Contributions to the string topology of product manifolds

Dissertation

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Abstract

The thesis discusses several aspects of string topology presented by Chas and Sullivan in [5] on the homology of the free loop space of a closed oriented manifold. After an introductory chapter we use a specific chain model for string topology defined by Irie [20] to perform the homotopy transfer to homology in a special case. We prove vanishing results and combine these with a theorem of Fukaya [13] to get the following result (cf. Theorem 4.17) as a corollary:

Theorem 0.1

A closed, oriented, spin Lagrangian submanifold

$$X \subset (\mathbb{C}^k, \omega_0)$$

for $k = n + m \geqslant 3$ *cannot be of the form* $M \times N$ *where* M, N *are smooth, closed and oriented manifolds of finite dimension* $\dim M = m \geqslant 0$ *and* $\dim N = n \geqslant 3$ *respectively with* M *simply connected and* N *admitting a Riemannian metric of negative sectional curvature.*

Earlier publications derived from the dissertation: –

Zusammenfassung

Die Arbeit behandelt verschiedene Aspekte der String Topologie, dargelegt von Chas und Sullivan in [5], auf der Homologie des freien Schleifenraums einer geschlossenen und orientierten Mannigfaltigkeit. Nach einem einführenden Kapitel benutzen wir ein konkretes Kettenmodell für String-Topologie von Irie [20], um in einem Spezialfall den Homotopie-Transfer auf Homologie durchzuführen. Die daraus resultierenden Verschwindungsresultate kombinieren wir mit einem Theorem von Fukaya [13] und erhalten folgenden Satz (vgl. Theorem 4.17):

Theorem 0.2

Eine geschlossene, orientierte, spin Lagrangesche Untermannigfaltigkeit

$$X \subset (\mathbb{C}^k, \omega_0)$$

für $k = n + m \geqslant 3$ kann nicht von der Form $M \times N$ sein, für M, N glatte, geschlossene und orientierte Mannigfaltigkeiten der Dimension $\dim M = m \geqslant 0$ beziehungsweise $\dim N = n \geqslant 3$, wobei M einfach zusammenhängend ist und N eine Riemannsche Metrik mit negativer Schnittkrümmung zulässt.

Aus dieser Dissertation hervorgegangene Vorveröffentlichungen: –

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Chapter 1

Introduction

1.1 History and motivation

Surprisingly, in mathematics difficult questions are sometimes much easier to handle when first complicating things. Like in modern tendencies in physics, that prefer to regard particles as strings rather than point-like, mathematicians try to understand properties of a space X by examining the space of loops on X. These mapping spaces $C^k(S^1, X)$ are commonly denoted by LX without further specifying $k \in \mathbb{N}_0$. From a topological point of view these are the same (cf. section 2 of [6]).

A way to better understand the geometry of LX is to use the language of algebra and try to understand its homology $H_*(LX)$ arising of a certain chain model. At least researchers in topology, Riemannian geometry, TFT/string theory and symplectic geometry may extract information of an understanding of $H_*(LX)$. Having this broader influence in mind it is justified to study the topology of free loop spaces. Our field of interest is symplectic geometry that poses the motivating question:

"What closed manifolds arise as Lagrangians submanifold of $\mathbb{C}^k$?"

In this thesis our contribution to that question is:

"A high-dimensional product manifold of a hyperbolic and a simply connected manifold does not arise as a Lagrangian submanifold of $\mathbb{C}^k$!"

In order to obtain such a result, we aim to understand the (co-)homology of the free loop space $H^*(LM)$ and $H_*(LM)$ respectively. For $H^*(LM)$ there is the cup product turning it into a ring. Further, as discovered by M. Chas and D. Sullivan, $H_*(LM)$ is not just a module but may be equipped with a BV-algebra structure. Comparably to the Pontryagin product for pointed loop spaces concatenation of loops at its basepoints provides a product $\bullet$, the *loop product*. Notice that basepoints do not coincide in general, thus one needs to incorporate the intersection product

$$H_*(M) \times H_*(M) \xrightarrow{\cap} H_{*-\dim M}(M)$$

in $ev_0(LM) = M$ yielding a product of degree $(-\dim M)$. One works with shifted homology

$$\mathbb{H}_*(LM) := H_{*+\dim M}(LM)$$

in order to get an algebra structure with a product of degree 0.
The BV-operator Δ of degree $+1$ is induced by the natural S^1-action on LM by moving the basepoints around the loops. The loop product and the BV-operator combine to a Lie bracket $\{\cdot,\cdot\}$ of degree $+1$, the *loop bracket*.
Erasing basepoints or putting basepoint markers everywhere along the loops yields maps

$$\begin{tikzcd} \mathbb{H}_*(LM) \arrow[r, bend left, "\mathcal{E}"] & H_*^{S^1}(LM) \arrow[l, bend left, "\mathcal{M}"] \end{tikzcd} .$$

where $\mathcal{M}\circ\mathcal{E}=\Delta$ and $\mathcal{E}\circ\mathcal{M}=0$.
Here $H_*^{S^1}(LM)$ arises via the Borel construction for equivariant homology. The $\mathcal{E}$rase and $\mathcal{M}$ark maps are used to transfer structure from $H_*(LM)$ to $H_*^{S^1}(LM)$ and vice versa. In particular the loop product descends to a Lie bracket $[\cdot,\cdot]$ on $H_*^{S^1}(LM)$, the *string bracket*.
In this thesis the notion *string topology* means dealing with the BV-algebra

$$(\mathbb{H}_*(LM),\bullet,\{\cdot,\cdot\},\Delta)$$

and the graded Lie algebra

$$(H_*^{S^1}(LM),[\cdot,\cdot])$$

for M^n being an n-dimensional manifold that is closed and oriented. Though the theory is defined for integer coefficients we mostly work with field coefficients. In particular in chapter 4 we use real coefficients. Here the notion *higher string topology* in turn stands for discussions concerning the A_∞/L_∞-algebra

$$(\mathbb{H}_*(LM),\{m_k\}_{k\geqslant 1}) \quad \text{and} \quad (\mathbb{H}_*(LM),\{\lambda_k\}_{k\geqslant 1})$$

where m_2 corresponds to the loop product and λ_2 corresponds to the loop bracket.

To be able to do string topology computations we may apply direct methods or drift into the world of algebra. Direct methods are very limited in a way that we may only discuss 'nice' spaces as the circle S^1, the n-torus T^n or surfaces of higher genus Σ_g^2. Here one actually sees how loops or strings interact. This insight is given up in order to get results when using concepts of algebra. In the thesis we use spectral sequences which are shortly recalled in appendix 5.4. Further concepts for doing computations would be Hochschild homology and Cyclic homology. These kind of approaches are not discussed here.

1.2 Motivation from symplectic geometry

An ongoing research project in symplectic geometry asks about the embeddability of closed Lagrangian manifolds into symplectic manifolds $(Y,\,\omega=d\lambda)$.
A submanifold $X\hookrightarrow Y$ is called *Lagrangian* if $\omega|_X=0$, so that $\lambda|_X$ is a closed 1-form. It is called *exact Lagrangian* if the cohomology class $[\lambda|_X]\in H^1(X;\mathbb{R})$ vanishes.

For the exact symplectic manifold $(\mathbb{C}^k, \omega_0)$ with $\omega_0 = d\lambda_0 = d(\sum_{i=1}^{k} x_i dy_i)$ and X closed we know that $X \hookrightarrow \mathbb{C}^k$ Lagrangian implies that

(*i*) $H^1(X; \mathbb{R}) \neq 0$ (Gromov, [16]).

(*ii*) X does not admit a Riemannian metric of negative sectional curvature (Viterbo, cf. [12]).

To prove (*i*), Gromov constructs a non-constant pseudo-holomorphic disk, which in particular is a smooth map $u : (D^2, \partial D^2) \to (\mathbb{C}^k, X)$ such that

$$0 < E(u) := \int_{D^2} u^*\omega_0 = \int_{S^1} u^*\lambda_0 \ ,$$

implying $0 \neq [\lambda_0|_X] \in H^1(X; \mathbb{R})$. In particular it follows that a Lagrangian submanifold of $\mathbb{C}^k$ cannot be simply connected.

For (*ii*) the authors in particular need that all non-constant geodesics are not contractible which is the case for negatively curved manifolds.

The techniques for proving (*i*) and (*ii*) are rather different and do not allow to exclude that a product $M \times N$ of a simply connected and a negatively curved manifold embeds as a Lagrangian submanifold into $\mathbb{C}^k$. In this thesis we aim to treat this special case. We use the work of Fukaya as input.

Fukaya's insight was that compactifications of moduli spaces may be understood in terms of algebraic equations in string topology. These equations in turn yield better obstructions against the Lagrangian embeddability. This approach combines the two different methods of proof into one strategy inspired by homological algebra and in particular by string topology. We briefly recall the author's ideas.

Pick an almost complex structure J compatible with ω_0 that is $J : T\mathbb{C}^k \to T\mathbb{C}^k$ with $J^2 = -1$, $\omega_0(v, Jv) > 0$ for all $v \neq 0$ and $\omega_0(Jv, Jw) = \omega_0(v, w)$ for all v, w. Further choose a class $a \in \pi_2(\mathbb{C}^k, X) \cong \pi_1(X)$. One expects the following moduli spaces to be finite dimensional manifolds:

moduli space	dimension
$\widetilde{\mathcal{M}}(a) := \{u \in C^\infty(\,(D^2, \partial D), (\mathbb{C}^k, X)\,) \mid [u] = a,\ \overline{\partial}_J u = 0\}$ parametrized J-holomorphic curves of class a	$k + \mu(a)$
$\mathcal{M}(a) := \widetilde{\mathcal{M}}(a)/\mathrm{Aut}(D^2, 1)$ unparametrized J-holomorphic curves of class a	$k + \mu(a) - 2$
$\mathcal{N}(a,t) := \{u \in C^\infty(\,(D^2, \partial D), (\mathbb{C}^k, X)\,) \mid [u] = a,\ \overline{\partial}_J u = \eta_t\}$ parametrized, perturbed J-holomorphic curves of class a	$k + \mu(a)$
$\mathcal{N}(a) := \bigcup_{t \in [0,1]} \mathcal{N}(a,t)$	$k + \mu(a) + 1$

Remark that $\overline{\partial}_J u = \frac{1}{2}(du + J \circ du \circ j) \in \Omega^{0,1}(D^2, u^*T\mathbb{C}^k) \cong \Omega^{0,1}(D^2, \mathbb{C}^k)$ is the antiholomorphic part of du and $\{\eta_t\}$ is a one parameter family of antiholomorphic one forms satisfying

- $\eta_0 = 0$ (so that $\mathcal{N}(a, 0) = \widetilde{\mathcal{M}}(a)$)
- η_1 such that $\mathcal{N}(a, 1) = \varnothing$ for all $a \in \pi_2(\mathbb{C}^k, X)$.

Recall that for the Maslov index μ one has $\mu(a) \in 2\,\mathbb{Z}$ since X is oriented. Later we work with a degree $(-\,k)$ shifted chain complex, where $\widetilde{\mathcal{M}}(a)$, $\mathcal{M}(a)$ and $\mathcal{N}(a,t)$ yield even dimensional data.

All stated moduli spaces come with an evaluation map

$$ev_1 : \text{'moduli space'} \to X$$

via $u \mapsto u(1)$ and $[u] \mapsto u(1)$, respectively. The second map defined on $\mathcal{M}(a)$ is well-defined since we only divide out the automorphisms that fix $1 \in \partial D^2$.

The spaces $\mathcal{M}(a)$ and $\mathcal{N}(a)$ are compactified by adding bubble trees of J-holomorphic curves. For details the reader is referred to [31] and especially chapter 4 therein. Remark that only disk bubbles and no sphere bubbles appear since $\pi_2(\mathbb{C}^k) = 0$. The resulting compact spaces are expected to have codimension one boundaries

$$\partial\,\overline{\mathcal{M}}(a) = \coprod_{a_1 + a_2 = a} \mathcal{M}(a_1) \times_X \mathcal{M}(a_2) \quad \text{and}$$
$$\partial\,\overline{\mathcal{N}}(a) = \underbrace{\mathcal{N}(a,1) \sqcup \mathcal{N}(a,0)}_{=\widetilde{\mathcal{M}}(a)} \sqcup \coprod_{a_1 + a_2 = a} \big(\mathcal{N}(a_1) \times_X \mathcal{M}(a_2) \ \sqcup\ \mathcal{M}(a_1) \times_X \mathcal{N}(a_2)\big)\ ,$$

where the fiber products are taken using the evaluation maps ev_1 described above.

Fukaya's insight was that these compactifications may be described in the language of string topology as follows. The evaluation map

$$\begin{aligned} ev : C^\infty(\,(D^2, \partial D), (\mathbb{C}^k, X)\,) &\longrightarrow LX \\ u &\longmapsto u|_{\partial D^2} \end{aligned}$$

induces a corresponding map for the above moduli spaces. It allows to interpret these moduli spaces as chains in a certain chain model $C_{*-k}(LX)$. Heuristically speaking when lifting the string topology operations defined by Chas and Sullivan to chain level one gets the following identities

$$\begin{aligned} \partial\,\mathcal{M} &= \frac{1}{2}\,\{\mathcal{M}, \mathcal{M}\}\,, \\ \partial\,\mathcal{N} &= \{\mathcal{N}, \mathcal{M}\} + [X] \end{aligned}$$

where $\mathcal{M} := \sum_{a\neq 0} \overline{\mathcal{M}}(a)$ is of even degree $|\mathcal{M}(a)| = k + \mu(a) - 2 - k = \mu(a) - 2$ and $\mathcal{N} := \sum_a \overline{\mathcal{N}}(a)$ is of even degree $|\mathcal{N}(a)| = k + \mu(a) - k = \mu(a)$ in $C_{*-k}(LX)$.
Remark that for $a \neq 0$ we have that $\widetilde{\mathcal{M}}(a)$ is a degenerate chain, factorizing over $\mathcal{M}(a)$. The remaining $\widetilde{\mathcal{M}}(0) = \mathcal{M}(0)$ just consists of constant J-holomorphic curves corresponding to the chain of constant loops $[X]$ in LX.

The infinite sums make sense when working with completions with respect to the action filtration $\{\mathcal{F}^l\}_{l\in\mathbb{Z}}$, with $\mathcal{F}^l \supset \mathcal{F}^{l+1}$ given by

$$\mathcal{F}^l := \mathcal{F}^l C_*(LX) := \{c \in C_*(LX)\,|\,\mathcal{A}(c_i) \geqslant l\}$$

where $c = \sum c_i$ and c_i with connected domain. Here the action $\mathcal{A}(c_i)$ is defined as follows. Having connected domains means that c_i is a chain in a path component $L^{\alpha_i}X$ of LX. Remark that

$$\alpha_i \in \pi_0(LX) \cong \tilde{\pi}_1(X) = \text{conjugacy classes of } \pi_1(X)\ .$$

For a smooth map $u : (D^2, \partial D^2) \to (\mathbb{C}^k, X)$ the action

$$\mathcal{A}(u) := \int_{D^2} u^*\omega_0 = \int_{S^1} u^*\lambda_0 = \int_{u|_{S^1}} \lambda_0$$

just depends on the class $[u] \in \tilde{\pi}_2(\mathbb{C}^k, X) \cong \tilde{\pi}_1(X)$. We thus define

$$\mathcal{A}(c_i) := \mathcal{A}(\alpha_i) := \mathcal{A}(u)$$

where $[u|_{S^1}] = \alpha_i$.

The action integral is additive when composing loops.
In the language of string topology this means that if $\{a, b\} \neq 0$, we have

$$\mathcal{A}(\{a, b\}) = \mathcal{A}(a) + \mathcal{A}(b)\ .$$

For the chain coming from the moduli space of non-constant holomorphic curves $\mathcal{M}$ we can apply proposition 4.1.4. of [31] and get

$$\mathcal{A}(ev_*(\mathcal{M})) > 0 \ .$$

Further for the chain $[X] \mathrel{\widehat{=}} (X \to LX)$ coming from the constant loops at each point of X we get

$$\mathcal{A}([X]) = 0 \ .$$

In full generality the observations are summarized as a theorem (see [25] for more details) proposed by Fukaya in [13].
Two difficulties are silently suppressed here. It is quite nontrivial to find an almost complex structure such that $\mathcal{M}$ and $\mathcal{N}$ are transversally cut out, and thus are manifolds, whose boundary can still be described as outlined above. Further since working with real coefficients one has to think about signs in the stated equations, resulting in a discussion about orientations of the involved moduli spaces.

Theorem 1.1 (*Thm. 6.1., Thm. 6.4. and Thm. 12.3. of [13]*)

For a closed, oriented, spin Lagrangian submanifold $X \subset \mathbb{C}^k$ there exists a completed, filtered, degree shifted complex $\widehat{C}_*(LX)$ with a filtered dg Lie algebra structure $(\partial, \{\cdot,\cdot\})$ implementing the Chas-Sullivan loop bracket on homology.

The moduli spaces yield chains $\mathcal{M}, \mathcal{N} \in \widehat{C}_*(LX)$ with $\mathcal{M} \in \widehat{C}_*(L^{\neq 0}X)$, which satisfy the following equations:

$$\partial\,\mathcal{M} = \frac{1}{2}\,\{\mathcal{M},\mathcal{M}\}\,, \tag{1.1}$$

$$\partial\,\mathcal{N} = \{\mathcal{N},\mathcal{M}\} + [X] \tag{1.2}$$

A suitable dg Lie algebra structure on chain level is introduced and discussed in Irie [20].

This theorem motivates the study of algebraic structures on $H_*(LX)$ in chapter 4 of this thesis. There the focus is laid on closed, oriented, finite dimensional Riemannian manifolds X arising as products $M \times N$ where M, N are assumed to be smooth, closed and oriented Riemannian manifolds of finite dimension $\dim M = m \geqslant 0$ respectively $\dim N = n \geqslant 3$. Further M is assumed simply connected and N has negative sectional curvature. To apply the arguments of Fukaya we need X to be spin. For the topological discussion presented in the text this assumption is negligible.

1.3 Results of the thesis

When nothing else is indicated we consider (co-)homology with coefficients in a field of characteristic 0. Goals of our study can be summarized as follows:

- How far can the vector space structure of $H_*(L(X_1 \times X_2)), H_*^{S^1}(L(X_1 \times X_2))$ be described in terms of the homology vector space structure of the separate factors?

- How can string topology operations on $H_*(L(X_1 \times X_2)), H_*^{S^1}(L(X_1 \times X_2))$ be described in terms of those on the homology of the separate factors?
- How can A_∞/L_∞-structures on $H_*(LX)$ be computed in specific examples?
- For which manifolds can we achieve appropriate vanishings result for the higher operation implying the non-embeddability as a Lagrangian submanifold into $\mathbb{C}^k$?

The following results are discusses in the thesis. The author remarks that not all are completely new but proofs of them are sometimes missing in the literature.

(i) **String topology of products**

It is explicitly proven that one has a Künneth type isomorphism of BV-algebras

$$H_*(L(M_1 \times M_2)) \cong H_*(LM_1) \otimes H_*(LM_1)$$

for M_i being finite dimensional smooth manifolds that are closed and oriented. Further by analysing the corresponding universal bundles we present a way of how the Euler class of the S^1-bundles

$$L(X_1 \times X_2) \longrightarrow (LX_1 \times LX_2)//S^1 \quad \text{and} \quad (LX_1 \times LX_2)//S^1 \longrightarrow LX_1//S^1 \times LX_2//S^1 ,$$

where $LX//S^1 := LX \times_{S^1} ES^1$, may be computed in terms of the Euler classes of the separate factors. Using the Serre spectral sequences gives a method to compute

$$H_*^{S^1}(L(X_1 \times X_2))$$

whenever the X_i are path-connected topological spaces. Unfortunately so far it is not clear how the string bracket may be computed in this set-up due to missing information about the $\mathcal{M}$ark and $\mathcal{E}$rase map for the product case.

(ii) **Higher structures in string topology**

We want to understand A_∞-/L_∞-algebra structures in string topology. Therefore we rely on the work of K. Irie [20]. In that article it is proven that when working with de Rham chains and real coefficients we get a Gerstenhaber algebra structure on chain level of LX. This structure in turn descends to the string topology structure on homology defined by Chas and Sullivan.

By applying the homotopy transfer construction this equips quasi-isomorphic chain complexes (as for example $H_*(LX)$) with an A_∞-/L_∞-algebra structure. We prove that for a product X of a simply connected and a hyperbolic manifold of dimension greater than 3 the corresponding higher operations on $H_*(LX)$ essentially vanish (c.f. theorem 4.15 and 4.16).

Using the arguments of Fukaya as a black box this yields an obstruction against the Lagrangian embeddability of X into $\mathbb{C}^k$, precisely speaking we prove:

Theorem 1.2

A closed, oriented, spin Lagrangian submanifold $X \subset (\mathbb{C}^k, \omega_0)$ for $k = n + m \geqslant 3$ can not be of the form

$$M \times N$$

where M, N are smooth, closed and oriented Riemannian manifolds of finite dimension $\dim M = m \geqslant 0$ respectively $\dim N = n \geqslant 3$, with M simply connected and N of negative sectional curvature.

1.4 Outline

As the results suggest the text consists of three parts:

- A general, geometry focused introduction to the world of string topology in chapter 2.
- An algebraic discussion of A_∞/L_∞-algebras in chapter 3. As an example we construct an A_∞-algebra structure on the homology of a complex C, where $H(C)$ is isomorphic to $H_{*+n}(LS^n)$ as an algebra for $n \geqslant 2$.
- A construction of the transfer of the dg Lie algebra structure on Irie's complex (cf. [20]) to homology in chapter 4. The arising vanishing results for a certain class of manifolds then yield theorem 1.2 as a corollary.

The first chapter can be seen as more introductory since many already known concepts are described. In chapter 3 we discuss A_∞/L_∞-structures in general and in particular for the homology of LS^n. This serves as a toy model for the general picture of higher string topology of product manifolds in the last chapter of this thesis. Chapter 4 forms the heart of the thesis in the sense that we discuss concepts that are necessary for addressing the motivating question of the present studies, namely the Lagrangian embeddability into $\mathbb{C}^k$.

Chapter 2

String topology

In this chapter we discuss basic notions of *string topology*. In particular we review algebraic operations on

$$H_*(LM) \quad \text{and} \quad H_*^{S^1}(LM)$$

where M is a finite dimensional smooth manifold that is closed and oriented. Throughout the chapter we closely follow the original work of Chas and Sullivan (cf. [5]). We recall their ideas with a slight focus on the geometrical perspective, meaning that we highlight why concepts only work for homology and may not be generalized to a chain level description. As the title of the thesis suggests we then pay attention to manifolds that arise as products $M = M_1 \times M_2$. The chapter then directly leads to section 4.1 where Irie's rigorous definition of string topology on the chain level is reviewed.

2.1 Topology of loop spaces

As outlined in the motivation we are interested in certain path/loop spaces. In the following we denote the standard interval $[0,1]$ by I and regard the one dimensional circle as $S^1 = \mathbb{R}/\mathbb{Z}$. Without further mention we require X to be path connected and having the homotopy type of a countable CW-complex.

Definition 2.1

For a given path-connected, pointed topological space (X, x_0) we consider

- *the path space*

$$P_{x_0}X := \{\gamma : I \xrightarrow{C^0} X \mid \gamma(0) = x_0\}$$

- *the based loop space, and its Moore version,*

$$\begin{aligned}\Omega_{x_0}X \equiv \Omega X &:= \{\gamma : S^1 \xrightarrow{C^0} X \mid \gamma(0) = \gamma(1) = x_0\} \\ \Omega^M_{x_0}X \equiv \Omega^M X &:= \{(\gamma, r) : [0,\infty) \xrightarrow{C^0} X \mid \forall\ t \geqslant r \in [0,\infty) : (\gamma,r)(0) = (\gamma,r)(t) = x_0\} \\ &\subset C^0([0,\infty), X) \times \mathbb{R}\end{aligned}$$

- *the free loop space, and its Moore version,*

$$\begin{aligned}LX :=& \{\gamma : S^1 \xrightarrow{C^0} X\} \\ L^M X :=& \{(\gamma, r) : [0,\infty) \xrightarrow{C^0} X \mid \forall\ t \geqslant r : (\gamma,r)(0) = (\gamma,r)(t)\}\end{aligned}$$

- *the homotopy orbit space or string space*

$$LX \times_{S^1} ES^1 \ .$$

Remark 2.2. *For the homotopy orbit space we quotient out the diagonal S^1-action. This is done by using ES^1, the total space of the universal bundle over BS^1, in order to get the circle acting freely on $LX \times ES^1$ and thus the quotient to be non-singular. Remark that the action $S^1 \curvearrowright LX$ via*

$$\gamma(\cdot) \to \gamma(\cdot + \theta)$$

for $\theta \in S^1$, $\gamma \in LX$ is not free since for example constant loops $\gamma_{x_0}(t) \equiv x_0 \in X$ are fixed points for all θ. For a short recap about classifying spaces and the Borel construction we refer to Appendix 5.2.

Lemma 2.3

We have deformation retractions

$$\Omega^{M}_{x_0} X \xrightarrow{\simeq} \Omega_{x_0} X \quad \text{and} \quad L^{M} X \xrightarrow{\simeq} LX \ .$$

Proof: The case for the pointed loop space is discussed in [3]. We describe the case for the free loop space that works analogously.
Remark that we have a homeomorphism

$$LX \cong \{(\gamma, r) \in L^{\mathrm{M}} X \ | r = 1\} =: L^{\mathrm{M}}_{=1} X$$

that is used for the following inclusions

$$L^{\mathrm{M}}_{=1} X \overset{\iota_2}{\hookrightarrow} L^{\mathrm{M}}_{\geqslant 1} X := \{(\gamma, r) \in L^{\mathrm{M}} X \ | r \geqslant 1\} \overset{\iota_1}{\hookrightarrow} L^{\mathrm{M}} X \ .$$

We deform in two steps from right to left.
A deformation retraction $H_1 : [0,1] \times L^{\mathrm{M}} X \to L^{\mathrm{M}} X$ for ι_1 is given by

$$H_1(s, (\gamma, r)) \equiv H_1^s((\gamma, r)) := \begin{cases} (\gamma, r+s) & \text{for} \quad r+s \leqslant 1\,, \\ (\gamma, 1) & \text{for} \quad r \leqslant 1 \text{ and } r+s \geqslant 1\,, \\ (\gamma, r) & \text{else}\,. \end{cases}$$

That is we have $H_1^s \circ \iota_1 = \mathrm{id}_{L^{\mathrm{M}}_{\geqslant 1} X}$ for all $s \in [0,1]$ and $\iota_1 \circ H_1^1 \simeq \mathrm{id}_{L^{\mathrm{M}} X}$ via H_t.
The space $L^{\mathrm{M}}_{\geqslant 1} X$ deformation retracts to $L^{\mathrm{M}}_{=1} X$ via H_2 given by reparameterizations of the form

$$H_2(s, (\gamma, r)) \equiv H_2^s((\gamma, r)) := (\gamma \circ h_{r,s}, (1-s)r + s)$$

where $h_{r,s}(t) := \frac{r}{(1-s)r+s} t$ reparametrizes γ. ■

In particular the Moore- and the ordinary loop space (based or free) have the same homotopy type and thus their homotopy and homology groups are isomorphic.

Remark that ΩX, $\Omega^M X$ are H-spaces, that is we get an induced algebra structure on $H_*(\Omega X)$, $H_*(\Omega^M X)$. The reader is referred to Appendix 5.3. The product for ΩX is simply the concatenation, whereas the product on $\Omega^M X$ is given by

$$(\gamma, r) * (\tau, s) = (\gamma * \tau, r + s) \ ,$$

where

$$\gamma * \tau(t) := \begin{cases} \gamma(t) & , \ 0 \leqslant t \leqslant r \\ \tau(t-r) & , \ r \leqslant t \leqslant r+s \end{cases} \quad .$$

Clearly i_1, i_2 in the proof above are H-maps and the homeomorphism relating ΩX and $\Omega^{\mathrm{M}} X$ is an H-equivalence. One easily checks that H_1^1, H_2^1 are H-maps, namely

$$\begin{aligned} & H_1^1((\gamma, r) * (\tau, s)) \\ =& \begin{cases} ((\gamma * \tau), 1) & , \ r+s \leqslant 1 \\ ((\gamma * \tau), r+s) & , \ r+s \geqslant 1 \end{cases} \sim \begin{cases} ((\gamma * \tau), 2) & , \ r, s < 1 \\ ((\gamma * \tau), r+1) & , \ r \geqslant 1, \, s \leqslant 1 \\ ((\gamma * \tau), 1+s) & , \ r \leqslant 1, \, s \geqslant 1 \\ ((\gamma * \tau), r+s) & , \ r, s \geqslant 1 \end{cases} \\ =& H_1^1((\gamma, r)) * H_1^1((\tau, s)) \end{aligned}$$

and

$$\begin{aligned} & H_2^1((\gamma, r) * (\tau, s)) = H_2^1(((\gamma * \tau), r+s)) \\ =& ((\gamma * \tau) \circ h_{r+s,1}, 1) \sim (\gamma \circ h_{r,1}, 1) * (\tau \circ h_{s,1}, 1) = H_2^1((\gamma, r)) * H_2^1((\tau, s)) \ . \end{aligned}$$

We conclude that we even have an algebra isomorphism

$$H_*(\Omega X) \cong H_*(\Omega^M X) \ . \tag{2.1}$$

As the headline of this chapter suggests we are interested in the topology of loop spaces and it thus does not matter if we work with the Moore version or not. The advantage of Moore loop spaces is provided by the fact that the concatenation operation is associative. The space of based Moore loops is a monoid with the constant loop x_0 being the neutral element. For non-Moore loops concatenation is only associative up to homotopy given by reparameterization.
We introduce the slightly less intuitive Moore version of the free loop space for defining operations (see chapter 4.4) for chains on LX. There we need that concatenating loops is strictly associative and thus defines an algebra structure on $C_*(LX)$.
To keep the presentation simple we mostly work with spaces of non-Moore loops ΩX and LX in this chapter.

As all considered loop spaces are mapping spaces $\mathrm{Map}(X, Y)$ of continuous maps between topological spaces X and Y, we equip them with the compact-open topology (see e.g. [29]). A subbase is given by open sets of the form $\{f \in \mathrm{Map}(X, Y) \mid f(K) \subset U\}$ for $K \subset X$ compact and $U \subset Y$ open.

These loop spaces are not only just topological spaces. Using J. Milnor's result (Corollary 2 in [32]) we know that for a topological space Y having the homotopy type of a countable CW-complex, the mapping space $\mathrm{Map}(X,Y)$ is of the homotopy type of a countable CW-complex if X is a compact metric space.

As a first approach to understand these spaces we think about their path-connected components, labelled by classes in $\pi_0(\cdot)$. Loop spaces are disjoint unions

$$\begin{aligned}
\Omega_{x_0}X &= \coprod_{[f]\in\pi_0(\Omega_{x_0}X)} \left(\Omega_{x_0}^{[f]}X := \{\gamma\in\Omega_{x_0}X \mid \gamma\sim_{x_0} f\}\right) \\
LX &= \coprod_{[f]\in\pi_0(LX)} \left(L^{[f]}X := \{\gamma\in LX \mid \gamma\sim f\}\right) ,
\end{aligned}$$

where we used based and free homotopies, respectively. For homology we get

$$\begin{aligned}
H_*(\Omega_{x_0}X) &\cong \bigoplus_{[f]\in\pi_0(\Omega_{x_0}X)} H_*(\Omega_{x_0}^{[f]}X) \\
H_*(LX) &\cong \bigoplus_{[f]\in\pi_0(LX)} H_*(L^{[f]}X) \\
H_*(LX\times_{S^1} ES^1) \equiv H_*^{S^1}(LX) &\cong \bigoplus_{[f]\in\pi_0(LX)} H_*^{S^1}(L^{[f]}X) .
\end{aligned}$$

Points in the loop space LX correspond to loops in X. We aim to understand how $\pi_0(LX)$ may be interpreted in terms of the fundamental group $\pi_1(X)$. For a short recollection of fundamental groups and homotopy theory in general the reader is referred to Appendix 5.1.

Two given based loops $f,g\in\Omega_{x_0}X$ are homotopic and thus define the same element of $\pi_0(\Omega_{x_0}X)$ if and only if there exists a path of based loops connecting them. The map

$$\begin{gathered}
H : I \longrightarrow \Omega_{x_0}X \\
H(0,t) = f(t) \ ; \ H(1,t) = g(t) \ ; \ H(s,0) = x_0 \ ,
\end{gathered}$$

is interpreted as a homotopy $H : I\times S^1 \to X$ implying $[f]=[g]\in\pi_1(X,x_0)$.

Next we want to understand $\pi_0(LX)$. This is done in two steps.

For points f,g in the same path-component of the free loop space LX we do not have $f(0)\neq g(0)$ in general and thus may not work with a based homotopy H with $H(s,0)=x_0$. But we require X to be path-connected and thus get a path h connecting $f(0)$ and $g(0)$. Since g and $h^{-1}gh$ are freely homotopic in X, we identify $\pi_0(LX)$ with the set of based loops modulo free homotopies that do not have to fix the base point x_0 (see figure 2.1).

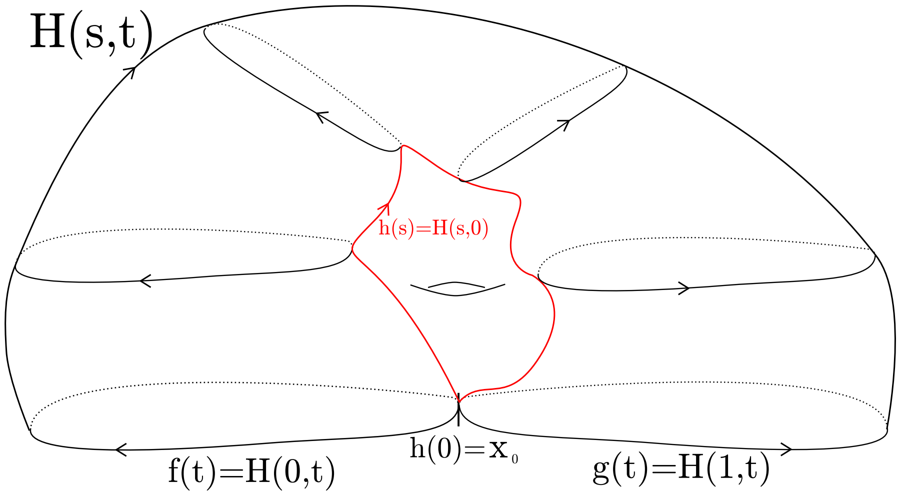


Figure 2.1: Free homotopy $H' : I \to LX$ connecting f and g

So suppose $f, g \in \Omega_{x_0}X$ are freely homotopic via

$$H' : I \longrightarrow LX \quad \widehat{=} \quad H : I \times S^1 \longrightarrow X$$
$$H(0,t) = f(t) \ ; \ H(1,t) = g(t) \ .$$

The path traversed by the base point $h(s) = H(s,0)$ is a loop in X that is $[h] \in \pi_1(X, x_0)$. We claim that $[f] = [h^{-1}gh] = [h]^{-1}[g][h]$ and thus get that loops in $\Omega_{x_0}X$ which are freely homotopic correspond to elements in $\pi_1(X, x_0)$ that are conjugate.
A homotopy is given by

$$\widetilde{H} : I \times S^1 \longrightarrow X$$
$$(s,t) \longmapsto \begin{cases} h(3t) & ; \ t \in [0, \frac{s}{3}] \\ H(s, \frac{t-\frac{s}{3}}{1-\frac{2s}{3}}) & ; \ t \in [\frac{s}{3}, 1-\frac{s}{3}] \\ h(3(1-t)) & ; \ t \in [1-\frac{s}{3}, 1] \end{cases} \quad .$$

Conversely for $[h^{-1}fh] = [g] \in \pi_1(X, x_0)$ we may use the homotopy yielding $h^{-1}fh \sim g$ to write down a free homotopy where the path of the basepoint is a closed loop in X. We thus get $[f] = [g] \in \pi_0(LX)$.

In total when assuming X to be path-connected we get

$$\pi_0(\Omega_{x_0}X) \xleftrightarrow{1:1} \pi_1(X) \tag{2.2}$$

and

$$\begin{aligned}\pi_0(LX) \overset{1:1}{\longleftrightarrow} \widetilde{\pi}_1(X) & \\ := \underbrace{\{[f] = [g] \mid \exists\, \gamma \in LX \;:\; f \sim \gamma^{-1} g \gamma\}}_{\text{conjugacy classes of } [f],[g] \in \pi_1(X)} & \\ \underbrace{=}_{\substack{\text{if } \pi_1(X) \\ \text{abelian}}} \pi_1(X)\ . &\end{aligned} \tag{2.3}$$

In order to get a better handling of our loop spaces we make use of the fact that they all fit into fibrations. We refer to Appendix 5.1 for a short summary of the most important facts of fibrations. For them we have many methods for deriving topological properties of the involved spaces, for example long exact homotopy sequences and spectral sequences (Appendix 5.4).

Definition; Lemma 2.4. The following maps are fibrations:

- path-loop fibration

$$\begin{tikzcd} \Omega X \arrow[r] & P_{x_0}X \arrow[d, "ev_1"] & \gamma \arrow[d, mapsto] \\ & X & \gamma(1) \end{tikzcd}$$

- loop-loop fibration

$$\begin{tikzcd} \Omega X \arrow[r] & LX \arrow[d, "ev_0"'] & \gamma \arrow[d, mapsto, "ev_0"] & \gamma_x(\cdot) = x \\ & X \arrow[u, bend right, dotted, "s"'] & \gamma(0) & x \arrow[u, mapsto, "s"'] \end{tikzcd}$$

- loop-string fibration

$$\begin{tikzcd} S^1 \arrow[r] & LX \times ES^1 \arrow[d] \\ & LX \times_{S^1} ES^1 \end{tikzcd}$$

Remark 2.5. *By using the long exact homotopy sequence (see e.g. Appendix 5.4) and that $P_{x_0}X$ is contractible we get that the homotopy groups of the involved spaces are given by*

$$\begin{aligned}\pi_i(\Omega^\alpha X) &\cong \pi_{i+1}(X) \\ \pi_i(L^0 X) &\cong \pi_i(\Omega^0 X) \oplus \pi_i(X) \cong \pi_{i+1}(X) \oplus \pi_i(X)\end{aligned}$$

for $i \geqslant 1$. Further

$$\pi_i(L^\alpha X \times_{S^1} ES^1) \cong \pi_i(L^\alpha X)$$

for $i \geqslant 3$.

Proof: We show that the stated maps are fibrations.

Denote the set of continuous maps $I \to X$ by X^I. Consider the associated fibration $p : E_\iota \to X$ to the map $\{x_0\} \xrightarrow{\iota} X$. As described in Appendix 5.1 its total space is given by

$$E_\iota = \{(x_0, \gamma) \in \{x_0\} \times X^I \mid \gamma(0) = x_0\} = P_{x_0}X =: PX \ .$$

Since the fibration map is of the form $p(x_0, \gamma) = \gamma(1)$ its general fiber is given by

$$p^{-1}(x) = \{(x_0, \gamma) \in \{x_0\} \times X^I \mid \gamma(0) = x_0, \gamma(1) = x\} \simeq \Omega_{x_0}X \ .$$

This shows that $PX \to X$ is a fibration. The construction of the associated fibration further yields $PX \simeq \{x_0\}$ which implies $\pi_{i\geqslant 0}(PX) = 0$.
Observe that the contractibility of the path space PX simplifies the long exact homotopy sequence for the path-loop fibration as follows

$$\cdots \to \pi_n(\Omega^\alpha X) \to 0 \to \pi_n(X) \to \pi_{n-1}(\Omega^\alpha X) \to 0 \to \cdots \to \pi_1(X) \to \pi_0(\Omega^\alpha X) \ .$$

Exactness directly implies $\pi_i(\Omega^\alpha X) = \pi_{i+1}(X)$ for $i \geqslant 1$.

We directly show that $X^I \xrightarrow{(ev_0, ev_1)} X \times X$ is a fibration. Consider the commuting diagram

$$\begin{tikzcd}
Y \times \{0\} \arrow[r, "g"] \arrow[d] & X^I \arrow[d, "(ev_0, ev_1)"] \\
Y \times I \arrow[r, "G=(G_1,G_2)"] & X \times X
\end{tikzcd} \ .$$

We define $\widetilde{G} : Y \times I \to X^I$ as

$$X \ni \widetilde{G}(y,t)(s) := \begin{cases} G_1(y, t-3s) & , \ 0 \leqslant s \leqslant \frac{t}{3} \\ g(y,0)(\frac{1}{3-2t}(3s-t)) & , \ \frac{t}{3} \leqslant s \leqslant 1 - \frac{t}{3} \\ G_2(y, 3(s-1)+t) & , \ 1 - \frac{t}{3} \leqslant s \leqslant 1 \end{cases}$$

and get that $\widetilde{G}(y,0) = g(y,0)$ and

$$(ev_0, ev_1) \circ \widetilde{G}(y,t) = (\widetilde{G}(y,t)(0), \widetilde{G}(y,t)(1)) = (G_1(y,t), G_2(y,t)) \ .$$

That is $X^I \xrightarrow{(ev_0, ev_1)} X \times X$ is a fibration.
Pulling back this fibration along the map $\Delta : X \to X \times X$ yields the loop-loop fibration. The existence of a global section $s : M \to LM$ implies that the long exact homotopy sequence for the loop-loop fibration splits. With $\pi_i(\Omega^\alpha X) = \pi_{i+1}(X)$ we get $\pi_i(L^0X) \cong \pi_{i+1}(X) \oplus \pi_i(X)$ for $i \geqslant 1$.

The map $LX \times ES^1 \to LX \times_{S^1} ES^1$ is a S^1-principal bundle and thus a fibration by construction. ∎

2.2 Operations on the homology of certain loop spaces

In the following discussion we replace X by M since we require the underlying space to carry the structure of a n-dimensional manifold M^n that is closed and oriented.
The standard reference for the following chapter is the original article [5]. When we define our operations we mostly refer to it. In our summary of the construction we keep a geometric focus, relying on ideas illustrated in [7]. This geometric approach helps in section 4.1 for a chain level description of string topology. For a strict homotopy theoretic construction the reader is referred to [11]. A general overview of both approaches and possible further developments is provided by [9].

Remark that the upcoming section does not claim mathematical preciseness. We aim to provide a schematic picture about the particular operations. For a detailed discussion of the operations on chain level we refer to [20] and chapter 4.

2.2.1 The commutative algebra $(\mathbb{H}_*(LM), \bullet)$

One easily defines an intersection product $\cap$ on $H_*(M)$ if M is a Poincaré duality space. This is done by dualizing the cup product with the help of Poincaré duality. This approach can not be used for defining such a product for the homology of free loop spaces LM.
But Poincaré duality is defined for M and we have an intersection product $\cap$ (of degree $-n$) on $H_*(M)$. We further have the Pontryagin product $\bullet$ (of degree 0) on $H_*(\Omega M)$. The theory for pointed loop spaces is relatively classical. Important results are stated in appendix 5.3.
As we have seen, the spaces LM, ΩM and M fit into the loop-loop fibration. Thus we may regard LM as a twisted product of M and ΩM and try to combine the two operations $\cap$ and $\bullet$ to define the so called loop product $\bullet$ (of degree $-n$) on $H_*(LM)$. We remark that similarly to the intersection product the loop product is defined on homology but on chain level only makes sense for transversal chains. We adopt the language of [5] and call such operations *transversally defined* on chain level.

In the following we work with coefficients in a field $\mathbf{k}$ of characteristic 0 (mostly $\mathbb{Q}$ or $\mathbb{R}$). It is possible to define the operations for $\mathbb{Z}$ coefficients. This is done in the stated references above.
Recall the theorem of R. Thom ([35]) about realizing homology classes by manifolds. For all classes $a \in H_i(M;\mathbb{Z})$ there exists $k \in \mathbb{N}$ such that $ka = f_*[K^i]$ where

$$f : K^i \to M$$

is a smooth map from a closed, oriented, i-dimensional manifold K.
This allows us to describe the intersection product for coefficients in the field $\mathbf{k}$ coefficient set-up as follows. Namely for $a \in H_i(M;\mathbf{k})$ and $b \in H_j(M;\mathbf{k})$ we get representing chains $f_a : K_a^i \to M$ and $f_b : K_b^j \to M$, that is

$$k_a \cdot a = (f_a)_*[K_a^i] \qquad \text{and} \qquad k_b \cdot b = (f_b)_*[K_b^i] \ . \tag{2.4}$$

Recall that we have

Proposition 2.6 (*Corollary 2.5 of [23]*)

Let $f : V \to M$, $g : W \to M$ be two maps between manifolds. Then there is a homotopy h_t of g such that $h_0 = g$ and $h_1 \pitchfork f$. In particular $[g] = [h_1]$ on homology $H(M)$.

After such a perturbation of f_b to $\widetilde{f}_b$ (by abuse of notation also denoted by f_b) we get transversality of the two maps, $f_a \pitchfork f_b$. By the implicit function theorem the space

$$K_{a\cdot b} \equiv K_a \times_M K_b := \{(k_a, k_b) \in K_a \times_M K_b \mid f_a(k_a) = \widetilde{f}_b(k_b)\}$$

is an oriented manifold of dimension $i + j - n$. This yields a chain

$$f_a \cap f_b : K_{a\cdot b} \to M \tag{2.5}$$

of degree $i + j - n$.

For details about which orientation is naturally assigned to $K_a \times_M K_b$ the reader is referred to chapter 8.2. of [14]. In the following we use their conventions. In order to understand sign issues we recap some properties of the orientation of fibre products. Reversing the orientation of some manifold X is as usually denoted by $-X$.

Lemma 2.7 (*Chapter 8.2. of [14]*)

For smooth oriented manifolds X_i and Y_j ($\partial Y_j = \emptyset$) one has orientation preserving diffeomorphisms between

(i) $\partial(X_1 \times_Y X_2)$ *and* $\partial X_1 \times_Y X_2 \sqcup (-1)^{\dim X_1 + \dim Y} X_1 \times_Y \partial X_2$

(ii) $(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3$ *and* $X_1 \times_{Y_1} (X_2 \times_{Y_2} X_3)$

(iii) $X_1 \times_{Y_1 \times Y_2} (X_2 \times X_3)$ *and* $(-1)^{\dim Y_2(\dim Y_1 + \dim X_2)}(X_1 \times_{Y_1} X_2) \times_{Y_2} X_3$

(iv) $X_1 \times_Y X_2$ *and* $\epsilon(f_1) \cdot \epsilon(f_2) \cdot \epsilon(g)\, X_1' \times_{Y'} X_2'$
induced by $\epsilon(f_i)$-oriented diffeomorphisms $X_i \xrightarrow{f_i} \epsilon(f_i)X_i'$ and an $\epsilon(g)$-oriented diffeomorphisms $Y \xrightarrow{g} \epsilon(g)Y'$ where $\epsilon(f_i), \epsilon(g) \in \{\pm 1\}$.

Remark that we assumed appropriate maps between (products of) X_i and Y_j such that expressions in the Lemma make sense. As shown in chapter 3.1 of [7] relation (iv) yields that the canonical twist map $X_1 \times X_2 \xrightarrow{\tau} X_2 \times X_1$ induces an orientation preserving diffeomorphism between

$$(v) \quad X_1 \times_Y X_2 \quad \text{and} \quad (-1)^{(\dim X_1 + \dim Y)(\dim X_2 + \dim Y)} X_2 \times_Y X_1 \ . \tag{2.6}$$

The importance of this relation is reflected in the fact that later all appearing products are graded commutative on homology.
For chains as defined in (2.5) defined above Lemma 2.7 yields the following relations

(i) $\partial(f_a \cap f_b) = \partial f_a \cap f_b + (-1)^{|f_a|} f_a \cap \partial f_b$

(ii) $(f_a \cap f_b) \cap f_c = f_a \cap (f_b \cap f_c)$

(iii) $f_a \cap (f_b \otimes f_c) = (-1)^{\dim M |f_2|}(f_a \cap f_b) \cap f_c$

(iv) $f_a \cap f_b = (-1)^{|f_a||f_b|} f_b \cap f_a$

where from now on we always use

$$|f_i| := \dim K_i - \dim M \ .$$

Since K_i is closed we get that $\partial(f_a \cap f_b) = (f_a \cap f_b)|_{\partial K_{a \cdot b}} = 0$ and thus the product defined above descends to homology. In total we define the *intersection product* $H_i(M; \mathbf{k}) \otimes H_j(M; \mathbf{k}) \to H_{i+j-n}(M; \mathbf{k})$ via

$$a \cap b := \frac{1}{k_a k_b}[f_a = f_b : K_a \times_M K_b \to M] \in H_{i+j-n}(M; k) \ . \tag{2.7}$$

Due to the appearing coefficients it is clear that this definition only works for coefficients in a field $\mathbf{k}$ of characteristic 0.
In total we get the well known fact that

$$\mathbb{H}_*(M; \mathbf{k}) := H_{*+\dim M}(M; \mathbf{k})$$

is an associative, graded commutative algebra with $|\cap| = 0$.

The discussion above is classical and can be generalized to define a product for the free loop space LM. Again remark that the discussion is possible for coefficients in a ring, but is simplified here by using coefficients in a field $\mathbf{k}$ of characteristic 0. We recall ideas presented in [5] and [7].

By using the loop-loop fibration

$$\begin{tikzcd} \Omega M \arrow[r, hook] & LM \arrow[d, "ev_0"'] \\ & M \arrow[u, bend right, dotted, "s"'] \end{tikzcd}$$

we regard LM as a twisted product of M and ΩM. Combining the intersection product $\cap$ on $H_*(M; \mathbf{k})$ and the Pontryagin product $\bullet$ generalizes the discussion above such that we get a product $\bullet$ of degree 0 on

$$\mathbb{H}_*(LM; \mathbf{k}) := H_{*+\dim M}(LM; \mathbf{k}) \ .$$

Given classes $a \in H_i(LM; \mathbf{k})$ and $b \in H_j(LM; \mathbf{k})$ are represented by continuous maps $f_a : K_a^i \to LM$ and $f_b : K_b^j \to LM$ from closed oriented manifolds K_a, K_b.
We choose the representatives such that $\overline{f_a} := ev_0 \circ f_a$ and $\overline{f_b} := ev_0 \circ f_b$ are smooth and mutually transversal in M. As in the discussion above this yields an $(i + j - n)$-chain

$$\overline{f_a} \cap \overline{f_b} : K_a \times_M K_b \to M \ .$$

Since $LM \to M$ is a fibration the perturbations can be lifted and we get that

$$f_a \cap f_b : K_a \times_M K_b \to LM \times_M LM$$

defines an $(i + j - n)$-chain.
For $(k_a, k_b) \in K_a \times_M K_b$ the base points $f_a(k_a)(0) = f_b(k_b)(0)$ coincide and we thus can concatenate the loops as in the definition of the Pontryagin product for the based loop space. In total this means that

$$f_a \bullet f_b : K_a \times_M K_b \to LM \tag{2.8}$$

where

$$f_a \bullet f_b\, (k_a, k_b)(t) := \begin{cases} f_a(k_a)(2t) & , \quad t \in [0, 1/2] \\ f_b(k_b)(2t - 1) & , \quad t \in [1/2, 1] \end{cases}$$

defines an $(i + j - n)$-chain in $LM = C^0(S^1, M)$.
Analogously as in the discussion of the intersection product one can then prove that:

Theorem 2.8 (*[5], section 2*)

$(\mathbb{H}_*(LM; \mathbf{k}), \bullet)$ is an associative, graded commutative algebra. The algebra unit is given by $e \equiv s_*([M]) \in \mathbb{H}_0(LM; \mathbf{k})$.

Remark 2.9. *Since the map*

$$M \xrightarrow{s} LM \xrightarrow{ev_0} M$$

is the identity, the corresponding chain representing e is transverse to all possible given chains. Thus $e \bullet a$ respectively $a \bullet e$ makes sense (even on chain level) for all $a \in \mathbb{H}_*(LM)$ and equals a since one concatenates with constant based loops. It follows that $e = s_*([M])$ is the algebra unit.

The reader should be aware of the fact that associativity on chain level only holds up to homotopy. This comes from the fact that concatenating pointed loops is only strictly associative when working with Moore loops. Similarly to equation (2.1) we have

$$\mathbb{H}_*(LM) \cong \mathbb{H}_*(L^M M) \tag{2.9}$$

as algebras. Analogously as above we have a loop product for the homology of the free Moore loop space when defining (2.8) as

$$f_a \bullet f_b(k_a, k_b)(t) := (f_a(k_a) * f_b(k_b))(t) \ ,$$

where we concatenate Moore loops. Taking the fiber product $K_1 \times_M K_2$ is independent of using Moore or non-Moore loops. The homotopy equivalence $L^M M \simeq LM$ only *involves reparameterizations of the given loops and thus the product structures on homology agree.*

The graded commutativity needs more attention, because the algebra $(H_*(\Omega M), \bullet)$ *is clearly not (graded) commutative. A schematic illustration of the loop product may be drawn as in figure 2.2.*

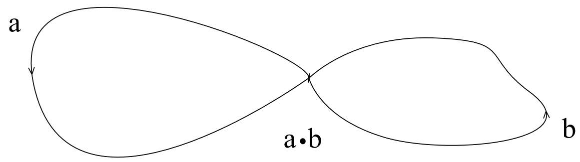


Figure 2.2: Illustration of the loop product $a \bullet b$

On chain level of LM *we need to define an operation* $f_a * f_b$ *whose boundary yields*

$$f_a \bullet f_b - (-1)^{|a||b|} f_b \bullet f_a \tag{2.10}$$

at least for chains representing homology classes. Pictorially this has to be considered as in figure 2.3. The construction of $*$ *is recalled in the next section 2.2.2.*

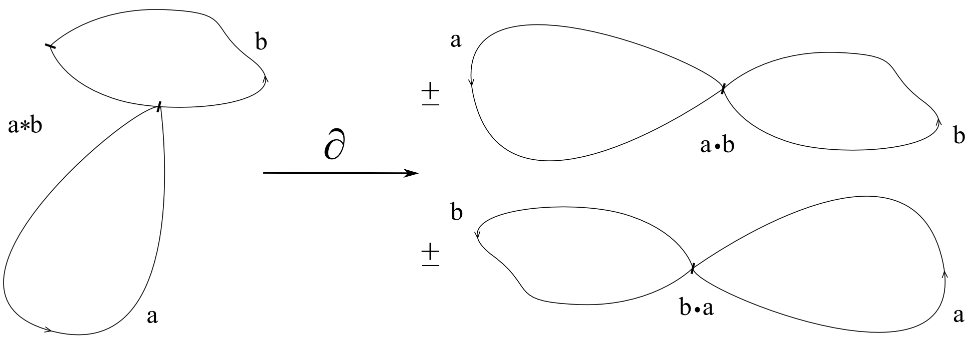


Figure 2.3: Graded commutativity of $\bullet$ on $\mathbb{H}_*(LM)$

2.2.2 The Gerstenhaber algebra $(\mathbb{H}_*(LM), \{\cdot,\cdot\}, \bullet)$

Extending the ideas of how the loop product is defined it is clear that a loop product where the 2nd basepoint is moving should have the following domain

$$K_a \times_M (I \times K_b) \ .$$

We review ideas for non-Moore loops and thus work with the standard interval $I = [0,1]$ instead of $\mathbb{R}_{\geqslant 0}$ as the time domain. For given homology classes

$$a \in H_i(LM; \mathbf{k}),\ b \in H_j(LM; \mathbf{k})$$

represented by closed manifolds K_a, K_b we get that

$$ev_0 : K_a \to M \ \text{ and } \ ev : I \times K_b \to M$$

are mutually transversal (after perturbation). That is $K_a \times_M (I \times K_b)$ is manifold of dimension $i + j + 1 - n$. Since this domain gets mapped to a family of based loops we again may concatenate and thus get a chain in LM. The operation

$$* : C_i(LM) \otimes C_i(LM) \to C_{i+j+1-n}(LM)$$

is transversally defined on chain level where

$$(f_a * f_b)(k_a, t, k_b)(s) := \begin{cases} f_b(k_b)(2s) & , \quad s \in [0, t/2] \\ f_a(k_a)(2s - t) & , \quad s \in [t/2, \frac{t+1}{2}] \\ f_b(k_b)(2s - 1) & , \quad s \in [\frac{t+1}{2}, 1] \end{cases} ,$$

for $(k_a, t, k_b) \in K_a \times_M (I \times K_b)$. Visualized in a schematic way it looks like the left side of figure 2.3.
By using the results of Lemma 2.7 we may examine $\partial(f_a * f_b)$. The geometric boundary of its domain is given by

$$\begin{aligned} &(-1)^{|f_a|}\partial\big(\overbrace{K_a \times_M (I \times K_b)}^{=:K_{a*b}}\big) \\ =&(-1)^{|f_a|}\big(K_{\partial a * b} + (-1)^{|f_a|}(K_a \times_M (\{1\} \times K_b) - K_a \times_M (\{0\} \times K_b) - K_{a * \partial b})\big) \\ =&(-1)^{|f_a|}K_{\partial a * b} + K_{a \bullet b} - (-1)^{|f_a||f_b|}K_{b \bullet a} - K_{a * \partial b} . \end{aligned}$$

On the one hand this proves the graded commutativity of the loop product on homology. But further it also yields that for representing cycles f_a, f_b one has

$$\begin{aligned} \partial(\overbrace{f_a * f_b - (-1)^{(|a|+1)(|b|+1)} f_b * f_a}^{(*)}) \overbrace{=}^{\partial f_a = \partial f_b = 0} &(-1)^{|a|}(f_a \bullet f_b - (-1)^{|a||b|} f_b \bullet f_a) + \\ &- (-1)^{(|a|+1)(|b|+1)}(-1)^{|b|}(f_b \bullet f_a - (-1)^{|b||a|} f_a \bullet f_b) = 0 . \end{aligned}$$

As shown in [5] the closed chain $(*)$ not only descends to homology but also defines a graded Lie algebra structure via

$$\{[f_a], [f_b]\} := [f_a * f_b - (-1)^{(|a|+1)(|b|+1)} f_b * f_a] .$$

Theorem 2.10 (*[5], chapter 4*)

$(\mathbb{H}_*(LM), \{\cdot,\cdot\})$ is a graded Lie algebra with $|\{\cdot,\cdot\}| = 1$. That is

(i) $\{a, b\} = -(-1)^{(|a|+1)(|b|+1)}\{b, a\}$ *(Symmetry)*

(ii) $\{a, \{b, c\}\} = \{\{a, b\}, c\} + (-1)^{(|a|+1)(|b|+1)}\{b, \{a, c\}\}$ *(Jacobi identity)* .

Further $\{\cdot,\cdot\}$ defines a derivation on the algebra $(\mathbb{H}_*(LM), \bullet)$

$$\{a, b \bullet c\} = \{a, b\} \bullet c + (-1)^{|b|(|a|-1)} b \bullet \{a, c\} .$$

Remark that a datum like $(\mathbb{H}_*(LM), \{\cdot,\cdot\}, \bullet)$ satisfying the stated properties is called a *Gerstenhaber algebra* in the literature.

2.2.3 The Batalin-Vilkovisky algebra $(\mathbb{H}_*(LM), \Delta, \bullet)$

In the last section we defined a 'basepoint moving loop product' $*$. Here we try to separate this into two operations namely the ordinary loop product (with fixed basepoints) and an operation Δ that models the moving of the basepoint. In particular Δ descends to homology and we get a BV-algebra structure whose informations could alternatively be used to prove Theorem 2.10.
As reflected in the loop-string fibration we have an action of S^1 on LM that rotates the basepoint. This defines a BV-operator of degree +1 on $C_*(LM)$ via $f_a \mapsto \Delta f_a$, where

$$\begin{aligned} \Delta f_a : \overbrace{S^1 \times K_a}^{=: K_{\Delta a}} &\to LM \\ (t, k_a) &\mapsto f_a(k_a)(t + \cdot) \ . \end{aligned} \tag{2.11}$$

Remark that this operation is fully defined and not just transversally on chain level. Since $\partial(S^1 \times K_a) = \partial S^1 \times K_a - S^1 \times \partial K_a$ by Lemma 2.7 we conclude that Δ descends to homology and we get an operation

$$\Delta : \mathbb{H}_*(LM) \longrightarrow \mathbb{H}_{*+1}(LM) \ .$$

Further on homology Δ is a differential, that is it squares to zero, $\Delta \circ \Delta \equiv 0$. This can be seen as follows. For an i-chain $f_a \in C_i(LM)$ applying the BV operator twice yields a degree $i+2$ chain $\Delta(\Delta f_a) : S^1 \times S^1 \times K_a \to LM$. However, it is a degenerate chain and thus homologous to zero since it factors through an $i+1$ chain

$$S^1 \times S^1 \times K_a \to S^1 \times K_a \to LM$$

via

$$\Delta(\Delta f_a)(s, t, k_a)(\cdot) = f_a(k_a)(s + t + \cdot) = \Delta f_a(s + t, k_a)(\cdot) \ .$$

As announced the following theorem states the fact that a combination the loop product $\bullet$ and the BV operator Δ yields the loop bracket $\{\cdot, \cdot\}$.

Theorem 2.11 ***([5], section 5)***

$(\mathbb{H}_*(LM), \bullet, \Delta)$ is a Batalin-Vilkovisky algebra with $|\Delta| = +1$. That is:

(i) *$(\mathbb{H}_*(LM), \bullet)$ is an associative, graded commutative algebra.*

(ii) $\Delta \circ \Delta = 0$

(iii) *The expression $(-1)^{|a|}\Delta(a \bullet b) - (-1)^{|a|}\Delta a \bullet b - a \bullet \Delta b$ is a derivation in each variable*

One easily checks that

$$(-1)^{|a|}\Delta(a \bullet b) - (-1)^{|a|}\Delta a \bullet b - a \bullet \Delta b =: \{a, b\} \tag{2.12}$$

defines a Lie bracket for $a, b \in \mathbb{H}_*(LM)$. In [5] the authors show that it coincides with the loop bracket defined above. So Theorem 2.11 can indeed be taken as a generalization of Theorem 2.10. In fact one may check that a Batalin-Vilkovisky algebra in general yields a Gerstenhaber algebra when defining the Lie bracket via

$$\{a, b\} := (-1)^{|a|}\Delta(a \bullet b) - (-1)^{|a|}\Delta a \bullet b - a \bullet \Delta b + a \bullet \Delta(1) \bullet b .$$

Remark that in our case the algebra unit 1 is represented by the constant loop at each point that is $f_1 : M \to LM$. We get that $\Delta(f_1)(t,p)(\cdot) = f_1(p)(\cdot + t) = f_1(p)(\cdot)$. That is $\Delta(f_1)$ is a degenerate chain and thus 0.
This leads to

$$\{a, 1\} = (-1)^{|a|}\Delta(a \bullet 1) - (-1)^{|a|}\Delta a \bullet 1 = (-1)^{|a|}\Delta a - (-1)^{|a|}\Delta a = 0$$

for all $a \in \mathbb{H}_*(LM)$.

2.2.4 A graded Lie bracket for $H_*^{S^1}(LM)$

We apply the Gysin sequence, see for example appendix 5.2, to the loop-string fibration $S^1 \hookrightarrow LM \times ES^1 \xrightarrow{\pi} LM \times_{S^1} ES^1$ and get the exact sequence

$$\cdots \longrightarrow \mathbb{H}_k(LM) \xrightarrow{\mathcal{E}} H^{S^1}_{k+n}(LM) \xrightarrow{\cap e} H^{S^1}_{k+n-2}(LM) \xrightarrow{\mathcal{M}} \mathbb{H}_{k-1}(LM) \longrightarrow \cdots$$

The maps π_*, π^* are called $\mathcal{M}$ark and $\mathcal{E}$rase since we think of LM as the space of loops marked by the basepoint whereas $LM \times_{S^1} ES^1$ presents the space of unmarked strings. $\mathcal{E}$ is just the induced map on homology thus can be interpreted as forgetting the basepoint. The degree +1 map $\mathcal{M}$ maps a family of strings to the particular S^1 fibres in the total space, that is it puts basepoints everywhere to the loops.

The Gysin sequence provides a possibility the 'go back and forth' between non-equivariant and equivariant homology. Precisely speaking one asks what happens with operations defined for one side when transferred to the other via

$$\begin{tikzcd} \mathbb{H}_*(LM) \arrow[r, bend left, "\mathcal{E}"] & H_*^{S^1}(LM) \arrow[l, bend left, "\mathcal{M}"] \end{tikzcd} .$$

When taking the identity maps

$$\mathrm{id}_{\mathbb{H}_*} : \mathbb{H}_*(LM) \to \mathbb{H}_*(LM)$$
$$\mathrm{id}_{H_*^{S^1}} : H_*^{S^1}(LM) \to H_*^{S^1}(LM) ,$$

these transfer to

$$\mathcal{E} \circ \mathrm{id}_{\mathbb{H}_*} \circ \mathcal{M} = 0 : H_*^{S^1}(LM) \to H_*^{S^1}(LM)$$
$$\mathcal{M} \circ \mathrm{id}_{H_*^{S^1}} \circ \mathcal{E} \overset{(*)}{=} \Delta : \mathbb{H}_*(LM) \to \mathbb{H}_*(LM) .$$

For $(*)$ remark that applying $\mathcal{M}$ to $\mathcal{E}(a)$ for a family of loops a we get back a but now with basepoints spread along the loops, that is Δa.
The BV operator on non-equivariant homology transfers to

$$\mathcal{E} \circ \Delta \circ \mathcal{M} = \underbrace{\mathcal{E} \circ (\mathcal{M}}_{0} \circ \underbrace{\mathcal{E}) \circ \mathcal{M}}_{0} = 0 \ .$$

With (2.12) we get for the loop bracket transferred to equivariant homology that

$$\mathcal{E} \circ \{\mathcal{M}(a), \mathcal{M}(b)\} = \mathcal{E}(\ \pm \ \Delta(\mathcal{M}(a) \bullet \mathcal{M}(b)) \ \mp \ \underbrace{\Delta(\mathcal{M}}_{0}(a)) \bullet b \ - \ a \bullet \underbrace{\Delta\mathcal{M}}_{0}(b) \ ) = 0 \ .$$

It remains to check what happens to the loop product $\bullet$. In fact it yields a non-trivial operation and surprisingly not a product but a bracket on non-equivariant homology:

Theorem 2.12 (*[5]*)

$(H_*^{S^1}(LM; \mathbf{k}), [\cdot, \cdot])$ is a graded Lie algebra, with bracket of degree $2-n$ defined by

$$[a, b] := (-1)^{|a|}\mathcal{E}(\mathcal{M}(a) \bullet \mathcal{M}(b)) \ , \tag{2.13}$$

where $|a| = \dim a - \dim M$. This means that graded commutativity

$$[a, b] = -(-1)^{|a||b|}[b, a]$$

and the graded Jacobi identity

$$[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$$

are satisfied.

2.3 Computational methods

It is mostly non-trivial to compute the vector space structure $H_*(LM; \mathbf{k})$ for a given topological space X. In the following we mostly work with coefficients in a field $\mathbf{k}$ of characteristic 0 and write $H_*(LM)$ for simplicity reasons. To derive string topology structures for smooth finite dimensional oriented closed manifolds as defined in section 2.2 is even harder. Exceptions are very well understood spaces as S^1, Lie groups or Eilenberg-MacLane spaces $K(G, 1)$. In the following we show how direct methods may already yield some information.
The following section about computations refers to methods presented in [1] and [7].

The circle S^1

Throughout the whole text spheres S^n appear all the time. We distinguish between the simply connected spheres $S^{n\geqslant 2}$ and the non-simply connected circle S^1.

The 1-sphere S^1 is the simplest closed manifold. For the point $\{pt\}$ one has a ring isomorphism $H_*(Lpt) \cong \mathbb{Z}$ and $H_*^{S^1}(Lpt) \equiv H_*(BS^1)$.

Recall that

$$H_*(LS^1;\mathbb{Z}) = \bigoplus_{\substack{n \in \tilde{\pi}_1(S^1) \\ = \pi_1(S^1) \cong \mathbb{Z}}} H_*(L^n S^1;\mathbb{Z})$$

that is we need to understand

$$L^n S^1 = \{\gamma : S^1 \to S^1 \,|\, \deg \ = n\}$$

consisting of loops with winding number n. Via its universal cover $\mathbb{R} \stackrel{exp}{\to} S^1$ a map $f \in L^n S^1$ lifts to a map

$$F \in \mathcal{F}_n = \{\Gamma : \mathbb{R} \to \mathbb{R} \,|\, \Gamma(t+1) - \Gamma(t) = n\} \ .$$

The lift F is unique up to translation by an integer and further homotopes to

$$G(t) = nt + m(F)$$

via

$$\begin{aligned} H : [0,1] \times \mathcal{F}_n &\to \mathcal{F}_n \\ (t, F) &\mapsto (1-s)F + s(nt + m(F)) \end{aligned}$$

where $m(F) := \int_0^1 (F(t) - nt)\, dt \in \mathbb{R}$.
Projecting this homotopy via exp yields a deformation retraction from $L^n S^1$ to the set of constant speed loops

$$L_c^n S^1 := \{\gamma_n : S^1 \to S^1 | d\gamma_n/dt = n\}$$

that wind around n-times and only differ by their basepoints $p \in S^1$. Remark that the homotopy is S^1-equivariant, meaning that the following diagram commutes

$$\begin{tikzcd} {[0,1] \times L_c^n S^1 \times S^1} \arrow[r, "S^1\text{-action}"] \arrow[d, "H"'] & {[0,1] \times L_c^n S^1} \arrow[d, "H"] \\ L_c^n S^1 \times S^1 \arrow[r, "S^1\text{-action}"] & L_c^n S^1 \end{tikzcd}$$

where the $(S^1 = \mathbb{R}/\mathbb{Z})$-action is given by

$$\begin{aligned} S^1 \times L_c^n S^1 &\to L_c^n S^1 \\ (\tau, f) &\mapsto f(\tau + \cdot) \ . \end{aligned}$$

The commutativity is provided by

$$m(F(\tau+\cdot)) = \int_0^1 (F(\tau+t)-nt)\,dt = \int_{\tau+0}^{\tau+1} (F(x)-n(x-\tau))\,dx = m(F)+n\tau$$

since $F(t)-nt$ is 1-periodic.

The evaluation at the basepoint $ev_0(\gamma_n) = \gamma_n(0)$ yields a homotopy equivalence

$$L_c^n S^1 \simeq S^1$$

that is also S^1-equivariant. Here the action of S^1, with coordinate τ, on S^1 is given by

$$(\tau, t) \mapsto [n\tau + t] \in \mathbb{R}/\mathbb{Z} \tag{2.14}$$

In total we get

$$H_*(LS^1;\mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} H_*(S^1;\mathbb{Z}) \ .$$

The generators of $H_*(S^1;\mathbb{Z}) \cong \mathbb{Z}\langle x,y\rangle$ ($|x|=0,\ |y|=1$) are similarly used for the free loop space homology of LS^1. We set

$$x_n : \{pt\} \to LS^1 \quad \text{and} \quad y_n : S^1 \to LS^1$$

where $x_n(pt)(t) = [nt] \in \mathbb{R}/\mathbb{Z}$ and $y_n(\tau)(t) = [nt+\tau] \in \mathbb{R}/\mathbb{Z}$ and get

$$\mathbb{H}_*(LS^1;\mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}\langle [x_n],[y_n]\rangle \quad \text{with} \quad |[x_n]| = -1, |[y_n]| = 0 \ .$$

We work with shifted degrees and thus the loop product $\bullet$ is of degree 0. By degree reasons we get

$$[x_i] \bullet [x_j] = 0 \ .$$

Since $ev_0 \circ y_j : S^1 \to S^1$ is a submersion, the products $x_i \bullet y_j$ and $y_i \bullet y_j$ are defined even on chain level.
The domain of $x_i \bullet y_j$ is $pt \times_{S^1} S^1 \simeq pt$. So concatenating at $t=0$ the loop that winds around i-times with the one winding around j-times yields

$$[x_i] \bullet [y_j] = [x_{i+j}] \ ,$$

that holds on chain level only up to reparameterization. For $y_i \bullet y_j$ it is similar except that now the domain is $S^1 \times_{S^1} S^1 \simeq S^1$. The resulting one dimensional family now is given by

$$[y_i] \bullet [y_j] = [y_{i+j}] \ .$$

So the algebra structure is fully understood and we deduce

$$\mathbb{H}_*(LS^1;\mathbb{Z}) = \Lambda_{\mathbb{Z}}(u) \otimes_{\mathbb{Z}} \mathbb{Z}\,[t, t^{-1}] \quad \text{with} \quad |u| = -1, |t| = 0 \ , \tag{2.15}$$

where $u \equiv [x_0]$, $t^i \equiv [y_i]$ and $ut^i \equiv [x_i]$. Remark that we already use the notation proposed by [10].

We conclude with the BV-algebra structure. On homology we get for the generator $[x_i] = ut^i$ that

$$(\Delta\, x_i)\,(\tau, t) = \big[i(\tau + t)\big] \ ,$$

so that $\Delta\, x_i$ is homologous to iy_i. Thus for $\mathbb{H}_*(LS^1;\mathbb{Z})$ the BV operator is fully determined by

$$\Delta\, ut^i = it^i \ , \tag{2.16}$$

that in turn yields a Gerstenhaber algebra with Lie bracket given by

$$\{ut^i, ut^j\} = (i-j)\, ut^{i+j} \ ; \ \{ut^i, t^j\} = -jt^{i+j} \ ; \ \{t^i, t^j\} = 0 \ . \tag{2.17}$$

The S^1-action is trivial on the component $L^0_c S^1 \simeq S^1 \subset LS^1$ containing the trivial loop. Further for $n \neq 0$ and the diagonal S^1-action on $\underbrace{L^n S^1}_{\simeq S^1} \times ES^1$, where the action on the first factor is as in (2.14), we get that

$$L^n S^1 \times_{S^1} ES^1 \simeq S^\infty/\mathbb{Z}_n$$

for $n \neq 0$. Here $S^\infty/\mathbb{Z}_n$ is the infinite lens space. See for example appendix 5.1 for a short review of its topological properties. Its homology groups are given by

$$H_i(S^\infty/\mathbb{Z}_n;\mathbb{Z}) = \begin{cases} \mathbb{Z} & ; \quad i = 0 \\ \mathbb{Z}_n & ; \quad i \text{ odd} \\ 0 & ; \quad \text{else} \end{cases} \ .$$

In total we get a $\mathbb{Z}$-module

$$\begin{aligned} H^{S^1}_*(LS^1;\mathbb{Z}) &= \bigoplus_{n\in\mathbb{Z}} H^{S^1}_*(L^n S^1;\mathbb{Z}) = \bigoplus_{n} H_*(L^n S^1 \times_{S^1} ES^1;\mathbb{Z}) \\ &= H_*(L^0 S^1 \times BS^1;\mathbb{Z}) \oplus \bigoplus_{n\neq 0} H_*(L^n S^1 \times_{S^1} ES^1;\mathbb{Z}) \\ &\cong H_*(S^1;\mathbb{Z}) \otimes H_*(BS^1;\mathbb{Z}) \oplus \bigoplus_{n\neq 0} H_*(S^\infty/\mathbb{Z}_n;\mathbb{Z}) \\ &\cong \Big(\bigoplus_{i\geqslant 0} H_*(S^1;\mathbb{Z})\langle c^i/i!\rangle\Big) \oplus \bigoplus_{n\neq 0} H_*(S^\infty/\mathbb{Z}_n;\mathbb{Z}) \ , \end{aligned}$$

where the generator $c \in H_2(BS^1;\mathbb{Z})$ is Kronecker dual to the Euler class $\widetilde{c} \in H^2(BS^1;\mathbb{Z})$ of the universal S^1-bundle $ES^1 \to BS^1$ and $H_*(BS^1;\mathbb{Z}) \cong \mathbb{Z}_{\text{div.}}[c]$ is the divided polynomial algebra, that is it is generated by monomials $\frac{c^i}{i!}$.

We simplify things by working with coefficients in a field $\mathbf{k}$ of characteristic 0 and get

$$H_i^{S^1}(LS^1;\mathbf{k}) \cong \begin{cases} \bigoplus\limits_{n\in\mathbb{Z}} \mathbf{k}\langle\, \alpha_n \,\rangle & , \quad i = 0 \\ \mathbf{k}\langle\, \alpha_0 \otimes c^j \,\rangle & , \quad i = 2j \neq 0 \\ \mathbf{k}\langle\, \mathbb{1}_{S^1} \otimes c^j \,\rangle & , \quad i = 2j+1 \end{cases} . \tag{2.18}$$

When working with shifted degrees the $\mathcal{M}$ark respectively the $\mathcal{E}$rase map have degrees $|\mathcal{M}| = 0$ and $|\mathcal{E}| = -1$. Due to (2.15) the non-equivariant homology of LS^1 is concentrated in degree -1 and 0. This means by construction

$$\mathcal{M}(\alpha_i) = it^i \ , \ \mathcal{E}(ut^i) = \alpha_i \ , \ \mathcal{E}(t^0) = \mathbb{1}_{S^1}$$

and zero else.
We end up with the string bracket of degree $2 - n = 1$ that is fully described by

$$\begin{aligned} [\alpha_i, \alpha_j] &= -\mathcal{E}(\mathcal{M}(\alpha_i) \bullet \mathcal{M}(\alpha_j)) = -\mathcal{E}(it^i \bullet jt^j) \overbrace{=}^{(2.15)} -ij\,\mathcal{E}(t^{i+j}) \\ &= \begin{cases} -ij\,\mathbb{1}_{S^1} & , \quad i+j = 0 \\ 0 & , \quad i+j \neq 0 \end{cases} \end{aligned} \tag{2.19}$$

because $\mathcal{E}(t^{i+j}) = \mathcal{E}(\mathcal{M}(\frac{\alpha_{i+j}}{i+j})) = 0$ if $i + j \neq 0$.

Eilenberg-MacLane spaces $K(\pi_1, 1)$

Recall that the loop-loop fibration yields an exact sequence

$$\cdots \to \pi_n(\Omega_{x_0}M) \to \pi_n(LM) \to \pi_n(M) \to \pi_{n-1}(\Omega_{x_0}M) \to \cdots , \tag{2.20}$$

for M path-connected. *Eilenberg-MacLane spaces* M with

$$\pi_n(M) = 0 \quad \text{for} \quad n \neq 1$$

are very attractive to be studied in the context of string topology. Examples of such spaces may be found in chapter 1.B. of [18]. Recall that we require M to be an n-dimensional closed and oriented manifold. The following examples shall be discussed:

(i) the circle S^1 (previously treated)

(ii) the torus T^n

(iii) manifolds of non-positive sectional curvature K

(iv) products of the stated examples (see chapter 2.4)

The torus T^n and products are easily understood in terms of string topology for the separate factors when we have the results of chapter 2.4 about string topology of product manifolds in general. In this way we will deduce the BV-algebra structure of $\mathbb{H}_*(LT^n)$.

Lemma 2.13

The S^1-equivariant homology of LT^n is given by

$$H_*^{S^1}(LT^n) \cong H_*(T^n) \otimes H_*(BS^1) \oplus \Big(\bigoplus_{\substack{(m_1,\dots,m_n)\in \\ \mathbb{Z}^n\setminus\{0\}}} H_*(T^{n-1}) \otimes H_*(ES^1/\mathbb{Z}_{\mathrm{ggT}(m_1,\dots,m_n)}) \Big) \ . \tag{2.21}$$

Proof: Again we follow [1] here.

As for the circle S^1 the homotopy equivalence $T^n \to L^0T^n$ is S^1-equivariant. We thus get

$$H_*(L^0T^n \times_{S^1} ES^1) \cong H_*(T^n \times_{S^1} ES^1) \cong H_*(T^n \times BS^1) \cong H_*(T^n) \otimes H_*(BS^1) \ .$$

Since T^n is a Lie group we have a product $\cdot$ and get a homeomorphism

$$\begin{aligned} L^0T^n &\longrightarrow L^{\alpha\neq 0}T^n \\ \gamma &\longmapsto a(\cdot)\cdot\gamma(\cdot) \end{aligned} \tag{2.22}$$

where $a : S^1 \to T^n$ is of constant speed and a representative of α.

As for the circle S^1 we get a homotopy equivalence

$$\{a\cdot\gamma_p \mid \gamma_p(t) \equiv p \in T^n\} =: aT^n \to L^{\alpha\neq 0}T^n$$

which is also S^1-equivariant. The S^1-action is given by

$$\begin{aligned} S^1 \times L^{\alpha\neq 0}T^n &\to L^{\alpha\neq 0}T^n \\ (\tau,\gamma) &\mapsto \gamma(\tau+\cdot) \end{aligned}$$

and

$$\begin{aligned} S^1 \times aT^n &\to aT^n \\ (\tau, a\cdot\gamma_p) &\mapsto a(\tau+\cdot)\cdot\gamma_p \end{aligned}$$

respectively.

We thereof get

$$L^{\alpha\neq 0}T^n \times_{S^1} ES^1 \simeq aT^n \times_{S^1} ES^1 \simeq aT^n/S^1 \times ES^1/\mathrm{Stab}(a) \simeq aT^{n-1} \times S^\infty/\mathbb{Z}_{\mathrm{ggT}(m_1,\dots,m_n)}$$

since the stabilizer $\mathrm{Stab}(a)$ of a in S^1 is given by $\mathbb{Z}_{\mathrm{ggT}(m_1,\dots,m_n)}$ when its class α is $(m_1,\dots,m_n) \in \mathbb{Z}^n$. Further $aT^{n-1} \simeq T^{n-1}$ since tori are Lie groups.

In total we get

$$H_*^{S^1}(L^{\alpha \neq 0}T^n) \cong H_*(T^{n-1}) \otimes H_*(S^\infty / \mathbb{Z}_{\mathrm{ggT}(m_1,\ldots,m_n)}) \ .$$

■

Since we understand the loop product it remains to understand the $\mathcal{M}$ark and $\mathcal{E}$rase map to compute the string bracket $[\cdot,\cdot]$ for $H_*^{S^1}(LT^n)$. Unfortunately we do not have a general answer and refer the reader to chapter 2.3.1 of [1], where the calculation is done for $n = 2$.

So how to compute things for manifolds with non-positive sectional curvature? The following proposition derives the module structure of homology.

Proposition 2.14

Let X be a path-connected topological $K(\pi_1, 1)$-space and $[f] = \alpha \in \pi_0(LX)$. Topologically one has

$$L^0 X \simeq X \quad \text{and} \quad L^{\alpha \neq 0} X \text{ is a } K\big(C_{[f]}(\pi_1(X))\,, 1\big) \text{ space} ,$$

where the subgroup

$$C_g(\pi_1(X)) = \{g' \in \pi_1(X) | g'g = gg'\}$$

is the centralizer of $g \in \pi_1(X)$. So for homology we have

$$H_*(LX) \cong H_*(X) \oplus \bigoplus_{0 \neq \alpha \in \tilde{\pi}_1(X)} H_*(K(C_{[f]}(\pi_1(X)), 1)) \ .$$

Corollary 2.15

If a Riemannian manifold M has sectional curvature $K(p, \sigma) < 0$ for all $p \in M$ and $\sigma \in T_pM$ then it is a $K(\pi_1, 1)$-space and further

$$C_{[f] \neq 0}(\pi_1(M)) \cong \mathbb{Z} \ .$$

This implies

$$LM \simeq M \sqcup \bigsqcup_{0 \neq \alpha \in \pi_0(LM)} S^1$$

yielding for homology

$$H_*(LM) \cong H_*(M) \oplus \bigoplus_{0 \neq \alpha \in \pi_0(LM)} H_*(S^1)$$
$$H_*^{S^1}(LM) \cong H_*(M) \otimes H_*(BS^1) \oplus \bigoplus_{0 \neq \alpha \in \pi_0(LM)} H_*(ES^1 / \mathbb{Z}_{n(\alpha)}) \ ,$$

where the free homotopy class α is the $n(\alpha)$-th iterate of a primitive homotopy class.

Proof of Proposition 2.14 : For X a $K(\pi_1, 1)$-space, (2.20) and the fact that we have a section $s : X \to L^0 X$ allows to deduce

$$\pi_1(L^0 X) \cong \pi_1(X) \oplus \pi_1(\Omega^0_{x_0} X) \underbrace{\cong}_{\text{path-loop fibration}} \pi_1(X) \oplus \pi_2(X) \cong \pi_1(X) \ .$$

Remark that the splitting exists only for the $\alpha = 0$ component. From remark 2.5 we see that $\pi_k(\Omega^\alpha_{x_0} X) \cong \pi_{k+1}(X) = 0$ for $k \geqslant 1$ and thus with (2.20) we deduce

$$\pi_k(L^\alpha X) = 0$$

for $k \geqslant 2$. By using the Whitehead theorem we get that the inclusion of constant loops $X \hookrightarrow L^0 X$ induces a homotopy equivalence

$$X \simeq L^0 X \quad \text{and thus} \quad H_*(L^0 X) \cong H_*(X) \ .$$

Since $\pi_k(L^\alpha X) = 0$ for $k \geqslant 2$ it remains to compute

$$\pi_1(L^\alpha X) \equiv \pi_1(L^\alpha X, f) \equiv \pi_1(LX, f) \ ,$$

for $[f] = \alpha \neq 0$.
Recall the result of [17] namely

$$\pi_1(LX, f) \cong C_{[f]}(\pi_1(X)) \ .$$

Remark when setting $\alpha = 0$ we get the previous result for $\pi_1(L^0 X) \cong \pi_1(X)$. The statement can be easily seen when considering the loop-loop fibration. Indeed, the exactness of

$$\underbrace{\pi_1(\Omega_{x_0} X, f)}_{\cong \pi_2(X) = 0} \longrightarrow \pi_1(LX, f) \xrightarrow{(ev_0)_*} \pi_1(X)$$

implies $\pi_1(LX, f) \cong \mathrm{im}((ev_0)_*)$. Remark that $\beta \in \mathrm{im}((ev_0)_*) \subset \pi_1(X)$ if and only if there is a map

$$b : S^1 \times S^1 \longrightarrow X$$

such that $b_0 = ev_0 \circ b : S^1 \times \{0\} \to X$ is a possible representative of β and further that $b|_{\{0\} \times S^1}$ represents $[f]$. Similar as in figure 2.1 this means that there is a based homotopy from $b_0 * f$ to $f * b_0$.
We thus get $[b_0][f] = [f][b_0]$ that is $\beta = [b_0] \in C_{[f]}(\pi_1(X))$ and therefore

$$\pi_1(LX, f) \cong C_{[f]}(\pi_1(X)) \ .$$

We conclude that $L^\alpha X$ is a $K(C_\alpha(\pi_1(X)), 1)$-space for $\alpha \neq 0$ and thus

$$H_*(L^\alpha X) \cong H_*(K(C_\alpha(\pi_1(X)), 1))$$

for $\alpha \neq 0$. ∎

Proof of Corollary 2.15 : It remains to think about the statement for X being a negatively curved manifold denoted by M. Due to the Theorem of Cartan-Hadamard (see e.g. [4]) we know that in this case the exponential map

$$exp_p : T_pM \to M$$

is a covering and thus $\pi_i(M) \cong \pi_i(\mathbb{R}^n) = 0$ for all $i \geqslant 2$. So M is a $K(\pi_1, 1)$ space.

So with the previous proposition it remains to compute

$$\pi_1(LX, f) \cong C_{[f]}(\pi_1(X))$$

for $[f] \neq 0$. Here we rely on methods presented in chapter 12 of [4].
For the universal covering $\pi : \widetilde{M} \to M$ we get that the group of covering transformations of $\widetilde{M}$ is isomorphic to $\pi_1(M)$ due to [28].
When combining Proposition 2.6 and Lemma 3.3 of [4] we get that under the stated isomorphism a nonzero element $[f] \in \pi_1(M)$ corresponds to a translation

$$F : \widetilde{M} \to \widetilde{M}$$

and there exists a unique geodesic $\widetilde{\gamma} \subset \widetilde{M}$ which is invariant under F, that is $F(\widetilde{\gamma}) = \widetilde{\gamma}$.
For $[g] \in C_{[f]}(\pi_1(X))$ the defining condition of the centralizer translates into

$$F(G(\widetilde{\gamma})) = G(F(\widetilde{\gamma})) = G(\widetilde{\gamma})$$

and by uniqueness we get $G(\widetilde{\gamma}) = \widetilde{\gamma}$.
This holds for all elements of $C_{[f]}(\pi_1(X))$ and thus Lemma 3.5 of [4] states that $C_{[f]}(\pi_1(X))$ is infinite cyclic, that is

$$\pi_1(LX, f) \cong C_{[f]}(\pi_1(X)) \cong \mathbb{Z}$$

for $[f] \neq 0$.

We deduce that $L^\alpha M$ is a $K(\mathbb{Z}, 1)$-space for $\alpha \neq 0$ and thus homotopy equivalent to S^1. If α is the n-th iteration of a primitive class, we can find a representative f for α of the form $f(t) = \gamma(nt)$. Then the homotopy equivalence is realized by the map

$$\begin{aligned} S^1 &\to L^\alpha M \\ \tau &\mapsto f(\tau + \cdot)\ . \end{aligned}$$

Remark that this map is S^1-equivariant for the S^1-actions

$$\begin{aligned} S^1 \times S^1 \to S^1 \ &;\ (s, \tau) \mapsto [ns + \tau] \in \mathbb{R}/\mathbb{Z} \\ S^1 \times L^\alpha M \to L^\alpha M \ &;\ (s, x) \mapsto x(s + \cdot) \end{aligned}$$

As in the discussion previous for T^n we thus get for $\alpha \neq 0$ that

$$L^\alpha M \times_{S^1} ES^1 \simeq S^1 \times_{S^1} ES^1 \simeq S^\infty/\mathbb{Z}_n$$

implying

$$H_*^{S^1}(L^\alpha M) \cong H_*(ES^1/\mathbb{Z}_n) \ .$$

For $L^0 M \simeq M$ by working in the simply connected cover of M we get an S^1-equivariant homotopy from contractible to trivial loops. Thus as in the previous discussion we get

$$H_*^{S^1}(L^0 M) \cong H_*(M \times_{S^1} ES^1) \cong H_*(M \times ES^1/S^1) \cong H_*(M) \otimes H_*(BS^1) \ .$$

■

So what do we know about the string topology operations for manifolds of negative sectional curvature?

Corollary 2.16

Let M be a manifold of negative sectional curvature of dimension $n \geqslant 3$. For the space

$$L^{\neq 0} M := \bigsqcup_{\alpha \neq 0} L^\alpha M$$

of non-contractible loops on M the loop product, the loop bracket and the string bracket vanish.

Proof: This holds by degree reasons. Du to the previous corollary

$$\mathbb{H}_*(L^{\neq 0} M) \cong \bigoplus_{0 \neq \alpha \in \pi_0(LM)} H_{*+n}(S^1)$$

is concentrated in degrees $-n$ and $-n+1$. When working with these shifted degrees the loop product is of degree 0 and the loop bracket is of degree 1.
The image of the loop product lives in degrees $-2n, -2n+1$ or $-2n+2$. To possibly get non-vanishing operations these degrees must be $-n$ or $-n+1$. This can only be satisfied for $2 \geqslant n \geqslant -1$, a contradiction.
The same consideration for the loop bracket yields $3 \geqslant n \geqslant 0$, but the $n = 3$ case can be excluded. The only non-trivially vanishing operation would be of the form $\{c, d\}$ with $|\{c, d\}| = -3$ for $|c| = |d| = -2$, but remark that $(ev_t)_* c$ and $(ev_t)_* d$ are degenerate chains and thus $\{c, d\} = 0$.
The string bracket is vanishing since $\mathcal{M}$ preserves the property of a loop to be non-contractible and further the loop product is 0.

■

The reader is referred to chapter 4.4 where we discuss how these effects already partially appear on chain level.

For the dimension 2 case we refer to chapter 2.3.2 of [1]. We know that a closed oriented surface M admits a hyperbolic structure if and only if $\chi(M) = 2 - 2g < 0$ (see e.g. Theorem 9.3.2. in [33]). Since we need orientability for the string topology operations we may focus on oriented surfaces of higher genus $\Sigma_{g>1}$ in the following. Working with coefficients in a field $\mathbf{k}$ of characteristic 0 yields

$$
\begin{aligned}
\mathbb{H}_{-2}(LM) &\cong H_0(M) \oplus \bigoplus_{0 \neq \alpha \in \tilde{\pi}_1(X)} H_*(S^1) \cong \mathbb{Z} \oplus \bigoplus_{0 \neq \alpha} \mathbf{k}\langle x_\alpha \rangle \\
\mathbb{H}_{-1}(LM) &\cong H_1(M) \oplus \bigoplus_{0 \neq \alpha \in \tilde{\pi}_1(X)} H_*(S^1) \cong H_1(M) \oplus \bigoplus_{0 \neq \alpha} \mathbf{k}\langle y_\alpha \rangle \\
\mathbb{H}_0(LM) &\cong \mathbf{k}\langle [M] \rangle \\
\mathbb{H}_k(LM) &= 0 \quad \text{for} \quad k \notin \{-2, -1, 0\}
\end{aligned}
$$

where we adopt the notation of the discussion of S^1, namely x_α is one loop and y_α is the S^1-family of loops in the class α. We know that $[M]$ is the unit for the loop product. By degree reasons ($|\bullet| = 0$) the remaining pairing to discuss is

$$\mathbb{H}_{-1}(LM) \otimes \mathbb{H}_{-1}(LM) \xrightarrow{\bullet} \mathbb{H}_{-2}(LM) \ .$$

For the BV operator we get $\Delta\, x_\alpha = c_\alpha y_\alpha$ and 0 else for $c_\alpha \in \mathbf{k}$ being the multiplicity of α.
When ignoring the constant loops L^0M we get for the S^1-equivariant homology

$$H_*^{S^1}(L^\alpha M) \cong H_0^{S^1}(L^\alpha M) \cong \bigoplus_{\alpha \neq 0} \mathbf{k}\langle \alpha \rangle \quad \text{for } \alpha \neq 0 \ .$$

Thus string topology is incorporated in

$$H_0^{S^1}(LM) \otimes H_0^{S^1}(LM) \xrightarrow{\mathcal{M}^{\otimes 2}} \mathbb{H}_{-1}(LM) \otimes \mathbb{H}_{-1}(LM) \xrightarrow{\bullet} \mathbb{H}_{-2}(LM) \xrightarrow{\mathcal{E}} H_0^{S^1}(LM) \xrightarrow{\mathcal{M}} \mathbb{H}_{-1}(LM)$$

where up to sign the composition of the first three arrows is the string bracket

$$[a, b] = (-1)^{|a|} \mathcal{E}(\mathcal{M}(a) \bullet \mathcal{M}(b)) \ .$$

Composing the last three arrows yields the loop bracket $\{\cdot, \cdot\}|_{\mathbb{H}_{-1}(LM)^{\otimes 2}}$ since $\Delta\, y_\alpha = 0$.
Recall that

$$\{a, b\} = (-1)^{|a|} \Delta(a \bullet b) - (-1)^{|a|} \Delta a \bullet b - a \bullet \Delta b \ .$$

Since for surfaces the string bracket $[\cdot, \cdot]$ is just the *Goldman bracket*

$$\{[\gamma_1], [\gamma_2]\} = \sum_{p \in \gamma_1 \cap \gamma_2} \mathrm{sgn}(p) [\gamma_1 *_p \gamma_2]$$

we conclude that

$$y_\alpha \bullet y_\beta = \sum_p \pm\, x_{\alpha *_p \beta}$$

when again ignoring the constant loops, that is $\alpha, \beta \neq 0$.

2.4 Products of manifolds

In terms of the algebraic structure defined in chapter 2.2 for the ordinary and the S^1-equivariant homology of LM_1 and LM_2 we show how these structures may be computed for loop spaces of the product manifold $M_1 \times M_2$.

2.4.1 BV structure of the non-equivariant loop space homology

We aim to understand the BV-algebra structure of $\mathbb{H}_*(L(M_1 \times M_2))$, where M_i are compact, oriented manifolds of dimension dim $M_i = d_i$.
For $M := M_1 \times M_2$ we have a homeomorphism $LM \simeq LM_1 \times LM_2$. It is provided by

$$\phi = (\phi_1, \phi_2) : \gamma \longmapsto (Lpr_1 \circ \gamma, Lpr_2 \circ \gamma) \ ,$$

where $Lpr_i : LM \to LM_i$ is the natural projection induced by projecting on one factor with $pr_i : M \to M_i$. As before we work with coefficients in a field $\mathbf{k}$ of characteristic 0. By the Künneth theorem for vector spaces we have

$$\begin{aligned} \mathbb{H}_k(L(M_1 \times M_2)) \equiv H_{k+d_1+d_2}(L(M_1 \times M_2)) &\cong \bigoplus_{i+j=k+d_1+d_2} H_i(LM_1) \otimes H_j(LM_2) = \\ &= \bigoplus_{i+j=k} H_{i-d_1}(LM_1) \otimes H_{j-d_2}(LM_2) \equiv \bigoplus_{i+j=k} \mathbb{H}_i(LM_1) \otimes \mathbb{H}_j(LM_2) \ . \end{aligned} \tag{2.23}$$

We want this relation to be an algebra isomorphism where the multiplication is given by the loop product as a degree $|\bullet| = 0$ morphism on shifted homology. This is indeed true and can be seen as follows. The considerations are inspired by the discussion of the loop product for Lie groups in [7]. Remark that we refer to chapter 3.2.1 of [1] where the formulas for the loop bracket and the BV operator for product manifolds are used for computational purposes.

In summary we get

Proposition 2.17

The BV-algebra operations of $\mathbb{H}_*(LM) \cong \mathbb{H}_*(LM_1) \otimes \mathbb{H}_*(LM_2)$ for a product manifold $M = M_1 \times M_2$ are given by

$$([x_1] \otimes [x_2]) \bullet ([y_1] \otimes [y_2]) = (-1)^{|x_2||y_1|}([x_1] \bullet [y_1]) \otimes ([x_2] \bullet [y_2]) \tag{2.24}$$

$$\Delta([x_1] \otimes [x_2]) = \Delta_1([x_1]) \otimes [x_2] + (-1)^{|x_1|+dim\, M_1} [x_1] \otimes \Delta_2([x_2]) \tag{2.25}$$

for the tensor product of the BV-algebras $\mathbb{H}_*(LM_1)$ and $\mathbb{H}_*(LM_2)$.

Proof of (2.24): For $i = 1, 2$ let $x_i : K_{x_i} \to LM_i$ and $y_i : K_{y_i} \to LM_i$ be given and consider the product chains

$$(x_1, x_2) : K_{x_1} \times K_{x_2} \to LM_1 \times LM_2$$
$$(y_1, y_2) : K_{y_1} \times K_{y_2} \to LM_1 \times LM_2 \ .$$

We may assume that $ev_0 \circ x_i$ and $ev_0 \circ y_i$ are mutually transversal in M_i for $i = 1, 2$. This implies that the fiber product $K_{x \bullet y} = K_x \times_M K_y$ may be written as an union of

$$
\begin{aligned}
&\{(k_{x_1}, k_{x_2}, k_{y_1}, k_{y_1}) | (ev_0 \circ x_i)(k_{x_i}) = (ev_0 \circ y_i)(k_{y_i})\} = \\
=&(K_{x_1} \times K_{x_2}) \times_{(M_1 \times M_2)} (K_{y_1} \times K_{y_2}) \\
\overset{(iii)}{=}&(-1)^{\dim M_2(\dim M_1 + \dim K_{y_1})}((K_{x_1} \times K_{x_2}) \times_{M_1} K_{y_1}) \times_{M_2} K_{y_2} \\
\overset{(v)}{=}&(-1)^{\dim M_2(\dim M_1 + \dim K_{y_1}) + \dim K_{x_1} \dim K_{x_2}}((K_{x_2} \times K_{x_1}) \times_{M_1} K_{y_1}) \times_{M_2} K_{y_2} \\
\overset{(ii)}{=}&(-1)^{\dim M_2(\dim M_1 + \dim K_{y_1}) + \dim K_{x_1} \dim K_{x_2}}(K_{x_2} \times (K_{x_1} \times_{M_1} K_{y_1})) \times_{M_2} K_{y_2} \\
\overset{(v)}{=}&(-1)^{\dim M_2(\dim M_1 + \dim K_{y_1}) + \dim K_{x_1} \dim K_{x_2} + \dim K_{x_2}(\dim K_{x_1} + \dim K_{y_1} + \dim M_1)} \\
&\qquad\qquad\qquad\qquad ((K_{x_1} \times_{M_1} K_{y_1}) \times K_{x_2}) \times_{M_2} K_{y_2} \\
\overset{(ii)}{=}&(-1)^{\dim K_{y_1}(\dim K_{x_2} + \dim M_2) + \dim M_1(\dim K_{x_2} + \dim M_2)}(K_{x_1} \times_{M_1} K_{y_1}) \times (K_{x_2} \times_{M_2} K_{y_2}) \\
=&(-1)^{(\dim K_{y_1} - \dim M_1)(\dim K_{x_2} - \dim M_2)}(K_{x_1} \times_{M_1} K_{y_1}) \times (K_{x_2} \times_{M_2} K_{y_2}) \ .
\end{aligned}
$$

Remark that we applied the results $(i) - (v)$ of Lemma 2.7 and write '=' if there exists an orientation preserving diffeomorphism. The resulting orientation preserving diffeomorphism

$$
K_{x \bullet y} \xrightarrow{\cong} \bigcup (-1)^{(\dim K_{y_1} - \dim M_1)(\dim K_{x_2} - \dim M_2)} K_{x_1 \bullet y_1} \times K_{x_2 \bullet y_2}
$$

fits into a commutative diagram of the form

$$
\begin{tikzcd}
K_{x \bullet y} \arrow[rr, "x \bullet y"] \arrow[d, "(-1)^{\cdots}"', "\cong"] & & LM \arrow[d, "\simeq"', "\phi"] \\
\bigcup K_{x_1 \bullet y_1} \times K_{x_2 \bullet y_2} \arrow[rr, "\sum (x_1 \bullet y_1, x_2 \bullet y_2)"'] & & LM_1 \times LM_2
\end{tikzcd}
\quad . \tag{2.26}
$$

Here the vertical maps are given by

$$
(x \bullet y)(k_x, k_y)(t) = \begin{cases} x(k_x)(2t) & , \ t \in [0, 1/2] \\ y(k_y)(2t-1) & , \ t \in [1/2, 1] \end{cases} ,
$$

$$
(x_1 \bullet y_1, x_2 \bullet y_2)(k_{x_1}, k_{y_1}, k_{x_2}, k_{y_2})(t) = \begin{cases} (x_1(k_{x_1})(2t), x_2(k_{x_2})(2t)) & , \ t \in [0, 1/2] \\ (y_1(k_{y_1})(2t-1), y_2(k_{y_2})(2t-1)) & , \ t \in [1/2, 1] \end{cases} .
$$

The commutativity of (2.26) implies that the loop product on the level of homology is given by

$$\begin{tikzcd}
\mathbb{H}_*(LM) \otimes \mathbb{H}_*(LM) \arrow[d, "\cong"] \\
\mathbb{H}_*(LM_1) \otimes \mathbb{H}_*(LM_2) \otimes \mathbb{H}_*(LM_1) \otimes \mathbb{H}_*(LM_2) \arrow[d, "(-1)^{(\dim K_{y_1} - \dim M_1)(\dim K_{x_2} - \dim M_2)}"] \\
\mathbb{H}_*(LM_1) \otimes \mathbb{H}_*(LM_1) \otimes \mathbb{H}_*(LM_2) \otimes \mathbb{H}_*(LM_2) \arrow[d, "\bullet \otimes \bullet"] \\
\mathbb{H}_*(LM_1) \otimes \mathbb{H}_*(LM_2) \arrow[d, "\cong"] \\
\mathbb{H}_*(LM) \ .
\end{tikzcd}$$

For homology classes $[x]$, $[y] \in \mathbb{H}_*(LM)$ the loop product $[x] \bullet [y]$ is therefore given by

$$([x_1] \otimes [x_2]) \bullet ([y_1] \otimes [y_2]) = (-1)^{|x_2||y_1|}([x_1] \bullet [y_1]) \otimes ([x_2] \bullet [y_2]) \ , \tag{2.27}$$

where $|\cdot|$ is the degree of an homogeneous element of the commutative graded algebra $(\mathbb{H}_*(LM), \bullet)$. In total we get that (2.23) is an algebra isomorphism with respect to the loop product. ■

It remains to derive how the BV operator Δ on $\mathbb{H}_*(LM)$ may be expressed in terms of Δ_i, the ones defined on $\mathbb{H}_*(LM_i)$.

Proof of (2.25): For $i = 1, 2$ let $x_i : K_{x_i} \to LM_i$ be given and consider the product chain

$$(x_1, x_2) : K_{x_1} \times K_{x_2} \to LM_1 \times LM_2 \ .$$

We have a T^2-action on (x_1, x_2) is given by

$$\begin{aligned} T : (S^1 \times S^1) \times (K_{x_1} \times K_{x_2}) &\longrightarrow LM_1 \times LM_2 \\ (s_1, s_2, k_{x_1}, k_{x_2}) &\longmapsto (x_1(k_{x_1})(\cdot + s_1), x_2(k_{x_2})(\cdot + s_2)) \ . \end{aligned}$$

The BV operator as an S^1-action on (x_1, x_2) in turn is given by the composition

$$\begin{aligned} \Delta(x_1, x_2) : S^1 \times (K_{x_1} \times K_{x_2}) &\overset{\mathrm{diag} \times \mathrm{id}}{\longrightarrow} (S^1 \times S^1) \times (K_{x_1} \times K_{x_2}) \overset{T}{\longrightarrow} LM_1 \times LM_2 \\ (s, k_{x_1}, k_{x_2}) &\longmapsto (s, s, k_{x_1}, k_{x_2}) \ . \end{aligned}$$

Further we have the separate S^1-actions

$$\begin{aligned} (\Delta x_1, x_2) : \overbrace{S^1 \times (K_{x_1} \times K_{x_2})}^{\cong (S^1 \times K_{x_1}) \times K_{x_2}} &\overset{\iota_1 \times \mathrm{id}}{\longrightarrow} (S^1 \times S^1) \times (K_{x_1} \times K_{x_2}) \overset{T}{\longrightarrow} LM_1 \times LM_2 \\ (s, k_{x_1}, k_{x_2}) &\longmapsto (s, 0, k_{x_1}, k_{x_2}) \ . \end{aligned}$$

and

$$(x_1,\Delta x_2): \quad \overbrace{S^1\times(K_{x_1}\times K_{x_2})}^{\cong(-1)^{\dim K_{x_1}}K_{x_1}\times(S^1\times K_{x_2})} \xrightarrow{\iota_2\times\mathrm{id}} (S^1\times S^1)\times(K_{x_1}\times K_{x_2}) \xrightarrow{T} LM_1\times LM_2$$
$$(s,k_{x_1},k_{x_2}) \longmapsto (0,s,k_{x_1},k_{x_2})\ .$$

The stated domains fit together such that

$$\Delta(x_1,x_2) - \big((\Delta x_1,x_2) + (-1)^{\dim K_{x_1}}(x_1,\Delta x_2)\big)$$

is a the boundary of T restricted to the triangle $D \subset T^2$ which is the projection of $\{(s_1,s_2)\in\mathbb{R}^2\,|\,0\leqslant s_1\leqslant 1,\ 0\leqslant s_2\leqslant 1,\ 0\leqslant s_1-s_2\leqslant 1\}$ under the projection $\mathbb{R}^2\to T^2$. This implies that

$$\Delta([x_1]\otimes[x_2]) = \Delta_1([x_1])\otimes[x_2] + (-1)^{|x_1|+\dim M_1}\,[x_1]\otimes\Delta_2([x_2])\ ,$$

for $[x_i]\in\mathbb{H}_*(LM_i)$ since $|x_1| = \dim K_{x_1} - \dim M_1$. ■

2.4.2 The structure of the S^1-equivariant free loop space homology

Extending the ideas of the last section we try to understand the graded Lie algebra structure of
$$(H_*^{S^1}\big(L(M_1\times M_2)\big),[\cdot,\cdot])\ ,$$
in terms of $(H_*^{S^1}(LM_i),[\cdot,\cdot])$ where M_i are oriented manifolds of dimension $\dim M_i = d_i$ and the Lie bracket is given by the string bracket.
Unfortunately we may not be that optimistic as in the last section. We derive a spectral sequence related result which provides a possibility to compute the module structure of $H_*^{S^1}(L(M_1\times M_2))$. It is still not clear how the string bracket for products is computed in terms of the underlying brackets for the separate factors.
We only present results for some specific cases where the string bracket is already vanishing for at least one of factors. In contrast to the loop product this does not in general imply the vanishing of the string bracket for $L(M_1\times M_2)$.

Basics facts about spectral sequences for fibre bundles are briefly reviewed in appendix 5.4. The essence of understanding the Leray-Serre spectral sequence of an S^n-bundle $E\to B$ lies in the topology of the base and the transgression map, that in turn is determined by the Euler class $e\in H^{n+1}(B)$.
For an S^1-bundle $Y\to X$ spectral sequence arguments yield the exact Gysin sequence

$$\cdots\longrightarrow H_k(Y)\longrightarrow H_k(X)\xrightarrow{\cap e} H_{k-2}(X)\longrightarrow H_{k-1}(Y)\longrightarrow\cdots.$$

For coefficients in a field of characteristic zero we deduce

$$H_k(Y)\cong \ker(H_k(X)\overset{\cap e}{\to} H_{k-2}(X))\oplus\mathrm{coker}(H_{k+1}(X)\overset{\cap e}{\to} H_{k-1}(X))\ .$$

Theorem 2.18

For $X = M_1 \times M_2$ the Euler classes of the bundles

$$\begin{tikzcd} S^1 \arrow[r] & LX \times_{S^1} ES^1 \arrow[d, "p"] \\ & (LM_1 \times_{S^1} ES^1) \times (LM_2 \times_{S^1} ES^1) \end{tikzcd} \quad , \quad \begin{tikzcd} S^1 \arrow[r] & L(M_1 \times M_2) \times ES^1 \simeq LX \arrow[d] \\ & LX \times_{S^1} ES^1 \end{tikzcd} .$$

are given by

$$\pm\,(\,\pi_1^*(e_1) - \pi_2^*(e_2)\,)$$

and

$$\pm\, p^*(\pi_1^*(e_1)) = \pm\, p^*(\pi_2^*(e_2))$$

respectively. Here $e_i \in H^2(LM_i \times_{S^1} ES^1)$ for $i = 1, 2$ are the Euler classes of the S^1-bundles

$$\begin{tikzcd} S^1 \arrow[r] & LM_i \times ES^1 \simeq LM_i \arrow[d] \\ & LM_i \times_{S^1} ES^1 \end{tikzcd}$$

and π_i are the projections

$$(LM_1 \times_{S^1} ES^1) \times (LM_2 \times_{S^1} ES^1) \xrightarrow{\pi_i} LM_i \times_{S^1} ES^1 \ .$$

The theorem is proven with the help of the universal bundles. A short summary of universal bundles and classifying spaces is given in Appendix 5.2.

Understanding the homology

$$H_*(LM_i) \quad \text{and} \quad H_*^{S^1}(LM_i) = H_*(LM_i \times_{S^1} ES^1)$$

means that we understand the Euler classes of the following bundles

$$\begin{tikzcd} S^1 \arrow[r] & LM_i \times ES^1 \arrow[r, "F_i"] \arrow[d] & S^\infty \arrow[d] & S^1 \arrow[l] \\ & LM_i \times_{S^1} ES^1 \arrow[r, "f_i"] & \mathbb{C}P^\infty & \end{tikzcd} .$$

By examining the Leray-Serre spectral sequence and using the contractibility of S^∞ we get

$$H^*(CP^\infty) \cong \mathbb{Z}[u]$$

where $u \in H^2(\mathbb{C}P^\infty)$ is the Euler class of $S^\infty \to \mathbb{C}P^\infty$. It yields the Euler class of the bundle on the left via pullback

$$e_i(LM_i) := e_i = f_i^*(u) \in H^2(LM_i \times_{S^1} ES^1)\ .$$

In the following we do not care about the explicit form of f_i.

Euler class of the S^1-bundle $(LM_1 \times LM_2) \times_{S^1} ES^1 \to (LM_1 \times_{S^1} ES^1) \times (LM_2 \times_{S^1} ES^1)$:

The Euler class $e \in H^2((LM_1 \times_{S^1} ES^1) \times (LM_2 \times_{S^1} ES^1))$ is given by

$$(f_1 \times f_2)^*(\widetilde{u}) \tag{2.28}$$

where $\widetilde{u}$ denotes the Euler class of the bundle on the right hand side of

$$\begin{tikzcd}
S^1 \arrow[r] & (LM_1 \times LM_2) \times_{S^1} ES^1 \arrow[rr] \arrow[d] & & S^\infty \times_{S^1} S^\infty \arrow[d] & S^1 \arrow[l] \\
& (LM_1 \times_{S^1} ES^1) \times (LM_2 \times_{S^1} ES^1) \arrow[rr, "f_1 \times f_2"] & & \mathbb{C}P^\infty \times \mathbb{C}P^\infty & .
\end{tikzcd} \tag{2.29}$$

Remark that

$$(LM_1 \times LM_2) \times_{S^1} ES^1 \to (LM_1 \times_{S^1} ES^1) \times (LM_2 \times_{S^1} ES^1)$$

mods out a complement of the diagonal $S^1 \xrightarrow{\Delta} T^2$. It remains to understand how $\widetilde{u}$ can be written in terms of u the Euler class of the S^1-bundle $S^\infty \to \mathbb{C}P^\infty$.

Since the bundle $S^\infty \to \mathbb{C}P^\infty$ arises as a direct limit of S^1-bundles

$$\begin{tikzcd}
\dots \arrow[r, hook] & S^{2n+1} \arrow[r] \arrow[d] & S^{2n+3} \arrow[r, hook] \arrow[d] & \dots \\
\dots \arrow[r, hook] & \mathbb{C}P^n \arrow[r] & \mathbb{C}P^{n+1} \arrow[r, hook] & \dots
\end{tikzcd}$$

and $H^*(\mathbb{C}P^n) \cong \mathbb{Z}[a]/(a^{n+1})$ $(|a| = 2)$ we conclude for the Euler class

$$\begin{aligned}
u &= e(S^\infty \to \mathbb{C}P^\infty) = e(S^3 \to \mathbb{C}P^1) = \\
&= e(\underbrace{S^3 \to S^2}_{\text{Hopf fibration}}) = a_2 \in H^*(S^2) \cong \mathbb{Z}[a_2]/(a_2^2)\ ,\ |a_2| = 2\ .
\end{aligned}$$

By the same reason the Euler class $\widetilde{u}$ equals the Euler class of $S^3 \times_{S^1} S^3 \to S^2 \times S^2$. Observe that we have a diagram of pullback bundles:

$$\begin{tikzcd}
& & S^3 \times S^3 \arrow[dd, bend left] & \Delta^*(S^3 \times S^3) \cong S^3 \times S^1 \arrow[l] \arrow[d] \arrow[dd, bend left] & T^2 \arrow[l, hook'] \\
S^1 \arrow[r, hook] & \iota_i^*(S^3 \times_{S^1} S^3) \arrow[r] \arrow[d, "p_i"] & S^3 \times_{S^1} S^3 \arrow[d] & \Delta^*(S^3 \times_{S^1} S^3) \arrow[l] \arrow[d] & S^1 \arrow[l, hook'] \\
& S^2 \arrow[r, "\iota_i"] & S^2 \times S^2 & S^2 \arrow[l, "\Delta"'] &
\end{tikzcd} \tag{2.30}$$

Here ι_i is the inclusion into the first respectively second factor and Δ is the diagonal map $x \mapsto (x,x)$. This yields for the total space and p_i that

$$S^1 \hookrightarrow \iota_i^*(S^3 \times_{S^1} S^3) \xrightarrow{p_1} S^2 \quad \equiv \quad \begin{cases} S^1 \hookrightarrow S^3 \times_{S^1} S^1 \xrightarrow{p_1} S^2 & ; \quad i = 1 \\ S^1 \hookrightarrow S^1 \times_{S^1} S^3 \xrightarrow{p_1} S^2 & ; \quad i = 2 \end{cases} .$$

The bundle on the left is thus a Hopf bundle with Euler class ± 1. For the Euler class $\tilde{u} = z_1 \oplus z_2 \in H^2(S^2 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z}$ we thus get $\iota_i^*(z_1 \oplus z_2) = \pm z_i = 1$ and conclude

$$e(S^3 \times_{S^1} S^3 \to S^2 \times S^2) = (\pm 1) \oplus (\pm 1) .$$

The total space

$$\Delta^*(S^3 \times S^3) = \{(x,y) \in S^3 \times S^3 \,|\, \exists\, \theta \in S^1 : \theta.x = y\}$$

of the T^2-bundle over S^2 can be identified with $S^3 \times S^1$. For this space the diagonal S^1-action is not a diagonal map but the Hopf map

$$S^3 \times S^1 \to \Delta^*(S^3 \times_{S^1} S^3) = S^2 \times S^1 .$$

We deduce that the bundles on the right hand side is trivial and therefore get

$$\Delta^*\big(e(S^3 \times_{S^1} S^3 \to S^2 \times S^2)\big) = \Delta^*\big((\pm 1) \oplus (\pm 1)\big) = \pm 1 \pm 1 \stackrel{!}{=} 0$$

since the Euler class of a trivial fibre bundles vanishes in general. In total we conclude

$$e\big(S^3 \times_{S^1} S^3 \to S^2 \times S^2\big) = \tilde{u} = \pm\,(1 \oplus -1) = \pm\,(u \oplus -u) . \tag{2.31}$$

Combining this with (2.28) we get that the Euler class of

$$(LM_1 \times LM_2) \times_{S^1} ES^1 \to (LM_1 \times_{S^1} ES^1) \times (LM_2 \times_{S^1} ES^1)$$

is given by

$$\begin{aligned} H^2((LM_1 \times_{S^1} ES^1) \times (LM_2 \times_{S^1} ES^1)) \ni \pm\, e = \pm\,(f_1 \times f_2)^*(\tilde{u}) = \\ = (f_1 \times f_2)^*(u \oplus -u) = f_1^* u \oplus -f_2^* u = \\ = \pi_1^*(e_1) - \pi_2^*(e_2) . \end{aligned} \tag{2.32}$$

Remark that

$$H^2((LM_1 \times LM_2) \times_{S^1} ES^1)) \ni p^*(\pi_1^*(e_1) - \pi_2^*(e_2)) = 0 \tag{2.33}$$

by the exactness of the Gysin sequence.

So as claimed in the beginning of this section the knowledge of the Euler class of

$$\begin{tikzcd} S^1 \arrow[r] & LM_i \times ES^1 \simeq LM_i \arrow[d] \\ & LM_i \times_{S^1} ES^1 \end{tikzcd} .$$

yields $H_*^{S^1}(LM_1 \times LM_2)$ by using (2.32) for the Leray-Serre spectral sequence of the left fibration of (2.29).

<u>Euler class of the S^1-bundle $L(M_1 \times M_2) \times ES^1 \to L(M_1 \times M_2) \times_{S^1} ES^1$:</u>

So far we presented a possibility to compute the module structure of

$$H_*^{S^1}(LM_1 \times LM_2)\ .$$

When discussing operations that arise as descended operations on $\mathbb{H}_*(LM)$ as described in section 2.2.4 we need to better understand the corresponding Gysin sequence for the loop-string fibration.
To be precise we need a concept of how the Euler class and the $\mathcal{M}$ark, $\mathcal{E}$rase map for

$$\begin{tikzcd} S^1 \arrow[r] & L(M_1 \times M_2) \times ES^1 \simeq LM_1 \times LM_2 \arrow[d] \\ & (LM_1 \times LM_2) \times_{S^1} ES^1 \end{tikzcd}$$

are computed in terms of the ones of

$$\begin{tikzcd} S^1 \arrow[r] & LM_i \times ES^1 \simeq LM_i \arrow[d] \\ & LM_i \times_{S^1} ES^1\ . \end{tikzcd}$$

At least for the Euler class we present a concept of how to compute it. We tie up to the considerations and notions from above.
Remark that since the S^3-bundle $S^3 \times_{S^1} S^3 \to S^2$ is trivial we have a homotopy equivalence

$$S^3 \times_{S^1} S^3 \simeq S^2 \times S^3 \simeq S^3 \times S^2\ .$$

Again we find the Hopf map and thus get for the Euler class

$$e\big(S^3 \times S^3 \to S^3 \times_{S^1} S^3\big) \in H^2(S^2) \otimes H^0(S^3) \cong H^0(S^3) \otimes H^2(S^2) \tag{2.34}$$

the class that that clearly arises when pulling back the Euler class of the Hopf bundle $S^3 \to S^2$ via the projection onto the first or the second S^2-factor.
This principle of the universal bundles is manifested in the loop-string fibration for products, namely its Euler class arises as a pulled back Euler class via

$$\begin{tikzcd} (LM_1 \times LM_2) \times ES^1 \arrow[r] \arrow[d] & LM_1 \times ES^1 \arrow[d] \\ (LM_1 \times LM_2) \times_{S^1} ES^1 \arrow[r, "\pi_1 \circ p"] & LM_1 \times_{S^1} ES^1 \end{tikzcd}$$

or analogously for LM_2. We conclude that the Euler class

$$e(\ L(M_1 \times M_2) \to (LM_1 \times LM_2) \times_{S^1} ES^1\ )$$

is given by

$$H^2((LM_1 \times LM_2) \times_{S^1} ES^1)) \ni \pm e = p^*(\pi_1^*(e_1)) = p^*(\pi_2^*(e_2)) \tag{2.35}$$

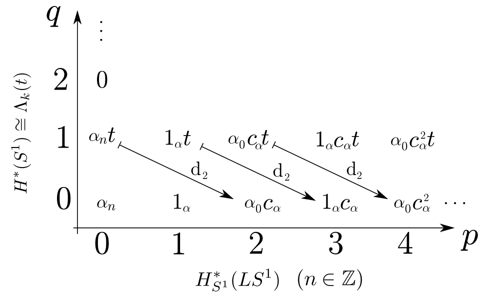
which is consistent with $p^*(\pi_1^*(e_1) - \pi_2^*(e_2)) = 0$.

Exemplifying computation for $T^2 = S^1 \times S^1$:

Described concepts are demonstrated for the torus T^2 as a product of two circles S^1. Recall that for a field $\mathbf{k}$ of characteristic 0 we have $\mathbf{k}$-vector spaces

$$H^*(LS^1) \cong \bigoplus_{n \in \mathbb{Z}} \mathbf{k}\langle a_n, A_n\rangle \quad \text{and} \quad H^*_{S^1}(LS^1) \cong \bigoplus_{i \geqslant 0} \mathbf{k}\langle \alpha_0 \otimes c_\alpha^i, 1_\alpha \otimes c_\alpha^i\rangle \oplus \bigoplus_{0 \neq n \in \mathbb{Z}} \mathbf{k}\langle \alpha_n\rangle\ ,$$

for generators of degree $|a_n| = |A_n| - 1 = 0$, $|\alpha_0 \otimes c_\alpha^i| = |1_\alpha \otimes c_\alpha^i| - 1 = 2i$ and $|\alpha_n| = 0$. Since we work with coefficients in a field we may equally work with homology or cohomology. As described in appendix 5.4 for the S^1-bundle $LS^1 \times ES^1 \to LS^1 \times_{S^1} ES^1$ the whole information of the Leray-Serre spectral sequence is encoded in the E_2-page:



We thus have for $x, c_\alpha \in H^*_{S^1}(LS^1)$ that

$$d_2 : x \otimes t \mapsto (x \cup c_\alpha) \otimes 1$$

and zero else. That is the Euler class of the stated bundle is $c_\alpha \in H^2_{S^1}(LS^1)$. With the spectral sequence we get the following generators for $H^*(LS^1)$:

$$[\alpha_n] = a_n\ ,\ [\alpha_{n \neq 0} t] = A_n\ ,\ [1_\alpha] = A_0\ .$$

By the considerations above we get that up to a sign the Euler class c of the S^1-bundle

$$LT^2 \times_{S^1} ES^1 \to (LS^1 \times_{S^1} ES^1) \times (LS^1 \times_{S^1} ES^1) \tag{2.36}$$

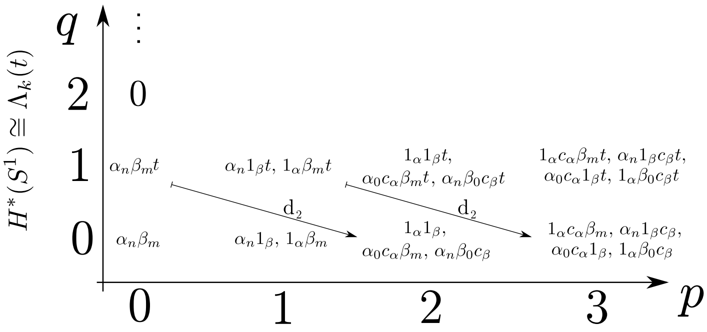
is given by

$$c_\alpha \otimes 1 - 1 \otimes c_\beta\ .$$

That in turn allows to compute $H^*_{S^1}(LT^2)$. Namely the E_2-page for (2.36) is given by where

$$d_2 : x \otimes t \to (x \cup c) \otimes 1$$

for $x \in H^*_{S^1}(LS^1) \otimes H^*_{S^1}(LS^1)$ and zero else. For the cohomology of the total space we get the following generators on the E_3-page and thus for $H^*_{S^1}(LT^2)$:



$H^*_{S^1}(LS^1) \otimes H^*_{S^1}(LS^1)$ $(n, m \in \mathbb{Z})$

generators	degree
$[\alpha_n\beta_m]$ for $n, m \geqslant 0$	0
$[\alpha_n\beta_m t]$ for $n \neq 0 \wedge m \neq 0$	1
$[\alpha_n 1_\beta]$, $[1_\alpha\beta_m]$ for $n, m \geqslant 0$	1
$[1_\alpha 1_\beta c^i_\alpha]$ for $i \geqslant 0$	$2i+2$
$[\alpha_0\beta_0 c^i_\alpha]$ for $i \geqslant 1$	$2i$
$[\alpha_0 1_\beta c^i_\alpha]$, $[1_\alpha\beta_0 c^i_\alpha]$ for $i \geqslant 1$	$2i+1$

Remark that

$$H^*_{S^1}(LT^2) \ni [x(c_\alpha \otimes 1 + 1 \otimes c_\beta)] = 2[xc_\alpha] = 2[xc_\beta]$$

due to (2.33).
We conclude that the S^1-equivariant (co-)homology vector space $H^{S^1}(LT^2)$ is given by

$$\big((\mathbf{k} \oplus \mathbf{k}^2 \oplus \mathbf{k}) \otimes \mathbf{k}[c_\alpha]\big) \oplus \bigoplus_{(n,m) \in \mathbb{Z}^2 \setminus 0} (\mathbf{k} \oplus \mathbf{k}) \cong H_*(T^2) \otimes H_*(BS^1) \oplus \bigoplus_{\mathbb{Z}^n \setminus \{0\}} H_*(T^1) \ .$$

which is consistent with (2.21).

So how do results apply for the Euler class of the S^1-bundle

$$(\overbrace{LS^1 \times LS^1}^{\simeq LT^2}) \times ES^1 \to LT^2 \times_{S^1} ES^1 \ . \tag{2.37}$$

Its Euler class is given by

$$c_\alpha \otimes 1 = 1 \otimes c_\beta \ .$$

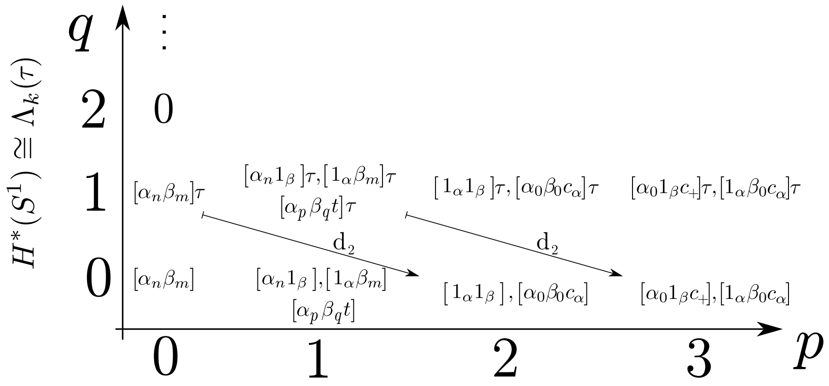
With the knowledge of $H^*_{S^1}(LT^2)$ we are now able to compute

$$H^*(LT^2) \cong H^*(LS^1) \otimes H^*(LS^1) \ .$$

Namely the E_2-page for (2.37) is given by
where

$$d_2 : [x] \otimes \tau \to [x \cup c_\alpha] \otimes 1$$

and zero else. For the cohomology of the total space we get the following generators on the E_3-page and thus for $H^*(LT^2)$:



$H^*_{S^1}(LT^2)$ $\quad (n, m \in \mathbb{Z})$ $(p, q \in \mathbb{Z}\backslash 0)$

<table>
<tr><th>generators</th><th>degree</th></tr>
<tr><td>$a_n b_m := [\alpha_n\beta_m]_2$</td><td>0</td></tr>
<tr><td>$A_p b_q := [\alpha_p\beta_p t]_2$ for $p \neq 0 \wedge q \neq 0$</td><td rowspan="2">1</td></tr>
<tr><td>$a_n B_m := [\alpha_n\beta_m]_2\tau$ for $n \neq 0 \vee m \neq 0$</td></tr>
<tr><td>$A_n b_0 := [\alpha_n 1_\beta]_2$, $A_0 b_m := [1_\alpha\beta_m]_2$</td><td>1</td></tr>
<tr><td>$A_p B_q := [\alpha_p\beta_q t]_2\tau$ for $p \neq 0 \wedge q \neq 0$</td><td rowspan="2">2</td></tr>
<tr><td>$A_p B_0 := [\alpha_n 1_\beta]_2\tau$, $A_0 B_q := [1_\alpha\beta_m]_2\tau$ for $n, m \neq 0$</td></tr>
<tr><td>$A_0 B_0 := [1_\alpha 1_\beta]_2$</td><td>2</td></tr>
</table>

In total this yields

$$H_*(LT^2) \cong \Big(\bigoplus_{n\in\mathbb{Z}} \mathbf{k}\langle a_n, A_n\rangle\Big) \otimes \Big(\bigoplus_{m\in\mathbb{Z}} \mathbf{k}\langle b_m, B_m\rangle\Big) \cong H^*(LS^1) \otimes H^*(LS^1) \ ,$$

which is consistent with proposition (2.17).

We conclude the chapter by remarking the fact that methods nicely apply to considerations concerning $LM \times LN$ when one Euler class is vanishing. This for example is the case if we consider the space of non-contractible loops on a manifold M with negative sectional curvature. Recall that corollary 2.15 yields that for a field $\mathbf{k}$ of characteristic 0 one has

$$H_*^{S^1}(L^{>0}M; \mathbf{k}) \cong \bigoplus_{0 \neq \alpha \in \widetilde{\pi}_1(M)} H_*(ES^1/\mathbb{Z}_n) \otimes \mathbf{k} \cong \bigoplus_{0 \neq \alpha \in \widetilde{\pi}_1(M)} H_0(ES^1/\mathbb{Z}_n) \otimes \mathbf{k}$$

which means that the Euler class of the loop-string fibration of LM is vanishing by degree reasons and further by the considerations above thus vanishes for

$$\begin{array}{c} (LM \times LN) \times ES^1 \\ \Big\downarrow \\ (LM \times N) \times_{S^1} ES^1 \end{array} \quad .$$

Products of manifolds where one factor has negative sectional curvature are further examined in chapter 4.4. The essence of why we are discussing these kinds of spaces lies in the fact that the topology of the space of non-contractible loops on them is so well understood. Recall that in our context questions posed by symplectic geometry and answered by using holomorphic curve theory only concern non-contractible loops.

Chapter 3

Homotopy algebras

Structures such as algebras or Lie algebras transfer from one complex to an isomorphic complex. If the complexes are just quasi-isomorphic (as for a formal chain complex and its homology) we get higher homotopy versions of algebras and Lie algebras namely A_∞-/L_∞-algebras. This transfer construction is summarized in section 3.1 for algebras and in section 3.3 for Lie algebras. Standard references are [21] and [27].
The concepts for algebras are then applied to the dg algebra

$$(C,d) = (\Lambda_{\mathbb{R}}(\alpha) \otimes_{\mathbb{R}} \mathbb{R}[\lambda], d)$$

where

$$H_*(C) \cong \mathbb{H}_*(LS^n)$$

as algebras for $n \geqslant 2$. We get higher string topology operations extending the loop product on $\mathbb{H}_*(LS^n)$ for $n \geqslant 2$.

In the following we always work with coefficients in $\mathbb{R}$.

3.1 The homotopy transfer construction for algebras

3.1.1 A_∞-algebras

We rely on the ideas and concepts presented by Kadeishvili in [21].

An A_∞-algebra consists of a graded vector space $C = \bigoplus\limits_{m\geqslant 0} C^m$ and operations

$$m_n : C^{\otimes n} \to C \quad , \quad n \geqslant 1$$

of degree $|m_n| = n-2$ (homological convention) such that

$$\sum_{k=0}^{n-1}\sum_{j=1}^{n-k}(-1)^{k+|a_1|+\ldots+|a_k|} m_{n-j+1}(a_1, ..., a_k, m_j(a_{k+1}, ..., a_{k+j}), ..., a_n) = 0 \tag{3.1}$$

for all $n \geqslant 1$. An equivalent approach is given by the bar construction.
On

$$TC[-1] := C[-1] \oplus C[-1] \otimes C[-1] \oplus ...$$

we have a coproduct η given by

$$\eta(a_1 \otimes ... \otimes a_k) := \sum_{i=0}^{k} (a_1 \otimes ... \otimes a_i) \otimes (a_{i+1} \otimes ... \otimes a_k) \quad , \quad |\eta| = 0 \ .$$

The operations $m_n : C^{\otimes n} \to C$ determine a coderivation $d : TC[-1] \to TC[-1]$, which is given by

$$d(a_1 \otimes ... \otimes a_n) := \sum_{k=0}^{n-1} \sum_{j=1}^{n-k} (-1)^{k+|a_1|+...+|a_k|} a_1 \otimes ... \otimes a_k \otimes m_j(a_{k+1} \otimes ... \otimes a_{k+j}) \otimes ... \otimes a_n \ .$$

For d being a coderivation means that

$$\eta \otimes d = (d \otimes \mathrm{id} + \mathrm{id} \otimes d) \circ \eta : C \to C \otimes C \ .$$

Lemma 3.1 (*[21] and chapter 3.6. of [22]*)

$(C, \{m_n\}_{n \geqslant 1})$ is an A_∞-algebra, that is (3.1) are satisfied for all $n \geqslant 1$, if and only if

$$d^2 = 0 \ ,$$

that is d is a differential on the coalgebra $TC[-1]$.

Remark 3.2. *A dga (A, μ, d) may be viewed as an A_∞-algebra when setting*

$$\widetilde{m}_1 := d, \ \widetilde{m}_2(a, b) := (-1)^{|a|} \mu(a \otimes b) = (-1)^{|a|} ab, \ \widetilde{m}_{k \geqslant 3} := 0 \ .$$

The sign factor is necessary so that the A_∞-relations (3.1) imply associativity and that d is a derivation and vice versa. Namely (3.1) *for $i = 2$ reads as*

$$\widetilde{m}_1(\widetilde{m}_2(a, b)) + \widetilde{m}_2(\widetilde{m}_1(a), b) + (-1)^{|a|+1} \widetilde{m}_2(a, \widetilde{m}_1(b)) = 0 \ ,$$

which according to the definition is equivalent to

$$(-1)^{|a|} d(ab) + (-1)^{|a|+1} dab - adb = 0 \ ,$$

that is

$$d(ab) = dab + (-1)^{|a|} adb \ .$$

Similarly (3.1) *for $i = 3$ reads as*

$$\widetilde{m}_2(\widetilde{m}_2(a, b), c) + (-1)^{|a|+1} \widetilde{m}_2(a, \widetilde{m}_2(b, c)) = 0 \ ,$$

which is equivalent to

$$(-1)^{2|a|+|b|}(ab)c + (-1)^{2|a|+|b|+1}a(bc) = 0 \ ,$$

or simply

$$(ab)c = a(bc) \ .$$

Conversely an A_∞-algebra $(B, \{m_n\}_{n\geqslant 1})$ is a dg algebra if $m_n = 0$ for all $n \geqslant 3$.

An A_∞-morphism between A_∞-algebras

$$\{f_n\}_{n\geqslant 1} : (C, \{m_n\}) \to (C', \{m'_n\})$$

consists of a collection of maps $\{f_n : C^{\otimes n} \to C'$ of degree $|f_n| = n-1\}$ satisfying certain relations. These relations can be expressed by saying that

$$f : TC[-1] \to TC'[-1]$$

where f is given by

$$f(a_1 \otimes ... \otimes a_n) := \sum_{t=1}^{n} \sum_{\{k_1,...,k_t \ | \ \sum k_i = n\}} f_{k_1}(a_1 \otimes ... \otimes a_{k_1}) \otimes ... \otimes f_{k_t}(a_{n-k_t+1} \otimes ... \otimes a_n)$$

is a differential coalgebra map in the bar construction, that is

$$f \circ d = d' \circ f \quad \text{and} \quad \eta' \circ f = f \otimes f \circ \eta \ .$$

Lemma 3.3 (*[21] and chapter 3.6. of [22]*)

The morphism f is a coalgebra map if and only if for all $n \geqslant 1$

$$\sum_{k=0}^{n-1}\sum_{j=1}^{n-k}(-1)^{k+|a_1|+...+|a_k|} f_{n-j+1}(a_1, ..., a_k, m_j(a_{k+1}, ..., a_{k+j}), a_{k+j+1}, ..., a_n) \tag{3.2}$$
$$= \sum_{t=1}^{n} \sum_{\{k_1,...,k_t \ | \ \sum k_i = n\}} \widetilde{m}'_t(f_{k_1}(a_1, ..., a_{k_1}), ..., f_{k_t}(a_{n-k_t+1}, ..., a_n)) \ .$$

The following subsection describes a procedure to transfer a given A_∞-algebra structure on A to a sub-complex $i(B) \subset A$. We need that $i : B \to A$ is a quasi-isomorphism. In the following we often assume B to be the homology of A with trivial differential. Further the A_∞-algebra A is mostly just a differential graded algebra, that is $m_n = 0$ for $n \geqslant 3$.

3.1.2 Homotopy transfer for dg algebras

Again we are following [21]. Let (C, μ, d) be a differential (graded) algebra, corresponding to an A_∞-algebra $(C, \{\widetilde{m}_n\}_{n\geqslant 1})$ with $m_n = 0$ for $n \geqslant 3$.

Remark that on $H(C)$ we always have the trivial A_∞-structure with $m_n = 0$ for $n \geqslant 3$ when setting $0 = f_n : H(C)^{\otimes n} \to C$ for all $n \geqslant 2$.

Further it is possible to define an A_∞-algebra structure

$$(H(C,d), \{m_n\}_{n\geqslant 1})$$

that is induced by (C, μ, d). This so called *homotopy transfer* construction is described in the stated reference. We recall the construction here in order to derive a feeling for the required formulas.

The construction described in [21] allows to write down an A_∞-algebra structure on $H(C)$ and further an A_∞-morphism

$$f = \{f_n\}_{n\geqslant 1} : (H(C), \{m_n\}_{n\geqslant 1}) \to (C, \{\widetilde{m}_n\}_{n\geqslant 1}) \ ,$$

where $f_1 = i : H(C) \to C$ is a chosen quasi-isomorphism. The morphism f is called an A_∞*-quasi-isomorphism*.

Theorem 3.4 (***Theorem 1 of [21]***)

Let (C, μ, d) be a differential algebra over $\mathbb{R}$. Then one gets an A_∞-algebra structure $\{m_n\}_{n\geqslant 1}$ on $H(C)$ such that $m_1 = 0$ and m_2 is the induced product on $H(C)$. Further one gets an A_∞-algebra morphism $f = \{f_n\}_{n\geqslant 1} : H(C) \to C$ such that f_1 is a quasi-isomorphism.

Proof: The dg algebra (C, μ, d) is an A_∞-algebra when setting

$$\widetilde{m}_1 := d, \ \widetilde{m}_2(a,b) := (-1)^{|a|}\mu(a \otimes b) = (-1)^{|a|}ab, \ \widetilde{m}_{k\geqslant 3} := 0$$

as before.

The theorem is proven by induction.

One starts by setting $m_1 = 0$ and defines f_1 to be a cycle choosing homomorphism i, which is possible since we assume $H(C)$ to be free. Then $f_1 m_1 = 0 = \widetilde{m}_1 f_1$ that is (3.2) is satisfied for $n = 1$.

For $n \geqslant 2$ the necessary relations of m_n and f_n required in (3.2) for $\{f_n\}_{n\geqslant 1}$ to be an A_∞-morphism translate into

$$f_1 m_n - U_n = \widetilde{m}_1 f_n \ , \tag{3.3}$$

where

$$U_n(a_1, ..., a_n) := \sum_{s=1}^{n-1} \widetilde{m}_2(f_s(a_1, ..., a_s), f_{n-s}(a_{s+1}, ..., a_n)) +$$
$$\sum_{k=0}^{n-2}\sum_{j=2}^{n-1} (-1)^{k+1+|a_1|+...+|a_k|} f_{n-j+1}(a_1, ..., a_k, m_j(a_{k+1}, ..., a_{k+j}), a_{k+j+1}, ..., a_n) \ .$$

Things are simplified in the described way since $\widetilde{m}_k = 0$ for $k \geqslant 3$.

For the inductive step $k-1 \mapsto k$ assume that all operations m_j and morphisms f_j are constructed for $1 \leqslant j \leqslant k-1$ and satisfy the relation (3.3) for $n \leqslant k-1$.

Equation (3.3) reads as

$$i \circ m_k - U_k = d \circ f_k \ . \tag{3.4}$$

Since U_k only involves operations m_j and morphisms f_j for $1 \leqslant j \leqslant k-1$ it is determined and one checks that

$$d(U_k(a_1, ..., a_k)) = 0$$

for $a_i \in H(C)$. We define

$$(p \circ U_k)(a_1, ..., a_k) = [U_k(a_1, ..., a_k)] =: m_k(a_1, ..., a_k) \ .$$

Then $(i \circ m_k)(a_1, ..., a_k)$ and $U_k(a_1, ..., a_k)$ are homologous for all $a_i \in H(C)$, and so for generators $x_1, ..., x_k$ of $H(C)$ we can choose $f_k(x_1, ..., x_k) \in C$ to be a chain whose boundary equals

$$(i \circ m_k - U_k)(x_1, ..., x_k) \ .$$

Linearly extending defines f_k. It remains to check that the operations m_k satisfy (3.1), which is proven in [21].

This completes the inductive step, and we conclude that $f = \{f_n\}_{n \geqslant 1}$ defined in this way is an A_∞-algebra morphism.

■

Remark that the constructed operations and morphisms are not unique in the sense that for each degree we chose f_1 and a homotopy f_n bounding $f_1 m_n - U_n$ for all $n \geqslant 1$.

To get more insight into the defined operations remark that for a formal dga (C, μ, d) we have the set-up

$$\begin{aligned} (H(C,d), d=0) \underset{p}{\overset{i}{\rightleftarrows}} (C, \mu, d) \circlearrowleft^h \\ p \circ i =& \mathrm{id} \\ i \circ p - \mathrm{id} =& dh + hd \end{aligned} \tag{3.5}$$

The first equation implies that p is surjective and i is injective. Both equations together say that i is a chain homotopy equivalence with inverse p.

The described non-uniqueness of the arising operations and morphisms is displayed in the global homotopy h. This can be seen as follows. Reinterpreting the stated recursive construction yields

$$m_1 = 0 \quad , \quad m_2(a,b) = p \circ \underbrace{\widetilde{m}_2(f_1(a), f_1(b))}_{=U_2(a,b)} = (-1)^{|a|} p(i(a) \cdot i(b))$$

and for (3.3) with $n = 2$ we get

$$i \circ m_2 - U_2(a, b) = (i \circ p - \mathrm{id})\widetilde{m_2}(i(a), i(b)) \overset{(3.5)}{=} (dh + hd)\widetilde{m_2}(i(a), i(b)) =$$
$$= d(h \circ \widetilde{m_2}(i(a), i(b))) =: d(\ f_2(a, b)\ )\ .$$

Here we used that d is a derivation and $d \circ i = 0$.
Continuing in that manner we end up with A_∞-operations on $H(C)$ of the form visualized in figure (3.1). A detailed description of this approach can be found in section 10.3.7. and in paricular theorem 10.3.8. of [27].

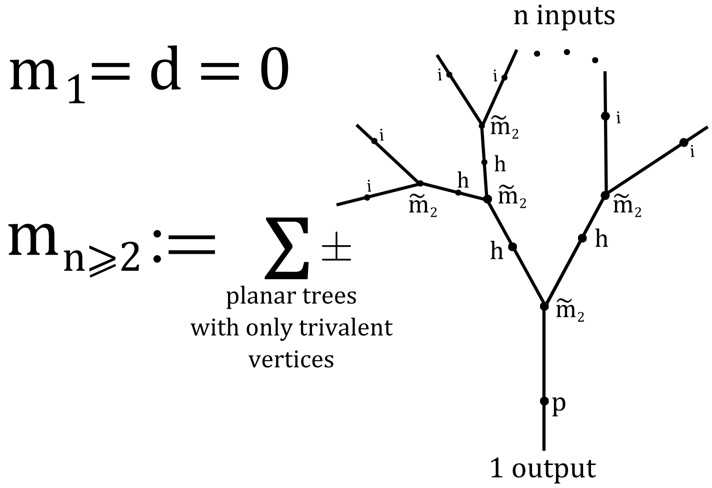


Figure 3.1: Visualization of higher operations

3.1.3 Homotopy transfer for a product of dg algebras

We continue to work with real coefficients. Recall that the tensor product $C = A \otimes B$ of two differential graded algebras (A, μ_A, d_A), (B, μ_B, d_B) is a differential graded algebra (C, μ_C, d_C), where

$$C^k = \bigoplus_{i+j=k} A^i \otimes B^j\ ,\ \mu(x \otimes y, x' \otimes y') := (-1)^{|y||x'|}\mu_A(x, x') \otimes \mu_B(y, y')\ ,$$
$$d(x \otimes y) := d_A(x) \otimes y + (-1)^{|x|} x \otimes d_B(y)\ .$$

For linear maps

$$f : A \to A' \quad \text{and} \quad g : B \to B'$$

of degree $|f| = p$ and $|g| = q$ the linear map $f \otimes g : A \otimes B \to A' \otimes B'$ of degree $|f \otimes g| = p + q$ is defined as

$$(f \otimes g)(a \otimes b) = (-1)^{|g||a|} f(a) \otimes g(b)\ .$$

The homotopy transfer construction described above allows to write down an induced A_∞-algebra structure on $H(C)$ and further an A_∞-morphism

$$\{f_n\} : (H(C), \{m_n\}) \to (C, \{\widetilde{m}_n\}) \ ,$$

where $f_1 : H(C) \to C$ is a quasi-isomorphism. One may ask whether the operations m_n and the morphisms f_n may be written in terms of the operations m_n^A, m_n^B and morphisms f_n^B arising when doing the construction for the separate factors A and B. In general this question is not easy to answer but the problem simplifies in the case when the induced A_∞-algebra on the homology of one factor is trivial. Here without loss of generality we assume that $H(A)$ is a graded algebra with vanishing higher operations $m_{k\geqslant 3} = 0$.

Lemma 3.5

If the A_∞-algebra structure on $H(A)$ obtained from the homotopy transfer does not have non-trivial higher operations, then the induced A_∞-algebra of $H(A\otimes B)$ is given up to sign by the induced A_∞-algebra of $H(B)$ with coefficients in the algebra $H(A)$.

Proof: Remark that for set-up (3.5) and thus for the homotopy construction we do not need that the left hand side is the homology of the right. We want to split the homotopy transfer into two steps and first discuss it for the following set-up:

$$H(A, d_A) \otimes B \underset{p=p_A\otimes \mathrm{id}|_B}{\overset{i=i_A\otimes \mathrm{id}|_B}{\rightleftarrows}} (A \otimes B, \mu, d) \circlearrowleft^{h=h_A\otimes \mathrm{id}|_B} \tag{3.6}$$

$$\begin{aligned} p \circ i &= \mathrm{id} \\ i \circ p - \mathrm{id} &= dh + hd \end{aligned}$$

The stated relations hold since

$$p \circ i = p(i_A \otimes \mathrm{id}_B) = p_A i_A \otimes \mathrm{id}_B = \mathrm{id}_A \otimes \mathrm{id}_B = \mathrm{id} \ ,$$

$$\begin{aligned} (dh + hd) &= d(h_A \otimes \mathrm{id}_B) + h(d_A \otimes \mathrm{id}_B + \mathrm{id}_A \otimes d_B) \\ &= d_A h_A \otimes \mathrm{id}_B - h_A \otimes d_B + h_A d_A \otimes \mathrm{id}_B + h_A \otimes d_B \\ &= (i_A p_A - \mathrm{id}_A) \otimes \mathrm{id}_B \\ &= ip - \mathrm{id} \ . \end{aligned}$$

Remark that now we have $m_1 \neq 0$ and thus if we define U_n as before (3.2) does not rewrite as (3.4) but

$$f_1 \circ m_1 = d \circ f_1 \equiv \widetilde{m}_1 \circ f_1$$

and

$$f_1 \circ m_n - U_n = d \circ f_n - \sum_{k=0}^{n-1} (-1)^k f_n \circ (\hat{\mathrm{id}}^k \otimes m_1 \otimes \mathrm{id}^{n-k-1})$$
$$= d \circ f_n - f_n \circ d_{(H(A)\otimes B)^{\otimes n}} =: \delta f_n \ ,$$

for $n \geqslant 2$ where $\hat{\mathrm{id}}(x) = (-1)^{|x|} x$ and $\widetilde{m}_1(x) = d_{A\otimes B}(x)$.

Doing the homotopy transfer construction for set-up (3.6) yields the following operations and morphisms for $H(A, d_A) \otimes B$:

$$U_2 = \widetilde{m}_2 \circ (f_1 \otimes f_1) \ ,$$

$$m_1 = \mathrm{id}|_{H(A,d_A)} \otimes d_B \ , \ m_2 = p \circ U_2 \ ,$$

$$f_1 = i_A \otimes \mathrm{id}|_B \ , \ f_2 = h \circ U_2 \ ,$$

where $\widetilde{m}_2(x, y) = (-1)^{|x|} \mu(x, y)$. This can be seen as follows:

The equation $f_1 \circ m_1 = d \circ f_1$ holds, since

$$\begin{aligned}
&(f_1 \circ m_1)([a] \otimes b) \\
&= \big(f_1 \circ (\mathrm{id}|_{H(A)} \otimes d_B)\big)([a] \otimes b) = (-1)^{-[a]} i_A([a]) \otimes d_B(b) \\
&\overset{d_A \circ i_A = 0}{=} (d \circ (i_A \otimes \mathrm{id}|_B))([a] \otimes b) = (d \circ f_1)([a] \otimes b) \ ,
\end{aligned}$$

and

$$i \circ m_2 - U_2 = d \circ f_2 - f_2(m_1 \otimes \mathrm{id}) + f_2(\hat{\mathrm{id}} \otimes m_1)$$

holds since

$$\begin{aligned}
&(i \circ m_2 - U_2)([a] \otimes b, [a'] \otimes b') \\
&= (f_1 \circ p - \mathrm{id}) \widetilde{m}_2(i_A[a] \otimes b, i_A[a'] \otimes b') = (dh + hd) \widetilde{m}_2(i_A[a] \otimes b, i_A[a'] \otimes b') \\
&= (d \circ f_2 + h \circ \widetilde{m}_1 \circ \widetilde{m}_2)(i_A[a] \otimes b, i_A[a'] \otimes b') \\
&\overset{(3.1)}{=} \big(d \circ f \ - \ (h \circ \widetilde{m}_2) \circ (\widetilde{m}_1 \otimes \mathrm{id}) \circ (f_1 \otimes f_1) + \\
&\qquad + (h \circ \widetilde{m}_2) \circ (\hat{\mathrm{id}} \otimes \widetilde{m}_1) \circ (f_1 \otimes f_1)\big)([a] \otimes b, [a'] \otimes b') \\
&\overset{f_1 m_1 = \widetilde{m}_1 f_1}{=} \big(d \circ f_2 \ - \ f_2 \circ (m_1 \otimes \mathrm{id}) + f_2 \circ (\hat{\mathrm{id}} \otimes m_1)\big)([a] \otimes b, [a'] \otimes b') \ .
\end{aligned}$$

Continuing in that manner we define higher ($k \geqslant 3$) operations and morphisms via

$$\begin{aligned}
U_k(([a_1]\otimes b_1),...,([a_k]\otimes b_k)) &: = (-1)^{\bigstar} U_k^A([a_1],...,[a_k])\otimes \mu_B(b_1,...,b_k)\\
f_k(([a_1]\otimes b_1),...,([a_k]\otimes b_k)) &: = (h\circ U_k)([a_1],...,[a_k])\otimes \mu_B(b_1,...,b_k)\\
m_k(([a_1]\otimes b_1),...,([a_k]\otimes b_k)) &: = (p\circ U_k)([a_1],...,[a_k])\otimes \mu_B(b_1,...,b_k)\\
&= (-1)^{\bigstar} m_k^A([a_1],...,[a_k])\otimes \mu_B(b_1,...,b_k)\ ,
\end{aligned}$$

where $\bigstar = |b_1|(|a_2|+...+|a_n|+n-1)+|b_2|(|a_3|+...+|a_n|+n-2)+...+|b_{n-1}|(|a_n|+1)$. Here m_k^A, U_k^A, f_k^A are operations and morphisms arising when doing the homotopy transfer construction for

$$H(A,d_A) \underset{p_A}{\overset{i_A}{\rightleftarrows}} (A,\mu_A,d_A) \circlearrowleft^{h_A}\quad .$$

We abbreviate $\mu_B(b_1,...,b_k) := \mu_B(...\mu_B(\mu_B(b_1,b_2),b_3),...),b_k)$ which is possible since we assume μ_B to be associative.

These definitions are justified since

$$\begin{aligned}
&(f_1\circ m_n - U_n)([a_1]\otimes b_1,...,[a_n]\otimes b_n)\\
=&(-1)^{\bigstar}(i\circ p-\mathrm{id})\big(U_n^A([a_1],...,[a_n])\otimes\mu_B(b_1,...,b_k)\big)\\
=&(-1)^{\bigstar}(dh+hd)\big(U_n^A([a_1],...,[a_n])\otimes\mu_B(b_1,...,b_k)\big)\\
=&df_n([a_1]\otimes b_1,...,[a_n]\otimes b_n)+\\
&+\sum_{k=1}^{n}(-1)^{\bigstar+|a_1|+...+|a_n|+n-2+|b_1|+...+|b_{k-1}|}h\big(U_n^A([a_1],...,[a_n])\otimes\mu_B(b_1,...,d_B(b_k),...,b_n)\big)\\
=&df_n([a_1]\otimes b_1,...,[a_n]\otimes b_n)+\\
&+\sum_{k=1}^{n}(-1)^{|a_1|+...+|a_k|+n-2+|b_1|+...+|b_{k-1}|+n-k}(hU_n)\big([a_1]\otimes b_1,...,[a_k]\otimes d_B(b_k),...,[a_n]\otimes b_n\big)\\
=&df_n([a_1]\otimes b_1,...,[a_n]\otimes b_n)+\\
&+\sum_{k=1}^{n}(-1)^{|a_1|+|b_1|+...+|a_{k-1}|+|b_{k-1}|-k}(hU_n)\big([a_1]\otimes b_1,...,m_1([a_k]\otimes b_k),...,[a_n]\otimes b_n\big)\\
=&\Big(df_n-\sum_{k=0}^{n-1}(-1)^k(f_n)(\underbrace{\hat{\mathrm{id}},...,\hat{\mathrm{id}}}_{k},m_1,\mathrm{id},...,\mathrm{id})\Big)([a_1]\otimes b_1,...,[a_n]\otimes b_n)\ .
\end{aligned}$$

By assumption $m_{k\geqslant 3}^A=0$ and thus we end up with a homotopy transferred A_∞-algebra structure on

$$H(A)\otimes B$$

of the form

$$m_1=\mathrm{id}|_{H(A)}\otimes d_B\ ,\ m_2=(p\circ\widetilde{m}_2)(i\otimes i)=\pm\, m_2^A\otimes\mu_B\ ,\ m_{k\geqslant 3}=0\ .$$

Since $m_2^A([a],[a']) = (-1)^{|[a]|}\mu_{H(A)}([a],[a'])$ we get

$$\begin{aligned} m_2([a]\otimes b,[a']\otimes b') =& (p\circ \widetilde{m}_2)(a\otimes b, a'\otimes b') \\ =& (-1)^{|a|+|b|+|b||a'|}\mu_{H(A)}([a],[a'])\otimes \mu_B(b,b') \ . \end{aligned}$$

That is up to sign the resulting A_∞-algebra structure is nothing but the differential graded algebra structure of (B,μ_B,d_B) with coefficients in the graded algebra $(H(A),\mu_{H(A)})$.

So it remains to think about the A_∞-algebra structure resulting of the homotopy transfer construction for

$$H(A)\otimes H(B,d) \underset{\mathrm{id}|_{H(A)}\otimes\, p_B}{\overset{\mathrm{id}|_{H(A)}\otimes\, i_B}{\rightleftarrows}} (H(A)\otimes B, \mu|_{H(A)}\otimes \mu|_B, \mathrm{id}|_{H(A)}\otimes d_B) \circlearrowleft^{\mathrm{id}|_{H(A)}\otimes\, h_B}$$

The stated relations hold since they hold for p_B, i_B, d_B, h_B by assumption. Thus the homotopy transfer affects only the second factor and we directly conclude that on $H(A)\otimes H(B)$ we have operations and morphisms of the form

$$\begin{aligned} m_n([a_1]\otimes b_1,...,[a_n]\otimes b_n) &:= \pm [a_1]\cdot ...\cdot [a_n]\otimes m_n^B([b_1],...,[b_n]) \\ f_n([a_1]\otimes b_1,...,[a_n]\otimes b_n) &:= \pm [a_1]\cdot ...\cdot [a_n]\otimes f_n^B([b_1],...,[b_n]) \ . \end{aligned}$$

■

3.2 Examples: A_∞-structures for $\mathbb{H}_*(LS^n)$

We keep working with coefficients in $\mathbb{R}$.

We exemplify the stated homotopy transfer construction for a dg algebra of the form

$$\Lambda_{\mathbb{R}}(\alpha)\otimes_{\mathbb{R}} \mathbb{R}[\lambda] \ ,$$

where $\Lambda_{\mathbb{R}}(\alpha) := \mathbb{R}[\alpha]/(\alpha^2)$, and discuss how the resulting higher operations may be considered as the higher loop product of simply connected spheres.

Theorem 3.6

Consider the dg algebras

$(A_n, d_{A_n}) \equiv (A, d_A) = (\Lambda_{\mathbb{R}}(\alpha)\otimes_{\mathbb{R}} \mathbb{R}[\lambda], 0)$ *for* $n \geqslant 3$ *odd*

and

$(B_n, d_{B_n}) \equiv (B, d_B) = (\Lambda_{\mathbb{R}}(\alpha)\otimes_{\mathbb{R}} \mathbb{R}[\lambda], d\alpha = 0, d\lambda = \alpha\lambda^2)$ *for* $n \geqslant 4$ *even* ,

where $|\alpha| = -n$, $|\lambda| = n-1$.

The A_∞-algebra on $H(A, d_A)$ obtained by homotopy transfer is trivial, that is it is A_∞-quasi-isomorphic to the dga

$$(H(A, d_A), \{m_1 = 0, m_2, m_{i\geqslant 3} = 0\}) \ .$$

The A_∞-algebra on $H(B, d_B)$ obtained by homotopy transfer is non-trivial, that is it is not A_∞-quasi-isomorphic to the dga

$$(H(B, d_B), \{m_1 = 0, m_2, m_{i\geqslant 3} = 0\}) \ .$$

Corollary 3.7

The A_∞-algebra on

$$H(A_{n_1} \otimes ... \otimes A_{n_k})$$

is trivial and the A_∞-algebra on

$$H(A_{n_1} \otimes ... \otimes A_{n_k} \otimes B_n)$$

is non-trivial for $n_i \geqslant 3$ odd, $n \geqslant 2$ even and $k \geqslant 1$.

Proof of Corollary: The corollary follows by theorem 3.6 combined with lemma 3.5. ■

Remark 3.8. *Note that the algebra $\Lambda_{\mathbb{R}}(\alpha)\otimes_{\mathbb{R}}\mathbb{R}[\lambda]$ viewed as an $\mathbb{R}$-module has additive generators of the form*

$$1,\ \lambda,\ \lambda^2,\ ...$$

and

$$\alpha,\ \alpha\lambda,\ \alpha\lambda^2,\ ...\ .$$

It has at most rank 1 in each degree for $n \geqslant 3$, since

$$k(n-1) - n = |\alpha\lambda^k| = |\lambda^l| = l(n-1)$$

$$\Leftrightarrow l = \frac{(k-1)(n-1)-1}{(n-1)} \in \mathbb{Z} \Leftrightarrow \frac{1}{(n-1)} \in \mathbb{Z} \ ,$$

which is not possible for $n \geqslant 3$.

Remark 3.9. *It is tempting to call the arising operations higher loop product of S^n for $n \geqslant 2$ since we have algebra isomorphisms*

$$H(A, d_A) \cong \Lambda_{\mathbb{R}}(a) \otimes_{\mathbb{R}} \mathbb{R}[u] \cong (\mathbb{H}_*(LS^n), \bullet) \quad \textit{for } n \geqslant 3 \textit{ odd}$$

with $[\alpha] = a$ and $[\lambda] = u$ and

$$H(B, d_B) \cong \Lambda_{\mathbb{R}}(b) \otimes_{\mathbb{R}} \mathbb{R}[a, v]/(a^2, ab, av) \cong (\mathbb{H}_*(LS^n), \bullet) \quad \textit{for } n \geqslant 2 \textit{ even}$$

with $[\alpha] = a$, $[\alpha\lambda] = b$ and $[\lambda^2] = v$. It remains to discuss whether (A, d_A) and (B, d_B) are indeed sub-algebras of a fully defined chain level string topology complex such as the one introduced in section 4. We postpone this discussion to section 4.3, where for spheres of odd dimension $n \geqslant 3$ we prove that this is indeed the case. For spheres of even dimension $n \geqslant 2$ we are only able to leave it as a conjecture.

Proof of Theorem 3.6 : In the following we construct the operations m_n on homology for $n \geqslant 1$ as suggested in section 10.3.7. and in particular theorem 10.3.8. of [27]. That is we use the approach via trees as visualized in figure 3.1.

We regard the algebra A as an A_∞-algebras when setting $\widetilde{m}_1 = d_A$, $\widetilde{m}_2(x,y) := (-1)^{|x|}\mu(x \otimes y)$ and $\widetilde{m}_n = 0$ for $n \geqslant 3$. The algebra multiplication μ is given by

$$\mu(\alpha \otimes \lambda^{k_1}, \alpha \otimes \lambda^{k_2}) = 0 \ , \ \mu(1 \otimes \lambda^{k_1}, 1 \otimes \lambda^{k_2}) = 1 \otimes \lambda^{k_1+k_2} \ ,$$
$$\mu(\alpha \otimes \lambda^{k_1}, 1 \otimes \lambda^{k_2}) = \alpha \otimes \lambda^{k_1+k_2} = \mu(1 \otimes \lambda^{k_1}, \alpha \otimes \lambda^{k_2})$$

since α and λ have different parity, so one of them is even, and $|1| = 0$.
Analogously we define $\{\widetilde{m}_n\}_{n\geqslant 1}$ for the algebra B.

<u>higher operations for $H(A, d_A)$</u>

The vanishing of the chain level boundary operator implies that

$$(H(A, d_A), 0) \cong (A, d_A)$$

which in turn allows to define $h = 0$. We get that the morphisms i, p are isomorphisms and that (3.5) is satisfied. Therefore

$$m_n = 0$$

for $n \geqslant 3$.

Since $d = 0$ we have $hd + dh = \mathrm{id} - i \circ p = 0$ for any map $h : A \to A$ of degree $+1$. The homotopy transfer would yield

$$(H(A, d_A), \{\widehat{m}_n\}_{n\geqslant 1}) \xrightarrow{\phi} (A, \{\widetilde{m}_n\}_{n\geqslant 1})$$

where $\widehat{m}_n$ is not necessarily vanishing for $n \geqslant 3$ and ϕ is an A_∞-quasi-isomorphism. Due to theorem 10.4.7. of [27] we know that we can construct an inverse A_∞-quasi-isomorphism

$$(H(A, d_A), \{\widehat{m}_n\}_{n\geqslant 1}) \xleftarrow{\psi} (A, \{\widetilde{m}_n\}_{n\geqslant 1}) \ .$$

We conclude that $(H(A, d_A), \{m_n\}_{n\geqslant 1})$ and $(H(A, d_A), \{\widehat{m}_n\}_{n\geqslant 1})$, arising when setting $h = 0$ and h arbitrary, respectively, are A_∞-quasi-isomorphic.

<u>higher operations for $H(B, d_B)$</u>

By remark 3.8 we know that we have just one generator in each degree of B and thus at most one generator in each degree of $H(B, d_B)$. That is the morphisms i and p are uniquely defined as

$$\alpha \overset{p}{\longmapsto} a \overset{i}{\longmapsto} \alpha \quad , \quad \alpha\lambda^{2k+1} \overset{p}{\longmapsto} bv^k \overset{i}{\longmapsto} \alpha\lambda^{2k+1} \quad , \quad \alpha\lambda^{2k} \overset{p}{\longmapsto} 0 \overset{i}{\longmapsto} 0 \quad ,$$
$$\lambda^{2k} \overset{p}{\longmapsto} v^k \overset{i}{\longmapsto} \lambda^{2k} \quad , \quad \lambda^{2k+1} \overset{p}{\longmapsto} 0 \overset{i}{\longmapsto} 0$$

for $k \geqslant 0$. We define the homotopy $h : B \to B$ of degree 1 as

$$h(\alpha\lambda^{2k}) := \lambda^{2k-1} \text{ (for } k>0) \quad ; \quad h \equiv 0 \text{ else.}$$

That is set-up (3.5) is provided. Namely $p \circ i = \mathrm{id}$ holds. Further $\mathrm{id} = i \circ p$ except for λ^{2k-1} and $\alpha\lambda^{2k}$ we have

$$\begin{aligned} (dh+hd)(\alpha\lambda^{2k}) &= (dh)(\alpha\lambda^{2k}) = d(\lambda^{2k-1}) = \alpha\lambda^{2k} = (\mathrm{id} - i\circ p)(\alpha\lambda^{2k}) \ , \\ (dh+hd)(\lambda^{2k-1}) &= (hd)(\lambda^{2k-1}) = h(\alpha\lambda^{2k}) = \lambda^{2k-1} = (\mathrm{id} - i\circ p)(\lambda^{2k-1}) \ , \end{aligned}$$

since $d(\alpha\lambda^{2k}) = 0$ and $d(\lambda^{2k-1}) = \alpha\lambda^{2k}$ which in turn follows by

$$d\lambda^k = \sum_{i=0}^{k-1}(-1)^i\lambda^i d(\lambda)\lambda^{k-i-1} = \sum_{i=0}^{k-1}(-1)^i\alpha\lambda^{k+1} = \begin{cases} 0 & \text{for } k = \text{even} \\ \alpha\lambda^{k+1} & \text{for } k = \text{odd} \ . \end{cases}$$

and thus

$$d(\alpha\lambda^k) = \alpha d(\lambda^k) = 0 \ .$$

Remark that the product $\widetilde{m}_2(x,y) = (-1)^{|x|}\mu(x,y)$ on B is commutative, that is we will not care about the order of the input elements in the following.
We use the approach via trees as visualized in figure 3.1 to construct the operations m_n.
To understand which trees produce a non-trivial output remark that only compositions of the following type occur for $k,l,r \geqslant 0$:

1)

$$(h\circ\mu)(i(x)\otimes i(y)) = \begin{cases} \lambda^{2k-1} & , \ \{x,y\} = \{a, v^k\} \\ 0 & , \ \text{else} \end{cases}$$

since $\mu(i(x)\otimes i(y))$ is of the form $\alpha\lambda^{2k}$ only for $\{x,y\} = \{a, v^k\}$.

2)

$$(h\circ\mu)(h(\cdot)\otimes i(x)) = (h\circ\mu)(\lambda^{2k-1}\otimes i(x)) = \begin{cases} h(\alpha\lambda^{2(k+l)}) = \lambda^{2(k+l)-1} & , \ x = bv^l \\ 0 & , \ \text{else} \end{cases}$$

since the output of h is either 0 or of the form λ^{2k-1} and $\mu(\lambda^{2k-1}\otimes i(x))$ is of the form $\alpha\lambda^{2k+2l}$ only for $x = bv^l$.

3)

$$(h\circ\mu)(h(\cdot)\otimes h(\cdot)) = (h\circ\mu)(\lambda^{2k-1}\otimes\lambda^{2l-1}) = 0 \ .$$

since the output of h is either 0 or of the form λ^r for r odd.

For possible non-trivial final outputs we have:

1)′
$$(p \circ \mu)(i(x) \otimes i(y)) = m_2(x,y)$$

since m_2 is the induced product on $H(B, d_B)$ due to theorem 3.4.

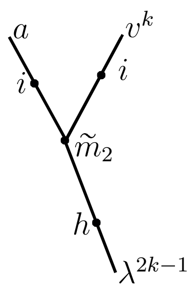
2)′
$$(p \circ \mu)(h(\cdot) \otimes i(x)) \;= (p \circ \mu)(\lambda^{2k-1} \otimes i(x)) = \begin{cases} p(\alpha\lambda^{2k-1}) = bv^{k-1} & , \; x = a \\ 0 & , \; \text{else} \end{cases}$$

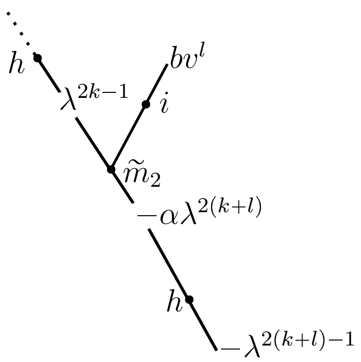
since $(p \circ \mu)(\lambda^{2k-1} \otimes i(x))$ is only non-zero for $i(x)$ being of the form $\alpha\lambda^{2l}$ or λ^{2l+1} but only $\alpha\lambda^0 = \alpha$ is in the image of i.

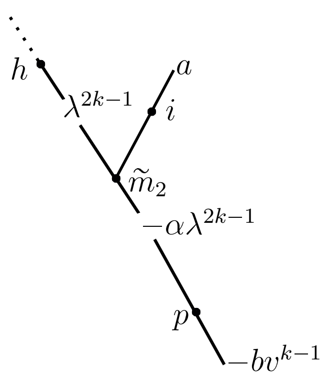
3)′
$$(p \circ \mu)(h(\cdot) \otimes h(\cdot)) = (p \circ \mu)(\lambda^{2k-1} \otimes \lambda^{2l-1}) = v^{k+l-1} \; .$$

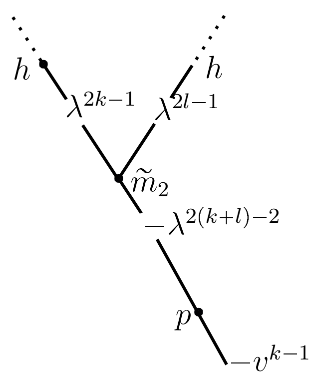
since the output of h is either 0 or of the form λ^r for r odd.

Visualizing this information we conclude that non-trivial trees may only be built by the following sub-trees and their mirrored version. For appearing signs recall that $\widetilde{m}_2(x,y) = (-1)^{|x|}\mu(x,y)$ and $|\lambda| = \text{odd}$, $|\alpha| = \text{even}$ in B.

1)


2)


2)′


3)′


Combining these four type of trees we deduce that only non-trivial trees as visualized in figure 3.2 occur (where $k, l > 0$ and $k_m, l_n \geqslant 0$).

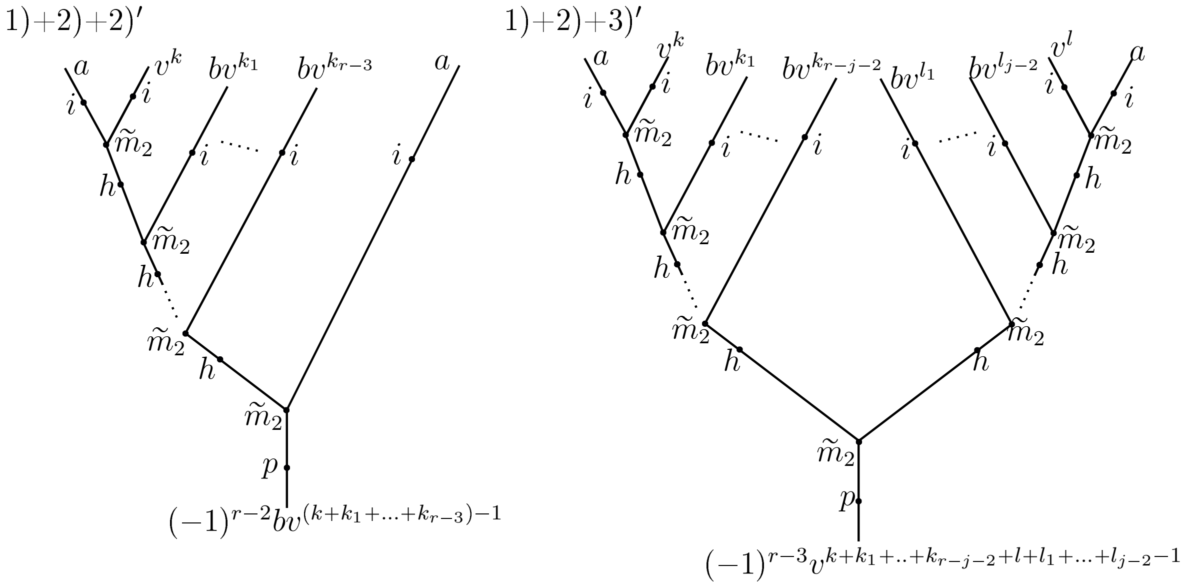


Figure 3.2: Possible higher operations for $\mathbb{H}_*(LS^n)$ for n even

Remark that we are free to interchange the two edges at each vertex and the corresponding trees also produce non-vanishing outputs.

For the tree on the right hand side we are free to distribute the edges with inputs of the form bv^{k_s} among the two main branches of the tree, and all resulting terms come with the same sign. This explains the factor $(i-3)$ in the formula (3.7) below, since this is the number of possible distributions.

In total we conclude that for the induced A_∞-algebra of $H_*(B, d_B)$ the following non-trivial induced higher $(i \geqslant 3)$ operations appear:

$$\boxed{\begin{aligned} m_i(a, v^k, bv^{k_1}, ..., bv^{k_{i-3}}, a) &= (-1)^{i-2}\ bv^{(k+k_1+\ldots+k_{i-3})-1} \\ m_i(a, v^k, bv^{k_1}, ..., bv^{k_{i-4}}, v^l, a) &= (-1)^{i-3}\cdot(i-3)\cdot v^{(k+l+k_1+\ldots+k_{i-4})-1} \end{aligned}} \tag{3.7}$$

where $k, l > 0$ and $k_j \geqslant 0$.

Remark the possibility of interchanging certain edges which yields the stated non-vanishing higher operations with a different order of the inputs.

It remains to prove that the A_∞-algebra structure $(H(B, d_B), \{m_n\}_{n\geqslant 1})$ is indeed non-trivial, that is there exists no A_∞-quasi-isomorphism

$$(H(B, d_B), \{m_n\}_{n\geqslant 1}) \xrightarrow{\phi} (H(B, d_B), \{\widehat{m}_n\}_{n\geqslant 1})$$

where m_n is of the form described above and $\widehat{m}_1 = 0$, $\widehat{m}_2 = m_2 = (p \circ \widetilde{m}_2)(i \otimes i)$ and $\widehat{m}_n = 0$ for $n \geqslant 3$.
This is proven by contradiction, that is we assume that there exists such an A_∞-quasi-isomorphism ϕ.

This implies that ϕ_1 is a quasi isomorphism of degree 0. Since $m_1 = \widehat{m}_1 = 0$ and we only have at most one generator in each degree we find that $\phi_1(a) = \lambda_a \cdot a$ and $\phi_1(b) = \lambda_b \cdot b$ with $\lambda_a, \lambda_b \in \mathbb{R}^*$.
Since ϕ is an A_∞-morphism, it satisfies (3.2), namely

$$\sum_{k=0}^{n-1}\sum_{j=1}^{n-k}(-1)^{k+|a_1|+\ldots+|a_k|}\phi_{n-j+1}(a_1,\ldots,a_k,m_j(a_{k+1},\ldots,a_{k+j}),a_{k+j+1},\ldots,a_n)$$
$$=\sum_{t=1}^{n}\sum_{\{k_1,\ldots,k_t \;|\; \sum k_i=n\}}\widehat{m}_t(\phi_{k_1}(a_1,\ldots,a_{k_1}),\ldots,\phi_{k_t}(a_{n-k_t+1},\ldots,a_n))\ .$$

For $(a_1, a_2, a_3) = (a, v, a)$ this reads as

$$\phi_1(m_3(a,v,a)) + \phi_2(m_2(a,v),a) + (-1)^{|a|+1}\phi_2(a,m_2(v,a))$$
$$= \widehat{m}_2(\phi_1(a),\phi_2(v,a)) + \widehat{m}_2(\phi_2(a,v),\phi_1(a))\ .$$

since $m_1 = \widehat{m}_1 = \widehat{m}_3 = 0$. Further $\phi_1 = \mathrm{id}$, $m_3(a,v,a) = -b$ and $m_2(a,v) = m_2(v,a) = 0$. Thus the equations writes as

$$-\lambda_b \cdot b = \lambda_a \cdot (\widehat{m}_2(a,\phi_2(v,a)) + \widehat{m}_2(\phi_2(a,v),a)) = \lambda_a \cdot (m_2(a,\phi_2(v,a)) + m_2(\phi_2(a,v),a))\ .$$

But this can not be the case because multiplication with a is zero in $H(B)$.

We conclude that such an A_∞-quasi-isomorphism ϕ to the trivial A_∞-algebra can not exist. ■

3.3 The homotopy transfer construction for Lie algebras

Notice the considerations about algebras and A_∞-algebras presented in section 3.1. Here we only recall ideas of Appendix A3. of [14], chapter 10. of [27] and section 4 of [25]. The interested reader is referred to these sources for more details.

An L_∞-algebra over $\mathbb{R}$ consists of a graded real vector space $C = \bigoplus_{m\in\mathbb{Z}} C^m$ and operations

$$\lambda_n : \Lambda^n C \to C \quad (n \geqslant 1)$$

of degree $|\lambda_n| = n - 2$ (homological convention) such that

$$\sum_{n_1+n_2=n+1}(-1)^{n_2}\sum_{\substack{\rho\in S_n\\ \rho(1)<\ldots<\rho(n_1)\\ \rho(n_1+1)<\ldots<\rho(n)}}\epsilon\cdot\lambda_{n_2}(\lambda_{n_1}(a_{\rho(1)},\ldots,a_{\rho(n_1)}),a_{\rho(n_1+1)},\ldots,a_{\rho(n)}) = 0 \quad (3.8)$$

where $\epsilon = \pm 1$ is determined by $a_1 \wedge \dots \wedge a_n = \epsilon\, a_{\rho(1)} \wedge \dots \wedge a_{\rho(n)}$.
Here $\Lambda^n C = T^n C/(a \otimes b + (-1)^{|a||b|} b \otimes a)$ denotes the nth exterior power of C and S_n is the symmetric group.

As for A_∞-algebras there is an equivalent approach in terms of a bar construction. The concept is equivalent to

$$(S(C[-1]), \eta, l)$$

being a differential coalgebra structure, that is $\widehat{l} \circ \widehat{l} = 0$. Precisely speaking on

$$S(C[-1]) := \bigoplus_{k \geqslant 1} S^k(C[-1]) := \bigoplus_{k \geqslant 1} (C[-1] \otimes \dots \otimes C[-1]) / \sim$$

where $S^n C = T^n C/(a \otimes b - (-1)^{|a||b|} b \otimes a)$, we define

$$l_k(c_1 \cdots c_r) = \sum_{\rho \in S_r} \pm \frac{1}{k!(r-k!)} (\sigma_1 \circ \lambda_k \circ \sigma_k^{-1})(c_{\rho(1)} \cdots c_{\rho(k)}) \otimes c_{\rho(k+1)} \otimes \dots \otimes c_{\rho(r)}$$

if $r \geqslant k$ and zero else. Here we use the isomorphism

$$\begin{aligned} (\Lambda^k C)[-k] &\xrightarrow{\sigma_k} S^k(C[-1]) \\ a_1 \wedge \dots \wedge a_k &\longmapsto (-1)^{\sum (k-i)|a_i|} a_1 \cdots a_k \end{aligned}$$

where we used degrees in C for $|a_i|$ for $1 \leqslant i \leqslant k$.

An L_∞-algebra morphism between L_∞-algebras $(C, \{\lambda_k\}_{k \geqslant 1})$ and $(C', \{\lambda'_k\}_{k \geqslant 1})$ is a sequence of maps $\{\phi_k : \Lambda^k C \to C\}_{k \geqslant 1}$ of degree $|\phi_k| = k - 1$ that satisfy

$$e^g \circ l = l' \circ e^g \tag{3.9}$$

in the bar construction, where

$$g_k := \sigma_1 \circ \phi_k \circ \sigma_k^{-1} : S^k(C[-1]) \to C'[-1]$$

and

$$e^g : S(C[-1]) \to S(C'[-1]) \; ; \; c_1 \cdots c_k \mapsto \sum_{k_1 + \dots k_r = k} \sum_\rho \pm \frac{1}{r! k_1! \dots k_r!} (g_{k_1} \otimes \dots \otimes g_{k_r})(c_{\rho(1)} \otimes \dots \otimes c_{\rho(k)}) \; .$$

Similar to the fact that differential graded algebras can be viewed as A_∞-algebras, it is possible to interpret a differential graded Lie algebra $(C, d, \{\cdot, \cdot\})$ as an L_∞-algebra. In fact we have

$$\{a, b\} = -(-1)^{|a||b|} \{b, a\}$$

and when setting

$$\lambda_1 := d, \;\; \lambda_2(a, b) := (-1)^{|a|} \{a, b\} \quad \text{and} \quad \lambda_{k \geqslant 3} := 0 \; ,$$

for $n = 1, 2, 3$ equation (3.8) translates into

$$\begin{aligned} d \circ d &= 0 \ , \\ d\{a, b\} &= \{da, b\} + (-1)^{|a|}\{a, db\} \ , \\ \{a, \{b, c\}\} &= \{\{a, b\}, c\} + (-1)^{|a||b|}\{b, \{a, c\}\} \ . \end{aligned}$$

For general L_∞-algebras the Jacobi identity just holds up to homotopy given by λ_3. So when passing down to homology via the boundary λ_1 Jacobi identity strictly holds. As for A_∞-algebras we can transfer L_∞-algebra structures from one complex C to a quasi-isomorphic complex B and thus in particular to homology $H_*(C)$. Generally speaking we want to transfer structure from C to a homotopy retract B.

Theorem 3.10 (*[27] Theorem 10.3.2., Theorem 10.3.8., Theorem 10.3.9.*)

Let (B, d_B) and (C, d_C) be chain complexes such that

$$\begin{tikzcd} (B, d_B) \arrow[r, shift left, "i"] & (C, d_C) \arrow[l, shift left, "p"] \arrow[loop right, "h"] \end{tikzcd}$$

$$\begin{aligned} p \circ i &= id_B \quad (p \text{ surjective and } i \text{ injective}) \\ i \circ p - id_C &= d_C h + h d_C \quad (i \text{ is chain homotopy equivalence}) \end{aligned}$$

Suppose $\{\widetilde{\lambda}_n\}_{n \geqslant 1}$ is an L_∞-algebra structure on C with $\widetilde{\lambda}_1 = d_C$. Then B is equipped with an induced L_∞-algebra structure $\{\lambda_n\}_{n \geqslant 1}$ with $\lambda_1 = d_B$ pictorially given by figure 3.3, and $\phi_1 := i$ extends to a morphism $\{\phi_n\}_{n \geqslant 1}$ of L_∞-algebras.

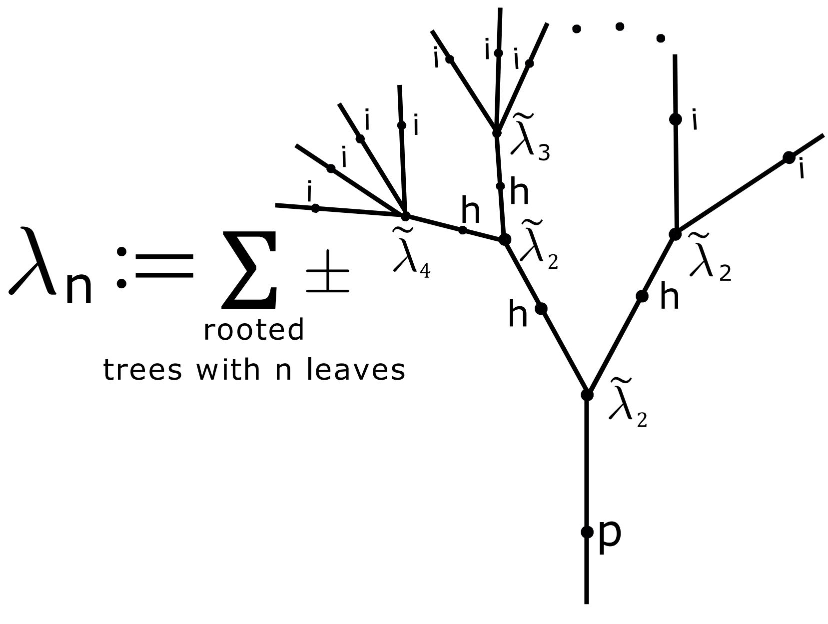


Figure 3.3: Transferring L_∞-algebras

Inspired by the work of Kadeishvili in [21] there is a recursive construction for λ_n without using the stated trees and the global homotopy h displayed on the inner edges above. In the following we prefer that approach since we do not want to specify the homotopy h.

As described in [8] the homotopy transfer may be done recursively without specifying the homotopy h in the case that we transfer to homology, namely for the set-up

$$(H(C,d_C), d=0) \underset{p}{\overset{i}{\rightleftarrows}} (C,d_C)\ .$$

This is done in much more generality in Theorem 6.1. of [8]. Restricting to the case of L_∞-algebras, it is proven that a given L_∞-algebra $(C,\{\widetilde{l}_k\}_{k\geqslant 1})$ and in particular a Lie algebra structure (that is $\widetilde{l}_k=0$ for $k\geqslant 3$) transfers to an L_∞-algebra structure $(H(C),\{l_k\}_{k\geqslant 1})$ and further that there exists an L_∞-algebra morphism

$$g: H(C)\to C$$

such that g_1 is a quasi-isomorphism. The morphism g is called a ∞-*quasi-isomorphism*. Analogously to equation (3.3) for the homotopy transfer for L_∞-algebras Lemma 2.9. of [8] yields a relation between the L_∞-algebra operations and the morphism g, namely

$$g_k\circ l_1-\widetilde{l}_1\circ g_k\ +\ g_1\circ l_k-\frac{1}{k!}\widetilde{l}_k\circ g_1^{\odot k}+R_k(g,l,\widetilde{l})=0 \tag{3.10}$$

where

$$\begin{aligned} &g_{k_1}\odot\cdots\odot g_{k_i}(c_1\cdots c_l)\\ &:=\sum_{\sigma\in S_k}\frac{\epsilon(\sigma)}{k_1!\cdots k_i!}g_{k_1}(c_{\sigma(1)}\cdots c_{\sigma(k_1)})\cdots g_{k_i}(c_{\sigma(k_1+\ldots+k_{i-1}+1)}\cdots c_{\sigma(k_1+\ldots+k_i)}) \end{aligned}$$

and $\epsilon(\sigma)$ is determined by $c_{\sigma(1)}\cdots c_{\sigma(k)}=\epsilon(\sigma)c_1\cdots c_k$. Here the morphisms $R_k(g,l,\widetilde{l})$ only contain components $l_{k'}$, $\widetilde{l}_{k'}$, $g_{k'}$ with $k'<k$. In particular $R_1=0$ and $R_2=0$ since $\widetilde{l}_1\circ g_1=0=l_1$.

Since $l_1=0$ equation (3.10) simplifies to

$$\widetilde{l}_1\circ g_k=g_1\circ l_k-\frac{1}{k!}\widetilde{l}_k\circ g_1^{\odot k}+R_k(g,l,\widetilde{l})$$

When we are in the special case that C is just a Lie algebra we have $\widetilde{l}_k=0$ for $k\geqslant 3$ and thus may write

$$\widetilde{l}_1\circ g_k\equiv d_c\circ g_k=g_1\circ l_k-\begin{cases}\widetilde{l}_1\circ g_1=d_c\circ i=0 & ,\quad k=1\\ \frac{1}{2}\widetilde{l}_2\circ g_1^{\odot 2} & ,\quad k=2\\ -R_k(g,l,\widetilde{l}) & ,\quad k\geqslant 3\end{cases} =: g_1\circ l_k-V_k$$

and analogously

$$\widetilde{\lambda}_1\ \circ\ \phi_k\equiv d_c\circ\phi_k=\phi_1\ \circ\ \lambda_k\ -\ \underbrace{\sigma_1\circ V_k\circ\sigma_k^{-1}}_{=:\mathcal{V}_k}\ . \tag{3.11}$$

Chapter 4

Higher string topology via homotopy transfer

Throughout this chapter we work with real coefficients and loop spaces consisting of smooth loops. When talking about string topology on chain level we have to specify which chain complex we are working with such that its homology yields the singular homology of LM. Further in full strictness string topology operations are only defined on the level of homology via homotopy theoretical considerations as in [11]. The definition of [5] namely by defining them geometrically on chain level and then let them descend to homology still lacks the specification which chain model one should use.
For performing the homotopy transfer construction later we have to think of how the initially only partially defined operations can be fully defined such that the chain complex becomes a differential graded algebra respectively differential graded Lie algebra. By using the work of Irie (cf. [20]) we get a chain level version of the loop product and the loop bracket. We then let these structures descend to homology $\mathbb{H}_*(LM)$ which yields an A_∞/L_∞-algebra structure on $\mathbb{H}_*(LM)$ for general closed and oriented manifolds M.

Remark that we rely on version v2 of Irie's work [20] in the following. The most recent version of this document is version v4 with the title "‘A chain level Batalin-Vilkovisky structure in string topology via de Rham chains"’.

A different and more algebraic approach would be to work in the language of operads. We are not discussing these methods here but refer to [24] and [36]. There it is proven that there exists a functor converting partial algebras into algebras such that both are quasi-isomorphic as partial algebras. Especially Theorem 2.7.3. of [36] states that the complex of chains of the free loop space can be equipped with a Lie algebra structure induced by the loop bracket. We do not pursue this approach since we are interested in actually computing the operations on chain level. This would be harder when working on that more algebraic level.

Finally when we understand the homotopy algebra structures, in particular the L_∞-algebra, we are able to use Fukaya's theorem 1.1 in a meaningful way and prove that a product of a hyperbolic manifold and a simply connected manifold does not embed as a Lagrangian submanifold into $\mathbb{C}^n$.

4.1 De Rham homology of LM

According the work of Irie in [20] for a given manifold M it is possible to define a chain complex for $LM = C^\infty(S^1, M)$ which becomes a differential graded algebra and a differential graded Lie algebra (with a degree $+1$ operator Δ) which further descends to the known BV-algebra structure on homology defined in [5]. We briefly recall the author's ideas. This is done in order to adapt ideas and then discuss the case for $L(N_{\text{SC.}} \times M_{K<0})$ in the next section where N is simply connected of dimension $n \geqslant 0$ and M has negative sectional curvature and is of dimension $m \geqslant 3$. As usually in string topology N and M are assumed to be closed and oriented.

In the following we refer to definitions and results of [20].

A *differentiable space* is a set X equipped with a *differentiable structure*

$$\mathcal{P}(X) := \{(U, \phi) \mid U \in \mathcal{U}, \phi : U \to X \text{ is a } plot\} ,$$

where $\mathcal{U} := \bigsqcup_{\substack{n \geqslant 1 \\ 0 \leqslant k \leqslant n}} \mathcal{U}_{n,k}$ and $\mathcal{U}_{n,k}$ is the set of k-dimensional oriented C^∞-submanifolds of $\mathbb{R}^n$ without boundary. The collection of plots $\{\phi : U \to X\}$ is required to have the following properties:

(*i*) If $\theta : U' \to U$ a C^∞-submersion for $U' \in \mathcal{U}$ and $(U, \phi) \in \mathcal{P}(X)$, then $(U', \phi \circ \theta) \in \mathcal{P}(X)$

(*ii*) If $\phi : U \to X$ is a map with $U \in \mathcal{U}$ such that there is an open covering $(U_\alpha)_{\alpha \in I}$ of U such that $(U_\alpha, \phi|_{U_\alpha}) \in \mathcal{P}(X)$ for all $\alpha \in I$, then $(U, \phi) \in \mathcal{P}(X)$.

A manifold M is a differentiable space by specifying $\phi : U \to M$ to be a plot if ϕ is smooth, that is

$$(U, \phi) \in \mathcal{P}(M) :\Leftrightarrow \phi \in C^\infty(U, M)$$

A subset $X_1 \xrightarrow{\iota} X_2$ of a differentiable space X_2 is a differentiable spaces by specifying a map $\phi : U \to X_1$ to be a plot if $(U, \iota \circ \phi) \in \mathcal{P}(X_2)$.

A map between differentiable spaces $f : X \to Y$ is *smooth* if $(U, \phi) \in \mathcal{P}(X)$ implies

$$(U, f \circ \phi) \in \mathcal{P}(Y) .$$

By definition the inclusion $X_1 \xrightarrow{\iota} X_2$ is a smooth map.

Two such maps f, g are *smoothly homotopic* if a smooth map $h : X \times \mathbb{R} \to Y$ exists such that

$$h(x, s) = \begin{cases} f(x) & , \quad s < 0 \\ g(x) & , \quad s > 1 \end{cases} .$$

Remark that we have a canonical differentiable structure on products of differentiable spaces. A map is a plot if all its projections are plots of the particular factors.

In the following we want to treat free loop spaces. For M a smooth closed and oriented manifold and $LM = C^\infty(S^1, M)$ we define a differentiable structure as follows:

$$(U, \phi) \in \mathcal{P}(LM) :\Leftrightarrow ev \circ \phi \in C^\infty(U \times S^1, M) \quad \text{where} \quad (ev \circ \phi)(u, t) := \phi(u)(t) \; .$$

Remark that by definition evaluation maps $LM \xrightarrow{ev_t} M$; $\gamma \mapsto \gamma(t)$ are thus smooth. Further the energy functional

$$\begin{aligned} E : LM &\to \mathbb{R} \\ \gamma &\mapsto \int_{S^1} |\dot{\gamma}|^2 \end{aligned} \tag{4.1}$$

is smooth for the differentiable structures defined above.

For a differentiable space $(X, \mathcal{P}(X))$ we define the *de Rham chain complex*

$$C_k^{\mathrm{dR}}(X) := \mathbb{R}\langle \mathcal{Z}_k(X) \rangle / Z_k(X) \quad (k \geqslant 0)$$

where the vector space $\mathbb{R}\langle \mathcal{Z}_k(X) \rangle$ is generated by the set

$$\mathcal{Z}_k(X) := \{(U, \phi, \omega) \mid (U, \phi) \in \mathcal{P}(X), \omega \in \Omega_c^{\dim U - k}(U)\} \; ,$$

where $\Omega_c^i(U)$ is the vector space of compactly supported i-forms on U.
We mod out the subspace $Z_k(X)$ generated by vectors:

- $a(U, \phi, \omega) - (U, \phi, a\omega)$ for $a \in \mathbb{R}$
- $(U, \phi, \omega) + (U, \phi, \omega') - (U, \phi, \omega + \omega')$
- $(U, \phi, \pi_! \omega) - (U', \phi \circ \pi, \omega)$, where $\pi_! : \Omega_c^r(U') \to \Omega_c^{r - \dim U' + \dim \mathrm{U}}(U)$ is the integration along the fiber defined for C^∞-submersions $\pi : U' \to U$

The linear degree -1 map

$$\partial\, [(U, \phi, \omega)] := [(U, \phi, d\omega)]$$

defines a boundary. We define *de Rham homology* as the homology

$$H_*^{\mathrm{dR}}(X) := H_*(C_*^{\mathrm{dR}}(X), \partial) \; .$$

An augmentation is given by $[(U, \phi, \omega)] \mapsto \int_U \omega$ for $[(U, \phi, \omega)] \in C_0^{\mathrm{dR}}(X)$
Smooth maps $f : X \to Y$ between differentiable spaces induce chain maps

$$f_*([(U, \phi, \omega)]) := [(U, f \circ \phi, \omega)] \; .$$

The de Rham chain complex is indeed functorial here since smoothly homotopic maps induce chain homotopic maps as shown in Proposition 2.5. of [20].

Next we want to compare Irie's construction with standard singular homology.

A map $\rho : \Delta^k \to X$ is *strongly smooth* if either $k = 0$ or if $k > 0$ and there exists a neighbourhood U of

$$\Delta^k = \{(t_1, ..., t_k) \in \mathbb{R}^k \mid 0 \leqslant t_1 \leqslant ... \leqslant t_k \leqslant 1\} \subset \mathbb{R}^k$$

and a smooth map $\overline{\rho} : U \to X$ such that $\overline{\rho}|_{\Delta^k} = \rho$. For a differentiable space we can define the chain complex of *strongly smooth maps*

$$S^{\mathrm{sm}}_*(X) \subset S_*(X) = \bigoplus_{k \geqslant 0} \mathbb{R} \left\langle \mathrm{Map}(\Delta^k, X) \right\rangle$$

as the sub-complex generated by strongly smooth maps inside the singular chain complex.

Lemma 4.1 (*e.g. Theorem 18.7 of [26]*)

For a smooth finite dimensional manifold X the inclusion

$$S^{sm}_*(X) \hookrightarrow S_*(X)$$

is a quasi-isomorphism. It yields an isomorphism

$$H^{sm}_*(X) \cong H_*(X) \ . \tag{4.2}$$

Remark that Δ^k carries the canonical structure of a differentiable space as it is a subset of $\mathbb{R}^k$.

Lemma 4.2 (*Lemma 2.6. and Proposition 3.2. of [20]*)

There exist $u_k \in C^{dR}_k(\Delta^k)$ for all $k \in \mathbb{N}_0$ such that the map

$$\begin{aligned} S^{sm}_k(X) &\to C^{dR}_k(X) \\ \sigma &\mapsto \sigma_*(u_k) \end{aligned}$$

for X a smooth finite dimensional manifold is a chain map and yields an isomorphism

$$H^{sm}_*(X) \cong H^{dR}_*(X) \tag{4.3}$$

that is not depending on the choice of $(u_k)_{k \geqslant 0}$.

When combining both Lemmas we conclude that de Rham homology computes real singular homology for finite dimensional smooth manifolds.

Proposition 4.3

For X a smooth finite dimensional manifold there exists an isomorphism

$$H^{dR}_*(X) \cong H_*(X) \ . \tag{4.4}$$

We want a similar result for free loop spaces of finite d-dimensional smooth Riemannian manifolds M that are closed and oriented. That is we want an isomorphism

$$H_*^{\mathrm{dR}}(LM) \cong H_*(LM) \ .$$

By choosing a strictly increasing sequence $(E_j)_{j\geqslant 1}$ such that $\lim_{j\to\infty} E_j = \infty$ we define the energy filtration of LM via

$$LM^{E_j} := \{\gamma \in LM \mid E(\gamma) < E_j\}$$

where we used the energy as defined in (4.1). Inclusion of subspaces $LM^{E_i} \hookrightarrow LM^{E_j}$ $(j > i)$ provides a directed system which in turn yields homomorphisms

$$\begin{aligned} \varinjlim_{j\to\infty} H_k(LM^{E_j}) &\to H_k(LM) \\ \varinjlim_{j\to\infty} H_k^{\mathrm{sm}}(LM^{E_j}) &\to H_k^{\mathrm{sm}}(LM) \\ \varinjlim_{j\to\infty} H_k^{\mathrm{dR}}(LM^{E_j}) &\to H_k^{\mathrm{dR}}(LM) \ . \end{aligned} \tag{4.5}$$

Remark that $(E(x) - E_j)_{j\geqslant 1}$ is a sequence of decreasing smooth functions, that is

$$(E(x) - E_1) \geqslant (E(x) - E_2) \geqslant \ldots \quad \text{for all } x \in LM \ ,$$

and $\lim_{j\to\infty}(E(x) - E_j) = -\infty$ for all $x \in LM$. Therefore results of chapter 2.7. of [20] can be applied.

Lemma 4.4 (*chapter 3.3. of [18]; Lemma 2.8. and Lemma 2.10. of [20]*)

For the loop space LM of a finite dimensional, closed and oriented Riemannian manifolds M with the energy filtration

$$LM^{E_j} := \{\gamma \in LM \mid E(\gamma) < E_j\}$$

the inclusion induces isomorphisms

$$\begin{aligned} \varinjlim_{j\to\infty} H_k(LM^{E_j}) &\to H_k(LM) \\ \varinjlim_{j\to\infty} H_k^{sm}(LM^{E_j}) &\to H_k^{sm}(LM) \\ \varinjlim_{j\to\infty} H_k^{dR}(LM^{E_j}) &\to H_k^{dR}(LM) \ . \end{aligned} \tag{4.6}$$

Proof (sketch): Represent a cycle in LM by singular simplices. The union of their images is a compact set in LM where the energy functional E attains a maximum E_{j_0} and thus the cycle is a cycle in $LM^{E_{j_0}}$. This proves surjectivity.
Injectivity follows similarly since a bounding chain in LM of a cycle in $LM^{E_{j_1}}$ is compact and thus lies in some $LM^{E_{j_2}}$ for $j_1 \leqslant j_2$. ∎

In order to prove that de Rham homology computes singular homology for free loop spaces of finite dimensional smooth manifolds it is therefore enough to show that

$$\varinjlim_{j\to\infty} H_k(LM^{E_j}) \leftarrow \varinjlim_{j\to\infty} H_k^{\mathrm{sm}}(LM^{E_j}) \to \varinjlim_{j\to\infty} H_k^{\mathrm{dR}}(LM^{E_j}) \tag{4.7}$$

are isomorphisms.

In [20] this is done by approximating the free loop space LM by finite dimensional smooth manifolds $F_N^E M$. By previous considerations (4.2) and (4.3) we know that we have isomorphisms

$$\varinjlim_{j\to\infty} H_k(F_{N_j}^{E_j} M) \leftarrow \varinjlim_{j\to\infty} H_k^{\mathrm{sm}}(F_{N_j}^{E_j} M) \to \varinjlim_{j\to\infty} H_k^{\mathrm{dR}}(F_{N_j}^{E_j} M)\ . \tag{4.8}$$

So it remains to clarify how the approximations $F_N^E M$ are defined and then to show that (4.7) is equivalent to (4.8).

Finite dimensional approximations of LM

Remark that M is equipped with a Riemannian metric, so that we can measure distances. We approximate a loop by a finite number of points on it, that is we define

$$\begin{aligned} F_N M &:= \{(x_0, ..., x_N) \in M^N \mid x_0 = x_N\}\ , \\ F_N^{E_0} M &:= \{x = (x_0, ..., x_N) \in F_N M \mid E(x) := N \cdot \sum_{0\leqslant j\leqslant N-1} d(x_j, x_{j+1})^2 < E_0\}\ . \end{aligned}$$

The approximations carry the canonical differentiable structure as subsets of M^N.

Lemma 4.5 (*Lemma 4.3. of [20]*)

For a sequence $E_j \to \infty$ of strictly increasing positive real numbers there exists a sequence $N_j \to \infty$ of integers such that the evaluation map

$$\begin{aligned} e_N : LM &\to LM_N \\ \gamma &\mapsto (\gamma(0), \gamma(1/N), \gamma(2/N), ..., \gamma(1)) \end{aligned} \tag{4.9}$$

induces an isomorphism

$$\varinjlim_{i\to\infty} H_*^{\#}(LM^{E_i}) \xrightarrow{\varinjlim_{i\to\infty} H_*^{\#}(e_{N_i})} \varinjlim_{i\to\infty} H_*^{\#}(F_{N_i}^{E_i} M) \qquad . \tag{4.10}$$

Here # either means 'de Rham homology' or 'smooth singular homology' or 'singular homology'.

The Lemma combined with the isomorphisms in (4.8) imply that

$$\varinjlim_{j\to\infty} H_k(LM^{E_j}) \leftarrow \varinjlim_{j\to\infty} H_k^{\mathrm{sm}}(LM^{E_j}) \to \varinjlim_{j\to\infty} H_k^{\mathrm{dR}}(LM^{E_j})$$

are isomorphisms. Since we already proved that $LM^{E_i} \hookrightarrow LM$ induces isomorphisms on homology for singular homology, smooth singular homology and de Rham homology we conclude that

$$H_k(LM) \leftarrow H_k^{\mathrm{sm}}(LM) \rightarrow H_k^{\mathrm{dR}}(LM)$$

are isomorphisms that is de Rham homology computes singular homology for free loop spaces of finite dimensional smooth Riemannian manifolds M that are closed and oriented.

Corollary 4.6

For M a smooth finite dimensional manifold there exists an isomorphism

$$H_*^{\mathrm{dR}}(LM) \cong H_*(LM) \ . \tag{4.11}$$

■

Proof of Lemma 4.5: The evaluation map

$$\begin{aligned} e_N : LM &\to LM_N \\ \gamma &\mapsto (\gamma(0), \gamma(1/N), \gamma(2/N), ..., \gamma(1)) \end{aligned} \tag{4.12}$$

is smooth by definition and $e_N(LM^{E_0}) \subset F_N^{E_0}M$ by using the Cauchy-Schwarz inequality, namely for $\gamma \in LM^{E_0}$ one has

$$\begin{aligned} E(e_N(\gamma)) &= N \sum_{0 \leqslant j \leqslant N-1} d(x_j, x_{j+1})^2 = N \sum_{0 \leqslant j \leqslant N-1} \left(\int_{\frac{i}{N}}^{\frac{i+1}{N}} |\dot{\gamma}| \right)^2 \leqslant N \sum_{0 \leqslant j \leqslant N-1} \left(\int_{\frac{i}{N}}^{\frac{i+1}{N}} 1^2 \right) \left(\int_{\frac{i}{N}}^{\frac{i+1}{N}} |\dot{\gamma}|^2 \right) \\ &= \sum_{0 \leqslant j \leqslant N-1} \left(\int_{\frac{i}{N}}^{\frac{i+1}{N}} |\dot{\gamma}|^2 \right) = \int_{S^1} |\dot{\gamma}|^2 < E_0 \ . \end{aligned}$$

For E_0 fixed we choose N_0 sufficiently large such that

$$\sqrt{E_0/N_0} < r_M$$

where r_M is the injectivity radius that is positive since M is closed.
Then

$$d(x_j, x_{j+1}) < \sqrt{\sum_{0 \leqslant j \leqslant N-1} d(x_j, x_{j+1})^2} = \sqrt{E_0/N_0} < r_M \ ,$$

so that there is a geodesic connecting x_j and x_{j+1} which we denote by $\gamma_{x_j, x_{j+1}}$. These geodesics will soon be further subdivided into m parts. We fix $m > 0$. For given energies $0 < E_0 < E_0'$ we choose $\delta > 0$ such that

$$(1 + \delta)^4 < E_0'/E_0 \ .$$

Our next goal is to define a map

$$g_0 : F_{N_0}^{E_0}M \to LM^{E_0'} \tag{4.13}$$

that is smooth (in the sense above) and continuous (in the sense of Whitney C^∞-topology). For that we need a map $\mu : [0,1] \to [0,1]$ that satisfies

(i) $0 \leqslant \mu'(t) \leqslant 1 + \delta$

(ii) $\mu(i/m) = i/m$

(iii) μ is constant near 0 and 1 .

Now we set $g_0(x_0, ..., x_{N_0}) = \gamma$ where

$$\gamma(t) = \begin{cases} \gamma_{x_0,x_1}(\mu(N_0 t - 0)) & ;\ t \in [0, 1/N_0] \\ \gamma_{x_1,x_2}(\mu(N_0 t - 1)) & ;\ t \in [1/N_0, 2/N_0] \\ \cdots & \\ \gamma_{x_{N_0-1},x_{N_0}}(\mu(N_0 t - (N_0 - 1))) & ;\ t \in [N_0 - 1/N_0, 1] \end{cases} .$$

Notice that property (i) of μ implies $E(\gamma) \leqslant (1+\delta)^2 E(x) = (1+\delta)^2 E_0 < E_0'$.

We define

$$\begin{aligned} i_m : F_{N_0}^{E_0}M &\xrightarrow{g_0} LM^{E_0'} \xrightarrow{e_{mN_0}} F_{mN_0}^{E_0}M \\ (x_0, ..., x_{N_0}) &\mapsto \gamma \mapsto \underbrace{(x_0, \gamma_{x_0,x_1}(1/m), ..., \gamma_{x_0,x_1}(1) = \gamma_{x_1,x_2}(0) = x_2, ..., ..., x_{N_0})}_{mN_0+1} . \end{aligned}$$

One further checks that

$$\begin{tikzcd} LM^{E_0} \arrow[r, "\text{incl.}"] \arrow[d, "e_{N_0}"'] & LM^{E_0'} \\ F_{N_0}^{E_0}M \arrow[ur, "g_0"'] & \end{tikzcd}$$

commutes up to homotopy. This is done in chapter 4 of [20]. Roughly speaking since N_0 is sufficiently large points of incl.(γ) and $(g_0 \circ e_{N_0}^{E_0})(\gamma)$ in $LM^{E_0'}$ can be connected by geodesics. This defines a smooth homotopy γ_s connecting these two loops. Further we have $E(\gamma_s) \leqslant (1+\delta)^4 E_0 < E_0'$.

We end up with smooth (in the sense above) and continuous (in the sense of Whitney C^∞-topology) maps fitting in the diagram

$$\begin{tikzcd} LM^{E_0} \arrow[r, "\text{incl.}"] \arrow[d, "e_{N_0}"'] & LM^{E_0'} =: LM^{E_1} \arrow[d, "e_{mN_0} =: e_{N_1}"] \\ F_{N_0}^{E_0}M \arrow[r, "i_m"'] \arrow[ur, "g_0"] & F_{mN_0}^{E_0'}M =: F_{N_1}^{E_1}M \end{tikzcd}$$

that commutes up to homotopy.
Continuing the construction inductively we get a sequence

$$\begin{tikzcd} \cdots \arrow[r] & LM^{E_j} \arrow[r] \arrow[d] & LM^{E_{j+1}} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & F^{E_j}_{N_j} M \arrow[r] \arrow[ur, "g_j"] & F^{E_{j+1}}_{N_{j+1}} M \arrow[r] \arrow[ur, "g_{j+1}"] & \cdots \end{tikzcd}.$$

In total we get an isomorphism

$$\varinjlim_{i\to\infty} H_*^{\#}(LM^{E_i}) \underset{\varinjlim_{i\to\infty} H_*^{\#}(g_i)}{\overset{\varinjlim_{i\to\infty} H_*^{\#}(e_{N_i})}{\rightleftarrows}} \varinjlim_{i\to\infty} H_*^{\#}(F^{E_i}_{N_i} M) \quad .$$

■

4.2 Chain level string topology of LM

We want to describe Irie's definition of string topology operations on $C_*^{\mathrm{dR}}(LM)$. In order to make such definitions one faces three issues:

(I) Concatenation of loops is not associative. That is on chain level we would not get an associative algebra. Further the fundamental class

$$[M] \mathrel{\hat{=}} [(M, s, f)]$$

(for $s : M \to LM$ section and $f \equiv 1 \in C^\infty(M, \mathbb{R})$) would not be a strict unit with respect to the loop product. We therefore want to work with Moore loops

$$L^M M \equiv \overline{LM} = \{(\gamma, T) \equiv \gamma_T \mid T \geqslant 0 \ ; \ \gamma \in C^\infty([0,T], M) \ ; \ \gamma(0) = \gamma(T)\} \ .$$

A differentiable structure for $\overline{LM}$ is defined as follows:

$$\begin{aligned} \big(U, \phi = (\gamma^\phi, T^\phi)\big) \in \mathcal{P}(\overline{LM}) :\iff \ & T^\phi \in C^\infty(U, \mathbb{R}) \text{ and} \\ & \{(u,t) \mid u \in U, 0 \leqslant t \leqslant T^\phi(u)\} \to M \\ & \qquad\qquad (u,t) \mapsto \gamma^\phi(u)(t) \\ & \text{extends to a smooth map on } U \times \mathbb{R} \end{aligned}$$

(II) When concatenating loops the derivatives may not fit and thus the resulting chain of loops is not an element of $C_*^{\mathrm{dR}}(LM)$. That is $C_*^{\mathrm{dR}}(LM)$ would not be closed under the loop product. We further restrict to

$$\begin{aligned} \overline{LM}_l := \{(\gamma, t_1, ..., t_l, T) \mid (\gamma, T) \in \overline{LM} \ ; \ 0 \leqslant t_1 \leqslant ... \leqslant t_l \leqslant T \ ; \\ \gamma^{(m)}(t) = 0 \text{ for } m \geqslant 1 \text{ and } t \in \{0, t_1, ..., t_l, T\}\} \subset \overline{LM} \times \Delta^l \end{aligned}$$

where $\cdot^{(m)}$ means taking the m-th derivative. A differentiable structure for $\overline{LM}_l$ is defined via

$$(U, \phi = (\gamma^\phi, t_1^\phi, ..., t_l^\phi, T^\phi)) \in \mathcal{P}(\overline{LM}_l) :\Leftrightarrow$$
$$(U, (\gamma^\phi, T^\phi)) \in \mathcal{P}(\overline{LM}) \text{ and } t_1^\phi, ..., t_l^\phi \in C^\infty(U, \mathbb{R}) \ .$$

Notice that evaluations $ev_{l,j} : \overline{LM}_l \to M$ are smooth in the sense of differentiable spaces, that is $ev_{l,j} \circ \phi \in C^\infty(U, M)$, where

$$(ev_{l,j} \circ \phi)(u) = \begin{cases} \phi(u)(0); \ j = 0 \\ \phi(u)(t_j); \ 1 \leqslant j \leqslant l \end{cases} .$$

Relying on Lemma 7.6. and 7.7. of [20] we get that for $p : [0,1] \to S^1 = [0,1]/0 \sim 1$ the maps

$$LM \xleftarrow{\text{pr}_1} LM \times \Delta^l \xleftarrow{\text{incl.}} \begin{array}{ccc} LM_l & \longrightarrow & \overline{LM}_l \\ (\gamma, t_1, ..., t_l) & \longmapsto & (\gamma \circ p, t_1, ..., t_l, 1) \end{array} \tag{4.14}$$

induce isomorphisms on de Rham homology. Here the differentiable structure of

$$LM_l = \{(\gamma, t_1, ..., t_l) \in LM \times \Delta^l \ |\gamma^{(m)}(t) = 0 \text{ for } m \geqslant 1 \text{ and } t \in \{0, t_1, ..., t_l\}\}$$

is the canonical one assigned to it as it is a subset of the differentiable space $LM \times \Delta^l$. Thus all maps of (4.14) are smooth.
The fact (4.14) combined with (4.11) yields isomorphisms

$$H_*^{\text{dR}}(\overline{LM}_l) \cong H_*^{\text{dR}}(LM \times \Delta^l) \cong H_*^{\text{dR}}(LM) \cong H_*(LM) \ . \tag{4.15}$$

(III) In order to intersect evaluated chains in M mutual transversality in M has to be given. This is guaranteed if we only allow chains whose evaluation to M is submersive. That is we define $\overline{LM}_{l,\text{reg}}$ to be the set $\overline{LM}_l$ where the differentiable structure is modified as follows:

$$(U, \phi) \in \mathcal{P}(\overline{LM}_{l,\text{reg}}) : \Longleftrightarrow \ (U, \phi) \in \mathcal{P}(\overline{LM}_l) \text{ and } ev_{l,j} \circ \phi \text{ is further a submersion} \tag{4.16}$$

For de Rham chains with respect to this differentiable structure intersection in M is fully defined. Since

$$\begin{array}{c} M \to LM \to M \\ p \mapsto \gamma_p \mapsto p \end{array}$$

is a submersion the chain given by the family of constant loops at p for all $p \in M$ is a regular chain.

In the following we work on the complex

$$\mathfrak{C}_*(LM) := \prod_{l\geqslant 0} C^{\mathrm{dR}}_{*+d+l}(\overline{LM}_{l,\mathrm{reg}}) \ .$$

Irie defines a de Rham chain level loop product. For $x, y \in \mathfrak{C}_*(LM)$ it is given by

$$(x \bullet y)_k := \sum_{l+m=k} \pm\, (c_{l,m})_* \,(\, x_l \times_M y_m \,) \ \in C^{\mathrm{dR}}_{*+d+k}(\overline{LM}_{k,\mathrm{reg}}) \ ,$$

where

$$x_l \times_M y_m := [(U \times_M V, \phi \times \psi, \omega \times \eta)]$$

for $x_l = [(U, \phi, \omega)] \in C^{\mathrm{dR}}_{*+d+l}(\overline{LM}_{l,\mathrm{reg}})$ and $y_m = [(V, \psi, \eta)] \in C^{\mathrm{dR}}_{*+d+m}(\overline{LM}_{m,\mathrm{reg}})$.

The chain map $c_{l,m}$ is defined by concatenating loops at time 0, that is

$$\begin{aligned} c_{l,m} : \overline{LM}_{l,\mathrm{reg}} \times_M \overline{LM}_{m,\mathrm{reg}} &\to \overline{LM}_{l+m,\mathrm{reg}} \\ \big((\gamma_1, \tau_1, ..., \tau_l, T_1), (\gamma_2, t_1, ..., t_m, T_2)\big) &\mapsto (\gamma, \tau_1, ..., \tau_l, T_1 + t_1, ..., T_1 + t_m, T_1 + T_2) \ , \end{aligned}$$

where $\gamma(t) := \begin{cases} \gamma_1(t) & , \ 0 \leqslant t \leqslant T_1 \\ \gamma_2(t - T_1) & , \ T_1 \leqslant t \leqslant T_1 + T_2 \end{cases}$.

The product is indeed fully defined since

$$U \times_M V = \{(u, v) \in U \times V \,|\, ev_{l,0} \circ \phi(u) = ev_{m,0} \circ \psi(v)\}$$

is a manifold due to the required regularity in (4.16).
Out of this loop product, Irie further defines a de Rham chain level loop bracket

$$\{x, y\}_k := (x * y)_k \pm (y * x)_k \ ,$$

where

$$(x * y)_k := \sum_{l+m=k} \pm\, \sigma\, (x_l \bullet \rho(y_m))$$

and σ, ρ are both induced maps that move the basepoints along the involved loops. Remark that after applying the de Rham chain level loop product or the de Rham chain level loop bracket the evaluation maps are still submersive in the sense above, that is $\mathfrak{C}_*(LM)$ is indeed closed under the defined operations.

Notice the trivial but important fact that if

$$\{\phi(u)(t_1) = \psi(v)(t_2)\} = \varnothing$$

in M for all times $t_i \in \mathbb{R}$ and $u \in U$ and $v \in V$, the de Rham loop product and the de Rham loop bracket both vanish already on chain level since $U \times_M V = \varnothing$ and thus

$$x_l \bullet y_m = 0 = x_l \bullet \rho(y_m) \ .$$

This is used in the following section to prove that both operations, and their higher versions, are essentially trivial for the homology of a particular class of manifolds.

Further both operations are related via

$$\{a, b \bullet c\} = \{a, b\} \bullet c + (-1)^{|b|(|a|+1)} b \bullet \{a, c\} \ . \tag{4.17}$$

Remark that when taking $b = c = [M]$ the algebra unit this yields

$$\begin{aligned}\{a, [M]\} = \{a, [M] \bullet [M]\} = \{a, [M]\} \bullet [M] + (-1)^{|[M]|(|a|+1)} [M] \bullet \{a, [M]\} = \\ = ((-1)^{|[M]|(|a|+1)} + 1)\{a, [M]\}\end{aligned}$$

that is either $2\{a, [M]\}$ or zero. In both cases we get

$$\{a, [M]\} = 0 \ . \tag{4.18}$$

The S^1-action on LM is also incorporated in the de Rham picture, namely Irie uses this action to define a degree $+1$ de Rham BV-operator Δ that commutes with the differential D. The operations on chain level introduced above descend to homology and combine to a BV-algebra structure:

Proposition 4.7 (***Theorem 1.2. of [20]***)

The de Rham loop product and the de Rham loop bracket turn the chain complex

$$\mathfrak{C}_*(LM) := \prod_{l \geqslant 0} C^{dR}_{*+d+l}(\overline{LM}_{l,reg})$$

into an associative non-commutative dg algebra and a dg Lie algebra with respect to the differential D where $d = \dim M$.

Further both operations and the de Rham BV-operator descend to homology and turn $H_*(\mathfrak{C}_*(LM))$ into a BV-algebra.

On homology there exists an isomorphism

$$H_*(\mathfrak{C}_*(LM)) \cong H_{*+d}(LM) = \mathbb{H}_*(LM) \tag{4.19}$$

as BV-algebras, where the BV-structure on $\mathbb{H}_*(LM)$ is the one defined by Chas and Sullivan in [5].

In section 4.4 we will use Irie's chain level operations and study the induced A_∞/L_∞-algebra structure on $\mathbb{H}_*(LM)$, for certain product manifolds M.

Before we do so we discuss how the dg algebra

$$(C, d) = (\Lambda_{\mathbb{R}}(\alpha) \otimes_{\mathbb{R}} \mathbb{R}[\lambda], d) \ ,$$

where $H(C, d) \cong \mathbb{H}_*(LS^n)$ as algebras, can be seen as a sub-algebra of Irie's dg algebra.

4.3 Chain level string topology of LS^n

We show that $(\Lambda_{\mathbb{R}}(\alpha) \otimes_{\mathbb{R}} \mathbb{R}[\lambda], 0)$ for $|\alpha| = -n$, $|\lambda| = n-1$ and $n \geqslant 3$ odd is a sub-algebra of Irie's chain complex. This yields that the considerations of section 3.2 actually compute the A_∞-operations extending the loop product on $\mathbb{H}_*(LS^n)$ for $n \geqslant 3$ odd.

When only considering the algebra structure given by the chain level loop product theorem 4.7 simplifies to:

Proposition 4.8 (*Section 5.3. of [20]*)

The de Rham loop product turns the chain complex

$$\mathcal{C}_*(LM) := C^{dR}_{*+d}(\overline{LM}_{0,reg})$$

into an associative non-commutative dg algebra with respect to the differential D where $d = \dim M$.

Further the operation descends to homology and there exists an isomorphism

$$H_*(\mathcal{C}_*(LM)) \cong H_{*+d}(LM) = \mathbb{H}_*(LM) \tag{4.20}$$

as algebras, where the algebra structure on $\mathbb{H}_*(LM)$ is provided by the loop product defined by Chas and Sullivan in [5].

In the following we want to show that for odd dimensional simply connected spheres things are quite simply to handle, namely:

Lemma 4.9

One can define $\alpha, \lambda \in \mathcal{C}_*(LM)$ with $|\alpha| = -n$, $|\lambda| = n-1$ for $n \geqslant 3$ odd such that

$$\iota : (\Lambda_{\mathbb{R}}(\alpha) \otimes_{\mathbb{R}} \mathbb{R}[\lambda], 0) \hookrightarrow (\mathcal{C}_*(LS^n), D)$$

is an inclusion as a sub-algebra

When combining this with theorem 3.6 we get as a corollary:

Theorem 4.10

For $n \geqslant 3$ odd the dg algebra

$$(\Lambda_{\mathbb{R}}(\alpha) \otimes_{\mathbb{R}} \mathbb{R}[\lambda], 0)$$

for $|\alpha| = -n$, $|\lambda| = n-1$ induce trivial higher operations for $\mathbb{H}_*(LS^n)$.

Further for $n \geqslant 3$ odd

$$\mathbb{H}_*(LS^n \times \cdots \times LS^n)$$

is equipped with trivial induced higher operations. ■

Proof of Lemma 4.9: We are considering S^n for $n \geqslant 3$ odd with $p_0 \in S^n$ fixed.
We set $\iota(\alpha) := a$ and $\iota(\lambda) := l$, where a, l are defined in the following.

Remark that $al^k \neq 0$ and $l^k \neq 0$ for $k \geqslant 1$ since

$$H(\Lambda_{\mathbb{R}}(\alpha) \otimes_{\mathbb{R}} \mathbb{R}[\lambda]) \cong \Lambda_{\mathbb{R}}([a]) \otimes_{\mathbb{R}} \mathbb{R}[[l]] \cong \mathbb{H}_*(LS^n)$$

and $[a][l]^k$ and $[l]^k$ for $k \geqslant 1$ generate the module $\mathbb{H}_*(LS^n)$.

So let us define a and l. Pick an embedding

$$\phi : D^n \to S^n$$

such that $\phi(D^n)$ is a neighbourhood of p_0. Further fix a volume form $\omega \in \Omega_c^n(D^n)$, that is $\int_{D^n} \omega = 1$. We define

$$a := [(D^n, \Phi, \omega)]$$

of degree $-n$ in $\mathcal{C}_*(LM)$ where

$$\begin{aligned} \Phi : D^n &\to LS^n \times \mathbb{R}_{\geqslant 0} \\ x &\mapsto (\gamma_{\phi(x)}, 0) \end{aligned}$$

and $\gamma_{\phi(x)}$ is the constant loop at $\phi(x)$. Remark that

$$(D^n, \Phi) \in \mathcal{P}(\overline{LS^n}_{0,\mathrm{reg}})$$

since $ev_0 \circ \Phi = \phi$ is an embedding and in particular a submersion.
We have

$$Da = [(D^n, \Phi, d\omega)] = [(D^n, \Phi, 0)] = 0$$

by degree reasons.

For the generator l recall that

$$H_*(\Omega_{p_0} S^n) \cong \mathbb{R}[u] \quad \text{with} \quad |u| = n - 1 \; .$$

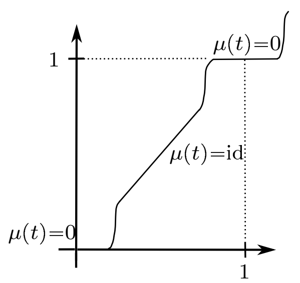
For $y_0 \in T_{p_0} S^n$ fixed the class u may be represented by

$$l_{p_0,y_0} : S^{n-1}_{y_0} = \{z \in S^n | \langle z, y_0 \rangle_{\mathbb{R}^{n+1}} = 0\} = S^{n-1} \to \Omega_{p_0} S^n \tag{4.21}$$

where

$$l_{p_0,y_0}(z)(t) = \frac{p_0 + z}{2} + (\cos 2\pi\mu(t)) \cdot \frac{p_0 - z}{2} + (\sin 2\pi\mu(t)) \cdot \sqrt{\frac{1 - \langle p_0, z\rangle}{2}} \; y_0$$

and $\mu : \mathbb{R}_{\geqslant 0} \mapsto \mathbb{R}_{\geqslant 0}$ smooth of the form:



We need the reparametrization μ such that

$$(l_{p_0,y_0}(z))^{(m)}(t) = 0$$

for $m \geqslant 1$ and $t \in \{0, 1, 2, ...\}$.

More can be found in chapter 3.7 of [7]. The described $(n-1)$-chain is visualized as:

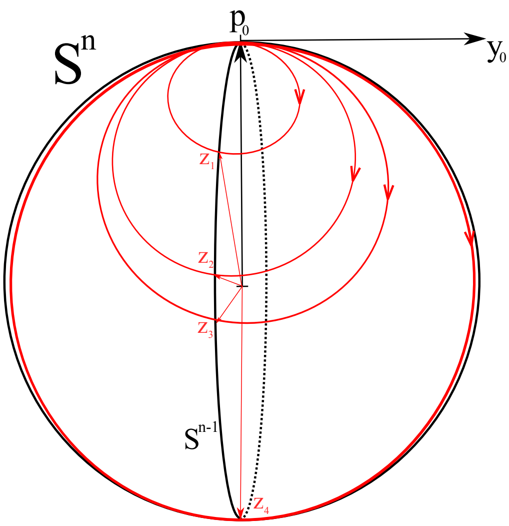


Figure 4.1: Representative l_{p_0,y_0} of the generator of $H_*(\Omega_{p_0}S^n) \cong \mathbb{Z}[u]$

The product is given by

$$l^k_{p_0,y_0} : \underbrace{S^{n-1}_{y_0} \times \cdots \times S^{n-1}_{y_0}}_{k} \to \Omega_{p_0}S^n \tag{4.22}$$

where

$$l^k_{p_0,y_0}(z_1, ..., z_k)(t) = \begin{cases} l_{p_0,y_0}(z_1)(\mu(t)) & , \quad 0 \leqslant t \leqslant 1 \\ \cdots & \\ l_{p_0,y_0}(z_k)(\mu(t)) & , \quad k-1 \leqslant t \leqslant k \end{cases} .$$

To extend this construction to the free loop space LS^n we make use of a fixed nowhere vanishing vector field ν on S^n. With $1 \in \Omega^0_c(S^n \times S^{n-1})$ constant we now define

$$l := [(S^n \times S^{n-1}, \Psi, 1)]$$

of degree $n-1$ in $\mathcal{C}_*(LM)$ where

$$\begin{aligned} \Psi : S^n \times S^{n-1} &\to LS^n \times \mathbb{R}_{\geqslant 0} \\ (p, z) &\mapsto (l_{p,\nu(p)}(z), 1) \ . \end{aligned}$$

Remark that

$$(S^n \times S^{n-1}, \Psi) \in \mathcal{P}(\overline{LS^n}_{0,\mathrm{reg}})$$

since $(ev_0 \circ \Psi)(p,z) = p$ meaning that $ev_0 \circ \Psi$ is a submersion.
We clearly have

$$Dl = [(S^n \times S^{n-1}, \Psi, D(1))] = [(S^n \times S^{n-1}, \Psi, 0)] = 0 \ .$$

It remains to check that $a \bullet a = 0$ and that $a \bullet l = l \bullet a$.

Recall that for de Rham chains we have

$$[(U, \phi, \pi_! \omega)] = [(U', \phi \circ \pi, \omega)] \ ,$$

where $\pi_! : \Omega_c^r(U') \to \Omega_c^{r - \dim U' + \dim \mathrm{U}}(U)$ is the integration along the fiber defined for C^∞-submersions $\pi : U' \to U$.

We use the concatenation $c := c_{0,0}$ defined in the previous section. This yields

$$\begin{aligned} a \bullet a = c_*(a \times_{S^n} a) &= c_*\big([(D^n, \Phi, \omega)] \times_{S^n} [(D^n, \Phi, \omega)]\big) \\ &= c_*\big([(D^n \times_{S^n} D^n, \Phi \times \Phi, \pi^*\omega \wedge \pi^*\omega)]\big) = [(D^n, \Phi, \omega \wedge \omega)] = 0 \end{aligned}$$

for $\pi : D^n \times_{S^n} D^n \to D^n$ since $\omega^2 \in \Omega_c^{2n}(D^n)$ vanishes.

We further have $a \bullet l = l \bullet a$ since

$$\begin{aligned} a \bullet l = c_*\big(a \times_{S^n} l\big) &= c_*\big([(D^n, \Phi, \omega)] \times_{S^n} [(S^n \times S^{n-1}, \Psi, 1)]\big) \\ &= c_*\big([(D^n \times_{S^n} (S^n \times S^{n-1}), \Phi \times \Psi, \pi^*\omega \wedge \pi^*1)]\big) \\ &= c_*\big([((S^n \times S^{n-1}) \times_{S^n} D^n, \Psi \times \Phi, \pi^*1 \wedge \pi^*\omega)]\big) \\ &= l \bullet a \end{aligned}$$

for the diffeomorphism $D^n \times_{S^n} (S^n \times S^{n-1}) \to (S^n \times S^{n-1}) \times_{S^n} D^n$ that in particular is a C^∞-submersion. Further $c_*(a \times_{S^n} l) = c_*(l \times_{S^n} a)$ since a is a family of constant Moore loops. ■

For even dimensional spheres this construction does not work since we do not find a nowhere vanishing vector field on these spheres. The existence of poles complicates the definition of l as a regular chain.

4.4 Higher string topology of product manifolds

In this section M, N are assumed to be smooth, closed and oriented Riemannian manifolds of finite dimension $\dim M = m \geqslant 0$ respectively $\dim N = n \geqslant 3$. Further M is simply connected and N has negative sectional curvature.
Recall that for

$$LN = \bigsqcup_{\alpha \in \pi_0(LN)} L^\alpha N$$

we derived that $L^\alpha N$ is a $K(\mathbb{Z}, 1)$ space for $\alpha \neq 0$ and $L^0 N \simeq N$. For $\alpha \neq 0$ the isomorphism

$$H_*(L^\alpha N) \cong H_*(S^1)$$

can be realized by choosing a representative

$$\gamma_\alpha : S^1 \to N$$

and considering

$$\begin{aligned}\Gamma_\alpha : S^1 = \mathbb{R}/\mathbb{Z} &\to LN\\ t &\to \gamma_\alpha(\cdot + 1/l \cdot t)\ .\end{aligned}$$

for l being the winding number of γ_α. For $\alpha = 0$ we have

$$\begin{aligned}\Gamma_0 : N &\to LN\\ p &\to \gamma_p\end{aligned}$$

where $\gamma_p(t) = p$ for all $t \in S^1$. We set $\operatorname{im}\Gamma_\alpha =: S^1_\alpha$ and $\operatorname{im}\Gamma_0 =: N$.

For $X := M \times N$ the free loop space LX thus topologically looks like

$$\begin{aligned}&L(M \times N)\\ &\cong LM \times LN = LM \times \bigsqcup_{\alpha \in \pi_0(LN)} L^\alpha N \overset{\text{Cor. (2.15)}}{\simeq} LM \times (N \sqcup \bigsqcup_{\substack{0 \neq \alpha \\ \in \tilde{\pi}_1(N) \cong \tilde{\pi}_1(M \times N)}} S^1_\alpha)\\ &\simeq (LM \times N) \sqcup (LM \times \bigsqcup_{\alpha \neq 0} S^1_\alpha)\ .\end{aligned}$$

Our goal is to transfer the structure defined above, namely the dg algebra structure and the dg Lie structure of

$$(\mathfrak{C}_*(X), D, \bullet, \{\cdot,\cdot\})$$

to an A_∞-algebra and an L_∞-algebra on homology

$$H(\mathfrak{C}_*(X), D) \cong \mathbb{H}_*(LX)\ .$$

The basic idea of the following construction is that we want the subspaces S^1_α to be disjoint implying that the A_∞/L_∞-algebra operations on homology are essentially zero.

The construction further yields an A_∞-algebra morphism f and an L_∞-algebra morphism ϕ that are ∞-quasi-isomorphism

$$\big(\mathbb{H}_*(LX), \{m_n\}_{n\geqslant 1}, \{\lambda_n\}_{n\geqslant 1}\big) \xrightarrow{f=\{f_n\}} \big(\mathfrak{C}_*(LX), \tilde{m}_1, \tilde{m}_2\big)\ .$$

and

$$\big(\mathbb{H}_*(LX), \{m_n\}_{n\geqslant 1}, \{\lambda_n\}_{n\geqslant 1}\big) \xrightarrow{\phi=\{\phi_n\}} \big(\mathfrak{C}_*(LX), \tilde{\lambda}_1, \tilde{\lambda}_2\big)\ .$$

where

$$\begin{aligned}\tilde{m}_1 :=& D,\ \tilde{m}_2(a,b) := (-1)^{|a|} a \bullet b\ ,\\ \tilde{\lambda}_1 :=& D,\ \tilde{\lambda}_2(a,b) := (-1)^{|a|} \{a,b\}\ .\end{aligned}$$

We work with the disk of radius r defined as $D^n(r) := \{x \in \mathbb{R}^n \mid |x| \leqslant r\}$ and set $D^n := D^n(1)$ in the following.

Proposition 4.11

Let $A \subset \tilde{\pi}_1(N)$ be the set of primitive nontrivial homotopy classes of loops in N. Then there exist curves γ_a in N indexed by $a \in A$ and closed tubular neighbourhoods $\mathcal{O}_a \supset \gamma_a$ with the following properties:

(i) *The curve γ_a represents the homotopy class a.*

(ii) *$\mathcal{O}_a$ is a smooth submanifold of N with boundary and is diffeomorphic to $S^1 \times D^{n-1}$ via a diffeomorphism $\phi_a : S^1 \times D^{n-1} \to \mathcal{O}_a$.*

(iii) *For $a \neq b$ the submanifolds $\mathcal{O}_a$ and $\mathcal{O}_b$ are disjoint.*

Remark that in particular $\mathcal{O}_a$ and $\mathcal{O}_{-a}$ are disjoint.

Proof: For $a \neq 0$ the curve $\gamma_a \subset N$ is chosen as a representative of a.

The manifold N is compact and thus $\pi_0(LN) \cong \tilde{\pi}_1(N)$ is countable. We choose a counting

$$A = \{a_1, a_2, ...\} \ .$$

Fix γ_{a_1} and the closed tubular neighbourhood $\mathcal{O}_{a_1}$ in N, which is possible due to corollary 2.3 of [23] for example. We have a diffeomorphism $\phi_{a_1} : S^1 \times D^{n-1} \to \mathcal{O}_{a_1}$.

We recursively isotope γ_{a_i} for $i \neq 1$ and use the same notation for the perturbed γ_{a_i}. Since we use isotopies the perturbed γ_{a_i} is still a representative of a_i.

For the inductive step assume that we have modified $\gamma_{a_1}, ..., \gamma_{a_k}$ and constructed disjoint closed neighbourhoods $\mathcal{O}_{a_1}, ..., \mathcal{O}_{a_k}$ satisfying (i) - (iii) of the proposition. Isotope $\gamma_{a_{k+1}}$ such that

$$\gamma_{a_{k+1}} \pitchfork \gamma_{a_1} \ , \ \gamma_{a_{k+1}} \pitchfork \gamma_{a_2} \, , ..., \ \gamma_{a_{k+1}} \pitchfork \gamma_{a_k} \ .$$

Such isotopies exist due Corollary IV.2.4 of [23] for example and the fact that the γ_{a_i}'s are smooth compact submanifolds. Since the curves γ_{a_i} are one dimensional and we assume N to be of dimension $m \geqslant 3$ this implies

$$\gamma_{a_i} \cap \gamma_{a_j} = \varnothing \quad \text{for } 0 < i, j \leqslant k+1 \text{ with } i \neq j \ .$$

By radially moving out we can achieve that $\gamma_{a_{k+1}}$ intersects $\mathcal{O}_{a_i}$ $(1 \leqslant i \leqslant k)$ only in $\partial \mathcal{O}_{a_i} =: B_{a_i} \cong S^1 \times S^{n-2}$ and that

$$\gamma_{a_{k+1}} \cap \mathring{\mathcal{O}}_{a_i} = \varnothing \ .$$

These submanifolds $B_{a_i}, B_{a_j} \subset N$ are disjoint, closed and compact, thus have positive pairwise distances $d_{i,j} > 0$. We fix disjoint open neighbourhoods U_{a_i} of B_{a_i} in M for $1 \leqslant i \leqslant k$. The B_{a_i}'s are diffeomorphic to $S^1 \times S^{n-2}$ and in particular hypersurfaces. We can achieve that

$$\gamma_{a_{k+1}} \cap \mathcal{O}_{a_i} = \varnothing \quad \text{for} \quad 1 \leqslant i \leqslant k$$

by perturbing $\gamma_{a_{k+1}}$ in U_{a_i}. After these perturbations for all $1 \leqslant i \leqslant k$ the submanifolds $\gamma_{a_{k+1}}$ and $\mathcal{O}_{a_i} \subset M$ are disjoint and have a distance $d_i > 0$. We thus can construct $\mathcal{O}_{a_{k+1}}$ as a closed tubular neighbourhood of $\gamma_{a_{k+1}}$, and in particular we can arrange

$$\mathcal{O}_{a_i} \cap \mathcal{O}_{a_j} = \varnothing \quad \text{for } 0 < i, j \leqslant k+1 \text{ with } i \neq j \ .$$

This concluded the inductive step and thus proves the proposition. ∎

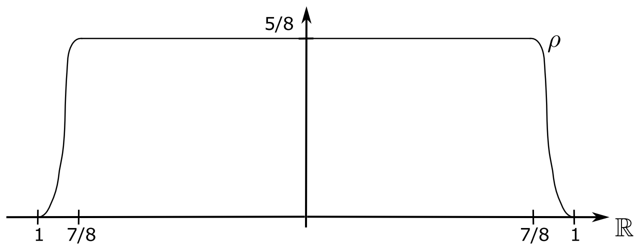
Remark 4.12. *We fix a smooth homotopy*

$$\begin{aligned} H : S^1 \times D^{n-1} \times [0,1] &\to S^1 \times D^{n-1} \\ (\tau, x, t) &\mapsto H_t(\tau, x) \end{aligned}$$

where $H_t(\tau, x)$ is the flow of the vector field

$$V(\tau, x) := \rho(|x|) \cdot \tfrac{\partial}{\partial x_{n-1}}$$

on $S^1 \times D^{n-1}$ at time t. Here $x = (x_1, ..., x_{n-1}) \in D^{n-1} \subset \mathbb{R}^{n-1}$ and ρ smooth is a cut-off function of the form



The homotopy H satisfies

(*i*) *$H_t \equiv id$ near $\partial(S^1 \times D^{n-1})$ for all t .*

(*ii*) *$H_0 = id$.*

(*iii*) *$H_1\big(S^1 \times D^{n-1}(1/4)\big) \cap \big(S^1 \times D^{n-1}(1/4)\big) = \varnothing$.*

Due to the work of Irie in [20] we know that the homology of the complex

$$\mathfrak{C}_*(L(M \times N))$$

is isomorphic to $\mathbb{H}_*(L(M \times N))$. Further the de Rham loop product and the de Rham loop bracket descend to homology and there they coincide with the loop product and the loop bracket respectively defined by Chas and Sullivan in [5].
We have

$$\mathfrak{C}_*(L(M \times N)) = \mathfrak{C}^0_*(L(M \times N)) \oplus \bigoplus_{a \in A} \mathfrak{C}^a_*(L(M \times N))$$

where $\mathfrak{C}^a_*(L(M \times N))$ contains chains in homotopy classes which are positive iterates of a and $\mathfrak{C}^0_*(L(M \times N))$ contains chains of contractible loops.

Remark the subcomplex $\mathfrak{C}'_* \subset \mathfrak{C}_*(L(M \times N))$ that splits as

$$\mathfrak{C}'_* := (\mathfrak{C}'_*)^0 \oplus \bigoplus_{a \in A} (\mathfrak{C}'_*)^a$$

where $(\mathfrak{C}'_*)^a \subset \mathfrak{C}^a_*(L(M \times N))$ contains all the chains whose loops are in

$$M \times \phi_a(S^1 \times D^{n-1})$$

and $(\mathfrak{C}'_*)^0 \subset \mathfrak{C}^0_*(L(M \times N))$ contains all the chains whose loops are contractible in $M \times N$ and further constant in N.

Lemma 4.13

The inclusion of the chain complex

$$\mathfrak{C}'_* \hookrightarrow \mathfrak{C}_*(L(M \times N))$$

induces an isomorphism on homology. In particular

$$H_*(\mathfrak{C}') \cong \mathbb{H}_*(L(M \times N)) \ .$$

Further $\mathfrak{C}'_*$ is closed under the de Rham loop product and the de Rham loop bracket defined in [20].

Proof: By proposition 4.11 for $a \in A$ and $\alpha = ka$ we have homotopy equivalences

$$LM \times \coprod_{k \geqslant 1} L^k(S^1 \times D^{n-1}) \stackrel{\mathrm{id} \times L\phi_a}{\longrightarrow} LM \times \coprod_{k \geqslant 1} L^{ka} N$$

and clearly

$$LM \times N \longrightarrow LM \times L^0 N \ .$$

The two complexes $\mathfrak{C}'_*$ and $\mathfrak{C}_*(L(M \times N))$ are the complexes of de Rham chains on the loop spaces on the left and the right respectively. By corollary 4.6 we thus get that the homology of these spaces is isomorphic.
By definition the homotopy equivalences are compatible with the de Rham loop product and the de Rham loop bracket. ■

Remark 4.14. *The lemma in general holds for $D^{n-1}(r)$ with $0 < r \leqslant 1$. For reasons of clarity in the upcoming proofs we highlight the radius as $\mathfrak{C}'_{*,r}$ if $r \neq 1$.*

The homotopy of remark 4.12 yields chain maps

$$h : \bigoplus_{a \in A} (\mathfrak{C}'_*)^a \to \bigoplus_{a \in A} (\mathfrak{C}'_*)^a$$

of degree 1 induced by H and

$$T : \bigoplus_{a \in A} (\mathfrak{C}'_*)^a \to \bigoplus_{a \in A} (\mathfrak{C}'_*)^a$$

of degree 0 induced by H_1. Further H_0 induces the identity on $\mathfrak{C}'_*$.
These relate to

$$Dh + hD = \mathrm{id} - T \tag{4.23}$$

by proposition 2.5. of [20] which guarantees that smoothly homotopic maps induce chain homotopic ones on $\mathfrak{C}'_*$.

The topological rewriting and simplification of the set-up will imply that

$$x \bullet y = 0 \quad \text{and} \quad \{x, y\} = 0$$

for $x, y \in \mathbb{H}_*(LX)$ being homology classes of loops in non-trivial conjugacy class components of $\widetilde{\pi}_1(M \times N)$ since the classes x, y can be either represented as families of loops that are disjoint in N due to *(iii)* of remark 4.12 and (4.23).

The following theorems state the generalization of this fact to the higher A_∞/L_∞-algebra operations $m_{k\geqslant 3}$ and $\lambda_{k\geqslant 3}$ on homology $\mathbb{H}_*(LX)$.
Remark that in the following we work with

$$\mathbb{H}_*(LX) = \mathbb{H}_*(L^0X) \oplus \bigoplus_{a \in A} \mathbb{H}_*(L^aX)$$

where $\mathbb{H}_*(L^aX) = \bigoplus_{\substack{\alpha = ka \\ \text{for } k \geqslant 1}} \mathbb{H}_*(L^\alpha X)$ when setting

$$L^aX := \coprod_{\substack{\alpha = ka \\ \text{for } k \geqslant 1}} L^\alpha X$$

for $\alpha \in \widetilde{\pi}_1(X) \cong \widetilde{\pi}_1(N)$.

In the following theorems we assume $X = M \times N$ and M, N to be smooth, closed and oriented Riemannian manifolds of finite dimension $\dim M = m \geqslant 0$ respectively $\dim N = n \geqslant 3$. Further M is simply connected and N has negative sectional curvature.

Theorem 4.15

The homotopy transfer construction for

$$\mathbb{H}_*(LX) \longrightarrow \mathfrak{C}_*(LX)$$

equips $\mathbb{H}_*(LX)$ with an A_∞-algebra structure $(\mathbb{H}_*(LX), \{m_k\}_{k\geqslant 1})$ and yields an A_∞-algebra morphism

$$f = \{f_k\}_{k\geqslant 1} : \mathbb{H}_*(LX) \longrightarrow \mathfrak{C}_*(LX)$$

such that:

(i) $m_1 \equiv 0$,

(ii) f_1 *is a cycle choosing homomorphism and in particular a quasi-isomorphism,*

(iii) m_2 *corresponds to the loop product, and*

(iv) $m_k(x_1, ..., x_k) = 0$

for $k \geqslant 2$ *whenever the inputs* x_i *are classes of families of loops that are non-contractible, that is* $x_i \in \bigoplus_{a\in A} \mathbb{H}_*(L^a X)$.

Theorem 4.16

The homotopy transfer construction for

$$\mathbb{H}_*(LX) \longrightarrow \mathfrak{C}_*(LX)$$

equips $\mathbb{H}_*(LX)$ *with an* L_∞*-algebra structure* $(\mathbb{H}_*(LX), \{\lambda_k\}_{k\geqslant 1})$ *and yields an* L_∞*-algebra morphism*

$$\phi = \{\phi_k\}_{k\geqslant 1} : \mathbb{H}_*(LX) \longrightarrow \mathfrak{C}_*(LX)$$

such that:

(i) $\lambda_1 \equiv 0$,

(ii) ϕ_1 *is a cycle choosing homomorphism and in particular a quasi-isomorphism,*

(iii) λ_2 *corresponds to the loop bracket, and*

(iv) $\lambda_k(y_1, ..., y_k) = 0$

for $k \geqslant 2$ *whenever the inputs are elements* $y_i \in \mathbb{H}_*(L^{a_i} X)$ *for primitive classes* $a_i \in A$ *which are not all equal.*

For proving the theorems we apply the homotopy transfer construction presented by Kadeishvili in [21] by recursively constructing the higher operations and morphisms.

Proof of theorem 4.15: We use the notation from section 3.

For the first operations we set

$$m_1 = 0 = U_1 \quad \text{and} \quad f_1 = \iota$$

where $\iota : \mathbb{H}_*(L^a X) \to (\mathfrak{C}'_{*,1/4})^a$ for $a \in A$ and $\iota : \mathbb{H}_*(L^0 X) \to (\mathfrak{C}'_*)^0$ are cycle choosing homomorphism. Thus equation (3.3) is satisfied, namely

$$U_1 - f_1 \circ m_1 = 0 = D \circ \iota = \widetilde{m}_1 \circ f_1 \ .$$

The operation $m_2 = [\widetilde{m}_2 \circ i^{\otimes 2}]$ on $\mathbb{H}_*(LX)$ is the loop product up to sign due to how

$$\widetilde{m}_2 := \pm \text{ de Rham loop product}$$

are constructed by Irie.

It remains to prove (iv). For general $k \geqslant 2$ we have

$$\begin{aligned} U_k(x_1, ..., x_k) = & \sum_{s=1}^{k-1} \widetilde{m}_2(f_s(x_1, ..., x_s), f_{k-s}(x_{s+1}, ..., x_k)) + \\ & + \sum_{i=0}^{k-2} \sum_{j=2}^{k-1} (-1)^{i+1+|x_1|+...+|x_i|} f_{k-j+1}(x_1, ..., x_i, m_j(x_{i+1}, ..., x_{i+j}), x_{i+j+1}, ..., x_k) \ . \end{aligned}$$

We will show that there exist maps

$$f_k : \mathbb{H}_*(LX)^{\otimes k} \longrightarrow \mathfrak{C}'_*$$

such that

$$D \circ f_k = \widetilde{m}_1 \circ f_k = U_k - f_1 \circ m_k = U_k$$

when acting on inputs x_i that are classes of families of loops that are non-contractible. This then yields

$$m_k(x_1, ..., x_k) = [U_k(x_1, ..., x_k)] = 0$$

for such inputs.
For the induction assume the stated assertion holds up to degree k. We perform the inductive step for $k \to k+1$.

Assume all operations and morphisms are constructed up to degree k. In the induction hypothesis we assume that the image of $f_k(x_1, ..., x_k)$ is contained in the support of $f_1(x_{k+1})$ when acting on $x_1, ..., x_{k+1}$ as in the condition for (iv). In particular we thus have

$$\widetilde{m}_2(f_k(x_1, ..., x_k), Tf_1(x_{k+1})) = 0 \ . \tag{4.24}$$

According to the definition of the loop product by Irie we know that for chains $x \in \mathfrak{C}'_*$ and x_i as in the condition for (iii) we have that the supports of

$$\widetilde{m}_2(f_1(x_i), x) \quad \text{and} \quad \widetilde{m}_2(Tf_1(x_i), x)$$

are contained in the support of $f_1(x_i)$ and $Tf_1(x_i)$ respectively. By (iii) of remark 4.12 we thus have

$$\widetilde{m}_2(f_1(x_i), Tf_1(x_i)) = 0 \ . \tag{4.25}$$

We define

$$f_{k+1} := (-1)^{d_1 + ... + d_k + k - 1}\, \widetilde{m}_2 \circ (f_k \otimes hf_1) \tag{4.26}$$

when acting on $\mathbb{H}_{d_1}(L^{a_{i_1}}X) \otimes ... \otimes \mathbb{H}_{d_{k+1}}(L^{a_{i_{k+1}}}X)$ for $a_1, ..., a_{i_{k+1}} \in A$. Remark that for $x_1, ..., x_{k+2}$ as in the condition for (iv) we get that the image of $f_{k+1}(x_1, ..., x_{k+1})$ is contained in the support of $f_1(x_{k+2})$

Due to he work of Kadeishvili in [21] we can define f_{k+1} for the remaining cases if at least one input is of $\mathbb{H}_*(L^0X)$ such that $D \circ f_{k+1} = U_{k+1} - f_1 \circ m_{k+1}$ implying m_{k+1} not necessarily zero. We do not want to prove something about these operations here.

It remains to show that

$$Df_{k+1} = U_{k+1}$$

when acting on $x_1, ..., x_{k+1}$ as in the condition for (iii). Recall the A_∞-operations in the case $\widetilde{m}_k = 0$ for $k \geqslant 3$, namely

$$\begin{aligned}
&\widetilde{m}_1 \circ \widetilde{m}_1 = 0 \ , \\
&\widetilde{m}_1(\widetilde{m}_2(x,y)) + \widetilde{m}_2(\widetilde{m}_1(x),y) + (-1)^{|x|+1}\widetilde{m}_2(x,\widetilde{m}_1(y)) = 0 \ , \\
&\widetilde{m}_2(\widetilde{m}_2(x,y),z) + (-1)^{|x|+1}\widetilde{m}_2(x,\widetilde{m}_2(y,z)) = 0 \ .
\end{aligned}$$

Since $|f_k| = k-1$ we get

$$\begin{aligned}
(-1)^{|x_1|+\cdots|x_k|+k-1}&(Df_{k+1})(x_1,...,x_{k+1}) \\
&= \big(D \circ \widetilde{m}_2 \circ (f_k \otimes hf_1)\big)(x_1,...,x_{k+1}) \\
&= \big(-\widetilde{m}_2 \circ ((Df_k)\otimes hf_1) - (-1)^{|x_1|+\cdots|x_k|+k}\widetilde{m}_2 \circ (f_k \otimes (Dhf_1))\big)(x_1,...,x_{k+1}) \ .
\end{aligned} \tag{4.27}$$

For the first summand we use the induction hypothesis

$$Df_j = U_j \ \text{ implying } \ m_j = 0$$

for $1 \leqslant j \leqslant k$. In particular we thus get

$$\begin{aligned}
Df_k = U_k &= \sum_{s=1}^{k-1} \widetilde{m}_2(f_s \otimes f_{k-s}) + \sum_{i=0}^{k-2}\sum_{j=2}^{k-1} \pm f_{k-j+1}(1^i \otimes m_j \otimes 1^{k-i-j}) \\
&= \sum_{s=1}^{k-1} \widetilde{m}_2(f_s \otimes f_{k-s}) \ .
\end{aligned}$$

For the second summand we use (4.23) and $D \circ f_1 \equiv 0$, that is

$$\widetilde{m}_2 \circ (f_k \otimes (Dhf_1)) = \widetilde{m}_2 \circ (f_k \otimes ((\mathrm{id}-T)f_1)) = \widetilde{m}_2 \circ (f_k \otimes f_1)$$

by (4.24). For (4.27) we deduce

$$\begin{aligned}
&(-1)^{|x_1|+\cdots|x_k|+k-1}(Df_{k+1})(x_1,...,x_{k+1}) \\
&\quad = \Big(-\sum_{s=1}^{k-1} \widetilde{m}_2 \circ (\widetilde{m}_2(f_s \otimes f_{k-s}) \otimes hf_1) - (-1)^{|x_1|+\cdots|x_k|+k}\widetilde{m}_2 \circ (f_k \otimes f_1)\Big)(x_1,...,x_{k+1}) \ .
\end{aligned}$$

Using that the de Rham loop product and thus $\widetilde{m}_2$ is associative implies

$$\begin{aligned}
&(-1)^{|x_1|+\cdots|x_k|+k-1}(Df_{k+1})(x_1,...,x_{k+1}) \\
&= \Big(\sum_{s=1}^{k-1}(-1)^{|x_1|+\cdots|x_s|+s}\widetilde{m}_2 \circ (f_s \otimes \widetilde{m}_2(f_{k-s} \otimes hf_1)) - (-1)^{|x_1|+\cdots|x_k|+k}\widetilde{m}_2 \circ (f_k \otimes f_1)\Big)(x_1,...,x_{k+1})
\end{aligned}$$

By definition (4.26) we have

$$\widetilde{m}_2 \circ (f_{k-s} \otimes hf_1) = (-1)^{|x_1'|+\ldots+|x_{k-s}'|+k-s-1} f_{k-s+1}$$

for $x'_i = x_{s+i}$, that is we get

$$
\begin{aligned}
&(-1)^{|x_1|+\cdots|x_k|+k-1}(Df_{k+1})(x_1,...,x_{k+1})\\
&= \Big(\sum_{s=1}^{k-1}(-1)^{|x_1|+\cdots|x_k|+k-1}\widetilde{m}_2\circ(f_s\otimes f_{k-s+1}) + (-1)^{|x_1|+\cdots|x_k|+k-1}\widetilde{m}_2\circ(f_k\otimes f_1)\Big)(x_1,...,x_{k+1})\\
&= (-1)^{|x_1|+\cdots|x_k|+k-1}\Big(\sum_{s=1}^{k-1}\widetilde{m}_2\circ(f_s\otimes f_{k-s+1}) + \widetilde{m}_2\circ(f_k\otimes f_1)\Big)(x_1,...,x_{k+1})\\
&= (-1)^{|x_1|+\cdots|x_k|+k-1}\Big(\sum_{s=1}^{k}\widetilde{m}_2\circ(f_s\otimes f_{k-s+1})\Big)(x_1,...,x_{k+1})
\end{aligned}
$$

which is

$$(-1)^{|x_1|+\cdots|x_k|+k-1}U_{k+1}(x_1,...,x_{k+1})$$

since $m_j|_{(\bigoplus_{a\in A}\mathbb{H}_*(L^aX))^{\otimes k}} = 0$ for $1\leqslant j\leqslant k$ by the induction hypothesis. ■

Proof of theorem 4.16 : We use the notation from section 3.3.
For the first operations we set

$$\lambda_1 = 0 = V_1 \quad\text{and}\quad \phi_1 = \iota\ ,$$

where $\iota:\mathbb{H}_*(L^aX)\to(\mathfrak{C}'_*)^a$ for $a\in A$ and $\iota:\mathbb{H}_*(L^0X)\to(\mathfrak{C}'_*)^0$ are cycle choosing homomorphisms. Thus equation (3.11) is satisfied, namely

$$\phi_1\circ\lambda_1 - V_1 = 0 = D\circ\iota = \widetilde{\lambda}_1\circ\phi_1\ .$$

The operation $\lambda_2 = [\widetilde{\lambda}_2\circ i^{\otimes 2}]$ on $\mathbb{H}_*(LX)$ is the loop bracket up to sign due to how

$$\widetilde{\lambda}_2 := \pm\ \text{de Rham loop bracket}$$

are constructed by Irie.

The recursive construction of Kadeishvili yields

$$V_2(x,y) = \widetilde{\lambda}_2(\phi_1(x),\phi_1(y))$$

for $x,y\in\mathbb{H}_*(LX)$. Therefore by construction for $a\neq b\in A$ and $x\in\mathbb{H}_*(L^aX)$ we get

$$V_2(x,y)\begin{cases} = 0 & ,\ \text{for } y\in\mathbb{H}_*(L^bX)\\ \in(\mathfrak{C}'_*)^a & ,\ \text{for } y\in\mathbb{H}_*(L^aX)\text{ or } y\in\mathbb{H}_*(L^0X)\end{cases}$$

since $(\mathfrak{C}'_*)^a$ and $\mathbb{H}_*(L^aX)$ contains families/classes of loops that are positive iterates of the primitive nontrivial homotopy class a.
For $\lambda_2 := [V_2]$ we get

$$\lambda_2(x,y)\begin{cases} = 0 & ,\ \text{for } y\in\mathbb{H}_*(L^bX)\\ \in\mathbb{H}_*(L^aX) & ,\ \text{for } y\in\mathbb{H}_*(L^aX)\text{ or } y\in\mathbb{H}_*(L^0X)\end{cases}$$

which allows to define ϕ_2 such that

$$\phi_2(x,y) \begin{cases} = 0 & , \quad \text{for } y \in \mathbb{H}_*(L^b X) \\ \in (\mathfrak{C}'_*)^a & , \quad \text{for } y \in \mathbb{H}_*(L^a X) \text{ or } y \in \mathbb{H}_*(L^0 X) \end{cases}$$

Analogously we get

$$\lambda_2(x,y) \in \mathbb{H}_*(L^0 X) \quad \text{and} \quad \phi_2(x,y) \in (\mathfrak{C}'_*)^0$$

for $x, y \in \mathbb{H}_*(L^0 X)$.
We end up with (3.11), namely

$$\phi_1 \circ \lambda_2 - V_2 = \widetilde{\lambda}_1 \circ \phi_2 \ .$$

It remains to prove (iii). We perform the inductive step for $k \to k+1$.

Assume all operations and morphisms are constructed up to degree k and that

$$\lambda_l(c_1, ..., c_l) \begin{cases} = 0 & , \quad (I) \\ \in \mathbb{H}_*(L^a X) & , \quad (II) \\ \in \mathbb{H}_*(L^0 X) & , \quad (III) \end{cases} \tag{4.28}$$

and

$$\phi_l(c_1, ..., c_l) \begin{cases} = 0 & , \quad (I) \\ \in (\mathfrak{C}'_*)^a & , \quad (II) \\ \in (\mathfrak{C}'_*)^0 & , \quad (III) \end{cases} \tag{4.29}$$

for all $1 \leqslant l \leqslant k$. Here condition (I) means

$$(I) \mathrel{\widehat{=}} \exists\, i, j \in \{1, ..., l\} \text{ such that } c_i \in \mathbb{H}_*(L^a X) \text{ and } c_j \in \mathbb{H}_*(L^b X) \text{ for } a \neq b \in A \ ,$$

(II) means

$$\begin{aligned} (II) \mathrel{\widehat{=}} \forall\, i \in \{1, ..., l\} & \text{ we either have } c_i \in \mathbb{H}_*(L^0 X) \text{ or } c_i \in \mathbb{H}_*(L^a X) \text{ for } a \in A \\ & \text{and there exists at least one } i_0 \in \{1, ..., l\} \text{ such that } c_{i_0} \in \mathbb{H}_*(L^a X) \end{aligned}$$

and (III) means

$$(III) \mathrel{\widehat{=}} \forall\, i \in \{1, ..., l\} \text{ we have } c_i \in \mathbb{H}_*(L^0 X) \ .$$

We prove that (4.28) and (4.29) hold for λ_{k+1} and ϕ_{k+1} which then proves (iii) of the theorem, namely that

$$\lambda_l(c_1, ..., c_l) = 0 \quad \text{if} \,\exists\, i, j \text{ such that } c_i \in \mathbb{H}_*(L^a X) \text{ and } c_j \in \mathbb{H}_*(L^b X) \text{ for } a \neq b \in A$$

and for all $l \geqslant 1$.

The recursive construction of Kadeishvili yields

$$
\begin{aligned}
V_{k+1}(c_1,...,c_{k+1}) = & \sum_{\substack{\sigma,\\ p+q=k+1}} \pm\, c_\sigma\, \widetilde{\lambda}_2(\phi_p(c_{\sigma(1)},...,c_{\sigma(p)}),\phi_q(c_{\sigma(p+1)},...,c_{\sigma(k+1)}))+ \\
& + \sum_{\substack{\tau,\\ 1<l<k+1}} \pm\, c_\tau\, \phi_{k-l+2}(\lambda_l(c_{\tau(1)},...,c_{\tau(l)}),c_{\tau(l+1)},...,c_{\tau(k+1)}) \ .
\end{aligned}
$$

The multiplicities, the signs and in particular the question which σ and τ are used, shuffles or permutations, is an important issue in general. We may bypass these questions since the statements above will hold independently for each summand.

Since only morphisms and operations of degree $\leqslant k$ are involved we apply the induction hypothesis and get

$$
V_{k+1}(c_1,...,c_{k+1}) \begin{cases} = 0 & , \ (I) \\ \in (\mathfrak{C}'_*)^a & , \ (II) \\ \in (\mathfrak{C}'_*)^0 & , \ (III) \end{cases}
$$

Since $\lambda_{k+1} := [V_{k+1}]$ we get

$$
\lambda_{k+1}(c_1,...,c_{k+1}) \begin{cases} = 0 & , \ (I) \\ \in \mathbb{H}_*(L^a X) & , \ (II) \\ \in \mathbb{H}_*(L^0 X) & , \ (III) \end{cases} \ .
$$

According to the definition of Kadeishvili ϕ_{k+1} is defined such that

$$
D\phi_{k+1}(c_1,...,c_{k+1}) := V_{k+1}(c_1,...,c_{k+1}) - \phi_1(\lambda_{k+1}(c_1,...,c_{k+1})) \ .
$$

Since ϕ_1 satisfies (4.29) we can choose ϕ_{k+1} such that

$$
\phi_{k+1}(c_1,...,c_{k+1}) \begin{cases} = 0 & , \ (I) \\ \in (\mathfrak{C}'_*)^a & , \ (II) \\ \in (\mathfrak{C}'_*)^0 & , \ (III) \end{cases} \ .
$$

This finishes the inductive step and proves (iii) namely that for (I) we have

$$
\lambda_{k+1}(c_1,...,c_{k+1}) = 0 \ .
$$

■

4.5 Obstruction against the Lagrangian embedding $X \hookrightarrow \mathbb{C}^d$

In this section we combine the results of the previous section with a result of Fukaya to prove:

Theorem 4.17

A closed, oriented, spin Lagrangian submanifold

$$X \subset (\mathbb{C}^d, \omega_0)$$

for $d = n + m \geqslant 3$ can not be of the form $M \times N$ where M, N are smooth, closed and oriented manifolds of finite dimension $\dim M = m \geqslant 0$ and $\dim N = n \geqslant 3$ respectively with M simply connected and N admitting a Riemannian metric of negative sectional curvature.

We prove theorem 4.17 by contradiction, that is we assume that

Assumption: (4.30)

$X = M \times N$ with the stated conditions embeds as a Lagrangian submanifold into $\mathbb{C}^d$.

We will prove that for such an X and the corresponding dg Lie algebra $\mathfrak{C}_*(LX)$ the chain of constant loops $[X] = [(X, s : X \to LX, 1)] \in \mathfrak{C}_*(LX)$ is not in the image of the twisted differential

$$D^a = D + \widetilde{\lambda}_2(\,\cdot\,, a)$$

where $a \in \mathfrak{C}_*(L^{\alpha \neq 0}X)$ is any Maurer Cartan element which is positive with respect to a suitable filtration. With Fukaya's theorem 4.21 (see below) we get the desired contradiction and hence a proof of Theorem 4.17.

To make sense of the intermediate statements we first need to discuss completions with respect to a given filtration.

For a smooth map $u : (D^2, \partial D^2) \to (\mathbb{C}^d, X)$ we have the action

$$\mathcal{A}(u) := \int\limits_{D^2} u^*\omega_0 = \int\limits_{D^2} u^* d\lambda_0 = \int\limits_{\partial D^2 = S^1} u^*\lambda_0 = \int\limits_{u(S^1)} \lambda_0 = \langle [\lambda_0|_X], u_*[S^1] \rangle \in \mathbb{R}$$

for $u_*[S^1] \in H_1(X; \mathbb{Z})$.

Indeed $\mathcal{A}$ only depends on the free relative homotopy class $[u]$ in $\widetilde{\pi}_2(\mathbb{C}^d, X)$.This holds since for a relative homotopy $h : (D^2 \times [0,1], \partial D^2 \times [0,1]) \to (\mathbb{C}^d, X)$ between $u = h_0$ and $u' = h_1$ we have

$$0 = \int\limits_{\partial(D^2 \times [0,1])} h^*\omega_0 = \int\limits_{\partial D^2 \times [0,1]} h^*\omega_0 - \int\limits_{D^2} h_1^*\omega_0 + \int\limits_{D^2} h_0^*\omega_0$$

which implies

$$\mathcal{A}(u') = \int\limits_{D^2} h_1^*\omega_0 = \int\limits_{D^2} h_0^*\omega_0 = \mathcal{A}(u)$$

since X is a Lagrangian submanifold and thus

$$\int\limits_{\partial D^2 \times [0,1]} h^*\omega_0 = \int\limits_{h(\partial D^2 \times [0,1]) \subset X} \omega_0 = 0\ .$$

Since $\mathbb{C}^d$ is contractible we get

$$\tilde{\pi}_2(\mathbb{C}^d, X) \cong \tilde{\pi}_1(X) \ .$$

Further for the path components $\pi_0(LX)$ of LX we have $\pi_0(LX) \cong \tilde{\pi}_1(X)$.

Lemma 4.18

A Lagrangian embedding $\iota_0 : X \to \mathbb{C}^d$ is isotopic via Lagrangian embeddings to $\iota_1 : X \to \mathbb{C}^d$ such that

$$[\iota^* \lambda_0] \in H^1(X; \mathbb{Z}) \subset H^1(X; \mathbb{R}) \ .$$

Proof: 1) For the Lagrangian submanifold $X \hookrightarrow (C^d, \omega_0 = d\lambda_0)$ we apply the Weinstein tubular neighbourhood theorem (cf. theorem 9.3 of [34]) that states:

There exist neighbourhoods U of X in $\mathbb{C}^d$ and V of X in $(T^*X, \omega = d\lambda)$, embedded as the zero section $s_0 : X \to T^*X$, and a diffeomorphism $\phi : V \to U$ such that $\phi^*\omega_0 = \omega$ and $\phi \circ s_0 = \iota_0$.

2) The Lagrangian submanifold $X \hookrightarrow (T^*X, \omega = d\lambda)$ can be isotoped in $V \subset T^*X$ (cf. proposition 3.4 of [34]) as follows:

For any closed one form μ on X the isotopy $s : [0,1] \times X \to T^*X$ given by

$$s_t(x) = (x, t\mu_x)$$

is a Lagrangian isotopy in T^*X and $s_1^*\lambda = \mu$.

3) We choose $\mu \in \Omega^1(X)$ closed such that $X_\mu \in V$ and

$$[\iota_0^* \lambda_0] + [\mu] \in \frac{1}{N} H^1(X; \mathbb{Z})$$

for some $N \in \mathbb{N}$. Since $\phi : V \to U$ is a symplectomorphism, we know that $\phi^*\lambda_0 - \lambda$ is a closed 1-form on T^*X.
Consider the isotopy s from step 2) and define $\iota_t : X \to \mathbb{C}^d$ as $\iota_t = \phi \circ s_t$.
Note that

$$s_0^*(\phi^*\lambda_0 - \lambda) = \iota_0^*\lambda_0$$

since λ vanishes along the zero section $s_0(X) \subset T^*X$.
On the other hand $s_1^*(\phi^*\lambda_0 - \lambda) = \iota_1^*\lambda_0 - \mu$. Since s_0 and s_1 are homotopic and μ is closed, we conclude that

$$[\iota_0^*\lambda_0] = [s_0^*(\phi^*\lambda_0 - \lambda)] = [s_1^*(\phi^*\lambda_0 - \lambda)] = [\iota_1^*\lambda_0 - \mu] = [\iota_1^*\lambda_0] - [\mu]$$

and so $[\iota_1^*\lambda_0] = [\iota_0^*\lambda_0] + [\mu] \in \frac{1}{N} H^1(X; \mathbb{Z})$.

4) Since the translation and multiplication with a real number does not change the property of a submanifold of $\mathbb{C}^d$ to be Lagrangian we can scale up the Lagrangian by a factor N and get $[\lambda_0|_X] \in H^1(X; \mathbb{Z})$. ■

Since we now may assume that $\mathcal{A}|_{\pi_0(LX)} \subset \mathbb{Z}$ this allows to equip

$$\mathfrak{C}_*(LX) = \bigoplus_{\alpha \in \pi_0(LX)} \mathfrak{C}_*(L^\alpha X)$$

with an integer filtration $\{\mathcal{F}^k \mathfrak{C}_*(LX)\}_{k \in \mathbb{Z}}$ with $\mathcal{F}^k \mathfrak{C}_*(LX) \supset \mathcal{F}^{k+1} \mathfrak{C}_*(LX)$ given by

$$\mathcal{F}^k \mathfrak{C}_*(LX) := \{c \in \mathfrak{C}_*(LX) \,|\, \mathcal{A}(c_i) \geqslant k\} \ .$$

where $c = \sum_i c_i$ and

$$\mathcal{A}(c_i) := \mathcal{A}(\alpha)$$

for $c_i \in \mathfrak{C}_*(L^\alpha X)$ with connected domain.
By construction the de Rham loop bracket and the boundary operator D preserve the filtration, that is

$$\{\mathcal{F}^{k_1} \mathfrak{C}_*(LX), \mathcal{F}^{k_2} \mathfrak{C}_*(LX)\} \subset \mathcal{F}^{k_1+k_2} \mathfrak{C}_*(LX) \quad \text{and} \quad D\mathcal{F}^k \mathfrak{C}_*(LX) \subset \mathcal{F}^k \mathfrak{C}_*(LX) \ .$$

It is a filtration on the index set $\pi_0(LX)$ and therefore the filtration descends to homology and we get $\{\mathcal{F}^k \mathbb{H}_*(LX)\}_{k \in \mathbb{Z}}$. This further allows to extend the operations to the completion

$$\widehat{\mathfrak{C}_*}(LX) := \{ \sum_{k \geqslant k_0 \in \mathbb{Z}}^{\infty} c_k \,|\, c_k \in \mathcal{F}^k \mathfrak{C}_*(LX) \}$$

and we get that $\widehat{\mathfrak{C}_*}(LX)$ is a dg Lie algebra with Lie bracket given by the de Rham loop bracket. The induced filtration of the completion is denoted by $\{\mathcal{F}^k \widehat{\mathfrak{C}}_*(LX)\}_{k \in \mathbb{Z}}$.

Remark 4.19. *An L_∞-algebra $(C, \{\lambda_k\}_{k \geqslant 1})$ is called filtered if for C there exists a filtration $\mathcal{F}^k C \supset \mathcal{F}^{k+1} C$ and the operations preserve that filtration, namely*

$$\lambda_l(\mathcal{F}^{k_1} C, ..., \mathcal{F}^{k_l} C) \subset \mathcal{F}^{k_1 + ... + k_l} C \ .$$

An L_∞-algebra morphism between filtered L_∞-algebras

$$(C, \{\lambda_k\}_{k \geqslant 1}) \xrightarrow{\{\phi_k\}_{k \geqslant 1}} (C', \{\lambda'_k\}_{k \geqslant 1})$$

is called filtered if the morphisms preserve that filtration, namely

$$\phi_l(\mathcal{F}^{k_1} C, ..., \mathcal{F}^{k_l} C) \subset \mathcal{F}^{k_1 + ... + k_l} C' \ .$$

The dg Lie algebra operations $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$ on Irie's complex $\mathfrak{C}_*(LX)$ preserve the filtration. We conclude that

$$(\widehat{\mathfrak{C}}_*(LX), \widetilde{\lambda}_1, \widetilde{\lambda}_2)$$

is a completed, filtered L_∞-algebra with $\widetilde{\lambda}_k = 0$ for $k \geqslant 3$.

In general Maurer-Cartan elements can be used to define twisted differentials that is:

Lemma 4.20 (***Lemma 4.8. in [25]***)

If $a \in \widehat{C}$ is a Maurer-Cartan element in a completed L_∞-algebra $(\widehat{C}, \{\lambda_k\}_{k \geqslant 1})$, that is

$$\sum_{k=1}^{\infty} (-1)^{\frac{(k-1)k}{2}} \frac{1}{k!} \lambda_k(a, ..., a) = 0 \ ,$$

the morphism $D^a : \widehat{C} \to \widehat{C}$ given by

$$D^a(b) := \sum_{k=1}^{\infty} (-1)^{\frac{(k-2)(k-1)}{2}} \frac{1}{(k-1)!} \lambda_k(b, a, ..., a)$$

is a differential.

Our main input from symplectic geometry is the following version of a result of Fukaya.

Theorem 4.21 (***cf. Fukaya [13]***)

Let X be a closed, oriented, spin Lagrangian submanifold $X \subset \mathbb{C}^d$, and let

$$(\widehat{\mathfrak{C}}_*(LX), \widetilde{\lambda}_1, \widetilde{\lambda}_2)$$

be the completion of the filtered, degree shifted Irie complex with its induced filtered dg Lie algebra structure.
Then there exist chains $a \in \widehat{\mathfrak{C}}_*(LX)$ with $\mathcal{A}(a) > 0$ and $b \in \widehat{\mathfrak{C}}_*(LX)$ satisfying:

$$\widetilde{\lambda}_1(a) + \frac{1}{2}\widetilde{\lambda}_2(a, a) = 0 \tag{4.31}$$

$$D^a(b) = \widetilde{\lambda}_1(b) + \widetilde{\lambda}_2(b, a) = [X] \tag{4.32}$$

All appearing operations and morphisms in the homotopy transfer construction for

$$\mathbb{H}_*(LX) \longrightarrow \mathfrak{C}_*(LX)$$

in the last section preserve the decomposition by homotopy classes of loops. Therefore we can do the same homotopy transfer construction as in the last section now for the filtered set-up and get:

Theorem 4.22

The homotopy transfer construction for

$$H_*(\widehat{\mathfrak{C}_*}(LX), D) \longrightarrow \widehat{\mathfrak{C}_*}(LX)$$

equips $H_*(\widehat{\mathfrak{C}_*}(LX), D)$ with a filtered L_∞-algebra structure $(H_*(\widehat{\mathfrak{C}_*}(LX)), \{\lambda_k\}_{k \geqslant 1})$ and yields a filtered L_∞-algebra morphism

$$\phi = \{\phi_k\}_{k \geqslant 1} : H_*(\widehat{\mathfrak{C}_*}(LX)) \longrightarrow \widehat{\mathfrak{C}_*}(LX)$$

such that:

(i) $\lambda_1 \equiv 0$

(ii) ϕ_1 *is a cycle choosing homomorphism and in particular a quasi-isomorphism*

(iii) λ_2 *corresponds to the loop bracket*

(iv) $\lambda_k(y_1, ..., y_k) = 0$

for $k \geqslant 2$ *where the inputs* $y_1, ..., y_k$ *are classes of loops in at least two different non-trivial conjugacy class components modulo positive iterations of loops.*

■

Since ϕ is an L_∞-quasi-isomorphism, that is ϕ_1 is a quasi-isomorphism, we may apply Theorem 10.4.7. of [27] and get an inverse L_∞-quasi-isomorphism

$$\left(H_*(\widehat{\mathfrak{C}_*}(LX)), \{\lambda_n\}_{n\geqslant 2}\right) \xleftarrow{\psi=\{\psi_n\}_{n\geqslant 1}} \left(\widehat{\mathfrak{C}_*}(LX), \{\widetilde{\lambda}_n\}_{n\geqslant 1}\right) .$$

The way this L_∞-morphism is constructed is described in Theorem 10.4.2. of [27]. The morphism ψ_k is constructed by applying

$$p : \widehat{\mathfrak{C}_*}(LX) \to H_*(\widehat{\mathfrak{C}_*}(LX)) \quad \text{and} \quad \phi_{i\leqslant k} : H_*(\widehat{\mathfrak{C}_*}(LX))^{\otimes i} \to \widehat{\mathfrak{C}_*}(LX)$$

in various combinations. Since the morphisms ϕ_i for all i and p preserve the stated filtration we get that $\psi = \{\psi_n\}_{n\geqslant 1}$ preserves the filtration that is

$$\psi_n(\mathcal{F}^{k_1}\widehat{\mathfrak{C}_*}(LX), ..., \mathcal{F}^{k_n}\widehat{\mathfrak{C}_*}(LX)) \subset \mathcal{F}^{k_1+...+k_n}H_*(\widehat{\mathfrak{C}_*}(LX)) .$$

The L_∞-quasi-isomorphism ψ is now used to transfer information from $\widehat{\mathfrak{C}_*}(LX)$ to $H_*(\widehat{\mathfrak{C}_*}(LX))$.

Lemma 4.23 (*Proposition 4.9. (1) in [25]*)

If $a \in \widehat{C}$ *is a Maurer-Cartan element in a completed* L_∞*-algebra* $(\widehat{C}, \{\widetilde{\lambda}_n\}_{n\geqslant 1})$, *that is*

$$\sum_{k=1}^{\infty} (-1)^{\frac{(k-1)k}{2}} \frac{1}{k!} \widetilde{\lambda}_n(a, ..., a) = 0 ,$$

and

$$\left(\widehat{C}, \{\widetilde{\lambda}_n\}_{n\geqslant 1}\right) \xrightarrow{\psi=\{\psi_n\}_{n\geqslant 1}} \left(H_*(\widehat{C}, \widetilde{\lambda}_1), \{\lambda_n\}_{n\geqslant 2}\right) .$$

is an L_∞*-quasi-isomorphism, then the element*

$$\overline{a} := \sum_{k=1}^{\infty} \frac{1}{k!} \psi_k(a, ..., a)$$

is a Maurer-Cartan element in $\left(H_*(\widehat{C}, \widetilde{\lambda}_1), \{\lambda_n\}_{n\geqslant 2}\right)$, *that is*

$$\sum_{k=2}^{\infty} (-1)^{\frac{(k-1)k}{2}} \frac{1}{k!} \lambda_k(\overline{a}, ..., \overline{a}) = 0 .$$

Maurer-Cartan elements can be used to define twisted differentials as described in lemma 4.20 that is

$$D^a(x) := \sum_{k=1}^{\infty} (-1)^{\frac{(k-2)(k-1)}{2}} \frac{1}{(k-1)!} \widetilde{\lambda}_k(x, a, ..., a) = \widetilde{\lambda}_1(x) + \widetilde{\lambda}_2(x, a)$$

and

$$D^{\overline{a}}(y) := \sum_{k=2}^{\infty} (-1)^{\frac{(k-2)(k-1)}{2}} \frac{1}{(k-1)!} \lambda_k(y, \overline{a}, ..., \overline{a})$$

define differentials on $\widehat{\mathfrak{C}_*}(LX)$ and $H_*(\widehat{\mathfrak{C}_*}(LX))$ respectively.

Recall the L_∞-quasi-isomorphism

$$\left(H_*(\widehat{\mathfrak{C}_*}(LX)), \{\lambda_n\}_{n \geqslant 2}\right) \xleftarrow{\psi = \{\psi_n\}_{n \geqslant 1}} \left(\widehat{\mathfrak{C}_*}(LX), \{\widetilde{\lambda}_n\}_{n \geqslant 1}\right) .$$

It actually gives rise to a chain map between

$$\left(\widehat{\mathfrak{C}_*}(LX), D^a \right) \quad \text{and} \quad \left(H_*(\widehat{\mathfrak{C}_*}(LX)), D^{\overline{a}} \right) ,$$

namely:

Lemma 4.24 (*Proposition 4.9. (2) in [25]*)

If $a \in \widehat{C}$ is a Maurer-Cartan element in a completed L_∞-algebra $(\widehat{C}, \{\widetilde{\lambda}_n\}_{n \geqslant 1})$ and

$$\overline{a} := \sum_{k=1}^{\infty} \frac{1}{k!} \psi_k(a, ..., a)$$

is the induced Maurer-Cartan element in $H_*(\widehat{C_*}, \widetilde{\lambda}_1)$, the map

$$\overline{\psi} : \left(\widehat{\mathfrak{C}_*}(LX), D^a \right) \longrightarrow \left(H_*(\widehat{\mathfrak{C}_*}(LX)), D^{\overline{a}} \right)$$

given by

$$\overline{\psi}(x) := \sum_{k=1}^{\infty} \frac{1}{(k-1)!} \psi_k(x, a, ..., a)$$

is a chain map between the complexes with their twisted differentials, that is

$$D^{\overline{a}} \circ \overline{\psi} = \overline{\psi} \circ D^a .$$

Recall that Fukaya's Theorem 4.21 gives a Maurer-Cartan element $a \in \mathcal{F}^1\widehat{\mathfrak{C}_*}(LX)$ and an element $b \in \widehat{\mathfrak{C}_*}(LX)$ such that

$$D^a(b) = [X] .$$

Applying Lemma 4.23 and Lemma 4.24 we obtain

$$D^{\overline{a}}(\overline{\psi}(b)) = \overline{\psi}([X]) .$$

It is equivalent to

$$\begin{aligned} D^{\overline{a}}\left(\overline{\psi}(b)\right) &= \sum_{k=2}^{\infty}(-1)^{\frac{(k-2)(k-1)}{2}}\frac{1}{(k-1)!}\lambda_k(\overline{\psi}(b),\overline{a},...,\overline{a}) \\ &= \sum_{k=1}^{\infty}\frac{1}{(k-1)!}\psi_k([X],a,...,a) = \psi_1([X]) + \psi_2([X],a) + \cdots \end{aligned}$$

Remark that $[X] \in \mathcal{F}^0\widehat{\mathfrak{C}_*}(LX)$ and $a \in \mathcal{F}^1\widehat{\mathfrak{C}_*}(LX)$. Since ψ preserves the filtration we get that

$$\begin{aligned} \mathcal{A}(\psi_1([X])) = 0 \quad , \quad & \overline{a} \in \mathcal{F}^1 H_*(\widehat{\mathfrak{C}_*}(LX)) \\ \text{and} \quad & \psi_k([X],a,...,a) \in \mathcal{F}^{k-1}H_*(\widehat{\mathfrak{C}_*}(LX)) \ . \end{aligned}$$

Since both ψ and λ preserve the filtration and $\psi_1 = p$ we get for

$$b \in \mathcal{F}^{l_b}\widehat{\mathfrak{C}_*}(LX)$$

that

$$\overline{\psi}(b) \in \mathcal{F}^{l_b}H_*(\widehat{\mathfrak{C}_*}(LX))$$

and further that

$$\lambda_k(\overline{\psi}(b),\overline{a},...,\overline{a}) \in \mathcal{F}^{l_b+k-1}H_*(\widehat{\mathfrak{C}_*}(LX)) \ .$$

We deduce that we need some summand of zero action in $D^{\overline{a}}\left(\overline{\psi}(b)\right)$ and so $\psi_1([X])$ must arise as the sum of elements of the form

$$\sum_k \pm\, c_k\ \lambda_k(x_k,\overline{a},...,\overline{a})$$

where we need x_k to have negative action since $\overline{a} \in \mathcal{F}^1 H_*(\widehat{\mathfrak{C}_*}(LX))$. But then x_k and any component of $\overline{a}$ cannot represent positive multiples of the same homotopy class. However, in that case (iv) theorem 4.22 yields

$$\lambda_k(x_k,\overline{a},...,\overline{a}) = 0$$

for $k \geqslant 2$.

We deduce that the assumption (4.30) was wrong and conclude that $X = M \times N$ with the properties as in theorem 4.17 does not embed as a Lagrangian submanifold into $\mathbb{C}^d$. ∎

Chapter 5

Appendix

Basic mathematical concepts and methods frequently used throughout the text are recalled. We assume the reader to be somehow familiar with the upcoming theory. Proofs are thus more or less completely omitted and referred to the literature. The specific literature we rely on is highlighted in each section.

Precisely speaking we recall the following:

1) How are higher homotopy groups defined, what are some of their properties and how may computations be done.

2) We recall the basic notions necessary to define the $H_*^{S^1}(LX)$ sloppily called the homology of the space of strings on X or more seriously speaking the S^1-equivariant homology of LX. We use the Borel construction of equivariant homology. For defining operations on $H_*^{S^1}(LX)$ we need and thus recap the Gysin sequence for a sphere bundle.

3) The pointed loop space is characterized as an H-space. In that sense the homology ring structure of the pointed loop space of the sphere S^n is recalled.

4) We briefly discuss the Leray-Serre spectral sequence for a fibration and as an example compute the cohomology ring structure of BS^1 with these methods. Further the exactness of the Gysin sequence for a sphere bundle is explained.

5.1 (Higher) Homotopy theory

In string topology one studies spaces of maps $S^1 \to M$. At least from a computational perspective it is essential to understand the fundamental group $\pi_1(M)$. Computations are possible since we have methods like long exact sequences or the Whitehead theorem. Both need the concept of higher homotopy theory that we shortly recap in the following. Mainly we rely on [18].

The concept *higher homotopy* ($n \geqslant 1$) is a covariant functor from the category of pointed topological spaces into the category of (abelian for $n \geqslant 2$) groups

$$\pi_n : \mathbf{Top}_\bullet \longrightarrow \mathbf{Grp} \ .$$

With $I = [0,1]$ it is defined on objects by

$$(X, x_0) \longmapsto \{\text{homotopy classes of (continuous) maps } f : (I^n, \partial I^n) \to (X, x_0)\} \ .$$

In that sense π_0 is regarded as

$$\pi_0(X) = \{\text{path-components of } X\}$$

with no boundary condition since $\partial I^0 = \varnothing$. We assume $n > 0$ in the following.

At some point it is helpful to work with a relative version of homotopy groups. For $A \subset X$ we define $\pi_n(X, A, x_0)$ to be the set of homotopy classes of maps

$$f : (I^n, \partial I^n, \overline{\partial I^n \backslash I^{n-1}}) \to (X, A, x_0) \ .$$

It may be regarded as a generalization of the previous definition since for $A = \{x_0\}$ we have

$$\pi_n(X, A, x_0) \equiv \pi_n(X, x_0) \ .$$

The compression lemma yields that $f \sim 0$ holds if $f(I^n) \subset A$. This lemma is enough to prove that there exists a long exact sequence

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0) \xrightarrow{j_*} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots , \tag{5.1}$$

where the homomorphisms are induced by the inclusions

$$i : (A, x_0) \hookrightarrow (X, x_0) \quad \text{and} \quad j : (X, x_0, x_0) \hookrightarrow (X, A, x_0) \ .$$

This in particular proves $\pi_2(\mathbb{C}^d, Y) \cong \pi_1(Y)$ since $\pi_k(\mathbb{C}^d) = 0$ for $k \geqslant 1$.
In contrast to the long exact sequence in homology the connecting homomorphism ∂ indeed comes from a map namely the restriction of

$$f : (I^n, \partial I^n, \overline{\partial I^n \backslash I^{n-1}}) \to (X, A, x_0)$$

to $f : (I^{n-1}, \partial I^{n-1}) \to (A, x_0)$.

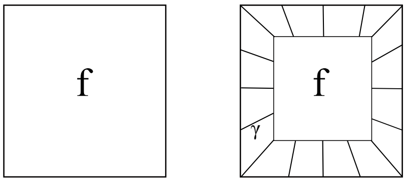
For X path connected we have $\pi_n(X, x_0) \cong \pi_n(X, x_1)$ for all $x_0, x_1 \in X$ and thus define $\pi_n(X) := \pi_n(X, x_0)$. Clearly to make sense of $\pi_n(X, A)$ we need A to be path-connected.
The naturally given action

$$\pi_1(X, x_0) \rightharpoondown \pi_n(X, x_0)$$

allows to consider $\pi_n(X, x_0)$ as a module over $\mathbb{Z}[\pi_1(X, x_0)]$. The stated action is given as follows. For representatives

$$f : (I^n, \partial I^n) \to (X, x_0) \quad \text{and} \quad \gamma : (I^1, \partial I^1) \to (X, x_0)$$

we define γf by shrinking the domain of f and inserting γ. It is visualized as :



Analogously to the absolute case we have an action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$. It is clear that $\pi_1(A, x_0)$ acts on each group of the long exact homotopy sequence (5.1) and further commutes with the homomorphism between them.

For computational purposes we highlight:

Theorem 5.1 (***Prop. 4.2. [18]***)

For a product space $\prod_\alpha X_\alpha$ we have isomorphisms $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ for all n.

In (co)homology many computations are possible since excision leads to a long exact sequence. For homotopy theory this is not true but alternatively long exact sequences arise for fibrations.

Definition 5.2

A fibration is a map $E \xrightarrow{p} B$ such that the homotopy lifting property (HLP) holds: Given maps $X \times [0,1] \xrightarrow{g_t} B$ and $X \times \{0\} \xrightarrow{\widetilde{g}_0} E$ such that

$$\begin{array}{ccccc}
(x,0) & & X \times \{0\} & \xrightarrow{\widetilde{g}_0} & E \\
\big\downarrow & & \big\downarrow i & & \big\downarrow p \\
(x,0) & & X \times [0,1] & \xrightarrow{g_t} & B
\end{array}$$

commutes, there exists a homotopy $X \times [0,1] \xrightarrow{\widetilde{g}_t} E$ such that $p \circ \widetilde{g}_t = g_t$ and $\widetilde{g}_t \circ i = \widetilde{g}_0$.
For $b_0 \in B$ and B path-connected the space $p^{-1}(b_0) =: F$ is called the fiber of the fibration.

Remark 5.3. *(i) Topologically we are allowed to speak of the fiber $F \subset E$ without specifying the corresponding basepoint since there exists a homotopy equivalence $p^{-1}(b_0) \cong p^{-1}(b_1)$ for all b_0, b_1 in the same path component of B.*

(ii) If B is path-connected and E is not the empty set then the map p is surjective.

(iii) Given a fibration $E \xrightarrow{p} B$, any map $B' \xrightarrow{\beta} B$ yields a pullback fibration $E' \xrightarrow{p'} B$, where $E' := B' \times_B E = \{(b', e) \in B' \times E \mid \beta(b') = p(e)\}$ and $p'(b', e) = b'$.

(iv) For B being Hausdorff and paracompact fiber bundles $F \hookrightarrow E \to B$ are always fibrations. Remark that throughout the thesis B is mostly assumed to be a manifold.

(v) If F is discrete, then a fiber bundle is a covering. A covering is a fiber bundle, if all fibres have the same cardinality.

A given map $f : A \to B$ allows to write down a fibration (called the *associated fibration*)

$$p : E_f \to B$$

with fiber F_f such that

$$\begin{tikzcd} A \arrow[dd, "f"'] \arrow[dr, "\iota", "\simeq"'] & \\ & E_f \arrow[dl, "p"] \\ B & \end{tikzcd}$$

commutes and

$$\begin{aligned} \iota : A &\to E_f \\ a &\mapsto (a, \gamma_{f(a)}) \ , \end{aligned}$$

where $\gamma_{f(a)}(t) = f(a)$ for all $t \in I$, is a homotopy equivalence. Here the total space is given by

$$E_f = \{(a, \gamma) \in A \times \underbrace{B^I}_{C^0(I,B)} \mid \gamma(0) = f(a)\}$$

and $p(a, \gamma) = \gamma(1)$ yielding a fiber

$$p^{-1}(b_0) = F_f = \{(a, \gamma) \in A \times B^I \mid \gamma(0) = f(a), \gamma(1) = b_0\} \ .$$

Assuming B to be path-connected, a fibration $E \to B$ with fiber F yields a long exact sequence

$$\cdots \to \pi_n(F, x_0) \xrightarrow{i_*} \pi_n(E, x_0) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F, x_0) \to \cdots \to \pi_0(F, x_0) \xrightarrow{\pi_0(i)} \pi_0(E, x_0) \ , \tag{5.2}$$

induced by (5.1), since $\pi_n(E, F, x_0) \xrightarrow{p_*} \pi_n(B)$ is an isomorphism.
For F discrete (e.g. for a covering) we have $\pi_{n \geqslant 1}(F) = 0$ and thus

- $\pi_n(E, x_0) \cong \pi_n(B)$ for all $n \geqslant 2$
- $\pi_1(p) : \pi_1(E, x_0) \to \pi_1(B)$ injective .

Lens spaces $\mathcal{L}(m; l_1, .., l_n) := S^{2n-1}/\mathbb{Z}_m$ $(n > 1)$ appear at one point of the text. For $m, l_1, ..., l_n \in \mathbb{N}$ fixed and $gcd(l_i, m) = 1$ for all $1 \leqslant i \leqslant n$, the quotient arises by modding out the action on $S^{2n-1} \subset \mathbb{C}^n$ given by

$$\begin{aligned} \mathbb{Z}_m \times S^{2n-1} &\longrightarrow S^{2n-1} \\ (k, (z_1, ..., z_n)) &\longmapsto (e^{\frac{2\pi i k l_1}{m}} z_1, ..., e^{\frac{2\pi i k l_n}{m}} z_n)) \ . \end{aligned}$$

The projection

$$S^{2n-1} \longrightarrow S^{2n-1}/\mathbb{Z}_m$$

serves as an example of a fibration with discrete fiber $\mathbb{Z}_m$. Using the stated long exact homotopy sequence and $\pi_0(S^{2n-1}) = \pi_1(S^{2n-1}) = 0$ we get

$$\pi_i(\mathcal{L}(m; l_1, .., l_n)) \cong \begin{cases} 0 & ;\ i = 0 \\ \mathbb{Z}_m & ;\ i = 1 \\ \pi_i(S^{2n-1}) & ;\ i \geqslant 2 \end{cases} . \tag{5.3}$$

Analogously we could do the same construction starting with the infinite dimensional sphere $S^\infty = \varinjlim_n S^{2n-1}$, yielding $\mathcal{L}(m; l_1, ..) := S^\infty/\mathbb{Z}_m$ and

$$\pi_i(\mathcal{L}(m; l_1, ..)) \cong \begin{cases} \mathbb{Z}_m & ,\ \ i = 1 \\ 0 & ,\ \ i \neq 1 \end{cases} . \tag{5.4}$$

That is $\mathcal{L}(m; l_1, ..)$ is an Eilenberg-MacLane space $K(\mathbb{Z}_m, 1)$.
For homology, remark that one may use a CW structure of $\mathcal{L}(m; l_1, .., l_n)$ respectively $\mathcal{L}(m; l_1, ..)$ given by one cell in each dimension and a boundary map alternating between 0 and multiplication by m. Its homology is thus given by

$$H_i(\mathcal{L}(m; l_1, .., l_n)) \cong \begin{cases} \mathbb{Z} & ,\ \ i = 0, 2n-1 \\ \mathbb{Z}_m & ,\ \ 0 < i < 2n-1 \ \wedge\ i \text{ odd} \\ 0 & ,\ \ \text{else} \end{cases} \tag{5.5}$$

and

$$H_i(\mathcal{L}(m; l_1, ..)) \cong \begin{cases} \mathbb{Z} & ,\ \ i = 0 \\ \mathbb{Z}_m & ,\ \ i \text{ odd} \\ 0 & ;\ \ \text{else} \end{cases} . \tag{5.6}$$

5.2 Universal bundles and Gysin sequence

We rely on discussions presented in [19] and [29]. For B paracompact and Hausdorff, a principal bundle is a fibration. So results of appendix 5.1 may be applied.
For $x_i \in G$ and $t_i \in [0, 1]$ we define

$$EG := \left\{ \langle x, t \rangle = (x_0, t_0, x_1, t_1, ...) \mid \sum t_i = 1, t_i \neq 0 \text{ for finitely many } i \right\} / \sim\ ,$$

where we mod out the equivalence relation

$$\langle x, t \rangle = \langle x', t' \rangle \ \Leftrightarrow\ \forall i : t_i = t'_i \ \wedge\ (t_i = t'_i > 0 :\ x_i = x'_i)\ .$$

The following important facts hold for EG:

- EG has a natural topology such that the G-action

$$\begin{aligned} EG \times G &\to EG \\ ([\langle x, t\rangle], g) &\mapsto [(x_0 g, t_0, x_1 g, t_1, ...)] \end{aligned}$$

  is continuous.

- The G-action on EG is free and thus

$$G \hookrightarrow EG \to BG := EG/G$$

  is a G-principal bundle by a theorem of Gleason (e.g. [15]).

- EG is contractible and thus $\pi_i(BG) \cong \pi_{i-1}(G)$ for $i \geqslant 1$.

The space BG is called the classifying space and $EG \to BG$ the universal bundle of G, since we have the following bijection

$$\begin{aligned} [X, BG] &\xrightarrow{\cong} \{G\text{-bundle over } X/\text{iso.}\} \\ [f] &\longmapsto \left\{f^* EG \to X\right\} \end{aligned}$$

between homotopy classes of maps and isomorphism classes of G-bundles over X. Since we mostly work with $G = S^1$ remark that $ES^1 \simeq S^\infty$ and thus $BS^1 \simeq \mathbb{C}P^\infty$. For a space X with (non-free) G-action we get a free diagonal G-action on $X \times EG$ and thus again by [15] a G-principal bundle

$$G \hookrightarrow X \times EG \to X \times_G EG \ . \tag{5.7}$$

Further the associated fibre bundle is given by

$$\begin{tikzcd} X \arrow[r] & X \times_G EG \arrow[d] \\ & EG/G = BG \end{tikzcd} \quad .$$

In the Borel construction the G-equivariant co-/homology of a space X is given by the co-/homology

$$H_G(X) := H(X \times_G EG) \ . \tag{5.8}$$

G-equivariant maps between spaces X, Y with G-action descend to maps between the G-equivariant co-/homology of X and Y. Further the homotopy property holds that is homotopic G-equivariant maps induce the same maps on co-/homology. The additivity property of non-equivariant co-/homology transfers to G-equivariant co-/homology since

$$\begin{aligned} H_G\left(\bigsqcup_\alpha X_\alpha\right) &= H\left((\bigsqcup_\alpha X_\alpha) \times_G EG\right) = H\left(\bigsqcup_\alpha (X_\alpha \times_G EG)\right) \\ &\cong \bigoplus_\alpha H(X_\alpha \times_G EG) = \bigoplus_\alpha H_G(X_\alpha) \ . \end{aligned}$$

Namely the co-/homology of a disjoint union of spaces is isomorphic to the direct sum of the co-/homology of the particular path-components.

The extreme cases are that G acts either freely or trivially on X. If the action is free we get a fibre bundle

$$\begin{tikzcd} EG \arrow[r] & X \times_G EG \arrow[d] \\ & X/G \end{tikzcd} ,$$

and thus since EG is contractible the G-equivariant co-/homology $H_G(X)$ is given by $H(X/G)$.
If the action is trivial we get $X \times_G EG \cong X \times EG/G$ and thus for field coefficients

$$H(X \times_G EG) \cong H(X) \otimes H(BG) \; .$$

In particular the coefficient group of G-equivariant co-/homology is given by

$$H_G(\mathrm{pt}) \cong H(BG) \; .$$

Recall that an oriented fibre bundle $E \xrightarrow{\pi} B$ with $F = S^n$ yield exact sequences (Gysin sequence)

$$\cdots \to H_i(E) \xrightarrow{\pi_*} H_i(B) \xrightarrow{\cap e} H_{i-n-1}(B) \xrightarrow{\pi^*} H_{i-1}(E) \to \cdots$$
$$\cdots \to H^i(E) \xrightarrow{\pi_*} H^{i-n}(B) \xrightarrow{\cup e} H^{i+1}(B) \xrightarrow{\pi^*} H^{i+1}(E) \to \cdots \; .$$

where we either take the cap respectively the cup product with the Euler class

$$e \in H^{n+1}(B) \; .$$

The morphism $\pi_* : H_*(E) \to H_*(B)$ and $\pi^* : H^*(B) \to H^*(E)$ are the induced maps of π on either homology or cohomology.
For homology

$$\pi^* : H_i(B) \to H_{i+n}(E)$$

is induced by the chain map mapping a cycle $x : K \to B$ to

$$x^*E = K \times_B E \to E$$

which is induced by the pullback fibration. It is indeed a cycle map since $F = S^n$ is closed.
For cohomology the map

$$\int_{S^n} = \pi_* : H^i(E) \to H^{i-n}(B)$$

is the dual map to the map just described. It is the integration along the fibre when working with compact de Rham forms and assuming that E, B are smooth finite dimensional manifolds. In general the cohomological Gysin sequence is easily constructed

out of the E_{n+1}-page of the Leray-Serre spectral sequence for the fibration $E \to B$ as described in appendix 5.4.
Exactness of the Gysin sequences implies

$$\pi_* \circ \pi^* = 0 \ ,$$

and further

$$\pi^* \circ \pi_* =: \Delta : H(E) \to H(E)$$

defines an operator of degree deg $\Delta = +$ dim F for homology and deg $\Delta = -$ dim F for cohomology.
Important for defining operations for strings out of operations for loops via the loop-string fibration is the fact that an operation $\theta : H(E)^{\otimes n} \to H(E)$ for the co-/homology of the total space defines an operation for the co-/homology of the base space via

$$\pm\, \pi_* \circ \theta \circ (\pi^*)^{\otimes n} : H(B)^{\otimes n} \to H(B) \ .$$

Remark that the discussion fits into the concept of equivariant co-/homology for the fiber being S^1 since it is both a Lie group (for equivariant co-/homology) and a sphere (for the Gysin sequence).

5.3 The based loop space

Without diving very deep into the world of loop spaces we recall some basic facts appearing in the text. That is we discuss its H-space structure and the resulting Pontryagin product for its homology. Some easy homology ring computations are recalled. Computational ambitions then directly lead us to spectral sequences, which are reviewed in Appendix 5.4.
Throughout the chapter we rely on concepts presented in [7] and [18].

Definition 5.4

An H-space is a pointed topological space (X, e) equipped with a continuous map $\mu : X \times X \to X$, such that the maps

$$\begin{aligned} X &\longrightarrow X \\ x &\longmapsto \mu(x, e) \\ x &\longmapsto \mu(e, x) \end{aligned}$$

are homotopic relative e to the identity $X \to X$.
A continuous map $f : X \to Y$ between H-spaces (X, e_X, μ_X), (Y, e_Y, μ_Y) is an H-map if

$$f \circ \mu_X \quad \text{and} \quad \mu_Y \circ f^{\times 2}$$

are homotopic relative e_Y. It is further an H-equivalence if there exists an H-map $g : Y \to X$ such that

$$g \circ f \quad \text{and} \quad f \circ g$$

are homotopic relative e_X respectively e_Y to the respective identity maps $\mathrm{id}_X, \mathrm{id}_Y$.

Standard examples of H-spaces are topological groups and based loop spaces.
For a pointed topological space X and

$$\Omega_{x_0} X \equiv \Omega X := \{\gamma \in C^0(S^1, X) \mid \gamma(0) = \gamma(1) = x_0\}$$

the multiplication $\mu(\gamma_1, \gamma_2) = \gamma_1 * \gamma_2$ is defined as the concatenation

$$\gamma_1 * \gamma_2\,(t) := \begin{cases} \gamma_1(2t) & , \quad 0 \leqslant t \leqslant 1/2 \\ \gamma_2(2t-1) & , \quad 1/2 \leqslant t \leqslant 1 \end{cases}$$

This multiplication is clearly only associative and unital up to homotopy given by reparameterization. The unit is given by the constant loop $t \mapsto x_0$.

In the following we work with coefficients in R, a field of characteristic 0. For the rest of this section we assume X and Y to be H-spaces. The H-space multiplication descends to a product on homology, the *Pontryagin product*

$$\bullet : H_*(X) \otimes H_*(X) \xrightarrow{\cong} H_*(X \times X) \xrightarrow{\mu_*} H_*(X)$$

and equips $H_*(X)$ with an algebra structure. The unit is given by $[e]$. Further one has a Künneth type isomorphism between algebras

$$H_*(X \times Y) \cong H_*(X) \otimes H_*(Y)$$

where the Pontryagin product on the tensor product is given by

$$(a \otimes b) \bullet (a' \otimes b') = (-1)^{|a'||b|}(a \bullet a') \otimes (b \bullet b') \ .$$

The cohomology $H^*(X)$ is equipped with the (cup-)product. The Pontryagin product provides a coproduct

$$\Delta : H^*(X) \xrightarrow{\mu^*} H^*(X \times X) \xrightarrow{\cong} H^*(X) \otimes H^*(X)$$

that is compatible with the product. In total we get that $H^*(X)$ is a commutative, associative Hopf algebra (without antipode). This combined with the Theorem of Hopf (cf. Theorem 3C.4. of [18]) then yields:

Theorem 5.5

Let R be a field of characteristic 0. If X is a path connected H-space whose cohomology $H^k(X; R)$ is finite dimensional for all k, then there is an algebra isomorphism

$$H^*(X; R) \cong \Lambda_R[x_1, x_2, ...] = R[x_1, x_2, ...]/(x_i x_j - (-1)^{|x_i||x_j|} x_j x_i) \ .$$

In particular for X finite dimensional we get

$$H^*(X; R) \cong \Lambda_R[x_1, ..., x_l,] \ ,$$

with $|x_i|$ =odd.

A discussion of the theory and a proof of the theorem can be found in [7].
Remark that H-equivalent spaces have isomorphic homology algebras.

<u>Examples:</u>

1) The circle S^1:

As shown in Lemma 2.4 the path-loop fibration yields $\pi_{k+1}(X) \cong \pi_k(\Omega_{p_0}X)$. Thus in particular for the circle we get

$$\pi_k(\Omega S^1) \cong \begin{cases} \mathbb{Z} & ; \quad k=0 \\ 0 & ; \quad \text{else} \end{cases} \qquad \text{and thus} \quad H_*(\Omega S^1;R) \cong \bigoplus_{l\in\mathbb{Z}} R\langle\,[l]\,\rangle\;,$$

where $[l]$ may be represented by $(t \mapsto e^{2\pi i l t}) \in \Omega S^1$. For the product we get

$$[l] \bullet [m] = [l+m]$$

and thus

$$H_*(\Omega S^1;R) \cong R\big[\,t,t^{-1}\,\big]$$

as algebras with $|t| = 0$.

2) The spheres $S^{n\geqslant 2}$:

Out of a given pointed topological space (X,e) we get an H-space $\Omega_e X$. We further get its 'free' H-space JX. The *James reduced product* is defined as

$$JX := \big(\coprod_{k\geqslant 1} X^k\,\big)/\sim\; = \big(\bigcup_{k\geqslant 1} J^kX\big)/\sim$$

where

$$\underbrace{(x_1,...,x_k)}_{\in X^k} \sim \underbrace{(x_1,...,x_i,e,x_{i+1},...,x_k)}_{\in X^{k+1}}$$

and

$$J^kX := X^k/(x_1,...,x_i,e,...x_k) \sim (x_1,...,e,x_i,...x_k)\;.$$

The H-space multiplication is defined as

$$\mu([x_1,...,x_k],[y_1,...,y_l]) = [x_1,...,x_k,y_1,...,y_l]$$

whereas the unit is given by $[e]$.
For spheres we take the standard cell decomposition $S^{n\geqslant 2} = e_0 \cup e_n \equiv e \cup_\partial D^n$. So the quotient map $X^k \to J^kX$ maps cells into cells, namely subcomplexes with one coordinate e are glued. That is

$$
\begin{aligned}
J^1X &= S^n = e_0 \cup e_n \\
J^2X &= S^n \times S^n/(x,e) \sim (e,x) = (e_0 \cup e_n) \times (e_0 \cup e_n)/(x,e) \sim (e,x) = \\
&= J^1X \cup_\partial (S^n - e)^{\times 2} = e_0 \cup e_n \cup e_{2n} \\
&\dots \\
J^kX &= J^{k-1}X \cup (S^n - e)^k = e_0 \cup e_n \cup \dots \cup e_{kn} \ .
\end{aligned}
$$

We deduce that $JS^n = e_0 \cup e_n \cup e_{2n} \cup \dots$ is a CW complex and by dimension reasons the cellular boundary map is 0 for $n \geqslant 2$. Therefore

$$
H_*(JS^n) \cong \begin{cases} R & , \quad * = i \cdot n \ (i \geqslant 0) \\ 0 & , \quad \text{else} \end{cases} \cong \bigoplus_i R\langle\, [e_{in}] \,\rangle \ .
$$

For computing the algebra structure with respect to the Pontryagin product we compute $[e_{in}] \bullet [e_{jn}]$. Represent the homology classes by

$$
i : (\Delta^{in}, \partial\Delta^{in}) \to (e_{in}, e_0) \ , \ j : (\Delta^{jn}, \partial\Delta^{jn}) \to (e_{jn}, e_0) \ .
$$

For the product we then get

$$
\begin{aligned}
\Delta^{in} \times \Delta^{jn} &\longrightarrow JS^n \times JS^n \xrightarrow{\ \mu\ } JS^n \\
(x,y) &\longmapsto (i(x), j(y)) \longmapsto i(x)j(y) \ ,
\end{aligned}
$$

that is on homology $[e_{in}] \bullet [e_{jn}] = [e_{(i+j)n}]$. We conclude with the Pontryagin algebra structure

$$
H_*(JS^{n \geqslant 2}) \cong R[\, u \,] \tag{5.9}
$$

with $|u| = n$. The James reduced product relates to pointed loop spaces as follows:

For a pointed topological space (X, e) we can defined its reduced suspension

$$
\Sigma X := X \times I/(X \times \partial I) \cup (e \times I)
$$

and get a map into its pointed loop spaces

$$
\begin{aligned}
\lambda : X &\to \Omega_{[e]}\Sigma X \equiv \Omega\Sigma X \\
x &\mapsto \lambda(x)(\cdot)
\end{aligned}
$$

where $\lambda(x)(t) := (t \mapsto [x,t])$. This generalizes to an map of H-spaces

$$
\begin{aligned}
\lambda : JX &\to \Omega\Sigma X \\
[x_1, \dots, x_k] &\mapsto (\lambda(x_1) * \dots * \lambda(x_k))(\cdot) \ .
\end{aligned}
$$

By Theorem 4J.1. of [18] this is further a weak homotopy equivalence for X being a CW complex. Since λ is compatible with the H-space products this then further yields

$$
H_*(JX) \cong H_*(\Omega\Sigma X)
$$

as algebras. For spheres we have a homeomorphism

$$\Sigma S^n \cong S^{n+1} \ .$$

This and result (5.9) are then used to prove that

$$H_*(\Omega S^{n+1}) \cong R\big[\, u \,\big] \text{ with } |u| = n \ , \tag{5.10}$$

where u is represented by an explicit cycle of loops in S^{n+1}, cf. section 4.3.

A more systematic method to compute (co-)homology groups and certain products also for free loop spaces is provided by spectral sequences which are briefly discussed in the next section.

5.4 Spectral sequences

We recall some basic facts about spectral sequences for a double positively graded complex. In the thesis we need them to do computations for fibrations and thus ideas are exemplified by means of the Leray-Serre spectral sequence.
All presented ideas can be found in detail in [2] or [30]. We also profit from ideas presented in [7].

For a graded R-module

$$K = \bigoplus_{n \geqslant 0} K^n$$

with a linear map D is a graded complex if $D(K^n) \subset K^{n+1}$ and $D^2 = 0$. So cohomology with respect to D is defined. It is a filtered complex if a (decreasing) filtration of sub-complexes

$$K = K_0 \supset K_1 \supset K_2 \supset \cdots$$

exists. If it is both graded and filtered one gets an induced filtration

$$K^n = K_0^n \supset K_1^n \supset K_2^n \supset \cdots$$

for each dimension n by setting $K_p^n := K_p \cap K^n$. Inclusion and projection induces an exact sequence

$$0 \to K_{p+1}^n \xrightarrow{i} K_p^n \xrightarrow{j} K_p^n / K_{p+1}^n \to 0$$

that can be reinterpreted as an exact triangle. Its long exact sequence on cohomology can also be written as an exact triangle

$$\begin{tikzcd}[column sep=small]
A_1 := \bigoplus_{n \geqslant 0, p \geqslant 0} H(K_p^n) \arrow[rr, "i_1 = i_*"] & & \bigoplus_{n \geqslant 0, p \geqslant 0} H(K_p^n) = A_1 \arrow[dl, "j_1 = j_*"] \\
& E_1 := \bigoplus_{n \geqslant 0, p \geqslant 0} H(K_p^n / K_{p+1}^n) \arrow[ul, "k_1 = k"] &
\end{tikzcd} \ . \tag{5.11}$$

It yields another well-defined exact triangle

$$\begin{tikzcd}
k_1(e) & A_2 := i_1(A_1) \arrow[rr, "i_2 := i_1|_{i_1(A_1)}"] & & A_2 \arrow[dl, "j_2"] & i_1(a) \arrow[dl, mapsto] \\
& {[e]} \arrow[ul, mapsto] & E_2 := H(E_1, d_1 := j_1 \circ k_1) \arrow[ul, "k_2"'] & {[j_1(a)]} & .
\end{tikzcd}$$

Deriving exact triangles from given ones can be done infinitely often.

A *spectral sequence* is a sequence of differential complexes (E_r, d_r) with $E_{r+1} = H(E_r, d_r)$. It *stabilizes* if $E_{l+1} = E_{l+2} = \cdots =: E_\infty$ and *converges to* $H(K)$ if

$$E_\infty \cong \bigoplus_p H(K)_p / H(K)_{p+1}$$

for the induced cohomology filtration $H(K) = H(K)_0 \supset H(K)_1 \supset H(K)_2 \supset \cdots$ given by $H(K)_p := (i_*)^p H(K_p)$. If $H(K)$ is a vector space over a field $\mathbf{k}$ we have

$$\bigoplus_p H(K)_p / H(K)_{p+1} \cong H(K) \ .$$

Theorem 5.6 (*e.g. Theorem 14.6. in [2]*)

If the filtration has finite length l_n, that is

$$K^n = K^n_0 \supset K^n_1 \supset \cdots \supset K^n_{l_n} \supset K^n_{l_n+1} = 0$$

for each dimension $n \geqslant 0$, the induced spectral sequence stabilizes and converges.

We are in the situation required in the theorem when considering a double (bigraded) complex

$$K = \bigoplus_{p,q \geqslant 0} K^{p,q}$$

with differentials

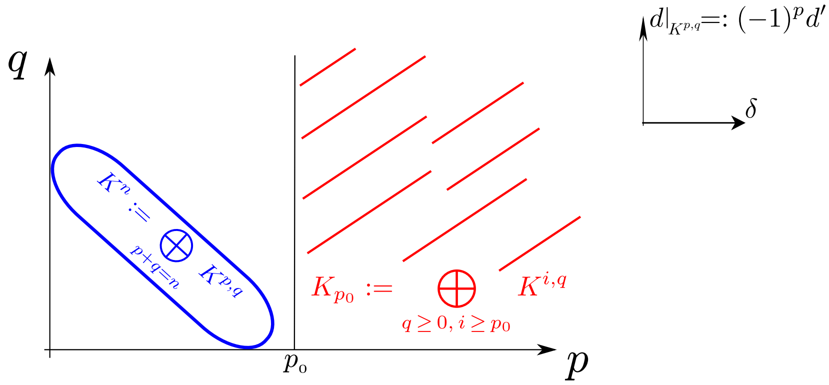
$$\delta : K^{p,q} \to K^{p+1,q} \quad \text{and} \quad d' : K^{p,q} \to K^{p,q+1}$$

such that $(d')^2 = 0$, $\delta^2 = 0$ and $d' \circ \delta = \delta \circ d'$.

It yields a single graded filtered complex $(K = \bigoplus_{n \geqslant 0} K^n, D)$ with $K^n := \bigoplus_{p+q=n} K^{p,q}$ and filtration $K_{p_0} := \bigoplus_{q \geqslant 0, i \geqslant p_0} K^{i,q}$ of finite length in each dimension. The definitions are best illustrated as in figure 5.1.

Finite length is given since $K^n_p = K^n \cap K_p = 0$ for $p > n$. For the E_1-page (E_1, d_1) we get

$$E_1 = \bigoplus_{p \geqslant 0} H(K_p / K_{p+1}, D) = \bigoplus_{p,q \geqslant 0} H(K^{p,q}, d') =: \bigoplus_{p,q \geqslant 0} E_1^{p,q}$$



$$D := \delta + d : K^n \to K^{n+1}$$

Figure 5.1: Induced single complex

since $\delta|_{K_p/K_{p+1}} = 0$. For $d_1 = j_1 \circ k_1 : H(K_p/K_{p+1}) \to H(K_{p+1}/K_{p+2})$ we get

$$[a] \mapsto [Da] = [\delta a]$$

since k_1 is the connecting homomorphism of the long exact sequence and thus

$$E_2^{p,q} = H^p(H^{\cdot,q}(K,d),\delta) \ .$$

This principle is manifested in the zig-zag Lemma as described in [2].

Lemma 5.7 ***(§14 of [2])***

For $x_0 \in K^{p,q}$ one has

$$[x_0]_{k+1} \in E_{k+1}^{p,q} \Leftrightarrow \exists\, k\text{-zig-zag } (x_0, ..., x_k)$$
$$\text{i.e. } dx_0 = 0,\ \delta x_l = (-1)^{p+1} dx_{l+1}\ (l < k)$$

and further

$$d_{k+1}[x_0]_{k+1} = [\delta x_k]_{k+1} \in E_{k+1}^{p+k+1,q-k} \ .$$

Our motivation for studying spectral sequences are (co-)homological computations for fibrations $F \hookrightarrow E \xrightarrow{\pi} B$ for F, E, B being CW-complexes and B being path-connected. For $\underline{U} = \{U_\alpha\}_{\alpha\in I}$ being a cover of B we define a double complex

$$K^{p+1,q} \xleftarrow{\delta} K^{p,q} \xrightarrow{d} K^{p,q+1} \ ,$$

with $K^{p,q} := C^p(\pi^{-1}(\underline{U}), C^q)$ being the p-th Čech cochain group with values in the presheaf of singular q-cochains. This set-up yields a spectral sequence. As described in the literature, if $\pi_1(B)$ acts trivially on $H^q(F)$ we have

$$H^q(\pi^{-1}(\underline{U})) \cong H^q(F)$$

if $\underline{U}$ is a good cover of B, that is it is locally finite and non-empty intersections $U_{\alpha_1} \cap \cdots \cap U_{\alpha_r}$ are diffeomorphic to $\mathbb{R}^n$.

Following chapter 5 of [30] for the corresponding spectral sequence we get:

- $E_1^{p,q} = C^p(\underline{U}, H^q(F))$
- $E_2^{p,q} = H^p(\underline{U}, H^q(F))$
- (E_r, d_r) converges to $H^*(E)$
- The universal coefficient theorem yields

$$E_2^{p,q} \cong H^p(B) \otimes H^q(F)$$

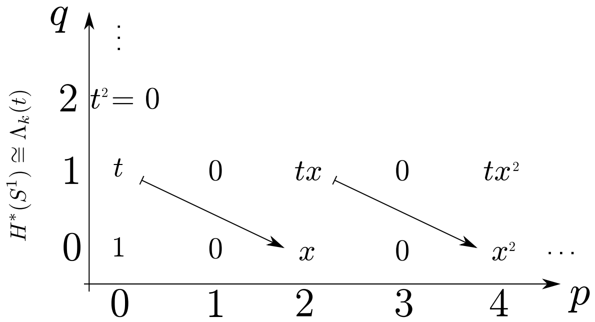
  if we use field coefficients and $H^q(F)$ is finite dimensional for all q.

Analogously we could work with chains instead of cochains and would get the same statements for homology. For the cup product on cohomology or the loop product on $\mathbb{H}_*(LX)$, the statement generalizes in a way such that module isomorphisms become algebra isomorphisms. For this we refer to [10] and [30]

As an example consider $S^\infty \to \mathbb{C}P^\infty$ as a realization of the universal S^1-bundle. Contractibility of S^∞ yields

$$E_{k\geqslant 3}^{p,q} = E_\infty^{p,q} = \begin{cases} 0 & , \quad (p,q) \neq (0,0) \\ \mathbf{k} & , \quad (p,q) = (0,0) \end{cases}$$

when using coefficients in a field $\mathbf{k}$. This follows by degree reasons whereas the E_2-page is given by



which implies

$$H^*(BS^1; \mathbf{k}) \cong H^*(\mathbb{C}P^\infty; \mathbf{k}) \cong \mathbf{k}\,[x]$$

with $|x| = 2$.

The class $x \in H^2(\mathbb{C}P^\infty; \mathbf{k})$ is known as the Euler class which can be easily defined for general sphere bundles using spectral sequences. The exactness of the previously mentioned Gysin sequence is also straightforward.
Both statements can be seen as follows:

In general for a fibration $F \hookrightarrow E \to B$ an element $\omega \in H^n(F)$ is called *transgressive* if

$$d_2(\omega) = \cdots = d_n(\omega) = 0 \ .$$

Since $p, q \geqslant 0$ we have $d_{k\geqslant n+2}(\omega) = 0$ by degree reasons. In this situation, the map $\omega \mapsto d_{n+1}(\omega)$ is called the *transgression map*.

For an oriented S^n-bundle $\pi_1(B)$ acts trivially on $H^q(S^n)$ and we have

$$E_2^{0,*} = H^*(S^n; \mathbf{k}) \cong \mathbf{k}\,[\omega]/(\omega^2) \ .$$

Thus by degree reasons $E_2 = \cdots = E_{n+1}$ and $E_{n+2} = \cdots = E_\infty$. So for computing $H^*(E)$ we just need to understand

$$d_{n+1}(\omega) =: e \in H^{n+1}(B) \ ,$$

called the Euler class of the bundle $E \to B$. We immediately get that a trivial sphere bundle has a vanishing Euler class.
In total the differential d_{n+1} on E_{n+1} is given by

$$\begin{aligned} H^p(B) \otimes H^n(S^n) &\longrightarrow H^{p+n+1}(B) \otimes H^0(S^n) \\ x \otimes \omega &\longmapsto (x \cup e) \otimes 1 \ . \end{aligned}$$

For coefficients in a field it yields $H^*(E) \cong \ker(\cdot \cup e) \oplus H^*(B)/\mathrm{im}(\cdot \cup e)$ which may be interpreted as

$$\cdots \to H^i(E) \xrightarrow{\pi_*} H^{i-n}(B) \xrightarrow{\cup e} H^{i+1}(B) \xrightarrow{\pi^*} H^{i+1}(E) \to \cdots$$

where π_* is the projection to $\ker(\cdot \cup e)$ and $\pi^* : H^*(B) \to H^*(B)/\mathrm{im}(\cdot \cup e)$. This is the already mentioned Gysin sequence that is clearly exact.

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Eidesstattliche Versicherung

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 6. November 2016

Johannes Huster

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