) \times H_L^2(H^*)$$

is given by the set of all (ϕ, σ) with $\phi(h) = tht^{-1}$ for some group-like element $t \in G(H)$ and $\sigma = d\eta$ for $\eta \in \text{Reg}_L^1(H^*)$. To see that the functor (F_ϕ, J^σ) is in this case trivial up to monoidal natural transformations we apply Lemma 1.3.9 for the element $x = \eta^{-1} \cdot t$ in H : We only have to check that indeed

$$\phi(h) = tht^{-1} = xhx^{-1}$$

since $\eta \in \text{Reg}_L^1(H^*)$ is by definition (convolution-) central in $(H^*)^* = H$, and that

$$\sigma = d(\eta^{-1}) = (d\eta)^{-1} = \Delta(\eta)(\eta^{-1} \otimes \eta^{-1}) = \Delta(x)(x^{-1} \otimes x^{-1})$$

since t is group-like. \square

In particular, the subgroup $\text{Aut}_{\text{Hopf}}(H)$ is mapped to a monoidal autoequivalences of the form $(F_\phi, J^{\text{triv}})$ given by a pullback of the H -module action along an $\phi \in \text{Aut}_{\text{Hopf}}(DG)$ and trivial monoidal structure (see Example 1.2.11). The subgroup $Z_L^2(H^*)$ is mapped to monoidal autoequivalences of the form (id, J^σ) , which act trivial on objects and morphisms but have a non-trivial monoidal structure given by J^σ (see Corollary 1.2.15). Note that according to Lemma 1.3.9 there are in general invertible $x \in H$ that are not group-likes but still give functors (F_ϕ, J^σ) that are trivial up to monoidal natural transformations and which are not zero in $\text{Out}_{\text{Hopf}}(H) \times H_L^2(H^*)$.

Corollary 1.3.11. We call the image of map (1.9) the group of *lazy monoidal autoequivalences* $\text{Aut}_{\text{mon},L}(H\text{-mod})$ and get a short exact sequence

$$0 \rightarrow \text{Int}(H)/\text{Inn}(H) \rightarrow \text{Out}_{\text{Hopf}}(H) \times \mathbb{H}_L^2(H^*) \rightarrow \text{Aut}_{\text{mon},L}(H\text{-mod}) \rightarrow 0 \quad (1.10)$$

The restriction is an embedding of groups $\text{Out}_{\text{Hopf}}(H) \rightarrow \text{Aut}_{\text{mon},L}(H\text{-mod})$ and we have a bijection of sets of cosets

$$\mathbb{H}_{aL}^2(H^*) \simeq \text{Aut}_{\text{mon},L}(H\text{-mod})/\text{Out}_{\text{Hopf}}(H) \quad (1.11)$$

Proof. Using Lemma 1.3.9, we see that the map

$$\text{Aut}_{\text{Hopf}}(H) \rightarrow \text{Aut}_{\text{mon},L}(H\text{-mod}); \phi \mapsto (F_\phi, J^1)$$

has as kernel automorphisms of the form $\phi = x \cdot \text{id}_H \cdot x^{-1}$ for some invertible $x \in H$ such that $\Delta(x) = x \otimes x$. This implies that $\phi \in \text{Inn}(H)$.

In order to show the bijection (1.11), we use the exact sequence (1.10) to define the map from right to left by $[\phi, \sigma] \mapsto [\sigma]$ which is well-defined because a different representative in $[\phi, \sigma]$ would be mapped to a 2-cocycle that differs by an almost lazy coboundary from σ . \square

Chapter 2

Dijkgraaf-Witten Theory

An oriented $(3, 2, 1)$ -extended TQFT is a symmetric monoidal weak 2-functor:

$$F : \text{Bord}_{3,2,1}^{\text{or}} \rightarrow 2\text{Vect}$$

where $\text{Bord}_{3,2,1}^{\text{or}}$ is the symmetric monoidal bicategory of oriented 3-cobordisms and 2Vect the symmetric monoidal bicategory of Kapranov-Voevodsky 2-vector spaces, thus objects of 2Vect are k -linear, abelian, semisimple categories, morphisms are k -linear functors and 2-morphisms are natural transformations. (See [KV94], [Mo11] and the Appendix of [BDSV15] for more details on 2Vect and other targets).

Oriented $(3, 2, 1)$ -extended TQFTs are classified by anomaly free modular tensor categories (by Thm. 2 in [BDSV15]), where a functor F corresponds to the anomaly free modular tensor category $F(S^1)$, which we also refer to as the category of bulk Wilson lines. For general modular tensor categories, such theories are called Reshetikhin-Turaev type theories. In the case the modular tensor category is $F(S^1) = Z(\mathcal{C})$, the Drinfeld center of some fusion category \mathcal{C} , such theories are called Turaev-Viro type theories. They are Dijkgraaf-Witten theories if $F(S^1) = Z(\text{Vect}_G^\omega)$ where Vect_G^ω is the category of G -graded vector spaces for some finite group G and non-trivial associativity constraints determined by 3-cocycles $\omega \in Z^3(G, k^\times)$. If $\omega = 1$ the Dijkgraaf-Witten theory is called untwisted. Alternatively, one can use the Reshetikhin-Turaev construction ([RT91]), which is essentially based on surgery on 3-manifolds along links, to define a Reshetikhin-Turaev type theory explicitly.

In the next section, we want to present an explicit construction of Dijkgraaf-Witten theories that is based on gauge theory for a finite structure group G and a trivial topological Lagrangian $\omega = 1$.

2.1 Gauge Theoretic Construction

In this section we want to recall an explicit construction of untwisted Dijkgraaf-Witten theories based on [Mo13]. For this we quickly give some of the basic notions needed for this construction.

For an n -dimensional manifold M and a topological group G , let $\text{Bun}_G(M)$ be the groupoid of principal G -bundles on M and let $\Pi_1(M)$ be the fundamental groupoid of M , hence the objects of $\Pi_1(M)$ are points in M and morphisms are homotopy classes of paths between two points. For two categories \mathcal{C} and \mathcal{D} we denote by $[\mathcal{C}, \mathcal{D}]$

the category of functors and natural transformations. For a group G and set A with a G -action ρ , the action groupoid $A//_{\rho}G$ is defined as follows: objects are elements of A and a morphism between two objects $a, b \in A$ is an element $g \in G$ such that $\rho(g, a) = b$. A groupoid is called finite if it has finitely many objects and morphisms and is called essentially finite if it is equivalent to a finite groupoid. A groupoid is called finitely generated if it has finitely many objects and if all morphisms are generated under composition by a finite set. Further, a groupoid is called essentially finitely generated if it is equivalent to a finitely generated groupoid. Consider from now on only finite groups G and oriented compact manifolds.

- Let \mathcal{G} be a groupoid. There is a non-canonical equivalence

$$\mathcal{G} \simeq \coprod_{[g] \in \pi_0(\mathcal{G})} *//\text{Aut}(g) \quad (2.1)$$

where $*$ denotes the one element set and $\pi_0(\mathcal{G})$ the set of isomorphism classes of \mathcal{G} . To construct one such equivalence ψ we choose a family of isomorphisms $\{\gamma_y : g \rightarrow y \mid y \in \mathcal{G}\}$ for the choice of an object g in a component $[g] \in \pi_0(\mathcal{G})$. For an object $y \in \mathcal{G}$ in the connected component of g we set $\psi(y) = *$ and for a morphism $f : y \rightarrow y'$ we set $\psi(f)$ to be the composition

$$g \xrightarrow{\gamma_y} y \xrightarrow{f} y' \xrightarrow{\gamma_{y'}^{-1}} g \in \text{Aut}(g)$$

That the embedding $\iota : \coprod_{[g] \in \pi_0(\mathcal{G})} *//\text{Aut}(g) \rightarrow \mathcal{G}$ is a left inverse of ψ is immediate. To see that ι is also a right inverse, notice that the family $\{\gamma_y : g \rightarrow y \mid y \in \mathcal{G}\}$ provides a natural equivalence $\iota \circ \psi \simeq \text{id}_{\mathcal{G}}$.

- In particular, we have

$$\Pi_1(M) \simeq \coprod_{[m] \in \pi_0(M)} *//\pi_1(M, m)$$

where $\pi_0(M)$ denotes the set of path connected components of M and $\pi_1(M, m)$ the fundamental group of M at the point $m \in M$.

- We have a canonical equivalence

$$\text{Bun}_G(M) \simeq [\Pi_1(M), G\text{-Tor}]$$

where $G\text{-Tor}$ is the category of G -torsors, hence G -sets X that $X \simeq G$ as G -sets (where G acts on itself by left multiplication). Here we assign a G -bundle $P \xrightarrow{p} M$ to the functor $\Pi_1(M) \rightarrow G\text{-Tor}$ that maps a point $m \in M$ to its fiber $p^{-1}(m)$ and maps a homotopy class of a path γ to the map $p^{-1}(m) \rightarrow p^{-1}(n); x \mapsto \bar{\gamma}_x(1)$, where $\bar{\gamma}_x : [0, 1] \rightarrow P$ is the unique lift of γ with $\bar{\gamma}_x(0) = x$. Note that by the isomorphism (2.1), we also have a non-canonical equivalence $G\text{-Tor} \simeq *//G$.

- Since M is compact, the groupoid $\Pi_1(M) \simeq \coprod_{[m] \in \pi_0(M)} *//\pi_1(M, m)$ is essentially finitely generated. Since in addition G is finite, $\text{Bun}_G(M) \simeq [\Pi_1(M), *//G]$ is essentially finite.

- Let \mathcal{G} be an essentially finite groupoid, then the category $[\mathcal{G}, \text{Vect}]$ is a k -linear, abelian, finite semisimple category. We are mainly interested in essentially finite groupoids $\text{Bun}_G(M)$ for some compact manifold M and finite group G .

We want to construct a symmetric monoidal weak 2-functor: $F : \text{Bord}_{3,2,1}^{\text{or}} \rightarrow 2\text{Vect}$. Dijkgraaf-Witten theory is a gauge theory with finite structure group G . The gauge fields are therefore principal G -bundles. Therefore we consider

$$\mathcal{A}(M) := \text{Bun}_G(M) \simeq [\Pi_1(M), G\text{-Tor}]$$

to be the groupoid of gauge fields and gauge transformations on an n -manifold M . Let Σ be an object in $\text{Bord}_{3,2,1}^{\text{or}}$, hence an oriented compact 1-dimensional manifold. Given the configuration space $\mathcal{A}(\Sigma)$, we need to construct a k -linear category. In the original (non-extended) Dijkgraaf-Witten theory we would consider the space of states to be $\text{Hom}_k(k\pi_0(\text{Bun}_G(M)), k)$, the vector space of linear functions from the k -span of $\pi_0(\text{Bun}_G(M))$ to k . In the extended case there is a natural candidate for a linear category, namely:

$$F(\Sigma) := [\mathcal{A}(\Sigma), \text{Vect}] \tag{2.2}$$

Let us calculate a couple of examples:

Example 2.1.1.

(i) Empty Set: It is easy to deduce from the bullet points above that the groupoid of G -bundles in this case is $\mathcal{A}(\emptyset) \simeq [\emptyset, */G] \simeq */*$ and hence

$$F(\emptyset) = [\mathcal{A}(\emptyset), \text{Vect}] \simeq [*/*, \text{Vect}] = \text{Vect}$$

F sends hence the monoidal unit $\emptyset \in \text{Bord}_{3,2,1}^{\text{or}}$ to the monoidal unit $\text{Vect} \in 2\text{Vect}$.

(ii) Circle: We choose a base point on S^1 and an equivalence $\Pi_1(S^1) \simeq */\pi_1(S^1) \simeq */\mathbb{Z}$. Further we have an equivalence

$$[*/\mathbb{Z}, */G] \simeq G//_{\text{ad}}G$$

where ad denotes the adjoint action of G on itself. This equivalence maps a functor $A : */\mathbb{Z} \rightarrow */G$ to the group element $A(1) \in G$. A natural transformation of functors $A \Rightarrow A'$ is determined by a group element $h \in G$ such that $A'(1) = hA(1)h^{-1}$, hence h is a morphism $A(1) \rightarrow A'(1)$ in $G//_{\text{ad}}G$. Therefore we have

$$\mathcal{A}(S^1) \simeq [*/\mathbb{Z}, */G] \simeq G//_{\text{ad}}G$$

Further, there is a canonical equivalence of abelian categories:

$$F(S^1) = [G//_{\text{ad}}G, \text{Vect}] \simeq DG\text{-mod} \tag{2.3}$$

Given a functor $V : G//_{\text{ad}}G \rightarrow \text{Vect}$, the assignment of objects of V corresponds to the structure of a G -graded vector space $\bigoplus_{g \in G} V(g)$, which in other words defines a right k^G -module. The assignment of morphisms of V corresponds to the k^G -action:

A morphism $g \rightarrow hgh^{-1}$ in $G//_{\text{ad}}G$ gives us linear map $V(h) : V(g) \rightarrow V(hgh^{-1})$, which defines a kG -action compatible with the G -grading, hence we have a DG -module.

Similarly, we get for a disjoint union of n circles:

$$F\left(\bigsqcup_{1,\dots,n} S^1\right) \simeq [(G//G)^n, \text{Vect}] \simeq DG^{\otimes n}\text{-mod}$$

(iii) Disc: We have $\Pi_1(D) \simeq */*$ and therefore $\mathcal{A}(D) = [*/*, */G] = */G$. Then we can calculate as above

$$[\mathcal{A}(D), \text{Vect}] \simeq [*/G, \text{Vect}] = \text{Rep}(G)$$

(iv) Pair of Pants: Let P be the pair of pants $P : S^1 \sqcup S^1 \rightarrow S^1$. Then the fundamental group $\pi_1(P)$ is the free product $\mathbb{Z} * \mathbb{Z}$ and

$$\mathcal{A}(P) \simeq [*/\pi_1(P), */G] \simeq [*/(\mathbb{Z} * \mathbb{Z}), */G] \simeq (G \times G)//G$$

where G acts on $G \times G$ by diagonal adjoint action. Therefore

$$[\mathcal{A}(P), \text{Vect}] = [(G \times G)//G, \text{Vect}]$$

is equivalent to the category of $(G \times G)$ -graded vector spaces together with a compatible G -action.

In (iii) and (iv) we have calculated the linearization $[\mathcal{A}(M), \text{Vect}]$ for two 2-dimensional manifolds M . This should not be confused with what the $(3, 2, 1)$ -extended TQFT $F : \text{Bord}_{3,2,1}^{\text{or}} \rightarrow 2\text{Vect}$ assigns to M seen as a 1-morphism in $\text{Bord}_{3,2,1}^{\text{or}}$. Rather, F assigns a linear functor to a 1-morphism as we will describe below. For this we need the categories $[\mathcal{A}(M), \text{Vect}]$.

Let us now consider 1-morphisms in $\text{Bord}_{3,2,1}^{\text{or}}$, hence let M be a cobordism between Σ_1 and Σ_2 . We can write it as a span:

$$\begin{array}{ccc} & M & \\ \iota'_1 \nearrow & & \nwarrow \iota'_2 \\ \Sigma_1 & & \Sigma_2 \end{array}$$

where for $i = 1, 2$ the maps ι'_i denote the respective embeddings. For an $i = 1, 2$ we get a functor $\iota_i : \mathcal{A}(M) \rightarrow \mathcal{A}(\Sigma_i)$ induced by the embedding ι_i which gives us a span of essentially finite groupoids:

$$\begin{array}{ccc} & \mathcal{A}(M) & \\ \iota_1 \searrow & & \swarrow \iota_2 \\ \mathcal{A}(\Sigma_1) & & \mathcal{A}(\Sigma_2) \end{array}$$

The functors ι_1, ι_2 induce the following two functors by precomposition:

$$\iota_i^* : [\mathcal{A}(\Sigma_i), \text{Vect}] \rightarrow [\mathcal{A}(M), \text{Vect}]; \Psi \mapsto \Psi \circ \iota_i$$

This is also referred to as the restriction or the pullback. One can show that ι_i^* has a left adjoint $(\iota_{i*}^L, \eta_i^L, \epsilon_i^L)$, where η_i^L is the unit and ϵ_i^L the counit of the adjunction $\iota_{i*}^L \dashv \iota_i^*$. The functor ι_{i*}^L assigns to a $\Phi : \mathcal{A}(M) \rightarrow \text{Vect}$ the left Kan extension of Φ along ι_i (see [Mo13]). ι_{i*}^L is also referred to as a push-forward of ι_i^* .

Using this we define the following linear functor for the morphism $M : \Sigma_1 \rightarrow \Sigma_2$:

$$F(M) := \iota_{2*}^L \circ \iota_1^* : [\mathcal{A}(\Sigma_1), \text{Vect}] \rightarrow [\mathcal{A}(\Sigma_2), \text{Vect}] \quad (2.4)$$

Note that this construction involves a choice, namely we are choosing the left adjoint instead of a right adjoint. However, as we will see later, the left and the right adjoint are naturally equivalent. Further, the chosen natural equivalence allows us to construct the TQFT on 2-morphisms of $\text{Bord}_{3,2,1}^{or}$, hence on diffeomorphism classes of 3-manifolds with corners (see [Mo13] for more details). The construction $\iota_{2*}^L \circ \iota_1^*$ is also referred to as the pull-push construction or sometimes the pull-push quantization. We can always pullback fiber bundles, even in more general situations than this. The push-forward, however is more subtle and it is not obvious that it exists. We are however in a special situation where the push-forward does exist and we would like to recall the explicit construction of the push-forward from [Mo11].

If we consider the isomorphism (2.1) we see that every groupoid decomposes into a disjoint union of groups (one-point groupoids). The pullbacks are then just restrictions of the representations of automorphism groups induced by the functors ι_i . In the case of modules over rings these are also called *restriction of scalars*. The left adjoint of a restriction is the induction of representations or *extension of scalars*. That way we can give an explicit formula for the push-forward: Given two essentially finite groupoids \mathcal{G}, \mathcal{H} and a functor $\iota : \mathcal{G} \rightarrow \mathcal{H}$, the left adjoint of the pullback $\iota^* : [\mathcal{H}, \text{Vect}] \rightarrow [\mathcal{G}, \text{Vect}]$ is a functor $\iota_*^L : [\mathcal{G}, \text{Vect}] \rightarrow [\mathcal{H}, \text{Vect}]$ that assigns to a functor $V : \mathcal{G} \rightarrow \text{Vect}$ the functor

$$\begin{aligned} \iota_*^L V : \mathcal{H} &\rightarrow \text{Vect} \\ h &\mapsto \bigoplus_{\substack{[g] \in \pi_0(\mathcal{G}) \\ \iota(g) \simeq h}} k\text{Aut}(h) \otimes_{k\text{Aut}(g)} V(g) \end{aligned} \quad (2.5)$$

where $\text{Aut}(g)$ acts on $\text{Aut}(h)$ by precomposition with $\iota : \mathcal{G} \rightarrow \mathcal{H}$ and then normal multiplication in $\text{Aut}(h)$. For η_i^L, ϵ_i^L see [Mo11] Sect. 4.3. Since we need to make non-canonical choices for this formula, this approach might not always be useful. However, we can use this approach in order to calculate some basic examples which already give rise to non-trivial algebraic structure. In particular, we will see that $F(S^1)$ can be naturally equipped with a monoidal structure that arises from the pair of pants and a braiding that arises from a 3-dimensional cobordism, as shown further below.

Example 2.1.2.

(i) Death of a circle: Consider the disc D as a cobordism $S^1 \rightarrow \emptyset$. We consider the span

$$\begin{array}{ccc} & D & \\ \nearrow \iota'_1 & & \nwarrow \iota'_2 \\ S^1 & & \emptyset \end{array}$$

We have already calculated in Example 2.1.1 the groupoids of gauge fields on S^1 , D and \emptyset . Hence the above span induces the following span

$$\begin{array}{ccc} & *//G & \\ \nearrow \iota_1 & & \nwarrow \iota_2 \\ G//G & & *//* \end{array} \quad \begin{array}{l} \iota_1(*) = e, \iota_1(g) = e \\ \iota_2(*) = *, \iota_2(g) = * \end{array}$$

This gives us a span:

$$\begin{array}{ccc} & \text{Rep}(G) & \\ \nearrow \iota_1^* & & \nwarrow \iota_2^* \\ DG\text{-mod} & & \text{Vect} \end{array}$$

The functor $\iota_1^* : DG\text{-mod} \rightarrow \text{Rep}(G)$ maps a DG -module M to the component M_e together with the kG -action from M . The functor $\iota_2^* : \text{Vect} \rightarrow \text{Rep}(G)$ maps a vector space V to the trivial representation V^{triv} and its left adjoint is the induction functor: $\iota_*^L : \text{Rep}(G) \rightarrow \text{Vect}; M \mapsto k \otimes_{kG} M$, where $k \otimes_{kG} M \simeq M^G$ are the G -invariants of M . Hence the pull-push construction is the following:

$$\begin{aligned} F(D) &= \iota_{2*}^L \iota_1^* : DG\text{-mod} \rightarrow \text{Vect}; \\ &M \mapsto M_e^G \end{aligned}$$

(ii) Birth of a circle: Consider the disc D as a cobordism $\emptyset \rightarrow S^1$. We abbreviate this calculation a bit since it is similar to the death of a circle. We get

$$\begin{array}{ccc} & *//G & \\ \nearrow \iota_1 & & \nwarrow \iota_2 \\ *//* & & G//G \end{array} \quad \begin{array}{l} \iota_1(*) = *, \iota_1(g) = * \\ \iota_2(*) = e, \iota_2(g) = e \end{array}$$

Here $\iota_1^* : \text{Vect} \rightarrow \text{Rep}(G)$ maps a vector space V to the trivial representation V^{triv} . The functor $\iota_2^* : DG\text{-mod} \rightarrow \text{Rep}(G)$ maps a DG -module M to M_e together with the kG -action induced from M . The left adjoint $\iota_{2*}^L : [*//G, \text{Vect}] \rightarrow [G//G, \text{Vect}]$ is given by mapping a functor $W : *//G \rightarrow \text{Vect}$ to the functor

$$\begin{aligned} \iota_{2*}^L W &: G//G \rightarrow \text{Vect} \\ g &\mapsto \bigoplus_{\substack{[h] \in \pi_0(*//G) \\ \iota_2(h) \simeq *}} k\text{Aut}(g) \otimes_{k\text{Aut}(h)} W(*) \end{aligned}$$

More explicitly: $\iota_{2*}^L : \text{Rep}(G) \rightarrow DG\text{-mod}$ maps a kG -module W to the kG -module W together with the trivial G -grading: $W_e = W$ and $W_g = 0$ for all $g \neq e$. The pull-push construction hence is the following functor:

$$F(D) = \iota_{2*}^L \iota_1^* : \text{Vect} \rightarrow DG\text{-mod}; \quad M \mapsto M^{triv}$$

where M^{triv} denotes the trivial DG -module.

(iii) Pair of Pants: Let us consider the pair of pants P as a cobordism $S^1 \sqcup S^1 \rightarrow S^1$. We already have calculated $\mathcal{A}(S^1 \sqcup S^1) = (G//G)^2$, $\mathcal{A}(P) = (G \times G)//G$ and $\mathcal{A}(S^1) = G//G$ in Example 2.1.1. Similarly to above, we have to consider the following span of groupoids:

$$\begin{array}{ccc} & (G \times G)//G & \\ \iota_1 \swarrow & & \searrow \iota_2 \\ (G//G) \times (G//G) & & G//G \end{array}$$

where the functors ι_1, ι_2 are given as follows:

$$\begin{aligned} \iota_1 : (G \times G)//G &\rightarrow (G//G) \times (G//G) \\ (r, s) &\mapsto (r, s) \\ g &\mapsto (g, g) \end{aligned}$$

$$\begin{aligned} \iota_2 : (G \times G)//G &\rightarrow (G//G) \\ (r, s) &\mapsto rs \\ g &\mapsto g \end{aligned}$$

Recall from Example 2.2 that $[(G//G)^2, \text{Vect}] \simeq DG \otimes DG\text{-mod}$ which is equivalent to $DG\text{-mod} \boxtimes DG\text{-mod}$ and that $[(G \times G)//G, \text{Vect}]$ is equivalent to the category of $(G \times G)$ -graded vector spaces with compatible G -action.

The pullback $\iota_1^* : [(G//G)^2, \text{Vect}] \rightarrow [(G \times G)//G, \text{Vect}]$ assigns to a pair of DG -modules M, N the $(G \times G)$ -graded vector space

$$\bigoplus_{(r,s) \in G \times G} M_r \otimes N_s \tag{2.6}$$

equipped with the diagonal G -action.

Now consider the pullback ι_2^* . By equation (2.5), the left adjoint $\iota_{2*}^L : [(G \times G)//G, \text{Vect}] \rightarrow [G//G, \text{Vect}]$ of ι_2^* assigns to a functor $W : (G \times G)//G \rightarrow \text{Vect}$ the functor:

$$\begin{aligned} \iota_{2*}^L W : G//G &\rightarrow \text{Vect} \\ g &\mapsto \bigoplus_{\substack{[r,s] \in \pi_0((G \times G)//G) \\ \iota_2(r,s) \simeq g}} k\text{Aut}(g) \otimes_{k\text{Aut}(r,s)} W(r, s) \end{aligned}$$

First note that $\text{Aut}(g) = C_G(g)$ is the centralizer of g and $\text{Aut}(r, s) = \text{Stab}_G(r, s) = \{\gamma \in G \mid r^\gamma = r, s^\gamma = s\}$ is the stabilizer of $(r, s) \in G \times G$ under the diagonal conjugation action. Further, we write $(G \times G)/G = \pi_0((G \times G)//G)$ for the set of orbits $[r, s]$ of the diagonal conjugation. We calculate further:

$$\begin{aligned} (\iota_{2*}^L W)(g) &\simeq \bigoplus_{\substack{[r,s] \in (G \times G)/G \\ rs \simeq g}} kC_G(g) \otimes_{k[\text{Stab}_G(r,s)]} W(r, s) \\ &\simeq \bigoplus_{\substack{[r,s] \in (G \times G)/G \\ rs \simeq g}} \left(\bigoplus_{C_G(g)/\text{Stab}_G(r,s)} W(r, s) \right) \end{aligned} \quad (2.7)$$

We want to show that, as a vector space, this is canonically isomorphic to

$$\bigoplus_{\substack{(r,s) \in G \times G \\ rs = g}} W(r, s)$$

To show this we need to compare the number of copies of $W(r, s)$ on both sides. For this we use $|\text{Stab}_G(r, s)||[r, s]| = |G|$ and $|C_G(g)||[g]| = |G|$ where $[g]$ is the conjugation class of g . We need to show that the direct sum in (2.7) has $|\{(r, s) \in G \times G \mid rs = g\}| = |G|$ copies of $W(r, s)$:

$$\begin{aligned} \sum_{\substack{[r,s] \in (G \times G)/G \\ rs \simeq g}} \frac{|C_G(g)|}{|\text{Stab}_G(r, s)|} &= \sum_{\substack{[r,s] \in (G \times G)/G \\ rs \simeq g}} \frac{|[r, s]|}{|[g]|} \\ &= \sum_{\substack{[r,s] \in (G \times G)/G \\ \forall (r', s') \in [r, s]: r' s' \in [g]}} \frac{|[r, s]|}{|[g]|} \\ &= \sum_{\substack{(r,s) \in (G \times G) \\ rs \in [g]}} \frac{1}{|[g]|} \\ &= |G| \end{aligned}$$

where $rs \simeq g$ means isomorphic in $G//G$. Thus the e.g. the first sum goes over all orbits $(G \times G)/G$ such that there exists a representative (r, s) and a $x \in G$ with $rs = g^x$ (hence $rs \in [g]$). This shows that

$$(\iota_{2*}^L W)(g) \simeq \bigoplus_{\substack{(r,s) \in G \times G \\ rs = g}} W(r, s)$$

Further, the functor $(\iota_{2*}^L W) : G//G \rightarrow \text{Vect}$ on morphisms gives the G -graded vector space $\bigoplus_{g \in G} \bigoplus_{\substack{(r,s) \in G \times G \\ rs = g}} W(r, s)$ a compatible G -action induced by the linear maps $W(\iota_2(\gamma)) = W(\gamma, \gamma) : W(r, s) \rightarrow W(\gamma r \gamma^{-1}, \gamma s \gamma^{-1})$.

Recall that the pullback ι_1^* assigns to two DG -modules M, N the $(G \times G)$ -graded vector space defined in (2.6). Thus, the pull-push functor $\iota_{2*}^L \iota_1^*$ assigns to a pair of DG -modules M, N its tensor product:

$$F(P) = \iota_{2*}^L \iota_1^* : DG\text{-mod} \boxtimes DG\text{-mod} \rightarrow DG\text{-mod}; \quad M \boxtimes N \mapsto M \otimes_k N$$

Therefore F evaluated at the pair of pants P gives the canonical monoidal product on $DG\text{-mod}$.

For the whole monoidal structure we need more data than just the monoidal product. In particular, we need a monoidal unit, associativity constraints and left/right constraints. These have to fulfill well-known coherence conditions. The monoidal unit is given by the birth of a circle $D : \emptyset \rightarrow S^1$ which induces a functor $F(D) : \text{Vect} \rightarrow DG\text{-mod}$. As seen in Example 2.2 $F(D)$ maps the ground field k to k^{triv} which is the monoidal unit in $DG\text{-mod}$. The rest of the data for the monoidal structure arises from 2-morphisms in $\text{Bord}_{3,2,1}^{\text{or}}$, hence 3-dimensional manifolds with corners, as we will remark below.

Let us now consider F on 2-morphisms in $\text{Bord}_{3,2,1}^{\text{or}}$, hence on 3-manifolds with corners. For this first note that the functor ι_i^* also has a right adjoint $(\iota_{i*}^R, \eta_i^R, \epsilon_i^R)$ defined analogously by the right Kan extension. Analogously to the left adjoint, we can give an explicit formula for the right adjoint following [Mo11]. Again, consider two essentially finite groupoids \mathcal{G}, \mathcal{H} and a functor $\iota : \mathcal{G} \rightarrow \mathcal{H}$. The right adjoint of the pullback $\iota^* : [\mathcal{H}, \text{Vect}] \rightarrow [\mathcal{G}, \text{Vect}]$ is a functor $\iota_*^R : [\mathcal{G}, \text{Vect}] \rightarrow [\mathcal{H}, \text{Vect}]$ that assigns a functor $V : \mathcal{G} \rightarrow \text{Vect}$ to the functor

$$\begin{aligned} \iota_*^R V : \mathcal{H} &\rightarrow \text{Vect} \\ h &\mapsto \bigoplus_{\substack{[g] \in \pi_0(\mathcal{G}) \\ \iota(g) \simeq h}} \text{Hom}_{k\text{Aut}(g)}(k\text{Aut}(h), V(g)) \end{aligned} \quad (2.8)$$

where $\text{Aut}(g)$ acts on $\text{Aut}(h)$ by precomposition with $\iota : \mathcal{G} \rightarrow \mathcal{H}$ and then multiplication in $\text{Aut}(h)$. For η_i^R, ϵ_i^R see [Mo11] Sect. 4.3.

Now it is essential for the construction of the Dijkgraaf-Witten theory that we can choose a natural equivalence between the left and right adjoints:

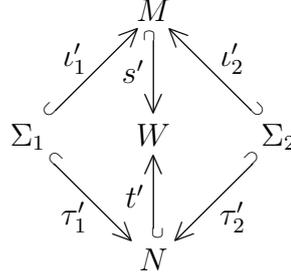
$$\iota_{i*}^R \xrightarrow{\mathcal{N}} \iota_{i*}^L$$

called the Nakayama isomorphism (see [Mo11], [FHLT10]). The property that the left and right adjoints are naturally equivalent is called *ambidexterity*. The importance of ambidexterity for extended TQFTs was first stressed in [FHLT10]. See also [LH14] for a more detailed exposition on ambidexterity in this context. Following [Mo11] we have an explicit formula for $\mathcal{N} = \{\mathcal{N}_V : \iota_{i*}^R V \rightarrow \iota_{i*}^L V \mid V \in [\mathcal{G}, \text{Vect}]\}$ as follows:

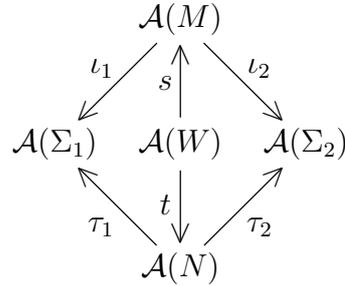
$$\begin{aligned} \mathcal{N}_V : \bigoplus_{\substack{[g] \in \pi_0(\mathcal{G}) \\ \iota(g) \simeq h}} \text{Hom}_{k\text{Aut}(g)}(k\text{Aut}(h), V(g)) &\rightarrow \bigoplus_{\substack{[g] \in \pi_0(\mathcal{G}) \\ \iota(g) \simeq h}} k\text{Aut}(h) \otimes_{k\text{Aut}(g)} V(g) \\ \bigoplus_{\substack{[g] \in \pi_0(\mathcal{G}) \\ \iota(g) \simeq h}} \phi_{[g]} &\mapsto \bigoplus_{\substack{[g] \in \pi_0(\mathcal{G}) \\ \iota(g) \simeq h}} \frac{1}{|\text{Aut}(g)|} \sum_{x \in \text{Aut}(h)} x^{-1} \otimes \phi_{[g]}(x) \end{aligned}$$

Consider the following 3-dimensional cobordism class: Let W be a 3-dimensional manifold with corners, M and N be 2-dimensional manifolds with boundary and

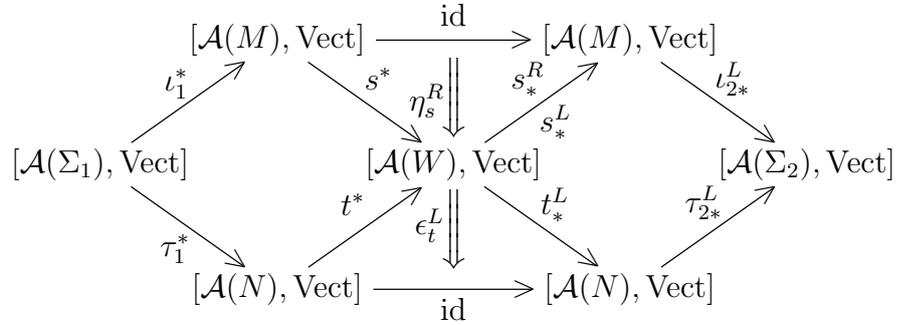
Σ_1 and Σ_2 be 1-dimensional closed manifolds. Further, W is a cobordism between M and N and M, N are cobordisms between Σ_1, Σ_2 . Hence we have a commuting diagram of manifolds



which induces a commuting diagram of essentially finite groupoids



As above all functors in this diagram induce restriction functors, which have left and right adjoints. This leads us to the following commuting diagram



Define now $F(W)$ to be the natural transformation that is given by composing the natural transformations of the above diagram from top to bottom:

$$F(W) : \underbrace{\iota_{2*}^L \text{id} \iota_1^*}_{F(M)} \xrightarrow{\eta_s^R} \iota_{2*}^L s_*^R s^* \iota_1^* \xrightarrow{\mathcal{N}} \iota_{2*}^L s_*^L s^* \iota_1^* = \tau_{2*}^L t_*^L t^* \tau_1^* \xrightarrow{\epsilon_t^L} \underbrace{\tau_{2*}^L \text{id} \tau_1^*}_{F(N)} \quad (2.9)$$

Where $\mathcal{N} : s_*^R \Rightarrow s_*^L$ is the chosen Nakayama isomorphism between the right and left adjoint of the pullback s^* . Now we have defined F on objects, morphisms and 2-morphisms of $\text{Bord}_{3,2,1}^{\text{or}}$. We cite the following theorem:

Theorem 2.1.3. ([Mo13] Prop. 2) The assignments (2.2), (2.4) and (2.9) define a symmetric monoidal weak 2-functor $F : \text{Bord}_{3,2,1}^{\text{or}} \rightarrow 2\text{Vect}$ and hence a $(3, 2, 1)$ -extended TQFT.

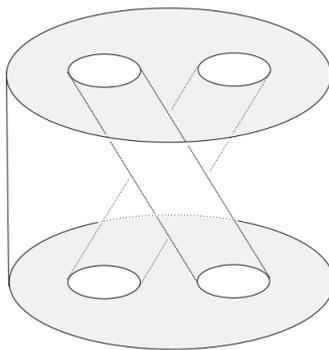
In Example 2.1.1 we have seen that $F(S^1) \simeq [G//G, \text{Vect}] \simeq DG\text{-mod} \simeq Z(\text{Vect}_G)$ as abelian categories. Additionally, we calculated in Example 2.1.2 that the pair of pants $P : S^1 \sqcup S^1 \rightarrow S^1$ endows the category $F(S^1)$ with a monoidal product $F(P) : F(S^1) \boxtimes F(S^1) \rightarrow F(S^1)$ that is equivalent to the canonical monoidal product on $DG\text{-mod}$.

The associativity constraint is a natural equivalence

$$(\cdot \otimes \cdot) \circ (\text{id} \boxtimes (\cdot \otimes \cdot)) \xrightarrow{\cong} (\cdot \otimes \cdot) \circ (\text{id} \boxtimes (\cdot \otimes \cdot))$$

The two functors on both sides correspond to a disc with three discs removed, hence the 3-dimensional cobordism giving the structure of associativity constraints is just the identity cobordism. Therefore, we have a trivial associativity constraint on $F(S^1)$ (since we have a trivial associativity constraint on Vect). Similarly, the left and right constraints are trivial cobordisms from the disc with one disc removed.

Further, let us consider the 3-manifold with corners W :



seen as a cobordism from the pair of pants (gray disc with two discs removed) to the pair of pants. Evaluating the TQFT F from Theorem 2.1.3 on this 3-manifold gives us a natural transformation $\cdot \otimes \cdot \Rightarrow \cdot \otimes^{op} \cdot$ and it turns out that this natural transformation is indeed the braiding on $DG\text{-mod}$ as in (1.2). Further, it is well-known that with this braiding $DG\text{-mod}$ is a modular tensor category. Thus the Dijkgraaf-Witten-theory evaluated on the circle gives a modular tensor category braided equivalent to $DG\text{-mod}$.

Note that one can show for a general $(3, 2, 1)$ -extended TQFT $F : \text{Bord}_{3,2,1}^{\text{or}} \rightarrow 2\text{Vect}$ that $F(S^1)$ is an anomaly free modular tensor category, without relying on any explicit gauge theoretic constructions (see [BDSV15]). The monoidal structure is induced by the pair of pants and the braiding by the same 3-dimensional manifold with corners as in the Dijkgraaf-Witten case (see the picture above). Due to Thm. 2 in [BDSV15] we moreover know that a TQFT $F : \text{Bord}_{3,2,1}^{\text{or}} \rightarrow 2\text{Vect}$ is uniquely (up to equivalence) determined by the anomaly free modular tensor category it assigns to the circle.

We should also mention that the Dijkgraaf-Witten theory F constructed in this section factors through the functor $\text{Bord}_{3,2,1}^{\text{or}} \rightarrow \text{Span}(\text{Grd})$, where $\text{Span}(\text{Grd})$ is the bicategory of groupoids, spans of groupoids and (equivalence classes of) spans of spans. This functor assigns a 1-manifold M to the groupoid of G -bundles $\text{Bun}_G(M)$, a 2-manifold with boundary to a span of groupoids and a 3-manifold with corners to a span of spans. This procedure can essentially be deduced from the discussion above, where we wrote down the spans and spans of spans but did not explicitly mention the functor $\text{Bord}_{3,2,1}^{\text{or}} \rightarrow \text{Span}(\text{Grd})$.

There exists a functor $\text{Span}(\text{Grd}) \rightarrow 2\text{Vect}$, the so-called 2-linearization. It assigns an essentially finite groupoid \mathcal{G} to its 2-linearization $[\mathcal{G}, \text{Vect}]$. It assigns a span of groupoids to its pull-push functor and a span of spans to the natural transformation determined by the Nakayama isomorphism. In [FHLT10], [Mo13] the functor $\text{Bord}_{3,2,1}^{\text{or}} \rightarrow \text{Span}(\text{Grd})$ is interpreted as the classical field theory and the 2-linearization $\text{Span}(\text{Grd}) \rightarrow 2\text{Vect}$ as the quantization of this classical field theory. In order for this procedure to be well-defined we need that left and right adjoints are equivalent. This property is called ambidexterity and also appears in a different but related context in [Lu13].

In the discussion above, we have thus presented an explicit gauge theoretic construction of a Reshetikhin-Turaev type theory based on the modular tensor category $Z(\text{Vect}_G) = DG\text{-mod}$ or equivalently a Turaev-Viro type theory based on Vect_G . For the equivalence between the Turaev-Viro theory based on a spherical fusion category \mathcal{C} and the Reshetikhin-Turaev theory based on the modular tensor category $Z(\mathcal{C})$, as extended TQFTs, we refer to [BK10].

Note that our discussion has not touched on defects or boundary conditions yet. Still it serves as a guide to the construction of Dijkgraaf-Witten theories than incorporate defects and boundary conditions. In the following section we describe parts of such a gauge theoretic construction, where we model the classical part of the gauge theory by groupoids of generalizations of relative bundles and use the pull-push construction to define the quantized theory.

2.2 Defects and the Transmission Functor

2.2.1 General Considerations

A general approach to the study of defects in Reshetikhin-Turaev type theories was presented in [FSV13]. Given two Reshetikhin-Turaev type theories based on modular tensor categories \mathcal{C} and \mathcal{D} respectively, living in two separated three dimensional regions, it is interesting to ask when topological surface defects between these theories exists and if they do exist, it is interesting to classify these defects. Based on [FSV13], such defects exists if and only if there exists a fusion category \mathcal{A} and a braided equivalence

$$\mathcal{C} \boxtimes \mathcal{D}^{\text{rev}} \simeq Z(\mathcal{A})$$

where \mathcal{D}^{rev} is the category \mathcal{D} with opposite braiding and where $Z(\mathcal{A})$ is the Drinfeld center of \mathcal{A} . In other words, \mathcal{C} and \mathcal{D} are Witt equivalent (are in the same Witt

class, see [DMNO13]). Further, assuming that such an equivalence does indeed exist, defects between these two theories are described by the bicategory of \mathcal{A} -module categories, bimodule functors and natural transformations.

We are interested in topological surface defects between a Dijkgraaf-Witten theory based on a finite group G and itself. In this case the modular categories determining the two theories are $\mathcal{C} = Z(\text{Vect}_G) = \mathcal{D}$ and thus we already have a canonical braided equivalence

$$Z(\text{Vect}_G) \boxtimes Z(\text{Vect}_G)^{rev} \simeq Z(\text{Vect}_{G \times G^{op}})$$

Topological surface defects are therefore given by $\text{Vect}_{G \times G^{op}}$ -module categories or equivalently Vect_G -bimodule categories. Invertible defects correspond to invertible Vect_G -bimodule categories. Due to [O03], we know that indecomposable Vect_G -bimodule categories correspond to pairs (H, ϕ) where H is a subgroup of $G \times G^{op}$ and where ϕ is a 2-cocycle on H (up to coboundary). Additionally, Corollary 3.6.3 of [Dav10] and Proposition 5.2 of [NR14] gives conditions on (H, ϕ) such that the corresponding indecomposable Vect_G -bimodule category is invertible (see Proposition 2.2.2).

An analysis of the gauge theoretic construction of $(3, 2, 1)$ -extended Dijkgraaf-Witten theories with boundary conditions and surface defects was done in [FSV14]. Such a theory is per definition a symmetric monoidal weak 2-functor $\text{Bord}_{3,2,1}^{dec} \rightarrow 2\text{Vect}$ where the manifolds in $\text{Bord}_{3,2,1}^{dec}$ carry extra decorations corresponding to defects and boundary conditions. In order to do an explicit gauge theoretic construction the notion of relative bundles was introduced (see [FSV14]). Here we are only interested in certain parts of this $(3, 2, 1)$ -theory, so we do not give, nor do we need, the full construction of this 2-functor at this point.

Let us consider the following situation: Assume we are given two manifolds M, N and a smooth map $j : N \rightarrow M$. We can pullback a G -bundle P_G on M to a G -bundle $j^*(P_G)$ on N . Additionally, if we have given a group homomorphism $\iota : H \rightarrow G$ then we can assign, using the so-called reduction of the structure group, an H -bundle P_H on N to the associated G -bundle $\text{Ind}_\iota(P_H) = P_H \times_H G$ on N . A relative (G, H) -bundle on (M, N, j) is defined to be a triple (P_G, P_H, α) where P_G is a G -bundle on M , P_H is an H -bundle on N and α is an isomorphism:

$$\alpha : \text{Ind}_\iota(P_H) \xrightarrow{\cong} j^*(P_G)$$

Relative bundles form an essentially finite groupoid, where a morphism of two relative bundles $(P_G, P_H, \alpha) \rightarrow (P'_G, P'_H, \alpha')$ is a pair (ϕ_G, ϕ_H) where $\phi_G : P_G \rightarrow P'_G$ is a morphism of G -bundles and $\phi_H : P_H \rightarrow P'_H$ is a morphism of H -bundles such that the following diagram commutes:

$$\begin{array}{ccc} \text{Ind}_\iota(P_H) & \xrightarrow{\alpha} & j^*(P_G) \\ \text{Ind}_\iota(\phi_H) \downarrow & & \downarrow j^*(\phi_G) \\ \text{Ind}_\iota(P'_H) & \xrightarrow{\alpha'} & j^*(P'_G) \end{array}$$

The notion of relative bundles can now be used to describe the following situation: We consider N to be the boundary of M and $j : N \hookrightarrow M$ an embedding. Further, imagine a Dijkgraaf-Witten theory living on M that is based on the data (G, ω) where G is a finite group and $\omega \in Z^3(G, k^\times)$ the topological Lagrangian of the theory. On the boundary N we have a boundary condition labeled by (H, η) where $\eta \in C^2(H, k^\times)$ is a cochain such that $d\eta = \iota^*\omega$. The pair (H, η) is also referred to as the Lagrangian data corresponding to the boundary condition.

Similarly, for defects, we consider relative (G_1, G_2, H) -bundles on (M_1, M_2, N, j_1, j_2) . In this case we have manifolds M_1, M_2 and two embeddings $\iota_i : N \hookrightarrow M_i$. A relative bundle is then the data

$$(P_{G_1}, P_{G_2}, P_H, \alpha_1, \alpha_2)$$

where P_{G_i} is a G_i -bundle on M_i , P_H an H -bundle on N and $\alpha_i : \text{Ind}_{\iota_i}(P_H) \xrightarrow{\cong} j_i^*(P_{G_i})$ are isomorphisms of G_i -bundles where $\iota_i : H \rightarrow G_i$ are group homomorphisms. Morphisms of such relative bundles are defined similarly as above. We have two Dijkgraaf-Witten theories based on (G_1, ω_1) and (G_2, ω_2) living on M_1, M_2 respectively with a defect N labeled by (H, η) where $\eta \in C^2(H, k^\times)$ is a cochain such that $d\eta = \iota_1^*\omega_1 \cdot (\iota_2^*\omega_2)^{-1}$.

With these notions we have described the classical part of the Dijkgraaf-Witten theory with boundary conditions and defects. Now we also need a 2-linearization that takes into account the relative bundles and the Lagrangian data of the defects/boundary conditions. We refer to [FSV14] for more details and examples on this and will only concentrate on a specific construction, namely the so-called transmission functor.

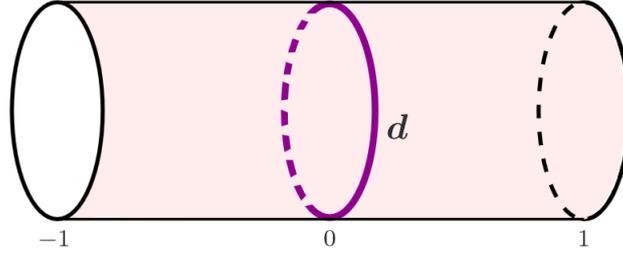
2.2.2 Transmission Functor

Before going to the explicit constructions in the Dijkgraaf-Witten model, let us describe the transmission functor for Reshetikhin-Turaev type theories with defects following [FPSV15] Sect. 2.2.

Assume we have a cylinder $S^1 \times [-1, 1]$ between two 3-dimensional regions that are separated by a topological surface defect labeled by d . On both sides of the defect surface we have a Reshetikhin-Turaev-type theory based on a modular tensor category \mathcal{C} . Since \mathcal{C} is modular, we have a canonical braided autoequivalence by Thm 7.10 in [Mü03]:

$$\mathcal{C} \boxtimes \mathcal{C}^{rev} \simeq Z(\mathcal{C})$$

Therefore topological surface defects are described by \mathcal{C} -module categories. Such a surface between these two regions intersects the cylinder transversely, say at $S^1 \times \{0\}$, and therefore produces a defect line along $S^1 \times \{0\}$:



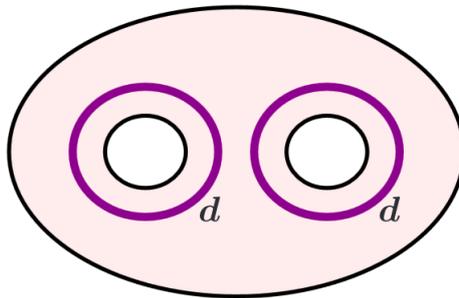
A TQFT with defects maps both circles to the modular tensor category \mathcal{C} , the category of bulk Wilson lines, and maps such a decorated cylinder to a linear functor

$$F_d : \mathcal{C} \rightarrow \mathcal{C}$$

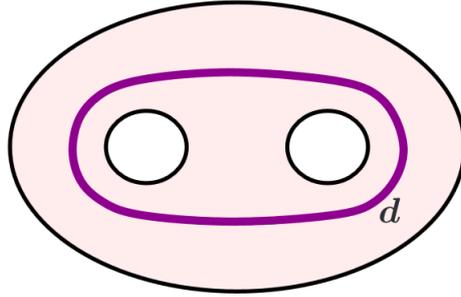
We call the functor F_d the *transmission functor* for the defect d . It describes how the type of a bulk Wilson line changes when it passes through the defect surface labeled by d . We want to show, in case d is invertible, that F_d is a braided autoequivalence of \mathcal{C} . If d is invertible, then there exists another defect d' and a transmission functor $F_{d'}$ such that the fusion of defects implies that $F_d \circ F_{d'} \simeq \text{id}_{\mathcal{C}} \simeq F_{d'} \circ F_d$. Therefore F_d is an autoequivalence of \mathcal{C} . We further want to show that the transmission functor F_d has a canonical monoidal structure and that F_d with this monoidal structure is braided. A monoidal structure on F_d is a natural transformation of the following two functors:

$$(\cdot \otimes \cdot) \circ (F_d \boxtimes F_d) \Rightarrow F_d \circ (\cdot \otimes \cdot)$$

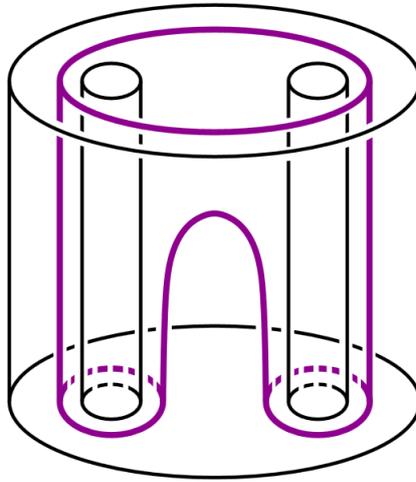
The functor $(\cdot \otimes \cdot) \circ (F_d \boxtimes F_d)$ corresponds to the result of gluing a disjoint union of two cylinders each decorated with d to the pair of pants along the two incoming boundary circles. The result is a decorated pair of pants P_1 as seen in the following picture:



The functor $F_d \circ (\cdot \otimes \cdot)$ corresponds to the result of gluing a pair of pants to the cylinder decorated with d along the outgoing boundary circle. The result is a decorated pair of pants P_2 as seen in the following picture:



Therefore, a monoidal structure on F_d correspond to a cobordism $P_1 \rightarrow P_2$. Let us consider the following 3-dimensional manifold with corners decorated with a defect surface:

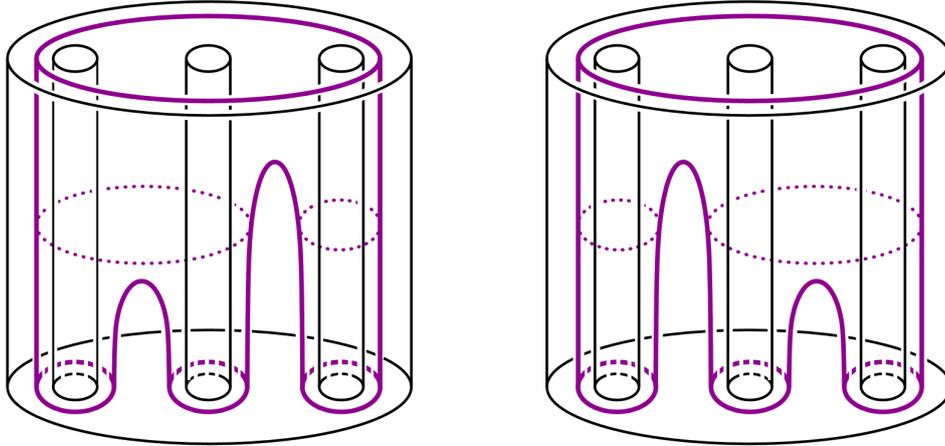


Read from bottom to top, this is indeed a cobordism from P_1 to P_2 and we argue that the TQFT maps this cobordism to a natural equivalence that defines a monoidal structure on the functor F_d . At this point it is essential that d is indeed invertible since the natural transformation would not be an equivalence for more general d . For d invertible, we indeed have an inverse, namely the decorated 3-cobordism from the last picture but taken upside down. To see that the result of the gluing is diffeomorphic to the identity, one has to use the fact that d is invertible, so the resulting defect surface becomes just two defect cylinders. Gluing the other way around, and using invertibility of d , there surface defect after gluing becomes just one defect cylinder.

In order to show that the above natural equivalence is indeed a monoidal structure on the functor F_d we have to check a coherence condition. Thus, we the following diagram of natural transformations has to commute:

$$\begin{array}{ccc}
 (\cdot \otimes \cdot \otimes \cdot) \circ (F_d \boxtimes F_d \boxtimes F_d) & \xrightarrow{\simeq} & (F_d \otimes \cdot) \circ (\cdot \otimes \cdot \boxtimes F_d) \\
 \simeq \Downarrow & & \simeq \Downarrow \\
 (\cdot \otimes F_d) \circ (F_d \boxtimes \cdot \otimes \cdot) & \xrightarrow{\simeq} & F_d \circ (\cdot \otimes \cdot \otimes \cdot)
 \end{array}$$

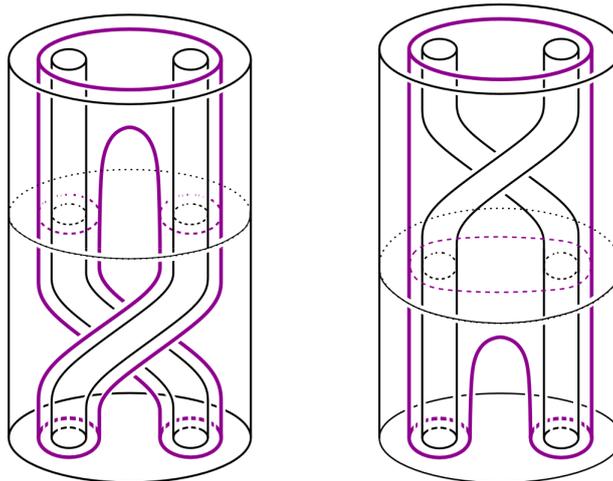
To show that this diagram commutes we consider the two cobordisms corresponding to the two natural equivalences:



and see that they are indeed diffeomorphic to each other. This implies that the diagram commutes and thus that the coherence condition is fulfilled. The next thing to check is that the monoidal structure as defined above is braided. For this we again need to check an equality of natural equivalences, namely the following diagram of natural transformations has to commute:

$$\begin{array}{ccc}
 (\cdot \otimes \cdot) \circ F_d & \xrightarrow{\simeq} & F_d \circ (\cdot \otimes \cdot) \\
 \simeq \downarrow & & \downarrow \simeq \\
 (\cdot \otimes^{op} \cdot) \circ F_d & \xrightarrow{\simeq} & F_d \circ (\cdot \otimes^{op} \cdot)
 \end{array}$$

For this we compare the following two 3-dimensional cobordisms:



and see that they are diffeomorphic to each other and thus the monoidal structure is indeed a braided autoequivalence of \mathcal{C} .

2.2.3 Transmission Functor for Dijkgraaf-Witten theories

With a gauge theoretic construction of a TQFT with defects we can explicitly calculate the transmission functor F_d . We sketched in the Subsection 2.2.1 how to describe the classical situation of a Dijkgraaf-Witten theory with defects with the notion of relative bundles. We now describe relative bundles and the 2-linearization procedure in the case of a cylinder decorated by an invertible defect d . Here on both sides of the defect we consider an untwisted Dijkgraaf-Witten theory based on the a finite group G . The invertible defect d is therefore given by an invertible Vect_G -bimodule category. By the results in [O03] we know that indecomposable Vect_G -bimodule categories are determined by pairs (H, η) where $H \subset G \times G^{op}$ a subgroup and $\eta \in Z^2(H, k^\times)$ a 2-cocycle (up to coboundary). Corollary 3.6.3 of [Dav10] (and Proposition 5.2 of [NR14]) give conditions on (H, ϕ) such that the corresponding indecomposable Vect_G -bimodule category is invertible (see Proposition 2.2.2).

First we want to determine the groupoid of relative (G, G, H) -bundles on (C_1, C_2, S^1) where $C_1 := S^1 \times [-1, 0]$, $C_2 := S^1 \times [0, 1]$ are cylinders and where S^1 denotes the defect circle labeled by (H, η) . This groupoid further depends on the obvious embeddings of the defect circle to the boundary of the cylinders ($i = 1, 2$)

$$j_i : S^1 \hookrightarrow C_i$$

and the group homomorphisms ($i = 1, 2$)

$$\iota_i : H \hookrightarrow G \times G^{op} \xrightarrow{p_i} G$$

where p_1 projects to the first factor and p_2 is the composition of the projection to the second factor postcomposed with the canonical group isomorphism $G^{op} \simeq G$. Let us denote this groupoid by $\text{Bun}_{G,G,H}(C_1, C_2, S^1)$. In the following claim we describe the objects and morphisms of this groupoid and demonstrate how one would proceed when calculating such groupoids.

Claim 2.2.1.

The groupoid $\text{Bun}_{G,G,H}(C_1, C_2, S^1)$ is equivalent to the action groupoid $H//_{ad}H$.

Proof. $\text{Bun}_{G,G,H}(C_1, C_2, S^1)$ has per definition as objects the data: A G -bundle P_G^1 on C_1 , a G -bundle P_G^2 on C_2 , an H -bundle P_H on S^1 and two isomorphisms of G -bundles

$$\alpha_i : \text{Ind}_{\iota_i}(P_H) \xrightarrow{\cong} j_i^*(P_G^i)$$

for $i = 1, 2$. As described in Example 2.1.1 we identify P_G^1, P_G^2 with functors $A_1 : \Pi_1(C_1) \rightarrow *//G$, $A_2 : \Pi_1(C_2) \rightarrow *//G$ respectively and we identify P_H with a functor $A_d : \Pi_1(S^1) \rightarrow *//H$. Then the inductions for $i = 1, 2$ corresponds to the functors

$$\text{Ind}_{\iota_i}(A_d) : \Pi_1(S^1) \rightarrow *//H \xrightarrow{\iota_i} *//G$$

and the pullbacks for $i = 1, 2$ correspond to the functors:

$$j_i^*(A_i) : \Pi_1(S^1) \xrightarrow{j_i} \Pi_1(C_i) \rightarrow *//G$$

We are choosing a base point on the circle S^1 and isomorphisms

$$\Pi_1(C_i) \simeq *//\pi_1(C_1) \simeq *//\mathbb{Z}$$

and

$$\Pi_1(S^1) \simeq *//\pi_1(S^1) \simeq *//\mathbb{Z}$$

Abusing notation a bit we write $A_i : *//\mathbb{Z} \rightarrow *//G$, $A_d : *//\mathbb{Z} \rightarrow *//H$ and similarly for induction and pullback. The A_i are determined by the elements $g_i = A_i(1) \in G$ and A_d is determined by the element $h = A_d(1) \in H$. Similarly, the inductions and pullbacks are determined by

$$\text{Ind}_{\iota_i}(A_d)(1) = \iota_i(h) \in G \quad j_i^*(A_i)(1) = g_i \in G$$

Further, we identify the isomorphism α_i of G -bundles with natural transformation of functors $\text{Ind}_{\iota_i}(A_d) \Rightarrow j_i^*(A_i)$ which are determined by morphisms in $G//G$. Hence, by abusing notation a bit, the isomorphisms α_1, α_2 of G -bundles are just $\alpha_1, \alpha_2 \in G$ such that

$$\iota_1(h) = \alpha_1^{-1}g_1\alpha_1 \quad \iota_2(h) = \alpha_2^{-1}g_2\alpha_2 \quad (2.10)$$

Objects of $\text{Bun}_{G,G,H}(C_1, C_2, S^1)$ are therefore tuples $(g_1, g_2, h, \alpha_1, \alpha_2) \in G^2 \times H \times G^2$ such that (2.10) holds. An isomorphism of relative bundles

$$(g_1, g_2, h, \alpha_1, \alpha_2) \xrightarrow{\simeq} (g'_1, g'_2, h', \alpha'_1, \alpha'_2)$$

is then a triple $(\gamma_1, \gamma_2, \beta) \in G \times G \times H$ such that:

$$g_1 = (g'_1)^{\gamma_1} \quad g_2 = (g'_2)^{\gamma_2} \quad h = (h')^{\beta} \quad (2.11)$$

$$\alpha_1\gamma_1 = \iota_1(\beta)\alpha'_1 \quad \alpha_2\gamma_2 = \iota_2(\beta)\alpha'_2 \quad (2.12)$$

The equations (2.11) follow from the fact that γ_i correspond to isomorphisms of G -bundles and equations (2.12) correspond to the compatibility diagram for morphisms of relative bundles (see Subsection 2.2.1).

We want to show that

$$\text{Bun}_{G,G,H}(C_1, C_2, S^1) \simeq H//_{ad}H$$

The direction $\text{Bun}_{G,G,H}(C_1, C_2, S^1) \rightarrow H//_{ad}H$ is just forgetting $g_1, g_2, \alpha_1, \alpha_2$ on objects and forgetting γ_1, γ_2 on morphisms. It is clear from the equations above that this is well-defined.

For the opposite direction we assign on objects

$$h \mapsto (\iota_1(h), \iota_2(h), h, 1, 1)$$

and on morphisms

$$\left(h \xrightarrow{\beta} \beta h \beta^{-1} \right) \mapsto \left((\iota_1(h), \iota_2(h), h, 1, 1) \xrightarrow{(\iota_1(\beta), \iota_2(\beta), \beta)} (\iota_1(\beta h \beta^{-1}), \iota_2(\beta h \beta^{-1}), \beta h \beta^{-1}, 1, 1) \right)$$

This is well-defined because (2.10), (2.11), (2.12) are fulfilled. Also it is clear that the composition these functors $H//_{ad}H \rightarrow \text{Bun}_{G,G,H}(C_1, C_2, S^1) \rightarrow H//_{ad}H$ is the identity on $H//_{ad}H$. On the other hand the composition

$$\text{Bun}_{G,G,H}(C_1, C_2, S^1) \rightarrow H//_{ad}H \rightarrow \text{Bun}_{G,G,H}(C_1, C_2, S^1)$$

is on objects and morphisms:

$$\begin{aligned} (g_1, g_2, h, \alpha_1, \alpha_2) &\mapsto (\iota_1(h), \iota_2(h), h, 1, 1) \\ (\gamma_1, \gamma_2, \beta) &\mapsto (\iota_1(\gamma_1), \iota_2(\gamma_2), \beta) \end{aligned}$$

This functor is indeed natural equivalent to the identity functor on $\text{Bun}_{G,G,H}(C_1, C_2, S^1)$ by the following natural transformation:

$$\{(g_1, g_2, h, \alpha_1, \alpha_2) \xrightarrow{(\alpha_1^{-1}, \alpha_2^{-1}, 1)} (\iota_1(h), \iota_2(h), h, 1, 1)\}$$

□

Now that we have the groupoid of relative bundles we need to go through a linearization/quantization procedure similar as in Section 2.1. Thus we have to consider the span from the circle to the circle over the cylinder decorated with the defect label (H, η) . This gives us the span of groupoids:

$$\begin{array}{ccc} & H//H & \\ \swarrow \iota_1 & & \searrow \iota_2 \\ G//G & & G//G \end{array}$$

where, by a slight abuse of notation, the functors $\iota_i : H//H \rightarrow G//G$ for $i = 1, 2$ are defined by mapping objects and morphisms according to the group homomorphisms $\iota_i : H \rightarrow G$ defined above. Given this span we have to construct a linear functor $[G//G, \text{Vect}] \rightarrow [G//G, \text{Vect}]$. As in Section 2.1 we have two pullbacks ($i = 1, 2$): $\iota_i^* : [G//G, \text{Vect}] \rightarrow [H//H, \text{Vect}]; W \mapsto W \circ \iota_i$ and two pushforwards

$$\iota_{i*}^L [H//H, \text{Vect}] \rightarrow [G//G, \text{Vect}]$$

given by equation (2.5). Hence for $V : H//H \rightarrow \text{Vect}$:

$$(\iota_{i*}^L V)(g) = \bigoplus_{\substack{[h] \in \pi_0(H//H) \\ \iota_i(h) \simeq g}} kC_G(g) \otimes_{kC_H(h)} V(h) \quad (2.13)$$

where $C_G(g), C_H(h)$ denotes the centralizers of the respective elements. Additionally, we have a twisting $\eta \in Z^2(H, k^\times)$ on the defect circle. As described in [Mo13] Sect. 5.4, we construct a functor

$$\phi_\eta : [H//H, \text{Vect}] \rightarrow [H//H, \text{Vect}]$$

depending on η . For this we first use the transgression map

$$H^2(H, k^\times) \rightarrow H^1(H//H, k^\times)$$

where for any groupoid \mathcal{G} we denote by $H^n(\mathcal{G}, k^\times)$ the n th *groupoid cohomology* of \mathcal{G} , which is per definition the n th cohomology group of the simplicial set $N\mathcal{G}$ (the nerve of \mathcal{G}). By [Wi08] Thm. 3 the transgression of η is the 1-cocycle $\omega_\eta \in H^1(H//H, k^\times)$ defined by

$$\omega_\eta : \text{Mor}(H//H) \rightarrow k^\times; \left(h \xrightarrow{\gamma} \gamma h \gamma^{-1} \right) \mapsto \frac{\eta(h, \gamma)}{\eta(\gamma, h)}$$

where $\text{Mor}(H//H)$ is the space of morphisms of the groupoid $H//H$. We simply write: $\omega_\eta(h, \gamma) = \eta(h, \gamma)/\eta(\gamma, h)$.

Now we can define the functor ϕ_η by assigning a functor $V : H//H \rightarrow \text{Vect}$ to the functor

$$\begin{aligned} \phi_\eta(V) : H//H &\rightarrow \text{Vect} \\ h &\mapsto V(h) \\ \left(h \xrightarrow{\gamma} \gamma h \gamma^{-1} \right) &\mapsto \omega_\eta V(\gamma) \end{aligned} \quad (2.14)$$

where $\omega_\eta V(\gamma)$ is the following linear map of vector spaces

$$V(h) \rightarrow V(\gamma h \gamma^{-1}); v \mapsto \omega_\eta(h, \gamma)v$$

We remark that this is equivalent to the following: Let $k[H//H]$ be the so-called groupoid algebra, which is spanned by morphisms of $H//H$ as a vector space together with the algebra structure that is given by the composition of morphisms, whenever the composition is well-defined and zero otherwise. There is a canonical equivalence $k[H//H]\text{-mod} \simeq [H//H, \text{Vect}]$. Further there is an algebra isomorphism

$$k[H//H] \rightarrow k[H//H]; \left(h \xrightarrow{\gamma} \gamma h \gamma^{-1} \right) \mapsto \omega_\eta(h, \gamma) \left(h \xrightarrow{\gamma} \gamma h \gamma^{-1} \right)$$

which induces a functor $k[H//H]\text{-mod} \rightarrow k[H//H]\text{-mod}$ by precomposition with this algebra isomorphism. The resulting functor is equivalent to ϕ_η .

The transmission functor for the defect (H, η) is thus the following pull-push construction:

$$F_{H,\eta} : [G//G, \text{Vect}] \xrightarrow{\iota_1^*} [H//H, \text{Vect}] \xrightarrow{\phi_\eta} [H//H, \text{Vect}] \xrightarrow{\iota_2^L} [G//G, \text{Vect}]$$

and can be explicitly calculated. $F_{H,\eta}$ maps a functor $W : G//G \rightarrow \text{Vect}$ to the functor $F_{H,\eta}W : G//G \rightarrow \text{Vect}$ given by:

$$(F_{H,\eta}W)(g) = \bigoplus_{\substack{[h] \in \pi_0(H//H) \\ \iota_2(h) \simeq g}} kC_G(g) \otimes_{kC_H(h)} W(\iota_1(h)) \quad (2.15)$$

where $\pi_0(H//H)$ is the set of all conjugacy classes $[h]$. The centralizer $C_H(h)$ acts on the centralizer $C_G(g)$ through $\iota_2 : H \rightarrow G$ and $C_H(h)$ acts on $W(\iota_1(h))$ by the action of the functor W on morphisms.

As shown in the Subsection 2.2.2 the transmission functor is a braided autoequivalence if the defect is invertible. Our defect is labeled by (H, η) which corresponds by the classification in [O03] to the indecomposable Vect_G -bimodule category $\mathcal{M}(H, \eta)$ consisting of vector spaces graded by the set of cosets G/H with the action of Vect_G induced by the action of G on G/H and the module category structure induced by η . From Corollary 3.6.3 of [Dav10] and Proposition 5.2 of [NR14] we recall the necessary and sufficient condition on (H, η) such that $\mathcal{M}(H, \eta)$ is an invertible Vect_G -bimodule category and hence the such that the defect is invertible:

Proposition 2.2.2. (Proposition 5.2 [NR14])

The Vect_G -bimodule category $\mathcal{M}(H, \eta)$ is invertible if and only if the following three conditions are satisfied:

- (i) $H(G \times \{1\}) = H(\{1\} \times G^{op}) = G \times G^{op}$
- (ii) $H_1 := H \cap (G \times \{1\})$ and $H_2 := H \cap (\{1\} \times G^{op})$ are abelian groups
- (iii) The following bicharacter is non-degenerate:

$$\omega_\eta|_{H_1 \times H_2} : H_1 \times H_2 \rightarrow k^\times; (h_1, h_2) \mapsto \frac{\eta(h_1, h_2)}{\eta(h_2, h_1)}$$

Let us call a pair (H, η) invertible if it corresponds to an invertible Vect_G -bimodule category $\mathcal{M}(H, \eta)$. The transmission functor $F_{H, \eta}$ corresponding to such an invertible pair (H, η) gives us a braided autoequivalence of $Z(\text{Vect}_G)$. Therefore, we have defined an assignment

$$\begin{aligned} \text{BrPic}(\text{Vect}_G) &\rightarrow \text{Aut}_{br}(Z(\text{Vect}_G)) \\ \mathcal{M}(H, \eta) &\mapsto F_{H, \eta} \end{aligned} \tag{2.16}$$

On the other hand, for any fusion category \mathcal{C} we can assign an invertible \mathcal{C} -bimodule category \mathcal{M} to a braided autoequivalence $\Phi_{\mathcal{M}} : Z(\mathcal{C}) \xrightarrow{\sim} Z(\mathcal{C})$ defined by the condition that there is an equivalence of \mathcal{C} -bimodule functors:

$$M \otimes \cdot \simeq \cdot \otimes \Phi_{\mathcal{M}}(M)$$

for all $M \in Z(\mathcal{C})$.

Proposition 2.2.3. (Theorem. 1.1 [ENO10])

The assignment

$$\begin{aligned} \text{BrPic}(\mathcal{C}) &\rightarrow \text{Aut}_{br}(Z(\mathcal{C})) \\ \mathcal{M} &\mapsto \Phi_{\mathcal{M}} \end{aligned} \tag{2.17}$$

is a group isomorphism.

It is natural to assume that for $\mathcal{C} = \text{Vect}_G$ and every invertible pair (H, η) there is a natural equivalence of monoidal functors:

$$\Phi_{\mathcal{M}(H, \eta)} \simeq F_{H, \eta}$$

In the forthcoming section, we show that in the case $G = A$ abelian, these functors are indeed equivalent.

2.3 Abelian Dijkgraaf-Witten Theory

Let us consider in this section the case of $G = A$ abelian. Recall from the Preliminaries that simple objects of DG -mod are the modules $\mathcal{O}_g^\rho := kG \otimes_{k\text{Cent}(g)} V$, where $[g] \subset G$ is a conjugacy class and $\rho : \text{Cent}(g) \rightarrow \text{GL}(V)$ an isomorphism class of an irreducible representation of the centralizer of a representative $g \in [g]$. Thus if $G = A$ is abelian then simple objects in DA -mod are in bijective correspondence with pairs (ρ, a) where $a \in A$ and ρ a 1-dimensional character of A . Denote the character group of A by \hat{A} . The tensor product of two DA -modules corresponding to (a, ρ) and (a', ρ') gives us a DA -module corresponding to $(a \cdot a', \rho \cdot \rho')$. Thus the group of isomorphism classes of simple object of DA -mod, where the group structure is given by the tensor product, is isomorphic to the group $\hat{A} \times A$. A autoequivalence of abelian categories preserves direct sums and is thus an autoequivalence of DA -mod is determined by its action on simple objects $\hat{A} \times A$. The corresponding assignment of simple objects $\phi : \hat{A} \times A \xrightarrow{\sim} \hat{A} \times A$ is a group automorphism if the autoequivalence is monoidal. Further, the monoidal structure corresponds to a 2-cocycle $\sigma \in Z^2(\hat{A} \times A, k^\times)$. This way we get a group isomorphism

$$\text{Aut}_{\text{mon}}(DA\text{-mod}) \simeq \text{Aut}(\hat{A} \times A) \rtimes H^2(\hat{A} \times A, k^\times)$$

Note that this also follows directly from the exact sequence (1.9), since we have no non-trivial inner and internal automorphisms of DA in the abelian case.

Let us now describe the braided situation which should also serve as a guide to understand the approach for the non-abelian case in the forthcoming chapters. We can identify every automorphism $\phi \in \text{Aut}(\hat{A} \times A)$ in an obvious way with a matrix

$$\begin{pmatrix} u & b \\ p & v \end{pmatrix} \quad \begin{array}{ll} u \in \text{End}(\hat{A}) & b \in \text{Hom}(A, \hat{A}) \\ p \in \text{Hom}(\hat{A}, A) & v \in \text{End}(A) \end{array}$$

and using the Künneth formula for group cohomology

$$H^2(\hat{A} \times A, k^\times) \simeq H^2(\hat{A}, k^\times) \times H^2(A, k^\times) \times P(A, \hat{A})$$

where $P(A, \hat{A})$ is the group of abelian bicharacters $A \times \hat{A} \rightarrow k^\times$. We can thus identify every $\sigma \in Z^2(\hat{A} \times A, k^\times)$ (up to coboundary) with a triple (α, β, λ) , where α is a group 2-cocycle on \hat{A} , β is a group 2-cocycle on A and $\lambda : A \times \hat{A} \rightarrow k^\times$ an abelian bicharacter. The pair

$$\left(\begin{pmatrix} u & b \\ p & v \end{pmatrix}, (\alpha, \beta, \lambda) \right) \in \text{Aut}(\hat{A} \times A) \rtimes H^2(\hat{A} \times A, k^\times)$$

corresponds to a *braided* autoequivalence if and only if the following conditions are fulfilled for all $a, a' \in A$ and $\chi, \rho \in \hat{A}$:

$$\beta(a, a') = \beta(a', a)b(a')(v(a)) \tag{2.18}$$

$$\alpha(\rho, \chi) = \alpha(\chi, \rho)u(\chi)(p(\rho)) \tag{2.19}$$

$$\lambda(a, \chi) = b(a)(p(\chi)) \tag{2.20}$$

$$\rho(a) = u(\rho)[v(a)]b(a)[p(\rho)] \tag{2.21}$$

This follows from evaluating the braiding diagram on simple DA -modules $(\rho, a), (\chi, a')$. The first three equations (2.18),(2.19),(2.20) imply that $\phi = (u, b, p, v)$ determines the 2-cocycles α, β uniquely up to coboundary and the bicharacter λ uniquely. Therefore the monoidal structure is determined by the action of the functor on simple objects. The last equation (2.21) gives a condition on the automorphism $\phi = (u, b, p, v)$ which is equivalent to the property:

$$q \circ \phi = q$$

where $q : \hat{A} \times A \rightarrow k^\times; (\rho, a) \mapsto \rho(a)$. We call such a ϕ an orthogonal automorphism of $\hat{A} \times A$ and denote the group of all orthogonal automorphisms of $\hat{A} \times A$ by $O_q(\hat{A} \times A)$. Therefore we have thus sketched the following group isomorphism:

$$\text{Aut}_{br}(DA\text{-mod}) \simeq O_q(\hat{A} \times A)$$

We refer to Sect. 10 of [ENO10] and Sect. 5 of [DN12] for more details on and properties of the group $O_q(\hat{A} \times A)$.

With this characterization of the group of braided autoequivalences of $DA\text{-mod}$ we want to show that for the assignment $\mathcal{M}(H, \eta) \mapsto F_{H, \eta}$, where $F_{H, \eta}$ is the transmission functor for an invertible defect (H, η) , is indeed the same as the assignment (2.17), confirming the claim in the abelian case. In order to show the equivalence we first decompose $O_q(\hat{A} \times A)$. This decomposition is a special case of Theorem 6.6.1 of this thesis.

Proposition 2.3.1.

The following are subgroups of $O_q(\hat{A} \times A)$:

- (i) $\mathcal{V} := \left\{ \begin{pmatrix} (v^{-1})^* & 0 \\ 0 & v \end{pmatrix} \mid v \in \text{Aut}(A) \right\}$
- (ii) $\mathcal{B} := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \text{Hom}(A, \hat{A}) \quad b(a)(a') = b(a')(a)^{-1} \forall a, a' \in A \right\}$
- (iii) $\mathcal{E} := \left\{ \begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \mid p \in \text{Hom}(\hat{A}, A) \quad \rho(p(\chi)) = \chi(p(\rho))^{-1} \forall \chi, \rho \in \hat{A} \right\}$

For each triple (Q, C, δ) , where Q, C are two subgroups of A such that $A = Q \times C$ and $\delta : C \xrightarrow{\cong} \hat{C}$ a group isomorphism, we define a partial dualization on C to be the following map:

$$(\chi_Q, \chi_C) \times (q, c) \xrightarrow{r_{Q, C, \delta}} (\chi_Q, \delta(c)) \times (q, \delta^{-1}(\chi_C)) \quad (2.22)$$

We denote the set of such maps $r_{Q, C, \delta}$ by \mathcal{R} and claim that this is a subset of $O_q(\hat{A} \times A)$. Taking the trivial decomposition $A = \{1\} \times A$ and a $\delta : A \xrightarrow{\cong} \hat{A}$, we call $r_{1, A, \delta}$ a full dualization on A . Then, we have $r_{1, A, \delta} \cdot \mathcal{E} \cdot r_{1, A, \delta}^{-1} = \mathcal{B}$

For each $\phi \in O_q(\hat{A} \times A)$ there exists an $r = r_{Q, C, \delta} \in O_q(\hat{A} \times A)$ such that

$$\phi \in \mathcal{V} \mathcal{B} r \mathcal{B}$$

Proof. All statements of this proposition follow from the Theorem 6.6.1. The fact that the subgroups \mathcal{V} , $\mathcal{B} \simeq \mathcal{E}$ together with the partial dualizations r are generators of $O_q(\hat{A} \times A)$ was also shown in Sect. 4 of [FPSV15]. \square

For the groups defined above, we have the obvious isomorphisms:

$$\begin{aligned} \mathcal{V} &\simeq \text{Aut}(A) \\ \mathcal{B} &\simeq \text{Hom}_{alt}(A, \hat{A}) := \left\{ b \in \text{Hom}(A, \hat{A}) \mid b(a)(a') = b(a')(a)^{-1} \forall a, a' \in A \right\} \\ \mathcal{E} &\simeq \text{Hom}_{alt}(\hat{A}, A) := \left\{ p \in \text{Hom}(\hat{A}, A) \mid \rho(p(\chi)) = \chi(p(\rho))^{-1} \forall \chi, \rho \in \hat{A} \right\} \end{aligned}$$

where the group structure on $\text{Hom}_{alt}(A, \hat{A})$ and $\text{Hom}_{alt}(\hat{A}, A)$ is point-wise multiplication. In order to show the equivalence of $\Phi_{\mathcal{M}(H, \eta)}$ and $F_{H, \eta}$ for all $\mathcal{M}(H, \eta) \in \text{BrPic}(\text{Vect}_A) \simeq O_q(\hat{A} \times A)$, it is enough to check that for all generators of $O_q(\hat{A} \times A)$. For this we use the results of [ENO10] Sec. 10 where an explicit isomorphism between invertible pairs (H, η) and group isomorphisms $O_q(\hat{A} \times A)$ is established (See also Rem. 10.5 in [ENO10]).

Proposition 2.3.2. We have the following correspondences between generators $(u, b, p, v) \in O_q(\hat{A} \times A)$ and invertible pairs (H, η) .

(i) To an $v \in \text{Aut}(A) \simeq \mathcal{V}$ corresponds the pair

$$H = \{(a, v(a)) \mid a \in A\} \quad \eta = 1$$

(ii) To an $b \in \text{Hom}_{alt}(A, \hat{A}) \simeq \mathcal{B}$ corresponds the subgroup

$$H = \{(a, a) \mid a \in A\} = H_{diag} \simeq A$$

and the 2-cocycle η on $H \simeq A$ uniquely (up to coboundary) determined by

$$\frac{\eta(a, a')}{\eta(a', a)} = b(a)(a')$$

(iii) To an $p \in \text{Hom}_{alt}(\hat{A}, A) \simeq \mathcal{E}$ corresponds the subgroup

$$H = \{(a, p(\chi)a) \mid a \in A, \chi \in \hat{A}\}$$

and η a 2-cocycle on H uniquely (up to coboundary) determined by

$$\frac{\eta((a, p(\chi)a), (a', p(\chi')a'))}{\eta((a', p(\chi')a'), (a, p(\chi)a))} = \chi(p(\chi'))$$

(iv) To a partial dualization $r_{Q, C, \delta} \in \mathcal{R}$ corresponds the subgroup

$$H = \{(qc, \delta^{-1}(\chi_C)q) \mid q \in Q, c \in C, \chi_C \in \hat{C}\} \simeq Q_{diag} \times C \times C$$

and η a 2-cocycle on $H \simeq Q_{diag} \times C \times C$ uniquely (up to coboundary) determined by

$$\frac{\eta((q, c_1, c_2), (q', c'_1, c'_2))}{\eta((q', c'_1, c'_2), (q, c_1, c_2))} = \frac{\delta(c'_2)(c_1)}{\delta(c_2)(c'_1)}$$

Proof. We use the explicit isomorphism between invertible pairs (H, η) and group isomorphisms $O_q(\hat{A} \times A)$ given in [ENO10] Sect. 10. Under this isomorphism an orthogonal automorphism $\phi = (u, b, p, v) \in O_q(\hat{A} \times A)$ corresponds to the subgroup $H \leq A \times A$ given by

$$H = \{(a, p(\chi)v(a)) \mid a \in A, \chi \in \hat{A}\}$$

and the 2-cocycle η on H determined uniquely (up to coboundary) by

$$\frac{\eta((a, p(\chi)v(a)), (a', p(\chi')v(a')))}{\eta((a', p(\chi')v(a')), (a, p(\chi)v(a)))} = \chi(a')^{-1}[u(\chi)b(a)][p(\chi')v(a')]$$

One can check that the right hand side of this equation is an abelian bicharacter on H and hence η is indeed determined up to coboundary. We refer to [ENO10] to check that (H, η) defined as above are indeed invertible, hence fulfill the conditions (i), (ii), (iii) of Proposition (2.2.2). Also we refer to Sect. 3 of [FPSV15] where invertibility of (H, η) was checked for the generators. Evaluating the given forms for generators (u, b, p, v) gives exactly the above subgroups and 2-cocycles. \square

It is easy to see that the multiplication of a full dualization $r_{1,A,\delta}$ with an $p \in E$ gives us an element in B and that $r_{1,A,\delta} \cdot E \cdot r_{1,A,\delta}^{-1} = B$. To see this on the level of invertible pairs (H, η) is not that easy. For this one has to calculate the relative Deligne tensor product of the respective Vect_G -bimodule categories, which indeed can be done using Proposition 3.16 in [ENO10], but it is of course more complicated than just multiplying two matrices.

Proposition 2.3.3. The functor $F_{H,\eta}$ agrees with $\Phi_{\mathcal{M}(H,\eta)}$ for all invertible pairs (H, η) .

Proof. Every pair (H, η) is determined uniquely by an orthogonal automorphism $\phi = (u, b, p, v) \in O_q(\hat{A} \times A)$. Then according to [ENO10] Sect. 10 we have

$$\Phi_{\mathcal{M}(H,\eta)}(\chi, a) = (u(\phi)b(a), p(\chi)v(a)) = \phi(\chi, a)$$

We now want to show that for every (H, η) the transmission functor acts on simple objects the same way. For this we first calculate the transmission functor $F_{H,\eta}$ on simple objects of $DA\text{-mod}$. The pullback $\iota_1^* : DA\text{-mod} \rightarrow DH\text{-mod}$ is given on simple objects by:

$$\iota_1^*(\chi, a) = \bigoplus_{h \in \iota_1^{-1}(a)} (\chi \circ \iota_1, h)$$

Recall that $\iota_1 : H \hookrightarrow A \times A \rightarrow A$ was the composition of the embedding to $A \times A$ and then projecting to the first factor. Further, the functor $\phi_\eta : DH\text{-mod} \rightarrow DH\text{-mod}$ defined in (2.14) acts on simple objects:

$$\phi_\eta(\zeta, h) = (\zeta \cdot \omega_\eta(h, \cdot), h)$$

and from (2.15). Recall that $\omega_\eta(h, h') = \eta(h, h')/\eta(h', h)$. Finally, the pushforward $\iota_{2*}^L : DH\text{-mod} \rightarrow DA\text{-mod}$ on simple objects is given by

$$\iota_{2*}^L(\zeta, h) = \bigoplus_{\substack{\rho \in \hat{A} \\ \zeta = \rho \circ \iota_2}} (\rho, \iota_2(h))$$

where $\iota_2 : H \hookrightarrow A \times A \rightarrow A$ is the composition of the embedding to $A \times A$ and then projecting to the second factor. So the transmission functor is given by the composition $F_{H,\eta} = \iota_{2*} \phi_\eta \iota_1^* : DA\text{-mod} \xrightarrow{\sim} DA\text{-mod}$ and acts on simple objects as follows:

$$F_{H,\eta}(\chi, a) = \bigoplus_{h \in \iota_1^{-1}(a)} \bigoplus_{\substack{\rho \in \hat{A} \\ \chi \circ \iota_1 \cdot \omega_\eta(h, \cdot) = \rho \circ \iota_2}} (\rho, \iota_2(h)) \quad (2.23)$$

Now we use that

$$H = \{(a, p(\chi)v(a)) \mid a \in A, \chi \in \hat{A}\}$$

and that η on H is determined by

$$\frac{\eta((a, p(\chi)v(a)), (a', p(\chi')v(a')))}{\eta((a', p(\chi')v(a')), (a, p(\chi)v(a)))} = \chi(a')^{-1} [u(\chi)b(a)] [p(\chi')v(a')]$$

(see proof of Proposition 2.3.2). Therefore we have

$$\iota_1^{-1}(a) = \{(a, p(\gamma)v(a)) \mid \gamma \in \hat{A}\} \subset H$$

and $\iota_2((a', p(\gamma')v(a'))) = p(\gamma')v(a')$. We evaluate the equation $\chi \circ \iota_1 \cdot \omega_\eta(h, \cdot) = \rho \circ \iota_2$ on both sides with elements $h' = (a', p(\gamma')v(a')) \in H$ and get:

$$\chi(a') \gamma(a')^{-1} [u(\gamma)b(a)] [p(\gamma')v(a')] = \rho(p(\gamma')v(a')) \quad \forall a' \in A, \gamma' \in \hat{A} \quad (2.24)$$

At this point we want to use the decomposition of $O_q(\hat{A} \times A)$ from Proposition 2.3.1. Let us first look at (H, η) corresponding to elements of

$$\begin{pmatrix} (v^{-1})^* & 0 \\ 0 & v \end{pmatrix} \in V \simeq \text{Aut}(V)$$

Then $\Phi_{H,\eta}(\chi, a) = ((v^{-1})^* \chi, v(a))$. The equation (2.24) reduces to just $\rho = (v^{-1})^*$ and since $p = 0$ there is only one element in the preimage $\iota_1^{-1}(a) = \{(a, v(a))\}$. So the transmission functor gives

$$F_{H,\eta}(\chi, a) = ((v^{-1})^* \chi, v(a)) = \Phi_{H,\eta}(\chi, a)$$

Let us now consider (H, η) corresponding to elements of

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in B \simeq \text{Hom}_{alt}(A, \hat{A})$$

Then $\Phi_{H,\eta}(\chi, a) = (\chi \cdot b(a), a)$. The equation (2.24) reduces to $\chi \cdot b(a) = \rho$ and since $p = 0$ there is only one element in the preimage $\iota_1^{-1}(a) = \{(a, a)\}$.

$$F_{H,\eta}(\chi, a) = (\chi \cdot b(a), a) = \Phi_{H,\eta}(\chi, a)$$

Now consider (H, η) corresponding to elements of

$$\begin{pmatrix} 1 & 0 \\ p & 1 \end{pmatrix} \in \mathcal{E} \simeq \text{Hom}_{alt}(\hat{A}, A)$$

Then $\Phi_{H,\eta}(\chi, a) = (\chi, p(\chi)a)$. We have $\iota_1^{-1}(a) = \{(a, p(\gamma)a) \mid \gamma \in \hat{A}\}$ and equation (2.24) reduces to $\chi = \rho$ and $p(\gamma) = p(\chi)$.

$$\begin{aligned} F_{H,\eta}(\chi, a) &= \bigoplus_{\substack{(a, p(\gamma)a) \in \iota_1^{-1}(a) \\ \rho \in \hat{A} \\ \chi = \rho, p(\gamma) = p(\chi)}} (\rho, p(\gamma)a) \\ &= (\chi, p(\chi)a) = \Phi_{H,\eta}(\chi, a) \end{aligned}$$

Now consider (H, η) corresponding to partial dualizations $r_{Q,C,\delta} \in \mathcal{R}$, then $\Phi_{H,\eta}(\chi_Q \chi_C, qc) = (\chi_Q \delta(c), q\delta^{-1}(\chi_C))$. We have $\iota_1^{-1}(qc) = \{(qc, \delta^{-1}(\gamma_C)q) \mid \gamma_C \in \hat{C}\}$ and equation (2.24) reduces to

$$\chi_C(c') \chi_Q(q') \gamma_C(c')^{-1} \delta(c) (\delta^{-1}(\gamma'_C)) = \rho_C(\delta^{-1}(\gamma'_C)) \rho_Q(q')$$

which is equivalent to

$$\chi_Q = \rho_Q \quad \chi_C = \gamma_C \quad \rho_C = \delta(c)$$

Therefore the transmission functor in this case:

$$\begin{aligned} F_{H,\eta}(\chi_Q \chi_C, qc) &= \bigoplus_{\substack{(qc, \delta^{-1}(\gamma_C)q) \in \iota_1^{-1}(qc) \\ \rho_C \in \hat{C}, \rho_Q \in \hat{Q} \\ \chi_Q = \rho_Q, \chi_C = \gamma_C, \rho_C = \delta(c)}} (\rho, \delta^{-1}(\gamma_C)q) \\ &= (\chi_Q \delta(c), \delta^{-1}(\chi_C)q) = \Phi_{H,\eta}(\chi_Q \chi_C, qc) \end{aligned}$$

For all $\chi_Q \in \hat{Q}$, $\chi_C \in \hat{C}$, $q \in Q$, $c \in C$.

So we have shown that $F_{H,\eta} = \Phi_{\mathcal{M}(H,\eta)}$ on all generators. Note that it was technically not necessary to do this both on \mathcal{B} and \mathcal{E} because $r_{1,A,\delta} \cdot E \cdot r_{1,A,\delta}^{-1} = B$. \square

At the end of this section, we want to mention the field theoretic interpretation of the subgroups \mathcal{V} , \mathcal{B} , \mathcal{E} and the partial dualizations \mathcal{R} . The input data of a Dijkgraaf-Witten theory is a pair (A, ω) where $\omega \in H^3(A, k^\times)$ was considered to be trivial. It is natural to expect that symmetries of the input data give us symmetries of the quantized theory. Indeed, as we have seen, the group $\mathcal{V} \simeq \text{Aut}(A)$ is a symmetry group and can be viewed as the symmetry of the stack of A -bundles Bun_A . This is a symmetry of both the classical as well as the quantized Dijkgraaf-Witten theory. In addition to A we also have the ingredient $\omega = 1$. Even though ω is trivial, it can have a non-trivial symmetry group. In fact, every element in $Z^2(A, k^\times)$ gives a gauge transformations of ω and hence it is natural to expect the symmetry group $H^2(A, k^\times)$. Indeed, we have $\mathcal{B} \simeq \text{Hom}_{alt}(A, \hat{A}) \simeq H^2(A, k^\times)$ as a subgroup of $\text{BrPic}(\text{Vect}_A)$. The elements of \mathcal{B} are symmetries of the classical theory as well as the quantized theory. If we view the Dijkgraaf-Witten theory as a σ -model with target BA , the classifying space of the group A , the trivial 3-cocycle ω has the role of a background field and elements of the group $\mathcal{B} \simeq H^2(A, k^\times)$ are interpreted as background field symmetries.

In addition to the subgroups \mathcal{V}, \mathcal{B} , which were both natural to expect from the input data, we have also partial dualizations. These are not visible on the classical level and appear only in the quantized Dijkgraaf-Witten theory as an interacting term between \mathcal{B} and \mathcal{E} symmetries. In physics literature these are called electric-magnetic dualities, as they exchange magnetic charges of anyons (bulk Wilson lines of the form $(1, a) \in \hat{A} \times A$) and electric charges of anyons (bulk Wilson lines of the form $(\chi, 1) \in \hat{A} \times A$) (see [BCKA13]). Generally, electric-magnetic dualities play a central role in gauge theories (see e.g [KaW07]).

For abelian Dijkgraaf-Witten theories, \mathcal{V}, \mathcal{B} and partial dualizations give a complete set of generators, since we get elements of \mathcal{E} by conjugation with a full dualization on A . Nevertheless, it is interesting to note that we have $\mathcal{E} \simeq \text{Hom}_{alt}(\hat{A}, A) \simeq \text{H}^2(\hat{A}, k^\times)$ as a symmetry group.

In the case of a non-abelian structure group G , we do not have a full dualization; we can only dualize on abelian (semi-)direct products and hence not on the whole group G . For this reason the symmetry group \mathcal{E} and \mathcal{B} are truly different in the non-abelian case (see Chapter 7).

Chapter 3

Monoidal Autoequivalences of $\text{Rep}(G)$

We want to determine the lazy cohomology on k^G for a finite group G . We obtain this by using Movshev's classification of k^G -Galois objects and the additional result in the form presented in Davydov [Dav01] and apply these to the lazy Galois objects. Further, we give an explicit 2-cocycle representing a lazy cohomology class on k^G .

3.1 Galois Algebras

For a finite group G , a left G -algebra is an associative algebra R together with a left G -action given by a homomorphism $G \rightarrow \text{Aut}(R)$, where $\text{Aut}(R)$ is the group of algebra automorphisms of R . A G -algebra is called *Galois*, if the algebra homomorphism

$$\theta : R \otimes kG \rightarrow \text{End}(R) \quad \theta(r \otimes g)(r') = r \cdot (g.r')$$

is an isomorphism. This is the same as saying that R is a k^G -comodule algebra such that the Galois map (see Definition 1.2.1) is an isomorphism

$$\begin{aligned} \beta_R : R \otimes_k R &\longrightarrow R \otimes_k k^G \\ r \otimes r' &\longmapsto \sum_{g \in G} r \cdot (g.r') \otimes e_g \end{aligned}$$

Therefore, a left G -Galois algebra is a right k^G -Galois object.

For any group S and any 2-cocycle $\eta \in Z^2(S, k^\times)$ we view the twisted group algebra $k_\eta S$ as a S -algebra with the associative multiplication given by $g \cdot h = \eta(g, h)gh$ and the left S -action $g.h = g \cdot h \cdot g^{-1}$. Twisted group algebras corresponding to cohomologous 2-cocycles are isomorphic as S -algebras. We pick an 2-cocycle $\eta \in Z^2(S, k^\times)$ such that $\eta(g, g^{-1}) = 1$. Every 2-cocycle is cohomologous to a 2-cocycle with this property:

For an $s \in S$ we define a group homomorphism

$$\text{Cent}_S(s) \rightarrow k^\times; t \mapsto \frac{\eta(s, t)}{\eta(t, s)}$$

where $\text{Cent}_S(s)$ is the centralizer of s in S . This is indeed a homomorphism, because for all $s \in S$ $t, t' \in \text{Cent}_S(s)$:

$$\eta(s, tt') = \frac{\eta(s, t)\eta(st, t')}{\eta(t, t')} = \frac{\eta(s, t)\eta(ts, t')}{\eta(t, t')} = \frac{\eta(s, t)\eta(s, t')\eta(t, st')}{\eta(t, t')\eta(t, s)}$$

where we used that $t \in \text{Cent}_S(s)$ in the second equality. We also have

$$\eta(tt', s) = \frac{\eta(t', s)\eta(t, t's)}{\eta(t, t')}$$

Putting these equations together and using that $t' \in \text{Cent}_S(s)$ we get:

$$\frac{\eta(s, tt')}{\eta(tt', s)} = \frac{\eta(s, t)\eta(s, t')\eta(t, st')\eta(t, t')}{\eta(t, t')\eta(t, s)\eta(t', s)\eta(t, t's)} = \frac{\eta(s, t)\eta(s, t')}{\eta(t, s)\eta(t', s)}$$

Therefore, the map defined above is indeed a group homomorphism. If this homomorphism is non-trivial for all $s \in S$, we call η non-degenerate.

Given a subgroup S of G and a S -algebra B , there is a natural way to construct a G -algebra by induction:

$$\text{ind}_S^G(B) := \{r : G \rightarrow B \mid r(sg) = s.r(g) \forall s \in S\}$$

which is an associative algebra with the pointwise multiplication of maps and has a left G -action given by $(g'.r)(g) = r(gg')$. We are now ready to present the first classification result.

Lemma 3.1.1. (Theorem 3.8 [Dav01])

Let k be an algebraically closed field of characteristic zero. Then there is a bijection between isomorphism classes of right k^G -Galois objects and conjugacy classes $(S, [\eta])$ where S is a subgroup of G and $[\eta] \in \text{H}^2(S, k^\times)$ for η a non-degenerate 2-cocycle. The isomorphism assigns to a conjugacy class $(S, [\eta])$ the isomorphism class of

$$R(S, \eta) := \{r : G \rightarrow k_\eta S \mid r(sg) = s.r(g) \forall s \in S, g \in G\}$$

with multiplication given by pointwise multiplication of functions and with a G -action given by $(g.r)(h) = r(hg)$.

Lemma 3.1.2. (Proposition 6.1 [Dav01])

Let $R(S, \eta)$ be a right k^G -Galois object as above and let $G' := \text{Aut}_G(R)$ be the group of G -algebra automorphisms. The following are equivalent:

- $R(S, \eta)$ is an $k^{G'}-k^G$ -Bigalois object with the natural left $k^{G'}$ -coaction resp. right G' -action $r.f = f(r)$.
- $|G'| = |G|$.
- S is an abelian normal subgroup of G and the class $[\eta]$ is G -invariant.

3.2 Lazy Cohomology of k^G

We want to describe all Bigalois objects with the property $G \cong G'$ from the characterization above. In particular, we determine for the induced Bigalois objects $R(S, \eta)$ the explicit 2-cocycle twist in the following sense:

Lemma 3.2.1. Let S be a normal abelian subgroup of G and $\eta \in Z^2(S)$ a non-degenerate 2-cocycle i.e. $\langle s, t \rangle := \eta(s, t)\eta(t, s)^{-1}$ is a non-degenerate alternating bicharacter. Further, assume for simplicity that $\eta(s, s^{-1}) = 1$ for all $s \in S$ (Note that this is always possible up to cohomology). Then there is an isomorphism of k^G -comodule algebras: ${}_{\alpha}(k^G) \simeq R(S, \eta)$ where $\alpha \in Z^2(k^G)$ is defined by

$$\alpha(f, f') = \frac{1}{|S|^2} \sum_{r, t, r', t' \in S} \eta(t, t') \langle t, r \rangle \langle t', r' \rangle f(r) f'(r')$$

and where ${}_{\alpha}(k^G)$ is k^G as a k^G -comodule and where the algebra structure is the one on k^G twisted with α .

Proof. We show that an isomorphism $\Phi : {}_{\alpha}(k^G) \mapsto R(S, \eta)$ is given by

$$\Phi : f \mapsto \frac{1}{|S|} \left(h \mapsto \sum_{t, r \in S} tf(rh) \langle t, r \rangle \right)$$

We check that $\Phi(f)$ is a kS -linear map:

$$\begin{aligned} \Phi(f)(sg) &= \frac{1}{|S|} \sum_{t, r \in S} tf(rsg) \langle t, r \rangle = \frac{1}{|S|} \sum_{t, r' \in S} tf(r'g) \langle t, r' \rangle \langle t, s \rangle^{-1} \\ &= \frac{1}{|S|} \sum_{t, r' \in S} (\eta(s, t) t \eta(t, s)^{-1}) f(r'g) \langle t, r' \rangle \\ &= \frac{1}{|S|} \sum_{t, r' \in S} \left(\eta(s, t) t \frac{\eta(st, s^{-1})}{\eta(t, 1) \eta(s, s^{-1})} \right) f(r'g) \langle t, r' \rangle \\ &= \frac{1}{|S|} \sum_{t, r' \in S} (s.t) f(r'g) \langle t, r' \rangle = s.\Phi(f)(g) \end{aligned}$$

Next we check that Φ is a kG -module morphism:

$$\Phi(h.f) = \Phi(f_2(h)f_1) = \Phi(g' \mapsto f(g'h)) = \frac{1}{|S|} (g \mapsto \sum_{t, r \in S} tf(rgh) \langle t, r \rangle) = h.\Phi(f)$$

Further, it is an algebra morphism: $\Phi(1)(g) = \frac{1}{|S|^2} \sum_{t,r \in S} t \langle t, r \rangle = \sum_{t \in S} t \delta_{t,1} = 1$

$$\begin{aligned}
(\Phi(f)\Phi(f'))(g) &= \frac{1}{|S|^2} \left(\sum_{t,r \in S} t f(rg) \langle t, r \rangle \right) \cdot_{\eta} \left(\sum_{t',r' \in S} t' f(r'g) \langle t', r' \rangle \right) \\
&= \frac{1}{|S|^2} \sum_{t,r,t',r' \in S} tt' \eta(t, t') \cdot f(rg) \langle t, r \rangle f'(r'g) \langle t', r' \rangle \\
&= \frac{1}{|S|^2} \sum_{r,t,r',t' \in S} \left(\frac{1}{|S|} \sum_{\tilde{t}, \tilde{r} \in S} \tilde{t} \langle \tilde{t}, \tilde{r} \rangle \langle t, \tilde{r}^{-1} \rangle \langle t', \tilde{r}^{-1} \rangle \right) \\
&\quad \cdot f_1(r) f_1'(r') \langle t, r \rangle \langle t', r' \rangle \eta(t, t') \cdot f_2(g) f_2'(g) \\
&= \frac{1}{|S|^3} \sum_{\tilde{t}, \tilde{r}, r, t, r', t' \in S} \tilde{t} \langle \tilde{t}, \tilde{r} \rangle \\
&\quad \cdot f_1(r) f_1'(r') \langle t, r \tilde{r}^{-1} \rangle \langle t', r' \tilde{r}^{-1} \rangle \eta(t, t') \cdot f_2(g) f_2'(g) \\
&\stackrel{\substack{r''=r\tilde{r}^{-1} \\ r'''=r'\tilde{r}^{-1}}}{=} \frac{1}{|S|^3} \sum_{\tilde{t}, \tilde{r}, r'', t, r''', t' \in S} \tilde{t} \langle \tilde{t}, \tilde{r} \rangle \\
&\quad \cdot f_1(r'') f_1'(r''') \langle t, r'' \rangle \langle t', r''' \rangle \eta(t, t') \cdot f_2(\tilde{r}) f_2'(\tilde{r}) \cdot f_3(g) f_3'(g) \\
&= \frac{1}{|S|} \sum_{\tilde{t}, \tilde{r} \in S} \tilde{t} \langle \tilde{t}, \tilde{r} \rangle \cdot \alpha(f_1, f_1') f_2(\tilde{r}g) f_2'(\tilde{r}g) = \Phi(f \cdot_{\alpha} f')(g)
\end{aligned}$$

We finally check bijectivity of Φ . We first note that for any coset $Sg \subset G$ the functions f which are nonzero only on Sg are sent to functions $\Phi(f)$ which are nonzero only on Sg . With the fixed representative g of a coset Sg we consider the basis e_{sg} for k^{Sg} . By construction this element is mapped to the following element in $\text{Hom}_{kS}(k[Sg], kS)$:

$$\Phi(e_{sg}) = \frac{1}{|S|} \left(s'g \mapsto \sum_{t,r \in S} t e_{sg}(r s'g) \langle t, r \rangle \right) = \frac{1}{|S|} \left(s'g \mapsto \sum_{t \in S} t \langle t, s \rangle \langle t, s' \rangle^{-1} \right)$$

This is by construction a Fourier transform with a non-degenerate form $\langle \cdot, \cdot \rangle$, which implies bijectivity. More explicitly, we show injectivity by considering $s' = 1$. Then we get elements $\sum_{t \in S} t \langle t, s \rangle$ in S , which are linearly independent by the assumed non-degeneracy. Now bijectivity follows, because source and target have dimension $|S|$. \square

In particular, the assumption in Lemma 3.1.2 is that the class $[\eta] \in H^2(S)$ is G -invariant and hence the alternating bicharacter $\langle \cdot, \cdot \rangle$ on S is G -invariant. Then the criterion in Example 1.3.5 (ii) implies that α is lazy if and only if it is G -invariant and we have:

Corollary 3.2.2. A lazy cocycle for k^G is up to cohomology the restriction of a G -invariant 2-cocycle η (not just an G -invariant class $[\eta]$) on a normal abelian subgroup S in the sense of Lemma 3.2.1.

Corollary 3.2.3. Let $\alpha \in Z_L^2(k^G)$ be a 2-cocycle with the additional property $\alpha(e_g, e_h) = \alpha(e_h, e_g)$. Then α is cohomologically trivial.

Proof. Since k^G is commutative, we have

$$d\nu(e_x, e_y) = \nu(e_{x_1})\nu(e_{y_1})\nu^{-1}(e_{x_2}e_{y_2}) = d\nu(e_y, e_x)$$

thus every 2-cocycle α' in the same cohomology class as α is also symmetric, hence $\alpha'(e_g, e_h) = \alpha'(e_h, e_g)$. By Lemma 3.2.1 every cohomology class has a representative induced by an $\eta \in Z^2(S)$ for a normal abelian subgroup S . One checks that such an η is given by

$$\eta(s, s') = \sum_{x, y \in S} \alpha(e_x, e_y) \langle x, s \rangle \langle y, s' \rangle$$

which implies that η is symmetric on S . Since S is abelian η is cohomologically trivial. Since any lazy 2-cocycle is up to cohomology the restriction of a 2-cocycle η on a normal abelian subgroup $S \subset G$ the symmetry condition implies that η is also symmetric, hence $\langle x, s \rangle = 1$ for all $x, s \in S$. But we know from 3.1.1 that η is non-degenerate hence $\langle x, s \rangle = 1$ for all x implies $s = 1$. This implies that the abelian subgroup S is trivial and hence α is cohomologically trivial. \square

Note that this can also be seen from the fact that the Galois object ${}_{\alpha}(k^G)$ is a commutative algebra, since k^G commutative and α symmetric. Such a Galois object is then trivial up to isomorphism.

Chapter 4

Hopf Automorphisms of the Drinfeld Double

4.1 Construction of Subgroups

In this chapter we give a description of $\text{Aut}_{\text{Hopf}}(DG)$. More precisely, we determine in Theorem 4.2.1 a decomposition of $\text{Aut}_{\text{Hopf}}(DG)$ into double cosets similar to the Bruhat-decomposition of a Lie group. If the direct abelian factors of G are not elementary abelian, then the reflections we need for a *double* coset decomposition are twisted, in particular they do not square to the identity. However, it is possible to give a coset decomposition based on non-twisted reflections.

Our results in this section rely on the approach [ABM12] Corollary 3.3 and on the works of Keilberg [Keil15]. He has determined a product decomposition (exact factorization) of $\text{Aut}_{\text{Hopf}}(DG)$ whenever G does not contain abelian direct factors. In [KS14] Keilberg and Schauenburg determined $\text{Aut}_{\text{Hopf}}(DG)$ in the general case, hence when G is allowed to have abelian direct factors using an approach that differs from ours.

Recall from Definition 1.1.1 the Hopf algebras kG , k^G and DG . We write $g \triangleright e_h = e_{ghg^{-1}}$ for the left action of kG on k^G and $g \triangleright h = ghg^{-1}$ for the left action of kG on itself.

Proposition 4.1.1. (Theorem 1.1, Corollary 1.2 [Keil15])

The underlying set of $\text{Aut}_{\text{Hopf}}(DG)$ is in bijection with the set of invertible matrices

$\begin{pmatrix} u & b \\ a & v \end{pmatrix}$ where

$u : k^G \rightarrow k^G$ is a Hopf algebra morphism
 $b : G \rightarrow \hat{G}$ is a group homomorphism
 $a : k^G \rightarrow kG$ is a Hopf algebra morphism
 $v : G \rightarrow G$ is a group homomorphism

fulfilling the following three additional conditions for all $f \in k^G$ and $g \in G$:

$$u(f_{(1)}) \times a(f_{(2)}) = u(f_{(2)}) \times a(f_{(1)}) \quad v(g) \triangleright u(f) = u(g \triangleright f) \quad v(g)a(f) = a(g \triangleright f)v(g) \quad (4.1)$$

This bijection maps such a matrix to the automorphism $\phi \in \text{Aut}_{\text{Hopf}}(DG)$ defined by:

$$\phi(f \times g) = u(f_{(1)})b(g) \times a(f_{(2)})v(g) \quad \forall f \in k^G \quad \forall g \in G$$

We equip this set of matrices with matrix multiplication where we use convolution to add and composition to multiply then the above bijection is a group homomorphism.

Example 4.1.2. For $G = A$ a finite abelian group we have $k^A \simeq k\hat{A}$ as Hopf algebras. Since Hopf algebra homomorphisms map group-likes to group-likes, u is determined by a group homomorphism of \hat{A} and a is determined by a group homomorphism $\hat{A} \rightarrow A$. Thus, we get an isomorphism between Hopf algebra automorphisms of $DA \simeq \hat{A} \times A$ and group automorphisms of $\hat{A} \times A$:

$$\text{Aut}_{\text{Hopf}}(DA) \simeq \text{Aut}(\hat{A} \times A)$$

where all maps u, b, a, v are group homomorphisms.

In the following we will often express the maps u, a, b in terms of a canonical basis in the following way

$$\begin{aligned} u(e_g) &= \sum_{h \in G} u(e_g)(h)e_h & b(g) &= \sum_{h \in G} b(g)(h)e_h \\ a(e_g) &= \sum_{h \in G} e_h(a(e_g))h =: \sum_{h \in G} a_g^h h \end{aligned}$$

We denote by e.g. $u^* : kG \rightarrow kG$ the dual map of $u : k^G \rightarrow k^G$, hence $e_h(u^*(g)) = u(e_h)(g)$ and similarly for a, b, v . An automorphism $\phi \in \text{Aut}_{\text{Hopf}}(DG)$ can then be given in the following way:

$$\phi(e_g \times h) = \sum_{x, y \in G} b(h)(x) a_{gu^*(x)}^{v(h)^{-1}y} (e_x \times y)$$

In the matrix notation $\begin{pmatrix} u & b \\ a & v \end{pmatrix}$ we use the following conventions:

- $u \equiv 0$ denotes the map $k^G \rightarrow k^G; e_g \mapsto (h \mapsto \delta_{g,1_G})$ and $u \equiv 1$ the identity map on k^G .
- $b \equiv 0$ denotes the map $kG \rightarrow k^G; g \mapsto 1_{k^G}$
- $a \equiv 0$ denotes the map $k^G \rightarrow kG; e_g \mapsto 1_G \delta_{g,1_G}$
- $v \equiv 0$ denotes the map $G \rightarrow G; g \mapsto 1_G$ and $v \equiv 1$ the identity map on G .

Lemma 4.1.3. (Section 2[Keil15])

Let $\begin{pmatrix} u & b \\ a & v \end{pmatrix} \in \text{Aut}_{\text{Hopf}}(DG)$. Then the following holds:

- The Hopf morphism a is uniquely determined by a group isomorphism $\hat{A} \cong B$ where A, B are abelian subgroups of G such that $\text{im}(a^*) = kA$ and $\text{im}(a) = kB$. The map a is then given by composing $k^G \hookrightarrow k^A \cong k\hat{A} \cong kB \hookrightarrow kG$.
- $A, B \leq Z(G)$, $G = Z(G)\text{im}(v)$, $\text{im}(a)\text{im}(v) = \text{im}(v)\text{im}(a)$.
- $u^* \circ v$ is a normal group homomorphism.
- The kernels of v and u^* are contained in an abelian direct factor of G .

We now introduce several important subgroups of $\text{Aut}_{\text{Hopf}}(DG)$. The first subgroup illustrates how a group automorphism of G induces an Hopf automorphism of DG .

Proposition 4.1.4. (Proposition 4.3 [Keil15])

There is a natural subgroup of $\text{Aut}_{\text{Hopf}}(DG)$ given by:

$$V := \left\{ \begin{pmatrix} (v^{-1})^* & 0 \\ 0 & v \end{pmatrix} \mid v \in \text{Aut}(G) \right\}$$

An element in V corresponds to the following automorphism of DG :

$$e_g \times h \mapsto e_{v(g)} \times v(h)$$

We obviously have an isomorphism of groups: $V \simeq \text{Aut}(G)$.

Then we have two group of 'strict upper triangular matrices' and 'strict lower triangular matrices' which come from the abelianization $G_{ab} = G/[G, G]$ and the center $Z(G)$ respectively.

Proposition 4.1.5. (Proposition 2.3.5 [Cour12])

There is a natural abelian subgroup of $\text{Aut}_{\text{Hopf}}(DG)$ given by:

$$B := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \text{Hom}(G_{ab}, \hat{G}_{ab}) \right\}$$

An element in B corresponds to the following automorphism of DG :

$$e_g \times h \mapsto b(h)(g) e_g \times h$$

We have an isomorphism of groups $B \cong \text{Hom}(G_{ab}, \hat{G}_{ab}) \cong \hat{G}_{ab} \otimes_{\mathbb{Z}} \hat{G}_{ab}$ where the Hom-space is equipped with the convolution product.

Proposition 4.1.6. (Proposition 4.1 and 4.2 [Keil15])

There is natural abelian subgroup of $\text{Aut}_{\text{Hopf}}(DG)$ given by:

$$E := \left\{ \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \mid a \in \text{Hom}(\widehat{Z(G)}, Z(G)) \right\}$$

An element in E corresponds to the following isomorphism of DG :

$$e_g \times h \mapsto \sum_{g_1 g_2 = g} e_{g_1} \times a(e_{g_2})h$$

We have an isomorphism $E \cong \text{Hom}(\widehat{Z(G)}, Z(G)) \cong Z(G) \otimes_{\mathbb{Z}} Z(G)$ where the Hom-space is equipped with the convolution product.

Proposition 4.1.7. (Proposition 4.5 [Keil15])

There is a subgroup of $\text{Aut}_{\text{Hopf}}(DG)$ given by:

$$V_c := \left\{ \begin{pmatrix} (v^{-1})^* & 0 \\ 0 & 1 \end{pmatrix} \mid v \in \text{Aut}_c(G) \right\}$$

where $\text{Aut}_c(G) := \{v \in \text{Aut}(G) \mid v(g)g^{-1} \in Z(G)\}$. An element in V_c corresponds to the following map

$$e_g \times h \mapsto e_{v(g)} \times h$$

We can now state the main result of [Keil15] which determines $\text{Aut}_{\text{Hopf}}(DG)$ in the case that G is purely non-abelian (i.e. has no direct abelian factors):

Theorem 4.1.8. (Theorem 5.7 [Keil15])

Let G be a purely non-abelian finite group. There is an exact factorization into subgroups

$$\begin{aligned} \text{Aut}_{\text{Hopf}}(DG) &\simeq E((V_c \rtimes V) \rtimes B) \\ &\simeq ((V_c \rtimes V) \rtimes B)E \end{aligned}$$

The main step in the proof is using the fact that $\ker(u^*)$ and $\ker(v)$ are contained in direct factors of G according to Lemma 4.1.3. Clearly, if G is purely non-abelian then u, v have to be invertible. This leads directly to the above decomposition.

The exact factorization fails to be true in the presence of direct abelian factors. In this case neither u nor v have to be invertible, but their kernels are still contained in an direct abelian factor. From this point of view it seems natural to introduce an additional class of automorphisms of DG that act on direct abelian factors of G . These will be maps that exchange an abelian factor $kC \subset k^G \rtimes kG$ with its dual $k^C \subset k^G \rtimes kG$.

Proposition 4.1.9. Let R_t be the set of all tuples (H, C, δ, ν) , where C is an abelian direct factor of G and H is a subgroup of G such that $G = H \times C$, $\delta : kC \xrightarrow{\sim} k^C$ a Hopf isomorphism and $\nu : C \rightarrow C$ a nilpotent homomorphism.

- (i) For (H, C, δ, ν) we define a *twisted reflection* on C to be $r_{H,C,\delta,\nu} : DG \rightarrow DG$ on C given by:

$$(f_H, f_C) \times (h, c) \mapsto (f_H, \delta(c)) \times (h, \delta^{-1}(f_C)\nu(c))$$

where $f_H \in k^H, f_C \in k^C, h \in H$ and $c \in C$. All twisted reflections $r_{H,C,\delta,\nu}$ are Hopf automorphisms.

- (ii) Denote by the subset $R \subset R_t$ those elements that have $\nu = 1_C$. We call the corresponding Hopf automorphisms $r_{H,C,\delta}$ a *reflections* on C .

Let $(H, C, \delta), (H', C', \delta')$ be two reflections. If there exist two group isomorphisms $f_1, f_2 : C \simeq C'$ such that $\delta = f_2^* \circ \delta' \circ f_1$, then there exists two group automorphisms $v_1, v_2 \in \text{Aut}(G) \simeq V$ (see Prop. 4.1.4) such that

$$r_{H,C,\delta} = \begin{pmatrix} v_2^* & 0 \\ 0 & v_2^{-1} \end{pmatrix} \cdot r_{H',C',\delta'} \cdot \begin{pmatrix} (v_1^{-1})^* & 0 \\ 0 & v_1 \end{pmatrix}$$

Before proving this proposition, we want to make some remarks:

- By a nilpotent homomorphism $\nu : C \rightarrow C$ we mean that $\nu^n = 1_C$ for some $n \in \mathbb{N}$. If we decompose C into cyclic groups and view ν as a matrix with respect to this decomposition, nilpotent means that all entries in this matrix are multiplicative non-invertible elements in these cyclic groups. In particular, for elementary abelian groups, all direct factors are of prime order and hence $\nu = 1_C$.

- Part (ii) of the proposition shows how we can go from one (non-twisted) reflection to another by multiplying with elements in $\text{Aut}(G)$. The condition $\delta = f_2^* \circ \delta' \circ f_1$ evaluated on group elements gives us:

$$\delta(c_1)(c_2) = \delta'(f_1(c_1))(f_2(c_2)) \quad \forall c_1, c_2 \in C$$

- The matrix for a twisted reflection (H, C, δ, ν) is given as follows:

$$\begin{pmatrix} p_{k^H} & \delta \circ p_C \\ \delta^{-1} \circ p_{k^C} & p_H + \nu \end{pmatrix}$$

where the maps $p_{k^H} : k^H \times k^C \rightarrow k^H, p_{k^C} : k^H \times k^C \rightarrow k^C, p_H : H \times C \rightarrow H$ and $p_C : H \times C \rightarrow C$ are projections. In order to simplify the notation, we sometimes abbreviate this matrix by

$$\begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H + \nu \end{pmatrix}$$

Proof. (i) In order to prove that a reflection is indeed an automorphism of DG we have to check bijectivity (which is clear) and the three equations (4.1). In this case we have:

$$\begin{aligned} u(e_{hc}) &= p_{k^H}(e_{hc}) = e_h \delta_{c,1} \\ b(hc) &= \delta \circ p_C(hc) = \delta(c) \\ a(e_{hc}) &= \delta^{-1} \circ p_{k^H}(e_{hc}) = \delta^{-1}(e_c) \delta_{h,1} \\ v(hc) &= p_C(hc) = c \end{aligned}$$

We check the first equation of (4.1). It holds, because for all $h \in H$, $c \in C$ we have:

$$\begin{aligned}
\sum_{\substack{h_1 h_2 = h \\ c_1 c_2 = c}} p_{k^H}(e_{h_1 c_1}) \times \delta^{-1} \circ p_{k^C}(e_{h_2 c_2}) &= \sum_{\substack{h_1 h_2 = h \\ c_1 c_2 = c}} \delta_{c_1,1} \delta_{h_2,1} e_{h_1} \times \delta^{-1}(e_{c_2}) \\
&= e_h \times \delta^{-1}(e_c) \\
&= \sum_{\substack{h_1 h_2 = h \\ c_1 c_2 = c}} \delta_{c_2,1} \delta_{h_1,1} e_{h_2} \times \delta^{-1}(e_{c_1}) \\
&= \sum_{\substack{h_1 h_2 = h \\ c_1 c_2 = c}} p_{k^H}(e_{h_2 c_2}) \times \delta^{-1} \circ p_{k^C}(e_{h_1 c_1})
\end{aligned}$$

The second equation of (4.1) holds, because for all $h \in H$, $c \in C$:

$$p_H(hc)\nu(c) \triangleright p_{k^H}(e_{h'c'}) = h\nu(c) \triangleright \delta_{c',1} e_{h'} = \delta_{c',1} e_{hh'h^{-1}}$$

is equal to

$$p_{k^H}(hc \triangleright e_{h'c'}) = p_{k^H}(e_{hh'h^{-1}c'}) = e_{hh'h^{-1}} \delta_{c',1}$$

The last equation of (4.1) also holds, since for all $h \in H$, $c \in C$:

$$h\nu(c)\delta^{-1} \circ p_{k^C}(e_{h'c'}) = h\nu(c)\delta^{-1}(e_{c'})\delta_{h',1}$$

is equal to

$$\delta^{-1} \circ p_{k^C}(e_{h'c'}) p_H(hc)\nu(c) = \delta^{-1}(e_{c'})\delta_{h',1} h\nu(c)$$

(ii) Let H, C, H', C' be subgroups of G such that $H \times C = G = H' \times C'$ and $f_1, f_2 : C \cong C'$ group isomorphisms such that $\delta = f_2^* \circ \delta' \circ f_1$. Note that we then have the identity $\delta = f_1^* \circ \delta' \circ f_2$, because for all $c_1, c_2 \in C$:

$$\begin{aligned}
\delta(c_1)(c_2) &= \delta(c_2)(c_1) \\
&= f_2^*(\delta'(f_1(c_2)))(c_1) \\
&= \delta'(f_1(c_2))(f_2(c_1)) \\
&= \delta'(f_2(c_1))(f_1(c_2)) \\
&= f_1^*(\delta'(f_2(c_2)))(c_1)
\end{aligned}$$

where we have used $\delta(c_1)(c_2) = \delta(c_2)(c_1) \forall c_1, c_2 \in C$ and similarly for δ' .

According to [Rem11], for a finite group G , the group of central automorphisms $\text{Aut}_c(G)$ acts transitively on the set of Krull-Schmidt decompositions of G . We have $H \times C = G = H' \times C'$. So we can find a central automorphism $\psi \in \text{Aut}_c(G)$ such that ψ restricted to H gives us an isomorphism $\psi|_H : H \xrightarrow{\cong} H'$ and ψ restricted to C gives us an isomorphism $\psi|_C : C \xrightarrow{\cong} C'$.

For $i = 1, 2$, we define $v_i \in \text{Aut}(G)$ by $v_i(hc) := \psi|_H(h)f_i(c)$ for $h \in H, c \in C$. Let us check that these fulfill statement in the proposition:

$$\begin{aligned}
& \begin{pmatrix} v_2^* & 0 \\ 0 & v_2^{-1} \end{pmatrix} \begin{pmatrix} p_{kH'} & \delta' \circ p_{C'} \\ \delta'^{-1} \circ p_{kC'} & p_{H'} \end{pmatrix} \begin{pmatrix} (v_1^{-1})^* & 0 \\ 0 & v_1 \end{pmatrix} \\
&= \begin{pmatrix} v_2^* \circ p_{kH'} \circ (v_1^{-1})^* & v_2^* \circ \delta' \circ p_{C'} \circ v_1 \\ v_2^{-1} \circ \delta'^{-1} \circ p_{kC'} \circ (v_1^{-1})^* & v_2^{-1} \circ p_{H'} \circ v_1 \end{pmatrix} \\
&= \begin{pmatrix} (\psi_H)^* \circ (\psi_H^{-1})^* \circ p_{kH} & f_2^* \circ \delta' \circ f_1 \circ p_{C'} \\ f_2^{-1} \circ \delta'^{-1} \circ (f_1^{-1})^* \circ p_{kC} & \psi_H^{-1} \circ \psi_H p_H \end{pmatrix} \\
&= \begin{pmatrix} p_{kH} & \delta \circ p_C \\ \delta^{-1} \circ p_{kC} & p_H \end{pmatrix}
\end{aligned}$$

where we have used $\delta = f_2^* \circ \delta' \circ f_1$ and $\delta = f_1^* \circ \delta' \circ f_2$ in the last equation. \square

4.2 Decomposition

Theorem 4.2.1. Let G be a finite group.

- (i) $\text{Aut}_{\text{Hopf}}(DG)$ is generated by the subgroups V, V_c, B, E and the set of reflections R .
- (ii) For every $\phi \in \text{Aut}_{\text{Hopf}}(DG)$ there is a twisted reflection $r = r_{H,C,\delta,\nu} \in R_t$ such that ϕ is an element in the double coset

$$[(V_c \rtimes V) \rtimes B] \cdot r \cdot [(V_c \rtimes V) \rtimes E]$$

- (iii) Two double cosets corresponding to (non-twisted) reflections $(H, C, \delta), (H', C', \delta')$ are equal if and only if there exists a group isomorphism $C \simeq C'$.
- (iv) For every $\phi \in \text{Aut}_{\text{Hopf}}(DG)$ there is a reflection $r = r_{H,C,\delta} \in R$ such that ϕ is an element in

$$r \cdot [B((V_c \rtimes V) \rtimes E)]$$

- (v) For every $\phi \in \text{Aut}_{\text{Hopf}}(DG)$ there is a reflection $r = r_{H,C,\delta} \in R$ such that ϕ is an element in

$$[((V_c \rtimes V) \rtimes B)E] \cdot r$$

Before we turn to the proof, we illustrate the statement of Theorem 4.2.1 on some examples:

Example 4.2.2. In the case G is purely non-abelian, there are no (non-trivial) reflections. We get the result of Theorem 4.1.8.

Example 4.2.3. Let $G = \mathbb{Z}_p^n$ and p a prime number. We fix a group isomorphism $\mathbb{Z}_p \simeq \widehat{\mathbb{Z}}_p$. We then have a group isomorphism

$$\text{Aut}_{\text{Hopf}}(DG) \simeq \text{Aut}(\mathbb{F}_p^n \times \mathbb{F}_p^n) \simeq \text{GL}_{2n}(\mathbb{F}_p)$$

The previously defined subgroups, as subgroups of $\text{GL}_{2n}(\mathbb{F}_p)$, are in this case:

- $V \simeq \text{GL}_n(\mathbb{F}_p) \simeq \left\{ \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \mid A \in \text{GL}_n(\mathbb{F}_p) \right\}$
- $V_c \simeq \text{GL}_n(\mathbb{F}_p) \simeq \left\{ \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mid A \in \text{GL}_n(\mathbb{F}_p) \right\}$
- $B \simeq \widehat{G}_{ab} \otimes_{\mathbb{Z}} \widehat{G}_{ab} = \mathbb{F}_p^{n \times n} \simeq \left\{ \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \mid B \in \mathbb{F}_p^{n \times n} \right\}$.
- $E \cong Z(G) \otimes_{\mathbb{Z}} Z(G) = \mathbb{F}_p^{n \times n} \simeq \left\{ \begin{pmatrix} 1 & 0 \\ E & 1 \end{pmatrix} \mid E \in \mathbb{F}_p^{n \times n} \right\}$.

The set R is very large: For each dimension $d \in \{0, \dots, n\}$ there is a unique isomorphism type $C \cong \mathbb{F}_p^d$. The possible subgroups of this type $C \subset G$ are the Grassmannian $\text{Gr}(n, d, G)$, the possible $\delta : C \rightarrow \widehat{C}$ are parametrized by $\text{GL}_d(\mathbb{F}_p)$ and in this fashion R can be enumerated.

On the other hand, we have only $n + 1$ representatives $r_{[C]}$ for each dimension d , given for example by permutation matrices

$$\left(\begin{array}{cc|cc} 0 & 0 & \mathbb{1}_d & 0 \\ 0 & \mathbb{1}_{n-d} & 0 & 0 \\ \hline \mathbb{1}_d & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{1}_{n-d} \end{array} \right)$$

One checks this indeed gives a decomposition of $\text{GL}_{2n}(\mathbb{F}_p)$ into V_cVB - V_cVE -cosets, e.g.

$$\begin{aligned} \text{GL}_4(\mathbb{F}_p) &= (V_cVB \cdot r_{[1]} \cdot V_cVE) \cup (V_cVB \cdot r_{[\mathbb{F}_p]} \cdot V_cVE) \cup (V_cVB \cdot r_{[\mathbb{F}_p^2]} \cdot V_cVE) \\ |\text{GL}_4(\mathbb{F}_p)| &= p^8 |\text{GL}_2(\mathbb{F}_p)|^2 + \frac{p^3 |\text{GL}_2(\mathbb{F}_p)|^4}{(p-1)^4} + p^4 |\text{GL}_2(\mathbb{F}_p)|^2 \\ &= p^8 (p^2 - 1)^2 (p^2 - p)^2 + \frac{p^3 (p^2 - 1)^4 (p^2 - p)^4}{(p-1)^4} + p^4 (p^2 - 1)^2 (p^2 - p)^2 \\ &= (p^4 - 1)(p^4 - p)(p^4 - p^2)(p^4 - p^3) \end{aligned}$$

It corresponds to a decomposition of the Lie algebra A_{2n-1} according to the $A_{n-1} \times A_{n-1}$ parabolic subsystem. Especially on the level of Weyl groups we have a decomposition as double cosets of the parabolic Weyl group

$$\begin{aligned} \mathbb{S}_{2n} &= (\mathbb{S}_n \times \mathbb{S}_n) \mathbb{1} (\mathbb{S}_n \times \mathbb{S}_n) \cup (\mathbb{S}_n \times \mathbb{S}_n) (1, 1+n) (\mathbb{S}_n \times \mathbb{S}_n) \cup \\ &\quad \cdots \cup (\mathbb{S}_n \times \mathbb{S}_n) (1, 1+n) (2, 2+n) \cdots (n, 2n) (\mathbb{S}_n \times \mathbb{S}_n) \\ \text{e.g. } |\mathbb{S}_4| &= 4 + 16 + 4 \end{aligned}$$

In this case, the full Weyl group \mathbb{S}_{2n} of $\text{GL}_{2n}(\mathbb{F}_p)$ is the set of all reflections (as defined above) that preserve a given decomposition $G = \mathbb{F}_p \times \cdots \times \mathbb{F}_p$.

Proof of Theorem 4.2.1.

(i) follows immediately from (iv).

(ii) From above we know that $\ker(v)$ is contained in an abelian direct factor G . The other factor can be abelian or not, but we can decompose it into a purely non-abelian factor times an abelian factor. Hence we arrive at the decomposition $G = H \times C$ where H is purely non-abelian and where $\ker(v)$ is contained in a direct abelian factor of C . Since C is a finite abelian group there is an $n \in \mathbb{N}$ and an isomorphism

$$C \cong C_1 \times \dots \times C_n$$

where C_i are cyclic groups of order $p_i^{k_i}$ for some prime numbers p_i and $k_i \in \mathbb{N}$ with $p_i^{k_i} \leq p_j^{k_j}$ for $i \leq j$.

A general Hopf automorphism $\phi \in \text{Aut}_{\text{Hopf}}(DG)$ can then be written in matrix form with respect to the decomposition $G = H \times C_1 \times C_2 \times \dots \times C_n$ as

$$\phi = \left(\begin{array}{ccc|ccc} u_{H,H} & \cdots & u_{C_n,H} & b_{H,H} & \cdots & b_{C_n,H} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u_{H,C_n} & \cdots & u_{C_n,C_n} & b_{H,C_n} & \cdots & b_{C_n,C_n} \\ \hline a_{H,H} & \cdots & a_{C_n,H} & v_{H,H} & \cdots & v_{C_n,H} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{H,C_n} & \cdots & a_{C_n,C_n} & v_{H,C_n} & \cdots & v_{C_n,C_n} \end{array} \right) \quad (4.2)$$

Let c_n be the generator of C_n . Since ϕ is an automorphism we know that the order of $\phi(c_n)$ is also $p_n^{k_n}$. This implies that one of the elements

$$b_{C_n,H}(c_n), \dots, b_{C_n,C_n}(c_n), v_{C_n,H}(c_n), \dots, v_{C_n,C_n}(c_n)$$

has order $p_n^{k_n}$. Then we can have three possible cases:

Case (1): One of the v_{C_n,C_n} or b_{C_n,C_n} is injective.

Case (2): One of the v_{C_n,C_m} or b_{C_n,C_m} is injective and $m < n$.

Case (3): One of the $v_{C_n,H}$ or $b_{C_n,H}$ is injective.

Case (1): If v_{C_n,C_n} is injective, then it has to be bijective (since C_n is finite). We can construct an element in B and an element in V_c :

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & v_{C_n,H}v_{C_n,C_n}^{-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & v_{C_n,C_n}^{-1} \end{array} \right) \left(\begin{array}{ccc|ccc} 1 & \cdots & 0 & 0 & \cdots & b_{C_n,H}v_{C_n,C_n}^{-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & \cdots & b_{C_n,C_n}v_{C_n,C_n}^{-1} \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array} \right) \quad (4.3)$$

Multiplying (4.2) from the *left* by (4.3) we eliminate the $2n$ -column. Similarly, multiplying with elements of E and V_c from the *right* we can eliminate the $2n$ -row.

$$\begin{array}{c} \xrightarrow{B, V_c} \\ \xrightarrow{\sim} \end{array} \left(\begin{array}{ccc|ccc} * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * & 0 \\ \hline * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * & 1 \end{array} \right) \xrightarrow{E, V_c} \left(\begin{array}{ccc|ccc} * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * & 0 \\ \hline * & \cdots & * & * & \cdots & * & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ * & \cdots & * & * & \cdots & * & 0 \\ \hline 0 & \cdots & 0 & 0 & \cdots & 0 & 1 \end{array} \right) \quad (4.4)$$

In the case that b_{C_n, C_n} is injective, hence bijective, we construct an element in V_c :

$$\left(\begin{array}{ccc|ccc} 1 & \cdots & b_{C_n, H} b_{C_n, C_n}^{-1} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & b_{C_n, C_n} b_{C_n, C_n}^{-1} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{array} \right) \quad (4.5)$$

Since the upper left quadrant of (4.5) is the dual of an automorphism that takes values in abelian factors of G it is indeed an element of V_c .

Multiplying (4.2) from the *left* by (4.5) we eliminate the upper half of the $2n$ -column and by multiplying with elements of E and V_c from the *right* we eliminate the n -row:

$$\begin{array}{c} \xrightarrow{B, V_c} \\ \xrightarrow{\sim} \end{array} \left(\begin{array}{ccc|ccc} * & \cdots & * & * & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & b_{C_n, C_n} \\ \hline * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \end{array} \right) \xrightarrow{E, V_c} \left(\begin{array}{ccc|ccc} * & \cdots & * & * & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & b_{C_n, C_n} \\ \hline * & \cdots & * & * & \cdots & * \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & \cdots & * & * & \cdots & * \end{array} \right) \quad (4.6)$$

Case (2): If $v_{C_n, C_m}(c_n)$ is injective it is also bijective, since $|C_m| \leq |C_n|$ by construction. Then we must have $p_n = p_m$ and $k_n = k_m$. Then let $w \in \text{Aut}_c(G)$ be an automorphism such that $w(C_n) = C_m$, $w(C_m) = C_n$ and identity elsewhere. Multiplying with $\begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \in V_c$ from the left returns us to Case (1) when v_{C_n, C_n} is invertible. Similarly, if b_{C_n, C_m} is injective, it has to be bijective because of the same order argument as above. Exchanging the C_n with C_m by an $\begin{pmatrix} w^* & 0 \\ 0 & 1 \end{pmatrix} \in V_c$ we return to the Case (1) when b_{C_n, C_n} is invertible.

Case (3): If $v_{C_n, H}$ is injective, we can also assume that $v_{C_n, H}$ is the only injective map in the last column of ϕ , else we choose Case (1) or Case (2). Since the whole matrix ϕ is invertible, there exists an inverse matrix $\phi^{-1} = \begin{pmatrix} u' & b' \\ a' & v' \end{pmatrix}$ and then the multiplication of the last right column of ϕ with the last upper row of ϕ^{-1} has to be 1. Therefore there has to be a homomorphism $v'_{H, C_n} : H \rightarrow C_n$ such that

$w := v'_{H,C_n} \circ v_{C_n,H} : C_n \rightarrow C_n$ is injective, and therefore bijective. We have an exact sequence

$$0 \rightarrow \ker(v'_{H,C_n}) \rightarrow H \xrightarrow{v'_{H,C_n}} C_n \rightarrow 0$$

which splits on the right via $v_{C_n,H} \circ w^{-1}$. Restricting to group-like objects $G(DG) = \hat{G} \times G$ the row $a_{H,H} a_{C_1,H} \dots v_{H,H} \dots v_{C_n,H}$ gives a surjection from $\hat{G} \times G$ to H . It maps central elements to central elements because it is surjective and hence the restriction to C_n , namely $v_{C_n,H}$, has central image. This implies that C_n is a direct abelian factor of H which is not possible, because H is purely non-abelian per construction.

Hence we end up with either the form (4.6) or the form (4.7). Now we inductively move on to $C_{n-1}, C_{n-2}, \dots, C_1$ where we permute parts with the non-invertible v 's to the right lower corner by multiplying with elements of V_c . Since $\ker(v)$ has trivial intersection with H per construction, the map $v_{H,H}$ is invertible. As in the Case (1) we can use row and column manipulation to get zeros below and above $v_{H,H}$ as well as left and right. Note that the elements we are constructing are always in V_c because either have abelian image per definition or are restrictions on abelian direct factors of surjections. Only $v_{H,H}, u_{H,H}$ do in general not induce V_c elements like that. But corresponding to the automorphism $v_{H,H}^{-1}$ there is a matrix in V . Multiplying with this matrix changes the remaining $u_{H,H}$ to $v_{H,H}^* \circ u_{H,H}$ and the $v_{H,H}$ to id_H .

Now we now consider a generator χ_n of \hat{C}_n and conclude from the fact that ϕ is an automorphism the analogous case differentiation from above but now for entries in the remaining u and a . With the same arguments as above we move through the columns corresponding to $\hat{C}_{n-1}, \dots, \hat{C}_1, \hat{H}$ and end up with a matrix of the following form:

$$\left(\begin{array}{ccc|ccc} u_{H,H} & 0 & 0 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_m \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k & 0 \\ 0 & 0 & A_m & 0 & 0 & V_m \end{array} \right) \quad (4.7)$$

where $k + m + 1 = n$, B_m, A_m are diagonal $m \times m$ -matrices with isomorphisms on the diagonal, I_k an $k \times k$ -identity matrix and V_m a $m \times m$ -matrix with non-invertible homomorphisms as entries. Further, since H is purely non-abelian and by Lemma 4.1.3 $\ker(u)$ is contained in an abelian direct factor we deduce that $u_{H,H}$ is an isomorphism. Also by Lemma 4.1.3 we know that the composition

$$\begin{pmatrix} u_{H,H}^* & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & V_m \end{pmatrix} = \begin{pmatrix} u_{H,H}^* & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

has to be a normal homomorphism, hence $u_{H,H}$ has to be a central automorphism. Therefore we get (4.7) with $u_{H,H} = 1$ by multiplying with the inverse in V_c . Our final step is normalizing the A_m by multiplying with an element in V_c corresponding to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & I_k & 0 \\ 0 & 0 & B_m^{-1}A_m^{-1} \end{pmatrix}$$

Hence we end up with a twisted reflection:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & B_m \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k & 0 \\ 0 & 0 & B_m^{-1} & 0 & 0 & V_m \end{array} \right) \quad (4.8)$$

(iii) Assume that the two double cosets corresponding to $r'_{H',C',\delta'}$ and $r_{H,C,\delta}$ are equal. Then there are $w, w', v, v' \in \text{Aut}(G)$, $a' \in \text{Hom}(\widehat{Z(G)}, Z(G))$, $b \in \text{Hom}(G, \widehat{G})$ such that

$$\begin{pmatrix} w^* & w^*b \\ 0 & v \end{pmatrix} \begin{pmatrix} \hat{p}_H & \delta \\ \delta^{-1} & p_H \end{pmatrix} = \begin{pmatrix} \hat{p}_{H'} & \delta' \\ \delta'^{-1} & p_{H'} \end{pmatrix} \begin{pmatrix} w'^* & 0 \\ v'a' & v' \end{pmatrix}$$

$$\begin{pmatrix} w^*\hat{p}_H + w^*b\delta^{-1} & w^*\delta + w^*bp_H \\ v\delta^{-1} & vp_H \end{pmatrix} = \begin{pmatrix} \hat{p}_{H'}w'^* + \delta'v'a' & \delta'v' \\ \delta'^{-1}w'^* + p_{H'}v'a' & p_{H'}v' \end{pmatrix}$$

Comparing entries implies $v \circ p_H = p_{H'} \circ v'$. Then $C = \ker(p_H) = \ker(v^{-1} \circ p_{H'} \circ v') = v'^{-1}(\ker(p_{H'})) = v'^{-1}(C')$. Hence v' restricted to C gives an isomorphism $C \simeq C'$.

On the other hand, if we have an isomorphism $v : C \xrightarrow{\sim} C'$, Proposition 4.1.9 (ii) with $f_1 := v : C \xrightarrow{\sim} C'$ and $f_2 := \delta'^{-1} \circ v'^{-1} \circ \delta$ implies that the double cosets corresponding to (H, C, δ) and (H', C', δ') have non-trivial intersection and are therefore equal.

(iv) Let ϕ be a general element in $\text{Aut}_{\text{Hopf}}(DG)$ as in (4.4). We use the same arguments as in (ii) to get to the case differentiation for every column. We can produce zeros in each row by multiplying ϕ with E and V_c from the right, except when we have invertible entries in a and u . In this case multiplying with E, V_c from the right can only produce zeros in u and a respectively. Hence we end up with:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & * & * & * \\ 0 & I_k & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & B_m \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_k & 0 \\ 0 & 0 & B_m^{-1} & * & * & * \end{array} \right) \quad (4.9)$$

where B_m is a diagonal $m \times m$ -matrix with isomorphisms on the diagonal, I_k an $k \times k$ -identity matrix and $k + m = n + 1$. Multiplying again from the right but now with elements of B (which was not allowed in (ii)) and elements of V_c we eliminate the $*$ which results in a (non-twisted) reflection.

(v) This is similar to above. We start with (4.4) an move from column to column as in (ii) and (iv). We identify the invertible entries and can clean up each *column* by multiplying with elements of B and V_c from the *left*. Finally as in (ii) we get a non-invertible twist V_m below B_m but contrary to (ii) we can multiply with E from the *left* and eliminate the V_m . Therefore we again end up with a (non-twisted) reflection. This concludes the proof of Theorem 4.2.1. \square

Chapter 5

Lazy Cohomology of the Dual Drinfeld Double

In Chapter 4 we have decomposed $\text{Aut}_{\text{Hopf}}(DG^*)$ into manageable and natural subgroups. Now we would like to decompose the group of lazy 2-cohomology classes $H_L^2(DG^*)$ into manageable subgroups as well (recall Definition 1.3.3 of lazy cohomology). These subgroups should, in some sense, match the decomposition of $\text{Aut}_{\text{Hopf}}(DG)$ as will become clear in Chapter 6.

For Hopf algebra tensor products such as $kG \otimes k^G$ the Kac-Schauenburg sequence (see [Schau02]) implies among others

$$\begin{aligned} H^2(kG \otimes k^G) &\simeq H^2(G, k^\times) \times P(kG, k^G) \times H^2(k^G) \quad \text{as sets} \\ H_L^2(kG \otimes k^G) &\simeq H^2(G, k^\times) \times P_L(kG, k^G) \times H_L^2(k^G) \quad \text{as groups} \end{aligned}$$

where $P(kG, k^G)$ is the group of bialgebra pairings $kG \times k^G \rightarrow k$ and $P_L(kG, k^G)$ the group of lazy bialgebra pairings (see Definition 5.0.6). This should be directly compared to the Künneth formula for topological spaces

$$H^2(X \times Y) \simeq H^2(X) \times (H^1(X) \otimes H^1(Y)) \times H^2(Y)$$

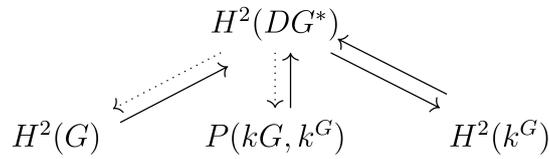
The lazy cohomology of DG is the same as the lazy cohomology of the tensor product $k^G \otimes kG$ (see e.g [Bich04] Corollary 4.11) because DG is the Doi twist of the tensor product. However, for the dual Drinfeld double DG^* , which is a Drinfeld twist and therefore has a modified coalgebra structure, such a formula might not be true anymore. In fact DG^* is characterized by the following exact sequence of Hopf algebras:

$$kG \xleftarrow{p} DG^* \xleftarrow{t} k^G \tag{5.1}$$

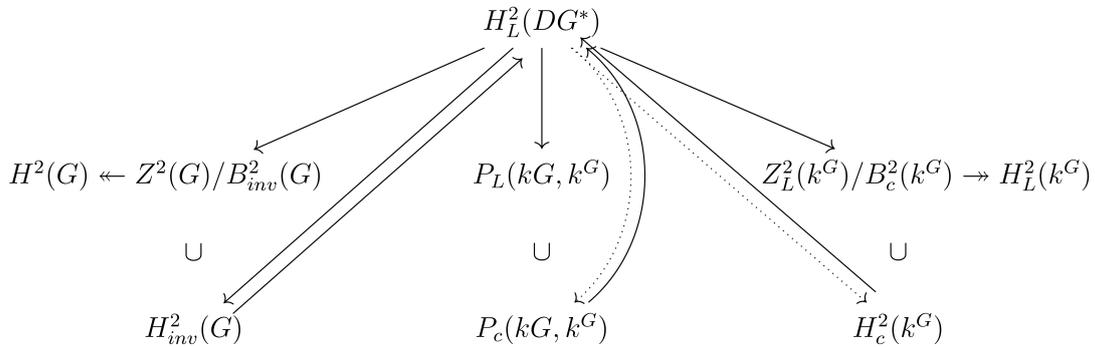
where we additionally have a natural splitting $s : DG^* \rightarrow k^G$, in particular s is a Hopf algebra map, and where the natural map $t : kG \rightarrow DG^*$ is an algebra map but not a coalgebra map. Considering this sequence, we would still hope that one can define suitable subgroups coming from kG , k^G and interactions and prove an exact factorization of $H_L^2(DG^*)$ into these subgroups. It is also natural to hope that

a similar approach decomposes $H^2(DG^*)$.

The following diagram gives a sketch of the idea:



Most of the arrows in the diagram above follow easily by functoriality of H^2 . The dashed arrows indicate the direction we were not able to achieve. Going from bottom-to-top gives us explicit subgroups (subsets) of $H^2(DG^*)$ and we would hope for a decomposition. The maps top-to-bottom that would be necessary to prove this should follow from a restriction of the cocycle, however since kG is not a Hopf subalgebra of DG^* not all restrictions are well-defined. For *lazy* 2-cocycles we are able to solve some of these problems. On the other hand, here we have to deal with the fact that lazy cohomology is *not functorial* which forces us to restrict the bottom-to-top maps on subgroups. One needs to prove that the images of the top-to-bottom maps are contained in these subgroups. Moreover, the lazy cohomology group $H_L^2(DG^*)$ is not a mere subset of $H^2(DG^*)$ but also a quotient by fewer coboundaries. In formulating the results, this is a tedious but not serious obstruction. We consider the following diagram



where $H_{inv}^2(G, k^\times)$ is the group of conjugation invariant 2-cocycles on G modulo conjugation invariant cochains (see Definition 5.0.2), $P_c(kG, k^G) \subset P_L(kG, k^G)$ the group of central bialgebra pairings (see Proposition 5.0.8) and $H_c^2(k^G)$ the group of central lazy 2-cocycles (see also Proposition 5.0.8).

These maps together with the following Lemmas are partial result that are needed to provide the full decomposition and are in addition necessary for our application in Chapter 6.

Let us start by collecting the following property:

Lemma 5.0.1. Let $\sigma \in Z^2(DG^*)$ and recall that we write $g^t = t^{-1}gt$ for $g, t \in G$. Then σ is lazy if and only if for all $g, h, x, y \in G$:

- If $gh = g^x h^y$ we have $\sigma(g \times e_x, h \times e_y) = \sigma(g^t \times e_{x^t}, h^t \times e_{y^t})$ for all $t \in G$
- If $gh \neq g^x h^y$ we have $\sigma(g \times e_x, h \times e_y) = 0$

Further, $\eta \in \text{Reg}^1(DG^*)$ is lazy if and only if for all $g, x \in G$:

- If $gx = xg$ then $\eta(g^t \times e_{x^t}) = \eta(g \times e_x)$
- If $gx \neq xg$ we have $\eta(g \times e_x) = 0$

Proof. The lazy condition is $\sigma * \mu = \mu * \sigma$ where $*$ is the convolution product. The left hand side evaluated on $(g \times e_x) \otimes (h \times y) \in DG^* \otimes DG^*$ is:

$$\sum_{\substack{x_1 x_2 = x \\ y_1 y_2 = y}} \sigma(g \times e_{x_1}, h \times e_{y_1})(g^{x_1} h^{y_1} \times e_{x_2} e_{y_2}) = \sum_t \sigma(g \times e_{xt^{-1}}, h \times e_{yt^{-1}})(g^{xt^{-1}} h^{yt^{-1}} \times e_t)$$

the right hand side evaluated on $(g \times e_x) \otimes (h \times y) \in DG^* \otimes DG^*$ is:

$$\sum_{\substack{x_1 x_2 = x \\ y_1 y_2 = y}} \sigma(g^{x_1} \times e_{x_2}, h^{y_1} \times e_{y_2})(gh \times e_{x_1} e_{y_1}) = \sum_t \sigma(g^t \times e_{t^{-1}x}, h^t \times e_{t^{-1}y})(gh \times e_t)$$

The equality between the left and right hand side above is equivalent to saying that for all $t, z, g, h, x, y \in G$:

$$\sigma(g \times e_x, h \times e_y) \delta_{z, g^x h^y} = \sigma(g^t \times e_{x^t}, h^t \times e_{y^t}) \delta_{z, gh}$$

Similarly an $\eta \in \text{Reg}^1(DG^*)$ is lazy iff: $\eta * \text{id} = \text{id} * \eta$. Then comparing the left hand side evaluated on $(g \times e_x \in DG^*)$:

$$\sum_{x_1 x_2 = x} \eta(g \times e_{x_1})(g^{x_1} \times e_{x_2}) = \sum_t \eta(g \times e_{xt^{-1}})(g^{xt^{-1}} \times e_t)$$

with the right hand side evaluated on $(g \times e_x \in DG^*)$:

$$\sum_{y_1 y_2 = x} \eta(g^{y_1} \times e_{y_2})(g \times e_{y_1}) = \sum_t \eta(g^t \times e_{t^{-1}x})(g \times e_t)$$

leads to the equation $\eta(g^t \times e_{x^t}) \delta_{g, z} = \eta(g \times e_x) \delta_{g^x, z}$. \square

It is natural to expect that, similarly as for the Künneth formula, $H_L^2(DG^*)$ decomposes in some way into parts that depend on kG , k^G and some sort of pairings between both. We start with the kG part:

Definition 5.0.2. Let $Z_{inv}^2(G, k^\times)$ be the set of those 2-cocycles $\beta \in Z^2(G, k^\times)$ that fulfill

$$\beta(g, h) = \beta(g^t, h^t) \quad \forall t \in G$$

and let $C_{inv}^1(G, k^\times)$ be the set of those maps $\eta : G \rightarrow k^\times$ such that $\eta(g) = \eta(g^t) \forall t \in G$ and $B_{inv}^2(G, k^\times) := d(C_{inv}^1(G, k^\times))$. We define the cohomology group of conjugation invariant cocycles to be the group

$$H_{inv}^2(G, k^\times) := Z_{inv}^2(G, k^\times) / B_{inv}^2(G, k^\times) \quad (5.2)$$

Lemma 5.0.3. There is a map (as sets)

$$\mathbb{H}_L^2(DG^*) \rightarrow \mathbb{H}_{inv}^2(G); \sigma \mapsto \beta_\sigma$$

defined by $\beta_\sigma(g, h) = \sigma(g \times 1, h \times 1)$. Further, it is a splitting of $\mathbb{H}_{inv}^2(G) \rightarrow \mathbb{H}_L^2(DG^*)$ mapping a conjugation invariant 2-cocycle $\beta \in \mathbb{Z}_{inv}^2(G)$ to $\sigma_\beta \in \mathbb{Z}_L^2(DG^*)$ defined by

$$\sigma_\beta(g \times e_x, h \times e_y) = \beta(g, h)\epsilon(e_x)\epsilon(e_y)$$

Proof. We apply the cocycle condition (1.6) for $a = g \times 1$, $b = h \times 1$ and $c = r \times 1$ for $g, h, r \in G$:

$$\sum_{t,s} \sigma(g \times e_t, h \times e_s)\sigma(g^t h^s \times 1, r \times 1) = \sum_{gh=g^s h^t} \sigma(g \times e_t, h \times e_s)\sigma(gh \times 1, r \times 1) \quad (5.3)$$

$$= \beta_\sigma(g, h)\beta_\sigma(gh, r) \quad (5.4)$$

$$= \sum_{s,z} \sigma(h \times e_s, r \times e_z)\sigma(g \times 1, h^s r^z \times 1) \quad (5.5)$$

$$= \beta_\sigma(h, r)\beta_\sigma(g, hr) \quad (5.6)$$

where we have used Lemma 5.0.1. This shows the 2-cocycle condition for β_σ . To show that β_σ is indeed in $\mathbb{Z}^2(G, k^\times)$, we argue that $\beta_\sigma(g, h) \neq 0$ for all $g, h \in G$. This is not clear because $g \times 1$ is not a group-like element in DG^* . Let σ^{-1} be the convolution inverse of σ . First we claim that $\beta_\sigma(g, g^{-1}) \neq 0$. Here we use again Lemma 5.0.1:

$$\begin{aligned} 1 &= \sigma * \sigma^{-1}(g \times 1, g^{-1} \times 1) = \sum_{x,y \in G} \sigma(g \times e_x, g^{-1} \times e_y)\sigma^{-1}(g^x \times 1, (g^{-1})^y \times 1) \\ &= \sum_{\substack{x,y \in G \\ 1=g^x (g^{-1})^y}} \sigma(g \times e_x, g^{-1} \times e_y)\sigma^{-1}(g^x \times 1, (g^{-1})^x \times 1) \\ &= \sum_{x,y \in G} \sigma(g \times e_x, g^{-1} \times e_y)\sigma^{-1}(g \times 1, (g^{-1}) \times 1) \\ &= \sigma(g \times 1, g^{-1} \times 1)\sigma^{-1}(g \times 1, (g^{-1}) \times 1) \end{aligned}$$

Using the cocycle condition we get:

$$\beta_\sigma(g, g^{-1}) = \beta_\sigma(g, g^{-1})\beta_\sigma(gg^{-1}, h) = \beta_\sigma(g^{-1}, h)\beta_\sigma(g, g^{-1}h)$$

Since $\beta_\sigma(g, g^{-1})$ is invertible we also have that $\sigma(g \times 1, h \times 1) = \beta_\sigma(g, h) \neq 0$ for all $g, h \in G$. Since σ is conjugation invariant so is β_σ .

If β is a 2-cocycle then it is obvious that σ_β is a 2-cocycle and it is lazy since it fulfills the conditions in Lemma 5.0.1. Also, the second part of Lemma 5.0.1 implies that a lazy coboundary in DG^* is mapped to a conjugation invariant coboundary on G . □

Lemma 5.0.4. There is a natural map $H_L^2(DG^*) \rightarrow H_L^2(k^G)$; $\sigma \mapsto \alpha_\sigma$ defined by

$$\alpha_\sigma(e_x, e_y) = \sigma(1 \times e_x, 1 \times e_y)$$

Further, α_σ restricted to the character group \widehat{G} gives a 2-cocycle in $H^2(\widehat{G}, k^\times)$.

Proof. Since the map ι in sequence (5.1) is Hopf it follows that $\alpha_\sigma \in Z^2(k^G)$. Comparing Example 1.3.5 with Lemma 5.0.1 its easy to see that $\alpha_\sigma \in Z_L^2(k^G)$. Also, elements $1 \times \chi$ for $\chi \in \widehat{G}$ are group-like elements in DG^* . Therefore the convolution invertibility of σ implies that α_σ is indeed a 2-cocycle in $Z^2(\widehat{G}, k^\times)$. \square

Definition 5.0.5. Let A, B be a finite dimensional bialgebras. The group of *bialgebra pairings* $P(A, B)$ consists of convolution invertible k -linear maps $\lambda : A \otimes B \rightarrow k$ such that for all $a, a' \in A, b, b' \in B$:

$$\begin{aligned} \lambda(aa', b) &= \lambda(a, b_{(1)})\lambda(a', b_{(2)}) & \lambda(a, bb') &= \lambda(a_{(1)}, b)\lambda(a_{(2)}, b') \\ \lambda(1, b) &= \epsilon_B(b) & \lambda(a, 1) &= \epsilon_A(a) \end{aligned}$$

Here we need the following special case:

Definition 5.0.6. The group of *bialgebra pairings* $P(kG, k^G)$ consists of convolution invertible k -linear maps $\lambda : kG \otimes k^G \rightarrow k$ such that for all $g, h \in G, f, f' \in k^G$:

$$\begin{aligned} \lambda(gh, f) &= \lambda(g, f_{(1)})\lambda(h, f_{(2)}) & \lambda(g, f * f') &= \lambda(g, f)\lambda(g, f') \\ \lambda(1, f) &= \epsilon_{k^G}(f) & \lambda(g, 1) &= \epsilon_{kG}(g) \end{aligned}$$

The subgroup of *lazy bialgebra pairings* $P_L(kG, k^G) \subset P(kG, k^G)$ consists of bialgebra pairings $\lambda : kG \otimes k^G \rightarrow k$ such that for all $g, x, t \in G$:

$$\lambda(g, e_x) = \lambda(tgt^{-1}, e_{txt^{-1}})$$

Bialgebra pairings $P(kG, k^G)$ are in bijection with the group $\text{Hom}(G, G)$, similarly lazy bialgebra pairings $P_L(kG, k^G)$ are in bijection with group homomorphisms $f \in \text{Hom}(G, G)$ that are conjugation invariant $f(g) = f(g^t)\forall g, t \in G$.

Lemma 5.0.7. There is a group homomorphism

$$\pi : H_L^2(DG^*) \rightarrow P_L(kG, k^G); \sigma \mapsto \lambda_\sigma$$

defined by

$$\lambda_\sigma(g, f) = \sigma^{-1}((1 \times f)_{(1)}, (g \times 1)_{(1)})\sigma((g \times 1)_{(2)}, (1 \times f)_{(2)}) \quad (5.7)$$

$$= \sum_{t, x, y \in G} f(xy)\sigma^{-1}(1 \times e_x, g \times e_t)\sigma(t^{-1}gt \times 1, 1 \times e_y) \quad (5.8)$$

Proof. Recall the coproduct in DG^* : $\Delta(g \times 1) = \sum_t (g \times e_t) \otimes (t^{-1}gt \times 1)$. We check that π is a well-defined group homomorphism. It is more convenient to be slightly more general here.

Let $a, b, c \in H$ for a Hopf algebra H and $\sigma \in Z_L^2(H)$. Consider the following map

$$\tau : H \times H \rightarrow k; \quad \tau(a, b) := \sigma^{-1}(b_{(1)}, a_{(1)})\sigma(a_{(2)}, b_{(2)})$$

Laziness $\sigma(b_{(1)}, c_{(1)})b_{(2)}c_{(2)} = b_{(1)}c_{(1)}\sigma(b_{(2)}, c_{(2)})$ implies that we can commute certain terms:

$$\sigma(b_{(1)}, c_{(1)})\sigma(a, b_{(2)}c_{(2)}) = \sigma(a, b_{(1)}c_{(1)})\sigma(b_{(2)}, c_{(2)}) \quad (5.9)$$

The 2-cocycle condition implies:

$$\sigma(c, ab) = \sigma^{-1}(a_{(1)}, b_{(1)})\sigma(c_{(1)}, a_{(2)})\sigma(c_{(2)}a_{(3)}, b_{(2)}) \quad (5.10)$$

$$\sigma^{-1}(ab, c) = \sigma^{-1}(a_{(1)}, b_{(1)}c_{(1)})\sigma^{-1}(b_{(2)}, c_{(2)})\sigma(a_{(2)}, b_{(3)}) \quad (5.11)$$

Now we can show that τ is multiplicative in the first argument:

$$\begin{aligned} & \tau(a_{(1)}, c_{(1)})\tau(b_{(1)}, c_{(2)})\tau^{-1}(a_{(2)}b_{(2)}, c_{(3)}) \\ &= \sigma^{-1}(c_{(1)}, a_{(1)})\sigma(a_{(2)}, c_{(2)})\sigma^{-1}(c_{(3)}, b_{(1)})\sigma(b_{(2)}, c_{(4)})\sigma^{-1}(a_{(3)}b_{(3)}, c_{(5)})\sigma(c_{(6)}, a_{(4)}b_{(4)}) \\ &\stackrel{(5.10)}{=} \sigma^{-1}(c_{(1)}, a_{(1)})\sigma(a_{(2)}, c_{(2)})\sigma^{-1}(c_{(3)}, b_{(1)})\sigma(b_{(2)}, c_{(4)}) \\ &\quad \sigma^{-1}(a_{(3)}, b_{(3)}c_{(5)})\sigma^{-1}(b_{(4)}, c_{(6)})\sigma(a_{(4)}, b_{(5)})\sigma^{-1}(a_{(5)}, b_{(6)})\sigma(c_{(7)}, a_{(6)})\sigma(c_{(8)}a_{(7)}, b_{(7)}) \\ &\stackrel{(5.9)}{=} \sigma^{-1}(c_{(1)}, a_{(1)})\sigma(a_{(2)}, c_{(2)})\sigma^{-1}(c_{(3)}, b_{(1)})\sigma^{-1}(a_{(3)}, b_{(2)}c_{(4)})\sigma(c_{(5)}a_{(4)}, b_{(3)})\sigma(c_{(6)}, a_{(5)}) \end{aligned}$$

If all coproduct terms of $a_{(1)}, a_{(2)}, \dots$ commute with all coproduct terms $c_{(1)}, c_{(2)}, \dots$ and similarly if all $b_{(1)}, b_{(2)}, \dots$ commute with all $c_{(1)}, c_{(2)}, \dots$, which will be the case for $H = DG^*$, we go on:

$$\begin{aligned} &= \sigma^{-1}(c_{(1)}, a_{(1)}) \underbrace{\sigma(a_{(2)}, c_{(2)})\sigma^{-1}(c_{(3)}, b_{(1)})\sigma^{-1}(a_{(3)}, c_{(4)}b_{(2)})\sigma(a_{(4)}c_{(5)}, b_{(3)})}_{\epsilon} \sigma^{-1}(c_{(6)}, a_{(5)}) \\ &= \epsilon(abc) \end{aligned}$$

Where we again used the cocycle condition in the middle. Note that $\tau(a, b) = \tau^{-1}(b, a)$ hence the above shown property of τ implies:

$$\tau(c, ab) = \tau(c_{(1)}, b)\tau(c_{(2)}, a)$$

Taking $H = DG^*$, $\sigma \in Z_L^2(DG^*)$, we get $\tau(g \times 1, 1 \times e_x) = \lambda_\sigma(g, e_x)$ for all $g, x \in G$ (compare with equation (5.7)). Note that the coproduct terms of $(1 \times e_x)$ and $(1 \times e_y)$ in DG^* stay in $1 \times k^G$ and therefore the coproduct terms of $(g \times 1)$ commute with all coproduct terms $(1 \times e_x)$ and $(1 \times e_y)$. Using this we show that λ_σ is multiplicative:

$$\begin{aligned}
\lambda_\sigma(g, e_x * e_y) &= \tau(g \times 1, (1 \times e_x)(1 \times e_y)) = \tau((g \times 1)_{(2)}, 1 \times e_x) \tau((g \times 1)_{(1)}, 1 \times e_y) \\
&= \sum_{t \in G} \tau(g \times e_t, 1 \times e_y) \tau(t^{-1}gt \times 1, 1 \times e_x) \\
&= \tau(g \times 1, 1 \times e_y) \tau(g \times 1, 1 \times e_x) \\
&= \lambda_\sigma(g, e_x) \lambda_\sigma(g, e_y)
\end{aligned}$$

Similarly:

$$\begin{aligned}
\lambda_\sigma(gh, e_x) &= \tau((g \times 1)(h \times 1), 1 \times e_x) = \sum_{x_1 x_2 = x} \tau(g \times 1, 1 \times e_{x_1}) \tau(h \times 1, 1 \times e_{x_2}) \\
&= \sum_{x_1 x_2 = x} \lambda_\sigma(g, e_{x_1}) \lambda_\sigma(h, e_{x_2})
\end{aligned}$$

The map also induces a well-defined map on cohomology, since for any $a, b \in H$

$$\begin{aligned}
(d\mu)^{-1}(b_{(1)}, a_{(1)}) d\mu(a_{(2)}, b_{(2)}) &= \mu(b_{(1)}a_{(1)}) \mu^{-1}(b_{(2)}) \mu^{-1}(a_{(2)}) \mu(a_{(3)}) \mu(b_{(3)}) \mu^{-1}(a_{(4)}b_{(4)}) \\
&= \mu(b_{(1)}a_{(1)}) \mu^{-1}(a_{(2)}b_{(2)})
\end{aligned}$$

and therefore $(d\mu)^{-1}(b_{(1)}, a_{(1)}) d\mu(a_{(2)}, b_{(2)}) = \epsilon(ab)$ if a, b commute. \square

Proposition 5.0.8.

- Let $P_c(kG, k^G) \subset P_L(kG, k^G)$ be the subgroup of central lazy bialgebra pairings. These are $\lambda \in P_L(kG, k^G)$ such that for $g \in G$: $\lambda(g, e_x) = 0$ if x not in $Z(G)$. Then there is a group homomorphism

$$P_c(kG, k^G) \rightarrow H_L^2(DG^*); \lambda \mapsto \sigma_\lambda = \lambda \circ (p \otimes s) \quad (5.12)$$

where p, s were defined in the splitting sequence (5.1). Note that $P_c(kG, k^G)$ is in bijection with $\text{Hom}(G, Z(G))$.

- Let $Z_c^2(k^G) \subset Z_L^2(k^G)$ be the subgroup of central lazy 2-cocycles. These are $\alpha \in Z_L^2(k^G)$: $\alpha(e_x, e_y) = 0$ if x or y not in $Z(G)$. We define $H_c^2(k^G)$ to be the quotient by central coboundaries $d\eta \in Z_c^2(k^G)$ for $\eta \in \text{Reg}^1(k^G)$ and denote by $B_c^2(k^G)$ the group of such coboundaries. Then there is a group homomorphism $H_c^2(k^G) \rightarrow H_L^2(DG^*); \alpha \mapsto \sigma_\alpha$ defined by

$$\sigma_\alpha(g \times e_x, h \times e_y) = \alpha(e_x, e_y) \epsilon(g) \epsilon(h)$$

Proof. The cocycle condition (equation (1.6)) for σ_λ reduces to

$$\lambda(x, e_y) \lambda(xy^{-1}zy, e_w) = \sum_{w_1 w_2 = w} \lambda(z, e_{w_1}) \lambda(x, e_y * e_{w_2})$$

for all $x, y, z, w \in G$. Using the properties of the pairing λ we check this equality:

$$\begin{aligned}
\lambda(x, e_y)\lambda(xy^{-1}zy, e_w) &= \sum_{w_1 w_2 = w} \lambda(x, e_y)\lambda(x, e_{w_1})\lambda(y^{-1}zy, e_{w_2}) \\
&= \sum_{w_1 w_2 = w} \lambda(x, e_y * e_{w_1})\lambda(y^{-1}zy, e_{w_2}) \\
&= \lambda(x, e_y)\lambda(y^{-1}zy, e_{y^{-1}w}) = \lambda(x, e_y)\lambda(z, e_{wy^{-1}}) \\
&= \sum_{w_1 w_2 = w} \lambda(z, e_{w_1})\lambda(x, e_y * e_{w_2})
\end{aligned}$$

The fact that $\lambda(g, e_x)$ is conjugation invariant and zero if x is not central ensures that σ_λ is lazy in $Z^2(DG^*)$ (see again Lemma 5.0.1). The fact that $\lambda \mapsto \sigma_\lambda$ is a homomorphism is straightforward to check. Similarly, since α is lazy on k^G this implies conjugation invariance and since α is central Lemma 5.0.1 implies that σ_α is lazy on DG^* . □

Conjecture 5.0.9. The group $H_L^2(DG^*)$ is generated by $H_c^2(k^G)$, $P_c(kG, k^G)$ and $H_{inv}^2(G, k^\times)$. Further, there is an exact factorization:

$$H_L^2(DG^*) = H_c^2(k^G)P_c(kG, k^G)H_{inv}^2(G, k^\times)$$

It is tempting to ask if every element in $H^2(DG^*)$ is a product of elements in $H^2(k^G)$, $P(kG, k^G)$ and $H^2(G, k^\times)$.

The following two Lemmas are needed for the proof of the main result, Theorem 6.6.1.

Lemma 5.0.10. A lazy 2-cocycle σ such that β_σ is cohomologically trivial in $H^2(G, k^\times)$, α_σ is cohomologically trivial in $Z^2(k^G)$ and $\lambda(g, e_x) = \epsilon(e_x)$ fulfills the property that it is cohomologically trivial in $H^2(DG^*)$, hence there is a chain $\zeta \in \text{Reg}_{aL}^1(DG^*)$ such that $\sigma = d\zeta$. There is no reason to assume that the coboundary ζ is lazy, hence σ is not necessarily cohomologically trivial in $H_L^2(DG^*)$.

Proof. Let $\beta_\sigma = d\nu$ and $\alpha_\sigma = d\eta$. Again, there is no reason to assume that these induce lazy coboundaries on DG^* using the maps defined above. However, we apply the 2-cocycle condition several times on σ by separating the kG and k^G parts:

$$\begin{aligned}
\sigma(g \times e_x, h \times e_y) &= \sigma((g \times 1)(1 \times e_x), h \times e_y) \\
&= \sum_{\substack{x_1 x_2 x_3 = x \\ y_1 y_2 = y}} \sigma^{-1}(g \times 1, 1 \times e_{x_1})\sigma(1 \times e_{x_2}, (h \times 1)(1 \times e_{y_1}))\sigma(g \times 1, (h \times 1)(1 \times e_{x_3} e_{y_2})) \\
&= \sum_{\substack{x_1 x_2 x_3 x_4 x_5 = x \\ y_1 y_2 y_3 y_4 y_5 = y}} \sigma^{-1}(g \times 1, 1 \times e_{x_1})\sigma^{-1}(1 \times e_{y_1}, h \times 1)\sigma(1 \times e_{x_2}, 1 \times e_{y_2}) \\
&\quad \sigma(1 \times e_{x_3} e_{y_3}, h \times 1)\sigma^{-1}(h \times 1, 1 \times e_{x_4} e_{y_4})\sigma(g \times 1, h \times 1)\sigma(gh \times 1, 1 \times e_{x_5} e_{y_5}) \\
&= \sum_{\substack{x_1 x_2 = x \\ y_1 y_2 = y}} \sigma^{-1}(g \times 1, 1 \times e_{x_1})\sigma^{-1}(1 \times e_{y_1}, h \times 1)d\nu(g, h)d\eta(e_{x_2}, e_{y_2})\sigma(gh \times 1, 1 \times e_{x_2} e_{y_2})
\end{aligned} \tag{5.13}$$

Now let $\mu(g \times e_x) := \sigma^{-1}(g \times 1, 1 \times e_x)$ and check that $\zeta := \mu * (\eta \otimes \nu)$ gives us the desired almost lazy coboundary:

$$\begin{aligned}
d\zeta(g \times e_x, h \times e_y) &= d(\mu * (\eta \otimes \nu))(g \times e_x, h \times e_y) \\
&= \sum_{\substack{x_1 x_2 = x \\ y_1 y_2 = y}} \mu * (\eta \otimes \nu)(g \times e_{x_1}) \mu * (\eta \otimes \nu)(h \times e_{y_1}) \mu * (\eta \otimes \nu)(g^{x_1} h^{y_1} \times e_{x_2} e_{y_2}) \\
&= \sum_{\substack{x_1 x_2 x_3 x_4 = x \\ y_1 y_2 y_3 y_4 = y}} \sigma^{-1}(g \times 1, 1 \times e_{x_1}) \nu(g) \eta(e_{x_2}) \sigma^{-1}(h \times 1, 1 \times e_{y_1}) \nu(h) \eta(e_{y_2}) \\
&\quad \sigma(g^{x_1 x_2} h^{y_1 y_2} \times 1, 1 \times e_{x_3} e_{y_3}) \nu(gh) \eta(e_{x_4} e_{y_4}) \\
&= \sum_{\substack{x_1 x_2 t = x \\ y_1 y_2 t = y}} \sigma^{-1}(g \times 1, 1 \times e_{x_1}) \sigma^{-1}(h \times 1, 1 \times e_{y_1}) d\nu(g, h) d\eta(e_{x_2}, e_{y_2}) \sigma(g^{xt^{-1}} h^{yt^{-1}} \times 1, 1 \times e_t) \\
&= \sum_{\substack{x_1 x_2 t = x \\ y_1 y_2 t = y}} \sigma^{-1}(g \times 1, 1 \times e_{x_1}) \sigma^{-1}(h \times 1, 1 \times e_{y_1}) d\nu(g, h) d\eta(e_{x_2}, e_{y_2}) \sigma(g^x h^y \times 1, 1 \times e_t)
\end{aligned} \tag{5.14}$$

Here we used the lazy property of σ as in Lemma 5.0.1. We also used that lazy implies equations of the form $\sigma(a_1, b_1) \eta(a_2 b_2) = \eta(a_1 b_1) \sigma(a_2 b_2)$ and we used that the pairing λ_σ is trivial, which implies that $\sigma(g \times 1, 1 \times e_x) = \sigma(1 \times e_x, g \times 1)$. Both (5.14) and (5.13) are equal, which proves the statement. Note that this equality in particular implies that ζ is almost lazy. We do not see a reason to assume that ζ is lazy. \square

At the end we want to present an interesting special case of the above decomposition. Let us call a 2-cocycle $\sigma \in Z_L^2(DG^*)$ *symmetric* if for all $g, t, h, s \in G$:

$$\sigma(g \times e_t, h^g \times e_s) = \sigma(h \times e_{gs(g^{-1})t}, g \times e_t) \tag{5.15}$$

This is motivated by the study of lazy *braided* monoidal autoequivalences of $DG\text{-mod}$ in Chapter 6, where such a 2-cocycle defines an autoequivalence of $DG\text{-mod}$ that is the identity on objects and morphisms, but has a non-trivial monoidal structure determined by σ .

Lemma 5.0.11. A symmetric lazy 2-cocycle on DG^* is cohomologically equivalent in $H^2(DG^*)$ (but not necessarily in $H_L^2(DG^*)$) to a lazy 2-cocycle in the image of map $Z_{inv}^2(G) \rightarrow Z_L^2(DG^*)$.

Proof. From the symmetry condition follows that $\beta_\sigma(g, h^g) = \beta_\sigma(h, g)$, that α_σ is symmetric and that $\lambda_\sigma = 1$. Multiplying σ from the left by $\sigma_{\beta_\sigma}^{-1}$ gives a 2-cocycle σ' that fulfills all the properties in Lemma 5.0.10, in particular $\beta_{\sigma'}$ cohomologically trivial in $H^2(G, k^\times)$, hence σ' is cohomologically trivial in $H^2(DG^*)$. \square

Chapter 6

Decomposition of $\text{Aut}_{br,L}(DG\text{-mod})$

Let us recall from Definition 1.1.2 that we denote by $\underline{\text{Aut}}_{mon}(H\text{-mod})$ (the objects of) the functor category of monoidal autoequivalences, by $\text{Aut}_{mon}(H\text{-mod})$ the group of isomorphism classes of monoidal autoequivalences and similarly for the braided versions $\underline{\text{Aut}}_{br}(H\text{-mod})$, $\text{Aut}_{br}(H\text{-mod})$. In the Preliminaries we have seen, that to every pair $(\phi, \sigma) \in \text{Aut}_{Hopf}(H) \times Z_L^2(H^*)$ we can assign an H^* -Bigalois object $\phi^*(\sigma H^*)$ and that this Bigalois object corresponds under the equivalence in Proposition 1.2.10 to the functor $(F_\phi, J^\sigma) \in \underline{\text{Aut}}_{mon}(H\text{-mod})$.

In this Chapter we consider $H = DG$ for a finite group G . We want to slightly modify the maps $(\phi, \sigma) \mapsto (F_\phi, J^\sigma)$ in order to match the formulas in Section 2.3 for the action of F_ϕ on simple modules. There is a group anti-automorphism (called flip in Def. 3.1 of [Keil15]):

$$\dagger : \text{Aut}_{Hopf}(DG) \xrightarrow{\cong} \text{Aut}_{Hopf}(DG); \begin{pmatrix} u & b \\ a & v \end{pmatrix} \mapsto \begin{pmatrix} v^* & b^* \\ a^* & u^* \end{pmatrix} \quad (6.1)$$

where $v^* : k^G \rightarrow k^G$ is the dual of $v : kG \rightarrow kG$, $u^* : kG \rightarrow kG$ is the dual of $u : kG \rightarrow kG$, and similarly $b^* : kG \rightarrow k^G$ and $a^* : k^G \rightarrow kG$. We precompose with this anti-automorphism when identifying a pair (ϕ, σ) with a monoidal functor (F_ϕ, J^σ) . Let us state this explicitly in the following definition:

Definition 6.0.1. We have the following map:

$$\begin{aligned} \Psi : \text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*) &\rightarrow \underline{\text{Aut}}_{mon}(DG\text{-mod}) \\ (\phi, \sigma) &\mapsto (F_\phi, J^\sigma) \end{aligned} \quad (6.2)$$

where F_ϕ assigns a left DG -module (M, ρ_L) to the left DG -module

$$(M, \rho_L \circ (\phi^\dagger \otimes_k \text{id}))$$

We denote this DG -module simply by ${}_\phi M$. The monoidal structure J^σ on the functor $F_\phi : DG\text{-mod} \rightarrow DG\text{-mod}$ is given by

$$\begin{aligned} J_{M,N}^\sigma : {}_\phi M \otimes_k {}_\phi N &\xrightarrow{\cong} {}_\phi(M \otimes_k N) \\ m \otimes n &\mapsto \sigma_1.m \otimes \sigma_2.m \end{aligned}$$

where we view the 2-cocycle $\sigma \in Z_L^2(DG^*)$ as $\sigma = \sigma_1 \otimes \sigma_2 \in DG \otimes_k DG$ leaving out the sum. This should not be confused with the Sweedler notation for a coproduct or a coaction.

Before going to the precise statements, let us motivate our definitions and our approach.

We are going to construct certain subsets of $\text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*)$ that will be denoted by $\mathcal{V}_L, \mathcal{E}_L, \mathcal{B}_L, \mathcal{R}_L$. Those subsets will have the following properties: The natural projection $\text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*) \rightarrow \text{Aut}_{Hopf}(DG)$ maps them to V, E, B, R from Chapter 4 and Ψ maps them to *braided* monoidal autoequivalences. Even though $\mathcal{V}_L, \mathcal{E}_L, \mathcal{B}_L$ are subgroups, one should rather consider of them as sets consisting of functors (or functor categories), because we do not identify functors that are monoidal equivalent yet. The sets $\mathcal{V}_L, \mathcal{E}_L, \mathcal{B}_L$ are too large for concrete calculations, essentially because the set of 2-cocycles is too large. Thus, we need to take quotients. First, two different pairs in $\text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*)$ give functors that are monoidal equivalent if they differ by a pair consisting of an *inner* Hopf automorphism (conjugation by a group-like element) and an exact 2-cocycle ($d\eta \in Z_L^2(DG^*)$, where η is lazy). This leads us to consider the quotient $\text{Out}_{Hopf}(DG) \times H_L^2(DG^*)$. Second, an invertible object $h \in DG$ that is not necessary group-like, produces an *internal* Hopf automorphism of DG by conjugation. It also produces an *almost lazy* exact 2-cocycle $d(h^*) \in Z_L^2(DG^*)$, hence the map $h^* : DG^* \rightarrow k$ is not lazy but $d(h^*)$ is lazy (recall Definition 1.3.3 and Corollary 1.3.11). Such pairs have the property that $\Psi(h^{-1} \cdot h, d(h^*))$ is monoidal equivalent to (id, J^{triv}) (see Lemma 1.3.9 and sequence 1.10). Identifying pairs that differ by an invertible (modulo group-like) object $h \in DG$ defines the group $\text{Aut}_{mon,L}(DG\text{-mod})$. Let us summarize this in the following definition:

Definition 6.0.2.

- We define the set $\underline{\text{Aut}}_{mon,L}(DG\text{-mod}) := \text{im}(\Psi) \simeq \text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*)$ to be the set of lazy monoidal autoequivalences.
- We define the group $\widetilde{\text{Aut}}_{mon,L}(DG\text{-mod}) := \text{Out}_{Hopf}(DG) \times H_L^2(DG^*)$
- We define $\text{Aut}_{mon,L}(DG\text{-mod})$ to be the image of

$$\widetilde{\text{Aut}}_{mon,L}(DG\text{-mod}) \rightarrow \text{Aut}_{mon}(DG\text{-mod})$$

or equivalently $\text{Out}_{Hopf}(DG) \times H_L^2(DG^*)$ modulo $\text{Int}(DG)/\text{Inn}(DG)$. See sequence 1.10.

- We define the braided versions as follows:

$$\underline{\text{Aut}}_{br,L}(DG\text{-mod}) := \underline{\text{Aut}}_{mon,L}(DG\text{-mod}) \cap \underline{\text{Aut}}_{br}(DG\text{-mod})$$

Further, let $\widetilde{\text{Aut}}_{br,L}(DG\text{-mod})$ be the image of

$$\underline{\text{Aut}}_{br,L}(DG\text{-mod}) \rightarrow \widetilde{\text{Aut}}_{mon,L}(DG\text{-mod})$$

and $\text{Aut}_{br,L}(DG\text{-mod})$ to be the image of

$$\widetilde{\text{Aut}}_{br,L}(DG\text{-mod}) \rightarrow \text{Aut}_{br}(DG\text{-mod})$$

- For $\mathcal{U} \subset \underline{\text{Aut}}_{br,L}(DG\text{-mod})$ we denote the respective images by $\widetilde{\mathcal{U}} \subset \widetilde{\text{Aut}}_{br,L}(DG\text{-mod})$ and $\mathcal{U} \subset \text{Aut}_{br,L}(DG\text{-mod})$.

$$\begin{array}{ccccc}
 & & \underline{\text{Aut}}_{mon} & \xrightarrow{\quad\quad\quad} & \text{Aut}_{mon} \\
 & \nearrow & \uparrow & & \nearrow \\
 \underline{\text{Aut}}_{br} & \xrightarrow{\quad\quad\quad} & \text{Aut}_{br} & \xrightarrow{\quad\quad\quad} & \text{Aut}_{br} \\
 & \searrow & \downarrow & & \searrow \\
 & & \underline{\text{Aut}}_{mon,L} & \xrightarrow{\quad\quad\quad} & \widetilde{\text{Aut}}_{mon,L} & \xrightarrow{\quad\quad\quad} & \text{Aut}_{mon,L} \\
 & & & & \uparrow & & \uparrow \\
 \underline{\text{Aut}}_{br,L} & \xrightarrow{\quad\quad\quad} & \widetilde{\text{Aut}}_{br,L} & \xrightarrow{\quad\quad\quad} & \text{Aut}_{br,L} & \xrightarrow{\quad\quad\quad} & \text{Aut}_{br,L} \\
 & & \nearrow & & \nearrow & & \nearrow \\
 & & \underline{\text{Aut}}_{mon,L} & \xrightarrow{\quad\quad\quad} & \widetilde{\text{Aut}}_{mon,L} & \xrightarrow{\quad\quad\quad} & \text{Aut}_{mon,L}
 \end{array}$$

6.1 General Considerations

Before constructing the subgroups mentioned above it is convenient for later to show some general properties of pairs $(\phi, \sigma) \in \text{Aut}_{Hopf}(DG) \times \mathbb{Z}_L^2(DG^*)$ that give us monoidal and braided monoidal functors (F_ϕ, J^σ) . The following Lemma follows essentially from Theorem 9.4 [Keil15] but we want to state and prove this in our notation.

Lemma 6.1.1. Let $\phi \in \text{Aut}_{Hopf}(DG)$ and recall from Proposition 4.1.1 that ϕ corresponds uniquely to a matrix

$$\begin{pmatrix} u & b \\ a & v \end{pmatrix}$$

where $\phi(f \times g) = u(f_{(1)})b(g) \times a(f_{(2)})v(g)$ for $f \in k^G$ and $g \in G$. Then the functor F_ϕ maps a DG -module M to the DG -module ${}_\phi M$ with the action:

$$(f \times g) \cdot {}_\phi m := \phi^\dagger(f \times g) \cdot m := (v^*(f_{(1)})b^*(g) \times a^*(f_{(2)})u^*(g)) \cdot m$$

F_ϕ has the following explicit form on simple DG -modules:

$$F_\phi(\mathcal{O}_g^\rho) = \mathcal{O}_{a(\rho')v(g)}^{(\rho \circ u^*)b(g)}$$

where we denote by $\rho' : \mathbb{Z}(G) \rightarrow k^\times$ the one-dimensional representation such that the any central element $z \in \mathbb{Z}(G)$ act in ρ by multiplication with the scalar $\rho'(z)$. In particular $\rho|_{\mathbb{Z}(G)} = \dim(\rho) \cdot \rho'$.

Proof. We first check that the G -coaction resp. k^G -action on $F_\phi(\mathcal{O}_g^\rho)$ is as asserted:

$$\begin{aligned}
\phi^*(e_x \times 1).(1 \otimes w) &= \sum_{kl=x} (v^*(e_k) \times a^*(e_l)).(1 \otimes w) \\
&= \sum_{kl=x, k', l'} v^*(e_k)(k')e_{l'}(a^*(e_l))(e_{k'} \times l').(1 \otimes w) \\
&= \sum_{kl=x, k', l'} e_k(v(k'))e_{l'}(a(e_{l'}))(e_{k'} \times l').(1 \otimes w) \\
&= \sum_{kl=x, l'} e_k(v(l'gl'^{-1}))e_{l'}(a(e_{l'}))l' \otimes w \\
&= \sum_{kl=x, l' \in Z(G)} e_k(v(g))e_{l'}(a(e_{l'}))\hat{\rho}(l')1 \otimes w \\
&= \sum_{v(g)l=x} \delta_{l, a(\hat{\rho})} 1 \otimes w \\
&= \delta_{x, v(g)a(\hat{\rho})} 1 \otimes w
\end{aligned}$$

For $y \in \text{Cent}(a(\rho')v(g))$ we check that the action is as asserted. We first collect the following facts:

- By definition $a(\rho') \in Z(G)$, so $y \in \text{Cent}(v(g))$.
- By Lemma 4.1.3 we have that $u^* \circ v$ is a normal group homomorphism, hence
$$(u^* \circ v)(u^*(y)gu^*(y)^{-1}) = u^*(y) \cdot u^*(v(g)) \cdot u^*(y)^{-1} = u^*(yv(g)y^{-1}) = u^*(v(g))$$
Therefore $[u^*(y), g] = u^*(y)gu^*(y)^{-1}g^{-1} \in \ker(u^* \circ v)$.
- Moreover by Lemma 4.1.3 we have that $\ker(u^*)$ is in a direct abelian factor C of G , hence the commutator $v([u^*(y), g]) = [v(u^*(y)), v(g)] \in \ker(u^*)$ is already equal to 1 (for we may consider the projection p_C of the commutator being equal to 1 but the projection is the identity on C). Thus $[u^*(y), g] \in \ker(v)$. By the same Lemma, $\ker(v)$ is in a direct abelian factor of G hence again the commutator $[u^*(y), g]$ is already equal 1. This finally shows $u^*(y) \in \text{Cent}(g)$.

Now we calculate using $u^*(y) \in \text{Cent}(g)$ and $b(hgh^{-1}) = b(g)$:

$$\begin{aligned}
\phi^*(1 \times y).(1 \otimes v) &= (b^*(y) \times u^*(y)).(1 \otimes v) \\
&= \sum_k b^*(y)(k)(e_k \times u^*(y)).(1 \otimes v) \\
&= \sum_k b(k)(y)e_k(u^*(y)gu^*(y)^{-1})(u^*(y) \otimes v) \\
&= b(u^*(y)gu^*(y)^{-1})(y)(u^*(y) \otimes v) \\
&= b(g)(y)(1 \otimes \rho(u^*(y))(v)) \\
&= 1 \otimes [(\rho \circ u^*)b(g)](y)v
\end{aligned}$$

Since these two actions characterize the simple DG -module we have verified the claim. □

Note that in the abelian case $G = A$ all simple objects are 1-dimensional \mathcal{O}_a^ρ for $a \in A$ and $\rho \in \hat{A}$. Then

$$F_\phi(\mathcal{O}_a^\rho) = \mathcal{O}_{a(\rho)v(a)}^{u(\rho)b(a)}$$

which fits together with the action on simple modules as in Section 2.3.

The functor $(F_\phi, J^\sigma) \in \underline{\text{Aut}}_{\text{mon}, L}(DG\text{-mod})$ is braided if and only if the following diagram commutes:

$$\begin{array}{ccc} {}_\phi M \otimes {}_\phi N & \xrightarrow{F_\phi(J_{M,N}^\sigma)} & {}_\phi(M \otimes N) \\ \downarrow c_{\phi M, \phi N} & & \downarrow F_\phi(c_{M,N}) \\ {}_\phi N \otimes {}_\phi M & \xrightarrow{F_\phi(J_{N,M}^\sigma)} & {}_\phi(N \otimes M) \end{array}$$

for all $M, N \in DG\text{-mod}$. This is equivalent to the fact that for all DG -modules M, N

$$R_2 \cdot \sigma_2 \cdot n \otimes R_1 \cdot \sigma_1 \cdot m = \sigma_1 \cdot \phi^*(R_2) \cdot n \otimes \sigma_2 \cdot \phi^*(R_1) \cdot m \quad (6.3)$$

holds for all $m \in M$ and $n \in N$. Where $R = R_1 \otimes R_2 = \sum_{x \in G} (e_x \times 1) \otimes (1 \times x)$ is the R -matrix of DG and c the braiding in $DG\text{-mod}$. As above, we identify $\sigma \in Z_L^2(DG^*)$ with an element $\sigma = \sigma_1 \otimes \sigma_2 \in DG \otimes DG$.

In Chapter 5 we have defined three subgroups of $Z_L^2(DG^*)$:

- $Z_{inv}^2(G, k^\times)$: group 2-cocycles $\beta \in Z^2(G, k^\times)$ such that $\beta(g, h) = \beta(g^t, h^t) \forall t, g \in G$ that are trivially extended to 2-cocycles on DG^* .
- $Z_c^2(k^G)$: 2-cocycles $\sigma \in Z^2(k^G)$ such that $\alpha(e_g, e_h) = 0$ if g or h not in $Z(G)$, extended trivially to DG^* .
- $P_c(k^G, k^G) \simeq \text{Hom}(G, Z(G))$: central bialgebra pairings $\lambda : k^G \times k^G \rightarrow k$ resp. group homomorphisms $G \rightarrow Z(G)$. These give 2-cocycles on DG^* as follows: $\sigma_\lambda(g \times e_x, h \times e_y) = \lambda(g, e_y)\epsilon(e_x)$.

On the other hand, $\beta_\sigma(g, h) = \sigma(g \times 1, h \times 1)$ defines a 2-cocycle in $Z_{inv}^2(G, k^\times)$. Further, for $\chi, \rho \in \hat{G}$ the map $\alpha_\sigma(\chi, \rho) := \sigma(1 \times \chi, 1 \times \rho)$ defines a 2-cocycle in $Z^2(\hat{G}, k^\times)$. Also, $\lambda_\sigma(g, f) := \sigma^{-1}((g \times 1)_1, 1 \times f_1)\sigma(1 \times f_2, (h \times 1)_2)$ defines a lazy bialgebra pairing in $P_L(k^G, k^G) \simeq \text{Hom}(G, G)$.

Lemma 6.1.2. Let $\phi \in \text{Aut}_{\text{Hopf}}(DG)$ be given as above by

$$\phi(f \times g) = u(f_1)b(g) \times a(f_2)v(g)$$

and $\sigma \in Z_L^2(DG^*)$ such that (F_ϕ, J^σ) is braided then the following equations have to hold for all $\rho, \chi \in \hat{G}$, $g, h \in G$:

$$\beta_\sigma(g, g^{-1}hg) = \beta_\sigma(h, g)b(h)(v(g)) \quad (6.4)$$

$$\alpha_\sigma(\rho, \chi) = \alpha_\sigma(\chi, \rho)u(\chi)(a(\rho)) \quad (6.5)$$

$$\lambda_\sigma(h, \chi) = b(h)(a(\chi)) \quad (6.6)$$

$$\rho(g) = u(\rho)[v(g)]b(g)[a(\rho)] \quad (6.7)$$

Proof. Evaluating equation (6.3) we get

$$\begin{aligned} & \sum_{g,t,h,d,k \in G} \sigma(g \times e_t, h \times e_d)(1 \times k).(e_h \times d).n \otimes (e_k \times 1).(e_g \times t).m \\ &= \sum_{g,t,h,d,k \in G} \sigma(g \times e_t, h \times e_d)(e_g \times t).\phi^*(1 \times k).n \otimes (e_h \times d).\phi^*(e_k \times 1).m \end{aligned}$$

The left hand side is equal to

$$\begin{aligned} & \sum_{g,t,h,d \in G} \sigma(g \times e_t, h \times e_d)(e_{ghg^{-1}} \times gd).n \otimes (e_g \times t).m \\ &= \sum_{g,t,h,d \in G} \sigma(g \times e_t, g^{-1}hg \times e_{g^{-1}d})(e_h \times d).n \otimes (e_g \times t).m \end{aligned}$$

and the right hand side is equal to

$$\begin{aligned} & \sum_{g,k,t,h,d,k_1,k_2=k} \sigma(g \times e_t, h \times e_d)(e_g \times t).(b^*(k) \times u^*(k)).n \otimes (e_h \times d).(v^*(e_{k_1}) \times a^*(e_{k_2})).m \\ &= \sum_{g,t,h,d,w,y,z,x} \sigma(g \times e_t, h \times e_d)a_x^w b(y)(v(z)w)(e_g \times t).(e_y \times u^*(v(z)w)).n \otimes (e_h \times d).(e_z \times x).m \\ &= \sum_{g,t,h,d,w,y,z,x} \sigma(g \times e_t, h \times e_d)a_x^w b(y)(v(z)w)(\delta_{g,tyt^{-1}}e_g \times tu^*(v(z)w)).n \otimes (\delta_{h,dzd^{-1}}e_h \times dx).m \\ &= \sum_{t,d,y,z,x,w} \sigma(y \times e_t, z \times e_d)b(y)(v(d^{-1}zd)w)a_x^w(e_y \times tu^*(v(d^{-1}zd)w)).n \otimes (e_z \times dx).m \\ &= \sum_{t,d,y,h,x,w} \sigma(h \times e_d, g \times e_t)b(h)(v(t^{-1}gt)w)a_x^w(e_h \times du^*(v(t^{-1}gt)w)).n \otimes (e_g \times tx).m \\ &= \sum_{\substack{t,h,g,d,x,w,d' \\ d=d'u^* \circ v(xt^{-1}gtx^{-1})u^*(w)}} \sigma(h \times e_{d'}, g \times e_{tx^{-1}})b(h)(v(g)w)a_x^w(e_h \times d).n \otimes (e_g \times t).m \end{aligned}$$

Here we have used several times that the homomorphism a is supported on $Z(G)$ and that b maps G to the character group \hat{G} which is abelian. We now that the above equality of the right and left hand side have to hold in particular for the regular DG -module and the elements $m = n = 1$. This implies:

$$\sigma(g \times e_t, h^g \times e_{g^{-1}d}) = \sum_{\substack{x,w,d' \\ d=d'u^*(w)(u^* \circ v)(g^t)}} \sigma(h \times e_{d'}, g \times e_{tx^{-1}})b(h)(v(g)w)a_x^w \quad (6.8)$$

for all $g, h, d, t \in G$ where $a_x^w = e_w(a(e_x))$. On the other hand, if equation (6.8) holds, then also the right and left hand side above are equal. Let us set $g = 1$, sum over all d , multiply with $\chi(t)$ for $\chi \in \hat{G}$ and sum over t in (6.8):

$$\begin{aligned} \sigma(1 \times \chi, h \times 1) &= \sum_{x,w,t} \chi(t)\sigma(h \times 1, 1 \times e_{tx^{-1}})b(h)(w)e_w(a(e_x)) \\ &= \sum_{x,t} \chi(t)\chi(x)\sigma(h \times 1, 1 \times e_t)b(h)((a(e_x))) \\ &= \sigma(h \times 1, 1 \times \chi)b(h)(a(\chi)) \end{aligned}$$

applying the convolution with σ^{-1} on both sides leads to equation (6.6). Further, we multiply both sides of equation (6.8) with $\rho(t), \chi(d)$ for some $\chi, \rho \in \widehat{G}$ and sum over all $t, d \in G$:

$$\sigma(g \times \rho, h^g \times \chi)\chi(g) = \sigma(h \times \chi, g \times \rho)\chi(a(\rho))\chi(u^* \circ v(g))b(h)(v(g)a(\rho))$$

Setting $\chi = 1 = \rho$ gives equation (6.4) and setting $g = 1 = h$ gives equation (6.5). On the other hand setting $g = h$ and $\rho = \chi$ and using equations (6.4),(6.5) we have:

$$\sigma(g \times \rho, g \times \rho)\rho(g) = \sigma(g \times \rho, g \times \rho)u(\rho)(v(g))b(g)(a(\rho))$$

This almost implies the last equation (6.7) but it is not yet clear that $\sigma(g \times \rho, g \times \rho)$ is never zero, since elements of the form $g \times \rho$ are not group-like in DG^* . However, we can argue as follows: Apply the 2-cocycle condition several times

$$\begin{aligned} \sigma(g \times e_x, h \times e_y) &= \sigma((g \times 1)(1 \times e_x), h \times e_y) \\ &= \sum_{\substack{x_1 x_2 x_3 = x \\ y_1 y_2 = y}} \sigma^{-1}(g \times 1, 1 \times e_{x_1})\sigma(1 \times e_{x_2}, (h \times 1)(1 \times e_{y_1}))\sigma(g \times 1, (h \times 1)(1 \times e_{x_3} e_{y_2})) \\ &= \sum_{\substack{x_1 x_2 x_3 x_4 x_5 = x \\ y_1 y_2 y_3 y_4 y_5 = y}} \sigma^{-1}(g \times 1, 1 \times e_{x_1})\sigma^{-1}(1 \times e_{y_1}, h \times 1)\sigma(1 \times e_{x_2}, 1 \times e_{y_2}) \\ &\quad \sigma(1 \times e_{x_3} e_{y_3}, h \times 1)\sigma^{-1}(h \times 1, 1 \times e_{x_4} e_{y_4})\sigma(g \times 1, h \times 1)\sigma(gh \times 1, 1 \times e_{x_5} e_{y_5}) \end{aligned} \tag{6.9}$$

which on characters gives:

$$\begin{aligned} \sigma(g \times \chi, h \times \rho) &= \sigma^{-1}(g \times 1, 1 \times \chi)\alpha_\sigma^{-1}(1 \times \rho, h \times 1)\alpha_\sigma(\chi, \rho)\lambda_\sigma(h, \chi\rho) \\ &\quad \cdot \beta_\sigma(g, h)\sigma(gh \times 1, 1 \times \chi\rho) \end{aligned}$$

since $\beta_\sigma \in Z^2(G, k^\times)$ and $\alpha_\sigma \in Z^2(\widehat{G}, k^\times)$ the only thing left is:

$$\begin{aligned} 1 &= (\sigma^{-1} * \sigma)(g \times 1, 1 \times \chi) = \sum_{\substack{t \in G \\ g^t = g}} \sigma^{-1}(g \times e_t, 1 \times \chi)\sigma(g^t \times 1, 1 \times \chi) \\ &= \sigma^{-1}(g \times 1, 1 \times \chi)\sigma(g \times 1, 1 \times \chi) \end{aligned}$$

Hence elements of the form $\sigma(g \times 1, 1 \times \chi)$ and $\sigma(1 \times \chi, g \times 1)$ are also non zero and it follows that $\sigma(g \times \rho, g \times \rho)$ is also never zero which proves equation (6.7). \square

We are going to use these equations in order to proof the Theorem 6.6.1. Also compare these equations with: (2.18),(2.19),(2.20) and (2.21).

6.2 Automorphism Symmetries

We have seen in Definition 4.1.4 that a group automorphism $v \in \text{Aut}(G)$ induces a Hopf automorphism in $V \subset \text{Aut}_{\text{Hopf}}(DG)$. We now show that automorphisms of G also naturally induce braided autoequivalences of $DG\text{-mod}$.

Proposition 6.2.1.

- (i) Consider the subgroup $\mathcal{V}_L := V \times \{1\}$ of $\text{Aut}_{Hopf}(DG) \times \mathbb{Z}_L^2(DG^*)$. For an element $(v, 1) \in \mathcal{V}_L$ the corresponding monoidal functor (F_v, J^{triv}) with trivial monoidal structure is given on simple objects by

$$F_v(\mathcal{O}_g^\rho) = \mathcal{O}_{v(g)}^{(v^{-1})^*(\rho)}$$

- (ii) Every (F_v, J^{triv}) is braided.

- (iii) Let $\tilde{\mathcal{V}}_L$ be the image of \mathcal{V}_L in $\text{Out}_{Hopf}(DG) \times \mathbb{H}_L^2(DG^*)$, then we have

$$\tilde{\mathcal{V}}_L \cong \text{Out}(G)$$

Proof. (i),(iii) Obvious from the above and Lemma 6.1.1.

(ii) Consider again equation (6.8) in the proof of Lemma 6.1.2. An element in $\text{Aut}_{Hopf}(DG) \times \mathbb{Z}_L^2(DG^*)$ is braided if and only if equation (6.8) is satisfied. For an element $(v, 1)$ it is easy to check that its true. (iv) The intersection of \mathcal{V}_L with the kernel $\text{Inn}(G) \times \mathbb{B}_L^2(DG^*)$ is clearly $\text{Inn}(G)$. \square

Example 6.2.2. The *extraspecial* p -group p_+^{2n+1} is a group of order p^{2n+1} generated by elements x_i, y_i for $i \in \{1, 2, \dots, n\}$ and the following relations. In particular $2_+^{2+1} = \mathbb{D}_4$.

$$x_i^p = y_i^p = 1 \quad [x_i, x_j] = [y_i, y_j] = [x_i, y_j] = 1, \text{ for } i \neq j \quad [x_i, y_i] = z \in Z(p_+^{2n+1})$$

Then the inner automorphism group is $\text{Inn}(G) \cong \mathbb{Z}_p^{2n}$ and the automorphism group is $\text{Out}(G) = \mathbb{Z}_{p-1} \times \text{Sp}_{2n}(\mathbb{F}_p)$ for $p \neq 2$ resp. $\text{Out}(G) = \text{SO}_{2n}(\mathbb{F}_2)$ for $p = 2$, see [Win72].

6.3 B-Symmetries

Now we want to characterize subgroups of $\text{Aut}_{br}(DG\text{-mod})$ corresponding to the lazy induction $\text{Aut}_{mon}(\text{Vect}_G) \rightarrow \text{Aut}_{br,L}(DG\text{-mod})$. One fact we need to understand for this is what trivial braided autoequivalences $(1, \beta)$ coming from Vect_G look like. If the group is abelian, then β has to be cohomologically trivial, which implies that the characterization of such elements is easy. On the other hand, if G is not abelian there are non-trivial cocycles β leading to non-trivial braided monoidal functors. For this we need the following:

Definition 6.3.1. Let G be a finite group. A cohomology class $[\beta] \in \mathbb{H}^2(G, k^\times)$ is called *distinguished* if one of the following equivalent conditions is fulfilled [Higgs87]:

- The twisted group ring $k_\beta G$ has the same number of irreducible representations as kG . Note that $k_\beta G$ for $[\beta] \neq 1$ has no 1-dimensional representations.

- The centers are of equal dimension $\dim Z(k_\beta G) = \dim Z(kG)$.
- All conjugacy classes $[x] \subset G$ are β -regular, i.e. for all $g \in \text{Cent}(x)$ we have $\beta(g, x) = \beta(x, g)$.

The conditions are clearly independent of the representing 2-cocycle β and the set of distinguished cohomology classes forms a subgroup $H_{dist}^2(G)$.

In fact, nontrivial distinguished classes are quite rare and we give in Example 6.3.5 a non-abelian group with p^9 elements which admits such a class.

In the following Proposition we construct \mathcal{B}_L which should be seen as a subset of the functors $\underline{\text{Aut}}_{br}(DG\text{-mod})$. This is of course a large set and we need to identify certain functors. For this reason, as described in the introduction, we consider the quotient $\tilde{\mathcal{B}}_L$, where we identify pairs that differ by inner Hopf automorphism and exact cocycles. The main property, as shown below, is that up to certain elements this quotient is isomorphic to the group of alternating homomorphisms $G_{ab} \rightarrow G_{ab}$. In order to get $\mathcal{B}_L \subset \underline{\text{Aut}}_{br}(DG\text{-mod})$ we need to consider the quotient of $\tilde{\mathcal{B}}_L$ by the kernel of $\tilde{\mathcal{B}}_L \rightarrow \underline{\text{Aut}}_{br}(DG\text{-mod})$.

Proposition 6.3.2.

- (i) The group $B \times Z_{inv}^2(G)$ is a subgroup of $\text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*)$. An element (b, β) corresponds to the monoidal functor (F_b, J^β) given by $F_b(\mathcal{O}_g^\rho) = \mathcal{O}_g^{\rho * b(g)}$ with monoidal structure

$$\begin{aligned} \mathcal{O}_g^{\rho * b(g)} \otimes \mathcal{O}_h^{\chi * b(h)} &\rightarrow F_b(\mathcal{O}_g^\rho \otimes \mathcal{O}_h^\chi) \\ (s_m \otimes v) \otimes (r_n \otimes w) &\mapsto \beta(g_m, h_n)(s_m \otimes v) \otimes (r_n \otimes w) \end{aligned}$$

where $\{s_m\}, \{r_n\} \subset G$ are choices of representatives of $G/\text{Cent}(g)$ and $G/\text{Cent}(h)$ respectively and where $g_m = s_m g s_m^{-1}, h_n = r_n h r_n^{-1}$.

- (ii) The subgroup \mathcal{B}_L of $B \times Z_{inv}^2(G)$ defined by

$$\mathcal{B}_L := \{(b, \beta) \in B \times Z_{inv}^2(G) \mid b(g)(h) = \frac{\beta(h, g)}{\beta(g, h)} \quad \forall g, h \in G\}$$

consists of all elements $(b, \beta) \in B \times Z_{inv}^2(G)$ such that (F_b, J^β) is a braided autoequivalence.

- (iii) Let $B_{alt} \cong \hat{G}_{ab} \wedge_{\mathbb{Z}} \hat{G}_{ab}$ be the subgroup of alternating homomorphisms of B , i.e. $b \in \text{Hom}(G_{ab}, \hat{G}_{ab})$ with $b(g)(h) = b(h)(g)^{-1}$. Then the following group homomorphism is well-defined and surjective:

$$\mathcal{B}_L \rightarrow B_{alt}; \quad (b, \beta) \mapsto b$$

- (iv) Let $\tilde{\mathcal{B}}_L$ be the image of \mathcal{B}_L in $\text{Out}_{Hopf}(DG) \times H_L^2(DG^*)$. Then we have a central extension

$$1 \rightarrow H_{dist, inv}^2(G) \rightarrow \tilde{\mathcal{B}}_L \rightarrow B_{alt} \rightarrow 1$$

where $H_{dist, inv}^2(G)$ is the cohomology group of conjugation invariant and distinguished cocycles.

Before we proceed with the proof, we give some examples:

Example 6.3.3. For $G = \mathbb{F}_p^n$ we have $B = \widehat{G} \otimes_{\mathbb{Z}} \widehat{G} = \mathbb{F}_p^{n \times n}$ respectively $B_{alt} = \widehat{G} \wedge \widehat{G} = \mathbb{F}_p^{\binom{n}{2}}$ the additive group of $n \times n$ -matrices resp. skew-symmetric $n \times n$ -matrices (for $p = 2$ we additionally demand all diagonal entries are zero).

For an abelian group there are no distinguished 2-cohomology-classes, hence $\widetilde{\mathcal{B}}_L \cong B_{alt}$ where $b \in B_{alt}$ corresponds to $(b, \beta) \in \mathcal{B}_L$ where β is any 2-cocycle with $\beta(g, h)\beta(h, g) = b(g)(h)$, which precisely determines a cohomology class $[\beta]$ in this case.

Example 6.3.4. For $G = \mathbb{D}_4 = \langle x, y \mid x^2 = y^2 = (xy)^4 = 1 \rangle$ we have $G_{ab} = \langle \bar{x}, \bar{y} \rangle \cong \mathbb{Z}_2^2$, $B = \text{Hom}(G_{ab}, \widehat{G}_{ab}) = \mathbb{Z}_2^{2 \times 2}$ and $B_{alt} = \{1, b\} \cong \mathbb{Z}_2$ with $b(\bar{x})(\bar{y}) = b(\bar{y})(\bar{x}) = -1$. It is known that $H^2(\mathbb{D}_4, k^\times) = \mathbb{Z}_2 = \{[1], [\alpha]\}$ and that the non-trivial 2-cocycles in the class $[\alpha]$ have a non-trivial restriction to the abelian subgroups $\langle x, z \rangle, \langle y, z \rangle \cong \mathbb{Z}_2^2$ of G . Especially $[\alpha]$ is not a distinguished 2-cohomology class. By definition of \mathcal{B}_L :

$$\mathcal{B}_L = \{(1, sym), (b, \beta \cdot sym)\}$$

where β is the pullback of any nontrivial 2-cocycle in G_{ab} with $\beta(x, y)\beta(y, x)^{-1} = -1$ and sym denotes any symmetric 2-cocycles. Especially $[\beta] = [1]$ as one checks on the abelian subgroups and thus by definition

$$\widetilde{\mathcal{B}}_L = \{(1, [1]), (b, [1])\} \cong \mathbb{Z}_2$$

However, these $(1, 1)$ and (b, β) , which are pull-backs of two different braided autoequivalences on G_{ab} , give rise to the *same* braided equivalence up to monoidal isomorphisms on G . Especially in this case we have a *non-injective* homomorphism.

$$\widetilde{\mathcal{B}}_L \rightarrow \text{Aut}_{mon}(DG\text{-mod})$$

More generally for the examples $G = p_+^{2n+1}$ we have B, B_{alt} as for the abelian group \mathbb{F}_p^{2n} , but (presumably) all braided autoequivalences in $\mathcal{B}_L(\mathbb{F}_p^{2n})$ pull back to a single trivial braided autoequivalence on G .

It is tempting to ask if in general the kernel of $\widetilde{\mathcal{B}}_L \rightarrow \text{Aut}_{mon}(DG\text{-mod})$ consist of those (b, β) for which $[\beta] = [1]$ i.e. if the remaining non-injectivity is controlled by the non-injectivity of the pullback $H^2(G_{ab}) \rightarrow H^2(G)$.

We give now an example where $\widetilde{\mathcal{B}}_L \rightarrow B_{alt}$ is not injective, thus we get a new 'kind' of a braided autoequivalences $(1, \beta)$ that would be trivial in the abelian case:

Example 6.3.5. In [Higgs87] p. 277 a group G of order p^9 with $H_{dist}^2(G) = \mathbb{Z}_p$ is constructed as follow: We start with the group \widetilde{G} of order p^{10} generated by x_1, x_2, x_3, x_4 of order p , all commutators $[x_i, x_j], i \neq j$ nontrivial of order p and central. Then \widetilde{G} is a central extension of $G := \widetilde{G}/\langle s \rangle$ where $s := [x_1, x_2][x_3, x_4]$. This central extension corresponds to a class of distinguished 2-cocycles $\langle \sigma \rangle = \mathbb{Z}_p = H_{dist}^2(G) = H^2(G)$. This is a consequence of the fact that s cannot be written as a single commutator. Further, we can find a conjugation invariant representative, because there is a conjugation invariant section $G \rightarrow \widetilde{G}$.

The conjugation invariant distinguished 2-cocycle β corresponds to a braided equivalence (id, J^β) trivial on objects. From $G_{ab} \cong \mathbb{Z}_p^4$, hence $B_{alt} = \mathbb{Z}_p^4 \wedge \mathbb{Z}_p^4 = \mathbb{Z}_p^6$ we have a central extension

$$1 \rightarrow \mathbb{Z}_p \rightarrow \tilde{\mathcal{B}}_L \rightarrow \mathbb{Z}_p^6 \rightarrow 1$$

In fact, we assume that the sequence splits and the braided autoequivalence (id, J^β) is the only nontrivial generator of the image $\tilde{\mathcal{B}}_L \rightarrow \text{Aut}_{br}(DG\text{-mod})$, since the pullback $\text{H}^2(G_{ab}) \rightarrow \text{H}^2(G)$ is trivial.

Proof of Lemma 6.3.2.

(i): Let us show that B acts trivially on $Z^2(G, k^\times)$, hence also on $Z_{inv}^2(G, k^\times)$:

$$\begin{aligned} b.\beta &= \sum_{x,y,g,h} ((\epsilon_{k^G \otimes k^G} \otimes \beta) * \epsilon_{k^G \otimes k^G})(x \times e_y, g \times e_h) \begin{pmatrix} 1 & b^* \\ 0 & 1 \end{pmatrix} (e_x \times y) \otimes \begin{pmatrix} 1 & b^* \\ 0 & 1 \end{pmatrix} (e_g \times h) \\ &= \sum_{x,g} \beta(x, g)(e_x \times 1) \otimes (e_g \times 1) = \beta \end{aligned}$$

For the action on simple DG -modules use Lemma 6.1.1. The rest of the statements are easy calculations.

(ii): Assume (F_b, J^β) is braided then according to Lemma 6.1.2 we get for $v = \text{id}$:

$$b(g)(h) = \beta(h, g)\beta(hgh^{-1}, h)^{-1} \quad \forall g, h \in G \quad (6.10)$$

Because β is closed we have: $1 = d\beta(h, gh^{-1}, h) = \frac{\beta(gh^{-1}, h)\beta(h, g)}{\beta(hgh^{-1}, h)\beta(h, gh^{-1})}$ and therefore:

$$b(g)(h) = \beta(h, g)\beta(hgh^{-1}, h)^{-1} = \beta^{-1}(gh^{-1}, h)\beta(h, gh^{-1})$$

$$\Leftrightarrow b(g)(h) = b(g)(h)b(h)(h) = b(gh)(h) = \beta^{-1}(g, h)\beta(h, g) \quad (6.11)$$

In the proof of Lemma 6.1.2 we also have shown that (F_b, J^β) is braided if and only if (6.8) holds. In this case where $\sigma(g \times e_x, h \times e_y) = \beta(g, h)\epsilon(e_x)\epsilon(e_y)$ it reduces to (6.10), hence (F_b, J^β) is braided. Since the product of braided autoequivalences is braided this also shows that \mathcal{B}_L is in fact a subgroup of $B \times Z_{inv}^2(G, k^\times)$.

(iii) By definition of \mathcal{B}_L , if $(b, \beta) \in \mathcal{B}_L$, then $b \in B_{alt}$. To show surjectivity, let $G_{ab} = G/[G, G]$ be the abelianization of G and $\hat{\beta}_b \in Z^2(G_{ab})$ an abelian 2-cocycle defined uniquely up to cohomology by $b(g)(h) = \hat{\beta}_b(h, g)\hat{\beta}_b(hgh^{-1}, h)^{-1} = \hat{\beta}_b(h, g)\hat{\beta}_b(g, h)^{-1}$ for $g, h \in G_{ab}$. Further, we have a canonical surjective homomorphism $\iota : G \rightarrow G_{ab}$ which induces a pullback $\iota^* : Z^2(G_{ab}) \rightarrow Z_{inv}^2(G, k^\times)$, hence we define $\beta_b := \iota^*\hat{\beta}_b$.

(iv) By (iii) the map $(b, \beta) \mapsto b$ is a group homomorphism $\mathcal{B}_L \rightarrow B_{alt}$ and this factorizes to a group homomorphism $\tilde{\mathcal{B}}_L \rightarrow B_{alt}$, since $(\text{Inn}(G) \times \text{B}^2(G)) \cap (B \times Z_{inv}^2(G, k^\times)) = 1$. The kernel of this homomorphism consists of all $(1, [\beta]) \in \tilde{\mathcal{B}}_L$, hence all $(1, [\beta])$ where $[\beta]$ has at least one representative β with $\beta(g, x) = \beta(gxg^{-1}, g)$ for all $g, x \in G$. We denote this kernel by K and note that it is central in \mathcal{B}_L .

It remains to show $K = \mathbb{H}_{dist}^2(G)$: Whenever $[\beta] \in K$ then there exists a representative β with $\beta(g, x) = \beta(gxg^{-1}, g)$ for all $g, x \in G$, in particular for any elements $g \in \text{Cent}(x)$, which implies any conjugacy class $[x]$ is β -regular and thus $[\beta] \in \mathbb{H}_{dist}^2(G)$. For the other direction we need a specific choice of representative: Suppose $[\beta] \in \mathbb{H}_{dist}^2(G)$ and thus all x are β -regular; by [Higgs87] Lm. 2.1(i) there exists a representative β with

$$\frac{\beta(g, x)\beta(gx, g^{-1})}{\beta(g, g^{-1})} = 1$$

for all β -regular x (i.e. here all x) and all g . An easy cohomology calculation shows indeed

$$\frac{\beta(g, x)}{\beta(gxg^{-1}, g)} = \frac{\beta(g, x)}{\beta(gxg^{-1}, g)} \cdot \frac{\beta(gx, g^{-1})\beta(gxg^{-1}, g)}{\beta(gx, 1)\beta(g, g^{-1})} = 1$$

hence $(1, \beta) \in \mathcal{B}_L$ by equation (6.10). □

6.4 E-Symmetries

It is now natural to construct a subgroup of $E \times Z_c^2(k^G)$ in a similar fashion. This construction corresponds to the lazy induction $\text{Aut}_{mon}(\text{Rep}(G)) \rightarrow \text{Aut}_{br}(DG\text{-mod})$. Unlike in the case of B -Symmetries, we do not need to consider some sort of distinguished cocycles. As we will see, being braided for elements of the form $(1, \alpha)$ already implies that the corresponding braided functor is trivial.

In the following Proposition we construct \mathcal{E}_L . As in the case of B -Symmetries, this set should be thought of as a collection of braided monoidal functors. Identifying pairs that differ by inner Hopf automorphisms and exact cocycles gives us $\tilde{\mathcal{E}}_L$. As shown below, the main statement is that this quotient is isomorphic to the group of alternating homomorphisms $Z(G) \rightarrow Z(G)$. In order to get the subgroup $\mathcal{E}_L \subset \text{Aut}_{br}(DG\text{-mod})$, we have to take the quotient of $\tilde{\mathcal{E}}_L$ by the kernel of $\tilde{\mathcal{E}}_L \rightarrow \text{Aut}_{br}(DG\text{-mod})$.

Proposition 6.4.1.

- (i) The group $E \times Z_c^2(k^G)$ is a subgroup of $\text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*)$. An element (a, α) corresponds to the monoidal functor (F_a, J^α) given on simple objects by $F_a(\mathcal{O}_g^\rho) = \mathcal{O}_{a(\rho')g}^\rho$, with the monoidal structure (in (v) we give easy representatives).

$$\begin{aligned} \mathcal{O}_{a(\hat{\rho})g}^\rho \otimes \mathcal{O}_{a(\hat{\chi})h}^\rho &\rightarrow F_a(\mathcal{O}_g^\rho \otimes \mathcal{O}_h^\chi) \\ (s_m \otimes v) \otimes (r_n \otimes w) &\mapsto \sum_{\substack{i,j;x \in \text{Cent}(g) \\ y \in \text{Cent}(h)}} \alpha(e_{s_i x s_m^{-1}}, e_{r_j y r_n^{-1}}) [s_i \otimes \rho(x)(v)] \otimes [r_j \otimes \chi(y)(w)] \end{aligned}$$

where we denote by $\rho' : Z(G) \rightarrow k^*$ the one-dimensional representation such that the any central element $z \in Z(G)$ act in ρ by multiplication with the

scalar $\rho'(z)$ and $\{s_i\}, \{r_j\} \subset G$ are choices of representatives of $G/\text{Cent}(g)$ and $G/\text{Cent}(h)$ respectively.

(ii) The subgroup $\mathcal{E}_L \subset E \times Z_c^2(k^G)$ defined by

$$\begin{aligned} \mathcal{E}_L := \{ & (a, \alpha) \in E \times Z_c^2(k^G) \mid \forall g, t, h \in G : \alpha(e_t, e_{ght}) = \alpha(e_t, e_{hg^{-1}t}) \\ & \alpha(e_t, e_h) = \sum_{x, y \in Z(G)} \alpha(e_{hy^{-1}}, e_{tx^{-1}}) e_y(a(e_x)) \} \end{aligned}$$

consists of all elements $(a, \alpha) \in E \times Z_c^2(k^G)$ such that the monoidal autoequivalence (F_a, J^α) is braided.

(iii) Let $E_{alt} \cong Z(G) \wedge Z(G)$ be the subgroup of alternating homomorphisms in $E = \text{Hom}(\widehat{Z(G)}, Z(G)) = Z(G) \otimes_{\mathbb{Z}} Z(G)$, i.e. the set of homomorphisms $a : \widehat{Z(G)} \rightarrow Z(G)$ with $\rho(a(\chi)) = \chi(a(\rho))^{-1}$ and $\chi(a(\chi))$ for all $\chi, \rho \in \widehat{Z(G)}$. Then the following group homomorphism is well-defined and surjective:

$$\mathcal{E}_L \rightarrow E_{alt}, \quad (a, \alpha) \mapsto a$$

(iv) Let $\tilde{\mathcal{E}}_L$ be the image of \mathcal{E}_L in $\text{Out}_{Hopf}(DG) \times H_L^2(DG^*)$, then the previous group homomorphism factorizes to an isomorphism

$$\tilde{\mathcal{E}}_L \cong E_{alt}$$

For each $a \in E_{alt}$ we have a representative functor (F_a, J^α) for a certain α obtained by pull-back from the center of G . More precisely, the functor is given by $F_a(\mathcal{O}_g^\rho) = \mathcal{O}_{a(\rho')g}^\rho$ and the monoidal structure given by a scalar

$$\begin{aligned} \mathcal{O}_{a(\rho')g}^\rho \otimes \mathcal{O}_{a(\chi')h}^\rho & \rightarrow F_a(\mathcal{O}_g^\rho \otimes \mathcal{O}_h^\chi) \\ m \otimes n & \mapsto \alpha'(\rho', \chi') \cdot (m \otimes n) \end{aligned}$$

where $\alpha' \in Z^2(\widehat{Z(G)})$ is any 2-cocycle in the cohomology class associated to the alternating form $a \in E_{alt}$ on the abelian group $\widehat{Z(G)}$.

Before we proceed to the proof we give some examples:

Example 6.4.2. For $G = \mathbb{D}_4 = \langle x, y \rangle, x^2 = y^2 = (xy)^4 = 1$ we have $Z(G) = \langle [x, y] \rangle \cong \mathbb{Z}_2$ and hence $E = \text{Hom}(\widehat{Z(G)}, Z(G)) = \mathbb{Z}_2$ and $E_{alt} = 1$. More generally for the examples $G = p_+^{2n+1}$ we have $E = \mathbb{Z}_p \otimes \mathbb{Z}_p = \mathbb{Z}_p$ and $E_{alt} = \mathbb{Z}_p \wedge \mathbb{Z}_p = 1$ and hence $\tilde{\mathcal{E}}_L = 1$.

Example 6.4.3. For the group of order p^9 in Example 6.3.5 we have $Z(G) = \mathbb{Z}_p^5$ generated by all commutators $[x_i, x_j], i \neq j$ modulo the relation $[x_1, x_2][x_3, x_4]$. Hence $E_{alt} = \mathbb{Z}_p^5 \wedge \mathbb{Z}_p^5 \cong \mathbb{Z}_p^{\binom{5}{2}} = \mathbb{Z}_p^{10}$ and respectively $\tilde{\mathcal{E}}_L = \mathbb{Z}_p^{10}$.

Proof of Proposition 6.4.1.

(i) Let us show that E acts trivially on $Z_c^2(k^G)$. For this we calculate:

$$\begin{aligned} a.\alpha &= \sum_{x,y,z,w} ((\alpha \otimes \epsilon_{kG \otimes kG}) * \epsilon_{kG \otimes kG})(x \times e_y, z \times e_w) \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} (e_x \times y) \otimes \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} (e_z \times w) \\ &= \sum_{y,w} \alpha(e_y, e_w) \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} (1 \times y) \otimes \begin{pmatrix} 1 & 0 \\ a^* & 1 \end{pmatrix} (1 \times w) \\ &= \sum_{y,w} \alpha(e_y, e_w) (1 \times y) \otimes (1 \times w) = \alpha \end{aligned}$$

For the action on simple DG -modules use Lemma 6.1.1. The rest are easy calculations.

(ii) Let $(a, \alpha) \in E \times Z_c^2(k^G)$. Then we use again the fact that (F_a, J^α) is braided if and only if equation (6.8) holds. In this case we have $\sigma(g \times e_x, h \times e_y) = \alpha(e_x, e_y)$ and (F_a, J^α) is braided iff for all $g, t, d \in G$:

$$\alpha(e_t, e_{gd}) = \sum_{h,x \in Z(G)} \alpha(e_{dh^{-1}(t^{-1}g^{-1}t)}, e_{tx^{-1}}) e_h(a(e_x)) \quad (6.12)$$

Setting $g = 1$ gives us the second defining equation of \mathcal{E}_L . Further, (6.12) is equivalent to

$$\alpha(e_t, e_{gdt^{-1}gt}) = \sum_{h,x \in Z(G)} \alpha(e_{dh^{-1}}, e_{tx^{-1}}) e_h(a(e_x)) \quad (6.13)$$

and therefore: $\alpha(e_t, e_{gdt^{-1}gt}) = \alpha(e_t, e_d)$ which is equivalent to the first defining equation of \mathcal{E}_L . Since the product of braided autoequivalences is braided this also shows that \mathcal{E}_L is in fact a subgroup of $E \times Z_c^2(k^G)$.

(iii) We first note that by equation (6.5) for $u = \text{id}$ we have $a \in E_{alt}$. We now show surjectivity: Since $Z(\widehat{G})$ is an abelian group there exists a unique (up to cohomology) 2-cocycle $\alpha \in H^2(\widehat{Z(G)})$ which can be pulled back to a 2-cocycle in $Z_c^2(k^G)$. Then (a, α) is in \mathcal{E}_L which proves surjectivity.

(iv) Before we show the isomorphism we obtain the description of the explicit representatives: In (iii) we constructed preimages (a, α) of each $a \in E_{alt}$ by pulling back a 2-cocycle $\alpha' \in Z^2(\widehat{Z(G)})$ in the cohomology class associated to a . We now apply the explicit formula in (i) and use that α is only nonzero on e_g, e_h with $g, h \in Z(G)$: Hence we have only nonzero summands for $s_m^{-1}s_i \in Z(G)$, hence $i = m$ and similarly $j = n$. Moreover, ρ, χ reduce on $Z(G)$ to one dimensional representations ρ', χ' . Evaluating the resulting sum we get the asserted form.

Next we note that the group homomorphism $\mathcal{E}_L \rightarrow E_{alt}$ in (iii) factorizes to a group homomorphism $\tilde{\mathcal{E}}_L \rightarrow E_{alt}$, since $(\text{Inn}(G) \times B_L^2(k^G)) \cap (E \times Z_c^2(k^G)) = 1$. The kernel

of this homomorphism consists of all $(1, [\alpha]) \in \tilde{\mathcal{E}}_L$, i.e. all classes $[\alpha]$ such that there exists a lazy representative $\alpha \in Z_c^2(k^G)$. Then, by definition of \mathcal{E}_L , the following is fulfilled for a pair $(1, \alpha) \in \mathcal{E}_L$:

$$\alpha(e_t, e_{ght}) = \alpha(e_t, e_{hg^{-1}t}) \quad \alpha(e_g, e_h) = \alpha(e_h, e_g)$$

By Corollary 3.2.3 a symmetric lazy cocycle $\alpha \in Z_c^2(k^G)$ is already cohomologically trivial. □

6.5 Partial Dualizations

Recall from Proposition 4.1.9 that R is the set of triples (H, C, δ) such that $G = H \times C$ and $\delta : kC \xrightarrow{\sim} k^C$ a Hopf isomorphism. Corresponding to that triple, we have defined a Hopf automorphism $r_{H,C,\delta}$ of DG that we called a reflection on C . In some sense, a reflection $r_{H,C,\delta}$ exchanges $kC \subset DG$ with $k^C \subset DG$ via δ (recall Proposition 4.1.9). We will identify the triple (H, C, δ) with the corresponding automorphism $r = r_{H,C,\delta}$ and the other way around.

Also, we have defined the group of central bialgebra pairings $P_c(kG, k^G)$ in Proposition 5.0.8, which is a subgroup of $Z_L^2(DG^*)$. In the following Proposition, we construct braided monoidal autoequivalences of $DG\text{-mod}$, where the action on objects is determined by a reflection and the monoidal structure is determined by a central bialgebra pairing.

Proposition 6.5.1.

- (i) Consider the subset $R \times P_c(kG, k^G)$ in $\text{Aut}_{\text{Hopf}}(DG) \times Z_L^2(DG^*)$. An element (r, λ) corresponds to the monoidal functor (F_r, J^λ) given on simple objects by $F_r(\mathcal{O}_{hc}^{\rho_H \rho_C}) = \mathcal{O}_{\delta^{-1}(\rho_C)h}^{\rho_H \delta(c)}$, where we decompose any group element and representation according to the choice $G = H \times C$ into $h \in H, c \in C$ resp. $\rho_H \in \text{Cent}_H(h)\text{-mod}, \rho_C \in \text{Cent}_C(c)\text{-mod}$. The monoidal structure is given by

$$\begin{aligned} \mathcal{O}_{\delta^{-1}(\rho_C)h}^{\rho_H \delta(c)} \otimes \mathcal{O}_{\delta^{-1}(\chi_C)h'}^{\chi_H \delta(c')} &\rightarrow F_r(\mathcal{O}_{hc}^{\rho_H \rho_C} \otimes \mathcal{O}_{h'c'}^{\chi_H \chi_C}) \\ (s_m \otimes v) \otimes (r_n \otimes w) &\mapsto \sum_{\substack{i \\ z \in \text{Cent}(hc)}} \lambda((h'c')_n, e_{s_i z s_m^{-1}})[s_i \otimes \rho(z)(v)] \otimes (r_n \otimes w) \end{aligned}$$

where $\{s_m\}, \{r_n\} \subset G$ are choices of representatives of $G/\text{Cent}(g)$ and $G/\text{Cent}(h)$ respectively and where $(h'c')_n = r_n h' c' r_n^{-1}$.

- (ii) Define the following set:

$$\mathcal{R}_L := \{(r_{H,C,\delta}, \lambda) \in R \times P_c(kG, k^G) \mid \lambda(hc, e_{h'c'}) = \delta_{c,c'} \delta_{h',1} \forall h, h' \in H, c, c' \in C\}$$

A pair $(F_{r_{H,C,\delta}}, J^\lambda)$ is a braided autoequivalence if and only if $(r_{H,C,\delta}, \lambda) \in \mathcal{R}_L$. We call the elements of \mathcal{R}_L partial dualizations of H on C .

(iii) For $(r_{H,C,\delta}, \lambda) \in \mathcal{R}_L$ the monoidal structure of $(F_{r_{H,C,\delta}}, J^\lambda)$ simplifies:

$$\begin{aligned} \mathcal{O}_{\delta^{-1}(\rho_C)h}^{\rho_H\delta(c)} \otimes \mathcal{O}_{\delta^{-1}(x_C)h'}^{x_H\delta(c')} &\rightarrow F_r(\mathcal{O}_{hc}^{\rho_H\rho_C} \otimes \mathcal{O}_{h'c'}^{x_Hx_C}) \\ m \otimes n &\mapsto \rho_C(c') \cdot (m \otimes n) \end{aligned}$$

Proof. (i) For the action on simple DG -modules use Lemma 6.1.1.

(ii) For $(r_{H,C,\delta}, \lambda) \in R \times P_c(kG, k^G)$ the functor (F_r, J^λ) is braided if and only if the equation (6.8) holds, where we have to consider the case $\sigma(g \times e_x, h \times e_y) = \lambda(g, e_y)\epsilon(e_x)$. Let us denote an element in the group $G = H \times C$ by $g = g_H g_C$ and recall that we write p_C, p_H for the obvious projections. Then we check equation (6.8) in this case:

$$\sum_{x,y,z \in G} \lambda(y^{-1}xy, e_z)(e_x \times y) \otimes (e_y \times z) = \sum_{x,y,z,w \in G} \delta_{y,w} \lambda(y^{-1}xy, e_z)(e_x \times y) \otimes (e_w \times z)$$

has to be equal to

$$\begin{aligned} &\sum_{w,y,g_1,g_2} \lambda(w, e_y)(1 \times y)(\delta^*((g_1g_2)_C) \times (g_1g_2)_H) \otimes (e_w \times 1)(e_{g_1} \circ p_H \times \delta^{-1*}(e_{g_2} \circ p_C)) \\ &= \sum_{x,y,g_1,g_2,w,z} \lambda(w, e_y)(1 \times y)\delta^*((g_1g_2)_C)(r)e_z(\delta^{-1*}(e_{g_2} \circ p_C)) \\ &\quad (e_x \times (g_1g_2)_H) \otimes (e_w \times 1)(e_{g_1} \circ p_H \times z) \\ &= \sum_{\substack{x,y,g_1,g_2,w,z \\ (g_2)_H=1}} \delta_{zH,1} \lambda(w, e_y)\delta(x_C)((g_1g_2)_C)e_{(g_2)_C}(\delta^{-1}(e_{zC})) \\ &\quad (1 \times y)(e_x \times (g_1g_2)_H) \otimes (e_w \times 1)(e_{g_1} \circ p_H \times z) \\ &= \sum_{\substack{w,y,g_1,g_2 \\ (g_2)_H=1, w_H=(g_1)_H}} \delta_{zH,1} \lambda(w, e_y)\delta(x_C)((g_2)_C)e_{(g_2) \circ p_C}(\delta^{-1}(e_{zC}))(e_{yx}y^{-1} \times y(g_1)_H) \otimes (e_w \times z) \\ &= \sum_{x,y,w,z} \delta_{zH,1} \lambda(w, e_{yw_H^{-1}})\delta(x_C)((\delta^{-1}(e_{zC}))) (e_x \times y) \otimes (e_w \times z) \end{aligned}$$

This is equivalent to saying that for all $x, y, w, z \in G$ the following holds:

$$\delta_{y,w} \lambda(y^{-1}xy, e_z) = \delta_{zH,1} \lambda(w, e_{yw_H^{-1}})\delta(x_C)((\delta^{-1}(e_{zC})))$$

So we see that (r, λ) fulfills this equation if and only if

$$\lambda(hc, e_{h'c'}) = \delta(c)(\delta^{-1}(e_{c'}))\epsilon(h)\epsilon(e_{h'}) = \delta_{c,c'}\epsilon(h)\epsilon(e_{h'})$$

for all $hc, h'c' \in H \times C$. Since for $c, c' \in C$ we have $\delta(c)(c') = \delta(c')(c)$, this also implies:

$$\delta(c)(\delta^{-1}(e_{c'})) = \sum_{x \in C} e_x(\delta^{-1}(e_{c'}))\delta(c)(x) = \sum_{x \in C} e_x(\delta^{-1}(e_{c'}))\delta(x)(c) = e_{c'}(c) = \delta_{c,c'}$$

This proves that $(r, \lambda) \in \mathcal{R}_L$.

(iii) This is a simple calculation using that C is abelian and then that $\lambda(hc, e_{h'c'}) = \delta_{c,c'}\epsilon(h)\epsilon(e_{h'})$ implies $i = m$ and only leaves the term $\delta_{c',z}$. \square

6.6 Statement and Proof of the Decomposition

Recall that we have defined certain characteristic elements of $\text{Aut}_{br}(DG\text{-mod})$ in the Propositions 6.2.1, 6.3.2, 6.4.1, 6.5.1 and showed how they can be explicitly calculated: We have that $\tilde{\mathcal{E}}_L$ is isomorphic to the group of alternating homomorphisms $\widehat{Z(G)} \rightarrow Z(G)$, that $\tilde{\mathcal{B}}_L$ is a central extension of the group of alternating homomorphisms on $G_{ab} \rightarrow \widehat{G_{ab}}$ and that \mathcal{R}_L is parametrized by decompositions $G = H \times C$ together with a Hopf isomorphism $\delta : kC \simeq k^C$. In the following Theorem, which is our main result, we show that these elements generate $\text{Aut}_{br}(DG\text{-mod})$ and that we even have a (double) coset decomposition.

Theorem 6.6.1.

- (i) Let $G = H \times C$ where H is purely non-abelian and C is *elementary* abelian. Then the subgroup of $\text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*)$ defined by

$$\underline{\text{Aut}}_{br,L}(DG\text{-mod}) := \{(\phi, \sigma) \in \text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*) \mid (F_\phi, J^\sigma) \text{ braided} \}$$

has the following decomposition into disjoint double cosets

$$\underline{\text{Aut}}_{br,L}(DG\text{-mod}) = \bigsqcup_{(r,\lambda) \in \mathcal{R}_L/\sim} \mathcal{V}_L \mathcal{B}_L \cdot (r, \lambda) \text{dReg}_{aL}^1(DG^*) \cdot \mathcal{V}_L \mathcal{E}_L$$

where two reflections $(r_{H,C,\delta}, \lambda)$ and $(r_{H',C',\delta'}, \lambda')$ are equivalent if and only if there exists a group isomorphism $C \simeq C'$.

Similarly, the quotient $\text{Aut}_{br,L}(DG\text{-mod})$ has a decomposition into double cosets

$$\text{Aut}_{br,L}(DG\text{-mod}) = \bigsqcup_{(r,\lambda) \in \mathcal{R}_L/\sim} \mathcal{V}_L \mathcal{B}_L \cdot (r, \lambda) \cdot \mathcal{V}_L \mathcal{E}_L$$

- (ii) Let G be a finite group with not necessarily elementary abelian direct factors. For every element $(\phi, \sigma) \in \underline{\text{Aut}}_{br,L}(DG\text{-mod})$ there exists a $(r, \lambda) \in \mathcal{R}_L$ such that (ϕ, σ) is in

$$(r, \lambda) \cdot [\mathcal{B}_L(\mathcal{V}_L \times \mathcal{E}_L)]$$

and similarly for $\underline{\text{Aut}}_{br,L}(DG\text{-mod})$.

- (iii) Let G be a finite group with not necessarily elementary abelian direct factors. For every element $(\phi, \sigma) \in \text{Aut}_{br,L}(DG\text{-mod})$ there exists a $(r, \lambda) \in \mathcal{R}_L$ such that (ϕ, σ) is in

$$[(\mathcal{V}_L \times \mathcal{B}_L)\mathcal{E}_L] \cdot (r, \lambda)$$

and similarly for $\underline{\text{Aut}}_{br,L}(DG\text{-mod})$.

Before we turn to the proof, we add some useful facts proven above. In the subsequent section we give examples and discuss how the Brauer-Picard group is described in this way.

- $(\phi, \sigma) \mapsto (F_\phi, J^\sigma)$ induces a group homomorphism to $\text{Aut}_{br}(DG\text{-mod})$ that factors through

$$\underline{\text{Aut}}_{br,L}(DG\text{-mod}) \rightarrow \widetilde{\text{Aut}}_{br,L}(DG\text{-mod})$$

- $(\phi, \sigma) \mapsto (F_\phi, J^\sigma)$ is still not necessarily injective, as Example 6.3.4 shows. The kernel is controlled by certain internal elements in DG (see exact sequence (1.10)).
- The group structure of $\widetilde{\text{Aut}}_{br,L}(DG\text{-mod})$ can be almost completely read off using the maps from $\widetilde{\mathcal{V}}_L, \widetilde{\mathcal{B}}_L, \widetilde{\mathcal{E}}_L, \widetilde{\mathcal{R}}_L$ to the known groups (resp. set) $\text{Out}(G), B_{alt}, E_{alt}, R$ in terms of matrices. Only $\widetilde{\mathcal{B}}_L \rightarrow B_{alt}$ is not necessarily a bijection in rare cases (in these cases additional cohomology calculations are necessary to determine the group structure).
- The decomposition of $\underline{\text{Aut}}_{br,L}(DG\text{-mod})$ is of course up to a coboundary in $\text{dReg}_{\text{SaL}}^1(DG^*)$ or equivalently up to a monoidal natural transformation.

Proof of Theorem 6.6.1.

(i) We start with a general element $(F_\phi, J^\sigma) \in \underline{\text{Aut}}_{br,L}(DG\text{-mod})$. We use the decomposition in Theorem 4.2.1 (ii) to write ϕ as a product of elements in V, V_c, B, E, R . Since G has only elementary abelian direct factors, the twist ν is zero. Thus, there exist $v', w' \in \text{Aut}(G) \simeq V$, $v, w \in \text{Aut}_c(G) \simeq V_c$, $b \in \text{Hom}(G, \hat{G}) \simeq B$, $a \in \text{Hom}(\widehat{Z(G)}, Z(G)) \simeq E$ and reflection $r_{H,C,\delta}$, where $G = H \times C$ and where $\delta : kC \xrightarrow{\cong} k^C$ a Hopf isomorphism, such that ϕ is given by the matrix:

$$\rightsquigarrow \begin{pmatrix} v'^{* -1} & 0 \\ 0 & v' \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} w'^{* -1} & 0 \\ 0 & w' \end{pmatrix} \quad (6.14)$$

Let us sketch the general procedure before going into details: The idea is to multiply (ϕ, σ) with specifically constructed elements of $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ from both sides in order to simplify the general form of the matrix corresponding to ϕ . Recall that for $(\phi, \sigma), (\phi', \sigma') \in \text{Aut}_{\text{Hopf}}(DG) \times \mathbb{Z}_L^2(DG^*)$:

$$(\phi, \sigma)(\phi', \sigma') = (\phi' \circ \phi, \sigma * \phi.\sigma')$$

where $\phi.\sigma' = (\phi \otimes \phi)(\sigma')$.

For the construction of these elements $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ we heavily make use of Lemma 6.1.2 and Propositions 6.3.2 (iii) and 6.4.1 (iii). This way we reduce (ϕ, σ) to (id, σ') for some $\sigma' \in \mathbb{Z}_L^2(DG^*)$. Since we multiply (multiplication is composition of monoidal functors) with braided functors, also (id, σ') is braided. Equation (6.8) implies that σ' is symmetric in the sense of equation (5.15) and from Lemma 5.0.11 thus follows that σ' is (up to almost lazy coboundary) a distinguished 2-cocycle on G in $\mathbb{Z}_{inv}^2(G)$ and therefore in \mathcal{B}_L , which proves the decomposition.

We will use the symbol \rightsquigarrow after an multiplication with $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ and warn that the u, v, b, a etc. before and after this multiplication, after \rightsquigarrow , are in general different. We do this because we want to avoid unreadable notation u', u'', u''' etc.

We are using the matrix notation for elements in $\text{Aut}_{\text{Hopf}}(DG)$ with respect to the product $DG = k^G \rtimes kG$ and for elements in $\text{Aut}(G), \text{Aut}_c(G), \text{Hom}(G, \hat{G}), \text{Hom}(Z(\hat{G}), Z(G))$ with respect to the direct product $G = H \times C$. For example, we write an $v \in \text{Aut}(H \times C)$ as the following matrix

$$\begin{pmatrix} v_{H,H} & v_{C,H} \\ v_{H,C} & v_{C,C} \end{pmatrix}$$

and similarly for the u, b, a .

First, we want to simplify the matrix (6.14) by lifting v', w' trivially (with trivial 2-cocycles) to elements in \mathcal{V}_L and then multiplying the inverses of these lifts with the matrix (6.14) such that it becomes

$$\rightsquigarrow \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad (6.15)$$

The 2-cocycle σ stays the same after this multiplication, since the 2-cocycles in \mathcal{V}_L are trivial. Note that V normalizes both V_c and E . With this step, we have 'eliminated' the $\mathcal{V}_L \simeq \text{Aut}(G)$ parts in ϕ . Further, we use the fact that the subgroup V_c normalizes the subgroup B and arrive at

$$\rightsquigarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad (6.16)$$

$$= \begin{pmatrix} v^* p_{k^H} + b\delta^{-1} + v^* \delta w a + b p_H w a & v^* \delta w + b p_H w \\ \delta^{-1} + p_H w a & p_H w \end{pmatrix} \quad (6.17)$$

Since (6.17) together with the 2-cocycle σ is braided, we deduce from Lemma 6.1.2 equation (6.4) that

$$1 = [\delta \circ p_C \circ w(g)(v \circ p_H \circ w(g))] \cdot [b \circ p_H \circ w(g)(p_H \circ w(g))] \quad (6.18)$$

for all $g \in G$. In particular, if we choose for $g := w^{-1}(h)$ with an arbitrary $h \in H$ equation (6.18) implies:

$$1 = b(h)(h) = b_{H,H}(h)(h) \quad (6.19)$$

which means that $b_{H,H} : H \rightarrow \hat{H}$ is alternating. Further, taking $g := w^{-1}(h, c)$ in equation (6.18), we get $\delta(c)(v(h)) = 1$ for all $c \in C, h \in H$, hence $v_{H,C} = 0$ (since δ is an isomorphism).

Now we take the inverse of (6.16) and argue analogously as above on the inverse matrix. We deduce that $a_{H,H}$ is alternating and that $(w^{-1})_{C,H} = 0$ and therefore that $w_{C,H} = 0$.

The alternating homomorphism $b_{H,H} : H \rightarrow \hat{H}$ can be trivially extended to an alternating homomorphism $G \rightarrow \hat{G}$:

$$b = \begin{pmatrix} b_{H,H} & 0 \\ 0 & 0 \end{pmatrix}$$

and similarly for $a_{H,H}$. Now we use Propositions 6.3.2 (iii) and 6.4.1 (iii): For these alternating homomorphisms a, b on G exist 2-cocycles $\beta_b \in Z_{inv}^2(G, k^\times)$ and $\alpha_a \in Z_c^2(k^G)$ such that $(b, \beta_b) \in \mathcal{B}_L$ and $(a, \alpha_a) \in \mathcal{E}_L$. Multiplying equation (6.16) with the inverses of (b, β_b) and (a, α_a) we simplify equation (6.16) to

$$\rightsquigarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad (6.20)$$

with $a = \begin{pmatrix} 0 & a_{C,H} \\ a_{H,C} & a_{C,C} \end{pmatrix}$, $b = \begin{pmatrix} 0 & b_{C,H} \\ b_{H,C} & b_{C,C} \end{pmatrix}$, $v = \begin{pmatrix} v_{H,H} & v_{C,H} \\ 0 & v_{C,C} \end{pmatrix}$ and $w = \begin{pmatrix} w_{H,H} & 0 \\ w_{H,C} & w_{C,C} \end{pmatrix}$ where the 2-cocycle σ changes to some other 2-cocycle σ' . With this step we 'eliminated' the ' H -part' in a and b .

The b and a can be simplified even further by using the fact that we can construct alternating $\tilde{b} = \begin{pmatrix} 0 & \tilde{b}_{C,H} \\ -\tilde{b}_{H,C} & 0 \end{pmatrix}$ with $\tilde{b}_{C,H}(c)(h) = -(b_{H,C}(h)(c))^{-1}$ and similarly an alternating \tilde{a} . For these maps there exists again 2-cocycles that lift them to elements in \mathcal{B}_L and \mathcal{E}_L respectively. As before, we multiply equation (6.20) with the inverses of the lifts and get:

$$\rightsquigarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad (6.21)$$

with $a = \begin{pmatrix} 0 & 0 \\ a_{H,C} & a_{C,C} \end{pmatrix}$, $b = \begin{pmatrix} 0 & b_{C,H} \\ 0 & b_{C,C} \end{pmatrix}$, $v = \begin{pmatrix} v_{H,H} & v_{C,H} \\ 0 & v_{C,C} \end{pmatrix}$ and $w = \begin{pmatrix} w_{H,H} & 0 \\ w_{H,C} & w_{C,C} \end{pmatrix}$

Now we commute the matrix corresponding to b to the right as follows:

$$\begin{pmatrix} 1 & \begin{pmatrix} 0 & b_{C,H} \\ 0 & b_{C,C} \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} = \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \begin{pmatrix} 0 & \tilde{b}_{C,H} \\ 0 & \tilde{b}_{C,C} \end{pmatrix} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \quad (6.22)$$

$$= \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \underbrace{\begin{pmatrix} \begin{pmatrix} 1 & \tilde{b}_{C,H}\delta^{-1} \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & 1 \end{pmatrix}}_{\in V_c} \underbrace{\begin{pmatrix} 1 & 0 \\ \begin{pmatrix} 0 & 0 \\ 0 & \delta^{-1}\tilde{b}_{C,C}\delta^{-1} \end{pmatrix} & 1 \end{pmatrix}}_{\in E} \quad (6.23)$$

By commuting the V_c elements in the decomposition to the right, multiplying with V as in the first step and then commuting back we thus arrived at the following form:

$$\rightsquigarrow \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p_{k^H} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad (6.24)$$

with $a = \begin{pmatrix} 0 & 0 \\ 0 & a_{C,C} \end{pmatrix}$, $v = \begin{pmatrix} v_{H,H} & v_{C,H} \\ 0 & v_{C,C} \end{pmatrix}$ and $w = \begin{pmatrix} w_{H,H} & 0 \\ w_{H,C} & w_{C,C} \end{pmatrix}$. Here we eliminated the $a_{H,C}$ part, similarly as the $b_{C,H}$ part, by commuting the corresponding matrix to the left, past through the reflection. This gives us again an element in V_c which we can absorb.

Now consider the inverse of (6.24):

$$\begin{pmatrix} p_{kH}(v^*)^{-1} & \delta \\ -ap_{kH}(v^*)^{-1} + w^{-1}\delta^{-1}v^{*-1} & -a\delta + w^{-1}p_H \end{pmatrix}$$

is again braided, hence we use as before Lemma 6.1.2 equation (6.4) to get:

$$1 = \delta(p_C(g))(a(\delta(p_C(g))w_{H,C}^{-1}(p_H(g)))) = \delta(g_C)(a_{C,C}(\delta(g_C)))\delta(g_C)(w_{H,C}^{-1}(g_H)) \quad (6.25)$$

Since this has to hold for all $g = g_H g_C \in H \times C$, we argue as before and get that $a_{C,C}$ is alternating and that $w_{H,C}^{-1} = 0$ and therefore $w_{H,C} = 0$. So we can eliminate the $a_{C,C}$ part by the same arguments as before. Using Lemma 6.1.2 equation (6.5) on (6.24) we deduce: $v_{C,H} = 0$. Since v is diagonal we can commute the matrix to the right through the reflection. We then get a product of a reflection $\delta' = v_{C,C}^* \circ \delta$, $H = H'$ and v . In other words, diagonal elements w.r.t a decomposition $G = H \times C$ of V_c normalize reflections $r_{H,C,\delta}$. We can lift any reflection to an element in \mathcal{R}_L according to Proposition 6.5.1 (iii). Thus we arrive at:

$$\rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & w \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \begin{pmatrix} w_{H,H} & 0 \\ 0 & w_{C,C} \end{pmatrix} \end{pmatrix} \quad (6.26)$$

Applying Lemma 6.1.2 equation (6.7) on (6.26) we get that $\chi(g) = \chi(w(g))$ for all $g \in G$, hence $w = \text{id}$.

During all of the above multiplications the 2-cocycle σ changed to some other 2-cocycle $\sigma' \in Z_L^2(DG^*)$ so that now we are left with the braided autoequivalence (id, σ') .

By Lemma 5.0.3 we know that $\beta_{\sigma'}(g, h) = \sigma'(g \times 1, h \times 1)$ defines a 2-cocycle on G . From equation (6.8) we deduce that if $(1, \sigma')$ is braided then

$$\begin{aligned} \sigma'(g \times 1, 1 \times e_x) &= \sigma'(1 \times e_x, g \times 1) \\ \sigma'(g \times 1, h^g \times 1) &= \sigma'(h \times 1, g \times 1) \\ \sigma'(1 \times e_x, 1 \times e_y) &= \sigma'(1 \times e_y, 1 \times e_x) \end{aligned}$$

this shows that $(1, \beta_{\sigma'}) \in \mathcal{B}_L$. We multiply $(1, \sigma')$ from the left with $(1, \sigma_{\beta_{\sigma'}}^{-1})$ where

$$\sigma_{\beta_{\sigma'}}(g \times e_x, h \times e_y) = \beta_{\sigma'}(g, h)\epsilon(e_x)\epsilon(e_y) = \sigma'(g \times 1, h \times 1)\epsilon(e_x)\epsilon(e_y)$$

and the resulting cocycle fulfills

$$\begin{aligned} \sigma_{\beta_{\sigma'}}^{-1} * \sigma'(g \times 1, h \times 1) &= \sum_{t,s \in G} \sigma_{\beta_{\sigma'}}^{-1}(g \times e_t, h \times e_s) \sigma'(g^t \times 1, h^s \times 1) \\ &= \sum_{t,s} \sigma^{-1}(g \times 1, h \times 1) \epsilon(e_t) \epsilon(e_s) \sigma(g^t \times 1, h^s \times 1) = 1 \end{aligned}$$

Denote the new cocycle again by σ' and note that σ' is trivial if restricted to $kG \times kG$, since we got rid of the 'distinguished kG -part'.

So (id, σ') is a braided autoequivalence with trivial restriction to $kG \times kG$. By equation (6.8), σ' is a symmetric 2-cocycle on DG^* as defined in equation (5.15). Lemma 5.0.11 implies that such a symmetric 2-cocycle is trivial up to an almost lazy coboundary that is not necessary lazy. Thus, there exists an $\eta \in \text{Reg}_{aL}^1(DG^*)$ such that $\sigma' = d\eta$.

(ii) By Theorem 4.2.1 (iv) we write

$$\phi = \begin{pmatrix} p_{kH} & \delta \\ \delta^{-1} & p_H \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \quad (6.27)$$

where we have already eliminated the V element, since it normalizes E and every V has a lift to \mathcal{Y}_L . Similarly, we know from Proposition 6.5.1 that every reflection r has a lift $(r, \lambda) \in \mathcal{R}_L$. Hence we multiply (ϕ, σ) with the inverse $(r, \lambda)^{-1}$ from the left so that ϕ changes to:

$$\rightsquigarrow \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^* & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} v^* + ba & b \\ a & 1 \end{pmatrix} \quad (6.28)$$

Since this element has to be braided, using Lemma 6.1.2 equation 6.4 together with (6.11) it follows that b is alternating on G . From Lemma 6.1.2 equation 6.7 follows that $v = \text{id}_G$ and then that a is alternating. Hence we can construct lifts to \mathcal{B}_L and \mathcal{E}_L as in (i) and multiplying with the corresponding inverses just leaves us with a $(1, \sigma')$. As in (i) we get rid of the distinguished part and then this is a trivial autoequivalence (up to natural transformation).

The proof of (iii) is completely analogous to (ii). □

Chapter 7

Examples and the full Brauer-Picard group

As described in the introduction, we have the following two group homomorphisms to $\text{Aut}_{br}(DG\text{-mod})$:

$$\text{Ind}_{\text{Vect}_G} : \text{Aut}_{mon}(\text{Vect}_G) \rightarrow \text{Aut}_{br}(DG\text{-mod})$$

$$\text{Ind}_{\text{Rep}(G)} : \text{Aut}_{mon}(\text{Rep}(G)) \rightarrow \text{Aut}_{br}(DG\text{-mod})$$

The images give us subgroups of $\text{Aut}_{br}(DG\text{-mod})$. Further, we have defined the set of partial dualizations \mathcal{R} that interact between the two subgroups $\text{im}(\text{Ind}_{\text{Vect}_G})$ and $\text{im}(\text{Ind}_{\text{Rep}(G)})$ (see Introduction and the next section).

Conjecture 7.0.1. The subgroups $\text{im}(\text{Ind}_{H\text{-mod}})$, $\text{im}(\text{Ind}_{H^*\text{-mod}})$ together with partial dualizations \mathcal{R} generate the group $\text{Aut}_{br}(H\text{-mod})$. Further, $\text{Aut}_{br}(H\text{-mod})$ decomposes into an (ordered) product of $\text{im}(\text{Ind}_{H\text{-mod}})$, $\text{im}(\text{Ind}_{H^*\text{-mod}})$ and \mathcal{R} .

We now discuss the results of this thesis for several classes of groups G in the lazy case and also verify that the decomposition we propose in Conjecture 7.0.1 is true for the full Brauer-Picard group (not necessary lazy). Let us first summarize the approach in [NR14] to study $\text{Aut}_{br}(DG\text{-mod})$.

Definition 7.0.2. Let \mathcal{C} be a modular tensor category. A subcategory $\mathcal{D} \subset \mathcal{C}$ is called Lagrangian if the spherical twist of \mathcal{C} is the identity on \mathcal{D} and $\mathcal{D} = \mathcal{D}'$, where

$$\mathcal{D}' := \{X \in \mathcal{C} \mid c_{Y,X} \circ c_{X,Y} = \text{id}_{X \otimes Y} \ \forall Y \in \mathcal{D}\}$$

is the Müger centralizer of \mathcal{D} and $\{c_{X,Y} : X \otimes Y \xrightarrow{\cong} Y \otimes X \mid X, Y \in \mathcal{C}\}$ is the braiding of \mathcal{C} .

An isotropic subcategory \mathcal{D} is necessary symmetric monoidal and the condition $\mathcal{D} = \mathcal{D}'$ means that \mathcal{D} is the 'maximal' symmetric monoidal subcategory of \mathcal{C} .

Let $\mathbb{L}(G)$ be the set (lattice) of Lagrangian subcategories $\mathcal{L} \subset DG\text{-mod}$. Lagrangian subcategories of $DG\text{-mod}$ are parametrized by pairs (N, μ) , where N is a normal

abelian subgroup of G and $\mu \in H^2(N, k^\times)$ a G -invariant 2-cohomology class on N . The Lagrangian subcategory associated to a pair (N, μ) is denoted by $\mathcal{L}_{N, \mu}$ and is generated, as an abelian category, by the following simple objects (see Sect. 7 [NR14]):

$$\mathcal{L}_{N, \mu} := \langle \mathcal{O}_g^\chi \mid g \in N, \chi(h) = \mu(g, h)\mu(h, g)^{-1} \forall h \in N \rangle$$

Let further

$$\mathbb{L}_0(G) := \{ \mathcal{L} \in \mathbb{L}(G) \mid \mathcal{L} \simeq \text{Rep}(G) \text{ as a braided fusion category} \}$$

The group $\text{Aut}_{br}(DG\text{-mod})$ acts on the set (lattice) of fusion subcategories of $DG\text{-mod}$ and on $\mathbb{L}(G)$, where the subset $\mathbb{L}_0(G)$ is invariant under this action.

Proposition 7.0.3. (Proposition 7.6 [NR14])

The action of $\text{Aut}_{br}(DG\text{-mod})$ on $\mathbb{L}_0(G)$ is transitive.

Proposition 7.0.4. (Corollary 6.9, Lemma 6.10 [NR14])

The stabilizer of the standard Lagrangian subcategory $\mathcal{L}_{1,1} = \text{Rep}(G)$ is the image of the induction

$$\text{Ind}_{\text{Vect}_G} : \text{Aut}_{mon}(\text{Vect}_G) \rightarrow \text{Aut}_{br}(DG\text{-mod})$$

Moreover, we have a group isomorphism: $\text{im}(\text{Ind}_{\text{Vect}_G}) \simeq \text{Out}(G) \times H^2(G, k^\times)$. Putting both together, we have a group isomorphism:

$$\text{Stab}(\text{Rep}(G)) \simeq \text{Out}(G) \times H^2(G, k^\times)$$

Corollary 7.0.5. (Corollary 7.7 [NR14])

For a finite group G we have

$$|\text{Aut}_{br}(DG\text{-mod})| = |H^2(G, k^\times)| |\text{Out}(G)| |\mathbb{L}_0(G)|$$

As we will explain further below, with the statements above, one can determine $\text{Aut}_{br}(DG\text{-mod})$ for certain groups G as done in [NR14]. The main purpose of this section is to show that, in these examples for G , the subgroups $\mathcal{B}, \mathcal{E}, \mathcal{V}$ together with the partial dualizations \mathcal{R} act transitively on $\mathbb{L}_0(G)$, which implies that they are indeed generators of $\text{Aut}_{br}(DG\text{-mod})$ confirming the Conjecture 7.0.1.

7.1 General Considerations on Partial Dualizations

For each triple (Q, N, δ) where G is a semi-direct product $G = Q \rtimes N$, N a normal abelian subgroup of G , $\delta : kN \xrightarrow{\sim} k^N$ a G -invariant (under conjugation action) Hopf isomorphism, we obtain an element $r_{Q, N, \delta} := \Omega \in \text{Aut}_{br}(DG\text{-mod})$, where Ω is the braided autoequivalence given in Thm 3.20 in [BLS15]: We have a decomposition

of kG as a Radford biproduct $kG = kQ \ltimes kN$, where N is a normal subgroup of G , kN is a Hopf algebra in $DQ\text{-mod}$, where kQ acts on kN by conjugation and where the kQ -coaction on kN is trivial. In our notation, the braided autoequivalence $r_{Q,N,\delta} : DG\text{-mod} \xrightarrow{\cong} DG\text{-mod}$ assigns a DG -module M to a DG -module $r_{Q,N,\delta}(M)$ where $r_{Q,N,\delta}(M)$ is M as a k -vector space and where the DG -action on $r_{Q,N,\delta}(M)$ is given by postcomposing with the following algebra isomorphism of $DG = D(Q \ltimes N)$:

$$DG \ni (f_Q, f_N) \times (q, n) \mapsto (f_Q, \delta(n)) \times (q, \delta^{-1}(f_N)) \in DG$$

where $f_Q \in k^Q$, $f_N \in k^N$, $q \in Q$ and $n \in N$. Essentially, this is the reflection as defined in Proposition 4.1.9 but since we do not have a direct product of Q and N , we do not have a coalgebra isomorphism as we would have in the lazy case. We are denoting the DG -action on M by $((f_Q, f_N) \times (q, n)) \cdot m \in M$ for $m \in M$ and the DG -action on $r_{Q,N,\delta}(M)$ by $((f_Q, f_N) \times (q, n)) \cdot r m = ((f_Q, \delta(n)) \times (q, \delta^{-1}(f_N))) \cdot m$.

We now show how partial dualization $r_{Q,N,\delta}$ act on irreducible representations \mathcal{O}_1^X of DG and hence on subcategory $\mathcal{L}_{1,1}$. We need this in order to check that the group generated by $\mathcal{B}, \mathcal{E}, \mathcal{V}$ and \mathcal{R} acts transitively on $\mathbb{L}_0(G)$. Since $\mathbb{L}_0(G)$ is the orbit of $\mathcal{L}_{1,1} = \text{Rep}(G)$ and since $\mathcal{L}_{1,1}$ is generated by the simple objects \mathcal{O}_1^X , we only need the action of partial dualizations on simple objects of the form \mathcal{O}_1^X for some irreducible character χ on G .

Since $r_{Q,N,\delta}$ is a monoidal autoequivalence, it sends simple objects to simple objects. Therefore, for each irreducible character χ on G there exists a conjugacy class $[g] \subset G$ and an irreducible character ρ on $\text{Cent}(g)$ such that $r_{Q,N,\delta}(\mathcal{O}_1^X) = \mathcal{O}_g^\rho$. We have $\dim(\chi) = |[g]| \cdot \dim(\rho)$. We want to determine $[g]$ and ρ .

Clifford's theorem (see e.g. Page 70, Theorem 4.1 in [Gor07]) states that the restriction of an irreducible character χ to a normal subgroup N of G decomposes into a direct sum of irreducible N -characters χ_i with the same multiplicity $e \in \mathbb{N}$:

$$\chi|_N = e \sum_{i=1}^t \chi_i$$

where the χ_i form a G -orbit under conjugation action on N and hence on $\text{Rep}(N)$. The group $Q = G/N$ acts on χ_i by conjugation in the argument and the subgroups $I_i \subset G/N = Q$ fixing a χ_i are called the inertia subgroups. We have $[Q : I_i] = t$.

Since N is abelian, we obtain 1-dimensional representations $\chi_i \in \hat{N}$ forming a G -conjugacy class. Then $n_i := \delta^{-1}(\chi_i) \in N$ are all conjugate to each other in G . Fix one representative $n_i = \delta^{-1}(\chi_i)$ in this conjugacy class and the corresponding inertia subgroup $I_i \subset Q$. Further, since δ is G -conjugation invariant we also have the formula $\text{Cent}(n_i) = N \rtimes I_i$.

We have a decomposition via Clifford's theorem $\mathcal{O}_1^X = \bigoplus_{j=0}^t T_j \otimes M_j$, where the M_j are 1-dimensional k -vector spaces with an N -action given by χ_j and where T_j is an e -dimensional k -vector space with trivial N -action. Since the partial dualization

preserves vector spaces, we also have a decomposition as vector spaces: $r_{Q,N,\delta}(\mathcal{O}_1^X) = \mathcal{O}_g^\rho = \bigoplus_{j=0}^t T_j \otimes M_j$. We calculate the k^N -action (kN -coaction) on \mathcal{O}_g^ρ : Let $t \otimes m_j \in T_j \otimes M_j$ then the modified k^N -action:

$$\begin{aligned} e_{n_i \cdot r}(t \otimes m_j) &= \delta^{-1}(e_{n_i}) \cdot (t \otimes m_j) = \chi_j(\delta^{-1}(e_{n_i}))(t \otimes m_j) \\ &= e_{n_i}(\delta^{-1}(\chi_j))(t \otimes m_j) = e_{n_i}(n_j)(t \otimes m_j) = \delta_{i,j}(t \otimes m_j) \end{aligned}$$

On the other hand the k^Q -action (kQ -coaction) on \mathcal{O}_g^ρ stays the same, which is trivial here. Hence, we have shown: $[g] = [n_i]$.

Now calculate the action of $\text{Cent}(n_i) = N \rtimes I_i$ on $T_i \otimes M_i$. Let $n \in N$ and note that the k^G -action on \mathcal{O}_1^X is trivial since $[[1]] = 1$. Hence the kN -action on $\mathcal{O}_{n_i}^\rho$ is trivial. For $q \in I_i \subset Q = G/N$: $q \cdot r(t \otimes m_i) = (q \cdot t) \otimes m_i$ where Q acts on T_i since $T_i \otimes M_i$ is an I_i -submodule. Thus, ρ is the character on $N \rtimes I_i$ which is the trivial extension of the I_i -representation T_e . Overall we get

$$r_{Q,N,\delta}(\mathcal{O}_1^X) = \mathcal{O}_{n_i}^{T_i}$$

7.2 General considerations on non-lazy induction

We now turn to the subgroups of $\text{Aut}_{br}(DG\text{-mod})$ defined to be the images of the functors

$$\begin{aligned} \text{Ind}_{\text{Vect}_G} &: \text{Aut}_{mon}(\text{Vect}_G) \rightarrow \text{Aut}_{br}(DG\text{-mod}) \\ \text{Ind}_{\text{Rep}(G)} &: \text{Aut}_{mon}(\text{Rep}(G)) \rightarrow \text{Aut}_{br}(DG\text{-mod}) \end{aligned}$$

We already know (e.g. from [NR14]):

$$\text{im}(\text{Ind}_{\text{Vect}_G}) = \text{Aut}_{mon}(\text{Vect}_G) = \text{Out}(G) \rtimes \text{H}^2(G, k^\times)$$

The subgroup $\text{im}(\text{Ind}_{\text{Rep}(G)})$ is much harder to compute. The group $\text{Aut}_{mon}(\text{Rep}(G))$ is parametrized by pairs (N, α) where N is an abelian subgroup of G and α belongs to a G -invariant cohomology class (see [Dav01]). The subgroup of lazy monoidal autoequivalences corresponds to all pairs where α is G -invariant even as a 2-cocycle.

Remark 7.2.1. An interesting example appears when we consider $G = \mathbb{Z}_2^{2n} \rtimes \text{Sp}_{2n}(2)$ where $\text{Sp}_{2n}(2)$ is the symplectic group over \mathbb{F}_2 . There is a pair (N, α) such that the associated functor is a monoidal equivalence

$$F_{N,\alpha} : \text{Rep}(\mathbb{Z}_2^{2n} \rtimes \text{Sp}_{2n}(2)) \xrightarrow{\sim} \text{Rep}(\mathbb{Z}_2^{2n} \cdot \text{Sp}_{2n}(2))$$

The groups $\mathbb{Z}_2^{2n} \rtimes \text{Sp}_{2n}(2)$ and $\mathbb{Z}_2^{2n} \cdot \text{Sp}_{2n}(2)$ are isomorphic only for $n = 1$. Namely, they are both isomorphic to \mathbb{S}_4 . See Example 7.6 in [Dav01]. This leads to a nontrivial and non-lazy monoidal autoequivalence, which leads to a non-trivial non-lazy braided autoequivalence of DS_4 , see the example below.

For any $F \in \text{Aut}_{mon}(\text{Rep}(G))$, we want to determine the image

$$E_F := \text{Ind}_{\text{Rep}(G)}(F) \in \text{Aut}_{br}(DG\text{-mod})$$

Unfortunately, it is not easy to calculate E_F explicitly, since it depends on the isomorphism $\text{BrPic}(\mathcal{C}) \rightarrow \text{Aut}_{br}(Z(\mathcal{C}))$. In [NR14] equations (16),(17), the image of the induction $\text{Ind}_{\text{Vect}_G}$ was worked out, but we are also interested in the image of $\text{Ind}_{\text{Rep}(G)}$ which seems to be harder. We can easily derive at least an necessary condition. From [ENO10] we know that given an invertible \mathcal{C} -bimodule category ${}_c\mathcal{M}_c$ the corresponding braided autoequivalence $\Phi_{\mathcal{M}} \in \text{Aut}_{br}(Z(\mathcal{C}))$ is determined by the condition that there exists a isomorphism of \mathcal{C} -bimodule functors $Z \otimes \cdot \simeq \cdot \otimes \Phi_{\mathcal{M}}(Z)$ for all $Z \in Z(\mathcal{C})$. In our case ${}_c\mathcal{M}_c = {}_F\mathcal{C}_c$ and $\Phi_{\mathcal{M}} = E_F$. This implies for $(V, c), (V', c') \in Z(\mathcal{C})$

$$E_F(V, c) = (V', c') \Rightarrow F(V) \otimes X \cong X \otimes V' \quad \forall X \in \mathcal{C}$$

In particular, we have $F(V) \simeq V'$. For $\mathcal{C} = \text{Rep}(G)$ this implies moreover:

$$E_F(\mathcal{O}_g^{\chi}) = \mathcal{O}_{g'}^{\chi'} \Rightarrow F(\text{Ind}_{\text{Cent}(g)}^G(\chi)) \cong \text{Ind}_{\text{Cent}(g')}^G(\chi')$$

Thus, possible images of E_F are determined by the character table of G and induction-restriction table with $\text{Cent}(g), \text{Cent}(g')$. We continue for the special case $g = 1$ to determine the possible images $E_F(\mathcal{O}_1^{\chi})$ and hence $E_F(\mathcal{L}_{1,1})$. Our formula above implies:

$$F(\chi) = \text{Ind}_{\text{Cent}(g')}^G(\chi')$$

In particular, $\text{Ind}_{\text{Cent}(g')}^G(\chi')$ has to be irreducible.

7.2.1 Elementary abelian groups

For $G = \mathbb{Z}_p^n$ with p a prime number. We fix an isomorphism $\mathbb{Z}_p \simeq \widehat{\mathbb{Z}}_p$. We know that

$$\text{BrPic}(\text{Rep}(\mathbb{Z}_p^n)) \simeq \text{O}_{2n}^+(\mathbb{F}_p)$$

where $\text{O}_{2n}^+(\mathbb{F}_p) := \text{O}_{2n}(\mathbb{F}_p, q)$ is the group of invertible $2n \times 2n$ matrices invariant under the form:

$$q : \mathbb{F}_p^n \times \mathbb{F}_p^n \rightarrow \mathbb{F}_p : (k_1, \dots, k_n, l_1, \dots, l_n) \mapsto \sum_{i=1}^n k_i l_i$$

For abelian groups, all 2-cocycles over DG are lazy and the results of this article gives a product decomposition of $\text{BrPic}(\text{Rep}(\mathbb{Z}_p^n))$.

- $\mathcal{V} \cong \text{GL}_n(\mathbb{F}_p) \simeq \left\{ \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} \mid A \in \text{GL}_n(\mathbb{F}_p) \right\} \subset \text{O}_{2n}(\mathbb{F}_p, q)$
- $\mathcal{B} \cong B_{alt} \cong \left\{ \begin{pmatrix} \mathbb{1}_n & B \\ 0 & \mathbb{1}_n \end{pmatrix} \mid B = -B^T, B_{ii} = 0, B \in \mathbb{F}_p^{n \times n} \right\} \subset \text{O}_{2n}(\mathbb{F}_p, q)$
- $\mathcal{E} \cong E_{alt} \cong \left\{ \begin{pmatrix} \mathbb{1}_n & 0 \\ E & \mathbb{1}_n \end{pmatrix} \mid E = -E^T, E_{ii} = 0, E \in \mathbb{F}_p^{n \times n} \right\} \subset \text{O}_{2n}(\mathbb{F}_p, q)$

The set \mathcal{R}_L / \sim consists of $n+1$ representatives $r_{[C]}$, one for each possible dimension d of a direct factor $\mathbb{F}_p^d \cong C \subset G$, and $r_{[C]}$ is an actual reflection on the subspace C with a suitable monoidal structure determined by the pairing λ . Especially the generator $r_{[C]}$ conjugates \mathcal{B} and \mathcal{E} . In this case the double coset decomposition is a variant of the Bruhat decomposition of $O_{2n}(\mathbb{F}_p, q)$.

It is interesting to discuss how, in this example, our subgroups act on the Lagrangian subcategories and to see that this action is indeed transitive. $\mathbb{L}_0(G) = \mathbb{L}(G)$ is parametrized by pairs $(N, [\mu])$ where N is a subvector space of \mathbb{F}_p^n and $[\mu] \in \mathbb{H}^2(N, k^\times)$ is uniquely determined by an alternating bilinear form $\langle \cdot, \cdot \rangle_\mu$ on N given by $\langle g, h \rangle_\mu = \mu(g, h)\mu(h, g)^{-1}$. Let N' be the orthogonal complement, so $\mathbb{F}_p^n = N \oplus N'$. We have

$$\mathcal{L}_{N,\mu} = \left\langle \mathcal{O}_g^{\chi_{N'}(g, \cdot)} \mid g \in N, \chi_{N'} \in \widehat{N}' \right\rangle \quad \mathcal{L}_{1,1} = \left\langle \mathcal{O}_1^\chi \mid \chi \in \widehat{G} \right\rangle$$

- Elements in $\widetilde{\mathcal{V}}_L = \text{Out}(G) = \text{GL}_n(\mathbb{F}_p)$ stabilize $\mathcal{L}_{1,1}$.
- For any δ , a partial dualization $r_N \in \mathcal{R}_L$ on N maps $\mathcal{L}_{1,1}$ to $\mathcal{L}_{N,1}$.
- $b \in \text{Hom}_{\text{alt}}(G, \widehat{G}) \simeq \widetilde{\mathcal{B}}_L$ acts by $\mathcal{O}_g^\chi \mapsto \mathcal{O}_g^{\chi \cdot b(g, \cdot)}$. In particular, it stabilizes $\mathcal{L}_{1,1}$ and sends $\mathcal{L}_{N,1} \mapsto \mathcal{L}_{N,\beta|_N}$ where $\beta \in \mathbb{Z}^2(G, k^\times)$ is uniquely (up to coboundary) determined by $b(g, h) = \beta(g, h)\beta(h, g)^{-1}$.
- $a \in \text{Hom}_{\text{alt}}(\widehat{G}, G) \simeq \widetilde{\mathcal{E}}_L$ acts by $\mathcal{O}_g^\chi \mapsto \mathcal{O}_{a(\chi)g}^\chi$. In particular, it sends $\mathcal{L}_{1,1}$ to $\mathcal{L}_{N,\eta}$ with $N = \text{im}(a)$ being the image of a and $\eta \in \mathbb{Z}^2(N, k^\times)$ is uniquely (up to coboundary) determined by $\eta(n, n')\eta(n', n)^{-1} = \chi(n')$ where $a(\chi) = n$ and $n' \in N = \text{im}(a)$. For another χ' with $a(\chi') = n$ we have $\chi(n') = \chi(a(\rho)) = \rho(a(\chi))^{-1} = \rho(a(\chi')) = \chi'(n)$.

We see that can get every $\mathcal{L}_{N,\mu} \in \mathbb{L}_0(G)$ with suitable combinations of elements of our subgroups applied to $\mathcal{L}_{1,1}$.

7.2.2 Simple groups

Let G be a simple group, then our result returns

- $\widetilde{\mathcal{V}}_L = \text{Out}(G)$
- $\widetilde{\mathcal{B}}_L = \widehat{G}_{ab} \wedge \widehat{G}_{ab} = 1$
- $\widetilde{\mathcal{E}}_L = Z(G) \wedge Z(G) = 1$
- $\mathcal{R}_L = 1$

hence the only *lazy* autoequivalences are induced by outer automorphisms of G .

We have no normal abelian subgroups except $\{1\}$ and hence the only Lagrangian subcategory is $\mathcal{L}_{1,1}$ and the stabilizer $\text{Out}(G) \times \mathbb{H}^2(G, k^\times)$ is equal to $\text{Aut}_{br}(DG\text{-mod})$.

Observe that in this example we obtain also a decomposition of the full Brauer-Picard group and our Conjecture 7.0.1 is answered positively: Namely, $\text{Aut}_{br}(DG\text{-mod})$ is equal the image of the induction $\text{Ind}_{\text{Vect}_G}$, while the other subgroups are trivial.

7.2.3 Lie groups and quasisimple groups

Lie groups over finite fields $G(\mathbb{F}_q)$, $q = p^k$ have (with small exceptions) the property $G_{ab} = 1$ and there are no semidirect factors. On the other hand, they may contain a nontrivial center $Z(G)$. This is comparable to their complex counterpart, where the center of the simply-connected form $Z(G_{sc}(\mathbb{C}))$ is equal to the fundamental group $\pi_1(G_{ad}(\mathbb{C}))$ of the adjoint form with no center $Z(G_{ad}(\mathbb{C})) = 1$. In exceptional cases for q , the maximal central extension may be larger than $\pi_1(G_{ad}(\mathbb{C}))$. Similarly, we could consider central extensions of sporadic groups G ; these appear in any insolvable group as part of the Fitting group.

Definition 7.2.2. A group G is called *quasisimple* if it is a perfect central extension of a simple group:

$$Z \rightarrow G \rightarrow H \quad Z = Z(G), \quad [G, G] = G$$

As long as $H^2(Z, \mathbb{C}^\times) = 1$, e.g. because Z is cyclic, there is no difference to the simple case. *Nontrivial* $\tilde{\mathcal{E}}_L$ -terms appear as soon as $H^2(Z, \mathbb{C}^\times) \neq 1$. This is the case for $D_{2n}(\mathbb{F}_q) = \mathrm{SO}_{4n}(\mathbb{F}_q)$ (for q odd or $q = 2$) where we have $\pi_1(G_{ad}(\mathbb{C})) = \mathbb{Z}_2 \times \mathbb{Z}_2$ and in some other (exceptional) cases. We consider all universal perfect central extensions where $H^2(Z, \mathbb{C}^\times) \neq 1$:

Z	H	$\tilde{\mathcal{E}}_L$
$\mathbb{Z}_2 \times \mathbb{Z}_2$	$D_{2n}(\mathbb{F}_q)$	\mathbb{Z}_2
$\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_3$	$A_2(\mathbb{F}_{2^2})$	\mathbb{Z}_4
$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_4$	${}^2A_3(\mathbb{F}_{3^2})$	\mathbb{Z}_3
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	${}^2A_5(\mathbb{F}_{2^2})$	\mathbb{Z}_2
$\mathbb{Z}_2 \times \mathbb{Z}_2$	${}^2B_2(\mathbb{F}_{2^3})$	\mathbb{Z}_2
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$	${}^2E_6(\mathbb{F}_{2^2})$	\mathbb{Z}_2

The upper indices denote the order of the automorphism by which the so-called Steinberg groups are defined. $\mathrm{Out}(H)$ typically consists of scalar- and Galois-automorphisms of the base field \mathbb{F}_q , extended by the group of Dynkin diagram automorphisms; for example D_4 we have the triality automorphisms \mathbb{S}_3 . Note further that any automorphism on G preserves the center Z , hence it factors to an automorphism in H . The kernel of this group homomorphism $\mathrm{Out}(G) \rightarrow \mathrm{Out}(H)$ is trivial, since all elements in Z are products of commutators of G elements. We have $\mathrm{Out}(G) \cong \mathrm{Out}(H)$ where surjectivity follows from G being a universal central extension. For G as above, the following holds:

- $\tilde{\mathcal{V}}_L = \mathrm{Out}(H)$
- $\tilde{\mathcal{B}}_L = \widehat{G}_{ab} \wedge \widehat{G}_{ab} = 1$
- $\tilde{\mathcal{E}}_L = (\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_k) \wedge (\mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_k) = \mathbb{Z}_n$ for $\mathrm{gcd}(n, k) = 1$ where $n \in \{2, 3, 4\}$ as indicated in the above table.
- $\mathcal{R}_L = 1$, as there are no direct factors of G .

Hence $\text{Aut}_{br,L}(DG\text{-mod}) \simeq \text{Out}(H) \rtimes \mathbb{Z}_n$.

Claim 7.2.3. The decomposition we proposed in Conjecture 7.0.1 is also true for the full Brauer-Picard group for the G above. More precisely

$$\begin{aligned} \text{BrPic}(\text{Rep}(G)) &= \text{im}(\text{Ind}_{\text{Vect}_G}) \cdot \text{im}(\text{Ind}_{\text{Rep}(G)}) \cdot \mathcal{R} \\ &= \text{Out}(G) \rtimes \text{H}^2(G, k^\times) \cdot \mathbb{Z}_n \cdot 1 \end{aligned}$$

- $\text{im}(\text{Ind}_{\text{Vect}_G}) = \text{H}^2(G, k^\times)$
- $\text{im}(\text{Ind}_{\text{Rep}(G)}) \simeq \tilde{\mathcal{E}} \simeq \tilde{\mathcal{E}}_L \simeq \mathbb{Z}_n$
- No reflections, as there is no semidirect decomposition of G .

Proof. Let N be a normal abelian subgroup of G . Then the image $\pi(N)$ of the surjection $\pi : G \rightarrow H$ is a normal abelian subgroup of H . Since H is simple and non-abelian, N has to be a subgroup of the center $\ker(\pi) = Z = Z(G)$. Further, since $\widehat{G} = \widehat{G_{ab}} = 1$, the only 1-dimensional simple object in $\mathcal{L}_{1,1}$ is \mathcal{O}_1^1 . On the other hand, if $[\mu] \in \text{H}^2(N, k^\times)$ is degenerate on a non-trivial N , hence if there exists a $n \in N$ such that $\mu(n, \cdot)\mu(\cdot)^{-1} = 1$, then $\mathcal{L}_{N,\mu}$ has at least two (non-isomorphic) 1-dimensional simple objects, namely \mathcal{O}_1^1 and \mathcal{O}_n^1 . This implies, all $\mathcal{L}_{N,\mu} \in \mathbb{L}_0(G)$ with $N \neq 1$ must have a non-degenerate μ .

Recall that an element in $\text{im}(\text{Ind}_{\text{Rep}(G)})$ determined by an $a \in \text{Hom}_{alt}(\widehat{Z}, Z)$ sends \mathcal{O}_1^χ to $\mathcal{O}_{a(\chi')}^\chi$ where $\chi' : Z \rightarrow k^\times$ the 1-dimensional character determined by χ restricted to Z . Given a pair $(N, [\mu])$ where N is a normal central subgroup of G and μ non-degenerate we give $a \in \text{Hom}_{alt}(\widehat{Z}, Z)$ such that $a(\mathcal{L}_{1,1}) = \mathcal{L}_{N,\mu}$. Since μ is non-degenerate, $b : N \xrightarrow{\sim} \widehat{N}$ defined by $b(n)(n') = \mu(n, n')\mu^{-1}(n', n)$ is bijective. We claim that $a \in \text{Hom}_{alt}(\widehat{Z}, Z)$ defined by $a(\chi) := b^{-1}(\chi|_N)$ does the job. Using that N is a central normal subgroup, we see that $N = \text{im}(a)$. Also, μ is indeed the cocycle determined by a because: $\chi(n') = b(n, n') = \mu(n, n')\mu^{-1}(n', n)$ for $a(\chi) = n$ and all $n' \in N$. This proves that $a(\mathcal{L}_{1,1}) = \mathcal{L}_{N,\mu}$. The action of $\text{Hom}_{alt}(\widehat{Z}, Z) \simeq \mathbb{Z}_n$ on $\mathbb{L}_0(G)$ is therefore indeed transitive. All elements of \mathbb{Z}_n act differently on $\mathbb{L}_0(G)$ and therefore $\tilde{\mathcal{E}}_L \simeq \mathcal{E}_L$. Also, in this case, the lazy elements $\mathcal{E}_L \simeq \mathbb{Z}_n$ already give all $\text{im}(\text{Ind}_{\text{Rep}(G)}) \simeq \mathbb{Z}_n$. The only non-lazy terms come from $\text{im}(\text{Ind}_{\text{Vect}_G}) = \text{H}^2(G, k^\times)$ in the stabilizer. □

7.2.4 Symmetric group \mathbb{S}_3

For $G = \mathbb{S}_3$ the following holds

- $\tilde{\mathcal{V}}_L = \text{Out}(\mathbb{S}_3) = 1$
- $\tilde{\mathcal{B}}_L = \hat{\mathbb{S}}_3 \wedge \hat{\mathbb{S}}_3 = \mathbb{Z}_2 \wedge \mathbb{Z}_2 = 1$
- $\tilde{\mathcal{E}}_L = Z(\mathbb{S}_3) \wedge Z(\mathbb{S}_3) = 1$
- $\mathcal{R}_L = 1$, as there are no direct factors.

Hence our result implies that there are no *lazy* braided autoequivalences of $DS_3\text{-mod}$. We now discuss the full Brauer-Picard group of \mathbb{S}_3 which was computed in [NR14] Sec. 8.1: We have the Lagrangian subcategories $\mathcal{L}_{1,1}$, $\mathcal{L}_{\langle(123)\rangle,1}$ and stabilizer $\text{Out}(\mathbb{S}_3) \rtimes H^2(\mathbb{S}_3, k^\times) = 1$. Hence $\text{Aut}_{br}(DS_3\text{-mod}) = \mathbb{Z}_2$.

Claim 7.2.4. The decomposition we proposed in Conjecture 7.0.1 is also true for the full Brauer-Picard group of \mathbb{S}_3 . More precisely

$$\text{BrPic}(\text{Rep}(\mathbb{S}_3)) = \text{im}(\text{Ind}_{\text{Vect}_G}) \cdot \text{im}(\text{Ind}_{\text{Rep}(G)}) \cdot \mathcal{R} = 1 \cdot 1 \cdot \mathbb{Z}_2$$

- $\text{im}(\text{Ind}_{\text{Vect}_G}) = 1$
- $\text{im}(\text{Ind}_{\text{Rep}(G)}) = 1$
- Reflections \mathbb{Z}_2 , generated by the partial dualizations r_N on the semidirect decomposition $\mathbb{S}_3 = \mathbb{Z}_3 \rtimes \mathbb{Z}_2$ with abelian normal subgroup $N = \mathbb{Z}_3$. More precisely r interchanges $\mathcal{L}_{1,1}$, $\mathcal{L}_{\langle(123)\rangle,1}$, the action on \mathcal{O}_1^χ is made explicit in the proof.

Proof. First, $\text{im}(\text{Ind}_{\text{Vect}_G})$ is the stabilizer $\text{Out}(\mathbb{S}_3) \rtimes H^2(\mathbb{S}_3, k^\times) = 1$. Second, [Dav01] states that $\text{Aut}_{mon}(\text{Rep}(G))$ is a subset of the set of pairs consisting of an abelian normal subgroup and a *non-degenerate* G -invariant cohomology class on this subgroup. The only nontrivial normal abelian subgroup for \mathbb{S}_3 is cyclic and hence there is no such pair, thus $\text{im}(\text{Ind}_{\text{Rep}(\mathbb{S}_3)}) = 1$.

We apply the general considerations in Section 7.1: The Clifford decomposition of the restrictions $\text{triv}|_N$, $\text{sgn}|_N$, $\text{ref}|_N$ to $N = \mathbb{Z}_3$ is $1, 1, \zeta \oplus \zeta^2$ respectively. In the last case \mathbb{Z}_2 is acting by interchanging the summands (resp. by Galois action), the inertia group being trivial. We get $r(\mathcal{O}_1^{\text{ref}}) = \mathcal{O}_{(123)}^1$ and the partial dualization r maps

$$\mathcal{L}_{1,1} = \langle \mathcal{O}_1^{\text{triv}}, \mathcal{O}_1^{\text{sgn}}, \mathcal{O}_1^{\text{ref}} \rangle \longmapsto \mathcal{L}_{\langle(123)\rangle,1} = \langle \mathcal{O}_1^{\text{triv}}, \mathcal{O}_1^{\text{sgn}}, \mathcal{O}_{(123)}^1 \rangle$$

□

7.2.5 Symmetric group \mathbb{S}_4

For $G = \mathbb{S}_4$ the following holds:

- $\tilde{\mathcal{V}}_L = \text{Out}(\mathbb{S}_4) = 1$
- $\tilde{\mathcal{B}}_L = \hat{\mathbb{S}}_4 \wedge \hat{\mathbb{S}}_4 = \mathbb{Z}_2 \wedge \mathbb{Z}_2 = 1$
- $\tilde{\mathcal{E}}_L = Z(\mathbb{S}_4) \wedge Z(\mathbb{S}_4) = 1$
- $\mathcal{R}_L = 1$, as there are no direct factors.

Hence, your result implies that there are no *lazy* braided autoequivalences of $DS_4\text{-mod}$.

The full Brauer-Picard group of \mathbb{S}_4 was computed in Sec. 8.2. [NR14]. Denote the standard irreducible representations of \mathbb{S}_4 by triv , sgn , ref2 , ref3 , $\text{ref3} \otimes \text{sgn}$.

	1	(12)(34)	(12)	(1234)	(123)
triv	1	1	1	1	1
sgn	1	1	-1	-1	1
ref2	2	2	0	0	-1
ref3	3	-1	1	-1	0
ref3 \otimes sgn	3	-1	-1	1	0

There is a unique abelian normal subgroup $N = \{1, (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. We have three Lagrangian subcategories $\mathcal{L}_{1,1}, \mathcal{L}_{N,1}, \mathcal{L}_{N,\mu}$ for $N \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. The stabilizer is $\text{Out}(\mathbb{S}_4) \rtimes \text{H}^2(\mathbb{S}_4, k^\times) = \mathbb{Z}_2$. In particular, $\text{Aut}_{br}(D\mathbb{S}_4\text{-mod})$ has order 6. One checks, that the nontrivial $[\beta] \in \text{H}^2(\mathbb{S}_4, k^\times)$ restricts to the nontrivial $[\mu]$ on N , hence

$$[\beta] : \mathcal{L}_{1,1}, \mathcal{L}_{N,1}, \mathcal{L}_{N,\mu} \longmapsto \mathcal{L}_{1,1}, \mathcal{L}_{N,\mu}, \mathcal{L}_{N,1}$$

and by order and injectivity, we have $\text{Aut}_{br}(D\mathbb{S}_4\text{-mod}) \cong \mathbb{S}_3$.

Claim 7.2.5. The decomposition we proposed in Conjecture 7.0.1 is also true for the full Brauer-Picard group of \mathbb{S}_4 . More precisely

$$\begin{aligned} \text{BrPic}(\text{Rep}(\mathbb{S}_4)) &= \text{im}(\text{Ind}_{\text{Vect}_G}) \cdot \text{im}(\text{Ind}_{\text{Rep}(G)}) \cdot \mathcal{R} \\ &= \mathbb{Z}_2 \cdot \mathbb{Z}_2 \cdot \mathbb{Z}_2 = \mathbb{S}_3 \end{aligned}$$

- $\text{im}(\text{Ind}_{\text{Vect}_G}) = \mathbb{Z}_2$ generated by the nontrivial cohomology class $[\beta]$ of \mathbb{S}_4 with action on $\mathbb{L}_0(G)$ described above. Note that $[\beta]$ restricts to the unique nontrivial cohomology class $[\mu]$ on N .
- $\text{im}(\text{Ind}_{\text{Rep}(G)}) = \mathbb{Z}_2$ generated by the non-lazy monoidal autoequivalence F of $\text{Rep}(\mathbb{S}_4)$, described in detail in Sect. 8 of [Dav01]. $E_F \in \text{Aut}_{br}(DG\text{-mod})$ interchanges $\mathcal{L}_{1,1}, \mathcal{L}_{\langle(123)\rangle,\mu}$.
- Reflections $\mathcal{R} \cong \mathbb{Z}_2$, generated by the reflection $r = r_N$ on the semidirect decomposition $\mathbb{S}_4 = N \rtimes \mathbb{S}_3$ with abelian kernel N . More precisely r interchanges $\mathcal{L}_{1,1}, \mathcal{L}_{\langle(123)\rangle,1}$.

Proof. The stabilizer $\text{im}(\text{Ind}_{\text{Vect}_G})$ and its action on $\mathbb{L}_0(G)$ has already been calculated. To compute $\text{im}(\text{Ind}_{\text{Rep}(\mathbb{S}_4)})$ note that $\text{Aut}_{mon}(\text{Rep}(\mathbb{S}_4))$ has been explicitly computed in Sect. 8 of [Dav01]: Since there is only one ontrivial normal subgroup $N = \mathbb{Z}_2 \times \mathbb{Z}_2$ and only one (up to coboundary) non-degenerate 2-cocycle μ on N , which is G -invariant *only* as a cohomology class $[\mu]$. In [Dav01] it is shown that this gives rise to a (non-lazy) monoidal autoequivalence F of $\text{Rep}(\mathbb{S}_4)$ such that $F(\text{ref3}) = \text{ref3} \otimes \text{sgn}$ which corresponds to mapping $[(12)]$ to $[(1234)]$. This automorphism is visible as a symmetry of the character table.

We compute the action of $E_F \in \text{im}(\text{Ind}_{\text{Rep}(\mathbb{S}_4)})$ on all \mathcal{O}_1^χ . First, $\chi = \text{triv}, \text{sgn}, \text{ref2}$ restricted to N are trivial representations. Second, the possible images

$$E_F(\mathcal{O}_1^{\text{ref3}}) = \mathcal{O}_g^\chi, \quad E_F(\mathcal{O}_1^{\text{ref3} \otimes \text{sgn}}) = \mathcal{O}_{g'}^{\chi'}$$

belong to the G -conjugacy classes in N , i.e. $g, g' = 1$ or $g, g' = (12)(34)$. They have to fulfill the characterization outlined in general considerations above, namely:

$$F(\text{ref}) = \text{ref} \otimes \text{sgn} \stackrel{!}{=} \text{Ind}_{\text{Cent}(g)}^G(\chi) \quad F(\text{ref} \otimes \text{sgn}) = \text{ref} \stackrel{!}{=} \text{Ind}_{\text{Cent}(g')}^G(\chi')$$

Assume $g = g' = 1$. This implies that $E_F(\mathbb{L}_0(G)) = \mathbb{L}_0(G)$ and thus E_F is in the stabilizer, which is $\text{Out}(\mathbb{S}_4) \times \text{H}^2(\mathbb{S}_4, k^\times)$. This is not possible since E_F acts nontrivial on objects and does not come from an automorphism of G . Therefore we take $g, g' = (12)(34)$ and consider

$$F(\text{ref3}) = \text{ref3} \otimes \text{sgn} \stackrel{!}{=} \text{Ind}_{\text{Cent}(12)(34)}^G(\chi) \quad F(\text{ref3} \otimes \text{sgn}) = \text{ref3} \stackrel{!}{=} \text{Ind}_{\text{Cent}(12)(34)}^G(\chi')$$

where $\text{Cent}(12)(34) = \langle (12), (13)(24) \rangle \cong \mathbb{D}_4$. The character table quickly returns the only possible χ, χ' :

$$E_F(\mathcal{O}_1^{\text{ref3}}) = \mathcal{O}_{(12)(34)}^{(-)} \quad E_F(\mathcal{O}_1^{\text{ref3} \otimes \text{sgn}}) = \mathcal{O}_{(12)(34)}^{(+-)}$$

where $(++)$, $(--)$, $(+-)$, $(-+)$ are the four 1-dimensional irreducible representations of $\mathbb{D}_4 = \langle (12), (13)(24) \rangle$ where the first generator acts by the first ± 1 in the bracket and the second generator by the second ± 1 . We see that $\chi|_N$ and $\chi'|_N$ are nontrivial, hence in $\mathcal{L}_{N,\mu}$ for μ nontrivial and

$$\begin{aligned} E_F : \mathcal{L}_{1,1} &= \left\langle \mathcal{O}_1^{\text{triv}}, \mathcal{O}_1^{\text{sgn}}, \mathcal{O}_1^{\text{ref2}}, \mathcal{O}_1^{\text{ref3}}, \mathcal{O}_1^{\text{ref3} \otimes \text{sgn}} \right\rangle \\ \longmapsto \mathcal{L}_{N,\mu} &= \left\langle \mathcal{O}_1^{\text{triv}}, \mathcal{O}_1^{\text{sgn}}, \mathcal{O}_1^{\text{ref2}}, \mathcal{O}_{(12)(34)}^{(-)}, \mathcal{O}_{(12)(34)}^{(+-)} \right\rangle \end{aligned}$$

We finally calculate the action of the partial dualization r on the decomposition $\mathbb{S}_4 = N \rtimes \mathbb{S}_3$. The general considerations in Section 7.1 imply the following for the images $r(\mathcal{O}_1^\chi)$: Since $\chi = \text{triv}, \text{sgn}, \text{ref2}$ restricted to N are trivial, these are fixed. For $\chi = \text{ref3}, \chi' = \text{ref3} \otimes \text{sgn}$ the restrictions are easily determined by the character table to be

$$\chi|_N = \chi'|_N = (-+) \oplus (+-) \oplus (--)$$

which returns via $\delta : kN \rightarrow k^N$ precisely the conjugacy class $[(12)(34)]$ and the inertia subgroup is $I = N \rtimes \langle (12) \rangle$. To see the action on the centralizer, we restrict the representations χ, χ' to I and extend it trivially to $I = \text{Cent}(12)(34) = \langle (12), (13)(24) \rangle \cong \mathbb{D}_4$ yielding finally:

$$\begin{aligned} r(\mathcal{O}_1^{\text{ref}}) &= \mathcal{O}_{(12)(34)}^{(++)} \quad r(\mathcal{O}_1^{\text{ref} \otimes \text{sgn}}) = \mathcal{O}_{(12)(34)}^{(-+)} \\ r : \mathcal{L}_{1,1} &= \left\langle \mathcal{O}_1^{\text{triv}}, \mathcal{O}_1^{\text{sgn}}, \mathcal{O}_1^{\text{ref2}}, \mathcal{O}_1^{\text{ref3}}, \mathcal{O}_1^{\text{ref3} \otimes \text{sgn}} \right\rangle \\ \longmapsto \mathcal{L}_{N,1} &= \left\langle \mathcal{O}_1^{\text{triv}}, \mathcal{O}_1^{\text{sgn}}, \mathcal{O}_1^{\text{ref2}}, \mathcal{O}_{(12)(34)}^{(++)}, \mathcal{O}_{(12)(34)}^{(-+)} \right\rangle \end{aligned}$$

□

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Summary

The study of the Brauer-Picard group $\text{BrPic}(\text{Vect}_G) \simeq \text{Aut}_{br}(DG\text{-mod})$ is motivated by the study of symmetries of Dijkgraaf-Witten theories with a finite structure group G and the classification of group extensions of fusion categories (see Introduction). The goal of this thesis is to gain a firm understanding and calculational control over the structure of $\text{BrPic}(\text{Vect}_G)$ by decomposing it into natural and easy to handle subgroups; analogous to the Bruhat decomposition of a group with a Tits-system. This can be seen as the beginning of a structure theory of the Brauer-Picard group.

In Theorem 6.6.1 we provide a decomposition of the subgroup of braided lazy autoequivalences $\text{Aut}_{br,L}(DG\text{-mod}) \subset \text{Aut}_{br}(DG\text{-mod})$ into $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L$ and \mathcal{R}_L that is based on the decomposition of the group of Hopf automorphisms $\text{Aut}_{Hopf}(DG)$ in Theorem 4.2.1. Let us summarize the main ideas:

By Definition 6.0.2, braided autoequivalences of $DG\text{-mod}$ are given by

$$\underline{\text{Aut}}_{br,L}(DG\text{-mod}) = \{ (\phi, \sigma) \in \text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*) \mid (F_\phi, J^\sigma) \text{ braided} \}$$

where $Z_L^2(DG^*)$ is the group of lazy 2-cocycles on DG^* . In Theorem 4.2.1 we prove a decomposition of $\text{Aut}_{Hopf}(DG)$ into natural subgroups $V \simeq \text{Aut}(G)$, $V_c \simeq \text{Aut}_c(G)$, $B \simeq \text{Hom}(G, \hat{G})$, $E \simeq \text{Hom}(\widehat{Z(G)}, Z(G))$ and the set of reflections R , consisting of triples (H, C, δ) where $G = H \times C$ for C abelian and $\delta : kC \xrightarrow{\sim} k^C$ a Hopf isomorphism. In Chapter 6 we combine trivial 2-cocycles on DG^* with V to construct \mathcal{V}_L , conjugation invariant 2-cocycles $Z_{inv}^2(G, k^\times)$ with B to construct \mathcal{B}_L , central 2-cocycles $Z_c^2(k^G)$ with E to construct \mathcal{E}_L and central bialgebra pairings $P_c(kG, k^G)$ with R to construct partial dualizations \mathcal{R}_L . In the Propositions 6.2.1, 6.3.2, 6.4.1 and 6.5.1 we give a characterization of the corresponding images $\tilde{\mathcal{V}}_L, \tilde{\mathcal{B}}_L, \tilde{\mathcal{E}}_L, \tilde{\mathcal{R}}_L$ under the quotient map $\underline{\text{Aut}}_{br,L}(DG\text{-mod}) \rightarrow \text{Out}_{Hopf}(DG) \times H_L^2(DG^*)$. We claim in Theorem 6.6.1 that the constructed $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L$ and \mathcal{R}_L give a useful decomposition of $\underline{\text{Aut}}_{br,L}(DG\text{-mod})$ and similarly for the quotients $\tilde{\mathcal{V}}_L, \tilde{\mathcal{B}}_L, \tilde{\mathcal{E}}_L, \tilde{\mathcal{R}}_L$ and the images $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ under the quotient map $\widetilde{\text{Aut}}_{br,L}(DG\text{-mod}) \rightarrow \text{Aut}_{br,L}(DG\text{-mod})$.

To show that the decompositions in Theorem 6.6.1 (i), (ii) and (iii) hold, we start with a pair $(\phi, \sigma) \in \underline{\text{Aut}}_{br,L}(DG\text{-mod})$ and use Theorem 4.2.1 to decompose ϕ . Using Lemma 6.1.2, we conclude certain properties on the components of the decomposition of ϕ and use these properties together with Propositions 6.3.2 (iii) and 6.4.1 (iii) to construct lifts to $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ so that multiplying (ϕ, σ) with these lifts reduces (ϕ, σ) to $(1, \sigma')$. Finally, we use Lemma 5.0.11 to show that σ' is, up to an almost lazy coboundary, a distinguished conjugation invariant 2-cocycle $\beta' \in Z_{inv}^2(G, k^\times)$.

Since $(1, \beta') \in \mathcal{B}_L$ by definition, we have proven that the decompositions in Theorem 6.6.1 indeed holds.

Thus, Theorem 6.6.1 and Theorem 4.2.1 are the main results of this thesis. Secondary results of this thesis are

- The construction of $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ and the characterization of its quotients $\tilde{\mathcal{V}}_L, \tilde{\mathcal{B}}_L, \tilde{\mathcal{E}}_L, \tilde{\mathcal{R}}_L$ in the Propositions 6.2.1, 6.3.2, 6.4.1, and 6.5.1.
- Partial results on the decomposition of $H_L^2(DG^*)$ in Chapter 5. In particular, Proposition 5.0.8, Lemma 5.0.10 and Lemma 5.0.11.

In addition, using Theorem 6.6.1 in the case $G = A$ abelian, we prove in Proposition 2.3.3 that the functor

$$\Phi_{\mathcal{M}(H,\eta)} : Z(\text{Vect}_A) \xrightarrow{\cong} Z(\text{Vect}_A)$$

from Theorem 1.1 [ENO10] (see also Theorem 2.2.3) is equivalent to the transmission functor

$$F_{H,\eta} : Z(\text{Vect}_A) \xrightarrow{\cong} Z(\text{Vect}_A)$$

(see equation 2.15) for all invertible pairs (H, η) . This result gives a field theoretic interpretation of the isomorphism $\text{BrPic}(\text{Vect}_A) \simeq \text{Aut}_{br}(DA\text{-mod})$.

Zusammenfassung

Die Untersuchung der Brauer-Picard Gruppe $\text{BrPic}(\text{Vect}_G) \simeq \text{Aut}_{br}(DG\text{-mod})$ ist motiviert durch Symmetrien von Dijkgraaf-Witten Theorien mit einer endlichen Strukturgruppe G und der Klassifikation von Gruppenerweiterungen von Fusion-kategorien (siehe Einleitung). Unser Ziel ist es ein festes Verständnis und rechnerische Kontrolle über die Struktur der Gruppe $\text{BrPic}(\text{Vect}_G)$ zu erlangen, indem wir diese in natürliche und einfach zu handhabende Untergruppen zerlegen; analog zur Bruhat-Zerlegung einer Gruppe mit Tits-System. Dies kann als Anfang einer Strukturtheorie der Brauer-Picard Gruppe angesehen werden.

In Theorem 6.6.1 zerlegen wir die Untergruppe der verzopften 'lazy' Autoäquivalenzen $\text{Aut}_{br,L}(DG\text{-mod}) \subset \text{Aut}_{br}(DG\text{-mod})$ in $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L$ und \mathcal{R}_L , basierend auf der Zerlegung der Gruppe der Hopf-Automorphismen $\text{Aut}_{Hopf}(DG)$ in Theorem 4.2.1. Im Folgenden fassen wir kurz zusammen wie man $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ konstruiert und die Zerlegung in Theorem 6.6.1 beweist.

In Definition 6.0.2 haben wir verzopfte 'lazy' Autoäquivalenzen von $DG\text{-mod}$ definiert als

$$\underline{\text{Aut}}_{br,L}(DG\text{-mod}) = \{(\phi, \sigma) \in \text{Aut}_{Hopf}(DG) \times Z_L^2(DG^*) \mid (F_\phi, J^\sigma) \text{ verzopft} \}$$

hierbei bezeichnen wir die Gruppe der 'lazy' 2-Kozykel auf DG^* mit $Z_L^2(DG^*)$. In Theorem 4.2.1 beweisen wir die Zerlegung von $\text{Aut}_{Hopf}(DG)$ in natürliche Untergruppen $V \simeq \text{Aut}(G)$, $V_c \simeq \text{Aut}_c(G)$, $B \simeq \text{Hom}(G, \hat{G})$, $E \simeq \text{Hom}(\overline{Z}(\hat{G}), Z(G))$ und die Menge der Reflexionen R , die aus Tripeln (H, C, δ) besteht mit $G = H \times C$ für abelsches C und $\delta : kC \xrightarrow{\sim} k^C$ ein Hopf-Isomorphismus. In Kapitel 6 kombinieren wir triviale 2-Kozykel auf DG^* mit V um \mathcal{V}_L zu konstruieren, konjugationsinvariante 2-Kozykel $Z_{inv}^2(G, k^\times)$ mit B um \mathcal{B}_L zu konstruieren, zentrale 2-Kozykel $Z_c^2(k^G)$ mit E um \mathcal{E}_L zu konstruieren und zentrale Bialgebra-Paarungen $P_c(kG, k^G)$ mit Reflexionen R um partielle Dualisierungen \mathcal{R}_L zu konstruieren. In den Propositionen 6.2.1, 6.3.2, 6.4.1 und 6.5.1 geben wir eine Charakterisierung der zugehörigen Bilder $\tilde{\mathcal{V}}_L, \tilde{\mathcal{B}}_L, \tilde{\mathcal{E}}_L, \tilde{\mathcal{R}}_L$ unter der Quotientenabbildung $\underline{\text{Aut}}_{br,L}(DG\text{-mod}) \rightarrow \text{Out}_{Hopf}(DG) \times H_L^2(DG^*)$ an. Wir behaupten in Theorem 6.6.1, dass die konstruierten $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ eine sinnvolle Zerlegung von $\underline{\text{Aut}}_{br,L}(DG\text{-mod})$ geben, dass genauso die Quotienten $\tilde{\mathcal{V}}_L, \tilde{\mathcal{B}}_L, \tilde{\mathcal{E}}_L, \tilde{\mathcal{R}}_L$ eine sinnvolle Zerlegung von $\widetilde{\text{Aut}}_{br,L}(DG\text{-mod})$ geben und dass die Bilder $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ unter der Abbildung $\widetilde{\text{Aut}}_{br,L}(DG\text{-mod}) \rightarrow \text{Aut}_{br,L}(DG\text{-mod})$ eine sinnvolle Zerlegung von $\text{Aut}_{br,L}(DG\text{-mod})$ geben.

Um zu zeigen, dass die Zerlegungen in Theorem 6.6.1 (i), (ii) und (iii) gelten, starten

wir mit einem Paar $(\phi, \sigma) \in \underline{\text{Aut}}_{br,L}(DG\text{-mod})$ und benutzen Theorem 4.2.1 um ϕ zu zerlegen. Mit Lemma 6.1.2 können wir bestimmte Eigenschaften der Komponenten der Zerlegung von ϕ zeigen und zusammen mit den Propositionen 6.3.2 (iii) und 6.4.1 (iii) bestimmte Liftungen in $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ konstruieren, sodass die Multiplikation von (ϕ, σ) mit diesen Liftungen das Element (ϕ, σ) zu $(1, \sigma')$ reduziert. Dann benutzen wir Lemma 5.0.11 um zu zeigen, dass σ' , bis auf einen 'almost lazy' Korand, ein 'distinguished' konjugationsinvarianter 2-Kozykel $\beta' \in Z_{inv}^2(G, k^\times)$ ist. Weil nach Definition $(1, \beta') \in \mathcal{B}_L$, ist die in Theorem 6.6.1 behauptete Zerlegung bewiesen.

Hauptergebnisse der vorliegenden Arbeit sind Theorem 6.6.1 und Theorem 4.2.1. Sekundärergebnisse dieser Arbeit sind

- Die Konstruktion von $\mathcal{V}_L, \mathcal{B}_L, \mathcal{E}_L, \mathcal{R}_L$ und die Charakterisierung der Quotienten $\tilde{\mathcal{V}}_L, \tilde{\mathcal{B}}_L, \tilde{\mathcal{E}}_L, \tilde{\mathcal{R}}_L$ in den Propositionen 6.2.1, 6.3.2, 6.4.1, und 6.5.1.
- Teilresultate zu der Zerlegung von $H_L^2(DG^*)$ in Kapitel 5. Insbesondere, Proposition 5.0.8, Lemma 5.0.10 und Lemma 5.15.

Außerdem, benutzen wir Theorem 6.6.1 um in Proposition 2.3.3 zu zeigen, dass im Fall $G = A$ abelsch, der Funktor

$$\Phi_{\mathcal{M}(H,\eta)} : Z(\text{Vect}_A) \xrightarrow{\cong} Z(\text{Vect}_A)$$

aus Theorem 1.1 [ENO10] (siehe auch Theorem 2.2.3) equivalent ist zum Transmissions Funktor

$$F_{H,\eta} : Z(\text{Vect}_A) \xrightarrow{\cong} Z(\text{Vect}_A)$$

(see equation 2.15) für alle invertierbare Paare (H, η) . Dieses Ergebnis gibt uns eine feldtheoretische Interpretation des Isomorphismus $\text{BrPic}(\text{Vect}_A) \simeq \text{Aut}_{br}(DA\text{-mod})$.

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den

Unterschrift