Structural properties of dense graphs with high odd girth and packing of minor-closed families of graphs

D i s s e r t a t i o n

zur Erlangung des Doktorgrades

an der

Fakultät für Mathematik, Informatik und Naturwissenschaften

am Fachbereich Mathematik

der Universität Hamburg

vorgelegt von

Silvia Messuti

Hamburg

2016

Tag der Disputation: 07.07.2016

Folgende Gutachter empfehlen die Annahme der Dissertation:

Prof. Mathias Schacht, PhD

Prof. Dr. Anusch Taraz

Contents

Acknowledgements

First and foremost I would like to thank my advisor Mathias Schacht for everything I accomplished throughout these four years. His contribution goes far beyond this thesis. He taught me how to write maths with a clear and objective style, how to represent my work in words and pictures, how to talk about my results in a fascinating and involving way, how to evaluate people's work with fairness and an open mind, how to deal with the dark times in work and life. Working with him has been a truly rewarding experience.

My sincerest gratitude goes to Vojtěch Rödl for his time and care. I appreciate his honesty and accuracy while pushing me to write precise and easily understandable mathematics. I feel honoured to have worked with him.

Throughout the last year I was supported through a doctoral completion fellowship of the University of Hamburg. I am greatly indebted to Anusch Taraz for providing an evaluation of my doctoral project. I would also like to thank him for serving on my PhD committee and for his inspiring and enjoyable talks.

I would like to express my gratitude to the people in the Discrete Mathematics group at UHH for providing a friendly and joyful environment. In particular I would like to thank the "extremal people" for the fruitful discussions around maths and beyond, and Christian Reiher for his thoughtful comments and suggestions.

I thank the DAAD for the opportunity to visit the Combinatorics research group at the University of São Paulo, and Yoshiharu Kohayakawa and his colleagues for their hospitality.

I am especially grateful to János Körner, who introduced me to extremal combinatorics and encouraged me to pursue a PhD. I will never thank him enough for his tireless support.

Finally, I thank Vito and my family for their love and understanding.

CHAPTER 1

Introduction

A typical question in graph theory investigates the relationship between structural properties of a graph and its invariants. In extremal graph theory we are interested in the quantitative aspects of this dependence, for example the maximum or minimum number of edges for which a certain property is satisfied, and how the graphs with exactly such number of edges look like.

Extremal graph theory is a branch of discrete mathematics whose origin is usually set in 1941, when Turán proved his celebrated theorem on K_r-free graphs. In the last few years, many advanced results have been proved and new techniques have been developed, including methods that have their roots in other branches of mathematics, like algebra and probability theory. In this thesis we introduce two new results that deal with different aspects of extremality in relationships between graphs.

A central part of extremal graph theory investigates the structural properties of graphs that do not contain a given subgraph. Turán's theorem is a prime example. It establishes that the maximal number of edges a K_r-free graph may have is the number of edges of the complete (almost) balanced $(r-1)$-partite graph and that this is actually the only K_r-free graph attaining such many edges. The case where triangles are forbidden gave rise to further questions that set the basis of our first result. In fact, we extended a theorem on triangle-free graphs to the case where small odd cycles are not contained in the graph. In Theorem 5 we determine the minimum degree that allows a graph with a given odd girth to be homomorphically mapped into its smallest odd cycle and in Theorem 6 we study the structure of the extremal graphs for this property. We give a brief history of the problem and introduce our contribution in Section 1.1.

We remark that here we started from a (forbidden) substructure and studied how this affects the overall appearance of a larger graph. On the other hand, one may start from a large complete graph and investigate which families of graphs can be found as edge disjoint subgraphs in it. These questions are called packing problems and their difficult increases as the number of edges of the graphs we want to pack approaches that of the hosting complete graph. In particular when all edges are used we call such a packing *perfect*. For example, a well-known and still open conjecture of Gyárfás asks for a perfect packing of n trees having all possible orders from 1 to n into K_n. In this thesis we extend a recent result that solves an asymptotic version of this conjecture for trees with bounded maximum degree. In fact, Theorem 9 establishes a similar statement that involves graphs from a minor closed family with bounded maximum degree. We refer the reader to Section 1.2 for a more detailed description of our result and the research around Gyárfás' conjecture.

Notation. Throughout this thesis we consider finite and simple graphs without loops and for any notation not defined here we refer to the textbooks [**12**, **15**, **24**]. As usual, $V(G)$ and $E(G)$ denote the vertex set and the edge set of a graph G respectively, with their cardinalities indicated by $v(G)$ and $e(G)$. The degree of a vertex v, i.e., the number of edges having v as an endpoint, is denoted by $d(v)$, while $\delta(G)$, $d(G)$, and $\Delta(G)$ signify the minimum, average, and maximum degree of G respectively. Finally, $\chi(G)$ designates the *chromatic number* of G, i.e., the minimum number of colours with which we may label the vertices of G in such a way that any two adjacent vertices have different colours.

§1.1. Graph homomorphism

A large branch of extremal graph theory studies sufficient conditions for given graphs F and G that force the existence of a subgraph isomorphic to F in G. In this type of problems, the number of edges of G is a natural parameter to consider. Let $\mathrm{ex}(n, F)$ be the maximum number of edges that a graph G of order n not containing F as a subgraph may have. The case when F is a clique of size r, meaning that G does not contain a set of r vertices any two of which are joined by an edge, was settled by Turán [53] in 1941 and is considered the starting point of extremal graph theory.

Theorem 1 (Turán). *For any graph G with $n \equiv \ell \pmod{r-1}$ vertices and $0 \leqslant \ell \leqslant r-1$*

$$\mathrm{ex}(n, K_r) = \frac{1}{2}\left(1 - \frac{1}{r-1}\right)(n^2 - \ell^2) + \binom{\ell}{2}.$$

Moreover, the only K_r-free graph with n vertices and $\mathrm{ex}(n, K_r)$ edges is the Turán graph $T(n,r)$*, i.e., the $(r-1)$-partite graph where any two partition classes differ by at most one in size and there is an edge between two vertices if and only if they belong to distinct partition classes.*

While this exact number of edges gives us a precise description of an extremal K_r-free graph, it is impossible to grasp the structure of a K_r-free graph with fewer edges by this information alone, since we don't know how those edges are distributed among the vertices. Considering the minimum degree allows us to characterise a broader range of K_r-free graphs.

In this sense, a direct consequence of Turán's theorem is that a graph G with minimum degree larger than $\frac{r-2}{r-1}n$ must contain K_r [57]. In some cases when $(r-1)$ does not divide n, a graph with minimum degree exactly $\frac{r-2}{r-1}n$ may have chromatic number larger than $r - 1 = \chi(T(n,r))$. Since the Turán graph has the maximum number of edges, one may guess that a sharper degree condition could mark this change in the structure. In fact, the following is true [9].

THEOREM 2 (Andrasfái, Erdős, and Sós). *Let $r \geqslant 3$. For any n-vertex graph G at most two of the following properties can hold:*

(1) $K_r \nsubseteq G$,
(2) $\delta(G) > \frac{3r-7}{3r-4}n$,
(3) $\chi(G) \geqslant r$.

The extremal graph here is unique in the sense that, when $(3r-4) \mid n$, there exists a unique K_r-free graph with n vertices, minimum degree exactly $\frac{3r-7}{3r-4}n$ and chromatic number r. This graph has vertex set

$$V = V_1 \dot\cup \ldots \dot\cup V_{r-3} \dot\cup U_0 \dot\cup \ldots \dot\cup U_4$$

where

$$|V_i| = \frac{3n}{3r-4} \quad \text{and} \quad |U_j| = \frac{n}{3r-4}$$

for $i = 1, \ldots, r-3$ and $j = 0, \ldots, 4$, and its edge set contains all pairs $\{x, y\}$ such that $x \in V_i$ and $y \notin V_i$ or $x \in U_j$ and $y \in U_{j+1 (\bmod 5)} \cup U_{j-1 (\bmod 5)}$.

In the triangle case we have that if a K_3-free graph G with n vertices has minimum degree $\frac{2n}{5} < \delta(G) \leqslant \frac{n}{2}$, then G is bipartite, and if $5 \mid n$ and $\delta(G) = \frac{2n}{5}$ then G is either bipartite, or it is a balanced blow-up of C_5, i.e., a graph where the vertex set is partitioned into five classes of the same size and any two vertices from classes V_i and $V_{i+1 (\bmod 5)}$ are joined by an edge. One may thus expect that if a graph G has minimum degree lower than but close to $\frac{2n}{5}$, then G has a structure "similar" to that.

More precisely, we say that a graph G is *homomorphic* to a graph H if there exists a map $\phi\colon V(G) \to V(H)$ such that $\{\phi(u), \phi(w)\} \in E(H)$ whenever $\{u, w\} \in E(G)$. Graph homomorphisms are strictly related to chromatic numbers. In fact, if there exists a colouring of G with k colours, then there exists a homomorphism into K_k where each vertex of K_k is a colour, and if there exists a homomorphism from G to K_k one can colour all vertices of G mapped to the same vertex of H with the same colour.

We have seen that if G is K_3-free and has minimum degree at least $\frac{2n}{5}$ then G is homomorphic to C_5 (note that a bipartite graph is homomorphic to any graph with at least one edge). Häggkvist [36] showed that such a phenomenon already happens when the minimum degree is larger than $\frac{3n}{8}$.

Theorem 3 (Häggkvist). *Any n-vertex, K_3-free graph G with minimum degree $\delta(G) > \frac{3n}{8}$ is homomorphic to C_5.*

The degree condition here is best possible, since there exists a graph with minimum degree exactly $\frac{3n}{8}$ that is not homomorphic to C_5. This graph is the balanced blow-up of the cycle of length eight $a_0 \dots a_7 a_0$ with additional edges $\{a_i, a_{i+4 (\mathrm{mod}\, 8)}\}$ for $i = 0, \dots, 3$. It is denoted by M_8 and named the *Möbius ladder* of order eight.

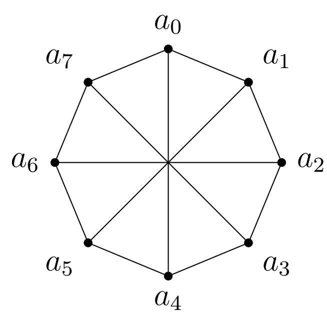


Figure 1. The Möbius Ladder M_8.

We may further refine the problem and ask whether there exists a minimum degree condition that guarantees that a K_3-free graph is homomorphic to M_8, then look at the extremal graph that is not homomorphic to M_8, study the minimum degree condition for which a K_3-free graph is homomorphic to that, and so on.

Let F_ℓ be a cycle of length $3\ell - 1$ with additional edges joining vertices whose distance in the cycle is $3j + 1$ for any $j = 1, \dots, \lfloor \frac{\ell-1}{2} \rfloor$. Note that for any $\ell \geqslant 1$, F_ℓ is an ℓ-regular graph with chromatic number three and $F_{\ell+1}$ contains F_ℓ as a subgraph but it is not homomorphic to it. With this notation in mind, we remark that Theorems 2 and 3 establish the degree condition for the existence of a homomorphism into F_1 and F_2 respectively, and in general we would like to argue that any K_3-free graph with minimum degree larger than $\frac{(\ell+1)n}{3\ell+2}$ is homomorphic to F_ℓ for every $\ell \geqslant 1$. In fact, this is true for $1 \leqslant \ell \leqslant 9$.

THEOREM 4 (Chen, Jin, and Koh). *Let $1 \leqslant \ell \leqslant 9$. Any n-vertex K_3-free graph G with minimum degree $\delta(G) > \frac{(\ell+1)n}{3\ell+2}$ is homomorphic to F_ℓ. Moreover, for each such ℓ there exists an extremal graph with minimum degree exactly $\frac{(\ell+1)n}{3\ell+2}$ which is homomorphic to $F_{\ell+1}$ but not to F_ℓ.*

For $\ell > 9$ graphs with larger chromatic number appear. In fact, Häggkvist [36] showed that there exists a K_3-free graph with minimum degree exactly $\frac{10n}{29}$ that contains the Grötzsch graph (see Figure 2) as a subgraph. Since the Grötzsch graph has chromatic number four, a graph containing it cannot be homomorphic to any F_ℓ since they all have chromatic number three.

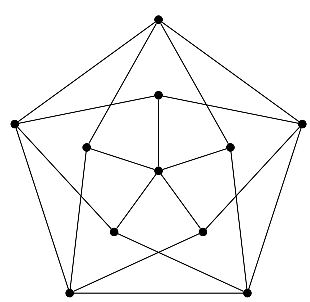


FIGURE 2. The Grötzsch graph.

This result disproved a conjecture of Erdős and Simonovits stating that any K_3-free graph with minimum degree larger than $\frac{n}{3}$ has chromatic number three. The value $\frac{n}{3}$ reflects the existence of graphs with arbitrarily large chromatic number and minimum degree $\left(\frac{1}{3} - \varepsilon\right) n$ for any $\varepsilon > 0$. The structure of the graphs F_ℓ seems to sustain this choice, since $\frac{1}{3}$ is limit of the degree of F_ℓ divided by its number of vertices for $\ell \to \infty$.

In fact, Chen, Jin, and Koh [22] showed that containing the Grötzsch graph is the only obstacle for a triangle-free graph with minimum degree larger than $\frac{n}{3}$ to be homomorphic to some F_ℓ and, hence, have chromatic number at most three. The problem posed by Erdős and Simonovits was thoroughly investigated in [17,42,52] and finally settled by Brandt and Thomassé [19], proving that K_3-free graphs with minimum degree larger than $\frac{n}{3}$ have chromatic number at most four.

In this thesis we establish the starting point for a generalisation of this theory to a broader class of graphs. The *odd girth* of a graph is defined as the length of its smallest odd cycle. Hence, since triangles are cycles of length three, triangle-free graphs have odd girth at least five. Our aim is to find the minimum degree conditions that help describe the structure of graphs with larger odd girth. In this sense, we generalised Theorem 3 to graphs of any odd girth [46].

THEOREM 5. *For every integer $k \geqslant 2$ and for every n-vertex graph G the following holds. If G has minimum degree $\delta(G) > \frac{3n}{4k}$ and G has odd girth at least $2k+1$, then G is homomorphic to C_{2k+1}.*

As in the triangle-free case, the minimum degree here is best possible, as the Möbius ladder of order $4k$ shows. We provide a detailed characterisation of the extremal graphs in the following theorem.

THEOREM 6. *For every integer $k \geqslant 2$ and for every n-vertex graph G with minimum degree $\delta(G) = \frac{3n}{4k}$ and odd girth at least $2k+1$ the following holds. If G is not homomorphic to C_{2k+1} then G is a blow-up of M_{4k} with vertex partition $A_0, \ldots, A_{4k-1}$. Furthermore,*

- *if $3 \nmid k$ then G is a balanced blow-up, i.e., $|A_0| = \cdots = |A_{4k-1}| = \frac{n}{4k}$;*
- *if $3 \mid k$ then there exist $\varrho_0, \varrho_1, \varrho_2 \in \{\frac{z}{3} : z \in \mathbb{Z}\}$ with $\varrho_0 + \varrho_1 + \varrho_2 = 0$ such that $|A_i| = \frac{n}{4k} + \varrho_{i (mod\, 3)}$ for $i = 0, \ldots, 4k-1$.*

The proofs of both theorems, together with a discussion on the open questions in the area will be the subject of Chapter 2.

§1.2. Graph packing

The problem of finding a certain subgraph in a larger graph naturally extends to the case where we require many subgraphs at the same time. More precisely, given a sequence of graphs $(G_1, \ldots, G_t)$, we say that it packs into some graph H if there exist edge-disjoint subgraphs $H_1, \ldots, H_t \subseteq H$ with H_i isomorphic to G_i for every $i \in [t]$. In packing problems we are interested in characterising those classes of graphs $\mathcal{G}_1, \ldots, \mathcal{G}_t$ such that $G_i \in \mathcal{G}_i$ and $(G_1, \ldots, G_t)$ packs into a given graph H.

In the simplest instance of this problem we are given two n-vertex graphs G_1 and G_2 and study the conditions that allow such graphs to be packed into K_n. A simple counting argument by Sauer and Spencer [51] shows that this is possible if $e(G_1)e(G_2) < \binom{n}{2}$. Bollobás and Eldridge [13] studied a more specific case, i.e., when one of the graphs has less than $\frac{n}{2}$ edges. In this case, for sufficiently large n, (G_1, G_2) packs into K_n if $e(G_1) \leqslant \alpha n$ with $0 < \alpha < \frac{1}{2}$ and $e(G_2) \leqslant \frac{1}{2}(1-2\alpha)n^{3/2}$.

The following example shows that the exponent in $n^{3/2}$ is best possible. For fixed α, let $s = (2\alpha n)^{\frac{1}{2}}$, $G_1 = K_s \cup \overline{K_{n-s}}$, and $G_2 = \overline{T(n,s)}$, thus $e(G_1) \leqslant \alpha n$ and $e(G_2) \leqslant n^{3/2}$. Since G_2 is the union of $s-1$ complete graphs and G_1 contains a clique on s vertices, they cannot pack into K_n. This example suggests that graphs containing vertices with high degree may be difficult to pack. In fact, if we consider graphs with a bounded maximum degree, then a larger number of edges is allowed. In this sense, Sauer and Spencer [51] showed that (G_1, G_2) packs into K_n if $\Delta(G_1)\Delta(G_2) < \frac{n}{2}$.

Can we replace the $\frac{1}{2}$ factor with something better? Let $d_1 \leqslant d_2 < n$ such that $(d_1+1)(d_2+1) \geqslant n+2$, set $G_1 = d_2 K_{d_1+1} \cap K_{d_1-1}$ and $G_2 = d_1 K_{d_2+1} \cap K_{d_2-1}$, and suppose that (G_1, G_2) packs into K_n. Then each K_{d_1+1} component of G_1 would use at most one vertex in each of the d_1 components of G_2 that are isomorphic to K_{d_2+1} and, hence, at least one vertex from the K_{d_2-1} component of G_2. Since

K_{d_2-1} has fewer vertices than the number of K_{d_1+1} components in G_1, such a packing cannot exist.

Bollobás and Eldridge [13] and Catlin [21] conjectured that this example is best possible, and therefore a packing exists if $(\Delta(G_1)+1)(\Delta(G_2)+1) \leqslant n+1$. Some special cases were proved in [1, 4, 14, 21, 23] and a solution for large n was recently claimed by Kun. We also remark that such a conjecture is related to the well-known Hajnal-Szemerédi theorem [38], which states that any graph with maximum degree Δ has a colouring with $(\Delta+1)$ colours in which any two colour classes differ by at most one in size. In fact, suppose G_2 is the union of r cliques of size $\frac{n}{r}$ (here we assume $r \mid n$ for simplicity) and, hence, it has maximum degree $\frac{n}{r}-1$. Then $(\Delta(G_1)+1)(\Delta(G_2)+1) \leqslant n+1$ implies that $\Delta(G_1) \leqslant r-1$, therefore G_1 has r independent sets of size $\frac{n}{r}$ that can host G_2.

Let us now consider packing problems that involve a larger number of graphs. The following conjecture was formulated by Gyárfás in 1976 and it is referred to as the Tree Packing Conjecture [35].

CONJECTURE 7. *Any sequence $\mathcal{T} = (T_1, \ldots, T_n)$ of trees of order $v(T_i) = i$ for $i \in [n]$ packs into K_n.*

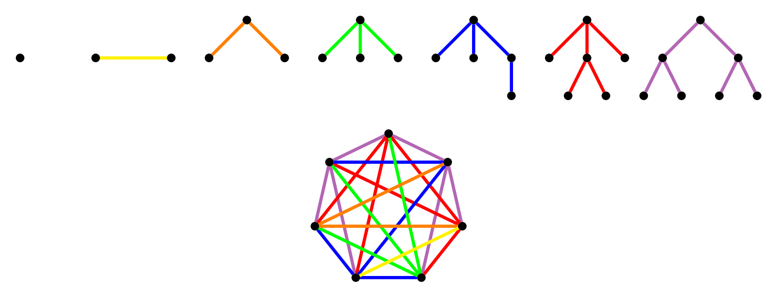


FIGURE 3. A packing of $(T_1, \ldots, T_7)$ into K_7.

The simplicity of the statement and the fact that the packing of these sequence of trees into K_n would be perfect make this conjecture an appealing problem. Some special cases of Conjecture 7 were verified (see, e.g., the survey [39] and

[33]). Gyárfás and Lehel [35] showed that the conjecture holds when all but two of the trees in the sequence are stars, and when each tree is either a star or a path (see also [56]). The case when at most one of the trees has diameter more than three was proved by Hobbs, Bourgeois, and Kasiraj [40]. Other cases concerning restrictions on the structure of the trees were investigated by Dobson [25–27] and by Roditty [50].

Another line of research concerns the packing of subsequences of $\mathcal{T}$. In this sense, Bollobás [11] showed that $(T_1, \ldots, T_k)$ packs into K_n if $k < \frac{n}{\sqrt{2}}$. About the other endpoint of $\mathcal{T}$, it was shown by Hobbs, Bourgeois, and Kasiraj [40] that (T_{n-2}, T_{n-1}, T_n) packs into K_n, while Balogh and Palmer [10] proved that $(T_k, \ldots, T_n)$ with $k > n - \frac{n^{1/4}}{10}$ packs into K_{n+1}.

A related conjecture was formulated and studied by Gerbner, Keszegh, and Palmer [31]. This states that $\mathcal{T}$ packs into any n-chromatic graph, and it was proved to hold in the case when all but three of the trees are stars. Another conjecture by Hobbs [39] states that $\mathcal{T}$ packs into the complete bipartite graph $K_{n-1,\lceil n/2\rceil}$. This holds if each of the trees is either a star or a path (see [56] and [39] for the case when n is even and odd, respectively). Yuster [55] proved this conjecture for a subsequence of $\mathcal{T}$, i.e., $(T_1, \ldots, T_k)$ with $k < \sqrt{5/8}n$, improving the previously best-known bound on k by Caro and Roditty [20].

As we have seen for packing problems involving two graphs, a bounded maximum degree allows for a more efficient use of the edges of the hosting graph. In fact, Böttcher, Hladký, Piguet, and Taraz [16] showed that with such a restriction on the trees, the graph obtained by adding only εn vertices to K_n is sufficient to accommodate n trees of order at most n when n is sufficiently large (since we provide asymptotical results, we will omit floors and ceilings in the following).

Theorem 8 (Böttcher, Hladký, Piguet, and Taraz). *For any $\varepsilon > 0$ and any $\Delta \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$ the following holds for every $t \in \mathbb{N}$. If $\mathcal{T} = (T_1, \ldots, T_t)$ is a sequence of trees satisfying* *(a)* $\Delta(T_i) \leqslant \Delta$ *and* $v(T_i) \leqslant n$ *for every* $i \in [t]$*, and*

(b) $\sum_{i=1}^{t} e(T_i) \leqslant \binom{n}{2}$,

then $\mathcal{T}$ *packs into* $K_{(1+\varepsilon)n}$.

In the proof of Theorem 8 the trees are cut into equally sized forests that are packed with a randomized procedure into a large complete subgraph of $K_{(1+\varepsilon)n}$ and then the remaining vertices are used to correct collisions. By splitting the trees in a different way we managed to extend this result to graphs from any non-trivial minor-closed family.

THEOREM 9. *For any* $\varepsilon > 0$, $\Delta \in \mathbb{N}$, *and any non-trivial minor-closed family* $\mathcal{G}$ *there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geqslant n_0$ *the following holds for every integer* $t \in \mathbb{N}$*. If* $\mathcal{F} = (F_1, \ldots, F_t)$ *is a sequence of graphs from* $\mathcal{G}$ *satisfying*

(a) $\Delta(F_i) \leqslant \Delta$ *and* $v(F_i) \leqslant n$ *for every* $i \in [t]$*, and*

(b) $\sum_{i=1}^{t} e(F_i) \leqslant \binom{n}{2}$,

then $\mathcal{F}$ *packs into* $K_{(1+\varepsilon)n}$.

Actually, we established a more general result that concerns the packing of (δ, s)-separable graphs. Such graphs have the property that by removing a δ-proportion of the vertices the resulting components have size at most s, where s is a small constant.

THEOREM 10. *For any* $\varepsilon > 0$ *and* $\Delta \in \mathbb{N}$ *there exists* $\delta > 0$ *such that for every* $s \in \mathbb{N}$ *and any* (δ, s)*-separable family* $\mathcal{G}$ *there exists* $n_0 \in \mathbb{N}$ *such that for every* $n \geqslant n_0$ *the following holds. If* $\mathcal{F} = (F_1, \ldots, F_t)$ *is a sequence of graphs from* $\mathcal{G}$ *satisfying*

(a) $\Delta(F_i) \leqslant \Delta$ *and* $v(F_i) \leqslant n$ *for every* $i \in [t]$*, and*

(b) $\sum_{i=1}^{t} e(F_i) \leqslant \binom{n}{2}$,

then $\mathcal{F}$ *packs into* $K_{(1+\varepsilon)n}$.

In fact, our strategy consists in removing the separator from each tree, packing the resulting components into a large complete subgraph of $K_{(1+\varepsilon)n}$ using some classical results and then use the remaining vertices of $K_{(1+\varepsilon)n}$ to embed the separators and reconnect the components. The details of this procedure will be discussed in Chapter 3.

CHAPTER 2

Graphs with given odd girth and large minimum degree

The material presented in this chapter is largely based on the paper *On the structure of graphs with given odd girth and large minimum degree* [46], joint work with Mathias Schacht. Similar results were obtained by Brandt and Ribe-Baumann.

§2.1. Homomorphisms of graphs with given odd girth

A *homomorphism* from a graph G into a graph H is a mapping

$$\phi\colon V(G) \to V(H)$$

with the property that $\{\phi(u), \phi(w)\} \in E(H)$ whenever $\{u, w\} \in E(G)$. We say that G is *homomorphic* to H if there exists a homomorphism from G into H. Furthermore, a graph G is a *blow-up* of a graph H, if there exists a surjective homomorphism ϕ from G into H, but for any proper supergraph of G on the same vertex set the mapping ϕ is not a homomorphism into H anymore. In particular, a graph G is homomorphic to H if and only if it is a subgraph of a suitable blow-up of H. Moreover, we say a blow-up G of H is *balanced* if the homomorphism ϕ signifying that G is a blow-up has the additional property that $|\phi^{-1}(u)| = |\phi^{-1}(u')|$ for all vertices u and u' of H.

Homomorphisms can be used to capture structural properties of graphs. For example, a graph is k-colourable if and only if it is homomorphic to K_k. Furthermore, many results in extremal graph theory establish relationships between the minimum degree of a graph and the existence of a given subgraph. The following theorem of Andrásfai, Erdős and Sós [9] is a classical result of that type.

THEOREM 11 (Andrásfai, Erdős & Sós). *For every integer $r \geqslant 3$ and for every n-vertex graph G the following holds. If G has minimum degree $\delta(G) > \frac{3r-7}{3r-4}n$ and G contains no copy of K_r, then G is $(r-1)$-colourable.* □

In the special case $r = 3$, Theorem 11 states that every triangle-free n-vertex graph with minimum degree greater than $\frac{2n}{5}$ is homomorphic to K_2. Several extensions of this result and related questions were studied. In particular, Häggkvist [36] showed that triangle-free graphs $G = (V, E)$ with $\delta(G) > \frac{3|V|}{8}$ are homomorphic to C_5. In other words, such a graph G is a subgraph of suitable blow-up of C_5. This can be viewed as an extension of Theorem 11 for $r = 3$, since balanced blow-ups of C_5 show that the degree condition $\delta(G) > \frac{2|V|}{5}$ is sharp there. Strengthening the assumption of triangle-freeness to graphs of higher odd girth, allows us to consider graphs with a more relaxed minimum degree condition. In this direction Häggkvist and Jin [37] showed that graphs $G = (V, E)$ which contain no odd cycle of length three and five and with minimum degree $\delta(G) > \frac{|V|}{4}$ are homomorphic to C_7.

We generalize those results to arbitrary odd girth, where we say that a graph G has *odd girth* at least g, if it contains no odd cycle of length less than g.

THEOREM 12. *For every integer $k \geqslant 2$ and for every n-vertex graph G the following holds. If G has minimum degree $\delta(G) > \frac{3n}{4k}$ and G has odd girth at least $2k+1$, then G is homomorphic to C_{2k+1}.*

Note that the degree condition given in Theorem 12 is best possible as the following example shows. For an even integer $r \geqslant 6$ we denote by M_r the so-called *Möbius Ladder* (see, e.g., [34]), i.e., the graph obtained by adding all diagonals to a cycle of length r, where a diagonal connects vertices of distance $\frac{r}{2}$ in the cycle (Figure 4). One may check that M_{4k} has odd girth $2k+1$, but it is not homomorphic to C_{2k+1}. Moreover, M_{4k} is 3-regular and, consequently, balanced blow-ups of M_{4k} show that the degree condition in Theorem 12 is best possible when n is divisible by $4k$.

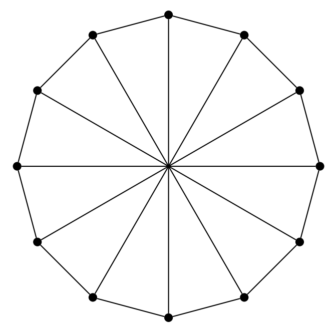


FIGURE 4. The Möbius Ladder M_{4k} for $k = 3$.

In the following we will denote the vertices of M_{4k} by $a_0, \dots, a_{4k-1}$, where $a_0 a_1 \dots a_{4k-1} a_0$ is a $4k$-cycle and all other edges of M_{4k} are in the form $\{a_i, a_{i+2k}\}$ (where the indices are taken modulo $4k$). Similarly, we will denote the vertex classes of a blow-up of M_{4k} by $A_0, \dots, A_{4k-1}$.

If G has minimum degree exactly $\frac{3n}{4k}$ and $3 \nmid k$, then clearly $4k \mid n$. In this case we will thus show that if G is not homomorphic to C_{2k+1}, then it is a balanced blow-up of M_{4k}. In the case when $3 \mid k$ we will show that if G is not homomorphic to C_{2k+1}, then there exist $\varrho_0, \varrho_1, \varrho_2 \in \{\frac{z}{3} : z \in \mathbb{Z}\}$ with $\varrho_0 + \varrho_1 + \varrho_2 = 0$ such that each vertex class of the blow-up has one neighbouring class having size $\frac{n}{4k} + \varrho_0$, one having size $\frac{n}{4k} + \varrho_1$, and one having size $\frac{n}{4k} + \varrho_2$.

THEOREM 13. *For every integer $k \geqslant 2$ and for every n-vertex graph G with minimum degree $\delta(G) = \frac{3n}{4k}$ and odd girth at least $2k+1$ the following holds. If G is not homomorphic to C_{2k+1} then G is a blow-up of M_{4k} with vertex partition $A_0, \dots, A_{4k-1}$. Furthermore,*

- *if $3 \nmid k$ then G is a balanced blow-up, i.e., $|A_0| = \dots = |A_{4k-1}| = \frac{n}{4k}$;*
- *if $3 \mid k$ then there exist $\varrho_0, \varrho_1, \varrho_2 \in \{\frac{z}{3} : z \in \mathbb{Z}\}$ with $\varrho_0 + \varrho_1 + \varrho_2 = 0$ such that $|A_i| = \frac{n}{4k} + \varrho_{i (mod 3)}$ for $i = 0, \dots, 4k-1$.*

We also remark that Theorem 12 implies that every graph with odd girth at least $2k+1$ and minimum degree bigger than $\frac{3n}{4k}$ contains an independent set of size at least $\frac{kn}{2k+1}$. This affirmatively answers a question of Albertson, Chan, and Haas [2].

§2.2. Proof of the main results

In this section we prove Theorem 12 and Theorem 13. Our main technical tool is Proposition 14 (see below), that gives some preliminary results on edge-maximal graphs that satisfy the assumptions of the theorems. We say that a graph G with odd girth at least $2k+1$ is *edge-maximal* if adding any edge to G (by keeping the same vertex set) yields an odd cycle of length at most $2k-1$. We denote by $\mathcal{G}_{n,k}$ all edge-maximal n-vertex graphs satisfying the assumptions of Theorem 13, i.e., for integers $k \geqslant 2$ and n we set

$$\mathcal{G}_{n,k} = \{G = (V, E)\colon\ |V| = n\,,\ \delta(G) \geqslant \tfrac{3n}{4k}\,,$$
$$\text{and } G \text{ is edge-maximal with odd girth } 2k+1\}\,.$$

Moreover, for n and k we define $\mathcal{G}^{>}_{n,k}$ as the subset of $\mathcal{G}_{n,k}$ satisfying the degree condition with strict inequality, i.e.,

$$\mathcal{G}^{>}_{n,k} = \{G \in \mathcal{G}_{n,k} : \delta(G) > \tfrac{3n}{4k}\}.$$

Proposition 14 states that graphs from $\mathcal{G}_{n,k}$ have a very simple structure.

Proposition 14. *For all integers $k \geqslant 2$ and n and for every $G \in \mathcal{G}_{n,k}$ one of the following holds:*

- *G is bipartite;*
- *G is a blow-up of C_{2k+1};*
- *G is a blow-up of M_{4k} and $\delta(G) = \frac{3n}{4k}$.*

The proof of Proposition 14 will be given in Section 2.4.

Proof of Theorem 12. Let G be a graph with n vertices, odd girth at least $2k+1$, and minimum degree $\delta(G) > \frac{3n}{4k}$. Consider an edge-maximal supergraph G' of G. Since $G' \in \mathcal{G}^{>}_{n,k}$, Proposition 14 implies that either G' is bipartite or it is a blow-up of C_{2k+1} and in both cases it follows that G is homomorphic to C_{2k+1}. □

Proof of Theorem 13. Let G be a graph with n vertices, odd girth at least $2k+1$ and minimum degree $\delta(G) = \frac{3n}{4k}$ and, similarly to the proof above, let G' be a supergraph of G from $\mathcal{G}_{n,k} \smallsetminus \mathcal{G}^{>}_{n,k}$. We may assume that G' is not bipartite and it is not a blow-up of C_{2k+1}, therefore, by Proposition 14, G' is a blow-up of M_{4k} with vertex classes $A_0, \ldots, A_{4k-1}$ and for each vertex $a_i \in A_i$ we have $N(a_i) = A_{i-1} \cup A_{i+1} \cup A_{i+2k}$.

First we show that all vertices of G' have degree exactly $\frac{3n}{4k}$. In fact, if the vertices in some vertex class have degree strictly larger than $\frac{3n}{4k}$, then we obtain the following contradiction:

$$3n = 4k\frac{3n}{4k} < \sum_{i=0}^{4k-1} |N(A_i)| = \sum_{i=0}^{4k-1} |A_{i-1}| + |A_{i+1}| + |A_{i+2k}| = 3\sum_{i=0}^{4k-1} |A_i| = 3n\,.$$

Note that this implies that $G' = G$, therefore G is a blow-up of M_{4k}.

It is left to show that the vertex classes of the blow-up are either balanced or have size $|A_i| = \frac{n}{4k} + \varrho_{i(\bmod 3)}$ for some $\varrho_0, \varrho_1, \varrho_2 \in \{\frac{z}{3} : z \in \mathbb{Z}\}$ with $\varrho_0 + \varrho_1 + \varrho_2 = 0$. Let $\varrho_i = |A_i| - \frac{n}{4k}$ for $i \in \{0, \ldots, 4k-1\}$. Below we prove that $\varrho_i = \varrho_{i(\bmod 3)}$.

Since each vertex has degree precisely $\frac{3n}{4k}$, for every $i = \{0, \ldots, 4k-1\}$ it holds $\varrho_{i-1} + \varrho_{i+1} + \varrho_{i+2k} = 0$. Moreover, A_{i+1} and A_{i+2k} are also adjacent to A_{i+2k+1}, whose third neighbouring class is A_{i+2k+2}. This implies that

$$\varrho_{i+2k+2} = 0 - \varrho_{i+1} - \varrho_{i+2k} = \varrho_{i-1}$$

and by shifting the indices we obtain that

$$\varrho_i = \varrho_{i+2k+3}$$

for every $i \in \{0, \ldots, 4k-1\}$.

We want to show that $\varrho_i = \varrho_{i(\mathrm{mod}\,3)}$ for every $i \in \{0, \ldots, 4k-1\}$. Therefore, it suffices to prove that the following linear congruence has a solution

$$(2k+3) \cdot x \equiv 3(\mathrm{mod}\,4k)\,. \tag{1}$$

This happens when $t = \gcd(2k+3, 4k) \mid 3$. Let r and $s \in \mathbb{N}$ such that $2k+3 = rt$ and $4k = st$. It follows that $2(2k+3) - 4k = 6 = (2r-s)t$, meaning that $t \mid 6$. We can exclude the cases $t = 2$ and $t = 6$ since $2k+3$ is odd. Consequently, $t \in \{1, 3\}$ and, hence, $t \mid 3$, which shows that the linear congruence (1) has a solution and therefore $\varrho_i = \varrho_{i+3}$ for every $i \in \{0, \ldots, 4k-1\}$. This already yields the desired conclusion for the case $3 \mid k$.

If $3 \nmid k$, then we can even show that $t = 1$. In fact, since $t \mid 2k+3$, having $t = 3$ would imply $t \mid 2k$ and, hence, $3 = t \mid k$, which contradicts the assumption on k. Consequently, in this case $t = 1$ and the linear congruence

$$(2k+3) \cdot x \equiv 1(\mathrm{mod}\,4k)$$

has a solution, implying $\varrho_{i+1} = \varrho_i$ for every $i \in \{0, \ldots, 4k-1\}$. Since the sum $\varrho_{i-1} + \varrho_{i+1} + \varrho_{i+2k}$ must be zero, we obtain $\varrho_i = 0$. $\square$

§2.3. Forbidden subgraphs

In this section we introduce two lemmas, Lemmas 15 and 17 below, needed for the proof of Proposition 14, which is described in Section 2.4. Roughly speaking, in each lemma we show that certain configurations cannot occur in graphs from $\mathcal{G}^{>}_{n,k}$ and if they occur in graphs from $\mathcal{G}_{n,k} \smallsetminus \mathcal{G}^{>}_{n,k}$, then this implies the existence of a subgraph isomorphic to M_{4k}.

For k fixed, we say that an odd cycle is *short* if its length is at most $2k-1$. A chord in a cycle of even length $2j$ is a *diagonal* if it joins two vertices at distance j in the cycle. Given a walk W we define its *length* $\ell(W)$ as the number of edges, each counted as many times as it appears in the walk. Hence, the lengths of paths and cycles coincide with their number of edges. We will also say that a path/cycle/walk is odd (even) if its length is odd (even).

2.3.1. Cycles of length six with precisely one diagonal. We denote by Φ (Figure 5) the graph obtained from C_6 by adding exactly one diagonal, i.e., $V(\Phi) = \{a_i\colon\ 0 \leqslant i \leqslant 5\} \subseteq V$ and

$$E(\Phi) = \{\{a_i, a_{i+1(\bmod 5)}\}\colon\ 0 \leqslant i \leqslant 5\} \cup \{a_1, a_4\}\,.$$

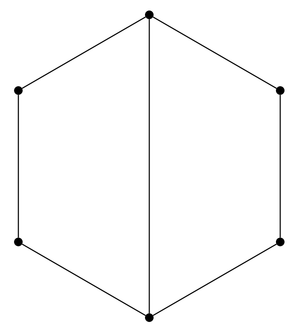
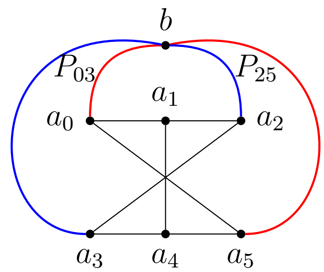


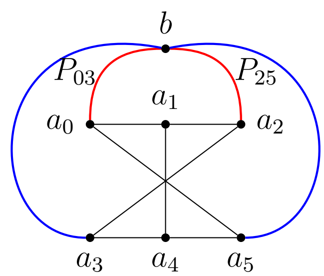
FIGURE 5. The graph Φ.

LEMMA 15. *For all integers $k \geqslant 2$ and n and for every $G \in \mathcal{G}_{n,k}$ the following holds. Either G does not contain an induced copy of Φ or G contains a copy of M_{4k} and $\delta(G) = \frac{3n}{4k}$.*

PROOF. Suppose that $G = (V, E)$ contains Φ in an induced way. Note that the chords of the C_6 in Φ which are not diagonals would create triangles in G so assuming that Φ is induced in G gives us only information concerning the non-existing two diagonals. Since G is edge-maximal, the non-existence of the diagonal between a_0 and a_3 must be forced by the existence of an even path P_{03} which, together with $\{a_0, a_3\}$, would yield an odd cycle of length at most $2k - 1$. Consequently, the length of P_{03} is at most $2k - 2$. Since a_0 and a_3 have distance three in Φ, a shortest path between them in Φ, together with P_{03}, results in a closed walk with odd length at most $2k + 1$. Recall that any odd closed walk is either an odd cycle or it contains a shorter odd cycle, it follows that P_{03} has length exactly $2k-2$ and its inner vertices are not in Φ. The same argument can be applied to the other missing diagonal between a_2 and a_5 to show that there exists another even path P_{25} of length $2k-2$ whose inner vertices are disjoint from $V(\Phi)$.



(A) W_{05} (red) and W_{23} (blue).



(B) W_{02} (red) and W_{35} (blue).

FIGURE 6. The paths P_{03} and P_{25} are vertex disjoint.

We now show that P_{03} and P_{25} are vertex disjoint. Suppose that they are not and let b be the first vertex in P_{03} which is also a vertex of P_{25}, i.e., b is the only vertex from $a_0P_{03}b$ which is also contained in P_{25}. Consider the walks

$$W_{05} = a_0P_{03}bP_{25}a_5$$

and

$$W_{23} = a_2P_{25}bP_{03}a_3\,,$$

where we follow the notation from [24], i.e., W_{05} is the walk in G which starts at a_0 and follows the path P_{03} up to the vertex b from which the walk continues on the path P_{25} up to the vertex a_5 (Figure 6a). Since W_{05} and W_{23} consist of the same edges (with same multiplicities) as P_{03} and P_{25} their lengths sum up to $4k-4$. Consequently, one of the walks, say W_{05}, has length at most $2k-2$. If W_{05} is even, then, together with the edge $\{a_0, a_5\}$, it yields an odd closed walk of length at most $2k-1$ and hence a short odd cycle. Otherwise, if W_{05} and W_{23} are odd, then also the walks

$$W_{02} = a_0 P_{03} b P_{25} a_2$$

and

$$W_{35} = a_3 P_{03} b P_{25} a_5$$

(Figure 6b) have an odd length. This implies that one of them, say W_{02}, has odd length at most $2k-3$. Together with the path $a_0a_1a_2$ this results into a closed walk with odd length at most $2k-1$ which yields the existence of a short odd cycle. Consequently, we derive a contradiction from the assumption that P_{03} and P_{25} are not vertex-disjoint.

Having established that $V(P_{03}) \cap V(P_{25}) = \varnothing$, we deduce that G contains the following graph Φ' consisting of a cycle of length $4k$

$$a_0a_1a_2P_{25}a_5a_4a_3P_{03}a_0$$

with three diagonals $\{a_0, a_5\}$, $\{a_1, a_4\}$ and $\{a_2, a_3\}$ (Figure 7).

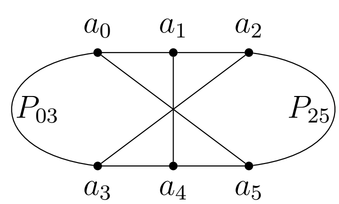
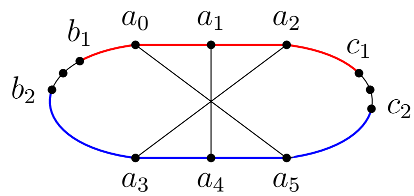
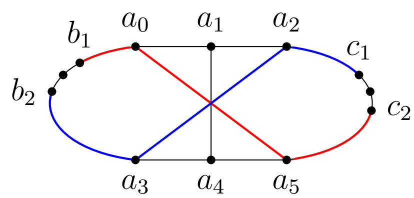


FIGURE 7. The graph Φ'.

We now show that no vertex in G can be joined to four vertices in Φ'. Suppose, for a contradiction, that there exists a vertex x in G such that $|N_G(x) \cap V(\Phi')| \geqslant 4$. Recall that x can be joined to at most two vertices of a cycle of length $2k+1$ and,



(A) P_{11} (red) and P_{22} (blue).



(B) P_{12} (red) and P_{21} (blue).

FIGURE 8. Each vertex of G can have at most three neighbours in Φ'.

if so, then these vertices must have distance two in that cycle. Since each of the three diagonals splits the cycle of length $4k$ of Φ' into two cycles of length $2k+1$, we have that x cannot have more than four neighbours in Φ'. Moreover, the only way to pick four neighbours is to choose two vertices from each of these cycles and none from their intersection, i.e. the ends of the diagonals. By applying this argument to each of the three diagonals, we infer that no vertex from $V(\Phi)$ can be a neighbour of x, therefore two neighbours b_1 and b_2 are some inner vertices of P_{03} and the two other neighbours c_1 and c_2 are inner vertices of P_{25}. Consider the vertex disjoint paths

$$P_{11} = b_1 P_{03} a_0 a_1 a_2 P_{25} c_1$$

and

$$P_{22} = b_2 P_{03} a_3 a_4 a_5 P_{25} c_2$$

(Figure 8a). Since b_1 and b_2 as well as c_1 and c_2 have distance two on the cycle of length $4k$ in Φ', both path lengths have the same parity and their lengths sum up to $4k-4$. If both lengths are odd, one must have length at most $2k-3$ and, together with x, this yields a short odd cycle. If, on the other hand, both lengths are even, then the paths

$$P_{12} = b_1 P_{03} a_0 a_5 P_{25} c_2$$

and

$$P_{21} = b_2 P_{03} a_3 a_2 P_{25} c_1$$

(Figure 8b) have odd length. Since their lengths sum up to $4k-6$, together with x, this yields the existence of a short odd cycle. Therefore, every vertex of G is joined to at most three vertices of Φ'.

If $G \in \mathcal{G}^{>}_{n,k}$, then this leads leads to the following contradiction

$$3n = 4k\frac{3n}{4k} < \sum_{u \in V(\Phi')} |N_G(u)| = \sum_{x \in V} |N_G(x) \cap V(\Phi')| \leqslant 3|V| = 3n\,. \tag{2}$$

Hence, G does not contain Φ as an induced subgraph.

If $G \in \mathcal{G}_{n,k} \setminus \mathcal{G}^{>}_{n,k}$ then it follows directly from 2 that each vertex of G has exactly three neighbours in Φ'. Let us denote the vertices of P_{03} and P_{25} as follows:

$$P_{03} = a_0 u_{2k-3} \ldots u_1 a_3$$

and

$$P_{25} = a_2 v_1 \ldots v_{2k-3} a_5\,.$$

We want to show that G contains M_{4k}. As we observed above, the cycle

$$a_0 a_1 a_2 a_3 v_1 \ldots v_{2k-3} a_5 a_4 a_3 u_1 \ldots u_{2k-3} a_0$$

has length $4k$ and contains three diagonals $\{a_0, a_3\}$, $\{a_1, a_4\}$, and $\{a_2, a_5\}$. It is then left to show that also the diagonals $\{u_i, v_i\}$ with $i = 0, \ldots, 2k-3$ are edges of G. Note that all these vertices have degree two in Φ', so they must all have one more neighbour in $V(\Phi')$ in the graph G. In particular, they cannot have any vertex of Φ as neighbour since these vertices have already degree three, so there exists a matching of the vertices of P_{03} with the vertices of P_{25}. Suppose that there exist $i, j \in \{1, \dots, 2k-3\}$ with $i \neq j$ such that $\{u_i, v_j\}$ is an edge of G. Two cases may occur. If i and j have the same parity, then the paths

$$P_{ij1} = u_i P_{03} a_0 a_1 a_2 P_{25} v_j$$

and

$$P_{ij2} = u_i P_{03} a_3 a_4 a_5 P_{25} v_j$$

(Figure 9a) have both even length, and since their lengths sum up to $4k$ and they cannot have the same length ($i \neq j$), one of them has length at most $2k-2$. Such a path, together with the edge $\{u_i, v_j\}$, yields a short cycle.

If i and j have a different parity, then the paths

$$P_{ij3} = u_i P_{03} a_0 a_5 P_{25} v_j$$

and

$$P_{ij4} = u_i P_{03} a_3 a_2 P_{25} v_j$$

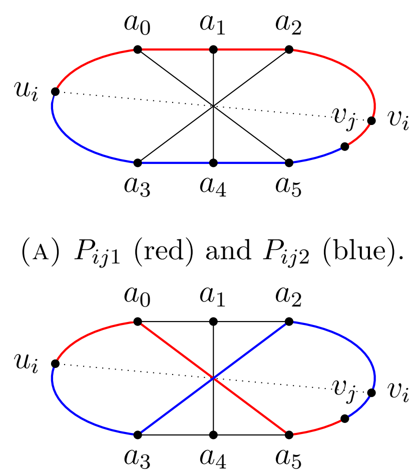
(Figure 9b) have both even length, and since their lengths sum up to $4k-2$, one of them has length at most $2k-2$ and together with the edge $\{u_i, v_j\}$ it yields a short cycle. It follows that the edges $\{u_i, v_i\}$ are contained in G, giving rise to a copy of M_{4k}. □

2.3.2. Tetrahedra with odd faces. In the next lemma we will consider graphs from the following family, which can be viewed as the family of tetrahedra with three faces formed by cycles of length $2k+1$, i.e., a particular *odd subdivision* of K_4 (see, e.g., [**30**]).

Definition 16 (($2k+1$)-tetrahedra). *Given $k \geqslant 2$ we denote by $\mathcal{T}_k$ the set of graphs T consisting of*

(*i*) *one cycle C_T with three* branch vertices *a_T, b_T, and $c_T \in V(C_T)$,*

(*ii*) *a* center vertex *z_T outside C_T, and*



(A) P_{ij1} (red) and P_{ij2} (blue).

(B) P_{ij3} (red) and P_{ij4} (blue).

FIGURE 9. Every vertex u_i is adjacent to the vertex v_i.

(*iii*) *internally vertex disjoint paths (called* spokes*)* P_{az}, P_{bz}, P_{cz} *connecting the branch vertices with the center.*

Furthermore, we require that each cycle in T *containing* z *and exactly two of the branch vertices must have length* $2k+1$ *and two of the spokes have length at least two.*

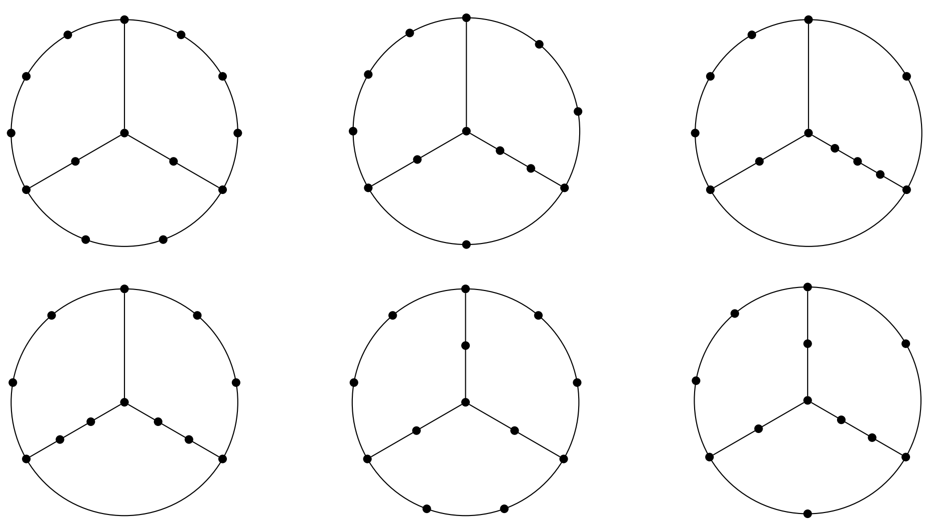


FIGURE 10. The family $\mathcal{T}_k$ for $k = 3$.

It follows from the definition that for $T \in \mathcal{T}_k$ we have that the cycle C_T has odd length and if $T \subseteq G$ for some $G \in \mathcal{G}_{n,k}$, then T consists of at least $4k$ vertices. In fact, the length of C_T equals the sum of the lengths of the three cycles containing z minus twice the sum of the lengths of the spokes. Since all three cycles containing z have an odd length, the length of C_T must be odd as well. In particular, if $T \subseteq G$ for some $G \in \mathcal{G}_{n,k}$, then the length of C_T must be at least $2k+1$. Summing up the lengths of all four cycles, counts every vertex twice, except the branch vertices and the center vertex, which are counted three times. Consequently,

$$|V(T)| \geqslant \frac{1}{2}\big(4 \cdot (2k+1) - 4\big) = 4k \tag{3}$$

for every $T \in \mathcal{T}_k$ with $T \subseteq G$ for some $G \in \mathcal{G}_{n,k}$.

We will also use the following further notation. For a cycle containing distinct vertices u, v, and w we denote by P_{uvw} the unique path on the cycle with endvertices u and w which contains v and, similarly, we denote by $P_{u\bar{v}w}$ the path from u to w which does not contain v.

For a tetrahedron $T \in \mathcal{T}_k$ we denote by C_{ab} the cycle containing z and the two branch vertices a and b. Similarly, we define C_{ac} and C_{bc}. Note that the union of two cycles, for instance C_{ab} and C_{ac}, contains an even cycle

$$C_{ab} \oplus C_{ac} = C_{ab} \cup C_{ac} - P_{az} = aP_{abz}zP_{zca}a\,, \tag{4}$$

where P_{abz} is a path on the cycle C_{ab} and P_{zca} a path on the cycle C_{ac}. Clearly, the length of $C_{ab} \oplus C_{ac}$ equals

$$\ell(C_{ab} \oplus C_{ac}) = \ell(C_{ab}) + \ell(C_{ac}) - 2\ell(P_{az}) = 4k + 2 - 2\ell(P_{az})\,. \tag{5}$$

LEMMA 17. *For all integers $k \geqslant 2$ and n and for every $G \in \mathcal{G}_{n,k}$ the following holds. Either G does not contain any $T \in \mathcal{T}_k$ as a (not necessarily induced) subgraph, or G contains a copy of M_{4k} and $\delta(G) = \frac{3n}{4k}$.*

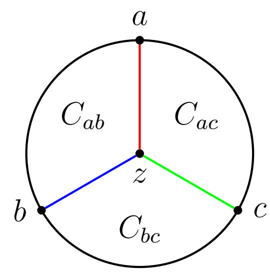


FIGURE 11. A tetrahedron T, with C_T in black, P_{az} in red, P_{bz} in blue, and P_{cz} in green.

PROOF. Suppose that $G = (V, E)$ contains a graph from $\mathcal{T}_k$. Fix that graph $T \in \mathcal{T}_k$ contained in G having the shortest length of C_T. We shall prove that no vertex in G can be joined to four vertices in T.

Suppose that there exists a vertex $x \in V$ such that $|N_G(x) \cap V(T)| \geqslant 4$ and fix four of those neighbours. Since T consists of the union of three cycles of length $2k+1$ one of those cycles must contain exactly two of these neighbours. This implies that we can either pick two of those cycles which contain the four neighbours (see Claim 18 below), or we have at least two ways to pick two such cycles which contain exactly three neighbours (see Claim 19 below).

Recall that the vertices on the spokes belong to two cycles and the center z belongs to all three cycles C_{ab}, C_{ac}, and C_{bc}. If z is a neighbour of x, then one more neighbour z' must be on a spoke, because it must have distance two from z and T has at least two spokes of length at least two. This means that two cycles already have two neighbours z and z', and the third cycle already has one neighbour, namely z. Therefore there cannot be two more neighbours of x in T. A similar argument shows that at most two neighbours of x can lie on all the spokes of T all together.

Before we proceed to analyze the two main cases, note that x can also be a vertex in T. It is easy to check that x cannot be z, since it would have three neighbours on the three spokes, which we just excluded. Furthermore, x cannot be one of the branch vertices. Indeed, suppose $x = a$. Then three neighbours y_1, y_2, y_3 of a are placed at distance one from a on $P_{a\bar{z}b}$, P_{az} and $P_{a\bar{z}c}$ respectively, and a neighbour y_4 can only be on $\mathring{P}_{b\bar{z}c}$, the interior of $P_{b\bar{z}c}$. Consider the paths

$$P_{24} = y_2 P_{az} z P_{zbc} y_4$$

and

$$P'_{24} = y_2 P_{az} z P_{zcb} y_4$$

(Figure 12a). Since the subpaths $zP_{zbc}y_4$ and $zP_{zcb}y_4$ cover the cycle C_{bc}, which has length $2k + 1$, the lengths of the paths P_{24} and P'_{24} have different parity. Suppose that P_{24} has odd length. Let

$$P_{34} = y_3 P_{a\bar{z}c} c P_{c\bar{z}b} y_4$$

(Figure 12b) and note that $C_{ac} \oplus C_{bc} = ay_2P_{24}y_4P_{34}y_3a$. Then both P_{24} and P_{34} have length $2k - 1$, because

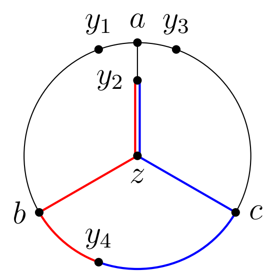
$$\ell(P_{24}) + \ell(P_{34}) = \ell(C_{ac} \oplus C_{bc}) - 2 \overset{(5)}{=} 4k - 2\ell(P_{cz}) \leqslant 4k - 2$$

and together with x each of the paths P_{24} and P_{34} create an odd cycle. The graph obtained from T by replacing the cycle C_{ab} with the cycle $ay_2P_{24}y_4a$ of length $2k + 1$ results in a graph $T' \in \mathcal{T}_k$, with branch vertices a, y_4, and c and center z (Figure 12c). Since the spoke P_{bz} of T is replaced by the larger spoke

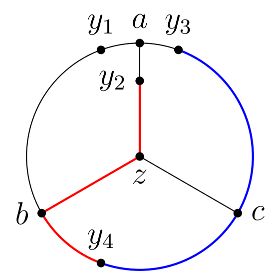
$$P_{y_4 z} = y_4 P_{cbz} z$$

in T', we have that the cycle $C_{T'}$ has shorter length than C_T. This contradicts the choice of $T \subseteq G$.

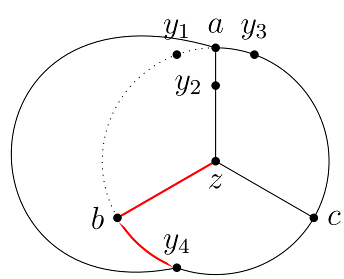
Summarizing the above, from now on we can assume that $x \in V \backslash \{z, a, b, c\}$. Moreover, if $x \in V(T)$, then x lies in one of the cycles C_{ab}, C_{ac}, or C_{bc} and two neighbours of x in T among the four we consider are direct neighbours of x on this cycle. We now study the aforementioned main cases in Claim 18 and Claim 19 below.



(A) P_{24} (red) and P'_{24} (blue).



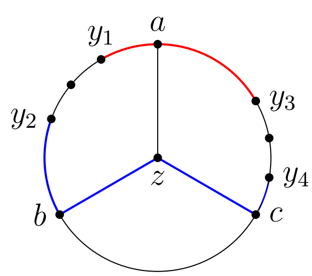
(B) P_{24} (red) and P_{34} (blue).



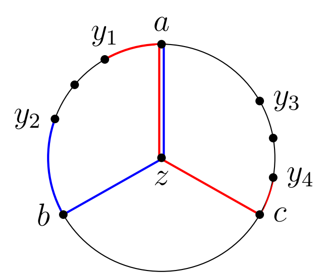
(C) T' with the spoke $P_{y_4 z}$ in red.

FIGURE 12. The vertex x cannot be a branch vertex.

CLAIM 18. *The four neighbours of x in T cannot be contained in only two of the cycles C_{ab}, C_{ac}, and C_{bc}.*



(A) P_{13} (red) and P'_{24} (blue).



(B) P_{14} (red) and P_{23} (blue).

FIGURE 13. The neighbours of x in the case when they are contained in two cycles.

Suppose C_{ab} and C_{ac} contain four neighbours of x. Then the spoke P_{az} shared by both cycles does not contain any neighbour of x. Let

$$y_1, y_2 \in N_G(x) \cap \mathring{P}_{abz}$$

and

$$y_3, y_4 \in N_G(x) \cap \mathring{P}_{acz}$$

where y_1 and y_3 are the neighbours of x coming first on the respective paths (P_{abz} and P_{acz}) starting at a. Consider the paths

$$P_{13} = y_1 P_{zba} a P_{acz} y_3$$

and

$$P_{24} = y_2 P_{abz} z P_{zca} y_4$$

(Figure 13a). Since the neighbours in the same $(2k+1)$-cycle have distance two and $\ell(C_{ab} \oplus C_{ac})$ is even, we infer that P_{13} and P_{24} have the same parity and

$$\ell(P_{13}) + \ell(P_{24}) = 2(2k+1) - 2\ell(P_{az}) - 4 \leqslant 4k - 4\,.$$

If P_{13} and P_{24} have odd length, then one of them must have length at most $2k-3$, thus, together with x, it yields the existence of a short odd cycle. This implies that P_{13} and P_{24} have even length. Consequently, the paths

$$P_{14} = y_1 P_{zba} a P_{az} z P_{zca} y_4$$

and

$$P_{23} = y_2 P_{abz} z P_{az} a P_{acz} y_3$$

(Figure 13b) have odd length and we have that

$$\ell(P_{14}) + \ell(P_{23}) = 2(2k+1) - 4 = 4k - 2\,.$$

Therefore, because of the odd girth of G, they must have both length $2k-1$.

Suppose that one path, say P_{14}, has no endpoints inside the spokes P_{bz} and P_{cz} (here the branch vertices b and c are allowed to be neighbours of x) and

x itself is not a vertex of P_{bz} and P_{cz}. In this case consider the $(2k+1)$-cycle $C_{y_1c} = xy_1P_{14}y_4x$. As a result the graph obtained from T by replacing C_{ac} with C_{y_1c} is a graph $T' \in \mathcal{T}_k$ with $\ell(C_{T'}) < \ell(C_T)$, since the spoke P_{az} is replaced by the longer spoke

$$P_{y_1z} = y_1P_{baz}z\,'$$

This contradicts the choice of T.

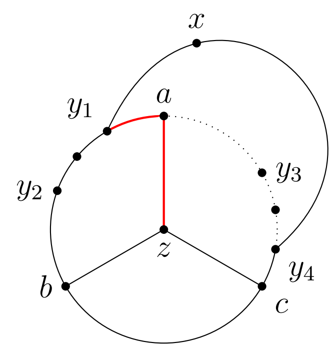


FIGURE 14. T' with the spoke P_{y_1z} in red.

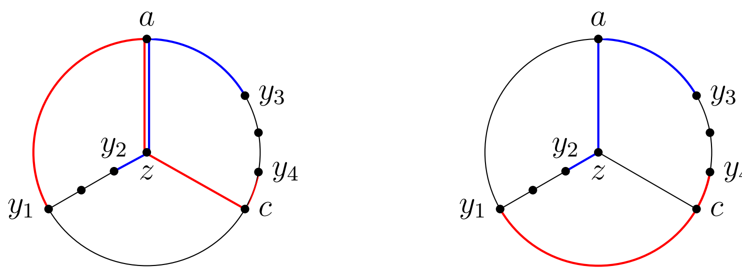
Furthermore, if x would be on one of the spokes P_{bz} or P_{cz}, then it must lie on P_{bz} since otherwise x would lie between y_3 and y_4 and then y_4 would be contained in the interior of P_{cz}, which we excluded here. Since we also excluded that x is a branch vertex, we arrive at the situation that $y_1 = b$ and both y_2 and x are inside P_{bz} (Figure 15a). Hence, the four neighbours of x are also contained in the cycle $C_{ac} \oplus C_{bc}$, which also contains P_{23}. Next we consider the path

$$P'_{14} = y_1P_{b\bar{z}c}cP_{c\bar{z}a}y_4$$

in $C_{ac} \oplus C_{bc}$ (Figure 15b). Since $\ell(C_{ac} \oplus C_{bc})$ is even and $\ell(P_{23})$ is odd we have $\ell(P'_{14}) = \ell(C_{ac} \oplus C_{bc}) - \ell(P_{23}) - 4$ is also odd. Recalling, that $\ell(P_{23}) = 2k-1$ we obtain

$$\ell(P'_{14}) = 2(2k+1) - 2\ell(P_{cz}) - \ell(P_{23}) - 4 = 2k - 1 - 2\ell(P_{cz}) \leqslant 2k-3\,.$$

Hence, we arrive at the contradiction that P'_{14} together with x yields a short odd cycle in G.



(A) P_{14} (red) and P'_{23} (blue). (B) P'_{14} (red) and P_{23} (blue).

FIGURE 15. The neighbours of x in the case when x is contained in a spoke.

Thus both of the paths P_{13} and P_{24} must have an end vertex on one of the spokes P_{bz} and P_{cz}. If both paths have an end vertex on the same spoke, say P_{bz}, then we can repeat the last argument (considering P'_{14}).

Therefore, it must be that both P_{bz} and P_{cz} contain one neighbour of x each, namely y_2 and y_4. Since y_2 and y_4 are in the same $(2k+1)$-cycle C_{bc}, they also have distance two in T. This means that T contains a path $y_1by_2zy_4$ which, together with x, results in cycle $xy_1by_2zy_4x$ of length six. Note that the diagonal $\{y_2, x\}$ is present (Figure 16).

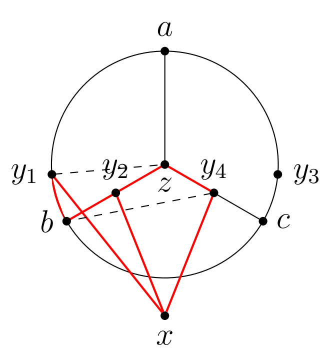


FIGURE 16. $G[V(T) \cup x]$ contains a copy of Φ.

Owing to Lemma 15, two cases may occur. If G contains an induced copy of Φ, then we know that G has minimum degree $\frac{3n}{4k}$ and it contains M_{4k}, hence we are done. If G does not contain Φ as an induced subgraph, then at least one of the other diagonals $\{y_1, z\}$ and $\{b, y_4\}$ must be an edge of G. But both these edges are chords in cycles (C_{ab} and C_{bc}) of length $2k+1$, which contradicts the odd girth assumption on G. This concludes the proof of Claim 18.

Claim 19. *Three neighbours of x in T cannot be contained in only two of the cycles C_{ab}, C_{ac}, and C_{bc}.*

Let $T \subseteq G$ chosen in the beginning of the proof violate the claim. First, we will show that we may assume that T also has the following properties:

(A) all four neighbours of x are contained in C_T,

(B) the two cycles can be chosen in such a way, that the spoke shared by them contains no neighbour of x and has length at least two, and

(C) the cycle containing one neighbour of x has the property that this neighbour is not one of the two branch vertices contained in that cycle.

Owing to Claim 18 we know that any pair of two out of the three cycles C_{ab}, C_{ac}, and C_{bc} contains at most three of the four neighbours of x in T. Consequently, the spokes P_{az}, P_{bz}, and P_{cz} all together can contain at most one neighbour of x. Suppose v is a neighbour of x on the spoke P_{az}. Since we already showed that z cannot be a neighbour of x, property (A) follows, by showing that v is not contained in $\mathring{P}_{az}$, the interior of P_{az}. If $v \neq a$, then the two neighbours y_1 and y_2 of x contained in C_{ab} and C_{ac} would have distance two from v. Consequently, v would have to be a neighbour of a in P_{az} and y_1 and y_2 would also have to be neighbours of a in T (Figure 17). Hence, replacing a by x would give a rise to a subgraph $T' \in \mathcal{T}_k$ of G, where x is a branch vertex. This yields a contradiction as shown before Claim 18 and, hence, property (A) must hold.

Furthermore, if none of the neighbours is a branch vertex, then one cycle would contain two neighbours and the other two would contain one neighbour

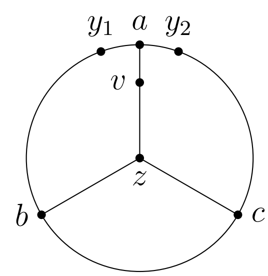


FIGURE 17. The neighbours of x in the case when one of them is contained in $\mathring{P}_{az}$. Note that this configuration yields Figure 12c.

each (Figure 18). Since at least two spokes have length at least two, we can select two cycles containing three neighbours in such a way that properties (B) and (C) hold.

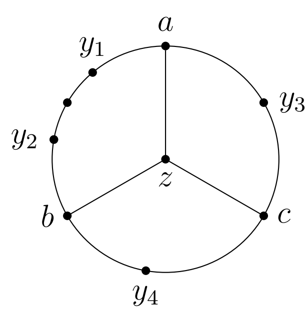


FIGURE 18. The neighbours of x in the case when none of them is a branch vertex.

If one neighbour is a branch vertex, say b, then the two cycles C_{ab} and C_{bc} contain two neighbours and C_{ac} contains one neighbour of x (Figure 19). In particular the spokes P_{az} and P_{cz} contain no neighbour and one of them has length at least two. This implies that we can select one of the cycles C_{ab} or C_{bc} together with C_{ac} such that properties (B) and (C) also hold in this case.

Without loss of generality, we may therefore assume that the cycle C_{ab} contains two neighbours y_1 and $y_2 \in P_{a\bar{z}b} \backslash \{a\}$ (where y_1 is closer to a and y_2 is closer to b), that the cycle C_{ac} contains one neighbour $y_3 \in \mathring{P}_{a\bar{z}c}$, and that the spoke P_{az} has

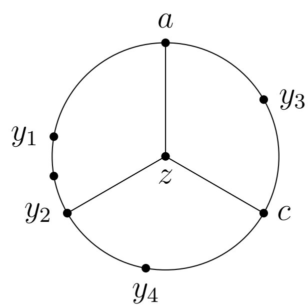


FIGURE 19. The neighbours of x in the case when one of them is a branch vertex.

length at least two. In $C_{ab} \oplus C_{ac}$ we consider the paths

$$P_{13} = y_1 P_{b\bar{z}a} a P_{a\bar{z}c} y_3$$

and

$$P_{23} = y_2 P_{abz} z P_{zca} y_3$$

(Figure 20a). Since P_{az} has length at least two, we have that

$$\ell(P_{13}) + \ell(P_{23}) = 2(2k+1) - 2\ell(P_{az}) - 2 \leqslant 4k - 4\,.$$

Therefore, if P_{13} and P_{23} have odd length, then one has length at most $2k-3$ and, together with x, it yields the existence of a short odd cycle. This implies that P_{13} and P_{23} have even length. Consequently, the paths

$$P'_{13} = y_1 P_{baz} z P_{zca} y_3$$

and

$$P'_{23} = y_2 P_{abz} z P_{zac} y_3$$

(Figure 20b) have odd length, and we have that

$$\ell(P'_{13}) + \ell(P'_{23}) = 2(2k+1) - 2 = 4k\,.$$

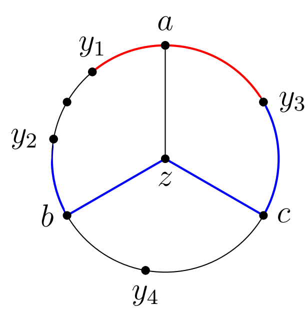
Therefore, one of these paths, say P'_{23} has length $2k-1$. Set

$$C_{23} = xy_2 P'_{23} y_3 x$$

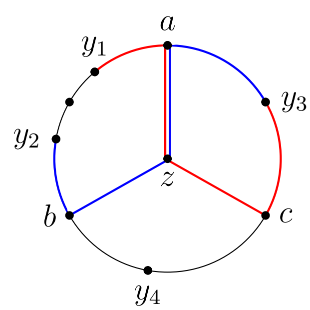
(Figure 20c). The graph T' obtained from T by replacing C_{ab} with C_{23} is a again member of $\mathcal{T}_k$. Since the spoke P_{az} is replaced by the longer spoke

$$P_{y_3z} = y_3 P_{caz} z \, ,$$

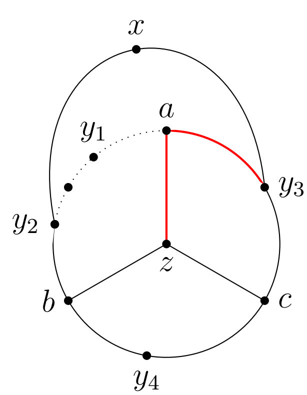
we have $\ell(C_{T'}) < \ell(C_T)$. This contradicts the minimal choice of T, and concludes the proof of Claim 19.



(A) P_{13} (red) and P_{23} (blue).



(B) P'_{13} (red) and P'_{23} (blue).



(C) T' with the spoke P_{y_3z} in red.

FIGURE 20. The vertex x cannot be a branch vertex.

Claim 19 yields that if G does not contain Φ, then every vertex x in G is joined to at most three vertices of T. Recall that every $T \in \mathcal{T}_k$ with $T \subseteq G$ consists of at least $4k$ vertices (see (3)). Similarly, as in the proof of Lemma 15 (see 2), we obtain the following contradiction for graphs $G \in \mathcal{G}^{>}_{n,k}$.

$$3n = 4k\frac{3n}{4k} < \sum_{u \in V(T)} |N_G(u)| = \sum_{x \in V} |N_G(v) \cap V(T)| \leqslant 3|V| = 3n\,.$$

On the other hand, if $G \in \mathcal{G}_{n,k} \smallsetminus \mathcal{G}^{>}_{n,k}$, then each vertex of G must have exactly three neighbours in T.

It is then left to show that in this case G contains M_{4k}. First we show that one spoke of T has length one. Suppose not, then let u be the vertex adjacent to z on a spoke, say P_{az}. Since u must have three neighbours in T, there exists some vertex u' in T such that $\{u, u'\}$ is an edge of G. Since u is contained in both C_{ab} and C_{ac}, the vertex u' must lie on the path $\mathring{P}_{b\bar{z}c}$. Then one of the paths

$$P_u = uP_{azb}P_{b\bar{z}c}u'$$

and

$$P'_u = uP_{azc}P_{c\bar{z}b}u'$$

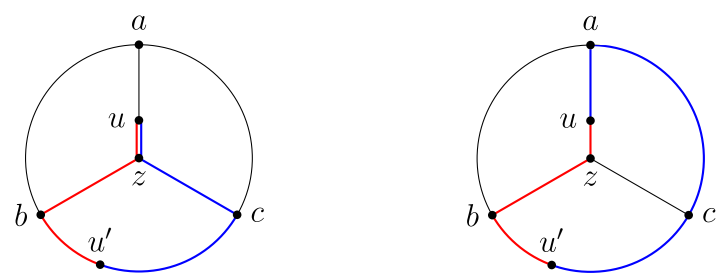
must have even length, and without loss of generality we can assume it is P_u. Then also the path

$$P''_u = uP_{zac}cP_{c\bar{z}b}u'$$

has even length, since its union with P_u is the cycle $C_{ac} \oplus C_{bc}$. Moreover, since the spoke P_{cz} contains at least two edges (because we assumed no spoke has length one), we have $\ell(C_{ac} \oplus C_{bc}) \leqslant 4k - 2$ and consequently one of the even paths P_u and P''_u has length $2k - 2$, thus yielding a short cycle with $\{u, u'\}$. We have thus shown that P_{az} has length one.

By definition of $\mathcal{T}$, the spokes P_{bz} and P_{cz} have length at least two. Let b' be the vertex adjacent to z on P_{bz}. Since each vertex of G has exactly three neighbours in $V(T)$, then there exists some vertex $b'' \in V(T)$ such that $\{b', b''\} \in E$. Since b' is contained in both C_{ab} and C_{bc}, then b'' must lie on the path $\mathring{P}_{a\bar{z}c}$. The paths

$$P_{b'} = b'zP_{zac}b''$$



(A) P_u (red) and P'_u (blue). (B) P_u (red) and P''_u (blue).

FIGURE 21. The spoke P_{az} has length one.

and

$$P'_{b'} = b'zP_{zca}b''$$

have different parity and their lengths sum up to $2k+3$. Hence, b'' must have distance at least $2k$ on the even path. If $P_{b'}$ is even, then b'' is a vertex of the spoke P_{cz}, yielding that $\{b', b''\}$ is a chord of C_{bc}. This implies that $P'_{b'}$ is even and b'' must be the vertex at distance $2k$ on $P'_{b'}$, i.e., the vertex adjacent to a on $P_{a\bar{z}c}$, since the vertex at distance $2k+2$ is already a neighbour of b' (i.e., z). Similarly, denoting by c' the vertex adjacent to z on the spoke P_{cz}, we find that its third neighbour c'' can only be the vertex adjacent to a in the path $P_{a\bar{z}b}$.

Note that the cycle $c''ab''b'zc'c''$ has length six, and the diagonal $\{a, z\}$ is an edge of G. Moreover, the diagonals $\{c'', b'\}$ and $\{b'', c'\}$ are chords of C_{ab} and C_{ac} respectively, hence they are not contained in $E(G)$ since they would close short odd cycles (Figure 22). Therefore, Φ is contained in G and, owing to Lemma 15, M_{4k} is also contained in G. □

§2.4. PROOF OF PROPOSITION 14

In this section we deduce Proposition 14 from Lemmas 15 and 17. Let $G = (V, E)$ be a graph from $\mathcal{G}_{n,k}$. Me may assume that G is not bipartite. Owing to Lemma 15 and Lemma 17, two cases may occur:

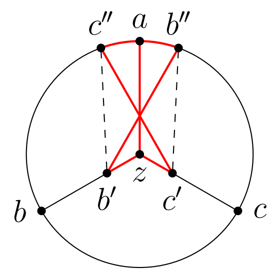


FIGURE 22. G contains a copy of Φ.

- Case 1: G does not contain Φ as an induced subgraph and any graph from $\mathcal{T}_k$ as a (not necessarily induced) subgraph;
- Case 2: G contains a copy of M_{4k} and has minimum degree exactly $\frac{3n}{4k}$.

Case 1. We shall prove that in this case G is a blow-up of C_{2k+1}. First we observe that G contains a cycle of length $2k+1$. Indeed, suppose for a contradiction that for some $\ell > k$ a cycle $C = a_0 \dots a_{2\ell}$ is a smallest odd cycle in G. Since G is edge-maximal, the non-existence of the chord $\{a_0, a_{2k}\}$ is due to the fact that it creates an odd cycle of length at most $2k-1$. Therefore a_0 and a_{2k} are linked by an even path P of length at most $2k-2$ which, together with the path $P' = a_{2k}a_{2k+1}\dots a_{2\ell}a_0$ yields the existence of an odd closed walk and, hence, of an odd cycle, of length at most $2\ell - 1$, which contradicts the minimal choice of C.

Let B be a vertex-maximal blow-up of a $(2k+1)$-cycle contained in G. Let $A_0, \dots, A_{2k}$ be its vertex classes, labeled in such a way that every edge of B is contained in $E_G(A_i, A_{i+1})$ for some $i \in \{0, \dots, 2k\}$. Here and below addition in the indices of A is taken modulo $2k+1$. Clearly, the sets $A_0, \dots, A_{2k}$ are independent sets in G. We will show $B = G$.

Suppose, for a contradiction, that there exists a vertex $x \in V \backslash V(B)$. Owing to the odd girth assumption on G, the vertex x can have neighbours in at most two of the vertex classes of B and if there are two such classes, then they must be of the form A_{i-1} and A_{i+1} for some $i = 0, \dots, 2k$.

First consider the case when x has neighbours in two classes and let $a_{i-1} \in A_{i-1}$ and $a_{i+1} \in A_{i+1}$ be two of such neighbours. In order to show that $x \in A_i$, we shall prove that x is joined to all the vertices from A_{i-1} and to all the vertices from A_{i+1}. Suppose that this is not the case and there is some vertex $b_{i-1} \in A_{i-1}$, which is not a neighbour of x. The argument for the other case, when there is such a vertex in A_{i+1} is identical.

Fix vertices $a_{i-2} \in A_{i-2}$ and $a_i \in A_i$ arbitrarily. This way we fixed a cycle

$$C = xa_{i+1}a_ib_{i-1}a_{i-2}a_{i-1}x$$

of length six in G. Owing to the choice of b_{i-1} the diagonal $\{x, b_{i-1}\}$ is missing in C. Moreover, the diagonal $\{a_{i+1}, a_{i-2}\}$ is also not present, since together with a path from a_{i-2} to a_{i+1} through the vertex classes

$$A_{i-3}, \dots, A_1, A_0, A_{2k-1}, \dots, A_{i+2}$$

it would create an odd cycle of length $2k-1$. On the other hand, since B is a blow-up, the edge $\{a_i, a_{i-1}\}$ is contained in $B \subseteq G$, which is a diagonal in C (Figure 23). Consequently, precisely one diagonal of C is present, which contradicts Lemma 15. Therefore, such a vertex b_{i-1} cannot exist, thus yielding $x \in V(B)$.

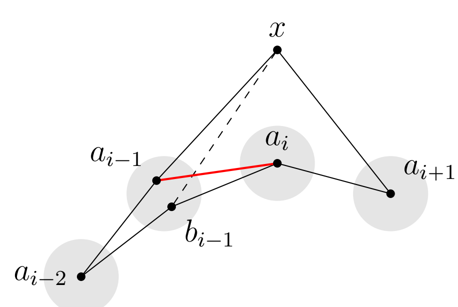


FIGURE 23. G contains an induced copy of Φ.

Now suppose that x has vertices in one class of the blow-up and fix some neighbour a_i of x in A_i. Moreover, for every $j \neq i$ fix a vertex $a_j \in A_j$ arbitrarily. Since B is a blow-up of C_{2k+1} those vertices span a cycle $C = a_0a_1 \dots a_{2k}a_0$ of length $2k+1$. Moreover, since x has no neighbours in $A_{i-2} \cup A_{i+2}$, the vertex x is neither joined to a_{i-2} nor to a_{i+2}.

The edge-maximality of $G \in \mathcal{G}_{n,k}$ implies the existence of paths $P_{a_{i-2}x}$ and $P_{xa_{i+2}}$ in G with an even length of at most $2k-2$. Under all choices of such paths we pick two which minimize the number of edges together with C, i.e., we pick paths $P_{a_{i-2}x}$ and $P_{xa_{i+2}}$ of even length at most $2k-2$ such that

$$E(C) \cup E(P_{a_{i-2}x}) \cup E(P_{xa_{i+2}})$$

has minimum cardinality and we set

$$T = C \cup P_{a_{i-2}x} \cup P_{xa_{i+2}} \subseteq G\,.$$

We shall show that T is a tetrahedron from $\mathcal{T}_k$ with center vertex a_i. Hence, Lemma 17 gives rise to a contradiction and no such vertex x can exist.

Owing to the path $xa_ia_{i-1}a_{i-2}$ of length three the path $P_{a_{i-2}x}$ must have length $2k-2$. Similarly, $a_{i+2}a_{i+1}a_ix$ yields that $P_{xa_{i+2}}$ has length $2k-2$. Moreover, $P_{a_{i-2}x}$ and $P_{a_{i+2}x}$ are disjoint from $\{a_{i-1}, a_i, a_{i+1}\}$. We set

$$C' = a_{i-2}P_{a_{i-2}x}xa_ia_{i-1}a_{i-2}$$

and

$$C'' = a_{i+2}a_{i+1}a_ixP_{xa_{i+2}}a_{i+2}$$

(Figure 24). We just showed that C' and C'' both have length $2k+1$. In order to show that T is a tetrahedron we have to show that the cycles C, C', and C'' intersect pairwise in spokes with center a_i.

Consider the intersection P of the cycles C' and C''. We will show that P is a path with one end vertex being a_i. Indeed every vertex in $a \in V(P) \smallsetminus \{a_i\}$ is a vertex in the paths $P_{a_{i-2}x}$ and $P_{xa_{i+2}}$. Owing to the minimal choice of $P_{a_{i-2}x}$ and $P_{xa_{i+2}}$ it suffices to show that a has the same distance to x in both paths.

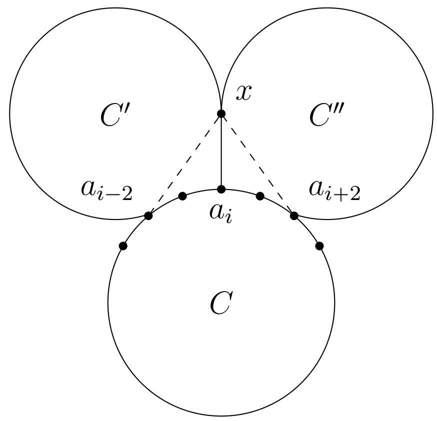


FIGURE 24. The structure arising from the assumption that x has only neighbours in A_i.

Suppose the distances have different parity. This implies that the closed walks

$$aP_{a_{i-2}x}xP_{xa_{i+2}}a$$

and

$$a_ia_{i-1}a_{i-2}P_{a_{i-2}x}aP_{xa_{i+2}}a_{i+2}a_{i+1}a_i$$

have odd length. Since those walks cover the edges (with multiplicity) of C' and C'' with the only exception of $\{x, a_i\}$, the sum of their lengths is $\ell(C') + \ell(C'') - 2$. Hence, one of the closed walks would have an odd length of at most $2k-1$, which yields a contradiction. If the distances between a and x are different, but have the same parity, then replacing the longer path by the shorter one in the corresponding cycle yields an odd cycle of length at most $2k-1$. This again contradicts the assumptions on G and, hence, $P = C' \cap C''$ is indeed a path with end vertex a_i.

In the same way one shows that $C \cap C'$ and $C \cap C''$ are paths with end vertex a_i. Since those two paths contain $a_ia_{i-1}a_{i-2}$ and $a_{i+2}a_{i+1}a_i$, respectively, their length is at least two. Therefore, T is a tetrahedron from $\mathcal{T}_k$ with center a_i and spokes $C' \cap C''$, $C \cap C'$, and $C \cap C''$.

This contradicts the assumption of Case 1, hence there is no vertex $v \in V$ with neighbours in only one vertex class of B. Moreover, since G is edge maximal, it is also connected, therefore there are no vertices with no neighbours in B. This implies $B = G$.

Case 2. We will prove that in this case G is a blow-up of M_{4k}. Recall that G has minimum degree $\frac{3n}{4k}$. First we show that any vertex of G is adjacent to exactly three vertices in every copy of M_{4k} contained in G. Moreover for every vertex $x \in V(G)$, there exists a vertex a_i in M_{4k} having the same neighbours as x in M_{4k} (here and below we take indices modulo $4k$).

CLAIM 20. *For every copy of M_{4k} contained in G and every vertex x of G there exists $i \in \{0, \ldots, 4k-1\}$ such that $N_G(x) \cap V(M_{4k}) = \{a_{i-1}, a_{i+1}, a_{i+2k}\}$.*

PROOF. First note that each diagonal splits M_{4k} into two cycles of length $2k+1$. Since each such cycle can contain at most two neighbours of x, we have $|N_G(x) \cap V(M_{4k})| \leqslant 4$. Suppose x has four neighbours and let a_j be one of these. The diagonal $\{a_j, a_{j+2k}\}$ splits the graph M_{4k} in two $(2k+1)$-cycles, but since a_j is contained in both, one of such cycles contains at least three neighbours of x, thus creating a short odd cycle. This shows that $|N_G(x) \cap V(M_{4k})| \leqslant 3$. Moreover, the minimum degree condition on G yields:

$$3n = 4k\frac{3n}{4k} = \sum_{u \in V(M_{4k})} |N_G(u)| = \sum_{x \in V} |N_G(x) \cap V(M_{4k})| \leqslant 3n$$

which implies $|N_G(x) \cap V(M_{4k})| = 3$ for every $x \in V(G)$.

For the proof of the claim it is left to show that the three neighbours of x in M_{4k} are the same neighbours of some vertex of M_{4k}. Let a_j be one of the neighbours of x and consider the $(2k+1)$-cycles in M_{4k} defined by the diagonal $\{a_j, a_{j+2k}\}$. Each such cycle must contain one of the other neighbours, and both such vertices must have distance two from a_j in each cycle. Hence, one of the candidates must be chosen in $\{a_{j+2}, a_{j+2k-1}\}$ and the other in $\{a_{j-2}, a_{j+2k+1}\}$ (Figure 25), which gives rise to four cases. In three of the cases we find a vertex of M_{4k} with the same neighbourhood of x in M_{4k}. In fact,

- $N_G(x) \cap V(M_{4k}) = \{a_j, a_{j+2k-1}, a_{j+2k+1}\} \Rightarrow i = j + 2k$;

- $N_G(x) \cap V(M_{4k}) = \{a_j, a_{j+2}, a_{j+2k+1}\} \Rightarrow i = j + 1$;
- $N_G(x) \cap V(M_{4k}) = \{a_j, a_{j-2}, a_{j+2k-1}\} \Rightarrow i = j - 1$.

In the remaining case $N_G(x) \cap V(M_{4k}) = \{a_j, a_{j-2}, a_{j+2}\}$ there is no suitable i, however, the diagonal $\{a_{j+2}, a_{j+2k+2}\}$ defines a $(2k + 1)$-cycle that contains all the three vertices, which yields a contradiction and the claim follows. □

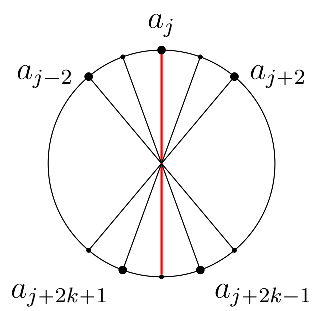


FIGURE 25. The possible neighbours of x in M_{4k}.

We now consider the largest blow-up B of M_{4k} in G with vertex classes denoted by $A_0, \ldots, A_{4k-1}$, where each vertex class A_i is completely adjacent to the classes A_{i-1}, A_{i+1}, and A_{i+2k}, and show that $B = G$.

First note that each vertex class is an independent set, otherwise triangles would be contained in G. Consider a vertex $x \in V(G)$. If x has neighbours in four (or more) vertex classes, then there exists a copy of M_{4k} in which x has four (or more) neighbours, which is impossible by Claim 20. For the same reason, x cannot have neighbours in at most two classes. Moreover, if x has a neighbour in the vertex class A_j, then it must have all vertices of A_j as neighbours, since otherwise we could take a copy of M_{4k} containing a vertex of A_j that is not adjacent to x and in such a copy of M_{4k} the vertex x would have less than three neighbours. Summarizing, Claim 20 implies that for any vertex x there exists $i \in \{0, \ldots, 4k - 1\}$ such that x has the whole vertex classes A_{i-1}, A_{i+1}, and A_{i+2k} as neighbours, hence $x \in V(B)$ and consequently $B = G$.

§2.5. Open questions

It would be interesting to study the situation when we further relax the degree condition in Theorem 13. It seems plausible that if G has odd girth at least $2k+1$ and $\delta(G) \geqslant (\frac{3}{4k}-\varepsilon)n$ for sufficiently small $\varepsilon > 0$, then the graph G is homomorphic to M_{4k}. In fact, this seems to be true until $\delta(G) > \frac{4n}{6k-1}$. At this point blow-ups of the $(6k-1)$-cycle with all chords connecting two vertices of distance $2k$ in the cycle added, would show that this is best possible. For $k = 2$ such a result was proved by Chen, Jin, and Koh [22], for $k = 3$ it was obtained by Brandt and Ribe-Baumann [18] and recently Ebsen et al.[1] extended these results to $k \geqslant 2$.

More generally, for $\ell \geqslant 2$ and $k \geqslant 3$ let $F_{\ell,k}$ be the graph obtained from a cycle of length $(2k-1)(\ell-1)+2$ by adding all chords which connect vertices with distance of the form $j(2k-1)+1$ in the cycle for some $j = 1, \dots, \lfloor(\ell-1)/2\rfloor$. Note that $F_{2,k} = C_{2k+1}$ and $F_{3,k} = M_{4k}$. For every $\ell \geqslant 2$ the graph $F_{\ell,k}$ is ℓ-regular, has odd girth $2k+1$, and it has chromatic number three. Moreover, $F_{\ell+1,k}$ is not homomorphic to $F_{\ell,k}$, but contains it as a subgraph.

A possible generalization of the known results would be the following: *if an n-vertex graph G has odd girth at least $2k+1$ and minimum degree bigger than $\frac{\ell n}{(2k-1)(\ell-1)+2}$, then it is homomorphic to $F_{\ell-1,k}$.* However, this is known to be false for $k = 2$ and $\ell > 10$, since such a graph G may contain a copy of the Grötzsch graph which (due to having chromatic number four) is not homomorphically embeddable into any $F_{\ell,2}$. However, in some sense this is the only exception for that statement. In fact, with the additional condition $\chi(G) \leqslant 3$ the statement is known to be true for $k = 2$ (see, e.g., [22]). To our knowledge it is not known if a similar phenomenon happens for $k > 2$ and it would be interesting to study this further.

[1] Personal communication

The discussion above motivates the following question, which asks for an extension of the result of Łuczak for triangle-free graphs from [43]. Note that for fixed k the degree of $F_{\ell,k}$ divided by its number of vertices tends to $\frac{1}{2k-1}$ as $\ell \to \infty$.

QUESTION 21. *Is it true that every n-vertex graph with odd girth at least $2k+1$ and minimum degree at least $(\frac{1}{2k-1}+\varepsilon)n$ can be mapped homomorphically into a graph H which also has odd girth at least $2k+1$ and $V(H)$ is bounded by a constant $C = C(\varepsilon)$ independent of n?*

A different formulation of this problem is related to chromatic thresholds. A question of Andrásfai [8] started the investigation on the minimum degree condition that forces bounded chromatic number in F-free graphs (where F is a graph itself). More precisely, the *chromatic threshold* of a given graph F is defined as

$$\begin{aligned}\delta_\chi(F) = \inf\{\alpha \in [0,1] : \exists k \in \mathbb{N} \text{ such that } \chi(G) \leqslant k \\ \forall G \text{ with } F \nsubseteq G \text{ and } \delta(G) \geqslant \alpha|V(G)|\}\,.\end{aligned}$$

Some special cases were studied in [28, 32, 47, 52]. In [44] Łuczak and Thomassé proved that $\delta_\chi(F) \notin (0, \frac{1}{3})$ for all graphs F, and finally Allen et al. [3] settled the question by showing that for every graph F we have

$$\delta_\chi(F) \in \left\{\frac{r-3}{r-2}, \frac{2r-5}{2r-3}, \frac{r-2}{r-1}\right\}.$$

One can ask for a graph F, what is the minimum degree that allows an F-free graph G to be homomorphic to a smaller graph H which is also F-free. This leads to the definition of *homomorphism threshold*:

$$\begin{aligned}\delta_{\mathrm{hom}}(F) = \inf\{\alpha \in [0,1] : \exists k \in \mathbb{N} \text{ such that} \\ \forall G \text{ with } F \nsubseteq G \text{ and } \delta(G) \geqslant \alpha|V(G)| \\ \exists H \text{ with } F \nsubseteq H,\ |V(H)| \leqslant k, \text{ and } G \xrightarrow{\mathrm{hom}} H\}\,.\end{aligned}$$

The aforementioned result of Łuczak [43] solves this problem for $F = K_3$ and it was extended to cliques by Goddard and Lyle [32], who showed $\delta_\chi(K_r) = \delta_{\mathrm{hom}}(K_r)$ for every $r \geqslant 3$.

The definition of homomorphism thresholds naturally extends to families of graphs, i.e., given a family $\mathcal{F}$, one considers graphs G and H that do not contain any member of $\mathcal{F}$ as subgraphs. If we denote by $\mathcal{C}_{2\ell+1}$ the family $\{C_3, C_5, \ldots, C_{2\ell+1}\}$, Question 21 is equivalent to the following one.

Question 22. *Is it true that* $\delta_{hom}(\mathcal{C}_{2\ell+1}) = \frac{1}{2\ell+1}$ *for* $\ell \geqslant 2$*?*

We conjecture that the answer to this question is positive.

CHAPTER 3

Packing minor-closed families of graphs

The material presented in this chapter is widely based on the paper *Packing minor-closed families of graphs* [45], joint work with Vojtěch Rödl and Mathias Schacht.

§3.1. The Tree Packing Conjecture

Given graphs H and F, an *F-packing* of H is a collection of edge-disjoint subgraphs of H that are isomorphic to F. This definition naturally extends to sequences of graphs. In particular, we say that $\mathcal{F} = (F_1, \dots, F_t)$ *packs* into H if there exist edge-disjoint subgraphs $H_1, \dots, H_t \subseteq H$ with H_i isomorphic to F_i for every $i \in [t]$. Gyárfás' tree packing conjecture [35] initiated a lot of research and asserts the following for the case where H is a complete graph and $\mathcal{F}$ is a sequence of trees.

Conjecture 23. *Any sequence of trees $(T_1, \dots, T_n)$ with $v(T_i) = i$ for $i \in [n]$ packs into K_n.*

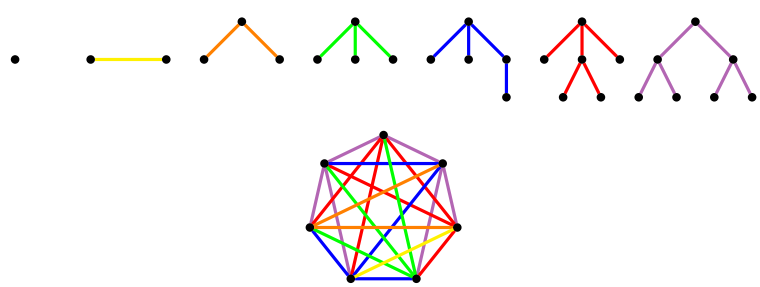


Figure 26. A packing of $(T_1, \dots, T_7)$ into K_7.

The difficulty of this conjecture lies in the fact that it asks for a perfect packing, i.e., a packing where all the edges of K_n are used, since each tree has $e(T_i) = i-1$ edges and hence $\sum_{i\in[n]} e(T_i) = \binom{n}{2}$. Although some special cases were proven (see, e.g., [**39**] and the references in [**16**]), this conjecture is still widely open.

Recently, Böttcher, Hladký, Piguet, and Taraz [**16**] showed that a restricted approximate version holds. More precisely, they considered a host graph with slightly more than n vertices and trees with bounded maximum degree, while relaxing the assumption on the number of vertices of each tree.

Theorem 24 (Böttcher, Hladký, Piguet, and Taraz). *For any $\varepsilon > 0$ and any $\Delta \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$ such that for any $n \geqslant n_0$ the following holds for every $t \in \mathbb{N}$. If $\mathcal{T} = (T_1, \dots, T_t)$ is a sequence of trees satisfying*

(a) $\Delta(T_i) \leqslant \Delta$ and $v(T_i) \leqslant n$ for every $i \in [t]$, and

(b) $\sum_{i=1}^{t} e(T_i) \leqslant \binom{n}{2}$,

then $\mathcal{T}$ packs into $K_{(1+\varepsilon)n}$.

In case $(1+\varepsilon)n$ is not an integer, we should talk about $K_{\lfloor(1+\varepsilon)n\rfloor}$. However, since we provide asymptotical results, we will omit floors and ceilings here. The proof of Theorem 24 is based on a randomized embedding strategy, which draws some similarities to the semirandom nibble method (see e.g. [**6**]). Inspired by the result in [**16**], we obtained a somewhat simpler proof of Theorem 24, which extends from sequences of trees to sequences of graphs contained in any non-trivial minor-closed class.

Theorem 25. *For any $\varepsilon > 0$, $\Delta \in \mathbb{N}$, and any non-trivial minor-closed family $\mathcal{G}$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ the following holds for every integer $t \in \mathbb{N}$. If $\mathcal{F} = (F_1, \dots, F_t)$ is a sequence of graphs from $\mathcal{G}$ satisfying*

(a) $\Delta(F_i) \leqslant \Delta$ and $v(F_i) \leqslant n$ for every $i \in [t]$, and

(b) $\sum_{i=1}^{t} e(F_i) \leqslant \binom{n}{2}$,

then $\mathcal{F}$ packs into $K_{(1+\varepsilon)n}$.

In the following we will consider graphs that do not contain isolated vertices. In fact, such vertices can easily be embedded after larger components just by picking any vertex of $K_{(1+\varepsilon)n}$ that has not been used before for the same graph. In the proof we split the graphs F_i into smaller pieces by removing a *small* separator, i.e., a small subset of the vertex set. We discuss these concepts and a generalisation of Theorem 25 in the next section.

§3.2. Main technical result

We shall establish a generalisation of Theorem 25 for graphs with small separators (see Theorem 29 below). In fact, the Separator Theorem of Alon, Seymour, and Thomas [5] will provide the connection between Theorem 25 and slightly more general Theorem 29.

THEOREM 26 (Alon, Seymour, and Thomas). *For every non-trivial minor-closed family of graphs $\mathcal{G}$ there exists $c_{\mathcal{G}} > 0$ such that for every graph $G \in \mathcal{G}$ there exists $U \subseteq V(G)$ with $|U| \leqslant c_{\mathcal{G}}\sqrt{n}$ such that every component of $G - U$ has order at most $n/2$.*

The graphs we consider in our main result satisfy the following property.

DEFINITION 27. *Given $\delta > 0$ and $s \in \mathbb{N}$, a (δ, s)-separation of a graph $G = (V, E)$ with minimum degree $\delta(G) \geqslant 1$ is a pair $(U, \mathcal{C})$ satisfying*

(i) $U \subseteq V$, $|U| \leqslant \delta v(G)$ and

(ii) $\mathcal{C} = G[V \smallsetminus U]$, i.e., the subgraph of G induced on $V \smallsetminus U$, has the property that each component of $\mathcal{C}$ has order at least two and at most s.

We refer to U as the separator*, and to $\mathcal{C}$ as the* component graph *of G.*

Note that, for technical reasons that will become clear later (see equation (18)), in *(ii)*. we only allow components of size at least two. Although the removal of a separator could induce components of size one, such a separator U^0 of G may yield at most $\Delta|U^0|$ components of size one, because in our setting we only deal with graphs G of bounded degree $\Delta(G) \leqslant \Delta$. This allows us to add these "few" vertices to U^0 without enlarging it too much, and ensure that the resulting set U complies with the definition above.

Definition 28. *A family $\mathcal{G}$ of graphs with minimum degree at least one is (δ, s)-separable if every $G \in \mathcal{G}$ admits a (δ, s)-separation.*

We will deduce Theorem 25 from the following result, in which the condition of $\mathcal{G}$ being minor-closed is replaced by the more general property of being (δ, s)-separable.

Theorem 29. *For any $\varepsilon > 0$ and $\Delta \in \mathbb{N}$ there exists $\delta > 0$ such that for every $s \in \mathbb{N}$ and any (δ, s)-separable family $\mathcal{G}$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ the following holds. If $\mathcal{F} = (F_1, \ldots, F_t)$ is a sequence of graphs from $\mathcal{G}$ satisfying*

(a) $\Delta(F_i) \leqslant \Delta$ and $v(F_i) \leqslant n$ for every $i \in [t]$, and

(b) $\sum_{i=1}^{t} e(F_i) \leqslant \binom{n}{2}$,

then $\mathcal{F}$ packs into $K_{(1+\varepsilon)n}$.

As mentioned above, Theorem 25 easily follows from Theorem 29. First we show that for any non-trivial minor-closed family $\mathcal{G}$ and any $\delta > 0$ there is some s such that $\mathcal{G}$ is (δ, s)-separable. Then we use this fact to deduce Theorem 25.

Given a graph $G \in \mathcal{G}$ of order n with minimum degree $\delta(G) \geqslant 1$ and maximum degree $\Delta(G) \leqslant \Delta$, we apply Theorem 26 to all components of G that have some size r_0 with $\frac{n}{2} \leqslant r_0 \leqslant n$. Since there are at most two such components, at most two applications of Theorem 26 lead to a separator of size at most $2c_{\mathcal{G}}n^{1/2}$ and a set of components all of which have order less than $n/2$. We then apply Theorem 26 to all components of G that have some size r_1 with $\frac{n}{4} \leqslant r_1 < \frac{n}{2}$ and obtain another separator of size at most $4c_{\mathcal{G}}\left(\frac{n}{2}\right)^{1/2}$. At this point all components have order less than $n/4$. Again, we apply Theorem 26 to all components of some size

r_2 with $\frac{n}{8} \leqslant r_2 < \frac{n}{4}$, add at most $8c_{\mathcal{G}}\left(\frac{n}{4}\right)^{1/2}$ more vertices to the separator, and so on. After $i > 0$ such iterations we obtain a separator $U^0 \subseteq V(G)$ such that

$$|U^0| \leqslant 2c_{\mathcal{G}}n^{1/2}+4c_{\mathcal{G}}\left(\frac{n}{2}\right)^{1/2}+\cdots+2^i c_{\mathcal{G}}\left(\frac{n}{2^{i-1}}\right)^{1/2} < 2c_{\mathcal{G}}n^{1/2}\cdot\frac{\sqrt{2}^{\,i}-1}{\sqrt{2}-1} < 6c_{\mathcal{G}}n^{1/2}\,2^{i/2}$$

and each component of $G - U^0$ has order at most $n/2^i$. For given $\delta > 0$ we can apply this with

$$i = \left\lfloor 2\log_2\left(\frac{\delta n^{1/2}}{6c_{\mathcal{G}}(\Delta+1)}\right)\right\rfloor$$

and obtain a separator U_0 of order at most $\delta n/(\Delta+1)$, and a set of components all of which have order at most $72c_{\mathcal{G}}^2(\Delta+1)^2/\delta^2$. Note that some of the components in $G - U^0$ may have size one. However, owing to the maximum degree of G there are at most $\Delta|U^0|$ such components. By defining U as the separator of size at most δn obtained from U^0 by adding all these degenerate components of order one, we have shown that the non-trivial minor-closed family $\mathcal{G}$ is (δ, s)-separable with $s = 72c_{\mathcal{G}}^2(\Delta+1)^2/\delta^2$. Applying Theorem 29 with this s yields Theorem 25.

The rest of this paper is devoted to the proof of Theorem 29. In Section 3.3 we introduce some definitions and state two technical lemmas that are used in the proof of the theorem, which is given at the end of the section. Resolvable and almost resolvable decompositions, which we will use to construct our packing, are introduced in Section 3.4. Finally, the two technical lemmas, Lemma 32 and Lemma 33, are proved in Sections 3.5 and 3.6, respectively.

§3.3. Proof of the main result

The following notation will be convenient.

Definition 30. *Let $\mathcal{G}$ be a family of graphs. A t-tuple of graphs $\mathcal{F} = (F_1, \ldots, F_t)$ with $F_i \in \mathcal{G}$ and $i \in [t]$ is called a $(\mathcal{G}, n, \Delta)$-sequence if*

- *(a) $\Delta(F_i) \leqslant \Delta$ and $v(F_i) \leqslant n$ for every $i \in [t]$, and*
- *(b) $\sum_{i=1}^t e(F_i) \leqslant \binom{n}{2}$.*

We will consider $(\mathcal{G}, n, \Delta)$-sequences with the following additional properties:

- $\mathcal{G}$ will be a (δ, s)-separable family and
- each graph F_i will be associated with a fixed (δ, s)-separation $(U_i, \mathcal{C}_i)$.

Note that, since we are only considering graphs F_i that do not contain isolated vertices, we have $v(F_i) \leqslant 2e(F_i)$ and, hence,

$$\sum_{i=1}^{t} |U_i| \leqslant \sum_{i=1}^{t} \delta v(F_i) \leqslant \delta \sum_{i=1}^{t} 2e(F_i) \overset{(b)}{\leqslant} 2\delta \binom{n}{2} < \delta n^2 .$$

For a simpler notation we will often suppress the dependence on U_i when we refer to a $(\mathcal{G}, n, \Delta)$-sequence $(F_1, \ldots, F_t)$, since the separator U_i will be always clear from the context. In a component C from $\mathcal{C}_i$ we distinguish the set of vertices that are connected to the separator U_i and refer to this set as the *boundary* ∂C of C

$$\partial C = V(C) \cap N(U_i),$$

where as usual $N(U_i)$ denotes the union of the neighbours in F_i of the vertices in U_i.

Moreover, for a component graph $\mathcal{C}_i$ we consider the union of the boundary sets of its components and set

$$\partial \mathcal{C}_i = \bigcup \{\partial C \colon\ C \text{ component in } \mathcal{C}_i\}.$$

Note that

$$|\partial \mathcal{C}_i| \leqslant \sum_{u \in U_i} d(u) \leqslant |U_i| \Delta \leqslant \delta n \Delta . \tag{6}$$

For the proof of Theorem 29 we shall pack a given $(\mathcal{G}, n, \Delta)$-sequence into $K_{(1+\varepsilon)n}$. The vertices of the host graph $K_{(1+\varepsilon)n}$ will be split into a large part X of order $(1 + \xi)n$ for some carefully chosen $\xi = \xi(\varepsilon, \Delta) > 0$, and a small part $Y = V \setminus X$. We will pack the graphs $\{\mathcal{C}_i\}_{i\in[t]}$ into the clique spanned on X and the sets $\{U_i\}_{i\in[t]}$ into Y. For this, we shall ensure that the vertices representing the boundary $\partial \mathcal{C}_i$ will be appropriately connected to the vertices representing the separator U_i. Having this in mind we will make sure that each vertex of X

will only host a few boundary vertices. In fact, since every edge of the complete bipartite graph induced by X and Y can be used only once in the packing, each vertex $x \in X$ can be used at most $|Y|$ times as boundary vertex for the packing of the sequence $\{\mathcal{C}_i\}_{i\in[t]}$. This leads to the following definition.

Definition 31. *For every $i \in [t]$, let $F_i = (V_i, E_i)$ be graphs with separators $U_i, \subseteq V_i$ and component graphs $\mathcal{C}_i = F_i[V_i \setminus U_i]$. For a family of injective maps $\boldsymbol{f} = \{f_i\}_{i\in[t]}$ with $f_i \colon V(\mathcal{C}_i) \to X$ and for $x \in X$ we define the* boundary degree *of x with respect to $\boldsymbol{f}$ by*

$$d^{\partial}_{\boldsymbol{f}}(x) = \left|\{i \in [t] \colon f_i^{-1}(x) \in \partial\mathcal{C}_i\}\right| .$$

*We call such a family of maps b-*balanced *for some $b \in \mathbb{R}$ if $d^{\partial}_{\boldsymbol{f}}(x) \leqslant b$ for every $x \in X$.*

Theorem 29 follows from Lemma 32 and Lemma 33 below. Lemma 32 yields a balanced packing of the component graphs $\{\mathcal{C}_i\}_{i\in[t]}$ into the clique spanned by X with $|X| \leqslant (1+\xi)n$.

Lemma 32. *For any $\xi > 0$ and $\Delta \in \mathbb{N}$ there exists $\delta > 0$ such that for every $s \in \mathbb{N}$ and any (δ, s)-separable family $\mathcal{G}$ there exists $n_0 \in \mathbb{N}$ such that if $\mathcal{F}$ is a $(\mathcal{G}, n, \Delta)$-sequence with $n \geqslant n_0$, then there exists a (ξn)-balanced packing of the component graphs $\{\mathcal{C}_i\}_{i\in[t]}$ of all members of $\mathcal{F}$ into $K_{(1+\xi)n}$.*

Once we have a balanced packing of $\{\mathcal{C}_i\}_{i\in[t]}$ into $K_{(1+\xi)n}$, the next lemma allows us to extend it to a packing of $\mathcal{F} = (F_1, \ldots, F_t)$ into a slightly larger clique of size $(1+\varepsilon)n$.

Lemma 33. *For any $\varepsilon > 0$ and $\Delta \in \mathbb{N}$, there exist $\xi > 0$ and $\delta > 0$ such that for every s and any (δ, s)-separable family $\mathcal{G}$ there exists n_0 such that for any $n \geqslant n_0$ the following holds. Suppose there exists a (ξn)-balanced packing of the component graphs $\{\mathcal{C}_i\}_{i\in[t]}$ associated with a $(\mathcal{G}, n, \Delta)$-sequence $\mathcal{F}$ into $K_{(1+\xi)n}$. Then there exists a packing of $\mathcal{F}$ into $K_{(1+\varepsilon)n}$.*

We postpone the proofs of Lemma 32 and Lemma 33 to Section 3.5 and Section 3.6. Here we describe the proof of our main Theorem based on these two lemmas.

3.3.1. Proof of Theorem 29. We will first fix all involved constants. Note that Theorem 29 and Lemma 33 have a similar quantification. Hence, for the proof of Theorem 29, we may apply Lemma 33 with ε and Δ from Theorem 29 and obtain ξ and δ'. Then Lemma 32 applied with ξ and Δ yields a constant δ''. For Theorem 29 we set $\delta = \min\{\delta', \delta''\}$. After displaying δ for Theorem 29 we are given some $s \in \mathbb{N}$ and a (δ, s)-separable family $\mathcal{G}$.

With constants chosen as above, we can apply Lemma 32 for a $(\mathcal{G}, n, \Delta)$-sequence $\mathcal{F}$ which then asserts that the assumptions of Lemma 33 are fulfilled. Finally, the conclusion of Lemma 33 yields Theorem 29. □

§3.4. Resolvable and almost resolvable decompositions

In the proof of Lemma 32 we will construct a packing of $\{\mathcal{C}_i\}_{i\in[t]}$ into K_X. The components of each graph will be grouped by isomorphism types and those from the same type will be packed into complete subgraphs of K_X. In this process we have to obey two main constraints. First, we want to use the space efficiently. For that it would be useful to have a packing of the components that covers (almost) all edges of the host graphs. Second, the components of a given graph $\mathcal{C}_i$ must be packed vertex-disjointly. Hence, we would like the host graph to contain disjoint copies of a given isomorphism type that cover (almost) all its vertices. This leads to resolvable decompositions and the concepts discussed below.

Given graphs H and F, a resolvable F-decomposition of H is an edge disjoint partition of H into F-factors. Note that for the existence of a perfect F-packing of H it is required that $e(F)$ divides $e(H)$, and the existence of an F-factor requires that $v(F)$ divides $v(H)$, hence both conditions are necessary for the existence of a resolvable F-decomposition. Let $H = K_n$. In the special case when F is also a clique, Ray-Chaudhuri and Wilson [49] showed that these necessary conditions are actually sufficient.

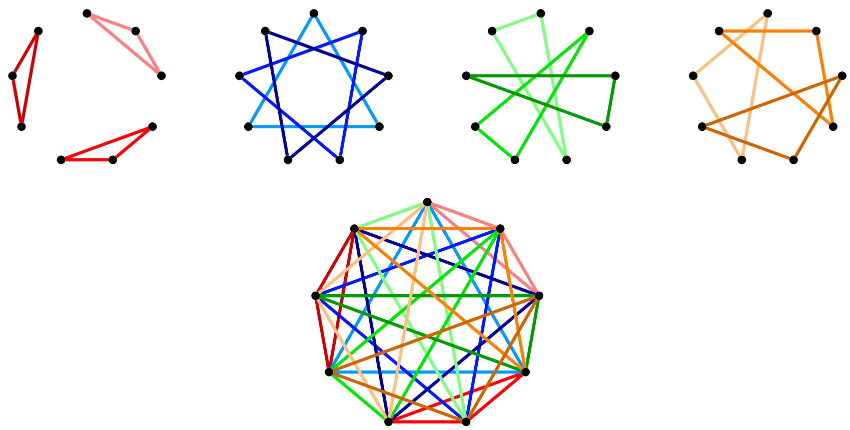


FIGURE 27. A resolvable K_3-decomposition of K_9.

THEOREM 34 (Ray-Chaudhury and Wilson). *For every $m \geqslant 2$ there exists n_0 such that if $n \geqslant n_0$ and $n \equiv m \pmod{m(m-1)}$, then K_n admits a resolvable K_m-decomposition.*

Note for future reference that a K_m-factor of K_n contains $\frac{n}{m}$ vertex disjoint cliques of order m, and a resolvable K_m-decomposition of K_n is a collection of

$$\frac{\binom{n}{2}}{\binom{m}{2}}\frac{m}{n} = \frac{n-1}{m-1}$$

edge disjoint K_m-factors.

For a general graph F, some additional conditions must be satisfied for the existence of an F-decomposition. Let $gcd(F)$ denote the greatest common divisor of the degree sequence of F. If an F-decomposition of K_n exists, then we have that $gcd(F)$ divides $n-1$, in addition to $e(F)$ divides $\binom{n}{2}$. In fact, Wilson [54] showed that for n sufficiently large these two necessary conditions are also sufficient.

THEOREM 35 (Wilson). *For every graph F there exists n_0 such that if $n \geqslant n_0$, $e(F)$ divides $\binom{n}{2}$, and $gcd(F)$ divides $n-1$, then K_n admits an F-decomposition.*

For general F, resolvable decompositions do not necessarily exists (for example it is easy to see that there is no n for which resolvable $K_{1,3}$-decompositions of K_n exist). Therefore, instead of F-factors, we consider F-matchings, i.e., sets of vertex disjoint copies of F.

Definition 36. *An* (F,η)-factorization *of* K_ℓ *is a collection of* F*-matchings of* K_ℓ *such that*

(1) each matching has size at least $(1-\eta)\frac{\ell}{v(F)}$*, and*
(2) these matchings together cover all but at most $\eta\binom{\ell}{2}$ *edges of* K_ℓ*.*

From these two properties we deduce that the number t of F-matchings in an (F,η)-factorization satisfies

$$(1-\eta)\frac{(\ell-1)v(F)}{2e(F)} \leqslant t \leqslant \frac{(\ell-1)v(F)}{2e(F)}.$$

Also note that any $(F,0)$-factorization of K_ℓ is a resolvable F-decomposition of K_ℓ. We will then use the following approximate result, which can be deduced from [29] and [48] (see also [7]).

Theorem 37. *For every* F *and* $\eta>0$ *there exists* ℓ_0 *such that for every* $\ell \geqslant \ell_0$ *there exists an* (F,η)*-factorization of* K_ℓ*.*

§3.5. Packing the components

The crucial part in the proof of Theorem 29 is Lemma 32, which we are going to prove in this section. In Lemma 32 we are given a $(\mathcal{G},n,\Delta)$-sequence $(F_1,\dots,F_t)$ of graphs from a (δ,s)-separable family $\mathcal{G}$ with fixed separations $(U_i,\mathcal{C}_i)$ associated with each F_i. Our goal will be to construct a (ξn)-balanced packing of the component graphs $\{\mathcal{C}_i\}_{i\in[t]}$ into K_N, with $N=(1+\xi)n$.

The packing of $\{\mathcal{C}_i\}_{i\in[t]}$ will make use of a resolvable K_m-decomposition of K_N (actually we will use a somewhat more complicated auxiliary structure which we will describe in Section 3.5.1) and will be realized in two steps: the *assignment* phase and the *balancing* phase.

- In the assignment phase we consider a K_m-decomposition of K_N and then describe which components of each $\mathcal{C}_i$ are assigned to which copies of K_m.
- In the balancing phase we ensure that the mapping from components of each $\mathcal{C}_i$ into copies of K_m from K_N will form a (ξn)-balanced packing as promised in Lemma 32.

Below we outline the main ideas of these two steps. We start with the assignment phase first. The balancing phase will be discussed in Section 3.5.3.

3.5.1. Outline of the assignment phase. The purpose of the assignment phase is to produce a "preliminary packing" of each $\mathcal{C}_i$, $i = 1, \ldots, t$ into some K_m-factor. We recall that each component graph $\mathcal{C}_i$ consists of several components each with at most s vertices and maximum degree at most Δ. Moreover, in each component C we distinguish the set ∂C of vertices that are connected to the separator U_i.

We define an *isomorphism type* S as a pair (R, B) where R is a graph on at most s labeled vertices and maximum degree at most Δ, and B is a subset of the vertices of R. Let $\mathcal{S} = (S_1, \ldots, S_\sigma)$ be the enumeration of all isomorphism types $S_j = (R_j, B_j)$, such that

$$\frac{e(R_1)}{v(R_1)} \geqslant \cdots \geqslant \frac{e(R_\sigma)}{v(R_\sigma)}. \tag{7}$$

The definition of $\mathcal{S}$ yields

$$\sigma \leqslant 2^{\binom{s}{2}} \cdot 2^s \leqslant 2^{s^2}. \tag{8}$$

For every component C of $\mathcal{C}_i$ there exists an isomorphism type $S_j = (R_j, B_j) \in \mathcal{S}$ such that there exists a graph isomorphism $\varphi \colon V(C) \to V(R_j)$ with the additional property that $\varphi(\partial C) = B_j$. Therefore, we can describe the structure of a component graph $\mathcal{C}_i$ as a disjoint union

$$\mathcal{C}_i = \bigcup_{S \in \mathcal{S}} \nu_i(S) \cdot S$$

where $\nu_i(S)$ denotes the number of components isomorphic to S contained in $\mathcal{C}_i$. In the rest of the paper we will simplify the notation and refer to S as a graph.

The assignment procedure makes use of further decomposition layers. In fact, for each copy of K_m appearing in the resolvable decomposition of K_N we consider a resolvable K_ℓ-decomposition of such a copy of K_m. Each resolution class consisting of $\frac{m}{\ell}$ disjoint copies of K_ℓ will be reserved for some isomorphism class S and the copies of S coming from various $\mathcal{C}_i$ will be then packed into each such K_ℓ. Since we consider K_m-decomposition of K_N, K_ℓ-decomposition of K_m, and S-decomposition of K_ℓ for each $S \in \mathcal{S}$, we will refer to such structure as *three layer decomposition* and motivate its use below.

3.5.2. The three layer decomposition. We begin our discussion with the simpler case when all components in all the component graphs $\mathcal{C}_i$ are isomorphic to a given graph S and argue why even in this simpler case at least two layers are required. Then we look at the general case, where the component graphs consist of more different isomorphism types, and explain the use of three layers.

3.5.2.1. *One layer.* In the case where all components in $\{\mathcal{C}_i\}_{i\in[t]}$ are isomorphic to a single graph S, a straightforward way to pack $\{\mathcal{C}_i\}_{i\in[t]}$ into K_N would be the following. Suppose there exists a resolvable S-decomposition of K_N. Then, by assigning the components of a graph $\mathcal{C}_i$ to copies of S from the same S-factor, we ensure that the components within each component graph are packed vertex-disjointly.

With this approach, however, we might end up not covering many edges of K_N (and consequently not being able to find a packing of the graphs $\mathcal{C}_i$). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be component graphs with strictly more than $N/2$ vertices. Once we assign the components of $\mathcal{C}_1$ to an S-factor of K_N, we cannot use the other copies of S in the same S-factor to accomodate the components of $\mathcal{C}_2$. In fact, at least one component of $\mathcal{C}_2$ would not fit in that S-factor and we would have to use a copy of S from another S-factor. We would have to ensure that this copy of S is vertex disjoint from those already used for $\mathcal{C}_2$ in the previous S-factor, and an obvious way to get around this would be to embed all components of $\mathcal{C}_2$ in a new S-factor all together. However, this would be very wasteful and if many (for example $\Omega(n)$) graphs $\mathcal{C}_i$ would be of size strictly larger than $N/2$, then we would not be able to pack all $\mathcal{C}_i$ into K_N in such a straightforward way. We remedy this situation by introducing an additional layer.

3.5.2.2. *Two layers.* For an appropriately chosen integer m, suppose there exist a resolvable K_m-decomposition of K_N and a resolvable S-decomposition of K_m. Note that, with this additional decomposition layer at hand, we can address the issue raised above more easily. In fact, we fix a K_m-factor of K_N and use sufficiently many K_m's of this K_m-factor to host the components of $\mathcal{C}_1$, all of which are isomorphic to S by our assumption. The remaining K_m's of the factor can host the first part of $\mathcal{C}_2$. We then "wrap around" and reuse the K_m's containing copies of S from $\mathcal{C}_1$ by selecting a new S-factor inside these K_m's to host the second part of $\mathcal{C}_2$. This way the components of $\mathcal{C}_1$ and $\mathcal{C}_2$ are packed edge disjointly and the components of $\mathcal{C}_2$ (resp. $\mathcal{C}_1$) are in addition vertex disjoint, as required for a packing. We can continue this process to pack $\mathcal{C}_3, \mathcal{C}_4, \ldots$ until the fixed K_m-factor of K_N is fully used. Then we continue with another K_m-factor and so on.

This procedure will work if all components of each $\mathcal{C}_i$ are isomorphic to a single S. Let us note however that in case $\mathcal{C}_i$ contains components of different isomorphism types two layers may not be sufficient. This is because we would have to select S-factors for different graphs S within K_m and there seems to be no obvious way to achieve this in a two layer decomposition. Instead we will introduce a third layer, which will give us sufficient flexibility to address this issue.

3.5.2.3. *Three layers.* Here we give an outline and describe how a three layer structure can be used to address the general problem. The details will follow in section 3.5.4.1. Consider a resolvable K_m-decomposition $\mathscr{D}^{m,N}$ of K_N, a resolvable K_ℓ-decomposition $\mathscr{D}^{\ell,m}$ of K_m, and resolvable S-decompositions $\mathscr{D}^{S,\ell}$ of K_ℓ for every $S \in \mathcal{S}$ (in fact the last assumption will never be used in its full strength, we will use Theorem 37 instead). We view resolvable decompositions as collections of factors. We write $\mathscr{D}^{m,N} = \{\mathcal{D}_1^{m,N}, \dots, \mathcal{D}_{\frac{N-1}{m-1}}^{m,N}\}$, where $\mathcal{D}_j^{m,N}$ is a K_m-factor of K_N for $j = 1, \dots, \frac{N-1}{m-1}$.

Suppose now we are given graphs $\mathcal{C}_1, \dots, \mathcal{C}_t$, $\mathcal{C}_i = \bigcup_{S\in\mathcal{S}} \nu_i(S) \cdot S$. We will proceed greedily processing the $\mathcal{C}_i$'s one by one. In each step we will work with one fixed K_m-factor $\mathcal{D}_j^{m,N} = \mathcal{D}_{\text{current}}^{m,N}$ of K_N which will be used repeatedly as long as "sufficiently many" edges of such factor are available. For example, $\mathcal{D}_1^{m,N}$ will host $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_a$ for some $a < t$, then $\mathcal{D}_2^{m,N}$ will host $\mathcal{C}_{a+1}, \mathcal{C}_{a+2}, \dots, \mathcal{C}_b$ for some $a < b < t$, and so on. Once we run out of available edges in factor $\mathcal{D}_{\text{current}}^{m,N}$ we will move that factor in the set $\mathscr{D}_{\text{used}}^{m,N} \subseteq \mathscr{D}^{m,N}$ of factors the edges of which were already assigned to previous $\mathcal{C}_i$'s and select a new factor $\mathcal{D}_{\text{current}}^{m,N} \in \mathscr{D}^{m,N} \setminus \mathscr{D}_{\text{used}}^{m,N}$ which we will continue to work with.

We outline the assignment within a K_m of the current K_m-factor. For each $K_m \in \mathcal{D}_{\text{current}}^{m,N}$ we consider a resolvable decomposition $\mathscr{D}^{\ell,m} = \mathscr{D}^{\ell,m}(K_m)$ of such a K_m. Again some factors in that decomposition might have already been completely used. Among those which were not completely used yet, we specify σ of such "current" factors $\mathcal{D}_S^{\ell,m}$, each ready to be used to embed copies of S in the current particular step. Since K_ℓ admits resolvable S-decompositions for every $S \in \mathcal{S}$, each $\mathcal{D}_S^{\ell,m}$ corresponds to $\frac{(\ell-1)v(S)}{2e(S)} = t(S)$ S-factors of K_m which we may denote by $\mathcal{D}_1^{S,\ell,m}, \dots, \mathcal{D}_{t(S)}^{S,\ell,m}$. At each step, in every K_m we will only use one of such S-factors, which we denote by $\mathcal{D}_{\text{current}}^{S,\ell,m}$. A set of components of $\mathcal{C}_i$ that are going to be assigned to an S-factor of a K_m will be referred to as a *chunk*.

With this structure in mind we are able to describe our greedy assignment procedure. Assume that in the assignment procedure the graphs $\mathcal{C}_1, \dots, \mathcal{C}_{i-1}$ were already processed and that $\mathcal{C}_i = \bigcup_{S\in\mathcal{S}} \nu_i(S) \cdot S$. The assignment of $\mathcal{C}_i$ will consist of the following four steps which we discuss in detail in Section 3.5.4.1.

(*i*) For every isomorphism type $S \in \mathcal{S}$, partition the $\nu_i(S)$ components into as few as possible chunks of size at most $\frac{m}{v(S)}$.

(*ii*) For every $S \in \mathcal{S}$, select $\frac{\nu_i(S)v(S)}{m}$ copies of K_m from the current K_m-factor $\mathcal{D}^{m,N}_{\text{current}}$ and match each such K_m with a chunk of components isomorphic to S.

(*iii*) For every $S \in \mathcal{S}$ and for each chunk of type S, assign the components in the chunk to the S-factor $\mathcal{D}^{S,\ell,m}_{\text{current}}$ of K_m. The copies of S will cover $m\frac{e(S)}{v(S)}$ edges of K_m.

(*iv*) Prepare for the assignment of the next component graph.

This procedure leads to a packing of $\{\mathcal{C}_i\}_{i\in[t]}$ into K_N if we do not run out of K_m-factors during the process, and in the proof we shall verify this. Assuming this for the moment, the procedure above yields a preliminary packing which can be encoded by functions $\boldsymbol{f} = \{f_i\}_{i\in[t]}$, with $f_i\colon V(\mathcal{C}_i) \to V(K_N)$.

3.5.3. Outline of the balancing phase. In this section we will outline how the preliminary packing $\boldsymbol{f}$ obtained in the assignment phase is used to realize a (ξn)-balanced packing of $\{\mathcal{C}_i\}_{i\in[t]}$ into K_N. Further detail will be given in Section 3.5.4.2.

Note that so far we did not consider the boundary degrees of the vertices of K_N and, in fact, $\boldsymbol{f}$ is not guaranteed to be balanced. However, the layered structure of the assignment will allow us to fix this by using the following degrees of freedom. Firstly, the $\frac{N}{m}$ K_m's in any of the $\frac{N-1}{m-1}$ K_m-factors from $\mathscr{D}^{m,N}$ can be permuted independently for each K_m-factor. Since any component graph is assigned to a single K_m-factor, the resulting mappings remain injective and the embedding of the $\mathcal{C}_i$'s stays pairwise edge disjoint. Secondly, each K_m can be embedded into K_N in $m!$ possible ways by permuting its vertices. There are

$$\left(\left(\frac{N}{m}\right)! \times (m!)^{\frac{N}{m}}\right)^{\frac{N-1}{m-1}}$$

such choices in total and each of them leads to a packing of the component graphs $\{\mathcal{C}_i\}_{i\in[t]}$.

We will pick one of such choices uniformly at random, and show that with positive probability each vertex of K_N is used as a boundary vertex approximately the same number of times. Since the sum of the boundary degrees is at most $\Delta\delta n^2 \leqslant \xi n^2/2$ (see (6)), this leads to a (ξn)-balanced packing $\boldsymbol{g}$ of $\{\mathcal{C}_i\}_{i\in[t]}$ into K_N.

3.5.4. Proof of Lemma 32. Given ξ and Δ, set

$$\delta = \frac{\xi}{2\Delta} \tag{9}$$

and let $\mathcal{G}$ be a (δ, s)-separable family, for some $s \in \mathbb{N}$. We apply Theorem 37 with

$$\eta = \xi/8 \tag{10}$$

and fix an integer $\ell > s^2$ satisfying that for every $S \in \mathcal{S}$ there exists an (S, η)-factorization of K_ℓ. Let $m \in \mathbb{N}$ such that

$$m > 16\sigma\ell/\xi \tag{11}$$

and there exists a resolvable K_ℓ-decomposition of K_m (see Theorem 34). Similarly, let

$$n_0 > \max\{4m^2/\xi, 2^{2m}\} \tag{12}$$

such that for any $n \geqslant n_0$ satisfying the necessary congruence property there exists a resolvable K_m-decomposition of K_n. Having defined n_0, we are given a $(\mathcal{G}, n, \Delta)$-sequence $\mathcal{F} = (F_1, \ldots, F_t)$ for some $n \geqslant n_0$. We show that there exists a (ξn)-balanced packing of the family of component graphs $\{\mathcal{C}_i\}_{i\in[t]}$ into K_N, for any N with $(1 + \frac{\xi}{2})n \leqslant N \leqslant (1 + \xi)n$ such that K_N admits a K_m-decomposition. Since $n \geqslant n_0 \geqslant \frac{4m^2}{\xi}$, such N indeed exist.

3.5.4.1. *The assignment phase.* Next we elaborate on the outline given in Sections 3.5.1 and 3.5.2. First we describe the auxiliary structure we are going to use followed by the actual assignment procedure.

The auxiliary structure. For each $S \in \mathcal{S}$ let $\mathscr{D}^{S,\ell}$ be a fixed (S,η)-factorization of K_ℓ (see Definition 36). Let $\mathscr{D}^{\ell,m}$ be an arbitrarily chosen resolvable K_ℓ-decomposition of K_m. Similarly, for the given N, denote by $\mathscr{D}^{m,N}$ an arbitrarily chosen resolvable K_m-decomposition of K_N.

At each point of time in the assignment procedure we will work with one K_m-factor which we refer to as the *current K_m-factor* $\mathcal{D}^{m,N}_{\text{current}} \in \mathscr{D}^{m,N}$. Each K_m of the current K_m-factor is decomposed into K_ℓ-factors using $\mathscr{D}^{\ell,m}$. Moreover, in every K_m, for every $S \in \mathcal{S}$ we pick a K_ℓ-factor which we denote by $\mathcal{D}^{\ell,m}_S$. We refer to $\mathcal{D}^{\ell,m}_S$ as the *current K_ℓ-factor for S*. We then apply Theorem 37 to all K_ℓ's in such a K_ℓ-factor and obtain (S,η)-factorizations for every K_ℓ in $\mathcal{D}^{\ell,m}_S$. Note that we can arbitrarily fix an S-matching in each K_ℓ of $\mathcal{D}^{\ell,m}_S$ and obtain an S-matching of K_m of size at least

$$(1-\eta)\frac{\ell}{v(S)}\frac{m}{\ell} = (1-\eta)\frac{m}{v(S)}. \tag{13}$$

This way we set up $t(S)$ edge disjoint S-matchings of K_m contained in $\mathcal{D}^{\ell,m}_S$, for

$$(1-\eta)\frac{(\ell-1)v(S)}{2e(S)} \leqslant t(S) \leqslant \frac{(\ell-1)v(S)}{2e(S)},$$

which we denote by $\mathcal{D}^{S,\ell,m}_1, \ldots, \mathcal{D}^{S,\ell,m}_{t(S)}$. Each of these S-matchings cover at least $(1-\eta)\frac{m}{\ell}\binom{\ell}{2}$ edges of the K_ℓ's in $\mathcal{D}^{\ell,m}_S$.

Every such structure will be used until it is considered *full* according to the following definition.

Definition 38. *A K_ℓ-factor $\mathcal{D}^{\ell,m}_S$ is full when all its S-matchings have been used. A K_m is full when there exists an isomorphism type $S \in \mathcal{S}$ such that $\mathcal{D}^{\ell,m}_S$ is full and any other K_ℓ-factor is either full or reserved to another isomorphism type. A K_m-factor is full when one of its K_m's is full.*

The assignment procedure. We now give the details of the four steps outlined in Section 3.5.2.3 for the assignment for the graph $\mathcal{C}_i = \bigcup_{S\in\mathcal{S}} \nu_i(S) \cdot S$.

We assume that the graphs $\mathcal{C}_1, \ldots, \mathcal{C}_{i-1}$ have already been assigned and that the current K_m-factor $\mathcal{D}^{m,N}_{\text{current}} = \mathcal{D}^{m,N}_j$ is not full.

(*i*) For each isomorphism type $S \in \mathcal{S}$ we group the $\nu_i(S)$ copies of S into as few as possible chunks of size at most $(1-\eta)\frac{m}{v(S)}$ (note that this matches the size of an S-matching of K_m, as given in (13)) The correction factor $(1-\eta)$ here addresses the fact that we deal with (S,η)-factorizations and not with resolvable S-decompositions. The number $\mu_i(S)$ of chunks required for the $\nu_i(S)$ components of type S is hence given by

$$\mu_i(S) = \left\lceil \frac{\nu_i(S) \cdot v(S)}{(1-\eta)m} \right\rceil. \tag{14}$$

(*ii*) We order the K_m's in the current K_m-factor $\mathscr{D}^{m,N}_{\text{current}}$ according to the number of edges that have already been assigned to it. We start with the one in which the least number of edges have been used. We then assign the $\mu_i(S_1)$ chunks of type S_1 to the first $\mu_i(S_1)$ K_m's in that order and continue in the natural way, that is, the $\mu_i(S_2)$ chunks of type S_2 are assigned to the next $\mu_i(S_2)$ K_m's, and so on. Since the members of $\mathcal{S}$ are ordered non-increasingly according to their densities (see 7), this way we will ensure that the K_m's in the current K_m-factor are used in a balanced way, which is essential to leave only little waste.

(*iii*) Once we have determined which chunk goes to which K_m, we have to assign the components S of the chunk to their copies in the corresponding K_m. In the chosen K_m we assign the components of the chunk to $\mathcal{D}^{S,\ell,m}_{\text{current}}$. Such a matching exists because we assumed that the current K_m-factor $\mathcal{D}^{m,N}_j$ is not full. Note that, independently of the precise number of components in the chunk, we use an entire S-matching in all the K_ℓ's of the current K_ℓ-factor for S for the assignment of this chunk.

(*iv*) After we have assigned the components of $\mathcal{C}_i$ we prepare for the assignment of $\mathcal{C}_{i+1}$. In each K_m, for every isomorphism type S, we check whether the current K_ℓ-factor for S is full. If it is, two cases may arise. In the first case there exists another K_ℓ-factor in the K_m that has not been reserved for any $S \in \mathcal{S}$ yet. Then, we apply Theorem 37 with S and η to all copies of K_ℓ in such a K_ℓ-factor and this factor becomes the current K_ℓ-factor for S, i.e., $\mathcal{D}_S^{\ell,m}$ in that K_m. In the second case, all K_ℓ-factors are either full or have been reserved for some $S' \in \mathcal{S}$ with $S' \neq S$, hence we cannot set up a new K_ℓ-factor for S. This implies that the K_m and the K_m-factor are full (see Definition 38). Since we assigned the components of $\mathcal{C}_i$ to the least used K_m's in the K_m-factor, we are ensured that at this point all the K_m's in $\mathcal{D}^{m,N}_{\text{current}}$ are almost completely used. At this point we add $\mathcal{D}^{m,N}_{\text{current}}$ to $\mathscr{D}^{m,N}_{\text{used}}$ and set $\mathcal{D}^{m,N}_{\text{current}} = \mathcal{D}^{m,N}_{j+1}$.

The assignment phase yields a packing. We shall verify that the procedure yields a correct assignment. For that we have to show that any component graph $\mathcal{C}_i$ "fits" into K_N, and that we do not run out of K_m-factors while iterating the four steps for all graphs in $\{\mathcal{C}_i\}_{i\in[t]}$.

We first show that every $\mathcal{C}_i$ fits into one K_m-factor. Recall that in Step (i)) the copies isomorphic to some $S \in \mathcal{S}$ are split into chunks of size at most $(1-\eta)\frac{m}{v(S)}$ and each chunk is assigned to an S-matching of $\mathcal{D}_S^{\ell,m}$. At this point some vertices may not be used for one of the following two reasons:

(V1) We always reserve a whole S-matching $\mathcal{D}^{S,\ell,m}_{\text{current}}$ for each chunk, even though some chunks may contain only a few copies of S. In the worst case where only one copy of S is contained in the chunk we may waste $m - v(S) \leqslant m$ vertices and in principle this could happen for every isomorphism type $S \in \mathcal{S}$. However, since such a "rounding error" occurs at most once for each isomorphism type, we may waste at most σm vertices for this reason.

(V2) We cannot guarantee that the S-matchings which we are using are perfect S-factors. However, from Theorem 37 it follows that each matching covers at least

$$(1-\eta)\frac{m}{v(S)}v(S) = (1-\eta)m$$

vertices of K_m. Therefore the number of uncovered vertices in the K_m-factor due to this imperfection is at most $\eta m \frac{N}{m} = \eta N$.

Hence $\mathcal{C}_i$ fits into one K_m-factor if we ensure that $v(\mathcal{C}_i) + \sigma m + \eta N \leqslant N$, which follows from

$$v(\mathcal{C}_i) + \sigma m + \eta N \leqslant n + \sigma m + \eta N \leqslant (1 + \frac{\xi}{2})n \leqslant N,$$

due to (10), (11), and (12).

It is left to show that $\frac{N-1}{m-1}$ K_m-factors are sufficient to host all the graphs from $\{\mathcal{C}_i\}_{i\in[t]}$. For that, we shall bound the number of unused edges in each K_m-factor. At the point when a K_m becomes full, all its K_ℓ-factors, except for the current K_ℓ-factors $\mathcal{D}_S^{\ell,m}$ for each isomorphism type $S \in \mathcal{S}$, have been used in the assignment. This leads to the following cases.

(E1) The current K_ℓ-factor $\mathscr{D}_S^{\ell,m}$ for a given isomorphism type S may not have been used at all and hence all its $\binom{\ell}{2}\frac{m}{\ell}$ edges are not used in the assignment.

(E2) Owing to Theorem 37, in a used K_ℓ-factor, up to at most $\eta\binom{\ell}{2}\frac{m}{\ell}$ edges are not covered by the S-matchings.

Hence the total number of edges that are not used in a full K_m can be bounded by

$$\left(\sigma + \eta\frac{m-1}{\ell-1}\right)\binom{\ell}{2}\frac{m}{\ell}.$$

It is left to establish a similar estimate for the other K_m's in the K_m-factor. Recall that we declared the whole K_m-factor to be full as soon as one K_m was full. Since all components of any $\mathcal{C}_i \subseteq F_i$ have bounded maximum degree Δ, in each step up to at most $\frac{m\Delta}{2}$ edges are reserved in any K_m of the current K_m-factor. Owing to the balanced selection of the K_m's within the current K_m-factor (see Step (*ii*)) we have that the number of used edges over all K_m's in $\mathscr{D}^{m,N}_{\text{current}}$ differs by at most

$\frac{m\Delta}{2}$. Consequently, the number of unused edges in any K_m at the point when the K_m-factor is declared full is at most

$$\left(\sigma + \eta\frac{m-1}{\ell-1}\right)\binom{\ell}{2}\frac{m}{\ell} + \frac{m\Delta}{2}.$$

Using this estimate for all $\binom{N}{2}/\binom{m}{2}$ of the K_m in the K_m-decomposition of K_N leads to a total of unused edges of at most

$$\left(\sigma\frac{\ell-1}{m-1} + \eta + \frac{\Delta}{m-1}\right)\binom{N}{2} < 2\eta\binom{N}{2},$$

where we used (10), (11), and $\Delta < \sigma$. Furthermore, since by $N \geqslant (1+\frac{\xi}{2})n$ we have

$$\binom{n}{2} + 2\eta\binom{N}{2} \leqslant \binom{N}{2},$$

we have shown that we do not run out of K_m-factors and, hence, the assignment procedure yields a preliminary packing of $\{\mathcal{C}_i\}_{i\in[t]}$.

For the proof of Lemma 32 we have to show not only that there exists such a packing but also that there is a balanced one. This will be the focus of the next phase.

3.5.4.2. *The balancing phase.* In the assignment phase we have constructed a preliminary packing $\boldsymbol{f}$ of $\{\mathcal{C}_i\}_{i\in[t]}$ into the K_m-factors of K_N as described in Section 3.5.1. We now construct a (ξn)-balanced packing $\boldsymbol{h}$ by the following random process consisting of two parts. Firstly, we randomly permute the $\frac{N}{m}$ K_m's in each K_m-factor independently and we will denote the resulting packing by $\boldsymbol{g}$. Secondly, for each K_m, we pick a random permutation of its vertices. As we already noted in Section 3.5.3, any such permutation yields a packing of $\{\mathcal{C}_i\}_{i\in[t]}$ into K_N.

It is left to show that with positive probability each vertex v of K_N has boundary degree with respect to $\boldsymbol{h}$ bounded by ξn. Recall from Definition 31 that the *boundary degree* with respect to $\boldsymbol{f}$ of a vertex v is defined by

$$d^{\partial}_{\boldsymbol{f}}(v) = |\{i \in [t] \colon f_i^{-1}(v) \in \partial\mathcal{C}_i\}|.$$

For a K_m of the K_m-decomposition of K_N and a vertex v of K_m we consider the *relative boundary degree*

$$d^{\partial}_{\boldsymbol{f}}(v, K_m) = |\{i \in [t] \colon f_i \text{ assigns some components of } \mathcal{C}_i \text{ to } K_m \text{ and } f_i^{-1}(v) \in \partial\mathcal{C}_i\}|$$

Clearly, $\sum d^{\partial}_{\boldsymbol{f}}(v, K_m) = d^{\partial}_{\boldsymbol{f}}(v)$, where the sum runs over all K_m from the K_m-decomposition of K_N that contain v. For each K_m we define its *label* as the monotone sequence of the relative boundary degrees of its vertices. Since these labels of the K_m's consist of relative boundary degrees, such a label is invariant under permutations of the vertices of a K_m and it is invariant under permutations of the K_m's within its K_m-factor. Moreover, the label of a K_m is determined by the isomorphism types $S \in \mathcal{S}$ it hosts, because each type S consists of a labelled graph R with a set of boundary vertices B. Since in the assignment phase we assigned an isomorphism type to a whole K_ℓ-factor, the number of possible labels is bounded by $|\mathcal{S}|^{(m-1)/(\ell-1)} = \sigma^{(m-1)/(\ell-1)} < 2^m$ (see (8) and the choice of $\ell > s^2$).

For every K_m-factor $\mathscr{D}^{m,N}_j$, let $\alpha_j(A)$ be the number of K_m's with label A in $\mathscr{D}^{m,N}_j$ and define

$$\alpha(A) = \sum_{j=1}^{\frac{N-1}{m-1}} \alpha_j(A).$$

We call a label *common* if $\alpha(A) \geqslant \frac{\eta}{2^m}\frac{N(N-1)}{m(m-1)}$ and *rare* otherwise. Note that the total number of K_m's having a rare label is bounded by $\eta\frac{N(N-1)}{m(m-1)}$, therefore

$$\sum_{A \text{ common}} \alpha(A) \geqslant (1-\eta)\frac{N(N-1)}{m(m-1)}\,. \tag{15}$$

We use these labels to show that each vertex in K_N hosts roughly the same amount of boundary vertices. For that we first prove that an arbitrary vertex is contained in approximately the *expected* number of K_m's of a given common label. For a vertex v of K_N and a common label A we denote by $X^{v,A}$ the number of K_m's containing v that have label A. Note that $X^{v,A}$ is the sum of $\frac{N-1}{m-1}$ indicator variables $X^{v,A}_j$, where $X^{v,A}_j = 1$ if the K_m from the K_m-factor $\mathscr{D}^{m,N}_j$ adjacent to v has label A. The probability that this happens is then given by $\frac{\alpha_j(A)}{N/m}$. By applying Chernoff's inequality ((2.9) in [41]) we obtain

$$\mathbb{P}\left(|X^{v,A} - \mathbb{E}X^{v,A}| > \eta\mathbb{E}X^{v,A}\right) < 2\exp\left(-\frac{\eta^2\mathbb{E}X^{v,A}}{3}\right) < 2\exp\left(-\frac{\eta^2}{3}\frac{m}{N}\alpha(A)\right).$$

Consequently, the probability that one of the common labels appears too many or too few times among the K_m's containing some vertex is bounded by

$$\sum_{v\in V(K_N)}\sum_{A \text{ common}} 2\exp\left(-\frac{\eta^2}{3}\frac{m}{N}\alpha(A)\right) < N2^{m+1}\exp\left(-\frac{\eta^3(N-1)}{2^m\cdot 3(m-1)}\right) < 1\,,$$

where we used that common labels A are defined through $\alpha(A) \geqslant \frac{\eta}{2^m}\frac{N(N-1)}{m(m-1)}$ in the first inequality. Therefore, with positive probability, all vertices are *balanced* in the sense that the occurrences of every common label among the K_m's incident to each vertex roughly agree in proportion with the occurrences of that label in the decomposition.

We fix such permutation of the K_m's and the corresponding numbers $X^{v,A}$ for every vertex v and every label A. Let $\boldsymbol{g}$ be the corresponding packing of $\{\mathcal{C}_i\}_{i\in[t]}$ into K_N. As a consequence, we get that for every vertex v the number of K_m's with common labels attached to it satisfies

$$\begin{aligned}\sum_{A \text{ common}} X^{v,A} &\geqslant \sum_{A \text{ common}} (1-\eta)\frac{m}{N}\alpha(A) = (1-\eta)\frac{m}{N}\sum_{A \text{ common}}\alpha(A)\\ &\overset{(15)}{\geqslant} (1-\eta)\frac{m}{N}(1-\eta)\frac{N(N-1)}{m(m-1)} = (1-\eta)^2\frac{N-1}{m-1} \geqslant (1-2\eta)\frac{N-1}{m-1}.\end{aligned}$$

We also obtain an upper bound on the number of K_m's with rare labels for every vertex v

$$\sum_{A \text{ rare}} X^{v,A} = \frac{N-1}{m-1} - \sum_{A \text{ common}} X^{v,A} \leqslant 2\eta\,\frac{N-1}{m-1}. \tag{16}$$

Next we show that randomly permuting the vertices of each K_m in the K_m-decomposition of K_N for the random packing $\boldsymbol{g}$ ensures that the boundary degrees in each K_m are evenly distributed. Let $d^{\partial}_{\boldsymbol{h}}(v, A)$ be the sum of the boundary degrees of the vertex v within the K_m's labelled by A and containing v. Clearly,

$$d^{\partial}_{\boldsymbol{h}}(v) = \sum_{A} d^{\partial}_{\boldsymbol{h}}(v, A).$$

We denote by $A(j)$ the j-th element of the degree sequence A and set $\beta(A) = \frac{1}{m}\sum_{j=1}^{m} A(j)$ as the average degree in A. Since $\alpha(A)$ is the number of K_m's with label A in the K_m-decomposition of K_N and $m\beta(A) = \sum_{j=1}^{m} A(j)$ is the sum of the relative boundary degrees of the vertices of such a K_m, for later reference we note

$$\sum_{A} m\beta(A)\alpha(A) = \sum_{A} \alpha(A) \sum_{j=1}^{m} A(j) = \sum_{i=1}^{t} |\partial \mathcal{C}_i|. \tag{17}$$

For a moment we ignore the K_m's with rare labels, since owing to (16) their contribution will be negligible, and consider only those that have a common label. We first show that for a vertex v of K_N and a common label A, $d^{\partial}_{\boldsymbol{h}}(v, A)$ is in the range $(1 \pm \eta)\beta(A)X^{v,A}$ with high probability. Let $Y_j^{v,A}$ be the number of K_m's labelled by A in which v gets boundary degree $A(j)$. By applying Chernoff's inequality we obtain

$$\mathbb{P}\left(\left|Y_j^{v,A} - \frac{X^{v,A}}{m}\right| > \eta\frac{X^{v,A}}{m}\right) < 2\exp\left(-\frac{\eta^2}{3}\frac{X^{v,A}}{m}\right)$$

for every $j \in [m]$. This implies that with probability $1 - 2m\exp\left(-\frac{\eta^2}{3}\frac{X^{v,A}}{m}\right)$ we have

$$d^{\partial}_{\boldsymbol{h}}(v, A) = \sum_{j=1}^{m} A(j)Y_j^{v,A} = \sum_{j=1}^{m} A(j)(1 \pm \eta)\frac{X^{v,A}}{m} = (1 \pm \eta)\beta(A)X^{v,A}.$$

By summing over all common labels, we have that with positive probability there exist permutations for every K_m of the K_m-decomposition of K_N for which all vertices have roughly the expected boundary degree. More precisely, the probability that there exists a misbehaving vertex is bounded by

$$\begin{aligned}\sum_{v\in V(K_N)}\sum_{A\text{ common}} 2m\exp\left(-\frac{\eta^2}{3}\frac{X^{v,A}}{m}\right) &< N2^{m+1}m\exp\left(-\frac{\eta^2}{3}(1-\eta)\frac{\alpha(A)}{N}\right)\\ &\leqslant N2^{m+1}m\exp\left(-\frac{\eta^3}{2^m\cdot 3}(1-\eta)\frac{N-1}{m(m-1)}\right)\\ &< 1\,,\end{aligned}$$

where the first inequality follows from $\boldsymbol{g}$ being a packing in which $X^{v,A}$ is close to its expected value for every $v \in V(K_n)$ and the second inequality follows from the definition of common labels. Therefore, the contribution of the K_m's with common labels for each vertex v is at most

$$\begin{aligned}\sum_{A\text{ common}} d^{\partial}_{\boldsymbol{h}}(v,A) &\leqslant \sum_{A\text{ common}}(1+\eta)\beta(A)X^{v,A}\\ &\leqslant \sum_{A\text{ common}}(1+\eta)\left[\beta(A)(1+\eta)\frac{m}{N}\alpha(A)\right]\\ &= (1+\eta)^2\frac{1}{N}\sum_{A\text{ common}}(m\beta(A)\alpha(A))\\ &\overset{(17)}{\leqslant} (1+\eta)^2\frac{1}{N}\sum_{i=1}^{t}|\partial\mathcal{C}_i|\\ &\overset{(6),(9)}{\leqslant} (1+\eta)^2\frac{1}{N}\frac{\xi}{2}n^2.\end{aligned}$$

Owing to (16), a vertex can be incident to at most $2\eta\frac{N-1}{m-1}$ K_m's with rare labels. Since no component of any $\mathcal{C}_i$ consists of a single isolated vertex (see *(ii)* in Definition 27), the largest relative boundary degree of any vertex in such a K_m can be at most $m-1$ and we infer that

$$d^{\partial}_{\boldsymbol{h}}(v) \leqslant (1+\eta)^2\frac{1}{N}\frac{\xi}{2}n^2 + 2\eta\frac{N-1}{m-1}(m-1) < \left(\frac{(1+\eta)^2}{1+\xi/2}\frac{\xi}{2}+2\eta(1+\xi)\right)n \overset{(10)}{<} \xi n \tag{18}$$

for every $v \in V(K_N)$, thus proving Lemma 32. ∎

§3.6. Packing the separators

In this section we prove Lemma 33. The Lemma asserts that a balanced packing of $\{\mathcal{C}_i\}_{i\in[t]}$ into $K_{(1+\xi)n}$ can be extended to a packing of $\{F_i\}_{i\in[t]}$ in $K_{(1+\varepsilon)}$. For that we have to show that we can embed the separators $\{U_i\}_{i\in[t]}$ in an appropriate way. Roughly speaking, we will show that a simple greedy strategy will work in here.

3.6.1. Proof of Lemma 33. Given ε and Δ, set

$$\xi = \frac{\varepsilon}{12\Delta^2} \qquad \text{and} \qquad \delta = \frac{\varepsilon^2}{72\Delta^2}.$$

Let $s \in \mathbb{N}$ and let $\mathcal{G}$ be a (δ, s)-separable family. For sufficiently large n let $\mathcal{F} = (F_1, \ldots, F_t)$ be a $(\mathcal{G}, n, \Delta)$-sequence and suppose that there exists a (ξn)-balanced packing of the component graphs $\{\mathcal{C}_i\}_{i\in[t]}$ into a clique of order $(1+\xi)n$. Fix a partition $X\dot\cup Y$ of the vertex set of $K_{(1+\varepsilon)n}$, where $|X| = (1+\xi)n$, and denote by K_X, K_Y, and $K_{X,Y}$ the complete subgraphs induced on X and on Y, and the complete bipartite subgraph between X and Y, respectively. Let $\boldsymbol{h} = \{h_i\}_{i\in[t]}$ with

$$h_i \colon V(\mathcal{C}_i) \to X$$

be a (ξn)-balanced packing of $\{\mathcal{C}_i\}_{i\in[t]}$ into K_X. We shall use K_Y to embed $\{U_i\}_{i\in[t]}$, and $K_{X,Y}$ for the necessary connections. It is easy to see that if the following conditions are satisfied then the resulting map is a packing of $\mathcal{F}$ into $K_{(1+\varepsilon)n}$:

- (P1) for every $i \in [t]$, the vertices of U_i are mapped injectively into Y;
- (P2) each edge in $K_{X,Y}$ is used at most once;
- (P3) each edge in K_Y is used at most once.

Note that we will embed $\sum_{i\in[t]} |U_i| \leqslant \delta n^2$ vertices into Y, therefore some vertices in Y will be used at least $\frac{\sum_{i\in[n]} |U_i|}{|Y|} \leqslant \frac{\delta n^2}{|Y|}$ times. However, we will ensure that each vertex in Y is used at most $3\frac{\delta n^2}{|Y|}$ times. The packing of $\mathcal{F}$ into $K_{(1+\varepsilon)n}$ will be expressed by a family of functions $\overline{\boldsymbol{h}} = \{\overline{h}_i\}_{i\in[t]}$ with

$$\overline{h}_i \colon V(F_i) \to X \dot{\cup} Y$$

where $\overline{h}_i$ extends h_i from $V(\mathcal{C}_i)$ to $V(F_i)$. For a vertex $v \in V(\mathcal{C}_i)$, we set $\overline{h}_i(v) = h_i(v) \in X$ for any $i \in [n]$. For the vertices in the separators $\{U_i\}_{i\in[t]}$ we will fix their image $\overline{h}_i(v)$ in Y one by one in a greedy way, starting with vertices of U_1.

At each step we embed a vertex $u \in U_i$ into Y, assuming that all vertices of U_j with $j < i$ and possibly some (at most $|U_i| - 1 < \delta n$) vertices of U_i were already embedded. Let $N_{\mathcal{C}_i}(u)$ be the neighbourhood of u in $\mathcal{C}_i$, and $N_{U_i}(u)$ the neighbourhood of u in U_i both of size at most Δ. Suppose so far we made sure that every vertex in Y was used at most $3\frac{\delta n^2}{|Y|}$ times. We will embed u in such a way that (P1), (P2), and (P3) are obeyed (see (P1′), (P2′), and (P3′) below), and afterwards each vertex of Y is still used at most $3\frac{\delta n^2}{|Y|}$ times. This will show that $\boldsymbol{h}$ can be extended to a packing $\overline{\boldsymbol{h}}$ of $\mathcal{F}$ and conclude the proof. Having this in mind we note:

(P1′) The vertices of U_i have to be embedded injectively into Y and, hence, up to at most $|U_i| - 1 < \delta n$ vertices of Y may not be used for the embedding of u.

(P2′) Since every edge in $K_{X,Y}$ can be used at most once, we require $\overline{h}_i(u) \neq \overline{h}_j(u')$ for every vertex $u' \in U_j$ with $\overline{h}_j(N_{\mathcal{C}_j}(u')) \cap \overline{h}_i(N_{\mathcal{C}_i}(u)) \neq \varnothing$. Let $x \in \overline{h}_i(N_{\mathcal{C}_i}(u))$. Owing to the (ξn)-balancedness of the packing $\{h_i\}_{i\in[t]}$, x hosts at most ξn vertices from $\bigcup_{k\in[t]} \partial\mathcal{C}_k$ and each of them has at most Δ neighbours in some U_k for $k \in [t]$. Assuming that all of them have already been embedded into Y, we obtain at most $\Delta\xi n$ forbidden vertices for each of the up to at most Δ neighbours of u in $\mathcal{C}_i$. Hence, the total number of forbidden options for $\overline{h}_i(u)$ in Y is at most $\Delta^2\xi n$.

(P3′) Note that K_Y also hosts the edges contained in the separator U_i and every edge of K_Y may be used at most once. Suppose that there exists a vertex u' from U_j with $j < i$ such that $\overline{h}_i(N_{U_i}(u)) \cap \overline{h}_j(N_{U_j}(u')) \neq \varnothing$.

Then $\overline{h}_i(u)$ must avoid $\overline{h}_j(u')$ for any such u', because at least one edge between this vertex and the image of the neighbours of u is already used. Since by our assumption every vertex in the set $\overline{h}_i(N_{U_i}(u))$ hosts at most $3\frac{\delta n^2}{|Y|}$ vertices embedded so far, and since $\Delta(F_j) \leqslant \Delta$, there are at most $\Delta \cdot 3\frac{\delta n^2}{|Y|}|N_{U_i}(u)| \leqslant 3\Delta^2\delta n^2/|Y|$ such restrictions.

Since up to now every vertex $y \in Y$ was used at most $3\frac{\delta n^2}{|Y|}$ times for the embedding, by denoting with $Y_u \subseteq Y$ the set of candidates for the embedding of u, we obtain

$$|Y_u| \geqslant |Y| - \left(\delta n + \Delta^2 \xi n + 3\Delta^2 \frac{\delta n^2}{|Y|}\right) \geqslant |Y| - \frac{\varepsilon}{4} n > \frac{|Y|}{2}.$$

Since we have to embed at most $\sum_{i \in [t]} |U_i| \leqslant \delta n^2$ vertices in total, at any time some vertex $y \in Y_u$ was used at most

$$\frac{\delta n^2}{|Y|/2} < 3\frac{\delta n^2}{|Y|} - 1$$

times, and this vertex we choose for $\overline{h}_i(u)$. We have thus shown that at each round we can always pick one vertex in Y such that all the edges needed to connect the vertex we want to embed to all its neighbour are available and it was used before at most $3\frac{\delta n^2}{|Y|} - 1$ times. This completes the proof of the lemma. $\square$

Bibliography

[1] M. Aigner and S. Brandt, *Embedding arbitrary graphs of maximum degree two*, J. London Math. Soc. (2) **48** (1993), no. 1, 39–51. ↑1.2

[2] M. O. Albertson, L. Chan, and R. Haas, *Independence and graph homomorphisms*, J. Graph Theory **17** (1993), no. 5, 581–588. ↑2.1

[3] P. Allen, J. Böttcher, S. Griffiths, Y. Kohayakawa, and R. Morris, *The chromatic thresholds of graphs*, Adv. Math. **235** (2013), 261–295. ↑2.5

[4] N. Alon and E. Fischer, *2-factors in dense graphs*, Discrete Math. **152** (1996), no. 1-3, 13–23. ↑1.2

[5] N. Alon, P. Seymour, and R. Thomas, *A separator theorem for nonplanar graphs*, J. Amer. Math. Soc. **3** (1990), no. 4, 801–808. ↑3.2

[6] N. Alon and J. H. Spencer, *The probabilistic method*, Third, Wiley-Interscience Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2008. With an appendix on the life and work of Paul Erdős. ↑3.1

[7] N. Alon and R. Yuster, *Every H-decomposition of K_n has a nearly resolvable alternative*, European J. Combin. **21** (2000), no. 7, 839–845. ↑3.4

[8] B. Andrásfai, *Über ein Extremalproblem der Graphentheorie*, Acta Math. Acad. Sci. Hungar. **13** (1962), 443–455. ↑2.5

[9] B. Andrásfai, P. Erdős, and V. T. Sós, *On the connection between chromatic number, maximal clique and minimal degree of a graph*, Discrete Math. **8** (1974), 205–218. ↑1.1, 2.1

[10] J. Balogh and C. Palmer, *On the tree packing conjecture*, SIAM J. Discrete Math. **27** (2013), no. 4, 1995–2006. ↑1.2

[11] B. Bollobás, *Some remarks on packing trees*, Discrete Math. **46** (1983), no. 2, 203–204. ↑1.2

[12] ———, *Modern graph theory*, Graduate Texts in Mathematics, vol. 184, Springer-Verlag, New York, 1998. ↑1

[13] B. Bollobás and S. E. Eldridge, *Packings of graphs and applications to computational complexity*, J. Combin. Theory Ser. B **25** (1978), no. 2, 105–124. ↑1.2

[14] B. Bollobás, A. Kostochka, and K. Nakprasit, *Packing d-degenerate graphs*, J. Combin. Theory Ser. B **98** (2008), no. 1, 85–94. ↑1.2

[15] J. A. Bondy and U. S. R. Murty, *Graph theory*, Graduate Texts in Mathematics, vol. 244, Springer, New York, 2008. ↑1

[16] J. Böttcher, J. Hladký, D. Piguet, and A. Taraz, *An approximate version of the tree packing conjecture*. Israel Journal of Mathematics, to appear. ↑1.2, 3.1, 3.1

[17] St. Brandt, *A 4-colour problem for dense triangle-free graphs*, Discrete Math. **251** (2002), no. 1-3, 33–46. Cycles and colourings (Stará Lesná, 1999). ↑1.1

[18] St. Brandt and E. Ribe-Baumann, *Graphs of odd girth 7 with large degree*, European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2009), 2009, pp. 89–93. ↑2.5

[19] St. Brandt and St. Thomassé, *Dense triangle-free graphs are four colorable: A solution to the Erdős-Simonovits problem*, J. Combin. Theory Ser. B. to appear. ↑1.1

[20] Y. Caro and Y. Roditty, *A note on packing trees into complete bipartite graphs and on Fishburn's conjecture*, Discrete Math. **82** (1990), no. 3, 323–326. ↑1.2

[21] P. A. Catlin, *Embedding subgraphs under extremal degree conditions*, Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977), 1977, pp. 139–145. Congressus Numerantium, No. XIX. ↑1.2

[22] C. C. Chen, G. P. Jin, and K. M. Koh, *Triangle-free graphs with large degree*, Combin. Probab. Comput. **6** (1997), no. 4, 381–396. ↑1.1, 2.5

[23] B. Csaba, A. Shokoufandeh, and E. Szemerédi, *Proof of a conjecture of Bollobás and Eldridge for graphs of maximum degree three*, Combinatorica **23** (2003), no. 1, 35–72. Paul Erdős and his mathematics (Budapest, 1999). ↑1.2

[24] R. Diestel, *Graph theory*, Fourth, Graduate Texts in Mathematics, vol. 173, Springer, Heidelberg, 2010. ↑1, 2.3.1

[25] E. Dobson, *Packing almost stars into the complete graph*, J. Graph Theory **25** (1997), no. 2, 169–172. ↑1.2

[26] ———, *Packing trees into the complete graph*, Combin. Probab. Comput. **11** (2002), no. 3, 263–272. ↑1.2

[27] ———, *Packing trees of bounded diameter into the complete graph*, Australas. J. Combin. **37** (2007), 89–100. ↑1.2

[28] P. Erdős and M. Simonovits, *On a valence problem in extremal graph theory*, Discrete Math. **5** (1973), 323–334. ↑2.5

[29] P. Frankl and V. Rödl, *Near perfect coverings in graphs and hypergraphs*, European J. Combin. **6** (1985), no. 4, 317–326. ↑3.4

[30] A. M. H. Gerards, *Homomorphisms of graphs into odd cycles*, J. Graph Theory **12** (1988), no. 1, 73–83. ↑2.3.2

[31] D. Gerbner, B. Keszegh, and C. Palmer, *Generalizations of the tree packing conjecture*, Discuss. Math. Graph Theory **32** (2012), no. 3, 569–582. ↑1.2

[32] W. Goddard and J. Lyle, *Dense graphs with small clique number*, J. Graph Theory **66** (2011), no. 4, 319–331. ↑2.5

[33] R. L. Graham, M. Grötschel, and L. Lovász (eds.), *Handbook of combinatorics. Vol. 1, 2*, Elsevier Science B.V., Amsterdam; MIT Press, Cambridge, MA, 1995. ↑1.2

[34] R. K. Guy and F. Harary, *On the Möbius ladders*, Canad. Math. Bull. **10** (1967), 493–496. ↑2.1

[35] A. Gyárfás and J. Lehel, *Packing trees of different order into K_n*, Combinatorics (proc. fifth hungarian colloq., keszthely, 1976), 1978, pp. 463–469. ↑1.2, 1.2, 3.1

[36] R. Häggkvist, *Odd cycles of specified length in nonbipartite graphs*, Graph theory (Cambridge, 1981), 1982, pp. 89–99. ↑1.1, 1.1, 2.1

[37] R. Häggkvist and G. P. Jin, *Graphs with odd girth at least seven and high minimum degree*, Graphs Combin. **14** (1998), no. 4, 351–362. ↑2.1

[38] A. Hajnal and E. Szemerédi, *Proof of a conjecture of P. Erdős*, Combinatorial theory and its applications, II (Proc. Colloq., Balatonfüred, 1969), 1970, pp. 601–623. ↑1.2

[39] A. M. Hobbs, *Packing trees*, Proceedings of the Twelfth Southeastern Conference on Combinatorics, Graph Theory and Computing, Vol. II (Baton Rouge, La., 1981), 1981, pp. 63–73. ↑1.2, 3.1

[40] A. M. Hobbs, B. A. Bourgeois, and J. Kasiraj, *Packing trees in complete graphs*, Discrete Math. **67** (1987), no. 1, 27–42. ↑1.2

[41] S. Janson, T. Łuczak, and A. Rucinski, *Random graphs*, Wiley-Interscience Series in Discrete Mathematics and Optimization, Wiley-Interscience, New York, 2000. ↑3.5.4.2

[42] G. P. Jin, *Triangle-free four-chromatic graphs*, Discrete Math. **145** (1995), no. 1-3, 151–170. ↑1.1

[43] T. Łuczak, *On the structure of triangle-free graphs of large minimum degree*, Combinatorica **26** (2006), no. 4, 489–493. ↑2.5, 2.5

[44] T. Łuczak and St. Thomassé, *Coloring dense graphs via vc-dimension*, submitted. ↑2.5

[45] S. Messuti, V. Rödl, and M. Schacht, *Packing minor closed families of graphs*, submitted. ↑3, 3.6.1

[46] S. Messuti and M. Schacht, *On the structure of graphs with given odd girth and large minimum degree*, J. Graph Theory **80** (2015), no. 1, 69–81. ↑1.1, 2, 3.6.1

[47] V. Nikiforov, *Chromatic number and minimum degree of k_r-free graphs*, preprint (2010). ↑2.5

[48] N. Pippenger and J. Spencer, *Asymptotic behavior of the chromatic index for hypergraphs*, J. Combin. Theory Ser. A **51** (1989), no. 1, 24–42. ↑3.4

[49] D. K. Ray-Chaudhuri and R. M. Wilson, *The existence of resolvable block designs*, Survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971), 1973, pp. 361–375. ↑3.4

[50] Y. Roditty, *Packing and covering of the complete graph. III. On the tree packing conjecture*, Sci. Ser. A Math. Sci. (N.S.) **1** (1988), 81–85. ↑1.2

[51] N. Sauer and J. Spencer, *Edge disjoint placement of graphs*, J. Combin. Theory Ser. B **25** (1978), no. 3, 295–302. ↑1.2

[52] C. Thomassen, *On the chromatic number of triangle-free graphs of large minimum degree*, Combinatorica **22** (2002), no. 4, 591–596. ↑1.1, 2.5

[53] P. Turán, *Eine Extremalaufgabe aus der Graphentheorie*, Mat. Fiz. Lapok **48** (1941), 436–452. ↑1.1

[54] R. M. Wilson, *Decompositions of complete graphs into subgraphs isomorphic to a given graph*, Proceedings of the Fifth British Combinatorial Conference (Univ. Aberdeen, Aberdeen, 1975), 1976, pp. 647–659. ↑3.4

[55] R. Yuster, *On packing trees into complete bipartite graphs*, Discrete Math. **163** (1997), no. 1-3, 325–327. ↑1.2

[56] S. Zaks and C. L. Liu, *Decomposition of graphs into trees*, Proceedings of the Eighth Southeastern Conference on Combinatorics, Graph Theory and Computing (Louisiana State Univ., Baton Rouge, La., 1977), 1977, pp. 643–654. Congressus Numerantium, No. XIX. ↑1.2

[57] K. Zarankiewicz, *Sur les relations synétriques dans l'ensemble fini*, Colloquium Math. **1** (1947), 10–14. ↑1.1

Appendix

Summary/Zusammenfassung

We present two results that concern different aspects of extremal graph theory. In the first part we study minimum degree conditions for which a graph with given odd girth is homomorphic to its smallest odd cycle. This is motivated by a classical result of Andrásfai, Erdős, and Sós which states that every n-vertex graph with odd girth at least $2k+1$ and minimum degree larger than $\frac{2n}{2k+1}$ is bipartite. Since the cycle C_{2k+1} is an extremal graph for this problem, we asked whether a weaker degree condition implies the existence of a homomorphism into C_{2k+1}. We show that this happens for any n-vertex graph with odd girth $2k+1$ and minimum degree larger than $\frac{3n}{4k}$ and give a detailed description of the extremal graphs.

The second part of our work is dedicated to a packing problem that has its roots in Gyárfás' Tree Packing Conjecture. This conjecture states that a sequence of n trees $(T_1, \ldots, T_n)$ with $v(T_i) = i$ packs into K_n. An asymptotic version of this conjecture in which trees with bounded maximum degree are packed into $K_{(1+o(1))n}$ was recently proved. We generalise this result from sequences of trees to sequences of graphs from any non-trivial minor-closed class.

Wir stellen zwei Ergebnisse vor, die verschiedene Aspekte der extremalen Graphentheorie betreffen. Im ersten Teil untersuchen wir Minimalgradbedingungen für die ein Graph mit gegebener ungerader Taillenweite homomorph zu dem kleinsten ungeraden Kreis ist, den er enthält. Diese Frage ist durch ein Resultat von Andrásfai, Erdős, und Sós motiviert, welches besagt, dass jeder Graph auf n Ecken mit ungerader Taillenweite mindestens $2k+1$ und Minimalgrad größer als $\frac{2n}{2k+1}$ bipartit ist. Da der Kreis auf $2k+1$ Ecken ein extremaler Graph für dieses Problem ist, untersuchen wir ob eine schwächere Minimalgradbedingung die Existenz eines Homomorphismus in C_{2k+1} impliziert. Wir zeigen, dass dies für

Graphen auf n Ecken mit ungerader Taillenweite $2k+1$ und Minimalgrad größer als $\frac{3n}{4k}$ gilt und beschreiben die extremalen Graphen im Detail.

Im zweiten Teil untersuchen wir ein Packungsproblem, dass seinen Ursprung in der Baumpackungsvermutung von Gyárfás hat. Diese Vermutung besagt, dass jede Folge von Bäumen $(T_1, \ldots, T_n)$ mit $v(T_i) = i$ sich in den K_n packen lässt. Eine asymptotische Variante dieser Vermutung, in der Bäume mit beschränktem Maximalgrad in $K_{(1+o(1))n}$ gepackt werden, wurde kürzlich gezeigt. Wir verallgemeinern dieses Resultat auf Folgen von Graphen mit beschränktem Maximalgrad aus jeder beliebigen nicht-trivialen unter Minorenbildung abgeschlossenen Graphenklasse.

Publications related to this thesis

Articles

- S. Messuti, M. Schacht, *On the structure of graphs with given odd girth and large minimum degree*, Journal of Graph Theory, 80(1), 2015, 69-81
- S. Messuti, V. Rödl, M. Schacht, *Packing minor-closed families of graphs into complete graphs*, submitted to Journal of Combinatorial Theory Series B

Extended abstracts

- S. Messuti, M. Schacht, *On the structure of graphs with given odd girth and large minimum degree*, Proceedings of EuroComb 2013, vol. 16, CRM Series, Ed. Norm., Pisa, 521-526
- S. Messuti, V. Rödl, M. Schacht, *Packing grids into complete graphs*, proceedings of the 9th International Colloquium on Graph Theory and Combinatorics, Grenoble, 2014
- S. Messuti, V. Rödl, M. Schacht, *Packing minor closed families of graphs*, Proceedings of EuroComb 2015, vol. 49 series Electron. Notes Discrete Math., 651-659

Declaration on my contributions

Chapter 2 is mainly based on the paper *On the structure of graphs with given odd girth and large minimum degree* [46], on which I worked together with my PhD supervisor Mathias Schacht. He introduced me to this problem and suggested that Theorem 12 could hold. I read the relevant literature and together we discussed possible strategies for the proof of the theorem. In particular, I developed the proofs of the two technical lemmas that led to Lemmas 15 and 17 here. These lemmas are crucial for the proof of Theorem 13, which describes the extremal case and which I established on my own while writing this dissertation.

Chapter 3 is based on the paper *Packing minor-closed families of graphs into complete graphs* [45], which is joint work with Vojtěch Rödl and Mathias Schacht. We discussed some ideas for an alternative proof of Theorem 24 which could possibly work for other classes of graphs while Vojtěch Rödl was visiting UHH. We first solved the problem for grids, which became the object of an extended abstract I wrote and presented at ICGT 2014. The subsequent work that led to Theorem 25 was mainly carried out here in Hamburg by Mathias Schacht and me. We contributed a fair share in the proof of the main result.

I drafted the first version of both papers, which were later proof-read and refined together with my coauthors.

Declaration on oath/Eidesstattliche Erklärung

I hereby declare, on oath, that I have written the present dissertation by my own and have not used other than the acknowledged resources and aids.

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 15.03.2016

Silvia Messuti