Quantization of
Super Teichmüller Spaces

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To my lovely parents and my dear brother

Distance and nearness are attributes of bodies
The journeyings of spirits are after another sort.
You journeyed from the embryo state to rationality
without footsteps or stages or change of place,
The journey of the soul involves not time and place.
And my body learnt from the soul its mode of journeying,
Now my body has renounced the bodily mode of journeying.
It journeys secretly and without form, though under a form.

Jallaludin Rumi (1207-73), The Masnavi

Abstract

The quantization of the Teichmüller spaces of Riemann surfaces has found important applications to conformal field theory and $\mathcal{N} = 2$ supersymmetric gauge theories. We construct a quantization of the Teichmüller spaces of super Riemann surfaces, using coordinates associated to the ideal triangulations of super Riemann surfaces.

A new feature is the non-trivial dependence on the choice of a spin structure which can be encoded combinatorially in a certain refinement of the ideal triangulation. We construct a projective unitary representation of the groupoid of changes of refined ideal triangulations. Therefore, we demonstrate that the dependence of the resulting quantum theory on the choice of a triangulation is inessential.

In the quantum Teichmüller theory, it was observed that the key object defining the Teichmüller theory has a close relation to the representation theory of the Borel half of $U_q(sl(2))$. In our research we observed that the role of $U_q(sl(2))$ is taken by quantum superalgebra $U_q(osp(1|2))$. A Borel half of $U_q(osp(1|2))$ is the super quantum plane. The canonical element of the Heisenberg double of the quantum super plane is evaluated in certain infinite dimensional representations on $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$ and compared to the flip operator from the Teichmüller theory of super Riemann surfaces.

Zusammenfassung

Die Quantisierung der Teichmüller-Räume von Riemannflächen hat wichtige Anwendungen in konformen Feldtheorien und in $\mathcal{N} = 2$ supersymmetrischen Eichtheorien gefunden. Wir konstruieren eine Quantisierung der Teichmüller-Räume von super-Riemannschen Flächen, unter Verwendung von Koordinaten, die mit den idealen Triangulationen der super-Riemannschen Flächen assoziiert sind.

Ein neues Merkmal ist die nichttriviale Abhängigkeit von der Wahl der Spinstruktur, welche kombinatorisch in einer gewissen Verfeinerung der idealen Triangulationen kodiert werden kann. Wir konstruieren eine projektive unitäre Darstellung des Gruppoids der Änderungen der verfeinerten idealen Triangulationen. Dadurch zeigen wir, dass die Abhängigkeit der resultierenden Quantentheorie von der Wahl der Triangulation nicht wesentlich ist.

In der Quanten-Teichmüller-Theorie wurde beobachtet, dass der entscheidende Bestandteil der Teichmüller-Theorie in enger Verbindung mit der Darstellungstheorie der Borelhälfte der $U_q(sl(2))$ steht. Bei unserer Forschung haben wir beobachtet, dass die Rolle der $U_q(sl(2))$ von einer Quanten-Superalgebra übernommen wird. Eine Borelhälfte der $U_q(osp(1|2))$ ist die Quanten-Superebene. Das kanonische Element des Heisenbergdoppels der Quanten-Superebene wird in einer bestimmten unendlichdimensionalen Darstellung auf $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$ ausgewertet und mit dem Flip-Operator der Teichmüller-Theorie von super-Riemannflächen vergleichen.

Declaration

I herewith declare, on oath, that I have produced this thesis without the prohibited assistance of third parties and without making use of aids other than those specified. This thesis has not been presented previously in identical or similar form to any other German or foreign examination board. This thesis is based on the publication [1], as well as [2] which is in preparation.

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den Unterschrift

Contents

Chapter 1

Introduction

Einstein's theory of gravity and quantum field theory have proven to be appropriate frameworks to explain some of the observed features of physics, from elementary particles like electrons and protons to cosmology and the evolution of the universe. There remain however unresolved fundamental problems. String theory may be offering answers to many of these questions, such as the unification of all interactions, including gravity, and the physics of strongly interacting quantum field theories.

Low-energy limits of string theory can often be identified with some quantum field theories. One may expect the existence of a low-energy limit of string theory with a certain amount of supersymmetry, but there is no known quantum field theory the limit could correspond to. This expectation has led to a striking prediction in the mid 1990's: There exists a class of six-dimensional interacting conformal quantum field theories known as $(2,0)$-theories [3, 4]. Although little is known about these theories, their existence leads to a geometric description of many supersymmetric field theories in lower dimensions.

Families of four dimensional quantum field theories with $\mathcal{N} = 2$ supersymmetry can be described by means of compactification from the six-dimensional $(2,0)$-theory on spaces of the form $M^4 \times \Sigma$, where Σ is a Riemann surface of genus g with n punctures. This description allows us to relate the main features of the four-dimensional physics to geometric structures on Σ. It seems supersymmetric field theories offer a promising starting point to better understand the non-perturbative phenomena in quantum field theory and by studying different choices of Σ, one can obtain a large class of four dimensional quantum field theories and predict some results from their physics [5, 6].

In addition to the significance of the $(2,0)$-theory for the study of quantum field theories, this theory also plays a role in the remarkable duality conjecture proposed by Alday, Gaiotto and Tachikawa (AGT) in 2009 [7]. AGT established a relation between four-dimensional quantum field theory and correlation functions of a two-dimensional quantum field theory, the so-called Liouville theory (see [8] for a review). Liouville theory is a two dimensional non-rational conformal field theory, where conformal symmetry implies that correlation functions can be represented in a holomorphically factorized form. Liouville theory has the following action [1]

$$S = \int d^2z(\partial\phi\bar{\partial}\phi + \pi\mu e^{2b\phi}), \tag{1.1}$$

where, μ is a cosmological constant and $b \in \mathbb{R}$ is Liouville coupling constant.

Under the AGT correspondence, instanton partition functions [9], which encode non-perturbative effects of $\mathcal{N} = 2$ theories with $SU(2)$ gauge groups, can be expressed in terms of the conformal blocks, the holomorphic blocks of correlations functions, of Liouville conformal field theory on Riemann surface Σ.

Furthermore, the expectation values of certain loop observables in four-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories coincide with the expectation values of natural observables in the quantum theory of moduli spaces of flat connections[2] (see [10] for a review). On the other hand, Liouville conformal blocks are naturally related to certain wave-functions in the quantum theory obtained by quantising the moduli spaces of flat $PSL(2,\mathbb{R})$-connections on certain Riemann surfaces Σ [11]. To explain these relations we need the proper mathematical terminology.

The Teichmüller spaces $\mathcal{T}(\Sigma)$ are the spaces of deformations of complex structures on Riemann surfaces Σ. As there is a unique metric of constant curvature -1 associated with each complex structure, one may identify the Teichmüller spaces with the spaces of deformations of metrics with constant curvature -1. Such metrics naturally define flat $PSL(2,\mathbb{R})$-connections on Σ, relating the Teichmüller spaces to the moduli spaces $\mathcal{M}_{flat}(\Sigma)$ of flat $PSL(2,\mathbb{R})$-connections. The Teichmüller spaces appear as one of the components in moduli of flat $SL(2,\mathbb{R})$ connections [12, 13].

From classical uniformization theorem, there exists a unique constant negative curvature metric on the Riemann surface Σ. In a complex coordinate z, such a metric has the form $ds^2 = e^{2b\phi}dzd\bar{z}$, with ϕ being a solution of the Liouville equation $\partial\bar{\partial}\phi = \mu e^{2\phi}dzd\bar{z}$, which coincides with the equation of motion for the Liouville equation (1.1)[3]. Due to the close connections between Liouville theory and the theory of Riemann surfaces, quantum Liouville theory turns out to have a geometric interpretation as describing the quantization of theories of spaces of two-dimensional metrics with constant negative curvature. Moreover, Verlinde conjectured that the space of conformal blocks in quantum Liouville theory can be identified with the Hilbert spaces obtained by the quantization of Teichmüller spaces of Riemann surfaces [14]. The relation between Liouville theory and quantum Teichmüller theory was established by Teschner in [15, 16][4]. Therefore, there exist relations between quantized moduli spaces of flat $PSL(2,\mathbb{R})$-connections, quantum Teichmüller theory and conformal field theory.

At this point we continue the motivation for studying the supersymmetric version of the picture we outlined above and replace all the basic ingredients by the theories which

[1] This theory has central charge $c = 1 + 6Q^2$, where $Q = b + b^{-1}$.

[2] The space of isomorphism classes of flat G-bundles modulo gauge transformations.

[3] This is the classical equivalence between Liouville and Teichmüller theory.

[4] One can show that the Hilbert spaces of two theories and the mapping class group actions are equivalent.

are established on super Riemann surfaces. It was shown recently that there are generalizations of AGT where super Liouville theory appears instead of ordinary Liouville theory [17]. It seems likely that such generalizations are related to the quantum theory of super Riemann surfaces in a way that is analogous to the relations between gauge, Liouville and the quantum Teichmüller theories [10]. This bring us to the strong motivation to focus our attention on the quantization of super Teichmüller spaces.

Beyond the motivation arising from the supersymmetric gauge theory, topological quantum field theories (TQFT's) are another important motivation for the research presented in this thesis. They give an example for a fruitful interplay between mathematics and physics. TQFT's basic concepts formalize properties that one can expect for a quantum field theory defined by some path integral. Chern-Simons theory is a prominent example of a topological quantum field theory[5]. It describes a non-abelian gauge theory on a three dimensional space manifold. There exists a partial equivalence between Chern Simons theory on three manifolds with boundary and a certain conformal field theory, the so-called WZW model, living on the boundary of these three manifolds [6]. The Chern-Simons theory on a compact spatial manifold gives rise to a finite dimensional Hilbert space which turns out to be isomorphic to the space of conformal blocks of a WZW model.

For Chern-Simons theory on a three-dimensional manifold of the form $M = \mathbb{R}\times\Sigma$, where $\mathbb{R}$ is the time line, the classical phase space is the space of flat connections on Σ. Chern-Simons theory with a compact gauge group G is well studied because of its applications to knot theory and three dimensional topology. Further interesting examples of 3d TQFTs arise from Chern-Simons theories having a non-compact gauge group. The relation between Chern-Simons theories and moduli spaces of flat connections becomes richer when the holonomy of the flat connections takes values in non-compact groups like $G = SL(2,\mathbb{R})$ or $G = SL(2,\mathbb{C})$. The relevant conformal field theories are then non-rational, having continuous families of primary fields (see [20] for a recent review of some of these relations, and [21, 22] for recent progress on Chern-Simons theory with a complex gauge group). Also, the study of Chern-Simons theories associated to non-compact groups appears to have various profound links with three-dimensional hyperbolic geometry [11, 23], [24–26].

Quantum Chern-Simons theory is obtained by quantizing the phase space and therefore quantum Teichmüller theory is a useful tool for studying the quantization of $SL(2,\mathbb{R})$ Chern-Simons theory [27]. In the case which is currently best understood one is dealing with a connected component of the moduli space of flat $PSL(2,\mathbb{R})$-connections on Σ which is isomorphic to the Teichmüller space of Riemann surfaces [12, 13]. Relevant observables acquire the geometric interpretation of quantized geodesic length functions.

[5]It has the action $S = (k\int_\Sigma tr(A\wedge A + \frac{2}{3}A\wedge A\wedge A)$, where k is related to the coupling constant. A is a gauge field, a Lie algebra valued one form.

[6]Ref [14, 18] argued that physical wave functions obeying Gauss law constrains of $SL(2,\mathbb{R})$ Chern Simons theory are Virasoro conformal blocks and provide the quantization of the Teichmüller space of the surface Σ in [15, 19].

Our motivation is to expand the resulting picture to the cases where the groups are replaced by supergroups and to find the quantum super Teichmüller theory. The constructed quantum super Teichmüller would be a starting point for finding the quantization of super Chern-Simons theory for the non-compact supergroup $G = OSp(1|2)$.

Witten realized the relation between non-compact Chern-Simons theory and 2+1 quantum gravity [28][7]. Moreover, the relation of Teichmüller theory with $(2+1)$-dimensional gravity with negative cosmological constant has been already discussed in literature [14, 29]. Such a relation indicates that the super Teichmüller theory may also play an analogous role for $(2 + 1)$-dimensional supergravity and it would be an interesting direction of research.

Another motivation for the study of super Teichmüller theory comes from super string perturbation theory. Understanding super string perturbation theory requires the understanding of subtleties of the superalgebraic geometry of super Riemann surfaces. The Teichmüller theory has an interesting and rich generalization provided by the deformation theory of super Riemann surfaces. Initially motivated by superstring perturbation theory, there has been a lot of research (reviewed in [30]) on the complex analytic theory of super Teichmüller spaces. There exists a uniformization theorem for super Riemann surfaces, describing super Riemann surfaces as quotients of the super upper half plane by discrete subgroups of $OSp(1|2)$ [31]. This provides us with an alternative picture of super Teichmüller theory similar to the perspective on ordinary Teichmüller theory offered by hyperbolic geometry. The theory of super Riemann surfaces should lead to interesting generalizations of two and three dimensional hyperbolic geometry, currently much less developed than the corresponding theories for ordinary Riemann surfaces. This may be expected to lead to a new class of invariants of three manifolds in the future.

Before explaining our approach for the quantization of super Teichmüller theory, we now give some background about quantum Teichmüller theory and the role of quantum groups in this subject.

Quantum Teichmüller theory

Quantization of Teichmüller spaces is a deformation of the algebra of functions on these spaces. Teichmüller spaces of punctured surfaces have been quantized during the 1990s in two different but essentially equivalent ways by Fock and Chekhov [32, 33] and in parallel by Kashaev [34].

Ordinary Teichmüller theory is based on a suitable collection of coordinates associated to the triangles forming a certain type of triangulation[8] of the Riemann surface. One essential ingredient in this theory are the coordinates associated to the triangles. The

[7]Three dimensional Einstein gravity with negative cosmological constant can be performed as a Chern Simons gauge theory with gauge group $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

[8]This type of triangulations is called ideal triangulations. Such a triangulation can be defined by a maximal set of geodesic arcs intersecting only at the punctures of $\Sigma_{g,n}$ representing their start- and end-points. Such a collection of arcs decomposes the surface Σ into a collection of triangles.

spaces of functions on Teichmüller spaces have natural Poisson structures which can be used to formulate quantization problems.

This quantized theory leads to projective infinite dimensional unitary representations of the mapping class groups of punctured surfaces [35], where the projective factor is related to the Virasoro central charge in quantum Liouville theory [36]. The mapping class group is a discrete group of symmetries of the Teichmüller spaces. The action of operators generating the mapping class groups can be constructed using quantum groups as the mathematical tools. Quantum groups have been found to be relevant in conformal field theory, where fusion matrices are realized as 6j symbols for representations of the associated quantum groups. The quantum group structure of Teichmüller theory is consistent with the representation theoretical approach to quantum Liouville theory [37, 38].

At this point we want to comment on the role of quantum groups, as algebraic tools to reach the goal of this thesis. Afterwards, we will continue the details of constructing quantum Teichmüller theory by defining appropriate coordinates on the Riemann surfaces.

Drinfeld [39] and Jimbo [40] have defined certain types of Hopf algebras [9], known as quantum groups, for any finite dimensional complex simple Lie algebra g and more generally for any Kac-Moody algebra. The quantum group $U_q(g)$ is a deformation of the universal enveloping algebra $U(g)$ for a nonzero complex parameter q. The methods coming from the representation theory of quantum groups have found a wide range of applications in mathematical and theoretical physics.

Moreover, quantum groups are quasi-triangular Hopf algebras. A Hopf algebra $\mathcal{A}$ is called quasi-triangular if there exists an element $R \in \mathcal{A} \otimes \mathcal{A}$, the so-called universal R-matrix. Initially, this element has been developed in the context of quantum integrable systems, where it was shown that the R matrix satisfies the so-called Yang-Baxter equation [41, 42]

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{1.2}$$

The universal R-matrix is a canonical element of quantum groups and can be obtained using the Drinfeld double construction. The Drinfeld double construction takes an arbitrary Hopf algebra and its dual and creates a quasi-triangular Hopf algebra which has a R-matrix. From a given Hopf algbera one can make another double construction, called Heisenberg double construction [43]. It admits a canonical element $S \in \mathcal{A} \otimes \mathcal{A}$ similar to the R matrix. However,it satisfies not the Yang-Baxter equation, but the pentagon equation

$$S_{12}S_{13}S_{23} = S_{23}S_{12}. \tag{1.3}$$

Using Heisenberg doubles one can obtain the representations of Drinfeld doubles, because one can embed the elements of the Drinfeld double into a tensor square of the Heisenberg

[9] A Hopf algebra is a bialgebra which satisfies particular axioms.

double [43]. In our research, Heisenberg doubles appear in the context of quantum Teichmüller theory of Riemann surfaces.

Now we return to the Teichmüller theory and explain the suitable coordinates with the aim of quantizing such spaces. As mentioned, there exist useful systems of coordinates associated to a triangulation of Σ, if Σ has at least one puncture. Kashaev assigned a pair of variables (p_i, q_i) to each triangle i, the so-called Kashaev coordinates. The space of these coordinates is equipped with a Poisson structure.

One can transform any two triangulations to each other by a finite composition of elementary transformations ω_{ij}. The flip transformation ω_{ij} changes a quadrilateral, which is formed by two triangles, by replacing the common edge by the opposite diagonal of the quadrilateral as it is illustrated in figure 1.1.

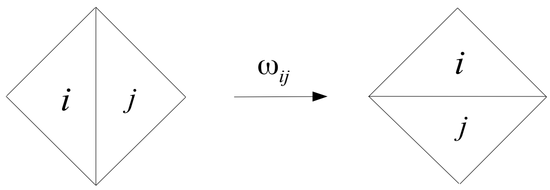


FIGURE 1.1: The flip transformation ω_{ij} rotates clockwise the diagonal.

In quantum Teichmüller theory, Kashaev assigned a Hilbert space $\mathcal{H}_i \backsimeq L^2(\mathbb{R})$ to each triangle of the triangulation. In this theory, Kashaev coordinates become operators p_i,q_i which are the position and momentum self adjoint operators respectively and satisfy the Heisenberg commutation relation $[\mathsf{p}, \mathsf{q}] = \frac{1}{2\pi}$. The classical transformations ω_{ij} is represented by a flip operator which is denoted by $\mathsf{T}_{ij} : \mathcal{H}_i \otimes \mathcal{H}_j \to \mathcal{H}_i \otimes \mathcal{H}_j$. The role of this operator is to describe how the coordinates change at the quantum level.

A basic issue to address in any approach based on triangulations is to demonstrate the independence of the resulting quantum theory from the choice of triangulation. This can be done by constructing unitary operators relating the quantum theories associated to any two given triangulations. Being unitary equivalent, one may identify the quantum theories associated to two different triangulations as different representations of one and the same quantum theory.

Let us finally note that the flip operators $\mathsf{T}_{12}^{(i)}$ have an interesting interpretation within the Heisenberg double construction. The canonical element of Heisenberg double of the Borel half of $U_q(sl(2,\mathbb{R}))$ [44] in quantum groups language can be identified with the flip operator T_{ij}. The constructed operator is unitary and it generates a projective representation of the Ptolemy groupoid describing the transition between different triangulations. The Ptolemy groupoid includes a particular relation, called pentagon [34, 44],

$$\mathsf{T}_{jk}\mathsf{T}_{ik}\mathsf{T}_{ij} = \mathsf{T}_{ij}\mathsf{T}_{jk}. \tag{1.4}$$

According to Kashaev, each operator T_{ij} is expressed as follows:

$$\mathsf{T}_{ij} = e_{\mathrm{b}}(\mathsf{q}_i + \mathsf{p}_j - \mathsf{q}_j)e^{-2\pi i \mathsf{p}_i \mathsf{q}_j}, \tag{1.5}$$

where the Faddeev's quantum dilogarithm e_{b} [45, 46] is a particular special function defined as

$$e_{\mathrm{b}}(z) = \exp\left(\int_C \frac{e^{-2izw}}{\sinh(w\mathrm{b})\sinh(w/\mathrm{b})}\frac{dw}{4w}\right), \tag{1.6}$$

and which can be regarded as a quantization of the Roger's dilogarithm. Faddeev's quantum dilogarithm [47] finds its origins and applications in quantum integrable systems [15, 48–50] and it has already been used in formal state-integral constructions of invariants of three manifolds in the following works [23, 25, 26, 51, 52]. The Faddeev's quantum dilogarithm also found applications in conformal field theory, topological field theory and hyperbolic geometry.

Super Teichmüller theory and quantum supergroup

The super Teichmüller theory is the Teichmüller theory of super Riemann surfaces. For the classical super Teichmüller theory, Penner and Zeitlin [53] recently provided a super symmetric version of the so-called Penner-λ-length coordinate [54] which has a connection to super Minkowski geometry. Bouschbacher [55] provided other coordinates by using a different treatment of spin structures and based upon quite a different approach using so-called shear coordinates (Fock coordinates). He constructed shear coordinates for punctured super Riemann surfaces equipped with an ideal triangulation and defined a super Poisson structure on this space using these coordinates.

In the super Teichmüller spaces, in addition to even coordinates associated to edges of the underlying triangulation one may define additional odd coordinates associated to the triangles. Assigning the so-called Kasteleyn orientations to the edges of a triangle allows one to parametrize the choices of spin structures on super Riemann surfaces. The additional orientation data assigned to a triangulation are used to provide an unambiguous definition of the signs of the odd coordinate.

We used shear coordinates as our coordinates on super Teichmüller space. Our approach for quantizing is similar to the one used by Kashaev [34] for the case of ordinary Teichmüller theory based on a suitable collection of coordinates associated to the triangles forming an ideal triangulation of the surface. As for the ordinary case, the super flip operator T_{ij} also has a quantum groups meaning. Our main idea is to replace the Borel half of $U_q(sl(2,\mathbb{R}))$ of the ordinary case, by a suitable quantum superalgebra, the Borel half of $U_q(osp(1|2))$ and establish the quantization.

Before explaining our approach to quantization and presenting our main results, we briefly give a background of the superalgebra $osp(1|2)$ and its role in super Liouville theory.

The superalgebra $osp(1|2)$ is a graded extension of the $sl(2)$ algebra and was first introduced by Kulish in [56]. The simplest non-rational supersymmetric CFT theory is $\mathcal{N} = 1$ supersymmetric Liouville theory, which is related to the superalgebra $osp(1|2)$. This algebra appears as part of $\mathcal{N} = 1$ super conformal symmetry.

The finite dimensional representations of superalgebra $osp(1|2)$ and Racah-Wigner coefficients have already been studied in the literature [57]. Also, super conformal symmetry can be realized in terms of free fields [58–60]. This free fields representation can be used to construct conformal blocks and their behavior under braiding and fusion can be expressed by a quantum deformation of the universal enveloping algbera of $osp(1|2)$. For the series of representation, the Clebsch-Gordan and Racah-Wigner coefficients for the quantum deformed algebras $U_q(osp(1|2))$ have been determined in [61]. Here, it was shown that the associated Racah-Wigner coefficients agree with the fusion matrix in the Neveu-Schwarz sector of $\mathcal{N} = 1$ supersymmetric Liouville field theory.

Approach and summary of main results

Same as for the ordinary case, the symplectic structure of super Teichmüller spaces gives the possibility of canonical quantization [33, 34]. In what follows, we will present our approach for quantizing super Teichmüller theory and summarize our main results.

An important new feature is the dependence of the super Teichmüller theory on the choices of spin structures. Following the approach of Cimansoni and Reshetikhin [62, 63], we encode the choices of spin structures into combinatorial data, Kasteleyn orientations, suitably adapted to the triangulations of our interest.

We assign the Hilbert space $\mathcal{H}_i \backsimeq L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$ to each triangle. Therefore, the Hilbert space associated to the entire super Riemann surface is the tensor product of the spaces for each triangle. In addition to a pair of even variables (q_i, p_i) assigned to each ideal triangle, we introduce an odd variable ξ_i. The collection of these variables is called super Kashaev coordinates. The super Kashaev coordinates get quantized to linear operators on the Hilbert spaces $\mathcal{H}_i$. The coordinates p_i and q_i are replaced by operators satisfying canonical commutation relations and are represented on $L^2(\mathbb{R})$ as multiplication and differentiation operators. The odd coordinate ξ_i becomes an operator acting on $\mathcal{H}_i$ of the form

$$\xi_i = \sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\kappa_i, \qquad \kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbb{C}^{1|1}, \tag{1.7}$$

where $q = e^{i\pi b^2}$ and the quantization constant $\hbar$ is related to b as $\hbar = 4\pi b^2$.

The unitary operators representing changes of triangulations, generate a projective representation of the super Ptolemy groupoid describing the transitions between suitably refined triangulations equipped with Kasteleyn orientations. Suitable choices of orientation on the triangulations lead to different types of super flip operators.

In our results there exist eight possible superflips $T^{(1)}_{ij} \ldots T^{(8)}_{ij}$ and they can be related to each other. The superflip $T^{(1)}_{ij}$ is the one which satisfies the pentagon relation by itself and has the following form

$$\mathsf{T}^{(1)}_{ij} = \frac{1}{2}\Big[\, f_+(\mathsf{q}_i + \mathsf{p}_j - \mathsf{q}_j)\mathbb{I} \otimes \mathbb{I} - i f_-(\mathsf{q}_i + \mathsf{p}_j - \mathsf{q}_j)\,\kappa \otimes \,\kappa\Big] e^{-i\pi \mathsf{p}_i \mathsf{q}_j}, \tag{1.8}$$

where, p,q are the position and momentum self adjoint operators respectively and satisfy the Heisenberg commutation relation $[\mathsf{p}, \mathsf{q}] = \frac{1}{2\pi}$. The two functions f_+, f_- are constructed out of quantum dilogarithm functions and κ is a two by two matrix in $\mathbb{C}^{1|1}$. We also generalize the Ptolemy groupoid relations, including the pentagon relation to the supersymmetric case.

In a similar manner, as for the ordinary case, the flip operator $\mathsf{T}_{ij}^{(1)}$ is found to coincide with the canonical element of the Heisenberg double of the Borel half of $U_q(osp(1|2))$, which is evaluated in certain infinite dimensional representations on $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$. An ongoing project is to construct the basis and dual basis of Heisenberg double and check the identification of the canonical element S with the super flip operator $\mathsf{T}_{ij}^{(1)}$.

There exists also another related project to this thesis regarding the Drinfeld double of $U_q(osp(1|2))$. Using the structure of quantum super Teichmüller theory, we already derived the braiding operator and related R matrix for the quantum groups, Borel half of $U_q(osp(1|2))$. The ongoing project is to check the properties of the R matrix and find the canonical element of Drinfeld double and identify that with our proposed R matrix.

Overview

This thesis is based on the preprint [1] and forthcoming [2] and it is organized as follows.

In **Chapter 2**, we review ordinary Teichmüller theory of Riemann surfaces and its quantization. First we discuss how to parametrize the Teichmüller space using sets of coordinates associated to a triangulation. This triangulation has natural analogues in the case of super Teichmüller theory. Afterwards, we proceed to discuss the quantization of this theory and the projective representation of the Ptolemy groupoid relating the Hilbert spaces assigned to different triangulations.

Chapter 3 includes the introduction of the fundamentals of quantum groups. We introduce the basic notions of Drinfeld and Heisenberg doubles. We use this background for understanding the construction of the flip operator in the ordinary Teichmüller theory. This knowledge will be also useful for calculating the R matrix in chapter 8.

In **Chapter 4** we introduce the notion of a quantum plane and the Heisenberg double of the Borel half of $U_q(sl(2))$. We study the Kashaev representation of the latter. This representation has been shown to be relevant in the quantization of the Teichmüller theory. We explain the steps of an ongoing project to find the basis of the continuous version of the Heisenberg double of the Borel half of $U_q(sl(2))$.

In **Chapter 5**, we discuss classical super Teichmüller theory. In order to encode the choices of spin structure we refine the triangulations into graphs called hexagonalizations. Such graphs with chosen Kasteleyn orientations can be used to define super analogues of the shear coordinates. Changes of hexagonalizations define an analogue of the super Ptolemy groupoid which can be characterized in terms of generators and relations.

Chapter 6 describes the quantization of the classical super Teichmüller theory. We define operators representing analogues of the coordinates used in the work of Fock and Kashaev, respectively. These operators generate the super Ptolemy groupoid describing changes of triangulations. The relations of the super Ptolemy groupoid follow from identities satisfied by suitable variants of Faddeev's quantum dilogarithm.

In **Chapter 7** our goal is to generalize the construction involving Heisenberg double algebras (which allowed us to obtain the canonical element identified with a flip operator of the Teichmüller theory) to the case of the super Teichmüller theory. We start with an introduction to quantum supergroups and we focus on the quantum supergroup $U_q(osp(1|2))$. We explain the steps of an ongoing project to find the basis of the continuous version of the Heisenberg double of the Borel half of $U_q(osp(1|2))$.

Chapter 8 starts with a review of how one can derive the R matrix in the ordinary Teichmüller theory from a geometric point of view and how to check the defining its properties. Then we explain the geometric aspect of R matrix in super Teichmüller theory.

Chapter 2

Ordinary Teichmüller theory

The problem of classifying different structures on Riemann surfaces was of interest from the early on. Bernhard Riemann stated that for a compact Riemann surface of genus $g \geq 2$ the space $\mathcal{M}_{g,0}$ of different conformal structures has a complex dimension $3g - 3$, where the space $\mathcal{M}_{g,n}$ is the Riemann's moduli space of flat connections on punctured Riemann surfaces $\Sigma_{g,n}$. Given that Riemann surfaces can be equivalently defined using either complex analytic or algebra-geometric methods, the Riemann's moduli spaces can be studied in terms of generators and relations extensively from an algebraic geometry point of view. During the late 1930s, Teichmüller followed an analytic approach by using quasiconformal mapping and he defined new, but closely related, spaces called the Teichmüller spaces $\mathcal{T}_{g,n}$.

In order to prepare for the case of super Teichmüller theory, we found it useful to briefly review relevant background on the Teichmüller spaces of deformations of complex structures on Riemann surfaces in this chapter. In the first section we describe relevant background on the classical Teichmüller space. We define ideal triangulations of Riemann surfaces and, within this combinatorial framework, we study Penner coordinates [54], Fock coordinates [32] and Kashaev coordinates [34], which provide us with different parametrizations of the Teichmüller space and the symplectic structure on that. We also study how those coordinates transform under the changes of triangulations of Riemann surfaces, like flips and rotations. For a more comprehensive review we reference [16].

Later in the second section, we study the quantization of Teichmüller theory. We present the operatorial realization of Kashaev and Fock coordinates, as well as the transformations of them under the change of triangulations. We also present the quantum generators of Ptolemy groupoid.

2.1 Classical Teichmüller theory

In the following we will consider two-dimensional surfaces $\Sigma_{g,n}$ with genus $g \geq 0$ and $n \geq 1$ punctures having $2g - 2 + n > 0$. Useful starting points for the quantization of the Teichmüller spaces are the coordinates introduced by Penner [54], and their relatives used in the works of Fock [32], Chekhov and Fock [33] and Kashaev [34]. Using these coordinates one may define an essentially canonical quantization of the Teichmüller spaces which will be expressed in section 2.2.

2.1.1 Riemann surfaces

Here we will shortly recall some facts about Riemann surfaces. A Riemann surface $\Sigma_{g,n}$ is a 1-dimensional complex connected manifold with genus g and n punctures (i.e. the holes with vanishing length) with biholomorphic transition functions. Equivalently, one can define Riemann surfaces as 2-dimensional manifolds equipped with a conformal structure, that is an equivalence class of metrics identified by the property of being related by conformal transformations. We will be interested in a particular sub-class of Riemann surfaces — those having a hyperbolic structure, i.e. those with a metric of constant negative curvature equal to -1.

It is a well known result (dating back to Koebe and Poincaré) that every Riemann surface is conformally equivalent to either the Riemann sphere, the unit disk or the upper half-plane, depending on its curvature, known as a uniformization theorem. The uniformization theorem states that Riemann surfaces $\Sigma_{g,n}$ can be represented as quotients of the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ equipped with the Poincaré metric $ds^2 = \frac{dyd\bar{y}}{(\text{Im}(y))^2}$ by discrete subgroups Γ of $PSL(2,\mathbb{R})$ called Fuchsian groups[1]

$$\Sigma_{g,n} \equiv \mathbb{H}/\Gamma. \tag{2.1}$$

We may represent the points on $\Sigma_{g,n}$ as points in a fundamental domain D in the upper-half plane on which Γ acts properly discontinuously. The n punctures of $\Sigma_{g,n}$ will be represented by a collection of points on the boundary of $\mathbb{H}$ which can be identified with the projective real line $\mathbb{RP}^1$. Figure 2.1 illustrates the uniformization of a once-punctured torus $\Sigma_{1,1}$.

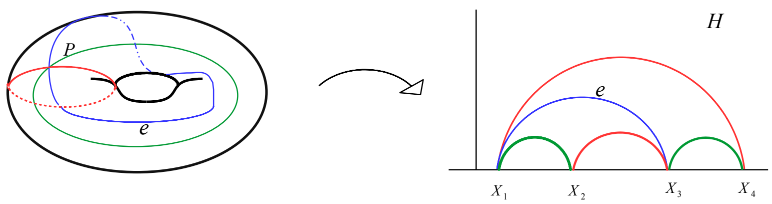


FIGURE 2.1: Realization of a quadrilateral laying on a Riemann surface on the upper half plane.

[1]Discrete subgroups of $PSL(2,\mathbb{R})$ having no elliptic elements.

The Teichmüller space $\mathcal{T}_{g,n}$ of Riemann surfaces $\Sigma_{g,n}$ can be identified with the connected component in

$$\mathcal{T}(\Sigma_{g,n}) = \mathcal{T}_{g,n} = \{\psi : \pi_1(\Sigma_{g,n}) \to PSL(2,\mathbb{R})\}/PSL(2,\mathbb{R}), \qquad (2.2)$$

that contains all Fuchsian representations ψ. The group $PSL(2,\mathbb{R})$ acts on representations ψ by conjugation,

$$\mathbb{H}/\Gamma \simeq \mathbb{H}/\Gamma', \qquad \text{iff} \quad \Gamma' = g\Gamma g^{-1}, \quad g \in PSL(2,\mathbb{R}).$$

2.1.2 Ideal triangulations and fat graphs

In order to study Teichmüller spaces, we need to define local coordinates. There are several ways to do that. Useful sets of coordinates for the Teichmüller spaces can be associated to ideal triangulations of $\Sigma_{g,n}$. Such a triangulation can be defined by a maximal set of geodesic arcs intersecting only at the punctures of $\Sigma_{g,n}$ representing their start- and endpoints. Such a collection of arcs decomposes the surface $\Sigma_{g,n}$ into a collection of triangles. An ideal triangulation τ of Riemann surface $\Sigma_{g,n}$ is defined by $3(2g-2+n)$ arcs, called edges, and has $2(2g-2+n)$ triangles.

The examples of 4-punctures sphere $\Sigma_{0,4}$ and of 1-punctured torus $\Sigma_{1,1}$ are illustrated in figures 2.2 and 2.3.

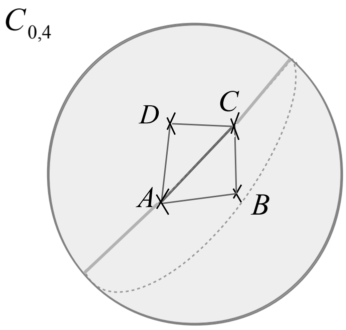


FIGURE 2.2: An ideal triangulation of $\Sigma_{0,4}$.

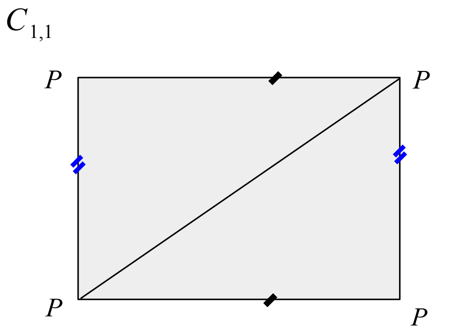


FIGURE 2.3: An ideal triangulation of $\Sigma_{1,1}$.

We will consider ideal triangulation τ of Riemann surfaces and associated to them a dual tri-valent graph, the so-called fat graphs $\varphi(\tau)$ and assign coordinates in a manner such that they transform appropriately under the change of triangulation. An example of a fat graph is illustrated in figure 2.4 and 2.5.

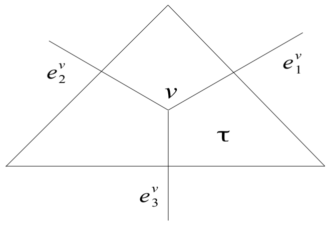


FIGURE 2.4: An ideal triangle with a dual fat graph.

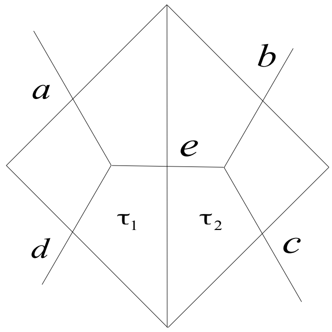


FIGURE 2.5: Two adjacent triangles and the dual fat graph.

2.1.3 Penner coordinates

We want to parametrize $\mathcal{T}_{g,n}$ using ideal triangulations of Riemann surfaces. In order to do that we will take a point p in $\mathcal{T}_{g,n}$ and a triangulation τ and assign coordinates to the edges of triangulation. Penner [54] first introduced such coordinates.

For any surface $\Sigma_{g,n}$ with $n \geq 0$, take the trivial $\mathbb{R}^{>0}$ bundle over $\mathcal{T}_{g,n}$ called decorated Teichmüller space and denoted by $\tilde{\mathcal{T}}_{g,n}$.

Given any point p in the decorated Teichmüller space and ideal triangulation on Σ, the Penner coordinate $l_e(p)$ is defined as the hyperbolic length of the segment δ of each edge e that lies between two horocycles h surrounding the punctures p that e connects. Triangulation of once puncture torus is illustrated in figure 2.6.

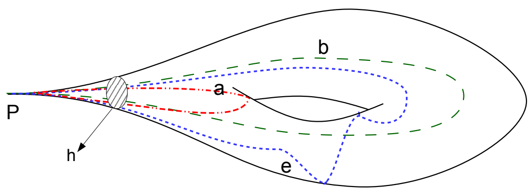


FIGURE 2.6: Triangulation of once-puncture torus.

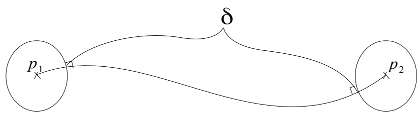


FIGURE 2.7: Length of geodesy between two horocycles.

Then $l(e) = e^{\pm\delta/2}$, while the plus sign is for the case that two horocycles do not intersect and minus sign otherwise (figure 2.7). There are variants of the Penner coordinates which were introduced by Fock and Kashaev in terms of the Penner coordinates as we will discuss next.

2.1.4 Shear coordinates (Fock coordinates)

Let us consider a model of the Riemann surface $\Sigma_{g,n}$ on the upper-half plane. Then, the ideal triangulation will be given by hyperbolic triangles with vertices on the boundary of the upper-half plane.

Useful sets of coordinates may be assigned to the edges of an ideal triangulation by assigning to an edge e separating two triangles as illustrated in figure 2.1 the cross-ratio

$$e^{-z_e} = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_4)(x_2 - x_3)}, \tag{2.3}$$

formed out of the points x_1, x_2, x_3, x_4 on representing the corners of the quadrilateral decomposed into two triangles by the edge e. The resulting set of $6g - 6 + 3n$ coordinate functions may be used to get a system of coordinates for Teichmüller space by taking into account the relations $\sum_{e \in E(P_i)} z_e = 0$, where $E(P)$ is the set of edges ending in puncture P. This combination is in fact a conformal invariant, i.e. is invariant under the action of the $PSL(2, \mathbb{R})$ on the upper half plane $\mathbb{H}$ given by Möbius transformations

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \to \frac{az + b}{cz + d}. \tag{2.4}$$

Moreover, if one uses the action of $PSL(2, \mathbb{R})$ to transform points with coordinates x_1, x_2, x_3, x_4 in a way such that three of them are mapped to the points $0, -1, \infty$, then the last one is mapped to the point with coordinate given by equation (2.3).

We can assign those coordinates to the edges of triangulations in the following way: the quadrilateral composed of points $x_1, \ldots, x_4$ can be triangulated into two triangles, with a common edge e connecting the points x_1 and x_3. We can assign to this edge a conformal cross-ratio given by (2.3)

$$e \to z_e. \tag{2.5}$$

This assignement gives us the Fock coordinates.

For an ideal triangulation of a Riemann surface $\Sigma_{g,n}$ we have $3(2g-2+n)$ edges, therefore, we have the same number of coordinates z_e assigned to the edges of the triangulation, or, equivalently, the edges of the fat graph dual to this triangulation. However, not all of those coordinates are independent of each other — there are in fact constraints imposed on them. In order to specify them, we consider paths along the edges of the fat graph.

Through the properties of the spaces with constant, negative metrics each closed curve can be homotopically deformed into a closed geodesic, and that one can be related to a curve of minimal length along the edges of the fat graph. For every closed curve c, corresponding to a sequence of edges $e_1(c), \ldots, e_{m_c}(c)$ on the fat graph, the following combination of Fock coordinates is not linearly independent

$$f_{\varphi,c} = \sum_{i=0}^{m_c} z_{e_i(c)} = 0. \tag{2.6}$$

As already mentioned, Fock coordinates are a variant of Penner coordinates. The dependence of the Penner coordinates on the choice of horocycles drops out in the Fock coordinates. For two adjacent triangles, Fock coordinate z_e is defined by the following equation, where the labeling follows figure 2.5

$$z_e = l_a + l_c - l_b - l_d. \tag{2.7}$$

2.1.5 Weil-Petersson form

A set of Fock coordinates assigned to an ideal triangulation of a Riemann surface $\Sigma_{g,n}$ subjected to the constraints parametrises the Teichmüller space $\mathcal{T}_{g,n}$. The Teichmüller space provided a symplectic structure described by a Weil-Petersson form. This Poisson bracket on the space of unconstrained Fock coordinates reduces to the Weil-Petersson one under the imposition of those constraints. It has however a particularly simple description

$$\{,\}_{WP} = \sum_{e,f\in E} n_{e,f} z_e z_f \frac{\partial}{\partial z_e}\frac{\partial}{\partial z_f}, \tag{2.8}$$

where $n = \pm 2, \pm 1, 0$ and E is the set of edges of the ideal triangulation under consideration. The number $n_{e,f}$ depends on the mutual position of the edges e and f inside the fat graph. If those edges do not share a common vertex or one of them is a loop, then $n_{e,f} = 0$. If that is not the case and the edges meet at two vertices, then if the edge f is the first one to the right of the edge e with respect to the orientation to the surface then $n_{e,f} = 2$; if it is to the left — $n_{e,f} = -2$; if any of those is not the case — $n_{e,f} = 0$. Otherwise, if the edge f is the first one to the right of the edge e at the common vertex then $n_{e,f} = 1$; if it is to the left — $n_{e,f} = -1$. Shortly, where n_{ef} is the number of times e and f meet in a common end-point P, counted positively if f is the first edge reached from e upon going around P in clockwise direction, counted negatively otherwise. We can write the Poisson bracket among the coordinate functions as

$$\{z_e, z_f\}_{WP} = n_{e,f}. \tag{2.9}$$

2.1.6 Changes of triangulations and the flip map

We used the ideal triangulations of Riemann surface in defining the coordinates on Teichmüller space $\mathcal{T}_{g,n}$. Definition of Teichmüller space does not involve triangulations, therefore, it is necessary to connect the parametrizations based on different triangulations of the same Riemann surface to each other. It can be shown that two ideal triangulations of the same Riemann surface can be connected by a sequence of elementary moves, which are permutations (vw) and flips ω_{vw}.

A permutation (vw) just exchanges the labels of triangles dual to vertices v and w of the associated fat graph. The flip ω_{vw} changes the triangulation of a quadrilateral composed of two triangles dual to v and w. We illustrate this map in figure 2.8.

The definition of the shear coordinates z_e was based on the choice of an ideal triangulation and changing the ideal triangulation defines new coordinates z'_e that can be

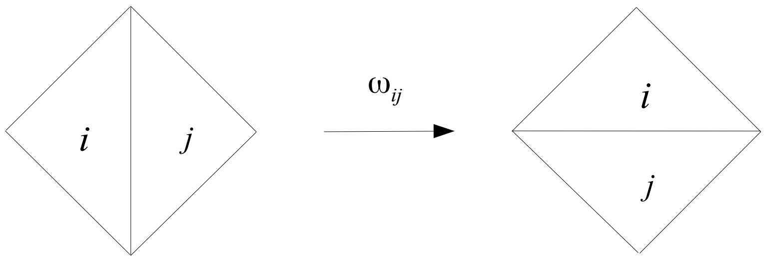


FIGURE 2.8: A flip map ω_{ij}.

expressed in terms of the coordinates z_e. Indeed, using the explicit expression (2.3), the change of triangulation induces the following change of coordinates

$$\begin{aligned} e^{z_1'} &= e^{z_1}(1+e^{z_e}), \\ e^{z_4'} &= e^{z_4}(1+e^{-z_e})^{-1}, \end{aligned} \qquad e^{z_e'} = e^{-z_e}, \qquad \begin{aligned} e^{z_2'} &= e^{z_2}(1+e^{-z_e})^{-1}, \\ e^{z_3'} &= e^{z_3}(1+e^{z_e}), \end{aligned} \tag{2.10}$$

leaving all other coordinates unchanged. The notation, involving the numbering of the edges of quadrilaterals, is explained in figure 2.9.

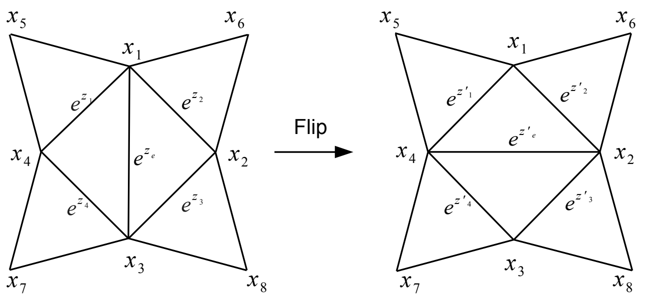


FIGURE 2.9: A transformation of Fock coordinates under a flip.

2.1.7 Kashaev coordinates

Up to this moment we considered the Fock coordinates, attached to the edges of an ideal triangulation, to parametrize Teichmüller space. However, as we have seen, the symplectic form of those coordinates is not particularly suitable when it comes to the quantization. As a particularly useful starting point for quantization it has turned out to be useful to describe the Teichmüller spaces by means of a set of coordinates associated to the triangles (or, alternatively, to the vertices of the associated fat graph) rather than the edges of an ideal triangulation, called Kashaev coordinates [34].

We shall label the triangles Δ_v by $v = 1, \ldots, 4g-4+2n$ and in order to define them, it is necessary to consider a refined version of triangulations, which we will call decorated triangulations. In every triangle of an ideal triangulation τ we distinguish one particular vertex, called a marked corner. To this decorated triangualation τ we associate a decorated fat graph $\varphi(\tau)$, that is a dual tri-valent graph with a cyclic ordering on the half-edges incident on each vertex, fixed by the decorated corners. An example of a decorated fat graph is illustrated in figure 2.10.

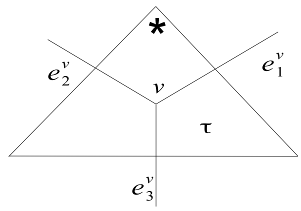


FIGURE 2.10: A decorated triangle with a dual fat graph.

According to figure 2.10 we label the edges that emanate from the vertex v by e_i^v, $i = 1, 2, 3$. Kashaev introduced pairs of variables (q_v, p_v) for each vertex v of a decorated fat graph $\varphi(\tau)$, as

$$(q_v, p_v) = (l_3 - l_2, l_1 - l_2). \tag{2.11}$$

As we have $2(2g - 2 + n)$ of those vertices in our fat graph, in total there will be $4(2g - 2 + n)$ Kashaev coordinates, parametrising a space isomorphic to $\mathbb{R}^{4(2g-2+n)}$, which we will call a Kashaev space.

A pair of variables (p_v, q_v) were assigned to each triangle (Kashaev coordinates) allowing us to recover the variables z_e (Fock variables). The Fock coordinate associated to an edge e of a fat graph is expressed in terms of Kashaev coordinates associated to vertices v, w of that fat graph, where the edge e connects the vertices v and w. Explicitly, we can write

$$z_e = \tilde{z}_{e,v} + \tilde{z}_{e,w}, \qquad \tilde{z}_{e,v} = \begin{cases} p_v & \text{if} \quad e = e_1^v, \\ -q_v & \text{if} \quad e = e_2^v, \\ q_v - p_v & \text{if} \quad e = e_3^v. \end{cases} \tag{2.12}$$

where e_i^v are the edges surrounding triangle Δ_v counted by $i = 1, 2, 3$ in counter-clockwise order such that e_i^3 is opposite to the distinguished corner, as illustrated in figure 2.10. The space $\mathbb{R}^{4(2g-2+n)}$ will be equipped with a Poisson structure defined by

$$\begin{aligned} \{p_v, p_w\} &= 0, \\ \{q_v, q_w\} &= 0, \end{aligned} \qquad \{p_v, q_w\} = \delta_{v,w}. \tag{2.13}$$

It can be shown that the Poisson structure of Kashaev coordinates given by (2.13) induces the Poisson structure on shear coordinates (2.9) via (2.12). However, it is clear that there is substantially too many Kashaev coordinates when compared with the dimension of Teichmüller space.

One may then describe the Teichmüller space using the Hamiltonian reduction of $\mathbb{R}^{4(2g-2+n)}$ with Poisson bracket (2.13) with respect to a suitable set of constraints h_γ labeled by $\gamma \in H_1(\Sigma_{g,n}, \mathbb{Z})$, and represented as linear functions in the (p_v, q_v) [34]. The functions z_e defined via (2.12) satisfy $\{h_\gamma, z_e\} = 0$ for all edges e and all $\gamma \in H_1(\Sigma_{g,n}, \mathbb{Z})$ and may therefore be used to get coordinates for the subspace defined by the constraints.

More extensively, every graph geodesic can be represented as a sequence of edges, but since each edge is an ordered pair of vertices of the fat graph, it can be just as well represented by an ordered sequence of vertices. For a closed curve γ we will denote the corresponding vertices as $v_i, i = 0, \ldots, m_\gamma$ where $v_0 = v_{m_\gamma}$, and corresponding edges as $e_i, i = 1, \ldots, m_\gamma$. Then to the closed curve $\gamma \in H_1(\Sigma_{g,n}, \mathbb{Z})$ we can assign a combination of Kashaev variables:

$$h_\gamma = \sum_{i=1}^{m_\gamma} u_i, \tag{2.14}$$

where

$$u_i = \omega_i \begin{cases} -q_{v_i} & \text{if} \quad \{e_i, e_{i+1}\} = \{e_3^{v_i}, e_3^{v_i}\} \\ p_{v_i} & \text{if} \quad \{e_i, e_{i+1}\} = \{e_2^{v_i}, e_3^{v_i}\} \\ q_{v_i} - p_{v_i} & \text{if} \quad \{e_i, e_{i+1}\} = \{e_1^{v_i}, e_2^{v_i}\}, \end{cases} \tag{2.15}$$

with the numbering of edges which is given according to figure 2.10, and $\omega_i = +1$ if the arcs connecting edges e_i and e_{i+1} turn around the vertex v_i in the counterclockwise fashion (with respect to the orientation of the surface) and $\omega_i = -1$ if not. Then, the constrains which described the embedding of the Teichmüller space into $\mathbb{R}^{4(2g-2+n)}$ are $h_\gamma = 0$, for every curve γ.

Change of Kashaev coordinates under the change of triangulation

One may define changes of Kashaev coordinates associated to any changes of ideal triangulations preserving the Poisson structure, and inducing the changes of shear coordinates (2.10) via (2.12). Having equipped the ideal triangulations with an additional decoration represented by the numbering of the triangles Δ_v and the choice of a distinguished corner in each triangle forces us to consider an enlarged set of elementary transformations relating arbitrary decorated ideal triangulations. Elementary transformations are the flips ω_{vw}, the rotations ρ_v and the permutations (vw).

Flips ω_{vw}, change the triangulation of a quadrilateral composed of two triangles dual to v and w — this flip however differs from the undecorated type by the fact that the triangles have distinguished vertices. The rotations ρ_v rotate the marked corner in a counter-clockwise fashion by 120° in a triangle dual to the vertex v. The first two are illustrated in figures 2.11 and 2.12, respectively, while the permutation (uv) simply exchanges the labels of the triangles u and v.

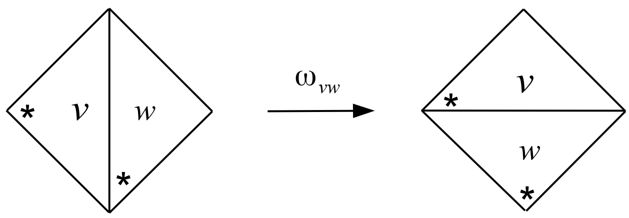


FIGURE 2.11: The transformation ω_{vw}.

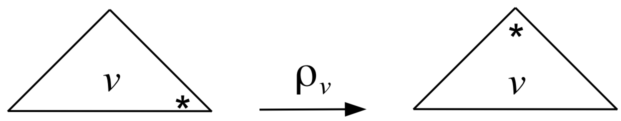


FIGURE 2.12: The transformation ρ_v.

The change of coordinates associated to the transformation ρ_v is given as

$$\rho_v^{-1} : (q_v, p_v) \to (p_v - q_v, -q_v), \tag{2.16}$$

while under a flip ω_{vw} the transformation of Kashaev coordinates is realized by

$$\omega_{vw}^{-1} : \begin{cases} (U_v, V_v) \to (U_v U_w, U_v V_w + V_v), \\ (U_w, V_w) \to (U_w V_v (U_v V_w + V_v)^{-1}, V_w (U_v V_w + V_v)^{-1}), \end{cases} \tag{2.17}$$

where we denote $U_v \equiv e^{q_v}$ and $V_v = e^{p_v}$.

2.1.8 Classical Ptolemy groupoid

The transformations between decorated ideal triangulations generate a groupoid that can be described in terms of generators and relations. As we mentioned above, any two decorated triangulations of the same Riemann surface can be related by a finite sequence of permutations (vw), flips ω_{vw} and rotations ρ_v. Any sequence of elementary transformations returning to its initial point defines a relation. A basic set of relations implying all others is known to be the following

$$\rho_v \circ \rho_v \circ \rho_v = id_v, \tag{2.18a}$$
$$(\rho_v^{-1} \rho_w) \circ \omega_{vw} = \omega_{wv} \circ (\rho_v^{-1} \rho_w), \tag{2.18b}$$
$$\omega_{wv} \circ \rho_v \circ \omega_{vw} = (vw) \circ (\rho_v \rho_w), \tag{2.18c}$$
$$\omega_{v_1 v_2} \circ \omega_{v_3 v_4} = \omega_{v_3 v_4} \circ \omega_{v_1 v_2}, \quad v_i \neq v_j, i \neq j, \tag{2.18d}$$
$$\omega_{vw} \circ \omega_{uw} \circ \omega_{uv} = \omega_{uv} \circ \omega_{vw}. \tag{2.18e}$$

The first equation implies simply that the threefold application of the rotation ρ_v on the same triangle returns the decorated vertex to the same position while, the second expresses the fact that the flips for unconnected quadrilaterals commute. The pentagon relation (2.18e) illustrated in figure 2.13 is of particular importance, while the relations (2.18a)-(2.18c) describe changes of the decorations. The other two equations are shown in figures 2.14 and 2.15, respectively.

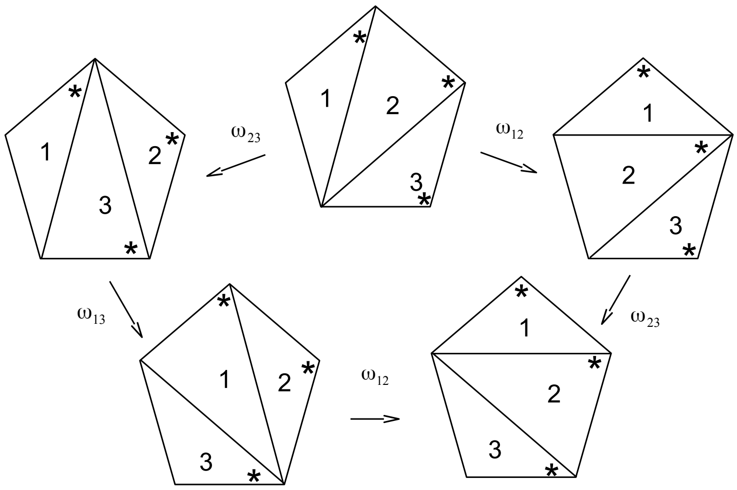


FIGURE 2.13: The pentagon equation.

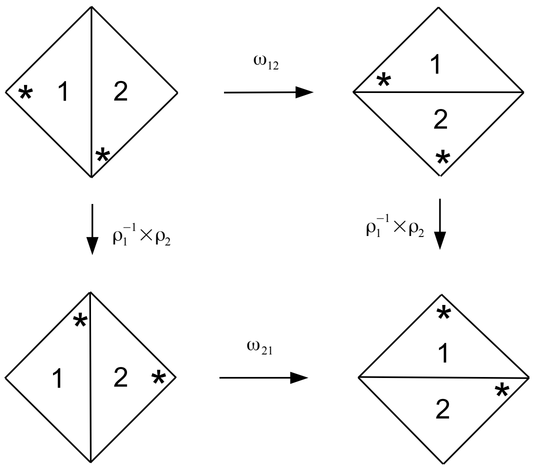


FIGURE 2.14: A pictorial representation of the 2$^{\text{nd}}$ equation of (2.18).

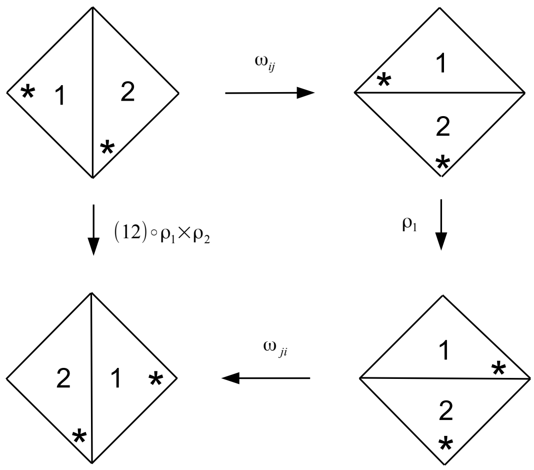


FIGURE 2.15: A pictorial representation of the 3$^{\text{th}}$ equation of (2.18).

2.2 Quantum Teichmüller theory

In section 2.1.4 we studied Fock coordinates, defined in term of conformal cross-ratios, and their properties. We introduced another useful set of coordinates which parametrise the Teichmüller spaces, called Kashaev coordinates in section 2.1.7. In this section we aim to provide a quantization of Teichmüller in terms of those coordinates.

Quantization of the Teichmüller theory of punctured Riemann surfaces was developed by Kashaev in [34] and independently by Fock and Chekhov in [32, 33], and utilized the Faddeev's quantum dilogarithm function in an essential way. Because of the functional relations of the quantum dilogarithm, the rational transformations of the Fock coordinates are ensured on the quantum level. A representation of mapping class groups can be constructed using the realization of the elementary Ptolemy groupoid transformations relations (2.1.8) and are expressed in terms of self adjoint operators (check [16, 34–36, 44] for more details).

2.2.1 Canonical quantization of Kashaev coordinates

The idea of quantization comes from theoretical physics. Quantizing a symplectic manifold, one considers a 1-parameter family of deformations of the algebra of functions on this manifold, which is called algebra of observables. This deformed algebra is non-commutative in general and realized as an algebra of operators on some Hilbert space, and a deformation parameter (denoted usually $\hbar$ or h) is known as Planck constant.

If the classical space is realized as a larger manifold subjected to constraints, it is possible to either first execute the constraints and then quantise the theory, or to quantise the unconstrained theory and impose the constraints directly on the quantum level.

Now, we want to perform a quantization of Teichmüller space. The quantization is particularly simple in terms of the Kashaev coordinates, because they are canonically conjugate. We will associate a Hilbert space $\mathcal{H}_v = L^2(\mathbb{R})$ with each face of a decorated triangulation and a Hilbert space associated to the entire triangulation is a multiplication of $N = 2(2g-2+n)$ of those spaces

$$\mathcal{H} = \bigotimes_{v=1}^{4g-4+2n} \mathcal{H}_v. \tag{2.19}$$

Then, the Kashaev coordinates, which previously were just canonically conjugate variables on $\mathbb{R}^{4(2g-g+n)}$, get quantized to a set of self adjoint operators $(p_v, q_v), v = 4g-4+2n$, have the following commutation relations

$$[\mathsf{p}_v, \mathsf{q}_w] = \frac{1}{2\pi i}\delta_{vw}, \qquad \begin{aligned}[\mathsf{q}_v, \mathsf{q}_w] = 0,\\ [\mathsf{p}_v, \mathsf{p}_w] = 0,\end{aligned} \tag{2.20}$$

and act on the Hilbert space as multiplication and differentiation.

Then, we can immediately introduce the quantized version of coordinate functions h_γ and Fock coordinates z_e as the self-adjoint operators h_γ and z_e on $\mathcal{H}$ respectively.

The result would be very similar to the classical one in (2.14) and (2.12) and is obtained by just replacing classical Kashaev coordinates with their quantum counterparts in those expressions. It can be shown that the resulting commutation relation satisfies,

$$\left[\mathsf{z}_e, \mathsf{z}'_e\right] = \frac{1}{2\pi i}\left\{z_e, z'_e\right\}_{WP}. \tag{2.21}$$

A quantum version of the Hamiltonian reduction procedure can be defined describing Hilbert space and algebra of observables of the quantum theory of Teichmüller spaces in terms of the quantum theory defined above. There exists a way to impose the constrains in the quantum theory. One can use those constrains to define the physical Hilbert space out of the tensor product Hilbert space introduced in (2.19). The treatment to produce the physical space discussed in the following references [16, 34].

2.2.2 Changes of triangulations and quantum Ptolemy grupoid

Here, we will consider a quantized realization of maps changing the triangulation τ of a Riemann surface Σ. The move ρ_v rotating the distinguished vertex of a triangle v is realized by an operator $\mathsf{A}_v : \mathcal{H}_v \to \mathcal{H}_v$

$$\mathsf{A}_v = e^{i\pi/3} e^{-i3\pi \mathsf{q}_v^2} e^{-i\pi(\mathsf{p}_v+\mathsf{q}_v)^2}. \tag{2.22}$$

One can show that it, as expected, cubes to the identity operator

$$\mathsf{A}_v^3 = id_v.$$

Operator A is unitary and is characterized by the equations

$$\mathsf{A}\mathsf{q}\mathsf{A}^{-1} = -\mathsf{p}, \qquad \mathsf{A}^{-1}\mathsf{q}\mathsf{A} = \mathsf{p} - \mathsf{q}, \tag{2.23}$$
$$\mathsf{A}\mathsf{p}\mathsf{A}^{-1} = \mathsf{q} - \mathsf{p}, \qquad \mathsf{A}^{-1}\mathsf{p}\mathsf{A} = -\mathsf{q}. \tag{2.24}$$

The flips get represented by unitary operators $\mathsf{T}_{vw} : \mathcal{H}_v \otimes \mathcal{H}_w \to \mathcal{H}_v \otimes \mathcal{H}_w$ defined as

$$\mathsf{T}_{vw} = e_{\mathrm{b}}(\mathsf{q}_v + \mathsf{p}_w - \mathsf{q}_w) e^{-2\pi i \mathsf{p}_v \mathsf{q}_w}, \tag{2.25}$$

where b is a parameter such that Planck's constant $\hbar = 2\pi b^2$, and e_{b} is a quantum dilogarithm function defined as

$$e_{\mathrm{b}}(x) = \exp\left[\int_{\mathbb{R}_i 0} \frac{\mathrm{d}w}{w} \frac{e^{-2ixw}}{4\sinh(wb)\sinh(w/b)}\right], \tag{2.26}$$

and it is related to the Double sine function as it is explained in appendix A.

In the literature, the T operator is expressed in terms of the function g_{b}, which is related to e_{b} as

$$g_{\mathrm{b}}(e^{2\pi b z}) = e_{\mathrm{b}}(z). \tag{2.27}$$

The quantized version of the transformation of the shear coordinates takes the form

$$\begin{aligned}
\mathsf{T}_{vw}^{-1} e^{2\pi b \mathsf{z}_1'} \mathsf{T}_{vw} &= e^{\pi b \mathsf{z}_1}(1 + e^{2\pi \mathsf{z}_e}) e^{\pi b \mathsf{z}_1}, \\
\mathsf{T}_{vw}^{-1} e^{2\pi b \mathsf{z}_2'} \mathsf{T}_{vw} &= e^{\pi b \mathsf{z}_2}(1 + e^{-2\pi \mathsf{z}_e})^{-1} e^{\pi b \mathsf{z}_2}, \\
\mathsf{T}_{vw}^{-1} e^{2\pi b \mathsf{z}_3'} \mathsf{T}_{vw} &= e^{\pi b \mathsf{z}_3}(1 + e^{2\pi \mathsf{z}_e}) e^{\pi b \mathsf{z}_3}, \\
\mathsf{T}_{vw}^{-1} e^{2\pi b \mathsf{z}_4'} \mathsf{T}_{vw} &= e^{\pi b \mathsf{z}_4}(1 + e^{-2\pi \mathsf{z}_e})^{-1} e^{\pi b \mathsf{z}_4}, \\
\mathsf{T}_{vw}^{-1} e^{2\pi b \mathsf{z}_e'} \mathsf{T}_{vw} &= e^{-2\pi b \mathsf{z}_e},
\end{aligned} \tag{2.28}$$

assuming that T_{vw} represents the flip depicted in figure 2.9 with decoration introduced in figure 2.11. The equations (2.28) provide the quantization of (2.17), and we can recover the classical transformation by taking the limit $q = e^{i\pi b^2} \to 1$.

The operators T_{uv} and A_v generate a projective representation of the Ptolemy groupoid characterized by the set of relations

$$\mathsf{A}_1^3 = \mathrm{id}_1, \tag{2.29}$$
$$\mathsf{T}_{23}\mathsf{T}_{13}\mathsf{T}_{12} = \mathsf{T}_{12}\mathsf{T}_{23}, \tag{2.30}$$
$$\mathsf{A}_2\mathsf{T}_{12}\mathsf{A}_1 = \mathsf{A}_1\mathsf{T}_{21}\mathsf{A}_2, \tag{2.31}$$
$$\mathsf{T}_{21}\mathsf{A}_1\mathsf{T}_{12} = \zeta\mathsf{A}_1\mathsf{A}_2\mathsf{P}_{(12)}, \tag{2.32}$$

where, $\zeta = e^{\pi i c_b{}^2/3}$ and $c_b = \frac{i}{2}(b + b^{-1})$. The permutation $\mathsf{P}_{(12)} : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_2 \otimes \mathcal{H}_1$ is defined as the operator acting as $\mathsf{P}_{(12)}(v_1 \otimes v_2) = v_2 \otimes v_1$ for all $v_i \in \mathcal{H}_i$.

In the following we show the proof of quantum Ptolemy groupoid.

proof of equation (2.30): After substituting the operator T_{12}, the right- and left-hand side has the form:

$$\begin{aligned}
RHS &= \mathsf{T}_{12}\mathsf{T}_{23} = e_{\mathrm{b}}(q_1 + p_2 - q_2)e^{-2\pi i p_1 q_2} e_{\mathrm{b}}(q_2 + p_3 - q_3)e^{-2\pi i p_2 q_3}\\
&= e_{\mathrm{b}}(q_1 + p_2 - q_2)e_{\mathrm{b}}(q_2 + p_3 - q_3)e^{-2\pi i p_1 q_2} e^{-2\pi i p_2 q_3}\\
&= e_{\mathrm{b}}(P)e_{\mathrm{b}}(X)e^{-2\pi i p_1 q_2} e^{-2\pi i p_2 q_3},\\
LHS &= \mathsf{T}_{23}\mathsf{T}_{13}\mathsf{T}_{12} = e_{\mathrm{b}}(q_2 + p_3 - q_3)e^{-2\pi i p_2 q_3} e_{\mathrm{b}}(q_1 + p_3 - q_3)e^{-2\pi i p_1 q_3} e_{\mathrm{b}}(q_1 + p_2 - q_2)e^{-2\pi i p_1 q_2}\\
&= e_{\mathrm{b}}(X)e_{\mathrm{b}}(q_1 + p_3 - q_3 + p_2)e^{-2\pi i p_2 q_3} e_{\mathrm{b}}(P - q_3)e^{-2\pi i p_1 q_3} e^{-2\pi i p_1 q_2}\\
&= e_{\mathrm{b}}(X)e_{\mathrm{b}}(X + P)e_{\mathrm{b}}(P) \underbrace{e^{-2\pi i p_2 q_3} e^{-2\pi i p_1 q_3} e^{-2\pi i p_1 q_2}}_{e^{-2\pi i p_1 q_2} e^{-2\pi i (p_2 - p_1) q_3} e^{-2\pi i p_1 q_3}},
\end{aligned}$$

where $P = q_1 + p_2 - q_2$, $X = q_2 + p_3 - q_3$. As we see these equations reduce to the pentagon for quantum dilogarithm with X, P such that $[P, X] = \frac{1}{2\pi i}$. The proof of pentagon relation for quantum dilogarithm is explained in appendix A.

proof of equation (2.31): It is straight forward by inserting the operators.

proof of equation (2.32): This equation can be written as

$$\mathsf{A}_2^{-1}\mathsf{A}_1^{-1}\mathsf{T}_{21}\mathsf{A}_1\mathsf{T}_{12} = \zeta P_{(12)}. \tag{2.33}$$

By using equations (2.23),(2.24) in the left hand side of the above equation and then inserting A and T we get the first line of the following relation and then by using the properties of e_b functions we have,

$$\begin{aligned}
LHS =& e^{i\pi/3} e^{-i\pi(p_2+q_2)^2} e^{-3\pi i q_2^2} e^{2\pi i (p_1 - q_1)p_2} e^{2\pi i p_2 q_1} g_b^{-1}(e^{2\pi b(-p_1 + q_1 - q_2)}) g_b^{-1}(e^{2\pi b(q_2 + p_1 - q_1)})\\
=& e^{i\pi/3} e^{i\pi/6} e^{i\pi c_b{}^2/3} \times e^{-i\pi(p_2+q_2)^2} e^{-3\pi i q_2^2} e^{2\pi i (q_1 - p_1) q_2} e^{2\pi i p_2 q_1} e^{-\pi i (q_2 + p_1 - q_1)^2}.
\end{aligned}$$

Using the evaluation of the matrix element of the exponential part we obtain,

$$\begin{aligned}
LHS &= e^{\frac{i\pi}{3}} e^{\frac{i\pi}{6}} e^{\frac{i\pi c_b{}^2}{3}} < x_1, q_2 \mid e^{-i\pi(p_2+q_2)^2} e^{-3\pi i q_2^2} e^{2\pi i (q_1 - p_1) q_2} e^{2\pi i p_2 q_1} e^{-\pi i (q_2 + p_1 - q_1)^2} \mid x_1{}', q_2' >\\
&= e^{\frac{i\pi}{3}} e^{\frac{i\pi}{6}} e^{\frac{i\pi c_b{}^2}{3}} \int dx_1{}'' dq_2'' e^{-3\pi i q_2''^2} e^{2\pi i q_2'' x_1''} \underbrace{< x_1 q_2 | e^{-i\pi(p_2+q_2)^2} | x_1{}'' q_2'' >}_{I} \underbrace{< x_1'' q_2'' | e^{2\pi i p_2 q_2 | x_1' q_2' >}}_{II},
\end{aligned}$$

where the details of the calculation of part I and II are explained as follows.

By using $e^{-\pi i q^2} p e^{-\pi i q^2} = p + q$ first and later $\int dk e^{-\pi i k^2} e^{2\pi i x k} = C_1 e^{\pi i x^2}$ we can write

$$I =< x_1 q_2 | e^{-i\pi(p_2+q_2)^2} | x_1'' q_2'' >= e^{-i\pi({q_2}^2 - {q_2''}^2)} < q_2 | e^{-i\pi {p_2}^2} | q_2'' >$$
$$= C_1 e^{-i\pi({q_2}^2 - {q_2''}^2)} e^{\pi(q_2 - {q_2''}^2)} = C_1 e^{-2i\pi {q_2''}^2} e^{-2\pi i q_2 {q_2''}^2},$$

where $C_1 = e^{\frac{-\pi i}{4}}$ because if we identify $e^{\frac{\pi i}{4}} q_2'' \equiv k$ then we have $dq_2'' = e^{\frac{-\pi i}{4}} dk$.

For part II we need to use the fact that $< x_1'', q_2'' | q_1, p_2 >= e^{\pi i (2 x_1'' q_1 - {q_1}^2)} e^{2\pi i q_2'' p_2}$ and also $< q_1, p_2 | x_1', q_2' >= e^{\pi i (q_1^2 - 2 x_1 q_1)} e^{-2\pi i p_2 q_2'}$.

So we derive:

$$II =< x_1'' q_2'' | e^{2\pi i p_2 q_2} | x_1' q_2' >= \int dp_2 dq_1 e^{2\pi i p_2 q_1} < x_1'' q_2'' | q_1 p_2 >< q_1 p_2 | x_1' q_2' >$$
$$= \int dq_1 e^{2\pi i (x_1'' - x_1')} \underbrace{\int dp_2 e^{2\pi i p_2 (q_1 + q_2'' - q_2')}}_{\delta(q_1 + q_2'' - q_2')} = e^{2\pi i (q_2' q_2'')(x_1'' - x_1')}.$$

Therefore, we have the result for the exponential part of the left hand side of the equation (2.33) by using Gaussian integral in the second line:

$$LHS = e^{i\pi/3} e^{i\pi/6} e^{i\pi {c_b}^2/3} \times e^{2\pi i q_2'(x_1 - x_1')} e^{-\pi i (q_2' - x_1')^2} \underbrace{\int dq_2'' e^{-\pi q_2''} e^{2\pi i q_2''(x_1 - q_2 - (x_1 - x_1'))}}_{e^{-\pi i (q_2'' - (x_1' - q_2))^2 - (x_1' - q_2)^2}}$$
$$= \underbrace{e^{i\pi/3} e^{i\pi/6} e^{i\pi {c_b}^2/3} C_1^2}_{constant=\zeta} e^{\pi i (q_2^2 - {q_2'}^2)} e^{2\pi i (q_2' x_1 - q_2 x_1')}.$$

For the right hand side of the equation (2.33) we have:

$$RHS = < x_1 q_2 | P(12) | x_1' q_2' >=< x_1 | q_2' >< q_2 | x_1' >= e^{2\pi i (x_1 q_2' - q_2 x_1')} e^{\pi i (q_2^2 - {q_2'}^2)}.$$

Comparison of LHS and RHS completes our proof.

The quantized flip transformation has an interesting relation with quantum groups theory. Kashaev [44] has shown that one can identify the flip operator T with the canonical element of the Heisenberg double of the quantum plane, the Borel half of $U_q(sl(2))$, evaluated on particular infinite-dimensional representations. Moreover, the rotation operator A_v is an algebra automorphism of this Heisenberg double. In chapter 4 we will show these relations more extensively but before that, in the next chapter we will give the basic definitions of quantum groups theory.

Chapter 3

Quantum groups, Drinfeld double and Heisenberg double

The ideas of symmetry and invariance play a very important role in mathematics and physics, and group theory structure is the most natural language for describing symmetries. Quantum groups and Hopf algebras are the natural generalizations of groups. Quantum groups first appeared in the " Inverse scattering method ", exactly solvable lattice models and low dimensional topology, developed by Fadeev and his collaborators in Leningrad school (for historical remarks look at [64]).

Beyond the physical models, quantum group was realized independently by V. G. Drinfeld [39] and M. Jimbo [40] as a Hopf algebra. Drinfeld also showed that quantum groups have the universal R-matrix which establishes a relation with the representation of braid groups, the so-called Yang-Baxter equation. The universal R-matrices for all quantum groups have been obtained in explicit form by Krillov and Reshetikhin [65].

Quantum groups provide a systematic way to construct the solution of Yang-Baxter equation and consequently build the new integrable lattice model. Quantum groups also have a significant contributions in conformal field theory [66] and they also play an important role in the recent developments in knot theory.

In this chapter we give a brief review of quantum groups. Afterwards, we present the Drinfeld double construction of quasi-triangular Hopf algebra and Heisenberg double related to that and present few examples. More details about these topics explained in many nice references such as [67–69].

3.1 Quantum groups

In this section after a brief explanation about algebra and coalgebra, we present the necessary notation for Hopf algebras and focus on the quasi-triangular Hopf algebra, called quantum group.

3.1.1 Algebras, bialgebras and Hopf algebras

In definitions of algebra, coalgebra and Hopf algebra we will consider the ground field k, where k can be considered to be field $\mathbb{C}$ of complex number or $\mathbb{R}$ of real number.

Definition 1. Considering k as a field and A as a vector space, the unital associative algebra is a triple (A, m, η), where $m : A \otimes A \to A$ is the multiplication map and $\eta : k \to A$ is the unital map and they satisfy the axioms of associativity and unitality

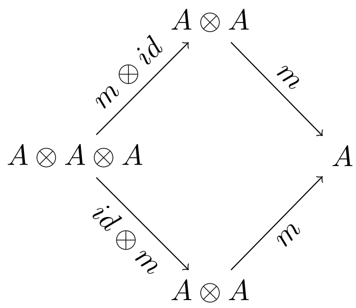
$$m(m \otimes id) = m(id \otimes m), \tag{3.1}$$

$$m(\eta \otimes id) = id = m(id \otimes \eta). \tag{3.2}$$

Each element of algebra A can be expressed as a linear combination of basis element e_i. For any two elements e_i and e_j we can define their multiplication in the form

$$m : A \otimes A \to A \Rightarrow e_i.e_j = m_{ij}^k e_k,$$

where m_{ij}^k is certain set of complex numbers with the condition that $m_{ij}^l m_{lk}^n = m_{ij}^n m_{lk}^l \equiv m_{ijk}^n$, which is equivalent to the condition of associativity for the algebra A as $(e_i e_j)e_k = e_i(e_j e_k)$. The axioms of associativity and unitality can also be summarized by the following commutative diagrams, respectively,



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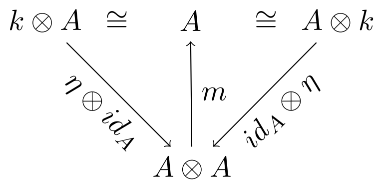

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One of the advantages of the diagrammatic language used here is that for the coalgebra definition one can reverse the direction of all arrows.

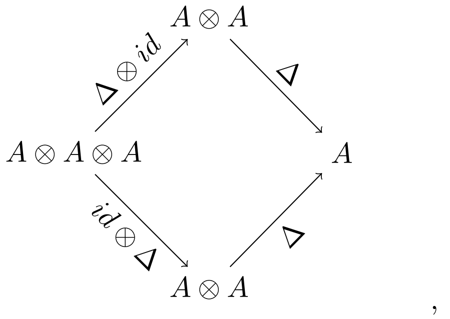
Definition 2. For comultiplication map $\Delta : A \to A \otimes A$ and counital map $\epsilon : A \to k$, coassociative coalgebra is defined as a triple (A, Δ, ϵ), such that the following axioms are satisfied

$$(\Delta \otimes id)\Delta = m(id \otimes \Delta), \tag{3.3}$$

$$(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta. \tag{3.4}$$

For $e_i \in A$, there is a notation for Δ, $\Delta(e) = \Sigma_{i,j} e_i \otimes e_j$, where the right hand side is the formal sum denoting an element of $A \otimes A$.

The axioms of coassociativity and counitality can also be summarized by the following commutative diagrams



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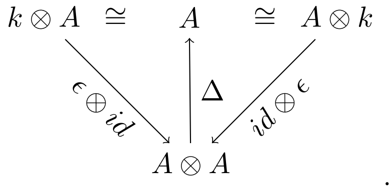

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Now we are prepared to introduce the main concept in the theory of quantum group, namely Hopf algebra.

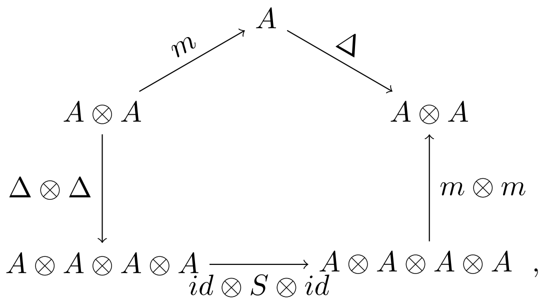
Definition 3. By considering (A, m, η) as an unital algebra and (A, Δ, ϵ) as a counital coassociative coalgebra, a bialgebra is a collection $(A, m, \delta, \eta, \epsilon)$ where algebra and coalgebra are compatible with each other by holding the following axioms:

$$\Delta m = (m \otimes m)(id \otimes \sigma)(\Delta \otimes \Delta), \tag{3.5}$$

$$\Delta \eta = \eta \otimes \eta, \tag{3.6}$$

$$\epsilon m = \epsilon \otimes \epsilon, \tag{3.7}$$

$$\epsilon \eta = id. \tag{3.8}$$



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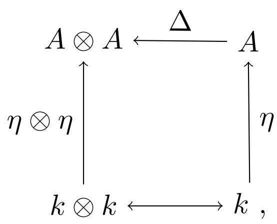

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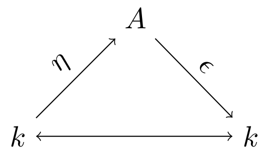

These axioms state that Δ and ϵ are homomorphism of algebras (or m and η are homomorphism of coalgebras) so it means

$$(g \otimes h)(h' \otimes g') = gg' \otimes hh', \tag{3.9}$$

$$\Delta(gh) = \Delta(g)\Delta(h), \qquad \Delta(1) = 1 \otimes 1, \tag{3.10}$$

$$\epsilon(gh) = \epsilon(g) \otimes \epsilon(h), \qquad \epsilon(1) = 1, \tag{3.11}$$

for all $g, g', h, h' \in A$. $\epsilon(1) = 1$ is automatic as k is a field.

Definition 4. By considering $(A, m, \delta, \eta, \epsilon)$ as a bialgebra, a Hopf algebra can be defined as a collection $(A, m, \eta, \Delta, \epsilon, S)$, where the linear antipode map $S : A \to A$ satisfies an extra axiom as,

$$\begin{array}{ccccc} A\otimes A & & \xrightarrow{id\otimes S} & & A\otimes A \\ \Delta\downarrow & & & & \downarrow m \\ A & \xrightarrow{\epsilon} & C & \xrightarrow{\eta} & A \\ \Delta\downarrow & & & & \uparrow m \\ A\otimes A & & \xrightarrow[S\otimes id]{} & & A\otimes A \end{array} \qquad m(S\otimes id)\Delta = m(id\otimes S)\Delta = \eta\epsilon. \quad (3.12)$$

The antipode of Hopf algebra is unique and has the properties,

$$S(gh) = S(h)S(g), \qquad S(1) = 1, \tag{3.13}$$

$$(S\otimes S)\Delta h = \sigma\Delta S h, \qquad \epsilon S h = \epsilon h, \tag{3.14}$$

where the tensor flip σ will be used as the operator of transposition, $\sigma(g\otimes h) = h \otimes g$ for all $g, h \in A$.

3.1.2 Duality

We mentioned that the axioms of coalgebra can be derived by inversing the arrows and interchanging Δ, ϵ with m, η. From the symmetry we can consider the dual linear space and conclude that for every Hopf algebra A, there is a dual Hopf algebra A^* built on the dual vector space.

Definition 5. Two Hopf algebras A, A^* are dually paired by a map $\langle , \rangle : A\otimes A^* \to k$ if

$$\langle \phi\psi, h\rangle = \langle \phi\otimes\psi, \Delta h\rangle, \qquad \langle \phi, hg\rangle = \langle \Delta\phi, h\otimes g\rangle, \tag{3.15}$$

$$\langle S\phi, h\rangle = \langle \phi, Sh\rangle, \tag{3.16}$$

$$\epsilon(h) = \langle 1, h\rangle, \qquad \epsilon(\phi) = \langle \phi, 1\rangle, \tag{3.17}$$

for all $g, h \in A$ and $\phi, \psi \in A^*$.

A Hopf algbera is commutative if it is commutative as an algbera and it is cocommutative if it is cocommutative as a coalgbera. The dual of commutative (cocommutative) Hopf algebra is commutative (cocommutative) and vice versa. For commutative or cocommutative Hopf algbera, we have $S^2 = id$. Here we express two examples of Hopf algebras for finite group G which are dual to each other.

Example 1(Functional algebra $\mathcal{F}(G)$): Let G be a finite group with identity e and $\mathcal{F}(G) = \{f : G \to k\}$ denote the set of functions on G with values in k. This has the structure of a commutative Hopf algebra with algebra structure:

$$(\lambda.\phi)(u) = \lambda.(\phi(u)),$$
$$m(\phi\psi)(u) = \phi(u)\psi(u),$$
$$\eta(\lambda)(u) = \lambda 1,$$

where, $\phi, \psi \in k(G), u, v \in G, \lambda \in k$ with following properties

$$
\begin{aligned}
&(\Delta\phi)(u,v) = \phi(uv), && \Delta : \mathcal{F}(G) \to \mathcal{F}(G) \times \mathcal{F}(G),\\
&\epsilon(\phi\psi)(u) = \phi(u)\psi(u), && \epsilon : \mathcal{F}(G) \to k\\
&(S\phi)(u) = \phi(u^{-1}).
\end{aligned}
$$

Example 2 (Group algebra $k[G]$): Let G be a finite group and $k[G]$ generated by G, i.e. $\{a = \sum_{u\in G} a(u)e_u\}$ where $\{e_u\}$ denotes the basis and $\lambda \in k$. Where $u, v \in G \subset k[G]$ we have the following list of properties:

- Product: $(\sum_u \lambda_1 u)(\sum_v \lambda_2 v) = \sum_{u,v} \lambda_1\lambda_2(uv),$
- Coproduct: $\Delta : k[G] \to k[G] \times k[G]), \quad \Delta(u) = u \otimes u,$
- Counite: $\epsilon : k[G] \to k, \quad \epsilon(u) = 1 = e,$
- Antipode : $S(u) = u^{-1}.$

The Hopf algebras $\mathcal{F}(G)$ and $k[G]$ are dual to each other such that $\langle , \rangle : \mathcal{F}(G) \otimes k[G] \to k$, where $\phi, \psi \in \mathcal{F}(G)$, $h \in k[G]$, $u \in G \subset k[G]$. Thus one can shows

$$
\begin{aligned}
&\langle \phi, u\rangle = \phi(\sum_u h(u)u) = \sum h(u)\phi(u),\\
&\langle \phi \otimes \psi, \Delta(u)\rangle = \langle \phi \otimes \psi, u \otimes u\rangle = \phi(u)\psi(u) = (\phi\psi)(u) = \langle \phi\psi, u\rangle.
\end{aligned}
$$

Therefore, we have $\mathcal{F}(G)^* = k[G]$, $k[G]^* = \mathcal{F}(G)$ and it follows that one's algbera structure corresponds to the other's coalgebra.

3.1.3 Quasi-triangular (braided) Hopf algebras and universal R- matrix

Quantum groups have an additional important structure which is not present in a general Hopf algebra, called the quasi triangular structure.

Definition 6. For a bialgebra $(A, m, \eta, \Delta, \epsilon)$ we call an invertible element $R \in \sum_i a_i \otimes b_i \in A \otimes A$ a universal R-matrix if it satisfies

$$\Delta^{op}(a) = R\Delta(a)R^{-1}, \tag{3.18}$$

$$(id \otimes \Delta)R = R_{13}R_{12}, \tag{3.19}$$

$$(\Delta \otimes id)R = R_{13}R_{23}, \tag{3.20}$$

where $\Delta^{op} = \sigma\Delta,\ a \in A$ and σ is a flip map. Also we have

$$\begin{aligned} R_{12} &= \sum_i a_i \otimes b_i \otimes 1 = R \otimes 1, \\ R_{23} &= \sum_i 1 \otimes a_i \otimes b_i = 1 \otimes R, \\ R_{13} &= \sum_i a_i \otimes 1 \otimes b_i. \end{aligned} \tag{3.21}$$

A Hopf algbera A with a quasi triangular structure is called a quasi-triangular Hopf algbera. The universal R-matrix satisfies the following identities

$$(\epsilon \otimes id)R = (id \otimes \epsilon)R = 1, \tag{3.22}$$

$$(S \otimes id)R = (id \otimes S^{-1})R = R^{-1}, \tag{3.23}$$

$$(S \otimes S)R = R. \tag{3.24}$$

Also the element $R^{op} = \sigma R$ is a universal quantum R-matrix for the Hopf algbera A^*.

Proposition 1. Universal R-matrix satisfies the quantum Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}. \tag{3.25}$$

which can be proven by using properties (3.18)-(3.20) in one line. Where $\sigma_{12} = \sigma \otimes id$ we have

$$\begin{aligned} R_{12}R_{13}R_{23} &= R_{12}(\Delta \otimes id)R = (\Delta^{op} \otimes id)(R)R_{12} \\ &= \sigma_{12}(R_{13}R_{23})R_{12} = R_{23}R_{13}R_{12}. \end{aligned} \tag{3.26}$$

A Hopf algebra A is called a quasi-triangular Hopf algebra, if for $A \otimes A$ there exists the universal R-matrix R. The main definition of this chapter is the definition of quantum group which is defined as a non-cocommutative quasi-triangular Hopf algebra.

Definition 7. Let $\mathfrak{g}$ be Lie algebra with universal algebra $\mathcal{U}(\mathfrak{g})$. The quantum enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ is an associative algebra generated by x_i, y_i, K_i, K_i^{-1} with relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1, \tag{3.27}$$

$$[K_i, K_j] = 0, \tag{3.28}$$

$$[x_iy_j, y_jx_i] = \delta_{i,j}\frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}}, \tag{3.29}$$

$$K_ix_j = q^{\frac{A_{ij}}{2}}x_jK_i, \quad K_iy_j = q^{-\frac{A_{ij}}{2}}y_jK_i, \tag{3.30}$$

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \binom{1-A_{ij}}{k}_q x_i^{1-A_{ij}-k}x_jx_i^k = 0, \qquad i \neq j, \tag{3.31}$$

$$\sum_{k=0}^{1-A_{ij}} (-1)^k \binom{1-A_{ij}}{k}_q y_i^{1-A_{ij}-k}y_jy_i^k = 0, \qquad i \neq j, \tag{3.32}$$

where $\binom{n}{k}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$ is a q-binomial coefficient and also

$$[n]_q! = [n]_q[n-1]_q \dots [2]_q[1]_q, \tag{3.33}$$

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{3.34}$$

Theorem 1. Let $\mathcal{U}_q(\mathfrak{g})$ be an algebra generated by x_i, y_i, K_i, K_i^{-1} with appropriate relations. Then $(\mathcal{U}_q(\mathfrak{g}), \Delta, \epsilon, S)$ with

$$\Delta(K_i) = K_i \otimes K_i, \tag{3.35}$$

$$\Delta(x_i) = x_i \otimes K_i + K_i^{-1} \otimes x_i, \tag{3.36}$$

$$\Delta(y_i) = y_i \otimes K_i + K_i^{-1} \otimes y_i, \tag{3.37}$$

$$\epsilon(K_i) = 1, \quad \epsilon(x_i) = \epsilon(y_i) = 0, \tag{3.38}$$

$$S(K_i) = K_i^{-1}, \quad S(x_i) = -q_i x_i, \quad S(y_i) = -q_i^{-1} y_i, \tag{3.39}$$

is a non-cocommutative Hopf algebra.
One can easily show that $\Delta x_i, \Delta K_i, \Delta y_i$ satisfy the defining relations of $\mathcal{U}_q(\mathfrak{g})$ and the axioms for the generators are verified. We know that for any Lie algebra $\mathfrak{g}$, we can present it by the generators and relations between them. The $(\mathcal{U}_q(\mathfrak{g}), \Delta, \epsilon, S)$ is a quantum group as expressed above and there exists the universal R-matrix R for that (For reference consult [67]).

3.2 Drinfeld double and Heisenberg double

Drinfeld double

We already explained the quasi-triangular (braided) Hopf algebras which satisfy Yang Baxter equation and we want to find such Hopf algebras. There exists a quantum double construction [39, 70, 71] presented by Drinfeld which builds a quasi triangular Hopf algebra out of an arbitrary Hopf algebra. We can consider Hopf algbera A and its dual A^* with opposite comultiplication. The algebraic tensor product of them can be made into a quasi-triangular Hopf algebra.

Definition 8. Let A be a Hopf algebra with base element E_α and A^* be its dual Hopf algebra with base element E^α with multiplication and comultiplication as

$$E_\alpha E_\beta = m^\gamma{}_{\alpha\beta} E_\gamma, \qquad E^\alpha E^\beta = \mu_\gamma^{\alpha\beta} E^\gamma, \tag{3.40}$$

$$\Delta(E_\alpha) = \mu_\alpha^{\beta\gamma} E_\beta \otimes E_\gamma, \qquad \Delta(E^\alpha) = m^\alpha_{\gamma\beta} E^\beta \otimes E^\gamma, \tag{3.41}$$

$$S(E_\alpha) = S_\alpha^\beta E_\beta, \qquad S(E^\alpha) = (S^{-1})^\alpha_\beta E^\beta. \tag{3.42}$$

As a remark regarding the notation, for reasons of simplicity we will write the elements $1 \otimes E_\alpha$ and $E^\alpha \otimes 1$ of Drinfeld double double as E_α and E^α, respectively.

Drinfeld double is defined as a vector space $D(A) = A \otimes A^*$ which satisfies

$$E_\alpha E_\beta = m^\gamma_{\alpha\beta} E_\gamma, \tag{3.43}$$

$$E^\alpha E^\beta = \mu^{\alpha\beta}_\gamma E^\gamma, \tag{3.44}$$

$$\mu^{\sigma\gamma}_\alpha m^\beta_{\gamma\rho} E_\sigma E^\rho = m^\beta_{\rho\gamma} \mu^{\gamma\sigma}_\alpha E^\rho E_\sigma. \tag{3.45}$$

The two initial Hopf algebras are two subalgebras of the larger Hopf algebra which can be constructed from them. The new Hopf algebra has universal R-matrix because of the existence of the multiplication of algbera and comultiplication of its dual. Moreover, the canonical element $R = E_\alpha \otimes E^\alpha$ satisfies Yang-Baxter relation (3.25) and can be shown as

$$\begin{aligned}
R_{12}R_{13}R_{23} &= (E_\alpha \otimes E^\alpha \otimes 1)(E_\beta \otimes 1 \otimes E^\beta)(1 \otimes E_\delta \otimes E^\delta) = \\
&= E_\alpha E_\beta \otimes E^\alpha E_\delta \otimes E^\beta E^\delta = E_\sigma \otimes m^\sigma_{\alpha\beta}\mu^{\beta\delta}_\rho E^\alpha E_\delta \otimes E^\rho = \\
&= E_\sigma \otimes m^\sigma_{\beta\alpha}\mu^{\delta\beta}_\rho E_\delta E^\alpha \otimes E^\rho = m^\sigma_{\beta\alpha} E_\sigma \otimes E_\delta E^\alpha \otimes \mu^{\delta\beta}_\rho E^\rho = \\
&= E_\beta E_\alpha \otimes E_\delta E^\alpha \otimes E^\delta E^\beta = (1 \otimes E_\delta \otimes E^\delta)(E_\beta \otimes 1 \otimes E^\beta)(E_\alpha \otimes E^\alpha \otimes 1) = \\
&= R_{23}R_{13}R_{12}.
\end{aligned}$$

Heisenberg double

As we explained, the Drinfeld double construction takes an arbitrary Hopf algebra and a Hopf algebra dual to it and produce a quantum group. There exists another construction, the so-called Heisenberg double [43]. The pentagon equation in Heisenberg, as it is shown below, has the similar role as Yang Baxter equation has in the Drinfeld double.

$$S_{12}S_{13}S_{23} = S_{23}S_{12}, \tag{3.46}$$

It was shown in [43] that the solution for Yang Baxter equation (3.25) can be obtained from solutions for pentagon equation (3.46).

Definition 9. Lets consider a bialgebra A spanned by the basis vectors e_α and the bialgebra A^* spanned by the basis vectors e^α.

The Heisenberg double $H(A)$ is an algebra as a vector space $H(A) \cong A \oplus A^*$ with multiplication and comultiplication on subalgebras A, A^* as

$$e_\alpha e_\beta = m^\gamma_{\alpha\beta} e_\gamma, \qquad e^\alpha e^\beta = \mu^{\alpha\beta}_\gamma e^\gamma \tag{3.47}$$

$$\Delta(e_\alpha) = \mu^{\beta\gamma}_\alpha e_\beta \otimes e_\gamma, \qquad \Delta(e^\alpha) = m^\alpha_{\beta\gamma} e^\beta \otimes e^\gamma \tag{3.48}$$

and

$$e_\alpha e^\beta = m^\beta_{\rho\gamma} \mu^{\gamma\sigma}_\alpha e^\rho e_\sigma. \tag{3.49}$$

As a remark regarding the notation, for reasons of simplicity we will write the elements $1 \otimes e_\alpha$ and $e^\alpha \otimes 1$ of Heisenberg double as e_α and e^α, respectively.

Bialgbera A^* is dual to A under the duality bracket $\langle , \rangle : A \times A^* \to \mathbb{C}$ defined on the basis by

$$\langle e_\alpha, e^\beta \rangle = \delta_\alpha^\beta.$$

This bracket exchange product and coproduct as follows:

$$\begin{aligned}\langle e_\alpha, e^\beta e^\gamma \rangle &= \mu_\tau^{\beta\gamma} \langle e_\alpha, e^\tau \rangle = \mu_\tau^{\beta\gamma} \delta_\alpha^\tau = \mu_\alpha^{\rho\sigma} \delta_\rho^\beta \delta_\sigma^\gamma = \mu_\alpha^{\rho\sigma} \langle e_\rho \otimes e_\sigma, e^\beta \otimes e^\gamma \rangle \\ &= (\Delta(e_\alpha), e^\beta \otimes e^\gamma \rangle, \\ \langle e_\alpha e_\beta, e^\gamma \rangle &= \langle e_\alpha \otimes e_\beta, \Delta(e^\gamma) \rangle.\end{aligned}$$

Indeed, in the case of Heisenberg algebras there is no comultiplication that would be compatible with the product defined above and at the same time agree with comultiplication Δ and Δ^* on subalgebras A and A^*, respectively.

Therefore, our Heisenberg double is only an algebra and not a Hopf algebra. The canonical element $S = e_\alpha \otimes e^\alpha$ satisfies equation (3.46), and can be shown as

$$\begin{aligned}S_{12}S_{13}S_{23} &= (e_\alpha \otimes e^\alpha \otimes 1)(e_\beta \otimes 1 \otimes e^\beta)(1 \otimes e_\gamma \otimes e^\gamma) = e_\alpha e_\beta \otimes e^\alpha e_\gamma \otimes e^\beta e^\gamma = \\ &= m_{\alpha\beta}^\rho e_\rho \otimes e^\alpha e_\gamma \otimes \mu_\sigma^{\beta\gamma} e^\sigma = e_\rho \otimes m_{\alpha\beta}^\rho \mu_\sigma^{\beta\gamma} e^\alpha e_\gamma \otimes e^\sigma = e_\rho \otimes e_\sigma e^\rho \otimes e^\sigma = \\ &= (1 \otimes e_\rho \otimes e^\rho)(e_\sigma \otimes e^\sigma \otimes 1) = S_{23}S_{12}.\end{aligned}$$

Examples

Here we want to make few examples about the Heisenberg double and Drinfeld double [43]. For finite group G we have group algbera $k[G]$ as a Hopf algebra denoted by A. For $\{e_g\}$ as the basis we have multiplication and comultiplications

$$m(e_g, e_h) = e_{gh} \Rightarrow m_{\alpha\beta}^\gamma = \delta_{\alpha\cdot\beta}^\gamma, \qquad e^g e^h = \mu_\gamma^{gh} e^\gamma = e^g \delta_h^g,$$

$$\Delta(e^g) = \sum_{h \in G} e^h \otimes e^{h^{-1}g}, \qquad \Delta(e_g) = e_g \otimes e_g \Rightarrow \mu_\alpha^{\beta\gamma} = \delta_\alpha^\beta \delta_\alpha^\gamma,$$

$$e_g e^h = \delta_{\rho\cdot\gamma}^h \delta_g^\gamma \delta_g^\sigma e^\rho e_\sigma = \delta_{\rho\cdot g}^h e^\rho e_g = e^{hg^{-1}} e_g.$$

Then multiplicative unitary is given by $S = \sum_\alpha e_\alpha \otimes e^\alpha$.

As the second example we consider polynomial ring $\mathbb{C}[x]$ by $e_m = \frac{x^m}{m!}$ as a normalized basis and $e^n = \bar{x}^n$ as a dual basis. Therefore,

$$\begin{aligned}e_n e_m &= \frac{(n+m)!}{n!m!} e_{n+m} = \binom{n+m}{n} e_{n+m} \Rightarrow m_{n,m}^\gamma = \binom{n+m}{n} \delta_{n+m}^\gamma, \\ e^n e^m &= \sum_{s=m}^{\infty} \delta_{s-m}^n e^s = e^{n+m}, \\ \Delta(e_n) &= \sum_{k=0}^{n} e_{n-k} \otimes e_k \Rightarrow \mu_n^{k,l} = \delta_{n-l}^k \Theta(n-l)\Theta(l), \\ \Delta(e^n) &= \sum_{k=0}^{n} \binom{n}{k} e^{n-k} \otimes e^k,\end{aligned}$$

where $\Theta = \begin{cases} \delta^k_{n-l}, & 0 \leq l \leq n \\ 0, & l > n \end{cases}$, and the coproduct are $\Delta(x) = x \otimes 1 + 1 \otimes x$ and follows up that $\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} \otimes x^k$.

The Heisenberg double has $e_n \otimes e^m$ as basis and one can find

$$e_n e^m = \sum_{s=0}^{n} \binom{m}{n-s} e^{m-n+s} e_s,$$
$$x\bar{x} = e_1 e^1 = e^0 e_0 + e^1 e_1 = 1 + \bar{x}x \Rightarrow x\bar{x} - \bar{x}x = 1,$$

and the canonical element encodes as

$$S = e_m \otimes e^m = \sum_{m=0}^{\infty} \frac{1}{m!} x^m \otimes \bar{x}^m = \exp(x \otimes \bar{x}).$$

Relation of Drinfeld double and Heisenberg double

Using Heisenberg doubles one can find the representations of Drinfeld doubles, because one can embed the elements of the Drinfeld double into a tensor square of Heisenberg double [43]. As we mentioned before the coproduct of the Hopf algebra structure of A and its dual A^* are not algebra homomorphism of the multiplication on $H(A)$. Let us have a Heisenberg double $H(A)$ defined as before. Moreover, we define another Heisenberg double $\tilde{H}(A)$ generated by basis vectors $\{\tilde{e}_\alpha, \tilde{e}_\beta\}$ with

$$\tilde{e}_\alpha \tilde{e}_\beta = m^\gamma_{\alpha\beta} \tilde{e}_\gamma, \qquad \Delta(\tilde{e}_\alpha) = \mu^{\beta\gamma}_\alpha \tilde{e}_\beta \otimes \tilde{e}_\gamma, \tag{3.50}$$
$$\tilde{e}^\alpha \tilde{e}^\beta = \mu^{\alpha\beta}_\gamma \tilde{e}^\gamma, \qquad \Delta(\tilde{e}^\alpha) = m^\alpha_{\beta\gamma} \tilde{e}^\beta \otimes \tilde{e}^\gamma, \tag{3.51}$$
$$\tilde{e}^\beta \tilde{e}_\alpha = \mu^{\sigma\gamma}_\alpha m^\beta_{\gamma\rho} \tilde{e}_\sigma \tilde{e}^\rho, \tag{3.52}$$

with canonical element $\tilde{S} = \tilde{e}_\alpha \otimes \tilde{e}^\alpha$, which satisfies the reversed pentagon equation:

$$\tilde{S}_{12}\tilde{S}_{23} = \tilde{S}_{23}\tilde{S}_{13}\tilde{S}_{12}. \tag{3.53}$$

Using $H(A)$ and $\tilde{H}(A)$ one can define Drinfeld double $D(A)$, which as a vector space $D(A) \subset H(A) \otimes \tilde{H}(A)$ has the elements

$$E_\alpha = \mu^{\beta\gamma}_\alpha e_\beta \otimes \tilde{e}_\gamma, \qquad E^\alpha = m^\alpha_{\gamma\beta} e^\beta \otimes \tilde{e}^\gamma. \tag{3.54}$$

They satisfy the following defining relations

$$E_\alpha E_\beta = m^\gamma_{\alpha\beta} E_\gamma, \qquad E^\alpha E^\beta = \mu^{\alpha\beta}_\gamma E^\gamma, \qquad \mu^{\sigma\gamma}_\alpha m^\beta_{\gamma\rho} E_\sigma E^\rho = m^\beta_{\rho\gamma} \mu^{\gamma\sigma}_\alpha E^\rho E_\sigma. \tag{3.55}$$

The first two statements can be easily proven by using the compatibility condition:

$$\Delta \circ m = (m \otimes m)(id \otimes \sigma \otimes id)(\Delta \otimes \Delta), \qquad m^\gamma_{\alpha\beta} \mu^{\sigma\rho}_\gamma = \mu^{\delta\epsilon}_\alpha \mu^{\eta\xi}_\beta m^\sigma_{\delta\eta} m^\rho_{\epsilon\xi}, \tag{3.56}$$

one can show

$$E_\alpha E_\beta = \mu_\alpha^{\pi\rho}\mu_\beta^{\sigma\tau}(e_\pi \otimes \tilde{e}_\rho)(e_\sigma \otimes \tilde{e}_\tau) = \mu_\alpha^{\pi\rho}\mu_\beta^{\sigma\tau} m_{\pi\sigma}^{\mu} m_{\rho\tau}^{\nu} e_\mu \otimes \tilde{e}_\nu = m_{\alpha\beta}^{\gamma}\mu_\gamma^{\mu\nu} e_\mu \otimes \tilde{e}_\nu = m_{\alpha\beta}^{\gamma} E_\gamma,$$
$$E^\alpha E^\beta = \mu_\gamma^{\alpha\beta} E^\gamma,$$

and in addition using associativity and coassociativity

$$(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta, \qquad \mu_\alpha^{\gamma\beta}\mu_\gamma^{\rho\sigma} = \mu_\alpha^{\rho\gamma}\mu_\gamma^{\sigma\beta}, \tag{3.57}$$
$$m(m \otimes id) = m(id \otimes m), \qquad m_{\alpha\beta}^{\delta} m_{\delta\gamma}^{\sigma} = m_{\alpha\delta}^{\sigma} m_{\beta\gamma}^{\delta}, \tag{3.58}$$

one shows

$$\begin{aligned}
&\mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^{\beta} E_\sigma E^\rho = \mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^{\beta} \mu_\sigma^{ab}(e_a \otimes \tilde{e}_b) m_{dc}^{\rho}(e^c \otimes \tilde{e}^d) = \\
&= \mu_\alpha^{\sigma\gamma} m_{\gamma\rho}^{\beta} \mu_\sigma^{ab} m_{dc}^{\rho} m_{rg}^{c} \mu_a^{gs} e^r e_s \otimes \tilde{e}_b \tilde{e}^d = (\mu_\alpha^{\sigma\gamma}\mu_\sigma^{ab})(m_{\gamma\rho}^{\beta} m_{dc}^{\rho}) m_{rg}^{c} \mu_a^{gs} e^r e_s \otimes \tilde{e}_b \tilde{e}^d = \\
&= (\mu_\alpha^{a\sigma}\mu_\sigma^{b\gamma})(m_{\rho c}^{\beta} m_{\gamma d}^{\rho}) m_{rg}^{c} \mu_a^{gs} e^r e_s \otimes \tilde{e}_b \tilde{e}^d = \mu_\alpha^{a\sigma} m_{\rho c}^{\beta} m_{rg}^{c} \mu_a^{gs} e^r e_s \otimes (\mu_\sigma^{b\gamma} m_{\gamma d}^{\rho} \tilde{e}_b \tilde{e}^d) = \\
&= (\mu_\alpha^{a\sigma}\mu_a^{gs})(m_{\rho c}^{\beta} m_{rg}^{c}) e^r e_s \otimes \tilde{e}^\rho \tilde{e}_\sigma = (\mu_\alpha^{ga}\mu_a^{s\sigma})(m_{cg}^{\beta} m_{\rho r}^{c})(e^r \otimes \tilde{e}^\rho)(e_s \otimes \tilde{e}_\sigma) = \\
&= \mu_\alpha^{ga} m_{cg}^{\beta}(m_{\rho r}^{c} e^r \otimes \tilde{e}^\rho)(\mu_a^{s\sigma} e_s \otimes \tilde{e}_\sigma) = \mu_\alpha^{ga} m_{cg}^{\beta} E^c E_a.
\end{aligned}$$

Now, we can consider the canonical element $R = E_\alpha \otimes E^\alpha$ which is said to satisfy Yang-Baxter relation

$$R_{1\tilde{1},2\tilde{2}} R_{1\tilde{1},3\tilde{3}} R_{2\tilde{2},3\tilde{3}} = R_{2\tilde{2},3\tilde{3}} R_{1\tilde{1},3\tilde{3}} R_{1\tilde{1},2\tilde{2}}, \tag{3.59}$$

It can be shown that one can express the R-matrix by canonical elements S, $\tilde{S}$, S',S'',

$$R_{12,34} = S''_{14} S_{13} \tilde{S}_{24} S'_{23} \qquad \text{where, } S' = \tilde{e}_\alpha \otimes e^\alpha, \;\; S'' = e_\alpha \otimes \tilde{e}^\alpha.$$

We contribute a short proof of Kashaev's machinery here

$$\begin{aligned}
&S''_{14} S_{13} \tilde{S}_{24} S'_{23} = (e_\alpha \otimes 1 \otimes 1 \otimes \tilde{e}^\alpha)(e_\beta \otimes 1 \otimes e^\beta \otimes 1)(1 \otimes \tilde{e}_\gamma \otimes 1 \otimes \tilde{e}^\gamma)(1 \otimes \tilde{e}_\delta \otimes e^\delta \otimes 1) = \\
&= e_\alpha e_\beta \otimes \tilde{e}_\gamma \tilde{e}_\delta \otimes e^\beta e^\delta \otimes \tilde{e}^\alpha \tilde{e}^\gamma = m_{\alpha\beta}^{a} e_a \otimes m_{\gamma\delta}^{b} \tilde{e}_b \otimes \mu_c^{\beta\delta} e^c \otimes \mu_d^{\alpha\gamma} \tilde{e}^d = \\
&= \mu_d^{\alpha\gamma}\mu_c^{\beta\delta} m_{\alpha\beta}^{a} m_{\gamma\delta}^{b} e_a \otimes \tilde{e}_b \otimes e^c \otimes \tilde{e}^d = m_{dc}^{\gamma}\mu_\gamma^{ab} e_a \otimes \tilde{e}_b \otimes e^c \otimes \tilde{e}^d = \\
&= (\mu_\gamma^{ab} e_a \otimes \tilde{e}_b) \otimes (m_{dc}^{\gamma} e^c \otimes \tilde{e}^d) = E_\alpha \otimes E^\alpha = R_{12,34}.
\end{aligned} \tag{3.60}$$

3.3 $U_q(sl(2))$ as an example

We consider $U_q(sl(2,\mathbb{R}))$ as an example of quantum groups. $\mathcal{U}_q(sl(2))$ is generated by K, K^{-1}, E, F and satisfies the relations

$$\begin{aligned}
&KK^{-1} = K^{-1}K = 1, \\
&[E,F] = -\frac{K^2 - K^{-2}}{q - q^{-1}}, \\
&KE = qEK, \quad KF = q^{-1}FK,
\end{aligned} \tag{3.61}$$

and with coproduct and unitality such that

$$\Delta(K) = K \otimes K,$$
$$\Delta(x) = E \otimes K + K^{-1} \otimes E, \qquad \Delta(y) = F \otimes K + K^{-1} \otimes F, \tag{3.62}$$
$$\epsilon(K) = 1, \quad \epsilon(E) = \epsilon(F) = 0,$$

and following antipode

$$S(K) = K^{-1}, \quad S(E) = -qE, \quad S(F) = -qF. \tag{3.63}$$

The Casimir operator C has the following form

$$C = FE - \frac{qK^2 + q^{-1}K^{-2} - 2}{(q - q^{-1})^2}, \tag{3.64}$$

and it can be shown easily that $[C, K] = [C, E] = [C, F] = 0$.

One can consider $\mathcal{U}(sl(2))$ as a classical limit of $\mathcal{U}_q(sl(2))$. We can take $K = e^{hH}, q = e^h$ and take the limit $q \to 1$, which gives the following relations:

$$[E, F] = -2H, \quad [H, E] = E, \quad [H, F] = -F, \tag{3.65}$$

which are the relations for algbera $U(sl(2))$. The other structures have the limits as follows

$$\Delta(u) = u \otimes 1 + 1 \otimes u, \qquad \epsilon(u) = 0, \qquad S(u) = -u,$$

where $u = H, E, F$.

Heisenberg double of $\mathcal{U}_q(sl(2, \mathbb{R}))$

We want to construct Drinfeld double of the algebra, Borel half $U_q(\mathcal{B}))$ of $U_q(sl(2))$. This algbera has two generators H and E with the following relations

$$[H, E] = E, \tag{3.66}$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H, \tag{3.67}$$
$$\Delta(E) = E \otimes e^{hH} + 1 \otimes E, \tag{3.68}$$

Considering $q = e^{-h}$ we can have the relation $K = e^{hH}$ and it brings the following basis element for the algebra

$$e_{m,n} = \frac{1}{m!(q)_n} H^m E^n, \tag{3.69}$$

where $(q)_n$ is q-factorial as defined in equation (3.70)

$$(q)_n = (1 - q)...(1 - q^n), n > 0, \qquad (q)_0 = 1 \tag{3.70}$$

and we can compute multiplication of the generators as

- $E^nK = q^nKE^n$,
- $E^nH^m = (H-n)^mE^n, \sum_{m=0}^{\infty}\frac{1}{m!}h^mE^nH^m = \sum_{m=0}^{\infty}\frac{1}{m!}h^mH^mE^n\sum_{k=0}^{\infty}\frac{1}{k!}(-nh)^k =$

$= \sum_{m=0}^{\infty}\frac{1}{m!}h^m\sum_{k=0}^{m}\binom{m}{k}(-n)^{m-k}H^kE^n = \sum_{m=0}^{\infty}\frac{1}{m!}h^m(H-n)^mE^n$,

- $H^mE^nH^lE^k = \sum_{j=0}^{l}\binom{l}{j}(-n)^{l-j}H^{m+j}E^{n+k}$.

It brings us to the multiplication for the basis elements

$$e_{m,n}e_{l,k} = \sum_{j=0}^{l}\binom{m+j}{j}\binom{n+k}{k}_q\frac{(-n)^{l-j}}{(l-j)!}e_{m+j,n+k}, \tag{3.71}$$

where $\binom{n}{k}_q = \frac{(q)_n}{(q)_k(q)_{n-k}}$. One can also find the comultiplication

- $\Delta(H^n) = \sum_{k=0}^{n}\binom{n}{k}H^{n-k}\otimes H^k$,
- $\Delta(E^n) = \sum_{k=0}^{n}f(n,k)E^{n-k}\otimes e^{(n-k)hH}E^k$,

and for one higher order we have

- $\Delta(E^{n+1}) = \sum_{k=0}^{n+1}f(n+1,k)E^{n+1-k}\otimes e^{(n+1-k)hH}E^k =$

$$= \sum_{k=0}^{n}f(n,k)E^{n-k}\otimes e^{(n-k)hH}E^k(E\otimes e^{hH}+1\otimes E) =$$
$$= f(n,0)E^{n+1}\otimes e^{(n+1)hH} + f(n,n)1\otimes E^{n+1}+$$
$$+\sum_{k=1}^{n}E^{n+1-k}\otimes e^{(n+1-k)hH}E^k(f(n,k)q^k+f(n,k-1)).$$

Therefore, $f(n+1,0) = f(n,0)$, $f(n+1,n+1) = f(n,n)$, $f(n+1,k) = f(n,k)q^k + f(n,k-1)$, for $0<k<n+1$. These properties are satisfied by q-symbol.

Therefore,

- $\Delta(E^n) = \sum_{k=0}^{n} \binom{n}{k}_q E^{n-k} \otimes e^{(n-k)hH} E^k,$
- $\Delta(H^n E^m) = \sum_{k=0}^{n} \binom{n}{k} H^{n-k} \otimes H^k \sum_{l=0}^{m} \binom{m}{l}_q E^{m-l} \otimes e^{(m-l)hH} E^l =$

$$= \sum_{k=0}^{n} \sum_{l=0}^{m} \binom{n}{k} \binom{m}{l}_q H^{n-k} E^{m-l} \otimes H^k e^{(m-l)hH} E^l =$$

$$= \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{p=0}^{\infty} \binom{n}{k} \binom{m}{l}_q \frac{1}{p!} (m-l)^p h^p H^{n-k} E^{m-l} \otimes H^{k+p} E^l,$$

and it gives the comultiplication as

$$\Delta(e_{n,m}) = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{p=0}^{\infty} \binom{k+p}{k} (m-l)^p h^p e_{n-k,m-l} \otimes e_{k+p,l}. \tag{3.72}$$

Now we consider the dual algebra A^*, which is generated by the elements $\bar{H}$ and F

$$[\bar{H}, F] = -hF, \tag{3.73}$$
$$\Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H}, \tag{3.74}$$
$$\Delta(F) = F \otimes e^{-\bar{H}} + 1 \otimes F, \tag{3.75}$$
$$\bar{K} = e^{b\bar{H}}, \ \bar{K}F = \tilde{q}F\bar{K}, \quad \tilde{q} = e^{-hb}. \tag{3.76}$$

The multiplication and comultiplication has the form

$$E^{m,n} E^{l,k} = \sum_{j=0}^{l} \binom{l}{j} (n)^{l-j} h^{l-j} E^{m+j,n+k}, \tag{3.77}$$
$$\Delta(E^{n,m}) = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{p=0}^{\infty} \binom{n}{k} \binom{m}{l}_q \frac{(-m+l)^p}{p!} E^{n-k,m-l} \otimes E^{k+p,l}. \tag{3.78}$$

It is clear that we can identify

$$e^{n,m} = \bar{H}^n F^m. \tag{3.79}$$

Since the bases are dual to each other by comparing multiplications and comultiplications we have

$$m^{r,s}_{m,n;l,k} = \sum_{j=0}^{l} \binom{m+j}{j} \binom{n+k}{k}_q \frac{(-n)^{l-j}}{(l-j)!} \delta_{r,m+j} \delta_{s,n+k} =$$
$$= \binom{r}{r-m} \binom{n+k}{k}_q \frac{(-n)^{l-r+m}}{(l-r+m)!} \Theta(r-m)\Theta(l-r+m)\delta_{s,n+k},$$

and

$$\mu_{r,s}^{m,n;l,k} = \sum_{j=0}^{l} \binom{l}{j} (n)^{l-j} h^{l-j} \delta_{r,m+j} \delta_{s,n+k} =$$
$$= \binom{l}{r-m} (n)^{l-r+m} h^{l-r+m} \Theta(r-m)\Theta(l-r+m)\delta_{s,n+k}.$$

By using equation (3.47) we can check the permutation relations for Heisenberg double $H(U_q(\mathcal{B}))$,

$$\begin{aligned}
H\bar{H} &= e_{1,0}e^{1,0} = m^{1,0}_{a,b;c,d}\mu^{c,d;e,f}_{1,0} e^{a,b} e_{e,f} = m^{1,0}_{a,0;c,0}\mu^{c,0;e,0}_{1,0} e^{a,0} e_{e,0} = \\
&= m^{1,0}_{0,0;1,0}\mu^{1,0;0,0}_{1,0} e^{0,0} e_{0,0} + m^{1,0}_{1,0;0,0}\mu^{0,0;1,0}_{1,0} e^{1,0} e_{1,0} = 1 + \bar{H}H, \\
E\bar{H} &= e_{0,1}e^{1,0} = m^{1,0}_{a,b;c,d}\mu^{c,d;e,f}_{0,1} e^{a,b} e_{e,f} = m^{1,0}_{a,0;c,0}\mu^{c,0;e,1}_{0,1} e^{a,0} e_{e,1} = \\
&= m^{1,0}_{1,0;0,0}\mu^{0,0;0,1}_{0,1} e^{1,0} e_{0,1} = e^{1,0} e_{0,1} = \bar{H}E,
\end{aligned}$$

and in the same way

$$HF - FH = -F, \qquad EF - FE = (1-q)q^{-H}.$$

Now we can consider the canonical element:

$$S = \sum_{n,m} e_{n,m} e^{n,m} = \sum_{n,m} \frac{1}{n!(q)_m} H^n E^m \otimes \bar{H}^n F^m = \sum_{n,m} \frac{1}{n!(q)_m} (H \otimes \bar{H})^n (E \otimes F)^m =$$
$$= \exp(H \otimes \bar{H})(E \otimes F; q)_\infty^{-1},$$

which follows from q-binomial formula (see appendix E). We used the fact that, $(x;q)^{-1} = \sum_{k=0}^{\infty} \frac{x^k}{(q)_k}$, the proof of that can be found in appendix E.

In the next chapter we will show how one can consider a bialgebra A spanned by the basis vectors $\{e(\alpha)\}$, where the basis is of infinite dimension. Afterwards, we will define all the objects in analogous way as in the finite dimensional case, replacing all sums with integrals over the spectrum.

$$\sum_{\alpha} \to \int \mathrm{d}\alpha.$$

Chapter 4

Non-compact quantum groups

The basic language of functional analysis is assumed as a background knowledge for this chapter. We start with a brief explanation of the quantum plane which is the simplest example of a non-compact quantum group. By taking complex powers of the generators as unbounded operators, we can define a C*-algebraic version of the Drinfeld-Jimbo quantum groups. Afterwards, we consider the non-compact version of Heisenberg double of the Borel half of $U_q(sl(2))$. We use the self dual representations of Heisenberg double and evaluate the canonical element which in particular satisfies the pentagon equation.

The discussion of entities that we provide in this chapter will be a introduction to the supersymmetric case discussed in chapter 7.

4.1 Quantum plane

A quantum plane $\mathcal{A}_q$ is a Hopf *-algebra which is the Borel half of a q-deformed universal enveloping algebra $U_q(sl(2))$. It is generated by elements A, A^{-1}, B such that they satisfy q commutation relation

$$AB = q^2 BA, \tag{4.1}$$

where, the deformation parameter is $q = \exp[i\pi b^2]$, with a real *-structure

$$A^* = A, \qquad B^* = B.$$

In addition, one has a compatible coproduct,

$$\Delta(A) = A \otimes A, \qquad \Delta(B) = B \otimes A + 1 \otimes B, \tag{4.2}$$

counit and antipode

$$\epsilon(A) = 1, \qquad \epsilon(B) = 0, \tag{4.3}$$

$$S(A) = A^{-1}, \qquad S(B) = qB. \tag{4.4}$$

Furthermore, to deal with non-compact quantum groups, we are interested in a normed *-algebra, the so called C*-algebra. We also need language of multiplier C*- algebra to define a natural coproduct.

A Banach space is a vector space V over the field of real or complex numbers, which is equipped with a norm and which is complete[1] with respect to that norm. This means that for every Cauchy sequence x_n in V, there exists an element x in V such that

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

A Banach algebra is a algebra which is a Banach space under a norm such as

$$\| xy \| \leq \| x \| \| y \| .$$

An C*-algebra is a Banach *-algebra $\mathcal{A}$ satisfying the C*-axiom:

$$\| xx^* \| = \| x \|^2 \qquad \text{for all } x \in \mathcal{A}.$$

For a compact group G, any function on G can be approximated, with respect to sup norm, by polynomial functions in the generators. But in the non-compact case, functions vanishing (decay) at infinity are not well approximated (with respect to sup norm) by polynomial generators. In both cases the convergence is defined using the sup-norm.

In the non-compact case we need to deal with unbounded operators and functional calculus for self-adjoint operators is the main technical tools. The operators A and B of the quantum plane will be represented as unbounded operators $\pi(A)$ and $\pi(B)$. We impose that operators $\pi(A)$ and $\pi(B)$ be positive self-adjoint to avoid the problems related to the self-adjointness of the coproduct and well-definedness of the algebra on C^*-algebra level which are discussed in the literature.

Definition 10. let X, Y be positive self-adjoint operators. According to [72], an integrable representation for the relation $XY = q^2YX$ means that for every real number s and t, we have $X^{is}Y^{it} = q^{-2st}Y^{it}X^{is}$, as the relation between the unitary operators.

One can realize operators mentioned above by means of an integrable representation π using the pairs of canonically commuting operators.

$$\pi(A) = e^{2\pi bx} = X, \quad \pi(B) = e^{2\pi bp} = Y, \tag{4.5}$$

which act on $\mathcal{H} = L^2(\mathbb{R})$ and $p = \frac{1}{2\pi i}\frac{d}{dx}$, so $X^{ib^{-1}s}f(x) = e^{2\pi isx}f(x)$, $Y^{ib^{-1}t}f(x) = e^{2\pi ipt}f(x) = f(x+t)$.

For positive, unbounded operators on $L^2(\mathbb{R})$, the domain for X is given by

$$D_X = \{f(x) \in L^2(\mathbb{R}) : e^{2\pi bx}f(x) \in L^2(\mathbb{R})\},$$

[1]In analysis, a space M is called complete (or a Cauchy space) if every Cauchy sequence in M converges in M.

and the domain for Y is given by the Fourier transform of D_X. Therefore, we can obtain various functions in X and Y. For any function defined on $x > 0 \in \mathbb{R}$ such that $|f(x)| = 1$, $f(X)$ will be a unitary operator [72].

Every commutative algebra of operators on a Hilbert space can be represented as the algebra of functions. By considering positive operators we can define a broad class of functions on $\mathcal{A}_q$ by using Mellin transform[2], and in this way we can define the C^*-algebra of "functions on the quantum plane vanishing at infinity" for $\mathcal{A}_q$, which is expressed as,

$$C_\infty(\mathcal{A}_q) := \left\{\mathcal{A}^\infty(\mathcal{A}_q)\right\}^{\text{norm closure}}, \tag{4.6}$$

where

$$\mathcal{A}^\infty(\mathcal{A}_q) = \text{Linear span of}\left\{f(X,Y)\right\}, \tag{4.7}$$

and

$$f(X,Y) = \int_{\mathbb{R}}\int_{\mathbb{R}+i0} F_1(s)F_2(t)X^{ib^{-1}s}Y^{ib^{-1}t}dsdt\},$$

where $F_1(s)$ is entire analytic in s and $F_2(t)$ is meromorphic in t with possible simple poles at

$$t = -ibn - i\frac{m}{b}, \qquad n, m = 0, 1, 2, \ldots \tag{4.8}$$

and norm given by

$$\| \ f(X,Y) \ \|^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} +i0 \mid F_1(s)F_2(t) \mid^2 dsdt. \tag{4.9}$$

According to [73] by using Mellin transform, the space $\mathcal{A}^\infty(\mathcal{A}_q)$ can be written as,

$$\mathcal{A}^\infty(\mathcal{A}_q) := \text{Linear span of} \ \ g(logX)\int_{\mathbb{R}+i0} F_2(t)Y^{ib^{-1}1t}dt,$$

where $g(x)$ is entire analytic and it has a rapid decay in $x \in \mathbb{R}$. $F_2(t)$ is a smooth function in t with rapid decay.

In addition to algebra of functions, we need to know how to find the coproduct for an arbitrary, non-algebraic element in the noncompact case and hence we need to introduce an additional object, called multiplicative unitary. Multiplicative unitaries are fundamental object in the theory of quantum groups in C^*-setting. Multiplicative unitary is the map which encode all the structure maps of quantum group. The Possible difficulties dealing with non-integer powers of generators are avoided by using multiplicative unitary. According to Woronowicz [74], multiplicative unitary is defined as follows

[2]Let $f(x)$ be a continues function on the half line. The Mellin transform of $f(x)$ is defined as, $(\mathcal{M}f)(s) = \int_0^\infty x^{s-1}f(x)dx$.

Definition 11. A unitary element $S \in \mathcal{A} \otimes \mathcal{A}$ is called a multiplicative unitary if it satisfies pentagon equation

$$S_{12}S_{13}S_{23} = S_{23}S_{12}, \tag{4.10}$$

multiplicative unitary encodes the the information of coproduct as

$$S^*(1 \otimes x)S = \Delta(x) \quad x \in \mathcal{A}. \tag{4.11}$$

The pentagon equation implies the coassociativity of the defined coproduct. We refer to [75] for more extended discussions.

According to a proposition 6.7 in [73] and [74] the multiplicative unitary for quantum plane is given by,

$$S = g_b^{-1}(Y^{-1} \otimes sq^{-1}YX^{-1})e^{log(qXY^{-1})\otimes logX^{-1}} \in C_\infty(\mathcal{A}_q) \otimes C_\infty(\mathcal{A}_q), \tag{4.12}$$

where g_b is Fadeev's quantum dilogarithm function as it is defined in equation (2.27).

4.2 Heisenberg double of $U_q(sl(2))$ with continuous basis

In this chapter we aim to introduce a continuous version of the Heisenberg double of the Borel half of $U_q(sl(2))$. We want to introduce it and consider a particular infinite-dimensional integrable representation thereof, with a special focus on the canonical element S. We describe this particular Heisenberg double keeping in mind the fact that in previous chapters some elements of it, as well as the canonical element S have shown up in the study of the Teichmüller theory of Riemann surfaces as an operatorial representations of shear variables and flip operator T.

The Heisenberg double construction give a particular algebra from two copies of a Hopf algebra in a specific way. Indeed, one can find two mutually dual subalgebras of Heisenberg double, which are isomorphic as algebras to the initial pair of Hopf algebra. This two subalgebras are algebras, but not Hopf algebras.

Kashaev has shown that the Heisenberg double of the Borel half $\mathcal{B}(U_q(sl(2)))$, which we will be denoting by $\mathcal{HD}(\mathcal{B}(U_q(sl(2))))$, can be defined as an algebra generated by the four elements. The $\mathcal{HD}^+$ subalgebra is generated by H and E^+ and $\mathcal{HD}^-$ subalgebra generated by $\hat{H}$ and E^-. They satisfying commutation relations as follows

$$\begin{aligned}
&[H,\hat{H}] = \frac{1}{2\pi i}, \qquad [E^+,E^-] = (q-q^{-1})e^{2\pi b H}, \\
&[H,E^\pm] = \mp ibE^+, \qquad [\hat{H},E^+] = 0, \qquad [\hat{H},E^-] = +ibE^-,
\end{aligned} \tag{4.13}$$

where $q = e^{i\pi b^2}$ for a parameter b such that $b^2 \in \mathbb{R}/\mathbb{Q}$, with the real *-structure, i.e.

$$H^* = H, \quad \hat{H}^* = \hat{H}, \quad (E^\pm)^* = E^\pm. \tag{4.14}$$

In addition, the coproduct for the generators $H, \hat{H}, E^{\pm}$ can presented as

$$\begin{aligned} \Delta(H) &= 1 \otimes H + H \otimes 1, \\ \Delta(\hat{H}) &= 1 \otimes \hat{H} + \hat{H} \otimes 1, \\ \Delta(E^+) &= E^+ \otimes e^{2\pi b H} + 1 \otimes E^+, \\ \Delta(E^-) &= E^- \otimes e^{-2\pi b \hat{H}} + 1 \otimes E^-. \end{aligned} \tag{4.15}$$

The canonical element S can be written in terms of the generators of $\mathcal{HD}$

$$S = \exp(2\pi i H \otimes \hat{H}) g_{\mathrm{b}}^{-1}(E^+ \otimes E^-), \tag{4.16}$$

This canonical element is expressed in terms of Faddeev's quantum dilogarithm in the same way as the multiplicative unitary (4.12) of the quantum plane.

$$g_{\mathrm{b}}(x) = \frac{\bar{\zeta}_{\mathrm{b}}}{G_{\mathrm{b}}(\frac{Q}{2} + \frac{1}{2\pi i b}\log x)},$$

where, $\bar{\zeta}_{\mathrm{b}} = \exp[\frac{i\pi}{4} + \frac{i\pi}{12}(b^2 + b^{-2})]$ and we have the following properties,

$$g_{\mathrm{b}}(e^{2\pi b r}) = \int \mathrm{d}t\, e^{2\pi i t r} \frac{e^{-i\pi t^2}}{G_{\mathrm{b}}(Q + it)}, \quad g_{\mathrm{b}}^{-1}(e^{2\pi b r}) = \int \mathrm{d}t\, e^{2\pi i t r} \frac{e^{-\pi t Q}}{G_{\mathrm{b}}(Q + it)}.$$

We can make those subalgebras into two mutually dual Hopf-subalgebras by assigning a coproduct in the following way by using a canonical element S

$$\Delta(u) = S^{-1}(1 \otimes u))S, \quad \text{where } u = H, E^+, \tag{4.17}$$

$$\Delta(v) = S(v \otimes 1)S^{-1}, \quad \text{where } v = \hat{H}, E^-. \tag{4.18}$$

Kashaev representation

Here we review the representations of the Heisenberg double $\mathcal{HD}(\mathcal{B}(U_q(sl(2))))$ considered in [44], that have been shown to be relevant in the quantization of the Teichmüller theory. We introduce the infinite dimensional representations $\pi : \mathcal{HD}(\mathcal{B}(U_q(sl(2)))) \to Hom(L^2(\mathbb{R}))$ with the action of the generators given by

$$\begin{aligned} H &= p, & E^+ &= e^{2\pi b q}, \\ \hat{H} &= q, & E^- &= e^{2\pi b(p-q)}, \end{aligned} \tag{4.19}$$

where $[p, q] = \frac{1}{2\pi i}$ are operators on $L^2(\mathbb{R})$. All of them are positive self-adjoint operators on $L^2(\mathbb{R})$. The canonical element S (4.16) evaluated on this representation is as follows

$$S = \exp(2\pi i H \otimes \hat{H}) g_{\mathrm{b}}^{-1}(E^+ \otimes E^-) = e^{2\pi i p_1 q_2} g_{\mathrm{b}}^{-1}(e^{2\pi b(q_1 + p_2 - q_2)}). \tag{4.20}$$

Using the quantum dilogarithm properties, one can easily confirm that this canonical element bring us to the same coproduct as expressed in relations (4.15), which is shown here

$$
\begin{aligned}
\Delta(H) &= Ad(S^{-1})(1\otimes H) = S^{-1}(1\otimes H)S = \\
&= g_{\mathrm{b}}(E^{+}\otimes E^{-})e^{-2\pi i H\otimes\hat{H}}(1\otimes H)e^{2\pi i H\otimes\hat{H}}g_{\mathrm{b}}^{-1}(E^{+}\otimes E^{-}) = \\
&= g_{\mathrm{b}}(E^{+}\otimes E^{-})(1\otimes H + H\otimes 1)g_{\mathrm{b}}^{-1}(E^{+}\otimes E^{-}) = 1\otimes H + H\otimes 1, \\
\Delta(E^{+}) &= Ad(S^{-1})(1\otimes H) = S^{-1}(1\otimes E^{+})S = \\
&= g_{\mathrm{b}}(E^{+}\otimes E^{-})(1\otimes E^{+})g_{\mathrm{b}}^{-1}(E^{+}\otimes E^{-}) = \\
&= g_{\mathrm{b}}(e^{2\pi b(q_1+p_2-q_2)})e^{2\pi b q_2}g_{\mathrm{b}}^{-1}(e^{2\pi b(q_1+p_2-q_2)}) = \\
&= e^{\pi b q_2}g_{\mathrm{b}}(e^{-i\pi b^2}e^{2\pi b(q_1+p_2-q_2)})g_{\mathrm{b}}^{-1}(e^{+i\pi b^2}e^{2\pi b(q_1+p_2-q_2)})e^{\pi b q_2} = \\
&= e^{\pi b q_2}(1+e^{2\pi b(q_1+p_2-q_2)})e^{\pi b q_2} = \\
&= e^{2\pi b q_2} + e^{2\pi b(q_1+p_2)} = 1\otimes E^{+} + E^{+}\otimes e^{2\pi b H}, \\
\Delta(\hat{H}) &= Ad(S)(\hat{H}\otimes 1) = S(\hat{H}\otimes 1)S^{-1} = \\
&= e^{2\pi i H\otimes\hat{H}}g_{\mathrm{b}}^{-1}(E^{+}\otimes E^{-})(\hat{H}\otimes 1)g_{\mathrm{b}}(E^{+}\otimes E^{-})e^{-2\pi i H\otimes\hat{H}} = \\
&= e^{2\pi i H\otimes\hat{H}}(\hat{H}\otimes 1)e^{-2\pi i H\otimes\hat{H}} = 1\otimes\hat{H} + \hat{H}\otimes 1, \\
\Delta(E^{-}) &= Ad(S)(E^{-}\otimes 1) = S(E^{-}\otimes 1)S^{-1} = \\
&= e^{2\pi i H\otimes\hat{H}}g_{\mathrm{b}}^{-1}(e^{2\pi b(q_1+p_2-q_2)})e^{2\pi b(p_1-q_1)}g_{\mathrm{b}}(e^{2\pi b(q_1+p_2-q_2)})e^{-2\pi i H\otimes\hat{H}} = \\
&= e^{2\pi i H\otimes\hat{H}}e^{\pi b(p_1-q_1)}g_{\mathrm{b}}^{-1}(e^{+i\pi b^2}e^{2\pi b(q_1+p_2-q_2)})g_{\mathrm{b}}(e^{-i\pi b^2}e^{2\pi b(q_1+p_2-q_2)})e^{\pi b(p_1-q_1)}e^{-2\pi i H\otimes\hat{H}} = \\
&= e^{2\pi i H\otimes\hat{H}}(e^{2\pi b(p_1-q_1)} + e^{2\pi b(p_1+p_2-q_2)}))e^{-2\pi i H\otimes\hat{H}} = \\
&= e^{2\pi i H\otimes\hat{H}}(E^{-}\otimes 1 + e^{2\pi b H}\otimes E^{-})e^{-2\pi i H\otimes\hat{H}} = 1\otimes\hat{H} + \hat{H}\otimes e^{-2\pi b\hat{H}},
\end{aligned}
$$

where we used the properties of quantum dilogarithm

$$g_{\mathrm{b}}(e^{2\pi b x}) = e_{\mathrm{b}}(x), \qquad g_{\mathrm{b}}(e^{-i\pi b^2}x) = (1+x)g_{\mathrm{b}}(e^{+i\pi b^2}x). \tag{4.21}$$

In addition, there exists an algebra automorphism A which is also an operator in Teichmüller theory. It is an operator for changing the place of decorated corner of the triangle in Teichmüller context (read more about in [44]) and it was defined as

$$A = e^{-i\pi/3}e^{3\pi i q^2}e^{i\pi(p+q)^2}. \tag{4.22}$$

This automorphism acts in particular on the momentum and position operators as

$$AqA^{-1} = (p-q), \qquad ApA^{-1} = -q. \tag{4.23}$$

By the adjoint action of this automorphism one can define new elements $\tilde{u} = AuA^{-1} \in Hom(L^2(\mathbb{R}))$ which generate another representation of the Heisenberg double in question.

$$
\begin{aligned}
\tilde{H} &= -q, & \tilde{\hat{H}} &= (p-q), \\
\tilde{E}^{+} &= e^{2\pi b(p-q)}, & \tilde{E}^{-} &= e^{-2\pi b p}.
\end{aligned}
$$

Basis elements of the Heisenberg double of the Borel half of $U_q(sl(2))$

This section is part of a ongoing project and our goal is to introduce the continuous version of Heisenberg double of the Borel half of $U_q(sl(2))$ by finding the appropriate bases. Because we are taking a complex power of the generators for defining the bases, we present them in a form that makes explicit their positive and negative definite parts, $H = +H_+ - H_= \sum_{\epsilon=\{+,-\}} \epsilon H_\epsilon$.

We have the following candidate as the possible bases for subalgebras $\mathcal{HD}^+$ and $\mathcal{HD}^-$

$$e(s,t,\epsilon) = \frac{1}{2\pi}\Gamma(-is)G_{\rm b}(-it)q^{-(ib^{-1}t)^2}e^{\epsilon\pi s/2}(2\pi|H|)^{is}\Theta(\epsilon H)(E^+)^{ib^{-1}t}, \tag{4.24}$$

$$\hat{e}(s,t,\epsilon) = e^{-\pi s\delta_{\epsilon,-}}|\hat{H}|^{is}\Theta(\epsilon H)(E^-)^{ib^{-1}t}. \tag{4.25}$$

where $\Theta(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$ and the special function $G_{\rm b}$.

The bases should satisfy the duality relation between them and they normalized such that

$$\langle e(s,t,\epsilon), \hat{e}(s',t',\epsilon')\rangle = \delta(s,s')\delta(t,t')\delta_{\epsilon,\epsilon'},$$

As we see, our algebra has an infinite number of basis elements. The multiplication of those elements is

$$e(s,t,\epsilon)e(s',t',\epsilon') = \sum_\omega \int \mathrm{d}\sigma\mathrm{d}\tau m(s,t,\epsilon,s',t',\epsilon';\sigma,\tau,\omega)e(\sigma,\tau,\omega), \tag{4.26}$$

$$\hat{e}(s,t,\epsilon)\hat{e}(s',t',\epsilon') = \sum_\omega \int \mathrm{d}\sigma\mathrm{d}\tau \mu(\sigma,\tau,\omega;s,t,\epsilon,s',t',\epsilon')\hat{e}(\sigma,\tau,\omega). \tag{4.27}$$

Now, given that one has the following

$$e(s,t,\epsilon)\hat{e}(s',t',\epsilon') = \sum_{\omega,\omega',\omega''} \int \mathrm{d}\sigma\mathrm{d}\sigma'\mathrm{d}\sigma''\mathrm{d}\tau\mathrm{d}\tau'\mathrm{d}\tau'' m(\sigma,\tau,\sigma',\tau';s',t')\mu(s,t;\sigma',\tau',\sigma'',\tau'',\epsilon)\times$$
$$\times\, \hat{e}(\sigma,\tau,\epsilon)e(\sigma'',\tau'').$$

The canonical element of Heisenberg double which satisfies pentagon equation (4.10) can be expressed as

$$S = \sum_\epsilon \int \mathrm{d}s\mathrm{d}t\, e(s,t,\epsilon)\otimes\hat{e}(s,t,\epsilon). \tag{4.28}$$

As we defined only multiplication of those elements (by means of the commutator), the subalgebras $\mathcal{HD}^\pm$ are algebras, but not Hopf algebras. However, we can make those subalgebras into two mutually dual Hopf-subalgebras by assigning a coproduct in the following way by using a canonical element S

$$\Delta(e(s,t,\epsilon)) = S^{-1}(1\otimes e(s,t,\epsilon))S, \tag{4.29}$$

$$\Delta(\hat{e}(s,t,\epsilon)) = S(\hat{e}(s,t,\epsilon)\otimes 1)S^{-1}. \tag{4.30}$$

Chapter 5

Classical super Teichmüller spaces

In the previous chapter we reviewed the Teichmüller theory of Riemann surfaces. The aim of this chapter is to present the basics of super Teichmüller theory, the Teichmüller theory of super Riemann surfaces. Of particular importance will be the coordinates for the super Teichmüller spaces introduced in [55]. These coordinates are closely related to the analogue of Penner's coordinates recently introduced in [53].

In this section, following [55], we will first review the basic notions of super Riemann surfaces and super Teichmüller spaces. We will then consider the definition of two sets of coordinates on this space, called Fock coordinates and Kashaev coordinates.

In order to define such coordinates we will need to refine the triangulations used to define coordinates for the ordinary Teichmüller spaces into certain graphs called hexagonalizations. Assigning the so-called Kasteleyn orientations to the edges of a hexagonalization allows one to parametrise the choices of spin structures on super Riemann surfaces. In addition to even coordinates associated to edges of the underlying triangulation one may define additional odd coordinates associated to the triangles. The additional orientation data assigned to a hexagonalization are used to provide an unambiguous definition of the signs of the odd coordinates. We can use the edges and faces of the hexagonalization with those additional structures to assign the supersymmetric analogues of Fock coordinates.

We will discuss the transformations of coordinates induced by changes of hexagonalizations. The result of the elementary operation of changing the diagonal in a quadrangle called flip will now depend on the choice of Kasteleyn orientation. We will furthermore need to consider an additional operation relating different hexagonalizations called push-out. This operation relates different Kasteleyn orientations describing the same spin structure. The relations that have to be satisfied by these transformations define a generalization of Ptolemy groupoid that will be called super Ptolemy groupoid.

5.1 Super Riemann surfaces

The concept of supermanifolds was settled by mathematicians after several years. This new concept got more attention from physicists after Wess and Zumino presented their famous first supersymmetric field theories, which was the starting point for the application of supergeometry in physics. The most famous applications are superstring theory and supergravity.

The supermanifolds are a generalization of ordinary manifolds in which the notion of variables has been extended to include the anti-commutative, Grasmannian variables,

$$\theta_i\theta_j = -\theta_j\theta_i, \tag{5.1}$$

which imply ${\theta_i}^2 = 0$. There are many ways in which that kind of local description can be incorporated into the picture of differential geometry, which were shown to be equivalent to each other, like DeWitt's approach which mimics ordinary differential geometry, swapping $\mathbb{R}^n$ spaces with graded $\mathbb{R}^{n|m}$, or the one based on sheaves of functions on non-graded manifolds [76, 77]. Here, we will use the former.

Our goal for this section is to study super Riemann surfaces, and in order to do that let us start with the notion of a particular case of G-graded vector spaces, that is $\mathbb{Z}_2$-graded vector spaces, called super vector spaces. An object of this kind $\mathcal{V}$, besides having the normal axioms of a vector space, is endowed with grading, i.e. it decomposes into a direct sum of subspaces,

$$\mathcal{V} = \mathcal{V}_0 \oplus \mathcal{V}_1. \tag{5.2}$$

The elements belonging to $\mathcal{V}_0$ are called even, and those belonging to $\mathcal{V}_1$ called odd.

Moreover, there exist a function $|| : \mathcal{V} \to G$, called a degree, assigning a group element $\alpha \in G$ to each homogeneous element $v \in V_\alpha$:

$$|v| = \alpha. \tag{5.3}$$

This naturally extends to the notion of a G-graded algebra $\mathcal{A}$ — $\mathcal{A}$ is endowed with a multiplication that is compatible with the grading in the following way the multiplication of two homogeneous elements $a, b \in \mathcal{A}$ of degrees $|a| = \alpha, |b| = \beta$ will have a degree being the sum of the degrees of the constituents, i.e. $|ab| = \alpha + \beta$, and

$$\mathcal{A}_\alpha\mathcal{A}_\beta \subset \mathcal{A}_{\alpha+\beta} \tag{5.4}$$

In particular, we are interested in the case when the group G is $\mathbb{Z}_2$, then those $\mathbb{Z}_2$-graded vector spaces are called super vector spaces and $\mathbb{Z}_2$-graded algebras (superalgebras), which naming convention we will use from this point on. Moreover, we call the elements a such that $|a| = 0 \mod 2$ even, and those such that $|a| = 1 \mod 2$ called odd. A superalgebra is super commutative if

$$ab = (-1)^{|a||b|}ba \quad \text{for } a, b \in A. \tag{5.5}$$

The basic example of a superalgebra is an algebra of Grassmanians $G_n(\mathbb{R})$ generated by anti-commutative elements θ_i,

$$G_n(\mathbb{R}) = \{\theta_1, \ldots, \theta_n, |\forall i, j : \theta_i\theta_j = -\theta_j\theta_i\}, \quad n \in \mathbb{N} \tag{5.6}$$

which decomposes into the even and odd subspaces,

$$G_n(\mathbb{R}) = (G_n(\mathbb{R}))_0 \oplus (G_n(\mathbb{R}))_1. \tag{5.7}$$

We can define a superspace $\mathbb{R}^{n|m}$ defined as a product of $\mathbb{R}^n$ and $\mathbb{R}^m$ as even and odd subspaces generated by elements $z_1, \ldots, z_n, \theta_1, \ldots, \theta_m$,

$$\mathbb{R}^{n|m} = \{z_1, \ldots, z_n, \theta_1, \ldots, \theta_m, |\forall i, j : z_iz_j = x_jz_i, z_i\theta_j = \theta_jz_i, \theta_i\theta_j = -\theta_j\theta_i\}. \tag{5.8}$$

We can also define a reduction $\sharp : \mathbb{R}^{n|m} \to \mathbb{R}^n$ which maps the superspace into an even subspace by setting all odd generators θ_i to 0, i.e. the image $(z_1^\sharp, \cdots, z_n^\sharp)$ of an element of $\mathbb{R}^{n|m}$ is defined as follows:

$$(z_1, \cdots, z_n|\theta_1, \cdots, \theta_m) \mapsto (z_1^\sharp, \cdots, z_n^\sharp). \tag{5.9}$$

The superspace $\mathbb{R}^{n|m}$ can be endowed with a topology, known as DeWitt topology, using the reduction map: the subset $U \subseteq \mathbb{R}^{n|m}$ is open if and only if there exists an open subset $V \subseteq \mathbb{R}^n$ such that one is an image of another through the reduction map $U = \sharp^{-1}(V)$. Then one can define a $n|m$-(real-)dimensional supermanifold M using the superspaces $\mathbb{R}^{n|m}$ in the same way as one uses $\mathbb{R}^n$ to define n-dimensional manifolds in ordinary differential geometry.

5.1.1 The super upper half plane and its symmetries

We will begin by introducing the basic group-theoretic and geometric background for the definition of the super Teichmüller spaces and for constructing convenient coordinates on these spaces.

The coordinates on the two-dimensional super plane $\mathbb{R}^{2|1}$ can be assembled in column or row-vectors $(x_1, x_2|\theta)$ with $x_i \in \mathbb{R}$, $i = 1, 2$, and θ being an element of a Grassmann algebra satisfying $\theta^2 = 0$. The elements of the subgroup $OSp(1|2)$ of the group of linear transformations of $\mathbb{R}^{2|1}$ may be represented by $(2|1) \times (2|1)$ matrices of the form

$$g = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}. \tag{5.10}$$

When the matrix elements are elements of a Grassmann algebra, they satisfy the relations

$$ad - bc - \alpha\beta = 1, \tag{5.11}$$
$$e^2 + 2\gamma\delta = 1, \tag{5.12}$$
$$\alpha e = a\delta - c\gamma, \tag{5.13}$$
$$\beta e = b\delta - d\gamma. \tag{5.14}$$

A natural map from $OSp(1|2)$ to $SL(2,\mathbb{R})$ may be defined by mapping the odd generators to zero. The image of $g \in OSp(1|2)$ under this map will be denoted as $g^\sharp \in SL(2,\mathbb{R})$.

The super upper half-plane is defined as $\mathbb{H}^{1|1} = \{(z,\theta) \in \mathbb{C}^{1|1} : \mathrm{Im}(z) > 0\}$. $OSp(1|2)$ acts on the super upper half plane $\mathbb{H}^{1|1}$ by generalized Möbius transformations of the form

$$z \longrightarrow z' = \frac{az + b + \gamma\theta}{cz + d + \delta\theta}, \tag{5.15}$$
$$\theta \longrightarrow \theta' = \frac{\alpha z + \beta + e\theta}{cz + d + \delta\theta}. \tag{5.16}$$

The one-point compactification of the boundary of $\mathbb{H}^{1|1}$ is the super projective real line denoted by $\mathbb{P}^{1|1}$. Elements of $\mathbb{P}^{1|1}$ may be represented as column or row vectors $(x_1, x_2|\theta)$ with $x_i \in \mathbb{R}$, $i = 1,2$ modulo overall multiplication by non-vanishing real numbers. Considering vectors $(x_1, x_2|\theta)$ with $x_i \in \mathbb{R}$, $i = 1,2$ modulo overall multiplication by non-vanishing *positive* numbers defines a double cover $\mathbb{S}^{1|1}$ of $\mathbb{P}^{1|1}$.

There are two types of invariants generalising the cross-ratio present in the ordinary case. To a collection of four points with coordinates $P_i = (x_i|\theta_i)$, $i = 1,\ldots,4$ one may assign a super conformal cross-ratio

$$e^{-z} = \frac{X_{12}X_{34}}{X_{14}X_{23}}, \tag{5.17}$$

where $X_{ij} = x_i - x_j - \theta_i\theta_j$. To a collection of three points $P_i = (x_i|\theta_i)$, $i = 1,2,3$ one may furthermore be tempted to assign an odd (pseudo-) invariant via,

$$\xi = \pm\frac{x_{23}\theta_1 + x_{31}\theta_2 + x_{12}\theta_3 - \frac{1}{2}\theta_1\theta_2\theta_3}{(X_{12}X_{23}X_{31})^{\frac{1}{2}}}, \tag{5.18}$$

where $x_{ij} = x_i - x_j$. Due to the appearance of a square-root one can use the expression in (5.18) to define ξ up to a sign. [1]

[1] Alternatively, we can use the infinitesimal action, using the form of generators $H, E^\pm, F^\pm$ of $osp(1|2)$ Lie superalgebra:

$$\pi_h(E^+) = \partial_x, \qquad \pi_h(E^-) = -x^2\partial_x - x\theta\partial_\theta + 2xh,$$
$$\pi_h(F^+) = \frac{1}{2}(\partial_\theta + \theta\partial_x), \qquad \pi_h(F^-) = -\frac{1}{2}x(\partial_\theta + \theta\partial_x) + \theta h,$$
$$\pi_h(H) = -x\partial_x - \frac{1}{2}\theta\partial_\theta + h,$$

In order to arrive at an unambiguous definition one needs to fix a prescription for the definition of the sign of ξ. A convenient way to parametrize the choices involved in the definition of the odd invariant uses the so-called Kasteleyn orientations of the triangles in $\mathbb{H}^{1|1}$ with corners at P_i, $i = 1, 2, 3$. A Kasteleyn orientation of a polygon embedded in an oriented surface is an orientation for the sides of the polygon such that the number of sides oriented against the induced orientation on the boundary of the polygon is odd.

A Kasteleyn orientation of triangles with three corners at $P_i \in \mathbb{P}^{1|1}$, $i = 1, 2, 3$ may then by used to define lifts of the points $P_i \in \mathbb{P}^{1|1}$ to points $\hat{P}_i$ of its double cover $\mathbb{S}^{1|1}$ for $i = 1, 2, 3$ as follows. We may choose an arbitrary lift of P_1, represented by a vector $(x_1, y_1|\theta_1) \in \mathbb{R}^{2|1}$. If the edge connecting P_i to P_1 is oriented from P_1 to P_i, $i = 2, 3$, we will choose lifts of P_i represented by vectors $(x_i, y_i|\theta_i) \in \mathbb{R}^{2|1}$ such that $\text{sgn}\big(\det\big(\begin{smallmatrix} x_1 & x_i \\ y_1 & y_i \end{smallmatrix}\big)\big) = -1$, while in the other case P_i will be represented by vectors $(x_i, y_i|\theta_i) \in \mathbb{R}^{2|1}$ satisfying $\text{sgn}\big(\det\big(\begin{smallmatrix} x_1 & x_i \\ y_1 & y_i \end{smallmatrix}\big)\big) = 1$. By means of $OSp(1|2)$-transformations one may then map $\hat{P}_i$, $i = 1, 2, 3$ to a triple of points Q_i of the form $Q_1 \simeq (1, 0|0)$, $Q_3 \simeq (0, -1|0)$, and $Q_2 \simeq \pm(1, -1|\xi)$. This allows us to finally define the odd invariant associated to a triangle with corners P_i, $i = 1, 2, 3$, and the chosen Kasteleyn orientation of its sides to be equal to ξ if $Q_2 \simeq (1, -1|\xi)$, and equal to $-\xi$ if $Q_2 \simeq -(1, -1|\xi)$.

5.1.2 Super Teichmüller spaces

After this summary of basic notions of supergeometry, in the following we can discuss the notion of super Riemann surfaces. Analogously to the non-graded case, a super Riemann surface $\Sigma_{g,n}$ is a 1-dimensional complex supermanifold with g denoting the genus and n the number of punctures.

For our goals it will be most convenient to simply define super Riemann surfaces as quotients of the super upper half plane by suitable discrete subgroups of Γ of $OSp(1|2)$. This approach is related to the complex-analytic point of view reviewed in [30] by an analogue of the uniformization theorem proven in [31].

We will use a supersymmetric equivalent of the uniformization theorem, which holds for super Riemann surfaces. To do that however we will use not the upper-half plane as a model surface, but a supersymmetric version: a super upper half-plane $\mathbb{H}^{1|1}$.

A discrete subgroup of Γ of $OSp(1|2)$ such that $\Gamma^\sharp$ is a Fuchsian group is called a super Fuchsian group. Super Riemann surfaces of constant negative curvature will be defined

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with $h = 0$, that generate the $OSp(1|2)$ group and satisfy defining (anti-)commutation relations

$$\begin{aligned}
&[H, E^\pm] = \pm E^\pm, \quad [H, F^\pm] = \pm\frac{1}{2}F^\pm, \\
&[E^+, E^-] = 2H, \quad [E^\pm, F^\mp] = -F^\pm, \\
&\{F^+, F^-\} = \frac{1}{2}H, \quad \{F^\pm, F^\pm\} = \pm\frac{1}{2}E^\pm.
\end{aligned}$$

It is clear that $OSp(1|2)$ has $PSL(2, \mathbb{R})$ as a subgroup generated by the even generators $H, E^\pm$.

as quotients of the super upper half-plane $\mathbb{H}^{1|1}$ by a super Fuchsian group Γ,

$$\Sigma_{g,n} \equiv \mathbb{H}^{1|1}/\Gamma. \tag{5.19}$$

In fact, a super Fuchsian group is a finitely generated discrete subgroup of $OSp(1|2)$ which reduces to a Fuchsian group. $OSp(1|2)$ is the group of automorphisms of $\mathbb{H}^{1|1}$ under which the metric is invariant.

The points of a super Riemann surface may be represented by the points of a fundamental domain D on the super upper-half plane on which Γ acts properly discontinuous. Super Riemann surfaces with n punctures have fundamental domains D touching the boundary $\mathbb{P}^{1|1}$ of $\mathbb{H}^{1|1}$ in d distinct points P_i, $i = 1, \dots, d$.[2]

We can finally define the super Teichmüller space $\mathcal{ST}_{g,n}$ of super Riemann surfaces $\Sigma_{g,n}$ of genus g with n punctures. It can be represented as the quotient

$$\mathcal{ST}_{g,n} = \left\{ \rho : \pi_1(\Sigma_{g,n}) \to OSp(1|2) \right\} / \, OSp(1|2), \tag{5.20}$$

where ρ is a discrete representation of fundamental group $\pi_1(\Sigma_{g,n})$ into $OSp(1|2)$ whose image is super Fuchsian.

There is always an ordinary Riemann surface $\Sigma^{\sharp}_{g,n}$ associated to each super Riemann surface, defined as quotient of the upper half plane $\mathbb{H}$ by $\Gamma^{\sharp}$. Notions like ideal triangulations will therefore have obvious counterparts in the theory of super Riemann surfaces.

Interestingly, while the Teichmüller is connected, the super Teichmüller space has multiple connected components, whose number is given by the number of spin structures (which will be discussed in the subsequent section) that a super Riemann surface admits. Moreover, each of those components is diffeomorphic to $\mathbb{R}^{6g-6+2m|4g-g+2m+m_{\mathrm{R}}}$, where m_{R} is a number of Ramond punctures and $m_{\mathrm{NS}} = m - m_{\mathrm{R}}$ is a number of Neveu-Schwartz punctures.

5.2 Hexagonalization and Kasteleyn orientations

Similarly to the ordinary Teichmüller spaces, the parametrization of super Teichmüller spaces introduced in [55] relies on ideal triangulations of super Riemann surfaces. It will be based on the even and odd invariants of the group $OSp(1|2)$ that we defined in Section 5.1.1. However, as noted there, one needs to introduce additional data to define the odd invariants unambiguously. The extra data must allow us to define the lifts of the punctures $P_i \in \mathbb{P}^{1|1}$ to points $\hat{P}_i$ on its double cover $\mathbb{S}^{1|1}$. Note that the even part of $\mathbb{P}^{1|1}$ is the real projective line $\mathbb{RP}^1$ with group of automorphisms $PSL(2, \mathbb{R})$, while the even part of $\mathbb{S}^{1|1}$ is a double cover of $\mathbb{RP}^1$ with group of automorphisms $SL(2, \mathbb{R})$. Lifting the vertices of a triangulation of $\mathbb{H}^{1|1}$ to $\mathbb{S}^{1|1}$ should therefore be accompanied with a lift of the Fuchsian group $\Gamma^{\sharp} \subset PSL(2, \mathbb{R})$ to a subgroup of $SL(2, R)$. It is known that the

[2]When pairs of points get identified by the action of the group Γ we will have $d \neq n$.

definition of such a lift depends on the choice of a spin structure on Σ [78]. Therefore, we need to introduce a suitable refinement of an ideal triangulation which will allow us to encode the extra data defining a spin structure.

The parametrization of spin structure on Riemann surfaces used in [55] is based on results of Cimasoni, Reshetikhin [62, 63] using Kasteleyn orientations. To begin with, let us first introduce the notion of a hexagonalization. The starting point will be an ideal triangulation of a surface Σ. Around each puncture let us cut out a small disc, giving a surface Σ_{b} with n holes. The parts of any two edges bounding a triangle in Σ which are contained in Σ_{b} will then be connected by an arc in the interior of Σ_b. The resulting hexagon has a boundary consisting of "long" edges coming from the edges of the original triangulation, and "short" edges represented by the arcs connecting the long edges. The procedure is illustrated in figure 5.1.

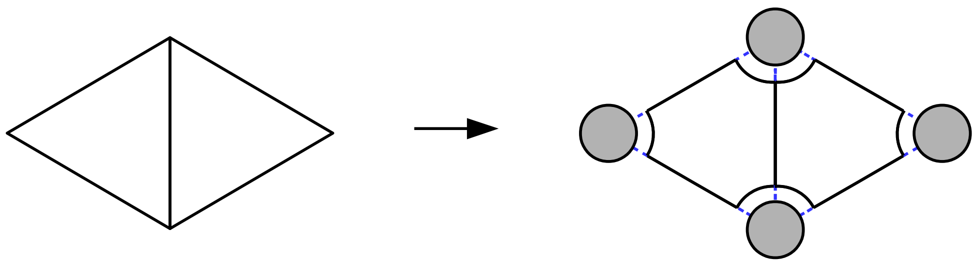

FIGURE 5.1: Hexagonalization.

Let us finally introduce another set of edges called dimers connecting the vertices of the hexagons with the boundary of Σ_{b}. The dimers are represented by dashed lines in Figure 5.1. The resulting graph will be called a hexagonalization of the given ideal triangulation.

The next step is to introduce a Kasteleyn orientation on the hexagonalization defined above. It is given by an orientation of the boundary edges of the hexagons such that for every face of the resulting graph the number of edges oriented against the orientation of the surface is odd. It then follows from Theorem 1 in [63] that the choice of the spin structure can be encoded in the choice of a Kasteleyn orientation on a hexagonalization.[3]

Different Kasteleyn orientations may describe the same spin structure. Two Kasteleyn orientations are equivalent in this sense if they are related by the reversal of orientations of all the edges meeting at the same vertex, as illustrated in Figure 5.2.

The equivalence classes of Kasteleyn orientations related by this operation are in one-to-one correspondence to the spin structures on Σ.

In order to represent a hexagonalization with Kasteleyn orientation graphically we will find it convenient to contract all short edges to points, and marking the corners of the resulting triangle coming from short edges with orientation against the orientation of

[3] The hexagonalizations constructed above are special cases of what is called surface graph with boundary in [62, 63]. The formulation of Theorem 1 in [63] makes use of the notion of a dimer configuration on a surface graph with boundary. In our case the dimer configuration is given by the set of edges connecting the corners of the hexagons with the boundary shown as dashed lines in Figure 5.1.

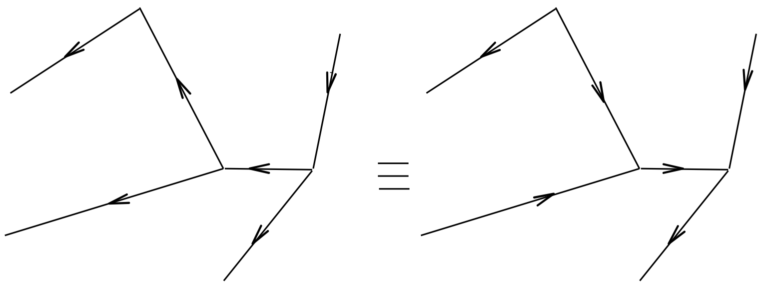

FIGURE 5.2: Equivalence between the Kasteleyn orientations.

the underlying surface by dots. An illustration of this procedure is given in Figures 5.3 and 5.4 below.

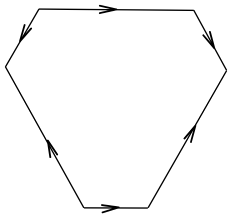

FIGURE 5.3: A hexagon with Kasteleyn orienations.

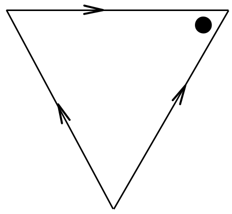

FIGURE 5.4: A representation of the figure 5.3 by a dotted triangle.

This amounts to representing the data encoded in a hexagonalization with Kasteleyn orientations in a triangulation carrying an additional decoration given by the choice of orientations for the edges, and by marking some corners with dots. The data graphically represented by dotted triangulations will be referred to as oriented hexagonalizations in the rest of this text.

As a final remark for this part, for dotted triangles the move reversing the orientation of all edges which meet in the same vertex has a direct generalization, as pictured in figure 5.5. In this picture we have the short edges of few hexagons next to each other. The short edges can collapse to the common projected vertex.

5.3 Coordinates of the super Teichmüller spaces

In order to define coordinates for the super Teichmüller spaces let us consider super Riemann surfaces $\Sigma_{g,n} \equiv \mathbb{H}^{1|1}/\Gamma$ with $n \geq 1$ punctures. $\Sigma_{g,n}$ can be represented by a polygonal fundamental domain $D \subset \mathbb{H}^{1|1}$ with a boundary represented by a collection of arcs pairwise identified with each other by the elements of Γ. The corners of the fundamental domains $P_i = (x_i|\theta_i)$, $i = 1, \ldots, d$ of D will be located on the boundary $\mathbb{P}^{1|1}$ of $\mathbb{H}^{1|1}$. An ideal triangulation of the underlying Riemann surface $\Sigma^{\sharp}_{g,n}$ induces a triangulation of the super Riemann surface with vertices represented by the corners $P_i = (x_i|\theta_i)$, $i = 1, \ldots, d$. Following [55] we will in the following assign even coordinates to the edges of a dotted triangulation, and odd variables to the triangles themselves.

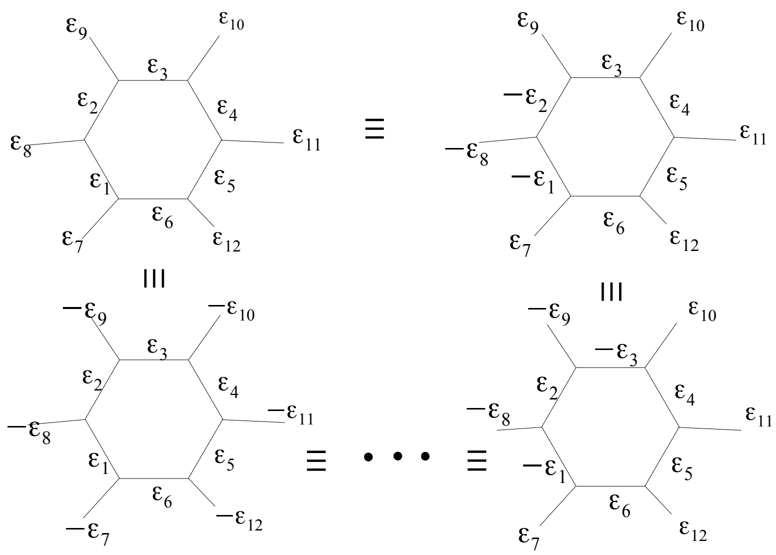

FIGURE 5.5: Equivalence between the Kasteleyn orientations for hexagons sharing the same projected vertex.

In order to define the coordinates associated to edges let us assume that the edge e represents the diagonal in a quadrangle with corners at $P_i = (x_i|\theta_i) \in \mathbb{P}^{1|1}$, $i = 1, \ldots, 4$ connecting P_2 and P_4. One may then define the even variable z_e assigned to the edge e to be given by the even super conformal cross-ratio defined in (5.17).

In order to define the odd Fock variables let us consider a hexagonalization decorated with a Kasteleyn orientation. We may triangulate each hexagon as shown in Figure 5.6.

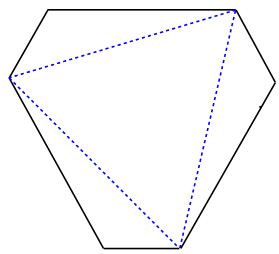

FIGURE 5.6: A hexagon and its underlying triangle.

Note that the orientation on the sides of the hexagon induces a canonical Kasteleyn orientation on each of the triangles appearing in this triangulation of the hexagon. We may therefore apply the definition of odd invariant given in Section 5.1.1 to the corners of the inner triangle drawn with blue, dashed sides in Figure 5.6. As the hexagons of the considered hexagonalization are in one-to-one correspondence with the triangles Δ of a dotted triangulation we will denote the resulting coordinates by ξ_Δ.

Poisson structure A super Poisson algebra is a superalgebra A with grading of $x \in A$ denoted as $|x|$, which has a super Poisson bracket $\{.,.\}_{ST} : A \times A \to A$ which satisfies:

- $\{x, y\}_{ST} = -(-1)^{|x||y|}\{y, x\}$ graded skew-symmetric,
- $\{x, \{y, z\}\} + (-1)^{|x|(|y|+|z|)}\{y, \{z, x\}\} + (-1)^{|z|(|x|+|y|}\{z, \{x, y\}\}$ super Jacobi identity,
- $\{x, yz\} = \{x, y\}z + (-1)^{|x||y|}y\{x, z\}$ super Leibniz's rule,

The super Teichmüller space is parametrized by $3(2g - 2 + n)$ even coordinates and $2(2g - 2 + n)$ odd coordinates. As in the non-graded case, the super Teichmüller space furnishes a symplectic structure given by a super Poisson bracket [79],

$$\{,\}_{ST} = \sum_{i,j \in E(\eta(\Sigma))} \varepsilon^{ij} x_i x_j \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} - \frac{1}{2}\sum_{k \in F(\eta(\Sigma))} \frac{\overleftarrow{\partial}}{\partial \xi_k}\frac{\overrightarrow{\partial}}{\partial \xi_k}, \tag{5.21}$$

where the numbers ϵ^{ij} are defined in the same way as in the non-graded case, and depend on the way the edges e_i and e_j meet with each other. Moreover, the odd differentials act as follows on the coordinates

$$\frac{\overleftarrow{\partial}\,\xi_j}{\partial \xi_i} = \frac{\overrightarrow{\partial}\,\xi_j}{\partial \xi_j} = \delta_{ij}, \qquad \frac{\overleftarrow{\partial}\,x_j}{\partial \xi_i} = \frac{\overrightarrow{\partial}\,x_j}{\partial \xi_i} = 0, \tag{5.22}$$

and anticommute with the odd coordinates

$$\frac{\overrightarrow{\partial}}{\partial \xi_i}(\xi_1 \dots \xi_i \dots \xi_k) = (-1)^{i-1}(\xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_k), \tag{5.23}$$

$$\frac{\overleftarrow{\partial}}{\partial \xi_i}(\xi_1 \dots \xi_i \dots \xi_k) = (-1)^{k-i}(\xi_1 \dots \xi_{i-1}\xi_{i+1} \dots \xi_k). \tag{5.24}$$

For comparing with the ordinary case we can write the super Poisson structure [55] with non-trivial Poisson brackets among the coordinate functions as

$$\{z_e, z_f\}_{\rm ST} = n_{ef}\,, \qquad \{\xi_v, \xi_w\}_{\rm ST} = \frac{1}{2}\delta_{vw}\,, \tag{5.25}$$

where the numbers n_{ef} are defined in the same way as in ordinary Teichmüller theory. This defines the Poisson-structure we aim to quantise.

5.4 Super Ptolemy groupoid

The coordinates that we use to parametrise the super Teichmüller space depend on the choice of the dotted triangulation. It is therefore necessary to determine how those coordinates transform under the moves that change the dotted triangulations of the Riemann surfaces. In addition to the supersymmetric analogue of the flip operation changing the diagonal in a quadrilateral we now need to consider an additional move describing a change of Kasteleyn orientation which leaves the spin structure unchanged.

The groupoid generated by the changes of dotted triangulations will be called super Ptolemy groupoid. We will now offer a description in terms of generators and relations.

5.4.1 Generators

Push-out As we discussed previously, the reversal of Kasteleyn orientations of all the edges that meet in the same vertex encodes equivalent spin structure to the beginning one. Therefore, we can consider a pair of two hexagons that meet along one long edge, and study a move that applies this orientation reversal on one of the vertices common to both hexagons, as in figure 5.7. We will call this move a (left) push-out. As for the action on the odd invariants, a push-out leaves the one of the left hexagon unchanged, but it changes the sign for the one on the right, and it does not change any of the even invariants.

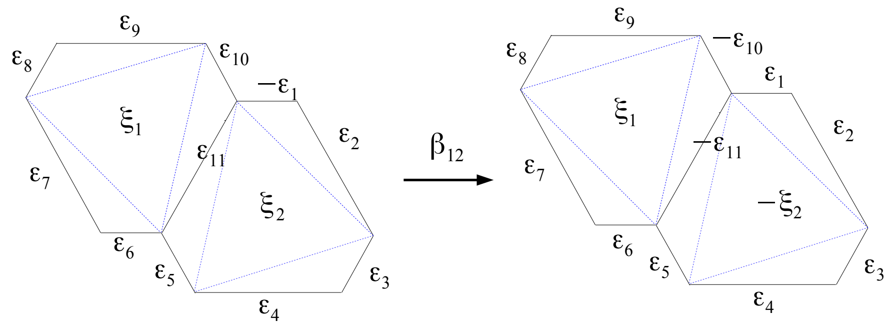

Figure 5.7: A (left) push-out.

To make this figure more clear we should explain that the notation ϵ_i on each edge means the correspondence edge i can have any orientation: clockwise or counterclockwise One should consider that ϵ_i has to be chosen in such way that the Kasteleyn orientation is satisfied. The reason for choosing this notation is that we show how the orientations change after the action of the generators.

In terms of dotted triangles, one can pictorially represent this move as in figure 5.8. As we see, we can interpret the push-out as a change of Kasteleyn orientation that moves dots (or equivalently, short edges oriented against the orientation of the surface) from one dotted triangle to another. Therefore, we can obtain the transformations of coordinates assigned to dotted triangles by composing the transformations of coordinates of triangles without any dots with an appropriate sequence of push-outs.

Moreover, we can define an inverse of a (left) push-out, which we will call a right push-out. We represent it in figure 5.9. On the odd invariants, it acts in the same way as the left push-out: the invariant of the left hexagon stays the same, while the invariant of the right changes sign.

If one would consider dotted triangles instead of hexagons, one can represent the right push-out as in figure 5.10.

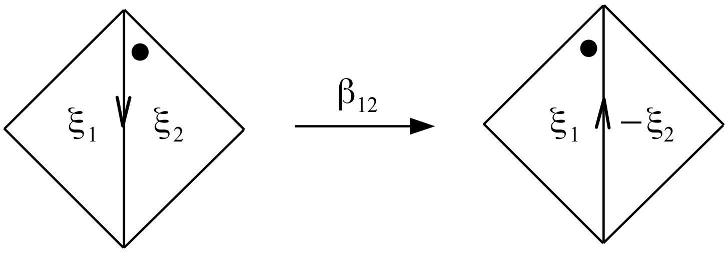

FIGURE 5.8: The pictorial representation of a (left) push-out on triangles with one dot.

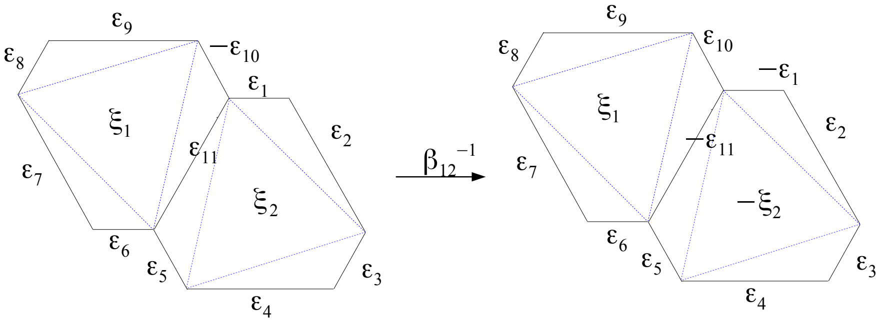

FIGURE 5.9: A right push-out.

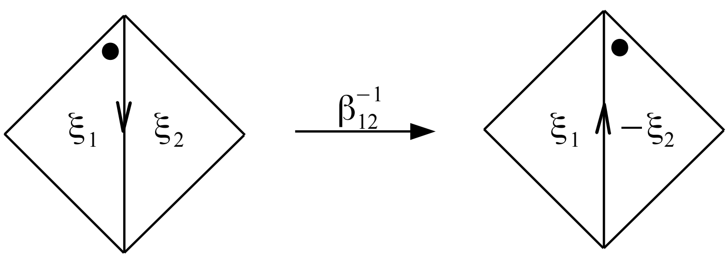

FIGURE 5.10: The pictorial representation of a right push-out on triangles with one dot.

Superflips We furthermore need to consider the flip operation describing the change of diagonal in a quadrilateral. The effect of this operation will in general depend on the assignment of Kasteleyn orientations. An example is depicted in Figure 5.11.

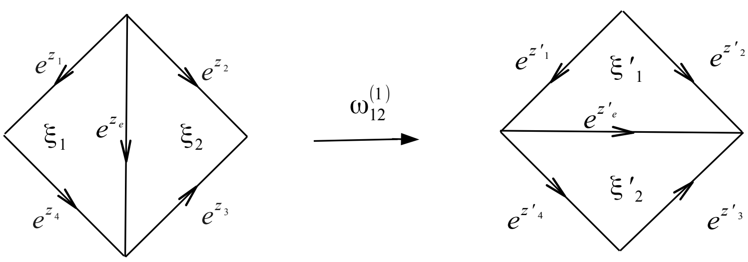

FIGURE 5.11: The flip $\omega_{12}^{(1)}$.

The change of even Fock coordinates may be represented as [55]

$$\begin{aligned}
e^{z'_e} &= e^{-z_e}, \\
e^{z'_1} &= e^{\frac{z_1}{2}}(1+e^{z_e}-\xi_1\xi_2 e^{\frac{z_e}{2}})e^{\frac{z_1}{2}}, \\
e^{z'_2} &= e^{\frac{z_2}{2}}(1+e^{-z_e}-\xi_1\xi_2 e^{-\frac{z_e}{2}})^{-1}e^{\frac{z_2}{2}}, \\
e^{z'_3} &= e^{\frac{z_3}{2}}(1+e^{z_e}-\xi_1\xi_2 e^{\frac{z_e}{2}})e^{\frac{z_3}{2}}, \\
e^{z'_4} &= e^{\frac{z_4}{2}}(1+e^{-z_e}-\xi_1\xi_2 e^{-\frac{z_e}{2}})^{-1}e^{\frac{z_4}{2}},
\end{aligned} \tag{5.26}$$

As we mentioned superflip is a map which relates two different ways of triangulating a quadrilateral. In the case of super Teichmüller theory, the triangles here should be interpreted as dotted triangles, that is hexagons with Kasteleyn orientations. To reduce the number of cases to be considered in the statement of the transformation of the odd coordinates one may first note that the push-out operation allows one to reduce the most general case to the case of undotted triangles. There are different ways of assigning Kasteleyn orientations to the long edges. It is easy to check that there are 8 possible ways of assigning Kasteleyn orientations in this case. In figure 5.12 we present the full list of all of possible superflips.

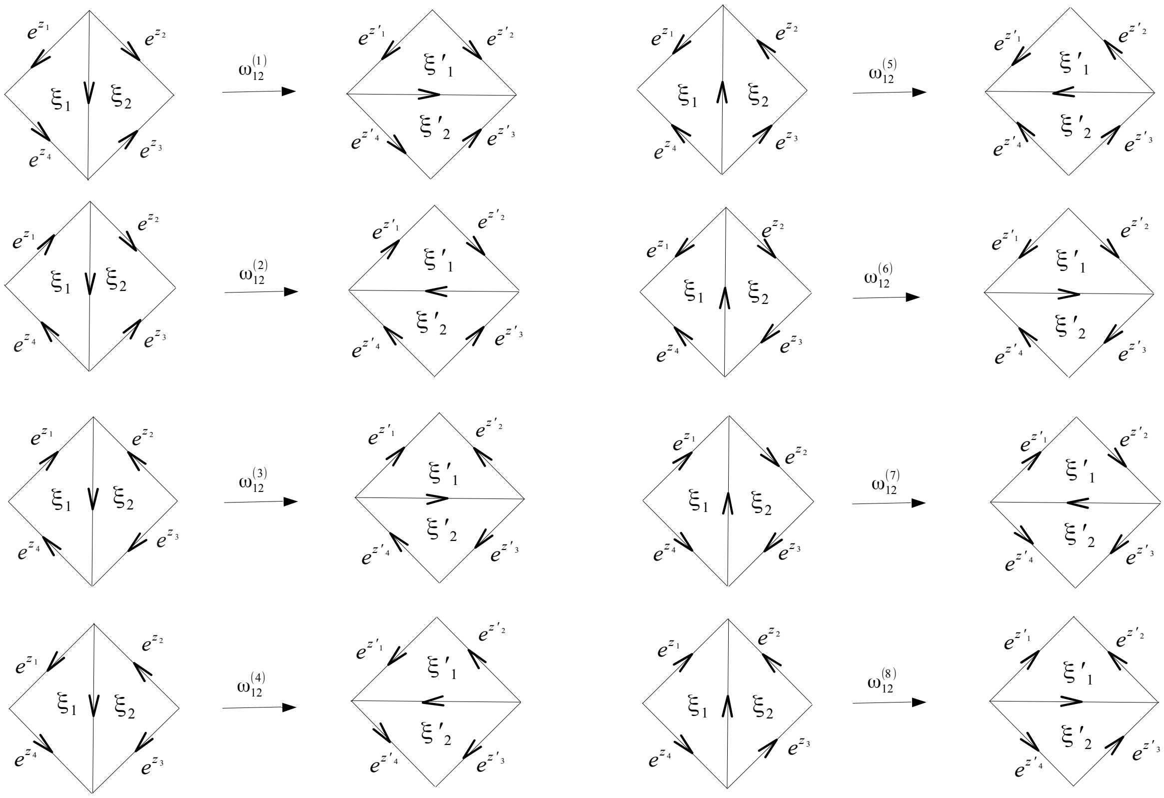

FIGURE 5.12: Superflips for quadrilaterals without dots; cases 1-8.

As a remark when we consider Kashaev type coordinates (in 5.5) it is necessary to use the decorated version of dotted triangulations. In the case of the quadrilaterals relevant for the flip map, decorated vertices should be chosen always as in figure 5.18.

Let us begin by considering the operation $\omega^{(1)}$ depicted in Figure 5.11. One then finds the following change of coordinates [55]

$$\begin{aligned} e^{\frac{z_1'}{2}}\xi_1' &= e^{\frac{z_1}{2}}(\xi_1 + \xi_2 e^{\frac{z_e}{2}}), \\ e^{\frac{z_1'}{2}}\xi_2' &= e^{\frac{z_1}{2}}(-\xi_1 e^{\frac{z_e}{2}} + \xi_2). \end{aligned} \tag{5.27}$$

μ operator As a useful book-keeping device for generating the expressions in the other cases let us introduce an operation μ_v that reverses the orientations of the two long edges entering a common vertex of a dotted triangulation. This operation is graphically represented in Figure 5.13. It is easy to see that this will induce a sign change in the definition of the odd invariant.

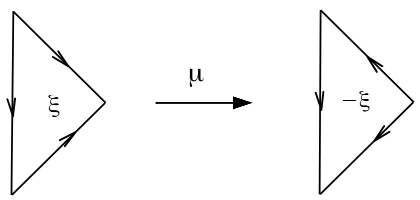

FIGURE 5.13: The operation μ.

The coordinate transformations induced by flips with other assignments of Kasteleyn orientations can then be obtained from the case of $\omega^{(1)}$ with the help of the operations μ_v. An example is represented by Figure 5.14.

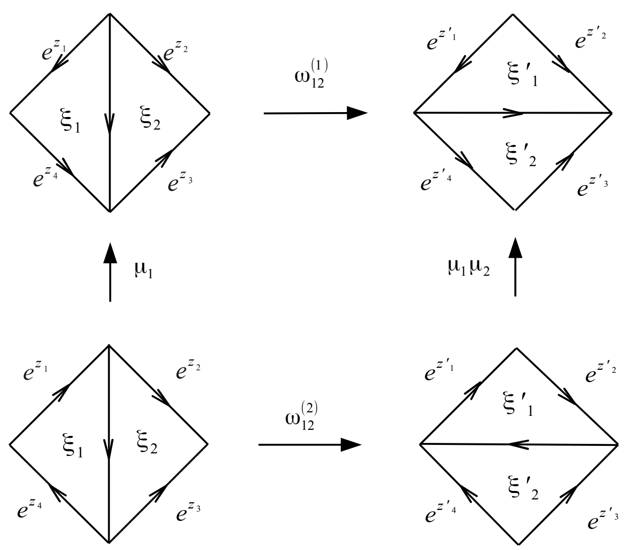

FIGURE 5.14: Different flips are related by application of transformations μ.

5.4.2 Relations

The changes of oriented hexagonalizations define a groupoid generalising the Ptolemy groupoid. In the following we are going to discuss the relations characterising this groupoid which will be called super Ptolemy groupoid.

It is clear that all relations of the super Ptolemy groupoid reduce to relations of the ordinary Ptolemy groupoid upon forgetting the decorations furnished by the Kasteleyn orientations. This fact naturally allows us to distinguish a few different types of relations.

To begin with, let us consider the relations reducing to the pentagon relation of the Ptolemy groupoid. The super Ptolemy groupoid will have various relations differing by the choices of Kasteleyn orientations. Considering first the case where all short edges are oriented with the orientation of the surface we have 16 possible pentagon relations:

$$
\begin{array}{ll}
\omega_{12}^{(1)}\omega_{23}^{(1)} = \omega_{23}^{(1)}\omega_{13}^{(1)}\omega_{12}^{(1)}, & \omega_{12}^{(1)}\omega_{23}^{(6)} = \omega_{23}^{(6)}\omega_{13}^{(6)}\omega_{12}^{(4)}, \\
\omega_{12}^{(5)}\omega_{23}^{(8)} = \omega_{23}^{(8)}\omega_{13}^{(5)}\omega_{12}^{(5)}, & \omega_{12}^{(6)}\omega_{23}^{(7)} = \omega_{23}^{(7)}\omega_{13}^{(6)}\omega_{12}^{(5)}, \\
\omega_{12}^{(2)}\omega_{23}^{(1)} = \omega_{23}^{(1)}\omega_{13}^{(2)}\omega_{12}^{(2)}, & \omega_{12}^{(8)}\omega_{23}^{(8)} = \omega_{23}^{(1)}\omega_{13}^{(8)}\omega_{12}^{(8)}, \\
\omega_{12}^{(4)}\omega_{23}^{(5)} = \omega_{23}^{(5)}\omega_{13}^{(5)}\omega_{12}^{(4)}, & \omega_{12}^{(5)}\omega_{23}^{(3)} = \omega_{23}^{(3)}\omega_{13}^{(4)}\omega_{12}^{(6)}, \\
\omega_{12}^{(3)}\omega_{23}^{(4)} = \omega_{23}^{(7)}\omega_{13}^{(3)}\omega_{12}^{(2)}, & \omega_{12}^{(7)}\omega_{23}^{(7)} = \omega_{23}^{(4)}\omega_{13}^{(7)}\omega_{12}^{(8)}, \\
\omega_{12}^{(6)}\omega_{23}^{(2)} = \omega_{23}^{(2)}\omega_{13}^{(1)}\omega_{12}^{(6)}, & \omega_{12}^{(7)}\omega_{23}^{(2)} = \omega_{23}^{(5)}\omega_{13}^{(2)}\omega_{12}^{(7)}, \\
\omega_{12}^{(5)}\omega_{23}^{(6)} = \omega_{23}^{(3)}\omega_{13}^{(7)}\omega_{12}^{(6)}, & \omega_{12}^{(3)}\omega_{23}^{(5)} = \omega_{23}^{(2)}\omega_{13}^{(8)}\omega_{12}^{(3)}, \\
\omega_{12}^{(1)}\omega_{23}^{(3)} = \omega_{23}^{(6)}\omega_{13}^{(3)}\omega_{12}^{(7)}, & \omega_{12}^{(4)}\omega_{23}^{(4)} = \omega_{23}^{(4)}\omega_{13}^{(4)}\omega_{12}^{(1)}.
\end{array}
\tag{5.28}
$$

The remaining cases can always be reduced to the cases listed above using the push-out operation. In Figure 5.15 we present graphically one of the 16 possibilities listed above.

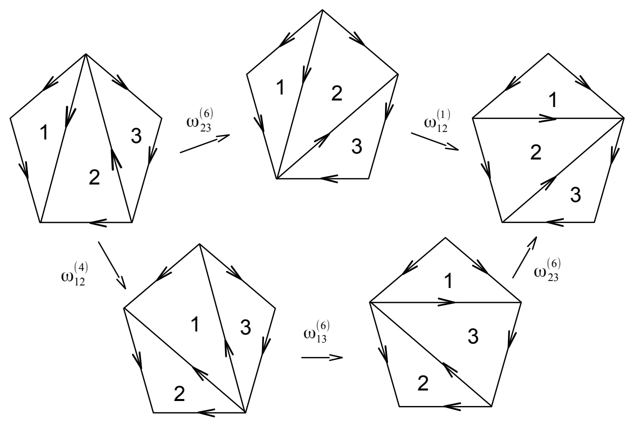

FIGURE 5.15: An example of one of the possible superpentagon relations.

Other relations reduce to trivial relations upon forgetting the orientation data. Some of these relations describe how the push-out operations relate flips with different orientation data. Such relations are

$$(\omega_{23}^{(i)})^{-1}\beta_{43}\beta_{32}\beta_{21} = \beta_{42}\beta_{21}(\omega_{23}^{(j)})^{-1}, \tag{5.29}$$

where i, j can be the following pairs $(5,8),(8,5),(6,7),(7,6),(1,2),(2,1),(3,4),(4,3)$ and

$$\omega_{23}^{(i)}\beta_{43}\beta_{32}\beta_{21} = \beta_{43}\beta_{31}\omega_{23}^{(j)}, heorem \tag{5.30}$$

where i, j can be the following pairs $(5,4),(4,5),(1,6),(6,1),(2,7),(7,2),(8,3),(3,8)$. An example for this type of relation is illustrated in Figure 5.16.

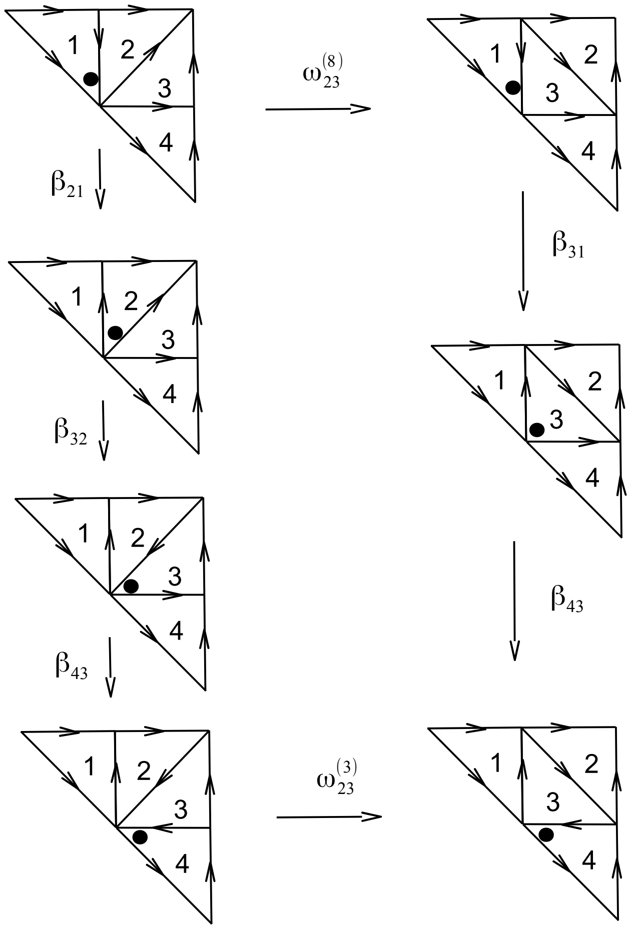

FIGURE 5.16: First type of relation between a flip and a push-out.

There are further relations reducing to the commutativity of the flip operations applied to two quadrilaterals which do not share a triangle, including

$$\omega_{34}^{(i)}\beta_{23}(\omega_{12}^{(j)})^{-1}\beta_{23}^{-1} = \beta_{24}(\omega_{12}^{(j)})^{-1}\beta_{24}^{-1}\omega_{34}^{(i)}, \tag{5.31}$$

$$(\omega_{34}^{(i)})^{-1}\beta_{13}\omega_{12}^{(j)}\beta_{23}^{-1} = \beta_{13}\omega_{12}^{(j)}\beta_{23}^{-1}(\omega_{34}^{(i)})^{-1}, \tag{5.32}$$

$$(\omega_{34}^{(i)})^{-1}\beta_{23}(\omega_{12}^{(j)})^{-1}\beta_{23}^{-1} = \beta_{23}(\omega_{12}^{(j)})^{-1}\beta_{23}^{-1}(\omega_{34}^{(i)})^{-1}, \tag{5.33}$$

$$\omega_{34}^{(i)}\beta_{13}\omega_{12}^{(j)}\beta_{23}^{-1} = \beta_{14}\omega_{12}^{(j)}\beta_{24}^{-1}\omega_{34}^{(i)}, \tag{5.34}$$

where the $i, j = 1, \ldots, 8$ depends on the Kasteleyn orientation of the graph from which the relation has been derived. Examples of these relations are represented in Figure 5.17.

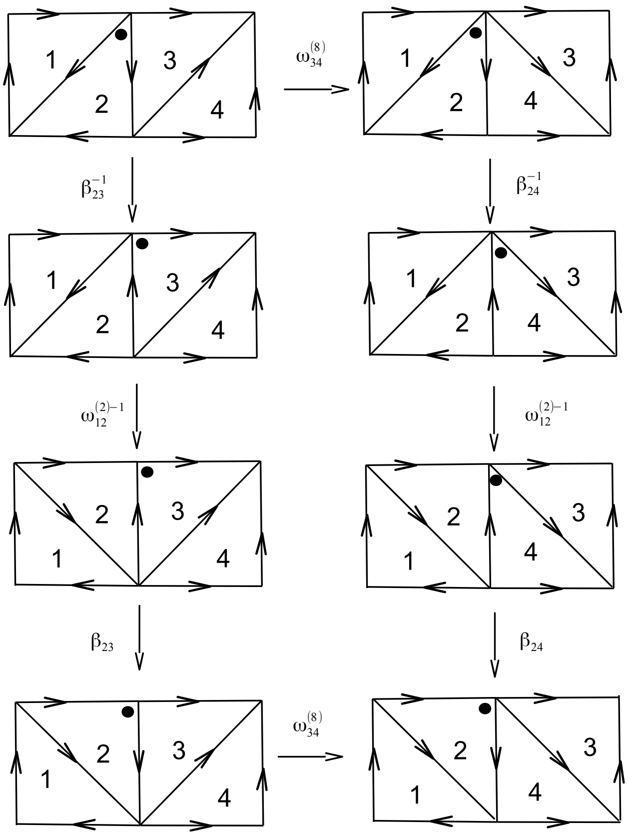

FIGURE 5.17: Second type of relation between a flip and a push-out.

It seems plausible that the completeness of the relations discussed above can be reduced to the corresponding result for the ordinary Ptolemy groupoid. This result, as pointed out in [33], follows from the cell decomposition of the Teichmüller space which can be defined with the help of Penner's coordinates [54].

5.5 Kashaev type coordinates

It will furthermore be useful to introduce analogues of the Kashaev coordinates in the case of super Teichmüller theory. Such coordinates will be associated to oriented hexagonalizations carrying an additional piece of decoration obtained by marking a distinguished short edge in each hexagon. Oriented hexagonalizations equipped with such a decoration will be called decorated hexagonalizations in the following.

In addition to a pair of even variables (q_v, p_v) assigned to each ideal triangle Δ_v, we now need to introduce an odd variable ξ_v. The collection of these variables parameterizing points in $\mathbb{R}^{4(2g-2+n)|2(2g-2+n)}$, which we will name super Kashaev space, will be called super Kashaev coordinates. The non-trivial Poisson brackets defining the Poisson structure on this space are

$$\{p_v, q_w\}_{\mathrm{ST}} = \delta_{v,w}, \qquad \{\xi_v, \xi_w\}_{\mathrm{ST}} = \frac{1}{2}\delta_{v,w}, \tag{5.35}$$

with all other Poisson brackets among the variables (q_v, p_v, ξ_v) being trivial.

The super Teichmüller spaces can be characterized within $\mathbb{R}^{8g-8+4n|4g-4+2n}$ by using the Hamiltonian reduction with respect to a set of constraints that is very similar to the one used in ordinary Teichmüller theory described in [34]. One may, in particular, recover the even Fock coordinates in a way that is very similar to (2.12), while the odd variables simply coincide.

The transformations relating different decorated hexagonalizations will induce changes of super Kashaev coordinates. Such transformations will generate a decorated version of the super Ptolemy groupoid. The set of generators becomes as in the case of ordinary Teichmüller theory enriched by the operation (vw) exchanging the labels associated to two adjacent triangles, and the rotations ρ_v of the distinguished short edge. The rotation ρ_v will be represented as

$$\rho_v^{-1} : (q_v, p_v, \xi_v) \to (p_v - q_v, -q_v, \xi_v). \tag{5.36}$$

The operation (vw) maps (q_v, p_v, ξ_v) to (q_w, p_w, ξ_w) and vice-versa. The flip $\omega_{vw}^{(1)}$, presented in the figure 5.18, is realized by

$$(\omega_{vw}^{(1)})^{-1} : \begin{cases} (U_v, V_v) \to (U_vU_w, U_vV_w + V_v - U_v^{\frac{1}{2}}V_w^{\frac{1}{2}}V_v^{\frac{1}{2}}\xi_v\xi_w), \\ (U_w, V_w) \to (U_wV_v(U_vV_w + V_v - U_v^{\frac{1}{2}}V_w^{\frac{1}{2}}V_v^{\frac{1}{2}}\xi_v\xi_w)^{-1}, \\ \qquad V_w(U_vV_w + V_v - U_v^{\frac{1}{2}}V_w^{\frac{1}{2}}V_v^{\frac{1}{2}}\xi_v\xi_w)^{-1}), \end{cases} \tag{5.37}$$

for the even variables and

$$(\omega_{vw}^{(1)})^{-1} : \begin{cases} \xi_v \to \frac{V_v^{\frac{1}{2}}\xi_v + U_v^{\frac{1}{2}}V_w^{\frac{1}{2}}\xi_w}{\sqrt{V_v + U_vV_w - U_v^{\frac{1}{2}}V_w^{\frac{1}{2}}V_v^{\frac{1}{2}}\xi_v\xi_w}}, \\ \xi_w \to \frac{V_v^{\frac{1}{2}}\xi_w - U_v^{\frac{1}{2}}V_w^{\frac{1}{2}}\xi_v}{\sqrt{V_v + U_vV_w - U_v^{\frac{1}{2}}V_w^{\frac{1}{2}}V_v^{\frac{1}{2}}\xi_v\xi_w}}, \end{cases} \tag{5.38}$$

for the odd ones, where we denote $U_v \equiv e^{q_v}$ and $V_v \equiv e^{p_v}$. The action of the rest of flips [4] can be obtained by the application of appropriate operations μ_v, as explained previously.

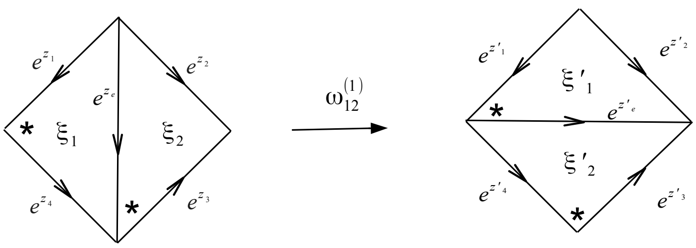

FIGURE 5.18: A flip $\omega^{(1)}$ on decorated triangulation.

[4]The flips transforming Kashaev coordinates relate decorated versions of quadrilaterals. Therefore, to represent flips of Kashaev coordinates one should add decoration to all the figures in 5.12 in the same places as in the figure 5.18.

Chapter 6

Quantization of super Teichmüller theory

In this section we will consider the quantization of the Teichmüller spaces of super Riemann surfaces. The coordinate functions defined in chapter 5 will become linear operators acting on a Hilbert space. The transformations which relate different hexagonalizations, like flips and push-outs, will be represented by linear operators T and B, respectively. We are going to discuss the relations satisfied by these operators, defining a projective representation of the super Ptolemy groupoid. We take a collection of equations (6.26), (6.32)-(6.37), (6.38)-(6.40) as the defining relations for the quantum super Ptolemy grupoid.

6.1 Quantization of super Kashaev space

The Hilbert space associated to a decorated hexagonalization of a super Riemann surface will be defined as follows. To each hexagon Δ_v (or equivalently each dotted triangle) we associate a Hilbert space $\mathcal{H}_v \simeq L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$. Then, the Hilbert space associated to the entire super Riemann surface is the tensor product of the spaces for each hexagon:

$$\mathcal{H} = \bigotimes_{v \in I} \mathcal{H}_v. \tag{6.1}$$

We will frequently use the corresponding leg-numbering notation: If O is an operator on $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$, we may define O_v to be the operator $\mathsf{O}_v = 1 \otimes \cdots \otimes 1 \otimes \underset{v-\mathrm{th}}{\mathsf{O}} \otimes 1 \otimes \cdots \otimes 1\,.$

The super Kashaev coordinates get quantized to linear operators on the Hilbert spaces $\mathcal{H}_v$. The coordinates p_v and q_v are replaced by operators satisfying canonical commutation relations

$$[\mathsf{p}_v, \mathsf{q}_w] = \frac{1}{\pi i}\delta_{vw}, \qquad [\mathsf{q}_v, \mathsf{q}_w] = 0, \qquad [\mathsf{p}_v, \mathsf{p}_w] = 0, \tag{6.2}$$

and are represented on $L^2(\mathbb{R})$ as multiplication and differentiation operators. In the classical limit $b \to 0$, the operators $2\pi b\mathsf{p}$ and $2\pi b\mathsf{q}$ give their classical counterparts p and q appropriately. The odd coordinate ξ becomes an operator acting on $\mathcal{H}$ of the form

$$\xi_i = \sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\kappa_i, \tag{6.3}$$

where κ is a $(1|1) \times (1|1)$ matrix acting on $\mathbb{C}^{1|1}$

$$\kappa = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{6.4}$$

and where $q = e^{i\pi b^2}$ and the quantization constant $\hbar$ is related to b as $\hbar = 4\pi b^2$.

Note that ξ satisfies $\xi^2 = q^{\frac{1}{2}} - q^{-\frac{1}{2}} = i\pi b^2 + \mathcal{O}(b^4)$, thereby reproducing both the relation $\xi^2 = 0$ and the Poisson bracket $\{\xi, \xi\} = \frac{1}{2}$ in the classical limit $b \to 0$.

Moreover, the formula (2.12), with the super coordinates replacing the ordinary ones, has an obvious counterpart in the quantum theory, defining self-adjoint even operators z_e satisfying

$$[\mathsf{z}_e, \mathsf{z}_{e'}] = \frac{1}{\pi i}\{z_e, z_{e'}\}_{\mathrm{ST}}\,. \tag{6.5}$$

The operators $2\pi b\mathsf{z}_e$ give in the classical limit the even shear coordinates z_e.

The redundancy of the parametrization in terms of Kashaev type coordinates can be described using a quantum version of the Hamiltonian reduction characterising the super Teichmüller spaces within $\mathbb{R}^{8g-8+4n|4g-4+2n}$. This procedure is very similar to the case of the usual Teichmüller theory described in [34, 36], as explained in chapter 2.

6.2 Generators of the super Ptolemy groupoid

We will now construct a quantum realization of the coordinate transformations induced by changing the decorated hexagonalization η of a super Riemann surface $\Sigma_{g,n}$. The coordinate transformations will be represented by operators $\mathsf{U}_{\eta'\eta} : \mathcal{H}_\eta \to \mathcal{H}_{\eta'}$ representing the change of the hexagonalization η to η' in the following way. Let $\{w^\imath; \imath \in \mathcal{I}_\eta\}$ be a complete set of coordinates defined in terms of a hexagonalization η. If η' is another hexagonalization one may in our case express the coordinates $\{\tilde{w}^\jmath; \jmath \in \mathcal{I}_{\eta'}\}$ associated to η' as functions $w'^\jmath = W^\jmath_{\eta'\eta}(\{w^\imath; \imath \in \mathcal{I}_\eta\})$ of the coordinates $w^\imath$. If $\mathsf{w}^\imath$ and $\mathsf{w}'^\jmath$ are the operators associated to $w_\imath$ and $w'_\jmath$, respectively, we are first going to define quantized versions of the changes of coordinate functions $\mathsf{W}^\jmath_{\eta'\eta}(\{\mathsf{w}_\imath; \imath \in \mathcal{I}_\eta\})$ which reduce to the functions $W^{\eta'\eta}_\jmath$ in the classical limit. Unitary operators $\mathsf{U}_{\eta'\eta}$ representing these changes of coordinates on the quantum level are then required to satisfy

$$\mathsf{U}^{-1}_{\eta'\eta} \cdot \mathsf{w}'^\jmath \cdot \mathsf{U}_{\eta'\eta} = \mathsf{W}^\jmath_{\eta'\eta}(\{\mathsf{w}_\imath; \imath \in \mathcal{I}_\eta\})\,. \tag{6.6}$$

This requirement is expected to characterize the operators $\mathsf{U}_{\eta'\eta}$ uniquely up to normalization. We are now going to construct the operators $\mathsf{U}_{\eta'\eta}$ for all pairs η and η' related by generators of the super Ptolemy groupoid.

6.2.1 "Flip" operator T

Of particular interest are the cases where η and η' are related by the flip operation changing the diagonal in a triangulation. We will begin by constructing operators $\mathsf{T}^{(i)}_{vw}$: $\mathcal{H}_v \otimes \mathcal{H}_w \to \mathcal{H}_v \otimes \mathcal{H}_w$, $i = 1, \dots, 8$ representing the super flips of hexagonalizations listed in chapter 5, with decorated vertices placed in appropriate places. In order to cover the remaining cases one may use the push-out operation, as will be discussed later. A useful starting point will be the operator $\mathsf{T}^{(1)}_{12}$ corresponding to the operation $\omega^{(1)}_{12}$ depicted in figure 5.18. Following the discussion around (6.6) above, we will require the following for the even coordinates

$$\begin{aligned}
{\mathsf{T}^{(1)}_{12}}^{-1} e^{2\pi b \mathsf{z}'_1} \mathsf{T}^{(1)}_{12} &= e^{\pi b \mathsf{z}_1}(1 + e^{2\pi b \mathsf{z}_e} - e^{\pi b \mathsf{z}_e} \xi_1 \xi_2) e^{\pi b \mathsf{z}_1}, \\
{\mathsf{T}^{(1)}_{12}}^{-1} e^{2\pi b \mathsf{z}'_2} \mathsf{T}^{(1)}_{12} &= e^{\pi b \mathsf{z}_2}(1 + e^{-2\pi b \mathsf{z}_e} - e^{-\pi b \mathsf{z}_e} \xi_1 \xi_2)^{-1} e^{\pi b \mathsf{z}_2}, \\
{\mathsf{T}^{(1)}_{12}}^{-1} e^{2\pi b \mathsf{z}'_3} \mathsf{T}^{(1)}_{12} &= e^{\pi b \mathsf{z}_3}(1 + e^{2\pi b \mathsf{z}_e} - e^{\pi b \mathsf{z}_e} \xi_1 \xi_2) e^{\pi b \mathsf{z}_3}, \\
{\mathsf{T}^{(1)}_{12}}^{-1} e^{2\pi b \mathsf{z}'_4} \mathsf{T}^{(1)}_{12} &= e^{\pi b \mathsf{z}_4}(1 + e^{-2\pi b \mathsf{z}_e} - e^{-\pi b \mathsf{z}_e} \xi_1 \xi_2)^{-1} e^{\pi b \mathsf{z}_4}, \\
{\mathsf{T}^{(1)}_{12}}^{-1} e^{2\pi b \mathsf{z}'_e} \mathsf{T}^{(1)}_{12} &= e^{-2\pi b \mathsf{z}_e},
\end{aligned} \tag{6.7a}$$

and for the odd ones we require

$$\begin{aligned}
{\mathsf{T}^{(1)}_{12}}^{-1} e^{\pi b \mathsf{z}'_1} {\xi_1}' \mathsf{T}^{(1)}_{12} &= e^{\frac{1}{2}\pi b \mathsf{z}_1}(\xi_1 + e^{\pi b \mathsf{z}_e} \xi_2) e^{\frac{1}{2}\pi b \mathsf{z}_1}, \\
{\mathsf{T}^{(1)}_{12}}^{-1} e^{\pi b \mathsf{z}'_1} {\xi_2}' \mathsf{T}^{(1)}_{12} &= e^{\frac{1}{2}\pi b \mathsf{z}_1}(-e^{\pi b \mathsf{z}_e} \xi_1 + \xi_2) e^{\frac{1}{2}\pi b \mathsf{z}_1}.
\end{aligned} \tag{6.7b}$$

The labelling of variables is the one introduced in Figure 5.18, and the definition of the variables z_e in terms of the Kashaev type variables uses the same conventions as introduced in Section 2.1.7 above.

An operator $\mathsf{T}^{(1)}_{12}$ satisfying (6.7) can be constructed as follows

$$\mathsf{T}^{(1)}_{12} = \frac{1}{2}\Big[f_+(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2) - i f_-(\mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2)\, \kappa_1\, \kappa_2 \Big] e^{-i\pi \mathsf{p}_1 \mathsf{q}_2}. \tag{6.8}$$

The operator $\mathsf{T}^{(1)}_{12}$ is unitary and satisfies (6.7) if $f_\pm(x) := e_{\rm R}(x) \pm e_{\rm NS}(x)$ with $e_{\rm NS}(x)$ and $e_{\rm R}(x)$ being special functions satisfying $|e_{\rm NS}(x)| = 1$ and $|e_{\rm R}(x)| = 1$ for $x \in \mathbb{R}$, together with the functional relations

$$\begin{aligned}
e_{\rm R}\left(x - \frac{ib^{\pm 1}}{2}\right) &= (1 + i e^{\pi b^{\pm 1} x}) e_{\rm NS}\left(x + \frac{ib^{\pm 1}}{2}\right), \\
e_{\rm NS}\left(x - \frac{ib^{\pm 1}}{2}\right) &= (1 - i e^{\pi b^{\pm 1} x}) e_{\rm R}\left(x + \frac{ib^{\pm 1}}{2}\right).
\end{aligned}$$

Functions $e_{\rm NS}(x)$ and $e_{\rm R}(x)$ satisfying these properties can be constructed as

$$e_{\rm R}(x) = e_{\rm b}\left(\frac{x + i(b - b^{-1})/2}{2}\right) e_{\rm b}\left(\frac{x - i(b - b^{-1})/2}{2}\right), \tag{6.9}$$

$$e_{\rm NS}(x) = e_{\rm b}\left(\frac{x + c_{\rm b}}{2}\right) e_{\rm b}\left(\frac{x - c_{\rm b}}{2}\right), \tag{6.10}$$

where $e_{\mathrm{b}}(x)$ is Faddeev's quantum dilogarithm function defined by the following integral representation

$$e_{\mathrm{b}}(x) = \exp\left[\int_{\mathbb{R}+i0} \frac{\mathrm{d}w}{w} \frac{e^{-2ixw}}{4\sinh(wb)\sinh(w/b)}\right]. \tag{6.11}$$

In the following some details on the verification of the quantized coordinate transformations (6.7) are given here.

First, we present the transformations of the quantized shear coordinates under the flip that is given by the map $\mathsf{T}^{(1)}$. For the quadrilaterals on the figure 5.18, the even shear coordinates assigned to the edges are expressed as the operators on the $(L^2(\mathbb{R})\otimes\mathbb{C}^{1|1})^{\otimes 2}$

$$\mathsf{Z}_e = e^{2\pi b(\mathsf{q}_v - \mathsf{p}_v + \mathsf{p}_w)}\mathbb{I}_2, \qquad \mathsf{Z}'_e = e^{2\pi b(-\mathsf{q}_v + \mathsf{q}_w - \mathsf{p}_w)}\mathbb{I}_2, \tag{6.12}$$

$$\mathsf{Z}_1 = e^{2\pi b \mathsf{p}_v}\mathbb{I}_2, \qquad \mathsf{Z}'_1 = e^{2\pi b \mathsf{p}_v}\mathbb{I}_2, \tag{6.13}$$

$$\mathsf{Z}_2 = e^{2\pi b(\mathsf{q}_w - \mathsf{p}_w)}\mathbb{I}_2, \qquad \mathsf{Z}'_2 = e^{2\pi b(\mathsf{q}_v - \mathsf{p}_v)}\mathbb{I}_2, \tag{6.14}$$

$$\mathsf{Z}_3 = e^{-2\pi b \mathsf{q}_w}\mathbb{I}_2, \qquad \mathsf{Z}'_3 = e^{-2\pi b \mathsf{q}_w}\mathbb{I}_2, \tag{6.15}$$

$$\mathsf{Z}_4 = e^{-2\pi b \mathsf{q}_v}\mathbb{I}_2, \qquad \mathsf{Z}'_4 = e^{2\pi b \mathsf{p}_w}\mathbb{I}_2, \tag{6.16}$$

and the odd coordinates

$$\xi_1 = \sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,\kappa\otimes\mathbb{I}_2, \qquad \xi'_1 = \sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,\kappa\otimes\mathbb{I}_2, \tag{6.17}$$

$$\xi_2 = \sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,\mathbb{I}_2\otimes\kappa, \qquad \xi'_2 = \sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,\mathbb{I}_2\otimes\kappa. \tag{6.18}$$

Those operators satisfy the algebraic relations as follows

$$\begin{aligned}
&[\mathsf{Z}_e, \mathsf{Z}_1] = (1 - q^{-4})\mathsf{Z}_e\mathsf{Z}_1, && [\mathsf{Z}_e, \mathsf{Z}_2] = (1 - q^{+4})\mathsf{Z}_e\mathsf{Z}_2,\\
&[\mathsf{Z}_e, \mathsf{Z}_3] = (1 - q^{-4})\mathsf{Z}_e\mathsf{Z}_3, && [\mathsf{Z}_e, \mathsf{Z}_4] = (1 - q^{+4})\mathsf{Z}_e\mathsf{Z}_4,\\
&[\mathsf{Z}_1, \mathsf{Z}_4] = (1 - q^{-4})\mathsf{Z}_1\mathsf{Z}_4, && [\mathsf{Z}_2, \mathsf{Z}_3] = (1 - q^{+4})\mathsf{Z}_2\mathsf{Z}_3,\\
&[\mathsf{Z}_1, \mathsf{Z}_2] = [\mathsf{Z}_1, \mathsf{Z}_3] = [\mathsf{Z}_2, \mathsf{Z}_4] = [\mathsf{Z}_3, \mathsf{Z}_4] = 0, && [\mathsf{Z}_\alpha, \xi_i] = 0,\\
&\{\xi_1, \xi_2\} = 0, && \{\xi_i, \xi_i\} = 2\sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,1\otimes 1.
\end{aligned}$$

Setting $q = e^{i\hbar/4}$ one can see that those commutation relations reproduce the classical Poisson bracket given by equation (5.25).

As an example, let us consider the transformation of the even variable $\mathsf{Z}'_1 = e^{2\pi b \mathsf{z}'_1}$:

$$
\begin{aligned}
&\mathsf{T}^{(1)-1}_{vw}\mathsf{Z}'_1\mathsf{T}^{(1)}_{vw} = \\
&= \frac{1}{4}e^{\pi b \mathsf{p}_v}[(e^{-1}_{\mathrm{NS}}(u+ib) + e^{-1}_{\mathrm{R}}(u+ib))\mathbb{I}_2\otimes\mathbb{I}_2 - i(e^{-1}_{\mathrm{R}}(u+ib) - e^{-1}_{\mathrm{NS}}(u+ib))\kappa\otimes\kappa]\times \\
&\times[(e_{\mathrm{NS}}(u-ib) + e_{\mathrm{R}}(u-ib))\mathbb{I}_2\otimes\mathbb{I}_2 - i(e_{\mathrm{R}}(u-ib) - e_{\mathrm{NS}}(u-ib))\kappa\otimes\kappa]e^{\pi b \mathsf{p}_v} = \\
&= \frac{1}{2}e^{\pi b \mathsf{p}_v}\left\{[e^{-1}_{\mathrm{NS}}(u+ib)e_{\mathrm{NS}}(u-ib) + e^{-1}_{\mathrm{R}}(u+ib)e_{\mathrm{R}}(u-ib)]\mathbb{I}_2\otimes\mathbb{I}_2 + \right. \\
&\left. -i[e^{-1}_{\mathrm{R}}(u+ib)e_{\mathrm{R}}(u-ib) - e^{-1}_{\mathrm{NS}}(u+ib)e_{\mathrm{NS}}(u-ib)]\kappa\otimes\kappa\right\}e^{\pi b \mathsf{p}_v} = \\
&= e^{\pi b \mathsf{p}_v}\left\{[1+e^{2\pi b(\mathsf{q}_v+\mathsf{p}_w-\mathsf{q}_w)}]\mathbb{I}_2\otimes\mathbb{I}_2 + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})e^{\pi b(\mathsf{q}_v+\mathsf{p}_w-\mathsf{q}_w)}\kappa\otimes\kappa\right\}e^{\pi b \mathsf{p}_v} = \\
&= \mathsf{Z}_1^{\frac{1}{2}}\left\{(1+\mathsf{Z}_e)\mathbb{I}_2\otimes\mathbb{I}_2 + (q^{-\frac{1}{2}} - q^{\frac{1}{2}})\mathsf{Z}_e^{\frac{1}{2}}\kappa\otimes\kappa\right\}\mathsf{Z}_1^{\frac{1}{2}} = \\
&= \mathsf{Z}_1^{\frac{1}{2}}\left\{(1+\mathsf{Z}_e)\mathbb{I}_2\otimes\mathbb{I}_2 - \mathsf{Z}_e^{\frac{1}{2}}\xi_1\xi_2\right\}\mathsf{Z}_1^{\frac{1}{2}},
\end{aligned}
$$

where we denoted $u = \mathsf{q}_v + \mathsf{p}_w - \mathsf{p}_v$ and used two times the shift relation of the quantum dilogarithm

$$
\begin{aligned}
e_{\mathrm{R}}(x-ib) &= (1 - i(q^{\frac{1}{2}} - q^{-\frac{1}{2}})e^{\pi b x} + e^{2\pi b x})e_{\mathrm{R}}(x+ib), \\
e_{\mathrm{NS}}(x-ib) &= (1 + i(q^{\frac{1}{2}} - q^{-\frac{1}{2}})e^{\pi b x} + e^{2\pi b x})e_{\mathrm{NS}}(x+ib).
\end{aligned}
$$

We can obtain the transformation property of the odd variable ξ'_1

$$
\begin{aligned}
&\mathsf{T}^{(1)-1}_{vw}\mathsf{Z}'^{\frac{1}{2}}_1\xi'_1\mathsf{T}^{(1)}_{vw} = \sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\,\mathsf{T}^{(1)-1}_{vw}(e^{\pi b \mathsf{p}_v}\kappa\otimes\mathbb{I}_2)\mathsf{T}^{(1)}_{vw} = \frac{1}{4}\sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}e^{\pi b \mathsf{p}_v}\times \\
&\times[(e^{-1}_{\mathrm{NS}}(u+ib) + e^{-1}_{\mathrm{R}}(u+ib))\mathbb{I}_2\otimes\mathbb{I}_2 - i(e^{-1}_{\mathrm{R}}(u+ib) - e^{-1}_{\mathrm{NS}}(u+ib))\kappa\otimes\kappa]\times \\
&\times[(e_{\mathrm{NS}}(u) + e_{\mathrm{R}}(u))\mathbb{I}_2\otimes\mathbb{I}_2 - i(e_{\mathrm{R}}(u) - e_{\mathrm{NS}}(u))\kappa\otimes\kappa]\,\kappa\otimes\mathbb{I}_2 = \\
&= \frac{1}{2}\sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}e^{\pi b \mathsf{p}_v}\left\{[e^{-1}_{\mathrm{NS}}(u+ib)e_{\mathrm{R}}(u) + e^{-1}_{\mathrm{R}}(u+ib)e_{\mathrm{NS}}(u)]\mathbb{I}_2\otimes\mathbb{I}_2 + \right. \\
&\left. -i[e^{-1}_{\mathrm{R}}(u+ib)e_{\mathrm{NS}}(u) - e^{-1}_{\mathrm{NS}}(u+ib)e_{\mathrm{R}}(u)]\kappa\otimes\kappa\right\}\,\kappa\otimes\mathbb{I}_2 = \\
&= \sqrt{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}e^{\pi b \mathsf{p}_v}\left\{\mathbb{I}_2\otimes\mathbb{I}_2 - q^{\frac{1}{2}}e^{\pi b(\mathsf{q}_v+\mathsf{p}_w-\mathsf{p}_v)}\kappa\otimes\kappa\right\}\,\kappa\otimes\mathbb{I}_2 = \\
&= \mathsf{Z}_1^{\frac{1}{2}}(\xi_1 + q^{\frac{1}{2}}\mathsf{Z}_e^{\frac{1}{2}}\xi_2) = \mathsf{Z}_1^{\frac{1}{4}}(\xi_1 + \mathsf{Z}_e^{\frac{1}{2}}\xi_2)\mathsf{Z}_1^{\frac{1}{4}}.
\end{aligned}
$$

In this case we used the shift property of the quantum dilogarithm as well. In the analogous way, one can obtain the transformation properties of the rest of Fock variables.

The appearance of Z_1 in the transformation property of odd coordinates is just illusory, and it is caused by our choice of using square roots of operators. Indeed, we can rewrite the square root as

$$\left(e^{\sqrt{2}\pi b z_1}(1+e^{2\sqrt{2}\pi b z_e}-(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{-1}e^{\sqrt{2}\pi b z_e}\xi_1\xi_2)e^{\sqrt{2}\pi b z_1}\right)^{-\frac{1}{2}}=$$
$$=\frac{1}{2}\left\{[e_{\mathrm{NS}}^{-1}(z_e)e_{\mathrm{NS}}(z_e+ib)+e_{\mathrm{R}}^{-1}(z_e)e_{\mathrm{R}}(z_e+ib)]+\right.$$
$$\left.-\frac{i}{q-q^{-1}}[e_{\mathrm{R}}^{-1}(z_e)e_{\mathrm{R}}(z_e+ib)-e_{\mathrm{NS}}^{-1}(z_e)e_{\mathrm{NS}}(z_e+ib)]\xi_1\xi_2\right\}e^{-\sqrt{2}\pi b z_1}.$$

It is clear that even coordinate z_1 cancels out from the transformations if we use this formula for the square root. However, the quantum transformations are written in terms of quantum dilogarithms, and their behaviour in the classical limit is less clear in this form.

Moreover, using the facts from the functional analysis the inverse of the square root of this variable is given by

$$T_{vw}^{(1)-1}Z'^{-\frac{1}{2}}_1T_{vw}^{(1)}=\left(Z_1^{\frac{1}{2}}\left\{(1+Z_e)\mathbb{I}_2\otimes\mathbb{I}_2-(q^{\frac{1}{2}}+q^{-\frac{1}{2}})^{-1}Z_e^{\frac{1}{2}}\xi_1\xi_2\right\}Z_1^{\frac{1}{2}}\right)^{-\frac{1}{2}}.$$

Using this fact, we can obtain the transformation property of the odd variable ξ_1'

$$T_{vw}^{(1)-1}\xi_1'T_{vw}^{(1)}=\sqrt{q-q^{-1}}T_{vw}^{(1)-1}Z'^{-\frac{1}{2}}_1T_{vw}^{(1)}T_{vw}^{(1)-1}(e^{\sqrt{2}\pi b p_v}\xi\otimes\mathbb{I}_2)T_{vw}^{(1)}=$$
$$=\frac{1}{4}\sqrt{q-q^{-1}}\left(Z_1^{\frac{1}{2}}\left\{(1+Z_e)\mathbb{I}_2\otimes\mathbb{I}_2-(q^{\frac{1}{2}}+q^{-\frac{1}{2}})^{-1}Z_e^{\frac{1}{2}}\xi_1\xi_2\right\}Z_1^{\frac{1}{2}}\right)^{-\frac{1}{2}}e^{\sqrt{2}\pi b p_v}\times$$
$$\times[(e_{\mathrm{NS}}^{-1}(u+ib)+e_{\mathrm{R}}^{-1}(u+ib))\mathbb{I}_2\otimes\mathbb{I}_2-i(e_{\mathrm{R}}^{-1}(u+ib)-e_{\mathrm{NS}}^{-1}(u+ib))\xi\otimes\xi]\times$$
$$\times[(e_{\mathrm{NS}}(u)+e_{\mathrm{R}}(u))\mathbb{I}_2\otimes\mathbb{I}_2-i(e_{\mathrm{R}}(u)-e_{\mathrm{NS}}(u))\xi\otimes\xi]\,\xi\otimes\mathbb{I}_2=$$
$$=\frac{1}{2}\sqrt{q-q^{-1}}\left(Z_1^{\frac{1}{2}}\left\{(1+Z_e)\mathbb{I}_2\otimes\mathbb{I}_2-(q^{\frac{1}{2}}+q^{-\frac{1}{2}})^{-1}Z_e^{\frac{1}{2}}\xi_1\xi_2\right\}Z_1^{\frac{1}{2}}\right)^{-\frac{1}{2}}e^{\sqrt{2}\pi b p_v}\times$$
$$\times\left\{[e_{\mathrm{NS}}^{-1}(u+ib)e_{\mathrm{R}}(u)+e_{\mathrm{R}}^{-1}(u+ib)e_{\mathrm{NS}}(u)]\mathbb{I}_2\otimes\mathbb{I}_2+\right.$$
$$\left.-i[e_{\mathrm{R}}^{-1}(u+ib)e_{\mathrm{NS}}(u)-e_{\mathrm{NS}}^{-1}(u+ib)e_{\mathrm{R}}(u)]\xi\otimes\xi\right\}\,\xi\otimes\mathbb{I}_2=$$
$$=\sqrt{q-q^{-1}}\left(Z_1^{\frac{1}{2}}\left\{(1+Z_e)\mathbb{I}_2\otimes\mathbb{I}_2-(q^{\frac{1}{2}}+q^{-\frac{1}{2}})^{-1}Z_e^{\frac{1}{2}}\xi_1\xi_2\right\}Z_1^{\frac{1}{2}}\right)^{-\frac{1}{2}}e^{\sqrt{2}\pi b p_v}\times$$
$$\times\left\{\mathbb{I}_2\otimes\mathbb{I}_2-q^{\frac{1}{2}}e^{\pi b\sqrt{2}(q_v+p_w-p_v)}\xi\otimes\xi\right\}\,\xi\otimes\mathbb{I}_2=$$
$$=\left(Z_1^{\frac{1}{2}}\left\{(1+Z_e)\mathbb{I}_2\otimes\mathbb{I}_2-(q^{\frac{1}{2}}+q^{-\frac{1}{2}})^{-1}Z_e^{\frac{1}{2}}\xi_1\xi_2\right\}Z_1^{\frac{1}{2}}\right)^{-\frac{1}{2}}Z_1^{\frac{1}{2}}(\xi_1+q^{\frac{1}{2}}Z_e^{\frac{1}{2}}\xi_2).$$

In this case we used the shift property of the quantum dilogarithm as well. In the analogous way, one can obtain the transformation properties of the rest of Fock variables in question.

6.2.2 "Change of orientations" operator M

As a useful tool for describing the definition of the remaining operators $\mathsf{T}_{12}^{(i)}$, $i = 2, \dots, 8$, we will introduce an operator $\mathsf{M}_v : \mathcal{H}_v \to \mathcal{H}_v$ representing the change of orientations μ_v in an undotted triangle shown in the figure 5.13. The operator M_v is associated by our conventions concerning tensor products introduced above to the operator M on $\mathbb{C}^{1|1}$ which can be represented by the matrix

$$\mathsf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.19}$$

The operator M_v squares to identity $\mathsf{M}_v^2 = \mathrm{id}_v$ and acts on the odd invariant as

$$\mathsf{M}_v^{-1} \cdot \xi_v \cdot \mathsf{M}_v = -\xi_v. \tag{6.20}$$

One should note that the operation μ_v relates Kasteleyn orientations describing inequivalent spin structures, in general.

It is easy to see that the flips $\omega_{12}^{(i)}$, $i = 2, \dots, 8$ can be represented as compositions of the flip $\omega_{12}^{(1)}$ with operations μ_v. We will define the corresponding operators $\mathsf{T}_{12}^{(i)}$, $i = 2, \dots, 8$ by taking the corresponding product of the operators M_v with the operator $\mathsf{T}_{12}^{(1)}$. To give an example, let us note that the flip $\omega^{(2)}$ can be represented by the sequence of operations shown in figure 6.1. This leads us to define the operator $\mathsf{T}_{12}^{(2)}$ as

$$\mathsf{T}_{12}^{(2)} = \mathsf{M}_1 \mathsf{M}_2 \mathsf{T}_{12}^{(1)} \mathsf{M}_1. \tag{6.21}$$

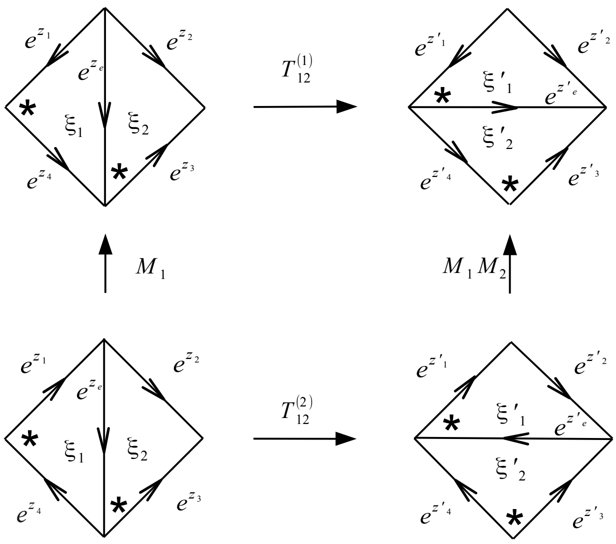

FIGURE 6.1: By using operators M we can find the map between the second superflip and the first one.

All other operators $\mathsf{T}_{12}^{(i)}$, $i = 3,\ldots,8$ associated to the flips $\omega^{(i)}$, $i = 3,\ldots,8$ can be defined in the same way.

$$\begin{aligned} \mathsf{T}_{12}^{(3)} &= \mathsf{T}_{12}^{(1)}\mathsf{M}_1\mathsf{M}_2 & \mathsf{T}_{12}^{(4)} &= \mathsf{M}_1\mathsf{M}_2\mathsf{T}_{12}^{(1)}\mathsf{M}_2 \\ \mathsf{T}_{12}^{(5)} &= \mathsf{M}_1\mathsf{T}_{12}^{(1)}\mathsf{M}_1 & \mathsf{T}_{12}^{(6)} &= \mathsf{M}_2\mathsf{T}_{12}^{(1)}\mathsf{M}_1\mathsf{M}_2 \\ \mathsf{T}_{12}^{(7)} &= \mathsf{M}_1\mathsf{T}_{12}^{(1)}\mathsf{M}_2 & \mathsf{T}_{12}^{(8)} &= \mathsf{M}_1\mathsf{T}_{12}^{(1)}\mathsf{M}_1\mathsf{M}_2. \end{aligned} \tag{6.22}$$

The operations considered up to now were associated to triangles that do not have corners marked with dots. As noted above, one may always locally reduce to this case by using the push-out operation. The push-out β will be represented by an operator $\mathsf{B}_{uv} : \mathcal{H}_u \otimes \mathcal{H}_v \to \mathcal{H}_u \otimes \mathcal{H}_v$ defined as follows

$$\mathsf{B}_{uv} = \mathrm{id}_u\,\mathsf{M}_v. \tag{6.23}$$

With the help of the operator B_{uv} one may now define all operators associated with the flips relating dotted triangles.

6.2.3 "Super permutation" operator $\Pi_{(12)}^{(i)}$

We furthermore need to define operators $\Pi_{(12)}^{(i)}$, $i = 1,\ldots,8$ representing the exchange (uv) of labels assigned to two adjacent triangles when the Kastelyn orientation is the one of the initial configurations of the flips $\omega_{12}^{(i)}$ depicted in Figure 5.12. By using the operators M_v one may reduce the definition to the case $i = 1$ in a way closely analogous to the definition of the $\mathsf{T}_{12}^{(i)}$, $i = 2,\ldots,8$ in terms of $\mathsf{T}_{12}^{(1)}$. In order to define the operator $\Pi_{(12)}^{(1)}$ let us represent $\mathcal{H}_1 \otimes \mathcal{H}_2$ as $L^2(\mathbb{R}^2) \otimes \mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1}$, and let

$$\Pi_{(12)}^{(1)} = (\mathsf{P}_{\mathrm{b}} \otimes \mathbb{I}_2 \otimes \mathbb{I}_2)(\mathrm{id} \otimes \mathsf{P}_{\mathrm{f}})\,, \quad \text{where} \quad \mathsf{P}_{\mathrm{f}} = (\mathbb{I}_2 \otimes \mathsf{M})(\mathbb{I}_2 \otimes \mathbb{I}_2 + \kappa \otimes \kappa), \tag{6.24}$$

with respect to this factorization, where P_{b} acts on functions of two variables as $\mathsf{P}_{\mathrm{b}}f(x_1,x_2) = f(x_2,x_1)$. One may note that P_{f} is not the standard permutation operator on $\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1}$ satisfying $\mathsf{P}_{\mathrm{f}}(\eta_1 \otimes \eta_2)\mathsf{P}_{\mathrm{f}} = \eta_2 \otimes \eta_1$ for arbitrary $\eta_1, \eta_2 \in \mathrm{End}(\mathbb{C}^{1|1})$ (one can find this calculation in appendix D).

However, the operator P_{f} squares to the identity and satisfies $\mathsf{P}_{\mathrm{f}}(\xi \otimes \mathbb{I}_2)\mathsf{P}_{\mathrm{f}} = \mathbb{I}_2 \otimes \xi$ and $\mathsf{P}_{\mathrm{f}}(\mathbb{I}_2 \otimes \xi)\mathsf{P}_{\mathrm{f}} = \xi \otimes \mathbb{I}_2$. This means that the operator P_{f} correctly represents the permutation on the the sub-algebra of $\mathrm{End}(\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1})$ generated by $\mathbb{I}_2 \otimes \xi$ and $\xi \otimes \mathbb{I}_2$. This is the algebra of operators on $\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1}$ relevant for the quantization of the super Teichmüller theory. The reason for adopting a non-standard representation of the permutation on this sub-algebra will become clear when we discuss the relations of the super Ptolemy groupoid.

6.2.4 "Rotating the distinguished vertex" operator A

We finally need to define an operator A_v representing the move rotating the distinguished vertex of a dotted triangle as shown in figure 2.12. The operator $\mathsf{A}_v : \mathcal{H}_v \to \mathcal{H}_v$ will be defined as

$$\mathsf{A}_v = e^{i\pi/3} e^{-i3\pi \mathsf{q}_v^2/2} e^{-i\pi(\mathsf{p}_v+\mathsf{q}_v)^2/2} \mathbb{I}_2. \tag{6.25}$$

Let us finally note that the flip operators $\mathsf{T}_{12}^{(i)}$ have an interesting interpretation within the representation theory of the Heisenberg double of the quantum super plane, which will be elaborated in chapter 7. The flip operator $\mathsf{T}_{12}^{(1)}$ is found to coincide with the canonical element of the Heisenberg double of the quantum super plane (which is a Borel half of $U_q(osp(1|2))$), evaluated in certain infinite-dimensional representations on $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$.

6.3 Quantum super Ptolemy groupoid

We are now going to describe essential steps in the verification that the operators defined previously generate a representation of the super Ptolemy groupoid.

6.3.1 Superpentagon equation

Of particular interest are the generalizations of the pentagon relation. Using the push-out operation one can always reduce to relations involving only undotted triangles. As noted previously, one needs to check the following set of relations,

$$\begin{aligned}
\mathsf{T}_{12}^{(1)}\mathsf{T}_{23}^{(1)} &= \mathsf{T}_{23}^{(1)}\mathsf{T}_{13}^{(1)}\mathsf{T}_{12}^{(1)}, & \mathsf{T}_{12}^{(6)}\mathsf{T}_{23}^{(2)} &= \mathsf{T}_{23}^{(2)}\mathsf{T}_{13}^{(1)}\mathsf{T}_{12}^{(6)}, \\
\mathsf{T}_{12}^{(5)}\mathsf{T}_{23}^{(8)} &= \mathsf{T}_{23}^{(8)}\mathsf{T}_{13}^{(5)}\mathsf{T}_{12}^{(5)}, & \mathsf{T}_{12}^{(6)}\mathsf{T}_{23}^{(7)} &= \mathsf{T}_{23}^{(7)}\mathsf{T}_{13}^{(6)}\mathsf{T}_{12}^{(5)}, \\
\mathsf{T}_{12}^{(2)}\mathsf{T}_{23}^{(1)} &= \mathsf{T}_{23}^{(1)}\mathsf{T}_{13}^{(2)}\mathsf{T}_{12}^{(2)}, & \mathsf{T}_{12}^{(8)}\mathsf{T}_{23}^{(8)} &= \mathsf{T}_{23}^{(1)}\mathsf{T}_{13}^{(8)}\mathsf{T}_{12}^{(8)}, \\
\mathsf{T}_{12}^{(4)}\mathsf{T}_{23}^{(5)} &= \mathsf{T}_{23}^{(5)}\mathsf{T}_{13}^{(5)}\mathsf{T}_{12}^{(4)}, & \mathsf{T}_{12}^{(5)}\mathsf{T}_{23}^{(3)} &= \mathsf{T}_{23}^{(3)}\mathsf{T}_{13}^{(4)}\mathsf{T}_{12}^{(6)}, \\
\mathsf{T}_{12}^{(3)}\mathsf{T}_{23}^{(4)} &= \mathsf{T}_{23}^{(7)}\mathsf{T}_{13}^{(3)}\mathsf{T}_{12}^{(2)}, & \mathsf{T}_{12}^{(7)}\mathsf{T}_{23}^{(7)} &= \mathsf{T}_{23}^{(4)}\mathsf{T}_{13}^{(7)}\mathsf{T}_{12}^{(8)}, \\
\mathsf{T}_{12}^{(1)}\mathsf{T}_{23}^{(6)} &= \mathsf{T}_{23}^{(6)}\mathsf{T}_{13}^{(6)}\mathsf{T}_{12}^{(4)}, & \mathsf{T}_{12}^{(7)}\mathsf{T}_{23}^{(2)} &= \mathsf{T}_{23}^{(5)}\mathsf{T}_{13}^{(2)}\mathsf{T}_{12}^{(7)}, \\
\mathsf{T}_{12}^{(5)}\mathsf{T}_{23}^{(6)} &= \mathsf{T}_{23}^{(3)}\mathsf{T}_{13}^{(7)}\mathsf{T}_{12}^{(6)}, & \mathsf{T}_{12}^{(3)}\mathsf{T}_{23}^{(5)} &= \mathsf{T}_{23}^{(2)}\mathsf{T}_{13}^{(8)}\mathsf{T}_{12}^{(3)}, \\
\mathsf{T}_{12}^{(1)}\mathsf{T}_{23}^{(3)} &= \mathsf{T}_{23}^{(6)}\mathsf{T}_{13}^{(3)}\mathsf{T}_{12}^{(7)}, & \mathsf{T}_{12}^{(4)}\mathsf{T}_{23}^{(4)} &= \mathsf{T}_{23}^{(4)}\mathsf{T}_{13}^{(4)}\mathsf{T}_{12}^{(1)}.
\end{aligned} \tag{6.26}$$

One may first observe that all of these relations follow from the pentagon equation that involves only $\mathsf{T}^{(1)}$. As an example let us consider the pentagon equation represented by Figure 6.2, corresponding to the equation

$$\mathsf{T}_{12}^{(6)}\mathsf{T}_{23}^{(2)} = \mathsf{T}_{23}^{(2)}\mathsf{T}_{13}^{(1)}\mathsf{T}_{12}^{(6)}.$$

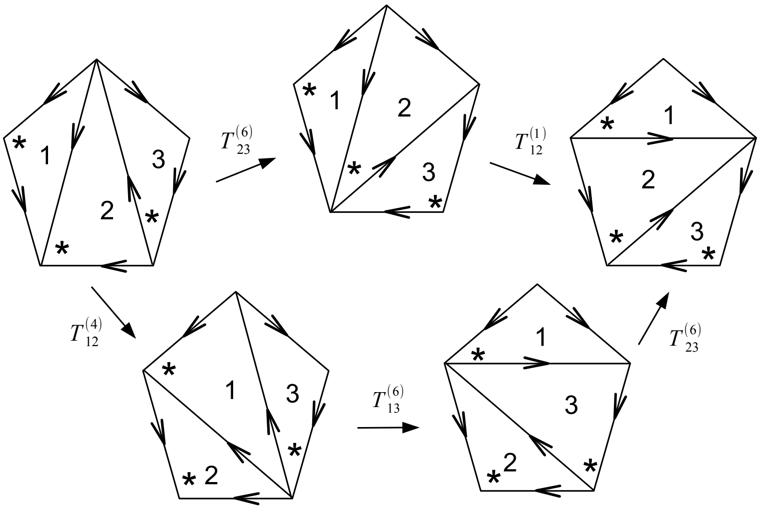

FIGURE 6.2: One of the pentagon equations.

Using the relations between $\mathsf{T}^{(1)}$ and other flips, we can rewrite it

$$(\mathsf{M}_2\mathsf{T}^{(1)}_{12}\mathsf{M}_1\mathsf{M}_2)(\mathsf{M}_2\mathsf{M}_3\mathsf{T}^{(1)}_{23}\mathsf{M}_2) = (\mathsf{M}_2\mathsf{M}_3\mathsf{T}^{(1)}_{23}\mathsf{M}_2)\mathsf{T}^{(1)}_{13}(\mathsf{M}_2\mathsf{T}^{(1)}_{12}\mathsf{M}_1\mathsf{M}_2),$$

which is just a pentagon for $\mathsf{T}^{(1)}$, given the fact that $\mathsf{M}_1\mathsf{M}_2\mathsf{T}^{(i)}_{12}\mathsf{M}_1\mathsf{M}_2 = \mathsf{T}^{(i)}_{12}$ for all i.

In order to verify the pentagon equation for $\mathsf{T}^{(1)}$ one may note that by straightforward calculations one may reduce the validity of this relations to the following identities

$$f_+(\mathsf{p})f_+(\mathsf{x}) = f_+(\mathsf{x})f_+(\mathsf{x}+\mathsf{p})f_+(\mathsf{p}) - if_-(\mathsf{x})f_-(\mathsf{x}+\mathsf{p})f_-(\mathsf{p}), \tag{6.27a}$$

$$f_+(\mathsf{p})f_-(\mathsf{x}) = -if_+(\mathsf{x})f_-(\mathsf{x}+\mathsf{p})f_-(\mathsf{p}) + f_-(\mathsf{x})f_+(\mathsf{x}+\mathsf{p})f_+(\mathsf{p}), \tag{6.27b}$$

$$f_-(\mathsf{p})f_+(\mathsf{x}) = f_+(\mathsf{x})f_+(\mathsf{x}+\mathsf{p})f_-(\mathsf{p}) - if_-(\mathsf{x})f_-(\mathsf{x}+\mathsf{p})f_+(\mathsf{p}), \tag{6.27c}$$

$$f_-(\mathsf{p})f_-(\mathsf{x}) = if_+(\mathsf{x})f_-(\mathsf{x}+\mathsf{p})f_+(\mathsf{p}) - f_-(\mathsf{x})f_+(\mathsf{x}+\mathsf{p})f_-(\mathsf{p}), \tag{6.27d}$$

with x and p being self-adjoint operators satisfying the relations

$$[\mathsf{p},\mathsf{x}] = \frac{1}{i\pi}.$$

The relations (6.27) follow from integral identities satisfied by the special functions $e_{\mathrm{NS}}(x)$ and $e_{\mathrm{R}}(x)$. Here we provide a proof of them and will show that theses equations are equivalent to the analogs (B.19) of the Ramanujan summation formula which have been derived in [80].

These formulae can be rewritten in terms of $e_{\rm NS}(x)$ and $e_{\rm R}(x)$ as follows,

$$\int dx e^{-\pi i x(u+c_{\rm b})}\left(\frac{e_{\rm NS}(x+c_{\rm b})}{e_{\rm NS}(x+v)}+\frac{e_{\rm R}(x+c_{\rm b})}{e_{\rm R}(x+v)}\right)=2\chi_0\frac{e_{\rm NS}(v+u+c_{\rm b})}{e_{\rm NS}(v)e_{\rm NS}(u)}, \tag{6.28a}$$

$$\int dx e^{-\pi i x(u+c_{\rm b})}\left(\frac{e_{\rm NS}(x+c_{\rm b})}{e_{\rm NS}(x+v)}-\frac{e_{\rm R}(x+c_{\rm b})}{e_{\rm R}(x+v)}\right)=2\chi_0\frac{e_{\rm R}(v+u+c_{\rm b})}{e_{\rm NS}(v)e_{\rm R}(u)}, \tag{6.28b}$$

$$\int dx e^{-\pi i x(u+c_{\rm b})}\left(\frac{e_{\rm NS}(x+c_{\rm b})}{e_{\rm R}(x+v)}+\frac{e_{\rm R}(x+c_{\rm b})}{e_{\rm NS}(x+v)}\right)=2\chi_0\frac{e_{\rm R}(v+u+c_{\rm b})}{e_{\rm R}(v)e_{\rm NS}(u)}, \tag{6.28c}$$

$$\int dx e^{-\pi i x(u+c_{\rm b})}\left(\frac{e_{\rm NS}(x+c_{\rm b})}{e_{\rm R}(x+v)}-\frac{e_{\rm R}(x+c_{\rm b})}{e_{\rm NS}(x+v)}\right)=2\chi_0\frac{e_{\rm NS}(v+u+c_{\rm b})}{e_{\rm R}(v)e_{\rm R}(u)}, \tag{6.28d}$$

where $\chi_0 = e^{-i\pi(1-c_b^2)/6}$. Taking the limit $v \to -\infty$ we can obtain the Fourier transforms

$$\begin{aligned}\tilde{f}_+(u) &= \int dx e^{-\pi i x u}(e_{\rm R}(x)+e_{\rm NS}(x)) = e^{-i\pi c_{\rm b}u}\frac{2\chi_0}{e_{\rm NS}(u-c_{\rm b})} = \\ &= 2\chi_0^{-1}e^{-i\pi u^2/2}e_{\rm NS}(c_{\rm b}-u),\\ \tilde{f}_-(u) &= \int dx e^{-\pi i x u}(e_{\rm R}(x)-e_{\rm NS}(x)) = -e^{-i\pi c_{\rm b}u}\frac{2\chi_0}{e_{\rm R}(u-c_{\rm b})} = \\ &= 2i\chi_0^{-1}e^{-i\pi u^2/2}e_{\rm R}(c_{\rm b}-u).\end{aligned}$$

Then, we can consider the matrix elements of the operators $f_r(\mathsf{X})f_s(\mathsf{P}+\mathsf{X})$ between (generalized) eigenstates $\langle p|$ and $|p'\rangle$ of the operator P with eigenvalues p and p', respectively:

$$\Xi_{rs} = \langle p|f_r(\mathsf{X})f_s(\mathsf{P}+\mathsf{X})|p'\rangle, \tag{6.29}$$

for $r,s=+,-$ and $[\mathsf{P},\mathsf{X}]=\frac{1}{i\pi}$. We have

$$\begin{aligned}\langle p|f_r(\mathsf{X})f_s(\mathsf{P}+\mathsf{X})|p'\rangle &= \int dp''\langle p|f_r(\mathsf{X})|p''\rangle\langle p''|f_s(\mathsf{P}+\mathsf{X})|p'\rangle = \\ &= \int dp'' e^{i\pi(p''^2-p'^2)/2}\tilde{f}_r(p-p'')\tilde{f}_s(p''-p'),\end{aligned}$$

where we used the identity between the matrix element of an arbitrary function g and its Fourier transform $\tilde{g}$

$$\langle p|g(\mathsf{X})|p'\rangle = \tilde{g}(p-p')\,,$$

and the fact that

$$g(\mathsf{X}+\mathsf{P}) = e^{\frac{i\pi}{2}\mathsf{P}^2}g(\mathsf{X})e^{-\frac{i\pi}{2}\mathsf{P}^2}.$$

Let us consider in detail the case $r=+, s=+$. Then we can write, using (6.28),

$$\begin{aligned}\Xi_{++} &= \int dp'' e^{\frac{i\pi}{2}(p''^2-p'^2)}\frac{e_{\rm NS}(p'-p''+c_{\rm b})}{e_{\rm NS}(p-p''-c_{\rm b})}e^{-\frac{i\pi}{2}(p''-p')^2}e^{-i\pi c_{\rm b}(p-p'')} = \\ &= e^{-i\pi c_{\rm b}(p-p')}\int dx e^{-i\pi x(p'+c_{\rm b})}\frac{e_{\rm NS}(x+c_{\rm b})}{e_{\rm NS}(x+p-p'-c_{\rm b})} = \\ &= \chi_0 e^{-i\pi c_{\rm b}(p-p')}\frac{1}{e_{\rm NS}(p-p'-c_{\rm b})}\left(\frac{e_{\rm NS}(p)}{e_{\rm NS}(p')}+\frac{e_{\rm R}(p)}{e_{\rm R}(p')}\right).\end{aligned}$$

Therefore

$$f_+(\mathsf{X})f_+(\mathsf{X}+\mathsf{P}) = e_{\mathrm{NS}}(\mathsf{P})f_+(\mathsf{X})e_{\mathrm{NS}}^{-1}(\mathsf{P}) + e_{\mathrm{R}}(\mathsf{P})f_+(\mathsf{X})e_{\mathrm{R}}^{-1}(\mathsf{P}).$$

If one repeats the calculations for other possibilities, the case $r=-, s=-$ gives

$$f_-(\mathsf{X})f_-(\mathsf{X}+\mathsf{P}) = -i(e_{\mathrm{NS}}(\mathsf{P})f_+(\mathsf{X})e_{\mathrm{NS}}^{-1}(\mathsf{P}) - e_{\mathrm{R}}(\mathsf{P})f_+(\mathsf{X})e_{\mathrm{R}}^{-1}(\mathsf{P})),$$

while $r=+, s=-$

$$f_+(\mathsf{X})f_-(\mathsf{X}+\mathsf{P}) = -i(e_{\mathrm{R}}(\mathsf{P})f_-(\mathsf{X})e_{\mathrm{NS}}^{-1}(\mathsf{P}) - e_{\mathrm{NS}}(\mathsf{P})f_-(\mathsf{X})e_{\mathrm{R}}^{-1}(\mathsf{P})),$$

and $r=-, s=+$

$$f_-(\mathsf{X})f_+(\mathsf{X}+\mathsf{P}) = e_{\mathrm{R}}(\mathsf{P})f_-(\mathsf{X})e_{\mathrm{NS}}^{-1}(\mathsf{P}) + e_{\mathrm{NS}}(\mathsf{P})f_-(\mathsf{X})e_{\mathrm{R}}^{-1}(\mathsf{P}).$$

Combining those relations one can easily obtain the system

$$f_+(\mathsf{P})f_+(\mathsf{X}) = f_+(\mathsf{X})f_+(\mathsf{X}+\mathsf{P})f_+(\mathsf{P}) - if_-(\mathsf{X})f_-(\mathsf{X}+\mathsf{P})f_-(\mathsf{P}), \tag{6.30a}$$
$$f_+(\mathsf{P})f_-(\mathsf{X}) = -if_+(\mathsf{X})f_-(\mathsf{X}+\mathsf{P})f_-(\mathsf{P}) + f_-(\mathsf{X})f_+(\mathsf{X}+\mathsf{P})f_+(\mathsf{P}), \tag{6.30b}$$
$$f_-(\mathsf{P})f_+(\mathsf{X}) = f_+(\mathsf{X})f_+(\mathsf{X}+\mathsf{P})f_-(\mathsf{P}) - if_-(\mathsf{X})f_-(\mathsf{X}+\mathsf{P})f_+(\mathsf{P}), \tag{6.30c}$$
$$f_-(\mathsf{P})f_-(\mathsf{X}) = if_+(\mathsf{X})f_-(\mathsf{X}+\mathsf{P})f_+(\mathsf{P}) - f_-(\mathsf{X})f_+(\mathsf{X}+\mathsf{P})f_-(\mathsf{P}), \tag{6.30d}$$

Combining these relations one can easily obtain the system (6.27) which was observed to imply the pentagon equation satisfied by $\mathsf{T}_{12}^{(1)}$. One can also see another way for proving these equations following the approach in [50] in appendix C.

6.3.2 Relations between push-outs and superflips operators

The quantum Ptolemy groupoid is defined by relations besides the superpentagon. There is an equation satisfied by a push-out

$$\mathsf{B}_{n,1}\mathsf{B}_{1,2}\ldots\mathsf{B}_{n-1,n} = \mathsf{M}_1\mathsf{M}_2\cdots\mathsf{M}_n, \tag{6.31}$$

for all $n \geq 2$, which comes from figure 6.3, where we consider a collection of hexagons meeting in the same vertex (a collection of vertices in $S^{1|1}$ that project to the same point in $\mathbb{P}^{1|1}$). Then, we can move the dot around this vertex until we arrive at the same hexagon, and then relate this hexagonalization to the initial one by reversing the orientation on the edges. This relation is an easy consequence of the definitions.

Further relations involve both flips and push-outs. It suffices to consider relations involving only triangles with one dot as other cases can be reduced to this one using push-outs. We found that the following relations between operators $\mathsf{T}_{23}^{(i)}$ and $\mathsf{T}_{23}^{(j)}$ for different values of i and j are satisfied:

$$(\mathsf{T}_{23}^{(i)})^{-1}\mathsf{B}_{43}\mathsf{B}_{32}\mathsf{B}_{21} = \mathsf{B}_{42}\mathsf{B}_{21}(\mathsf{T}_{23}^{(j)})^{-1}, \tag{6.32}$$

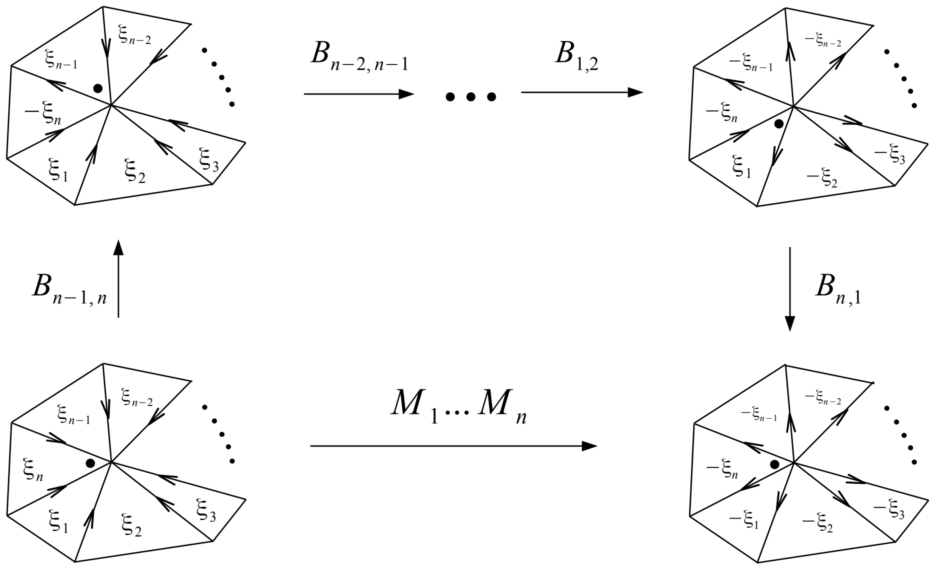

FIGURE 6.3: Relation for push-out.

where the pairs $(i, j) = (5, 8), (8, 5), (6, 7), (7, 6), (1, 2), (2, 1), (3, 4), (4, 3)$,

$$\mathsf{T}_{23}^{(i)}\mathsf{B}_{43}\mathsf{B}_{32}\mathsf{B}_{21} = \mathsf{B}_{43}\mathsf{B}_{31}\mathsf{T}_{23}^{(j)}, \tag{6.33}$$

where the pairs $(i, j) = (5, 4), (4, 5), (1, 6), (6, 1), (7, 2), (2, 7), (3, 8), (8, 3)$.
Another set of relations involves the operators $\mathsf{T}_{34}^{(i)}$ and $\mathsf{T}_{12}^{(j)}$ associated to two different pairs of triangles:

$$\mathsf{T}_{34}^{(i)}\mathsf{B}_{23}(\mathsf{T}_{12}^{(j)})^{-1}(\mathsf{B}_{23})^{-1} = \mathsf{B}_{24}(\mathsf{T}_{12}^{(j)})^{-1}(\mathsf{B}_{24})^{-1}\mathsf{T}_{34}^{(i)}, \tag{6.34}$$

$$(\mathsf{T}_{34}^{(i)})^{-1}\mathsf{B}_{13}\mathsf{T}_{12}^{(j)}(\mathsf{B}_{23})^{-1} = \mathsf{B}_{13}\mathsf{T}_{12}^{(j)}(\mathsf{B}_{23})^{-1}(\mathsf{T}_{34}^{(i)})^{-1}, \tag{6.35}$$

$$(\mathsf{T}_{34}^{(i)})^{-1}\mathsf{B}_{23}(\mathsf{T}_{12}^{(j)})^{-1}(\mathsf{B}_{23})^{-1} = \mathsf{B}_{23}(\mathsf{T}_{12}^{(j)})^{-1}(\mathsf{B}_{23})^{-1}(\mathsf{T}_{34}^{(i)})^{-1}, \tag{6.36}$$

$$\mathsf{T}_{34}^{(i)}\mathsf{B}_{13}\mathsf{T}_{12}^{(j)}(\mathsf{B}_{23})^{-1} = \mathsf{B}_{14}\mathsf{T}_{12}^{(j)}(\mathsf{B}_{24})^{-1}\mathsf{T}_{34}^{(i)}, \tag{6.37}$$

where the $i, j, k, l, m = 1, \ldots, 8$ depends on the Kasteleyn orientation of the graph from which the relation has been derived. Examples of these relations are represented diagrammatically in figures 5.16 and 5.17, with decorated vertices assigned appropriately. All the relations (6.34) can be reduced to the obvious identity $\mathsf{T}_{34}^{(i)}\mathsf{T}_{12}^{(i)} = \mathsf{T}_{12}^{(i)}\mathsf{T}_{34}^{(i)}$.

6.3.3 Relations between superflips and A operator

We finally need to discuss the relations of the super Ptolemy groupoid involving the operator A. We find that the following relations are satisfied

$$\mathsf{A}_1^3 = \mathrm{id}_1, \tag{6.38}$$

$$\mathsf{A}_2\mathsf{T}_{12}^{(i)}\mathsf{A}_1 = \mathsf{A}_1\mathsf{T}_{21}^{(i)}\mathsf{A}_2, \tag{6.39}$$

$$\mathsf{T}_{21}^{(j)}\mathsf{A}_1\mathsf{T}_{12}^{(k)} = \zeta_s\,\mathsf{A}_2\mathsf{A}_1\Pi_{(12)}^{(k)}, \tag{6.40}$$

where $i = 1, \ldots, 8$, the pairs $(j,k) = (4,1), (7,2), (2,3), (5,4), (8,5), (3,6), (6,7), (1,8)$, and $\zeta_s = e^{\frac{\pi i}{4}} e^{-i\pi(1+c_b^2)/6}$. It is the operator $\Pi^{(1)}_{(12)}$ defined in (6.24) which appears in (6.40) for $i=1$, explaining why we adopted this definition for $\Pi^{(1)}_{(12)}$.

Here I provide a computation of (6.40) for $(j,k) = (4,1)$, which involves the operator $\Pi^{(1)}$ permuting our observables. Explicitly, we consider the relation

$$\zeta_s \Pi^{(1)}_{(12)} = \mathsf{A}_2^{-1}\mathsf{A}_1^{-1}\mathsf{T}^{(4)}_{21}\mathsf{A}_1\mathsf{T}^{(1)}_{12}. \tag{6.41}$$

The relation between two superflips is as follows

$$\mathsf{T}^{(4)}_{12} = \mathsf{M}_1\mathsf{M}_2\mathsf{T}^{(1)}_{12}\mathsf{M}_2. \tag{6.42}$$

Let us denote $\alpha = \mathsf{q}_1 + \mathsf{p}_2 - \mathsf{q}_2$ and $\beta = \mathsf{q}_2 + \mathsf{p}_1 - \mathsf{q}_1$. Using that, the flips are expressed as

$$\begin{aligned}
\mathsf{T}^{(1)}_{12} &= \frac{1}{2}[(e_{\mathrm{R}}(\alpha) + e_{\mathrm{NS}}(\alpha))\mathbb{I}_2 \otimes \mathbb{I}_2 - i(e_{\mathrm{R}}(\alpha) - e_{\mathrm{NS}}(\alpha))\kappa \otimes \kappa]e^{-\pi i \mathsf{p}_1\mathsf{q}_2},\\
\mathsf{T}^{(1)}_{21} &= \frac{1}{2}[(e_{\mathrm{R}}(\beta) + e_{\mathrm{NS}}(\beta))\mathbb{I}_2 \otimes \mathbb{I}_2 + i(e_{\mathrm{R}}(\beta) - e_{\mathrm{NS}}(\beta))\kappa \otimes \kappa]e^{-\pi i \mathsf{p}_2\mathsf{q}_1}.
\end{aligned}$$

In addition, lets recall that A acts on p and q as

$$\mathsf{A}^{-1}\mathsf{q}\mathbb{I}_2\mathsf{A} = (\mathsf{p}-\mathsf{q})\mathbb{I}_2, \qquad \mathsf{A}^{-1}\mathsf{p}\mathbb{I}_2\mathsf{A} = -\mathsf{q}\mathbb{I}_2.$$

Using those formulas, we can evaluate the right hand side of (6.41)

$$\begin{aligned}
\mathrm{RHS} = {} & \frac{1}{4}\mathsf{A}_2^{-1}\mathsf{A}_1^{-1}\mathsf{M}_2\mathsf{M}_1[(e_{\mathrm{R}}(\alpha) + e_{\mathrm{NS}}(\alpha))\mathbb{I}_2 \otimes \mathbb{I}_2 + i(e_{\mathrm{R}}(\alpha) - e_{\mathrm{NS}}(\alpha))\kappa \otimes \kappa]\times\\
& \times \mathsf{M}_1 e^{-\pi i \mathsf{p}_1\mathsf{q}_2}\mathsf{A}_1[(e_{\mathrm{R}}(\beta) + e_{\mathrm{NS}}(\beta))\mathbb{I}_2 \otimes \mathbb{I}_2 - i(e_{\mathrm{R}}(\beta) - e_{\mathrm{NS}}(\beta))\kappa \otimes \kappa]e^{-\pi i \mathsf{p}_1\mathsf{q}_2} =\\
= {} & \frac{1}{4}\mathsf{A}_2^{-1}\mathsf{M}_2[(e_{\mathrm{R}}(\mathsf{q}_2-\mathsf{p}_1) + e_{\mathrm{NS}}(\mathsf{q}_2-\mathsf{p}_1))\mathbb{I}_2 \otimes \mathbb{I}_2 - i(e_{\mathrm{R}}(\mathsf{q}_2-\mathsf{p}_1) - e_{\mathrm{NS}}(\mathsf{q}_2-\mathsf{p}_1))\kappa \otimes \kappa]\times\\
& \times [(e_{\mathrm{R}}(\mathsf{p}_1-\mathsf{q}_2) + e_{\mathrm{NS}}(\mathsf{p}_1-\mathsf{q}_2))\mathbb{I}_2 \otimes \mathbb{I}_2 - i(e_{\mathrm{R}}(\mathsf{p}_1-\mathsf{q}_2) - e_{\mathrm{NS}}(\mathsf{p}_1-\mathsf{q}_2))\kappa \otimes \kappa]\\
& \times e^{-\pi i \mathsf{p}_2(\mathsf{p}_1-\mathsf{q}_1)}e^{-\pi i \mathsf{p}_1\mathsf{q}_2} =\\
= {} & \frac{1}{2}\mathsf{A}_2^{-1}\mathsf{M}_2[(e_{\mathrm{NS}}(\mathsf{q}_2-\mathsf{p}_1)e_{\mathrm{NS}}(-\mathsf{q}_2+\mathsf{p}_1) + e_{\mathrm{R}}(\mathsf{q}_2-\mathsf{p}_1)e_{\mathrm{R}}(-\mathsf{q}_2+\mathsf{p}_1))\mathbb{I}_2 \otimes \mathbb{I}_2 +\\
& - i(-e_{\mathrm{NS}}(\mathsf{q}_2-\mathsf{p}_1)e_{\mathrm{NS}}(-\mathsf{q}_2+\mathsf{p}_1) + e_{\mathrm{R}}(\mathsf{q}_2-\mathsf{p}_1)e_{\mathrm{R}}(-\mathsf{q}_2+\mathsf{p}_1))\kappa \otimes \kappa]\times\\
& \times e^{-\pi i \mathsf{p}_2(\mathsf{p}_1-\mathsf{q}_1)}e^{-\pi i \mathsf{p}_1\mathsf{q}_2} =\\
= {} & \frac{1}{2}e^{i\pi c_b^2/2}e^{-\pi(1+2c_b^2)/3}\mathsf{A}_2^{-1}\mathsf{M}_2[(e^{i\pi(-\mathsf{q}_2+\mathsf{p}_1)^2/2} + ie^{i\pi(-\mathsf{q}_2+\mathsf{p}_1)^2/2})\mathbb{I}_2 \otimes \mathbb{I}_2 +\\
& - i(-e^{i\pi(-\mathsf{q}_2+\mathsf{p}_1)^2/2} + ie^{i\pi(-\mathsf{q}_2+\mathsf{p}_1)^2/2})\kappa \otimes \kappa]e^{-\pi i \mathsf{p}_2(\mathsf{p}_1-\mathsf{q}_1)}e^{-\pi i \mathsf{p}_1\mathsf{q}_2} =\\
= {} & \frac{1+i}{2}e^{i\pi c_b^2/2}e^{-i\pi(1+2c_b^2)/3}\mathsf{M}_2[\mathbb{I}_2 \otimes \mathbb{I}_2 + i\kappa \otimes \kappa]\times\\
& \times \underbrace{\mathsf{A}_2^{-1}e^{i\pi(-\mathsf{q}_2+\mathsf{p}_1)^2/2}e^{-\pi i \mathsf{p}_2(\mathsf{p}_1-\mathsf{q}_1)}e^{-\pi i \mathsf{p}_1\mathsf{q}_2}}_{e^{-i\pi/3}e^{i\pi/2}\mathsf{P}_{\mathrm{b}}} =\\
= {} & e^{\frac{i\pi}{4}}e^{-i\pi(1+c_b^2)/6}\mathsf{M}_2[\mathbb{I}_2 \otimes \mathbb{I}_2 + \kappa \otimes \kappa]\mathsf{P}_{\mathrm{b}} = \zeta_s\mathsf{P}_{\mathrm{f}}\mathsf{P}_{\mathrm{b}} = \zeta_s\Pi^{(1)}_{12} = \mathrm{LHS},
\end{aligned}$$

which gives us the left hand side of the formula.

Chapter 7

Quantum supergroups, Heisenberg double and Drinfeld double

In chapters 3 and 4 we have seen how to construct the Heisenberg double for non-graded case with finite and infinite basis. In this chapter we aim to generalize that for the graded case. We prove that the canonical element of the Heisenberg double of the Borel half of $U_q(osp(1|2))$ evaluated on the self-dual representations can be identified with the flip operators of the quantized super Teichmüller theory of super Riemann surfaces. The details for calculating the flip operators of the quantized super Teichmüller theory of super Riemann surfaces were explained in the previous chapter. Finding the basis of Heisenberg double of $U_q(osp(1|2))$ is part of an ongoing project which we will partially explain.

7.1 Quantum supergroups

Quantum supergroups are the generalizations of quantum groups [81–83]. They have a natural connection with supersymmetric integrable lattice models and super conformal field theories. We give some basic definitions related to the quantum supergroup in this section.

Let us choose a field K. A $\mathbb{Z}_2$ graded vector space A over K is the direct sum of two vector spaces,

$$A = A_0 \oplus A_1. \tag{7.1}$$

To each element $a \in A_i \subset A$, $i = 0$ or 1, we assign a grading $|a|$, we call a even if $|a| = 0$ and odd if $|a| = 1$. We call A_0 and A_1 even and odd subspaces of A, respectively. Suppose $A = A_0 \oplus A_1$ is superalgebra or $\mathbb{Z}_2$ graded algebra. We call $A \otimes B$ tensor product of superalgebras A and B with the multiplication defined as,

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|} a_1 a_2 \otimes b_1 b_2, \quad a_i \in A, \quad b_i \in B. \tag{7.2}$$

Definition 12. The unital associative $\mathbb{Z}_2$ graded algebra is a triple (A, m, η) where $A = A_0 \oplus A_1$ is a vector space. m is multiplication map and η is unital map and they satisfy

$$m(m \otimes id) = m(id \otimes m),$$
$$m(\eta \otimes id) = id = m(id \otimes \eta).$$

if $a \in A_i, b \in A_j$ then $m(a, b) \in A_{i+j}$, where $i, j \in \mathbb{Z}_2$. They are similar axioms as the axioms in the non-graded case (equations (3.1), (3.2)).

Definition 13. The counital coassociative $\mathbb{Z}_2$ graded coalgebra is a triple (A, Δ, ϵ) where $A = A_0 \oplus A_1$ is a vector space. Δ is comultiplication map and ϵ is counital map and they satisfy the same axioms

$$(\Delta \otimes id)\Delta = m(id \otimes \Delta),$$
$$(\epsilon \otimes id)\Delta = id = (id \otimes \epsilon)\Delta.$$

and $|a| = |\Delta(a)|$. They are similar axioms as the axioms in the non-graded case (equations (3.3), (3.4)).

Definition 14. Let A be a $\mathbb{Z}_2$ graded algebra with multiplication m and unit η, and at the same time a $\mathbb{Z}_2$ graded coalgebra with comultiplication Δ and counit ϵ. A is called $\mathbb{Z}_2$ graded bialgebra when one of the following condition is satisfied; m and η are $\mathbb{Z}_2$ graded algebra homomorphism or Δ and ϵ are $\mathbb{Z}_2$ graded coalgebra homomorphism.

A $\mathbb{Z}_2$ graded bialgebra become a Hopf algebra if includes homomorphism $S : A \to A$ with following axiom

$$m(id \otimes S).\Delta = m(S \otimes is).\Delta = \eta\epsilon, \tag{7.3}$$

with some properties such as,

$$\Delta S = \Sigma(S \otimes S)\Delta, \qquad S\eta = \eta, \tag{7.4}$$
$$m(S \otimes S)\Sigma = Sm, \qquad \epsilon S = \epsilon, \tag{7.5}$$

where $\Sigma : A \otimes A \to A \otimes A, \quad \Sigma(a \otimes b) = (-1)^{|a||b|} b \otimes a.$

Definition 15. A Lie superalgebra is a superalgebra with commutator bracket $[,]$ which follows axioms,

$$[a, b] = -(-1)^{|a||b|} ba, \quad \text{(anticommutativity)} \tag{7.6}$$
$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|} [b, [a, c]], \quad \text{(Jacobbi identity)} \tag{7.7}$$

A Lie superalgebra is called commutative if $[a, b] = 0$ (more details about Lie superalgebra can be found in [57, 84–86]).

Let g be a Lie superalgebra, the $\mathbb{Z}_2$ graded Hopf algebra $\mathcal{U}(\mathfrak{g})$ admits one-parameter deformation namely the quantum supergroups [82, 87]. The $\mathcal{U}_q(\mathfrak{g})$ is a superalgebra generated by x_i, y_i, K_i, K_i^{-1} with appropriate relations [82, 87].

7.2 Graded Drinfeld double

As we explained in chapter 3, the quantum double construction takes a Hopf algbera A with a bijective antipode and provides the quasi-triangular Hopf algbera $\mathcal{D}(A)$ which includes A and its dual A^*, as two Hopf subalgebras. This new Hopf algbera can be also built from non-commutative and non-cocommutative Hopf algebras. It also provides the universal R-matrices which are the solution of Yang-Baxter equation. Here we explain the generalized Drinfeld's quantum double construction for the graded case.

Lets have a bialgebra A with multiplication m and comultiplication Δ and basis $\{E_\alpha\}$,

$$E_\alpha E_\beta = m^{\gamma}_{\alpha\beta} E_\gamma, \tag{7.8}$$
$$\Delta(E_\alpha) = \mu^{\beta\gamma}_{\alpha} E_\beta \otimes E_\gamma. \tag{7.9}$$

Additionally, we define a dual bialgebra A^* which is isomorphic to A as a vector space and with multiplication and co-multiplication given below. By using the bracket $\langle , \rangle : A \times A^* \to \mathbb{C}$ we have

$$\langle m(a,b), c\rangle = \langle a \otimes b, \Sigma\Delta^*(c)\rangle, \tag{7.10}$$
$$\langle \Delta(a), c \otimes d\rangle = \langle a, m^*(c,d)\rangle, \tag{7.11}$$
$$\langle a \otimes b, c \otimes d\rangle = (-1)^{|b||c|}\langle a, c\rangle\langle b, d\rangle, \tag{7.12}$$

where $a, b \in A$, $c, d \in A^*$ and $\Sigma(a \otimes b) = (-1)^{|a||b|} b \otimes a$. Using the dual basis $\{E^\alpha\}$ on A^* defined such as

$$\langle E_\alpha, E^\beta\rangle = \delta^\beta_\alpha,$$

one can write

$$E^\alpha E^\beta = (-1)^{|\alpha||\beta|}\mu^{\alpha\beta}_{\gamma} E^\gamma, \tag{7.13}$$
$$\Delta(E^\alpha) = m^{\alpha}_{\gamma\beta} E^\beta \otimes E^\gamma. \tag{7.14}$$

Then, one can define Drinfeld double $\mathcal{D}(A) \cong A \oplus A^*$ as a vector space, with multiplication and comultiplication given as above on the subspaces A, A^*. One can extend the bracket in the following way:

$$\langle a, c\rangle = (-1)^{|a||c|}\langle c, a\rangle,$$
$$\langle a, b\rangle = \langle c, d\rangle = 0, \qquad a, b \in A,\ c, d \in A^*.$$

In order to make $\mathcal{D}(A)$ into a bialgebra one has to define a product between E_α's and E^β's, which can be done in the following way

$$(-1)^{|\gamma||\sigma|+|\rho||\sigma|}\mu_\alpha^{\sigma\gamma}m_{\gamma\rho}^{\beta}E_\sigma E^\rho = (-1)^{|\rho||\gamma|}m_{\rho\gamma}^{\beta}\mu_\alpha^{\gamma\sigma}E^\rho E_\sigma.$$

Theorem 2. Let A be a proper $\mathbb{Z}_2$ graded Hopf algebra then $R = E_\alpha \otimes E^\alpha$ defines a universal R matrix for the double $\mathcal{D}(A)$, which is said to satisfy

$$\Sigma\Delta(a)R = R\Delta(a), \tag{7.15}$$
$$(\Delta \otimes id)R = R_{13}R_{23}, \tag{7.16}$$
$$(id \otimes \Delta)R = R_{13}R_{12}, \tag{7.17}$$
$$(T\Delta \otimes id)R = R_{23}R_{13}, \tag{7.18}$$
$$(id \otimes \Sigma\Delta)R = R_{12}R_{13}, \tag{7.19}$$

The equation (7.15) can be proven as follows

$$\begin{aligned}
R\Delta(E_i) &= (E_\alpha \otimes E^\alpha)\mu_i^{\beta\gamma}E_\beta \otimes E_\gamma = (-1)^{|\beta||\alpha|}\mu_i^{\beta\gamma}E_\alpha E_\beta \otimes E^\alpha E_\gamma = \\
&= E_\delta \otimes ((-1)^{|\beta||\alpha|}\mu_i^{\beta\gamma}m_{\alpha\beta}^{\delta}E^\alpha E_\gamma) = E_\delta \otimes ((-1)^{|\beta||\gamma|+|\alpha||\gamma|}\mu_i^{\gamma\beta}m_{\beta\alpha}^{\delta}E_\gamma E^\alpha) = \\
&= (-1)^{|\beta||\gamma|+|\alpha||\gamma|}\mu_i^{\gamma\beta}E_\beta E_\alpha \otimes E_\gamma E^\alpha = (-1)^{|\beta||\gamma|}\mu_i^{\gamma\beta}(E_\beta \otimes E_\gamma)(E_\alpha \otimes E^\alpha) = \\
&= \Sigma(\mu_i^{\gamma\beta}(E_\gamma \otimes E_\beta))R = \Sigma\Delta(E_i)R,
\end{aligned}$$

and analogously for $a = E^i$. The rest can be easily proven as,

$$\begin{aligned}
(\Delta \otimes id)R &= (\Delta \otimes id)(E_\alpha \otimes E^\alpha) = \mu_\alpha^{\beta\gamma}E_\beta \otimes E_\gamma \otimes E^\alpha = \\
&= E_\beta \otimes E_\gamma \otimes \mu_\alpha^{\beta\gamma}E^\alpha = (-1)^{|\beta||\gamma|}E_\beta \otimes E_\gamma \otimes E^\beta E^\gamma = \\
&= (E_\beta \otimes 1 \otimes E^\beta)(1 \otimes E_\gamma \otimes E^\gamma) = R_{13}R_{23}, \\
(id \otimes \Delta)R &= (id \otimes \Delta)(E_\alpha \otimes E^\alpha) = E_\gamma E_\beta \otimes E^\beta \otimes E^\gamma = \\
&= (E_\gamma \otimes 1 \otimes E^\gamma)(E_\beta \otimes E^\beta \otimes 1) = R_{13}R_{12}, \\
(\Sigma\Delta \otimes id)R &= (\Sigma\Delta \otimes id)(E_\alpha \otimes E^\alpha) = (\Sigma \otimes id)(\mu_\alpha^{\beta\gamma}E_\beta \otimes E_\gamma \otimes E^\alpha) = \\
&= E_\gamma \otimes E_\beta \otimes (-1)^{|\beta||\gamma|}\mu_\alpha^{\beta\gamma}E^\alpha = E_\gamma \otimes E_\beta \otimes E^\beta E^\gamma = \\
&= (1 \otimes E_\beta \otimes E^\beta)(E_\gamma \otimes 1 \otimes E^\gamma) = R_{23}R_{13}, \\
(id \otimes \Sigma\Delta)R &= (id \otimes \Sigma\Delta)(E_\alpha \otimes E^\alpha) = (-1)^{|\beta||\gamma|}m_{\beta\gamma}^{\alpha}E_\alpha \otimes E^\beta \otimes E^\gamma = \\
&= (-1)^{|\beta||\gamma|}E_\beta E_\gamma \otimes E^\beta \otimes E^\gamma = (E_\beta \otimes E^\beta \otimes 1)(E_\gamma \otimes 1 \otimes E^\gamma) = R_{12}R_{13}.
\end{aligned}$$

From above, it follows that the Yang-Baxter relation is satisfied

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

with the following proof

$$\begin{aligned}R_{12}R_{13}R_{23} &= (E_\alpha \otimes E^\alpha \otimes 1)(E_\beta \otimes 1 \otimes E^\beta)(1 \otimes E_\gamma \otimes E^\gamma) = \\ &= (-1)^{|\beta||\gamma|+|\alpha||\beta|} E_\alpha E_\beta \otimes E^\alpha E_\gamma \otimes E^\beta E^\gamma = (-1)^{|\beta||\gamma|+|\gamma||\alpha|} m^\sigma_{\beta\alpha} E_\sigma \otimes E_\gamma E^\alpha \otimes \mu^{\gamma\beta}_\rho E^\rho = \\ &= (1 \otimes E_\gamma \otimes E^\gamma)(E_\beta \otimes 1 \otimes E^\beta)(E_\alpha \otimes E^\alpha \otimes 1) = \\ &= R_{23}R_{13}R_{12}.\end{aligned}$$

7.3 Graded Heisenberg double

In this section we explain the generalized Heisenberg double construction for the graded case. Lets take a bialgebra A with multiplication m and comultiplication Δ and basis $\{e_\alpha\}$,

$$e_\alpha e_\beta = m^\gamma_{\alpha\beta} e_\gamma, \qquad \Delta(e_\alpha) = \mu^{\beta\gamma}_\alpha e_\beta \otimes e_\gamma.$$

Additionally, we define a dual bialgebra A^* which is isomorphic to A as a vector space and with multiplication and comultiplication given using the bracket $\langle , \rangle : A \times A^* \to \mathbb{C}$

$$\begin{aligned}\langle m(a,b), c\rangle &= \langle a \otimes b, \Delta^*(c)\rangle, \\ \langle \Delta(a), c \otimes d\rangle &= \langle a, m^*(c,d)\rangle, \\ \langle a \otimes b, c \otimes d\rangle &= (-1)^{|b||c|}\langle a, c\rangle\langle b, d\rangle,\end{aligned}$$

where $a, b \in A$, $c, d \in A^*$. Using the dual basis $\{e^\alpha\}$ on A^* defined by

$$\langle e_\alpha, e^\beta\rangle = \delta^\beta_\alpha,$$

one can write

$$e^\alpha e^\beta = (-1)^{|\alpha||\beta|}\mu^{\alpha\beta}_\gamma e^\gamma, \qquad \Delta(e^\alpha) = (-1)^{|\beta||\gamma|} m^\alpha_{\beta\gamma} e^\beta \otimes e^\gamma.$$

Then, one can define Heisenberg double $H(A) \cong A \oplus A^*$ as a vector space, with multiplication and comultiplication given as above on the subspaces A, A^*. One can extend the bracket in the following way

$$\begin{aligned}\langle a, c\rangle &= (-1)^{|a||c|}\langle c, a\rangle, \\ \langle a, b\rangle &= \langle c, d\rangle = 0,\end{aligned}$$

where $a, b \in A$, $c, d \in A^*$. In order to make $H(\mathcal{A})$ into bialgebra one has to define a product between e_α's and e^β's, which can be done in the following way

$$(-1)^{|\alpha||\beta|} e_\alpha e^\beta = (-1)^{|\rho||\gamma|} m^\beta_{\rho\gamma}\mu^{\gamma\sigma}_\alpha e^\rho e_\sigma.$$

As previously, there is no coproduct compatible with the above. Then canonical element $S = e_\alpha \otimes e^\alpha$ satisfies pentagon relation

$$S_{12}S_{13}S_{23} = S_{23}S_{12},$$

7.4 Relation of graded Drinfeld double and graded Heisenberg double

After this brief reminder on Heisenberg and Drinfeld double, we are prepared to define an algebra map between tensor square of Heisenberg double and Drinfeld double. Let us have a Heisenberg double $H(\mathcal{A})$ defined like previously. Moreover, let us define another Heisenberg double $\tilde{H}(\mathcal{A})$ generated by basis vectors $\{\tilde{e}_\alpha, \tilde{e}_\beta\}$ with

$$\tilde{e}_\alpha\tilde{e}_\beta = m^\gamma_{\alpha\beta}\tilde{e}_\gamma, \qquad \tilde{e}^\alpha\tilde{e}^\beta = (-1)^{|\alpha||\beta|}\mu^{\alpha\beta}_\gamma\tilde{e}^\gamma,$$
$$\Delta(\tilde{e}_\alpha) = \mu^{\beta\gamma}_\alpha\tilde{e}_\beta\otimes\tilde{e}_\gamma, \qquad \Delta(\tilde{e}^\alpha) = (-1)^{|\beta||\gamma|}m^\alpha_{\beta\gamma}\tilde{e}^\beta\otimes\tilde{e}^\gamma,$$
$$\tilde{e}^\beta\tilde{e}_\alpha = (-1)^{|\sigma||\rho|+|\sigma||\gamma|}\mu^{\sigma\gamma}_\alpha m^\beta_{\gamma\rho}\tilde{e}_\sigma\tilde{e}^\rho,$$

which canonical element $\tilde{S} = \tilde{e}_\alpha\otimes\tilde{e}^\alpha$ satisfies "reversed" pentagon equation:

$$\tilde{S}_{12}\tilde{S}_{23} = \tilde{S}_{23}\tilde{S}_{13}\tilde{S}_{12}.$$

Using $H(\mathcal{A})$ and $\tilde{H}(\mathcal{A})$ one can map elements of Drinfeld double $D(\mathcal{A})$, which as a vector space $D(\mathcal{A})\subset H(\mathcal{A})\otimes\tilde{H}(\mathcal{A})$ in the following way

$$E_\alpha = \mu^{\beta\gamma}_\alpha e_\beta\otimes\tilde{e}_\gamma, \qquad E^\alpha = m^\alpha_{\gamma\beta}e^\beta\otimes\tilde{e}^\gamma,$$

which satisfy the relations

$$E_\alpha E_\beta = m^\gamma_{\alpha\beta}E_\gamma, \qquad E^\alpha E^\beta = (-1)^{|\alpha||\beta|}\mu^{\alpha\beta}_\gamma E^\gamma,$$

which can be easily proven by using the compatibility condition

$$\Delta\circ m = (m\otimes m)(id\otimes T\otimes id)(\Delta\otimes\Delta),$$

which in terms of coordinates is

$$m^\gamma_{\alpha\beta}\mu^{\sigma\rho}_\gamma = (-1)^{|\epsilon||\eta|}\mu^{\delta\epsilon}_\alpha\mu^{\eta\xi}_\beta m^\sigma_{\delta\eta}m^\rho_{\epsilon\xi}.$$

One shows:

$$\begin{aligned}
E_\alpha E_\beta &= \mu^{\pi\rho}_\alpha\mu^{\sigma\tau}_\beta(e_\pi\otimes\tilde{e}_\rho)(e_\sigma\otimes\tilde{e}_\tau) = \\
&= (-1)^{|\rho||\sigma|}\mu^{\pi\rho}_\alpha\mu^{\sigma\tau}_\beta e_\pi e_\sigma\otimes\tilde{e}_\rho\tilde{e}_\tau = (-1)^{|\rho||\sigma|}\mu^{\pi\rho}_\alpha\mu^{\sigma\tau}_\beta m^\mu_{\pi\sigma}m^\nu_{\rho\tau}e_\mu\otimes\tilde{e}_\nu = m^\gamma_{\alpha\beta}\mu^{\mu\nu}_\gamma e_\mu\otimes\tilde{e} = \\
&= m^\gamma_{\alpha\beta}E_\gamma,\\
E^\alpha E^\beta &= m^\alpha_{\rho\pi}m^\beta_{\tau\sigma}(e^\pi\otimes\tilde{e}^\rho)(e^\sigma\otimes\tilde{e}^\tau) = (-1)^{|\rho||\sigma|}m^\alpha_{\rho\pi}m^\beta_{\tau\sigma}e^\pi e^\sigma\otimes\tilde{e}^\rho\tilde{e}^\tau = \\
&= (-1)^{|\rho||\sigma|+|\pi||\sigma|+|\rho||\tau|}m^\alpha_{\rho\pi}m^\beta_{\tau\sigma}\mu^{\pi\sigma}_\mu\mu^{\rho\tau}_\nu e^\mu\otimes\tilde{e}^\nu = (-1)^{|\alpha||\beta|}m^\gamma_{\nu\mu}\mu^{\alpha\beta}_\gamma e^\mu\otimes\tilde{e}^\nu = \\
&= (-1)^{|\alpha||\beta|}\mu^{\alpha\beta}_\gamma E^\gamma.
\end{aligned}$$

It can be shown that one can express the R matrix by canonical elements $S, \tilde{S}, S' = \tilde{e}_\alpha\otimes e^\alpha, S'' = e_\alpha\otimes\tilde{e}^\alpha$:

$$R_{12,34} = S''_{14}S_{13}\tilde{S}_{24}S'_{23}$$

which goes in the following way

$$
\begin{aligned}
&S''_{14}S_{13}\tilde{S}_{24}S'_{23} = (e_\alpha \otimes 1 \otimes 1 \otimes \tilde{e}^\alpha)(e_\beta \otimes 1 \otimes e^\beta \otimes 1)(1 \otimes \tilde{e}_\gamma \otimes 1 \otimes \tilde{e}^\gamma)(1 \otimes \tilde{e}_\delta \otimes e^\delta \otimes 1) = \\
&= (-1)^{(|\gamma|+|\delta|)(|\beta|+|\alpha|)+|\alpha||\delta|} e_\alpha e_\beta \otimes \tilde{e}_\gamma \tilde{e}_\delta \otimes e^\beta e^\delta \otimes \tilde{e}^\alpha \tilde{e}^\gamma = \\
&= (-1)^{(|\gamma|+|\delta|)(|\beta|+|\alpha|)+|\alpha||\delta|+|\beta||\delta|+|\alpha||\gamma|} m^a_{\alpha\beta} e_a \otimes m^b_{\gamma\delta}\tilde{e}_b \otimes \mu^{\beta\delta}_c e^c \otimes \mu^{\alpha\gamma}_d \tilde{e}^d = \\
&= (-1)^{|\gamma||\beta|} \mu^{\alpha\gamma}_d \mu^{\beta\delta}_c m^a_{\alpha\beta} m^b_{\gamma\delta} e_a \otimes \tilde{e}_b \otimes e^c \otimes \tilde{e}^d = m^\gamma_{dc}\mu^{ab}_\gamma e_a \otimes \tilde{e}_b \otimes e^c \otimes \tilde{e}^d = \\
&= (\mu^{ab}_\gamma e_a \otimes \tilde{e}_b) \otimes (m^\gamma_{dc} e^c \otimes \tilde{e}^d) = E_\alpha \otimes E^\alpha = R_{12,34},
\end{aligned}
$$

Our copies of Heisenberg double differ only when it comes to the product between $\mathcal{A}$ and $\mathcal{A}^*$, and not on the grading. Therefore, we make an assumption that $|e_\alpha| = |\tilde{e}_\alpha|$ in the above equation. This calculation is a generalization of equation (3.60).

7.5 Heisenberg double of the Borel half $U_q(osp(1|2))$

We reviewed the supersymmetric extension of quantum group in the previous sections. In this section we consider the Heisenberg double of the Borel half of $\mathcal{U}_q(osp(1|2))$ and a class of its self-dual representations. We start from the compact case and move to the non-compact version of Heisenberg double. We prove that the Heisenberg double canonical element evaluate on these representations can be identified with flip operator of qunatized Teichmüller theory of super Riemann surface which we already derived in equation (6.8).

7.5.1 Supergroup $U_q(osp(1|2))$

We call $osp(1|2)$ the simplest rank-one orthogonal symplectic Lie superalgebra. $osp(1|2)$ is special among superalgebras due to its similarity to Lie algebra $sl(2) \subset osp(1|2)$. It contains three even $K, E^{\pm}$ and two odd generators $v^{\pm}$ with following relations

$$
\begin{aligned}
&[H, E^{(\pm)}] = \pm E^{(\pm)}, && [E^+, E^-] = 2K, \\
&[H, v^{(\pm)}] = \frac{1}{2} v^{(\pm)}, && [E^{\pm}, v^{(\mp)}] = v^{(\pm)}, \quad [E^{\pm}, v^{(\pm)}] = 0 \\
&\{v^{(+)}, v^{(-)}\} = -\frac{1}{2} H, && \{v^{(\pm)}, v^{(\pm)}\} = \pm \frac{1}{2} E^{\pm}
\end{aligned}
\tag{7.20}
$$

The quantum superalgebra $\mathcal{U}_q(osp(2|1))$ studied in [86] and [56]. It is generated by $K, K^{-1}, v^{(+)}, v^{(-)}$ satisfying relations:

$$
\begin{aligned}
&K v^{(\pm)} = q^{\pm\frac{1}{2}} v^{(\pm)} K, \\
&\{v^{(+)}, v^{(-)}\} = -\frac{K^2 - K^{-2}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},
\end{aligned}
\tag{7.21}
$$

with the comultiplication

$$
\Delta(K) = K \otimes K, \qquad \Delta(v^{(\pm)}) = K \otimes v^{(\pm)} + v^{(\pm)} \otimes K^{-1}. \tag{7.22}
$$

There is a grading as explain in equation (7.2) and in this case $degK = 0, \;\; degv^{(\pm)} = 1$. One can check the comultiplication preserves the algebra structure:

$$\Delta(K)\Delta(v^{(\pm)}) = Kv^{(\pm)} \otimes 1 + K^2 \otimes Kv^{(\pm)} = q^{\pm\frac{1}{2}}\Delta(v^{(\pm)})\Delta(K),$$

Then by easy calculation one can check

$$\{\Delta(v^{(+)}), \Delta(v^{(-)})\} = K^2 \otimes \{v^{(+)}, v^{(-)}\} + \{v^{(+)}, v^{(-)}\} \otimes K^{-2} = \\ = -\frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}(\Delta(K)^2 - \Delta(K)^{-2}).$$

One can check the classical limit of $\mathcal{U}_q(osp(2|1))$ when we take the limit $q \to 1$, brings us to algbera (7.20). The self dual conjugate series of representations of this group are studied in [88], [61].

7.5.2 Heisenberg double of the Borel half of $U_q(osp(1|2))$

The Borel half of $U_q(osp(1|2))$ has H, $v^{(+)}$ as generators

$$[H, v^{(+)}] = v^{(+)},$$
$$\Delta(H) = H \otimes 1 + 1 \otimes H, \qquad \Delta(v^{(+)}) = v^{(+)} \otimes e^{hH} + 1 \otimes v^{(+)},$$

We have $q = e^{-h}$, $K = e^{hH}$ and therefore, it is easy to see $Kv^{(+)} = q^{-1}v^{(+)}K$. Therefore, we can check

$$v^{(+)^n}K = q^nKv^{(+)^n}, \qquad v^{(+)^n}H^m = (H-n)^m v^{(+)^n},$$

and it is easy to get

$$H^m v^{(+)^n} H^l v^{(+)^k} = \sum_{j=0}^{l} \binom{l}{j} (-n)^{l-j} H^{m+j} v^{(+)^{n+k}}.$$

As for the coproduct

$$\Delta(H^n) = \sum_{k=0}^{n} \binom{n}{k} H^{n-k} \otimes H^k,$$
$$\Delta(v^{(+)^n}) = \sum_{k=0}^{n} \binom{n}{k}_{-q} v^{(+)^{n-k}} \otimes e^{(n-k)hH} v^{(+)^k}.$$

Here we explain briefly how to find the coproduct explicitly

$$\Delta(v^{(+)^n}) = \sum_{k=0}^{n} f(n,k) v^{(+)^{n-k}} \otimes e^{(n-k)hH} v^{(+)^k},$$
$$\Delta(v^{(+)^{n+1}}) = \sum_{k=0}^{n+1} f(n+1,k) v^{(+)^{n+1-k}} \otimes e^{(n+1-k)hH} v^{(+)^k} =$$
$$= \sum_{k=0}^{n} f(n,k) v^{(+)^{n-k}} \otimes e^{(n-k)hH} v^{(+)^k} (v^{(+)} \otimes e^{hH} + 1 \otimes v^{(+)}) =$$
$$= f(n,0) v^{(+)^{n+1}} \otimes e^{(n+1)hH} + f(n,n) 1 \otimes v^{(+)^{n+1}} +$$
$$+ \sum_{k=1}^{n} v^{(+)^{n+1-k}} \otimes e^{(n+1-k)hH} v^{(+)^k} (f(n,k)(-q)^k + f(n,k-1)).$$

Therefore, by comparing the factors we get $f(n,k) = \binom{n}{k}_{-q}$. The algbera A will have the basis as follows

$$\tilde{e}_{m,n} = \frac{1}{m!(-q)_n} H^m v^{(+)^n}, \qquad \text{where } (q)_n = (1-q)...(1-q^n)$$

for having the exact basis we still have to fix the normalization, as explained later in this chapter.

The multiplication and comultiplication for those elements are

$$\tilde{e}_{m,n} \tilde{e}_{l,k} = \sum_{j=0}^{l} \binom{m+j}{j} \binom{n+k}{k}_{-q} \frac{(-n)^{l-j}}{(l-j)!} \tilde{e}_{m+j,n+k},$$
$$\Delta(\tilde{e}_{n,m}) = \sum_{k=0}^{n} \sum_{l=0}^{m} \sum_{p=0}^{\infty} \binom{k+p}{k} (m-l)^p h^p \tilde{e}_{n-k,m-l} \otimes \tilde{e}_{k+p,l}.$$

On the other hand we have the dual algebra A^* which is generated by $\bar{H}$, $v^{(-)}$ with following relations

$$[\bar{H}, v^{(-)}] = -h v^{(-)},$$
$$\Delta(\bar{H}) = \bar{H} \otimes 1 + 1 \otimes \bar{H}, \qquad \Delta(v^{(-)}) = v^{(-)} \otimes e^{-\bar{H}} + 1 \otimes v^{(-)}.$$

We know $\tilde{q} = e^{-hb}$ and we have $\bar{K} = e^{b\bar{H}}$ so it is easy to check $\bar{K} v^{(-)} = \tilde{q} v^{(-)} \bar{K}$ and one can compute the multiplication

$$v^{(-)^n} \bar{H}^m = (\bar{H} + hn)^m v^{(-)^n},$$
$$\bar{H}^m v^{(-)^n} \bar{H}^l v^{(-)^k} = \sum_{j=0}^{l} \binom{l}{j} (n)^{l-j} h^{l-j} \bar{H}^{m+j} v^{(-)^{n+k}},$$

and the coproduct

$$\Delta(\bar{H}^n) = \sum_{k=0}^{n} \binom{n}{k} \bar{H}^{n-k} \otimes \bar{H}^k,$$
$$\Delta(v^{(-)^n}) = \sum_{k=0}^{n} \binom{n}{k}_{-q} v^{(-)^{n-k}} \otimes e^{-(n-k)\bar{H}} v^{(-)^k}.$$

We identify the dual basis as

$$e^{n,m} = \bar{H}^n v^{(-)^m}. \tag{7.23}$$

Now, we have to find our duality bracket and, after that, fix the normalization to obtain orthonormal basis on the dual. Suppose

$$\langle \tilde{e}_{n,m}, e^{k,l} \rangle = g(n,m)\delta_n^k \delta_m^l.$$

Then,

$$\langle \tilde{e}_{m,n}\tilde{e}_{l,k}, e^{a,b} \rangle = \binom{a}{a-m}\binom{n+k}{k}_{-q} \frac{(-n)^{l-a+m}}{(l-a+m)!}\delta^b_{n+k}\Theta(a-m)f(a,b),$$
$$\langle \tilde{e}_{m,n} \otimes \tilde{e}_{l,k}, \Delta(e^{a,b}) \rangle = \binom{a}{a-m}\binom{b}{k}_{-q} \frac{(-b+k)^{l-a+m}}{(l-a+m)!}\delta^{b-k}_{n}\Theta(a-m)(-1)^{k(b-k)}f(m,n)f(l,k),$$

So by comparing these two we can get the normalization as

$$e_{m,n} = \frac{1}{g(m,n)}\tilde{e}_{m,n} = (-1)^{-n^2/2}\frac{1}{m!(-q)_n}H^m v^{(+)^n}. \tag{7.24}$$

Now by having two bases 7.23 and 7.24 in hand we can consider the canonical element:

$$\begin{aligned}
S &= \sum_{n,m} e_{n,m}e^{n,m} = \sum_{n,m}(-1)^{-m^2/2}\frac{1}{n!(-q)_m}H^n v^{(+)^m} \otimes \bar{H}^n v^{(-)^m} = \\
&= \sum_{n,m}(-1)^{-m^2/2}\frac{1}{n!(-q)_m}(H \otimes \bar{H})^n v^{(+)^m} \otimes v^{(-)^m} = \\
&= \exp(H \otimes \bar{H})\sum_{n=0}^{\infty}(-1)^{-\frac{n^2}{2}}(-1)^{\frac{n(n-1)}{2}}\frac{1}{(-q)_n}(v^{(+)} \otimes v^{(-)})^n = \\
&= \exp(H \otimes \bar{H})\sum_{n=0}^{\infty}(-1)^{-\frac{n}{2}}\frac{1}{(-q)_n}(v^{(+)} \otimes v^{(-)})^n = \\
&= \exp(H \otimes \bar{H})(-iv^{(+)} \otimes v^{(-)}; -q)^{-1}_{\infty}.
\end{aligned}$$

In this example we had the basis of infinite dimension. In the next section we consider a $\mathbb{Z}_2$-graded bialgebra $\mathcal{A}$ spanned by the basis vectors $\{e(\alpha)\}$, where the basis is of infinite dimension. From this we define all the objects in analogous way as in the finite dimensional case, replacing all sums with integrals over the spectrum.

$$\sum_{\alpha} \to \int \mathrm{d}\alpha.$$

7.5.3 Heisenberg double of the Borel half of $U_q(osp(1|2))$ with continuous basis

This section is devoted to the study of the Heisenberg double of the Borel half of $U_q(osp(1|2))$ (which we will be calling a quantum superplane). It will be shown later that this Heisenberg double is related to the quantization of the Teichmüller theory of super Riemann surfaces. We will consider infinite dimensional representations of the aformentioned algebra on $L^2(\mathbb{R}) \otimes \mathbb{C}^{1|1}$ with focus on canonical element S. Moreover, we will present the way how $U_q(osp(1|2))$ can be embedded in the tensor square of Heisenberg doubles.

We want to introduce the Heisenberg double of the Borel half of $U_q(osp(1|2))$, with an intention to study the infinite dimensional representations thereof. The Heisenberg double of $\mathcal{B}(U_q(osp(1|2)))$, which will be denoted $\mathcal{HD}(\mathcal{B}(U_q(osp(1|2))))$ (or $\mathcal{SHD}$ from now on), can be defined as an algebra generated by the even elements H and $\hat{H}$ and the odd elements $v^{(+)}$ and $v^{(-)}$ satisfying (anti-)commutation relations

$$\begin{aligned} &[H,\hat{H}] = \frac{1}{\pi i}, && \{v^{(+)}, v^{(-)}\} = e^{\pi b H}(e^{\pi i b^2/2} + e^{-\pi i b^2/2}) \\ &[H, v^{(+)}] = -ibv^{(+)}, && [H, v^{(-)}] = ibv^{(-)}, \\ &[\hat{H}, v^{(+)}] = 0, && [\hat{H}, v^{(-)}] = +ibv^{(-)}, \end{aligned} \tag{7.25}$$

where $q = e^{i\pi b^2}$ for a parameter b such that $b^2 \in \mathbb{R}/\mathbb{Q}$.

Moreover, this algebra is equipped with the real *-structure, i.e.

$$H^* = H, \quad \hat{H}^* = \hat{H}, \quad {v^{(+)}}^* = v^{(+)}, \quad {v^{(-)}}^* = v^{(-)}. \tag{7.26}$$

As usual, there are two interesting subalgebras to consider. We can define two mutually dual subalgebras $\mathcal{SHD}^+$ and $\mathcal{SHD}^-$, which are isomorphic to the Borel half of $U_q(osp(1|2))$. We define $\mathcal{SHD}^+$ as being generated by the generators $\{H, v^{(+)}\}$ with the basis elements $e(s,t,\epsilon,n)$. Moreover, we define $\mathcal{SHD}^-$ as being generated by the generators $\{\hat{H}, v^{(-)}\}$ with the basis elements $\hat{e}(s,t,\epsilon,n)$.

For the purposes of defining the basis elements, we would like to decompose the odd generators $v^{(+)}, v^{(-)}$ into even and odd parts

$$v^{(+)} = V^+X, \qquad v^{(-)} = V^-Y,$$

where V^+, V^- are graded even, X, Y are graded odd, and they satisfy following commutation relations

$$\begin{aligned} &[V^+, X] = 0, && [V^+, Y] = 0, \\ &[V^-, X] = 0, && [V^-, Y] = 0. \end{aligned}$$

Then, we have the candidates for the basis elements of the Heisenberg double as

$$e(s,t,\epsilon,n) = N(s,t,\epsilon,n)(|H|)^{is}\Theta(\epsilon H)(V^+)^{ib^{-1}t}X^n, \tag{7.27}$$
$$\hat{e}(s,t,\epsilon,n) = e^{-\pi s\delta_{\epsilon,-}}(|\hat{H}|)^{is}\Theta(\epsilon\hat{H})(V^-)^{ib^{-1}t}Y^n, \tag{7.28}$$

where $N(s,t,n)$ is the normalization such that

$$N(s,t,\epsilon,n) = \frac{1}{2\pi}\frac{\zeta_0^{-1}}{2}\Gamma(-is)e^{\epsilon\pi s/2}(\pi)^{is}e^{-\frac{1}{2}\pi btQ}iG_{n+1}^{-1}(Q+it), \tag{7.29}$$

and $H = H_+ - H_-$ and $\hat{H} = \hat{H}_+ - \hat{H}_-$ is a decomposition of generators $H, \hat{H}$ into positive operators $H_\epsilon, \hat{H}_\epsilon$ for $\epsilon = \pm$.

One can make those subalgebras into two mutually dual Hopf-subalgebras by assigning a coproduct using the adjoint action of the element S,

$$\Delta(e(s,t,\epsilon,n)) = S^{-1}(1\otimes e(s,t,\epsilon,n))S, \tag{7.30}$$
$$\Delta(\hat{e}(s,t,\epsilon,n)) = S(\hat{e}(s,t,\epsilon,n)\otimes 1)S^{-1}, \tag{7.31}$$

where S is a canonical element defined as

$$S = \sum_{\epsilon=\pm}\sum_{n=0,1}\int \mathrm{d}s\mathrm{d}t\, e(s,t,\epsilon,n)\otimes\hat{e}(s,t,\epsilon,n). \tag{7.32}$$

As the coproduct of the arbitrary element of subalgebras can be derived from the coproducts form the generators, we present them below

$$\begin{aligned}
\Delta(H) &= 1\otimes H + H\otimes 1,\\
\Delta(\hat{H}) &= 1\otimes\hat{H} + \hat{H}\otimes 1,\\
\Delta(v^{(+)}) &= v^{(+)}\otimes e^{\pi bH} + 1\otimes v^{(+)},\\
\Delta(v^{(-)}) &= v^{(-)}\otimes e^{-\pi b\hat{H}} + 1\otimes v^{(-)}.
\end{aligned} \tag{7.33}$$

Moreover, the canonical element S satisfies the graded pentagon equation

$$S_{12}S_{13}S_{23} = S_{23}S_{12}. \tag{7.34}$$

There is an ongoing project to fix these basis elements and find the normalization.

7.5.4 Representations of the Heisenberg double of the Borel half of $U_q(osp(1|2))$

In this section, we want to introduce the infinite dimensional representations $\pi : \mathcal{HD} \to Hom(L^2(\mathbb{R})\otimes\mathbb{C}^{1|1})$ of the Heisenberg double of quantum superplane. The generators are represented as the following operators

$$\begin{aligned}
H &= p\mathbb{I}_2, & \hat{H} &= q\mathbb{I}_2,\\
v^{(+)} &= e^{\pi bq}\kappa, & v^{(-)} &= e^{\pi b(p-q)}\kappa,
\end{aligned} \tag{7.35}$$

where $[p,q]=\frac{1}{\pi i}$ are operators on $L^2(\mathbb{R})$, $\mathbb{I}_2$ is a $(1|1)\times(1|1)$ identity matrix and

$$\kappa=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

The canonical element S in equation (7.32) evaluated on the representation (7.35) has the form

$$\begin{aligned}S=\frac{1}{2}\{&[e_{\mathrm{R}}^{-1}(q_1+p_2-p_1)+e_{\mathrm{NS}}^{-1}(q_1+p_2-p_1)]\mathbb{I}_2\otimes\mathbb{I}_2+\\ &-i[e_{\mathrm{R}}^{-1}(q_1+p_2-p_1)-e_{\mathrm{NS}}^{-1}(q_1+p_2-p_1)]\kappa\otimes\kappa\}\,e^{i\pi p_v q_w}.\end{aligned}\tag{7.36}$$

Comparing this result with superflip operators in super Teichmüller theory (6.8) shows that we can identify this canonical element with the flip operator of quantized Teichmüller theory of super Riemann surfaces.

We have the candidate for the basis elements

$$e(s,t,\epsilon,n)=N(s,t,\epsilon,n)(|p|)^{is}\Theta(\epsilon p)(e^{\pi bq})^{ib^{-1}t}\kappa^n,\tag{7.37}$$

$$\hat{e}(s,t,\epsilon,n)=e^{-\pi\frac{s}{2}\delta_{\epsilon,-}}(|q|)^{is}\Theta(\epsilon q)(e^{\pi b(p-q)})^{ib^{-1}t}\kappa^n,\tag{7.38}$$

and $N(s,t,\epsilon,n)=e^{\epsilon\pi s/4}\frac{1}{2\pi}\frac{\zeta_0^{-1}}{2}\Gamma(-is)(\pi)^{is}e^{-\frac{1}{2}\pi btQ}iG_{n+1}^{-1}(Q+it)$ is the normalization.

The problem of finding the coproduct of the Cartan part of the algebra is of the same type as in the non-supersymmetric case. But we are able to find the coproduct of the odd generators as it is explained below.

Let define e_s as follows for simplicity of our calculations

$$e_s(x)=\frac{1}{2}[(e_{\mathrm{R}}(x)+e_{\mathrm{NS}}(x))1\otimes 1-i(e_{\mathrm{R}}(x)-e_{\mathrm{NS}}(x))\kappa\otimes\kappa].\tag{7.39}$$

One can define the coproduct

$$\begin{aligned}\Delta((e^{\pi bq})^{it}\kappa)&=S^{-1}(1\otimes(e^{\pi bq_2})^{it}\kappa)S=e_s(q_1+p_2-q_2)(e^{\pi bq_2})^{it}(1\otimes\kappa)e_s^{-1}(q_1+p_2-q_2)=\\ &=\frac{b^2}{4}\int d\tau_1 d\tau_2 e^{-i\pi b^2\tau_1^2/2}\left\{\frac{1\otimes 1}{G_{\mathrm{NS}}(Q+ib\tau_1)}+\frac{\kappa\otimes\kappa}{G_{\mathrm{R}}(Q+ib\tau_1)}\right\}e^{i\pi b\tau_1(q_1+p_2-q_2)}\times\\ &\quad\times(e^{\pi bq_2})^{it}(1\otimes\kappa)e^{-\pi b\tau_2 Q/2}\left\{\frac{1\otimes 1}{G_{\mathrm{NS}}(Q+ib\tau_2)}+\frac{i\kappa\otimes\kappa}{G_{\mathrm{R}}(Q+ib\tau_2)}\right\}e^{i\pi b\tau_2(q_1+p_2-q_2)}\\ &\overset{\text{reflection}}{=}\frac{\zeta_0^{-2}b^2}{4}\int d\tau_1 d\tau_2 e^{\pi b(\tau_1-\tau_2)Q/2}\{G_{\mathrm{NS}}(-ib\tau_1)1\otimes 1+iG_{\mathrm{R}}(-ib\tau_1)\kappa\otimes\kappa\}\times\\ &\quad\times(1\otimes\kappa)\left\{\frac{1\otimes 1}{G_{\mathrm{NS}}(Q+ib\tau_2)}+\frac{i\kappa\otimes\kappa}{G_{\mathrm{R}}(Q+ib\tau_2)}\right\}e^{i\pi b\tau_1(q_1+p_2-q_2)}(e^{\pi bq_2})^{it}e^{i\pi b\tau_2(q_1+p_2-q_2)}=\\ &=\frac{\zeta_0^{-2}b^2}{4}\int d\tau_1 d\tau_2 e^{\pi b(\tau_1-\tau_2)Q/2}\left\{\left(\frac{G_{\mathrm{NS}}(-ib\tau_1)}{G_{\mathrm{NS}}(Q+ib\tau_2)}-\frac{G_{\mathrm{R}}(-ib\tau_1)}{G_{\mathrm{R}}(Q+ib\tau_2)}\right)1\otimes\kappa+\right.\\ &\quad\left.-i\left(\frac{G_{\mathrm{NS}}(-ib\tau_1)}{G_{\mathrm{R}}(Q+ib\tau_2)}-\frac{G_{\mathrm{R}}(-ib\tau_1)}{G_{\mathrm{NS}}(Q+ib\tau_2)}\right)\kappa\otimes 1\right\}e^{i\pi b\tau_1(q_1+p_2-q_2)}(e^{\pi bq_2})^{it}e^{i\pi b\tau_2(q_1+p_2-q_2)}\end{aligned}$$

$\tau_2\to\tau,\tau_1\to t+\tau$ by using reflection formula and using Ramanujan formula we get

$$\Delta((e^{\pi bq})^{it}\kappa) = \frac{\zeta_0^{-3}b}{2}\int d\tau e^{\pi b\tau(Q+2ibt)/2-i\pi b^2\tau^2/2}\left\{\frac{G_{\rm NS}(-ib\tau)G_{\rm R}(Q+ibt)}{G_{\rm R}(-ib\tau+Q+ibt)}1\otimes\kappa+\right.$$
$$\left.+i\frac{G_{\rm R}(-ib\tau)G_{\rm R}(Q+ibt)}{G_{\rm NS}(-ib\tau+Q+ibt)}\kappa\otimes 1\right\}(e^{\pi bq_2})^{i(t-\tau)}(e^{\pi b(q_1+p_2)})^{i\tau}=$$
$$=\frac{\zeta_0^{-1}b}{2}\int d\tau e^{i\pi b^2\tau(t-\tau)}\left\{\frac{G_{\rm R}(Q+ibt)}{G_{\rm NS}(Q+ib\tau)G_{\rm R}(-ib\tau+Q+ibt)}1\otimes\kappa+\right.$$
$$\left.+\frac{G_{\rm R}(Q+ibt)}{G_{\rm R}(Q+ib\tau)G_{\rm NS}(-ib\tau+Q+ibt)}\kappa\otimes 1\right\}(e^{\pi bq_2})^{i(t-\tau)}(e^{\pi b(q_1+p_2)})^{i\tau}$$

Now, one can repeat this computations for the even elements as shown

$$\Delta((e^{\pi bq})^{it}) = S^{-1}(1\otimes(e^{\pi bq_2})^{it})S = e_s(q_1+p_2-q_2)(e^{\pi bq_2})^{it}e_s^{-1}(q_1+p_2-q_2)=$$
$$=\frac{\zeta_0^{-1}b}{2}\int d\tau e^{i\pi b^2\tau(t-\tau)}\left\{\frac{G_{\rm NS}(Q+ibt)}{G_{\rm NS}(Q+ib\tau)G_{\rm NS}(-ib\tau+Q+ibt)}1\otimes 1+\right.$$
$$\left.+\frac{G_{\rm NS}(Q+ibt)}{G_{\rm R}(Q+ib\tau)G_{\rm R}(-ib\tau+Q+ibt)}\kappa\otimes\kappa\right\}(e^{\pi bq_2})^{i(t-\tau)}(e^{\pi b(q_1+p_2)})^{i\tau}.$$

We present generators in a form that makes explicit their positive and negative definite parts as $p = p_+ - p_- = \sum_{\epsilon=\pm}\epsilon p_\epsilon$ and $q = q_+ - q_- = \sum_{\epsilon=\pm}\epsilon q_\epsilon$.

The goal is to find the coproduct for the Cartan part and derive the normalization. In order to find the normalization one should compute the multiplication and comultiplication of a basis elements and compare the multiplication coefficients $m, \hat{m}$ and comultiplication coefficients $\mu, \hat{\mu}$ and require that the normalization factor ensures that

$$\mu(s,t,\epsilon,n;\sigma,\tau,\omega,\nu,\sigma',\tau',\omega',\nu') = (-1)^{|\nu||\nu'|}\hat{m}(\sigma,\tau,\omega,\nu,\sigma',\tau',\omega',\nu';s,t,\epsilon,n), \quad (7.40)$$
$$\hat{\mu}(s,t,\epsilon,n;\sigma,\tau,\omega,\nu,\sigma',\tau',\omega',\nu') = (-1)^{|\nu||\nu'|}m(\sigma,\tau,,\omega,\nu,\sigma',\omega',\tau',\nu';s,t,\epsilon,n). \quad (7.41)$$

In addition, there exists an algebra automorphism A

$$A = e^{-i\pi/3}e^{\frac{3}{2}\pi iq^2}e^{\frac{1}{2}i\pi(p+q)^2}U, \quad (7.42)$$

with a matrix U such that $[U,\kappa] = 0$. This automorphism acts in particular on the momentum and position operators

$$A(q\mathbb{I}_2)A^{-1} = (p-q)\mathbb{I}_2, \qquad A(p\mathbb{I}_2)A^{-1} = -q\mathbb{I}_2.$$

Then, by the adjoint action of this automorphism one can define new elements $\tilde{e}(s,t,\epsilon,n), \tilde{\hat{e}}(s,t,\epsilon,n) \in Hom(L^2(\mathbb{R})\otimes\mathbb{C}^{1|1})$

$$\tilde{e}(s,t,\epsilon,n) = Ae(s,t,\epsilon,n)A^{-1}, \qquad \tilde{\hat{e}}(s,t,\epsilon,n) = A\hat{e}(s,t,\epsilon,n)A^{-1}$$

which generate another representation of the Heisenberg double,

$$\tilde{H} = -q\mathbb{I}_2, \qquad \tilde{\hat{H}} = (p-q)\mathbb{I}_2,$$
$$\tilde{v}^{(+)} = e^{\pi b(p-q)}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad \tilde{\hat{v}}^{(+)} = e^{-\pi bp}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

Chapter 8

Braiding and R-matrices

In this chapter we explain how to derive the R-matrix in the Teichmüller theory and define the associated quantum group structure introduced by Kashaev. In the first part, we start with the ordinary case. It contains derivation of the R matrix, while revealing the associated quantum group structure and proving the properties of the R-matrix.

The goal is to generalize it in the supersymmetric case. The results obtained in this chapter are part of the ongoing project. We explain our Ansatz for the R matrix for super Teichmüller theory. Our goal is to check the properties of R-matrix for our result and show that it is the canonical element of the Drinfeld double $U_q(osp(1|2))$.

8.1 Non-supersymmetric case

We consider a compact connected orientable Riemann surface Σ. Let $Homeo(\Sigma, \partial\Sigma)$ denote the group of orientation-preserving homeomorphisms restricting to the identity on the boundary $\partial\Sigma$, and let $Homeo_0(\Sigma, \partial\Sigma)$ denote the normal subgroup of homeomorphisms that are isotopic to the boundary.

Definition 16. The mapping class group of Σ is the quotient group

$$MCG(\Sigma) := Homeo(\Sigma, \partial\Sigma)/Homeo_0(\Sigma, \partial\Sigma) \tag{8.1}$$

Briefly, the mapping class group is a discrete group of symmetries of the space. Mapping class groups are generated by Dehn twists along simple closed curves. A Dehn twist is a homeomorphism $\Sigma \to \Sigma$. A Dehn twist on a surface obtained by cutting the surface along a curve giving one of the boundary components a 2π counter-clockwise twist, and gluing the boundary components back together is illustrated in figure 8.1.

In variant literatures, there are other notations for the mapping class group, for instance: MCG, and $\Gamma_{g,n}$. As a general rule, mapping class group refers to the group of homotopy classes of homeomorphisms of a surface, but there are plenty of variations.

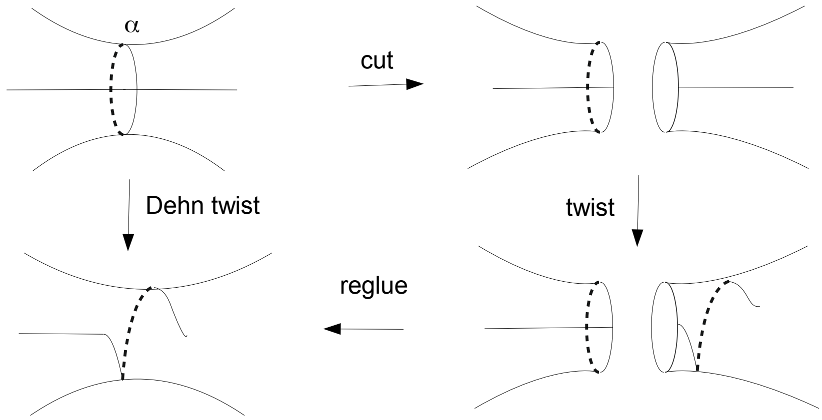

FIGURE 8.1: Dehn twist homeomorphism.

Kashaev showed [44] how the braiding of triangulations of a disk with two interior and two boundary marked points can be derived by a sequence of elementary transformations. Let α be a simple closed curve on Σ. Moreover, square of the braiding is Dehn twist along the associated contour like α.

By using the construction which explained in chapter 2 and considering operators A and T, the corresponding quantum braiding operator is shown in figure 8.2.

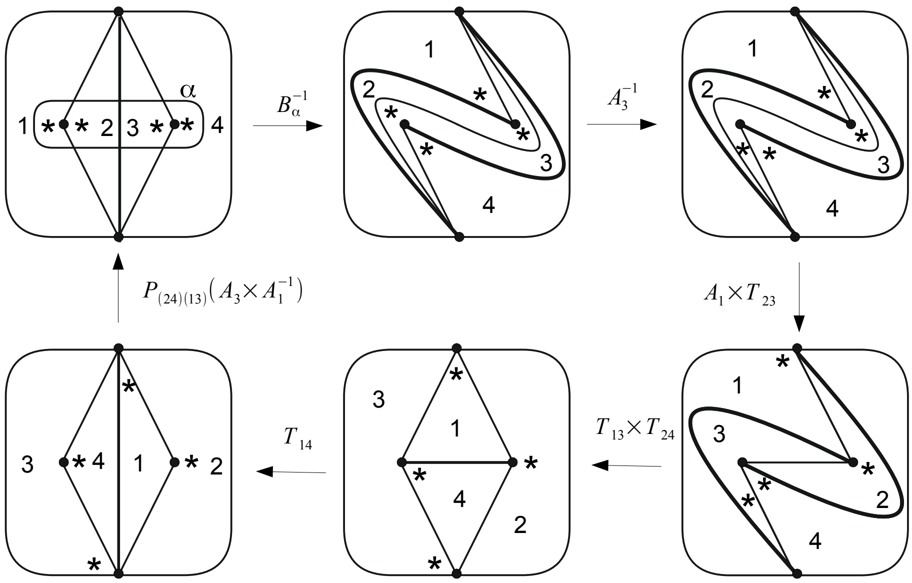

FIGURE 8.2: Braiding along contour α followed by a sequence of transformations brings one back to the initial triangulation τ.

The corresponding quantum braiding operator has the following form:

$$\mathsf{B}_\alpha \backsimeq \mathsf{P}_{(13)(24)}\mathsf{R}_{1234}, \tag{8.2}$$

with

$$\mathsf{R}_{12,34} = \mathsf{A}_1^{-1}\mathsf{A}_3\mathsf{T}_{41}\mathsf{T}_{31}\mathsf{T}_{42}\mathsf{T}_{32}\mathsf{A}_1\mathsf{A}_3^{-1}. \tag{8.3}$$

where the q-exponential property of the quantum dilogarithm

$$g_{\mathrm{b}}(u)g_{\mathrm{b}}(v) = g_{\mathrm{b}}(u+v), \tag{8.4}$$

for $uv = q^2vu$, $q = e^{i\pi b^2}$, and

$$e_{\mathrm{b}}(x) = g_{\mathrm{b}}(e^{2\pi bx}). \tag{8.5}$$

We can write $\mathsf{R}_{12,34}$ as follows,

$$\begin{aligned}
&\mathsf{R}_{12,34} = \mathsf{A}_1^{-1}\mathsf{A}_3 e^{2\pi i p_4 q_1} e_{\mathrm{b}}^{-1}(q_4+p_1-q_1) e^{2\pi i p_3 q_1} e_{\mathrm{b}}^{-1}(q_3+p_1-q_1)\times \\
&\times e^{2\pi i p_4 q_2} e_{\mathrm{b}}^{-1}(q_4+p_2-q_2) e^{2\pi i p_3 q_2} e_{\mathrm{b}}^{-1}(q_3+p_2-q_2)\mathsf{A}_1\mathsf{A}_3^{-1} = \\
&= A_1^{-1}A_3 e^{2\pi i p_4 q_1} e^{2\pi i p_3 q_1} e_{\mathrm{b}}^{-1}(q_4+p_1-q_1+p_3) e_{\mathrm{b}}^{-1}(q_3+p_1-q_1)\times \\
&\times e^{2\pi i p_4 q_2} e^{2\pi i p_3 q_2} e_{\mathrm{b}}^{-1}(q_4+p_2-q_2+p_3) e_{\mathrm{b}}^{-1}(q_3+p_2-q_2) A_1 A_3^{-1} = \\
&= \mathsf{A}_1^{-1}\mathsf{A}_3 e^{2\pi i p_4 q_1} e^{2\pi i p_3 q_1} e^{2\pi i p_4 q_2} e^{2\pi i p_3 q_2} e_{\mathrm{b}}^{-1}(q_4+p_1-q_1+p_3-q_2)\times \\
&\times e_{\mathrm{b}}^{-1}(q_3+p_1-q_1-q_2) e_{\mathrm{b}}^{-1}(q_4+p_2-q_2+p_3) e_{\mathrm{b}}^{-1}(q_3+p_2-q_2)\mathsf{A}_1\mathsf{A}_3^{-1} = \\
&\overset{(8.4)}{=} A_1^{-1}A_3 e^{2\pi i(p_4+p_3)(q_1+q_2)}\times \\
&\times g_{\mathrm{b}}^{-1}(e^{2\pi b(q_4+p_1-q_1+p_3-q_2)} + e^{2\pi b(q_3+p_1-q_1-q_2)} + e^{2\pi b(q_4+p_2-q_2+p_3)} + e^{2\pi b(q_3+p_2-q_2)}) A_1 A_3^{-1} = \\
&= e^{2\pi i(p_4-q_3)(-p_1+q_2)} g_{\mathrm{b}}^{-1}\Big(e^{2\pi b(q_4+q_1-q_3-q_2)} + e^{2\pi b(p_3-q_3+q_1-q_2)} + e^{2\pi b(q_4+p_2-q_2-q_3)} + e^{2\pi b(p_3-q_3+p_2-q_2)}\Big),
\end{aligned} \tag{8.6}$$

where, A and T are defined in (2.22), (2.25) respectively.

As an outcome of Ptolemy groupoid relations, $\mathsf{R} \in L^2(\mathbb{R})$ solves the Yang-Baxter equation,

$$\mathsf{R}_{1234}\mathsf{R}_{1256}\mathsf{R}_{3456} = \mathsf{R}_{3456}\mathsf{R}_{1256}\mathsf{R}_{1234}. \tag{8.7}$$

R_{1234} can also be written as follows

$$\mathsf{R}_{1234} = \mathsf{R} = \mathsf{T}_{1\check{4}}\mathsf{T}_{13}\mathsf{T}_{42}\mathsf{T}_{\hat{3}2}. \tag{8.8}$$

The following convention introduced by Kashaev will help us for further calculations:

$$a_{\hat{k}} \equiv \mathsf{A}_k a_k \mathsf{A}_k^{-1}, \quad a_{\check{k}} \equiv \mathsf{A}_k^{-1} a_k \mathsf{A}_k, \tag{8.9}$$

and some properties follow up:

$$a_k = a_{\hat{\hat{\hat{k}}}} = a_{\check{\check{\check{k}}}}, \quad a_{\hat{\hat{k}}} = a_{\check{k}}, \qquad a_{\check{\check{k}}} = a_{\hat{k}} \tag{8.10}$$

$$\mathsf{T}_{12} = \mathsf{T}_{\hat{2}\check{1}}, \quad \mathsf{T}_{\hat{1}2} = \mathsf{T}_{\hat{2}1} \tag{8.11}$$

$$\mathsf{T}_{12}\mathsf{T}_{2\hat{1}} = \zeta \mathsf{P}_{(12\hat{1})} \tag{8.12}$$

$$\mathsf{P}_{(kl\ldots m\hat{k})} \equiv \mathsf{A}_k\mathsf{P}_{(kl\ldots m)}, \qquad \mathsf{P}_{(kl\ldots m\check{k})} \equiv \mathsf{A}_k^{-1}\mathsf{P}_{(kl\ldots m)} \tag{8.13}$$

where $(kl\ldots m) : k \to l \to \ldots \to m \to k$ is cyclic permutation.

By using three times the pentagon relation such as $\mathsf{T}_{1\hat{2}}\mathsf{T}_{1\check{4}}\mathsf{T}_{\hat{2}\check{4}} = \mathsf{T}_{\hat{2}\check{4}}\mathsf{T}_{1\hat{2}}$ and $\mathsf{T}_{\hat{2}\check{4}}\mathsf{T}_{\hat{2}3}\mathsf{T}_{\check{4}3} = \mathsf{T}_{\check{4}3}\mathsf{T}_{\hat{2}\check{4}}$ we get,

$$\begin{aligned} \mathsf{R} &= (\mathsf{T}_{1\hat{2}})^{-1}(\mathsf{T}_{1\hat{2}})\mathsf{T}_{1\check{4}}\mathsf{T}_{13}\mathsf{T}_{42}\mathsf{T}_{\hat{3}2} = \mathsf{T}_{1\hat{2}}^{-1}\mathsf{T}_{1\hat{2}}\mathsf{T}_{1\check{4}}\mathsf{T}_{\hat{2}\check{4}}\mathsf{T}_{13}\mathsf{T}_{\hat{2}3} = \\ &= \mathsf{T}_{1\hat{2}}^{-1}\mathsf{T}_{\hat{2}\check{4}}\mathsf{T}_{1\hat{2}}\mathsf{T}_{13}\mathsf{T}_{\hat{2}3} = \mathsf{T}_{1\hat{2}}^{-1}\mathsf{T}_{\hat{2}\check{4}}\mathsf{T}_{\hat{2}3}\mathsf{T}_{1\hat{2}} = \mathsf{T}_{1\hat{2}}^{-1}\mathsf{T}_{\hat{2}\check{4}}\mathsf{T}_{\hat{2}3}\mathsf{T}_{\check{4}3}\mathsf{T}_{\check{4}3}{}^{-1}\mathsf{T}_{1\hat{2}} = \\ &= Ad(\mathsf{T}_{1\hat{2}}^{-1}\mathsf{T}_{\check{4}3})\mathsf{T}_{\hat{2}\check{4}}. \end{aligned} \tag{8.14}$$

Comparing this result with equation (8.6), one can see how the R matrix which was derived from four T operators can be written in terms of five T operators by using the adjoint of two operators on the third one. Also comparing this result with equation (3.60) shows the relation with the canonical elements of Heisenberg double.

We conclude that the R matrix can be written as

$$\mathsf{R} = \sum_a \mathsf{E}_a \otimes \mathsf{E}^a \tag{8.15}$$

where,

$$\mathsf{E}_a = \mathsf{E}_a \otimes 1 = Ad(\mathsf{A}_2\mathsf{T}_{12}^{-1})(1 \otimes e_a) = Ad(\mathsf{A}_2)\Delta(e_a), \tag{8.16a}$$

$$\mathsf{E}^a = 1 \otimes \mathsf{E}^a = Ad(\mathsf{A}_2\mathsf{T}_{21}^{-1})(1 \otimes e^a) = Ad(\mathsf{A}_2^{-1})\Delta'(e^a). \tag{8.16b}$$

This bring us to the fact that Drinfeld double basis elements can be built from the Heisenberg double's basis elements e_α and e^α.

Drinfeld double of the Borel half of $U_q(sl(2))$

We already mentioned how to get R matrix from basis elements in equations (8.16). There exists the Hopf algebra G_φ which is composed of those elements and we want to connect it to the quasi-triangular Hopf algebra of $U_q(sl(2))$.

For the Hopf algebra G_φ we have generators

$$g_{12} = p_1 - q_2, \qquad g_{21} = p_2 - q_1, \tag{8.17}$$

$$f_{12} = e^{2\pi b(q_1-q_2)} + e^{2\pi b(p_2-q_2)}, \qquad f_{21} = e^{2\pi b(q_2-q_1)} + e^{2\pi b(p_1-q_1)}, \tag{8.18}$$

that satisfy the commutation relations

$$[g_{nm}, f_{nm}] = -ibf_{nm}, \qquad [g_{mn}, f_{nm}] = ibf_{nm},$$

$$e^{i\alpha g_{nm}}f_{nm} = f_{nm}e^{i\alpha g_{nm}}e^{ib(-i\alpha)}, \qquad e^{i\alpha g_{mn}}f_{nm} = f_{nm}e^{i\alpha g_{mn}}e^{ib(+i\alpha)},$$

and have the coproduct

$$\Delta(g_{12}) = g_{12} \otimes 1 + 1 \otimes g_{12}, \qquad \Delta(g_{21}) = g_{21} \otimes 1 + 1 \otimes g_{21}, \tag{8.19}$$

$$\Delta(f_{12}) = f_{12} \otimes e^{2\pi b g_{12}} + 1 \otimes f_{12}, \qquad \Delta(f_{21}) = e^{2\pi b g_{21}} \otimes f_{21} + f_{21} \otimes 1.$$

Using (8.15) we can write the R-matrix as follows

$$R_{12,34} = e^{-2\pi i g_{12}\otimes g_{21}} g_{\mathrm{b}}^{-1}(f_{12}\otimes f_{21}). \tag{8.20}$$

The coproduct on G_φ can be defined using a twist as follows,

$$\Delta_\varphi = Ad(e^{i\varphi(g_{21}\otimes g_{12} - g_{12}\otimes g_{21})})\Delta. \tag{8.21}$$

Using this definition, we can define the new coproduct on generators,

$$\begin{aligned}
\Delta_\varphi(g_{12}) &= \Delta(g_{12}), \qquad \Delta_\varphi(f_{12}) = f_{12}\otimes e^{2\pi b g_{12}} e^{-\varphi b(g_{12}+g_{21})} + e^{\varphi b(g_{12}+g_{21})}\otimes f_{12},\\
\Delta_\varphi(g_{21}) &= \Delta(g_{21}), \qquad \Delta_\varphi(f_{21}) = e^{2\pi b g_{21}} e^{-\varphi b(g_{12}+g_{21})}\otimes f_{21} + f_{21}\otimes e^{\varphi b(g_{12}+g_{21})}.
\end{aligned}$$

There exists an algebra homomorphism $U_q(sl(2)) \longrightarrow G_\varphi$ such that,

$$\begin{aligned}
K &= e^{\pi b(q_{12}-g_{21}/2)}, \\
E &= e^{-\pi b(c_b+g_{21})}\frac{f_{21}}{q-q^{-1}}, \qquad F = \frac{f_{12}}{q-q^{-1}} e^{\pi b(c_b - g_{12})}.
\end{aligned} \tag{8.22}$$

We can check that this gives a proper representation of $U_q(sl(2))$ and they satisfy the commutation relations.

Then, using the algebra map that expresses the generators of $U_q(sl(2))$ in terms of generators of G_φ we get

$$\begin{aligned}
\Delta_\varphi(K) &= K\otimes K,\\
\Delta_\varphi(E) &= e^{-\pi b g_{21}} e^{2\pi b g_{21}} e^{-\varphi b(g_{12}+g_{21})}\otimes E + E\otimes e^{-\pi b g_{21}} e^{\varphi b(g_{21}+g_{12})},\\
\Delta_\varphi(F) &= F\otimes e^{2\pi b g_{12}} e^{-\varphi b(g_{12}+g_{21})} e^{-\pi b g_{12}} + e^{\varphi b(g_{12}+g_{21})} e^{-\pi b g_{12}}\otimes F.
\end{aligned}$$

For $\varphi = \frac{\pi}{2}$ the algebra map becomes the Hopf algebra map and

$$\Delta_{\frac{\pi}{2}}(K) = K\otimes K, \quad \Delta_{\frac{\pi}{2}}(E) = K^{-1}\otimes E + E\otimes K, \quad \Delta_{\frac{\pi}{2}}(F) = F\otimes K + K^{-1}\otimes F. \tag{8.23}$$

It is easy to check that, since $\Delta(g_{nm}) = g_{nm}\otimes 1 + 1\otimes g_{nm}$ and $\epsilon(g_{nm}) = 0$. The twist $F = e^{i\varphi(g_{21}\otimes g_{12} - g_{12}\otimes g_{21})}$ satisfies properties in below,

$$(F\otimes 1)(\Delta\otimes 1)F = (1\otimes F)(1\otimes\Delta)F, \qquad (\epsilon\otimes id)F = (id\otimes\epsilon)F = 1.$$

and the twisted R-matrix is as follows

$$R_F = F^t R F^{-1}. \tag{8.24}$$

If one show that R satisfies the R-matrix properties, then it immediately follows that the twisted R-matrix also satisfies them. Therefore, in the following we examine the properties of the R-matrix, such as quasi -triangularity and transposition of the coproduct.

First, lets consider the quasi-triangularity property

$$(\Delta\otimes 1)R = R_{13}R_{23}. \tag{8.25}$$

We have two ways of proving this property. The first approach is straightforward by using the q-binomial formula for $u = f_{12}\otimes e^{2\pi b g_{12}}\otimes f_{21}$ and $v = 1\otimes f_{12}\otimes f_{21}$. Since $uv = q^{-2}vu$ we have,

$$\begin{aligned}(\Delta\otimes 1)R &= (\Delta\otimes 1)e^{-2\pi i(g_{12}\otimes g_{21})}g_{\mathrm{b}}^{-1}(f_{12}\otimes f_{21}) = e^{-2\pi i(\Delta(g_{12})\otimes g_{21})}g_{\mathrm{b}}^{-1}(\Delta(f_{12})\otimes f_{21}) = \\ &= e^{-2\pi i(g_{12}\otimes 1+1\otimes g_{21})\otimes g_{21}}g_{\mathrm{b}}^{-1}(f_{12}\otimes e^{2\pi b g_{12}}\otimes f_{21}+1\otimes f_{12}\otimes f_{21}) = \\ &= e^{-2\pi i(g_{12}\otimes 1+1\otimes g_{21})\otimes g_{21}}g_{\mathrm{b}}^{-1}(u+v) \overset{(8.4)}{=} e^{-2\pi i(g_{12}\otimes 1+1\otimes g_{21})\otimes g_{21}}g_{\mathrm{b}}^{-1}(u)g_{\mathrm{b}}^{-1}(v) = \\ &= e^{-2\pi i(g_{12}\otimes 1\otimes g_{21})}g_{\mathrm{b}}^{-1}(ue^{2\pi b(1\otimes g_{12}\otimes 1)})e^{-2\pi i(1\otimes g_{12}\otimes g_{21})}g_{\mathrm{b}}^{-1}(v) = \\ &= e^{-2\pi i(g_{12}\otimes 1\otimes g_{21})}g_{\mathrm{b}}^{-1}(f_{12}\otimes 1\otimes f_{21})e^{-2\pi i(1\otimes g_{12}\otimes g_{21})}g_{\mathrm{b}}^{-1}(1\otimes f_{12}\otimes f_{21}) = \\ &= R_{13}R_{23},\end{aligned}$$

As the second proof, one can use the Fourier transform of the quantum dilogarithm,

$$b\int \mathrm{d}t e^{2\pi i b t r}\frac{e^{-\pi b t Q}}{G_{\mathrm{b}}(Q+ibt)} = g_{\mathrm{b}}^{-1}(e^{2\pi b r}).$$

Then by considering the fact that the coproduct is given as follows,

$$\Delta(f_{12}^{it}) = b\int \mathrm{d}\tau\frac{G_{\mathrm{b}}(Q+ibt)}{G_{\mathrm{b}}(Q+ib\tau)G_{\mathrm{b}}(-ib\tau+Q+ibt)}f_{12}^{i\tau}\otimes(e^{2\pi b g_{12}}f_{12})^{i(t-\tau)},$$

Therefore, we can check the quasi-triangularity properties,

$$\begin{aligned}&(\Delta\otimes 1)R = (\Delta\otimes 1)e^{-2\pi i(g_{12}\otimes g_{21})}g_{\mathrm{b}}^{-1}(f_{12}\otimes f_{21}) = \\ &= e^{-2\pi i(\Delta(g_{12})\otimes g_{21})}(\Delta\otimes 1)g_{\mathrm{b}}^{-1}(f_{12}\otimes f_{21}) = \\ &= e^{-2\pi i(g_{12}\otimes 1+1\otimes g_{21})\otimes g_{21}}(\Delta\otimes 1)\, b\int \mathrm{d}t(f_{12}\otimes f_{21})^{it}\frac{e^{-\pi b t Q}}{G_{\mathrm{b}}(Q+ibt)} = \\ &= e^{-2\pi i(g_{12}\otimes 1+1\otimes g_{21})\otimes g_{21}}\, b\int \mathrm{d}t\frac{e^{-\pi b t Q}}{G_{\mathrm{b}}(Q+ibt)}\Delta(f_{12}^{it})\otimes f_{21}^{it} \\ &= e^{-2\pi i(g_{12}\otimes 1+1\otimes g_{21})\otimes g_{21}}\, b^2\int \mathrm{d}t\mathrm{d}\tau\frac{e^{-\pi b t Q}}{G_{\mathrm{b}}(Q+ib\tau)G_{\mathrm{b}}(-ib\tau+Q+ibt)}f_{12}^{i\tau}\otimes f_{12}^{i(t-\tau)}(e^{2\pi b g_{12}})^{i\tau}\otimes f_{21}^{it} = \\ &\overset{\substack{t\to t+\tau \\ \mathrm{d}t\to\mathrm{d}t}}{=} e^{-2\pi i(g_{12}\otimes 1+1\otimes g_{21})\otimes g_{21}}\int\frac{\mathrm{d}\tau b e^{-\pi b\tau Q}}{G_{\mathrm{b}}(Q+ib\tau)}(f_{12}^{i\tau}\otimes(e^{2\pi b g_{12}})^{i\tau}\otimes f_{21}^{i\tau})\int\frac{\mathrm{d}t b e^{-\pi b t Q}}{G_{\mathrm{b}}(Q+ibt)}(1\otimes f_{12}^{it}\otimes f_{21}^{it}) = \\ &= e^{-2\pi i(g_{12}\otimes 1+1\otimes g_{21})\otimes g_{21}}\, g_{\mathrm{b}}^{-1}(f_{12}\otimes e^{2\pi b g_{12}}\otimes f_{21})\, g_{\mathrm{b}}^{-1}(1\otimes f_{12}\otimes f_{21}) = \\ &= e^{-2\pi i(g_{12}\otimes 1\otimes g_{21})}g_{\mathrm{b}}^{-1}(f_{12}\otimes 1\otimes f_{21})e^{-2\pi i(1\otimes g_{12}\otimes g_{21})}g_{\mathrm{b}}^{-1}(1\otimes f_{12}\otimes f_{21}) = R_{13}R_{23}.\end{aligned}$$

The other quasi-triangularity equation $(1\otimes\Delta)R = R_{12}R_{13}$ goes analogously.

The other important property of the R-matrix is the following one

$$R\Delta(u) = \Delta'(u)R, \tag{8.26}$$

where u is arbitrary generator. The equation is obviously satisfied for $u = g_{12}$ or g_{21}.

In order to prove it for other generators, we need to find their appropriate representations.

$$\begin{aligned} f_{12} &= w_{\rm b}(-q_1+p_2)e^{\pi b(q_1+p_2-2q_2)}w_{\rm b}(q_1-p_2) = \\ &= e^{\frac{\pi b}{2}(q_1+p_2-2q_2)}w_{\rm b}(-q_1+p_2+\frac{ib}{2})w_{\rm b}(q_1-p_2+\frac{ib}{2})e^{\frac{\pi b}{2}(q_1+p_2-2q_2)} = \\ &= e^{\frac{\pi b}{2}(q_1+p_2-2q_2)}2\cosh(\pi b(q_1-p_2))e^{\frac{\pi b}{2}(q_1+p_2-2q_2)} = \\ &= e^{\frac{\pi b}{2}(q_1+p_2-2q_2)}(e^{\pi b(q_1-p_2))}+e^{-\pi b(q_1-p_2))})e^{\frac{\pi b}{2}(q_1+p_2-2q_2)} = \\ &= e^{2\pi b(q_1-q_2)}+e^{2\pi b(p_2-q_2)}, \end{aligned}$$

where the quantum dilogarithm $w_{\rm b}$ defined as

$$w_{\rm b}(x) := e^{\frac{\pi i}{2}(\frac{Q^2}{4}+x^2)}G_{\rm b}(\frac{Q}{2}-ix)$$

The properties of special function is shown in appendix A. In the same way one can find

$$f_{21} = w_{\rm b}(-q_2+p_1)e^{\pi b(p_1+q_2-2q_1)}w_{\rm b}(q_2-p_1) = e^{2\pi b(q_2-q_1)}+e^{2\pi b(p_1-q_1)}.$$

Then we can compute

$$\begin{aligned} &[f_{12}, f_{21}^{\alpha}] = [w_{\rm b}(-q_1+p_2)e^{\pi b(q_1+p_2-2q_2)}w_{\rm b}(q_1-p_2), w_{\rm b}(-q_2+p_1)e^{\alpha\pi b(p_1+q_2-2q_1)}w_{\rm b}(q_2-p_1)] = \\ &= \frac{w_{\rm b}(-q_1+p_2)w_{\rm b}(-q_2+p_1+ib)}{w_{\rm b}(-q_1+p_2-ib)w_{\rm b}(-q_2+p_1)}e^{\pi b(g_{12}+g_{21})}f_{21}^{\alpha-1}+ \\ &- \frac{w_{\rm b}(-q_2+p_1-ib(\alpha-1))}{w_{\rm b}(-q_2+p_1-ib\alpha)}\frac{w_{\rm b}(-q_1+p_2+ib\alpha)}{w_{\rm b}(-q_1+p_2+ib(\alpha-1))}e^{\pi b(g_{12}+g_{21})}f_{21}^{\alpha-1} = \\ &= \left(\frac{w_{\rm b}(-q_2+p_1+ib)w_{\rm b}(-q_1+p_2)}{w_{\rm b}(-q_2+p_1)w_{\rm b}(-q_1+p_2-ib)} - \frac{w_{\rm b}(-q_2+p_1-ib(\alpha-1))w_{\rm b}(-q_1+p_2+ib\alpha)}{w_{\rm b}(-q_2+p_1-ib\alpha)w_{\rm b}(-q_1+p_2+ib(\alpha-1))}\right)e^{\pi b(g_{12}+g_{21})}f_{21}^{\alpha-1} \\ &= (2\sin(\pi b^2))^2\left([\frac{Q}{2b}-\frac{i(-q_2+p_1)}{b}]_q[\frac{Q}{2b}+\frac{i(-q_1+p_2)}{b}]_q+\right. \\ &\left.-[\frac{Q}{2b}-\alpha-\frac{i(-q_2+p_1)}{b}]_q[\frac{Q}{2b}-\alpha+\frac{i(-q_1+p_2)}{b}]_q\right)e^{\pi b(g_{12}+g_{21})}f_{21}^{\alpha-1} = \\ &= (2\sin(\pi b^2))^2[\alpha]_q[\frac{Q}{b}-\alpha+\frac{i}{b}(-q_1+p_2-(-q_2+p_1))]_q e^{\pi b(g_{12}+g_{21})}f_{21}^{\alpha-1} = \\ &= (2\sin(\pi b^2))^2[\alpha]_q[\alpha-1+\frac{1}{ib}(-q_1+p_2+q_2-p_1)]_q e^{\pi b(g_{12}+g_{21})}f_{21}^{\alpha-1} = \\ &= (2\sin(\pi b^2))^2[\alpha]_q[\alpha-1+\frac{1}{ib}(g_{21}-g_{12})]_q e^{\pi b(g_{12}+g_{21})}f_{21}^{\alpha-1}, \end{aligned}$$

where we used the properties of q-numbers,

$$[t]_q = \frac{\sin(\pi b^2 t)}{\sin(\pi b^2)}, \qquad [-t]_q = -[t]_q, \qquad [t+b^{-2}]_q = -[t]_q,$$
$$[x]_q[y]_q - [x-\alpha]_b[y-\alpha]_q = [\alpha]_q[x+y-\alpha]_q.$$

Now, as a check, we can look what kind of identity we get for $U_q(sl(2))$ from the above. First, let us note that

$$(e^{-\pi b g_{21}} f_{21})^{\alpha} = q^{-\frac{\alpha(\alpha-1)}{2}} e^{-\alpha\pi b g_{21}} f_{21}^{\alpha}.$$

Then,

$$[F, E^{\alpha}] = \frac{e^{-\pi b c_b(\alpha-1)}}{(q-q^{-1})^{\alpha+1}}[f_{12}e^{-\pi b g_{12}}, (e^{-\pi b g_{21}} f_{21})^{\alpha}] =$$
$$= -[\alpha]_q[\alpha - 1 + \frac{1}{ib}(g_{21} - g_{12})]_q E^{\alpha-1} = [\alpha]_q[-\alpha + 1 + 2H]_q E^{\alpha-1},$$

where we identify $H = -\frac{1}{2ib}(g_{21} - g_{12})$ and $K = q^H$ and one can follow the proof of theorem 3 in [89]. The difference in sign can be explained by noticing that our definitions of E and F differ from those by [89] which we denote here as $\tilde{E}, \tilde{F}$, in the following way

$$E = -i\tilde{E}, \qquad F = +i\tilde{F}.$$

Therefore, one will get

$$R\Delta(\tilde{E}) - \Delta'(\tilde{E})R = R(\tilde{E}_1 K_2 + K_1^{-1}\tilde{E}_2) - (\tilde{E}_1 K_2^{-1} + K_1\tilde{E}_2)R = 0$$

and the proof of equation (8.26) is complete.

Coproduct of Drinfeld double

We want to find an operatorial expression for the coproduct of the Drinfeld double defined in terms of Heisenberg double. The basis elements E^{α}, E_{α} of Drinfeld double are defined in terms of the generators e_{α}, e^{α} of the Heisenberg double as follows

$$E_a = Ad(A_2 T_{12}^{-1})(1 \otimes e_a), \qquad E^a = Ad(A_2^{-1} T_{21})(1 \otimes e^a).$$

The coproducts of both Heisenberg double and Drinfeld double agree with each other, and are defined in terms of coefficients m, μ as

$$\Delta(e_a) = \mu_a^{bc} e_b \otimes e_c, \qquad \Delta(E_a) = \mu_a^{bc} E_b \otimes E_c,$$
$$\Delta(e^a) = m_{bc}^a e^b \otimes e^c, \qquad \Delta(E^a) = m_{bc}^a E^b \otimes E^c.$$

In these section we use the Einstein summation convention, since we consider the continuous basis, we insert an integral instead of a summation over the variables.

We can calculate the coproduct of the basis of Drinfeld double in two ways, first expanding the E_a element on the left hand side, and after that expanding elements $E_b \otimes E_c$ on the right hand side

$$\begin{aligned}\Delta(E_a) &= \mu_a^{bc} E_b \otimes E_c = \mu_a^{bc} Ad(A_2 T_{12}^{-1} A_4 T_{34}^{-1})(1 \otimes e_a \otimes 1 \otimes e_b)\\ &= \mu_a^{bc} Ad(A_2 A_4)\Delta(e_a) \otimes \Delta(e_b) = (*)\\ \Delta(E_a) &= \Delta(Ad(A_2 T_{12}^{-1})(1 \otimes e_a)) =\\ &= \Delta(Ad(A_2)\Delta(e_a)) = \mu_a^{bc}\Delta(Ad(A_2)(e_b \otimes e_c)) = (**)\end{aligned}$$

Then, setting both sides to be equal $(*) = (**)$ we get

$$\Delta(Ad(A_2)(e_b \otimes e_c)) = Ad(A_2 A_4)\Delta(e_a) \otimes \Delta(e_b).$$

We know that the coproduct on one half of Heisenberg double is defined in terms of the canonical element T

$$\Delta_H(u) = T(1 \otimes u)T^{-1}, u \in \{e_a\}.$$

Then, we want to find an operator U which encodes the coproduct on the half of the Drinfeld double

$$\Delta_D(u_1 \otimes u_2) = U(1 \otimes u_1 \otimes 1 \otimes u_2)U^{-1}, u \in \{E_a\}.$$

We see that for $U = A_2 A_4 T_{12}^{-1} T_{34}^{-1} A_4^{-1}$ we get the right coproduct

$$\begin{aligned}\Delta_D(Ad(A_2)e_b \otimes e_c) &= A_2 A_4 T_{12}^{-1} T_{34}^{-1} A_2 (1 \otimes e_b \otimes 1 \otimes e_c) A_2^{-1} T_{12} T_{34} A_4^{-1} A_2^{-1}\\ &= A_2 A_4 \Delta(e_b) \otimes \Delta(e_c) A_4^{-1} A_2^{-1}.\end{aligned}$$

The calculation for the other half of the Drinfeld double, i.e. the generators E^a, gives

$$\begin{aligned}\Delta(Ad(A_2^{-1})(e^c \otimes e^b)) &= Ad(A_2^{-1} A_4^{-1})\Delta'(e^b) \otimes \Delta'(e^c) =\\ &= A_2^{-1} A_4^{-1} P_{(12)(34)} \Delta'(e^c) \otimes \Delta'(e^b) A_4 A_2 P_{(12)(34)}.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\Delta_D(X_{(1)} \otimes X_{(2)}) &= Ad(A_2 A_4 T_{12}^{-1} T_{34}^{-1} A_4^{-1})(1 \otimes X_{(1)} \otimes 1 \otimes X_{(2)}),\\ \Delta_D(\hat{X}_{(1)} \otimes \hat{X}_{(2)}) &= Ad(A_2^{-1} A_4^{-1} T_{12} T_{34} A_1)(\hat{X}_{(1)} \otimes 1 \otimes \hat{X}_{(2)} \otimes 1),\end{aligned}$$

where $E_a = X_{(1)} \otimes X_{(2)}$, $E^a = \hat{X}_{(1)} \otimes \hat{X}_{(2)}$ (we suppress the sum over terms here).

8.2 Supersymmetric case

In the supersymmetric case in order to find braiding one needs to consider Kasteleyn orientations on the edges. We can classify the braiding depending on the type of two interior vertexes. Following the rule we explained in chapter 5 for distinguishing the type of puncture as Ramond (R) or Neveu-Schwartz (NS), if two punctures are Ramond, then there is no dot next to them. If one puncture is R and the other one is NS, it means there is one dot next to the NS punctures. For the remaining case that the two punctures are

NS, there will be one dot next to each of them. The last case is the only one where the homogeneous Yang-Baxter equation has the possibility to be satisfied . By homogeneous Yang-Baxter we mean just one type of R matrix can be involved in the equation.

We summarize the 8 possible starting points for finding the braiding in figure 8.3, where decorated vertex have the same place as it was shown in figure 8.2. Figure 8.4 (which

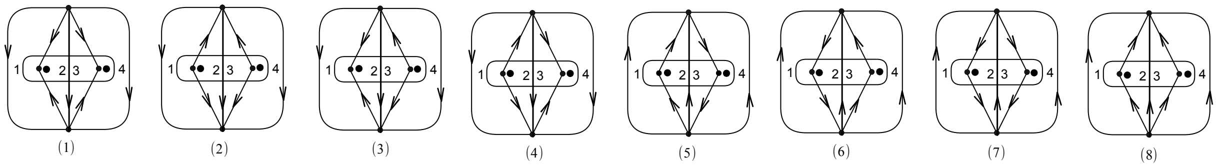

FIGURE 8.3: Eight possible orientations to find braiding with two NS-punctures.

presents one of the possible cases) shows how the braiding of triangulation of a disk with two interior and two boundary marked points can be removed by a sequence of elementary transformations of the graded Ptolemy groupoid. Using the operators $T^{(i)}_{mn}, A_v, B_{kl}$ the corresponding quantum braiding operator has the form

$$B^{(i)}_{\alpha} \equiv \Pi_{(13)(24)} R^{(i)}, \qquad \text{where } i = 1, \dots, 8 \tag{8.27}$$

One can consider all the braiding for orientations different than the one in figure 8.4 and show that

$$\begin{aligned}
R^{(1)} =& A_1^{-1} A_3 B_{34} T^{(8)}_{41} B_{21} T^{(2)}_{42} B_{21} B_{34} T^{(8)}_{31} B_{21} T^{(2)}_{32} B_{21} A_1 A_3^{-1} \\
R^{(2)} =& A_1^{-1} A_3 B_{34} T^{(6)}_{41} B_{21} T^{(6)}_{42} B_{21} B_{34} T^{(4)}_{31} B_{21} T^{(4)}_{32} B_{21} A_1 A_3^{-1} \\
R^{(3)} =& A_1^{-1} A_3 B_{34} T^{(3)}_{41} B_{21} T^{(7)}_{42} B_{21} B_{34} T^{(7)}_{31} B_{21} T^{(3)}_{32} B_{21} A_1 A_3^{-1} U_1 U_2 U_3 U_4 \\
R^{(4)} =& A_1^{-1} A_3 B_{34} T^{(1)}_{41} B_{21} T^{(1)}_{42} B_{21} B_{34} T^{(1)}_{31} B_{21} T^{(1)}_{32} B_{21} A_1 A_3^{-1} U_1 U_2 U_3 U_4 \\
R^{(5)} =& A_1^{-1} A_3 B_{34} T^{(4)}_{41} B_{21} T^{(4)}_{42} B_{21} B_{34} T^{(6)}_{31} B_{21} T^{(6)}_{32} B_{21} A_1 A_3^{-1} U_1 U_2 U_3 U_4 \\
R^{(6)} =& A_1^{-1} A_3 B_{34} T^{(2)}_{41} B_{21} T^{(8)}_{42} B_{21} B_{34} T^{(2)}_{31} B_{21} T^{(8)}_{32} B_{21} A_1 A_3^{-1} U_1 U_2 U_3 U_4 \\
R^{(7)} =& A_1^{-1} A_3 B_{34} T^{(7)}_{41} B_{21} T^{(3)}_{42} B_{21} B_{34} T^{(3)}_{31} B_{21} T^{(7)}_{32} B_{21} A_1 A_3^{-1} \\
R^{(8)} =& A_1^{-1} A_3 B_{34} T^{(5)}_{41} B_{21} T^{(5)}_{42} B_{21} B_{34} T^{(5)}_{31} B_{21} T^{(5)}_{32} B_{21} A_1 A_3^{-1},
\end{aligned} \tag{8.28}$$

Using the construction described in chapter 6 and rewriting $B_{ij} = 1 \otimes U_j$ one can rewrite (8.28) in the adjoint form as

$$\begin{aligned}
R^{(1)} &= Ad(A_1^{-1} A_3 U_2) T^{(1)}_{41} T^{(1)}_{42} T^{(1)}_{31} T^{(1)}_{32} \qquad & R^{(2)} &= Ad(A_1^{-1} A_3 U_2) T^{(1)}_{41} T^{(1)}_{42} T^{(1)}_{31} T^{(1)}_{32} \\
R^{(7)} &= Ad(A_1^{-1} A_3 U_3) T^{(1)}_{41} T^{(1)}_{42} T^{(1)}_{31} T^{(1)}_{32} \qquad & R^{(8)} &= Ad(A_1^{-1} A_3 U_3) T^{(1)}_{41} T^{(1)}_{42} T^{(1)}_{31} T^{(1)}_{32},
\end{aligned}$$

Yang-Baxter One can find the pictorial representation of the Yang-Baxter equation. It is obvious that we always have the equation with the form

$$R^{(i)}_{12} R^{(j)}_{13} R^{(i)}_{23} = R^{(j)}_{23} R^{(i)}_{13} R^{(j)}_{12}, \tag{8.29}$$

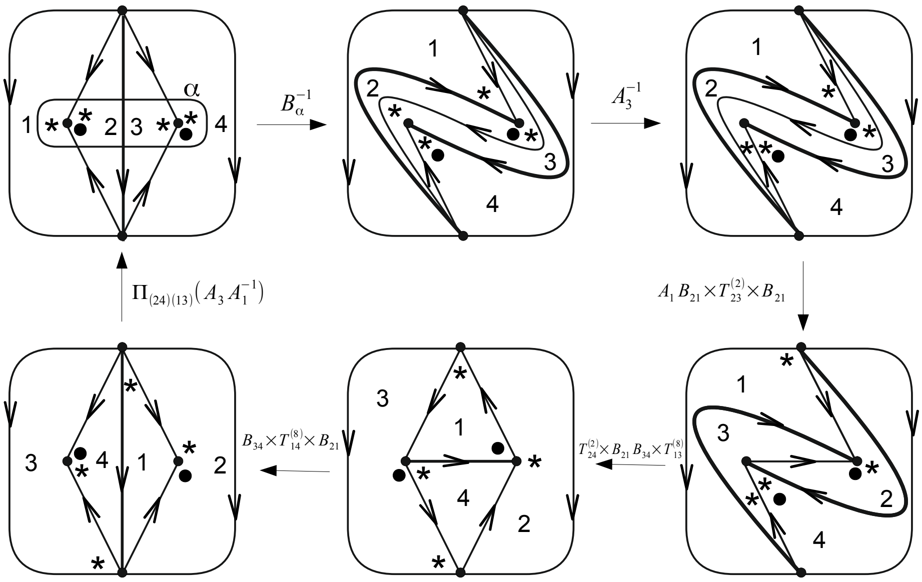

FIGURE 8.4: Braiding along contour α followed by a sequence of transformations brings one back to the initial triangulation τ.

We still focus on the case when both puncture are NS. There are few combinations of i and j can satisfy the Yang-Baxter equation. Between all the possible equations there are just four cases that i can be equal to j–one example is illustrated in figure 8.5.

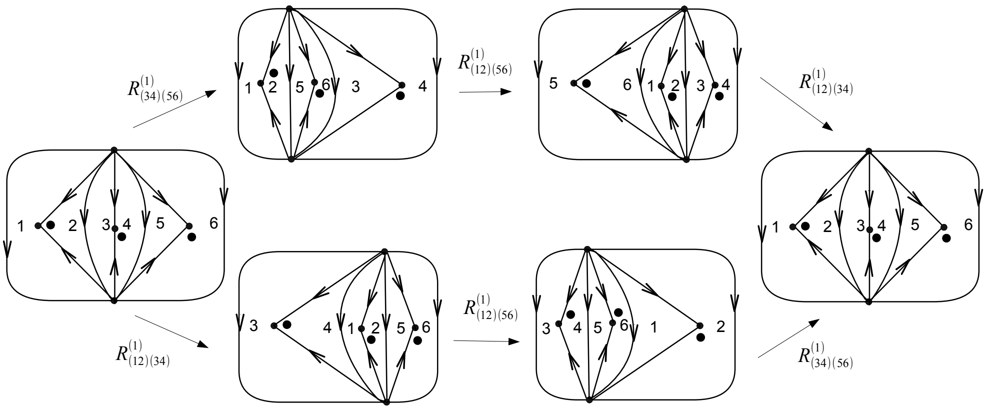

FIGURE 8.5: One possible Yang-Baxter equation with two NS-punctures.

The possible pairs for two NS punctures are
$(i, j) = (1, 1), (2, 2), (7, 7), (8, 8), (3, 4), (3, 2), (4, 1), (4, 3), (5, 7), (5, 6), (6, 8).$

Chapter 9

Conclusions and outlook

We used a similar approach to that of Kashaev [34] in the case of ordinary Riemann surfaces and generalized this result to the supersymmetric case to construct a quantization of the Teichmüller theory of super Riemann surfaces. The independence of the resulting quantum theory with respect to changes of triangulations was demonstrated by constructing a unitary projective representation of the super Ptolemy groupoid including superpentagon relations.

We identified coordinates on the quantum super Teichmüller space with elements of the Heisenberg double. The resulting quantum theory is identified with the quantum theory of the Teichmüller spaces of super Riemann surfaces. The goal of an ongoing project is to construct bases of the canonical element of the Heisenberg double.

The canonical element of the Heisenberg double is expressed in terms of particular functions called supersymmetric quantum dilogarithm. These resulting functions are of the same type as those in the 6j symbols of super Liouville theory. We anticipate that this similarity brings the possibility of a correspondence between quantum super Teichmüller and conformal blocks of super Liouville theory.

Kashaev derived the R-matrix, associated with braidings in the mapping class groups, in terms of the non-compact quantum dilogarithm, which first has been suggested by Faddeev as the universal $U_q(sl(2))$ R-matrix for the corresponding modular double. Kashaev established that the more general formula directly follows from the embedding of the Drinfeld doubles of Hopf algebras into tensor product of two Heisenberg doubles in [43] and he presented a geometrical interpretation [44]. In the super Teichmüller theory we already derived the geometrical view of the R matrix associated with braidings in the mapping class groups. The R matrix is derived in terms of the non-compact super quantum dilogarithm. The goal of an ongoing project is to find how this R matrix follows from the canonical embedding of the Drinfeld doubles Hopf superalgebras.

There are a number of issues which would be interesting to investigate as follow up work. It is known that ordinary Teichmüller theory is closely related to non-supersymmetric Liouville theory [20]. In particular, the spaces of Liouville conformal blocks and the spaces of states of Teichmüller theory of Riemann surfaces can be identified [14] and carry unitary equivalent representations of the mapping class group. In the case of $\mathcal{N}=1$ supersymmetric Liouville theory, the mapping class group representation for genus 0 can be represented using the fusion and braiding matrices,and has been investigated in [90, 91]. It would be interesting to study more closely the mapping class group representation defined by the representation of the super Ptolemy groupoid constructed in this text, and relate it to $\mathcal{N}=1$ supersymmetric Liouville theory.

Moreover, ordinary Teichmüller theory is the connected component of the space of $SL(2,\mathbb{R})$-valued flat connections on a Riemann surface Σ, and therefore closely related to $SL(2,\mathbb{R})$-Chern-Simons theory on $\Sigma\times\mathbb{R}$. The goal of one ongoing project is to investigate the connections between the quantum super Teichmüller theory described here and the quantum $OSp(1|2)$-Chern-Simons theory. One can note that a topological field theory on a 3-dimensional manifold can be constructed by using Teichmüller theory [92]. One can associate the flip operator of Teichmüller space to a tetrahedron of the triangulated 3 manifold. Since the flip operator is the canonical element of Heisenberg double, it satisfies the pentagon relation. Therefore, the partition function obtained by gluing tetrahedra together does not change by choosing a different triangulation of the three-manifold. This means that there exists an invariant under the 2-3 Pachner move, which follows from the pentagon like identity. An important implication of this thesis is that, one can use the supersymmetric flip operator to derive the invariant of spin three-manifolds from super Teichmüller theory. We can further anticipate that the recent work of Kapustin and Gaiotto [93] and also Petronio and Benedetti [94] might help us to find the proper way of encoding the spin structure. Then, one needs to show that the partition function is invariant under different ideal triangulations of hyperbolic spin three-manifolds.

Another direction where one may use the result which was presented here, is the study of integrability and quantum discrete super Liouville model. Liouville theory is interesting due to its connection with noncritical string theory [95] and two-dimensional quantum gravity [96]. It is an example of nonrational CFTs [8, 97] and has relation to the quantized Teichmüller spaces of Riemann surfaces [15, 98]. Integrable lattice regularization of quantum Liouville theory has been studied in the '80 [98], and later on in [99, 100]. The model was developed more recently by Kashaev and Faddeev [50]. According to [50], the model describes the region corresponding to the strongly coupled regime ($1<c<25$ where, c is the Virasoro central charge of the Liouville theory). Then, in the context of the discrete Liouville model, it was shown that the N-th power of the light-cone evolution operator of the model can be interpreted in pure geometrical terms within quantum Teichmüller theory as the Dehn twist operator. Another possible research direction based on this dissertation can be understanding the geometric realization of Dehn twist in the formalism of super Teichmüller theory and derive the light-cone evolution operator in the super case.

Appendix A

Non-compact quantum dilogarithm

Quantum dilogarithm plays a key role in this project. Here we review the non-compact quantum dilogarithm and its most important properties. We collected the different definitions of relative special functions which the reader may face in the related references of this thesis.

The basic building block for the class of special functions to be considered is the Double Gamma function introduced by Barnes [101]. The Double Gamma function is defined as

$$log\Gamma_2(z|\omega) := \Big(\frac{\partial}{\partial s}\sum_{m_1,m_2\in\mathbb{Z}_{\geq 0}}(z+m_1\omega_1+m_2\omega_2)^{-s}\Big)_{s=0},$$

and there exists the definition:

$$\Gamma_b(x) := \Gamma_2(x|b,b^{-1}).$$

For $\mathfrak{Re}x > 0$ it admits an integral representation

$$\log\Gamma_{\mathrm{b}}(x) = \int_0^\infty \frac{\mathrm{d}t}{t}\left[\frac{e^{-xt}-e^{-\frac{Q}{2}t}}{(1-e^{-t\mathrm{b}})(1-e^{-\frac{t}{\mathrm{b}}})} - \frac{\left(\frac{Q}{2}-x\right)^2}{2e^t} - \frac{\frac{Q}{2}-x}{t}\right],$$

where $Q = \mathrm{b}+\frac{1}{\mathrm{b}}$. One can analytically continue Γ_{b} to a meromorphic function defined on the entire complex plane $\mathbb{C}$. The most important property of Γ_{b} is its behavior with respect to shifts by $\mathrm{b}^{\pm}$,

$$\Gamma_{\mathrm{b}}(x+\mathrm{b}) = \frac{\sqrt{2\pi}\mathrm{b}^{\mathrm{b}x-\frac{1}{2}}}{\Gamma_{\mathrm{b}}(bx)}\Gamma_{\mathrm{b}}(x) \quad , \quad \Gamma_{\mathrm{b}}(x+\mathrm{b}^{-1}) = \frac{\sqrt{2\pi}\mathrm{b}^{-\frac{\mathrm{b}}{x}+\frac{1}{2}}}{\Gamma_{\mathrm{b}}(\frac{x}{\mathrm{b}})}\Gamma_{\mathrm{b}}(x)\ . \tag{A.1}$$

These shift equations allow us to calculate residues of the poles of $\Gamma_{\rm b}$. When $x \to 0$, for instance, one finds

$$\Gamma_{\rm b}(x) = \frac{\Gamma_{\rm b}(Q)}{2\pi x} + O(1). \tag{A.2}$$

From Barnes' Double Gamma function we can build other important special functions,

$$\Upsilon_{\rm b}(x) := \frac{1}{\Gamma_{\rm b}(x)\Gamma_{\rm b}(Q-x)}, \tag{A.3}$$

$$S_{\rm b}(x) := \frac{\Gamma_{\rm b}(x)}{\Gamma_{\rm b}(Q-x)}, \tag{A.4}$$

$$G_{\rm b}(x) := e^{-\frac{i\pi}{2}x(Q-x)}S_{\rm b}(x), \tag{A.5}$$

$$w_{\rm b}(x) := e^{\frac{\pi i}{2}(\frac{Q^2}{4}+x^2)}G_{\rm b}(\frac{Q}{2}-ix), \tag{A.6}$$

$$g_{\rm b}(x) := \frac{\zeta_b}{G_{\rm b}(\frac{Q}{2}+\frac{1}{2\pi ib}logx)}, \tag{A.7}$$

We shall often refer to the function $S_{\rm b}$ as double sine function. It is defined by the following integral representation,

$$\log S_{\rm b}(x) = \int_0^\infty \frac{dt}{it}(\frac{\sin 2xt}{2\sinh bt \sinh b^{-1}t} - \frac{x}{t}) \tag{A.8}$$

The $S_{\rm b}$ function is meromorphic with poles and zeros in

$$S_{\rm b}(x) = 0 \Leftrightarrow x = Q + nb + mb^{-1}, \qquad n, m \in \mathbb{Z}_{\geq 0} \ ,$$

$$S_{\rm b}(x)^{-1} = 0 \Leftrightarrow x = -nb - mb^{-1}, \qquad n, m \in \mathbb{Z}_{\geq 0} \ .$$

Other most important properties for this text are as follows:

Functional equation(Shift): $S_{\rm b}(x - ib/2) = 2\cosh(\pi bx)S_{\rm b}(x + ib/2)$ (A.9)

Self-duality: $S_{\rm b}(x) = S_{1/{\rm b}}(x)$ (A.10)

Inversion relation(Reflection): $S_{\rm b}(x)S_{\rm b}(-x) = 1$ (A.11)

Unitarity: $\overline{S_{\rm b}(x)} = 1/S_{\rm b}(\overline{x})$ (A.12)

Residue: $\text{res}_{x=c_b}S_{\rm b}(x) = e^{-\frac{i\pi}{12}}(1-4c_b{}^2)(2\pi i)^{-1}$ (A.13)

In addition, from the definition of $G_{\rm b}$ (A.5) and the shift property of Barnes' double Gamma function it is easy to derive the following shift and reflection properties of $G_{\rm b}$,

$$G_{\rm b}(x + {\rm b}) = (1 - e^{2\pi i{\rm b}x})G_{\rm b}(x) \ , \tag{A.14}$$

$$G_{\rm b}(x)G_{\rm b}(Q - x) = e^{\pi ix(x-Q)} \ . \tag{A.15}$$

The Fadeev's quantum dilogarithm function is defined by the following integral representation

$$e_{\rm b}(x) = \exp\left[\int_{\mathbb{R}_{i0}} \frac{{\rm d}w}{w}\frac{e^{-2ixw}}{4\sinh(wb)\sinh(w/b)}\right], \tag{A.16}$$

and it is related to the Double sine function in a way as follows

$$e_{\rm b}(x) = e^{\frac{\pi i}{2}x^2} e^{\frac{-\pi i}{24}(2-Q^2)} S_{\rm b} = \ A G_{\rm b}^{-1}(-ix + \frac{Q}{2}), \tag{A.17}$$

where

$$A \ = \ e^{-i\pi(1-4c_{\rm b}^2)/12} \quad , \quad c_{\rm b} = iQ/2 \ . \tag{A.18}$$

The function $e_b(x)$ introduced by the name of "quantum dilogarithm" in [45], "quantum exponential function" in [102] and Hyperbolic G function in [103]. This function has similar properties like shift and reflection relations. The properties which it satisfies are as follows:

Simple poles and zeros:

$$\begin{aligned} \text{poles} =& \{iQ/2 + imb + inb^{-1}, \quad m, n \in \mathbb{Z}_{\geq 0}\} \\ \text{zeros} =& \{-iQ/2 - imb - inb^{-1}, \quad m, n \in \mathbb{Z}_{\geq 0}\} \end{aligned}$$

The asymptotic behavior of the function $e_{\rm b}$ along the real axis

$$e_{\rm b}(z) = \begin{cases} 1 & , x \to -\infty \\ e^{-i\pi(1+2c_b^2)/6} e^{i\pi x^2} & , x \to +\infty \end{cases} \tag{A.19}$$

Functional equation(Shift): $$e_{\rm b}\left(x - \frac{ib^{\pm 1}}{2}\right) = (1 + e^{2\pi b^{\pm 1} x}) e_{\rm b}\left(x + \frac{ib^{\pm 1}}{2}\right), \tag{A.20}$$

Inversion relation(Reflection): $$e_{\rm b}(x) e_{\rm b}(-x) = e^{-i\pi(1+2c_b^2)/6} e^{i\pi x^2}, \tag{A.21}$$

Residue: $$\text{res}_{x=c_b} e_{\rm b} = (2\pi i)^{-1} \tag{A.22}$$

Product representation $$e_b(z) = \frac{(-qe^{2\pi z b}; q^2)_\infty}{(-\tilde{q}e^{2\pi z b^{-1}}; \tilde{q}^2)_\infty}, \qquad Imb^2 > 0 \tag{A.23}$$

where $q = e^{i\pi b^2}, \quad \tilde{q} = e^{-i\pi b^{-2}}, \quad (x, q)_\infty = \Pi_{k=0}^{\infty}(1 - q^k x).$

One can find the graphs of quantum dilogarithm and visualization of its analytic and asymptotic behaviors in [104].

Appendix B

Supersymmetric non-compact quantum dilogarithm

When discussing the supersymmetric Teichmüller theory we need the following additional special functions

$$\Gamma_1(x) = \Gamma_{\rm NS}(x) = \Gamma_{\rm b}\left(\frac{x}{2}\right)\Gamma_{\rm b}\left(\frac{x+Q}{2}\right), \tag{B.1}$$

$$\Gamma_0(x) = \Gamma_{\rm R}(x) = \Gamma_{\rm b}\left(\frac{x+{\rm b}}{2}\right)\Gamma_{\rm b}\left(\frac{x+{\rm b}^{-1}}{2}\right). \tag{B.2}$$

Furthermore, let us define

$$\begin{aligned} S_1(x) &= S_{\rm NS}(x) = \tfrac{\Gamma_{\rm NS}(x)}{\Gamma_{\rm NS}(Q-x)}, \qquad & G_1(x) &= G_{\rm NS}(x) = \zeta_0 e^{-\frac{i\pi}{4}x(Q-x)} S_{\rm NS}(x), \\ S_0(x) &= S_{\rm R}(x) = \tfrac{\Gamma_{\rm R}(x)}{\Gamma_{\rm R}(Q-x)}, \qquad & G_0(x) &= G_{\rm R}(x) = e^{-\frac{i\pi}{4}}\zeta_0 e^{-\frac{i\pi}{4}x(Q-x)} S_{\rm R}(x), \end{aligned} \tag{B.3}$$

where $\zeta_0 = \exp(-i\pi Q^2/8)$. As for $S_{\rm b}$, the functions $S_0(x)$ and $S_1(x)$ are meromorphic with poles and zeros in

$$\begin{aligned} S_0(x) = 0 &\Leftrightarrow x = Q + nb + mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m+n \in 2\mathbb{Z}+1, \\ S_1(x) = 0 &\Leftrightarrow x = Q + nb + mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m+n \in 2\mathbb{Z}, \\ S_0(x)^{-1} = 0 &\Leftrightarrow x = -nb - mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m+n \in 2\mathbb{Z}+1, \\ S_1(x)^{-1} = 0 &\Leftrightarrow x = -nb - mb^{-1}, \quad n, m \in \mathbb{Z}_{\geq 0}, m+n \in 2\mathbb{Z}. \end{aligned}$$

As in the previous subsection, we want to state the shift and reflection properties of the functions G_1 and G_0,

$$G_\nu(x + b^{\pm 1}) = (1 - (-1)^\nu e^{\pi i {\rm b}^{\pm 1} x}) G_{\nu+1}(x), \tag{B.4}$$

$$G_\nu(x) G_\nu(Q-x) = e^{\frac{i\pi}{2}(\nu-1)} \zeta_0^2 e^{\frac{\pi i}{2}x(x-Q)}. \tag{B.5}$$

where $\nu = 0, 1$.

We define the supersymmetric analogues of Fadeev's quantum dilogarithm function as

$$e_{\mathrm{R}}(x) = e_{\mathrm{b}}\left(\frac{x + i(b - b^{-1})/2}{2}\right) e_{\mathrm{b}}\left(\frac{x - i(b - b^{-1})/2}{2}\right), \tag{B.6}$$

$$e_{\mathrm{NS}}(x) = e_{\mathrm{b}}\left(\frac{x + c_{\mathrm{b}}}{2}\right) e_{\mathrm{b}}\left(\frac{x - c_{\mathrm{b}}}{2}\right), \tag{B.7}$$

and relate them to the double sine function in a way as follows

$$e_{\nu}(x) = \ A^2 G_{\nu}^{-1}(-ix + \frac{Q}{2}), \tag{B.8}$$

with a constant A as defined in eq. (A.18). The shift and reflection relations that it satisfies are as follows

$$e_{\mathrm{R}}\left(x - \frac{ib^{\pm 1}}{2}\right) = (1 + ie^{\pi b^{\pm 1} x}) e_{\mathrm{NS}}\left(x + \frac{ib^{\pm 1}}{2}\right), \tag{B.9}$$

$$e_{\mathrm{NS}}\left(x - \frac{ib^{\pm 1}}{2}\right) = (1 - ie^{\pi b^{\pm 1} x}) e_{\mathrm{R}}\left(x + \frac{ib^{\pm 1}}{2}\right), \tag{B.10}$$

$$e_{\mathrm{NS}}(x) e_{\mathrm{NS}}(-x) = e^{i\pi c_{\mathrm{b}}^2/2} e^{-i\pi(1+2c_b^2)/3} e^{i\pi x^2/2}, \tag{B.11}$$

$$e_{\mathrm{R}}(x) e_{\mathrm{R}}(-x) = e^{i\pi/2} e^{i\pi c_{\mathrm{b}}^2/2} e^{-i\pi(1+2c_b^2)/3} e^{i\pi x^2/2}. \tag{B.12}$$

Asymptotically, the functions e_1 and e_0 behave as

$$e_{\mathrm{NS}}(z) = \begin{cases} 1 & , x \to -\infty \\ e^{i\pi c_{\mathrm{b}}^2/2} e^{-i\pi(1+2c_b^2)/3} e^{i\pi x^2/2} & , x \to +\infty \end{cases} \tag{B.13}$$

$$e_{\mathrm{R}}(z) = \begin{cases} 1 & , x \to -\infty \\ e^{i\pi/2} e^{i\pi c_{\mathrm{b}}^2/2} e^{-i\pi(1+2c_b^2)/3} e^{i\pi x^2/2} & , x \to +\infty \end{cases} \tag{B.14}$$

Also, we know that for non-commutative variables P, X such that $[\mathsf{P}, \mathsf{X}] = \frac{1}{\pi i}$ they satisfy four pentagon relations

$$f_+(\mathsf{P}) f_+(\mathsf{X}) = f_+(\mathsf{X}) f_+(\mathsf{X} + \mathsf{P}) f_+(\mathsf{P}) - i f_-(\mathsf{X}) f_-(\mathsf{X} + \mathsf{P}) f_-(\mathsf{P}), \tag{B.15}$$

$$f_+(\mathsf{P}) f_-(\mathsf{X}) = -i f_+(\mathsf{X}) f_-(\mathsf{X} + \mathsf{P}) f_-(\mathsf{P}) + f_-(\mathsf{X}) f_+(\mathsf{X} + \mathsf{P}) f_+(\mathsf{P}), \tag{B.16}$$

$$f_-(\mathsf{P}) f_+(\mathsf{X}) = f_+(\mathsf{X}) f_+(\mathsf{X} + \mathsf{P}) f_-(\mathsf{P}) - i f_-(\mathsf{X}) f_-(\mathsf{X} + \mathsf{P}) f_+(\mathsf{P}), \tag{B.17}$$

$$f_-(\mathsf{P}) f_-(\mathsf{X}) = i f_+(\mathsf{X}) f_-(\mathsf{X} + \mathsf{P}) f_+(\mathsf{P}) - f_-(\mathsf{X}) f_+(\mathsf{X} + \mathsf{P}) f_-(\mathsf{P}), \tag{B.18}$$

where $f_{\pm}(x) = e_{\mathrm{R}}(x) \pm e_{\mathrm{NS}}(x)$. Those pentagon equations can be equivalently expressed as the supersymmetric analogues of Ramanujan summation formulae

$$\sum_{\sigma=0,1} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\tau}{i} (-1)^{\rho_\beta \sigma} e^{\pi i \tau \beta} \frac{G_{\sigma+\rho_\alpha}(\tau + \alpha)}{G_{\sigma+1}(\tau + Q)} = 2\zeta_0^{-1} \frac{G_{\rho_\alpha}(\alpha) G_{1+\rho_\beta}(\beta)}{G_{\rho_\alpha+\rho_\beta}(\alpha + \beta)}. \tag{B.19}$$

which have been derived in [80].

One can use the connection between $e_{\mathrm{R}}, e_{\mathrm{NS}}$ and $G_{\mathrm{R}}, G_{\mathrm{NS}}$ to prove the Ramanujan formulae based on the results from [80]. Here we show the details for the proof of one of those equations

$$
\begin{aligned}
LHS &= \int_{\mathbb{R}} \left(\frac{e_{\mathrm{NS}}(x+u)}{e_{\mathrm{NS}}(x+v)} + \frac{e_{\mathrm{R}}(x+u)}{e_{\mathrm{R}}(x+v)} \right) e^{\pi i w x} \mathrm{d}x = \\
&= \int \left(\frac{G_{\mathrm{NS}}(-i(x+v)+Q/2)}{G_{\mathrm{NS}}(-i(x+u)+Q/2)} + \frac{G_{\mathrm{R}}(-i(x+v)+Q/2)}{G_{\mathrm{R}}(-i(x+u)+Q/2)} \right) e^{\pi i w x} \mathrm{d}x = \\
&= \int_{i\mathbb{R}} \left(\frac{G_{\mathrm{NS}}(\tau - iv + Q/2)}{G_{\mathrm{NS}}(\tau - iu + Q/2)} + \frac{G_{\mathrm{R}}(\tau - iv + Q/2)}{G_{\mathrm{R}}(\tau - iu + Q/2)} \right) e^{\pi i \tau (iw)} \frac{\mathrm{d}\tau}{i} = \\
&= \int_{i\mathbb{R}} \left(\frac{G_{\mathrm{NS}}(\tau + (Q/2 - iv) + (Q/2 + iu))}{G_{\mathrm{NS}}(\tau + Q)} + \frac{G_{\mathrm{R}}(\tau + (Q/2 - iv) + (Q/2 + iu))}{G_{\mathrm{R}}(\tau + Q)} \right) \times \\
&\qquad \times e^{\pi i \tau (iw)} e^{\pi i (Q/2 + iu)(iw)} \frac{\mathrm{d}\tau}{i} = \\
&= 2e^{-i\pi c_{\mathrm{b}}^2/2} \frac{G_{\mathrm{NS}}(iw) G_{\mathrm{NS}}(Q + iu - iv)}{G_{\mathrm{NS}}(Q + iu - iv + iw)} e^{-\pi w (Q/2 + iu)} = \\
&= 2e^{-i\pi c_{\mathrm{b}}^2/2} \frac{A^4}{A^2} \frac{e_{\mathrm{NS}}(iQ - u + v - w - c_{\mathrm{b}})}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}}) e_{\mathrm{NS}}(iQ - u + v - c_{\mathrm{b}})} e^{-\pi w (Q/2 + iu)} = \\
&= 2e^{-i\pi c_{\mathrm{b}}^2/2} A^2 \frac{e_{\mathrm{NS}}(-u + v - w + c_{\mathrm{b}})}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}}) e_{\mathrm{NS}}(-u + v + c_{\mathrm{b}})} e^{-\pi w (Q/2 + iu)} = \\
&= \frac{e_{\mathrm{NS}}(-u + v - w + c_{\mathrm{b}})}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}}) e_{\mathrm{NS}}(-u + v + c_{\mathrm{b}})} e^{-\pi i w (u - c_{\mathrm{b}})} \left(2e^{-i\pi c_{\mathrm{b}}^2/2} A^2 \right) = RHS,
\end{aligned}
$$

where $\bar{B} = 2e^{-i\pi c_{\mathrm{b}}^2/2} A^2$ and $B = 2e^{i\pi c_{\mathrm{b}}^2/2} A^{-2}$.

Appendix C

Pentagon and superpentagon relation

In the first part of this appendix we explain the proof of the pentagon relation which was explained in [50] more extensively. In the second part we follow their line for proving superpentagon relations.

C.1 Pentagon identity

In appendix A we explained the properties of quantum dilogarithm functions. In this part first we express the Ramanujan formula based on which, we find the Fourier transformation of the quantum dilogarithm. The Fourier transformation will help us prove the pentagon relation afterwards. At the very end we explain the proof of the Ramanujan formula.

According to [44], the Ramanujan summation formula states that

$$\int \mathrm{d}x e^{2\pi i x(w-c_{\mathrm{b}})} \frac{e_{\mathrm{b}}(x+a)}{e_{\mathrm{b}}(x-c_{\mathrm{b}})} = e^{i\pi(1-4c_{\mathrm{b}}^2)/12} \frac{e_{\mathrm{b}}(a)e_{\mathrm{b}}(w)}{e_{\mathrm{b}}(a+w-c_{\mathrm{b}})}. \tag{C.1}$$

First by complex conjugating we have

$$\int \mathrm{d}x e^{-2\pi i x(w-c_{\mathrm{b}})} \frac{e_{\mathrm{b}}(x+c_{\mathrm{b}})}{e_{\mathrm{b}}(x+\bar{a})} = e^{-i\pi(1-4c_{\mathrm{b}}^2)/12} \frac{e_{\mathrm{b}}(\bar{a}+\bar{w}+c_{\mathrm{b}})}{e_{\mathrm{b}}(\bar{a})e_{\mathrm{b}}(\bar{w})},$$

with the use of the fact that $\overline{e_{\mathrm{b}}(x)} = e_{\mathrm{b}}^{-1}(\bar{x})$ and change of variables $x \to x - c_{\mathrm{b}} + u$,$w \to -w + c_{\mathrm{b}}$ and $\bar{a} \to v - u + c_{\mathrm{b}}$ on can rewrite the formula as

$$\int \mathrm{d}x e^{2\pi i x w} \frac{e_{\mathrm{b}}(x+u)}{e_{\mathrm{b}}(x+v)} = e^{-i\pi(1-4c_{\mathrm{b}}^2)/12} e^{-2\pi i w(u-c_{\mathrm{b}})} \frac{e_{\mathrm{b}}(v-u-w+c_{\mathrm{b}})}{e_{\mathrm{b}}(v-u+c_{\mathrm{b}})e_{\mathrm{b}}(-w-c_{\mathrm{b}})}, \tag{C.2}$$

Fourier transform

By using the Ramanujan formula (C.2), taking a limit $v \to -\infty$ one gets

$$\int \mathrm{d}x e^{2\pi i x w} e_{\mathrm{b}}(x+u) = e^{-i\pi(1-4c_{\mathrm{b}}^2)/12} e^{-2\pi i w(u-c_{\mathrm{b}})} \frac{1}{e_{\mathrm{b}}(-w-c_{\mathrm{b}})},$$

and after setting $u=0$:

$$\int \mathrm{d}x e^{2\pi i x w} e_{\mathrm{b}}(x) = e^{-i\pi(1-4c_{\mathrm{b}}^2)/12} e^{2\pi i w c_{\mathrm{b}}} \frac{1}{e_{\mathrm{b}}(-w-c_{\mathrm{b}})}.$$

Then, Fourier transform of the quantum dilogarithm is [50]

$$\begin{aligned}\phi_+(w) = \int e_{\mathrm{b}}(x) e^{2\pi i w x} \mathrm{d}x = \quad &(C.3)\\ = e_{\mathrm{b}}^{-1}(-w-c_{\mathrm{b}}) e^{2\pi i w c_{\mathrm{b}}} e^{-i\pi(1-4c_{\mathrm{b}}^2)/12} = \\ = e_{\mathrm{b}}(w+c_{\mathrm{b}}) e^{-\pi(w+c_{\mathrm{b}})^2} e^{+i\pi(1+2c_{\mathrm{b}}^2)/6} e^{2\pi i w c_{\mathrm{b}}} e^{-i\pi(1-4c_{\mathrm{b}}^2)/12} = \\ = e_{\mathrm{b}}(w+c_{\mathrm{b}}) e^{-i\pi w^2} e^{i\pi(1-4c_{\mathrm{b}}^2)/12}. \quad &(C.4)\end{aligned}$$

Then, the inverse transform is

$$e_{\mathrm{b}}(x) = \int \mathrm{d}y \phi_+(y) e^{-2\pi i x y}.$$

Moreover, one can take the limit $u \to -\infty$ of (C.2),

$$\phi_-(w) = \int e_{\mathrm{b}}^{-1}(x) e^{2\pi i w x} \mathrm{d}x = e^{i\pi w^2 - i\pi(1-4c_{\mathrm{b}}^2)/12} \frac{1}{e_{\mathrm{b}}(-w-c_{\mathrm{b}})}.$$

The inverse transform is

$$(e_{\mathrm{b}}(x))^{-1} = \int \mathrm{d}y \phi_-(y) e^{-2\pi i x y}.$$

Pentagon identity

Consider operators X, P which canonically commute $[P, X] = \frac{1}{2\pi i}$. The pentagon identity states that

$$e_{\mathrm{b}}(P) e_{\mathrm{b}}(X) = e_{\mathrm{b}}(X) e_{\mathrm{b}}(X+P) e_{\mathrm{b}}(P). \quad (C.5)$$

In order to prove that, one first performs a Fourier transform

$$\begin{aligned}
LHS &= e_{\rm b}(P)e_{\rm b}(X) = \int \mathrm{d}x\mathrm{d}y\phi_+(x)e^{-2\pi i xP}\phi_+(y)e^{-2\pi i yX} = \\
&= \int \mathrm{d}x\mathrm{d}y\phi_+(x)\phi_+(y)e^{-2\pi i xP}e^{-2\pi i yX} = \int \mathrm{d}x\mathrm{d}y\phi_+(x)\phi_+(y)e^{-2\pi i yX}e^{-2\pi i xP}e^{2\pi i yx} \\
RHS &= e_{\rm b}(X)e_{\rm b}(X+P)e_{\rm b}(P) = \\
&= \int \mathrm{d}x\mathrm{d}y\mathrm{d}z\phi_+(x)\phi_+(z)\phi_+(y)e^{-2\pi i yX}e^{-2\pi i z(X+P)}e^{-2\pi i xP} = \\
&= \int \mathrm{d}x\mathrm{d}y\mathrm{d}z\phi_+(x-z)\phi_+(z)\phi_+(y-z)e^{-2\pi i(y-z)X}e^{-2\pi i z(X+P)}e^{-2\pi i(x-z)P} \\
&= \int \mathrm{d}x\mathrm{d}y\mathrm{d}z\phi_+(x-z)\phi_+(z)\phi_+(y-z)e^{-2\pi i yX}e^{-2\pi i xP}e^{i\pi z^2}.
\end{aligned}$$

Now we try to show the left and right hand side are equal.

$$\int \mathrm{d}x\mathrm{d}y\phi_+(x)\phi_+(y)e^{-2\pi i yX}e^{-2\pi i xP}e^{2\pi i xy} = \int \mathrm{d}x\mathrm{d}y\mathrm{d}z\phi_+(x-z)\phi_+(z)\phi_+(y-z)e^{-2\pi i yX}e^{-2\pi i xP}e^{i\pi z^2}.$$

We can drop first the integrations and then multiply by $e^{-2\pi i yu}$ and integrate over y.

$$\begin{aligned}
\phi_+(x)\phi_+(y)e^{-2\pi i yX}e^{-2\pi i xP}e^{2\pi i xy} &= \int \mathrm{d}z\phi_+(x-z)\phi_+(z)\phi_+(y-z)e^{-2\pi i yX}e^{-2\pi i xP}e^{i\pi z^2}. \\
\int \mathrm{d}y\phi_+(x)\phi_+(y)e^{2\pi i y(x-u)} &= \int \mathrm{d}y\mathrm{d}z\phi_+(x-z)\phi_+(z)\phi_+(y-z)e^{i\pi z^2}e^{-2\pi i yu}.
\end{aligned}$$

We can use the identities for the inverse Fourier transforms for $\phi_+(y)$ and $\phi_+(y-z)$. Therefore, we have

$$\phi_+(x)e_{\rm b}(u-x) = \int \mathrm{d}z\phi_+(x-z)\phi_+(z)e_{\rm b}(u)e^{i\pi z^2}e^{-2\pi i zu}. \tag{C.6}$$

Next, we use the Fourier transforms (C.4) for all ϕ_+ functions on the LHS and RHS of (C.6)

$$\begin{aligned}
\frac{e_{\rm b}(u-x)}{e_{\rm b}(-x-c_{\rm b})}e^{2\pi i x c_{\rm b}}e^{-i\pi(1-4c_{\rm b}^2)/12} &= e_{\rm b}(u)\int \mathrm{d}z e^{-2\pi i uz}e^{2\pi i(x-z)c_{\rm b}}\frac{e_{\rm b}(z+c_{\rm b})}{e_{\rm b}(z-x-c_{\rm b})} \\
\frac{e_{\rm b}(u-x)}{e_{\rm b}(-x-c_{\rm b})e_{\rm b}(u)}e^{-i\pi(1-4c_{\rm b}^2)/12} &= \int \mathrm{d}z e^{-2\pi i z(u+c_{\rm b})}\frac{e_{\rm b}(z+c_{\rm b})}{e_{\rm b}(z-x-c_{\rm b})}
\end{aligned}$$

If we rewrite $u \to c_{\rm b}$, $v \to -x-c_{\rm b}$ and $w \to -u-c_{\rm b}$ then we derivethe Ramanujan summation formula (C.2). Therefore, we observe that pentagon and Ramanujan summation can be derived from each other.

C.2 Super pentagon identity

In this section we want to generalize the supersymmetric equivalents of pentagon. We start from the Ramanujan summation formula and the Fourier transforms of super dilogarithm functions and by using them we will be able to prove the superpentagon relation.

Ramanujan summation formulae

Being inspired by [80], the Ramanujan summation formulae state that

$$\int \mathrm{d}x e^{\pi i x(w-c_\mathrm{b})}\left(\frac{e_\mathrm{NS}(x+a)}{e_\mathrm{NS}(x-c_\mathrm{b})}+\frac{e_\mathrm{R}(x+a)}{e_\mathrm{R}(x-c_\mathrm{b})}\right)=B\frac{e_\mathrm{NS}(a)e_\mathrm{NS}(w)}{e_\mathrm{NS}(a+w-c_\mathrm{b})} \tag{C.7}$$

$$\int \mathrm{d}x e^{\pi i x(w-c_\mathrm{b})}\left(\frac{e_\mathrm{NS}(x+a)}{e_\mathrm{NS}(x-c_\mathrm{b})}-\frac{e_\mathrm{R}(x+a)}{e_\mathrm{R}(x-c_\mathrm{b})}\right)=B\frac{e_\mathrm{NS}(a)e_\mathrm{R}(w)}{e_\mathrm{R}(a+w-c_\mathrm{b})} \tag{C.8}$$

$$\int \mathrm{d}x e^{\pi i x(w-c_\mathrm{b})}\left(\frac{e_\mathrm{R}(x+a)}{e_\mathrm{NS}(x-c_\mathrm{b})}+\frac{e_\mathrm{NS}(x+a)}{e_\mathrm{R}(x-c_\mathrm{b})}\right)=B\frac{e_\mathrm{R}(a)e_\mathrm{NS}(w)}{e_\mathrm{R}(a+w-c_\mathrm{b})} \tag{C.9}$$

$$\int \mathrm{d}x e^{\pi i x(w-c_\mathrm{b})}\left(\frac{e_\mathrm{R}(x+a)}{e_\mathrm{NS}(x-c_\mathrm{b})}-\frac{e_\mathrm{NS}(x+a)}{e_\mathrm{R}(x-c_\mathrm{b})}\right)=B\frac{e_\mathrm{R}(a)e_\mathrm{R}(w)}{e_\mathrm{NS}(a+w-c_\mathrm{b})}, \tag{C.10}$$

where

$$B=2e^{i\pi c_\mathrm{b}^2/2}e^{i\pi(1-4c_b^2)/6}.$$

As in the non-supersymmetric case, we can use complex conjugation and use such relations as $\overline{e_\mathrm{R}(x)}=e_\mathrm{R}^{-1}(\bar{x})$ and $\overline{e_\mathrm{NS}(x)}=e_\mathrm{NS}^{-1}(\bar{x})$. Next, by changing variables $x\to x-c_\mathrm{b}+u$, $w\to -w+c_\mathrm{b}$ and $\bar{a}\to v-u+c_\mathrm{b}$ one can rewrite the Ramanujan summation as

$$\int \mathrm{d}x e^{i\pi x w}\left(\frac{e_\mathrm{NS}(x+u)}{e_\mathrm{NS}(x+v)}+\frac{e_\mathrm{R}(x+u)}{e_\mathrm{R}(x+v)}\right)=\bar{B}e^{-\pi i w(u-c_\mathrm{b})}\frac{e_\mathrm{NS}(v-u-w+c_\mathrm{b})}{e_\mathrm{NS}(v-u+c_\mathrm{b})e_\mathrm{NS}(-w-c_\mathrm{b})}$$

$$\int \mathrm{d}x e^{i\pi x w}\left(\frac{e_\mathrm{NS}(x+u)}{e_\mathrm{NS}(x+v)}-\frac{e_\mathrm{R}(x+u)}{e_\mathrm{R}(x+v)}\right)=\bar{B}e^{-\pi i w(u-c_\mathrm{b})}\frac{e_\mathrm{R}(v-u-w+c_\mathrm{b})}{e_\mathrm{NS}(v-u+c_\mathrm{b})e_\mathrm{R}(-w-c_\mathrm{b})}$$

$$\int \mathrm{d}x e^{i\pi x w}\left(\frac{e_\mathrm{NS}(x+u)}{e_\mathrm{R}(x+v)}+\frac{e_\mathrm{R}(x+u)}{e_\mathrm{NS}(x+v)}\right)=\bar{B}e^{-\pi i w(u-c_\mathrm{b})}\frac{e_\mathrm{R}(v-u-w+c_\mathrm{b})}{e_\mathrm{R}(v-u+c_\mathrm{b})e_\mathrm{NS}(-w-c_\mathrm{b})}$$

$$\int \mathrm{d}x e^{i\pi x w}\left(\frac{e_\mathrm{NS}(x+u)}{e_\mathrm{R}(x+v)}-\frac{e_\mathrm{R}(x+u)}{e_\mathrm{NS}(x+v)}\right)=\bar{B}e^{-\pi i w(u-c_\mathrm{b})}\frac{e_\mathrm{NS}(v-u-w+c_\mathrm{b})}{e_\mathrm{R}(v-u+c_\mathrm{b})e_\mathrm{R}(-w-c_\mathrm{b})},$$

Fourier transform

The Fourier transforms of quantum dilogarithm are

$$\phi_+^\mathrm{NS}(w)=\int e_\mathrm{NS}(x)e^{\pi i w x}\mathrm{d}x, \qquad \phi_+^\mathrm{R}(w)=\int e_\mathrm{R}(x)e^{\pi i w x}\mathrm{d}x. \tag{C.11}$$

By using the Ramanujan formula (C.11), taking a limit $v \to -\infty$ one gets

$$\int \mathrm{d}x e^{i\pi x w} \left(e_{\mathrm{NS}}(x+u) + e_{\mathrm{R}}(x+u)\right) = \bar{B} e^{-\pi i w (u - c_{\mathrm{b}})} \frac{1}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}})},$$
$$\int \mathrm{d}x e^{i\pi x w} \left(e_{\mathrm{NS}}(x+u) - e_{\mathrm{R}}(x+u)\right) = \bar{B} e^{-\pi i w (u - c_{\mathrm{b}})} \frac{1}{e_{\mathrm{R}}(-w - c_{\mathrm{b}})},$$

and after setting $u = 0$:

$$\int \mathrm{d}x e^{i\pi x w} \left(e_{\mathrm{NS}}(x) + e_{\mathrm{R}}(x)\right) = \bar{B} e^{\pi i w c_{\mathrm{b}}} \frac{1}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}})},$$
$$\int \mathrm{d}x e^{i\pi x w} \left(e_{\mathrm{NS}}(x) - e_{\mathrm{R}}(x)\right) = \bar{B} e^{\pi i w c_{\mathrm{b}}} \frac{1}{e_{\mathrm{R}}(-w - c_{\mathrm{b}})},$$

and by summing and reducing them

$$2 \int \mathrm{d}x e^{i\pi x w} e_{\mathrm{NS}}(x) = \bar{B} e^{\pi i w c_{\mathrm{b}}} \left(\frac{1}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}})} + \frac{1}{e_{\mathrm{R}}(-w - c_{\mathrm{b}})} \right),$$
$$2 \int \mathrm{d}x e^{i\pi x w} e_{\mathrm{R}}(x) = \bar{B} e^{\pi i w c_{\mathrm{b}}} \left(\frac{1}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}})} - \frac{1}{e_{\mathrm{R}}(-w - c_{\mathrm{b}})} \right).$$

Additionally, one can rewrite

$$\begin{aligned}
\phi_+^{\mathrm{NS}}(w) &= \frac{1}{2} \bar{B} e^{\pi i w c_{\mathrm{b}}} \left(\frac{1}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}})} + \frac{1}{e_{\mathrm{R}}(-w - c_{\mathrm{b}})} \right) = \\
&= \frac{1}{2} \bar{B} e^{i\pi(1 - c_{\mathrm{b}}^2)/3} e^{-i\pi w^2/2} \left(e_{\mathrm{NS}}(w + c_{\mathrm{b}}) - i e_{\mathrm{R}}(w + c_{\mathrm{b}})\right), \\
\phi_+^{\mathrm{R}}(w) &= \frac{1}{2} \bar{B} e^{\pi i w c_{\mathrm{b}}} \left(\frac{1}{e_{\mathrm{NS}}(-w - c_{\mathrm{b}})} - \frac{1}{e_{\mathrm{R}}(-w - c_{\mathrm{b}})} \right) = \\
&= \frac{1}{2} \bar{B} e^{i\pi(1 - c_{\mathrm{b}}^2)/3} e^{-i\pi w^2/2} \left(e_{\mathrm{NS}}(w + c_{\mathrm{b}}) + i e_{\mathrm{R}}(w + c_{\mathrm{b}})\right).
\end{aligned}$$

By taking the limit $u \to -\infty$ we can get

$$\phi_-^{\mathrm{NS}}(w) + \phi_-^{\mathrm{R}}(w) = \int e^{i\pi w x} \left(e_{\mathrm{NS}}^{-1}(x) + e_{\mathrm{R}}^{-1}(x)\right) = \bar{B} e^{i\pi w^2/2} e_{\mathrm{NS}}^{-1}(-w - c_{\mathrm{b}}),$$
$$\phi_-^{\mathrm{NS}}(w) - \phi_-^{\mathrm{R}}(w) = \int e^{i\pi w x} \left(e_{\mathrm{NS}}^{-1}(x) - e_{\mathrm{R}}^{-1}(x)\right) = i\bar{B} e^{i\pi w^2/2} e_{\mathrm{R}}^{-1}(-w - c_{\mathrm{b}}),$$

which can be rewritten

$$\phi_-^{\mathrm{NS}}(w) = \int e^{i\pi w x} e_{\mathrm{NS}}^{-1}(x) = \frac{1}{2} \bar{B} e^{i\pi w^2/2} \left(e_{\mathrm{NS}}^{-1}(-w - c_{\mathrm{b}}) + i e_{\mathrm{R}}^{-1}(-w - c_{\mathrm{b}})\right),$$
$$\phi_-^{\mathrm{R}}(w) = \int e^{i\pi w x} e_{\mathrm{R}}^{-1}(x) = \frac{1}{2} \bar{B} e^{i\pi w^2/2} \left(e_{\mathrm{NS}}^{-1}(-w - c_{\mathrm{b}}) - i e_{\mathrm{R}}^{-1}(-w - c_{\mathrm{b}})\right).$$

The inverse transforms are

$$(e_{\mathrm{b}}(x))^{\pm 1} = \int \mathrm{d}y \phi_{\pm}(y) e^{-\pi i x y}.$$

Pentagon identity

We want to reverse engineer the pentagon from the Ramanujan summation formulae. Lets start with the first equation (C.7):

$$\int \mathrm{d}z e^{i\pi zw}\left(\frac{e_{\mathrm{NS}}(z+c_{\mathrm{b}})}{e_{\mathrm{NS}}(z-x-c_{\mathrm{b}})}+\frac{e_{\mathrm{R}}(z+c_{\mathrm{b}})}{e_{\mathrm{R}}(z-x-c_{\mathrm{b}})}\right)=const\frac{e_{\mathrm{NS}}(u-x)}{e_{\mathrm{NS}}(-x-c_{\mathrm{b}})e_{\mathrm{NS}}(u)}.$$

Using the Fourier transform of the inverse, we get

$$\begin{aligned}LHS &= const\, e_{\mathrm{NS}}(u-x)[\phi_+^{\mathrm{NS}}(x)+\phi_+^{R}(x)],\\ RHS &= const\, e_{\mathrm{NS}}(u)\int \mathrm{d}z e^{i\pi zw}\left([\phi_+^{\mathrm{NS}}(z)+\phi_+^{\mathrm{R}}(z)][\phi_+^{\mathrm{NS}}(x-z)+\phi_+^{\mathrm{R}}(x-z)]+\right.\\ &\quad \left.-i[-\phi_+^{\mathrm{NS}}(z)+\phi_+^{\mathrm{R}}(z)][\phi_+^{\mathrm{NS}}(x-z)-\phi_+^{\mathrm{R}}(x-z)]\right).\end{aligned}$$

Therefore, we have an equation

$$\begin{aligned}\phi_+^{\mathrm{NS}}(u-x)[\phi_+^{\mathrm{NS}}(x)+\phi_+^{R}(x)]e^{i\pi xy} = const\int \mathrm{d}z\phi_+^{\mathrm{NS}}(u)\left([\phi_+^{\mathrm{NS}}(z)+\phi_+^{\mathrm{R}}(z)][\phi_+^{\mathrm{NS}}(x-z)+\phi_+^{\mathrm{R}}(x-z)]+\right.\\ \left.-i[-\phi_+^{\mathrm{NS}}(z)+\phi_+^{\mathrm{R}}(z)][\phi_+^{\mathrm{NS}}(x-z)-\phi_+^{\mathrm{R}}(x-z)]\right)e^{i\pi z^2/2}.\end{aligned}$$

The exponentials are half of those of the non-supersymmetric case, and since we want to use the formula

$$e^{\lambda xP}e^{\lambda yX}=e^{\lambda(y-z)X}e^{\lambda z(X+P)}e^{\lambda(x-z)P}e^{\frac{-\lambda^2 k}{4\pi i}(2xy-z^2)},$$

where $[P,X]=\frac{k}{2\pi i}$. To have appropriate exponentials to take the inverse of the Fourier transform, we have to choose $k=2$ and $\lambda=-i\pi$. Therefore, the first equation is of the form

$$\begin{aligned}[e_{\mathrm{NS}}(P)+e_{\mathrm{R}}(P)]e_{\mathrm{NS}}(X) &= const\, e_{\mathrm{NS}}(X)\left([e_{\mathrm{NS}}(X+P)+e_{\mathrm{R}}(X+P)][e_{\mathrm{NS}}(P)+e_{\mathrm{R}}(P)]+\right.\\ &\quad \left.-i[-e_{\mathrm{NS}}(X+P)+e_{\mathrm{R}}(X+P)][e_{\mathrm{NS}}(P)-e_{\mathrm{R}}(P)]\right),\end{aligned} \tag{C.12}$$

where the constant can be determined exactly. Therefore, summarizing all the rest of pentagons we have

$$\begin{aligned}[e_{\mathrm{NS}}(P)+e_{\mathrm{R}}(P)]e_{\mathrm{R}}(X) &= const\, e_{\mathrm{R}}(X)\left([e_{\mathrm{NS}}(X+P)+e_{\mathrm{R}}(X+P)][e_{\mathrm{NS}}(P)+e_{\mathrm{R}}(P)]+\right.\\ &\quad \left.+i[-e_{\mathrm{NS}}(X+P)+e_{\mathrm{R}}(X+P)][e_{\mathrm{NS}}(P)-e_{\mathrm{R}}(P)]\right),\end{aligned} \tag{C.13}$$

$$\begin{aligned}[e_{\mathrm{NS}}(P)-e_{\mathrm{R}}(P)]e_{\mathrm{R}}(X) &= const\, e_{\mathrm{NS}}(X)\left(-i[-e_{\mathrm{NS}}(X+P)+e_{\mathrm{R}}(X+P)][e_{\mathrm{NS}}(P)+e_{\mathrm{R}}(P)]+\right.\\ &\quad \left.+[e_{\mathrm{NS}}(X+P)+e_{\mathrm{R}}(X+P)][e_{\mathrm{NS}}(P)-e_{\mathrm{R}}(P)]\right),\end{aligned} \tag{C.14}$$

$$\begin{aligned}[e_{\mathrm{NS}}(P)-e_{\mathrm{R}}(P)]e_{\mathrm{NS}}(X) &= const\, e_{\mathrm{R}}(X)\left(-i[-e_{\mathrm{NS}}(X+P)+e_{\mathrm{R}}(X+P)][e_{\mathrm{NS}}(P)+e_{\mathrm{R}}(P)]+\right.\\ &\quad \left.-[e_{\mathrm{NS}}(X+P)+e_{\mathrm{R}}(X+P)][e_{\mathrm{NS}}(P)-e_{\mathrm{R}}(P)]\right).\end{aligned} \tag{C.15}$$

C.3 Proof of Ramanujan formulas

Ordinary case In order to prove Ramanujan formulas, let us define

$$\Psi(a,w)=\int \mathrm{d}x e^{2\pi i x(w-c_b)}\frac{e_b(x+a)}{e_b(x-c_b)},\qquad \Phi(a,w)=\frac{e_b(a)e_b(w)}{e_b(a+w-c_b)}. \tag{C.16}$$

One can consider the shift equations

$$\Psi(a-\frac{ib}{2},w)=\int \mathrm{d}x e^{2\pi i x(w-c_b)}\frac{e_b(x+a-\frac{ib}{2})}{e_b(x-c_b)}=\Psi(a+\frac{ib}{2},w)+e^{2\pi ab}\Psi(a+\frac{ib}{2},w-ib).$$

$$\Psi(a+\frac{ib}{2},w)-\Psi(a+\frac{ib}{2},w-ib)=\int \mathrm{d}x e^{2\pi i x(w-c_b)}(1-e^{2\pi bx})\frac{e_b(x+a+ib/2)}{e_b(x-c_b)}=$$

$$=\int \mathrm{d}x e^{2\pi i x(w-c_b)}\frac{e_b(x+a+ib/2)}{e_b(x-c_b+ib)}=e^{2\pi b(w-c_b)}\Psi(a-\frac{ib}{2},w),$$

since

$$e_b^{-1}(x-c_b)=(1+e^{2\pi b(x-c_b+ib/2)})^{-1}e_b^{-1}(x-c_b+ib)=(1-e^{2\pi bx})^{-1}e_b^{-1}(x-c_b+ib).$$

So, eventually

$$(1+e^{2\pi b(w+a-c_b)})\Psi(a-\frac{ib}{2},w)=(1+e^{2\pi ab})\Psi(a+\frac{ib}{2},w),$$

$$(1+e^{2\pi b(w+a-c_b)})\Psi(a+\frac{ib}{2},w-ib)=(1-e^{2\pi(w-c_b)})\Psi(a+\frac{ib}{2},w).$$

On the other hand, it is easy to check that

$$\Phi(a-\frac{ib}{2},w)=\frac{e_b(a-ib/2)e_b(w)}{e_b(a+w-c_b)}=\frac{1+e^{2\pi ab}}{1+e^{2\pi b(a+w-c_b)}}\Phi(a+\frac{ib}{2},w),$$

$$\Phi(a+\frac{ib}{2},w-ib)=\frac{e_b(a+ib/2)e_b(w-ib)}{e_b(a+w-c_b-ib/2)}=\frac{1-e^{2\pi b(w-c_b)}}{1+e^{2\pi b(a+w-c_b)}}\Phi(a+\frac{ib}{2},w).$$

Taking $b\to b^{-1}$ one gets an additional set of shift equations. Given that one has two doubly periodic functions with the same equations, they have to be equal up to a constant $\Psi(a,w)=C\Phi(a,w)$. Then, one can fix C by evaluating the expression on particular a and w.
Moreover, on can use the connection between e_b and G_b to prove the Ramanujan formula based on the results from [37].

$$LHS=\int_{\mathbb{R}}\frac{e_b(x+u)}{e_b(x+v)}e^{2\pi iwx}\mathrm{d}x=\int\frac{G_b(-i(x+v)+Q/2)}{G_b(-i(x+u)+Q/2)}e^{2\pi iwx}\mathrm{d}x=$$

$$=\int_{i\mathbb{R}}\frac{G_b(\tau-iv+Q/2)}{G_b(\tau-iu+Q/2)}e^{2\pi i\tau(iw)}\frac{\mathrm{d}\tau}{i}=\frac{G_b(iw)G_b(Q+iu-iv)}{G_b(Q+iu-iv+iw)}e^{-2\pi w(Q/2+iu)}=$$

$$=\frac{A^2}{A}\frac{e_b(iQ-u+v-w-c_b)}{e_b(-w-c_b)e_b(iQ-u+v-c_b)}e^{-2\pi w(Q/2+iu)}=$$

$$=\frac{e_b(-u+v-w+c_b)}{e_b(-w-c_b)e_b(-u+v+c_b)}e^{-2\pi iw(u-c_b)}e^{-i\pi(1-4c_b^2)/12}=RHS,$$

where, $\quad Ae^{-2\pi w(Q/2+iu)}=e^{-2\pi iw(u-c_b)}e^{-i\pi(1-4c_b^2)/12}.$

Supersymmetric case : In order to prove the Ramanujan formulas in the supersymmetric case , let us define

$$\Phi^+_{\rm NS}(a,w)=\frac{e_{\rm NS}(a)e_{\rm NS}(w)}{e_{\rm NS}(a+w-c_{\rm b})},\qquad \Phi^-_{\rm NS}(a,w)=\frac{e_{\rm NS}(a)e_{\rm R}(w)}{e_{\rm R}(a+w-c_{\rm b})},$$
$$\Phi^+_{\rm R}(a,w)=\frac{e_{\rm R}(a)e_{\rm NS}(w)}{e_{\rm R}(a+w-c_{\rm b})},\qquad \Phi^-_{\rm R}(a,w)=\frac{e_{\rm R}(a)e_{\rm R}(w)}{e_{\rm NS}(a+w-c_{\rm b})}.$$

$$\Psi^+_{\rm NS}(a,w)=\int {\rm d}x e^{\pi ix(w-c_{\rm b})}\left(\frac{e_{\rm NS}(x+a)}{e_{\rm NS}(x-c_{\rm b})}+\frac{e_{\rm R}(x+a)}{e_{\rm b}(x-c_{\rm R})}\right),$$
$$\Psi^-_{\rm NS}(a,w)=\int {\rm d}x e^{\pi ix(w-c_{\rm b})}\left(\frac{e_{\rm NS}(x+a)}{e_{\rm NS}(x-c_{\rm b})}-\frac{e_{\rm R}(x+a)}{e_{\rm b}(x-c_{\rm R})}\right),$$
$$\Psi^+_{\rm R}(a,w)=\int {\rm d}x e^{\pi ix(w-c_{\rm b})}\left(\frac{e_{\rm R}(x+a)}{e_{\rm NS}(x-c_{\rm b})}+\frac{e_{\rm NS}(x+a)}{e_{\rm b}(x-c_{\rm R})}\right),$$
$$\Psi^-_{\rm R}(a,w)=\int {\rm d}x e^{\pi ix(w-c_{\rm b})}\left(-\frac{e_{\rm R}(x+a)}{e_{\rm NS}(x-c_{\rm b})}+\frac{e_{\rm NS}(x+a)}{e_{\rm b}(x-c_{\rm R})}\right),$$

One can consider the shift equations (omitting intermediate steps)

$$\Psi^\pm_{\rm NS}(a-\frac{ib}{2},w)=\pm\Psi^\pm_{\rm R}(a+\frac{ib}{2},w)\pm ie^{\pi ab}\Psi^\mp_{\rm R}(a+\frac{ib}{2},w-ib),$$
$$\Psi^\pm_{\rm R}(a+\frac{ib}{2},w)+\Psi^\mp_{\rm R}(a+\frac{ib}{2},w-ib)=e^{\pi b(w-c_{\rm b})}\Psi^\pm_{\rm NS}(a-\frac{ib}{2},w),$$
$$\Psi^\pm_{\rm R}(a-\frac{ib}{2},w)=\pm\Psi^+_{\rm NS}(a+\frac{ib}{2},w)\pm ie^{\pi ab}\Psi^\mp_{\rm NS}(a+\frac{ib}{2},w-ib),$$
$$\Psi^\pm_{\rm NS}(a+\frac{ib}{2},w)-\Psi^\mp_{\rm NS}(a+\frac{ib}{2},w-ib)=e^{\pi b(w-c_{\rm b})}\Psi^\pm_{\rm R}(a-\frac{ib}{2},w).$$

since

$$e^{-1}_{\rm NS}(x-c_{\rm b})=(1-ie^{\pi b(x-c_{\rm b}+ib/2)})^{-1}e^{-1}_{\rm R}(x-c_{\rm b}+ib)=(1-e^{\pi bx})^{-1}e^{-1}_{\rm R}(x-c_{\rm b}+ib),$$
$$e^{-1}_{\rm R}(x-c_{\rm b})=(1+ie^{\pi b(x-c_{\rm b}+ib/2)})^{-1}e^{-1}_{\rm NS}(x-c_{\rm b}+ib)=(1+e^{\pi bx})^{-1}e^{-1}_{\rm NS}(x-c_{\rm b}+ib).$$

So, eventually

$$(1\mp ie^{\pi b(w+a-c_{\rm b})})\Psi^+_{\rm NS/R}(a-\frac{ib}{2},w)=(1\mp ie^{\pi ab})\Psi^+_{\rm R/NS}(a+\frac{ib}{2},w),$$
$$(1\pm ie^{\pi b(w+a-c_{\rm b})})\Psi^-_{\rm NS/R}(a-\frac{ib}{2},w)=-(1-ie^{\pi ab})\Psi^-_{\rm R/NS}(a+\frac{ib}{2},w),$$
$$(1\mp ie^{\pi b(w+a-c_{\rm b})})\Psi^\mp_{\rm R}(a+\frac{ib}{2},w-ib)=-(1\mp e^{\pi(w-c_{\rm b})})\Psi^\pm_{\rm R}(a+\frac{ib}{2},w),$$
$$(1\pm ie^{\pi b(w+a-c_{\rm b})})\Psi^\mp_{\rm NS}(a+\frac{ib}{2},w-ib)=(1\mp e^{\pi(w-c_{\rm b})})\Psi^\pm_{\rm NS}(a+\frac{ib}{2},w),$$

On the other hand, it is easy to check that

$$\begin{aligned}
\Phi^{+}_{\mathrm{NS/R}}(a-\frac{ib}{2},w) &= \frac{1\mp ie^{\pi ab}}{1\mp ie^{\pi b(a+w-c_{\mathrm{b}})}}\Phi^{+}_{\mathrm{R/NS}}(a+\frac{ib}{2},w),\\
\Phi^{-}_{\mathrm{NS/R}}(a-\frac{ib}{2},w) &= -\frac{1\mp ie^{\pi ab}}{1\pm ie^{\pi b(a+w-c_{\mathrm{b}})}}\Phi^{-}_{\mathrm{R/NS}}(a+\frac{ib}{2},w),\\
\Phi^{\pm}_{\mathrm{NS}}(a+\frac{ib}{2},w-ib) &= \frac{1\pm e^{\pi b(w-c_{\mathrm{b}})}}{1\mp ie^{\pi b(a+w-c_{\mathrm{b}})}}\Phi^{\mp}_{\mathrm{NS}}(a+\frac{ib}{2},w),\\
\Phi^{\pm}_{\mathrm{R}}(a+\frac{ib}{2},w-ib) &= -\frac{1\pm e^{\pi b(w-c_{\mathrm{b}})}}{1\pm ie^{\pi b(a+w-c_{\mathrm{b}})}}\Phi^{\mp}_{\mathrm{R}}(a+\frac{ib}{2},w),
\end{aligned}$$

Taking $b\to b^{-1}$ one gets an additional set of shift equations. Given that one has two doubly periodic functions with the same equations, they have to be equal up to a constant:

$$\Psi^{\pm}_{\mathrm{NS}}(a,w)=C\Phi^{\pm}_{\mathrm{NS}}(a,w),\quad \Psi^{\pm}_{\mathrm{R}}(a,w)=C\Phi^{\pm}_{\mathrm{R}}(a,w),$$

Then, one can fix C by evaluating the expression on particular a and w.

Moreover, on can use the connection between $e_{\mathrm{R}},e_{\mathrm{NS}}$ and $G_{\mathrm{R}},G_{\mathrm{NS}}$ to prove the Ramanujan formulae based on the results from [80].

$$\begin{aligned}
LHS &= \int_{\mathbb{R}}\left(\frac{e_{\mathrm{NS}}(x+u)}{e_{\mathrm{NS}}(x+v)}+\frac{e_{\mathrm{R}}(x+u)}{e_{\mathrm{R}}(x+v)}\right)e^{\pi iwx}\mathrm{d}x=\\
&=\int\left(\frac{G_{\mathrm{NS}}(-i(x+v)+Q/2)}{G_{\mathrm{NS}}(-i(x+u)+Q/2)}+\frac{G_{\mathrm{R}}(-i(x+v)+Q/2)}{G_{\mathrm{R}}(-i(x+u)+Q/2)}\right)e^{\pi iwx}\mathrm{d}x=\\
&=\int_{i\mathbb{R}}\left(\frac{G_{\mathrm{NS}}(\tau-iv+Q/2)}{G_{\mathrm{NS}}(\tau-iu+Q/2)}+\frac{G_{\mathrm{R}}(\tau-iv+Q/2)}{G_{\mathrm{R}}(\tau-iu+Q/2)}\right)e^{\pi i\tau(iw)}\frac{\mathrm{d}\tau}{i}=\\
&=\int_{i\mathbb{R}}\left(\frac{G_{\mathrm{NS}}(\tau+(Q/2-iv)+(Q/2+iu))}{G_{\mathrm{NS}}(\tau+Q)}+\frac{G_{\mathrm{R}}(\tau+(Q/2-iv)+(Q/2+iu))}{G_{\mathrm{R}}(\tau+Q)}\right)\times\\
&\qquad\times e^{\pi i\tau(iw)}e^{\pi i(Q/2+iu)(iw)}\frac{\mathrm{d}\tau}{i}=\\
&=2e^{-i\pi c_{\mathrm{b}}^2/2}\frac{G_{\mathrm{NS}}(iw)G_{\mathrm{NS}}(Q+iu-iv)}{G_{\mathrm{NS}}(Q+iu-iv+iw)}e^{-\pi w(Q/2+iu)}=\\
&=2e^{-i\pi c_{\mathrm{b}}^2/2}\frac{A^4}{A^2}\frac{e_{\mathrm{NS}}(iQ-u+v-w-c_{\mathrm{b}})}{e_{\mathrm{NS}}(-w-c_{\mathrm{b}})e_{\mathrm{NS}}(iQ-u+v-c_{\mathrm{b}})}e^{-\pi w(Q/2+iu)}=\\
&=2e^{-i\pi c_{\mathrm{b}}^2/2}A^2\frac{e_{\mathrm{NS}}(-u+v-w+c_{\mathrm{b}})}{e_{\mathrm{NS}}(-w-c_{\mathrm{b}})e_{\mathrm{NS}}(-u+v+c_{\mathrm{b}})}e^{-\pi w(Q/2+iu)}=\\
&=\frac{e_{\mathrm{NS}}(-u+v-w+c_{\mathrm{b}})}{e_{\mathrm{NS}}(-w-c_{\mathrm{b}})e_{\mathrm{NS}}(-u+v+c_{\mathrm{b}})}e^{-\pi iw(u-c_{\mathrm{b}})}\left(2e^{-i\pi c_{\mathrm{b}}^2/2}A^2\right)=RHS,
\end{aligned}$$

where

$$\bar{B}=2e^{-i\pi c_{\mathrm{b}}^2/2}A^2,\qquad B=2e^{i\pi c_{\mathrm{b}}^2/2}A^{-2},$$

Three additional equations can be proven in the same way.

Appendix D

Permutation

In section 6.2.3 we mentioned that the $\mathsf{P_f}$ in equation (6.24) is not the standard permutation operator on $\mathbb{C}^{1|1} \otimes \mathbb{C}^{1|1}$ satisfying $\mathsf{P_f}(\eta_1 \otimes \eta_2)\mathsf{P_f} = \eta_2 \otimes \eta_1$ for arbitrary $\eta_1, \eta_2 \in \mathrm{End}(\mathbb{C}^{1|1})$. In this appendix we want to show how to calculate the standard permutation operator on $\mathbb{C} \otimes \mathbb{C}$ in the supercase.

One can use the Pauli matrices as the bases : Hom $(\mathbb{C}^{1|1}) = \mathrm{span}(\overline{\sigma_i})$ =span$(\mathbb{I}, \sigma_i)$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{D.1}$$

We consider permutation as

$$P(12) = \Sigma a_{ij} \overline{\sigma_i} \otimes \overline{\sigma_j} \tag{D.2}$$

while the bases have grading as : $\mid \overline{\sigma_1} \mid = \mid \overline{\sigma_4} \mid = 0, \mid \overline{\sigma_2} \mid = \mid \overline{\sigma_3} \mid = 1, (-1)^{\delta_{i,2}} = (-1)^{|i|}$ and satisfy the following relations:

$$\overline{\sigma_1} e_i = e_i, \quad \overline{\sigma_2} e_i = e_{i+1}, \quad \overline{\sigma_3} e_i = i(-1)^{\delta_{i,2}} e_{i+1}, \qquad \overline{\sigma_4} e_i = (-1)^{\delta_{i,2}} e_i,$$

We can choose the basis for calculating $P(12)$ depending on the Pauli's matrices as

$$\overline{\sigma_1} = \mathbb{I}, \qquad \overline{\sigma_2} = \sigma_1, \qquad \overline{\sigma_3} = \sigma_2, \quad \overline{\sigma_4} = \sigma_3.$$

The permutation can be calculated as

$$\Sigma a_{ij} \overline{\sigma_i} \otimes \overline{\sigma_j} = \Sigma_{kl} a_{kl} (\overline{\sigma_k} \otimes \overline{\sigma_l})(e_i \otimes e_j) = (-1)^{|i||j|}(e_j \otimes e_i) \qquad i, j = 1, 2$$

then we get:

$\Sigma_{kl} a_{kl} (\overline{\sigma_k}.e_i) \otimes (\overline{\sigma_l}.e_j)(-1)^{|e_i||\overline{\sigma_l}|} = a_{11}(e_i \otimes e_j) + a_{12}(e_i \otimes e_j + 1)(-1)^{|e_i|} +$
$a_{13}(e_i \otimes i(-1)^{|j|} e_{j+1})(-1)^{|e_i|} + a_{14}(e_i \otimes (-1)^{|j|} e_j) + a_{21}(e_{i+1} \otimes e_j) + a_{22}(e_{i+1} \otimes e_{j+1})(-1)^{|e_i|} +$
$a_{23}(e_{i+1} \otimes i(-1)^{|j|} e_{j+1})(-1)^{|e_i|} + a_{24}(e_{i+1} \otimes (-1)^{|j|} e_j) + a_{31}((-1)^{|i|} i e_{i+1} \otimes e_j) +$
$a_{32}((-1)^{|i|} i e_{i+1} \otimes e_{j+1})(-1)^{|e_i|} + a_{33}((-1)^{|i|} i e_{i+1} \otimes i(-1)^{|j|} e_{j+1})(-1)^{|e_i|} +$
$a_{34}((-1)^{|i|} i e_{i+1} \otimes (-1)^{|j|} e_j) + a_{41}((-1)^{|i|} i e_i \otimes e_j) + a_{42}((-1)^{|i|} e_i \otimes e_{j+1})(-1)^{|e_i|} +$
$a_{43}((-1)^{|i|} e_i \otimes (-1)^{|j|} i e_{j+1})(-1)^{|e_i|} + a_{44}((-1)^{|i|} e_i \otimes (-1)^{|j|} e_j)$

we have 16 equations for different choices of i and j. By solving these equations we get the following non zero coefficients: $a_{14} = 1/2, \quad a_{23} = i/2, \quad a_{32} = -i/2, \quad a_{41} = 1/2.$

Then, the permutation it found to be

$$
\begin{aligned}
2P(12) &= \overline{\sigma_1} \otimes \overline{\sigma_4} + i\overline{\sigma_2} \otimes \overline{\sigma_3} - i\overline{\sigma_3} \otimes \overline{\sigma_2} + \overline{\sigma_4} \otimes \overline{\sigma_1} \\
&= \mathbb{I} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \mathbb{I} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\end{aligned}
$$

By knowing that

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} a\alpha & a\beta & b\alpha & b\beta \\ a\gamma & a\delta & b\gamma & b\delta \\ c\alpha & -c\beta & d\alpha & -b\beta \\ -c\gamma & c\delta & -d\gamma & b\delta \end{pmatrix}. \tag{D.3}
$$

Then permutation is

$$
P(12) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{D.4}
$$

and satisfies the necessary properties

$$
\begin{aligned}
&P(12)e_1 \otimes e_1 = e_1 \otimes e_1, \qquad & & P(12)e_3 \otimes e_3 = e_3 \otimes e_3, \\
&P(12)e_2 \otimes e_2 = e_2 \otimes e_2, & P(12)^2 = 1, \qquad & P(12)e_4 \otimes e_4 = -e_4 \otimes e_4.
\end{aligned} \tag{D.5}
$$

Appendix E

q-binomial

We introduce and prove some formulas for the so called q-analysis, which are useful for the construction of quantum group, Drinfeld double and Heisenberg double of $U_q(sl(2))$, Drinfeld double and Heisenberg double of $U_q(osp(1|2))$.

Lemma 1. q-binomial formula: If A and B are elements of an algebra obeying $BA = qAB$ then

$$(A+B)^n = \sum_{m=0}^{n} \binom{n}{m}_q A^m B^{n-m},$$

where

$$\binom{n}{m}_q = \frac{(n)_q!}{(m)_q!(n-m)_q!}, \qquad (n)_q = \frac{1-q^n}{1-q}.$$

We suppose that the q-integer $(m)_q$ is non zero for $0 < m < n$. By convention $\binom{n}{0}_q = 1$

Proof. Assuming the result for $(A+B)^{n-1}$ we have

$$\begin{aligned}
(A+B)^{n-1}(A+B) &= \sum_{m=0}^{n-1} \binom{n-1}{m}_q A^m B^{n-1-m}(A+B) \\
&= \sum_{m=0}^{n-1} q^{n-1-m} \binom{n-1}{m}_q A^{m+1} B^{n-1-m} + \sum_{m=0}^{n-1} \binom{n-1}{m}_q A^m B^{n-m} \\
&= \sum_{m=1}^{n} q^{n-m} \binom{n-1}{m-1}_q A^m B^{n-m} + \sum_{m=0}^{n-1} \binom{n-1}{m}_q A^m B^{n-m} \\
&= A^n + B^n + \sum_{m=1}^{n-1} \left(q^{n-m} \binom{n-1}{m-1}_q + \binom{n-1}{m}_q \right) A^m B^{n-m}
\end{aligned}$$

Then by using the identity $q^{n-m}(m)_q + (n-m)_q = (n)_q$ we can see the proof is complete. $\square$

We present a short proof of

$$\binom{n}{k}_q = \frac{(q)_n}{(q)_k (q)_{n-k}}, \tag{E.1}$$

where, $(q)_n = (1-q)...(1-q^n)$.

where we use the fact that

$$\begin{aligned}
&(x;q)^{-1} = \prod_{k=0}^{\infty} \frac{1}{1-xq^k} = \prod_{k=0}^{\infty}\sum_{l=0}^{\infty}(xq^k)^l = \\
&= (1+x+x^2+...)(1+xq+x^2q^2+...)...(1+xq^k+x^2q^{2k}+...)... = \\
&= 1 + x(1+q+q^2+...) + x^2(1+q+...+q^k+...+q^2+q^3+...+q^4+q^5+...)+... = \\
&= 1 + \frac{x}{1-q} + x^2\frac{1}{1-q}(1+q^2+q^4+...)+... = 1+\frac{x}{1-q}+x^2\frac{1}{1-q}\frac{1}{1-q^2}+... = \\
&= 1 + \sum_{k=1}^{\infty}\frac{x^k}{\prod_{l=1}^{k}(1-q^l)} = \sum_{k=0}^{\infty}\frac{x^k}{(q)_k}.
\end{aligned}$$

q-binomial in terms of G_b functions In the non-supersymmetric case, we have the following form of the Ramanujan summation formula

$$\int \mathrm{d}\tau e^{-2\pi\tau\beta}\frac{G_\mathrm{b}(\alpha+i\tau)}{G_\mathrm{b}(Q+i\tau)} = \frac{G_\mathrm{b}(\alpha)G_\mathrm{b}(\beta)}{G_\mathrm{b}(\alpha+\beta)}. \tag{E.2}$$

From this we can get the Fourier transforms. First, by taking $\alpha \to +\infty$ and using

$$G_\mathrm{b}(x) \to \bar{\zeta}_\mathrm{b}, \qquad \Im(x) \to +\infty, \tag{E.3}$$

where $\zeta_\mathrm{b}^{-1} = \bar{\zeta}_\mathrm{b}$ (because ζ_b is a pure phase), we get

$$\begin{aligned}
&\int \mathrm{d}\tau e^{-2\pi\tau\beta}\frac{\bar{\zeta}_\mathrm{b}}{G_\mathrm{b}(Q+i\tau)} = G_\mathrm{b}(\beta), \\
&\int \mathrm{d}(bt)e^{-2\pi(bt)(\frac{Q}{2}-ir)}\frac{1}{G_\mathrm{b}(Q+ibt)} = \zeta_\mathrm{b}G_\mathrm{b}(\frac{Q}{2}-ir), \\
&b\int \mathrm{d}te^{2\pi ibtr}\frac{e^{-\pi btQ}}{G_\mathrm{b}(Q+ibt)} = \zeta_\mathrm{b}G_\mathrm{b}(\frac{Q}{2}-ir) = g_\mathrm{b}^{-1}(e^{2\pi br}) = e_\mathrm{b}^{-1}(r).
\end{aligned}$$

Now, using the complex conjugation property of the G_b

$$\overline{G_\mathrm{b}(x)} = e^{-i\pi\bar{x}(\bar{x}-Q)}G_\mathrm{b}(\bar{x}), \tag{E.4}$$

we can complex conjugate the previous expression to get

$$\begin{aligned}
&b\int \mathrm{d}te^{-2\pi ibtr}\frac{e^{-\pi btQ}}{\overline{G_\mathrm{b}(Q+ibt)}} = \bar{\zeta}_\mathrm{b}\overline{G_\mathrm{b}(\frac{Q}{2}-ir)}, \\
&b\int \mathrm{d}te^{-2\pi ibtr}\frac{e^{-\pi btQ}}{G_\mathrm{b}(Q-ibt)}e^{-i\pi(Q-ibt)(Q-Q+ibt)} = \bar{\zeta}_\mathrm{b}G_\mathrm{b}(\frac{Q}{2}+ir)e^{i\pi(\frac{Q}{2}+ir)(Q-\frac{Q}{2}-ir)}, \\
&b\int \mathrm{d}te^{2\pi ibtr}\frac{e^{-i\pi b^2t^2}}{G_\mathrm{b}(Q+ibt)} = \bar{\zeta}_\mathrm{b}\frac{1}{G_\mathrm{b}(\frac{Q}{2}-ir)} = g_\mathrm{b}(e^{2\pi br}) = e_\mathrm{b}(r).
\end{aligned}$$

Now, we can proceed with the calculation. We start from the shift property

$$e_{\mathrm{b}}(x-\frac{ib}{2})=e_{\mathrm{b}}(x+\frac{ib}{2})(1+e^{2\pi bx}).$$

Using that, we define $u=e^{2\pi bq}$, $v=e^{2\pi bp}$ such that $uv=q^2vu$ with $q=e^{i\pi b^2}$. Then

$$e_{\mathrm{b}}(p-q)ue_{\mathrm{b}}^{-1}(p-q)=e^{\pi bq}(1+e^{2\pi b(p-q)})e^{\pi bq}=u+v.$$

Therefore,

$$\begin{aligned}(u+v)^{it}&=e_{\mathrm{b}}(p-q)u^{it}e_{\mathrm{b}}^{-1}(p-q)=b^2\int \mathrm{d}\tau_1\mathrm{d}\tau_2 e^{-i\pi b^2\tau_1^2}\frac{e^{2\pi ib\tau_1(p-q)}}{G_{\mathrm{b}}(Q+ib\tau_1)}u^{it}e^{-\pi b\tau_2 Q}\frac{e^{2\pi ib\tau_2(p-q)}}{G_{\mathrm{b}}(Q+ib\tau_2)}=\\&=b^2\int \mathrm{d}\tau_1\mathrm{d}\tau_2 e^{-i\pi b^2\tau_1^2}G_{\mathrm{b}}(-ib\tau_1)e^{-i\pi(Q+ib\tau_1)(Q+ib\tau_1-Q)}e^{-\pi b\tau_2 Q}\frac{e^{2\pi ib\tau_1(p-q)}u^{it}e^{2\pi ib\tau_2(p-q)}}{G_{\mathrm{b}}(Q+ib\tau_2)}=\\&=b^2\int \mathrm{d}\tau_1\mathrm{d}\tau_2 e^{-\pi b(\tau_2-\tau_1)Q}\frac{G_{\mathrm{b}}(-ib\tau_1)}{G_{\mathrm{b}}(Q+ib\tau_2)}e^{2\pi ib\tau_1(p-q)}e^{2\pi ibtq}e^{2\pi ib\tau_2(p-q)}=(*)\end{aligned}$$

and because $e^{2\pi ib\tau_1 p}e^{-2\pi ib\tau_1 q}=e^{\frac{-(2\pi ib\tau_1)^2}{4\pi i}}e^{2\pi ib\tau_2(p-q)}=e^{-\pi ib^2\tau_1^2}e^{2\pi ib\tau_2(p-q)}$ and similar expression for τ_2 we have

$$(*)=b^2\int \mathrm{d}\tau_1\mathrm{d}\tau_2 e^{-\pi b(\tau_2-\tau_1)Q}e^{-\pi ib^2(\tau_1+\tau_2)^2}e^{2\pi ib^2t\tau_1}\frac{G_{\mathrm{b}}(-ib\tau_1)}{G_{\mathrm{b}}(Q+ib\tau_2)}u^{i(t-\tau_1-\tau_2)}v^{i(\tau_1+\tau_2)}$$

and after the change of the integration variable $\tau=\tau_1+\tau_2$, Therefore,

$$(*)=b^2\int \mathrm{d}\tau\mathrm{d}\tau_2 e^{-2\pi b\tau_2(Q+ibt)+\pi b\tau(Q+2ibt)-\pi ib^2\tau^2}\frac{G_{\mathrm{b}}(-ib\tau+ib\tau_2)}{G_{\mathrm{b}}(Q+ib\tau_2)}u^{i(t-\tau)}v^{i\tau}$$

Now, using the Ramanujan summation formula with $\alpha=-ib\tau$ and $\beta=Q+ibt$ we get

$$\begin{aligned}(*)&=b\int \mathrm{d}\tau e^{\pi b\tau(Q+2ibt)-\pi ib^2\tau^2}\frac{G_{\mathrm{b}}(-ib\tau)G_{\mathrm{b}}(Q+ibt)}{G_{\mathrm{b}}(-ib\tau+Q+ibt)}u^{i(t-\tau)}v^{i\tau}=\\&=b\int \mathrm{d}\tau e^{2\pi ib^2t\tau-2\pi ib^2\tau^2}\frac{G_{\mathrm{b}}(Q+ibt)}{G_{\mathrm{b}}(Q+ib\tau)G_{\mathrm{b}}(-ib\tau+Q+ibt)}u^{i(t-\tau)}v^{i\tau}=b\int \mathrm{d}\tau\binom{t}{\tau}_{\mathrm{b}}u^{i(t-\tau)}v^{i\tau},\end{aligned}$$

where we set

$$\binom{t}{\tau}_{\mathrm{b}}=b\int \mathrm{d}\tau e^{2\pi ib^2\tau(t-\tau)}\frac{G_{\mathrm{b}}(Q+ibt)}{G_{\mathrm{b}}(Q+ib\tau)G_{\mathrm{b}}(-ib\tau+Q+ibt)}.$$

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