On the Relation between Quantized Teichmüller Spaces and the Free Field Quantization of Liouville Theory

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Summary

We investigate how the quantum Teichmüller theory is encoded in the free field quantization of Liouville theory. We show that the 4-point conformal blocks of Liouville theory contain the eigenstates of length operators of quantum Teichmüller theory in a certain representation. After giving a heuristic explanation for this observation, we formulate a conjectural generalization for n punctures and define a representation of the algebra of quantized shear coordinates for the n-punctured sphere on the Liouville Hilbert space. We present calculations in the attempt to prove the conjecture for $n = 5$. Finally, we show that, on the classical level, each Teichmüller space may be identified with a quotient of the Liouville phase space and investigate how an analogous reduction may be realized on the quantum level.

Zusammenfassung

Wir untersuchen wie die Quanten-Teichmüllertheorie in der Freifeldquantisierung der Liouvilletheorie dekodiert ist. Wir zeigen, dass die 4-Punkt konformen Blöcke der Liouvilletheorie die Eigenzustände von Längenoperatoren der Quanten-Teichmüllertheorie in einer bestimmten Darstellung enthalten. Nachdem wir eine heuristische Erklärung für diese Beobachtung geliefert haben, formulieren wir eine Vermutung für die Verallgemeinerung dieser Beobachtung auf n Punktierungen und definieren eine Darstellung der Algebra der quantisierten Scherkoordinaten für die n-punktierte Sphäre auf dem Hilbertraum der Liouvilletheorie. Wir präsentieren Rechnungen in dem Versuch, die Vermutung für $n = 5$ zu beweisen. Schließlich zeigen wir auf der klassischen Ebene, dass jeder Teichmüllerraum mit einem Quotienten des Phasenraums der Liouvilletheorie identifiziert werden kann und untersuchen wie eine analoge Reduktion auf der Quantenebene realisiert werden kann.

Contents

1 Introduction

Conformal field theory (CFT) in two dimensions is known to be an integral part of string theory. Since Polyakov recognized Liouville theory to be the effective theory of the non-critical ($D \neq 26$) bosonic string [28], a lot of effort has been put towards the development and solution of this CFT (early works are [7, 8, 14]). An exact formula for the three point function of the exponentiated Liouville field was proposed in [9, 44] and derived in [35, 37]. Due to the conformal symmetry this allows in principle to compute all correlation functions of the theory, which amounts to a solution of the theory.

Besides the relation with bosonic string theory via two-dimensional quantum gravity (the Liouville field φ is the conformal factor of the metric $g_{ab} = e^{\varphi}\eta_{ab}$), quantum Liouville theory has revealed many connections to different physical and mathematical subjects. Most prominent on the physical side are the connections to $N = 2$ supersymmetric gauge theories conjectured in [3] and proven in [40], where the conformal blocks of Liouville theory are related to the so-called instanton partition functions of the gauge theory. On the mathematical side, one may name relations to the representation theory of quantum groups [29, 41] and the quantum Teichmüller theory [36, 38, 40], which may be outlined as the quantum theory of Riemann surfaces. Concerning the latter, it has been found that conformal blocks, defined in genus zero as the n-point function of the exponentiated Liouville field, coincide with eigenstates of the so called length operators in a certain representation as holomorphic functions on Teichmüller space. This statement has been proven by showing that the monodromies and asymptotic behavior of both objects (which have been computed independently) coincide, and using the fact that a function with these properties is unique. Therefore, the proof has not brought much insight into the origin of that relation, which is the subject of the present work.

Starting on the classical level, there seem to be two slightly different approaches to understand the connection between Liouville theory and Teichmüller spaces. The first one starts from the observation that the conformal factor of the hyperbolic metric on a given Riemann surface satisfies the *euclidean* Liouville equation with certain boundary conditions. This may be used to embed the Teichmüller space into the phase space of euclidean Liouville theory, defined on an annulus, by embedding this annulus into each Riemann surface. This approach appears to be promising by the results of Takhtajan and Zograf [33] who related the Weil-Petersson symplectic form on Teichmüller space to the euclidean Liouville action. On the quantum level, this approach is affirmed by the observation that the conformal Ward identity relates the quantized stress-energy tensor of Liouville theory to the corresponding operator on Teichmüller space [10].

The second approach is to understand each Teichmüller space $\mathcal{T}_{g,n}$ as a quotient of the (chiral) phase space of Minkowskian Liouville theory. The latter may be identified with the chiral part of the conformal symmetry group, $\mathcal{P}_{\text{chir.}} = \text{Diff}^+(S^1)/\text{Rot}(S^1)$, where $\text{Diff}^+(S^1)$ denotes the group of orientation preserving diffeomorphisms of the unit circle S^1 and $\text{Rot}(S^1) \simeq S^1$ the subgroup of rigid rotations. Let Σ be a Riemann surface of genus g with n punctures. One may act with an element α of $\text{Diff}^+(S^1)$ on Σ by cutting out a punctured disc from Σ, changing its boundary parametrization with α (where rigid rotations leave the conformal equivalence class of the punctured disc invariant), and then regluing the surface according to the new boundary parametrization. Let $\text{Hol}(\Sigma)$ denote the subgroup of diffeomorphisms that leave the conformal equivalence class of Σ invariant (which are identified with holomorphic self-mappings of Σ). The quotient of the infinite dimensional space $\text{Diff}^+(S^1)/S^1$ by $\text{Hol}(\Sigma)$ is then naturally identified with a subset of the finite dimensional moduli space $\mathcal{M}_{g,n}$ of type (g, n) Riemann surfaces (respectively a subset of $\mathcal{T}_{g,n}$, when Σ is equipped with a marking).

Since we are chiefly interested in Minkowskian Liouville theory, we will primarily follow the second approach. (Nevertheless, we will also employ the first approach on a heuristic level by associating analytic continuations of Liouville observables with observables on Teichmüller space as inspired by the conformal Ward identity.) The aim of the present work is to investigate in what way the identification of Teichmüller space as a quotient of the Liouville phase space carries over to the quantum theory in the framework of the free field quantization of Liouville theory. In other words, we want to identify quantum Teichmüller theory as a "subtheory" of quantum Liouville theory, in the sense that the Teichmüller Hilbert space $\mathcal{H}_T$ becomes a subspace of the chiral Liouville Hilbert space $\mathcal{H}_L^{\text{chir.}}$ and the algebra of Teichmüller operators $\mathcal{A}_T$ is identified with a subalgebra of the algebra of chiral Liouville operators $\mathcal{A}_L^{\text{chir.}}$. The resulting picture could then be outlined by the phrase "reduction commutes with quantization", and diagrammatically represented as follows:

$$\begin{tikzcd}
\mathrm{Diff}^+(S^1)/S^1 \arrow[r, "\text{quant.}"] \arrow[d, "\text{reduction}"'] & (\mathcal{H}_L^{\text{chir.}}, \mathcal{A}_L^{\text{chir.}}) \arrow[d, "\text{reduction}"] \\
\mathrm{Hol}(\Sigma)\backslash\mathrm{Diff}^+(S^1)/S^1 \subset \mathcal{T}_{g,n} \arrow[r, "\text{quant.}"'] & (\mathcal{H}_T, \mathcal{A}_T)
\end{tikzcd}$$

The most obvious way to perform the reduction on the quantum level is in a "coherent state representation" of Liouville theory, a quantization scheme where the Hilbert space consists of holomorphic functions on the phase space. In such a representation $\mathcal{H}_T$ would be naturally identified as the functions on $\mathcal{P}_{\text{chir.}}$ which are invariant under the action of $\mathrm{Hol}(\Sigma)$, and can thus be identified as holomorphic functions on (a subset of) $\mathcal{T}_{g,n}$. This would correspond to a coherent state representation of quantum Teichmüller theory. Such a representation has been defined in [40] and is the one in which eigenstates of length operators are related to conformal blocks of Liouville theory, defined (amongst others) in the framework of the free field quantization. One of the tasks that are left is therefore to define a coherent state representation of Liouville theory from the free field representation, i.e., to associate with each state in $\mathcal{H}_L^{\text{chir.}}$ a holomorphic function on $\mathrm{Diff}^+(S^1)/S^1$. This is naturally done with the help of the representation of the Virasoro algebra obtained in free field quantization of Liouville theory, which yields a projective representation of $\mathrm{Diff}^+(S^1)$.

Much of this picture is already known, in particular the classical theories and their quantization in different schemes. As this material is not always available in the form we need it, and for the convenience of the reader, we will review it in the first sections. The reduction process has been indicated in the literature [40, 39] but not yet performed explicitly. Here we will make a number of new contributions, both on the classical and on the quantum level.

This work is structured as follows. In Section 2 we will expose the classical Liouville theory and its solution in terms of the free field, which will be the starting point for quantization in Section 5. In Section 3 we will introduce the classical Teichmüller spaces and suitable coordinate sets on them such as the shear coordinates, which will be the basis for the different quantization schemes introduced in the next section.

Having defined conformal blocks of Liouville theory in Section 6, we will observe in Section 6.3 that in a certain reordering of operators in 4-point conformal blocks there appear the eigenfunctions of length operators of quantum Teichmüller theory. This leads one to identify a certain operator of Liouville theory with a quantized shear coordinate on Teichmüller space. A first heuristic explanation for this identification, based on the conformal Ward identity, will be given in Section 7.1. In order to formulate a conjectural generalization of this observation for the n-point conformal block, we will define in Section 7.2 a representation of the algebra $\mathcal{A}_n$ of quantized shear coordinates for the n-punctured sphere on the space of Liouville operators. This will lead to the definition of a representation of $\mathcal{A}_n$ on the dual $\mathcal{H}_L^*$ of the Liouville Hilbert space $\mathcal{H}_L$ (where $\mathcal{H}_L^* \simeq \mathcal{H}_L$). The next task will be to investigate whether $\mathcal{H}_T$ can be identified with a subset of $\mathcal{H}_L^*$ which is invariant under this representation, such that the action of the quantized shear coordinates on this subspace corresponds to their action on $\mathcal{H}_T$. A natural candidate for this is the space $\mathrm{CB}(\Sigma)$ of elements $\langle\Psi|$ of $\mathcal{H}_L^*$ with the invariance property

$$\forall \eta \in \mathrm{Vect}(\Sigma): \quad \langle\Psi|\mathbf{T}_\eta \;=\; 0, \tag{1.1}$$

where $\mathrm{Vect}(\Sigma)$ denotes, modulo technical details, the space of holomorphic vector fields on Σ (which is the tangent space to $\mathrm{Hol}(\Sigma)$) and $\mathbf{T}_\eta$ is the operator that represents η in quantum Liouville theory. We will then find that at least a subset of the quantized shear coordinates leaves $\mathrm{CB}(\Sigma)$ invariant. The observation of Section 6.3 will help us to argue that at least in one case the action of a shear coordinate on $\mathrm{CB}(\Sigma) \simeq \mathcal{H}_T$ coincides with its action on $\mathcal{H}_T$ in the coherent state representation. (By the conjectural generalization for the n-punctured sphere the same would be true for a large class of shear coordinates on $\mathcal{T}_{0,n}$.) This may be seen as a first evidence for the compatibility at the quantum level of the two approaches to the relation between Liouville and Teichmüller theory described above.

2 Classical Liouville theory

In this section we introduce the classical Liouville theory and its solution in terms of the free field, as well as the characterization of its phase space in terms of coadjoint orbits of the Virasoro-Bott group. Although this is all known material, the given representation of the Liouville phase space has, to the authors knowledge, not appeared in the literature so far.

2.1 Solution of the Liouville equation

Classical Liouville theory is a two dimensional field theory defined by the action

$$S_L = \frac{1}{\pi}\int\limits_{-\infty}^{\infty} dt \int\limits_{0}^{2\pi} d\sigma \left(\frac{1}{16}\left((\partial_t\varphi)^2 - (\partial_\sigma\varphi)^2\right) - e^{\varphi}\right), \tag{2.1}$$

where the Liouville field φ is a real smooth function on the cylinder $\mathbb{R}\times S^1$ and one identifies S^1 with $\mathbb{R}/2\pi\mathbb{Z}$. Hence we require the periodic boundary conditions $\varphi(t,\sigma+2\pi) = \varphi(t,\sigma)$. The Euler-Lagrange equation for this action is the (Minkowskian) Liouville equation

$$\partial_+\partial_-\varphi = -2e^{\varphi}, \tag{2.2}$$

where $x^\pm := t \pm \sigma$, $\partial_\pm = \frac{\partial}{\partial x^\pm}$. The first step to its solution is the observation that

$$\partial_- T_+ = \partial_+ T_- = 0, \tag{2.3}$$

where

$$T_\pm \;:=\; \tfrac{1}{4}(\partial_\pm\varphi)^2 - \tfrac{1}{2}\partial_\pm^2\varphi, \tag{2.4}$$

if φ satisfies the Liouville equation. Thus $\exp(-\frac{1}{2}\varphi(x^+,x^-))$ for fixed x^- respectively for fixed x^+ are solutions to the Hill's equations

$$\partial_\pm^2 f^\pm(x^\pm) = T_\pm(x^\pm) f^\pm(x^\pm). \tag{2.5}$$

Let f_i^+, respectively f_i^- $(i=1,2)$, be two linearly independent solutions to these equations. Then the functions

$$d^\pm := f_1^\pm \partial_\pm f_2^\pm - f_2^\pm \partial_\pm f_1^\pm \tag{2.6}$$

are constants, $d^\pm \in \mathbb{R}$. Since

$$0 \;\neq\; \partial_\pm \frac{f_2^\pm}{f_1^\pm} \;=\; \frac{d^\pm}{(f_1^\pm)^2} \tag{2.7}$$

one has $d^\pm \neq 0$. Therefore $f_i^\pm$ can be normalized such that

$$f_1^\pm \partial_\pm f_2^\pm - f_2^\pm \partial_\pm f_1^\pm \;=\; 1. \tag{2.8}$$

Then there exists a matrix $C = (C_{ij})_{i,j=1,2}$ such that

$$e^{-\frac{1}{2}\varphi(x^+,x^-)} \;=\; \sum_{i,j=1}^{2} f_i^+(x^+) C_{ij} f_j^-(x^-) \;=\; f^+(x^+)\cdot C \cdot f^-(x^-), \tag{2.9}$$

where $f^+ := (f_1^+, f_2^+)$ and $f^- := (f_1^-, f_2^-)^t$. By the linear transformation $f^+ \to f^+ \cdot C^{-1}$ it can be achieved that $C = 1$ (it follows from (2.2) that $\det C = 1$). Conversely, given any two pairs of smooth functions $f^\pm$ satisfying (2.8), then

$$\varphi(x^+,x^-) \;=\; -2\log\left(f_1^+(x^+)f_1^-(x^-) + f_2^+(x^+)f_2^-(x^-)\right) \tag{2.10}$$

is a solution to the Liouville equation. However, we also have to consider the boundary conditions.

As φ is periodic, $\varphi(x^+ + 2\pi, x^- - 2\pi) = \varphi(x^+,x^-)$, so are $T_+(x^+)$ and $T_-(x^-)$. It follows that $f_i^\pm(x^\pm \pm 2\pi), i = 1,2$, are also two linearly independent solutions to (2.5) satisfying (2.8), and therefore related to $f_i^\pm(x^\pm)$ by an $\mathrm{SL}(2,\mathbb{R})$ transformation $M^\pm$:

$$f^\pm(x^\pm \pm 2\pi) = M^\pm \cdot f^\pm(x^\pm), \qquad M^\pm \in \mathrm{SL}(2,\mathbb{R}). \tag{2.11}$$

Furthermore, it follows from (2.10) and the periodicity of φ that $(M^+)^t \cdot M^- = 1$. Also, there is some freedom in the choice of the $f_i^\pm$: If we change

$$f^\pm \;\to\; Q^\pm f^\pm, \quad Q^\pm \in \mathrm{SL}(2,\mathbb{R}), \tag{2.12}$$

with $(Q^+)^t \cdot Q^- = 1$, then φ remains invariant. The monodromy matrices $M^\pm$ transform as $M^\pm \to Q^\pm M^\pm (Q^\pm)^{-1}$. Therefore only the conjugacy class of $M^\pm$ is fixed. It has been shown [26, 4] that for φ to be non-singular, M^+ and M^- have to be hyperbolic elements of $\mathrm{SL}(2,\mathbb{R})$, which means that $|\mathrm{tr} M^\pm| > 2$. By using the freedom (2.12), one may then bring $M^\pm$ to the form

$$M^\pm = s_\pm \begin{pmatrix} e^{\mp\pi p} & 0 \\ 0 & e^{\pm\pi p} \end{pmatrix}, \qquad p > 0,\; s_\pm \in \{1,-1\}. \tag{2.13}$$

We will now explicitly describe the phase space $\mathcal{P}$ of Liouville theory, defined as the set of non-singular solutions to the Liouville equation on the cylinder. To this end, let us define

$$A^\pm(x^\pm) \;:=\; \frac{f_2^\pm(x^\pm)}{f_1^\pm(x^\pm)}. \tag{2.14}$$

It can be shown [4] that $f_1^\pm$ has no zeros if φ is non-singular and the monodromy matrix is of the form (2.13). Thus the functions $A^\pm$ are also non-singular. By (2.7) they are monotonically increasing, $\partial_\pm A^\pm = (f_1^\pm)^{-2} > 0$, and we can define

$$\varphi_F^\pm := \log(\partial_\pm A^\pm). \tag{2.15}$$

Having brought the monodromy matrices to the form (2.13), we have

$$A^\pm(x^\pm \pm 2\pi) = e^{\pm 2\pi p} A^\pm(x^\pm). \tag{2.16}$$

This implies $\varphi_F^\pm(x^\pm \pm 2\pi) = \varphi_F^\pm(x^\pm) \pm 2\pi p$ and therefore

$$\varphi_F(x^+, x^-) \ := \ \varphi_F^+(x^+) + \varphi_F^-(x^-) \tag{2.17}$$

is a periodic solution to the free wave equation $\partial_+\partial_-\varphi_F = 0$ (called a free field). Going backwards, we can reconstruct $A^\pm$ from $\varphi_F^\pm$ by the formula

$$A^\pm(x^\pm) = \frac{1}{e^{2\pi p} - 1} \int_0^{2\pi} dy \ e^{\varphi_F^\pm(y + x^\pm)}, \tag{2.18}$$

and $f_i^\pm$ from $A^\pm$ by

$$f_1^\pm = (\partial_\pm A^\pm)^{-\frac{1}{2}}, \qquad f_2^\pm = A^\pm \ (\partial_\pm A^\pm)^{-\frac{1}{2}}. \tag{2.19}$$

Thus we have shown that every periodic non-singular solution of the Liouville equation is of the form

$$\varphi(x^+, x^-) = \log \frac{\partial_+ A^+ \partial_- A^-}{(1 + A^+ A^-)^2}, \tag{2.20}$$

with $A^\pm$ given by (2.18). Note that with the help of (2.20) and $\partial_+^2 A^+ = (\partial_+\varphi_F)\partial_+ A^+$ one can express $T_\pm$ in terms of the free field as

$$T_\pm \ = \ \tfrac{1}{4}(\partial_\pm\varphi_F)^2 - \tfrac{1}{2}\partial_\pm^2\varphi_F. \tag{2.21}$$

Conversely, given a periodic free field φ_F with left and right moving parts $\varphi_F^\pm(x^\pm)$ then (2.20) (together with (2.18)) defines a solution of the Liouville equation on the cylinder.[1] Thus we have constructed a surjective map $R : \mathcal{P}_F \to \mathcal{P}$ from the phase space of free field theory to that of Liouville theory.[2] However, this map is not one-to-one but two-to-one as φ is invariant under the transformation $f_1^\pm \to f_2^\pm$, $f_2^\pm \to -f_1^\pm$, which corresponds to $A^\pm \to -(A^\pm)^{-1}$ and $p \to -p$. This transformation defines a map $S : \mathcal{P}_F \to \mathcal{P}_F$ satisfying $S^2 = \mathrm{Id}$. One could now define an equivalence relation on $\mathcal{P}_F$ by $\varphi_F \sim \varphi'_F :\Leftrightarrow \varphi'_F = S(\varphi_F)$. Then one would obtain a bijection $\tilde{R} : \mathcal{P}_F/_\sim \to \mathcal{P}$. Another option is to restrict R to the space $\mathcal{P}_F^+$ of free fields with positive zero mode p. Then it has an inverse $W : \mathcal{P} \to \mathcal{P}_F^+$ (which has been explicitly described above).

The Liouville phase space $\mathcal{P}$ carries a canonical Poisson bracket coming from the action (2.1). Defining the conjugate momenta at $t = 0$

$$\Pi(\sigma) := \frac{\delta S_L}{\delta(\partial_t\varphi(\sigma))} = \frac{1}{8\pi}\partial_t\varphi(\sigma), \tag{2.22}$$

it is given by

$$\{\Pi(\sigma), \varphi(\sigma')\} = \delta(\sigma - \sigma'). \tag{2.23}$$

With regard to quantization of Liouville theory, we want to use functions on $\mathcal{P}_F$, composed with the map W, as functions on $\mathcal{P}$. A convenient set of functions on $\mathcal{P}_F$ is obtained by the Fourier expansion

$$\varphi_F(x^+, x^-) \ = \ q + p(x^+ + x^-) + i \sum_{n \neq 0} \frac{1}{n} \left(a_n e^{-inx^+} + b_n e^{-inx^-} \right). \tag{2.24}$$

It has been shown that [26]

$$\{\varphi_F(\sigma) \circ W, \partial_t\varphi_F(\sigma') \circ W\} \ = \ 8\pi\delta(\sigma - \sigma'), \tag{2.25}$$

[1] Periodicity of φ_F implies that there exists $p \in \mathbb{R}$ such that $\varphi_F^\pm(x^\pm \pm 2\pi) = \varphi_F^\pm(x^\pm) \pm 2\pi p$.

[2] The Liouville field really depends only on φ_F, since adding a constant to φ_F^+ and its negative to φ_F^- leaves it invariant.

that is, the Poisson brackets of the free field, viewed as functions on $\mathcal{P}$, coincide with the canonical Poisson brackets on $\mathcal{P}_F$. In terms of the Fourier modes q, p, a_n, b_n appearing in (2.24), viewed as functions on $\mathcal{P}$, equation (2.25) becomes

$$\{p, q\} = -2, \quad \{a_n, a_m\} = -2in\delta_{n+m,0}, \quad \{b_n, b_m\} = -2in\delta_{n+m,0}, \tag{2.26}$$

while all other brackets vanish. This will be the starting point for quantization in Section 5.

2.2 The $\mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$ symmetry

Let φ be a solution to the Liouville equation with corresponding functions $A^\pm$. One can then act with elements α^+, α^- of $\mathrm{Diff}^+(S^1)$, the group of orientation preserving diffeomorphisms of the circle, on $A^\pm$ according to

$$A^\pm \;\to\; A^\pm_{\alpha^\pm} := A^\pm \circ \alpha^\pm. \tag{2.27}$$

Here we represent elements of $\mathrm{Diff}^+(S^1)$ by monotonically increasing smooth functions $\alpha : \mathbb{R} \to \mathbb{R}$ that are compatible with the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z} \simeq S^1$. These are the α that satisfy $\alpha(x + 2\pi) = \alpha(x) + 2\pi$. (Although the shifts $\alpha_n(x) = x + 2\pi n,\ n \in \mathbb{Z}$, all represent the identity on S^1, they act non-trivial on $A^\pm$. Nevertheless, their action on φ will be trivial.) The corresponding action on φ is given by

$$\varphi(x^+, x^-) \to \varphi_{\alpha^+,\alpha^-}(x^+, x^-) = \varphi(\alpha^+(x^+), \alpha^-(x^-)) + \log(\partial_+\alpha^+) + \log(\partial_-\alpha^-), \tag{2.28}$$

yielding a new solution to the Liouville equation. Thus $\mathrm{Diff}^+(S^1)\times\mathrm{Diff}^+(S^1)$ acts as a symmetry group on the phase space of Liouville theory. This symmetry is known as the conformal symmetry, a term that becomes clear in the Wick rotated theory, i.e., when fields like $A^+(x^+)$ are analytically continued to imaginary time (see Sect. 5).

Let us calculate how the functions $T_\pm$ defined in (2.4) transform. To this end, it is useful to observe that $T_\pm$ is the Schwarzian derivative[3] of $A^\pm$:

$$T_\pm \;=\; S(A^\pm), \qquad S(f) :=\; -\frac{1}{2}\frac{f'''}{f'} + \frac{3}{4}\left(\frac{f''}{f'}\right)^2. \tag{2.29}$$

A little bit more computation then yields

$$T_\pm \;\to\; T_\pm^{\alpha^\pm} \;=\; S(A^\pm \circ \alpha^\pm) \;=\; (\partial_\pm\alpha^\pm)^2 T_\pm \circ \alpha^\pm + S(\alpha^\pm). \tag{2.30}$$

Let $\mathrm{Vect}(S^1)$ denote the space of smooth vector fields on S^1, which is the tangent space or Lie algebra of $\mathrm{Diff}^+(S^1)$. Let $\epsilon > 0$, $[-\epsilon, \epsilon] \ni t \mapsto \alpha_t$ a smooth curve in $\mathrm{Diff}^+(S^1)$ with $\alpha_0 = \mathrm{Id}$. Then $\xi = \xi(x)\partial_x \in \mathrm{Vect}(S^1)$ with $\xi(x) := \frac{\partial}{\partial t}\alpha_t(x)\big|_{t=0}$ is the tangent vector at $t = 0$ to that curve. From (2.30), one can compute the action of ξ on T (which stands for T_+ or T_-):

$$\delta_\xi T(x) \;:=\; \frac{\partial}{\partial t}T^{\alpha_t}\bigg|_{t=0} \;=\; \xi(x)T'(x) + 2\xi'(x)T(x) - \frac{1}{2}\xi'''(x). \tag{2.31}$$

This transformation behavior is known as the coadjoint action of the universal central extension of $\mathrm{Vect}(S^1)$. The next section is devoted to this concept.

[3]The Schwarzian derivative is often defined in a way that differs by a factor of -2 from (2.29).

2.3 Coadjoint orbits

2.3.1 Definition of the coadjoint representation

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Then the adjoint action of $\mathfrak{g}$ on itself is given by

$$\mathrm{ad}_v(w) = [v, w], \qquad v, w \in \mathfrak{g}. \tag{2.32}$$

By the Jacobi identity this defines a representation of $\mathfrak{g}$ on itself (corresponding to the action of G on itself by conjugation). The coadjoint action of $\mathfrak{g}$ on its dual $\mathfrak{g}^*$ is defined by

$$\mathrm{ad}^*_v(b)(w) = -b([v, w]), \qquad v, w \in \mathfrak{g}, b \in \mathfrak{g}^*. \tag{2.33}$$

This definition was made in such a way that the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$ is invariant under the $\mathfrak{g}$ action, meaning that

$$(\mathrm{ad}^*_v(b))(w) + b(\mathrm{ad}_v(w)) = 0. \tag{2.34}$$

It follows that (2.33) also defines a representation of $\mathfrak{g}$. The orbits of the corresponding action Ad^* of G on $\mathfrak{g}^*$ (the 'exponentiation' of (2.33)) are called coadjoint orbits of the group G. They are usually characterized by the following method. To every $b \in \mathfrak{g}^*$ belongs an orbit $W_b = \{\mathrm{Ad}^*_g(b) \mid g \in G\}$ (and every orbit is of this form). Find $g_0 \in G$ such that $b_0 = \mathrm{Ad}^*_{g_0}(b)$ has a particularly simple form and determine the subgroup G_0 of G that leaves b_0 invariant. Then the map $G \to W_b,\ g \mapsto \mathrm{Ad}^*_g(b_0)$, provides the identification of W_b with the quotient G/G_0.

2.3.2 Symplectic structure

The coadjoint orbits carry a unique symplectic structure that is invariant under the G-action [42]. It is defined as follows: Let $W \subset \mathfrak{g}^*$ be a coadjoint orbit and $a, a' \in \mathfrak{g}^*$ two tangent vectors to W at a point $b \in W$. Then there exist $v, v' \in \mathfrak{g}$ such that $\mathrm{ad}^*_v(b) = a,\ \mathrm{ad}^*_{v'}(b) = a'$ and we define the 2-form ω at b by

$$\omega_b(a, a') := b([v, v']). \tag{2.35}$$

This is well defined since the right hand side is invariant under shifts $v \to v+u$ and $v' \to v'+u'$ with $\mathrm{ad}^*_u(b) = \mathrm{ad}^*_{u'}(b) = 0$. It is not difficult to show that ω is non-degenerate and closed and thus defines a symplectic structure on W. Let us take a moment to verify the G-invariance of ω. Since the action of G on $\mathfrak{g}^*$ is by linear maps, the vectors a and a' also transform in the coadjoint representation of G. Infinitesimally, the action of some $w \in \mathfrak{g}$ on a is $\delta_w a = \mathrm{ad}^*_w(a)$ and similar for a'. This has to correspond to a transformation $\delta_w v$ of v. Since $a(u) = -b([v, u])$ for all $u \in \mathfrak{g}$, the relation that determines $\delta_w v$ is

$$\forall u \in \mathfrak{g}: \quad \mathrm{ad}^*_w(a)(u) \;=\; -\mathrm{ad}^*_w(b)([v, u]) - b([\delta_w v, u]). \tag{2.36}$$

By the definition (2.33) and with the help of the Jacobi identity, one finds that $\delta_w v = [w, v]$ (up to some $u \in \mathfrak{g}$ with $\mathrm{ad}^*_u(b) = 0$). Employing the Jacobi identity once again, we find that

$$\delta_w(\omega_b(a, a')) \;=\; \mathrm{ad}^*_w(b)([v, v']) + b([\delta_w v, v']) + b([v, \delta_w v']) \;=\; 0, \tag{2.37}$$

which shows the (infinitesimal) G-invariance of ω.

The symplectic form ω corresponds to a Poisson bracket $\{\cdot,\cdot\}_{\mathrm{O}}$ on W that resembles the Lie bracket on $\mathfrak{g}$ in the following way. Each $v \in \mathfrak{g}$ corresponds to a linear function Φ_v on $\mathfrak{g}^*$ defined by $\Phi_v(b) = b(v)$. These functions can be restricted to W and their Poisson brackets are given by [42]

$$\{\Phi_u, \Phi_v\}_{\mathrm{O}} \;=\; \Phi_{[u,v]} \qquad (u, v \in \mathfrak{g}). \tag{2.38}$$

To prove (2.38), let us find the hamiltonian vector field X_u on W that is generated by Φ_u, i.e., $\{\Phi_u, f\}_{\mathrm{O}} = X_u(f)$ for any function f on W. By the definition of the Poisson bracket this is the vector field with the property

$$d\Phi_u(Y) = \omega(Y, X_u) \tag{2.39}$$

for all vector fields Y. Equation (2.39) can be checked point-wise at each $b \in W$. As before, a tangent vector a at b can be represented by some $w \in \mathfrak{g}$ with $\mathrm{ad}^*_w(b) = a$. Then we have

$$d\Phi_u|_b(a) = D_a(\Phi_u) = \Phi_u(a) = -b([w,u]) = \omega_b(a, -\mathrm{ad}^*_u(b)), \tag{2.40}$$

where D_a denotes the derivative in the direction of a at b and the second equation holds since Φ_u is linear. Thus we have found $X_u(b) = -\mathrm{ad}^*_u(b)$, i.e., Φ_u is the generating function for the negative coadjoint action of u on W. Then immediately follows

$$\{\Phi_u, \Phi_v\}_{\mathrm{O}}(b) \;=\; D_{-\mathrm{ad}^*_u(b)}(\Phi_v) \;=\; -\Phi_v(\mathrm{ad}^*_u(b)) \;=\; b([u,v]) \;=\; \Phi_{[u,v]}(b). \tag{2.41}$$

Given a basis $\{v_i\}$ of $\mathfrak{g}$, the Lie bracket may be written in the form $[v_i, v_j] = \sum_k f^k_{ij} v_k$ with structure constants f^k_{ij}. As Φ_u is linear in u, it follows that

$$\{\Phi_i, \Phi_j\}_{\mathrm{O}} = \sum_k f^k_{ij}\Phi_k, \qquad \Phi_i := \Phi_{v_i}. \tag{2.42}$$

From this it is clear that quantization[4] of the coadjoint orbits (which is possible only for some orbits) should give rise to representations of G on a Hilbert space. In fact, it is known [20, 19] that these are unitary irreducible representations and that the coadjoint orbits can be used to classify all such representations of G. This method is known as Kirillov-Kostant representation theory.

2.3.3 Coadjoint orbits of the Virasoro-Bott group

Let us now consider the case where G is the Virasoro-Bott group. The latter is the Lie group with Lie algebra $\widehat{\mathrm{Vect}}(S^1) := \mathrm{Vect}(S^1) \oplus i\mathbb{R}c$, the universal central extension of $\mathrm{Vect}(S^1)$, where $\mathrm{Vect}(S^1)$ is the space of smooth vector fields on S^1 and c is the central element (usually identified with a real number). The Lie bracket between two vector fields $\xi = \xi(\sigma)\frac{\partial}{\partial\sigma}, \zeta = \zeta(\sigma)\frac{\partial}{\partial\sigma}$ is given by

$$[\xi, \zeta] \;=\; (\xi(\sigma)\zeta'(\sigma) - \zeta(\sigma)\xi'(\sigma))\frac{\partial}{\partial\sigma} + \frac{ic}{24\pi}\int_0^{2\pi} d\sigma \xi(\sigma)\zeta'''(\sigma). \tag{2.43}$$

One often considers the complexification $\widehat{\mathrm{Vect}}^{\mathbb{C}}(S^1) = \widehat{\mathrm{Vect}}(S^1) \otimes \mathbb{C}$, known as the Virasoro algebra.[5] It has generators $L_n := ie^{in\sigma}\frac{\partial}{\partial\sigma} + \frac{c}{24}\delta_{n,0}$ ($n \in \mathbb{Z}$) (and c), in terms of which (2.43) becomes

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.44}$$

The (real) dual space $\widehat{\mathrm{Vect}}(S^1)^*$ can be identified with the direct sum of the space of quadratic differentials on S^1 and the one-dimensional space spanned by an element $\hat{c}$ dual to c, i.e.

[4] see Section 4.1 for a definition of this word

[5] Strictly speaking, the Virasoro algebra $\mathfrak{V} = \mathrm{Span}\{L_n\}_{n\in\mathbb{Z}}$ consists of only finite sums.

$\hat{c}(c) = 1, \hat{c}(\xi) = 0$ for all $\xi \in \mathrm{Vect}(S^1)$. An element $(t,b) := t(\sigma)(d\sigma)^2 - ib\hat{c}$ ($b \in \mathbb{R}$) of this space acts on an element $(\xi, a) := \xi(\sigma)\frac{\partial}{\partial\sigma} + iac$ of $\widehat{\mathrm{Vect}}(S^1)$ as

$$(t,b)((\xi,a)) = \int_0^{2\pi} t(\sigma)\xi(\sigma)d\sigma + ba. \tag{2.45}$$

The invariance of this pairing implies the following coadjoint action of $(\zeta, e) \in \widehat{\mathrm{Vect}}(S^1)$ on $(t,b) \in \widehat{\mathrm{Vect}}(S^1)^*$:

$$\mathrm{ad}^*_{(\zeta,e)}(t,b) \;=\; (\zeta \cdot t' + 2\zeta' \cdot t + \tfrac{b}{24\pi}\zeta''', 0). \tag{2.46}$$

So b is constant on a coadjoint orbit. For $b = -6$, the variation of t coincides with that of $t := \frac{1}{2\pi}T$ under the action of $\mathrm{Vect}(S^1)$ as given in (2.31) (b is fixed by the requirement that the Poisson bracket on the orbit coincides with the physical Poisson bracket on $\mathcal{P}$). As this transformation of T corresponds to a simple change of variables of the quotient $A := f_1/f_2$ of two solutions f_1, f_2 to the Hill's equation $f''(x) = T(x)f(x)$, it is customary to identify $\widehat{\mathrm{Vect}}(S^1)^*$ for fixed b with the space of Hill's operators $H_t = \partial_\sigma^2 - t(\sigma)$ (also known as projective connections).

We will now investigate what kind of coadjoint orbit T can lie on. The coadjoint orbits of the Virasoro group have been extensively studied and classified in the literature [21, 42, 4]. A first classification is made by the trace of the monodromy matrix M assigned to the Hill's equation. As explained earlier, we are interested only in the hyperbolic case where $|\mathrm{tr}M| > 2$. Since $A' > 0$ and $A(x + 2\pi) = e^{2\pi p}A(x)$ with $p > 0$, we also have $\lim_{x\to\infty} A(x) = \infty$ and $\lim_{x\to-\infty} A(x) = 0$, and as a consequence $A > 0$. It follows that the function

$$\alpha(x) := \frac{1}{p}\log(A(x)) \tag{2.47}$$

defines an element of $\mathrm{Diff}^+(S^1)$. Acting with its inverse α^{-1} on A we get

$$A_{\alpha^{-1}}(x) = e^{px}, \tag{2.48}$$

and thus

$$T_{\alpha^{-1}} = S(A_{\alpha^{-1}}) = \frac{p^2}{4}. \tag{2.49}$$

So T_+ and T_- always lie on an orbit $\mathcal{W}_p$ that contains the constant function $T_p := p^2/4$. This orbit can be characterized by the subgroup of $\mathrm{Diff}^+(S^1)$ that leaves T_p invariant. The (infinitesimal) invariance condition

$$0 = \delta_\xi T_p = \frac{p^2}{2}\xi' - \frac{1}{2}\xi''', \tag{2.50}$$

together with periodicity of ξ, implies that ξ is constant. As the constant vector fields on S^1 generate the subgroup of rigid rotations of S^1 (which can be identified with S^1), the coadjoint orbit of T is isomorphic to $\mathrm{Diff}^+(S^1)/S^1$ via the map $T \mapsto [\alpha^{-1}]$ (we adopt the convention that S^1 acts on $\mathrm{Diff}^+(S^1)$ from the left). The space of pairs of functions (T_+, T_-) assigned to solutions of the Liouville equation can thus be identified with $(\mathrm{Diff}^+(S^1)/S^1) \times (\mathrm{Diff}^+(S^1)/S^1) \times \mathbb{R}_{>0}$, where the last factor contains p. On the other hand, a pair (T_+, T_-) carries all the information about the Liouville field φ except for the constant q in (2.24). (This is seen by considering the subset of $\mathrm{SL}(2,\mathbb{R})$ transformations (2.12) that leave the monodromy matrices of the form (2.13) invariant; it consists of the matrices $\mathrm{diag}(a^{-1}, a)$ with $a \in \mathbb{R}$, which transform A as $A \to a^2 A$.) Therefore we may describe the phase space of Minkowskian Liouville theory as

$$\mathcal{P} \;\simeq\; (\mathrm{Diff}^+(S^1)/S^1) \times (\mathrm{Diff}^+(S^1)/S^1) \times \mathbb{R}_{>0} \times \mathbb{R}. \tag{2.51}$$

Let us take a look at the Poisson structure on the coadjoint orbit $\mathrm{Diff}^+(S^1)/S^1$. The Virasoro generators L_n correspond to linear functions $l_n := -i\Phi_{L_n}$ on $\widehat{\mathrm{Vect}}(S^1)^*$. When restricted to a coadjoint orbit with $b = -6$, they are, according to (2.45), defined by

$$l_n(t) = \int d\sigma\, e^{in\sigma} t(\sigma) + \frac{1}{4}\delta_{n,0}. \tag{2.52}$$

According to (2.42) and (2.44), their Poisson algebra is

$$\{l_m, l_n\}_{\mathrm{O}} = -i(m-n)l_{m+n} - \frac{i}{2}(m^3 - m)\delta_{m+n,0}, \tag{2.53}$$

where we have used $\Phi_c = -ib = 6i$. By the relation $t = \frac{1}{2\pi}T_+$, the l_n are the Fourier modes of T_+ (up to the shift in l_0). By (2.21) and (2.24), they are given by

$$l_n \;=\; \frac{1}{4}\sum_{k\in\mathbb{Z}} a_k a_{n-k} + \frac{1}{2} i n a_n + \frac{1}{4}\delta_{n,0}. \tag{2.54}$$

Their (physical) Poisson algebra as derived from (2.26) takes the form

$$\{l_m, l_n\} \;=\; -i(m-n)l_{m+n} - \frac{i}{2}(m^3 - m)\delta_{m+n,0}. \tag{2.55}$$

Since the Poisson brackets $\{l_m, l_n\}_{\mathrm{O}}$ and $\{l_m, l_n\}$ coincide, and the a_n can be in principle expressed in terms of the l_n via the solution of the Hill's equation, it follows that the Poisson bracket on $\mathrm{Diff}^+(S^1)/S^1$ that is obtained from the coadjoint orbit method coincides with the physical Poisson bracket obtained from the Liouville action. Quantization of Liouville theory can therefore be described as quantization of coadjoint orbits of the Virasoro-Bott group.

3 Teichmüller spaces

In this section we define the classical Teichmüller spaces for punctured Riemann surfaces and Riemann surfaces with holes. We introduce the shear coordinates, hyperbolic length functions, and a set of complex coordinates for genus zero. This is known material, only the definition of the monodromy matrices M_c has been slightly refined.

3.1 Definition of Teichmüller spaces

Let us recall that a Riemann surface is a topological space equipped with a complex structure, i.e., the equivalence class of an atlas of $\mathbb{C}$-valued charts with holomorphic transition functions. Two Riemann surfaces Σ and Σ' are said to be conformally equivalent or isomorphic ($\Sigma \simeq \Sigma'$) if there exists a biholomorphic map $\Phi : \Sigma \to \Sigma'$. A Riemann surface of genus g with n punctures (sometimes called marked points) or shortly of type (g, n) is one that is conformally equivalent to a compact Riemann surface with g handles from which n points are removed. We will sometimes also consider bordered Riemann surfaces or Riemann surface with holes, but we postpone the precise definition until Section 9. Then we can define the moduli space $\mathcal{M}_{g,n}$ as the set of equivalence classes of Riemann surfaces of type (g, n). Alternatively, one can fix

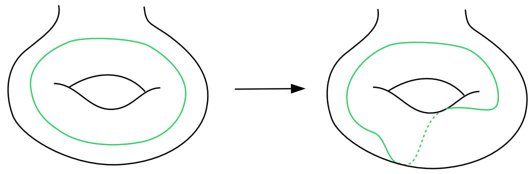

Figure 1: Dehn twist

a Riemann surface or simply a 2-dimensional real manifold Σ of type (g,n) and consider the space $\mathcal{C}(\Sigma)$ of complex structures on it. Then we have

$$\mathcal{M}_{g,n} = \mathcal{M}(\Sigma) = \mathcal{C}(\Sigma)/\mathrm{Diff}(\Sigma), \tag{3.1}$$

where $\mathrm{Diff}(\Sigma)$ is the group of diffeomorphisms of Σ that naturally acts on $\mathcal{C}(\Sigma)$. These two models for $\mathcal{M}_{g,n}$ are equivalent by the fact that all 2-dimensional real manifolds of type (g,n) are diffeomorphic.

Closely related is the Teichmüller space

$$\mathcal{T}_{g,n} = \mathcal{T}(\Sigma) = \mathcal{C}(\Sigma)/\mathrm{Diff}_0(\Sigma). \tag{3.2}$$

where $\mathrm{Diff}_0(\Sigma)$ is the group of diffeomorphisms of Σ that are isotopic to the identity. The last requirement makes $\mathcal{T}(\Sigma)$, as opposed to $\mathcal{M}(\Sigma)$, simply connected.

Another definition uses the concept of a marking, which is a set $\{a_j, b_j, c_k\}_{j=1,\dots,g}^{k=1,\dots,n-1}$ of canonical generators[6] of the fundamental group of Σ. A conformal equivalence between two marked Riemann surfaces is a biholomorphic map that transforms the markings into each other. Then $\mathcal{T}_{g,n}$ is the set of equivalence classes of marked Riemann surfaces of type (g,n).

The relation between $\mathcal{M}(\Sigma)$ and $\mathcal{T}(\Sigma)$ is

$$\mathcal{M}(\Sigma) = \mathcal{T}(\Sigma)/\mathrm{MCG}(\Sigma) \tag{3.3}$$

where $\mathrm{MCG}(\Sigma) = \mathrm{Diff}(\Sigma)/\mathrm{Diff}_0(\Sigma)$ is called the mapping class group of Σ. Since $\mathcal{T}(\Sigma)$ is simply connected, this implies that $\mathcal{T}_{g,n}$ is (isomorphic to) the universal covering space of $\mathcal{M}_{g,n}$, a term that will be defined in the next section. Intuitively, the Teichmüller space keeps track of the so called Dehn twists which generate $\mathrm{MCG}(\Sigma)$ and have the effect of twisting a handle (Fig. 1).

3.2 Universal covering spaces and uniformization

Let X be a topological space and $p \in X$ a fixed point.[7] A path in X is a continuous map $\gamma : [0,1] \to X$. Two paths γ_1, γ_2 with $\gamma_1(0) = \gamma_2(0),\ \gamma_1(1) = \gamma_2(1)$ in X are called homotopic if there exists a continuous map $H : [0,1] \times [0,1] \to X$ such that $H(t,0) = \gamma_1(t),\ H(t,1) = \gamma_2(t)$ for all $t \in [0,1]$ and $H(0,s) = \gamma_1(0),\ H(1,s) = \gamma_1(1)$ for all $s \in [0,1]$. (Intuitively, H is a continuous deformation of γ_1 into γ_2 that leaves the end points fixed.) We define the universal covering space $\tilde{X}$ of X by[8]

$$\tilde{X} := \{[\gamma] \,|\, \gamma \text{ path in } X \text{ with } \gamma(0) = p\}. \tag{3.4}$$

[6]Here 'canonical' means that each pair (a_j, b_j) consists of two loops associated with one handle while the c_k are the loops that encircle one puncture only.

[7]A detailed exposition of the content of this section, including proofs, can be found in [17, Ch. 2].

[8]The usual, more abstract definition uses the concept of covering spaces. Then $\tilde{X}$ would be a (universal) covering space of X with projection $\pi : \tilde{X} \to X,\ [\gamma] \mapsto \gamma(1)$.

The crucial property of $\tilde{X}$ is that it is simply connected (i.e., every closed path in $\tilde{X}$ is homotopic to a constant path). If X is a Riemann surface, then $\tilde{X}$ is also a Riemann surface with the complex structure that is induced by the projection $\pi : \tilde{X} \to X,\ [\gamma] \mapsto \gamma(1)$.
The fundamental group $\pi_1(X,p)$ of X at p is defined by

$$\pi_1(X,p) \;=\; \{[C]\,|\,C \text{ path in } X \text{ with } C(0)=C(1)=p\}. \tag{3.5}$$

The universal covering transformation group $\tilde{\Gamma}$ consists of the maps

$$[C]_* :\; \tilde{X} \to \tilde{X},\; [C]_*([\gamma]) := [C\cdot\gamma], \qquad [C] \in \pi_1(X,p), \tag{3.6}$$

where $C \cdot \gamma$ denotes the concatenation of C and γ. Obviously $\tilde{\Gamma}$ is isomorphic to $\pi_1(X,p)$. One has also the isomorphism $X \simeq \tilde{X}/\tilde{\Gamma}$. If $X = \Sigma$ is a Riemann surface, then $\tilde{\Sigma}/\tilde{\Gamma}$ is also a Riemann surface and this becomes a conformal equivalence

$$\Sigma \;\simeq\; \tilde{\Sigma}/\tilde{\Gamma}. \tag{3.7}$$

The last statement is particularly useful in combination with the following classical theorem by Klein, Poincaré and Koebe.

Uniformization theorem. *Every simply connected Riemann surface is conformally equivalent to either the complex plane $\mathbb{C}$, the Riemann sphere $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ or the upper half plane $\mathbb{H} := \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}$.*

Recall that $\bar{\mathbb{C}}$ comes by definition with the complex structure that is defined by the atlas consisting of the two charts $\mathrm{Id} : \mathbb{C} \to \mathbb{C}$ and $J : \bar{\mathbb{C}}\backslash\{0\} \to \mathbb{C},\ z \mapsto z^{-1}, \infty \mapsto 0$. Furthermore $\mathbb{C}, \bar{\mathbb{C}}$ and $\mathbb{H}$ are mutually nonequivalent, but $\mathbb{H}$ is conformally equivalent to the unit disk $D := \{z \in \mathbb{C} \mid |z| < 1\}$.

Let Σ be a Riemann surface. Applying the uniformization theorem to the universal covering surface $\tilde{\Sigma}$, we obtain a biholomorphic map $A :\; \tilde{\Sigma} \tilde{\to} \Delta$, where $\Delta \in \{\mathbb{C}, \bar{\mathbb{C}}, \mathbb{H}\}$, called the uniformization map, and by (3.7) an isomorphism

$$\Sigma \;\simeq \Delta/\Gamma, \tag{3.8}$$

where $\Gamma := A\tilde{\Gamma}A^{-1}$ is a subgroup of the automorphism group $\mathrm{Aut}(\Delta)$ and also isomorphic to $\pi_1(\Sigma,p)$. The elements of Γ are called the monodromies of A.

The Riemann surfaces we are interested in are those with $N := 2g-2+n > 0$, referred to as the hyperbolic type. In this case we always have $\Delta = \mathbb{H}$ and Γ is a Fuchsian group, i.e., a discrete subgroup of $\mathrm{Aut}(\mathbb{H})$. The latter is the group of real Möbius transformations which is isomorphic to $\mathrm{PSL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})/\{\pm 1\}$. (the automorphism groups of standard Riemann surfaces are described in Appendix D). Given two Fuchsian groups Γ and Γ', the Riemann surfaces $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Gamma'$ are conformally equivalent if and only if there exists $g \in \mathrm{Aut}(\mathbb{H})$ such that $\Gamma' = g\Gamma g^{-1}$. This corresponds to the freedom $A \to g \circ A$ we have in the choice of A. Therefore, the conformal equivalence class of Σ is characterized by the conjugacy class of Γ within $\mathrm{Aut}(\mathbb{H})$.

Uniformization of Σ also endows it with a unique hyperbolic metric (i.e., a metric of constant curvature -1). Namely, $\mathbb{H}$ carries the hyperbolic Poincaré metric

$$ds_P^2 := \frac{dz d\bar{z}}{(\Im z)^2}, \tag{3.9}$$

which is invariant under $\mathrm{Aut}(\mathbb{H})$, in particular invariant under Γ. Therefore it induces a hyperbolic metric ds^2 on $\Sigma \simeq \mathbb{H}/\Gamma$. This metric can be used to introduce coordinates on $\mathcal{T}(\Sigma)$, such as the hyperbolic lengths of closed geodesics on Σ.

3.3 The euclidean Liouville equation

In terms of a coordinate z on Σ, the hyperbolic metric is

$$ds^2 = \frac{\partial A \bar{\partial} \bar{A}}{(\Im A)^2} dz d\bar{z}, \qquad \partial \equiv \frac{\partial}{\partial z}, \ \bar{\partial} \equiv \frac{\partial}{\partial \bar{z}}. \tag{3.10}$$

Writing this in the form $ds^2 = e^{\varphi_e} dz d\bar{z}$, one finds that the real valued function

$$\varphi_e = \log \frac{\partial A \bar{\partial} \bar{A}}{(\Im A)^2} \tag{3.11}$$

which we will call the scaling factor (of the hyperbolic metric), is a solution to the *euclidean* Liouville equation

$$\partial \bar{\partial} \varphi_e = \tfrac{1}{2} e^{\varphi_e}. \tag{3.12}$$

The analysis of this equation is in some way analogous to the solution of the Minkowskian Liouville equation (2.2). For instance, one can define the function (cf. (2.4), the different normalization is conventional)

$$T(z) := \partial^2 \varphi_e - \tfrac{1}{2} (\partial \varphi_e)^2, \tag{3.13}$$

which is holomorphic ($\bar{\partial} T = 0$) if φ_e satisfies (3.12). In order to recover φ_e from T, one has to find two linearly independent holomorphic solutions f_i ($i = 1, 2$) to the Hill's equation (cf. (2.5))

$$(\partial^2 + \tfrac{1}{2} T(z)) f(z) = 0. \tag{3.14}$$

In general the f_i will have monodromies, i.e., they are analytic functions defined on the universal cover of Σ. The same is true for the function[9] (cf. (2.14))

$$A(z) = \frac{f_2(z)}{f_1(z)}. \tag{3.15}$$

For generic T, the monodromies of (f_1, f_2) are in $\mathrm{SL}(2, \mathbb{C})$. But if T is chosen such that the monodromies of (f_1, f_2) are in $\mathrm{SL}(2, \mathbb{R})$, then φ_e as defined in (3.11) is a solution to (3.12) on Σ. In order for φ_e to be the scaling factor of the hyperbolic metric and A the uniformization map, φ_e also has to satisfy certain boundary conditions at the punctures [33]. Namely, if there is a puncture at $z = z_0$, then

$$\varphi_e(z) = -2 \log |z - z_0| - 2 \log |\log |z - z_0|| + O(1) \text{ as } z \to z_0. \tag{3.16}$$

This asymptotic behavior is called a parabolic singularity (because the monodromy M_0 of A around z_0 is a parabolic element of $\mathrm{SL}(2, \mathbb{R})$, characterized by $|\mathrm{tr} M_0| = 2$) and corresponds to

$$T(z) = \tfrac{1}{2} (z - z_0)^{-2} + O((z - z_0)^{-1}) \text{ as } z \to z_0. \tag{3.17}$$

The relation between the euclidean and the Minkowskian Liouville equation is slightly non-trivial. One may observe the following: Let $\varphi(x^+, x^-)$ be a solution to (2.2). Now suppose that φ can be analytically continued to complex values of x^+, x^-. This would correspond to an analytic continuation of the functions A^+, A^- in (2.20), so the extended φ would still be a solution to (2.2), with the only difference that $\partial_\pm$ is now interpreted as the *holomorphic* derivative by $x^\pm$. Then define for complex $w = \tau + i\sigma$ ($\tau, \sigma \in \mathbb{R}$)

$$\hat{\varphi}_e(w, \bar{w}) := \varphi(-iw, -i\bar{w}) + \log(4). \tag{3.18}$$

[9] The relation between $A(z)$ and $T(z)$ is $T = -2S(A)$, cf. (2.29).

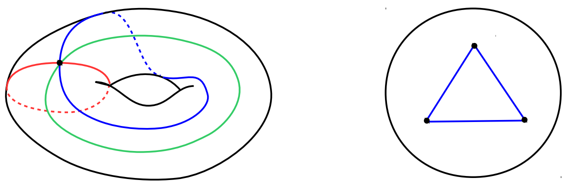

Figure 2: Ideal triangulations of the one-punctured torus and the 3-punctured sphere

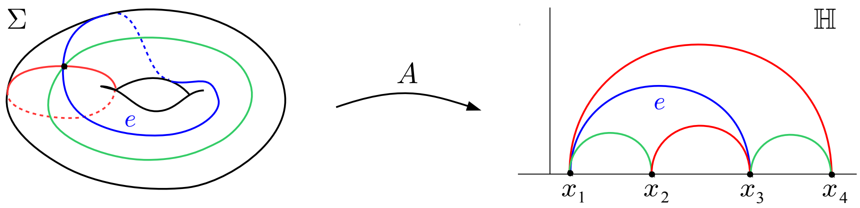

Figure 3: Uniformization of the one-punctured torus

This is known as the Wick rotation $\tau = it$. The function $\hat{\varphi}_e$ would already satisfy (3.12), but it is conventional to change the variable to $z = e^w$ (which nicely incorporates the periodic boundary conditions). Then

$$\varphi_e(z, \bar{z}) := \hat{\varphi}_e(\log(z), \log(\bar{z})) - \log(z\bar{z}) \tag{3.19}$$

is a solution to (3.12) on $C^* := \mathbb{C}\backslash\{0\}$. However, φ_e would not necessarily be a real valued function. Since $\partial_\tau \hat{\varphi}_e|_{\tau=0} = -i\partial_t \varphi|_{t=0}$, φ_e is real only if $\partial_t \varphi|_{t=0} = 0$.

Despite of this obstacle, there seems to be a connection between the quantum analog of the function A^+ and the uniformization map A in (quantum) Teichmüller theory as we will see in Section 5.

3.4 Triangulations and fat graphs

In this section we consider only hyperbolic Riemann surfaces Σ with at least one puncture. An ideal triangulation of Σ is the isotopy class of a set τ of disjoint curves (edges) in Σ starting and ending at the punctures that decompose Σ into triangles. Two examples are depicted in Figure 2. An ideal triangulation always consists of $3N = 6g - 6 + 3n$ edges (this is shown inductively by adding punctures or handles to the surface). These can be chosen as geodesics with respect to the hyperbolic metric on Σ. Then the preimages of an edge e under the canonical projection $\pi : \tilde{\Sigma} \to \Sigma$ will be mapped to geodesics in $\mathbb{H}$ (w.r.t. the metric (3.9)) by the uniformization map A, which we will simply call the images of e and which are mapped into each other by elements of the Fuchsian group Γ. Geodesics in $\mathbb{H}$ are half circles with ends lying on $\mathbb{R}$ or vertical lines going from the real axis to infinity (or segments of these lines).

Let us figure out what happens to the punctures under A. Strictly speaking, a puncture p_0 is not part of Σ, but one could choose a path γ in Σ that approaches p_0 (e.g. an edge in τ). This path has infinitely many lifts (i.e., preimages under the canonical projection) $\tilde{\gamma}_k$, $k = 1, 2, \ldots,$ in $\tilde{\Sigma}$. After choosing one $\tilde{\gamma}_k$, the limit of $A(p)$, as p goes along $\tilde{\gamma}_k$ is well defined and has to be a point of the real axis, the boundary of $\mathbb{H}$. We consider this point to be an image of p_0 under A. As a result of these considerations one can obtain a model of Σ by gluing hyperbolic triangles in $\mathbb{H}$, i.e., triangles whose edges are geodesics with vertices on the real axis (or at infinity). An example is depicted in Figure 3.

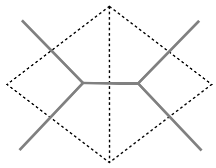

Figure 4: Two triangles of an ideal triangulation (dashed) and the dual fat graph

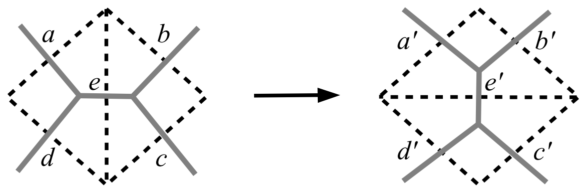

Figure 5: Flip along the edge e

Each ideal triangulation τ has a dual graph f called a fat graph. Associated to each triangle is a vertex of the fat graph, which is connected to three edges that cross the three edges of that triangle as illustrated in Figure 4. Thus the dual fat graph has also $3N$ edges, each of them associated to an edge in τ.

An ideal triangulation respectively a fat graph can be changed by elementary moves called flips. A flip ω_e along an edge e of τ (resp. along the dual edge of f) consists of a change of the diagonal in the quadrangle formed by the two triangles to which e belongs as depicted in Fig. 5 (this is not possible in the special case where these triangles coincide). It can be shown that any two ideal triangulations of the same surface can be converted into another by a sequence of flips [25]. The set of all ideal triangulations for a given surface together with all possible flips and sequences of flips form a groupoid, called the Ptolemy groupoid.

Let v be a vertex of f. Then for each element c in the fundamental group $\pi_1(\Sigma, v)$ there exists a unique closed path h_c on f of minimal length (i.e., there are no 180 degree turns except at the starting point v) that represents c. Also there exists a unique path g_c on f of minimal length which is in the free homotopy class corresponding to c.

The previous definitions can be adjusted to the case of bordered Riemann surfaces. An ideal triangulation is then the isotopy class of a set of disjoint curves connecting the boundary components (or holes) of Σ that decompose Σ into hexagons (these become triangles when the holes shrink to punctures). As before, a fat graph is the dual graph to an ideal triangulation.

3.5 Shear coordinates

With the help of the tools introduced in the previous section, we are now able to define a set of coordinates on Teichmüller space that are of particular interest to us. Here we work with the model (3.2) of $\mathcal{T}_{g,n}$, i.e., we have fixed a surface Σ with an ideal triangulation τ and consider different (equivalence classes of) complex structures on it. These correspond to different uniformization maps $A : \tilde{\Sigma} \to \mathbb{H}$.

To each edge e of τ we assign the shear coordinate w_e, first introduced in [13], as follows: Choose an image of e in $\mathbb{H}$ as in Fig. 3. This half circle delimits two hyperbolic triangles in $\mathbb{H}$ forming a hyperbolic quadrangle. The positions $x_1, \ldots, x_4 \in \mathbb{R} \cup \{\infty\}$ of the vertices of this

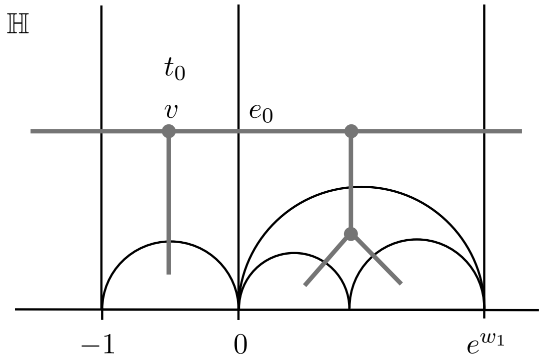

Figure 6: Constructing M_c

quadrangle determine the value of w_e as

$$w_e \; := \; \log \left| \frac{(x_4 - x_3)(x_2 - x_1)}{(x_4 - x_1)(x_3 - x_2)} \right| . \tag{3.20}$$

In fact, w_e does not depend on the choice of an image of e as the double cross-ratio in (3.20) is invariant under Möbius transformations, in particular under the monodromies of A. If $\check{e}$ is the edge in the dual fat graph which crosses e, then we also write $w_{\check{e}}$ for w_e.

From the set of $3N$ shear coordinates $\{w_e\}_{e\in\tau}$ associated to τ, it is possible to reconstruct the conjugacy class of the Fuchsian group Γ and thus the conformal equivalence class of Σ. To this end, assign to each $c \in \pi_1(\Sigma, v)$, an element $M_c \in \mathrm{SL}(2,\mathbb{R})$ as follows. First choose one of the three edges that emanate from v and denote it by e_0, which has to be the same for all $c \in \pi_1(\Sigma, v)$. The closed path h_c on the fat graph that represents c consists of a series $\{e_i\}_{i=1,\dots,r}$ of consecutive edges, where e_1 starts at v. Define s_0 to be 1 resp. -1 if e_1 is the edge that is next to e_0 at v in the counter-clockwise resp. clockwise direction, and 0 if $e_1 = e_0$. For $i = 1, \dots, r$, define s_i to be 1 if h_c turns left at the vertex that connects e_i with e_{i+1}, -1 if it turns right, and 0 if it makes a 180 degree turn (for $i = r$), where $e_{r+1} := e_1$. Then define the matrix

$$M_c \; := \; V^{s_0} V^{s_r} E(w_{e_r}) \dots V^{s_1} E(w_{e_1}) V^{-s_0}, \tag{3.21}$$

where $E(w)$ and V are given by

$$E(w) := \begin{pmatrix} 0 & e^{\frac{w}{2}} \\ -e^{-\frac{w}{2}} & 0 \end{pmatrix}, \qquad V := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.22}$$

The Fuchsian group $\hat{\Gamma} := \{M_c\}_{c\in\pi_1(\Sigma,v)}$ is then conjugacy equivalent to Γ and Σ is isomorphic to $\mathbb{H}/\hat{\Gamma}$. This construction was first given (in a slightly reduced form) in [12]. It can be explained as follows [34]. The ideal triangulation τ of Σ corresponds to a tessellation of the upper half plane by hyperbolic triangles (the preimage of τ under the canonical projection $\pi : \mathbb{H} \to \mathbb{H}/\Gamma \simeq \Sigma$). Given two hyperbolic triangles t, t' with numbered edges, the exists a unique element of $\mathrm{PSL}(2,\mathbb{R})$ that maps t onto t'. Let t_0 be a preimage in $\mathbb{H}$ of the triangle that contains v. Then each path h_c with $c \in \pi_1(\Sigma, v)$ is represented by a path $\tilde{h}_c$ in $\mathbb{H}$ (which is called a lift of h_c) that starts at the preimage of v that lies in t_0, which we will also denote by v, and ends on some $v' = g_c(v)$ in the triangle $t_0' = g_c(t_0)$, where $g_c \in \Gamma$. Then we have $\Gamma = \{g_c\}_{c\in\pi_1(\Sigma,v)}$. By changing $\Gamma \to g\Gamma g^{-1}$ with $g \in \mathrm{PSL}(2,\mathbb{R})$ (mapping the whole tessellation with g), it can be achieved that t_0 is the hyperbolic triangle with corners at $0, -1$ and ∞, such that the edge $e_0 \in f$ is the one that crosses the edge between 0 and ∞, as illustrated in Figure 6. Suppose that $e_1 = e_0$. Then the triangle one reaches by going from t_0

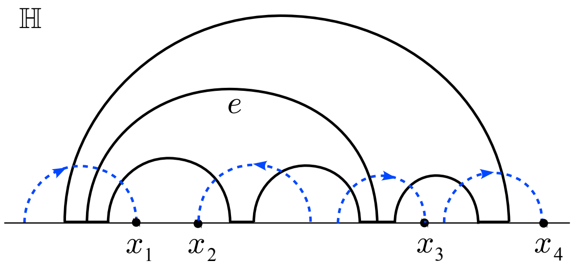

Figure 7: Definition of the shear coordinates for bordered Riemann surfaces

along e_1 is the triangle t_1 with corners at $0, \infty$ and e^{w_1}, where $w_1 := w_{e_1}$. The matrix $E(w_1)$ represents the Möbius transformation $\tau \mapsto -e^{w_1}\tau^{-1}$ that maps t_1 onto t_0. In order for e_2 to be mapped (topologically) onto e_0, one then has to rotate the corners of t_0. This is done by the map $g_V : \tau \mapsto -\tau^{-1} - 1$ or its inverse, that is represented by the matrix V^{s_1}. By going along $\tilde{h}_c$ and repeating these steps for each edge e_j, one constructs the matrix M_c that represents the map $(g_c)^{-1}$. If $e_1 \neq e_0$, one has to first rotate e_1 onto e_0 with $g_V^{-s_0}$ which amounts to conjugating M_c with V^{s_0}.

From these considerations we conclude that the map $\mathcal{M}(\Sigma) \to \mathbb{R}^{3N}$, $[\Xi] \mapsto \{w_e\}_{e\in\tau}$, where $[\Xi]$ is the equivalence class of a complex structure Ξ, is injective. In other words, the shear coordinates form a complete set of coordinates on $\mathcal{M}_{g,n}$. In fact they provide a complete set of coordinates on $\mathcal{T}(\Sigma)$, as a Dehn twist would effectively change the triangulation on Σ and therefore would be 'registered' by the w_e. However, there are constraints on the w_e associated to the punctures which originate from the fact that the elements M_k, $k = 1, \ldots, n$, of Γ that correspond to curves that encircle one puncture only are parabolic elements of $\mathrm{SL}(2, \mathbb{R})$. If we denote by $\tau_k \subset \tau$ the subset of edges that emanate from the k-th puncture, then we have

$$\sum_{e\in\tau_k} w_e = 0 \qquad (k = 1, \ldots, n). \tag{3.23}$$

Therefore, of the $3N$ shear coordinates only $3N - n = 6g - 6 + 2n$ are independent, which gives the dimension d of $\mathcal{T}_{g,n}$ as a real manifold.

In the case of bordered Riemann surfaces, the definition of the shear coordinates has to be modified [12]. In that case, each connected component $\partial_k\Sigma$ ($k = 1, \ldots, n$) of the boundary has to be endowed with an orientation. For each hole, there exists one closed geodesic g_k which is homotopic to $\partial_k\Sigma$, and the orientation of $\partial_k\Sigma$ induces an orientation of g_k. An edge e of an ideal triangulation is then part of two hexagons which together form an octagon in the upper half plane (Fig. 7). Four sides of this octagon are images of segments of boundary components and therefore segments of the real axis (they have replaced the points $x_1, \ldots, x_4$ in Fig. 3). To each of these sides corresponds an image of the closed geodesic g_k associated with the same boundary component. These are oriented geodesics in $\mathbb{H}$ that can be extended to geodesics that start and end on the real axis (the dashed half circles in Fig. 7). We define $x_1, \ldots, x_4$ to be their respective end points. Then w_e is defined just as in (3.20). When the holes shrink to punctures, the dashed half circles in Fig. 7 shrink to points on the real axis and the definition of w_e passes over into the definition for punctured surfaces. In the case of holes, the constraints (3.23) are replaced by

$$\sum_{e\in\tau_k} w_e = \pm l_k \qquad (k = 1, \ldots, s), \tag{3.24}$$

where l_k is the hyperbolic length of the closed geodesic that goes around the k-th hole. Note that as $l_k \to 0$, the hole becomes a puncture and (3.24) becomes again (3.23). It is clear that

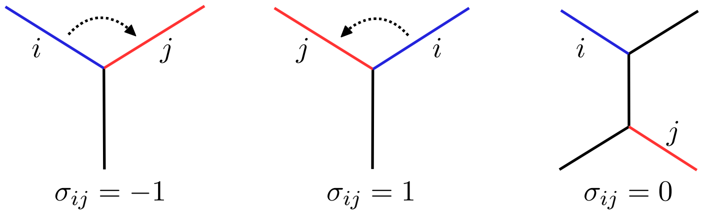

Figure 8: The definition of σ_{ij}

the shear coordinates do not capture all the information about the conformal equivalence class of the bordered Riemann surface C (as the moduli spaces of bordered Riemann surfaces are infinite dimensional). Rather they allow to reconstruct the equivalence class of the surface obtained by cutting off annuli from C along the geodesics g_k ($k = 1, \ldots, n$) [12]. Therefore they serve as a complete set of coordinates on the Teichmüller space of bordered Riemann surfaces with geodesic boundaries.

It is important to know the transformation behavior of the shear coordinates under a flip. Obviously, the flip ω_e as depicted in Fig. 5 affects only the shear coordinates associated to edges a, b, c, d of the fat graph that have a common vertex with e as well as w_e itself. From the definition (3.20) it follows that these transform as

$$\begin{array}{lll} e^{+w_{a'}} = e^{+w_a}(1+e^{+w_e}) & & e^{-w_{b'}} = e^{-w_b}(1+e^{-w_e}) \\ & w_{e'} = -w_e & \\ e^{-w_{d'}} = e^{-w_d}(1+e^{-w_e}) & & e^{+w_{c'}} = e^{+w_c}(1+e^{+w_e}) \end{array} . \tag{3.25}$$

These formulas also hold in the case of holes.

On the Teichmüller space $\mathcal{T}_{g,n}$ there exists a hermitian 2-form h_{WP} called the Weil-Petersson metric [17]. Associated with it is a symplectic form ω_{WP} and Poisson bracket $\{\cdot,\cdot\}_{\mathrm{WP}}$. It turns out that the Poisson bracket of two shear coordinates is relatively simple [12]: Given two edges e, e' of a fat graph, number their ends arbitrarily by $1, 2$. Then one has

$$\{w_e, w_{e'}\}_{\mathrm{WP}} = \sum_{i,j=1,2} \sigma_{ij}, \tag{3.26}$$

where σ_{ij} is -1 ($+1$) if the end j of e' is at the same vertex as the end i of e and in the (counter-) clockwise direction of the latter, and zero if the two ends do not meet (see Fig. 8). Thus $\{w_e, w_{e'}\}_{\mathrm{WP}}$ is an integer between 2 and -2. Note that (3.26) is consistent with (3.23) resp. (3.24) as $\sum_{e\in\tau_k} w_e$ Poisson commutes with all w_e.

3.6 Hyperbolic length functions

Given a simple (i.e., not self-intersecting) loop γ on our Riemann surface Σ, there is a unique closed geodesic $\hat{\gamma}$ that is in the same free homotopy class as γ. (The only exception is a loop going around one puncture only.) We will denote the hyperbolic length of this geodesic by l_γ, which is a function on Teichmüller space $\mathcal{T}_{g,n}$. Given the Fuchsian group Γ and an isomorphism $\Psi : \pi_1(\Sigma, p) \to \Gamma$, how to compute l_γ? The loop γ may start and end at a point p' different from our base point p. However, once we have fixed (the homotopy class of) a path μ going from p to p', γ corresponds to an element $c_\gamma := [\mu \cdot \gamma \cdot \mu^{-1}]$ of $\pi_1(\Sigma, p)$ and thus to an element $M_\gamma = \Psi(c_\gamma)$ of Γ. The freedom we have in the choice of μ corresponds to the conjugation of M_γ by another element of Γ. So l_γ has to depend only on the conjugacy class of M_γ (recall also the freedom $\Gamma \to g\Gamma g^{-1}$ with $g \in \mathrm{SL}(2,\mathbb{R})$). Since the elements of the universal covering transformation group $\tilde{\Gamma}$ have no fixed points [17], the fixed points of M_γ have to lie on $\mathbb{R} \cup \{\infty\}$. It follows that M_γ is a parabolic ($|\mathrm{tr}\, M_\gamma| = 2$) or hyperbolic

($|\mathrm{tr}\, M_\gamma| > 2$) element of $\mathrm{SL}(2,\mathbb{R})$. The parabolic case can be excluded by Theorem 2.22 in [17] applied to the compact Riemann surface $\bar{\Sigma}$ obtained by filling the punctures of Σ (M_γ is an element of a Fuchsian group for $\bar{\Sigma}$). So there exists $g \in \mathrm{SL}(2,\mathbb{R})$ such that $\tilde{M}_\gamma := gM_\gamma g^{-1}$ is of the form

$$\tilde{M}_\gamma = \pm \begin{pmatrix} a^{\frac{1}{2}} & 0 \\ 0 & a^{-\frac{1}{2}} \end{pmatrix}, \qquad a > 0, \tag{3.27}$$

which corresponds to $z \mapsto az$ in $\mathrm{Aut}(\mathbb{H})$. Now, it is easy to see that the shortest paths with respect to the Poincaré metric (3.9) between two equivalent points z and az in $\mathbb{H}$ (with variable z) are segments of the imaginary axis, e.g. the segment between i and ai. This path is a preimage of the geodesic $\hat{\gamma}$ in $\Sigma \simeq \mathbb{H}/\Gamma'$, where $\Gamma' := g\Gamma g^{-1}$, under the canonical projection $\mathbb{H} \to \mathbb{H}/\Gamma'$. So it has the same hyperbolic length as $\hat{\gamma}$, which is by (3.9) (use $y = \Im(z)$)

$$l_\gamma \;=\; \int_1^a \frac{1}{y} dy \;=\; \log(a). \tag{3.28}$$

Thus we obtain a relation

$$L_\gamma := 2\cosh(\tfrac{1}{2} l_\gamma) = a^{\frac{1}{2}} + a^{-\frac{1}{2}} = |\mathrm{tr}\tilde{M}_\gamma| = |\mathrm{tr} M_\gamma| \tag{3.29}$$

between l_γ and the trace of the matrix M_γ.

In order to express L_γ in terms of the shear coordinates introduced in the previous section, we have to find an element in the conjugacy class of M_γ. To this end, let g_γ be a closed path on the fat graph f consisting of a set of consecutive edges $e_1, \ldots, e_r \in f$ that is in the free homotopy class of γ (e.g. the one of minimal length, i.e., with no 180 degree turns). Then one can use the same prescription as in the previous section to define a matrix $\hat{M}_\gamma$ just as M_c in (3.21), that is conjugate to M_γ. Thus L_γ will be of the form

$$L_\gamma \;=\; |\mathrm{tr}\, \hat{M}_\gamma| \;=\; \sum_{\tau \in (\frac{1}{2}\mathbb{Z})^r} C(\tau) \exp\left(\sum_{i=1}^r \tau_i w_{e_i} \right), \tag{3.30}$$

where $C(\tau)$ are positive integers which are non-zero only for a finite number of $\tau \in (\frac{1}{2}\mathbb{Z})^r$.

3.7 Complex coordinates on Teichmüller space

There exists a way to define complex valued coordinates on $\mathcal{T}_{g,n}$ which is based on the gluing of Riemann surfaces by three-holed spheres [40]. Here we do not need the general scheme but only a simple variant which is applicable to the case of the n-punctured sphere. Since every sphere (i.e., a simply connected compact Riemann surface) is isomorphic to $\bar{\mathbb{C}}$, every n-punctured sphere is isomorphic to some $\Sigma = \bar{\mathbb{C}} \backslash \{z_1, \ldots, z_n\}$. The freedom of a complex Möbius transformation can be eliminated by putting z_{n-2}, z_{n-1}, z_n to 0, 1 and ∞. The positions $z_1, \ldots, z_{n-3}$ of the remaining punctures then serve as complex coordinates on the moduli space $\mathcal{M}_{0,n}$. Since there exists a natural projection from $\mathcal{T}_{0,n}$ to $\mathcal{M}_{0,n}$, $(z_1, \ldots, z_{n-3})$ is also a set of coordinates on $\mathcal{T}_{0,n}$ but not injective - moving z_i around z_j and back to its original position one reaches a different point in $\mathcal{T}_{0,n}$. This follows from the fact that $\mathcal{T}_{0,n}$ is the universal covering space of

$$\mathcal{M}_{0,n} \simeq \mathcal{Z}_n := \{(z_1, \ldots, z_{n-3}) \in \mathbb{C}^{n-3} | z_i \notin \{0,1\} \text{ and } z_i \neq z_j \text{ for } i \neq j\}. \tag{3.31}$$

Another set of complex coordinates on $\mathcal{T}_{0,n}$ can be defined using the following classical result by Poincaré [27]: On the n-punctured sphere $\Sigma = \bar{\mathbb{C}} \backslash \{z_1, \ldots, z_n\}$, with $z_j \neq \infty$ $(j = 1, \ldots, n)$, the function $T(z)$ defined in (3.13) is of the form[10]

$$T(z) \;=\; \sum_{j=1}^{n} \left(\frac{\delta_j}{(z - z_j)^2} + \frac{c_j}{z - z_j} \right), \tag{3.32}$$

where the $c_j \in \mathbb{C}$, called accessory parameters, depend on $z_1, \ldots, z_n$ and $\delta_j = \frac{1}{2}$ $(j = 1, \ldots, n)$. Furthermore, T has to be regular at infinity. Since a coordinate at $z = \infty$ is given by $\tilde{z} = z^{-1}$, and T transforms a under change of coordinates $z \to w(z)$ as (cf. (2.30))

$$T(z) \to T(w) \;=\; T(z(w)) \left(\frac{\partial z}{\partial w} \right)^2 - \tfrac{1}{2} S(z)(w), \tag{3.33}$$

this implies three constraints on the accessory parameters:

$$\sum_{j=1}^{n} c_j = 0, \qquad \sum_{j=1}^{n} (\delta_j + c_j z_j) = 0, \qquad \sum_{j=1}^{n} (2\delta_j z_j + c_j z_j^2) = 0. \tag{3.34}$$

The reason for introducing the 'weights' δ_j is that the Hill's equation (3.14) may also be solved for $T(z)$ with more general δ_j, e.g. $\delta_j = \frac{1}{2}(1 + \lambda_j^2)$ with $\lambda_j > 0$ (called hyperbolic weights). In the latter case, let us make the following ansatz for the asymptotic behavior of a pair of solutions[11] (f_1, f_2) around z_j

$$f_k(z) \;=\; (z - z_j)^{\mu_k} \big(1 + O(z - z_j)\big) \quad (k = 1, 2). \tag{3.35}$$

Then (3.14) yields

$$\mu_k(\mu_k - 1) = -\frac{1}{2}\delta_j = -\frac{1}{4}(1 + \lambda_j^2), \tag{3.36}$$

which has the solutions $\mu_1 = \frac{1}{2}(1 + i\lambda_j)$, $\mu_2 = \frac{1}{2}(1 - i\lambda_j)$. Then the monodromy M_j of $(f_1(z), f_2(z))$ as z is going around z_j in the counter-clockwise direction is given by

$$M_j \;=\; -\begin{pmatrix} e^{-\pi\lambda_j} & 0 \\ 0 & e^{\pi\lambda_j} \end{pmatrix}, \tag{3.37}$$

which is hyperbolic with $|\mathrm{tr} M_j| = 2\cosh(\pi\lambda_j)$. Then φ_e will not be defined on the whole of Σ, but will have infinitely many singular lines on which $\Im A(z) = 0$ around each puncture [16]. However, there will be one connected component of $\Sigma \backslash \{\text{singular lines}\}$ which has the same topology as Σ, i.e., it is an n-holed sphere. Let us denote this component by $\Sigma(Z; \delta)$, where $Z \equiv (z_1, \ldots, z_n), \delta \equiv (\delta_1, \ldots, \delta_n)$. It seems reasonable to think that A is the uniformization map of $\Sigma(Z; \delta)$. Then the geodesic on $\Sigma(Z; \delta)$ going around z_j (which then exists) will have hyperbolic length $l_j = 2\pi\lambda_j$.

The accessory parameters provide another set of complex coordinates on $\mathcal{T}_{0,n}$. Takhtajan and Zograf [33] where able to show that

$$\frac{\partial c_j}{\partial \bar{z}_k} = \frac{1}{2\pi} h_{\mathrm{WP}} \left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_k} \right) \qquad (j, k = 1, \ldots, n - 3), \tag{3.38}$$

[10] Originally, one puncture is at $z = \infty$ which results in a slightly different form of $T(z)$ and the constraints (3.34) (cf. [33]).

[11] We assume that the accessory parameters have been adjusted so that the monodromies of solutions to (3.14) are again in $\mathrm{SL}(2, \mathbb{R})$.

where $h_{\rm WP}$ is the Weil-Petersson metric.[12] From equation (3.38) follows that the Weil-Petersson symplectic form $\omega_{\rm WP}$ may be written as

$$\omega_{\rm WP} \;\equiv\; i\sum_{j,k=1}^{n-3} h_{\rm WP}(\partial_j,\bar\partial_k)\, dz_j\wedge d\bar z_k \;=\; 2\pi i\sum_{j=1}^{n-3} dz_j\wedge dc_j. \tag{3.39}$$

It follows that the Poisson bracket of the z_j and c_k is given by

$$\{z_j,z_k\}_{\rm WP}=0=\{c_j,c_k\}_{\rm WP},\quad \{z_j,c_k\}_{\rm WP}=\frac{1}{2\pi i}\delta_{j,k}\quad (j,k=1,\dots,n-3). \tag{3.40}$$

This result leads to a profound insight into the connection between conformal blocks in quantum Liouville theory and quantized Teichmüller theory, as we will see in Section 6.

4 Quantized Teichmüller spaces

Here we discuss the quantization of Teichmüller spaces, based on the Poisson algebra of the shear coordinates. We define quantized hyperbolic length functions and compute their eigenfunctions for the 4-holed sphere. Although the result is essentially known [38], a systematic computation of this kind has not been published before.

4.1 The operator algebra

Quantization of a symplectic manifold $\mathcal{P}$ (the phase space of the theory) can be roughly defined as constructing a linear map Q from a complete set of functions on $\mathcal{P}$ that is closed under the Poisson bracket to a set of (possibly unbound) operators on a Hilbert space $\mathcal{H}$ such that $[Q(f),Q(g)]=i\hbar Q(\{f,g\})$. This is possible only if $\mathcal{M}$ satisfies certain conditions which we will not discuss here. But the simple form of the Weil-Petersson Poisson bracket (3.26) suggests that it should be possible for $\mathcal{P}=\mathcal{T}_{g,n}$ and the set of shear coordinates $\{w_e\}$ assigned to an ideal triangulation of Σ. The quantized shear coordinates $\mathbf{w}_e=Q(w_e)$ should be self-adjoint operators satisfying the commutation relations

$$[\mathbf{w}_e,\mathbf{w}_{e'}] \;=\; 2\pi i b^2\{w_e,w_{e'}\}_{\rm WP}, \tag{4.1}$$

where we use b, defined by $b^2=\hbar$, as a quantization parameter.[13] The constraints (3.24) should also be realized in the quantum theory as

$$\sum_{e\in\tau_k}\mathbf{w}_e=\pm l_k \qquad (k=1,\dots,n). \tag{4.2}$$

(The operator on the l.h.s. commutes with all $\mathbf{w}_e$ and therefore should be represented as a multiplication operator in an irreducible representation.) Now, one can always find [34] $h:=d/2=3g-3+n$ pairs $(\mathbf{p}_j,\mathbf{x}_j)$ of (linearly independent) linear combinations of the $\mathbf{w}_e$ such that

$$[\mathbf{p}_j,\mathbf{x}_k]=\frac{1}{2\pi i}\delta_{j,k}. \tag{4.3}$$

One may represent these operators on $L^2(\mathbb{R}^h)$ by[14]

$$\mathbf{p}_j\psi(p)=p_j\psi(p),\qquad \mathbf{x}_j\psi(p)=\frac{i}{2\pi}\frac{\partial}{\partial p_j}\psi(p)\quad (j=1,\dots,h). \tag{4.4}$$

[12] Takhtajan and Zograf refer to the inner product on the *holomorphic* tangent space of $\mathcal{T}_{0,n}$ as the Weil-Petersson metric.

[13] The normalization is conventional. It means that we quantize the rescaled Poisson bracket $2\pi\{\cdot,\cdot\}_{\rm WP}$.

[14] Strictly speaking, they are defined only on subsets of $L^2(\mathbb{R}^h)$ as they destroy normalizability of some wave functions.

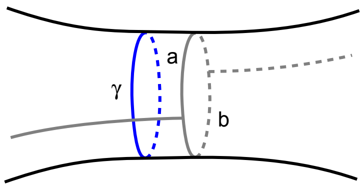

Figure 9: Simplest form of a fat graph (gray) in the vicinity of loop γ

The choice of the $(\mathbf{p}_j, \mathbf{x}_j)$ depends on the triangulation and includes some extra amount of freedom (i.e., that of a symplectic transformation), so that no general formula can be given. Therefore it is for some purposes more convenient to work with the so called Kashaev coordinates ([18], see also [34]), which are defined on a space that contains $\mathcal{T}_{g,n}$ as a linear subspace. But for our purposes the operators $\mathbf{p}_j, \mathbf{x}_j$ associated with the shear coordinates are sufficient.

Quantization of $\mathcal{T}_{g,n}$ should not depend on the choice of the triangulation. In other words, the quantized shear coordinates for different triangulations should be related in a way that is consistent with the algebras (4.1). These relations should also merge into the classical ones (3.25) in the limit $b \to 0$. The transformation of quantized shear coordinates under a flip ω_e as in Fig. 5 that satisfies these requirements is given by

$$\begin{aligned} e^{+\mathbf{w}_{a'}} &= e^{+\frac{1}{2}\mathbf{w}_a}(1+e^{+\mathbf{w}_e})e^{+\frac{1}{2}\mathbf{w}_a} \\ e^{-\mathbf{w}_{d'}} &= e^{-\frac{1}{2}\mathbf{w}_d}(1+e^{-\mathbf{w}_e})e^{-\frac{1}{2}\mathbf{w}_d} \end{aligned} \qquad \mathbf{w}_{e'} = -\mathbf{w}_e \qquad \begin{aligned} e^{-\mathbf{w}_{b'}} &= e^{-\frac{1}{2}\mathbf{w}_b}(1+e^{-\mathbf{w}_e})e^{-\frac{1}{2}\mathbf{w}_b} \\ e^{+\mathbf{w}_{c'}} &= e^{+\frac{1}{2}\mathbf{w}_c}(1+e^{+\mathbf{w}_e})e^{+\frac{1}{2}\mathbf{w}_c} \end{aligned} \,. \tag{4.5}$$

The relations (4.5) have been first stated in [12] in the form

$$\begin{aligned} \mathbf{w}_{a'} &= \mathbf{w}_a + \phi_b(+\mathbf{w}_e) \\ \mathbf{w}_{d'} &= \mathbf{w}_d - \phi_b(-\mathbf{w}_e) \end{aligned} \qquad \mathbf{w}_{e'} = -\mathbf{w}_e \qquad \begin{aligned} \mathbf{w}_{b'} &= \mathbf{w}_b - \phi_b(-\mathbf{w}_e) \\ \mathbf{w}_{c'} &= \mathbf{w}_c + \phi_b(+\mathbf{w}_e) \end{aligned} \,, \tag{4.6}$$

involving the special function ϕ_b (defined in Appendix A). Equivalence with (4.5) can be proven via the formulas

$$\mathbf{w}_{a'} = \left(e_b\Big(\frac{\mathbf{w}_e}{2\pi b}\Big)\right)^{-1} \mathbf{w}_a \; e_b\Big(\frac{\mathbf{w}_e}{2\pi b}\Big)\,, \qquad \mathbf{w}_{b'} = e_b\Big(-\frac{\mathbf{w}_e}{2\pi b}\Big)\,\mathbf{w}_b\,\left(e_b\Big(-\frac{\mathbf{w}_e}{2\pi b}\Big)\right)^{-1}\,, \tag{4.7}$$

and similar for $\mathbf{w}_{c'}$ and $\mathbf{w}_{d'}$, where the special function e_b is the 'quantum dilogarithm' defined in Appendix A. Note that equations (4.5), (4.6) and (4.7) only make sense in a representation as they involve functional calculus. Nevertheless, they turn a representation of an algebra (4.1) associated to a fat graph into a representation of all algebras associated to all possible fat graphs on Σ. Therefore we will simply speak of a representation in the following.

4.2 Length operators

In this section we will discuss quantum analogs $\mathbf{L}_\gamma$ of the hyperbolic length functions $L_\gamma = 2\cosh(l_\gamma/2)$. These length operators where first introduced and studied in [6], while a detailed and complete discussion can be found in [34]. They are required to have a number of properties, for example, $\mathbf{L}_\gamma$ should be self-adjoint with spectrum $[2,\infty[$ (the possible values of L_γ). The first step towards the definition of L_γ for all γ is to consider the simplest case where the fat graph in the vicinity of γ takes the 'standard form' depicted in Figure 9. In that case, one defines

$$\mathbf{L}_\gamma := 2\cosh(2\pi b\mathbf{p}) + e^{-2\pi b\mathbf{x}}, \qquad \text{where} \qquad \begin{aligned} \mathbf{p} &:= \tfrac{1}{4\pi b}(\mathbf{w}_b + \mathbf{w}_a) \\ \mathbf{x} &:= \tfrac{1}{4\pi b}(\mathbf{w}_b - \mathbf{w}_a) \end{aligned} \,. \tag{4.8}$$

Note that this becomes L_γ when one replaces $\mathbf{w}_a \to w_a,\ \mathbf{w}_b \to w_b$ (in the classical limit $b \to 0$) and the ordering of the operators is fixed by the requirement that $\mathbf{L}_\gamma^\dagger = \mathbf{L}_\gamma$. If the fat graph f can be brought to the standard form in the vicinity of γ by a sequence of flips, then (4.8) together with the transformation rules (4.5) resp. (4.7) define $\mathbf{L}_\gamma$ also for a representation associated with f. If this is not possible, one has to define $\mathbf{L}_\gamma$ in a recursive way as done in [34]. However, in this work we will only need the first definition.

It can be shown that two length operators $\mathbf{L}_{\gamma_1}$ and $\mathbf{L}_{\gamma_2}$ commute if the corresponding geodesics $\hat{\gamma}_1, \hat{\gamma}_2$ do not intersect. On Σ, a maximal set of mutually non-intersecting simple loops consists of $h = 3g - 3 + n$ elements (again, this is shown inductively by adding punctures or handles). The corresponding length operators $\mathbf{L}_1, \ldots, \mathbf{L}_h$ can be simultaneously diagonalized and the set of their common eigenfunctions provides a basis for $L^2(\mathbb{R}^h)$.

Let π_L be a representation of shear coordinates on $L^2(\mathbb{R}^h)$ as introduced in the previous section, where $L \equiv (l_1, \ldots, l_n)$ denotes the collection of constants in the constraints (4.2) associated with the punctures (holes). Let $\psi_{L,r}(p)$ be a family of common eigenfunctions of $\mathbf{L}_1, \ldots, \mathbf{L}_h$ in this representation, where $r \equiv (r_1, \ldots, r_h)$ parametrize the eigenvalues via $\mathbf{L}_j \psi_{L,r} = 2\cosh(2\pi b r_j)\psi_{L,r}$. Then one can construct from π_L a representation ρ_L on $L^2(\mathbb{R}^h)$, called a length representation, by the integral transformation

$$(R_L\psi)(r) := \langle \psi_{L,r} | \psi \rangle = \int_{\mathbb{R}^h} dp\ \overline{\psi_{L,r}(p)}\psi(p). \tag{4.9}$$

Any quantized shear coordinate $\mathbf{w}_e$ is then represented in ρ_L as $\rho_L(\mathbf{w}_e) := R_L \pi_L(\mathbf{w}_e) R_L^{-1}$. The representation ρ_L does not depend on the choice of π_L. The length representations can also be defined independently of the quantized shear coordinates as representations of the length operators based on the quantized Fenchel-Nielsen coordinates [40]. The latter are associated to so-called Moore-Seiberg graphs on Σ. The set of all Moore-Seiberg graphs on Σ and the moves that transform them into another form a groupoid, the Moore-Seiberg groupoid. By quantizing the Fenchel-Nielsen coordinates one naturally obtains a projective representation of the Moore-Seiberg groupoid associated with the length representations. Since each element of the mapping class group $\mathrm{MCG}(\Sigma)$ maps a given Moore-Seiberg graph onto another one, this induces a projective representation ϱ_L of $\mathrm{MCG}(\Sigma)$ on $L^2(\mathbb{R}^h)$.

For later use, let us already compute the eigenfunctions of $\mathbf{L}_\gamma$ as given in (4.8) in the representation where $\mathbf{p}$ is diagonal. We note that $\mathbf{L}_\gamma$ is only defined on wave functions $\psi(p)$ that have an analytic continuation to the strip $\{p \in \mathbb{C} \,|\, -ib \leq \Im(p) \leq 0\}$ (these form a dense subset of $L^2(\mathbb{R})$). Then the eigenvalue equation $\mathbf{L}_\gamma \Psi_r(p) = 2\cosh(2\pi b r)\Psi_r(p)$ is equivalent to

$$\Psi_r(p - ib) = 4\cosh(\pi b(p + r + \tfrac{i}{2b}))\cosh(\pi b(p - r + \tfrac{i}{2b}))\Psi_r(p). \tag{4.10}$$

It follows that

$$\Psi_r(p) = c(r) s_b(p + r + c_b) s_b(p - r + c_b), \qquad c_b := \frac{i}{2}(b + b^{-1}) \tag{4.11}$$

where the special function s_b is a close relative of e_b (see App. A) and $c(r)$ is a normalization factor.

4.3 The 4-holed sphere

Let us now apply the definitions of the previous sections to the 4-holed sphere, i.e., the Riemann sphere with four simply connected domains removed. A convenient fat graph f_0 is displayed in Figure 10, together with two different loops, the so called s and t-cycles (corresponding to the s and t-channel of scattering amplitudes). There are four constraints on the

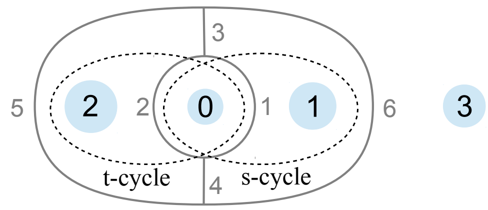

Figure 10: The fat graph f_0 for the 4-holed sphere

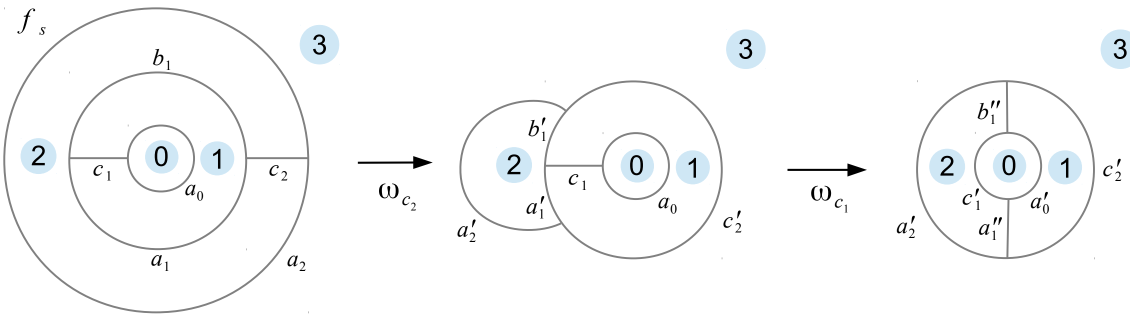

Figure 11: Transforming f_s into f_0

quantized shear coordinates $\mathbf{w}_1, \dots, \mathbf{w}_6$:

$$\begin{array}{rclcrcl} l_0 & = & \mathbf{w}_1 + \mathbf{w}_2 & \quad & l_1 & = & \mathbf{w}_1 + \mathbf{w}_3 + \mathbf{w}_4 + \mathbf{w}_6 \\ l_3 & = & \mathbf{w}_5 + \mathbf{w}_6 & & l_2 & = & \mathbf{w}_2 + \mathbf{w}_3 + \mathbf{w}_4 + \mathbf{w}_5 \end{array} . \tag{4.12}$$

Thus one is left with two linearly independent generators (which have to be non-commuting). Here we define

$$\mathbf{p} := \tfrac{1}{2\pi b}\mathbf{w}_2, \quad \mathbf{x} := \tfrac{1}{4\pi b}(\mathbf{w}_3 - \mathbf{w}_4). \tag{4.13}$$

With the help of (4.12) $\mathbf{x}$ may also be written as $\mathbf{x} = -(2\pi b)^{-1}(\mathbf{w}_4 + \frac{1}{4}(l_0 - l_1 - l_2 + l_3))$.

Let π be the representation on $L^2(\mathbb{R})$ in which $\mathbf{p}, \mathbf{x}$ act on $L^2(\mathbb{R})$ as in (4.4). Our goal is to compute the eigenfunctions of the length operators $\mathbf{L}_s$ and $\mathbf{L}_t$ corresponding to the s and t-cycle. The eigenfunctions of $\mathbf{L}_s$ and $\mathbf{L}_t$ are known in representations different from π, which correspond to fat graphs that have the standard form in the vicinity of the cycle. For the s-cycle, the appropriate fat graph f_s and its transformation into f_0 by two flips is shown in Figure 11. (For this purpose we have chosen a different naming of the edges. The name of an edge printed in bold will denote the corresponding shear coordinate in the following.) The representation π' in which we know the eigenfunctions $\Psi_r^{s'}$ of $\mathbf{L}_s$ is the one where

$$\mathbf{x}' := \tfrac{1}{4\pi b}(\mathbf{b}_1 - \mathbf{a}_1), \quad \mathbf{p}' := \tfrac{1}{4\pi b}(\mathbf{b}_1 + \mathbf{a}_1), \tag{4.14}$$

are represented in the standard way. Now suppose that there exists a (unitary) operator $\mathbf{U}$ such that $\mathbf{x} = \mathbf{U} \cdot \mathbf{x}' \cdot \mathbf{U}^{-1}$ and $\mathbf{p} = \mathbf{U} \cdot \mathbf{p}' \cdot \mathbf{U}^{-1}$. Then $\pi'(\mathbf{U})$ defines an intertwiner between the representations π and π' and $\Psi_r^s := \pi'(\mathbf{U}^{-1})\Psi_r^{s'}$ is an eigenfunction of $\mathbf{L}_s$ in the representation π. The operator $\mathbf{U}$ can be systematically constructed from the sequence of flips that transforms f_s into f_0 with the help of (4.7). In this process three types of operators corresponding to three types of transformations will contribute to $\mathbf{U}$:

1. flips realized by operators of the form $e_b(\mathbf{q} + c)$, where $\mathbf{q} \in \mathbb{R}\mathbf{x}' \cup \mathbb{R}\mathbf{p}', c \in \mathbb{R}$

2. symplectic transformations realized by operators of the form $e^{\pi i\mathbf{q}^2}$

3. shifts realized by operators of the form $e^{2\pi i\mathbf{q}}$

The best way to understand this is to work out an example like the given one. First we write down the constraints for f_s,

$$\begin{array}{rclrcl} l_0 &=& \mathbf{a}_0 & l_1 &=& \mathbf{a}_1+\mathbf{b}_1+2\mathbf{c}_1+\mathbf{a}_0 \\ l_3 &=& \mathbf{a}_2 & l_2 &=& \mathbf{a}_1+\mathbf{b}_1+2\mathbf{c}_2+\mathbf{a}_2 \end{array}. \tag{4.15}$$

It follows that $\mathbf{c}_1 = 2\pi b(-\mathbf{p}'+r_1-r_0),\ \mathbf{c}_2 = 2\pi b(-\mathbf{p}'+r_2-r_3)$, where $r_k = l_k/4\pi b$. Now we want to relate $\mathbf{x}',\mathbf{p}'$ to a new pair of conjugate operators $\mathbf{x}'',\mathbf{p}''$ which are linear combinations of the shear coordinates assigned to the intermediate fat graph. Since the first change of fat graph is the flip along c_2, ether $e_b(\frac{\mathbf{c}_2}{2\pi b})$ or $e_b(-\frac{\mathbf{c}_2}{2\pi b})$ has to be employed. As $\mathbf{a}_1$ and $\mathbf{b}_1$ transform differently under the flip, we first have to make $\mathbf{x}'$ a multiple of, say, $\mathbf{a}_1$ by the symplectic transformation

$$\mathbf{x}'\to\mathbf{x}'-\mathbf{p}' = -\frac{1}{2\pi b}\mathbf{a}_1, \quad \mathbf{p}'\to\mathbf{p}'. \tag{4.16}$$

This transformation is realized by conjugation with $\mathbf{S} := e^{-\pi i\mathbf{p}'^2}$. Then define

$$\mathbf{U}_1 \ := \ e_b\left(-\frac{\mathbf{c}_2}{2\pi b}\right) \ = \ e_b(\mathbf{p}'-r_2+r_3) \tag{4.17}$$

and

$$(\mathbf{x}'',\mathbf{p}'') := (\mathbf{U}_1\mathbf{S})(\mathbf{x}',\mathbf{p}')(\mathbf{U}_1\mathbf{S})^{-1} = (-\tfrac{1}{2\pi b}\mathbf{a}_1', -\tfrac{1}{2\pi b}\mathbf{c}_1+r_1-r_0). \tag{4.18}$$

Here we have written $\mathbf{p}''=\mathbf{p}'$ in terms of shear coordinates assigned to the intermediate fat graph. The next step is the flip along c_1 which is realized by $\mathbf{U}_2 := e_b(-\frac{\mathbf{c}_1}{2\pi b}) = e_b(\mathbf{p}'-r_1+r_0)$. This yields another pair of conjugate operators[15]

$$(\mathbf{x}^{(3)},\mathbf{p}^{(3)}) \ := \ \mathbf{U}_2(\mathbf{x}'',\mathbf{p}'')\mathbf{U}_2^{-1} \ = \ (-\tfrac{1}{2\pi b}\mathbf{a}_1'', \tfrac{1}{2\pi b}\mathbf{c}_1'-r_0+r_1). \tag{4.19}$$

Finally there are shifts $\mathbf{p}^{(3)}\to\mathbf{p} = \mathbf{p}^{(3)}+r_0-r_1$ and $\mathbf{x}^{(3)}\to\mathbf{x} = \mathbf{x}^{(3)}-\frac{1}{2}r$ with $r := r_0-r_1-r_2+r_3$ realized simultaneously by $\mathbf{T} := e^{2\pi i(r_1-r_0)\mathbf{x}^{(3)}}e^{-\pi i r\mathbf{p}^{(3)}}$. Altogether we have

$$(\mathbf{x},\mathbf{p}) \ = \ \mathbf{U}(\mathbf{x}',\mathbf{p}')\mathbf{U}^{-1}, \tag{4.20}$$

where

$$\mathbf{U} := \mathbf{T}\mathbf{U}_2\mathbf{U}_1\mathbf{S} = \mathbf{U}_2\mathbf{U}_1\mathbf{S}e^{2\pi i(r_1-r_0)\mathbf{x}'}e^{-\pi i r\mathbf{p}'}. \tag{4.21}$$

In the last step we have expressed $\mathbf{T}$ in terms of $\mathbf{x}',\mathbf{p}'$ as

$$\mathbf{T} \ = \ (\mathbf{U}_2\mathbf{U}_1\mathbf{S})e^{2\pi i(r_1-r_0)\mathbf{x}'}e^{-\pi i r\mathbf{p}'}(\mathbf{U}_2\mathbf{U}_1\mathbf{S})^{-1}. \tag{4.22}$$

Then we find as the eigenfunction of $\mathbf{L}_s$ with eigenvalue $2\cosh(2\pi b r_s)$

$$\begin{aligned} \Psi^s_{r_s}(p) &= \pi'(\mathbf{U}^{-1})\Psi^{s'}_{r_s}(p')|_{p'=p} \\ &= e^{\pi i r p}e^{\pi i(p+r_1-r_0)^2}(e_b(p+r_1-r_0-r_2+r_3))^{-1}(e_b(p))^{-1}\Psi^{s'}_{r_s}(p+r_1-r_0) \\ &= \lambda c(r_s)\cdot\frac{s_b(-p+r_0-r_1+r_2-r_3)s_b(-p)}{s_b(-p-r_s+r_0-r_1-c_b)s_b(-p+r_s+r_0-r_1-c_b)}, \end{aligned} \tag{4.23}$$

[15] Although the definitions of $\mathbf{p}''$ and $\mathbf{p}^{(3)}$ might seem redundant, they help to bring some systematics into the computation. This will prove very useful for more complicated computations like the one in Section 8.

where λ is an irrelevant phase factor and we have used $(s_b(x))^{-1} = s_b(-x)$. Similarly, the eigenfunction of $\mathbf{L}_t$ with eigenvalue $2\cosh(2\pi b r_t)$ can be computed to be

$$\Psi^t_{r_t}(p) = c(r_t) \cdot \frac{s_b(p - r_0 + r_1 - r_2 - r_3)s_b(p - 2r_0)}{s_b(p - r_t - r_0 - r_2 - c_b)s_b(p + r_t - r_0 - r_2 - c_b)}. \tag{4.24}$$

This result can also be derived from $\Psi^s_{r_s}$ by a symmetry consideration: Rotating the fat graph f_0 by 180 degree around the hole 0, amounts to exchanging s and t-cycle or

$$l_1 \leftrightarrow l_2, \quad w_1 \leftrightarrow w_2, \quad w_3 \leftrightarrow w_4, \quad w_5 \leftrightarrow w_6. \tag{4.25}$$

Since $\mathbf{w}_1 + \mathbf{w}_2 = l_0$ this induces $\mathbf{p} \to 2r_0 - \mathbf{p}$ and $\mathbf{x} \to -\mathbf{x}$. Therefore one obtains $\Psi^t_r(p)$ from $\Psi^s_r(p)$ by replacing $l_1 \to l_2, l_2 \to l_1,\ p \to 2r_0 - p$.

4.4 Coherent state representation

We have defined above a quantization of Teichmüller space $\mathcal{T}_{g,n}$ based on the Poisson algebra of the real-valued shear coordinates. It has been pointed out in [36] that one might also define a quantization of $\mathcal{T}_{0,n}$ which is based on the Poisson algebra (3.40) of the complex coordinates[16] z_j and c_j $(j = 1, \ldots, n-3)$. The idea can be illustrated by the simple example of the Poisson algebra $\{x, p\} = 1$ on $\mathbb{R}^2$. The corresponding Heisenberg algebra $[\mathbf{x}, \mathbf{p}] = i\hbar$ can be represented on $L^2(\mathbb{R})$ in the usual way. But one may also introduce complex coordinates $a = x + ip$ and $\bar{a} = x - ip$. Their Poisson algebra $\{a, \bar{a}\} = -2i$ becomes $[\mathbf{a}, \bar{\mathbf{a}}] = 2\hbar$ which can be represented on a Hilbert space $\mathcal{H}_{\text{hol.}}$ of *holomorphic* functions f by $\mathbf{a}f(a) = af(a)$ and $\bar{\mathbf{a}}f(a) = -2\hbar f'(a)$, known as the coherent state representation of quantum mechanics. Given that $\mathbf{a} = \mathbf{x} + i\mathbf{p}$, $\bar{\mathbf{a}} = \mathbf{x} - i\mathbf{p}$, one may ask for an intertwiner between these representations of the same algebra, i.e., a unitary map of the form

$$U : L^2(\mathbb{R}) \to \mathcal{H}_{\text{hol.}}, \quad U(\psi)(a) = \int_{\mathbb{R}} dx\; U(a,p)\psi(p), \tag{4.26}$$

that satisfies $U(\mathbf{x} + i\mathbf{p})U^{-1} = \mathbf{a}$ and $U(\mathbf{x} - i\mathbf{p})U^{-1} = \bar{\mathbf{a}}$. $f_p(a) := U(a,p)$ is then an eigenstate of $\mathbf{p}$ in the coherent state representation, and $\psi_a(p) := \overline{U(a,p)}$ is an eigenstate of $\mathbf{a}$ in the "real" representation.

Similarly, the Poisson algebra (3.40) corresponds to the following algebra[17] of the quantized $\mathbf{z}_j, \mathbf{c}_j$

$$[\mathbf{z}_j, \mathbf{z}_k] = 0 = [\mathbf{c}_j, \mathbf{c}_k], \qquad [\mathbf{z}_j, \mathbf{c}_k] = b^2\delta_{j,k} \quad (j,k = 1, \ldots, n-3). \tag{4.27}$$

This algebra may be represented on a space $\text{Hol}(\mathcal{T}_{0,n})$ of holomorphic functions on $\mathcal{T}_{0,n}$, represented as multivalued functions of $Z \equiv (z_1, \ldots, z_h)$, $h = n - 3$, by

$$\mathbf{z}_j\Psi(Z) := z_j\Psi(Z), \quad \mathbf{c}_j\Psi(Z) = -b^2\frac{\partial}{\partial z_j}\Psi(Z) \quad (j = 1, \ldots, h). \tag{4.28}$$

Let ρ_L, $L \equiv (l_1, \ldots, l_n)$, be a length representation of the algebra of quantized shear coordinates as introduced in Section 4.2. In contrast to the simple example given above, the complex coordinates z_j and c_j are highly complicated functions of the shear coordinates - an explicit formula is not even available. The reason is that the latter are defined with the help of the uniformization map - a function that has not yet been explicitly described (except for $g = 0,\ n = 3$). Thus there seems to be no way to construct quantum analogs of z_j and c_j inside

[16] Analogous complex coordinates exist for higher genus $g > 0$.

[17] Again, we quantize the rescaled Poisson bracket $2\pi\{\cdot,\cdot\}_{\text{WP}}$.

the representation ρ_L on $L^2(\mathbb{R}^h)$ that form a representation of the algebra (4.27). Therefore it makes a priori no sense to ask for an intertwiner between the coherent state representation (4.28) and the length representation (as they represent different algebras). Nevertheless, there is a natural representation of the mapping class group MCG(Σ), where $\Sigma = \bar{\mathbb{C}}\backslash\{z_1, \ldots, z_n\}$, on Hol($\mathcal{T}_{0,n}$) given by [40]

$$(M_\mu \Psi)(Z) \ := \ \Psi(\mu.Z) \quad (\mu \in \mathrm{MCG}(\Sigma)), \tag{4.29}$$

where $\mu.Z$ denotes the action of μ on $T_{0,n}$ as represented in the coordinates[18] Z. On the other hand, one has the representation ϱ_L of MCG(Σ) on $L^2(\mathbb{R}^h)$ associated with the length representations. For $\mu \in \mathrm{MCG}(\Sigma)$, $\varrho_L(\mu)$ is an integral transformation whose kernel we denote by $M_{L,\mu}(r, r')$, i.e.,

$$\varrho_L(\mu)\psi(r) = \int_{\mathbb{R}^h} dr' \ M_{L,\mu}(r, r')\psi(r'). \tag{4.30}$$

One may then ask for the existence of an (invertible) intertwiner between these representations of the form

$$\mathcal{F}_L : L^2(\mathbb{R}^h) \to \mathrm{Hol}(\mathcal{T}_{0,n}), \quad \mathcal{F}_L(\psi)(Z) = \int_{\mathbb{R}^h} dr \ \mathcal{F}_L(Z, r)\psi(r), \tag{4.31}$$

where $\mathcal{F}_L(Z, r)$ has to be holomorphic in Z. If such a map exists, it can be used to represent quantized shear coordinates and length operators on Hol($\mathcal{T}_{0,n}$) by

$$\rho_L^{\mathrm{hol.}}(\mathbf{w}_e) := \mathcal{F}_L \ \rho_L(\mathbf{w}_e) \ \mathcal{F}_L^{-1}, \qquad \rho_L^{\mathrm{hol.}}(\mathbf{L}_\gamma) := \mathcal{F}_L \ \rho_L(\mathbf{L}_\gamma) \ \mathcal{F}_L^{-1}. \tag{4.32}$$

Obviously $\Psi_{L,r}(Z) := \mathcal{F}_L(Z, r)$ would define a simultaneous eigenfunction of $\rho_L^{\mathrm{hol.}}(\mathbf{L}_k)$, $k = 1, \ldots, h$. In a similar way, one may use $\mathcal{F}_L$ to represent $\mathbf{z}_j$ and $\mathbf{c}_j$ on $L^2(\mathbb{R}^h)$. Furthermore, $\mathcal{F}_L$ would induce a scalar product on Hol($\mathcal{T}_{0,n}$) that turns it into a Hilbert space (where Hol($\mathcal{T}_{0,n}$) is simply defined as the image of $\mathcal{F}_L$).

The intertwining property $M_\mu \mathcal{F}_L = \mathcal{F}_L \varrho_L(\mu)$ may be written in terms of the integral kernels as

$$\mathcal{F}_L(\mu.Z, r) = \int dr' \ \mathcal{F}_L(Z, r') M_{L,\mu}(r', r). \tag{4.33}$$

Such an equation, which specifies the monodromies of a multi-valued holomorphic function, is known as a Riemann-Hilbert type problem. Its solution is unique when one fixes the asymptotic behavior at the boundary of $\mathcal{T}_{0,n}$ which is characterized by $z_i \to z_j$ for some $i, j \in \{1, \ldots, n-3\}$. In [40], a natural requirement for the asymptotics of $\mathcal{F}_L(Z, r)$ has been proposed. Now, the remarkable observation is that a solution to (4.33) with the required asymptotics exists and is given by the conformal blocks of Liouville theory which will be defined in Section 6.

5 Free field quantization of Liouville theory

One of the approaches to quantize Liouville theory is via the parametrization of the classical phase space $\mathcal{P}$ in terms of the free field as done in Section 2.1. In this approach one constructs the quantized Liouville field from the quantized free field in a way that resembles the classical relation between them. On the way, one defines important operators called chiral vertex operators. The latter allow to construct the conformal blocks, which are central objects of

[18] Of course, this is formal notation since the coordinates Z are invariant under the action of the mapping class group. Rather, $\Psi(\mu.Z)$ has to be understood as the analytic continuation of $\Psi(Z)$ along a closed path in $\mathcal{Z}_n$ (defined in (3.31)) which represents μ.

quantum Liouville theory (and any CFT). They have turned out to be related to eigenfunctions of length operators in quantum Teichmüller theory as defined in Section 4.2.

It is conventional to first rescale the free field as $\varphi_F = 2b\phi_F$ and then expand the quantized ϕ_F as

$$\phi_F(x^+, x^-) = \mathbf{q} + \mathbf{p}(x^+ + x^-) + i\sum_{n\neq 0} \frac{1}{n}\left(\mathbf{a}_n e^{-inx^+} + \mathbf{b}_n e^{-inx^-}\right). \tag{5.1}$$

The quantized Fourier modes will then satisfy the commutation relations (we display only non-vanishing commutators)

$$[\mathbf{q}, \mathbf{p}] = \frac{i}{2}, \quad [\mathbf{a}_n, \mathbf{a}_m] = \frac{n}{2}\delta_{n+m}, \quad [\mathbf{b}_n, \mathbf{b}_m] = \frac{n}{2}\delta_{n+m}. \tag{5.2}$$

Since ϕ_F is the quantum counterpart of a real field, we require also that $\mathbf{q}^\dagger = \mathbf{q},\ \mathbf{p}^\dagger = \mathbf{p}$ and $\mathbf{a}_n^\dagger = \mathbf{a}_{-n},\ \mathbf{b}_n^\dagger = \mathbf{b}_{-n}$, which will determine the scalar product on the Hilbert space. In the following we will concentrate on the chiral (left moving) part as the anti-chiral (right moving) part is completely analogous. The standard representation of the operators $\mathbf{a}_n$ is on the Fock space

$$\mathcal{F} := \mathrm{Span}\{\mathbf{a}_{n_r}\mathbf{a}_{n_{r-1}}\dots\mathbf{a}_{n_1}|0\rangle \mid r \in \mathbb{N}_0, n_r \leq n_{r-1}\dots \leq n_1 < 0\}, \tag{5.3}$$

where $|0\rangle$ is the ground state satisfying $a_n|0\rangle = 0$ for all $n > 0$. Therefore the natural representation of the algebra generated by $\{\mathbf{a}_n\}, \mathbf{p}, \mathbf{q}$ is on the Hilbert space

$$\mathcal{H}_{\text{chir}} := L^2(\mathbb{R}) \otimes \mathcal{F}. \tag{5.4}$$

If one defines

$$\mathcal{F}_p := \mathrm{Span}\{\mathbf{a}_{n_r}\dots\mathbf{a}_{n_1}|p\rangle \mid r \in \mathbb{N}_0, n_r \leq \dots \leq n_1 < 0\}, \tag{5.5}$$

where $\mathbf{p}|p\rangle = p|p\rangle,\ a_n|p\rangle = 0$ for all $n > 0$, then one has the isomorphism

$$\mathcal{H}_{\text{chir}} \tilde{\to} \int_{-\infty}^{\infty} dp\ \mathcal{F}_p, \quad \psi \otimes (\mathbf{a}|0\rangle) \mapsto \int_{-\infty}^{\infty} dp\ \psi(p)\mathbf{a}|p\rangle, \qquad \mathbf{a} \equiv \mathbf{a}_{n_r}\dots\mathbf{a}_{n_1}. \tag{5.6}$$

The previous discussion suggests that the natural candidate for the Hilbert space of quantum Liouville theory is

$$\mathcal{H} := L^2(\mathbb{R}) \otimes \mathcal{F} \otimes \bar{\mathcal{F}} \simeq \int_{-\infty}^{\infty} dp\ \mathcal{F}_p \otimes \bar{\mathcal{F}}_p, \tag{5.7}$$

where $\bar{\mathcal{F}}$ and $\bar{\mathcal{F}}_p$ are defined as $\mathcal{F}$ and $\mathcal{F}_p$, but replacing $\mathbf{a}_n \to \mathbf{b}_n$. However, we have not yet taken into account the fact that the Liouville phase space is not isomorphic to the phase space $\mathcal{P}_F$ of the free field, but to $\mathcal{P}_F^+$ defined by $p > 0$. But it is not clear how to realize this constraint in the quantum theory as operators of the form $e^{i\alpha\mathbf{q}}$ with $\alpha \in \mathbb{R}$ can always shift the $\mathbf{p}$ eigenvalue to the negative domain. The solution to this problem is to construct a quantum analog $\mathbf{S} : \mathcal{H} \to \mathcal{H}$ of the 'reflection map' S that sends p to $-p$ as done in [35]. The proper Liouville Hilbert space $\mathcal{H}_L$ is then the subspace of $\mathbf{S}$-invariant vectors and the operator algebra consists of operators in the algebra generated by $\{\mathbf{a}_n\}, \{\mathbf{b}_n\}, \mathbf{p}, \mathbf{q}$ that commute with $\mathbf{S}$.

The conformal symmetry group $G = \mathrm{Diff}^+(S^1) \times \mathrm{Diff}^+(S^1)$ appears in the quantum theory as a unitary projective representation of G on the Hilbert space. That is equivalent to having a representation of two copies (chiral and anti-chiral) of $\widehat{\mathrm{Vect}}(S^1)$. The Virasoro generators L_n of the chiral part are represented on $\mathcal{H}_{\mathrm{chir}}$ by

$$
\begin{aligned}
\mathbf{L}_0 &:= \mathbf{p}^2 + \frac{Q^2}{4} + 2\sum_{k=1}^{\infty} \mathbf{a}_{-k}\mathbf{a}_k, \\
\mathbf{L}_n &:= (2\mathbf{p} + inQ)\mathbf{a}_n + \sum_{k\neq 0,n} \mathbf{a}_k\mathbf{a}_{n-k} \qquad (n \neq 0),
\end{aligned}
\tag{5.8}
$$

where $Q = b + b^{-1}$. Equation (5.8) is known as the free field representation of the Virasoro algebra (2.44) with central charge $c = 1 + 6Q^2$. This representation decomposes into a one-parameter family of representations on $\int dp\, \mathcal{F}_p$ according to (5.6). The representation on $\mathcal{F}_p$ is a highest weight representation with weight $\Delta_p = p^2 + \frac{Q^2}{4}$ and is known to be isomorphic to the Verma module $\mathcal{V}_\alpha$ with $\alpha = \frac{Q}{2} + ip$ defined in Appendix B. It can be considered as the quantization of the coadjoint orbit $\mathcal{W}_p$ that has been introduced in Section 2.3.

It is also useful to combine the Virasoro generators to a chiral field

$$
\mathbf{T}_+(x^+) \;:=\; \sum_{n\in\mathbb{Z}} \hat{\mathbf{L}}_n\, e^{-inx^+}, \qquad \hat{\mathbf{L}}_n := \mathbf{L}_n - \frac{c}{24}\delta_{n,0}. \tag{5.9}
$$

A vector field $\xi = \xi(x^+)\partial_+ \in \mathrm{Vect}(S^1)$ with Fourier expansion $\xi(x^+) = \sum_{n\in\mathbb{Z}} \xi_n e^{inx^+}$ is then represented in the quantum theory by[19]

$$
\mathbf{T}_+[\xi] \;:=\; \frac{i}{2\pi}\int_0^{2\pi} dx^+\, \xi(x^+)\mathbf{T}_+(x^+) \;=\; i\sum_{n\in\mathbb{Z}} \xi_n \hat{\mathbf{L}}_n. \tag{5.10}
$$

Since $\mathbf{L}_n^\dagger = \mathbf{L}_{-n}$ and $\bar{\xi}_n = \xi_{-n}$, $\mathbf{T}_+[\xi]$ is anti-hermitian. Notice also that

$$
\mathbf{T}_+ \;=\; -Q\partial_+^2\phi_F \;+\; :(\partial_+\phi_F)^2: \;-\; \tfrac{1}{24}, \tag{5.11}
$$

where $:\;:$ denotes normal ordering, so that $b^2\mathbf{T}_+ \to -\frac{1}{2}\partial_+^2\varphi_F + \frac{1}{4}(\partial_+\varphi_F)^2 = T_+$ for $b \to 0$. Therefore $b^2\mathbf{T}_+$ can be considered as the quantum analog of T_+.

The representation of $\mathrm{Diff}^+(S^1)$ is constructed as follows. Let $f_t : S^1 \to S^1$, $t \in [0,1]$ be the integrated flow of ξ, i.e., the unique solution of

$$
\frac{\partial}{\partial t} f_t(\sigma) = \xi(f_t(\sigma)), \quad f_0(\sigma) = \sigma \quad (\sigma \in S^1). \tag{5.12}
$$

Then each f_t is an element of $\mathrm{Diff}^+(S^1)$, in particular $f[\xi] := f_1$. It will be represented by the unitary operator

$$
\mathbf{U}_{f[\xi]} \;:=\; e^{\mathbf{T}_+[\xi]}. \tag{5.13}
$$

5.1 Chiral vertex operators

In the mathematical sense, chiral vertex operators are maps $\mathbf{h}^{\alpha_2}_{\alpha_3,\alpha_1}(z)$ for $z \in \mathbb{C}^*$ (where $z = e^{ix^+}$, cf. Section 5.2) from the Verma module $\mathcal{V}_{\alpha_1}$ to $\mathcal{V}_{\alpha_3}$ with the properties

$$
\begin{aligned}
&(i) \quad [\mathbf{L}_n, \mathbf{h}^{\alpha_2}_{\alpha_3,\alpha_1}(z)] = z^n(z\partial_z + (n+1)\Delta_{\alpha_2})\mathbf{h}^{\alpha_2}_{\alpha_3,\alpha_1}(z), \\
&(ii) \quad \mathbf{h}^{\alpha_2}_{\alpha_3,\alpha_1}(z)e_{\alpha_1} = z^{\Delta_{\alpha_3}-\Delta_{\alpha_2}-\Delta_{\alpha_1}}(N(\alpha_3,\alpha_2,\alpha_1)e_{\alpha_3} + O(z)) \quad \text{for } z \to 0,
\end{aligned}
\tag{5.14}
$$

[19] Recall that $L_n = ie^{inx^+}\partial_+ + \frac{c}{24}\delta_{n,0}$, so $\hat{\mathbf{L}}_n$ is the quantum version of $ie^{inx^+}\partial_+$.

where e_α is the highest weight vector in $\mathcal{V}_\alpha$ and N is an arbitrary function of three variables.

We will define two kinds of operators from which we will construct the chiral vertex operators: normal ordered exponentials and screening charges. First we split off the left moving component of ϕ_F,

$$\phi_F^+(x^+) := \mathbf{q} + \mathbf{p}x^+ + \phi_<^+(x^+) + \phi_>^+(x^+), \tag{5.15}$$

where

$$\phi_<^+(x^+) := i\sum_{n<0} \frac{1}{n}\mathbf{a}_n e^{-inx^+}, \qquad \phi_>^+(x^+) := i\sum_{n>0} \frac{1}{n}\mathbf{a}_n e^{-inx^+}. \tag{5.16}$$

Then we can define the normal ordered exponentials

$$\mathbf{E}^\alpha(x) \ := \ e^{2\alpha\phi_<^+(x)} e^{2\alpha(\mathbf{q}+x\mathbf{p})} e^{2\alpha\phi_>^+(x)} \qquad (\alpha \in \mathbb{C}). \tag{5.17}$$

This may also be denoted as $\mathbf{E}^\alpha(x) \ = \ : e^{2\alpha\phi_F^+(x)} :$. The screening charges are

$$\mathbf{Q}(x) := \int_0^{2\pi} d\sigma \ \mathbf{E}^b(\sigma + x). \tag{5.18}$$

One has to be a little careful with the definition (5.17), as states in the image of $\mathbf{E}^\alpha(x)$ would have infinite norm, i.e., they are ill defined. However, as argued in [37], the analytical continuation of $\mathbf{E}^\alpha(x)$ to $\Im(x) > 0$ (i.e., negative Euclidean time $\tau = it$) is an unbounded operator defined on a dense subset of $\mathcal{H}$. Concerning the screening charges, they are densely defined unbounded operators even for $x \in \mathbb{R}$ as long as $2b^2 < 1$. Furthermore, they are positive operators since

$$\mathbf{E}^b(x) \ = \ \mathbf{E}_<^b(x)\big(\mathbf{E}_<^b(x)\big)^\dagger, \qquad \mathbf{E}_<^\alpha(x) := e^{2\alpha\phi_<^+(x)} e^{\alpha(\mathbf{q}+x\mathbf{p})} \tag{5.19}$$

so that

$$\langle\psi|\mathbf{Q}(x)\psi\rangle = \int_0^{2\pi} dx' \ ||\big(\mathbf{E}_<^b(x+x')\big)^\dagger\psi||^2 > 0 \tag{5.20}$$

for any ψ in the domain of definition of $\mathbf{Q}(x)$. Therefore one may take arbitrary (complex) powers of these operators.

The crucial property of the normal ordered exponentials and screening charges is their transformation behavior under diffeomorphisms, which is infinitesimally described by

$$[\mathbf{L}_n, \mathbf{E}^\alpha(x)] = e^{inx}(-i\partial_x + n\Delta_\alpha)\mathbf{E}^\alpha(x), \qquad [\mathbf{L}_n, \mathbf{Q}(x)] = -ie^{inx}\partial_x\mathbf{Q}(x), \tag{5.21}$$

where $\Delta_\alpha = \alpha(Q - \alpha)$ is the conformal weight. This may be equivalently written as

$$[\mathbf{T}_+[\xi], \mathbf{E}^\alpha(x)] \ = \ \big(\xi(x)\partial_x + \Delta_\alpha\xi'(x)\big)\,\mathbf{E}^\alpha(x), \quad [\mathbf{T}_+[\xi], \mathbf{Q}(x)] \ = \ \xi(x)\partial_x\mathbf{Q}(x). \tag{5.22}$$

Now we can define the chiral vertex operators[20]

$$\mathbf{h}_s^\alpha(x) := \mathbf{E}^\alpha(x)(\mathbf{Q}(x))^s, \qquad s \in \mathbb{C}. \tag{5.23}$$

Their most important properties are their transformation behavior under the Virasoro algebra,

$$[\mathbf{L}_n, \mathbf{h}_s^\alpha(x)] = e^{inx}(-i\partial_x + n\Delta_\alpha)\mathbf{h}_s^\alpha(x), \tag{5.24}$$

and their commutation relation with $\mathbf{p}$,

$$\mathbf{p}\ \mathbf{h}_s^\alpha(x) = \mathbf{h}_s^\alpha(x)\ (\mathbf{p} - i(\alpha + bs)). \tag{5.25}$$

So if $-i(\alpha + bs) \in \mathbb{R}$, then the restriction of $\mathbf{h}_s^\alpha(x)$ to $\mathcal{F}_p$ is a map

$$\mathbf{h}_s^\alpha(x)_p : \ \mathcal{F}_p \to \mathcal{F}_{p-i(\alpha+bs)}. \tag{5.26}$$

[20] Strictly speaking, the $\mathbf{h}_s^\alpha(x)$ are predecessors of the chiral vertex operators that will be introduced in the next section by analytic continuation of these.

5.2 Analytic continuation

An interesting aspect of the quantization of Liouville theory as described above is that it provides a representation, defined by (5.8), not only of $\widehat{\mathrm{Vect}}(S^1)$ but of the algebra of formal sums

$$\bar{\mathfrak{V}} := \Big\{ \sum_{n\in\mathbb{Z}} a_n L_n \,\Big|\, a_n \in \mathbb{C} \Big\}, \qquad L_n = ie^{in\sigma}\partial_\sigma, \tag{5.27}$$

of which both the Virasoro algebra $\mathfrak{V} := \mathrm{Span}\{L_n \mid n \in \mathbb{Z}\}$ and $\widehat{\mathrm{Vect}}(S^1)$ are subspaces. Another important subspace is the complexification

$$\mathrm{Vect}^{\mathbb{C}}(S^1) \;:=\; \{\xi(\sigma)\partial_\sigma \,|\, \xi \in C^\infty(S^1,\mathbb{C})\}, \tag{5.28}$$

where $C^\infty(S^1,\mathbb{C})$ is the space of smooth complex-valued functions on S^1. Although there exists no Lie group with Lie algebra $\mathrm{Vect}^{\mathbb{C}}(S^1)$ as proven by Lempert [22], this algebra is naturally associated with conformal (i.e., holomorphic) transformations of the complex plane or domains of the complex plane. To see this, note that an element

$$\eta = -\sum_{m\in\mathbb{Z}} \eta_m L_m = -\sum_{m\in\mathbb{Z}} i\eta_m e^{im\sigma}\partial_\sigma \;\in \mathrm{Vect}^{\mathbb{C}}(S^1) \tag{5.29}$$

acts on $C^\infty(S^1,\mathbb{C})$. From now on, we will identify S^1 with the unit circle $\{z \in \mathbb{C} \,|\, |z| = 1\}$. So given a function $f \in C^\infty(S^1,\mathbb{C})$ that can be analytically continued to a neighborhood of S^1, the corresponding action of η on $f(z)$, where $z = e^{i\sigma}$, is by the complex vector field

$$\eta(z)\partial_z = \sum_{m\in\mathbb{Z}} \eta_m z^{m+1}\partial_z, \tag{5.30}$$

which might be well-defined on a neighborhood of S^1 as well. In view of these considerations it is natural to analytically continue the chiral fields introduced in the previous section to complex $w = ix^+ = \tau + i\sigma$, where $\tau = it$ takes real values. Of particular interest are the chiral vertex operators $\mathbf{h}_s^\alpha(w)$; their explicit definition can be found in [37]. The second step is to exchange the coordinate w that lives on the complex cylinder $\mathbb{C}/2\pi i\mathbb{Z}$ for $z = e^w$ that lives on $\mathbb{C}^* := \mathbb{C}\backslash\{0\}$. Under this change of coordinates, the $\mathbf{h}_s^\alpha(w)$ naturally transform as

$$\mathbf{h}_s^\alpha(z) = z^{-\Delta_\alpha}\mathbf{h}_s^\alpha(w). \tag{5.31}$$

Let us define the Virasoro field

$$\mathbf{T}(z) := \sum_{m\in\mathbb{Z}} \mathbf{L}_m z^{-m-2}. \tag{5.32}$$

The operator that represents the infinitesimal conformal transformation by η is then given by

$$\mathbf{T}_\eta \;:=\; \frac{1}{2\pi i}\oint_C dz\,\eta(z)\mathbf{T}(z) \;=\; \sum_{m\in\mathbb{Z}} \eta_m \mathbf{L}_m, \tag{5.33}$$

where C is some curve in $\mathbb{C}^*$ that is homotopic to S^1, and which lies in the domain of definition of η (this allows to define $\mathbf{T}_\eta$ also for vector fields η which are defined on a domain that does not contain S^1). Note that if $\eta \in \mathrm{Vect}(S^1)$, i.e., if η is tangential to S^1, then $\mathbf{T}_\eta = \mathbf{T}_+[\eta]$, where $\mathbf{T}_+[\eta]$ was defined in (5.10). $\mathbf{T}_\eta$ acts on a field $\mathbf{V}(z)$ by

$$\delta_\eta \mathbf{V}(z) \;=\; [\mathbf{T}_\eta, \mathbf{V}(z)] \;=\; \mathbf{T}_\eta\mathbf{V}(z) - \mathbf{V}(z)\mathbf{T}_\eta. \tag{5.34}$$

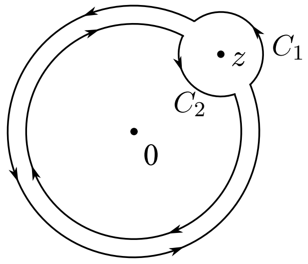

Figure 12: Integration contours for $\mathbf{T}_\eta$

By choosing in (5.33) two different contours C_1 and C_2 which are contained in the domain $|w| > |z|$ and $|w| < |z|$ respectively as depicted in Fig. 12, this may be rewritten as

$$\delta_\eta \mathbf{V}(z) \;=\; \frac{1}{2\pi i}\oint_{C(z)} dw\; \eta(w) R\{\mathbf{T}(w)\mathbf{V}(z)\} \tag{5.35}$$

where R denotes radial ordering and $C(z)$ is a contour that encircles z in the counter-clockwise direction.[21] The simplest possible transformation behavior of a field $\mathbf{V}(z)$ under the Virasoro algebra is given by $[\mathbf{L}_n, \mathbf{V}(z)] = z^{n+1}\partial_z \mathbf{V}(z)$, or equivalently

$$[\mathbf{T}_\eta, \mathbf{V}(z)] = \eta(z)\partial_z \mathbf{V}(z). \tag{5.36}$$

Such a field, which is called a primary chiral field of vanishing conformal weight, is given by $\mathbf{Q}(z)$, the analytic continuation of $\mathbf{Q}(x)$. The fields $\mathbf{h}_s^\alpha(z)$ are primary chiral fields of conformal weight Δ_α, which means

$$[\mathbf{L}_n, \mathbf{h}_s^\alpha(z)] \;=\; z^n(z\partial_z + (n+1)\Delta_\alpha)\mathbf{h}_s^\alpha(z), \tag{5.37}$$

or equivalently

$$[\mathbf{T}_\eta, \mathbf{h}_s^\alpha(z)] \;=\; \eta(z)\partial_z \mathbf{h}_s^\alpha(z) + \Delta_\alpha \partial_z \eta(z) \mathbf{h}_s^\alpha(z). \tag{5.38}$$

Yet another way of characterizing this transformation behavior is by the operator product expansion (OPE)

$$R\{\mathbf{T}(w)\mathbf{h}_s^\alpha(z)\} \;=\; \frac{\Delta_\alpha}{(w-z)^2}\mathbf{h}_s^\alpha(z) + \frac{1}{w-z}\partial_z \mathbf{h}_s^\alpha(z) + O(1) \quad \text{as } w \to z, \tag{5.39}$$

The right hand sides of (5.36) and (5.38) only make sense for vector fields η that are regular at z. Also the definition of $\mathbf{T}_\eta$ becomes ambiguous for a vector field η that has a pole at z, as it then depends on the choice of the integration contour C. Nevertheless, for such η it is natural to use (5.35) as a definition of $\delta_\eta \mathbf{V}(z)$. This leads to the definition of the so called descendants. These are fields $\mathbf{h}_s^\alpha(v|z)$ for every $v \in \mathcal{V}_\alpha$ which are linear in v. First one sets $\mathbf{h}_s^\alpha(e_\alpha|z) := \mathbf{h}_s^\alpha(z)$ for the highest weight state e_α. Then one recursively defines for $m < 0$

$$\begin{aligned} \mathbf{h}_s^\alpha(L_m v|z) &:= \delta_{(w-z)^{m+1}\partial_w} \mathbf{h}_s^\alpha(v|z) \\ &= \frac{1}{2\pi i}\oint_{C(z)} dw\; (w-z)^{m+1}\; R\{\mathbf{T}(w)\mathbf{h}_s^\alpha(v|z)\}. \end{aligned} \tag{5.40}$$

This obviously defines $\mathbf{h}_s^\alpha(v|z)$ for every $v \in \mathcal{V}_\alpha$.

[21] We use here without proof the fact that $R(\mathbf{T}(w)\mathbf{V}(z))$ is analytic in w.

Let us now investigate how this projective representation of $\mathrm{Vect}^{\mathbb{C}}(S^1)$ exponentiates.[22] Let $\eta \in \mathrm{Vect}^{\mathbb{C}}(S^1)$ be such that there exists the integrated flow

$$f_t : S^1 \to \mathbb{C}, \quad \frac{\partial}{\partial t} f_t(z) = \eta(f_t(z)), \quad f_0(z) = z, \quad t \in [0,1]. \tag{5.41}$$

If $\eta \notin \mathrm{Vect}(S^1)$, this means that η possesses an analytic continuation to some neighborhood of S^1. First we derive the transformation of the field $\mathbf{Q}(z)$ under the exponentiated action of $\mathbf{T}_\eta$, i.e., we compute[23]

$$\mathbf{Q}_t(z) := e^{t\mathbf{T}_\eta}\mathbf{Q}(z)e^{-t\mathbf{T}_\eta}. \tag{5.42}$$

$\mathbf{Q}_t(z)$ satisfies the differential equation

$$\frac{\partial}{\partial t}\mathbf{Q}_t(z) \;=\; [\mathbf{T}_\eta, \mathbf{Q}_t(z)] \tag{5.43}$$

with boundary condition $\mathbf{Q}_0(z) = \mathbf{Q}(z)$. On the other hand, using (5.41) and (5.36) one finds

$$\frac{\partial}{\partial t}\mathbf{Q}(f_t(z)) \;=\; \eta(f_t(z))(\partial_z \mathbf{Q})(f_t(z)) \;=\; [\mathbf{T}_\eta, \mathbf{Q}(f_t(z))], \tag{5.44}$$

and $\mathbf{Q}(f_0(z)) = \mathbf{Q}(z)$. Thus $\mathbf{Q}_t(z)$ and $\mathbf{Q}(f_t(z))$ satisfy the same differential equation with the same boundary condition, consequently $\mathbf{Q}_t(z) = \mathbf{Q}(f_t(z))$.

Similarly, let us define

$$\mathbf{T}_t(z) \;:=\; e^{t\mathbf{T}_\eta}\mathbf{T}(z)e^{-t\mathbf{T}_\eta} \tag{5.45}$$

and

$$\mathbf{T}(t,z) := (f_t'(z))^2\mathbf{T}(f_t(z)) - \frac{c}{6}S(f_t)(z), \tag{5.46}$$

where prime denotes the holomorphic z derivative[24] and S is the Schwarzian derivative (cf. (2.29))

$$S(f_t) \;=\; -\frac{1}{2}\frac{f_t'''}{f_t'} + \frac{3}{4}\left(\frac{f_t''}{f_t'}\right)^2. \tag{5.47}$$

Obviously $\mathbf{T}_t(z)$ satisfies the differential equation

$$\frac{\partial}{\partial t}\mathbf{T}_t(z) \;=\; [\mathbf{T}_\eta, \mathbf{T}_t(z)] \tag{5.48}$$

with boundary condition $\mathbf{T}_0(z) = \mathbf{T}(z)$. The field $\mathbf{T}(z)$ transforms under the Virasoro algebra as

$$[\mathbf{L}_n, \mathbf{T}(z)] \;=\; z^{n+1}\partial_z\mathbf{T}(z) + 2(n+1)z^n\mathbf{T}(z) + \frac{c}{12}(n^3-n)z^{n-2}. \tag{5.49}$$

It follows that

$$[\mathbf{T}_\eta, \mathbf{T}(z)] \;=\; \eta(z)\mathbf{T}'(z) + 2\eta'(z)\mathbf{T}(z) + \frac{c}{12}\eta'''(z), \tag{5.50}$$

and

$$[\mathbf{T}_\eta, \mathbf{T}(t,z)] \;=\; (f_t'(z))^2\left(\eta\cdot\mathbf{T}' + 2\eta'\cdot\mathbf{T} + \frac{c}{12}\eta'''\right)\circ f_t(z). \tag{5.51}$$

On the other hand one has

$$\frac{\partial}{\partial t}\mathbf{T}(t,z) \;=\; (f_t'(z))^2\eta(f_t(z))\mathbf{T}'(f_t(z)) + 2f_t'(z)(\eta\circ f_t)'(z)\mathbf{T}(f_t(z)) - \frac{c}{6}\frac{\partial}{\partial t}S(f_t)(z). \tag{5.52}$$

[22] Although the material of this section is essentially known, the author is not aware of a presentation of the following computations in the literature.

[23] If $\eta \notin \mathrm{Vect}(S^1)$, then $\exp(\mathbf{T}_\eta)$ is not unitary, but in general an unbound operator.

[24] Even if f_t admits no holomorphic continuation to a neighborhood of S^1, one may define f_t' using the angle coordinate σ on S^1, $z = e^{i\sigma}$, by $f_t'(z) := -ie^{-i\sigma}\partial_\sigma\hat{f}_t(\sigma)$.

From (5.47) we compute

$$
\begin{aligned}
\frac{\partial}{\partial t}S(f_t) &= -\frac{1}{2}\left(\frac{(\eta\circ f_t)'''}{f_t'} - \frac{f_t'''(\eta\circ f_t)'}{(f_t')^2}\right) + \frac{3}{2}\frac{f_t''}{f_t'}\left(\frac{(\eta\circ f_t)''}{f_t'} - \frac{f_t''(\eta\circ f_t)'}{(f_t')^2}\right) \\
&= -\frac{1}{2}\frac{((\eta'\circ f_t)f_t')'' - (\eta'\circ f_t)f_t'''}{f_t'} + \frac{3}{2}\frac{f_t''}{f_t'}\frac{((\eta'\circ f_t)f_t')' - (\eta'\circ f_t)f_t''}{f_t'} \\
&= -\frac{1}{2}(f_t')^2(\eta'''\circ f_t).
\end{aligned} \tag{5.53}
$$

Thus $\mathbf{T}(t,z)$ also satisfies the differential equation

$$
\frac{\partial}{\partial t}\mathbf{T}(t,z) \;=\; [\mathbf{T}_\eta, \mathbf{T}(t,z)] \tag{5.54}
$$

and the boundary condition $\mathbf{T}(0,z) = \mathbf{T}(z)$. This proves $\mathbf{T}_t(z) = \mathbf{T}(t,z)$, or

$$
e^{t\mathbf{T}_\eta}\mathbf{T}(z)e^{-t\mathbf{T}_\eta} \;=\; (f_t'(z))^2\mathbf{T}(f_t(z)) - \frac{c}{6}S(f_t)(z). \tag{5.55}
$$

In the same way one may show that the chiral vertex operators transform as

$$
e^{t\mathbf{T}_\eta}\mathbf{h}_s^\alpha(z)e^{-t\mathbf{T}_\eta} \;=\; (f_t'(z))^{\Delta_\alpha}\mathbf{h}_s^\alpha(f_t(z)). \tag{5.56}
$$

Equation (5.55) can also be interpreted as the natural transformation behavior of the Virasoro field under a change of variables. For example, using $x^+ = -i\log z$ and $S(x^+)(z) = -\frac{1}{4}z^{-2}$, one finds that

$$
\mathbf{T}(z) \;=\; -\mathbf{T}_+(x^+)\left(\frac{\partial x^+}{\partial z}\right)^2 - \frac{c}{6}S(x^+)(z), \tag{5.57}
$$

so that $-\mathbf{T}_+(x^+)$, when analytically continued to complex x^+, may be considered as the Virasoro field in the variable x^+. The latter equation also allows us to determine the classical analogon of $b^2\mathbf{T}(z)$:

$$
b^2\mathbf{T}(z) \underset{b\to 0}{\longrightarrow} -\left(\left(\frac{\partial x^+}{\partial z}\right)^2 T_+(x^+) + S(x^+)(z)\right) \;=\; -T_L(z), \tag{5.58}
$$

where $T_L(z)$ (the L, which stands for “Liouville”, is to distinguish it from $T(z)$ defined in (3.13)) is the natural form of the function T_+ in the variable z, cf. (2.30). E.g., let $A_L(z)$ denote the analytic continuation of $A_L(e^{ix^+}) = A^+(x^+)$ to a neighborhood of $S^1 = \{z \in \mathbb{C}\,|\,|z| = 1\}$. Then we have $T_L(z) = S(A_L)(z)$.

6 Conformal blocks

In this section we will introduce the physical notion of a conformal block; the mathematical one will be defined in Section 9. New results are the generalization of the conformal Ward identity (6.16), and the identification of eigenfunctions of length operators in a certain reordering of operators inside 4-point conformal blocks.

6.1 Physical definition of conformal blocks

A conformal block may be defined as the vacuum expectation value of n chiral vertex operators

$$
\mathcal{G}_s^\alpha(z_n,\dots,z_1) \;:=\; \langle \mathbf{h}_{s_n}^{\alpha_n}(z_n)\dots\mathbf{h}_{s_1}^{\alpha_1}(z_1)\rangle, \qquad \begin{matrix} s \equiv (s_n,\dots,s_1) \\ \alpha \equiv (\alpha_n,\dots,\alpha_1) \end{matrix}\;. \tag{6.1}
$$

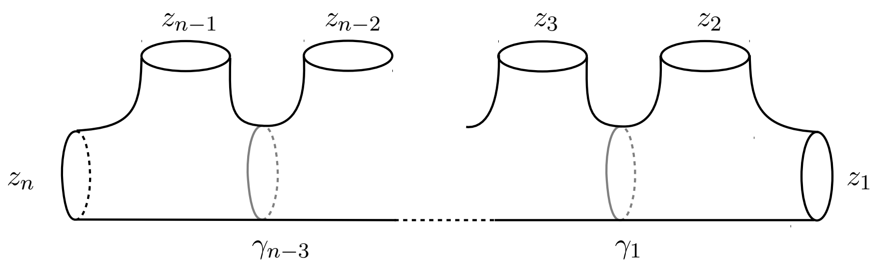

Figure 13: The n-holed sphere with a cut system

Here the vacuum expectation value of an operator $\mathbf{O}$ is defined by $\langle \mathbf{O} \rangle := \langle \Omega | \mathbf{O} | \Omega \rangle$, where $|\Omega\rangle$ is the vacuum state. This state satisfies by definition[25] $\mathbf{L}_n|\Omega\rangle = 0$ for all $n \geq -1$, so it is the highest weight state in a Virasoro representation $\mathcal{W}_0$ called the vacuum representation.

$\mathcal{G}_s^\alpha(z_n, \ldots, z_1)$ is a priori only well defined for $|z_n| > |z_{n-1}| > \cdots > |z_1|$ (since the vertex operators have to be radially ordered). Nevertheless, by analytic continuation it becomes a multivalued holomorphic function on the configuration space of n points on the complex plane. From (5.56) and $\mathbf{T}_\eta|\Omega\rangle = 0 = \langle\Omega|\mathbf{T}_\eta$ for all vector fields η on $\bar{\mathbb{C}}$ (cf. Section 9) follows the global $\mathrm{PSL}(2, \mathbb{C})$ invariance

$$\mathcal{G}_s^\alpha(z_n, \ldots, z_1) \;=\; \prod_{j=1}^{n} \left(f'(z_j)\right)^{\Delta_j} \mathcal{G}_s^\alpha(f(z_n), \ldots, f(z_1)), \tag{6.2}$$

where $\Delta_j := \Delta_{\alpha_j}$, for all complex Möbius transformations f. By (6.2), one may send z_{n-2}, z_{n-1}, z_n to $0, 1$ and ∞ (more precisely, take the limit $z_n \to \infty$ of $z_n^{2\Delta_n}\mathcal{G}_s^\alpha$). Then $\mathcal{G}_s^\alpha$ becomes a multivalued holomorphic function on the moduli space of the n-punctured sphere $\mathcal{M}_{0,n} \simeq \mathcal{Z}_n$ defined in equation (3.31), i.e. a holomorphic function on the universal cover of $\mathcal{M}_{0,n}$, which is the Teichmüller space $\mathcal{T}_{0,n}$. Therefore the $\mathcal{G}_s^\alpha$ are also referred to as genus zero conformal blocks. Conformal blocks for higher genus can be obtained via the gluing construction defined in [40], which is an application of the sewing construction that will be described in Section 9.

We have anticipated in Section 4.4 that conformal blocks of Liouville theory can be identified with the integral kernel $\mathcal{F}_L(Z, r)$ of the intertwiner between length and coherent state representation in quantum Teichmüller theory (respectively the length eigenfunctions $\Psi_{L,r}(Z) = \mathcal{F}_L(Z, r)$ in the coherent state representation). This can be precisely formulated as follows. Let γ_j for $j = 1, \ldots, n-3$ be the loop on the n-punctured sphere $\Sigma = \bar{\mathbb{C}} \backslash \{z_1, \ldots, z_n\}$ that encircles $z_1, \ldots, z_{j+1}$ (see Figure 13; the punctures are drawn as holes because they are effectively treated as such in the quantum theory). This is a maximal set of mutually non-intersecting simple loops on Σ. Let $L = (l_1, \ldots, l_h)$ denote the fixed hyperbolic lengths assigned to the holes and ρ_L the corresponding length representation as introduced in Section 4.2. A natural set of parameters for the conformal blocks $\mathcal{G}_s^\alpha(z_n, \ldots, z_1)$ coming from the gluing construction is given by

$$\beta_j = \sum_{k=1}^{j} (\alpha_k + bs_k) \quad (j = 1, \ldots, n-3). \tag{6.3}$$

After requiring $s_1 = s_n = 0$ and $\alpha_n = \beta_{n-3} + \alpha_{n-1} + bs_{n-2}$ (otherwise $\mathcal{G}_s^\alpha(z_n, \ldots, z_1) = 0$), one may then trade the parameters s for $\beta \equiv (\beta_h, \ldots \beta_1)$, $h = n-3$, and write $\mathcal{G}_\beta^\alpha(z_n, \ldots, z_1)$.

[25] The definition of $|\Omega\rangle$ requires some care since $\mathbf{L}_0|\Omega\rangle = 0$ implies $\mathbf{p}|\Omega\rangle = \pm i\frac{Q}{2}|\Omega\rangle$ which shows that it is not contained in the Liouville Hilbert space $\mathcal{H}_L$. Rather one can interpret its hermitian conjugate $\langle\Omega|$ as an element of $\mathcal{T}^*$, where $\mathcal{T} \subset \mathcal{H}$ is the subspace of states $|\psi\rangle$ with the property that $\langle p|\psi\rangle$ admits an analytic continuation to $p = -i\frac{Q}{2}$ and $\mathcal{T}^*$ is its dual [35].

Then

$$\mathcal{F}_{(l_1,\dots,l_n)}(z_1,\dots,z_h;r_1,\dots,r_h) \;:=\; \lim_{z_n\to\infty} z_n^{2\Delta_n}\,\mathcal{G}^{\alpha}_{\beta}(z_n,\dots,z_1), \tag{6.4}$$

where $(z_{n-1},z_{n-2})=(1,0)$ and

$$\alpha_j=\frac{Q}{2}+i\frac{l_j}{4\pi b}\quad (j=1,\dots,n),\qquad \beta_k=\frac{Q}{2}+ir_k\quad (k=1,\dots,h), \tag{6.5}$$

is the unique solution to the Riemann-Hilbert problem (4.33) with the required asymptotics. Therefore it is the intertwiner between the length representation ρ_L and the coherent state representation.

6.2 Conformal Ward identity

By definition of the vacuum state, $\mathbf{T}(z)|\Omega\rangle$ is regular at $z=0$ (even of order z^4 as $z\to 0$) and $\langle\Omega|\mathbf{T}(z)$ is regular at $z=\infty$ (even of order z^{-4} as $z\to\infty$). Therefore it follows from the OPE (5.39) that

$$\langle \mathbf{T}(z)\mathbf{h}^{\alpha_n}_{s_n}(z_n)\dots\mathbf{h}^{\alpha_1}_{s_1}(z_1)\rangle-\sum_{j=1}^{n}\left(\frac{\Delta_j}{(z-z_j)^2}+\frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right)\langle \mathbf{h}^{\alpha_n}_{s_n}(z_n)\dots\mathbf{h}^{\alpha_1}_{s_1}(z_1)\rangle, \tag{6.6}$$

where radial ordering is understood, is a bounded holomorphic function of $z\in\mathbb{C}$, which by Liouville's theorem is a constant. By taking the limit $z\to\infty$ one shows that this constant is zero. Thus one obtains the conformal Ward identity

$$\langle \mathbf{T}(z)\mathbf{h}^{\alpha_n}_{s_n}(z_n)\dots\mathbf{h}^{\alpha_1}_{s_1}(z_1)\rangle \;=\; \sum_{j=1}^{n}\left(\frac{\Delta_j}{(z-z_j)^2}+\frac{1}{z-z_j}\frac{\partial}{\partial z_j}\right)\langle \mathbf{h}^{\alpha_n}_{s_n}(z_n)\dots\mathbf{h}^{\alpha_1}_{s_1}(z_1)\rangle. \tag{6.7}$$

Let us denote the differential operator appearing on the right hand side of (6.7) by $\nabla_\alpha(z)$. Now, let us put $\alpha_j=\frac{Q}{2}+i\frac{l_j}{4\pi b}$ and $\delta_j=\frac{1}{2}(1+\lambda_j^2)$ where $\lambda_j=l_j/2\pi$ (cf. Section 3.7). Then we have

$$\Delta_j \;=\; \frac{Q^2}{4}+\left(\frac{l_j}{4\pi b}\right)^2, \tag{6.8}$$

so that for $b\to 0$

$$2b^2\nabla_\alpha(z)\longrightarrow\sum_{j=1}^{n}\left(\frac{\delta_j}{(z-z_j)^2}+\frac{2b^2}{z-z_j}\frac{\partial}{\partial z_j}\right). \tag{6.9}$$

Upon identifying

$$2b^2\frac{\partial}{\partial z_j}\equiv c_j \tag{6.10}$$

the right hand side of (6.9) becomes the function $T(z)$ associated with the hyperbolic metric on the n-holed sphere as given in (3.32). As already pointed out in [10], this means that $\mathbf{T}_T(z):=2b^2\nabla_\alpha(z)$ can be considered as a quantization of $T(z)$, as a function on Teichmüller space $\mathcal{T}_{0,n}$, in the coherent state representation[26]

$$\mathbf{z}_j\Psi(Z)=z_j\Psi(Z),\quad \mathbf{c}_j\Psi(Z)=2b^2\frac{\partial}{\partial z_j}\Psi(Z)\quad (j=1,\dots,n-3). \tag{6.11}$$

[26]To be precise, one has to first use the global $\mathrm{PSL}(2,\mathbb{C})$ invariance (6.2) to express the partial derivatives by z_{n-2},z_{n-1} and z_n in (6.7) in terms of the partial derivatives by $z_1,\dots,z_{n-3}$. Considering the $\mathrm{PSL}(2,\mathbb{C})$ invariance of $T(z)$, this should yield a differential operator $\tilde{\nabla}_\alpha(z)$ with the property that $2b^2\tilde{\nabla}_\alpha(z)\to T(z)$ for $b\to 0$.

The difference with (4.28) is marginal and presumably due to inconsistent conventions. This observation may be slightly generalized as follows. Split the Virasoro field as

$$\mathbf{T}_<(z) \ := \sum_{n\leq -2} \mathbf{L}_n z^{-n-2}, \qquad \mathbf{T}_>(z) \ := \sum_{n\geq -1} \mathbf{L}_n z^{-n-2} \tag{6.12}$$

and define a normal ordering for products of Virasoro fields inductively by

$$\begin{aligned} N\{\mathbf{T}(z_m)\ldots\mathbf{T}(z_1)\} &:= \mathbf{T}_<(z_m)N\{\mathbf{T}(z_{m-1})\ldots\mathbf{T}(z_1)\} + N\{\mathbf{T}(z_{m-1})\ldots\mathbf{T}(z_1)\}\mathbf{T}_>(z_m), \\ N\{\mathbf{T}(z_1)\} &:= \mathbf{T}(z_1). \end{aligned} \tag{6.13}$$

Let us compute

$$[\mathbf{h}_s^\alpha(z), \mathbf{T}_<(u)] \ = \ \left(\frac{\Delta_\alpha}{(u-z)^2} + \frac{1}{u-z}\frac{\partial}{\partial z}\right)\mathbf{h}_s^\alpha(z) \quad \text{for } |u|<|z| \tag{6.14}$$

and

$$[\mathbf{T}_>(u), \mathbf{h}_s^\alpha(z)] \ = \ \left(\frac{\Delta_\alpha}{(u-z)^2} + \frac{1}{u-z}\frac{\partial}{\partial z}\right)\mathbf{h}_s^\alpha(z) \quad \text{for } |u|>|z|. \tag{6.15}$$

Together with $\mathbf{T}_>(u)|\Omega\rangle = 0 = \langle\Omega|\mathbf{T}_<(u)$ this yields

$$\begin{aligned} &\left\langle N\left\{\vec{P}\left(\mathbf{T}(u_m),\ldots,\mathbf{T}(u_1)\right)\right\}\mathbf{h}_{s_n}^{\alpha_n}(z_n)\ldots\mathbf{h}_{s_1}^{\alpha_1}(z_1)\right\rangle \\ &\qquad = \ \vec{P}\left(\nabla_\alpha(u_m),\ldots,\nabla_\alpha(u_1)\right)\left\langle \mathbf{h}_{s_n}^{\alpha_n}(z_n)\ldots\mathbf{h}_{s_1}^{\alpha_1}(z_1)\right\rangle, \end{aligned} \tag{6.16}$$

where $\vec{P}$ is an ordered polynomial in m non-commuting variables. This observation will help us to explain at least heuristically the connections between conformal blocks and quantized Liouville theory that are described in the following.

6.3 Conformal blocks for the 4-punctured sphere - reordering of operators

In this section we will discuss a reordering of the operators appearing in the 4-point conformal blocks of Liouville theory, which reveals another aspect of the connection between these objects and the eigenstates of length operators in quantum Teichmüller space. This reordering has been performed in [35] and [37] in order to prove a formula for the braiding of chiral vertex operators. In fact, this braiding formula, which determines the monodromy of the conformal block, has been a crucial element in the proof of the fact that Liouville conformal blocks are the unique solution to the Riemann-Hilbert problem (4.33) with the required asymptotics.

The expressions we want to rewrite are $\mathbf{h}_{s_2}^{\alpha_2}(\sigma_2)\mathbf{h}_{s_1}^{\alpha_1}(\sigma_1)$ and $\mathbf{h}_{t_1}^{\alpha_1}(\sigma_1)\mathbf{h}_{t_2}^{\alpha_2}(\sigma_2)$. Matrix elements of these operators between highest weight states are associated with conformal blocks for the 4-punctured sphere, as is seen via the operator-state correspondence[27]

$$|p\rangle = \lim_{z\to 0}\mathbf{h}_0^\alpha(z)|\Omega\rangle, \qquad \langle p| = \lim_{z\to\infty} z^{2\Delta_\alpha}\langle\Omega|\mathbf{h}_0^{\bar{\alpha}}(z), \quad \alpha = \tfrac{Q}{2}+ip. \tag{6.17}$$

By(6.17) we have

$$\langle p_3|\mathbf{h}_{s_2}^{\alpha_2}(\sigma_2)\mathbf{h}_{s_1}^{\alpha_1}(\sigma_1)|p_0\rangle = z_2^{\Delta_2}z_1^{\Delta_1}\lim_{z_0\to 0}\lim_{z_3\to\infty} z_3^{2\Delta_3}\left\langle \mathbf{h}_0^{\bar{\alpha}_3}(z_3)\mathbf{h}_{s_2}^{\alpha_2}(z_2)\mathbf{h}_{s_1}^{\alpha_1}(z_1)\mathbf{h}_0^{\alpha_0}(z_0)\right\rangle, \tag{6.18}$$

where $z_j := e^{i\sigma_j}$ $(j=1,2)$, $\alpha_k := \frac{Q}{2}+ip_k$ $(k=0,3)$.

[27] We assume $\mathbf{a}_n|\Omega\rangle = 0$ for $n>0$. For the second equation in (6.17), note that $(\mathbf{E}^\alpha(z))^\dagger = \bar{z}^{-2\Delta_\alpha}\mathbf{E}^{\bar{\alpha}}(\bar{z}^{-1})$.

Let us consider the case $\sigma_2 > \sigma_1$. The central equation for reordering (which we do not prove here) is

$$\mathbf{E}^{\alpha}(\sigma)\mathbf{E}^{\alpha'}(\sigma') \;=\; e^{-2\pi i\alpha\alpha'\epsilon(\sigma-\sigma')}\mathbf{E}^{\alpha'}(\sigma')\mathbf{E}^{\alpha}(\sigma), \tag{6.19}$$

where $\epsilon(\sigma) := 1 + 2k$ if $2\pi k < \sigma < 2\pi(k+1)$. The idea is to split the operators $\mathbf{Q}(\sigma_j)$ by defining

$$\mathbf{Q}_1 := \int_{\sigma_1}^{\sigma_2} d\sigma\; \mathbf{E}^b(\sigma), \quad \mathbf{Q}_2 := \int_{\sigma_2}^{\sigma_1+2\pi} d\sigma\; \mathbf{E}^b(\sigma), \quad \mathbf{Q}_3 := \int_{\sigma_1+2\pi}^{\sigma_2+2\pi} d\sigma\; \mathbf{E}^b(\sigma). \tag{6.20}$$

This allows us to write

$$\mathbf{h}_{s_2}^{\alpha_2}(\sigma_2)\mathbf{h}_{s_1}^{\alpha_1}(\sigma_1) \;=\; \mathbf{E}^{\alpha_2}(\sigma_2)\mathbf{E}^{\alpha_1}(\sigma_1)(e^{-2\pi ib\alpha_1}\mathbf{Q}_2 + e^{-6\pi ib\alpha_1}\mathbf{Q}_3)^{s_2}(\mathbf{Q}_1+\mathbf{Q}_2)^{s_1} \tag{6.21}$$

and

$$\mathbf{h}_{t_1}^{\alpha_1}(\sigma_1)\mathbf{h}_{t_2}^{\alpha_2}(\sigma_2) \;=\; \mathbf{E}^{\alpha_1}(\sigma_1)\mathbf{E}^{\alpha_2}(\sigma_2)(e^{2\pi ib\alpha_2}\mathbf{Q}_1 + e^{-2\pi ib\alpha_2}\mathbf{Q}_2)^{t_1}(\mathbf{Q}_2+\mathbf{Q}_3)^{t_2}. \tag{6.22}$$

Like $\mathbf{Q}(\sigma)$, the $\mathbf{Q}_j$ $(j = 1,2,3)$ are positive self-adjoint operators. They have the following Weyl-type commutation relations

$$\begin{array}{l}\mathbf{Q}_1\mathbf{Q}_2 = q^2\mathbf{Q}_2\mathbf{Q}_1 \\ \mathbf{Q}_2\mathbf{Q}_3 = q^2\mathbf{Q}_3\mathbf{Q}_2\end{array} \qquad \mathbf{Q}_1\mathbf{Q}_3 = q^4\mathbf{Q}_3\mathbf{Q}_1, \quad q := e^{\pi ib^2}, \tag{6.23}$$

where we used (6.19) and for the last relation $\mathbf{Q}_3 = e^{2\pi b\mathbf{p}}\mathbf{Q}_1 e^{2\pi b\mathbf{p}}$ and $\mathbf{Q}_j\mathbf{p} = (\mathbf{p}+ib)\mathbf{Q}_j$. It is shown in Appendix C how to deal with complex powers of sums of such operators as appearing in (6.21) and (6.22). Here we define

$$\mathbf{x} := \frac{1}{4\pi b}(\log\mathbf{Q}_1 + \log\mathbf{Q}_2), \qquad \mathbf{t} := \frac{1}{2\pi b}(\log\mathbf{Q}_2 - \log\mathbf{Q}_1), \tag{6.24}$$

which satisfy $[\mathbf{x},\mathbf{t}] = \frac{i}{2\pi} = [\mathbf{x},\mathbf{p}]$ and $[\mathbf{t},\mathbf{p}] = 0$. We then apply equation (C.4) to each of the four terms of the form[28] $(\mathbf{U}+\mathbf{V})^s$ with $\mathbf{UV} = q^2\mathbf{VU}$. This yields

$$(\mathbf{Q}_1+\mathbf{Q}_2)^{s_1} = e^{2\pi bs_1(\mathbf{x}-\frac{1}{2}\mathbf{t})}\frac{e_b(\mathbf{t}-ibs_1)}{e_b(\mathbf{t})}, \tag{6.25}$$

$$(\mathbf{Q}_2+\mathbf{Q}_3)^{t_2} = e^{2\pi bt_2(\mathbf{x}+\frac{1}{2}\mathbf{t})}\;\frac{e_b(2\mathbf{p}-\mathbf{t}-ibt_2)}{e_b(2\mathbf{p}-\mathbf{t})} \tag{6.26}$$

$$(e^{2\pi ib\alpha_2}\mathbf{Q}_1 + e^{-2\pi ib\alpha_2}\mathbf{Q}_2)^{t_1} = e^{2\pi it_1\alpha_2}e^{2\pi bt_1(\mathbf{x}-\frac{1}{2}\mathbf{t})}\;\frac{e_b(\mathbf{t}-2i\alpha_2-ibt_1)}{e_b(\mathbf{t}-2i\alpha_2)}, \tag{6.27}$$

$$(e^{-2\pi ib\alpha_1}\mathbf{Q}_2 + e^{-6\pi ib\alpha_1}\mathbf{Q}_3)^{s_2} = e^{-2\pi ibs_2\alpha_1}e^{2\pi bs_2(\mathbf{x}+\frac{1}{2}\mathbf{t})}\;\frac{e_b(2\mathbf{p}-\mathbf{t}-2i\alpha_1-ibs_2)}{e_b(2\mathbf{p}-\mathbf{t}-2i\alpha_1)}. \tag{6.28}$$

Omitting some irrelevant prefactors, we then find

$$\mathbf{h}_{s_2}^{\alpha_2}(\sigma_2)\mathbf{h}_{s_1}^{\alpha_1}(\sigma_1) \;\sim\; \mathbf{E}^{\alpha_2}(\sigma_2)\mathbf{E}^{\alpha_1}(\sigma_1)\; e^{2\pi b(s_1+s_2)\mathbf{x}}\;\Upsilon^s(\mathbf{t},\mathbf{p}), \tag{6.29}$$

$$\mathbf{h}_{t_1}^{\alpha_1}(\sigma_1)\mathbf{h}_{t_2}^{\alpha_2}(\sigma_2) \;\sim\; \mathbf{E}^{\alpha_2}(\sigma_2)\mathbf{E}^{\alpha_1}(\sigma_1)\; e^{2\pi b(t_1+t_2)\mathbf{x}}\;\Upsilon^t(\mathbf{t},\mathbf{p}) \tag{6.30}$$

[28] Although the operators that include the prefactors are not necessarily self-adjoint, one can perform the steps of (C.4) in a completely analogous way. Alternatively, one may first take $\alpha_j \in i\mathbb{R}$ $(j = 1,2)$ and then analytically continue to generic α_j.

with

$$\Upsilon^s(t,p) = e^{\pi b(s_2-s_1)t}\,\frac{e_b(2p-t-2i\alpha_1-ib(s_1+s_2))e_b(t-ibs_1)}{e_b(2p-t-2i\alpha_1-ibs_1)e_b(t)}, \tag{6.31}$$

$$\Upsilon^t(t,p) = e^{\pi b(t_2-t_1)t}\,\frac{e_b(t-2i\alpha_2-ib(t_1+t_2))e_b(2p-t-ibt_2)}{e_b(t-2i\alpha_2-ibt_2)e_b(2p-t)}. \tag{6.32}$$

Finally, let us substitute e_b for s_b via $e_b(x) = e^{\frac{\pi i}{2}x^2-\frac{\pi i}{24}(2-Q^2)}s_b(x)$ and introduce the new variables $p_0, p_1, p_2, p_3, p_s, p_t$ via

$$\begin{array}{lll} p_0 := p, \quad p_3 := p_0 - i(\alpha_1+\alpha_2+b(s_1+s_2)), & & \alpha_1 = \dfrac{Q}{2}+ip_1, \\ p_s := p_0 - i(\alpha_1+bs_1), \quad p_t := p_0 - i(\alpha_2+bt_2), & & \alpha_2 = \dfrac{Q}{2}+ip_2 \end{array}. \tag{6.33}$$

Note that these are the $\mathbf{p}$ eigenvalues at the different insertion points in the conformal blocks $\langle p_3|\mathbf{h}^{\alpha_2}_{s_2}(\sigma_2)\mathbf{h}^{\alpha_1}_{s_1}(\sigma_1)|p_0\rangle$ and $\langle p_3|\mathbf{h}^{\alpha_1}_{t_1}(\sigma_1)\mathbf{h}^{\alpha_2}_{t_2}(\sigma_2)|p_0\rangle$. Assuming $t_1+t_2 = s_1+s_2$, we then obtain (again omitting all t-independent prefactors)

$$\Upsilon^s(t,p_0) \sim \frac{s_b(-t+p_0+p_1-p_2+p_3)s_b(-t)}{s_b(-t+p_s+p_0+p_1-c_b)s_b(-t-p_s+p_0+p_1-c_b)}, \tag{6.34}$$

$$\Upsilon^t(t,p_0) \sim \frac{s_b(t-p_0-p_1+p_2+p_3)s_b(t-2p_0)}{s_b(t+p_t-p_0+p_2-c_b)s_b(t-p_t-p_0+p_2-c_b)}. \tag{6.35}$$

Remarkably, the expressions on the r.h.s. of (6.34) and (6.35) coincide, up to constant factors, with the length eigenfunctions $\Psi^s_{r_s}$ and $\Psi^t_{r_t}$ in quantum Teichmüller theory of the 4-holed sphere as given in (4.23) and (4.24) upon using the identification of parameters

$$t = p, \qquad \begin{array}{lll} p_0 = r_0, & p_1 = -r_1, & p_s = r_s, \\ p_3 = -r_3, & p_2 = -r_2, & p_t = r_t \end{array}. \tag{6.36}$$

(Note that this identification corresponds to (6.5) with respect to the conformal blocks defined above.) This observation suggests that there exists a connection between the operator $\mathbf{t}$ defined in (6.24) and the operator $\mathbf{p}$ defined in (4.13). We will see in the following section that this is indeed the case.

7 A representation of the algebra of quantized shear coordinates

In this section we present a correspondence between Liouville operators appearing inside a conformal block and operators on quantum Teichmüller space, as inspired by the conformal Ward identity. This will allow us to explain heuristically the connection between the operator $\mathbf{t}$ and the quantized shear coordinate $\mathbf{p}$ discovered above. Then we construct two representations of the algebra of quantized shear coordinates by Liouville operators which affirm this correspondence. Finally we formulate a conjecture that generalizes the observation of the previous section to n punctures. All material of this section is original.

7.1 The correspondence

We have seen in Section 6.2 that the operator $2b^2\mathbf{T}(z)$, when inserted into a conformal block, can be associated with a quantized version $\mathbf{T}_T(z) = 2b^2\nabla_\alpha(z)$ of the function $T(z)$ for fixed z, as given in (3.32), on $\mathcal{T}_{0,n}$ via the dependence on the moduli $z_1, \ldots, z_{n-3}$. Considering also the classical limits, this may be schematically represented as follows:

$$\begin{tikzcd}
b^2\mathbf{T}(z) \arrow[d, "b\to 0"'] \arrow[r, leftrightarrow, dashed] & \frac{1}{2}\mathbf{T}_T(z) \arrow[d, "b\to 0"] \\
-T_L(z) & \frac{1}{2}T(z)
\end{tikzcd}$$

On a heuristic level, one may then argue that given a functional $\mathcal{F}$ and a Liouville operator $\mathbf{F}_L$ whose classical analogue is $\mathcal{F}[T_L]$, then $\mathbf{F}_L$, when appearing inside a conformal block, should be related to the quantum analog $\mathbf{F}_T$ of $\mathcal{F}[-\frac{1}{2}T]$ in Teichmüller theory.

In the previous section we saw that the operators inside a conformal block with four punctures can be reordered as

$$\left\langle \mathbf{h}_0^{\bar{\alpha}_3}(z_3)\mathbf{h}_{s_2}^{\alpha_2}(z_2)\mathbf{h}_{s_1}^{\alpha_1}(z_1)\mathbf{h}_0^{\alpha_0}(z_0)\right\rangle \sim \left\langle \mathbf{h}_0^{\bar{\alpha}_3}(z_3)\mathbf{E}^{\alpha_2}(z_2)\mathbf{E}^{\alpha_1}(z_1)e^{2\pi b(s_1+s_2)\mathbf{x}}\Upsilon^s(\mathbf{t},p_0)\mathbf{h}_0^{\alpha_0}(z_0)\right\rangle, \tag{7.1}$$

where the operator $\mathbf{t}$ was defined as

$$\mathbf{t} \;=\; \frac{1}{2\pi b}\Big(\log\int_{\sigma_2}^{\sigma_1+2\pi} d\sigma\;\mathbf{E}^b(\sigma) - \log\int_{\sigma_1}^{\sigma_2} d\sigma\;\mathbf{E}^b(\sigma)\Big). \tag{7.2}$$

Note that $\mathbf{E}^b(x) = \;:\exp(2b\phi_F^+(x)):$ is the quantized version of $\partial_+A^+(x) = \exp(\varphi_F^+(x))$. So we find that $\mathbf{t}$ corresponds to the classical observable

$$t \;=\; \frac{1}{2\pi b}\log\frac{A^+(\sigma_1+2\pi) - A^+(\sigma_2)}{A^+(\sigma_2)\;-\;A^+(\sigma_1)}. \tag{7.3}$$

So let us consider $A^+(\sigma)$ for each σ as a functional of T_+, defined as the ratio f_2^+/f_1^+ of two solutions to the Hill's equation $f''(x) = T_+(x)f(x)$, with diagonal monodromy fixed by (2.13). (Strictly speaking, $A^+(\sigma)$ as such is not a functional of T_+ because it is fixed only up to rescalings $A^+ \to \lambda A^+$ with $\lambda > 0$. Nevertheless, t is invariant under such rescalings.) In order to define $A^+(\sigma)$ as a functional of T_L, $A^+(\sigma) = A^+(\sigma)[T_L]$, recall that $A^+(\sigma) = A_L(e^{i\sigma})$ and by the relation $T_L = S(A_L)$, it follows that A_L is the quotient f_1/f_2 of two solutions to the complex Hill's equation

$$\partial_z^2 f(z) = T_L(z)f(z) \tag{7.4}$$

with diagonal monodromy around $z=0$. Let us assume for the moment that $\alpha_1 = \alpha_2 = \frac{Q}{2}$, $\alpha_j = \frac{Q}{2} + ip_j$ $(j=0,3)$, which corresponds to parabolic weights $\delta_1 = \delta_2 = \frac{1}{2}$ and hyperbolic weights $\delta_j = \frac{1}{2}(1+(2bp_j)^2)$ $(j=0,3)$ in

$$T(z) \;=\; \sum_{j=0}^{3}\left(\frac{\delta_j}{(z-z_j)^2} + \frac{c_j}{z-z_j}\right). \tag{7.5}$$

According to the discussion in sections 3.3 and 3.7, $A^+(\sigma)[-\frac{1}{2}T] = A(e^{i\sigma})$, where A is (presumably) a uniformization map of $\Sigma(Z,\delta)$, the Riemann sphere with two punctures at z_1, z_2 and two holes around z_0 and z_3, where $Z \equiv (z_0,\ldots,z_3)$, $\delta \equiv (\delta_0,\ldots,\delta_3)$. Since the monodromy of A_L around $z=0$ is diagonal, A should also have a diagonal monodromy; but as T has more than one singularity, it is not yet clear which one. But since $\mathbf{t}$ appears in the conformal block on the r.h.s. of (7.1) left to the first chiral vertex operator $\mathbf{h}_0^{\alpha_0}(z_0)$, it seems plausible that it is the monodromy around z_0, which is then given by $M_0 : A \mapsto e^{4\pi bp_0}A$ (this will become more clear by the discussion at the end of this section).

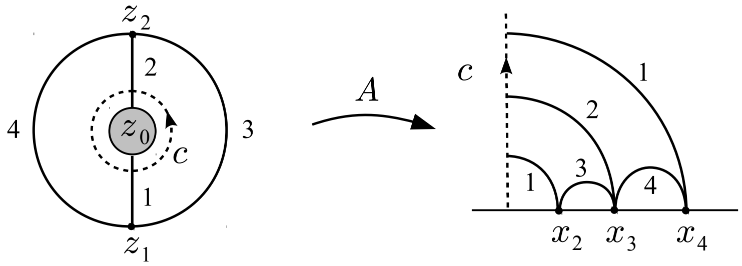

Figure 14: Expressing w_2 in terms of A

Let us now express the shear coordinate w_2 in terms of A, where 2 stands for an edge of the fat graph for the 4-holed sphere depicted in Fig. 10 and the relevant part of the triangulation is displayed in Fig. 14. As already stated in Section 3.6, the shortest geodesics in $\mathbb{H}$ connecting two points that are equivalent under M_0 (these are the ones that are projected to closed geodesics on $\mathbb{H}/M_0$), are segments of the imaginary axis. So an image of the closed geodesic c around z_0 under A is such a segment that will be extended as a geodesic to the positive imaginary axis (see Fig. 14). Therefore, if we endow c with a counter-clockwise orientation, the points $x_1, \ldots, x_4$ that determine w_2 according to (3.20) are

$$x_1 = \infty, \qquad x_2 = A(z_1), \qquad x_3 = A(z_2), \qquad x_4 = A(e^{2\pi i} z_1), \tag{7.6}$$

where $e^{2\pi i} z_1$ denotes the boundary point of the universal cover of $\Sigma(Z, \delta)$ which is reached by going from z_1 (which, by abuse of notation, denotes some point over z_1) around z_0 in the counter-clockwise direction[29]. This yields

$$w_2 = \log \left| \frac{A(e^{2\pi i} z_1) - A(z_2)}{A(z_2) - A(z_1)} \right|, \tag{7.7}$$

and we find that the functional of $t[-\frac{1}{2}T] \sim w_2/2\pi b$, which is the classical limit of the operator $\mathbf{p}$ defined (4.13). So we have indeed established, on a heuristic level, a connection between the operators $\mathbf{t}$ and $\mathbf{p}$, based on the conformal Ward identity, which supplements the more empirical connection found in the previous section.

It is also instructive to make such considerations for the zero mode $\mathbf{p}$, or rather $\mathbf{P} := 2\cosh(2\pi b\mathbf{p})$. The classical analogue of this operator is (recall that $2b\phi^F \simeq \varphi^F$).

$$P := 2\cosh(\pi p) \;=\; |\mathrm{tr} M^+|, \tag{7.8}$$

where M^+ is the monodromy of $(f_1^+, f_2^+)^t$. This observable can be explicitly expressed in terms of T_+ respectively T_L by the following consideration. The matrix function

$$H \;=\; \begin{pmatrix} f_1 & f_2 \\ f_1' & f_2' \end{pmatrix}, \tag{7.9}$$

where $f_j = f_j^+$, satisfies the differential equation

$$\partial_x H(x) \;=\; \Lambda(x) H(x), \qquad \Lambda(x) \;:=\; \begin{pmatrix} 0 & 1 \\ T_+(x) & 0 \end{pmatrix}. \tag{7.10}$$

This equation is solved by

$$H(x) \;=\; \mathcal{P} \exp\left(\int_0^x \Lambda(x') dx' \right) H(0), \tag{7.11}$$

[29]The notion of images of punctures under uniformization maps was discussed in Section 3.4.

where $\mathcal{P}\exp$ denotes the path ordered exponential defined by

$$\mathcal{P}\exp\left(\int_0^x \Lambda(x')dx'\right) \;=\; 1+\sum_{n=1}^{\infty}\int_0^x dx_1\int_0^{x_1}dx_2\ldots\int_0^{x_{n-1}}dx_n\,\Lambda(x_1)\ldots\Lambda(x_n). \tag{7.12}$$

Thus we find that $H(2\pi)=\tilde{M}^+H(0)$ where

$$\tilde{M}^+ \;=\; \mathcal{P}\exp\left(\int_0^{2\pi}\Lambda(x)dx\right). \tag{7.13}$$

Combining this with $H(2\pi)=H(0)(M^+)^t$ one concludes that $\tilde{M}^+=H(0)(M^+)^t(H(0))^{-1}$ ($H(x)$ is invertible by (2.8)). It follows that

$$P \;=\; |\mathrm{tr}\tilde{M}^+|. \tag{7.14}$$

Changing to coordinate z, this may be easily expressed in terms of T_L, yielding $P=P[T_L]$. Then $P[-\frac{1}{2}T]$ can be identified with

$$|\mathrm{tr}M_\gamma| \;=\; 2\cosh(l_\gamma/2), \tag{7.15}$$

where M_γ is the monodromy of the uniformization map A along a path γ, and l_γ is the associated length function. Again, the choice of γ is ambiguous from the outset. But one can argue that it depends on the insertion point of $\mathbf{P}$ into the conformal block. Namely, if $\mathbf{P}$ sits between $\mathbf{h}^{\alpha_{j+2}}_{s_{j+2}}(z_{j+2})$ and $\mathbf{h}^{\alpha_{j+1}}_{s_{j+1}}(z_{j+1})$ in (6.1), then suppose that it could be expressed in terms of $\mathbf{T}(z)$ in a way that quantizes the relation $P=P[T_L]$. Then the arguments of all $\mathbf{T}(z)$ appearing in this expression had to be confined inside the conformal block to $|z_j|>|z|>|z_{j-1}|$. This implies that the path ordered exponential appearing in this expression had to be defined on a path that runs in this region. This could only be the path γ_j that encircles $z_1,\ldots,z_{j+1}$. The quantized version of $2\cosh(l_{\gamma_j}/2)$ is the length operator $\mathbf{L}_j$. So what we have in mind is something similar to the relation (which follows from (6.16))

$$\langle\ldots\mathbf{h}^{\alpha_{j+2}}_{s_{j+2}}(z_{j+2})\hat{\mathbf{P}}_j\mathbf{h}^{\alpha_{j+1}}_{s_{j+1}}(z_{j+1})\ldots\rangle \;=\; \hat{\mathbf{L}}_j\langle\mathbf{h}^{\alpha_n}_{s_n}(z_n)\ldots\mathbf{h}^{\alpha_1}_{s_1}(z_1)\rangle, \tag{7.16}$$

where

$$\begin{aligned}\hat{\mathbf{P}}_j \;&:=\; N\left\{\mathrm{tr}\;\mathcal{P}\exp\left(\oint_{\gamma_j}dz\begin{pmatrix}0 & 1\\ -b^2\mathbf{T}(z) & 0\end{pmatrix}\right)\right\},\\ \hat{\mathbf{L}}_j \;&:=\; \mathrm{tr}\;\mathcal{P}\exp\left(\oint_{\gamma_j}dz\begin{pmatrix}0 & 1\\ -\frac{1}{2}\mathbf{T}_T(z) & 0\end{pmatrix}\right).\end{aligned} \tag{7.17}$$

If $\hat{\mathbf{P}}_j$ was the same as $\mathbf{P}$, then (7.16) would imply that the conformal block is an eigenfunction of $\hat{\mathbf{L}}_j$ (with eigenvalue $2\cosh(2\pi bp_j)$, $p_j=\Im\beta_j$), which looks like the length operator $\mathbf{L}_j$ in a coherent state representation. Unfortunately, we have $\hat{\mathbf{P}}_j\neq\mathbf{P}$ since $\mathbf{P}$, in contrast to $\hat{\mathbf{P}}_j$, commutes with $\mathbf{T}(z)$. In fact, one cannot hope to express $\mathbf{P}$ directly in terms of $\mathbf{T}$ since the center of the Virasoro algebra is spanned by c. Nevertheless, a direct proof of the relation between conformal blocks and eigenstates of length operators could be possible via the so-called loop operators [10, 2]. These involve the degenerate fields

$$\mathbf{f}_1(z):=\mathbf{E}^{-\frac{b}{2}}(z),\quad \mathbf{f}_2:=\mathbf{E}^{-\frac{b}{2}}(z)\mathbf{Q}(z), \tag{7.18}$$

which are quantum analogs of f_1 and f_2 satisfying the quantum Hill's equation

$$\partial^2\mathbf{f}_j(z)=-b^2N\{\mathbf{T}(z)\mathbf{f}_j(z)\} \tag{7.19}$$

with "diagonal" monodromy

$$\mathbf{f}_1(z)\to -e^{\frac{\pi i}{2}b^2}e^{-\pi b\mathbf{p}}\mathbf{f}_1(z)e^{-\pi b\mathbf{p}},\quad \mathbf{f}_2(z)\to -e^{\frac{\pi i}{2}b^2}e^{\pi b\mathbf{p}}\mathbf{f}_2(z)e^{\pi b\mathbf{p}} \tag{7.20}$$

around $z=0$. The author felt unable to concrete this idea.

7.2 The representations

We would like to formulate the observation of Section 6.3, the appearance of length eigenfunctions of quantum Teichmüller theory in 4-point conformal blocks, in a way that permits a (conjectural) generalization to the n-punctured sphere. Since the function Υ^s as appearing in (7.1) is evaluated on the *operator* $\mathbf{t}$, this object would be an eigenstate of the length operator $\mathbf{L}_s$ in a representation ρ of the algebra of shear coordinates which acts on a space of operators. In the case of four punctures, this space would simply be the space $\mathcal{F}(\mathbf{t})$ of functions f of $\mathbf{t}$ where we require f to be in the Schwartz space $\mathcal{S}(\mathbb{R})$. Elements of $\mathcal{F}(\mathbf{t})$ possess a Fourier expansion

$$f(\mathbf{t}) = \int_{-\infty}^{\infty} dx\, e^{-2\pi i x \mathbf{t}} \hat{f}(x), \tag{7.21}$$

where $\hat{f}$ is the inverse Fourier transform of f. Obviously we want to define

$$\rho(\mathbf{p}) f(\mathbf{t}) := \mathbf{t} f(\mathbf{t}), \tag{7.22}$$

where $\mathbf{p}$ was defined in (4.13) (not to be confused with the zero mode of quantum Liouville theory). Now, if $\mathbf{v}$ is an operator that satisfies $[\mathbf{v}, \mathbf{t}] = -ib$, then we can represent $\mathbf{x}$ (defined in (4.13)) by

$$\rho(\mathbf{x}) f(\mathbf{t}) := -\frac{1}{2\pi b}[\mathbf{v}, f(\mathbf{t})], \tag{7.23}$$

where $[\mathbf{v}, f(\mathbf{t})] = -ibf'(\mathbf{t})$ is again an element of $\mathcal{F}(\mathbf{t})$. Since the algebra $\mathcal{A}_{f_0}$ of shear coordinates for the 4-punctured sphere associated with the fat graph f_0 is generated by $\mathbf{p}, \mathbf{x}$ and 1, this would define (together with $\rho(1) = 1$) a representation of $\mathcal{A}_{f_0}$ on $\mathcal{F}(\mathbf{t})$. There are different possible choices for $\mathbf{v}$, e.g. one could define $\mathbf{v} = -\log \mathbf{Q}_2$. However, since Υ^s and Υ^t also depend on $\mathbf{p}$, we want $\mathbf{v}$ to commute with $\mathbf{p}$. Therefore we define

$$\mathbf{Q}_j^{\infty} := \int_{\gamma_j} dz\, \mathbf{E}^b(z), \qquad \gamma_j : [1, \infty) \to \mathbb{C},\ t \mapsto t \cdot z_j \quad (j = 1, 2), \tag{7.24}$$

and[30]

$$\mathbf{v} := \frac{1}{2}(\log \mathbf{Q}_1^{\infty} - \log \mathbf{Q}_2^{\infty} + 2\pi b \mathbf{p}). \tag{7.25}$$

These are rather formal definitions which still have to be given a precise meaning; in particular the existence of the implicit limit in (7.24) and the meaning of the logarithm in (7.25) - $\mathbf{Q}_j^{\infty}$ is neither positive nor self adjoint - is not yet established. Nevertheless, if a precise meaning exists, then it follows from the generalization[31] of (6.19),

$$\mathbf{E}^{\alpha_2}(z_2)\mathbf{E}^{\alpha_1}(z_1) \;=\; e^{-2\pi i \alpha_2 \alpha_1 \epsilon(\sigma_2 - \sigma_1)} \mathbf{E}^{\alpha_1}(z_1)\mathbf{E}^{\alpha_2}(z_2), \quad \sigma_j := \arg(z_j), \tag{7.26}$$

that $\mathbf{v}$ indeed satisfies $[\mathbf{v}, \mathbf{t}] = -ib$ and $[\mathbf{p}, \mathbf{v}] = 0$.

Note that the exponentiated operators are represented as

$$\rho(e^{2\pi b \mathbf{p}}) f(\mathbf{t}) = e^{2\pi b \mathbf{t}} f(\mathbf{t}), \qquad \rho(e^{2\pi b \mathbf{x}}) f(\mathbf{t}) = e^{-\mathbf{v}} f(\mathbf{t}) e^{\mathbf{v}} = f(\mathbf{t} + ib). \tag{7.27}$$

[30] The contribution with the zero mode $\mathbf{p}$ is only included in order to make the analogy with (quantized) shear coordinates more clear, as explained below. Moreover, it makes $\mathbf{v}$ commute with $\mathbf{x}$ defined in (6.24).

[31] Equation (7.26) also has to be interpreted with care: the left hand side (right hand side) is a priori only well-defined for $|z_2| > |z_1|$ ($|z_1| > |z_2|$). However, for $|z_2| > |z_1|$ the r.h.s. can be understood as the analytic continuation of the same expression with $|z_1| > |z_2|$.

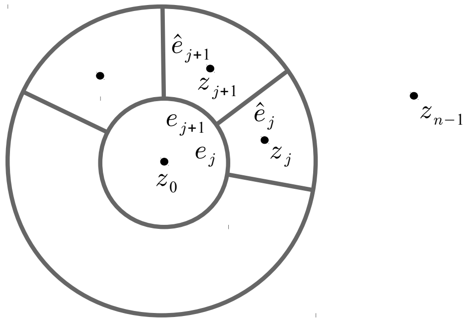

Figure 15: A fat graph for the n-punctured sphere

As usual, these operators are only defined on dense subspaces of $\mathcal{F}(\mathbf{t})$. The corresponding representation $\hat{\pi}$ on the Fourier transforms is given by

$$\hat{\pi}(\mathbf{p})\hat{f}(x) = \frac{1}{2\pi i}\hat{f}'(x), \qquad \hat{\pi}(\mathbf{x})\hat{f}(x) = x\hat{f}(x). \tag{7.28}$$

This is the standard representation on $\mathcal{S}(\mathbb{R})$ where $\mathbf{x}$ is diagonalized.

Let $\{|v_\nu\rangle\}_{\nu\in I_p}$ be an orthonormal basis for the $\mathbf{p}$ eigenspace $\mathcal{F}_p$, so that

$$\mathcal{P}_p := \sum_{\nu\in I_p} |v_\nu\rangle\langle v_\nu| \tag{7.29}$$

is a projection onto $\mathcal{F}_p$. We can then reformulate the observation of Section 6.3 in the following way:

$$\rho(\mathbf{L}_\flat)\big(\Upsilon^\flat(\mathbf{t},\mathbf{p})\big)\mathcal{P}_{p_0} \;=\; 2\cosh(2\pi b r_\flat)\Upsilon^\flat(\mathbf{t},\mathbf{p})\mathcal{P}_{p_0} \qquad (\flat = s,t) \tag{7.30}$$

where we have implicitly used the identification of parameters (6.36). The expression on the l.h.s. of (7.30) is well defined as we know how to express $\mathbf{L}_\flat$ in terms of $\mathbf{x}$ and $\mathbf{p}$ (and the dependence of $\Upsilon^\flat$ on $\mathbf{p}$ does not interfere with the definition of ρ). However, we can make the connection with conformal blocks resp. chiral vertex operators more explicit by defining

$$\begin{aligned}\Theta^{\alpha_1,\alpha_2}_{s_1,s_2}(\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{Q}_3) \;&:=\; \big(\mathbf{E}^{\alpha_2}(\sigma_2)\mathbf{E}^{\alpha_1}(\sigma_1)\big)^{-1}\;\mathbf{h}^{\alpha_2}_{s_2}(\sigma_2)\mathbf{h}^{\alpha_1}_{s_1}(\sigma_1)\\ &\sim e^{2\pi b(s_1+s_2)\mathbf{x}}\Upsilon^s(\mathbf{t},\mathbf{p}).\end{aligned} \tag{7.31}$$

Since $\mathbf{v}$ commutes with $\mathbf{x}$, we have (using again (6.36))

$$\rho(\mathbf{L}_s)\big(\Theta^{\alpha_1,\alpha_2}_{s_1,s_2}(\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{Q}_3)\big)\mathcal{P}_{p_0} \;=\; 2\cosh(2\pi b r_s)\Theta^{\alpha_1,\alpha_2}_{s_1,s_2}(\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{Q}_3)\mathcal{P}_{p_0}, \tag{7.32}$$

i.e., $\Theta^{\alpha_1,\alpha_2}_{s_1,s_2}(\mathbf{Q}_1,\mathbf{Q}_2,\mathbf{Q}_3)$ is also a kind of generalized eigenstate of $\rho(\mathbf{L}_s)$. An analogous statement holds for the t-channel.

In order to generalize (7.30) to n punctures (resp. holes), we will first define a representation ρ of the algebra $\mathcal{A}_n$ of shear coordinates associated with the fat graph that is depicted in Figure 15. First, we split the circle into $m := n-2$ segments at $\sigma_m > \cdots > \sigma_1 \in [0,2\pi)$ (we will associate $z_j = e^{i\sigma_j}$) and define

$$\mathbf{Q}_j := \int_{\sigma_j}^{\sigma_{j+1}} d\sigma\;\mathbf{E}^b(\sigma) \qquad (j = 0,1,\ldots,m), \tag{7.33}$$

where $\sigma_0 := \sigma_m - 2\pi,\ \sigma_{m+1} := \sigma_1 + 2\pi$. The operators

$$\mathbf{t}_j := \frac{1}{2\pi b}\left(\log \mathbf{Q}_j - \log \mathbf{Q}_{j-1}\right) \qquad (j = 1, \ldots, m) \tag{7.34}$$

satisfy the commutation relations

$$[\mathbf{t}_j, \mathbf{t}_k] \ = \ \frac{i}{2\pi}\left(\delta^{(m)}_{j+1,k} - \delta^{(m)}_{j-1,k}\right) \tag{7.35}$$

where $\delta^{(m)}_{j,k}$ is the m-periodic Kronecker delta. The representation ρ will be defined on the space $\mathcal{F}(\mathbf{t}_2, \ldots, \mathbf{t}_m)$ of functions of $\mathbf{t}_2, \ldots \mathbf{t}_m$ of the form

$$f(\mathbf{t}_2, \ldots, \mathbf{t}_m) \ = \ \int dx_2 \ldots dx_m\ \hat{f}(x_2, \ldots, x_m)\ e^{-2\pi i \sum_{j=2}^m x_j \mathbf{t}_j}, \tag{7.36}$$

where the generalized inverse Fourier transform $\hat{f}$ is in the Schwartz space $\mathcal{S}(\mathbb{R}^{m-1})$. On $\mathcal{F}(\mathbf{t}_2, \ldots, \mathbf{t}_m)$, we represent $\mathbf{w}_j := \mathbf{w}_{e_j}$ by

$$\rho(\mathbf{w}_j) f(\mathbf{t}_2, \ldots, \mathbf{t}_m) := 2\pi b f(\mathbf{t}_2, \ldots, \mathbf{t}_m)\mathbf{t}_j. \tag{7.37}$$

By (7.35) and (4.1), $[\rho(\mathbf{w}_j), \rho(\mathbf{w}_k)] = -(2\pi b)^2[\mathbf{t}_j, \mathbf{t}_k] = [\mathbf{w}_j, \mathbf{w}_k]$, so (7.37) (together with $\rho(1) = 1$) indeed defines a representation of the subalgebra of $\mathcal{A}_n$ generated by $\mathbf{w}_2, \ldots \mathbf{w}_m$ and 1. In order to represent also $\hat{\mathbf{w}}_j := \mathbf{w}_{\hat{e}_j}$ $(j = 1, \ldots, m)$, we define

$$\mathbf{Q}^\infty_j := \int_{\gamma_j} dz\ \mathbf{E}^b(z), \qquad \gamma_j : [1, \infty) \to \mathbb{C},\ t \mapsto t \cdot z_j \quad (j = 1, \ldots, m), \tag{7.38}$$

and

$$\mathbf{v}_j := \frac{1}{2}(\log \mathbf{Q}^\infty_j - \log \mathbf{Q}^\infty_{j+1}) \quad (j = 1, \ldots, m), \tag{7.39}$$

where $\mathbf{Q}^\infty_{m+1} := e^{2\pi b \mathbf{p}} \mathbf{Q}^\infty_1 e^{2\pi b \mathbf{p}}$, which corresponds to the analytic continuation of $\mathbf{E}^b(z)$ around 0. Then we have (at least formally)

$$[\mathbf{t}_j, \mathbf{v}_k] \ = \ ib\left(\delta^{(m)}_{j,k+1} - \delta^{(m)}_{j,k}\right), \tag{7.40}$$

So we define

$$\rho(\hat{\mathbf{w}}_j) f(\mathbf{t}_2, \ldots, \mathbf{t}_m) \ := \ -[\mathbf{v}_j, f(\mathbf{t}_2, \ldots, \mathbf{t}_m)] \quad (j = 1, \ldots, m). \tag{7.41}$$

The $\hat{\mathbf{w}}_j$ are actually not linear independent: one has $\sum_{j=1}^m \hat{\mathbf{w}}_j = \sum_{j=1}^m l_j - l_0 - l_{n-1}$. This constraint is not properly realized in the representation (7.41) (rather $\sum_j \rho(\hat{\mathbf{w}}_j) = 0$). Therefore we introduce

$$\mathbf{q}_j \ := \ \frac{1}{2\pi b}\mathbf{w}_j, \qquad \mathbf{x}_j \ := \ \frac{1}{4\pi b}\Big(\sum_{k=1}^{j-1} \hat{\mathbf{w}}_k - \sum_{k=j}^{m} \hat{\mathbf{w}}_k\Big) \quad (j = 2, \ldots, m), \tag{7.42}$$

so that $\{\mathbf{q}_j, \mathbf{x}_j\}_{j=2,\ldots,m} \cup \{1\}$ is a basis of $\mathcal{A}_n$ (as a Lie algebra) with commutation relations

$$[\mathbf{q}_j, \mathbf{q}_k] = \frac{i}{2\pi}(\delta_{j,k+1} - \delta_{j,k-1}), \qquad [\mathbf{q}_j, \mathbf{x}_k] = \frac{1}{2\pi i}\delta_{j,k}, \qquad [\mathbf{x}_j, \mathbf{x}_k] = 0. \tag{7.43}$$

Then we define

$$\rho(\mathbf{q}_j) := \frac{1}{2\pi b}\rho(\mathbf{w}_j), \qquad \rho(\mathbf{x}_j) \ := \ \frac{1}{4\pi b}\Big(\sum_{k=1}^{j-1} \rho(\hat{\mathbf{w}}_k) - \sum_{k=j}^{m} \rho(\hat{\mathbf{w}}_k)\Big). \tag{7.44}$$

This clearly defines a representation of $\mathcal{A}_n$. Putting

$$\mathbf{y}_j := \frac{1}{2}\Big(\sum_{k=1}^{j-1}\mathbf{v}_k - \sum_{k=j}^{m}\mathbf{v}_k\Big) = \frac{1}{2}(\log \mathbf{Q}_1^\infty - \log \mathbf{Q}_j^\infty) + \pi b \mathbf{p}, \tag{7.45}$$

it may be written as

$$\rho(\mathbf{q}_j)(\mathbf{O}) = \mathbf{O}\mathbf{t}_j, \qquad \rho(\mathbf{x}_j)(\mathbf{O}) := -\frac{1}{2\pi b}[\mathbf{y}_j, \mathbf{O}] \tag{7.46}$$

for $\mathbf{O} \in \mathcal{F}(\mathbf{t}_2, \ldots, \mathbf{t}_m)$ (or any other Liouville operator).

Notice also that

$$\bar{\rho}(\mathbf{q}_j) := \mathbf{t}_j, \qquad \bar{\rho}(\mathbf{x}_j) := \frac{1}{2\pi b}\mathbf{y}_j \qquad (j = 2, \ldots, m) \tag{7.47}$$

defines an anti-representation of the Lie algebra $\mathcal{A}_n$. Considering the adjoint $\mathbf{O}^\dagger$ of an operator $\mathbf{O}$ as an operator on the dual $\mathcal{H}_L^*$ of the Hilbert space (which is isomorphic to $\mathcal{H}_L$),

$$\mathbf{O}^\dagger :\ \mathcal{H}_L^* \to \mathcal{H}_L^*,\ \phi \to \phi \circ \mathbf{O}, \tag{7.48}$$

it is immediately seen that $\rho^*(\mathbf{a}) := (\bar{\rho}(\mathbf{a}))^\dagger$ for all $\mathbf{a} \in \mathcal{A}_n$ defines a representation of $\mathcal{A}_n$. Furthermore, in the classical limit we have

$$2\pi b \mathbf{t}_j \underset{\text{class.}}{\longrightarrow} \log \frac{A^+(\sigma_{j+1}) - A^+(\sigma_j)}{A^+(\sigma_j)\ -\ A^+(\sigma_{j-1})}, \tag{7.49}$$

so that $2\pi b \mathbf{t}_j$ can be considered as the Liouville analog of $\mathbf{w}_j$ in the line of thought pursued in Section 7.1. Similarly,

$$\mathbf{v}_j \underset{\text{class.}}{\longrightarrow} \frac{1}{2}\log \frac{A_L(\infty) - A_L(z_j)}{A_L(\infty) - A_L(z_{j+1})}, \tag{7.50}$$

where A_L denotes the analytic continuation of $A_L(e^{i\sigma}) := A^+(\sigma)$. (In the case of $\mathbf{v}_m$, the monodromy of A_z^+ around 0 is taken into account.) Thus one can consider $2\mathbf{v}_j$ as the Liouville analog of $\hat{\mathbf{w}}_j$ provided that z_{n-1} is send to infinity (the factor 2 is somewhat puzzling).

Let $\hat{\pi}$ denote the representation on the space of the $\hat{f}(x)$, $x \equiv (x_2, \ldots, x_m)$, (i.e., on $\mathcal{S}(\mathbb{R}^{m-1})$) that corresponds to ρ. One easily finds

$$\hat{\pi}(\mathbf{x}_j)\hat{f}(x) \ = \ x_j \hat{f}(x). \tag{7.51}$$

In order to compute $\hat{\pi}(\mathbf{w}_j)$, one may use the Baker-Campbell-Hausdorff formula to find

$$e^{-2\pi i \sum_{l=2}^m x_l \mathbf{t}_l} = e^{-2\pi i \sum_{l\neq j} x_l \mathbf{t}_l} e^{-2\pi i (\mathbf{t}_j + \frac{1}{2}(x_{j+1} - x_{j-1}))x_j}, \tag{7.52}$$

where it is understood that $x_1, x_{m+1} \equiv 0$. This implies

$$\hat{\pi}(\mathbf{q}_j)\hat{f}(x) \ = \ \left(\frac{1}{2\pi i}\frac{\partial}{\partial x_j} + \frac{1}{2}(x_{j-1} - x_{j+1})\right)\hat{f}(x). \tag{7.53}$$

Thus the operators $\mathbf{p}_j := \mathbf{q}_j + \frac{1}{2}(\mathbf{x}_{j+1} - \mathbf{x}_{j-1})$ $(j = 2, \ldots, m)$ are represented by $(2\pi i)^{-1}\frac{\partial}{\partial x_j}$.

7.3 The conjecture

We are now ready to formulate a conjectural generalization of the observation of Section 6.3 to n punctures respectively $m = n-2$ chiral vertex operators. Let $\sigma_m > \cdots > \sigma_1 \in [0, 2\pi)$ and define

$$\mathbf{Q}_j := \int_{\sigma_j}^{\sigma_{j+1}} d\sigma\, \mathbf{E}^b(\sigma), \qquad (j = 1, \ldots, 2m-1), \tag{7.54}$$

where $\sigma_{j+m} := \sigma_j + 2\pi$ $(j = 1, \ldots, m)$. Let $\pi \in S_m$ be a permutation. Then one may obviously reorder the product of m chiral vertex operators as

$$\prod_{j=m}^{1} \mathbf{h}_{s_{\pi(j)}}^{\alpha_{\pi(j)}}(\sigma_{\pi(j)}) \;=\; \Big(\prod_{j=m}^{1} \mathbf{E}^{\alpha_{\pi(j)}}(\sigma_{\pi(j)})\Big)\Theta^{\pi}_{\alpha,s}(\mathbf{Q}_1, \ldots, \mathbf{Q}_{2m-1}), \tag{7.55}$$

where $\alpha := (\alpha_1, \ldots, \alpha_m)$, $s := (s_1, \ldots, s_m)$, by using the Weyl-type commutation relations of $\mathbf{E}^{\alpha_j}(\sigma_j)$ with $\mathbf{Q}_k$.

On the Teichmüller side, the conformal block

$$\Big\langle \mathbf{h}_0^{\alpha_{n-1}}(z_{n-1})\mathbf{h}_{s_{\pi(m)}}^{\alpha_{\pi(m)}}(z_{\pi(m)}) \cdots \mathbf{h}_{s_{\pi(1)}}^{\alpha_{\pi(1)}}(z_{\pi(1)})\mathbf{h}_0^{\alpha_0}(z_0)\Big\rangle, \tag{7.56}$$

where $z_j = e^{i\sigma_j}$ $(j = 1, \ldots, m)$, is associated with the n-punctured sphere $\bar{\mathbb{C}}\backslash\{z_0, \ldots, z_{n-1}\}$ equipped with a set of mutually non-intersecting simple loops $\{\gamma_k\}_{k=1,\ldots,n-3}$, where γ_k is a loop that encircles $z_0, z_{\pi(1)}, \ldots, z_{\pi(k)}$ (cf. Section 6). The corresponding length operators $\mathbf{L}_k := \mathbf{L}_{\gamma_k}$ are part of the (exponentiated) algebra $\mathcal{A}_n$ of shear coordinates. Let $\beta_1, \ldots, \beta_{n-3}$ denote the intermediate momenta of the conformal block (7.56), as defined in Section 6, and $\alpha_0 = \frac{Q}{2} + ip_0$.

Conjecture 1. *The operators $\Theta^{\pi}_{\alpha,s}(\mathbf{Q}_1, \ldots, \mathbf{Q}_{2m-1})$ are generalized simultaneous eigenstates of the length operators $\mathbf{L}_1, \ldots, \mathbf{L}_{n-3}$ in the representation ρ defined above, in the sense that*

$$\rho(\mathbf{L}_k)\big(\Theta^{\pi}_{\alpha,s}(\mathbf{Q}_1, \ldots, \mathbf{Q}_{2m-1})\big)\mathcal{P}_{p_0} = 2\cosh(2\pi b r_k)\Theta^{\pi}_{\alpha,s}(\mathbf{Q}_1, \ldots, \mathbf{Q}_{2m-1})\mathcal{P}_{p_0}, \tag{7.57}$$

with the identification of variables given by (6.5).

Employing the relation $\mathbf{Q}_{j+m} = e^{2\pi b\mathbf{p}}\mathbf{Q}_j e^{2\pi b\mathbf{p}}$ $(j = 1, \ldots, m-1)$, one may express $\Theta^{\pi}_{\alpha,s}(\mathbf{Q}_1, \ldots, \mathbf{Q}_{2m-1})$ first in terms of $\mathbf{Q}_1, \ldots, \mathbf{Q}_m$ and $\mathbf{p}$ and then in terms of[32]

$$\mathbf{x} := \frac{1}{4\pi b}(\log \mathbf{Q}_1 + \log \mathbf{Q}_m), \qquad \mathbf{t}_k := \frac{1}{2\pi b}(\log \mathbf{Q}_k - \log \mathbf{Q}_{k-1}) \quad (k = 2, \ldots, m). \tag{7.58}$$

and $\mathbf{p}$. Finally one may use the techniques of Appendix C to extract the overall $\mathbf{x}$ dependence and obtain

$$\Theta^{\pi}_{\alpha,s}(\mathbf{Q}_1, \ldots, \mathbf{Q}_{2m-1}) \;=\; e^{2\pi b\left(\sum_{j=1}^{m} s_j\right)\mathbf{x}}\Upsilon^{\pi}_{\alpha,s}(\mathbf{t}_2, \ldots, \mathbf{t}_m, \mathbf{p}), \tag{7.59}$$

which implicitly defines $\Upsilon^{\pi}_{\alpha,s}(\mathbf{t}_2, \ldots, \mathbf{t}_m, \mathbf{p})$. Since $\mathbf{x}$ commutes with $\mathbf{y}_j$ $(j = 2, \ldots, m)$, by (7.46) the conjecture (7.57) is equivalent to

$$\rho(\mathbf{L}_k)\big(\Upsilon^{\pi}_{\alpha,s}(\mathbf{t}_2, \ldots, \mathbf{t}_m, p_0)\big) \;=\; 2\cosh(2\pi b r_k)\Upsilon^{\pi}_{\alpha,s}(\mathbf{t}_2, \ldots, \mathbf{t}_m, p_0). \tag{7.60}$$

Note that the operators $\Upsilon^{\pi}_{\alpha,s}(\mathbf{t}_2, \ldots, \mathbf{t}_m, p_0) \in \mathcal{F}(\mathbf{t}_2, \ldots, \mathbf{t}_m)$ are well defined since $\mathbf{p}$ commutes with $\mathbf{t}_2, \ldots, \mathbf{t}_m$.

[32] The inverse transformation is given by

$$\log \mathbf{Q}_k = \pi b\Big(\sum_{j=2}^{k}\mathbf{t}_j - \sum_{j=k+1}^{m}\mathbf{t}_j + 2\mathbf{x}\Big) \qquad (k = 1, \ldots, m).$$

8 The 5-punctured sphere

In this Section we present calculations in the attempt to prove Conjecture 1 for the case of the 5-punctured sphere and $\pi = \mathrm{Id}$. On the Liouville side, we will compute $\Upsilon^{\pi}_{\alpha,s}(\mathbf{t}_2,\mathbf{t}_3,p_0)$ as defined by (7.59) (with $\pi = \mathrm{Id}$) in the form (7.36), i.e., we will compute the generalized inverse Fourier transform of $\Upsilon^{\pi}_{\alpha,s}(\mathbf{t}_2,\mathbf{t}_3,p_0)$. On the Teichmüller side, we will compute the eigenfunctions of the length operators $\mathbf{L}_1, \mathbf{L}_2$ corresponding to $\pi = \mathrm{Id}$ in the representation $\hat{\pi}$ defined by (7.51) and (7.53). Conjecture 1 is true for this special case if and only if these coincide up to constant factors. All content of this section is original.

8.1 Operator reordering

Our aim is compute $\Upsilon^{\alpha}_{s}(\mathbf{t}_2,\mathbf{t}_3,\mathbf{p})$ in the form

$$\Upsilon^{\alpha}_{s}(\mathbf{t}_2,\mathbf{t}_3,\mathbf{p}) = \int dx_2 dx_3 \; e^{2\pi i(x_2\mathbf{t}_2 + x_3\mathbf{t}_3)} \; \hat{\Upsilon}^{\alpha}_{s}(x_2,x_3,\mathbf{p}). \tag{8.1}$$

We start with

$$\begin{aligned}\prod_{j=3}^{1} \mathbf{h}^{\alpha_j}_{s_j}(\sigma_j) &= \mathbf{E}^{\alpha_3}(\sigma_3)(\mathbf{Q}_3+\mathbf{Q}_4+\mathbf{Q}_5)^{s_3}\mathbf{E}^{\alpha_2}(\sigma_2)(\mathbf{Q}_2+\mathbf{Q}_3+\mathbf{Q}_4)^{s_2}\mathbf{E}^{\alpha_1}(\sigma_1)(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3)^{s_1} \\ &= \Big(\prod_{j=3}^{1}\mathbf{E}^{\alpha_j}(\sigma_j)\Big)\left(e^{-2\pi i b(\alpha_2+\alpha_1)}\mathbf{Q}_3 + e^{-2\pi i b(\alpha_2+3\alpha_1)}\mathbf{Q}_4 + e^{-6\pi i b(\alpha_2+\alpha_1)}\mathbf{Q}_5\right)^{s_3} \\ &\quad\times\left(e^{-2\pi i b\alpha_1}(\mathbf{Q}_2+\mathbf{Q}_3) + e^{-6\pi i b\alpha_1}\mathbf{Q}_4\right)^{s_2}(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3)^{s_1}.\end{aligned} \tag{8.2}$$

Let us begin with the simplest term $(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3)^{s_1}$. If we define $\mathbf{Q}_{23} := \mathbf{Q}_2+\mathbf{Q}_3$, then $\mathbf{Q}_1$ and $\mathbf{Q}_{23}$ are positive self-adjoint operators that satisfy $\mathbf{Q}_1\mathbf{Q}_{23} = q^2\mathbf{Q}_{23}\mathbf{Q}_1$. Therefore, by (C.6) and the Baker-Campbell-Hausdorff formula, we have

$$(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3)^{s_1} \;=\; \int dv\; \mathbf{Q}_1^{\frac{s}{2}-\frac{iv}{2b}}\mathbf{Q}_{23}^{\frac{iv}{b}}\mathbf{Q}_1^{\frac{s}{2}-\frac{iv}{2b}}\;\tilde{D}_{\frac{ib}{2}s_1}(v). \tag{8.3}$$

Then one can apply (C.6) to $\mathbf{Q}_{23}^{\frac{iv}{b}} = (\mathbf{Q}_2+\mathbf{Q}_3)^{\frac{iv}{b}}$ to obtain

$$(\mathbf{Q}_1+\mathbf{Q}_2+\mathbf{Q}_3)^{s_1} \;=\; \int dv\,dv'\; \mathbf{Q}(-ibs_1-v, v-v', v', 0)\;\tilde{D}_{\frac{ib}{2}s_1}(v)\tilde{D}_{-\frac{v}{2}}(v'), \tag{8.4}$$

where

$$\mathbf{Q}(u_1,u_2,u_3,x) \;:=\; e^{\frac{i}{b}\sum_{j=1}^{3}u_j\mathbf{q}_j + 2\pi i x\mathbf{p}}, \qquad \mathbf{q}_j = \log\mathbf{Q}_j. \tag{8.5}$$

The same technique can be applied to compute[33]

$$\begin{aligned}&\left(e^{-2\pi i b\alpha_1}(\mathbf{Q}_2+\mathbf{Q}_3)+e^{-6\pi i b\alpha_1}\mathbf{Q}_4\right)^{s_2} \\ &\quad= e^{-2\pi i b\alpha_1 s_2}\int du\,du'\;\mathbf{Q}(u', -ibs_2-u, u-u', 2u')\;e^{4\pi\alpha_1 u'}\tilde{D}_{\frac{ib}{2}s_2}(u)\tilde{D}_{-\frac{u}{2}}(u')\end{aligned} \tag{8.6}$$

and

$$\begin{aligned}&\left(e^{-2\pi i b(\alpha_2+\alpha_1)}\mathbf{Q}_3 + e^{-2\pi i b(\alpha_2+3\alpha_1)}\mathbf{Q}_4 + e^{-6\pi i b(\alpha_2+\alpha_1)}\mathbf{Q}_5\right)^{s_3} \\ &\quad= e^{-2\pi i b(\alpha_2+\alpha_1)s_3}\int dt\;dt'\;\mathbf{Q}(t-t', t', -ibs_3-t, 2t)\;e^{4\pi(\alpha_1 t+\alpha_2 t')}\tilde{D}_{\frac{ib}{2}s_3}(t)\tilde{D}_{-\frac{t}{2}}(t').\end{aligned} \tag{8.7}$$

[33] Although $\mathbf{Q}_3 + e^{-4\pi i b\alpha_1}\mathbf{Q}_4$ is self-adjoint only for $\alpha_1 \in i\mathbb{R}$, the formula for generic α_1 can be obtained by analytic continuation from this special case.

With the help of the formula

$$\mathbf{Q}(u_1,u_2,u_3,u_4)\mathbf{Q}(v_1,v_2,v_3,v_4) = e^{\pi i\sum_{j\neq k}\mathrm{sgn}(j-k)u_jv_k}\mathbf{Q}(u_1{+}v_1,u_2{+}v_2,u_3{+}v_3,u_4{+}v_4) \quad (8.8)$$

one may then write

$$\begin{aligned}\prod_{j=3}^{1}\mathbf{h}^{\alpha_j}_{s_j}(\sigma_j) \sim \Big(\prod_{j=3}^{1}\mathbf{E}^{\alpha_j}(\sigma_j)\Big)\int & d^6(t,t',u,u',v,v') \\ & \mathbf{Q}(t-t'+u'-v-ibs_1, t'-u+v-v'-ibs_2, -t+u-u'+v'-ibs_3, 2t+2u') \\ & \times e^{4\pi(\alpha_1 t+\alpha_2 t'+\alpha_1 u')+\pi i((t'-t)(v+u'-ibs_2)+t'(-u-v-v'-ibs_1)-(t+ibs_3)(-u-u'-v'-ibs_1-ibs_2))} \\ & \times e^{2\pi bt(s_1+s_2)+\pi i(uv-u'(v-v'+ibs_1)+ibs_2(v+v'))} \\ & \times \tilde{D}_{\frac{ib}{2}s_3}(t)\tilde{D}_{-\frac{t}{2}}(t')\tilde{D}_{\frac{ib}{2}s_2}(u)\tilde{D}_{-\frac{u}{2}}(u')\tilde{D}_{\frac{ib}{2}s_1}(v)\tilde{D}_{-\frac{v}{2}}(v').\end{aligned} \quad (8.9)$$

Following (7.58), we define

$$\mathbf{x} := \frac{1}{4\pi b}(\mathbf{q}_1+\mathbf{q}_3) \qquad \mathbf{t}_2 := \frac{1}{2\pi b}(\mathbf{q}_2-\mathbf{q}_1), \qquad \mathbf{t}_3 := \frac{1}{2\pi b}(\mathbf{q}_3-\mathbf{q}_2). \quad (8.10)$$

In order to write (8.9) in terms of $\mathbf{x},\mathbf{t}_2,\mathbf{t}_3$ and $\mathbf{p}$, we compute

$$\begin{aligned}\mathbf{Q}(u_1,u_2,u_3,x) &= e^{\pi i((-u_1+u_2+u_3)\mathbf{t}_2+(-u_1-u_2+u_3)\mathbf{t}_3+2(u_1+u_2+u_3)\mathbf{x})+2\pi ix\mathbf{p}} \\ &= e^{2\pi iu\mathbf{x}}e^{-\pi i\big((u_1-u_2-u_3)(\mathbf{t}_2+\frac{u}{4})+(u_1+u_2-u_3)(\mathbf{t}_3-\frac{u}{4})\big)}e^{2\pi ix(\mathbf{p}+\frac{u}{2})},\end{aligned} \quad (8.11)$$

where $u = u_1+u_2+u_3$. This yields

$$\begin{aligned}&\mathbf{Q}(t-t'+u'-v-ibs_1, t'-u+v-v'-ibs_2, -t+u-u'+v'-ibs_3, 2t+2u') \\ &\quad = e^{2\pi bs\mathbf{x}}e^{-2\pi i(t-t'+u'-v-\frac{ib}{2}(s_1-s_2-s_3))(\mathbf{t}_2-\frac{ib}{4}s)} \\ &\qquad \times e^{-2\pi i(t-u+u'-v'-\frac{ib}{2}(s_1+s_2-s_3))(\mathbf{t}_3+\frac{ib}{4}s)}e^{4\pi i(t+u')(\mathbf{p}-\frac{ib}{2}s)},\end{aligned} \quad (8.12)$$

where $s = s_1+s_2+s_3$. Therefore, in order to compute $\Upsilon^\alpha_s(\mathbf{t}_2,\mathbf{t}_3,\mathbf{p})$ in the form (8.1), we substitute t' and v' for[34]

$$x_2 \;:=\; t-t'+u'-v-\tfrac{ib}{2}(s_1-s_2-s_3), \quad x_3 \;:=\; t-u+u'-v'-\tfrac{ib}{2}(s_1+s_2-s_3). \quad (8.13)$$

Note that for the case of interest to us, where $\alpha_j,\beta_k \in \frac{Q}{2}+i\mathbb{R}$ according to (6.5), we have $x_2 \in \mathbb{R}-i\frac{Q}{4},\ x_3 \in \mathbb{R}+i\frac{Q}{4}$. However, if

$$\Upsilon^\alpha_s(\mathbf{t}_2,\mathbf{t}_3,p_0) = \int_{\mathbb{R}-i\frac{Q}{4}}dx_2\int_{\mathbb{R}+i\frac{Q}{4}}dx_3\; e^{2\pi i(x_2\mathbf{t}_2+x_3\mathbf{t}_3)}\;\hat{\Upsilon}^\alpha_s(x_2,x_3,p_0), \quad (8.14)$$

then $\hat{\Upsilon}^\alpha_s(x_2,x_3,p_0)$ still lives in the representation[35] $\hat{\pi}$. Then we find

$$\begin{aligned}\hat{\Upsilon}^\alpha_s(x_2,x_3,p_0) \sim e^{-\pi b\frac{s}{2}(x_2-x_3)}\int & d^4(t,u,u',v)\; e^{4\pi i(p_0-\frac{ib}{2}s)(t+u')+4\pi\alpha_1(t+u')+4\pi\alpha_2(t+u'-v-x_2)} \\ & \times e^{\pi i(u'-v-x_2-\frac{ib}{2}(s_1-s_2-s_3))(-t-u'-v+x_3+\frac{ib}{2}(-s_1+s_2-s_3))+\pi i(t+ibs_3)(t+2u'-x_3+\frac{ib}{2}(s_1+s_2+s_3))} \\ & \times e^{\pi it(-2ib(s_1+s_2))+\pi i(uv-u'(-t+u-u'+v+x_3+\frac{ib}{2}(3s_1+s_2-s_3)))+\pi i(t-u+u'+v-x_3)ibs_2} \\ & \times \tilde{D}_{\frac{ib}{2}s_3}(t)\tilde{D}_{-\frac{t}{2}}(t+u'-v-x_2-\tfrac{ib}{2}(s_1-s_2-s_3))\tilde{D}_{\frac{ib}{2}s_2}(u)\tilde{D}_{-\frac{u}{2}}(u') \\ & \times \tilde{D}_{\frac{ib}{2}s_1}(v)\tilde{D}_{-\frac{v}{2}}(t-u+u'-x_3-\tfrac{ib}{2}(s_1+s_2-s_3)).\end{aligned} \quad (8.15)$$

[34]The determinant of this invertible variable transformation is just a constant and thus insignificant to us.

[35]Alternatively, one may shift the integration contours in the end.

By (A.4), one has

$$\tilde{D}_\alpha(x) = \frac{w_b(2\alpha + c_b)w_b(x - c_b)}{w_b(x + 2\alpha + c_b)}, \tag{8.16}$$

where $w_b(x) = (s_b(x))^{-1}$. This allows to write (8.15) as

$$\begin{aligned}
\hat{\Upsilon}^\alpha_s(x_2, x_3, p_0) &\sim e^{-(4\pi\alpha_2+\pi bs)x_2+\pi bsx_3-\pi ix_2x_3} \\
&\times \int d^4(t,u,u',v)\; e^{\pi it(u'+x_2+4p_0-4i(\alpha_1+\alpha_2)+ib(-\frac{7}{2}s_1-\frac{5}{2}s_2-\frac{3}{2}s_3))+\pi iu(v-u'-ibs_2)} \\
&\times e^{\pi iu'(u'+4p_0-4i(\alpha_1+\alpha_2)+ib(-4s_1-s_3))+\pi iv(4i\alpha_2-x_3+\frac{ib}{2}s)} \\
&\times \frac{w_b(t+u'-v-x_2-\frac{ib}{2}(s_1-s_2-s_3)-c_b)}{w_b(t+ibs_3+c_b)} \cdot \frac{w_b(t-u+u'-x_3-\frac{ib}{2}(s_1+s_2-s_3)-c_b)}{w_b((t-u+u'-v-x_3-\frac{ib}{2}(s_1+s_2-s_3)+c_b)} \\
&\times \frac{w_b(u-u'-c_b)}{w_b(u+ibs_2+c_b)} \cdot \frac{w_b(v-u'+x_2+\frac{ib}{2}(s_1-s_2-s_3)-c_b)}{w_b(v+ibs_1+c_b)} \cdot w_b(u'-c_b).
\end{aligned} \tag{8.17}$$

Let us pick out the integration over u. It may be rewritten as

$$\begin{aligned}
&\int du\; e^{\pi iu(v-u'-ibs_2)} \frac{w_b(t-u+u'-x_3-\frac{ib}{2}(s_1+s_2-s_3)-c_b)}{w_b(t-u+u'-v-x_3-\frac{ib}{2}(s_1+s_2-s_3)+c_b)} \frac{w_b(u-u'-c_b)}{w_b(u+ibs_2+c_b)} \\
&= \int dx\; e^{-\pi i(v-u'-ibs_2)x}\; D_{\frac{v}{2}-c_b}(x+t+u'-\tfrac{v}{2}-x_3-\tfrac{ib}{2}(s_1+s_2-s_3)) \\
&\qquad \times D_{-\frac{1}{2}(u'+ibs_2)-c_b}(x+\tfrac{1}{2}u'-\tfrac{ib}{2}s_2),
\end{aligned} \tag{8.18}$$

where we used $D_\alpha(x) = D_\alpha(-x)$. By the identity (A.5), the integral can be performed to give

$$\begin{aligned}
&e^{\pi i(-v(-\frac{1}{2}u'+\frac{ib}{2}s_2)+(u'+ibs_2)(-t-u'+\frac{1}{2}v+x_3+\frac{ib}{2}(s_1+s_2-s_3)))} \\
&\quad \times w_b(v-c_b)w_b(-u'-ibs_2-c_b)w_b(u'-v+ibs_2+c_b) \\
&\quad \times \frac{w_b(-t-u'+v+x_3+\frac{ib}{2}(s_1-s_2-s_3)-c_b)}{w_b(-t+x_3+\frac{ib}{2}(s_1+s_2-s_3)+c_b)}.
\end{aligned} \tag{8.19}$$

Thus we obtain

$$\begin{aligned}
\hat{\Upsilon}^\alpha_s(x_2, x_3, p_0) &\sim e^{-(4\pi\alpha_2+\pi bs)x_2+\pi b(s_1+s_3)x_3-\pi ix_2x_3} \\
&\times \int dt\, du' dv\; e^{\pi it(x_2+4p_0-4i(\alpha_1+\alpha_2)+ib(-\frac{7}{2}s_1-\frac{7}{2}s_2-\frac{3}{2}s_3))} \\
&\times e^{\pi iu'(v+x_3+4p_0-4i(\alpha_1+\alpha_2)+ib(\frac{7}{2}s_1-\frac{1}{2}s_2-\frac{3}{2}s_3))+\pi iv(4i\alpha_2-x_3+\frac{ib}{2}s)} \\
&\times \frac{w_b(t+u'-v-x_2-\frac{ib}{2}(s_1-s_2-s_3)-c_b)\; w_b(t-x_3-\frac{ib}{2}(s_1+s_2-s_3)-c_b)}{w_b(t+ibs_3+c_b)\; w_b(t+u'-v-x_3-\frac{ib}{2}(s_1-s_2-s_3)+c_b)} \\
&\times \frac{w_b(v-c_b)\; w_b(u'-c_b)\; w_b(v-u'+x_2+\frac{ib}{2}(s_1-s_2-s_3)-c_b)}{w_b(v-u'-ibs_2-c_b)\; w_b(u'+ibs_2+c_b)\; w_b(v+ibs_1+c_b)}.
\end{aligned} \tag{8.20}$$

8.2 Length eigenfunctions

Our next task is to compute the common eigenfunctions of the length operators $\mathbf{L}_k = \mathbf{L}_{\gamma_k}$ ($k = 1, 2$), where γ_1, γ_2 are loops that encircle z_0 and z_1 respectively z_0, z_1 and z_2, in the representation $\hat{\pi}$ defined by (7.51) and (7.53). These are already known in a representation π' that is

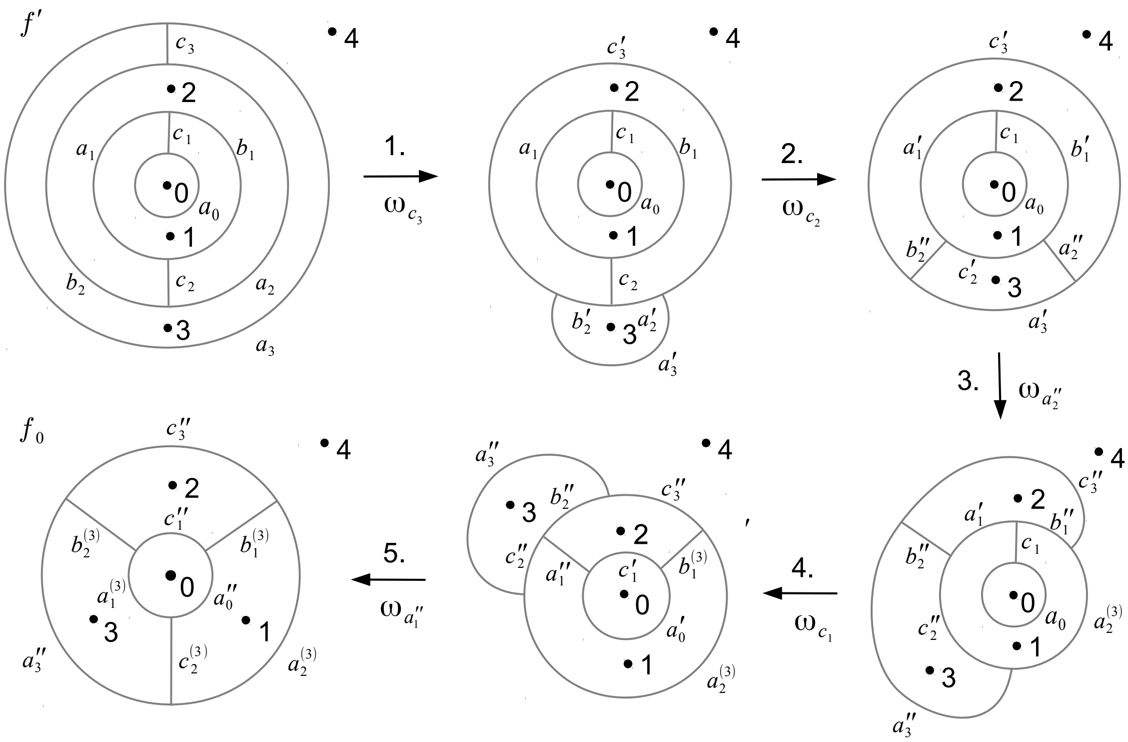

Figure 16: Transforming f' into f_0 by a sequence of flips

associated with the fat graph f' drawn in the upper left corner of Figure 16, namely the one where[36]

$$\begin{aligned} \mathbf{x}_2' &:= \frac{1}{4\pi b}(\mathbf{b}_1 - \mathbf{a}_1), \qquad & \mathbf{p}_2' &:= \frac{1}{4\pi b}(\mathbf{b}_1 + \mathbf{a}_1), \\ \mathbf{x}_3' &:= \frac{1}{4\pi b}(\mathbf{b}_2 - \mathbf{a}_2), \qquad & \mathbf{p}_3' &:= \frac{1}{4\pi b}(\mathbf{b}_2 + \mathbf{a}_2), \end{aligned} \tag{8.21}$$

are represented on $L^2(\mathbb{R})$ as in (4.4) (where $\mathbf{p}_j$ is diagonal). The representation $\hat{\pi}$ is associated with the fat graph f_0 drawn in the lower left corner of Figure 16 and corresponds to the pairs of conjugate operators

$$\begin{aligned} \mathbf{x}_2 &:= \frac{1}{4\pi b}\left(\mathbf{b}_1^{(3)} - \mathbf{b}_2^{(3)} - \mathbf{c}_2^{(3)}\right), \qquad & \mathbf{x}_3 &:= \frac{1}{4\pi b}\left(\mathbf{b}_1^{(3)} + \mathbf{b}_2^{(3)} - \mathbf{c}_2^{(3)}\right), \\ \mathbf{p}_2 &:= \frac{1}{2\pi b}\mathbf{c}_1'' + \frac{1}{2}\mathbf{x}_3, \qquad & \mathbf{p}_3 &:= \frac{1}{2\pi b}\mathbf{a}_1^{(3)} - \frac{1}{2}\mathbf{x}_2, \end{aligned} \tag{8.22}$$

where $\mathbf{x}_2, \mathbf{x}_3$ act diagonally while $\hat{\pi}(\mathbf{p}_j) = (2\pi i)^{-1}\frac{\partial}{\partial x_j}$. The operators $\mathbf{x}_j', \mathbf{p}_j'$ can be successively transformed into operators $\mathbf{x}_j^{(k)}, \mathbf{p}_j^{(k)}$ $(k = 2, \ldots, 6)$ associated with the intermediate fat graphs depicted in Figure 16 by conjugation with the following operators (where $\mathbf{U}_j$ generates the j-th flip and $\mathbf{S}_j$ a symplectic transformation; we omit the shift operators):

1. a) $\mathbf{S}_1 := e^{-\pi i (\mathbf{p}_3')^2}$

$$\mathbf{x}_3' \to \mathbf{x}_3' - \mathbf{p}_3' = -\frac{\mathbf{a}_2}{2\pi b}$$

   b) $\mathbf{U}_1 := e_b\left(-\frac{\mathbf{c}_3}{2\pi b}\right) = e_b(\mathbf{p}_3' - r_3 + r_4)$

[36]We start counting from 2 in order to be in harmony with previous naming conventions.

$$\mathbf{x}_2'' = \mathbf{x}_2', \quad \mathbf{p}_2'' = \mathbf{p}_2', \quad \mathbf{x}_3'' = -\frac{\mathbf{a}_2'}{2\pi b}, \quad \mathbf{p}_3'' = \frac{\mathbf{c}_3'}{2\pi b} = \mathbf{p}_3' - r_3 + r_4$$

2. a) $\mathbf{S}_2 := e^{-\pi i(\mathbf{p}_2'')^2}$

$$\mathbf{x}_2'' \to \mathbf{x}_2'' - \mathbf{p}_2'' = -\frac{\mathbf{a}_1}{2\pi b}$$

b) $\mathbf{U}_2 := e_b\left(-\frac{\mathbf{c}_2}{2\pi b}\right) = e_b(\mathbf{p}_2' + \mathbf{p}_3' - r_2)$

$$\mathbf{x}_2^{(3)} = -\frac{\mathbf{a}_1'}{2\pi b}, \qquad \mathbf{p}_2^{(3)} = -\frac{\mathbf{c}_1}{2\pi b} = \mathbf{p}_2' - r_0, \quad \mathbf{x}_3^{(3)} = -\frac{\mathbf{a}_2''}{2\pi b}, \qquad \mathbf{p}_3^{(3)} = \mathbf{p}_3''$$

3. $\mathbf{U}_3 := e_b\big(-\frac{\mathbf{a}_2''}{2\pi b}\big) = e_b(\mathbf{x}_3^{(3)})$

$$\mathbf{x}_2^{(4)} = \mathbf{x}_2^{(3)}, \quad \mathbf{p}_2^{(4)} = \mathbf{p}_2^{(3)}, \quad \mathbf{x}_3^{(4)} = \frac{\mathbf{a}_2^{(3)}}{2\pi b} = \mathbf{x}_3^{(3)}, \quad \mathbf{p}_3^{(4)} = \frac{\mathbf{c}_3''}{2\pi b}$$

4. $\mathbf{U}_4 := e_b\big(-\frac{\mathbf{c}_1}{2\pi b}\big) = e_b(\mathbf{p}_2' - r_0)$

$$\mathbf{x}_2^{(5)} = -\frac{\mathbf{a}_1''}{2\pi b}, \quad \mathbf{p}_2^{(5)} = \frac{\mathbf{c}_1'}{2\pi b} = \mathbf{p}_2^{(4)}, \quad \mathbf{x}_3^{(5)} = \mathbf{x}_3^{(4)}, \quad \mathbf{p}_3^{(5)} = \mathbf{p}_3^{(4)}$$

5. $\mathbf{U}_5 := e_b\big(-\frac{\mathbf{a}_1''}{2\pi b}\big) = e_b(\mathbf{x}_2^{(5)})$

$$\mathbf{x}_2^{(6)} = \frac{\mathbf{a}_1^{(3)}}{2\pi b} = \mathbf{x}_2^{(5)}, \quad \mathbf{p}_2^{(6)} = \frac{\mathbf{c}_1''}{2\pi b}, \quad \mathbf{x}_3^{(6)} = \mathbf{x}_3^{(5)} = \frac{\mathbf{a}_2^{(3)}}{2\pi b}, \quad \mathbf{p}_3^{(6)} = \mathbf{p}_3^{(5)} = \frac{\mathbf{c}_3''}{2\pi b}$$

Altogether we have

$$(\mathbf{x}_2^{(6)}, \mathbf{p}_2^{(6)}, \mathbf{x}_3^{(6)}, \mathbf{p}_3^{(6)}) \;=\; \mathbf{U}(\mathbf{x}_2', \mathbf{p}_2' - r_0, \mathbf{x}_3', \mathbf{p}_3' - r_3 + r_4)\mathbf{U}^{-1}, \tag{8.23}$$

where $\mathbf{U} := \mathbf{U}_5\mathbf{U}_4\mathbf{U}_3\mathbf{U}_2\mathbf{S}_2\mathbf{U}_1\mathbf{S}_1$. Therefore, if

$$\Psi'_{r_1,r_2}(p_2', p_3') = \prod_{j=1,2} c(r_j)\,\big(w_b(p_{j+1}' + r_j + c_b)\; w_b(p_{j+1}' - r_j + c_b)\big)^{-1} \tag{8.24}$$

denotes the eigenfunction of $\pi'(\mathbf{L}_1)$ and $\pi'(\mathbf{L}_2)$ (with eigenvalues $2\cosh(2\pi b r_j)$, cf. (4.11)), then

$$\Psi^{(6)}_{r_1,r_2}(p_2^{(6)}, p_3^{(6)}) \;:=\; \pi'(\mathbf{U}^{-1})\Psi'_{r_1,r_2}(p_2', p_3')|_{p_2'=p_2^{(6)}+r_0,\; p_3'=p_3^{(6)}+r_3-r_4} \tag{8.25}$$

is eigenfunction of $\pi^{(6)}(\mathbf{L}_j)$ $(j = 1, 2)$, where $\pi^{(6)}$ is the standard representation (4.4) associated with $(\mathbf{x}_j^{(6)}, \mathbf{p}_j^{(6)})_{j=2,3}$.

Finally, we need to figure out the symplectic transformation that transforms $(\mathbf{x}_j^{(6)}, \mathbf{p}_j^{(6)})$ into $(\mathbf{x}_j, \mathbf{p}_j)$. To this end, we write down the constraints

$$\begin{aligned} l_1 &= \mathbf{a}_0'' + \mathbf{c}_2^{(3)} + \mathbf{b}_1^{(3)} + \mathbf{a}_2^{(3)}, \\ l_2 &= \mathbf{c}_1'' + \mathbf{b}_1^{(3)} + \mathbf{b}_2^{(3)} + \mathbf{c}_3'', \\ l_3 &= \mathbf{a}_1^{(3)} + \mathbf{b}_2^{(3)} + \mathbf{c}_2^{(3)} + \mathbf{a}_3'', \end{aligned} \qquad \begin{aligned} l_0 &= \mathbf{a}_0'' + \mathbf{c}_1'' + \mathbf{a}_1^{(3)}, \\ l_4 &= \mathbf{a}_2^{(3)} + \mathbf{c}_3'' + \mathbf{a}_3''. \end{aligned} \tag{8.26}$$

Since

$$\mathbf{b}_1^{(3)} + \mathbf{b}_2^{(3)} + \mathbf{c}_2^{(3)} \;=\; \frac{1}{2}(l_1 + l_2 + l_3 - l_0 - l_4), \tag{8.27}$$

one may define $r_k := l_k/4\pi b$, $r := r_1 + r_2 + r_3 - r_0 - r_4$ and rewrite $\mathbf{x}_2, \mathbf{x}_3$ as

$$\mathbf{x}_2 = \frac{1}{2\pi b}\mathbf{b}_1^{(3)} - \frac{1}{2}r, \qquad \mathbf{x}_3 = -\frac{1}{2\pi b}\mathbf{c}_2^{(3)} + \frac{1}{2}r. \tag{8.28}$$

Then one easily finds

$$\begin{aligned}\mathbf{b}_1^{(3)} &= \mathbf{a}_1^{(3)} - \mathbf{a}_2^{(3)} - \mathbf{c}_3'' + 2\pi b(r_1 + r_2 - r_3 - r_0 + r_4),\\ \mathbf{c}_2^{(3)} &= \mathbf{c}_1'' + \mathbf{c}_3'' + 2\pi b(r_1 - r_2 + r_3 - r_0 - r_4),\end{aligned} \tag{8.29}$$

so that

$$\begin{aligned}\mathbf{x}_2 &= \mathbf{x}_2^{(6)} - \mathbf{x}_3^{(6)} - \mathbf{p}_3^{(6)} + c_2, \qquad & \mathbf{p}_2 &= \tfrac{1}{2}(\mathbf{p}_2^{(6)} - \mathbf{p}_3^{(6)}) + \tfrac{1}{2}c_3,\\ \mathbf{x}_3 &= -\mathbf{p}_2^{(6)} - \mathbf{p}_3^{(6)} + c_3, & \mathbf{p}_3 &= \tfrac{1}{2}(\mathbf{x}_2^{(6)} + \mathbf{x}_3^{(6)} + \mathbf{p}_3^{(6)}) - \tfrac{1}{2}c_2,\end{aligned} \tag{8.30}$$

where

$$c_2 = \tfrac{1}{2}(-r_0 + r_1 + r_2 - 3r_3 + 3r_4), \qquad c_3 = \tfrac{1}{2}(r_0 - r_1 + 3r_2 - r_3 + r_4). \tag{8.31}$$

We would like to realize this transformation (which is a symplectic transformation plus a shift) by means of an integral transformation

$$\mathbf{S} : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2), \quad (S\psi)(p_2, p_3) \;=\; \int dp_2^{(6)} dp_3^{(6)} S(p_2, p_3; p_2^{(6)}, p_3^{(6)})\psi(p_2^{(6)}, p_3^{(6)}). \tag{8.32}$$

More precisely, we want to find $S(p_2, p_3; p_2^{(6)}, p_3^{(6)})$ such that S is an intertwiner between the representations $\pi^{(6)}$ and π, where π is the standard representation (4.4) of $(\mathbf{x}_j, \mathbf{p}_j)_{j=2,3}$. The intertwining property $\pi(\mathbf{q})\mathbf{S} = \mathbf{S}\pi^{(6)}(\mathbf{q})$ for $\mathbf{q} \in \{\mathbf{x}_j, \mathbf{p}_j\}_{j=2,3}$ then yields the following system of algebraic and differential equations

$$\begin{aligned}0 &= \left(\tfrac{1}{2}(\tilde{p}_2 - \tilde{p}_3) + \tfrac{1}{2}c_3 - p_2\right) S(p_2, p_3; \tilde{p}_2, \tilde{p}_3),\\ 0 &= \left(-\frac{i}{4\pi}\left(\frac{\partial}{\partial \tilde{p}_2} + \frac{\partial}{\partial \tilde{p}_3}\right) + \frac{1}{2}\tilde{p}_3 - p_3 - \frac{1}{2}c_2\right) S(p_2, p_3; \tilde{p}_2, \tilde{p}_3),\\ 0 &= \left(-\frac{i}{2\pi}\left(\frac{\partial}{\partial \tilde{p}_2} - \frac{\partial}{\partial \tilde{p}_3} + \frac{\partial}{\partial p_2}\right) - \tilde{p}_3 + c_2\right) S(p_2, p_3; \tilde{p}_2, \tilde{p}_3),\\ 0 &= \left(-\tilde{p}_2 - \tilde{p}_3 + c_3 - \frac{i}{2\pi}\frac{\partial}{\partial p_3}\right) S(p_2, p_3; \tilde{p}_2, \tilde{p}_3).\end{aligned} \tag{8.33}$$

Introducing $\tilde{p}_\pm := \frac{1}{2}(\tilde{p}_2 \pm \tilde{p}_3)$, one finds the solution

$$\begin{aligned}S(p_2, p_3; \tilde{p}_2, \tilde{p}_3) &:= \delta(\tilde{p}_2 - \tilde{p}_3 - 2p_2 + c_3)\, e^{2\pi i(-\frac{1}{2}(\tilde{p}_+^2 + \tilde{p}_-^2) + \tilde{p}_+\tilde{p}_- + (2\tilde{p}_+ - c_3)p_3 + c_2(\tilde{p}_+ - \tilde{p}_-))}\\ &= \delta(\tilde{p}_2 - \tilde{p}_3 - 2p_2 + c_3)\, e^{2\pi i(2(\tilde{p}_3 + p_2 - c_3)p_3 + c_2\tilde{p}_3 - \frac{1}{2}\tilde{p}_3^2)}\end{aligned} \tag{8.34}$$

Let us now compute $\Psi^{(6)}_{r_1,r_2}(p_2^{(6)}, p_3^{(6)})$ as defined in (8.25). Using

$$\begin{aligned}\mathbf{U}_5 &= (\mathbf{U}_4\mathbf{U}_3\mathbf{U}_2\mathbf{S}_2\mathbf{U}_1\mathbf{S}_1)e_b(\mathbf{x}_2')(\mathbf{U}_4\mathbf{U}_3\mathbf{U}_2\mathbf{S}_2\mathbf{U}_1\mathbf{S}_1)^{-1},\\ \mathbf{U}_3 &= (\mathbf{U}_2\mathbf{S}_2\mathbf{U}_1\mathbf{S}_1)e_b(\mathbf{x}_3')(\mathbf{U}_2\mathbf{S}_2\mathbf{U}_1\mathbf{S}_1)^{-1},\end{aligned} \tag{8.35}$$

one finds

$$\mathbf{U}^{-1} = \left(e_b(\mathbf{x}_2')e_b(\mathbf{x}_3')\right)^{-1} \frac{e^{\pi i((\mathbf{p}_2')^2 + (\mathbf{p}_3')^2)}}{e_b(\mathbf{p}_2' - r_0)e_b(\mathbf{p}_2' + \mathbf{p}_3' - r_2)e_b(\mathbf{p}_3' - r_3 + r_4)}. \tag{8.36}$$

Generally speaking, a function $f(\mathbf{x}')$ of $\mathbf{x}' \equiv (\mathbf{x}'_2, \mathbf{x}'_3)$ is defined by $f(\mathbf{x}')\psi_{x'} = f(x')\psi_{x'}$ for eigenstates $\psi_{x'}$ of $\mathbf{x}'$ with eigenvalues $x' = (x_2, x_3)$ and the expansion of each wave function in terms of $\psi_{x'}$. In the representation π', one has $\psi_{x'}(p') = e^{-2\pi i x'\cdot p'}$, so this expansion is given by the Fourier transformation

$$\psi(p'_2, p'_3) = \int dx'_2\, dx'_3\; e^{-2\pi i x'\cdot p'}\,\hat{\psi}(x'_2, x'_3). \tag{8.37}$$

Then one finds

$$\pi'(f(\mathbf{x}'_2, \mathbf{x}'_3))\psi(p'_2, p'_3) \;=\; \int dp_2\, dp_3\; \hat{f}(p_2 - p'_2, p_3 - p'_3)\psi(p_2, p_3), \tag{8.38}$$

where

$$\hat{f}(p) := \int dx\; e^{2\pi i p\cdot x} f(x). \tag{8.39}$$

In our case we have

$$\begin{aligned} \Psi^{(6)}_{r_1,r_2}(p^{(6)}_2, p^{(6)}_3) \;&=\; \int dp'_2\, dp'_3\; f_b(p'_2 - p^{(6)}_2 - r_0) f_b(p'_3 - p^{(6)}_3 - r_3 + r_4) \\ &\times \frac{e^{\pi i((p'_2)^2 + (p'_3)^2)}}{e_b(p'_2 - r_0)e_b(p'_2 + p'_3 - r_2)e_b(p'_3 - r_3 + r_4)}\Psi'_{r_1,r_2}(p'_2, p'_3), \end{aligned} \tag{8.40}$$

where

$$f_b(x) \;:=\; \int dy\; e^{2\pi i x y}\,\frac{1}{e_b(y)} \;=\; e^{\frac{\pi i}{2}((x-c_b)^2 - c_b^2)}(w_b(x + c_b))^{-1}. \tag{8.41}$$

The last equation follows from

$$\int dy\; e^{2\pi i x y}\,\frac{e_b(y - 2\alpha)}{e_b(y)} \;=\; e^{-2\pi i \alpha x} w_b(2\alpha + c_b)\frac{w_b(x - 2\alpha - c_b)}{w_b(x + c_b)}, \tag{8.42}$$

which is another form of (A.4), in the limit $\alpha \to \infty$. Applying the transformation (8.32) to $\Psi^{(6)}_r$, $r \equiv (r_1, r_2)$, yields

$$\begin{aligned} \Psi_r(p_2, p_3) \;&=\; \int dp^{(6)}_3\; e^{2\pi i(2(p^{(6)}_3 + p_2 - c_3)p_3 + c_2 p^{(6)}_3 - \frac{1}{2}(p^{(6)}_3)^2)}\,\Psi^{(6)}_r(p^{(6)}_3 + 2p_2 - c_3, p^{(6)}_3) \\ &=\; c(r_1)c(r_2)\int dp\; e^{2\pi i(2(p + p_2 - c_3)p_3 + c_2 p - \frac{1}{2}p^2)} \\ &\quad\times \int dp'_2\, dp'_3\; f_b(p'_2 - p - 2p_2 + c_3 - r_0) f_b(p'_3 - p - r_3 + r_4)\; e^{\pi i((p'_2)^2 + (p'_3)^2)} \\ &\quad\times \frac{(e_b(p'_2 - r_0)e_b(p'_2 + p'_3 - r_2)e_b(p'_3 - r_3 + r_4))^{-1}}{w_b(p'_2 + r_1 + c_b)w_b(p'_2 - r_1 + c_b)w_b(p'_3 + r_2 + c_b)w_b(p'_3 - r_2 + c_b)}. \end{aligned} \tag{8.43}$$

This is the eigenfunction of $\pi(\mathbf{L}_j)$ $(j = 1, 2)$. The transition to the representation $\hat{\pi}$ is done by a Fourier transformation

$$\hat{\Psi}_r(x_2, x_3) \;=\; \int dp_2 \int dp_3\; e^{2\pi i(x_2 p_2 + x_3 p_3)}\;\Psi_r(p_2, p_3). \tag{8.44}$$

By trading the integration variable p for $x := -2(p - p_2 - c_3)$, $\Psi_r(p_2, p_3)$ may be written in the form

$$\Psi_r(p_2, p_3) \;=\; \int dx\; e^{-2\pi i x p_3}\;\tilde{\Psi}_r(p_2, x), \tag{8.45}$$

so that

$$\hat{\Psi}_r(x_2,x_3) \;=\; \int dp_2\, e^{2\pi i x_2 p_2}\; \tilde{\Psi}_r(p_2,x_3). \tag{8.46}$$

Using (8.41) and $e_b(x) \sim e^{\frac{\pi i}{2}x^2}\,(w_b(x))^{-1}$ one finds

$$\begin{aligned}
\hat{\Psi}_r(x_2,x_3) \;\sim\; & \int dp_2\; e^{2\pi i\left(x_2p_2-\frac{1}{2}(-\frac{1}{2}x_3-p_2+c_3-c_2)^2\right)}\\
&\times \int dp_2'\, dp_3'\; e^{\frac{\pi i}{2}\left((p_2'+\frac{1}{2}x_3-p_2-r_0-c_b)^2+(p_3'+\frac{1}{2}x_3+p_2-c_3-r_3+r_4-c_b)^2\right)}\\
&\times \frac{w_b(-p_2'-\frac{1}{2}x_3+p_2+r_0-c_b)}{w_b(p_3'+\frac{1}{2}x_3+p_2-c_3-r_3+r_4+c_b)}\; e^{\pi i\left((r_0+r_2)p_2'+(r_2+r_3-r_4)p_3'-p_2'p_3'\right)}\\
&\times \frac{w_b(p_2'-r_0)\; w_b(p_2'+p_3'-r_2)\; w_b(p_3'-r_3+r_4)}{w_b(p_2'+r_1+c_b)\; w_b(p_2'-r_1+c_b)\; w_b(p_3'+r_2+c_b)\; w_b(p_3'-r_2+c_b)}.
\end{aligned} \tag{8.47}$$

Collecting the p_2-dependent factors, one finds that the integration over p_2 may be performed with the help of (A.4),

$$\begin{aligned}
&\int dp_2\; e^{\pi i(2x_2-x_3-p_2'+p_3'+c_3-2c_2+r_0-r_3+r_4)p_2}\; \frac{w_b(p_2-p_2'-\frac{1}{2}x_3+r_0-c_b)}{w_b(p_2+p_3'+\frac{1}{2}x_3-c_3-r_3+r_4+c_b)}\\
&\quad = e^{\pi i\left(-x_2+\frac{1}{2}(x_3+p_2'-p_3')+\frac{1}{4}(5r_0-3r_1+r_2+3r_3-3r_4)\right)\left(p_3'-p_2'+\frac{1}{2}(r_0+r_1-3r_2-r_3+r_4)\right)}\\
&\quad\times w_b\left(-p_2'-p_3'-x_3+\tfrac{1}{2}(3r_0-r_1+3r_2+r_3-r_4)-c_b\right)\\
&\quad\times \frac{w_b(x_2+p_3'+\frac{1}{2}(r_0-r_1-r_2+r_3-r_4))}{w_b(x_2-x_3-p_2'+2r_0-r_1+r_2+r_3-r_4)}.
\end{aligned} \tag{8.48}$$

So we arrive at (dropping again all constant factors)

$$\begin{aligned}
\hat{\Psi}_r(x_2,x_3) \;\sim\; & e^{\pi i(-r_0-r_1+3r_2+r_3-r_4)x_2+\pi i\left(\frac{1}{2}(r_0-r_1+r_2+r_3-r_4)-c_b\right)x_3}\\
\times & \int dp_2'\, dp_3'\; e^{\pi i(x_2-r_0+r_1-r_3+r_4-c_b)p_2'+\pi i\left(x_3-x_2+\frac{1}{2}(r_0-r_1+r_2+3r_3-3r_4)-c_b\right)p_3'}\\
\times & \frac{w_b(p_2'-r_0)\; w_b(p_2'-x_2+x_3-2r_0+r_1-r_2-r_3+r_4)}{w_b(p_2'+r_1+c_b)\; w_b(p_2'-r_1+c_b)}\\
\times & \frac{w_b(p_2'+p_3'-r_2)\; w_b(p_3'-r_3+r_4)\; w_b(p_3'+x_2+\frac{1}{2}(r_0-r_1-r_2+r_3-r_4))}{w_b(p_2'+p_3'+x_3+\frac{1}{2}(-3r_0+r_1-3r_2-r_3+r_4)+c_b)w_b(p_3'+r_2+c_b)w_b(p_3'-r_2+c_b)}.
\end{aligned} \tag{8.49}$$

The author felt unable to match this with (8.20).

9 Systematic reduction

In this section we will investigate the systematic reduction of Liouville theory to Teichmüller theory as outlined in the introduction. To this end, we need to define rigged Riemann surfaces and the sewing of these. This will allow us to describe the reduction on the classical level.

In order to describe the reduction of the quantized theory, we will first introduce the mathematical notion of a conformal block. Then we will show how a conformal block naturally defines a holomorphic function on a certain subset of (enlarged) Teichmüller space. Thus we will obtain an identification of conformal blocks with states in a coherent state representation of quantum Teichmüller theory. This will allow us to identify certain operators of quantum Liouville theory, that act on the space of conformal blocks, with operators on quantum Teichmüller space. In this way, we will pick up some ideas which have been formulated in [40, ch. 12], see also [39]. Original material is also given by the quotient construction developed in Sections 9.3 and 9.4, and by Preposition 3.

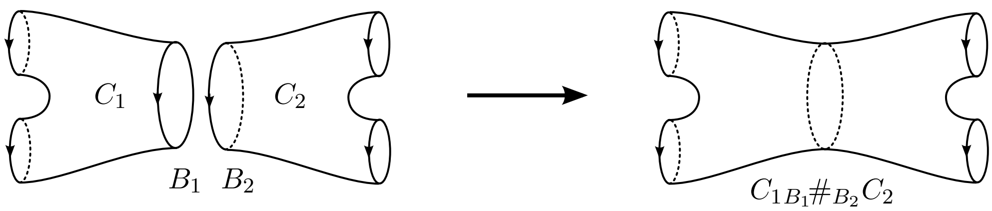

Figure 17: Sewing of two rigged Riemann surfaces

9.1 Rigged Riemann surfaces

A bordered Riemann surface C is by definition a connected Hausdorff space [1] equipped with a collection $\{\varphi_i\}_{i\in I}$ (called an atlas) of homeomorphisms φ_i (called charts or coordinates) from open sets $U_i \subset C$ into relatively open subsets of the closed upper half plane $\bar{\mathbb{H}} := \{z \in \mathbb{C} \,|\, \Im(z) \geq 0\}$, where $\bigcup_{i\in I} U_i = C$, such that $\varphi_i \circ \varphi_j^{-1}$ is a biholomorphic map for every $i, j \in I$ with $U_i \cap U_j \neq \emptyset$. More precisely, as in the case of ordinary Riemann surfaces, two atlases are called equivalent if their union is again an atlas, and one considers an equivalence class of atlases on C, called a complex structure. The boundary ∂C is defined as $\partial C := \bigcup_{i\in I} \varphi_i^{-1}(\mathbb{R} \cap \varphi_i(U_i))$ (we always assume $\partial C \neq \emptyset$) and does not depend on the choice of an atlas. We call C a bordered Riemann surface with n holes, or shortly an n-holed Riemann surface, if ∂C has n connected components $\partial_k C$ $(k = 1, \ldots, n)$ which are all homeomorphic to S^1.

We will now consider n-holed Riemann surfaces that are equipped with an additional structure called a rigging. This is a set of smooth boundary parametrizations $\phi_k : S^1 \to \partial_k C$ $(k = 1, \ldots, n)$. The boundary component $\partial_k C$ is called incoming or outgoing if ϕ_k is orientation reversing or orientation preserving respectively[37] (intuitively, if ϕ_k goes around the hole in the counter-clockwise or clockwise direction respectively). A holed Riemann surface with a rigging is called a rigged Riemann surface.

Let C_1 and C_2 be rigged Riemann surfaces with n_1 and n_2 holes respectively, an incoming boundary component $B_1 \subset \partial C_1$ and an outgoing boundary component $B_2 \subset \partial C_2$. Let $\phi_j : S^1 \to B_j$ $(j = 1, 2)$ be the boundary parametrizations. Then we denote by

$$C_{1B_1}\#_{B_2}C_2 \;:=\; (C_1 \sqcup C_2)/\sim \tag{9.1}$$

the disjoint union of C_1 and C_2, modulo the identification of points $B_1 \ni p_1 \sim p_2 \in B_2$ if and only if $\phi_1^{-1}(p_1) = \phi_2^{-1}(p_2)$ (we will omit the indices B_1, B_2 if there is no ambiguity). On $C_{1B_1}\#_{B_2}C_2$, there exists a unique complex structure that is compatible with the complex structures on C_1 and C_2 [1, 30]. Equipped with this complex structure, $C_{1B_1}\#_{B_2}C_2$ is again a rigged Riemann surface with $n_1 + n_2 - 2$ holes. We say it is the Riemann surface obtained by sewing C_1 and C_2 along B_1 and B_2 as depicted in Fig. 17. This procedure is also possible if $C_1 = C_2$, i.e., one may sew an incoming and an outgoing boundary component of the same surface.

Given a rigged Riemann surface C with n holes, one may sew to it n copies of $\bar{D}_0 := \bar{D}\backslash\{0\}$, equipped with the rigging $\mathrm{Id} : S^1 \to S^1$, along each of its boundary components, after making them incoming by precomposing ϕ_j with $J : S^1 \to S^1,\ z \mapsto z^{-1}$ if necessary. In this way, one obtains an n-punctured Riemann surface which comes with a local coordinate at each puncture (i.e., a chart that vanishes at the puncture), namely the standard coordinate $z = \mathrm{Id}$ of $\bar{D}_0$. Conversely, given an n-punctured Riemann surface Σ with local coordinate z_k $(k = 1, \ldots, m)$ at each puncture, such that D_0 is contained in the image of each z_k, and $z_k^{-1}(D_0) \cap z_l^{-1}(D_0) = \emptyset$

[37] with respect to the orientation that is induced by the complex structure

for all $k \neq l$ (in the following, we will always assume that local coordinates satisfy these conditions), we may define the rigged Riemann surface $C := \Sigma\backslash \bigcup_k z_k^{-1}(D_0)$, with rigging given by the $z_k^{-1}|_{S^1}$. Thus the concepts of rigged Riemann surface and punctured Riemann surface with local coordinates are widely exchangeable.

Two rigged Riemann surfaces C_1 and C_2 are called equivalent, $C_1 \simeq C_2$ if there exists a biholomorphic map $\Phi : C_1 \to C_2$ that respect the riggings, i.e., if $\phi_k^{(i)}$ $(k = 1, \ldots, n)$ are the boundary parametrizations of C_i, then[38] $\phi_k^{(2)} = \Phi|_{\partial_k C_1} \circ \phi_k^{(1)}$ $(k = 1, \ldots, n)$. The moduli space $\mathcal{M}_{g,n}^{\text{rigged}}$ is the set of equivalence classes of rigged Riemann surfaces of genus g with n holes.

9.2 The moduli spaces of (punctured) discs and annuli

Let $\mathcal{M}(D)$ and $\mathcal{M}(D_0)$ denote the moduli spaces of rigged discs and rigged punctured discs respectively, with outgoing boundaries. Let us define $D^* := \bar{\mathbb{C}}\backslash\bar{D}$ and

$$\mathrm{Hol}(D^*) := \left\{ g : D^* \to \bar{\mathbb{C}} \text{ univalent } \;\middle|\; \begin{array}{l} g(\infty) = \infty,\ g'(\infty) = 1,\ g''(\infty) = 0, \\ g \text{ has a smooth extension to } \partial D^* = S^1 \end{array} \right\}. \tag{9.2}$$

(Here $g^{(i)}(\infty) := \tilde{g}^{(i)}(0)$ with $\tilde{g}(z) := g(z^{-1})$.) With an element $g \in \mathrm{Hol}(D^*)$ we associate the disk $\Delta_g := \bar{\mathbb{C}}\backslash g(D^*)$ with the rigging $\phi := g|_{S^1}$. This provides a map

$$\Psi :\ \mathrm{Hol}(D^*) \to \mathcal{M}(D), \quad g \mapsto [\Delta_g], \tag{9.3}$$

The following consideration shows that Ψ is a bijection. Let $[\Delta] \in \mathcal{M}(D)$. The outer disk $\overline{D^*} := \bar{\mathbb{C}}\backslash D$ is a rigged Riemann surface endowed with the standard rigging Id_{S^1} which makes the boundary incoming. The surface $\overline{D^*}\#\Delta$ obtained by sewing $\overline{D^*}$ to Δ is a simply connected compact Riemann surface (called a sphere). By the uniformization theorem (stated in Section 3.2) it is isomorphic to the Riemann sphere $\bar{\mathbb{C}}$. Let $\Phi : \overline{D^*}\#\Delta \to \bar{\mathbb{C}}$ be an isomorphism. If $\tilde{\Phi}$ is another one, then $f := \Phi \circ \tilde{\Phi}^{-1}$ is an element of $\mathrm{Aut}(\bar{\mathbb{C}})$, which is the group of complex Möbius transformations $\text{Möb}(\mathbb{C})$ (cf. Appendix D). A simple calculation shows that there exists exactly one isomorphism $\Phi : \overline{D^*}\#\Delta \to \bar{\mathbb{C}}$ such that $g := \Phi|_{D^*}$ is an element of $\mathrm{Hol}(D^*)$, i.e., such that g satisfies the three normalization conditions (this may be achieved by adjusting the three complex parameters that characterize f). Then $\Delta \simeq \Phi(\Delta) = \bar{\mathbb{C}}\backslash\Phi(D^*) = \Delta_g$, i.e., $\Psi(g) = [\Delta]$.

In order to describe $\mathcal{M}(D_0)$, we basically only have to drop one normalization condition. So we define

$$\mathrm{Hol}_0(D^*) := \left\{ g : D^* \to \bar{\mathbb{C}} \text{ univalent } \;\middle|\; \begin{array}{l} g(\infty) = \infty,\ g'(\infty) = 1,\ 0 \notin \overline{g(D^*)}, \\ g \text{ has a smooth extension to } S^1 \end{array} \right\}. \tag{9.4}$$

To each $g \in \mathrm{Hol}_0(D^*)$, we associate the punctured disc $\Delta_g^x := \bar{\mathbb{C}}\backslash(g(D^*) \cup \{0\})$. Using exactly the same techniques as before,[39] one may then show that the map

$$\Psi_0 :\ \mathrm{Hol}_0(D^*) \to \mathcal{M}(D_0), \quad g \mapsto [\Delta_g^x], \tag{9.5}$$

is a bijection.

There exists yet another way to describe $\mathcal{M}(D)$ and $\mathcal{M}(D_0)$. This is due to the

Riemann mapping theorem. *Let U be a simply connected open non-empty subset of $\mathbb{C}$ which is not all of $\mathbb{C}$. Then there exists a biholomorphic map $f : U \to D$.*

[38] We implicitly assume the existence of a numbering of boundary components which is preserved by Φ.

[39] Given a punctured disc Δ^x, then $\overline{D^*}\#\Delta^x \simeq \mathbb{C}$. The automorphisms of $\mathbb{C}$ are the maps $z \mapsto az + b$ with $a \in \mathbb{C}^*,\ b \in \mathbb{C}$.

This enables one to establish the following isomorphisms

$$\mathcal{M}(D) \simeq \mathrm{Diff}^+(S^1)/\mathrm{PSU}(1,1), \qquad \mathcal{M}(D_0) \simeq \mathrm{Diff}^+(S^1)/S^1, \tag{9.6}$$

where the quotients are understood as quotients under the left action of the respective subgroups. The proof goes as follows. Let $[\Delta] \in \mathcal{M}(D)$. We already know that $[\Delta]$ is represented by exactly one $\Delta_g \simeq \Delta$ with $g \in \mathrm{Hol}(D^*)$. By the Riemann mapping theorem there exists a biholomorphic map $f : \Delta_g \backslash \partial\Delta_g \to D$. If we take a different $\tilde{f}$, then $f \circ \tilde{f}^{-1}$ is an automorphism of D. Therefore $\alpha := f \circ g|_{S^1}$ is an element of $\mathrm{Diff}^+(S^1)$ which is fixed by $[\Delta]$ up to post-composition with an element of $\mathrm{Aut}(D)$, which is conventionally identified with $\mathrm{PSU}(1,1)$ (cf. App. D). In other words, the map $\mathcal{M}(D) \to \mathrm{Diff}^+(S^1)/\mathrm{PSU}(1,1)$, $[\Delta] \mapsto [\alpha]$ is well defined. On the other hand, given $[\alpha] \in \mathrm{Diff}^+(S^1)/\mathrm{PSU}(1,1)$, then the disk D_α which is $\bar{D}$ but equipped with the boundary parametrization α defines an element of $\mathcal{M}(D_0)$ which is mapped to $[\alpha]$. Intuitively, every rigged disc (with outgoing boundary) is isomorphic to the unit disc with some boundary parametrization (i.e., an element of $\mathrm{Diff}^+(S^1)$), which is fixed up to the action of $\mathrm{PSU}(1,1)$.

For $\mathcal{M}(D_0)$ one can argue analogously, but the requirement that the puncture is mapped to 0 fixes two of the three real parameters of $\mathrm{PSU}(1,1)$ and one is left with $\mathrm{Aut}(D_0) \simeq S^1$.

To prove (9.6), we could have used the uniformization theorem directly (we have used it indirectly in the form of the identification $\mathcal{M}(D) \simeq \mathrm{Hol}(D^*)$). However, in this way we have encountered the so-called conformal welding, which is the map

$$W: \ \mathrm{Hol}(D^*) \to \mathrm{Diff}^+(S^1)/\mathrm{PSU}(1,1), \quad g \mapsto [\alpha], \tag{9.7}$$

where $\alpha := f \circ g|_{S^1}$ and $f : \Delta_g \backslash \partial\Delta_g \to D$ is the Riemann mapping.

Let us investigate how $\mathrm{Hol}(D^*)$ can be obtained by integrating the flow of vector fields in

$$\mathrm{Vect}_3(D^*) \ := \ \{\chi \in \mathrm{Vect}(D^*) \,|\, \tilde{\chi}(0) = 0, \ \tilde{\chi}'(0) = 0, \ \tilde{\chi}''(0) = 0\}, \tag{9.8}$$

where $\tilde{\chi}(\tilde{z}) := -\chi(\tilde{z}^{-1})\tilde{z}^2$. Obviously $\mathrm{Vect}_3(D^*)$ is the tangent space at the identity to $\mathrm{Hol}(D^*)$.

For $\chi \in \mathrm{Vect}^{\mathbb{C}}(S^1)$ that possess an integrated flow $f_t : S^1 \to \mathbb{C}$ $(0 \leq t \leq 1)$, denote by $D_\chi \subset \mathbb{C}$ the disk that is bounded by $f_1(S^1)$. Let $\mathcal{M}(D)' \subset \mathcal{M}(D)$ be the set of (equivalence classes of) discs of the form D_χ with $\chi \in \mathrm{Vect}_3(D^*)$. We conjecture that $\mathcal{M}(D)' = \mathcal{M}(D)$. A proof could go as follows. It has been shown [23] that the derivative of the conformal welding W at the identity is given by

$$\begin{aligned} &dW_{\mathrm{Id}}: \ \mathrm{Vect}_3(D^*) \to \mathrm{Vect}(S^1)/\mathfrak{psu}_{1,1}, \\ &\chi = \sum_{n\leq -2} \chi_n z^{n+1}\partial_z \ \mapsto \ i\Big(\sum_{n\geq 2} \bar{\chi}_n e^{in\sigma} - \sum_{n\leq -2} \chi_n e^{in\sigma}\Big)\partial_\sigma, \end{aligned} \tag{9.9}$$

where

$$\mathfrak{psu}_{1,1} = \left\{ \xi \in \mathrm{Vect}(S^1) \mid \xi = \sum_{n=-1}^{1} \xi_n e^{in\sigma}\partial_\sigma \right\} \tag{9.10}$$

is the Lie algebra of $\mathrm{PSU}(1,1)$. Let $g \in \mathrm{Hol}(D^*)$. Then there exists $\xi \in \mathrm{Vect}(S^1)$ such that $W(g) = [f_\xi]$, where $f_\xi \in \mathrm{Diff}^+(S^1)$ denotes the integrated flow of ξ. This ξ is fixed up to elements of $\mathfrak{psu}_{1,1}$, so the element $[\xi] \in \mathrm{Vect}(S^1)/\mathfrak{psu}_{1,1}$ is well-defined. Then we conjecture that g is the integrated flow of $\chi := dW_{\mathrm{Id}}^{-1}([\xi])$, so that $\Delta_g = D_\chi$.

Similarly, let us define

$$\mathrm{Vect}_2(D^*) \ := \ \{\chi \in \mathrm{Vect}(D^*) \,|\, \tilde{\chi}(0) = 0, \ \tilde{\chi}'(0) = 0\}. \tag{9.11}$$

This is obviously the tangent space to $\mathrm{Hol}_0(D^*)$ at the identity. So if $\chi \in \mathrm{Vect}_2(D^*)$ possesses an integrated flow f, then $f \in \mathrm{Hol}_0(D^*)$ and we may define the punctured disc $D^x_\chi := \Delta^x_f = \bar{\mathbb{C}} \backslash (f(D^*) \cup \{0\})$. Let $\mathcal{M}(D_0)'$ denote the set of all punctured discs of the form D^x_χ with $\chi \in \mathrm{Vect}_2(D^*)$. We conjecture that $\mathcal{M}(D_0)' = \mathcal{M}(D_0)$.

Interestingly, the spaces $\mathrm{Diff}^+(S^1)/\mathrm{PSU}(1,1)$ and $\mathrm{Diff}^+(S^1)/S^1$ are both coadjoint orbits of the Virasoro-Bott group (as introduced in Section 2.3). Furthermore, $\mathrm{Diff}^+(S^1)/S^1$ appears in the phase space of Liouville theory (2.51). In quantum Liouville theory, both spaces appear in a quantized version; the Virasoro representations $\mathcal{F}_p$ ($p \in \mathbb{R}$) are quantizations of $\mathrm{Diff}^+(S^1)/S^1$, and the vacuum representation $\mathcal{W}_0$ is the quantization of $\mathrm{Diff}^+(S^1)/\mathrm{PSU}(1,1)$.

The moduli space of rigged annuli $\mathcal{A} := \mathcal{M}^{\text{rigged}}_{0,2}$, were one boundary component is incoming and one is outgoing, is a semigroup that was introduced independently by Segal [32] and Neretin [24] (see also [31]). The semigroup structure is defined by sewing of annuli; namely, if $A_1, A_2 \in \mathcal{A}$, then $A_1 \cdot A_2 := A_1 \# A_2$, the annulus obtained by sewing the outgoing boundary component of A_2 to the incoming boundary component of A_1. $\mathcal{A}$ can be characterized [31] in a similar way as $\mathcal{M}(D)$ and $\mathcal{M}(D_0)$. Define

$$\mathrm{Hol}(D) := \{f : D \to \bar{\mathbb{C}} \text{ univalent} \mid f(0) = 0,\ f|_{S^1} \text{ is smooth}\}. \tag{9.12}$$

Let $f \in \mathrm{Hol}(D)$ and $g \in \mathrm{Hol}_0(D^*)$ such that

$$\overline{f(D)} \cap \overline{g(D^*)} \ = \ \emptyset. \tag{9.13}$$

Then one may associate with the pair (f, g) the annulus $A_{(f,g)} := \bar{\mathbb{C}} \backslash (f(D) \cup g(D^*))$ with rigging induced by f and g. If Λ denotes the set of pairs $(f, g) \in \mathrm{Hol}(D) \times \mathrm{Hol}_0(D^*)$ satisfying the additional condition (9.13), then the map

$$\Lambda \to \mathcal{A}, \quad (f, g) \mapsto [A_{(f,g)}] \tag{9.14}$$

is a bijection. This can be shown by sewing $\bar{D}_0$ to each annulus A along the incoming boundary component. In this way, one obtains a punctured disc $A \# \bar{D}_0$. Then there exists a unique $g \in \mathrm{Hol}_0(D^*)$ such that $A \# \bar{D}_0 \simeq \Delta^x_g$. The isomorphism $\Phi : A \# \bar{D}_0 \to \Delta^x_g$ is also unique and so is $f := \Phi|_{D_0} \in \mathrm{Hol}(D)$. Obviously we have $A \simeq A_{(f,g)}$.

The ability to represent each (punctured) disc and each annulus by a subset of the complex plane allows to explicitly describe the tangent spaces to $\mathcal{M}(D)$, $\mathcal{M}(D_0)$ and $\mathcal{A}$. Namely, given an element $[\Delta]$ of $\mathcal{M}(D)$, then a complex vector field χ on $\partial\Delta$ describes an infinitesimal deformation of Δ which is trivial in $\mathcal{M}(D)$ if χ can be analytically continued to Δ. Thus we have

$$T_{[\Delta]}\mathcal{M}(D) \ \simeq \ \mathrm{Vect}^{\mathbb{C}}(\partial\Delta)/\mathrm{Vect}(\Delta). \tag{9.15}$$

Similarly, the tangent space to $\mathcal{A}$ at $A \subset \mathbb{C}$ can be identified with the quotient [32] of $\mathrm{Vect}^{\mathbb{C}}(\partial A)$ by the subspace of vector fields that can be analytically continued into A,

$$T_{[A]} \ \simeq \ \mathrm{Vect}^{\mathbb{C}}(\partial A)/\mathrm{Vect}(A). \tag{9.16}$$

Note that a vector field on ∂A can be also described as a pair of vector fields on the incoming and outgoing boundary component of A respectively.

9.3 Quotient constructions

The sewing of rigged Riemann surfaces described above provides a natural way to define quotients of moduli (or Teichmüller) spaces of rigged Riemann surfaces. Although a more general setup exists, we will restrict our attention to quotients of the moduli spaces $\mathcal{M}(D)$, $\mathcal{M}(D_0)$ and $\mathcal{A}$.

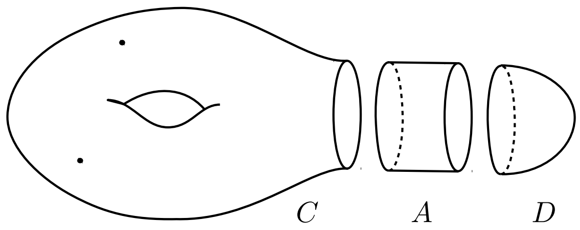

Figure 18: Sewing a punctured Riemann surface from three components

On $\mathcal{A}$, one may introduce right and left equivalence relations in the following way. Let C_R and C_L be (punctured) rigged Riemann surfaces with only one outgoing respectively one incoming boundary component. Let $A_1, A_2 \in \mathcal{A}$. Then we define

$$A_1 \sim_{C_R} A_2 \;:\Leftrightarrow\; A_1 \# C_R \simeq A_2 \# C_R, \qquad A_1 \;{}_{C_L}\!\!\sim A_2 \;:\Leftrightarrow\; C_L \# A_1 \simeq C_L \# A_2. \tag{9.17}$$

and

$$A_1 \;{}_{C_L}\!\!\sim_{C_R} A_2 \;:\Leftrightarrow\; C_L \# A_1 \# C_R \simeq C_L \# A_2 \# C_R. \tag{9.18}$$

These are obviously equivalence relations. In the following, we will focus on the simplest cases $U_R = \bar{D}$ and $U_R = \bar{D}_0$. Let us then consider the quotients $\mathcal{A}/\!\sim_{\bar{D}}$ and $\mathcal{A}/\!\sim_{\bar{D}_0}$, i.e., the sets of equivalence classes of $\sim_{\bar{D}}$ and $\sim_{\bar{D}_0}$. Let $\mathcal{M}(D)$ and $\mathcal{M}(D_0)$ denote the moduli spaces of rigged discs and punctured discs respectively, where the boundary is outgoing. Then there is a natural map

$$\Theta : \; \mathcal{A}/\!\sim_{\bar{D}} \to \mathcal{M}(D), \quad [A]_{\sim_{\bar{D}}} \mapsto [A \# \bar{D}], \tag{9.19}$$

and analogously $\Theta_0 : \; \mathcal{A}/\!\sim_{\bar{D}_0} \to \mathcal{M}(D_0)$. It follows immediately from the definition of $\sim_{\bar{D}}$ that Θ is well defined and injective. It is also surjective by the following argument: Let $[\Delta] \in \mathcal{M}(D)$. Since the internal $\Delta^\circ := \Delta \backslash \partial\Delta$ of Δ is open, there exists a univalent (i.e., holomorphic and injective) map[40] $\varphi : \; D \to \Delta^\circ$ such that $\varphi(D) \neq \Delta^\circ$. Then define the rigged annulus $A := \Delta \backslash \varphi(D)$ with rigging inherited from Δ and D (φ admits a smooth extension to $\partial D = S^1$). Since $\bar{D} \simeq \varphi(\bar{D})$ we have $A \# \bar{D} \simeq \Delta$, so $\Theta([A]_{\sim_{\bar{D}}}) = [\Delta]$. One can argue analogously for Θ_0 to obtain the isomorphisms (as sets)

$$\mathcal{A}/\!\sim_{\bar{D}} \simeq \mathcal{M}(D), \qquad \mathcal{A}/\!\sim_{\bar{D}_0} \simeq \mathcal{M}(D_0). \tag{9.20}$$

This is quite intuitive, as the sewing of a (punctured) disc to an annulus yields again a (punctured) disc. Now, let C (C_{-1}) be a rigged Riemann surface of genus g with n ($n-1$) punctures and one boundary component which is incoming. Then we may define the double quotients (pictorially represented in Fig. 18)

$$_{C}\!\sim\!\backslash \mathcal{A}/\!\sim_{\bar{D}}, \qquad _{C_{-1}}\!\sim\!\backslash \mathcal{A}/\!\sim_{\bar{D}_0} \tag{9.21}$$

in different ways which are all equivalent (i.e., the resulting sets are canonically isomorphic). For example, one may use the left-right equivalence $_{C}\!\sim_{\bar{D}}$ and define $_{C}\!\sim \backslash \mathcal{A}/\!\sim_{\bar{D}} := \mathcal{A}/_{C}\!\sim_{\bar{D}}$. Or one may define the left equivalence $_{C}\!\sim$ on $\mathcal{A}/\!\sim_{\bar{D}}$ by

$$\forall A_1, A_2 \in \mathcal{A} : \; [A_1]_{\sim_{\bar{D}}} \;{}_{C}\!\!\sim [A_2]_{\sim_{\bar{D}}} :\Leftrightarrow\; C \# A_1 \# \bar{D} \simeq C \# A_2 \# \bar{D}. \tag{9.22}$$

In the latter case the double quotients are naturally identified with the left quotients

$$_{C}\!\sim\!\backslash \mathcal{M}(D), \qquad _{C_{-1}}\!\sim\!\backslash \mathcal{M}(D_0), \tag{9.23}$$

[40] More precisely, one may pick out an arbitrary chart $\psi : U \to \mathbb{H}$ of Δ, then find a map $\rho : D \to \psi(U)\backslash\mathbb{R}$, of the form $z \mapsto az + b$ with $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ and define $\varphi := \psi^{-1} \circ \rho$.

where the definition of ${}_C\!\sim$ respectively ${}_{C_{-1}}\!\sim$ on $\mathcal{M}(D)$ and $\mathcal{M}(D_0)$ should be obvious. By the conformal welding isomorphisms (9.6), these may be understood as quotients of the coadjoint orbits $\mathrm{Diff}^+(S^1)/\mathrm{PSU}(1,1)$ and $\mathrm{Diff}^+(S^1)/S^1$ respectively. (Intuitively, these quotients are defined by sewing the unit disc with rigging given by an element of $\mathrm{Diff}^+(S^1)$ to C resp. C_{-1}.) In particular, the quotient ${}_{C_{-1}}\!\sim\backslash\mathrm{Diff}^+(S^1)/S^1$ defines the reduction of the (chiral) Liouville phase space to $\mathcal{M}_{g,n}$. Namely, the maps

$$\Xi:\ {}_C\!\sim\backslash\mathcal{M}(D)\to\mathcal{M}_{g,n},\ \ {}_C\!\sim[\Delta]\mapsto[C\#\Delta] \tag{9.24}$$

and $\Xi_0:\ {}_{C_{-1}}\!\sim\backslash\mathcal{M}(D_0)\to\mathcal{M}_{g,n}$ are well-defined and injective, and thus provide an identification of the left quotients (9.23) with subsets[41] of $\mathcal{M}_{g,n}$. When C is equipped with a marking, this induces a marking on the sewed surfaces and one obtains an identification with subsets of $\mathcal{T}_{g,n}$.

9.4 Infinitesimal quotient construction

Let us now investigate how the quotient construction looks like at the infinitesimal level. In the previous section we have explicitly described the tangent spaces to $\mathcal{M}(D)$ and $\mathcal{A}$. Now we would like to know which tangent vectors represent deformations within an equivalence class of ${}_C\!\sim$ and ${}_C\!\sim_{\bar{D}}$, i.e., trivial deformations in the quotients ${}_C\!\sim\backslash\mathcal{M}(D)$ and ${}_C\!\sim\backslash\mathcal{A}/\!\sim_{\bar{D}}$. Let us first consider ${}_C\!\sim\backslash\mathcal{M}(D)$.

We will use the following notation. For a (bordered) Riemann surface Σ with punctures, let $\mathrm{Vect}_0(\Sigma)$ denote the space of holomorphic vector fields on Σ that vanish at the punctures. If $\mathcal{M}$ and $\mathcal{N}$ are manifolds (e.g. Riemann surfaces), $f:\ \mathcal{M}\to\mathcal{N}$ is a smooth injective map, ξ a vector field on $\mathcal{N}$, denote by $f^*(\xi)$ the pullback of ξ by f, which is a vector field on $\mathcal{M}$ (more general, on the domain of f). Let ϕ_C be the boundary parametrization of C.

Preposition 1. *Let $\Delta\subset\mathbb{C}$ be a disc with boundary parametrization $\phi:S^1\to\partial\Delta$. Let $\varepsilon>0$, $\{\Delta_t\}_{t\in[-\varepsilon,\varepsilon]}$ a family of discs $\Delta_t\subset\mathbb{C}$ with boundary parametrizations ϕ_t that are smooth in t, such that $\Delta_0=\Delta$ and $C\#\Delta_t\simeq C\#\Delta$ for all $t\in[-\varepsilon,\varepsilon]$. This defines a smooth curve $t\mapsto[\Delta_t]$ in $\mathcal{M}(D)$ that runs within an equivalence class of ${}_C\!\sim$. Let $\eta:=\frac{\partial}{\partial t}\phi_t|_{t=0}\circ\phi^{-1}\in\mathrm{Vect}^{\mathbb{C}}(\partial\Delta)$ represent the tangent vector to that curve. Then there exist $\eta_\Delta\in\mathrm{Vect}(\Delta)$, $\eta_C\in\mathrm{Vect}_0(C)$ such that $\phi^*(\eta)=\phi_C^*(\eta_C)-\phi^*(\eta_\Delta)$.*

Sketch of proof. Since $C\#\Delta_t\simeq C\#\Delta$ for all $t\in[-\varepsilon,\varepsilon]$, and ϕ_t is smooth in t, we may assume that there exists a smooth family of conformal equivalences $\Phi_t:C\#\Delta_t\to C\#\Delta$ with $\Phi_0=\mathrm{Id}$. Smooth means in particular that there exist the derivatives[42]

$$\eta_\Delta(z):=\frac{\partial}{\partial t}\Phi_t(z)\bigg|_{t=0}\quad(z\in\Delta\backslash\partial\Delta),\quad \eta_C(z):=\frac{\partial}{\partial t}\Phi_t(z)\bigg|_{t=0}\quad(z\in C\backslash\partial C). \tag{9.25}$$

Note that $\eta_C(z)$ is not a complex number, but rather an element of the tangent space to C at z. Then η_Δ and η_C are holomorphic vector fields on $\mathring{\Delta}:=\Delta\backslash\partial\Delta$ and $\mathring{C}:=C\backslash\partial C$ respectively, that can be continuously extended to the boundaries. Since Φ_t leaves the punctures of C

[41] Ξ and Ξ_0 are not surjective as the following argument shows. Suppose Ξ was surjective. Then for all $[\Sigma]\in\mathcal{M}_{g,n}$ there would exist $[\Delta]\in\mathcal{M}(D)$ such that $\Sigma\simeq C\#\Delta$, in other words, there would exist a univalent map $\Phi:C\to\Sigma$. For $g=0$, C is always isomorphic to some $D\backslash\{y_1,\dots,y_n\}$ (where the rigging on $\partial C=S^1$ is generally not the standard one) and $\Sigma\simeq\bar{\mathbb{C}}\backslash\{z_1,\dots,z_n\}$. So if Ξ was surjective, then there would exist a univalent function $\Phi:D\to\bar{\mathbb{C}}$ for arbitrary prescribed values $z_k=\Phi(y_k)$. But this is not true as there are many inequalities for the values of a univalent function, see e.g. [15, Ch. 4].

[42] For each $z\in\Delta^\circ$, $\eta_\Delta(z)$ is well defined since there exists $\delta>0$ such that $\forall t\in[-\delta,\delta]:\Phi_t(z)\in\Delta^\circ$, which follows from openness of Δ° and continuity of ϕ_t and Φ_t.

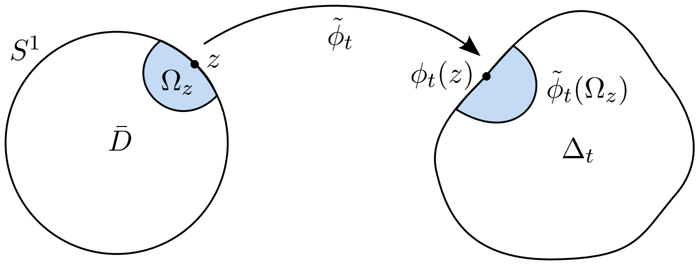

Figure 19: Illustrating the definition of $\tilde{\phi}_t$

invariant for every t, it follows that η_C vanishes at the punctures of C. The continuous extension η_C^1 of η_C to $\partial\Delta$ as a vector field on $C\#\Delta$ is formally given by

$$\forall z \in \partial\Delta: \quad \eta_C^1(z) = \lim_{w \to \phi_C(\phi^{-1}(z))} \eta_C(w). \tag{9.26}$$

Let $z \in S^1$. As $\phi_t(z)$ and $\phi_C(z)$ are identified on $C\#\Delta_t$ and Φ_t is in particular continuous at $\phi_t(z)$, we have

$$\forall t \in [-\varepsilon, \varepsilon]: \quad \lim_{\mathring{C} \ni w \to \phi_C(z)} \Phi_t(w) = \lim_{\mathring{\Delta}_t \ni w \to \phi_t(z)} \Phi_t(w). \tag{9.27}$$

We want to differentiate this equation by t and then set $t = 0$. (This involves mathematical subtleties that we ignore here.) For the l.h.s. this yields

$$\frac{\partial}{\partial t} \lim_{\mathring{C} \ni w \to \phi_C(z)} \Phi_t(w)\bigg|_{t=0} = \lim_{\mathring{C} \ni w \to \phi_C(z)} \frac{\partial}{\partial t}\Phi_t(w)\bigg|_{t=0} = \eta_C^1(\phi(z)). \tag{9.28}$$

To differentiate the r.h.s. of (9.27), let $\Omega_z \subset \bar{D}$ be a neighborhood of z in $\bar{D}$ and $\tilde{\phi}_t : \Omega_z \to \Delta_t$ for every $t \in [-\varepsilon, \varepsilon]$ be a continuous function such that $\tilde{\phi}_t = \phi_t$ on $S^1 \cap \Omega_z$, $\phi_t(\Omega_z \backslash S^1) \subset \mathring{\Delta}_t$ and $\tilde{\phi}_t(w)$ is smooth in t for every $w \in \Omega_z$ (as illustrated in Fig. 19). Then we obtain

$$\begin{aligned} \frac{\partial}{\partial t} \lim_{\Delta_t \ni w \to \phi_t(z)} \Phi_t(w) &= \frac{\partial}{\partial t} \lim_{w \to z} \Phi_t(\tilde{\phi}_t(w)) \\ &= \lim_{w \to z} \Big((\tfrac{\partial}{\partial t}\Phi_t)(\tilde{\phi}_t(w)) + (\partial\Phi_t)(\tilde{\phi}_t(w)) \tfrac{\partial}{\partial t}\tilde{\phi}_t(w) \Big), \end{aligned} \tag{9.29}$$

where $(\partial\Phi_t)(\tilde{\phi}_t(w))$ denotes the derivative of Φ_t at $\tilde{\phi}_t(w) \in \Delta_t$ (which for t small enough is a complex number). For $t = 0$ this becomes

$$\begin{aligned} \frac{\partial}{\partial t} \lim_{\Delta_t \ni w \to \phi_t(z)} \Phi_t(w)\bigg|_{t=0} &= \lim_{w \to z} \Big(\eta_D(\tilde{\phi}_0(w)) + \tfrac{\partial}{\partial t}\tilde{\phi}_t(w)|_{t=0} \Big) \\ &= \eta_\Delta(\phi(z)) + \eta(\phi(z)). \end{aligned} \tag{9.30}$$

Putting everything together, we find that $\eta(\phi(z)) = \eta_C^1(\phi(z)) - \eta_\Delta(\phi(z))$ for all $z \in S^1$, which implies

$$\eta = \eta_C^1 - \eta_\Delta|_{\partial\Delta}\,. \tag{9.31}$$

Since the boundaries of C and Δ are identified on $C\#\Delta$ via the map $\phi_C \circ \phi^{-1} : \partial\Delta \to \partial C$, we may write $\eta_C^1 = (\phi_C \circ \phi^{-1})^*(\eta_C) = (\phi^{-1})^*(\phi_C^*(\eta_C))$. Therefore, by applying ϕ^* to (9.31), one obtains $\phi^*(\eta) = \phi_C^*(\eta_C) - \phi^*(\eta_\Delta)$. $\square$

The generalization to the double quotient ${}_C{\sim}\backslash\mathcal{A}/{\sim}_{\bar{D}}$ is straightforward.

Preposition 2. *Let $A \subset \mathbb{C}$ be an annulus with boundary parametrizations $\phi_j : S^1 \to \partial_j A$ ($j = 1, 2$), where $\partial_1 A$ and $\partial_2 A$ are the incoming and outgoing boundary components respectively. Let $\{A_t\}_{t \in [-\epsilon,\epsilon]}$ be a smooth family of annuli $A_t \subset \mathbb{C}$ with riggings $(\phi_{1,t}, \phi_{2,t})$ such that $A_0 = A$ and $\forall t \in [-\epsilon, \epsilon] : C\#A_t\#\bar{D} \simeq C\#A\#\bar{D}$. Let $\eta_j := \frac{\partial}{\partial t}\phi_{j,t}\big|_{t=0} \circ \phi_j^{-1}$ ($j = 1, 2$) denote the vector fields on $\partial_1 A$ and $\partial_2 A$ that represent the tangent vector at $t = 0$ to the curve $t \mapsto [A_t]$ in $\mathcal{A}$. Then there exist $\eta_C \in \mathrm{Vect}_0(C)$, $\eta_A \in \mathrm{Vect}(A)$, $\eta_D \in \mathrm{Vect}(D)$ such that*

$$\phi_2^*(\eta_2) = \phi_C^*(\eta_C) - \phi_2^*(\eta_A), \qquad \phi_1^*(\eta_1) = \phi_1^*(\eta_A) - \eta_D|_{S^1}\,. \tag{9.32}$$

We omit the proof.

9.5 Mathematical definition of conformal blocks

In the mathematical sense, conformal blocks are maps between tensor products of Virasoro representations that are associated with rigged Riemann surfaces. But before we give the general definition, let us consider some examples from conformal field theory. The simplest examples are the vacuum state $|\Omega\rangle \in \mathcal{W}_0$ and its hermitian conjugate $\langle\Omega|$. Since $\mathbf{L}_n|\Omega\rangle = 0$ for all $n \geq -1$, we have

$$\forall \eta \in \mathrm{Vect}(D) : \ \mathbf{T}_\eta|\Omega\rangle = 0, \tag{9.33}$$

where D is again the unit disk. Similarly, since $\langle\Omega|\mathbf{L}_n = 0$ for all $n \leq 1$, and a vector field η transforms under the change of coordinate $z \to \tilde{z} = z^{-1}$ as $\eta(z) \to \tilde{\eta}(\tilde{z}) = -\eta(\tilde{z}^{-1})\tilde{z}^2$, one has

$$\forall \eta \in \mathrm{Vect}(D^*) : \ \langle\Omega|\mathbf{T}_\eta = 0. \tag{9.34}$$

We therefore consider $|\Omega\rangle$, or rather the map $\Phi_\Omega : \mathbb{C} \to \mathcal{W}_0, z \mapsto z|\Omega\rangle$, as a conformal block associated with the rigged Riemann surface $\bar{D}$ (the rigging is given by $\mathrm{Id} : S^1 \to S^1$), and $\langle\Omega|$ as a conformal block associated with $\overline{D^*} = D^* \cup S^1$.

The next example is the map

$$\Phi_z : \mathcal{V}_\alpha \to \mathcal{V}_{\alpha+bs}, \quad \Phi_z(v) := \mathbf{h}_s^\alpha(v|z)|\Omega\rangle, \quad z \in D. \tag{9.35}$$

Let $D_z := \bar{D}\backslash\{z\}$ and $\eta \in \mathrm{Vect}(D_z)$ (i.e., η may have a pole at z). Define

$$\mathbf{T}_\eta(z) \ := \ \oint_{C(0)} dw \; \eta(w+z)\mathbf{T}(w), \tag{9.36}$$

where $C(0)$ encircles the pole at $w = 0$. If $\eta(w) = \sum_{n\in\mathbb{Z}} \eta_n(z)(w-z)^{n+1}$ is the Laurent expansion of η around z, then $\mathbf{T}_\eta(z) = \sum_{n\in\mathbb{Z}} \eta_n(z)\mathbf{L}_n$. So it follows that

$$\begin{aligned} \mathbf{T}_\eta \circ \Phi_z(v) \ &= \ \oint_{S^1} \frac{dw}{2\pi i}\, \eta(w)\mathbf{T}(w)\mathbf{h}_s^\alpha(v|z)|\Omega\rangle \\ &= \ \oint_{C(z)+C(0)} \frac{dw}{2\pi i}\, \eta(w)R(\mathbf{T}(w)\mathbf{h}_s^\alpha(v|z))|\Omega\rangle \\ &= \ \mathbf{h}_s^\alpha(\mathbf{T}_\eta(z)v|z)|\Omega\rangle \ = \ \Phi_z \circ \mathbf{T}_\eta(z)(v), \end{aligned} \tag{9.37}$$

where in the second step we deformed the integration contour (where the integral over $C(0)$ vanishes) and in the third step we used (5.40). If one introduces the local coordinate $u(w) = w - z$ at the puncture $w = z$, then one may write $\mathbf{T}_\eta(z)$ as $T_{(u^{-1})^*(\eta)}$, where $g^*(\chi)(z) = \chi(g(z))(g'(z))^{-1}$ is the pullback of a vector field χ by a holomorphic function g. In this way, the definition of $\mathbf{T}_\eta(z)$ depends on the choice of a local coordinate at z. Therefore we consider Φ_z as a conformal block associated with the punctured rigged Riemann surface D_z with local coordinate u.

The last example will be the maps

$$\mathcal{F}^{\alpha}_{\Sigma,s}:\ \mathcal{V}_{\alpha_n}\otimes\cdots\otimes\mathcal{V}_{\alpha_1}\to\mathbb{C},\quad v_n\otimes\cdots\otimes v_1\mapsto\langle\mathbf{h}^{\alpha_n}_{s_n}(v_n|z_n)\ldots\mathbf{h}^{\alpha_1}_{s_1}(v_1|z_1)\rangle, \tag{9.38}$$

which are conformal blocks associated to the n-punctured sphere $\Sigma := \bar{\mathbb{C}}\backslash\{z_1,\ldots,z_n\}$ with local coordinates $u_k(z) := z - z_k$ $(k = 1,\ldots,n)$. The defining invariance property may here be written as

$$\forall\eta\in\mathrm{Vect}(\Sigma):\quad \sum_{k=1}^{n}\mathcal{F}^{\alpha}_{\Sigma,s}\circ\mathbf{T}^{(k)}_{(u_k^{-1})^*(\eta)} \ = \ 0, \tag{9.39}$$

where the upper index (k) denotes the action on the k-th tensor factor in $\bigotimes_{k=1}^{n}\mathcal{V}_{\alpha_k}$. The $\mathcal{F}^{\alpha}_{\Sigma,s}$ are related to the conformal blocks (in the physical sense) $\mathcal{G}^{\alpha}_{s}$ defined in Section 6 by

$$\mathcal{G}^{\alpha}_{s}(z_n,\ldots,z_1) \ = \ \mathcal{F}^{\alpha}_{\Sigma,s}(e_{\alpha_n}\otimes\cdots\otimes e_{\alpha_1}). \tag{9.40}$$

We are now ready to give a preliminary definition of a conformal block in the mathematical sense. Let C be a rigged Riemann surface with n incoming boundary components $\partial^{\mathrm{in}}_k C$ $(k = 1,\ldots,n)$ and m outgoing boundary components $\partial^{\mathrm{out}}_j C$ $(j = 1,\ldots,m)$. Let $\phi_k : S^1 \to \partial^{\mathrm{in}}_k C$ and $\psi_j : S^1 \to \partial^{\mathrm{out}}_j C$ be the boundary parametrizations. Furthermore, we assign to each boundary component a Verma module, characterized by α_k for $\partial^{\mathrm{in}}_k C$ and β_j for $\partial^{\mathrm{out}}_j C$. Then a conformal block associated with C is a linear map

$$\Phi_C:\ \bigotimes_{k=1}^{n}\mathcal{V}_{\alpha_k}\to\bigotimes_{j=1}^{m}\mathcal{V}_{\beta_j}, \tag{9.41}$$

where for $n = 0$ $(m = 0)$ it is understood that one replaces the tensor product on the left (right) hand side with $\mathbb{C}$, with the property

$$\forall\eta\in\mathrm{Vect}(C):\quad \sum_{k=1}^{m}\mathbf{T}^{(k)}_{\psi_k^*(\eta)}\circ\Phi_C \ = \ \sum_{k=1}^{n}\Phi_C\circ\mathbf{T}^{(k)}_{\phi_k^*(\eta)}. \tag{9.42}$$

Note that the parameters α_k and β_j play a minor role in this definition, as all Verma modules are isomorphic.

Considering the connection between holes and punctures with local coordinate discussed above, it is straightforward to generalize the definition of conformal blocks to punctured (rigged) Riemann surfaces with local coordinates. Namely, one would treat each puncture on C with local coordinate φ_j as an incoming hole and use $(\varphi_j^{-1})^*$ instead of ϕ_j^* in (9.42). If one also allows for "outgoing" punctures, i.e., punctures obtained from gluing D_0 to an outgoing boundary component $\partial^{\mathrm{out}}_j C$, one has to take into account that a local coordinate φ_j induces the reversed boundary parametrization $\psi_j\circ J$ on $\partial^{\mathrm{out}}_j C$. Therefore one would replace ψ_j^* in (9.42) by $(\varphi_j^{-1}\circ J)^*$.

The sewing procedure defined above is nicely incorporated in the definition of conformal blocks. For example, let C_j $(j = 1,2)$ be rigged Riemann surfaces with n_j incoming and m_j outgoing boundary components, such that $n_1 = m_2$. Let Φ_j be conformal blocks associated with C_j. Then $\Phi_1\circ\Phi_2$ is a conformal block associated with the surface $C_1\#C_2$, which is obtained by sewing all incoming boundary components of C_1 to all outgoing boundary components of C_2. This can be easily generalized to arbitrary $n_1 > 0$, $m_2 > 0$ and also to sewing of a rigged Riemann surface with itself.[43] This construction is naturally referred to as sewing of conformal blocks.

[43] If one would introduce an indexed basis for $\mathcal{V}_\alpha$ and represent conformal blocks by multi-indexed objects, then the sewing of an incoming with an outgoing boundary component would correspond to contraction of the corresponding indices.

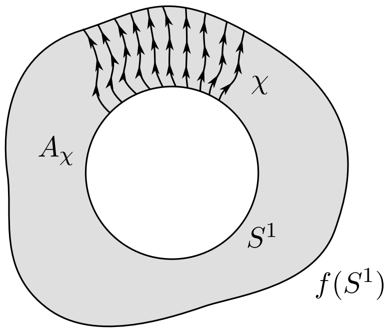

Figure 20: The definition of A_χ

The above definition applies to the examples given above, i.e., they are conformal blocks associated with the given surfaces in this sense. However, the following example shows that it is too naive (that is to say, it does not fully capture the structure of conformal field theory). Let $\chi \in \mathrm{Vect}^{\mathbb{C}}(S^1)$ be a vector field that possesses an integrated flow f_t ($t \in [0,1]$) such that $|f_1(z)| > 1$ for all $z \in S^1$. Define

$$\mathbf{U}_\chi : \mathcal{V}_\alpha \to \mathcal{V}_\alpha, \quad v \mapsto e^{-\mathbf{T}_\chi} v. \tag{9.43}$$

Then we associate with $\mathbf{U}_\chi$ the rigged annulus A_χ that is bounded by $\partial_1 A_\chi := S^1$ (incoming) and $\partial_2 A_\chi := f_1(S^1)$ (outgoing) with boundary parametrizations Id_{S^1} and $f := f_1$ (Figure 20). Let $\eta \in \mathrm{Vect}(A_\chi)$. Note that for $z \in S^1$

$$f^*(\eta)(z) \;=\; \frac{\eta(f(z))}{f'(z)} \tag{9.44}$$

and recall from Section 5.2 that

$$e^{\mathbf{T}_\chi}\mathbf{T}(z)e^{-\mathbf{T}_\chi} \;=\; (f'(z))^2\mathbf{T}(f(z)) - \frac{c}{6}S(f)(z). \tag{9.45}$$

So we compute

$$\begin{aligned} e^{\mathbf{T}_\chi}\mathbf{T}_{f^*(\eta)}e^{-\mathbf{T}_\chi} &= \oint_{S^1} \frac{dz}{2\pi i}\frac{\eta(f(z))}{f'(z)}\Big((f'(z))^2\mathbf{T}(f(z)) - \frac{c}{6}S(f)(z)\Big) \\ &= \oint_{f(S^1)} \frac{dw}{2\pi i}\eta(w)\mathbf{T}(w) + C[f,\eta], \\ &= \mathbf{T}_\eta + C[f,\eta], \end{aligned} \tag{9.46}$$

where

$$C[f,\eta] \;=\; -\frac{c}{6}\oint_{S^1}\frac{dz}{2\pi i}\frac{\eta(f(z))}{f'(z)}S(f)(z) \tag{9.47}$$

is a complex number. This number, which originates in the central extension of the Virasoro algebra, prevents $\mathbf{U}_\chi$ from being a conformal block associated with A_χ in the sense of the above definition. A way to overcome this obstacle is to introduce projective structures [43, Appendix]. (A projective structure on is an atlas $\{a_i\}_{i\in I}$ of charts such that $a_i \circ a_j^{-1} \in \mathrm{Möb}(\mathbb{C})$ on all non-empty overlaps). However, this solution is not suited for our purpose because it restricts the sewing of conformal blocks to rigged Riemann surfaces whose projective structures are compatible (i.e., there exists projective structure on the sewed surface that, when restricted, coincides with the projective structures on the original surfaces). Instead, we will find ways to avoid the appearance of the central extension by restricting ourselves to certain types of rigged Riemann surfaces.

9.6 Reduction on the quantum level

In this section we explicitly describe the reduction of Liouville theory to quantum Teichmüller theory (for genus zero) that was indicated in the introduction.

As before, let C be a rigged Riemann surface of genus g without punctures with only one boundary component which is incoming and is parametrized by $\phi_C : S^1 \to \partial C$. Let $\langle C|$ be a conformal block associated with C, which means that

$$\forall \eta \in \mathrm{Vect}(C): \quad \langle C|\mathbf{T}_{\phi_C^*(\eta)} = 0. \tag{9.48}$$

Let $\mathcal{A}' \subset \mathcal{A}$ denote the subset of annuli which are of the form A_χ which $\chi \in \mathrm{Vect}^{\mathbb{C}}(S^1)$. (There are reasons to assume that $\mathcal{A}'$ is a dense subset of $\mathcal{A}$). Now consider the function

$$\Theta_C : \ \mathcal{A}' \to \mathbb{C}, \quad A_\chi \mapsto \langle C|e^{-\mathbf{T}_\chi}|\Omega\rangle \tag{9.49}$$

What we would like to have is that Θ_C was constant on equivalence classes of the left-right equivalence ${}_C{\sim}_{\bar{D}}$. Then it would descend to a function on the double quotient ${}_C{\sim} \backslash \mathcal{A}'/\sim_{\bar{D}}$, which is naturally identified with a subset of the moduli space $\mathcal{M}_{g,0}$ (as discussed in Section 9.3). Provided that the coefficients χ_n from the Laurant expansion of χ define holomorphic coordinates on $\mathcal{M}_{g,0}$, Θ_C could be identified with a holomorphic function on a subset of $\mathcal{M}_{g,0}$, which may possess a (possibly multi-valued) analytic continuation to all of $\mathcal{M}_{g,0}$, which in turn defines a holomorphic function on $\mathcal{T}_{g,0}$. In Section 4.4 we have already encountered a quantization of $\mathcal{T}_{g,n}$ (there for $g = 0$, but which can be generalized to arbitrary g), in which the Hilbert space consists of holomorphic functions on Teichmüller space. Therefore, if the previous assumptions are true, then Θ_C could be identified with a state in a coherent state quantization of $\mathcal{T}_{g,0}$. Unfortunately, the existence of the central extension of the Virasoro algebra spoils this construction.

Preposition 3. *Let $s \mapsto A_s = A_{\chi_s}$, $s \in [-\epsilon, \epsilon]$ a smooth curve in $\mathcal{A}'$ through $\chi := \chi_0$. Let $f_{s,t} : S^1 \to \mathbb{C}$, $t \in [0,1]$, the integrated flow of χ_s and $f_t := f_{0,t}$. The pair of vector fields*

$$\eta_0(z) := 0, \qquad \eta_1(z) := \frac{\partial}{\partial s} f_{s,1}(f_1^{-1}(z))\bigg|_{s=0}, \tag{9.50}$$

defined on $\partial_1 A_\chi = S^1$ and $\partial_2 A_\chi = f_1(S^1)$ respectively, represents the tangent vector to the curve $s \mapsto A_s$. Then there exist constants κ_1, κ_2 such that

$$\frac{\partial}{\partial s} e^{-\mathbf{T}_{\chi_s}}\bigg|_{s=0} = \left(\mathbf{T}_{f_1^*(\eta_1)} + \kappa_1\right) e^{-\mathbf{T}_\chi} = e^{-\mathbf{T}_\chi} \left(\mathbf{T}_{\eta_1} + \kappa_2\right). \tag{9.51}$$

Note that in case η_1 admits an analytic continuation to A_χ, then $(-\eta_1|_{S^1}, 0)$ also represents the tangent vector to the curve $s \mapsto A_s$. For the proof we will need

Lemma 4. *Let $s \mapsto \mathbf{O}(s)$ be a differentiable curve in some operator space. Then*

$$\frac{\partial}{\partial s} e^{\mathbf{O}(s)} = \left(\int_0^1 dt\, e^{t\mathbf{O}(s)}\, \mathbf{O}'(s)\, e^{-t\mathbf{O}(s)}\right) e^{\mathbf{O}(s)}. \tag{9.52}$$

Proof. The operator valued function

$$\mathbf{Q}_1(s,t) := \frac{\partial}{\partial s} e^{t\mathbf{O}(s)} \tag{9.53}$$

satisfies the differential equation

$$\frac{\partial}{\partial t}\mathbf{Q}_1(s,t) \;=\; \mathbf{Q}_1(s,t)\mathbf{O}(s) + e^{t\mathbf{O}(s)}\mathbf{O}'(s) \tag{9.54}$$

with boundary condition $\mathbf{Q}_1(s,0) = 0$. On the other hand,

$$\mathbf{Q}_2(s,t) \;:=\; \left(\int_0^t dt'\, e^{t'\mathbf{O}(s)}\,\mathbf{O}'(s)\,e^{-t'\mathbf{O}(s)}\right) e^{t\mathbf{O}(s)} \tag{9.55}$$

satisfies the same differential equation and the same boundary condition. So $\mathbf{Q}_1(s,t) = \mathbf{Q}_2(s,t)$ for every s,t, which for $t=1$ is the claim. □

Proof of Preposition 3. By Lemma 4 we have

$$\left.\frac{\partial}{\partial s}e^{-\mathbf{T}_{\chi_s}}\right|_{s=0} \;=\; \left(\int_0^1 dt\, e^{-t\mathbf{T}_\chi}\left(\mathbf{T}_{\frac{\partial}{\partial s}\chi_s}\right)_{s=0} e^{t\mathbf{T}_\chi}\right) e^{-\mathbf{T}_\chi}. \tag{9.56}$$

By a computation that is very similar to (9.46) one finds

$$e^{-t\mathbf{T}_\chi}\,\mathbf{T}_{\partial_s\chi_s}\,e^{t\mathbf{T}_\chi} \;=\; \mathbf{T}_{f_t^*(\partial_s\chi_s)} - C[f_t,\partial_s\chi_s]. \tag{9.57}$$

Let us define the vector fields

$$\eta_t \;:=\; \left.\frac{\partial}{\partial s}f_{s,t}\circ f_t^{-1}\right|_{s=0}, \qquad t\in[0,1], \tag{9.58}$$

where η_t lives on $f_t(S^1)$. Let us compute (where prime again denotes the holomorphic z-derivative)

$$\begin{aligned}\frac{\partial}{\partial t}\left(f_t^*(\eta_t)\right) \;&=\; \frac{\partial}{\partial t}\left.\frac{\partial_s f_{s,t}}{f_t'}\right|_{s=0}\\ &=\; \frac{\partial_s(\chi_s\circ f_{s,t})}{f_t'} - \left.\frac{(\partial_s f_{s,t})(\chi\circ f_t)'}{(f_t')^2}\right|_{s=0}\\ &=\; \left.\frac{\partial_s\chi_s\circ f_{s,t}}{f_t'}\right|_{s=0} \;=\; f_t^*(\partial_s\chi_s)|_{s=0}.\end{aligned} \tag{9.59}$$

It follows immediately that

$$\frac{\partial}{\partial t}\mathbf{T}_{f_t^*(\eta_t)} \;=\; \left.\mathbf{T}_{f_t^*(\partial_s\chi_s)}\right|_{s=0}, \tag{9.60}$$

and thus, since $\eta_0 = 0$,

$$\int_0^1 dt\;\left.\mathbf{T}_{f_t^*(\partial_s\chi_s)}\right|_{s=0} \;=\; \mathbf{T}_{f_1^*(\eta_1)}. \tag{9.61}$$

Putting everything together, we have proved the first equation in (9.51) with

$$\kappa_1 \;=\; -\int_0^1 dt\, C[f_t,\partial_s\chi_s]|_{s=0}. \tag{9.62}$$

The second equation follows from (9.46). But in order to compute κ_2, it is more convenient to use a variant of equation (9.56),

$$\left.\frac{\partial}{\partial s}e^{-\mathbf{T}_{\chi_s}}\right|_{s=0} \;=\; e^{-\mathbf{T}_\chi}\int_0^1 dt\, e^{t\mathbf{T}_\chi}\left(\mathbf{T}_{\frac{\partial}{\partial s}\chi_s}\right)_{s=0} e^{-t\mathbf{T}_\chi}, \tag{9.63}$$

which yields $\kappa_2 = -\kappa_1$. $\square$

Now, let $s \mapsto A_s = A_{\chi_s}$ be a smooth curve in $\mathcal{A}'$ through $A := A_\chi = A_0$ such that $A_s {}_{\mathrm{C}}\!\sim_{\bar{D}} A$, i.e., $C\#A_s\#\bar{D} \simeq C\#A\#\bar{D}$, for all s. Let η_2 be the vector field on $\partial_2 A$ that represents (together with $\eta_1 = 0$) the tangent vector at $t = 0$ to that curve. By Preposition 2 there exist $\eta_C \in \mathrm{Vect}(C)$, $\eta_A \in \mathrm{Vect}(A)$, $\eta_D \in \mathrm{Vect}(D)$ such that

$$f^*(\eta_2) \;=\; \phi_C^*(\eta_C) - f^*(\eta_A), \qquad 0 \;=\; \eta_A|_{S^1} - \eta_D|_{S^1}\,. \tag{9.64}$$

This implies that $\mathbf{T}_{f^*(\eta_2)} = \mathbf{T}_{\phi_C^*(\eta_C)} - \mathbf{T}_{f^*(\eta_A)}$ and $\mathbf{T}_{\eta_A} = \mathbf{T}_{\eta_D}$. Using Preposition 3 and (9.46) we can then compute the directional derivative

$$\begin{aligned} \frac{\partial}{\partial s}\Theta_C(A_s)\bigg|_{s=0} &= \langle C|\, \tfrac{\partial}{\partial s} e^{-\mathbf{T}_{\chi_s}}|\Omega\rangle\big|_{s=0} \\ &= \langle C|\left(\mathbf{T}_{f^*(\eta_2)} + \kappa_1\right) e^{-\mathbf{T}_\chi}|\Omega\rangle \\ &= \langle C|\big(\mathbf{T}_{\phi_C^*(\eta_C)} - \mathbf{T}_{f^*(\eta_A)} + \kappa_1\big)e^{-\mathbf{T}_\chi}|\Omega\rangle \\ &= \langle C|e^{-\mathbf{T}_\chi}\big(- \mathbf{T}_{\eta_A} + \kappa_1 + C[f,\eta_A]\big)|\Omega\rangle \\ &= \big(\kappa_1 + C[f,\eta_A]\big)\langle C|e^{-\mathbf{T}_\chi}|\Omega\rangle. \end{aligned} \tag{9.65}$$

So we find that only the constants κ_1 and $C[f,\eta_A]$, which are both proportional to the central charge c, prevent Θ_C from being constant on equivalence classes of ${}_{\mathrm{C}}\!\sim_{\bar{D}}$. In order to overcome this obstacle, we observe that the central extension of the Virasoro algebra vanishes when one considers the subalgebra generated by $\mathbf{L}_n$ with $n \le -2$. Note that elements of $\mathrm{Vect}_3(D^*)$, defined in (9.8), are of the form

$$\chi \;=\; \sum_{n\le -2} \chi_n z^{n+1}\partial_z. \tag{9.66}$$

Let $s \mapsto \chi_s$ be a smooth curve in $\mathrm{Vect}_3(D^*)$ with $\chi = \chi_0$. Since $\mathbf{T}_{\chi_s}$ contains only $\mathbf{L}_n$ with $n \le -2$, it is almost obvious that κ_1 and κ_2 in (9.51) vanish in that case. The way to prove this rigorously is to show that the integrand in

$$C[f_t, \partial_s\chi_s]|_{s=0} \;=\; -\frac{c}{6}\oint_{S^1}\frac{dz}{2\pi i}\,\frac{\partial_s\chi_s(f_t(z))}{f_t'(z)}S(f_t)(z)\bigg|_{s=0} \tag{9.67}$$

possesses an analytic continuation to D^*. This is true since f_t possesses an analytic continuation to D^* (the integrated flow of χ) and $f_t'(z) \neq 0$ for all $z \in D^*$ since flow maps are always injective. Consequently, $C[f_t,\partial_s\chi_s]|_{s=0} = 0$, so $\kappa_1 = -\kappa_2 = 0$.

We want to represent each $D_\chi \in \mathcal{M}(D)'$ by the state $e^{\mathbf{T}_\chi}|\Omega\rangle$, even though this is not a conformal block associated with D_χ (again due to the central extension). Nevertheless, this "representation" has the following good property. Let $s \mapsto D_s = D_{\chi_s}$ be a smooth curve in $\mathcal{M}(D)'$ and f_s be the boundary parametrization of D_s with $f = f_0$. Then $\eta_1 := \frac{\partial}{\partial s} f_s \circ f^{-1}|_{s=0}$ represents the tangent vector at $s = 0$ to that curve and we have (according to the previous discussion)

$$\frac{\partial}{\partial s}e^{-\mathbf{T}_{\chi_s}}|\Omega\rangle\bigg|_{s=0} \;=\; \mathbf{T}_{f^*(\eta_1)}\, e^{-\mathbf{T}_{\chi_0}}|\Omega\rangle. \tag{9.68}$$

In order to get rid of the constant $C[f,\eta_A]$ in (9.65), consider the case where C is of the form $\overline{D^*}\backslash\{z_1,\ldots,z_n\}$ with $z_1,\ldots,z_n \in D^*$, rigging $\phi_C = \mathrm{Id}_{S^1}$ and local coordinates $\varphi_j(z) = z - z_j$ $(j = 1,\ldots,n)$. We already know that a conformal block associated with C is a linear map

$$\Phi_C : \bigotimes_{j=n}^{1}\mathcal{V}_{\alpha_j}\otimes\mathcal{V}_\alpha \to \mathbb{C}, \tag{9.69}$$

where V_{α_j} is assigned to z_j and $\mathcal{V}_\alpha$ to the boundary $\partial C = S^1$. The example from Liouville theory we have in mind are the maps

$$\bigotimes_{j=n}^{1} v_j \otimes v \;\mapsto\; \langle \Omega | \mathbf{h}_{s_n}^{\alpha_n}(v_n|z_n) \dots \mathbf{h}_{s_1}^{\alpha_1}(v_1|z_1)|v\rangle, \tag{9.70}$$

where $|v\rangle \in \mathcal{F}_p$ for $\alpha = \frac{Q}{2}+ip$ is the state of quantum Liouville theory associated with $v \in \mathcal{V}_\alpha$. What we would like to have is a linear map $\hat{\Phi}_C \in \mathcal{V}_\alpha^*$ with the property

$$\forall \eta \in \mathrm{Vect}_0(C): \quad \hat{\Phi}_C \circ \mathbf{T}_\eta = 0. \tag{9.71}$$

Let us define

$$\hat{\Phi}_C : \; \mathcal{V}_\alpha \to \mathbb{C}, \; v \mapsto \Phi_C(e_n \otimes \dots \otimes e_1 \otimes v), \tag{9.72}$$

where $e_j := e_{\alpha_j}$ denotes the highest weight state in $\mathcal{V}_{\alpha_j}$. In our example this would be

$$\langle C| \;=\; \langle \Omega | \mathbf{h}_{s_n}^{\alpha_n}(z_n) \dots \mathbf{h}_{s_1}^{\alpha_1}(z_1). \tag{9.73}$$

However, (9.71) is *not* fulfilled by $\hat{\Phi}_C$. Namely, let $\eta_C \in \mathrm{Vect}_0(C)$. Then η_C has a Taylor expansion $\eta_C(z) = \sum_{k=0}^\infty \eta_C^{(k+1)}(z_j)(z-z_j)^{k+1}$ around z_j so that

$$\mathbf{T}_{\varphi_j^*(\eta)} \;=\; \sum_{k=0}^{\infty} \eta_C^{(k+1)}(z_j)\mathbf{L}_k. \tag{9.74}$$

Then it follows that

$$\begin{aligned} \hat{\Phi}_C \circ \mathbf{T}_{\eta_C}(v) \;&=\; \Phi_C(e_n \otimes \dots \otimes e_1 \otimes \mathbf{T}_{\eta_C} v) \\ &=\; -\sum_{j=1}^n \Psi_C(e_n \otimes \dots \otimes \mathbf{T}_{\varphi_j^*(\eta_C)} e_j \otimes \dots \otimes e_1 \otimes v) \\ &=\; -\sum_{j=1}^n \Delta_{\alpha_j} \eta_C'(z_j) \hat{\Phi}_C(v), \end{aligned} \tag{9.75}$$

which means that $\hat{\Phi}_C \circ \mathbf{T}_{\eta_C}(v) = 0$ for all $\eta_C \in \mathrm{Vect}_1(C)$, if $\mathrm{Vect}_1(C)$ denotes the space of holomorphic vector fields on C that have a second order zero at each puncture.

The Riemann surface C has the nice property that every $\eta_C \in \mathrm{Vect}_0(C)$ is in particular a holomorphic vector field on D^*. Let us denote by $\mathrm{CB}(C)$ the space of all elements $\langle C| \in \mathcal{H}_L^*$ with the property $\langle C|\mathbf{T}_\eta = 0$ for all $\eta \in \mathrm{Vect}_1(C)$. Now assign to each $\langle C| \in \mathrm{CB}(C)$ the function

$$\Psi_{\langle C|} : \; \mathcal{M}(D)' \to \mathbb{C}, \quad D_\chi \mapsto \langle C| e^{-\mathbf{T}_\chi} |\Omega\rangle. \tag{9.76}$$

Suppose that $s \mapsto D_s = D_{\chi_s}$ runs within an equivalence class of ${}_C\!\sim$. By Preposition 1 this implies that there exist $\eta_C \in \mathrm{Vect}_0(C)$ and $\eta_\Delta \in \mathrm{Vect}(D_\chi)$ such that

$$f^*(\eta_1) \;=\; \eta_C|_{S^1} - f^*(\eta_\Delta). \tag{9.77}$$

Note that since each f_s admits an analytic continuation to D^*, the same is true for $f^*(\eta_1) = \partial_s f_s|_{s=0}/\partial_z f$. Therefore, by (9.77) (and $\eta_C \in \mathrm{Vect}(D^*)$), it must be also true for $f^*(\eta_\Delta)$. Then it follows that

$$C[f, \eta_\Delta] \;=\; -\frac{c}{6}\oint_{S^1} \frac{dz}{2\pi i} f^*(\eta_\Delta)(z)\, S(f)(z) \;=\; 0 \tag{9.78}$$

(since the integrand admits an analytic continuation to D^*) so that

$$\mathbf{T}_{f^*(\eta_\Delta)} e^{-\mathbf{T}_\chi}|\Omega\rangle \;=\; e^{-\mathbf{T}_\chi}\big(\mathbf{T}_{\eta_\Delta} + C[f,\eta_\Delta]\big)|\Omega\rangle \;=\; 0. \tag{9.79}$$

Therefore, by (9.68) and (9.77) we find that

$$\frac{\partial}{\partial s}\Psi_{\langle C|}(D_s)\bigg|_{s=0} \;=\; \langle C|\mathbf{T}_{\eta_C} e^{-\mathbf{T}_\chi}|\Omega\rangle \tag{9.80}$$

vanishes at least for $\eta_C \in \mathrm{Vect}_1(C)$. This may be interpreted as follows.

Let P be a puncture of a Riemann surface Σ. Let $\mathcal{O}_P^k$ denote the space of (germs of) holomorphic functions in a neighborhood of P that have a k-th order zero at P. Then a k-jet at P is an element of $\mathcal{O}_P^1/\mathcal{O}_P^{k+1}$ [11]. In a local coordinate z, a k-jet may be represented by $\sum_{j=1}^k a_j z^j$, so it is determined by k complex numbers a_j. Then a vector field $\eta_C \in \mathrm{Vect}(C)$ leaves a k-jet at z_j invariant if it has a k-th order zero at z_j. Let $\vec{C}$ be the same rigged Riemann surface as C, but equipped with a 1-jet at each puncture. Then one may interpret the result

$$\frac{\partial}{\partial s}\Psi_{\langle C|}(D_s)\bigg|_{s=0} \;=\; 0 \quad \text{if } \eta_C \in \mathrm{Vect}_1(C) \tag{9.81}$$

in the way that $\Psi_{\langle C|}$ is constant on equivalence classes of a left equivalence ${}_{\vec{C}}\!\sim$, which is defined by $\Delta_1 {}_{\vec{C}}\!\sim \Delta_2 \;:\Leftrightarrow\; \vec{C}\#\Delta_1 \simeq \vec{C}\#\Delta_2$ and $\vec{C}\#\Delta_1$, $\vec{C}\#\Delta_2$ are considered isomorphic if there exists a conformal equivalence that maps the 1-jets into each other. In this way, $\Psi_{\langle C|}$ descends to a function on the quotient ${}_{\vec{C}}\!\sim \backslash \mathcal{M}(D)'$, which in turn is identified with a subset of the moduli space $\vec{\mathcal{M}}_{0,n}$ of type $(0,n)$ punctured Riemann surfaces with 1-jets at each puncture.[44] When $\langle C|$ is of the form (9.73), this may be easily seen by using (5.56) to write

$$\Psi_{\langle C|}(D_\chi) \;=\; \left\langle \prod_{j=n}^{1} \big(f'(z_j)\big)^{\Delta_j}\, \mathbf{h}_{s_j}^{\alpha_j}(f(z_j)) \right\rangle. \tag{9.82}$$

Also note that $C\#D_\chi \simeq \bar{\mathbb{C}}\backslash f(\{z_1,\ldots,z_n\})$, where a conformal equivalence is given by (recall that $D_\chi = \bar{\mathbb{C}}\backslash f(D^*)$)

$$\hat{f}:\; C\#D_\chi \to \bar{\mathbb{C}}\backslash f(\{z_1,\ldots,z_n\}), \quad z \mapsto \begin{cases} f(z) \text{ for } z \in C \\ z \text{ for } z \in D_\chi \end{cases}. \tag{9.83}$$

Therefore, given $\chi_j \in \mathrm{Vect}_3(D^*)$ with integrated flow f_j $(j=1,2)$, $C\#D_{\chi_1} \simeq C\#D_{\chi_2}$ if and only if $f_1(z_j) = f_2(z_j)$ for $j=1,\ldots,n$, and $\vec{C}\#D_{\chi_1} \simeq \vec{C}\#D_{\chi_2}$ if and only if $f_1(z_j) = f_2(z_j)$, $f_1'(z_j) = f_2'(z_j)$ (the $\mathrm{PSL}(2,\mathbb{C})$ freedom is already fixed in the definition (9.8)). In the latter case it follows from (9.82) that $\Psi_{\langle C|}(D_{\chi_1}) = \Psi_{\langle C|}(D_{\chi_2})$.

In order to avoid the appearance of 1-jets, one could define $\mathrm{CB}_\alpha(C) \subset \mathrm{CB}(C)$ by (cf. (9.75))

$$\forall \eta \in \mathrm{Vect}_0(C):\; \langle C|\mathbf{T}_\eta \;=\; -\sum_{j=1}^n \Delta_j \eta'(z_j)\langle C|, \tag{9.84}$$

where $\Delta_j = \Delta_{\alpha_j}$, and assign to $\langle C| \in \mathrm{CB}_\alpha(C)$ the function

$$\tilde{\Psi}_{\langle C|}(D_\chi) \;:=\; \prod_{j=1}^n \big(f_\chi'(z_j)\big)^{-\Delta_j} \langle C|e^{-\mathbf{T}_\chi}|\Omega\rangle, \tag{9.85}$$

[44] $\vec{\mathcal{M}}_{g,n}$ is naturally seen as a vector bundle over $\mathcal{M}_{g,n}$, where the projection map $\pi: \vec{\mathcal{M}}_{g,n} \to \mathcal{M}_{g,n}$ is defined by forgetting the 1-jets.

where f_χ denotes the integrated flow of χ. $\tilde{\Psi}_{\langle C|}$ could be indeed identified as a holomorphic function on a subset of $\mathcal{M}_{0,n}$. Nevertheless, the prefactors in (9.85) seem to be somewhat artificial. Considering the connections with quantum Teichmüller theory, there could be indeed a natural way to include 1-jets in the latter. Namely, eigenstates of length operators in quantum Teichmüller space, written as $\psi_{L,r}(p)$ in a "real" representation or $\Psi_{L,r}(Z)$ in the coherent state representation, depend parametrically on the constants $L = (l_1, \ldots, l_n)$ that appear in the constraints (4.2) and have been identified with the hyperbolic lengths of the geodesics that encircle the holes. Instead, one could promote the l_j to "dynamical" variables and write $\psi_r(p, L)$ respectively $\Psi_r(Z, L)$. This would correspond to an enlargement of the Teichmüller space by two real dimensions per puncture, where one supplements each l_j by a conjugate observable k_j. (In the quantum theory, the k_j would become operators that shift the values of the l_j.) These variables could then be combined to define a 1-jet at each puncture.

In this way the enlarged Teichmüller space is identified with the Teichmüller space $\vec{\mathcal{T}}_{g,n}$ of type (g, n) Riemann surfaces with 1-jets at each puncture, which is naturally seen as a vector bundle over $\mathcal{T}_{g,n}$. The complex numbers a_j $(j = 1, \ldots, n)$ that represent the 1-jets for given local coordinates, would be (by definition) holomorphic functions on $\vec{\mathcal{T}}_{g,n}$ that define the complex structure on $\vec{\mathcal{T}}_{g,n}$ that is compatible with that on $\mathcal{T}_{g,n}$. It would then be natural to define a coherent state representation in which the Hilbert space $\mathrm{Hol}(\vec{\mathcal{T}}_{g,n})$ consists of holomorphic functions on $\vec{\mathcal{T}}_{g,n}$, written as $\Psi(Z, a)$ with $a = (a_1, \ldots, a_n)$. This would allow one to identify with each $\langle C| \in \mathrm{CB}(C)$ a state in $\mathrm{Hol}(\vec{\mathcal{T}}_{g,n})$, as defined by $\Psi_{\langle C|}$. In this way, one would be able to identify the enlarged quantum Teichmüller space with the subspace $\mathrm{CB}(C)$ of H_L^*.

This point of view has another advantage: it allows to replace the disc D, represented by the vacuum $|\Omega\rangle$, with the punctured disk D_0 with 1-jet, represented by the states $|p\rangle$. More precisely, let

$$|a\rangle \;=\; \int dp\; \psi_a(p)\, |p\rangle \qquad (a \in \mathbb{C}) \tag{9.86}$$

be eigenstates of the operator $\mathbf{a} = \mathbf{q} + i\mathbf{p}$ with eigenvalue a (cf. Section 4.4). Then we may associate with each $\langle \phi| \in \mathcal{H}_L^*$ a holomorphic "wave function"

$$\Psi_{0,\langle\phi|} : \; \mathcal{M}(D_0)' \times \mathbb{C} \to \mathbb{C}, \quad (D_\chi^x, a) \mapsto \langle \phi| e^{-\mathbf{T}_\chi} |a\rangle. \tag{9.87}$$

Provided that our conjecture $\mathcal{M}(D_0)' = \mathcal{M}(D_0)$ is true, then by $\mathcal{M}(D_0) \simeq \mathrm{Diff}^+(S^1)/S^1$, $\Psi_{0,\langle\phi|}$ can be identified with a holomorphic function on the chiral part[45] of the Liouville phase space (cf. (2.51))

$$\mathcal{P}_{\mathrm{chir.}} \;=\; (\mathrm{Diff}^+(S^1)/S^1) \times \mathbb{R} \times \mathbb{R}_{>0}, \tag{9.88}$$

where $a = x + ip$ is a holomorphic coordinate on $\mathbb{R} \times \mathbb{R}_{>0}$. In this way, one may define a coherent state representation of Liouville theory, where states are holomorphic functions on the Liouville phase space.

Let $\vec{C}_{-1} \;=\; D^* \backslash \{z_1, \ldots, z_{n-1}\}$ be an $n-1$ punctured rigged Riemann surface with 1-jets at each puncture. Note that $\mathbf{T}_\eta |a\rangle = 0$ for every $\eta \in \mathrm{Vect}_1(D_0)$. Therefore to each element $\langle C_{-1}|$ of $\mathrm{CB}(C_{-1})$ is associated a function $\Psi_{0,\langle C_{-1}|}$ that descends to the quotient

$$_{\vec{C}_{-1}}\!\sim \backslash \mathcal{M}(D_0)' \times \mathbb{C}, \tag{9.89}$$

which is naturally identified with a subset of $\vec{\mathcal{M}}_{0,n}$, where a defines the 1-jet at $z = 0$. In this way, $\mathrm{CB}(C_{-1})$ can be identified with a space of (locally defined) holomorphic functions

[45] One could easily include the anti-chiral part by incorporating in (9.87) an operator $\exp(\bar{\mathbf{T}}_\eta)$, where η is the vector field that generates the element of the anti-chiral copy of $\mathrm{Diff}^+(S^1)/S^1$ and $\bar{\mathbf{T}}_\eta = \sum_{m\in\mathbb{Z}} \eta_m \bar{\mathbf{L}}_m$ is the counterpart of $\mathbf{T}_\eta$ in the anti-chiral representation of the Virasoro algebra.

on $\vec{\mathcal{T}}_{0,n}$, which could be interpreted as the Hilbert space in a coherent state representation of enlarged quantum Teichmüller theory.

A question of interest to us is what states in quantum Teichmüller theory the standard conformal blocks

$$\langle C_r| \;:=\; \langle\Omega|\mathbf{h}_0^{\alpha_{n-1}}(z_{n-1})\mathbf{h}_{s_{n-2}}^{\alpha_{n-2}}(z_{n-2})\dots\mathbf{h}_{s_1}^{\alpha_1}(z_1), \tag{9.90}$$

parametrized by $r=(r_1,\dots,r_{n-3})$, where $\beta_k\equiv\frac{Q}{2}+ir_k=\alpha_{n-1}-\sum_{j=k+1}^{n-2}(\alpha_j+bs_j)$, are identified with. The most natural guess is of course that they are identified with eigenstates of length operators. This is supported by the computation

$$\begin{aligned}\Psi_{0,\langle C_{-1}|}(D_\chi^x,a) \;&=\; \langle\Omega|\mathbf{h}_{s_{n-1}}^{\alpha_{n-1}}(z_{n-1})\dots\mathbf{h}_{s_1}^{\alpha_1}(z_1)e^{-\mathbf{T}_\chi}|a\rangle\\ &=\; \psi_a(p_0)\prod_{j=1}^{n-1}(f'(z_j))^{\Delta_j}\lim_{z_0\to0}\langle\mathbf{h}_{s_{n-1}}^{\alpha_{n-1}}(f(z_{n-1}))\dots\mathbf{h}_{s_1}^{\alpha_1}(f(z_1))\mathbf{h}_0^{\alpha_0}(z_0)\rangle,\end{aligned} \tag{9.91}$$

where f denotes again the integrated flow of χ, and $\alpha_0\equiv\frac{Q}{2}+ip_0=\alpha_{n-1}-\sum_{j=1}^{n-2}(\alpha_j+bs_j)$, $s_{n-1}=0$. Since $C\#D_\chi\simeq\bar{\mathbb{C}}\backslash f(\{z_1,\dots,z_n\})$, the values $f(z_j)$, $j=1,\dots,n-3$, can be considered as holomorphic coordinates on $\mathcal{T}_{0,n}$ that correspond to the coordinates $Z=(z_1,\dots,z_{n-3})$ introduced before, while the values $f'(z_j)$ serve as coordinates on the fibers of the vector bundle $\vec{\mathcal{T}}_{0,n}$ over $\mathcal{T}_{0,n}$. Since the dependence of $\Psi_{0,\langle C_{-1}|}$ on the $f(z_j)$ is essentially the same as the dependence of $\mathcal{F}_L(Z,r)$, defined in (6.4), on Z, it seems plausible that $\Psi_{0,\langle C_{-1}|}$ is an eigenstate of the length operators $\mathbf{L}_1,\dots,\mathbf{L}_{n-3}$ in a coherent state representation of the enlarged quantum Teichmüller space.

Suppose that $\mathbf{O}$ is an operator of quantum Liouville theory with the property

$$\forall\eta\in\mathrm{Vect}_1(C):\quad [\mathbf{T}_\eta,\mathbf{O}]=0. \tag{9.92}$$

Then $\mathbf{O}$ would induce a map $\mathrm{CB}(C)\to\mathrm{CB}(C)$, $\langle C|\mapsto\langle C|\mathbf{O}$. By the discussion above, such a map could be identified with an operator on (enlarged) quantum Teichmüller space. We have in principle already encountered Liouville operators with the property (9.92) in Section 7.2; the only difference is that we had $z_j=e^{i\sigma_j}\in S^1$. This can be easily corrected by generalizing (7.33) as

$$\mathbf{Q}_j:=\int_{z_j}^{z_{j+1}}dz\,\mathbf{E}^b(z), \tag{9.93}$$

where the integral is performed along a path that crosses only the edge $\hat{e}_j$ in Fig. 15. This operator satisfies

$$[\mathbf{T}_\eta,\mathbf{Q}_j] \;=\; \eta(z_{j+1})\partial\mathbf{E}^b(z_{j+1})-\eta(z_j)\partial\mathbf{E}^b(z_j), \tag{9.94}$$

so that $[\mathbf{T}_\eta,\mathbf{Q}_j]=0$ for $\eta\in\mathrm{Vect}_0(C)$ (in particular for $\eta\in\mathrm{Vect}_1(C)$). Therefore also the operators $\mathbf{t}_j$, defined as in (7.34), satisfy (9.92), and thus possess an interpretation as Teichmüller operators. This observation strengthens the connection between these operators and the quantized shear coordinates found above.

The question that naturally arises then is whether the Teichmüller operator that $\mathbf{t}_j$ is identified with coincides with the (rescaled) quantized shear coordinate $\mathbf{q}_j$ that is represented by $\mathbf{t}_j^\dagger$ in the representation ρ^* defined in Section 7.2. This may be written as

$$\Psi_{0,\langle C|\mathbf{t}_j}\;\stackrel{?}{=}\;\rho^{\mathrm{hol.}}(\mathbf{q}_j)\Psi_{0,\langle C|}, \tag{9.95}$$

where $\rho^{\mathrm{hol.}}$ is the coherent state representation of enlarged Teichmüller space. For $n=4$, this conjecture is supported by the following argument. By the observation of Section 6.3 we may write the states

$$\langle C_r| \;=\; \langle\Omega|\mathbf{h}_0^{\alpha_3}(z_3)\mathbf{h}_{s_2}^{\alpha_2}(z_2)\mathbf{h}_{s_1}^{\alpha_1}(z_1), \tag{9.96}$$

where $\beta \equiv \frac{Q}{2} + ir = \alpha_3 - \alpha_2 - bs_2$, as

$$\langle C_r| \;=\; \langle\Omega|\mathbf{h}_0^{\alpha_3}(z_3)\mathbf{E}^{\alpha_2}(z_2)\mathbf{E}^{\alpha_1}(z_1)\; e^{2\pi b(s_1+s_2)\mathbf{x}}\; \tilde{\Upsilon}_r^s(\mathbf{t}, p_0), \tag{9.97}$$

where $p \mapsto \tilde{\Upsilon}_r^s(p, p_0)$ is an eigenfunction of $\mathbf{L}_s$ with eigenvalue $2\cosh(2\pi br)$ in the representation π_L based on shear coordinates with $L = (l_0, \ldots, l_3)$, $l_j := 4\pi b\Im\alpha_j$ $(j = 0, \ldots, 3)$. The action of the Teichmüller operator $\mathbf{p}$, related to $\mathbf{t}$ by $\rho^*(\mathbf{p}) = \mathbf{t}^\dagger$, on the properly normalized eigenfunctions Ψ_r^s may be written in the form of an integral transformation

$$\mathbf{p}\Psi_r^s(p) \;=\; p\Psi_r^s(p) \;=\; \int dr'\, p(r,r')\Psi_{r'}^s(p). \tag{9.98}$$

So if $\tilde{\Upsilon}_r^s$ and Ψ_r^s are related by $\tilde{\Upsilon}_r^s(p,p_0) \;=\; \lambda_L(r)\Psi_r^s(p)$, then we find that

$$\mathbf{p}\tilde{\Upsilon}_r^s(p,p_0) \;=\; \int dr'\, \tilde{p}(r,r')\tilde{\Upsilon}_{r'}^s(p,p_0), \tag{9.99}$$

where

$$\tilde{p}(r,r') \;=\; \frac{\lambda_L(r)}{\lambda_L(r')}p(r,r'). \tag{9.100}$$

Combined with (9.97), since $b(s_1+s_2) = \alpha_3 - \alpha_2 - \alpha_1 - \alpha_0$ does not depend on r, this implies that

$$\langle C_r|\mathbf{t} \;=\; \int dr'\, \tilde{p}(r,r')\langle C_{r'}|. \tag{9.101}$$

Let $\mathcal{F} : L^2(\mathbb{R}\times\mathbb{R}^4_{>0}) \to \mathrm{Hol}(\vec{\mathcal{T}}_{0,4})$ be the intertwiner between the representation π in which $\Psi_r^s(p,L)$ are the eigenfunctions of $\mathbf{L}_s$ and $\rho^{\mathrm{hol.}}$. By the discussion above, we expect $\Psi_{0,\langle C_r|}$ and $\mathcal{F}(\Psi_r^s)$ to be linearly dependent,

$$\Psi_{0,\langle C_r|} \;=\; \mu_L(r)\mathcal{F}(\Psi_r^s). \tag{9.102}$$

Then it follows by $\rho^{\mathrm{hol.}}(\mathbf{p}) = \mathcal{F}\,\mathbf{p}\,\mathcal{F}^{-1}$ that

$$\rho^{\mathrm{hol.}}(\mathbf{p})\Psi_{0,\langle C_r|} \;=\; \int dr'\, \frac{\mu_L(r)}{\mu_L(r')}p(r,r')\Psi_{0,\langle C_{r'}|}. \tag{9.103}$$

Therefore, given that $\lambda_L(r) = \mu_L(r)$, then we would conclude

$$\Psi_{0,\langle C_r|\mathbf{t}} \;=\; \rho^{\mathrm{hol.}}(\mathbf{p})\Psi_{0,\langle C_r|}. \tag{9.104}$$

Provided that Conjecture 1 is true, one could argue analogously for the operators $\mathbf{t}_j$. This would serve as a first evidence for the consistency of the representation ρ^* of quantized shear coordinates on $\mathcal{H}_L^*$ and the reduction picture developed in this section.

10 Conclusion and outlook

In this work we have systematically investigated the relation between the Liouville and Teichmüller quantum theories. We started this investigation from the observation that the eigenfunctions of length operators in quantum Teichmüller theory are decoded in the intrinsic structure of 4-point conformal blocks. Inspired by this observation we constructed a representation of the algebra of quantized shear coordinates on the dual of the Liouville Hilbert space and formulated a conjectural generalization of this observation to n punctures. (Here the precise definition of the operators $\mathbf{v}_j$ that are associated with the "spokes" of the fat graph requires some more attention.) All these definitions where inspired by a heuristic correspondence between Liouville and Teichmüller operators that is based on the conformal Ward identity.

In the last section we approached the problem from a slightly different angle by identifying (subsets of) the classical Teichmüller spaces with quotients of the Liouville phase space. Then we investigated how an analogous reduction may be performed on the quantum level, to obtain a statement of the type "quantization commutes with reduction". Indeed we were able to show that, modulo some technical difficulties, the (enlarged) quantum Teichmüller spaces may be identified with certain subspaces of the dual of the Liouville Hilbert space, and Teichmüller operators with Liouville operators that leave these subspaces invariant. Finally we could establish a first link between this picture and the representation of quantized shear coordinates defined before. In this part, there remain some open questions, e.g. whether the $\mathbf{v}_j$ can be also interpreted as Teichmüller operators in the reduction picture, and whether these operators coincide with the quantized shear coordinate they represent. If these questions can be answered affirmatively, then the reduction picture would be complete in the sense that the embedding of the space of Teichmüller operators into the space of Liouville operators would define a faithful representation of the former. Also the definition of the enlarged Teichmüller spaces requires some more work, as well as the generalization to higher genus.

Another subject for future projects could be the explicit description of the coherent state representation of quantum Teichmüller theory. In particular, it would be of interest to determine the quantum analogs of the functions $T(z)$ and compare them with the differential operator that appears in the conformal Ward identity.

A completely different approach to the relation between the Liouville and Teichmüller theories could employ the path integral formalism. Namely, the partition function

$$\mathcal{Z} \;=\; \int D\varphi\, e^{-b^{-2}S_e[\varphi]} \tag{10.1}$$

is expressed in terms of the *euclidean* Liouville action S_e. The connection between euclidean Liouville theory and Teichmüller theory seems to be even more concrete. Besides the fact that the conformal factor of the hyperbolic metric satisfies the euclidean Liouville equation, Takhtajan and Zograf [33] have related the Weil-Petersson symplectic form to the values of S_e evaluated on the conformal factor.

Summarizing, one finds that even though the connections between Liouville and Teichmüller theory are relatively simple to grasp on the classical level, it is quite challenging to precisely formulate them on the quantum level in an elementary way. This work should constitute a significant progress in this direction.

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Appendices

A Special functions

The function e_b, which is defined by

$$e_b(x) \;:=\; \exp\left(\int_{\mathbb{R}+i0} \frac{dw}{4w} \frac{e^{-2ixw}}{\sinh(bw)\sinh(b^{-1}w)}\right) \tag{A.1}$$

for $|\Im(x)| < Q/2$, $Q = b + b^{-1}$, and by analytic continuation otherwise, is known as Fadeev's quantum dilogarithm. It has the following properties:

(i) Zeros and poles: $(e_b(x))^{\pm 1} = 0 \;\Leftrightarrow\; x \in \mp\{c_b + inb + imb^{-1} \,|\, n, m \in \mathbb{Z}_{\geq 0}\}$

(ii) Functional equations: $e_b(x - \frac{ib^{\pm 1}}{2}) = \left(1 + e^{2\pi b^{\pm 1} x}\right) e_b(x + \frac{ib^{\pm 1}}{2})$

(iii) Self-duality: $e_b(x) = e_{b^{-1}}(x)$

(iv) Inversion relation: $(e_b(x))^{-1} = e^{\frac{\pi i}{12}(2-Q^2) - \pi i x^2} e_b(-x)$

(v) Asymptotic behavior ($x \in \mathbb{R}$):

$$e_b(x) \sim \begin{cases} 1 & \text{for } x \;\to +\infty \\ e^{\frac{\pi i}{12}(Q^2-2)} e^{\pi i x^2} & \text{for } x \;\to -\infty \end{cases}.$$

Closely related are the functions $s_b(x) := e^{-\frac{\pi i}{2}x^2} e^{\frac{\pi i}{24}(2-Q^2)} e_b(x)$ and $w_b(x) := (s_b(x))^{-1}$. These are more symmetric than e_b as seen in the inversion relation $(s_b(x))^{-1} = s_b(-x)$ and the functional equation

$$s_b\left(x - \tfrac{ib}{2}\right) \;=\; 2\cosh(\pi b x)\, s_b\left(x + \tfrac{ib}{2}\right). \tag{A.2}$$

One may also define

$$D_\alpha(x) \;:=\; \frac{w_b(x+\alpha)}{w_b(x-\alpha)}. \tag{A.3}$$

D_α satisfies the following integral identities [5]:

$$\int_{\mathbb{R}} dx\, e^{2\pi i x y}\, D_\alpha(x) \;=\; w_b(2\alpha + c_b) D_{-\alpha - c_b}(y) \tag{A.4}$$

and

$$\begin{aligned} &\int_{\mathbb{R}} dx\, e^{2\pi i(\alpha^* + \beta^*)x}\, D_\alpha(x-u)\, D_\beta(x-v) \\ &\quad = e^{2\pi i(v\alpha^* + u\beta^*)}\, w_b(2\alpha + c_b)\, w_b(2\beta + c_b)\, w_b(\alpha^* + \beta^*)\, D_{\alpha+\beta+c_b}(u-v), \end{aligned} \tag{A.5}$$

where $\alpha^* = -\alpha - c_b$.

In [12], Fock introduced the special function

$$\phi_b(x) \;=\; -\frac{\pi b^2}{2} \int_{\mathbb{R}+i0} dw \frac{e^{-iwx}}{\sinh(\pi w)\sinh(\pi b^2 w)}. \tag{A.6}$$

The relation with e_b is

$$\frac{\partial}{\partial z} \log e_b(z) \;=\; \frac{i}{b} \phi_b(2\pi b z). \tag{A.7}$$

B Highest weight representations of the Virasoro algebra

Highest weight representations of the Virasoro algebra $\mathfrak{V} := \mathrm{Span}\{L_n | n \in \mathbb{Z}\}$,

$$[L_m, L_n] \;=\; (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \tag{B.1}$$

are representations in which the L_0 eigenvalue is bounded from below. Since $L_0 + \bar{L}_0$ is the Hamiltonian in conformal field theories, all representations that appear in CFT's are of this type. As L_n with $n > 0$ lowers the L_0 eigenvalue, the highest weight vector e (i.e., the vector with lowest L_0 eigenvalue) has to satisfy $L_n e = 0$ for all $n > 0$. One can then classify by the L_0 eigenvalue of e.

There exist a standard family of highest weight representations on infinite dimensional vector spaces $\mathcal{V}_\alpha$ for $\alpha \in \mathbb{C}$, called Verma modules. $\mathcal{V}_\alpha$ is by definition the vector space with basis $\mathcal{B}_\alpha = \{v^\alpha_\nu | \nu \in \Lambda\}$, where

$$\Lambda = \{\nu = (\nu_k)_{k\in\mathbb{N}} \in (\mathbb{Z}_{\geq 0})^{\mathbb{N}} \mid \exists N \in \mathbb{N}\; \forall k > N : \; \nu_k = 0\} \tag{B.2}$$

is the space of finite sequences of non-negative integers. Thus all Verma modules are isomorphic and α serves only as a parameter to distinguish the different representations of $\mathfrak{V}$. If $e_\alpha = v^\alpha_0$, where 0 denotes the sequence that is identically zero, then the representation on $\mathcal{V}_\alpha$ is defined by

(i) $\forall n > 0 : \; L_n e_\alpha = 0$,

(ii) $L_0 e_\alpha = \Delta_\alpha e_\alpha$ where $\Delta_\alpha = \alpha(Q - \alpha)$,

(iii) $v^\alpha_\nu = (L_{-N})^{\nu_N} \dots (L_{-1})^{\nu_1} e_\alpha$ where $\nu_k = 0$ for all $k > N$.

So one may also write

$$\mathcal{V}_\alpha \;=\; \mathrm{Span}\{L_{n_r} \dots L_{n_1} e_\alpha \,|\, r \in \mathbb{N}_0, n_r \leq \dots \leq n_1 < 0\}. \tag{B.3}$$

Since $\Delta_{Q-\alpha} = \Delta_\alpha$, the representation on $\mathcal{V}_{Q-\alpha}$ is isomorphic to the one on $\mathcal{V}_\alpha$. If α is of the form $\alpha = \frac{Q}{2} + ip$ with $p \in \mathbb{R}$, it is sometimes replaced by the parameter p. Then one has $\Delta_p = \frac{Q^2}{4} + p^2$ and $\mathcal{V}_p \simeq \mathcal{V}_{-p}$ as a representation.

A representation of $\mathfrak{V}$ on a Hilbert space is called unitary if $L_n^\dagger = L_{-n}$. An sesquilinear form $(\cdot,\cdot)_\alpha$ on $\mathcal{V}_\alpha$ is uniquely defined by the requirements

(i) $(L_n v, w)_\alpha = (v, L_{-n} w)_\alpha$ for all $v, w \in \mathcal{V}_\alpha, \; n \in \mathbb{Z}$,

(ii) $(e_\alpha, e_\alpha)_\alpha = 1$.

$(\cdot,\cdot)_\alpha$ is non-degenerate if and only if $\alpha \notin \{\alpha_{r,s} = -\frac{1}{2}rb - \frac{1}{2}sb^{-1} \,|\, r, s \in \mathbb{N}_0\}$. If $(\cdot,\cdot)_\alpha$ is non-degenerate and positive definite, then one obtains a unitary highest weight representation of $\mathfrak{V}$ on $\mathcal{V}_\alpha$.

C Weyl-type operators

Let $\mathbf{U}, \mathbf{V}$ be positive self-adjoint operators that satisfy $\mathbf{UV} = q^2\mathbf{VU}$ with $q = e^{i\pi b^2}$. Our aim is to rewrite $(\mathbf{U} + \mathbf{V})^s$ for arbitrary $s \in \mathbb{C}$ in an elegant way. First we define $\mathbf{u} := \log \mathbf{U}$ and $\mathbf{v} := \log \mathbf{V}$. In order to compute $[u, v]$, consider the following equation:

$$e^{is\mathbf{u}} e^{it\mathbf{v}} \;=\; e^{-2\pi i b^2 st} e^{it\mathbf{v}} e^{is\mathbf{u}}. \tag{C.1}$$

Note that $e^{is\mathbf{u}}$ and $e^{it\mathbf{v}}$ for $s,t \in \mathbb{R}$ are well defined unitary operators. Taking the derivative of (C.1) by s and t and then setting $s = t = 0$, one finds that $[\mathbf{u}, \mathbf{v}] = 2\pi i b^2$. We may now introduce a pair of conjugate operators

$$\mathbf{x} := \frac{1}{2\pi b}\mathbf{u}, \quad \mathbf{p} := \frac{1}{2\pi b}(\mathbf{v} - \mathbf{u}) \tag{C.2}$$

that satisfy $[\mathbf{p}, \mathbf{x}] = (2\pi i)^{-1}$. With the help of the Baker-Campbell-Hausdorff formula, we can express $\mathbf{U}$ and $\mathbf{V}$ in terms of $\mathbf{x}$ and $\mathbf{p}$ as

$$\mathbf{U} = e^{2\pi b\mathbf{x}}, \quad \mathbf{V} = e^{\pi b\mathbf{x}} e^{2\pi b\mathbf{p}} e^{\pi b\mathbf{x}}. \tag{C.3}$$

This allows us to rewrite $(\mathbf{U} + \mathbf{V})^s$ in the following way

$$\begin{aligned}
(\mathbf{U} + \mathbf{V})^s &= \left(e^{\pi b\mathbf{x}}\left(1 + e^{2\pi b\mathbf{p}}\right)e^{\pi b\mathbf{x}}\right)^s \\
&= \left(e^{\pi b\mathbf{x}}\frac{e_b(\mathbf{p} - \frac{ib}{2})}{e_b(\mathbf{p} + \frac{ib}{2})}e^{\pi b\mathbf{x}}\right)^s = \left(e_b(\mathbf{p})e^{2\pi b\mathbf{x}}(e_b(\mathbf{p}))^{-1}\right)^s \\
&= e_b(\mathbf{p})e^{2\pi bs\mathbf{x}}(e_b(\mathbf{p}))^{-1} = e^{\pi bs\mathbf{x}}\frac{e_b(\mathbf{p} - \frac{ib}{2}s)}{e_b(\mathbf{p} + \frac{ib}{2}s)}e^{\pi bs\mathbf{x}} \\
&= e^{2\pi bs\mathbf{x}}\,\frac{e_b(\mathbf{p} - ibs)}{e_b(\mathbf{p})}.
\end{aligned} \tag{C.4}$$

It will be also useful to us to introduce the Fourier transform

$$\tilde{D}_\alpha(x) := \int_{-\infty}^{\infty} dy\; e^{-2\pi ixy}\,\frac{e_b(y - \alpha)}{e_b(y + \alpha)} \tag{C.5}$$

and write

$$(\mathbf{U} + \mathbf{V})^s = \int dx\; e^{s\mathbf{u} + \frac{i}{b}x(\mathbf{v} - \mathbf{u})}\;\tilde{D}_{\frac{ib}{2}s}(x). \tag{C.6}$$

D Automorphism groups

In this appendix we collect some basic knowledge about the automorphism groups of the standard Riemann surfaces $\mathbb{C}, \bar{\mathbb{C}}, \mathbb{H}, D$ and D_0, which can be found (including proofs) e.g. in [17]. We have

(i) For the complex plane $\mathbb{C}$

$$\mathrm{Aut}(\mathbb{C}) = \{z \mapsto az + b \,|\, a \in \mathbb{C}^*,\; b \in \mathbb{C}\}. \tag{D.1}$$

(ii) For the Riemann sphere $\bar{\mathbb{C}}$

$$\mathrm{Aut}(\bar{\mathbb{C}}) = \mathrm{M\ddot{o}b}(\mathbb{C}) := \left\{z \mapsto \frac{az + b}{cz + d}\,\middle|\, a, b, c, d \in \mathbb{C},\; ad - bc = 1\right\}. \tag{D.2}$$

$\mathrm{Aut}(\bar{\mathbb{C}})$ is therefore isomorphic to $\mathrm{PSL}(2, \mathbb{C}) := \mathrm{SL}(2, \mathbb{C})/\{\pm 1\}$, where

$$\mathrm{SL}(2, \mathbb{C}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\,\middle|\, a, b, c, d \in \mathbb{C},\; ad - bc = 1\right\}. \tag{D.3}$$

(iii) For the upper half plane $\mathbb{H}$

$$\mathrm{Aut}(\mathbb{H}) = \mathrm{M\ddot{o}b}(\mathbb{R}) := \left\{z \mapsto \frac{az + b}{cz + d}\,\middle|\, a, b, c, d \in \mathbb{R},\; ad - bc = 1\right\}. \tag{D.4}$$

Thus $\mathrm{Aut}(\mathbb{H})$ is isomorphic to $\mathrm{PSL}(2,\mathbb{R})$.

(iv) For the unit disc D

$$\mathrm{Aut}(D) \;=\; \Big\{ z \mapsto \frac{az+b}{\bar{b}z+\bar{a}} \,\Big|\, a,b \in \mathbb{C},\ |a|^2-|b|^2=1 \Big\}. \tag{D.5}$$

So $\mathrm{Aut}(D)$ is isomorphic to $\mathrm{PSU}(1,1)$.

(v) For the punctured unit disc D_0

$$\mathrm{Aut}(D_0) \;=\; \{ z \mapsto \lambda z \,|\, \lambda \in \mathbb{C}, |\lambda| = 1 \}. \tag{D.6}$$

Thus $\mathrm{Aut}(D_0)$ is isomorphic to S^1.

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Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

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