# Fiscal Policy in General Equilibrium

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**Dennis Wesselbaum**

aus Lübeck

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# Fiscal Policy in General Equilibrium

Approved by:

________________________________

Professor Anke Gerber, Committee Chair

________________________________

Professor Olaf Posch, First Reader

________________________________

Professor Mu-Chun Wang, Second Reader

________________________________

Professor Ester Faia, Second Supervisor

Date Approved: November 19, 2015

# PREFACE

The recent financial and economic crisis - known as the Great Recession - has shown that disturbances generated in financial markets can create large adverse spill-over effects towards the real economy: A turmoil in the financial and banking system generated a decrease in real and financial wealth which resulted in a drop in aggregate demand and economic activity.

Furthermore, a significant impact factor during the crisis was uncertainty that increased precautionary savings and a cautious behavior of private agents and banks. In particular, the Euro (debt) crisis was referred to a crisis in expectation about the sustainability of debt levels of several European countries. In the United States the so-called 2011 and 2013 debt ceiling crises raised the awareness about the structural budget imbalances. Along this line, given that many countries face high levels of debt and reveal problems due to high borrowing costs and a short maturity of debt, the role of government debt became a centre of attraction. In the words of former Chairman Bernanke (2010): "Amid all of the uncertainty surrounding the long-term economic and budgetary outlook, one certainty is that both current and future Congresses and Presidents will have to make some very tough decisions to put the budget back on a sustainable trajectory."

To counter the recessionary pressures governments and central banks around the world responded with all available measures. Many countries relied on tremendously large fiscal policy measures trying to counter the recessionary forces. For example, the United States economic stimulus package, the American Recovery and Reinvestment Act (ARRA, for short), was worth roughly 831 Billion U. S. Dollar in total. This Act follows the canonical countercyclical Keynesian viewpoint that a drop in aggregate demand can be offset by an increase in government spending.

At the same time, central banks around the world started reducing nominal interest rates close to the zero lower bound. Hence, they found themselves in a position in which the classical interest rate transmission channel was muted. As a consequence, many of them, e.g. the Federal Reserve Bank (FED, for short) and the Bank of England, engaged in so-called non-standard monetary policies, e.g., using tools that affect the composition of the central bank's balance sheet.

Overall, the crisis and the manifold, simultaneous policy responses resuscitated the interest in the design of the policy mix, the coordination of monetary and fiscal policy, as well as long-run effects of fiscal policy.[1] Although most developed countries established independent authorities that determine fiscal and monetary policies, interactions between those are omnipresent. Intuitively, each policy affects the effectiveness of the other and, hence, the overall effectiveness of the policy mix. Some prominent examples are credit crowding-out and wage-price spirals created by the effects of taxes on prices. Further, the expectation channel can have adverse effects on the stability of financial markets and the macroeconomy.

More technically, the root of those interactions can be traced to the government budget constraint. This identity creates a dynamic link between the budgetary position and interest rates. This equation is the key in the fiscal theory of the price level (FTPL, for short). In a nutshell, this theory claims that the price level is solely determined by government instruments. As a central point, again, the government budget constraint is interpreted as an equilibrium restriction that leads to price level changes when fiscal variables change. Put differently, the government chooses a strategy for fiscal policy, that is, it chooses the path of surplus and debt. Then, conditional on

[1] Those long-run effects could be created by supply-side fiscal measures or by demand-side measures that affect growth through an endogenous growth channel. Further, the non-standard monetary policy measures need not be neutral in the long-run as monetary measures working along the interest channel usually are.

this path of actions, the monetary authority chooses the path of the interest rate. Hence, the combination of fiscal and monetary policy (interactions) pins down the price path and a stable path is only achieved for some combination of policies. In conclusion, the FTPL points out that the monetarist point of view does only hold under strong assumptions on the behavior of the fiscal authority. However, monetary policy still controls the nominal interest rate and can create real effects.

A further dimension that has to be considered when talking about the policy mix is time-variability. More than monetary policy, fiscal policy is subject to swings in political preferences. While an often stressed example for a switch in preferences of central bankers is the era of Paul A. Volcker as Chairman of the FED. Examples for switches in the conduct of fiscal policy can easily be found in the history of the United States. Consider, for example, the military spending programs by the Presidents Johnson, Carter, Reagan, and G. W. Bush as well as the tax cuts by Ford and Reagan. Further, fiscal policy under President Clinton can be considered to be motivated by fiscal stabilization, while the Bush tax cuts in the early 2000's resuscitated the discretionary fiscal policy design. A dramatic consequence of this policy switch was the return of government deficits and a faster accumulation of government debt. Along this line, a growing body of empirical work shows that monetary as well as fiscal policy is subject to regime changes over time. Given the evidence for switches in fiscal and monetary policy, switches in the interactions between the two policies are an implication by the interrelatedness of both policies in the policy mix.

A different motivation for time-dependence of policy interactions are intertemporal financing implications. The way a given fiscal policy expenditure will be financed and the way the monetary authority behaves in the future drives the effects of current policies. Here, regime-switches can trigger wealth effects as well as intra- and intertemporal substitution effects. More importantly, those effects can be generated by the pure expectation of regime switches. Since switches in fiscal and monetary regimes occurred in the past and are likely to happen in the future, agents will form expectations about possible regimes and, hence, their decisions will depend on the implied probability distribution.

While the classical transmission channels of monetary policy are well understood, the effects and transmission mechanisms of fiscal policy, including government debt, as well as the interactions with monetary policy are controversially discussed. The revival of Keynesian stabilization policies during the Great Recession pursued by almost all governments of the developed world, the concerns about the sustainability of government budgets and the design of the policy mix of fiscal and monetary measures imply that more research on the behavior of fiscal policy and the interaction between fiscal and monetary policy is needed. A deeper understanding of those issues will be highly beneficial for policy makers today, to put the budget on a stable trajectory, and tomorrow, when a new recession needs to be dealt with.

The thesis is structured as follows. Chapter one identifies a new transmission channel for fiscal policy. Chapter two estimates, microfounds, and quantifies regime changes in the interaction between fiscal and monetary policy. Finally, chapter three develops a continuous time growth model to discuss the effects of implementation lags in government investment.

# SUMMARY

In the first of the three chapters of my dissertation I challenge the conventional Keynesian view that countercyclical fiscal policy stabilizes real variables over the business cycle. I present empirical evidence that government debt moves procyclical with output in the United States using a structural vector error correction model (VECM, for short). While this finding might be straightforward in terms of correlation, the SVECM evidence proofs that there is indeed a causal relationship. Then, I model fiscal policy via fiscal rules with feedback to endogenous variables. Calibrating those rules with coefficients in line with procyclical debt gives us sizably lower standard deviations compared to a model with coefficients that would generate countercyclical debt. The reason for this finding is a wealth channel that emerges in my model because of the introduction of a perpetual-youth structure. Hence, the Ricardian equivalence is broken and movements in debt affect household's wealth and, therefore, the consumption-leisure decision. This wealth channel proofed to be particularly important in the Great Recession and my analysis suggests a new way governments can generate wealth effects, by using government debt as an automatic stabilizer.

The main contribution of this chapter is theoretical. I show that government debt, as being a component of household's financial wealth, creates an additional wealth effect that has sizable effects on the business cycle. I interpret this channel as an additional automatic stabilizer of economic activity. Therefore, I present a new channel through which governments can influence cyclical fluctuations and contribute to macroeconomic stability. This channel emerges from combining Blanchard

(1985) - Yaari (1965) consumers and fiscal rules. The former implies that debt affects household decisions and the latter allows debt to be a function of output. The striking and provocative consequence is that classical (countercyclical) Keynesian fiscal policy destabilizes the business cycle in this basic framework. Remarkably, this channel plays a role for the propagation of all shocks that affect output and, hence, is important even in the absence of exogenous fiscal policy innovations.

The second chapter addresses the interactions between monetary and fiscal policy. Empirically, I show that the FED's policy is affected by the stance of fiscal policy. I do so by estimating a state-of-the art Taylor-type interest rate rule. Then, I estimate Markov-switching models allowing for time-varying transition probabilities showing that those interactions vary over time between accommodative and non-accommodative regimes. Along the theoretical dimension, I use a cheap talk game between central bank and government to microfound policy interactions and regime switches. Exogenous (or, potentially, endogenous) changes in the expectation of agents trigger policy shifts. For example, if a Ricardian government increase government spending this might trigger the expectation that the government becomes Non-Ricardian. Since debt matters for the conduct of monetary policy, the central bank reacts by changing its responsiveness to debt in the Taylor rule. Put differently, changes in the prior beliefs within this game, the pendant to the estimated Markov-switching probabilities, can trigger different outcomes and, hence, different weights in the Taylor-rule. This will have effects on the transmission of shocks and, hence, on the quantitative and qualitative results.

Lastly, I present a case study to show how to implement this cheap talk (or any type of finite sequential or simultaneous move) game in a state-space dynamic stochastic general equilibrium model (DSGE, for short), simulate this model, and discuss the effects of regime switches. My solution approach allows including the game structure explicitly in the model. That is to say, the sequential move game, its solution algorithm respectively, is directly implemented in the state-space of our model; something that is a novelty in DSGE modelling. This guarantees a high degree of flexibility for modelling those interactions, allows using this approach for a wide range of problems, and also allows the analysis of repeated games.

Chapter three, joint work with Olaf Posch from the University of Hamburg and Santanu Chatterjee from the University of Georgia, discusses the (growth) effects of implementation delays in the accumulation of the public capital stock. Government expenditures into public capital is considered superior to wasteful government consumption expenditures as they trigger supply-side effects. While developed countries use government investment expenditures to counter adverse effects of Recessions and to foster growth, developing countries use investment into public capital to remove the bottlenecks for economic growth. Public infrastructure programs, in particular, are subject to large implementation delays (or lags) due to the required planning, bidding, contracting, and construction process. We add to the literature on fiscal policy in endogenous growth models by building a stochastic endogenous growth model in continuous time with public capital. In this model, implementation lags generate uncertainty in the public capital accumulation process: the government continuously spends but the completion of the public investment project is unknown. We provide a numerical solution calibrated on the U. S. economy. We find that the implementation lags in the accumulation of public capital have sizable effects on agents' behavior. Then, we evaluate the effects of three policy reforms. We find that an increase in government expenditures raises the growth rate while an increase in the income tax rate reduces the growth rate. We then consider a policy reform exclusive to our model, namely a reallocation of government expenditures towards projects not associated with implementation lags. We find that such a policy increases the growth rate. While the effects are smaller compared to the increase in government spending, the main advantage of this policy reform is that it does not generate additional costs.

# SUMMARY

Das erste Kapitel meiner Dissertation hinterfragt den konventionellen keynesianischen Standpunkt, dass antizyklische Fiskalpolitik reale Variablen über den Konjunkturzyklus stabilisiert. Ich präsentiere empirische Ergebnisse eines strukturellen VECM Modells die zeigen, dass Staatsschulden in den Vereinigten Staaten prozyklisch mit dem Produktionsniveau verlaufen. Dieses Ergebnis mag wenig überraschend sein, wenn man Korrelationen betrachtet, allerdings zeigt die Analyse einen kausalen Zusammenhang auf. Anschließend modelliere ich Fiskalpolitik als Regeln, die durch endogene Modellvariablen beeinflusst werden. Die Koeffizienten dieser Regeln werden so kalibriert, dass sie mit dem prozyklischen Verhalten der Staatsschulden übereinstimmen. Das Ergebnis ist, dass die Standardabweichung wichtiger makroökonomischer Variablen deutlich niedriger ist, verglichen mit Regeln kalibriert auf antizyklischer Staatsverschuldung. Der Grund ist ein Vermögenseffekt, der in diesem Modell aufgrund der „perpetual-youth“ Struktur der Agenten entsteht. Daher hält die Ricardianische Äquivalenz nicht mehr und Veränderungen der Staatsschulden haben einen Effekt auf das Vermögen der Haushalte und, folgerichtig, auf ihre Konsum-Freizeit Entscheidung. Dieser Vermögenskanal war insbesondere in der Great Recession bedeutend. Meine Analyse zeigt einen neuen Weg für fiskalpolitische Entscheidungsträger auf, Vermögenseffekte zu erzeugen und zwar dadurch, das Staatsschulden als automatischer Stabilisator wirken.

Der Hauptbeitrag dieses Kapitels ist theoretisch. Ich zeige auf, das Staatsschulden, als Komponente des Vermögens der Haushalte, einen zusätzlichen Vermögenseffekt erzeugen, der signifikante Effekte auf den Konjunkturzyklus hat. Ich interpretiere

diesen Kanal als zusätzlichen automatischen Stabilisator ökonomischer Aktivität. Daher präsentiere ich einen neuen Kanal durch welchen die fiskalpolitischen Entscheidungsträger zyklische Fluktuation beeinflussen können und makroökonomische Stabilität unterstützen können. Dieser Kanal entsteht durch die Kombination von Blanchard (1985) - Yaari (1965) Konsumenten und fiskalpolitischen Regeln. Die Ersteren implizieren, dass Staatsschulden einen Effekt auf die optimalen Entscheidungen der Haushalte haben, während die Letzteren einen Zusammenhang zwischen Schulden und Produktionsniveau herstellen. Die provokante Konsequenz ist, dass klassische, antizyklische keynesianische Fiskalpolitik den Konjunkturzyklus in meinem Modellrahmen destabilisiert. Bemerkenswerterweise spielt dieser Kanal eine wichtige Rolle bei allen exogenen Schocks, die das Produktionsniveau beeinflussen. Daher ist dieser Kanal auch ohne exogene fiskalpolitische Innovationen bedeutend.

Das zweite Kapitel der Dissertation beschäftigt sich mit den Interaktionen von Geld- und Fiskalpolitik. Eine empirische Analyse zeigt, dass die Politik der FED von dem gegenwärtigen Stand der Fiskalpolitik beeinflusst wird. Zunächst schätze ich die Parameter einer Taylor-Zinsregel. Der Schwerpunkt der empirischen Analyse ist allerdings die Schätzung zweier Markov-switching Modelle mit zeitvariablen Übergangswahrscheinlichkeiten. Die Analyse zeigt, dass die Interaktion zwischen Geld- und Fiskalpolitik sich über die Zeit verändert und zwischen unterstützenden und nicht-unterstützenden Regimen wechselt.

Der theoretische Beitrag dieses Kapitel basiert auf der Anwendung eines Cheap Talk Spiels zwischen der Zentralbank und dem Staat um die Interaktion der Politiken sowie die Regimewechsel zu mikrofundieren. Exogene (potentiell endogene) Erwartungsänderungen lösen Politikänderungen (Regimewechsel) aus. Wenn zum Beispiel eine ricardianische Regierung die Staatsausgaben erhöht kann dies die Erwartungen auslösen, dass die Regierung sich nun nicht-ricardianisch verhält. Da die Staatsschulden einen Effekt auf die Geldpolitik haben, ändert die Zentralbank ihr

Reaktionsverhalten auf Schulden in der Taylor-Regel. Anders gesagt, Veränderungen in den Erwartungen dieses Spiels, dem Pendant der geschätzten Markov-switching Wahrscheinlichkeiten, kann in unterschiedlichen Gleichgewichten des Spiels führen und, daher, zu unterschiedlichen Reaktionskoeffizienten in der Taylor-Regel der Zentralbank. Dies hat einen Einfluss auf die Transmission von exogenen Innovationen und, daher, auf die qualitativen und quantitativen Effekte.

Zum Schluss entwickle ich ein Beispiel, dass zeigt, wie man das beschriebene Cheap Talk Spiel in einem State-Space DSGE Model einbauen und simulieren lässt. Mein Lösungsansatz erlaubt es, dass Spiel explizit in der State-Space Darstellung des Modells zu berücksichtigen. Dies ermöglicht eine hohe Flexiblität der Modellierung von Interaktionen, die über die Beschriebene hinausgeht.

Kapitel drei, ein gemeinsames Projekt mit Olaf Posch von der Universität Hamburg und Santanu Chatterjee von der University of Georgia, beschäftigt sich mit den Wachstumseffekten von Verzögerungen in der Umsetzung von staatlichen Investitionsprojekten. Staatsausgaben, die den öffentlichen Kapitalstock erhöhen werden als überlegen zu puren staatlichen Konsumausgaben angesehen, da sie Effekte entlang der Angebotsseite erzeugen. Während entwickelte Länder diese Ausgaben nutzen um die negativen Effekte von Rezession abzumildern und um ihr Wachstum zu stärken, nutzen Entwicklungsländer diese Ausgaben um die Voraussetzungen für Wachstum zu schaffen. Staatliche Investitionsprojekte sind insbesondere durch lange Umsetzungsverzögerungen gekennzeichnet durch die erforderliche Planung, Ausschreibung, Vertrags- und Bauphase.

Wir ergänzen die Literatur über Fiskalpolitik in endogenen Wachstumsmodellen in dem wir ein stochastisches, endogenes Wachstumsmodell in kontinuierlicher Zeit mit öffentlichen Kapital entwickeln. Dieses Model enthält darüber hinaus Verzögerungen in der Umsetzung von staatlichen Investitionsprojekten. Diese erzeugen eine

Unsicherheit bei den Agenten über den Akkumulationsprozess des öffentlichen Kapitalstocks. Während der Staat ununterbrochen Ausgaben tätigt, so ist die Umsetzung der Investitionsprojekte unbekannt.

Wir kalibrieren die Parameter auf die U. S. Wirtschaft und lösen das Modell numerisch. Wir zeigen, dass die Verzögerungen in dem Akkumulationsprozess des öffentlichen Kapitalstocks signifikante Effekte auf das Verhalten der Agenten haben. Des weiteren nutzen wir das Modell um die Effekte von drei politischen Reformen zu evaluieren. Ein Anstieg in den Staatsausgaben führt zu einem Anstieg der Wachstumsrate in unsere Modellökonomie, während ein Anstieg der Einkommenssteuer zu einer Senkung der Wachstumsrate führt. Zum Schluss betrachten wir eine Reform, die ausschließlich in unserem Modell betrachtet werden kann: eine Reallokation der Staatsausgaben zugunsten von Projekten, die keine Verzögerungen aufweisen. Wir zeigen, dass eine solche Reform die Wachstumsrate erhöht. Zwar sind die Effekte dieser Reform niedriger als die Effekte einer Erhöhung der Staatsausgaben, allerdings liegt der Vorteil dieser Reform darin, das sie keine neuen Ausgaben erzeugt.

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# CHAPTER I

# PROCYCLICAL DEBT AS AUTOMATIC STABILIZER

## *1.1 Introduction*

The Great Recession has shown that turmoils in financial markets can create significant adverse effects towards the real economy. Furthermore, a significant impact factor during the crisis was uncertainty that increased precautionary savings and generated distrust in the banking system. In particular, the Euro (debt) crisis was referred to a crisis in expectation about the sustainability of debt levels of many European countries. In the United States the so-called 2011 and 2013 debt ceiling crises raised the awareness about the structural budget imbalances. Along this line, given that many countries face high levels of debt and reveal problems due to high borrowing costs and a short maturity of debt, the role of government debt became a centre of attraction.

In this paper, we aim at investigating the relation between debt, debt policy, and the business cycle. We focus on the conditions under which debt policy stabilizes the business cycle. For this purpose, we build an otherwise canonical real business cycle model of the U. S. economy with Non-Ricardian agents. We use a perpetual-youth structure following the work of Blanchard (1985) and Yaari (1965) to break the Ricardian equivalence and fiscal rules to characterize the behavior of the fiscal authority. Those rules describe the evolution of taxes and government spending over the cycle and feature feedback on government debt and output. Therefore, they capture two major incentives for fiscal authorities, viz. to stabilize business cycle fluctuations and to keep debt on a sustainable path. Further, fiscal rules are tools that allow us to generate pro- and countercyclical government debt.

The main contribution of this paper is theoretical. We show that government debt, as being a component of household's financial wealth, creates an additional wealth effect that has sizable effects on the business cycle and that can be affected by the policy maker. Wealth effects proofed to be particularly important in the Great Recession where disturbances generated in financial markets created large adverse spill-over effects towards the real economy: A turmoil in the financial and banking system generated a decrease in real and financial wealth which resulted in a drop in aggregate demand and economic activity. We find that the debt channel in our model is an additional automatic stabilizer of economic activity. What needs to be stressed is that the automatic stabilizer component in the fiscal rules should not be mistaken with the (automatic) stabilizing effect of debt. Therefore, we present a new channel through which governments can influence cyclical fluctuations and contribute to macroeconomic stability.[1] Technically, this channel emerges from combining Blanchard (1985) - Yaari (1965) consumers and fiscal rules. The former implies that debt affects household decisions and the latter allows debt to be a function of output. The striking and provocative consequence is that classical (countercyclical) Keynesian fiscal policy destabilizes the business cycle in this basic framework. Remarkably, this channel plays a role for the propagation of all shocks that affect output and, hence, is important even in the absence of exogenous fiscal policy innovations. Along this line, the policy implication for a country exiting a recession, for example recently done in Greece, Portugal, and Spain, is that an increase in output accompanied by an increase in debt cannot, generally, be demonized.

In detail, the contribution of this paper is twofold. First, we present robust empirical evidence on the long-run relation of output and debt in the United States and estimate the parameters of the fiscal rules. We estimate a structural VECM model identified by long- and short-run restrictions to shed light on the relation between output and debt conditional on technology shocks. We do so because, empirically, technology shocks are main drivers of business cycles and are predominantly used in the Real Business Cycle (RBC, for short) paradigm. Our findings show that government debt is procyclical in output.

Second, we show that in our model, countercyclical debt creates larger volatilities of key macroeconomic variables and is hence destabilizing. This finding contradicts the canonical view of Keynesian fiscal policy as being able to counter adverse effects of economic recessions and, hence, stabilize the economy.

In order to develop some intuition for our result, assume that our economy is hit by a positive, mean-reverting technology shock. This shock will increases output

[1] Woodford (1995) uses a similar, though conceptionally different, effect in the FTPL. Changes in the real value of government debt generate changes in the lifetime budget constraint of private agents, i.e. a wealth effect, that drives aggregate demand. However, this only holds iff policy is "Non-Ricardian", that is to say that agents expect that the government does not adjust future budgets to neutralize this effect. In contrast, the channel presented in this paper does not rely on the violation of future government's budget constraints. The reason is that the Ricardian equivalence is broken by household's behavior and not by the government's policy.

and - as we have learned from our empirical exercise - government debt will co-move with output. As debt increases, households feel richer, because debt is net wealth in the perpetual-youth model. Consistently, this wealth effect affects the households' consumption-leisure decision and the labor supply schedule is shifted inwards, such that agents supply less labor. Fiscal policy can affect the size of the wealth effect steaming from the change in debt, namely by putting different weights on their two goals. We provide a robustness check on the parameters in the fiscal rule and document which parameter values would generate the procyclical result of low volatilities.

Finally, we would like to stress that our channel is not present in the standard Ricardian agent RBC model, such that in this environment fiscal policy has only negligible effects on the propagation of technology shocks.

The paper is structured as follows. The next section estimates the SVECM model and presents our empirical evidence. Section 3 develops the model while Section 4 discusses the differences between pro- and countercyclical fiscal policy. Section 5 provides a robustness check and section 6 briefly concludes.

## *1.2 Empirical Evidence*

This section provides empirical evidence about the (long-run) relation between output and debt over the U. S. business cycle. Bohn (1998) shows that debt policy in the United States was sustainable in that it satisfied an intertemporal budget constraint between 1916 and 1995. The government reduced the primary deficit if the debt-to-GDP ratio increased. Aghion and Marinescu (2008) use annual panel data from 1964 to 2005 for a set of OECD countries and regress public debt on the output gap using Ordinary Least Squares (OLS, for short). They find a countercyclical relation that becomes stronger over time for the United States.

Further, there is a growing literature on fiscal rules that also shed light on the relation between debt and output. Blanchard and Perotti (1999) use structural vector autoregression models.[2] They find that positive government spending shocks increase output, while positive tax shocks reduce output. Overall, fiscal multipliers are found to be small.

A different approach uses DSGE models to estimate the parameter of fiscal rules. Leeper and Yang (2008) show that in a stylized real business cycle model the response of the economy crucially depends on which fiscal instruments finances debt. Leeper

[2]Bayoumi and Eichengreen (1992) use long-run restrictions in order to distinguish between demand- and supply-side shocks in the context of fiscal rules.

et al. (2010a) extend this analysis and use Bayesian methods to determine the specifications of fiscal rules that feature endogenous feedback on output and debt. Finally, Leith and Wren-Lewis (2000) consider a model with Non-Ricardian agents and simple policy rules. They identify stable policy regimes in active and passive monetary, fiscal policy respectively.

We use a structural vector error correction model with long- and short-run restrictions. Hence, our approach takes into account that the variables are cointegrated. Further, we distinguish between government consumption and government investment as the latter usually only operates with lags and will have supply-side effects. In addition, as shown, for example, by Favero (2003) monetary and fiscal policy cannot be estimated separately and we therefore add the interest rate as the monetary policy's policy instrument to our estimation.

Technically, we consider a structural vector error correction model with cointegration rank $r$ (SVECM($p$)) with six variables

$$\mathbf{\Delta} y_t = \alpha\beta' y_{t-1} + \sum_{i=1}^{p-1} \mathbf{\Gamma}_i \Delta y_{t-i} + u_t, \tag{1}$$

where we assume that all variables are at most integrated of order one. Further, $y_t$ is a $(K \times 1)$ vector of observables, where $y_t = \left[Y_t, B_t, i_t, G_t^c, G_t^I, T_t\right]$. We consider the time series of output, government debt, the interest rate, government consumption and investment, and taxes. Further, $\alpha$ is a $(K \times r)$ dimensional loading coefficient matrix, and $\beta$ is a $(K \times r)$ dimensional matrix with the cointegrated vectors. Further, $\mathbf{\Gamma}_i$ are $(K \times K)$ dimensional coefficient matrices for all $i = 1, \ldots, p-1$. The reduced form errors are denoted by $u_t = \mathbf{B}\varepsilon_t$ and $\varepsilon_t$ are the structural shocks. The reduced form errors are assumed to be zero mean Gaussian white noise with time invariant covariance matrix $\Sigma_u = \mathbf{B}\mathbf{B}'$, while $\varepsilon_t \sim (0, \mathbf{I}_K)$. As shown by Lütkepohl (2005), using the Beveridge-Nelson MA representation of this process gives the long-run effects of $u_t$, given by the total impact matrix $\mathbf{C}(1)\mathbf{B}$. We need to impose $K(K-1)/2$ restrictions on $\mathbf{C}(1)\mathbf{B}$ and $\mathbf{B}$ to locally just-identify the structural innovations.

Hence, we impose the following restrictions on the long-run impact matrix $\mathbf{C}(1)\mathbf{B}$ and the contemporaneous impact matrix $\mathbf{B}$

$$\mathbf{C}(1)\mathbf{B} = \begin{bmatrix} * & * & 0 & * & * & * \\ * & 0 & * & * & * & * \\ * & * & * & 0 & 0 & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}, \mathbf{B} = \begin{bmatrix} * & * & * & * & * & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & * & 0 & 0 \\ * & * & * & * & * & 0 \\ * & * & * & * & * & * \end{bmatrix}. \tag{2}$$

The restrictions on the long-run impact matrix imply that the interest rate has no long-run effect on output. This should follow from the long-run neutrality of monetary policy. Further, we impose that government consumption and government investment do not have long-run effects on the interest rate.

All time series are obtained from the Bureau of Economic Analysis' NIPA tables and the St. Louis system FRED. We use a quarterly basis from 1966:Q1 to 2015:Q1 (197 observations) for the United States. For output, we use the Gross Domestic Product in Billion of U. S. Dollar (Table 1.1.5, line 1). Government debt, $B_t$, is total public debt (GFDEBTN) at the end of the quarter. The interest rate is measured by the effective federal funds rate (FEDFUNDS). Government consumption and investment are taken from Table 3.9.5 line 2 and 3 and are total government consumption expenditures and total government gross investment. Finally, taxes are total current tax receipts (Table 3.3, line 2).

Then, all time series are divided by the consumer price index (CPIAUCSL), are seasonally adjusted, and are written in log-terms (with the interest rate being the exception). A first and preliminary look at the data shows that the simple (unconditional) correlation between the linearly detrended time series for debt and GDP is 0.382. However, in order not to mistake correlation with causation, we need to have a more careful and systematic look at the relation between those two variables. A Johansen trace test points towards a maximum of three cointegrated vectors ($r = 3$) and the optimal lag length test (Hannan-Quinn and Schwarz Criterion) shows that one lag ($p = 1$) is optimal. We then estimate the SVECM using a maximum-likelihood estimator.

Figure 1 presents the impulse response functions for output and debt in response to the identified supply-side shock (the shock to GDP). Most importantly, we find that output and government debt move procyclical in response to the shock. Our measure of cyclicality is the simple correlation coefficient between the estimated impulse responses. We find that the model implies a correlation of 0.43 between output and debt on the business cycle frequency of 40 quarters.

In the robustness section we provide an analysis of the parameter combinations that generate pro- and countercyclical debt as well as their effects on the volatility. Hence, we offer an alternative to the empirical estimates and provide evidence on which parameter combinations support our empirical findings.

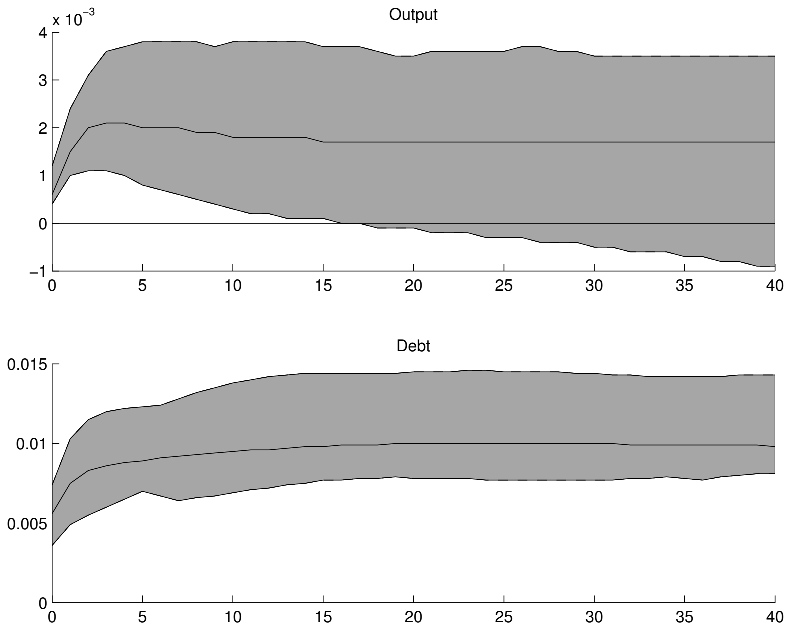


**Figure 1:** Impulse responses from SVECM estimation to a one s.d. shock in output. Grey area is the 90 percent bootstrapped Studentized Hall confidence interval.

## *1.3 The Model*

The description of our model economy proceeds in three steps and follows Prescott's narrative approach. First, we define the economy's preferences and technology and we then present the model's assumed market structure. Finally, we conclude by deriving the first-order necessary conditions and by defining the model's equilibrium.

### 1.3.1 Preferences and Technology

This section develops a dynamic, micro-founded model of the U. S. economy in discrete time. A period is assumed to be a quarter. Consumption and labor supply decisions are derived along the lines of the discrete time version of the Blanchard (1985) - Yaari (1965) perpetual-youth model.[3] Firms use a neoclassical production technology with constant-returns to scale to produce output on a perfectly competitive market. Output is produced using capital and labor services. Finally, we assume the presence of convex capital adjustment costs, in order to allow for a variable price of capital as in Christiano et al. (2005). The only source of uncertainty - disregarding the uncertainty about death - in our model is a mean-reverting shock to aggregate technology.

[3]The discrete time version of the Blanchard (1985) - Yaari (1965) model was first developed by Frenkel and Razin (1986).

Let us discuss the perpetual-youth structure of our economy. As in Blanchard (1985), we assume that there exists a constant probability of surviving, denoted by $\vartheta > 0$, that each agent faces throughout her lifetime. In turn, this implies an expected lifetime of $(1-\vartheta)^{-1}$. In addition, at any time $t$, a cohort of size $1-\vartheta$ is born, and total population is normalized to one. Therefore, our economy features a constant population with identical preferences. While agents are of different ages and wealth levels, they all face the same life horizon which implies that they are homogeneous with respect to the marginal propensity to consume. This homogeneity is a necessary condition in order to solve the aggregation problem, as shown by Blanchard (1985).

Perfectly competitive private markets provide insurance riskless through life insurance companies. Free entry and a zero profit condition imply that agents will pay a rate $1-\vartheta$ to receive one good contingent on their death. Since there is no bequest motive - and since negative bequests are ruled out - agents will contract to have all of their wealth returned to the life insurance company contingent on their death. The insurance company will equally distribute the wealth of the deceased to the survivors, by paying a fair premium.

#### *1.3.1.1 Households*

Given a representative agent out of generation $s$, let us denote consumption by, $C_t^s$, and hours worked by, $N_t^s$, then, the representative agents' preferences are given by the following expected von Neumann-Morgenstern utility function

$$\Gamma_t = \mathbb{E}_t \left\{ \sum_{j=0}^{\infty} (\vartheta\beta)^j \left[ \mathcal{U}_t\left(C_{t+j}^s\right) - \mathcal{V}_t\left(N_{t+j}^s\right) \right] \right\}, \tag{3}$$

where $\mathbb{E}_t$ denotes the expectation conditional on the information available at $t$ and $\beta \in (0,1)$ is the household's discount factor. Agents are assumed to have rational expectations, that is to say, the underlying probability distributions of the conditional mathematical expectations coincide with those implied by the model. Then, the single-period utility function, $\Gamma_0 : \mathbb{R}^2 \to \mathbb{R}$, satisfies the Inada conditions with goods and leisure, $L_t^s = 1 - N_t^s$, being normal. Furthermore, the utility function is compatible with the requirements of balanced growth. We assume that it is *CRRA* and make further use of the following specifications

$$\mathcal{U}_t(\cdot) = \ln(\cdot), \tag{4}$$

$$\mathcal{V}_t(\cdot) = \frac{(\cdot)^{1+\varphi}}{1+\varphi}, \tag{5}$$

where the inverse of the Frisch labor supply elasticity is given by $\varphi > 0$.

Note that we will assume that preferences over consumption are logarithmic for the reminder of this paper, as, for example, in Smets and Wouters (2002). This implies that we can find an intuitive expression for the marginal propensity to consume, which will depend only on the discount factor and the probability of surviving. In the general case of iso-elastic preferences the propensity of consume would depend on the expected real return on financial wealth (which we will explicitly define later on) and, moreover, would depend on the time of birth. Furthermore, as stated by Weil (1989), using utility functions that feature non-logarithmic preferences will offer no additional insight while, most importantly, would make the aggregation problem across the entire population impossible, because consumption would be non-linear in wealth.

#### *1.3.1.2 Technology*

Along the firm side of our economy, a representative firm uses capital, $K_t$, and labor services, $N_t$, as inputs for a Cobb-Douglas production function

$$Y_t = Z_t K_t^{\alpha} N_t^{1-\alpha}, \tag{6}$$

here, $\alpha \in [0, 1]$ is the capital elasticity in the production function and we denote the aggregate, Hicks-neutral technology shock by $Z_t$. A first-order autoregressive process determines its evolution

$$\ln Z_t = \rho_Z \ln Z_{t-1} + e_{Z,t}, \tag{7}$$

where $0 < \rho_Z < 1$ determines the degree of autocorrelation and its innovation is i.i.d. over time and Gaussian distributed,

$$e_{Z,t} \sim \mathcal{N}(0, \sigma_Z). \tag{8}$$

Households own the capital stock and rent it to the firm on a perfectly competitive market. The capital accumulation technology is given by

$$K_t = (1-\delta)K_{t-1} + \left[1 - S\left(\frac{I_t}{I_{t-1}}\right)\right] I_{t,} \tag{9}$$

where $I_t$ is investment and $S(\cdot)$ captures capital adjustment costs as in Christiano et al. (2005), which, in steady state, satisfies $S(\cdot) = 0$, $S'(\cdot) = 0$, and $S''(\cdot) > 0$. We add those adjustment costs in order to replicate more realistic asset price dynamics. This is particularly relevant in this framework as the capital price drives the real value of the capital stock and, therefore, households wealth.[4] Furthermore, $\delta > 0$ denotes the exogenous rate of capital depreciation.

[4]The results are robust to excluding capital adjustment costs.

#### 1.3.1.3 *Fiscal Authority*

We postulate that our fiscal authority follows a tax and a government spending rule to conduct fiscal policy as, for example, in Leeper et al. (2010a, 2010b). Those rules have endogenous feedback to lagged output and lagged government debt which allows us to model a dynamic response and to cover the two main objectives of fiscal policy.[5]

Let us spend some time to motivate those rules and to derive some intuition. Why should fiscal policy respond to those two variables? First the response to output is the usual automatic stabilizer component of fiscal policy described in the literature (see DeLong and Summers (1986) and Galí (1994)).[6]

Second, the budgetary position of the United States has deteriorated substantially over the past decades. Main driving forces have been short-run events (such as large spending programs and a sharp decline in tax revenues) and long-run trends. The share of the population receiving benefits from Social Security, Medicare, and Medicaid will keep increasing. Along this line, the implementation of fiscal rules with feedback to debt may help to structure the budget process and promote fiscal responsibility by constraining decisions about spending and taxes.

Formally, our fiscal authority issues bonds, provides government spending (that does not affect the marginal utility of private consumption), and uses lump sum taxes for redistribution purposes. However, only two of those instruments can be set independently, while the third follows from the equilibrium restriction. The equilibrium restriction on the fiscal authority's actions is

$$\frac{B_{t+1}}{R_t} = B_t + G_t - \tau_t. \tag{10}$$

The tax rule can - in log-linearized terms - be written as

$$\hat{\tau}_t = \tau_Y \hat{Y}_{t-1} + \tau_B \hat{B}_{t-1}. \tag{11}$$

Here, $\tau_B \in \mathbb{R}$ is the parameter governing the feedback on debt, and $\tau_Y \in \mathbb{R}$ is the coefficient on output. The former accounts for a debt stabilization goal of the fiscal authority, while the latter takes business cycle movements into account.

Then, we assume that the government spending rule - in log-linearized terms - follows

$$\hat{G}_t = \gamma_Y \hat{Y}_{t-1} + \gamma_B \hat{B}_{t-1}. \tag{12}$$

[5] Robustness checks reveal that our qualitative results are unaffected by assuming a contemporaneous relation.

[6] I wish to make a remark here: the automatic stabilizer component in the fiscal rules should not be mistaken with the (automatic) stabilizing effect of debt.

As before, $\gamma_Y \in \mathbb{R}$ accounts for the business cycle stabilization goal of our government and $\gamma_B \in \mathbb{R}$ captures the aim to stabilize debt.

At the end of this chapter we can draw the conclusion that fiscal policy, defined as the sequence of debt, spending, and taxes, affects the agents optimal allocation problems through three channels. First, debt is part of financial wealth which drives consumption (and leisure decisions). Second, taxes are an important factor for human wealth and therefore have an impact on the consumption/leisure decision. Finally, spending is a component of aggregate demand and therefore directly affects total output produced in our economy.

### 1.3.2 Market Structure

The model features four spot markets, namely the bond market, the capital market, the good market, and the labor market, the latter three being perfectly competitive. Then, we follow Mehra and Prescott (1980) and assume that only households own capital between quarters. At the beginning of each quarter, households sell capital to the representative firm. At the quarter's end, the firm sells all capital back to the households.

Furthermore, and confronted with the finiteness of agent's life and the accumulation process of capital, we assume that firms are long-lived. This requires the existence of an underlying stock market in order to pass firm ownership on to new agents. Our firm is a legal entity issuing equity shares, while its ownership is perfectly divisible across an unbounded sequence of finite-lived shareholders (i.e. households).

### 1.3.3 Optimization and Equilibrium

Optimization of all agents but the fiscal authority defines equilibrium. We start with the households utility maximization problem and continue with the firms profit maximization problem. We conclude with the aggregation problem and define the model's equilibrium.

#### *1.3.3.1 Households*

We assume that the economy begins with all households having identical financial wealth and consumption histories. This assumption assures that together with the optimal use of the available contingent claims markets, this homogeneity will continue. To be precise, agents have access to a full set of state-contingent Arrow-Debreu securities after their birth. Moreover, this allows us to only consider the consumption and savings decisions of a representative household. The representative household faces the following intertemporal budget constraint

$$\frac{B_{t+1}^s}{R_t} + Q_t K_{t+1}^s \leq A_t^s + W_t N_t^s - T_t^s - C_t^s, \tag{13}$$

where we define financial wealth, $A_t^s$, by

$$A_t^s = \frac{1}{\vartheta}\left[B_t^s + \left((1-\delta)Q_t + R_t^K\right)K_t^s\right], \tag{14}$$

where $Q_t$ represents the price of capital and $R_t^k$ is the nominal rental rate of capital. In addition, $B_t^s$ is a one-period government bond issued on a discount basis with an interest rate $R_t$.[7] The agent receives labor income $W_t N_t^s$ and has to pay lump sum taxes $T_t^s$ to the fiscal authority.

Further, there exists a transversality condition that prevents agents from going infinitely into debt

$$\lim_{t\to\infty}\left\{(\vartheta\beta)^t A_t^s\right\} \geq 0. \tag{15}$$

The unique solution to the concave optimization problem, maximizing eq. (3) subject to the constraint (13), is (13) with equality and - assuming that the solution is interior - the marginal conditions for consumption, investment, capital, and hours

$$\partial C_t^s \quad : \quad \frac{1}{C_t^s} = \zeta_t, \tag{16}$$

$$\partial I_t^s \quad : \quad \left\{\begin{array}{c} q_t\left[1 - S\left(\frac{I_t^s}{I_{t-1}^s}\right) - S'\left(\frac{I_t^s}{I_{t-1}^s}\right)\frac{I_t^s}{I_{t-1}^s}\right] \\ +\mathbb{E}_t\left[\beta\frac{\zeta_{t+1}}{\zeta_t}q_{t+1}S'\left(\frac{I_t^s}{I_{t-1}^s}\right)\left(\frac{I_{t+1}^s}{I_t^s}\right)^2\right] \end{array}\right\} = 1, \tag{17}$$

$$\partial K_t^s \quad : \quad q_t = \mathbb{E}_t\left\{\beta\frac{\zeta_{t+1}}{\zeta_t}\left[R_{t+1}^K + q_{t+1}(1-\delta)\right]\right\}, \tag{18}$$

$$\partial N_t^s \quad : \quad -\left(N_t^s\right)^{\varphi} + \zeta_t W_t = 0. \tag{19}$$

Here, $\zeta_t$, is the Lagrangian multiplier on the budget constraint. Now, let us define $H_t^s$ as human wealth given by

$$H_t^s = h_t^s + \mathbb{E}_t\left\{\sum_{j=1}^{\infty}\vartheta^j\left[\prod_{k=0}^{j-1}\frac{1}{R_{t+k}}\right]h_{t+j}^s\right\}, \tag{20}$$

where we use $h_t^s = W_t N_t^s - T_t^s$. Human wealth can be interpreted as the expected, discounted stream of labor incomes and profits net of taxes.

In the next step, the budget constraint can be re-written as

$$A_{t+1}^s = R_t\left[A_t^s - C_t^s + h_t^s\right]. \tag{21}$$

[7] In the United States, the maturity structure of government debt is fairly short: the median number is roughly two years.

Solving this equation forward and using the Euler equation one can find the equation for individual consumption,

$$C_t^s = (1 - \beta\vartheta) [A_t^s + H_t^s] . \tag{22}$$

This equation relates individual consumption to aggregate wealth, driven by financial and human wealth. As in Blanchard (1985), aggregate consumption is a linear function of total aggregate wealth. The household therefore consumes only a share of her financial wealth. This share is driven by the discount factor and the probability of surviving. We will later on come back to a more detailed discussion of this equation.

#### *1.3.3.2 Firms*

As we assumed that technology is constant returns to scale, we can focus on the solution to the optimization program of only one price taking firm.

This firm faces the cost minimization problem, viz. to choose the optimal input factor combination $\{K_t, N_t\}$ to produce a given output level, $Y_t$, and given their respective perfectly competitive prices. This problem is analogous to maximizing profits, hence the firm solves

$$\max_{\{N_t, K_t\}_{t=0}^{\infty}} \left\{ Y_t - W_t N_t - R_t^k K_t \right\} , \tag{23}$$

subject to the production frontier, eq. (6).

The solution to this sorting problem is an optimal capital-to-labor ratio,

$$\frac{K_t}{N_t} = \frac{\alpha}{1-\alpha} \frac{W_t}{R_t^K}, \tag{24}$$

i.e. a relation between payments and factors. Furthermore, we can find the standard expressions for the factor prices given by

$$W_t = \frac{(1-\alpha) Y_t}{N_t}, R_t^K = \frac{\alpha Y_t}{K_t}. \tag{25}$$

#### *1.3.3.3 Aggregation*

The aggregate value, $X_t$, of any individual variable, $X_t^s$, is obtained according to

$$X_t = (1 - \vartheta) \sum_{s=0}^{\infty} \vartheta^s X_t^s. \tag{26}$$

Here, $s$ refers to the generation born at period $t - s$. Then, aggregation of equations (21) and (22) over the generations alive at time $t$ gives

$$A_{t+1} = R_t [A_t - C_t + h_t] , \tag{27}$$

$$C_t = (1 - \beta\vartheta) [A_t + H_t] . \tag{28}$$

Using those two equations, one can derive an expression for aggregate consumption

$$C_t = \mathbb{E}_t \left\{ \frac{1}{R_t} \left[ \frac{\psi}{1-\psi}(1-\vartheta)A_{t+1} + \frac{\vartheta}{1-\psi}C_{t+1} \right] \right\}, \tag{29}$$

where $\psi = (1 - \beta\vartheta)$. Notice, that as in the Blanchard (1985) model, labor income is equally distributed across agents, which simplifies the wealth distribution since all agents have the same human wealth and ensures that we can solve the aggregation problem. This assumption implies that all agents have positive labor supply and the same productivity.

As we have seen in the derivation of the equation for individual consumption, the households' consumption decision is driven by three forces. First, as usual the interest rate impacts the households' intertemporal decision. Second, the expectation of future consumption weighted by the probability of surviving, also reflects the consumption smoothing incentive, which is the standard implication of the permanent income hypothesis. Third, and driven by the perpetual-youth structure of our model, financial wealth drives the consumption decision. The higher the probability of surviving, the smaller consumption will be, as the permanent income hypothesis implies that households will smooth consumption. It is also straightforward that in the Ricardian benchmark case, $\vartheta = 1$, only the standard elements, the interest rate and future consumption, will determine present consumption. However, if we assume that $\vartheta < 1$, the path of consumption is also driven by the dynamics of financial wealth. This idea goes back to the seminal contribution from Barro (1974), showing that under certain conditions, government bonds are net wealth for households. Note that financial wealth in our model is also driven by interest payments on capital.

#### *1.3.3.4 Equilibrium and Calibration*

A competitive equilibrium in our model is defined as follows.

**Definition**

A competitive equilibrium for given initial conditions, the stochastic process $\{Z_t\}$ and a set of prices $\{R_t, q_t, R_t^K, W_t\}$, is a tuple of processes for $\{C_t, A_t, I_t, K_t, N_t, Y_t, B_t, G_t, \tau_t\}$ such that

*1. Household optimality*

Given $\{W_t, R_t, R_t^K\}$, the processes for $\{C_t^s, A_t^s, I_t^s, N_t^s, K_t^s\}$ solve the optimization problem for any individual agent out of generation $s$, maximizing (3) subject to (13) and the transversality condition (15) holds.

*2. Aggregation*

Individual variables are transformed into aggregate variables according to (26).

*3. Profit maximization*

The process for $\{K_t, N_t\}$ solve the optimization problem, maximizing (23) subject to (6). Processes for $W_t$ and $R_t^K$ follow (25).

*4. Fiscal policy*

The processes for $\{B_t, G_t, \tau_t\}$ are determined by (12) and (11), while the government budget constraint, (10), holds with equality.

*5. Market clearing*

In equilibrium, factor and goods market clear and any feasible allocations are those satisfying

$$Y_t \geq C_t + I_t + G_t. \tag{30}$$

Then, the set of equations forming the rational expectation equilibrium is log-linearized around the non-stochastic steady state and solved by applying the Sims (2002) algorithm.

The calibration of the model is on a quarterly basis for the United States and parameter values are set according to stylized facts and the relevant literature.

We set the discount factor to $\beta = 0.998$. The probability of death, $1 - \vartheta$, is set to 0.015 as in Leith and Wren-Lewis (2000). We will provide a robustness check of this crucial parameter and discuss its role for business cycle fluctuations. According to the estimations from Leeper et al. (2010a), we set $\varphi = 2$, which implies a Frisch labor supply elasticity of 0.5. Then, we set the elasticity of output to capital, $\alpha$, to 0.3 which implies a labor share of 70 percent. The capital depreciation rate is set to 0.025, which is equal to a 10 percent annual depreciation rate. Tobin's q in steady state is set to unity. Steady state government consumption is set to 0.2 to match postwar U. S. data as shown in Baxter and King (1993).

The level of government debt in steady state, $B$, is set to 0.3396 as in Leeper et al. (2010a). This value is chosen because it coincides with the share of federal debt held by private domestic investors in the United States. Therefore, we ensure not to overestimate the effectiveness of our new channel, by assuming that all government bonds are held by households.

Then, steady state taxes are given by $\tau = G - \left[\frac{\beta-1}{\beta}\right] B$. The steady state capital rental rate follows $\bar{R}^k = \frac{1}{\beta} - (1 - \delta)$ and steady state aggregate technology, $\bar{Z}$, is set to unity. The autocorrelation of the technology shock is set to 0.9 and its variance is 0.0049, which matches the empirically observed volatility of U. S. GDP of 1.62 percent. Then, the steady state values for output, consumption, hours, and capital are given by the solution to the following linear system

$$
\begin{aligned}
\bar{Y} - \bar{C} - \bar{I} - \bar{G} &= 0, \\
\bar{Y} - \bar{K}^{\alpha} \bar{N}^{1-\alpha} &= 0, \\
\bar{R}^{K} - \alpha \frac{\bar{Y}}{\bar{K}} &= 0, \\
\bar{N}^{1+\varphi} - (1-\alpha) \frac{\bar{Y}}{\bar{C}} &= 0.
\end{aligned} \tag{31}
$$

Finally, we need to calibrate the four parameters governing the fiscal rules. We set the parameter of the tax rule to $\tau_B = 0.0014$ and $\tau_Y = 0.2077$. The parameters of the government spending rule are $\gamma_B = 0.0255$ and $\gamma_Y = 0.0077$. We find that taxes as well as spending react positively to changes in output and debt. Spending reacts stronger to changes in debt than taxes do (0.0255 vs. 0.0014). On the flipside, taxes react much stronger to variations in output (0.2077 vs. 0.0077). However, we find that the response of taxes to debt as well as the response of spending to output is insignificant. Therefore, taxes mainly respond to output changes, while government spending reacts mainly to changes in debt. Furthermore, the parameter values for the tax rule are at the lower bound of existing results by Leeper et al. (2010a, 2010b) ranging from -0.023 to 0.51 for the response of taxes to changes in debt, and 0.24 to 2.1 for the responsiveness of taxes to variations in output. However, the results for the government spending rule imply countercyclical movements. Here, the values range from -0.031 to -0.022 for the response of spending to changes in debt, and from -0.084 to -0.0064 for variations in output. Overall, we find that the responsiveness of taxes and spending is at the lower bound of the existing results obtained by applying Bayesian methods.

At the end of this section, we need to explain the way we calibrate the counterfactual, namely the countercyclical debt scenario. We multiply each value in the two fiscal rules by $-1$ to generate a countercyclical relationship between output and debt. Notice that our model is linear around its steady state. Therefore, the absolute size of fiscal policy effect is identical across regimes and we hence generate symmetric effects of pro- and countercyclical debt. Therefore, we exclude the possibility of creating results that are only driven by putting different (absolute) weights on the components in the fiscal rules. Further considerations on how to generate the counterfactual are given in the robustness section.

## *1.4 Discussion*

In the following, we want to discuss the response of our model to a mean-reverting, one percent favorable technology shock for two different calibrations of fiscal policy. In one case, the fiscal rules are calibrated such that debt moves procyclical, while in the other case debt moves countercyclical. The response of our model economy for those two cases is presented in Figure 2. Assume that our economy is hit by a positive, stationary technology shock. This shock will increase output and - as we have learned from our empirical analysis - debt will positively co-move with output. As debt increases, households feel richer, because debt is net wealth in the perpetual youth model. Consistently, this wealth effect affects the households' consumption-leisure decision and the labor supply schedule is shifted inwards, such that agents supply less labor compared to the countercyclical case. On the flipside, we observe that households consume less if debt moves procyclical over the cycle. The effects on consumption and leisure are additionally affected by the different paths of the interest rate. In the procyclical debt model, the interest rate increases on impact, creating incentives to shift consumption to the future. Further, the interest rate undershoots and converges from below to the steady state. We observe that the reaction of the interest rate is smaller in the procyclical compared to the countercyclical scenario. Coherently, we see that output deviations are smaller and less persistent in response to the shock, which implies jointly with the behavior of households, that investment activity is lower in the procyclical case. This spills over to a smaller build up of the capital stock and a less persistent adjustment process. Intuitively, this creates further repercussions for household's wealth since financial wealth is also driven by the value of the capital stock.

We can draw the conclusion that fiscal policy affects the size of the wealth effect steaming from the change in debt, namely by putting different weights on their two goals defined in the fiscal rules. Further, it affects the path of the interest rate and therefore affects the intertemporal allocation.

Our stabilizing result can nicely be inferred from Table 1. Here, we present the relative standard deviation of key variables for the two fiscal policies considered as well as the difference, $\Delta$, in percent.

As we have seen before, the economy with procyclical debt is significantly less volatile. We find that the volatility of output is 0.75 compared to 1.17 in the counter-cyclical case. This implies a stabilizing effect of 36 percent for output. The main difference can be found in the standard deviation of consumption. Since in our perpetual-youth model if policy affects debt, debt will effect household's wealth which

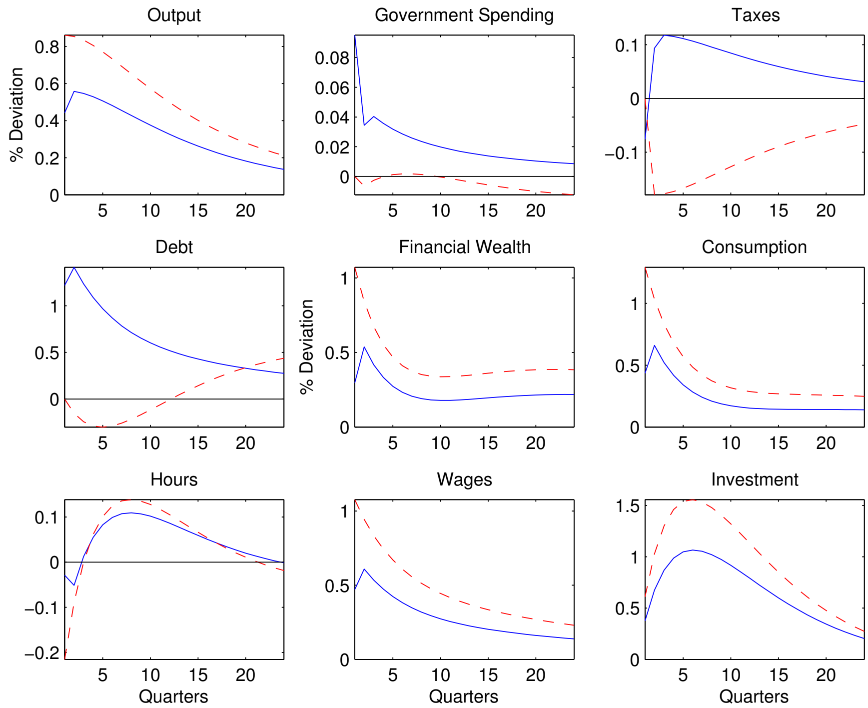


**Figure 2:** Model response to positive technology shock for counter- and procyclical debt. Horizontal axes measure quarters, vertical axes measure deviations from steady state. Blue, solid line is procyclical policy and red, dashed is countercyclical policy.

**Table 1:** Relative standard deviation for the two fiscal policies. Rstd is relative standard deviation with respect to output. Data values are taken from King and Rebelo (2000). Values for debt, spending, and taxes are based on own computations.

| | Data | Procyclical | Countercyclical | $\Delta$ |
|---|---|---|---|---|
| $std(Y)$ | 1.81 | 0.75 | 1.17 | -0.36 |
| $Rstd(C)$ | 0.74 | 0.77 | 0.94 | -0.18 |
| $Rstd(N)$ | 0.99 | 0.21 | 0.21 | 0 |
| $Rstd(I)$ | 2.93 | 2.05 | 1.91 | 0.07 |
| $Rstd(W)$ | 0.38 | 0.88 | 0.95 | -0.07 |
| $Rstd(G)$ | 0.07 | 0.08 | 0.03 | 2.67 |
| $Rstd(T)$ | 0.12 | 0.21 | 0.21 | 0 |
| $Rstd(B)$ | 2.80 | 2.01 | 1.06 | 1.90 |

directly drives the consumption/labor decision. We find that the standard deviation of consumption in the procyclical case is 0.77, while it is 0.94 in the countercyclical case. Furthermore, the second important dimension is labor supply. We find that the relative volatility stays roughly constant at 0.21. However, the absolute standard deviation of hours is significantly reduced (0.0016 vs. 0.0025). Here, we can identify the dampening effect of the wealth effect steaming from the procyclicality of government debt. This is further supported by exercises with different utility functions. If we shut down the wealth effect in the model, we observe much higher standard deviations of consumption and labor supply.[8] This proves that the positive wealth effect from government debt significantly effects the household's optimal allocation decision and explains why countercyclical policy is destabilizing.

Consequently, wages are less volatile since output and labor supply now move less volatile over the cycle. Standard deviation of investment is larger in the procyclical regime with 2.05 versus 1.91 in the countercyclical one. Finally, we observe that government spending is almost three times as volatile in the procyclical calibration, 0.08, as in the countercyclical example, 0.03. However, the volatility of taxes stays put at 0.21. Finally, the difference in volatility for government debt is large. In the procyclical case we obtain a standard deviation of 2.01, while in the countercyclical case, we obtain a value of 1.06. Hence, debt is almost twice as volatile, if fiscal policy is procyclical.

[8] To be precise, we use log-log preferences and preferences suggested by Greenwood et al. (1988) that generate a small short-run wealth effect.

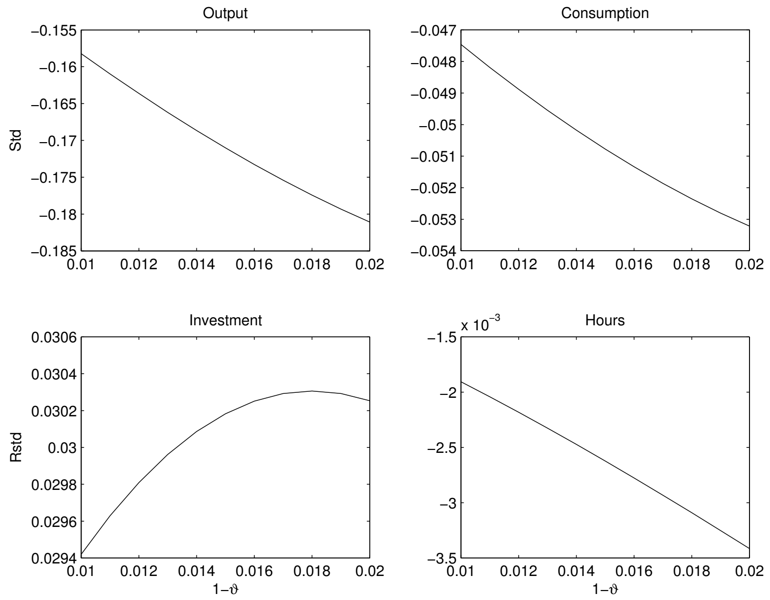


**Figure 3:** Difference in relative standard deviations w.r.t. output between pro- and countercyclical debt as a function of the probability $1 - \vartheta$. For output, we plot the standard deviation.

## *1.5 Robustness*

First, we want to provide a robustness check on the parameters in the fiscal rules. A central result relates to generating the counterfactual. While there should be no disagreement about the multiplication by $-1$, it is less clear that all parameters have to be changed. In fact, it is possible to generate countercyclical debt by multiplying less than all four parameters. However, the following holds: every combination of fiscal rule parameters multiplied by $-1$ that generate countercyclical debt, will generate larger second moments compared to the procyclical case. Hence, our results are robust to different ways to generate the counterfactual.

Next, we want to stress the importance of the assumption on the survival probability, $1 - \vartheta$. For this purpose, Figure 3 plots the difference in the relative standard deviations of key macro variables between pro- and countercyclical debt as a function of the probability to decease on the interval $[0.01, 0.02]$. To get an intuition for those values, consider that a death probability of 0.01 implies a lifetime of 100 periods, while a value of 0.2 implies a lifetime of 5 periods. Our baseline calibration of 0.015 results in an expected lifetime of roughly 70 periods. The figure shows that the difference between pro- and countercyclical debt for output, consumption, and hours stays negative over the interval. For investment, the opposite holds. Hence, we confirm that our result that procyclicality generates smaller volatilities of key variables holds independently from the value of the death probability. Further, we infer that a shorter lifetime (a larger value of $1 - \vartheta$) increases the difference between the two scenarios. This finding is in line with the intuition about the effects of fiscal policy for Non-Ricardian agents. The shorter the lifetime, the less likely it is that the agents have to face the higher financing burden and the more effective fiscal policy will be.

At the end of this section, we want to discuss a related, though different, question. How are the volatilities effected by different values of the fiscal rule parameters? Which parameter combinations replicate the procyclical debt results? To answer this question we plot the relative standard deviation of consumption as a function of six parameter pairs (see figure 4).

From the top left and the top right panel we observe a sharp decline of relative volatility in the lower-left corner of the $\tau_Y$-$\gamma_B$ plane, once $\gamma_B$ turns positive. Put differently, if government spending moves procyclical with debt, our observed debt procyclicality is in place and we observe the stabilizing effects on the real economy. Further, from the lower left panel we infer that a value smaller than roughly 0.03 of $\tau_B$ is required given the other parameter values. Once there is a small positive or even negative reaction in taxes to an increase in debt we observe the procyclicality of debt with a positive stabilization. The other parameter combinations show that a strong positive reaction of taxes to output further increase the stabilizing effect of debt. In contrast, $\gamma_Y$ has no sizable effects on the results.

We can conclude that the main parameter driving the results is the responsiveness of spending to debt ($\gamma_B$) and the responsiveness of taxes to debt ($\tau_B$).

## 1.6 Conclusion

The main contribution of this paper is theoretical. We show that government debt, as being a component of household's financial wealth, creates an additional wealth effect that has sizable effects on the business cycle. We interpret this channel as an additional automatic stabilizer of economic activity. Therefore, we present a new channel through which governments can influence cyclical fluctuations and achieve macroeconomic stability. The striking consequence is that classical (countercyclical) Keynesian fiscal policy destabilizes the business cycle in our framework.

In detail, this paper has two contributions. First, we systematically analyze the

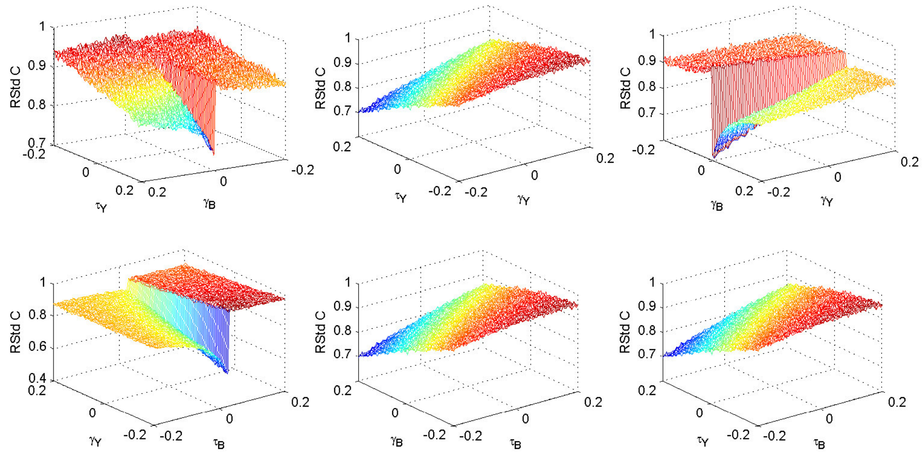


**Figure 4:** Relative standard deviation of consumption (RStd C) as a function of the fiscal rule parameters.

relation between output and government debt. For this purpose, we estimate a structural VECM identified by long-run restrictions to shed light on the underlying relationship between debt and output. Further, we estimate the parameters in fiscal rules describing the dynamics of spending and taxes. We find that debt is procyclical in output over the U. S. business cycle. Further, government spending is procyclical, while tax revenues are countercyclical.

Second, we build a Real Business Cycle model of the U. S. economy with Non-Ricardian agents. By implementing fiscal rules with endogenous feedback to output and debt, we are able to generate pro- and countercyclical fiscal policy. Further, fiscal rules allow us to write debt as a function of output, hence, creating an automatic stabilization role for debt. With those instruments, we show that standard deviations of key macroeconomic variables are significantly higher, if debt is countercyclical.

The intuition is an additional wealth effect that affects economic activity. The mechanism works as follows. In the perpetual-youth model, ceteris paribus, government debt is wealth from the household's perspective, because they are likely to not being affected by the higher tax burden of expansionary fiscal policy in the future. Higher productivity will increase output and - in the case of procyclical debt - debt will increase. This increase will lead to a rise in financial wealth of households. This additional wealth effect, which is not present in standard business cycle models, shifts the labor supply schedule inwards and agents supply less labor and consume more.

The implications for public policy are potentially severe and provocative. We have shown that in our framework the preferable policy instrument for business cycle stabilization is not the canonical, countercyclical Keynesian-type policy but, instead, procyclical policy. Further, fiscal policy can affect the size of the additional wealth effect steaming from the change in debt, namely by putting different weights on their two goals.

Finally, let us stress two limiting factors that should motivate future research. First, the interactions between fiscal and monetary policy rules should be analyzed to discuss the role of monetary policy. Besides implications for business cycle fluctuations, the channel might add to the discussion of fiscal determinations of the price level. We have seen that the discussed channel works for a all shocks that affect output. Hence, the combination of fiscal rules and Non-Ricardian agents allows fiscal policy to be of relevance for price level determination even for non-fiscal disturbances as stressed by the fiscal theory of the price level.

Moreover, our impulse response analysis shows that procyclical debt is associated with a smaller accumulation of the private capital stock. Therefore, we do find a trade-off between growth and the stabilization of the business cycle.

Lastly, a richer set of policy instruments, e.g. distortionary taxes and transfer payments, should be considered to allow for more detailed recommendations.

## 1.7 Technical Appendix

### 1.7.1 Cointegration

The Johansen cointegration test allows for deterministic and stochastic cointegration. We find that for both specifications of the test, we obtain three cointegrating vectors. The Johansen test uses the VEC($p$) process

$$\Delta y_t = Ay_{t-1} + \sum_{i=0}^{p} B_i \Delta y_{t-i} + CX + \varepsilon_t, \tag{32}$$

where $X$ contains exogenous terms that relate to deterministic trends. Then, our two hypothesis are

*Deterministic cointegration*

$Z\left(B^T y_{t-1} + g_0\right) + g_1$, where $Z$ is a matrix of error-correction speeds. Hence, there is an intercept in the cointegrating relation covered by $g_0$, as well as a linear trend covered by $g_1$.

*Stochastic cointegration*

$Z\left(B^T y_{t-1} + g_0 + g_2 t\right) + g_1$, where, additionally, there is a linear trend in the cointegrating relationship.

### 1.7.2 SVAR

This section estimates a bivariate structural Vector Autoregressive model (VAR, for short) with output and government debt using an $\mathbf{A}-\mathbf{B}$ model with long-run restrictions according to Galí (1992) and Breitung et al. (2004).

Our approach can be motivated by two observations. First, we are interested in the relation between output and debt conditional on a technology shock, as technology shocks are main drivers of business cycle fluctuations, see e.g. Fisher (2002) or Christiano et al. (2003). Therefore, we will feed a technology shock through the RBC model developed later in this paper to discuss the business cycle implications of the new channel we are emphasizing. As a consequence, we impose the restriction that output shocks do have long-run effects and shocks to government debt do not have long-run effects on output.

Further, empirical evidence has shown that at least some technology shocks have a unit root (see e.g. Galí (1999) or Shea (1999)), hence, they will affect the level of output in the long-run. Therefore, at least some of the innovations to output - we think of those technology shocks here or, more generally, supply-side shocks - will have permanent effects. This is the key assumption underlying our identifying restriction. Since

Second, besides characterizing the long-run dynamics of output and debt, we aim at estimating the parameters of fiscal rules. We will use those rules later on in our model to replicate the observed debt dynamics and discuss the estimation of those rules in the calibration section. Nevertheless, we want to stress that the estimation of rules, inherently, is related to a long-run perspective rather than a short-run one; which supports our identification approach.

Technically, we consider a structural form SVAR($p$)

$$\mathbf{A}y_t = \sum_{i=0}^{p} A_i^* y_{t-i} + \mathbf{B}\varepsilon_t, \tag{33}$$

where $y_t$ is a $(K \times 1)$ vector of observables and $u_t$ is a $K$-dimensional vector of residuals. Further, $A_i^* = \mathbf{A}A_i$ is a $(K \times K)$ dimensional coefficient matrix for all $i = 1, \ldots p$. We impose an additional assumption that allows us to identify the structural innovations $\varepsilon_t$ from the reduced form residuals $u_t$

$$\mathbf{A}u_t = \mathbf{B}\varepsilon_t, \tag{34}$$

where $\varepsilon_t \sim (0, \mathbf{I}_K)$ is Gaussian white noise and the covariance matrix is $\Sigma_u = \mathbf{A}^{-1}\mathbf{B}\mathbf{B}^T\left[\mathbf{A}^{-1}\right]^T$.

We need to impose $2K^2 - \frac{1}{2}K\left(K+1\right)$ restrictions to identify all $2K^2$ parameters of the $\mathbf{A}$ and $\mathbf{B}$ matrix. For large VAR systems the number of restrictions is quite large, which often leads to consideration of special cases, i.e. an $\mathbf{A}$ or $\mathbf{B}$ model. However, given our identification approach and the bivariate structure allows us to use the $\mathbf{A}-\mathbf{B}$ model with the following five restrictions

$$\mathbf{A} = \begin{bmatrix} 1 & 1 \\ * & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix}. \tag{35}$$

Intuitively, the zero restriction on $\mathbf{B}$ implies that debt innovations have no long-run effect on output.

The time series for output and debt are provided by the Bureau of Economic Analysis' NIPA on a quarterly basis from 1960:Q1 to 2008:Q3 (191 observations) for the United States. A first and preliminary look at the data shows that the simple (unconditional) correlation between the linearly detrended time series is 0.17. However, in order not to mistake correlation with causation, we need to have a more careful and systematic look at the relation between those two variables.

Figure 5 presents the impulse response functions for output and debt in response to the identified supply-side shock. Most importantly, we find that output and government debt move procyclical in response to the shock. Our measure of cyclicality is

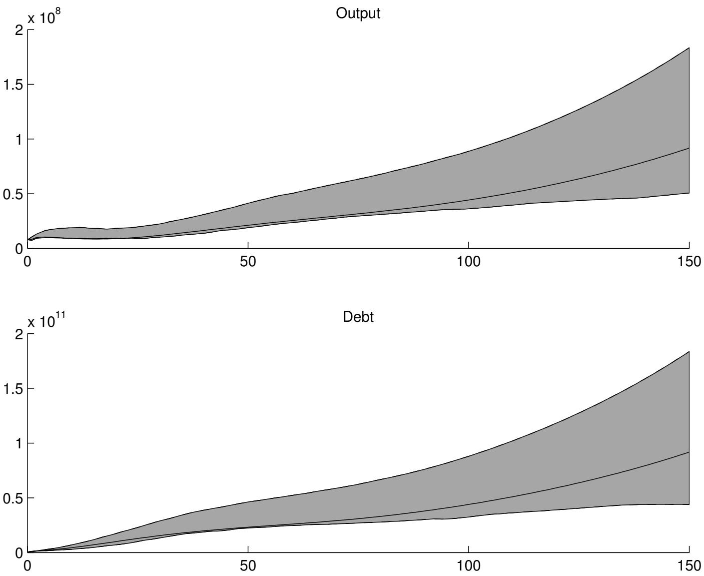


**Figure 5:** Impulse responses from SVAR estimation. Grey area is the 90 percent bootstrapped confidence interval.

the simple correlation coefficient between the estimated impulse responses. We find that the model implies a correlation of 0.52 between output and debt on the business cycle frequency of 32 quarters.

# CHAPTER II

# CHEAP TALK IN A NEW KEYNESIAN MODEL

## *2.1 Introduction*

The high inflation period of the late 1970s and early 1980s lead economic research to start modelling the interactions between monetary and fiscal policy as a coordination game between policy makers.[1] Although it is a common factor in developed countries that fiscal and monetary policies are determined by independent authorities, interactions between the two policies are ubiquitous. This seems intuitive as each policy affects the effectiveness of the other and, hence, the overall effect of the policy mix.[2] More technically, the root of those monetary and fiscal policy interactions can be traced to the government budget constraint as it creates a dynamic link between deficits, interest rates, and, hence, the path of debt. Along this line, research has shown that existence and uniqueness of a rational expectation equilibrium hinge upon the specific design of the policy mix.[3] Within this strategic environment authorities may have at least partially different objectives, differ in their perception about the effectiveness of fiscal and/or monetary policy tools, or differ in their forecasts and/or assessment of states of the economy.[4] Therefore, strategic considerations in the coordination of those policies play a key role in the design of the policy mix and in achieving macroeconomic policy goals. This is supported by various statements issued by Board members of the FED. For example, Powell (2013) states that "[...] fiscal sustainability and its interaction with monetary policy is certainly timely.[...]

[1]Seminal works include, but are not limited to, Pindyck (1976), Blinder (1982), Tabellini (1985, 1986), Alesina and Tabellini (1987), and Debelle and Fisher (1994). For coordination in the open economy setting see, for example, Hamada (1976), Canzoneri and Gray (1985), and Turnovsky et al. (1988). Games between monetary policy and economic agents were first considered by Kydland and Prescott (1977) and Barro and Gordon (1983).

[2]Some examples include credit crowding-out and wage-price spirals created by the effects of taxes on prices. Further, the expectation channel can have adverse effects on the stability of financial markets.

[3]See, for example, Sargent and Wallace (1981), Leeper (1991), and Benhabib et al. (2001).

[4]See, for example, Pindyck (1976), Blinder (1982), and Tabellini (1986).

accommodative monetary policy is often associated with successful fiscal consolidations.[...]", see also Bernanke (2013) and Yellen (2013a, 2013b).[5]

The seminal work by Sargent and Wallace (1981) shows that independent monetary policy is impossible, if the government runs deficits which creates the expectation that it might influence monetary policy in the future.[6] This opposes the canonical monetarist view of inflation arguing that only monetary policy controls the price level by controlling money supply. A flaw in this theory is that once the velocity of money is non-constant, the price level cannot be determined independently from other variables. Hence, the entire equilibrium path of the model matters and, ultimately, multiple equilibria can arise.

A new and different theory to determine the price level was developed mainly by Leeper (1991), Sims (1994, 1997), and Woodford (1994, 1995, 1998, 2001), namely the fiscal theory of the price level. In a nutshell, this theory claims that the price level is solely determined by government instruments. As a central point, again, the government budget constraint is interpreted as an equilibrium restriction that leads to price level changes when fiscal variables change. Put differently, the government chooses a strategy for fiscal policy, that is, it chooses the path of surplus and debt. Then, conditional on this path of actions, the monetary authority chooses the path of the interest rate. Hence, the combination of fiscal and monetary policy (interactions) pins down the price path and a stable path is only achieved for some combination of policies. In conclusion, the FTPL points out that the monetarist point of view does only hold under strong assumptions on the behavior of the fiscal authority.

The second dimension to this policy problem is time-variability. More than monetary policy, fiscal policy is subject to changes to swings in political preferences. While the era of Paul A. Volcker as Chairman of the FED is often stressed as an example for switches in preferences of central bankers, other examples for switches in the conduct of fiscal policy can be found easily in the history of the United States. Consider, for example, the military spending programs by the Presidents Johnson, Carter, Reagan, and G. W. Bush as well as the tax cuts by Ford and Reagan. Further, fiscal policy under President Clinton can be considered to be motivated by fiscal stabilization, while the Bush tax cuts in the early 2000's resuscitated the discretionary fiscal policy design.[7] A dramatic consequence of this policy switch was the return of

[5] For official statements by the Treasury see, for example, Lew (2013).

[6] From a game-theoretical viewpoint the game in Sargent and Wallace (1981) can be characterized as a game of chicken.

[7] The Omnibus Budget Reconciliation Act in 1993 contained a promise to establish a balanced budget.

government deficits and a faster government debt accumulation. Along this line, a growing body of empirical work shows that monetary as well as fiscal policy is subject to regime changes over time.[8] Given the evidence for switches in fiscal and monetary policy, switches in the interactions between the two policies are an implication by the interrelatedness of both policies in the policy mix.

A different motivation for time-dependence of policy interactions are intertemporal financing implications. The way a given fiscal policy expenditure will be financed and the way the monetary authority behaves in the future drives the effects of current policies. Here, regime-switches can trigger wealth effects as well as intra- and intertemporal substitution effects. Much more important, those effects can be generated by the pure expectation of regime switches. Since switches in fiscal and monetary regimes occurred in the past and are likely to happen in the future, agents will form expectations about possible regimes and, hence, their decisions will depend on this probability distribution. Given that agents form expectation about regimes and a credible commitment to keeping a given regime is not feasible, a regime-switching model seems to be the ultimate choice. While those models have been estimated using Markov-switching methods, there is no effort in microfounding switches and quantifying the effects of those switches.

In a strategic environment the fiscal authority can be seen as a sender of a strategic signal about the path of fiscal policy and the monetary authority as the receiver of this signal. While the problem so far implicitly assumed that information in this game is symmetrically distributed, this assumption might be too restrictive due to private information, for example, about the true level and future path of government debt. For example, the fiscal authority might be interested in claiming that debt is higher than it actually is, creating an incentive for the monetary authority to increase inflation in order to lower the real debt burden or to create an incentive to monetize debt. This can be achieved by setting lower interest rates and, in addition, lower interest rate would generate positive real effects, i.e. increase output and employment. On the other side, the government might be tempted to signal lower debt, indicating that is able to meet its debt obligations and having access to private capital markets to refinance its (future) debt.

We add to the empirical and theoretical literature on monetary and fiscal policy coordination. Empirically, we work along the lines of Davig and Leeper (2006, 2008, 2011) using Markov-switching models to characterize the interaction between monetary and fiscal policy. To the best of our knowledge, this paper is the first to estimate

[8]See, for example, Davig and Leeper (2006, 2007) and Chung et al. (2007).

those models including government surplus. The theoretical model is based upon the work by Tabellini (1985, 1986) using a Stackelberg-game between monetary and fiscal authority. In contrast, we consider an imperfect information setting. The theoretical as well as the empirical contributions are related to the work by Bianchi and Melosi (2013) and Bianchi and Ilut (2015), building a (Markov-switching) DSGE model with a rich monetary/fiscal policy mix. In contrast to our paper, in those two papers there is no effect of government debt in the Taylor-rule and no microfoundation of regime changes.

To be precise, this paper has several contributions, both empirical as well as theoretical ones. Starting with the empirical contributions we show that the stance of fiscal policy matters for the conduct of monetary policy. We estimate a single-equation Markov-switching model and a Markov-Switching Vector Autoregressive model (MS-VAR, for short) with time-varying probabilities and document frequent regime switches in the interaction between fiscal and monetary policy.

Along the theoretical dimension, we use a cheap talk game between central bank and government to microfound policy interactions and regime switches. Exogenous (or, in future research, endogenous) changes in the expectation of agents trigger policy shifts. For example, if a Ricardian government increase government spending this might trigger the expectation that the government becomes Non-Ricardian. We call a government Ricardian, if its path of debt is sustainable and it honors its debt obligations. Further, since debt matters for the conduct of monetary policy, the central bank reacts by changing its responsiveness to debt in the Taylor rule. Put differently, changes in the prior beliefs within this game, the pendant to the estimated Markov-switching probabilities, can trigger different outcomes and, hence, different weights in the Taylor-rule. This will have effects on the transmission of the shock and, hence, on the quantitative and qualitative results.

Lastly, we present a case study to show how to implement this cheap talk (or any type of finite sequential or simultaneous move) game in a state-space DSGE model. Our solution approach allows the inclusion the game structure explicitly in the model. That is to say, the sequential move game and its solution algorithm are directly implemented in the state-space of our model, something that is a novelty in DSGE modelling. This guarantees a high degree of flexibility for modelling those interactions and allows to use this approach for a wide range of problems and also allows the analysis of repeated games. Games are implicitly considered in DSGE models but, for example, only to the extent that a Ramsey problem is a Nash solution which gives us a set of equations that can be solved.[9] However, the game is not part of the state-space model and has de facto no direct impact on the solution.

The paper is structured as follows. The next section provides a literature review and section 3 provides empirical evidence about regime switches. Section 4 develops our model and introduces the cheap talk game. Section 5 discusses the simulation results and section 6 briefly concludes.

## *2.2 Literature Review*

### 2.2.1 Monetary Policy

The big contribution of Taylor's (1993) work is a simple description of the Federal Reserve's reaction function.[10] Taylor thereby assumes that there is a systematic and stable relationship between (short-term) interest rates - the instrument of monetary policy - and economic conditions that increases predictability (reducing uncertainty).[11] Of course, this rule is a simplification of what is a very complex undertaking: the conduct of monetary policy. Given the myriad of economic models, the large amount of data, the variety of economic advisors, it is straightforward that a simple Taylor rule will not capture all relevant features. In fact, the "true" rule describing monetary policy is likely to be a non-linear, asymmetric, time-varying, multivariate function on a large information set about past, current, and future (predicted) economic conditions. Nevertheless, it does rely on the assumption that the FED's policy is based on purposeful (the two pillars of price stability and maximum employment) behavior that systematically reacts to changes in the economy. This, for example, can be traced back to the findings by Kydland and Prescott (1977), Barro and Gordon (1983), and Rogoff (1985) showing that a rule-based approach to monetary policy is able to reduce the inflationary bias and increase the stability of the economy. The rule in this context can be interpreted as a credibility-enhancing commitment.

Over the last two decades, Taylor's contribution has been analyzed along the theoretical and the empirical dimension.

Research has shown that the simple Taylor (1993) rule can be derived in a model with nominal price frictions and a quadratic loss function for the central bank with

[9] See, for example, Faia (2008). Another example is Nash bargaining between firm and worker in search and matching models.

[10] Taylor used a parameterized version of an interest rate rule also discussed by Bryant et al. (1993) and Henderson and McKibbin (1993).

[11] See Svensson (2003) and Mishkin (2007) for a critique, claiming that simple Taylor rules basically are "too simple" and ignore relevant information.

feedback to inflation and output (see Svensson (2003)). It has also been shown that the difference in the degree of macroeconomic stability between an optimal and a simple rule is fairly small. Simple rules have practical and robustness advantages compared to optimal rules since they work in a variety of models. Further, Taylor (1993) also introduced researchers to the Taylor principle. This principle suggests that a central bank is able to stabilize macroeconomic variables by simply adjusting the interest rate more than one-for-one with inflation.

Along the empirical side, Taylor (1993) uses a calibrated version of his rule to show that it explains stylized facts for the United States from 1987 to 1992 extremely well. Then, numerous papers have been written over the last two decades dealing with the empirical performance of various specification of the Taylor rule.[12] The simple Taylor rule has been extended along the following three main lines.

First, Clarida et al. (1998, 2000) extend the standard Taylor rule by incorporating forward looking terms. Based on this work, it is sometimes assumed that a given Taylor rule features a four-quarter forward-looking inflation term. Similarly, Goodfriend (1991) and Rudebusch (1995) noticed that the interest rate is characterized by a gradual adjustment over time, giving rise to interest rate smoothing. It is now often assumed that the Taylor rule includes a lagged term, normally found to be large (around 0.9) and highly significant. However, this findings was forcefully challenged by Rudebusch (2002). He claims that the evidence in favour of interest rate smoothing on a quarterly frequency is only an illusion created by serially correlated deviations from the interest rate implied by the Taylor rule. While a direct proof of this hypothesis fails (likely due to a low power of the employed test), an indirect proof is put forward. This proof is based on the premise that smoothing should enable agents to forecast the interest rate with a high precision, but term structure regressions fail to generate this finding.

The second main line includes various other variables in the Taylor rule. Most prominently, asset prices have been added to an otherwise standard Taylor rule. Cecchetti et al. (2002) and Mishkin (2007) advocate the inclusion of measures for asset prices stressing that monetary policy is then able to effectively respond to asset price bubbles and achieve a higher degree of macroeconomic and financial stability. This result relies on very strong assumptions. Most importantly, the central bank has to measure non-fundamental deviations in asset prices and be able to clearly identify a

[12]See, for example, Judd and Rudebusch (1998), Clarida et al. (1999), and Taylor (1999) for an extensive overview.

bubble in real time. However, given the experience with the dot-com and the subprime crisis/housing bubble this assumption is at least questionable and calls for a rather cautious reaction to noisy asset prices. Along this line, Bernanke and Gertler (2001) show that monetary policy should only focus on movements in asset prices to the extent that they affect inflation expectations. Rigobon and Sack (2003) show that the stock market has a significant impact on interest rates. They find that monetary policy does respond to changes in stock prices to the extent that they affect the macroeconomy.

A different stream in the literature focuses on exchange rate movements. Clarida et al. (1998) show that monetary policy did respond to exchange rate movements, but this reaction was of small quantitative importance. Further, Lubik and Schorfheide (2007) find that some countries (Australia, New Zealand, and the U. K.) do not respond to exchange rate movements, while others (Canada, for example) do.

Finally, in recent years more and more research has been conducted on time-variability and non-linearities in Taylor rules. Starting with the latter, non-linearities can, in general, steam from underlying non-linearities in the economy (therefore, in the Phillips curve) or from non-linearities in central bank preferences.

A paper assuming the non-linearities are created by non-linearities in central bank preferences given a linear economy is the study by Cukierman and Muscatelli (2008). They use a parametric smoothing transition regression and find evidence for non-linearities in the U. S. (except the Volcker era) and U. K. Taylor rules. On the other hand, Dolado et al. (2005) assume non-linearities in the Phillips curve given a quadratic loss function of the central bank. They do not find evidence for non-linearities in the Taylor rule used by the FED but do find evidence for the European Central Bank (ECB, for short).

Then, there are several studies using a semi-parametric approach. Hayat and Mishra (2010) find that the FED does respond stronger to (expected) inflation rates between 8 and 10 percent. Conrad et al. (2010) find that non-linearities in the Taylor rule of the FED and the ECB are mainly driven by asymmetric preferences. They show that both central banks react more to positive than to negative deviations from the inflation target. Further, they find that this reaction does increase once a threshold level is reached.

For the remainder of this section, let us discuss the literature on time-variability in Taylor rules. Research along this line has challenged the viewpoint that parameters in the Taylor rule are stable over time and assumed that changes in monetary policy are endogenous. This agenda has been motivated, to a large extent, by the end of the

Great Inflation of the 1970's and the Great Moderation starting in the mid 80's.[13] From a more theoretical viewpoint, the perception alone that switches have occurred in the past and possibly occur in the future might be strong enough to generate macroeconomic effects through an expectation channel.

Relying on OLS and a subsample analysis Judd and Rudebusch (1998) show that monetary policy has been subject to changes over time. Clarida et al. (2000) and Orphanides (2004) find support for this viewpoint, stressing that policy was different during the pre-, post-Volcker era respectively. They show that monetary policy was accommodative pre-Volcker, not satisfying the Taylor principle and, hence, allowing for multiple equilibria.

The more recent literature on VAR's and maximum likelihood estimation supports the premise of time-variability in the conduct of monetary policy. Cogley and Sargent (2001) confirm the previous findings using a reduced form VAR with drifting coefficients. However, Sims (1999, 2001) and Sims and Zha (2006) using Markov-switching models show that the results are not robust to including heteroscedasticity. They find that most of the changes are driven by time-varying volatility of the innovation terms. Then, Cogley and Sargent (2005) proceed and include heteroscedasticity in the applied reduced form VAR. They find evidence for significant changes in the parameters describing monetary policy. Along this line, Boivin (2006) uses the medium-unbiased estimator with real time data allowing for heteroscedasticity and finds evidence for changes in the parameters on inflation and the output gap.[14] He finds that monetary policy likely did not satisfy the Taylor principle in the second half of the 1970's. Further, the changes put forward during the Volcker era occurred gradually, with the largest changes during 1980 and 1982. Starting in the middle of the 80's monetary policy reacted strongly to inflation but less strongly to output. Along this line, Kim and Nelson (2006) use a Heckman-type two-step estimator that accounts for endogeneity. They show that the FED's policy varies across the three subperiods 70's, 80's, and 90's.

Further, Clarida et al. (2000) and Lubik and Schorfheide (2004) show that monetary policy violated the Taylor principle during the 70's and, therefore, did not ensure determinacy. This lead to multiple equilibria and higher volatility in the macroeconomy. Those two papers do not include expectation effects created by past policy switches. Davig and Leeper (2007) show that the presence of policy switches have

[13] Lubik and Schorfheide (2004) and Boivin and Giannoni (2006) find that a New Keynesian model allowing for switches in the conduct of monetary policy generates the Great Moderation.

[14] Further evidence in favour of time-varying parameters is found, for example, by Fernández-Villaverde and Rubio-Ramirez (2008), Partouche (2007), Bianchi (2010), and Trecroci and Vassalli (2010).

crucial implication for determinacy. They show that a general Taylor principle applies, where the Taylor principle is satisfied in the long-run but not necessarily in the short-run. Along this line, Foerster (2013) shows that in periods of stable monetary policy, expectation of switches do create different outcomes depending on the switching type. While all of the papers cited so far assume that switches are exogenous, Davig and Leeper (2008) build a model featuring endogenous switches. In this model, the parameters in the Taylor rule are themselves functions of endogenous variables and do not follow stochastic processes. Switches are triggered, once some endogenous variables reach certain thresholds. They find significant expectation effects and asymmetric effects of symmetric shocks.

### 2.2.2 Fiscal Policy

The monetarist view of inflation argues that monetary policy controls the price level by controlling money supply. In the words of Friedman and Schwartz (1963): "Inflation is always and everywhere a monetary phenomenon". This basic insight can be inferred from the standard quantity theory equation

$$P_t = \frac{Y_t V_t}{M_t}, \tag{36}$$

where $V_t$ is the velocity of the (nominal) money stock, $M_t$, $P_t$ is the price level, and $Y_t$ is nominal output. The quantity theory of money assumes that the velocity is constant and, hence, the price level is proportional to the money stock. Inflation is then determined by the growth rates of money supply and output. However, there is a fundamental problem in this line of reasoning.

Since the velocity of money, in general, is not constant the price level cannot be determined independently from the other variables. Put differently, what matters is the entire equilibrium path of the model economy. Along this line, Kocherlakota and Phelan (1999) have shown that household's money demand largely depends on expectations about future inflation rates. This implies the existence of multiple equilibrium paths for the price level and, hence, multiple possible inflation rates. Therefore, controlling the money supply cannot be sufficient to uniquely determine the price path.

A new theory of the price level determination was developed mainly by Leeper (1991), Sims (1994, 1997), Woodford (1994, 1995, 1998, 2001), and Cochrane (1999). This fiscal theory of the price level claims that the price level is solely determined by government instruments, viz. nominal debt and surpluses. Similar to the quantity theory of money, the fiscal theory of the price level uses a simple accounting identity as a starting point, viz. the government's budget constraint

$$\text{Present Value of Real Surpluses} = \frac{\text{Nominal Debt}}{\text{Price Level}}. \tag{37}$$

Assume that the government commits to a sequence of fiscal variables, i.e. the present value of real surpluses. Then, given an initial condition for nominal debt, there exists a unique price level that fulfills (37). Hence, and in contrast to the quantity theory, the FTPL is able to pin down a unique path of the price level. Monetary policy does still have effect within the FTPL, as nominal debt depends on the interest rate.

On a more general note, Woodford (1995) introduces the concept of Non-Ricardian fiscal policy. In contrast to Ricardian policy, under Non-Ricardian policy the government's budget constraint does not have to hold for all price paths. But if the budget constraint is violated for a price path, this path cannot be an equilibrium, as, for example, markets would not clear. Hence, the government can purposely discard a price path by using its policy tools to ensure that the budget constraint does not hold.

Since its introduction the FTPL was controversially discussed, for example by McCallum (2001) and Buiter (2002).[15] The underlying issue is the role played by the government's budget constraint. There are two ways to understand this equation. Either it is an equilibrium restriction or it is a constraint on policy. The FTPL interprets it as an equilibrium restriction that would lead to price level changes, if (37) is violated. For example, assume that the left-hand side increases and leads to a violation of the equation. Then, the price level would decrease to increase the right-hand side, i.e. the nominal value of debt would increase. On the contrast, the monetarist view understands it as a constraint on policy. In contrast to price level changes, any violations to equation (37) would be undone by changes in fiscal policy. Put differently, while in the FTPL prices change, in the monetarist view the surplus or debt would adjust to ensure equality.

Finally, we turn to a review of the literature on time-variability in fiscal policy and on extensions of the FTPL. Various authors have worked on extending and applying the fiscal theory of the price level.[16] Bassetto (2002) uses a market microstructure of trading posts. In this setting, a bidding process between households and government determines prices on trading posts. Then, the economy is modelled using a game-theoretic approach, which allows to analyze the effects of out-of-equilibrium actions.

[15] Niepelt (2001) shows that the FTPL relies on the assumption of a positive initial stock of nominal debt. Weil (2002) shows that the FTPL does survive a reformulation paying tribute to the findings by Niepelt (2001).

[16] The FTPL is used in the open economy setting, for example, by Dupor (2000), Daniel (2001), and Mackowiak (2007).

It is shown that some strategies for the government exist where - like in the FTPL - fiscal policy solely pins down the price level. Further, Cochrane (2001) shows that the maturity structure of debt is important. He shows that if the government has long-term debt available, it can trade current inflation for future inflation. Further, the time series for surplus and debt generated from the optimal policy match the empirically observed ones for the United States.

Davig and Leeper (2006) estimate a Markov-switching model for monetary policy and fiscal policy for the United States. They find evidence in favor of regime switches in the fiscal-monetary policy mix.[17] Further, they show that the FTPL seems to be always present, as the estimation results imply that tax innovations always affect aggregate demand. They show that a tax cut of 1$ will raise output in the long-run by 0.76 - 1.02 U. S. Dollar depending on the policy regime and a tax cut by 2 percent of output will increase the long-run price level by 1.2 - 6.7 percent. The FTPL works along the expectation channel, that is, agents believe that fiscal policy might switch to an active/Non-Ricardian state even when current policy is passive/Ricardian. Then, Davig and Leeper (2011) focus on the government spending multiplier and, again, estimate Markov-switching fiscal and monetary policy rules. They find that the macroeconomic effects of fiscal policy depend on the current and the expected fiscal-monetary policy regime due to inter- and intratemporal substitution effects as well as wealth effects. Further, they confirm switches between active and passive policy regimes.

Bianchi and Melosi (2013) build a New-Keynesian model with a fiscal rule and Bayesian learning. In this model, the monetary and fiscal policy mix is described by a three-state Markov-switching process. They show that the low long-term interest rates as well as low inflation expectations could hide the true underlying risk of inflation in the United States. Further, Bianchi and Ilut (2015) estimate a Markov-switching DSGE model on the U. S. economy. The model features, i.a., fiscal rules and a maturity structure for government debt. They show that movements in U. S. inflation can be explained by shifts in the balance of power between monetary and fiscal authority. In contrast to our paper, in both papers there is no effect of government debt in the Taylor rule and no microfoundation of regime changes.

Favero and Monacelli (2005) estimate fiscal policy rules and a standard Taylor rule using a single-equation Markov-switching model to identify switches in the fiscal-monetary policy regime. They find significant evidence in favor of regime switches in fiscal policy and document that fiscal policy in the United States follows a systematic

[17] Other papers include Davig (2004), Davig and Leeper (2006, 2007), Chung et al. (2007), and Bianchi (2010).

rule. Further, they find no correlation in the switches of fiscal and monetary policy rules.

Historically, Woodford (1998, 2001) explains the U. S. policy between 1965 and 1989, as well as in the 1940's as being Non-Ricardian. Furthermore, Loyo (1999) explains the high inflation in Brazil in the 1970's and 1980's with the FTPL. He claims that the shift in monetary policy in 1985 aimed to reduce inflation by obeying the Taylor principle was not sufficient, as agents still believed that fiscal policy will be Non-Ricardian in the future. More recently, Bassetto (2006) applies the FTPL to Italy during the 1990's prior to joining the EMU. He argues that private expectation driven by fiscal news were the main source of movements in the exchange rate and, to a smaller extend, in inflation. Furthermore, Sims (2008) explains the high inflation rates during the 1970s and early 1980s with Non-Ricardian policy, while Cochrane (2009) uses Non-Ricardian policy to explain the financial crisis in the United States that triggered the Great Recession.

## *2.3 Empirical Evidence*

### 2.3.1 Data

We use seasonally adjusted, quarterly data from 1974:Q1 to 2012:Q4 (156 observations) for the United States obtained from the St. Louis FED system FRED. In detail, we use the time series for consumer price inflation (CPIAUCSL). Then, the time series for the real gross domestic product is in Billion of U. S. Dollar (GDP). Government spending is federal government consumption expenditures (FGCEXPQ027S). The time series for the budget surplus is the time series for federal government net operating surplus (FGOSNTQ027S) in Million of U. S. Dollar. Total debt is the total amount of public debt (GFDEBTN) in Million of U. S. Dollar.

Further, we use various control variables. We use the Chicago FED adjusted national financial conditions index (ANFCI) to control for risk, liquidity, and leverage in money and debt/equity markets. Positive values of this index imply tighter financial conditions than average. Then, we consider the spread between the returns of long-term and short-term bonds. The return of long-term bonds is measured by the bond buyer Go 20-Bond Municipal Bond index for states and local bonds (WSLB20) while the rate of return for short-term bonds is taken from the 3-month treasury bill (TB3MS). Finally, the interest rate is the effective federal funds rate (FEDFUNDS), which is not seasonally adjusted. The recession dummy is constructed from the NBER recession dates.

Then, for output and government debt we will use gap variables, denoted with a tilde, and defined as

$$\tilde{x} = 100\frac{x_{cycle}}{x_{trend}}, \tag{38}$$

where trend and cycle component ($x_{cycle}$ and $x_{trend}$) are generated by applying a Hodrick-Prescott (HP, for short) filter with smoothing parameter $\lambda = 1600$ to the original time series. The application of a HP filtered output gap follows many other studies and, in particular, Cúrdia et al. (2011) who show that the application of an HP filter performs particularly well in terms of fitting the data. We also use a Hodrick-Prescott filter for the government surplus and use both, the trend and the cycle component in our estimation. The trend component captures structural deviations from a balanced budget, while the cycle component captures the discretionary deviations.

### 2.3.2 Augmented Taylor Rule

In this section we want to establish the result that surplus has significant impact on the conduct of monetary policy; supporting the viewpoint of fiscal-monetary policy interactions. It serves the purpose to answer the basic question whether surplus has a significant impact on monetary policy or not.[18]

We follow the work by Clarida et al. (2000) who use Generalized Method of Moments (GMM, for short) to estimate the parameters of Taylor-type interest rate rules.[19] The main advantage of using GMM is that we do not need to make too many assumptions. Compared to the maximum likelihood estimator a key advantage is that we do not need to be precise about the underlying data generating process. Put differently, lower requirements on the structure imply that GMM can be used for various problems.

We estimate a Taylor-type interest rate rule augmented by surplus, controlling for recessions, and considering various instrumental variables. The specification employed is a widely used specifications taking into account various aspects of the conduct of monetary policy (see e.g. Clarida et al. (2000) and Cúrdia et al. (2011)). First, we postulate a simple linear relationship with a lagged dependent variable to account for partial adjustment in the underlying data generating process, i.e., interest rate smoothing by the FED. Second, we consider a one period lead for all variables to capture forward looking behavior and expectation effects. Formally, the equation is

[18] We could also consider government debt instead of surplus in the Taylor rule. We find that debt is a highly significant and robust variable as well.

[19] We also estimate a Taylor-type rule non-parametrically with surplus and obtain similar results compared to GMM.

given by

$$i_t = \alpha_1 i_{t-1} + (1-\alpha_1)\left[\alpha_2 + \alpha_3\pi_{t+1} + \alpha_4\tilde{y}_{t+1} + \alpha_5 s_{t+1} + \alpha_6 r_{t+1}\right]. \tag{39}$$

Technically, in order to control for potential heteroscedasticity we use a HAC variance-covariance matrix. Further, we use a Bartlett kernel to weight the covariances such that the variance-covariance matrix is positive semidefinite. The bandwidth is chosen based on the Variable Newey-West method. Our instrument set includes four lags of the interest rate, inflation, the output gap, surplus, the bond spread, the recession dummy, and the Chicago FED index. Compared to Clarida et al. (2000) we additionally use surplus, the recession dummy, and the Chicago FED index.

We add a recession dummy to the Taylor rule in order to control for a possible endogenous relation between fiscal policy actions (hence, surplus) and recessions. Further, this can be interpreted as an element that captures asymmetric responses of monetary policy.

Table 2 presents the results for the estimation with the cyclical component, the trend component of surplus respectively. We distinguish between cycle and trend component in order to capture the possibility that the FED considers short- as well as long-run fiscal dynamics. Further, we provide a baseline scenario without surplus and a subsample analysis.

**Table 2:** Taylor rule estimations. Significance levels: $^{***}: 1\%, ^{**}: 5\%, ^{*}: 10\%$.

| | Base | Cycle | Trend | Pre-Greenspan | Greenspan/Bernanke |
|---|---|---|---|---|---|
| $\alpha_1$ | $0.85^{***}$ (0.02) | $0.92^{***}$ (0.02) | $0.88^{***}$ (0.02) | $0.88^{***}$ (0.02) | $0.99^{***}$ (0.002) |
| $\alpha_2$ | $-0.65$ (0.86) | $2.94^{**}$ (1.30) | $-0.90$ (1.09) | $-0.66$ (3.21) | 0.29 (1.07) |
| $\alpha_3$ | $8.74^{***}$ (1.17) | $3.39^{***}$ (1.13) | $10.21^{***}$ (1.60) | $4.58^{***}$ (1.67) | $-7.51^{***}$ (1.75) |
| $\alpha_4$ | $1.80^{***}$ (0.32) | $6.01^{***}$ (1.38) | $2.21^{***}$ (0.59) | $-0.75$ (0.59) | $3.88^{***}$ (1.26) |
| $\alpha_5$ | – | $0.003^{***}$ (0.001) | $0.0003^{***}$ ($6.25e^{-5}$) | $-0.004^{**}$ (0.002) | $0.003^{**}$ (0.001) |
| $\alpha_6$ | $-9.37^{***}$ (2.63) | 1.08 (3.32) | $-12.99^{***}$ (4.15) | $15.66^{***}$ (3.60) | $-138.53^{***}$ (26.44) |
| $R^2_{adj}$ | 0.92 | 0.94 | 0.92 | 0.72 | 0.96 |
| $RMSE$ | 0.86 | 0.68 | 0.83 | – | – |
| $J-Test$ | 0.075 | 0.102 | 0.083 | 0.202 | 0.141 |

In the baseline scenario all coefficients, except the one on the intercept term, are statistically significant at the one percent level. As usual, we find a quite large coefficient on the lagged term, being 0.85. Further, we find large values on the inflation rate, far exceeding the value of one required by the Taylor principle to ensure determinacy. The weight on the output gap is close to 2. Finally, the weight on the recession dummy is estimated to be -9.37 which shows that the FED decreases the interest rate once they expect a recession. It also shows that monetary policy does respond asymmetrically to different states of the economy. In this setting, recessions are times during which monetary policy tends to be more loose than during booms.

The next step is to augment this rule by adding the cyclical component of government surplus. In this scenario, we again obtain a large and significant coefficient (0.92) on the lagged interest rate. The intercept term is now significant at the five percent level at a value of roughly 3. We further obtain a sizably lower coefficient on the inflation rate, at about 3.4. On the contrast, the weight on the output gap increases to 6. Then, we find that the coefficient on the cyclical surplus component is significant at the one percent level with a positive value of 0.003. This corresponds to an increase of the interest rate by three percentage points if surplus increases by one Billion Dollar.[20] In contrast to the baseline scenario the recession dummy is now insignificant.

This result is reversed when we consider the trend component of surplus instead of the cyclical component. In this scenario we again obtain a negative coefficient on the dummy (-12.99). At the same time, we find that the weight on the trend component is highly significant at a value of 0.0003. Here, an increase of surplus by one billion results in an increase of 0.3 percentage points. As before, the weight on the lagged interest rate is close to 0.9 and the weight on the inflation rate is large at a value of 10.21. The output gap enters the equation with a coefficient of 2.21, while the intercept is insignificant.

At the end of this section, we want to briefly motivate time-variability in the coefficients of the augmented Taylor rule as the next section will discuss regime switches over time. We split the sample into the time before and the time with Chairman Greenspan, who became Chairman in 1987. This breakpoint almost splits our sample in half, ensuring enough observations per subsample. One caveat needs to be mentioned. In the Greenspan/Bernanke period it is quite hard to reject the null of a unit root, such that the estimation results should be interpreted with care. What stands out is that the coefficient on surplus - we are using the cycle component here - is negative pre-Greenspan and is positive during the Greenspan/Bernanke years. This documents the time-variability in the interaction term between monetary and fiscal

[20] Recall that a low coefficient is expected as we can't write the time series for surplus in logarithmic terms.

policy. Again, the next section will discuss this issue in depth.

Further, we find that the model with the cyclical surplus component matches the data the best among the different specifications. In addition, it also generates the smallest root mean squared error (within-sample forecast) in forecasting the interest rate over the entire sample period. Hence, including a fiscal variable does increase the forecastability of monetary policy.

Finally, in GMM, as our equation is over-identified, we need to test whether the overidentifying restrictions are small or not. For all five estimations the J-test gives us values that correspond to p-values close to 0.99, such that the model is correctly specified and all our moment conditions hold.

### 2.3.3 Single-Equation Markov-Switching Model

In this section we want to extend the analysis in the previous section and analyze a Taylor rule augmented by a measure for fiscal policy using a single-equation Markov-switching model. We follow the line of research started by Davig and Leeper (2008) and augment this model by allowing for time-variation in transition probabilities. This endogenous approach to modelling regime switches is consistent with the systematic conduct of monetary policy. The weights in the Taylor rule should not stochastically vary over time but rather should be the result of the behavior of the monetary authority. The advantage of using a single-equation framework is the increase in the efficiency compared to a multivariate setting. Further, we are interested in characterizing the interactions between fiscal and monetary variables, in terms of switches in the weight attached to the fiscal stance.

We consider a single-equation setting with four explanatory variables - inflation, the output gap, surplus, and the recession dummy - while controlling for heteroscedasticity. Further, we allow the transition probabilities to vary over time. Formally, this model is given by

$$\begin{aligned} i_t &= \alpha_0 + \alpha_{1,S_t}\pi_t + \alpha_{2,S_t}\tilde{y}_t + \alpha_{3,S_t}s_t + \alpha_{4,S_t}r_t + \varepsilon_{S_t}, && (40) \\ \varepsilon_{S_t} &\sim \mathcal{N}\left(0, \sigma^2_{S_t}\right), && (41) \end{aligned}$$

where $S_t$ is the state in time $t$ and we assume two states. The innovations are Gaussian distributed with state-dependent variance $\sigma^2_{S_t}$, while the $\alpha$'s are the coefficients of the explanatory variables.

Further, there exists a time-varying transition matrix $\mathbb{P}_t$ that describes the likelihood of state changes,

$$\mathbb{P}_t = \begin{bmatrix} p_{11,t} & p_{12,t} \\ p_{21,t} & p_{22,t} \end{bmatrix}, \tag{42}$$

where $p_{ij,t}$ gives the probability from changing from state $i$ to state $j$ at time $t$.

Our estimation yields several interesting results. We find the following coefficients in the Taylor rule

$$
\begin{aligned}
i_t &= \underset{(0.24)}{2.87} + \begin{bmatrix} \underset{(0.21)}{0.21} \\ \underset{(0.33)}{3.27} \end{bmatrix} \pi_t + \begin{bmatrix} \underset{(0.10)}{1.14} \\ \underset{(0.21)}{0.03} \end{bmatrix} \tilde{y}_t + \begin{bmatrix} \underset{(0.0001)}{-0.002} \\ \underset{(0.0001)}{0.0003} \end{bmatrix} s_t + \begin{bmatrix} \underset{(0.57)}{0.58} \\ \underset{(0.69)}{1.74} \end{bmatrix} r_t + \varepsilon_t, && (43) \\
\varepsilon_t &\sim \mathcal{N}\left(0, \begin{bmatrix} \underset{(0.19)}{0.90} \\ \underset{(1.65)}{9.62} \end{bmatrix}\right), && (44)
\end{aligned}
$$

where robust standard errors are shown in parenthesis.

First of all, we can identify two regimes: an accommodative (regime 2 - lower row) and a non-accommodative regime (regime 1 - upper row). In the accommodative regime, monetary policy decreases the interest rate if surplus turns negative (debt increases). On the flipside, in the non-accommodative regime a negative surplus will lead to an increase of the interest rate. Further, we observe that the coefficient is roughly ten times larger in regime 1 compared to regime 2. This implies a stronger response to changes in the fiscal stance in the non-accommodative regime than in the accommodative regime. This seems intuitive as non-accommodative policy should be stronger to create incentives large enough to have the desired effects on the behavior of fiscal policy.

The intercept is highly significant and estimated to be 2.87. In regime 1, the coefficient on inflation is insignificant while it is 3.27 in regime 2. Further, the coefficient on the output gap is significant and 1.14 in regime 1 but is insignificant in the regime 2. Observe that the coefficient in regime 1 does violate the Taylor principle of a coefficient larger than one. However, as shown by Davig and Leeper (2007) monetary policy does not need to fulfil to the Taylor principle in every point in time if they fulfil Taylor principle in the long-run (the general Taylor principle). As already stressed, conclusions on determinacy can not simply be inferred from the coefficient on inflation, as it also depends on the design of fiscal policy (see, for example, Benhabib et al. (2001)).

Finally, the variance of the errors varies considerably across the two regimes. This supports the findings by Sims (1999, 2001) and Sims and Zha (2006) showing that monetary policy shocks are characterized by a large time-varying variance. We find a small value of 0.9 in the first regime and a roughly ten times larger value (9.62) in regime 2. Put differently, our findings show that monetary policy in the accommodative policy regime is more volatile compared to the non-accommodative regime.

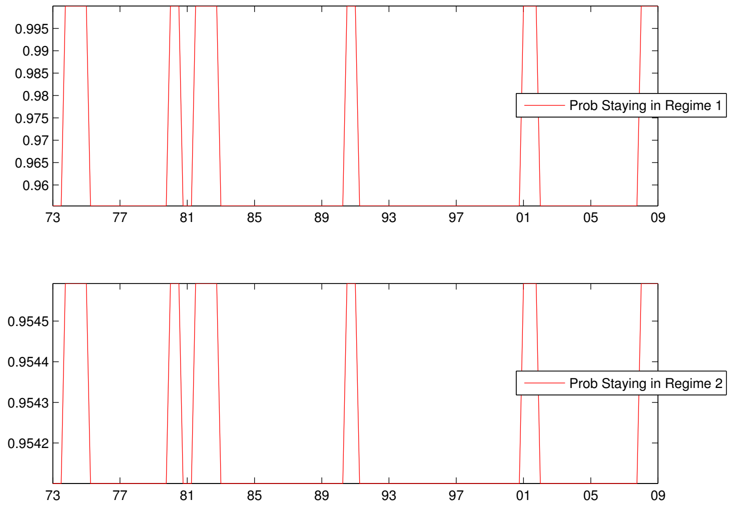


**Figure 6:** Time-varying transition probabilities.

The estimated transition matrix (for the last period) is

$$\mathbb{P}_t = \begin{bmatrix} 0.96 & 0.05 \\ 0.04 & 0.95 \end{bmatrix}. \tag{45}$$

Our results show that tight monetary policy (regime 2) will persist for a longer period of time. We find that the expected duration is 21.83 quarters for regime 2, while the expected duration for regime 1 is 27.21 quarters. This result is supported by the estimated transition matrix. We find that the probability to stay in regime 2 is 96 percent, while it is 92 percent for regime 1. This implies a larger probability to leave the regime for accommodative monetary policy (regime 1). The exit probability from regime 1, 2 respectively is eight percent, four percent respectively.

Figure 6 shows the estimated probabilities of staying in regime 1, 2 respectively over time. Overall, we observe eleven switches from a low probability of staying in the regime - around 95 percent for regime 1 and 2 - to the state with a high probability in staying in the regime - roughly 1 for regime 1 and 0.9546 for regime 2.The implied regime probabilities over time are shown in figure 7.

We find that for most of our observation period regime 2 prevailed. Further, over the entire time horizon six regime switches occurred. Between 1986 and 1995 we observe two local peaks of the probability for regime 1, while a switch did not occur. However, the probability for regime 1 was as high as 80 percent until regime 2 finally clearly took over. Towards the end of our sample we find three of our six switches. Those switches happen quite frequently with an average duration of a regime of about two years. Compared to the switches in the transition probabilities, we find

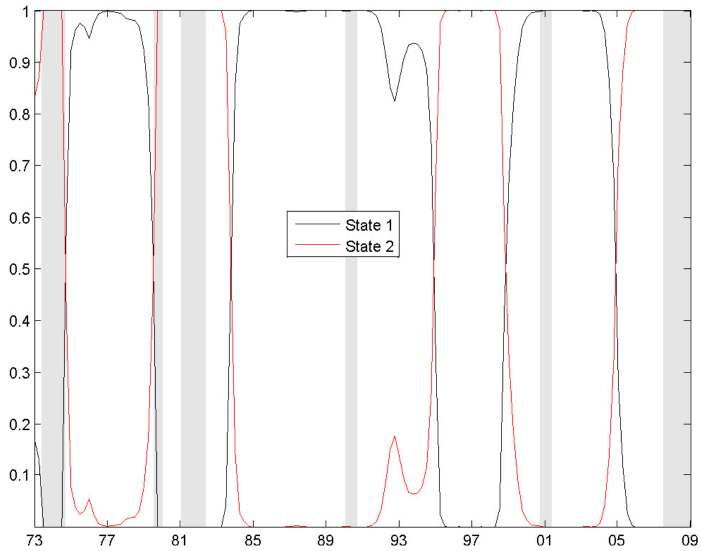


**Figure 7:** Markov-switching probabilities for both states. NBER recession dates are shaded in grey.

less frequent switches but observe that both type of switches are not correlated. Intuitively, regime switches do not necessarily depend on transition probabilities. Since the transition probabilities reflect the expectations of agents that a given regime prevails and are driven by other variables, they do not need to be correlated with regime switches. The results show that the low probability regime (around 0.95 for both regimes) is present for most of the observed time span. High probabilities only last for a few quarters and are quickly reversed.

At the beginning of our sample the accommodative regime (regime 2) was active until in 1975 the first switch occurred. Two events coincide with this switch: first, the U. S. economy left the severe recession of the early 1970's and President Ford's Tax Reduction Act became law. The latter resulted in an increase of public deficit of more than 120 Billion U. S. Dollar in 1975 and 1976.

The non-accommodative regime prevailed for about five years until the beginning of the first recession in the 1980's. At this time, President Reagan's policy change towards supply-side economics away from the famous Keynesian economics became common knowledge and it was expected (at least communicated) that the U. S. economy would benefit from lower tax rates as they were on the increasing part of the Laffer curve. However, this accommodative policy regime prevailed for only four years until 1984. In the previous year, the deficit reached a historic peak of six percent of GDP mainly driven by a large increase in military spending. Further, under the

Reagan administration the United States started to borrow internationally to finance the increased deficits (debt increased from 1 Trillion to 3 Trillion under Reagan). As a consequence, the U. S. turned from the largest creditor to the largest debtor in the world.

From 1985 to 1995 the non-accommodative regime was active. Then, in 1995 we observe a switch towards the accommodative regime. Under the Clinton legislation the Omnibus Budget Reconciliation Act of 1993 became law. It contained corporate tax cuts, tax cuts for low-income families, and tax increases for the wealthiest families. More importantly, the law contained a promise to establish a balanced budget (mainly via spending cuts). Four years later, we observe another switch back to the non-accommodative policy regime. This switch occurred at the same time the Medicare, Medicaid, and SCHIP Balanced Budget Refinement Act of 1999 became law (still under the Clinton administration). After the implementation of his law, social transfers (mainly health care spending) started to surge up to unprecedented levels.

For the end of our sample, we find that monetary policy is accommodative again. The final switch in our sample occurs in 2005, shortly before the financial crisis and the following recession. This is not surprising as the Great Recession lead the FED to drive down interest rates, while the government spending programs increased debt. Interestingly, the accumulation of debt was slowed by large seigniorage gains due to the quantitative easing programs. Here, the FED intentionally or unintentionally avoided a much stronger increase in government debt.

We can draw the conclusion that switches in the relation between monetary and fiscal policy occurred frequently in the history of the United States. Further, we find that all of our switches can be related to policy actions that significantly affected the future path of the fiscal budget. In light of our story, those policy actions are likely to change the expectations of monetary policy makers triggering switches in their behavior.

### 2.3.4 Markov-Switching Vector Autoregressive Model

In the previous section we consider a single-equation setting for two reasons: increased efficiency, and the fact that we are interested in the Taylor rule feedback parameter to government surplus. In this section we want to provide a robustness check and consider a larger model, namely a Markov-switching Vector Autoregressive Model. The advantage of this model is, as usual, that we do not have to specify endogenous (and exogenous) variables. In the following, we consider a four-variate Markov-switching

VAR($p$) model

$$\mathbf{Y}_t = \sum_{i=1}^{p} \mathbf{\Gamma}_{i,S_t} \mathbf{Y}_{t-i} + u_t, \tag{46}$$

where

$$u_t \sim \mathcal{N}\left(0, \mathbf{\Sigma}_{S_t}\right), \tag{47}$$

$$\mathbf{\Sigma}_{S_t} = \begin{bmatrix} \sigma^2_{1,S_t} & \sigma_{12,S_t} & \sigma_{13,S_t} & \sigma_{14,S_t} \\ - & \sigma^2_{2,S_t} & \sigma_{23,S_t} & \sigma_{24,S_t} \\ - & - & \sigma^2_{3,S_t} & \sigma_{34,S_t} \\ - & - & - & \sigma^2_{4,S_t} \end{bmatrix}. \tag{48}$$

The vector of variables is $\mathbf{Y}_t = [\tilde{y}_t, \pi_t, s_t, i_t]$ and $\mathbf{\Gamma}$ is the coefficient matrix for all $t$ and states $S_t$. Again we have a time-varying transition matrix $\mathbb{P}_t$. The best fit to the data is obtained with one lag, no intercept, and a time-varying transition matrix with endogenous feedback to the output gap and surplus.

Figure 8 presents the estimated switching probabilities. We confirm the finding of frequent regime switches over time. The average duration of regime 1 (the accommodative regime) is 4.5 quarters, while it is 2.5 quarters for regime 2. Again, we find a switch from a positive sign ($5.9547e^{-5}$) to a negative sign ($-7.0928e^{-4}$).

In comparison to the single-equation results, we observe more switches. However, most of the single-equation switches are also present in the MS-VAR results. In addition to these six shocks, we observe shocks in 1977, 1982, and 2011. In 1977 the economy was still recovering from the recession starting in 1973, the FED planed to increase money supply, and President Ford's tax cut and energy bill were implemented. The switch from accommodative to non-accommodative policy in 1982, occurred at the time the Tax Equity and Fiscal Responsibility Act of 1982 was implemented by President Reagan. During this period tax revenues dropped and raised concerns about the budget deficit. Finally, the switch towards the end of the sample in 2011 towards non-accommodative policy happens simultaneously with the 2011 debt ceiling crisis, which shifted attention towards the path of U. S. debt and its sustainability. In contrast, the shocks in 1995 and 1999 are not present in this version, although the probabilities over this time are close to 50:50, indicating that this period showed a significant degree of uncertainty about the relation between monetary and fiscal policy. Further, we do observe a volatile time period from 2004 to 2007. This is a time period during which the FED kept interest rates low, government debt started to increase, and the subprime lending crisis started to evolve. In addition, this time span was characterized by increasing levels of aggregate uncertainty, bond volatility, and monetary policy uncertainty (see, for example, Ulrich (2012)). Therefore, it appears that during this period there was a high degree of uncertainty about the true

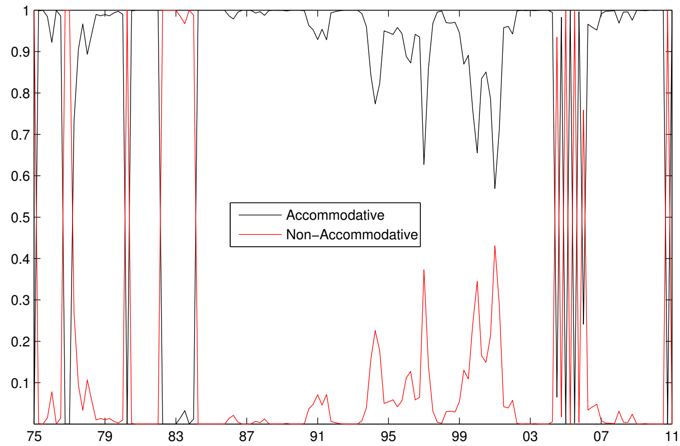


**Figure 8:** Markov-switching probabilities for the Markov-Switching VAR model.

reaction of monetary policy to fiscal policy.

Overall, the MS-VAR results support our previous findings. Compared to the single-equation model, we do observe more shocks and higher uncertainty about the true regime during the period of 1994 to 2002 and 2004 to 2007. This robustness check supports our finding that monetary policy switches frequently between accommodative and non-accommodative regimes. The switches are in line with important policy actions by the FED and the government.

## 2.4 The Model

We now develop a business cycle model for the U. S. economy. Time is discrete and a period is assumed to be a quarter. Our economy is populated by four agents: households, firms, a fiscal, and a monetary authority. Households derive utility from consuming an aggregate consumption basket of differentiated goods and providing labor to firms. Firms produce those goods using a concave production function in labor and face price adjustment costs à la Calvo (1983). Fiscal policy provides spending, collects lump-sum taxes, and issues bonds. Monetary policy sets the nominal interest rate according to a Taylor-type interest rate rule augmented by government deficit. Furthermore, the parameter on debt in this rule is determined by the outcome of a cheap talk game between fiscal and monetary policy and is subject to exogenous regime switches.

### 2.4.1 Households

We assume an infinitely lived representative household who seeks to maximize its utility given by

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U\left(C_t, N_t\right), \tag{49}$$

where $\mathbb{E}$ is the conditional expectation operator and $\beta \in (0,1)$ is the discount factor. Further, $U\left(C_t, N_t\right)$ is the single-period utility function in consumption, $C_t$, and labor, $N_t$, which is compatible with the requirements of balanced growth. We assume that it is separable in its arguments and, specifically, is given by

$$U\left(C_t, N_t\right) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}, \tag{50}$$

where $\sigma > 0$ is the intertemporal elasticity of substitution and $\varphi > 0$ is the inverse of the Frisch labor supply elasticity.

The consumption bundle is defined as

$$C_t = \left[\int_0^1 C_t\left(i\right)^{\frac{\varepsilon-1}{\varepsilon}} \mathbf{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{51}$$

where $\varepsilon > 0$ is the demand elasticity.

Then, the household faces the budget constraint

$$\int_0^1 P_t\left(i\right) C_t\left(i\right) \mathbf{d}i + Q_t B_t \leq B_{t-1} + \left(1-\tau_t\right) W_t N_t - T_t, \tag{52}$$

and a solvency constraint

$$\lim_{T\to\infty} \mathbb{E}_t\left[B_T\right] \geq 0, \ \forall t. \tag{53}$$

The minimum expenditure price index is given by

$$P_t = \left[\int_0^1 P_t\left(i\right)^{\frac{\varepsilon-1}{\varepsilon}} \mathbf{d}i\right]^{\frac{\varepsilon}{\varepsilon-1}}. \tag{54}$$

Further, $W_t$ is the nominal wage and $\tau_t > 0$ is the labor income tax rate. Dividend payments net of lump-sum taxes are denoted by $T_t$ and households buy $B_t$ one-period government bonds at a price $Q_t$. Later on, $i_t$ will be the interest rate defined as $i_t = -\log Q_t$.

We assume that the economy begins with all households having identical financial wealth and consumption histories. This assumption assures that together with the optimal use of the available contingent claims markets, this homogeneity will continue. To be precise, agents have access to a full set of state-contingent Arrow-Debreu securities. Moreover, this allows us to only consider the consumption and savings decisions of a representative household.

The unique solution to the concave optimization problem, maximizing (49) subject to (52) are - assuming that the solution is interior - the following three optimality conditions.

First, using (51) and (54) gives the household's demand schedule

$$C_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon} C_t. \tag{55}$$

Second, given the neoclassical character of the labor market, we obtain a second static equation for the labor supply schedule

$$\frac{N_t^{\varphi}}{C_t^{-\sigma}} = (1-\tau_t)\frac{W_t}{P_t}. \tag{56}$$

Lastly, we obtain an intertemporal optimality condition for the path of consumption

$$C_t^{-\sigma} = \mathbb{E}_t\left[\beta\frac{P_t}{P_{t+1}}\frac{1}{Q_t}C_{t+1}^{-\sigma}\right], \tag{57}$$

which is the well-known consumption Euler equation.

### 2.4.2 Firms

Along the supply-side of the model, we assume the existence of a continuum of monopolistically competitive firms with names $i \in [0,1]$ producing differentiated goods. All firms make us of the same production technology

$$Y_t(i) = Z_t N_t(i)^{1-\alpha}, \tag{58}$$

where $\alpha > 0$ and $Z_t$ is an aggregate Hicks-neutral technology shock that follows a first-order autoregressive process

$$\ln Z_t = \rho_Z \ln Z_{t-1} + \varepsilon_{Z,t}. \tag{59}$$

Its autocorrelation is determined by $1 > \rho_Z > 0$ and its innovations are i.i.d. over time and Gaussian distributed,

$$\varepsilon_{Z,t} \sim \mathcal{N}(0,\sigma_Z). \tag{60}$$

Firms maximize profits by setting prices subject to the discrete time version of the Calvo (1983) mechanism. Accordingly, in each period a firm faces a constant probability of being able to re-set its price, given by $1-\theta$. If a firm is allowed to re-set its price, it solves

$$\max_{P_t^*}\sum_{k=0}^{\infty}\theta^k\mathbb{E}_t\left\{Q_{t,t+k}\left[P_t^*Y_{t+k|t} - \Psi_{t+k}\left(Y_{t+k|t}\right)\right]\right\}, \tag{61}$$

$$s.t.$$

$$Y_{t+k|t} = \left[\frac{P_t^*}{P_{t+k}}\right]^{-\varepsilon} C_{t+k}, \tag{62}$$

where

$$Q_{t,t+k} = \beta^k \left[\frac{C_{t+k}}{C_t}\right]^{-\sigma} \left[\frac{P_t}{P_{t+k}}\right], \tag{63}$$

is the stochastic discount factor. Further, $\Psi_{t+k}(\cdot)$ is the cost function and $Y_{t+k|t}$ is output in period $t+k$ for a firm that was able to re-set its price in period $t$.

The first-order optimality condition for this problem is

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{Q_{t,t+k} Y_{t+k|t} \left[P_t^* - \mu \Xi_{t+k|t}\right]\right\} = 0, \tag{64}$$

where $\mu = \frac{\varepsilon}{\varepsilon-1}$ is the price mark-up over nominal marginal costs in $t+k$, which we denote by $\Xi_{t+k|t} = \frac{\partial \Psi_{t+k}\left(Y_{t+k|t}\right)}{\partial P_t^*}$.

As (64) will later become the well-known Phillips curve, it is useful to re-write this equation in terms of the inflation rate, $\Pi_{t,t+k} = \frac{P_{t+k}}{P_t}$,

$$\sum_{k=0}^{\infty} \theta^k \mathbb{E}_t \left\{Q_{t,t+k} Y_{t+k|t} \left[\frac{P_t^*}{P_{t-1}} - \mu \Upsilon_{t+k|t} \Pi_{t-1,t+k}\right]\right\} = 0, \tag{65}$$

where real marginal costs are defined as

$$\Upsilon_{t+k|t} = \frac{\Xi_{t+k|t}}{P_{t+k}}. \tag{66}$$

### 2.4.3 Fiscal Policy

Formally, our fiscal authority issues bonds, provides government spending (that does not affect the marginal utility of private consumption), and uses labor income taxes to generate revenues. However, only two of those instruments can be set independently, while the third follows from the equilibrium restriction. The equilibrium restriction on the fiscal authority's actions is

$$B_t + \tau_t W_t N_t = Q_{t-1} B_{t-1} + G_t, \tag{67}$$

where $G_t$ denotes government expenditures and $\tau_t W_t N_t$ are labor income revenues.

In order to generate an endogenous relation between fiscal and monetary policy, we assume that spending as well as taxes follow policy rules. This approach is needed as the alternative modelling scenario with exogenously determined processes would not allow effects from monetary policy on fiscal policy. If, instead, spending and taxes depend on endogenous variables this creates a transmission channel from monetary policy interventions to the conduct of fiscal policy.

For simplicity, we follow the work by Chung et al. (2007) and Davig and Leeper (2011) and assume that the fiscal rules have feedback to endogenous variables. We assume that the government has a cyclical target, output, and a structural target, government debt. Then, this rule in log-linear form can be written as

$$\hat{G}_t = -\kappa_g \hat{Y}_t - \kappa_B \hat{B}_t + u_t^g, \tag{68}$$
$$\hat{\tau}_t = -\zeta_g \hat{Y}_t - \zeta_B \hat{B}_t + u_t^\tau, \tag{69}$$

where

$$\ln u_t^g = \rho_g \ln u_{t-1}^g + \varepsilon_t^g, \tag{70}$$
$$\ln u_t^\tau = \rho_\tau \ln u_{t-1}^\tau + \varepsilon_t^\tau. \tag{71}$$

Further, we assume that all innovations are i.i.d. and Gaussian distributed, i.e. $\varepsilon_t^X \sim \mathcal{N}(0, \sigma_X), X \in (g, \tau)$.

### 2.4.4 Monetary Policy - Cheap Talk

First, we assume that the interest rate is determined by a canonical Taylor-type interest rate rule

$$\hat{\imath}_t = \rho + \phi_\pi \hat{\pi}_t + \phi_y \hat{y}_t + \phi_{B,S} \hat{B}_t, \tag{72}$$

where $\rho > 0$ is an intercept and $\phi_\pi > 0, \phi_y > 0$ is the policy weight on inflation, output respectively.

Further, $\phi_{B,S}$ is the weight attached to the change in the debt level. In contrast to the policy weight on inflation or output, the weight on debt switches between two regimes

$$\phi_{B,S} = \begin{cases} -0.002 & \text{if } S = 1, \\ 0.0003 & \text{if } S = 2. \end{cases} \tag{73}$$

The values of the coefficients for the two regimes is taken from our Markov-switching estimation.

So far, we have established that the interest rate is set according to some Taylor-type interest rate rule augmented by government debt. Then, we have shown that the interaction coefficient varies across two regimes. In the last section we developed a stylized New Keynesian model of the U. S. business cycle with monetary and fiscal policy being governed by feedback (Taylor-type) rules. What misses is a microfoundation of regime switches.

We build on the work by Tabellini (1985, 1986): simultaneous move games with corresponding open-loop solutions are an inadequate representation of the institutional settings in all developed countries. He argues that fiscal policy actions are characterized with a sizable implementation lag and are hence non-reversible in a

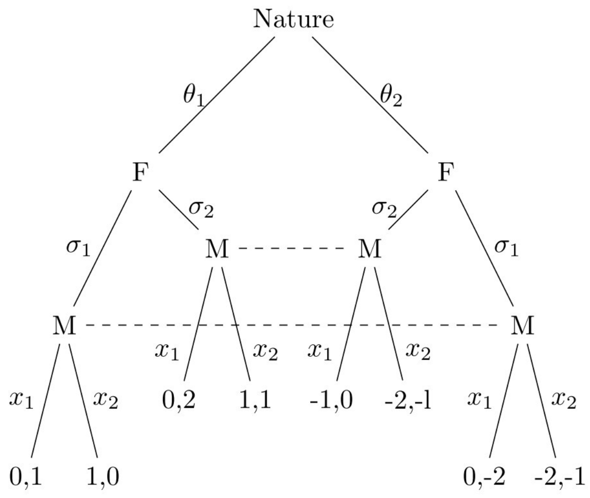


**Figure 9:** Extensive game with imperfect information between fiscal and monetary policy.

given period. Hence, he assumes that fiscal policy is the Stackelberg-leader and monetary policy is the Stackelberg-follower. In the following we will therefore model the interaction between monetary and fiscal policy makers as an extensive game. But, in contrast to the assumption in Tabellini (1985, 1986), we consider imperfect information. Again, imperfect information are introduced based upon the existence of private information for each policy. The introduction of imperfect information is the key in modelling regime switches. We use swings in expectations to generate switches in the policy parameter $\phi_{B,S}$. Technically, movements in the probability put on each state of the world by the players change the payoff matrix and, therefore, affect the equilibrium strategy.

The $2 \times 2 \times 2$ extensive game between fiscal (F) and monetary policy (M) is shown in figure 9 and proceeds as follows.

Nature (the choice player) chooses between two distinct states of the world, $\Theta = \{\theta_1, \theta_2\}$. We interpret those states as Ricardian, $\theta_1$, with probability $\mathbb{P}(\theta_1)$ and Non-Ricardian, $\theta_2$, with probability $\mathbb{P}(\theta_2)$. Those states might be related to elections, the economic outlook, or policy preferences. They are the key to the regime switches in the interest rate rule (72). For given probabilities $\mathbb{P}(\theta_1), \mathbb{P}(\theta_2)$ the game has a sequential equilibrium.[21] However, changes in the probability might trigger that the resulting

[21] There will always exists at least one equilibrium as Kreps and Wilson (1982) proof that every

sequential equilibrium is characterized by different strategies. Consider an intuitive example: if the prior probability for Non-Ricardian increases (put differently, if the monetary authority believes that fiscal policy is more likely to be Non-Ricardian) it is more likely that the monetary authorities wants to constrain fiscal policy action's by acting aggressively and raising interest rates. This increase will lower output (by the usual New-Keynesian reasoning) and hence fiscal policy will lower spending (via the fiscal policy rules).

Each of the policy maker has two strategies available. Fiscal policy can be Ricardian or Non-Ricardian $\Sigma = (\sigma_1, \sigma_2)$ and the monetary policy can be accommodative or non-accommodative $X = (x_1, x_2)$. The order of play is determined as follows: in the first place, nature chooses Ricardian or Non-Ricardian states. The fiscal authority observes this move and chooses a strategy. Then, the monetary authority observes the government's action but not nature's move.

The sequential equilibrium for this game can be found by transforming the extensive game into its normal form. In the following we consider two cases. First, we assume that the first state of the world is more likely, i.e. $\mathbb{P}(\theta_1) = 0.9$, and then consider the case in which the second scenario is more likely, i.e. $\mathbb{P}(\theta_2) = 0.9$. We will show that there exists a (unique) equilibrium in both cases. In the former scenario, the equilibrium strategy for the monetary policy is to be accommodative while in the latter scenario being non-accommodative is optimal. Changes in the probability, interpreted as changes in expectation by agents, then trigger the regime switches.

First, we make the assumption that all players believe $\mathbb{P}(\theta_1) = 0.9$ on state 1 and, for simplicity, we assume w.l.o.g. that $\mathbb{P}(\theta_2) = 1 - \mathbb{P}(\theta_1)$ for the remainder of this section.[22] Then, the usual considerations lead to the Nash equilibrium in the normal form (see figure 4). The strategy profile $\{(\sigma_1, \sigma_1), (x_1, x_1)\}$ is the unique Nash equilibrium of this game. We interpret this notation as playing $\sigma_1$ at the first information set (following $\theta_1$) and $\sigma_1$ at the second information set (following $\theta_2$). The optimal action for the second player is to play $x_1$ at her first information set (following action $\sigma_1$) and playing $x_1$ at her second information set (following $\sigma_2$).

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trembling hand equilibrium is a sequential equilibrium. Building on Selten (1975), trembling hand equilibria always exist for finite sequential games with perfect recall.

[22] For simplicity, we abstract from assuming a distribution function with mean and variance. An example with a normal distribution yields the same conclusions.

| | $x_1, x_1$ | $x_1, x_2$ | $x_2, x_1$ | $x_2, x_2$ |
|---|---|---|---|---|
| $\sigma_1$,$\sigma_1$ | $0, 0.7$ | $0.7, -0.1$ | $0, 0.7$ | $0.7, -0.1$ |
| $\sigma_1$,$\sigma_2$ | $-0.1, 0.9$ | $0.8, 0$ | $-0.2, 0.8$ | $0.7, -0.1$ |
| $\sigma_2$,$\sigma_1$ | $0, 1.6$ | $-0.2, 1.7$ | $0.9, 0.7$ | $0.7, 0.8$ |
| $\sigma_2$,$\sigma_2$ | $-0.1, 1.8$ | $-0.1, 1.8$ | $0.9, 0.8$ | $0.7, 0.8$ |

**Figure 10:** Normal form of the extensive game, $\mathbb{P}(\theta_1) = 0.9$, $\mathbb{P}(\theta_2) = 0.1$.

The next step is to find a system of beliefs that supports this strategy as a sequential equilibrium. We suggest that the system

$$\begin{aligned} \mu(\theta_1 \,|\sigma_1) &= 1, \mu(\theta_1 \,|\sigma_2) = 0, \\ \mu(\theta_2 \,|\sigma_1) &= 1, \mu(\theta_2 \,|\sigma_2) = 0, \end{aligned} \tag{74}$$

forms the beliefs of all players and take them as given from now on.

Then, at the information set following action $\sigma_1$, $x_1$ is the optimal action for player 2. This can be seen by comparing her payoff alternatives. Playing $x_1$ gives her a payoff of 1, while playing $x_2$ gives her a payoff of 0. Similarly, at the information set following action $\sigma_2$ the optimal response is to play $x_1$ for player 2. The payoff comparison gives 0 for playing $x_1$ and 0 for playing $x_2$.

Hence, we have shown that the system of beliefs supports the strategy profile $\{(\sigma_1, \sigma_1), (x_1, x_1)\}$ as part of a sequential equilibrium. The last step is to show that our assessment (the combination of strategy and belief system) is consistent. For this purpose, we need to find a sequence $\{(\mu_n, \sigma_n)\} \subseteq \Lambda_0$ in the subset $\Lambda_0$ of consistent assessments $\Lambda$, in which $\sigma \in \Sigma_0$ and beliefs are computed from $\mathbb{P}$ and $\sigma$ by Bayes' rule. Straightforward, $\sigma \in \Sigma_0$ is fulfilled. Therefore, we can suggest the following sequence of purely mixed behavioral strategies of player 1

$$\begin{aligned} \beta_1^n(\theta_1 \,|\sigma_1) &= \frac{1}{n}, \beta_1^n(\theta_1 \,|\sigma_2) = 1 - \frac{1}{n}, \\ \beta_1^n(\theta_2 \,|\sigma_1) &= \frac{1}{n}, \beta_1^n(\theta_2 \,|\sigma_2) = 1 - \frac{1}{n}. \end{aligned} \tag{75}$$

Next, we need to show that if beliefs for those sequences are pinned down via Bayes' rule they converge to the suggested system (74). Using Bayes' rule, we find that the system of posterior beliefs, $\mu^n$, is given by

$$\begin{aligned} \mu^n(\theta_1 \,|\sigma_1) &= \frac{\frac{1}{n}}{1 - \frac{1}{n} + \frac{1}{n}} = \frac{1}{n}, \mu^n(\theta_1 \,|\sigma_2) = \frac{1 - \frac{1}{n}}{1 - \frac{1}{n} + \frac{1}{n}} = 1 - \frac{1}{n}, \\ \mu^n(\theta_2 \,|\sigma_1) &= \frac{\frac{1}{n}}{\frac{1}{n} + 1 - \frac{1}{n}} = \frac{1}{n}, \mu^n(\theta_2 \,|\sigma_2) = \frac{1 - \frac{1}{n}}{\frac{1}{n} + 1 - \frac{1}{n}} = 1 - \frac{1}{n}. \end{aligned} \tag{76}$$

We can then show that for $n \to \infty$ the system of posterior beliefs converges to the suggested belief system. Formally, $\lim_{n\to\infty} \mu^n \to \mu$, such that

$$\begin{aligned} \lim_{n\to\infty} \mu^n (\theta_1 | \sigma_1) &= 1, \lim_{n\to\infty} \mu^n (\theta_1 | \sigma_2) = 0, \\ \lim_{n\to\infty} \mu^n (\theta_2 | \sigma_1) &= 1, \lim_{n\to\infty} \mu^n (\theta_2 | \sigma_2) = 0. \end{aligned} \tag{77}$$

We have shown that the strategy profile $\{(\sigma_1, \sigma_1), (x_1, x_1)\}$ together with the belief system (74) is a consistent (provided the existence of a sequence (76)) and sequentially rational assessment that, hence, supports a sequential equilibrium.

Next, we consider the case in which the second state of the world is more likely, $\mathbb{P}(\theta_1) = 0.1$. Given the new probabilities, we obtain a new payoff matrix shown in figure 5.

| | $x_1, x_1$ | $x_1, x_2$ | $x_2, x_1$ | $x_2, x_2$ |
|---|---|---|---|---|
| $\sigma_1$,$\sigma_1$ | $0, -1.7$ | $-1.7, -0.9$ | $0, -1.7$ | $-1.7, -0.9$ |
| $\sigma_1$,$\sigma_2$ | $-0.9, 0.1$ | $-0.8, 0$ | $-1.8, -0.8$ | $-1.7, -0.9$ |
| $\sigma_2$,$\sigma_1$ | $0, -1.6$ | $-1.8, -0.7$ | $0.1, -1.7$ | $-1.7, -0.8$ |
| $\sigma_2$,$\sigma_2$ | $-0.9, 0.2$ | $-0.9, 0.2$ | $-1.7, -0.8$ | $-1.7, -0.8$ |

**Figure 11:** Normal form of the extensive game, $\mathbb{P}(\theta_1) = 0.1$, $\mathbb{P}(\theta_2) = 0.9$.

In this game, the strategy profile $\{(\sigma_1, \sigma_1), (x_2, x_2)\}$ is the unique Nash equilibrium. One can show that this strategy is the (unique) sequential equilibrium provided the belief system

$$\begin{aligned} \mu (\theta_1 | \sigma_1) &= 1, \mu (\theta_1 | \sigma_2) = 0, \\ \mu (\theta_2 | \sigma_1) &= 1, \mu (\theta_2 | \sigma_2) = 0. \end{aligned} \tag{78}$$

To sum up, we consider two cases: first, we assume that the first state of the world, being Ricardian, is more likely, i.e. $\mathbb{P}(\theta_1) = 0.9$. Second, we considered the Non-Ricardian state to be more likely, $\mathbb{P}(\theta_2) = 0.9$. This change in the probabilities affects the payoff matrix of the game, ultimately resulting in different optimal strategies. Then, we have shown that for both games a unique sequential equilibrium exists. For the first case, the expectation of the Ricardian state results in an equilibrium in which monetary policy is accommodative. Then, changing the expectations, letting the Non-Ricardian state be more likely, leads monetary policy to switch its strategy and be non-accommodative. Hence, changes in the probability, interpreted as changes in expectation by agents, trigger regime switches.

At the end of this section let us emphasize that regime changes are triggered exogenously. Agents's expectations vary over time and, if the movements are larger enough, trigger the switch in the weight on surplus in the Taylor rule. Where do these expectation changes could come from? As we have discussed in the empirical part of the paper, policy actions by the monetary and the fiscal authority, elections, recessions, or policy actions or economic developments in other countries can affect agents' expectations.

#### 2.4.5 Equilibrium and Calibration

A competitive equilibrium for given initial conditions, the stochastic processes $\{u_t^g, u_t^\tau, Z_t\}$ and a set of prices $\{W_t\}$, is a tuple of processes for $\{B_t, C_t, i_t, G_t, \tau_t, N_t, P_t^*, Y_t\}$ such that

*1. Household optimality*

Given $\{W_t\}$, the processes for $\{C_t, N_t\}$ solve the optimization problem, maximizing (49) s.t. (52) and the solvency condition (53).

*2. Profit maximization*

The processes for $\{N_t, P_t^*\}$ maximize (61) s.t. (62).

*3. Fiscal policy*

The processes for $\{B_t, G_t, \tau_t, \}$ are determined by (68) and (69), while the government budget constraint, (67), holds with equality.

*4. Monetary policy*

The interest rate is determined by (72) and the imperfect information game determines (exogenous) regime switches. Further, the Taylor-type interest rate puts restrictions on the out-of-equilibrium dynamics of the model. However, it does not influence the steady state, as the steady state interest rate is pinned down by the Euler equation.

*5. Market clearing*

Aggregate output is defined as follows

$$Y_t = \left[ \int_0^1 Y_t\left(i\right)^{\frac{\varepsilon-1}{\varepsilon}} \mathbf{d}i \right]^{\frac{\varepsilon}{\varepsilon-1}}, \tag{79}$$

further the labor market clears

$$N_t = \int_0^1 N_t\left(i\right) \mathbf{d}i. \tag{80}$$

Then, the aggregate resource constraint is given by

$$Y_t = C_t + G_t. \tag{81}$$

The set of equations is log-linearized around the non-stochastic steady state. Notice that, as usual, equilibrium existence does not depend on the coefficients in the Taylor rule as the interest rate in equilibrium is pinned down by the Euler equation for consumption in steady state.

Finally, the solution of the game is based upon algorithmic game theory using a bimatrix solution algorithm in the normal form of the game 9. We treat the changes in expectations as an exogenous process. Since we don't know how expectations are created, we assume that they are generated by an exogenous process. Then, the algorithm uses the processes for agents' expectations as input and computes the equilibrium. Then, it maps the equilibrium strategy to one of the two possible values of the debt coefficient in the Taylor rule. Given the coefficient, the state-space system is solved using the usual methods. Future research will endogenize the regime switching process and analyze different expectation building processes.

The model is calibrated to match U. S. stylized facts. The intertemporal elasticity of substitution, $\sigma$, is set to 2 and the discount factor, $\beta$, is set to 0.99 such that we obtain an interest rate of 4 percent p.a.. Further, we assume a quadratic disutility of labor, $\varphi = 1$ and the demand elasticity is set to 6. Hours in steady state are calibrated to 1/3, which equals an average working day of eight hours. The elasticity of the production function is $\alpha = 1/3$. The probability to re-set prices is 2/3 implying an average price duration of three quarters. Monetary policy targets inflation with a parameter of 1.5 and output with a parameter value of 1. Government spending is set to 20 percent from output and debt is calibrated to be 34 percent of output in line with debt holdings of private agents in the United States. The parameters in the two fiscal rules are taken from the estimations by Leeper et al. (2010a). They estimate fiscal policy rules for the United States and report a debt coefficient of government spending, labor taxes respectively of -0.23, -0.05 respectively. Those values imply a stronger reaction of government spending to movements in debt. In contrast, the coefficient on output in the fiscal rules is -0.36 for the labor tax rate and -0.03 for government spending. The autocorrelation of the technology shock is set to 0.9.

## 2.5 Simulation Results

In the following we discuss the differences between the impulse responses to a permanent technology shock, a permanent increase of government spending shock respectively for the baseline case without policy switch(es) and the case with policy

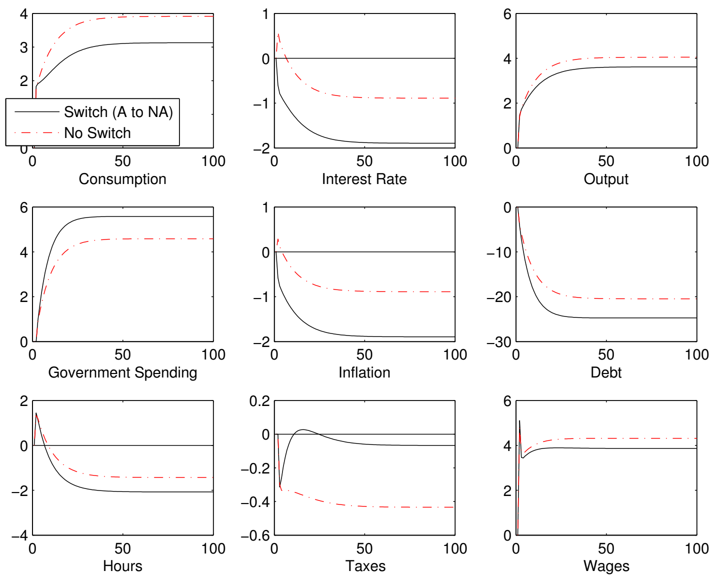


**Figure 12:** Impulse responses to a permanent technology shock. The black line is the switching scenario (from accommodative to non-accommodative). Horizontal axes measure quarters and vertical axes deviations from steady state.

switches. We start with a discussion of the impulse responses subject to only one policy shock. Then, we discuss the response to an anticipated policy switch. Finally, we discuss the response of our stylized model subject to multiple policy switches.

### 2.5.1 One Shock

Our first exercise is to analyze the adjustment path of our stylized model economy to a permanent increase in technology. We use the technology shock as a tool to drive the economy away from its initial steady state as regime switches in steady state will have no effect. Technology shocks seem to be a reasonable choice as they occur frequently and are considered to be a main driver of business cycle movements and economic growth.

We compare the baseline scenario without policy switch with the switching scenario, in which policy switches from accommodative (A) to non-accommodative (NA). The shock as well as the policy switch occur at time 0. Figure 12 presents the impulse response functions for key macroeconomic variables.

In the baseline scenario, the positive technology shock shifts the production frontier outwards and the representative firm produces more goods. Furthermore, higher productivity reduces the marginal costs of the firm which, in turn, allows the firm to set lower prices. As a consequence, inflation falls via the New Keynesian Phillips curve relation. Moreover, the increased productivity puts upward pressure on wages. This creates income and substitution effects. The net effect is a drop in hours worked. Put differently, the firm substitutes technology for labor. Consumption of households increases due to higher output and lower prices. The monetary authority puts a larger weight on inflation than on output and, hence, the interest rate decreases. The lower interest rate additionally increases consumption. Fiscal policy is countercyclical in the model where debt has a stronger effect on government spending but output has a stronger effect on taxes. Hence, spending increases, as debt decreases by more than output increases. For taxes the opposite result holds. Debt decreases due to lower interest rates and the higher tax base.

Next, we discuss the differences between the baseline scenario and the switching scenario. Here, we assume that the monetary authority now believes that the fiscal authority is Non-Ricardian and changes its behavior to create incentives for the fiscal authority to return to Ricardian behavior. The key difference between the two scenarios is the behavior of the interest rate. Recall that debt already decreases due to the higher labor tax base (wages increase by more than hours fall). Hence, since now the coefficient on debt is negative, we obtain an even larger drop in the interest rate. Therefore, debt decreases by even more which, via the fiscal rules, affects spending and taxes. Spending increases further, while the tax rate does not decrease as much as in the baseline scenario. The net effect is a further downward pressure on debt compared to the baseline scenario. Higher government spending crowds out private consumption and the net effect is a slight downward pressure on output. This spills over to lower wages affecting the consumption-leisure allocation.

So far, we considered a switch from accommodative to non-accommodative behavior. Figure 12 also presents this case, if we invert the labelling of the impulse response functions. In this scenario the monetary authority changes its beliefs about the fiscal authority from Non-Ricardian to Ricardian. We already discussed the transmission mechanisms at work. As expected, in the Non-Ricardian regime debt does not decrease as much as it does in the Ricardian regime. Output and consumption are higher. Agents work more and earn higher wages.

In conclusion, the monetary authority is able to create incentives to reduce debt compared to the baseline scenario. However, this it at the cost of lower output compared to the baseline scenario. Let us also emphasize that the differences are not just transitory, but are in fact permanent. The reason is that fiscal policy generates incentives that affect demand and supply side of the economy. Along the demand side, higher government spending increases aggregate demand but crowds out private consumption. The supply side is affected by higher taxes creating effects on the consumption-leisure decision.

### 2.5.2 Anticipation Effects

We have shown that regime switches in the interaction between monetary and fiscal policy are able to generate sizable short- and long-run effects. In this section we want to address possible anticipation effects of switches. Because the driving force of regime switches is changes in expectations about the character of fiscal policy, anticipation of those expectation changes play an important role for policy makers.

Figure 13 presents the impulse responses to a permanent, positive technology shock. We present the baseline scenario without switch, the already shown case with the unanticipated switch at time 0, and an anticipated switch. Agents in this economy anticipate that such a policy switch will occur in three periods. We find that the three quarters in which the interaction is accommodative leads to a higher level of output and consumption. Wages increase and households supply more labor. The monetary authority sets a lower interest rate while the fiscal authority - due to the accommodative monetary policy - accumulates more debt because of higher spending. When the regime switch materializes, the tax rate increases faster compared to the unanticipated scenario. Further, we observe that the non-accommodative monetary policy maker raises the interest rate and output and consumption undershoot the respective unanticipated saddle paths. Those adjustments take roughly five to ten quarters until the saddle paths overlap. The largest and most persistent difference is obtained for taxes. This can be explained by the larger respond of taxes to government debt.

Finally, we notice that the anticipation effects are fairly small for all variables at hand. This, of course, should at least partially be attributed to the fact that we consider a stylized model. For example, the observed differences in the tax rate will have larger effects in a model with Non-Ricardian agents. Nevertheless, even if those differences are fairly small in this stylized model, they should matter if we would perform a welfare analysis in a more involved model, as the paths of consumption and hours worked are affected.

### 2.5.3 Multiple Shocks

Figure 14 shows the response of the model to a positive, permanent technology shocks and two policy switches. The first switch occurs at the same time the technology shock hits and the second shock occurs after 8, 20 periods respectively.

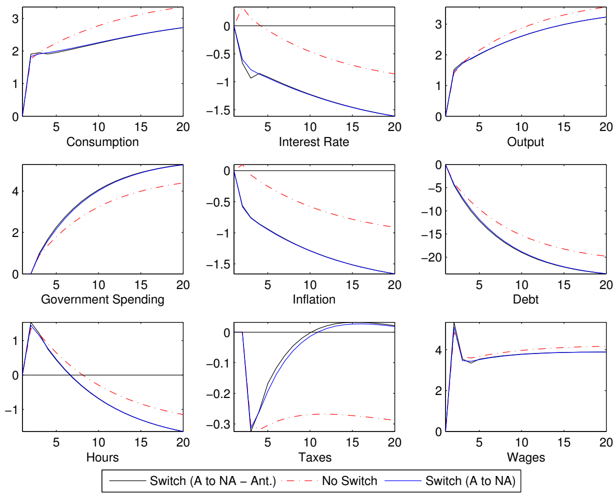


**Figure 13:** Impulse responses to a permanent technology shock. The black line is the switching scenario (from accommodative to non-accommodative). Horizontal axes measure quarters and vertical axes deviations from steady state.

First, we present the impulse responses for the scenario without any policy switch; already discussed in the previous sections. The scenario with a switch to non-accommodative policy and back to accommodative policy after eight quarters is presented with the solid, black line. For this scenario, we find that the impulse responses overlap with the one shock non-accommodative impulse responses for the first three periods until the anticipation effects kick in. Agents realize that policy will be accommodative and revise the previous (optimal) plans made under the non-accommodative policy regime. The anticipation of accommodative monetary policy, as we have seen in our previous discussion, leads towards a lower saddle path of consumption and output. This holds until one period after the policy switch. Further, we find that the interest rate is larger compared to the no switch scenario because output is smaller, inflation is larger, and debt is larger compared to the no switch scenario. The fiscal authority spends less and decreases tax rates by a larger amount. After the switch back to accommodative policy, the effects are reversed. Agents realize that plans aren't optimal any more and the lower levels of output and consumption call for a higher level of production, driven by more hours worked and lower wages. The interest rate decreases even further boosting output, consumption, and hours worked. For the fiscal authority, the accommodative policy allows a higher level of spending and lower taxes. The period of non-accommodative policy has sizable and fairly persistent effects on the adjustment paths.

In the last scenario the switch back to the accommodative regime occurs after

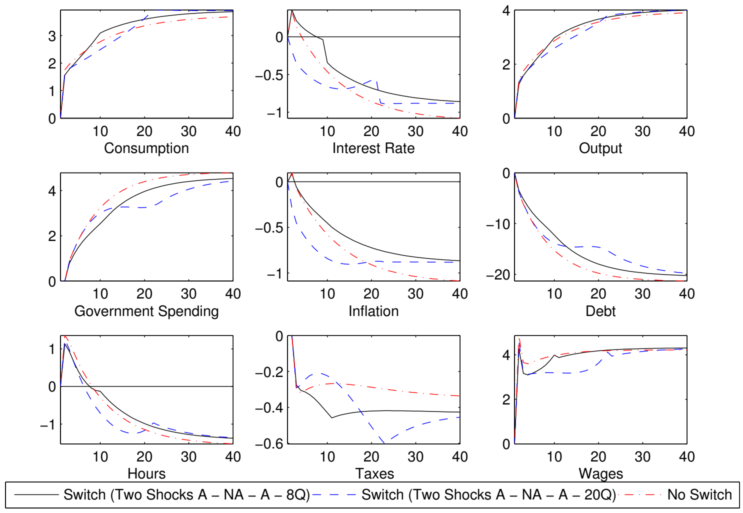


**Figure 14:** Impulse responses to a permanent technology shock. The black line is the switching scenario (from accommodative to non-accommodative). Horizontal axes measure quarters and vertical axes deviations from steady state.

20 periods. We observe that the impulse responses coincide with the already discussed one shock scenario until roughly ten periods before the policy switch back to accommodative behavior. Then, we observe that the anticipation of the policy switch towards accommodative behavior drives the impulse responses back to the no switch impulse responses. Because the non-accommodative policy regime prevails for a much longer time, the impulse responses are similar to the one shock switch scenario. After ten periods anticipation effects become visible. Agents realize that output and consumption are too low compared to the optimal levels under the accommodative policy regime. As a consequence, agents provide more labor and wages remain on a low level. The economy accumulates more debt which - since we are still in the non-accommodative regime - leads to a lower level of spending and higher taxes (compared to the no switch scenario). Once the policy regime finally switches those adverse effects disappear and output and consumption overshoot the no switch scenario. Higher labor supply, lower wages, and lower interest rate (due to the switch in the debt coefficient) boost economic activity and lead to a compensation of the "losses" during the non-accommodative regime.

In summation, multiple regime switches can have large and persistent effects on the adjustment of the model economy. Our findings show that the effects increase in the time between policy switches and that anticipation effects, in the context of multiple switches, are non-negligible.

## 2.6 Conclusion

Fiscal and monetary policies are determined by independent authorities. Nevertheless, interactions between the two policies are common as each policy affects the effectiveness of the other. For example, Sargent and Wallace (1981) have shown that existence and uniqueness of a rational expectation equilibrium hinges upon the specific design of the policy mix. Within this strategic environment authorities may have at least partially different objectives, differ in their perception about the effectiveness of fiscal and monetary policy tools, or differ in their forecasts of states of the economy. Hence, the coordination of those policies is subject to strategic actions and plays a key role in the design of the policy mix. This rather technical point of view is supported by statements of the FED, see, for example, Bernanke (2013) and Powell (2013).

Fiscal and monetary policy are subject to regime switches. More than monetary policy, fiscal policy is subject to changes to swings in political preferences. A prominent example for monetary policy switches is the era of Paul A. Volcker as Chairman of the FED. A recent example for a switch in the conduct of fiscal policy is the Bush tax cut in the early 2000's ending the fiscal stabilization doctrine by President Clinton. A dramatic consequence of this policy switch was the return of government deficits and a faster government debt accumulation. Given this anecdotal evidence for switches in fiscal and monetary policy, switches in the interactions between the two policies are an implication by the interrelatedness of both policies in the policy mix.

In a strategic environment the fiscal authority can be seen as a sender of a strategic signal about the path of fiscal policy and the monetary authority as the receiver of this signal. While the problem so far implicitly assumed that information in this game is symmetrically distributed, this assumption might be too restrictive due to private information, for example, about the true level and future path of government debt. For example, the fiscal authority might be interested in claiming that debt is higher as it actually is, creating an incentive for the monetary authority to increase inflation in order to lower the real debt burden or to create an incentive to monetize debt. This can be achieved by setting lower interest rates and, in addition, lower interest rate would generate positive real effects, i.e. increase output and employment. On the other side, the government might be tempted to signal lower debt, indicating that is able to meet its debt obligations and having access to private capital markets to refinance its (future) debt.

This paper has three contributions. First, we estimate an augmented Taylor-type interest rate rule. We find that government debt is a significant factor and increases forecastability in the United States. Then, we proceed and estimate Markov-switching models and document frequent regime switches in the interaction between fiscal and monetary policy.

Second, we use a cheap talk game between monetary and fiscal authority to microfound policy interactions. Regime switches are exogenously triggered by changes in the expectation of agents. For example, if a Ricardian government increase government spending this might trigger the expectation that the government becomes Non-Ricardian. Since debt matters for the conduct of monetary policy, the central bank reacts by changing its responsiveness to debt in the Taylor rule. Put differently, changes in the prior beliefs within this game, the pendant to the estimated Markov-switching probabilities, can trigger different outcomes and, hence, different weights in the Taylor rule. This will have effects on the transmission of the shock and, hence, on the quantitative and qualitative results.

Finally, we implement this cheap talk game in a state-space DSGE model. The sequential move game, its solution algorithm respectively, is directly implemented in the state-space of our model; something that is a novelty in DSGE modelling. This guarantees a high degree of flexibility for modelling those interactions and allows to use this approach for a wide range of problems and also allows the analysis of repeated games. In a case study, we simulate the impulse responses generated by a stylized New Keynesian model with fiscal policy with and without regime switches. We discuss the differences across the scenarios and show that anticipation effects are fairly small in this model.

# 2.7 Technical Appendix

### 2.7.1 Data

We start by describing the correlations between our variables as this might be interesting in interpreting the role of control variables. Fur this purpose, table 3 presents the unconditional correlations across key variables. Some findings stand out. For

**Table 3:** Correlation matrix.

| | *chi* | *exrate* | *infl* | *i* | *spend* | *spread* | *surplus* | *debt* | *output* | *M2* | *dow* |
|---|---|---|---|---|---|---|---|---|---|---|---|
| *chi* | 1 | 0.15 | 0.03 | 0.35 | −0.16 | −0.19 | 0.08 | −0.3 | 0.22 | −0.05 | −0.08 |
| *exrate* | – | 1 | 0.17 | 0.55 | 0.002 | −0.3 | 0.32 | −0.04 | −0.05 | 0.38 | −0.59 |
| *infl* | – | – | 1 | 0.64 | −0.18 | −0.7 | 0.04 | −0.17 | 0.26 | 0.05 | −0.51 |
| *i* | – | – | – | 1 | −0.13 | −0.85 | 0.21 | −0.32 | 0.36 | 0.17 | −0.71 |
| *spend* | – | – | – | – | 1 | 0.37 | 0.13 | 0.13 | −0.38 | −0.03 | −0.06 |
| *spread* | – | – | – | – | – | 1 | −0.22 | 0.18 | −0.52 | −0.09 | 0.49 |
| *surplus* | – | – | – | – | – | – | 1 | 0.03 | −0.11 | −0.15 | −0.27 |
| *debt* | – | – | – | – | – | – | – | 1 | −0.31 | −0.05 | −0.03 |
| *output* | – | – | – | – | – | – | – | – | 1 | −0.1 | 0.11 |
| *M2* | – | – | – | – | – | – | – | – | – | 1 | −0.16 |
| *dow* | – | – | – | – | – | – | – | – | – | – | 1 |

the variables related to financial markets, we find that the bond spread is negatively correlated with the Chicago FED financial indicator (-0.19) and positively correlated with the Dow Jones (0.49). The Dow Jones is negatively correlated with the Chicago FED financial indicator (-0.08). The exchange rate is negatively correlated with the bond spread (-0.3), the debt gap (-0.04), and the Dow Jones (-0.59). Further, there is a positive correlation between *chi* and the exchange rate (0.15).

The debt gap is negatively correlated with the *chi* indicator (-0.3) and the Dow Jones (-0.03) but positively correlated with the bond spread (0.18). Further, there is a negative comovement between the debt gap and the output gap (-0.31), the inflation rate (-0.17) respectively.

Next, we want to test the time series for non-stationarity. Formally, we test the assumption that a unit root is present using the augmented Dickey-Fuller test (ADF, for short), the Dickey-Fuller Generalized Least Squares test (DF-GLS, for short), and the Phillips-Perron test. Table 4 presents the test results for each variable. The test results indicate that the Dow Jones index, the exchange rate, and the surplus do have a unit root. As a consequence, we use the Hodrick-Prescott filter with $\lambda = 1600$ to generate a stationary series for those three variables.[23] Further, it is also hard to reject the null of a unit root for the interest rate. This particular finding is a known caveat of Taylor rule estimations (see e.g. Clarida et al. (1998) for a discussion). It

[23] We also use first-differenced series which leaves our qualitative results for almost all variables unaffected. However, there is one exception: the significance of the exchange rate does depend on the filtering choice.

**Table 4:** Unit root tests including an intercept. Significance levels: $^{***}:1\%,^{**}:5\%,^{*}:$ 10%.

| | $ADF$ | $DF-GLS$ | $Phillips-Perron$ |
|---|---|---|---|
| $chi$ | $-6.16^{***}$ | $-2.83^{***}$ | $-6.22^{***}$ |
| $exrate$ | $0.14$ | $1.05^{***}$ | $0.44$ |
| $infl$ | $-2.76^{*}$ | $-2.75^{***}$ | $-4.85^{***}$ |
| $i$ | $-2.12$ | $-2.11^{**}$ | $-2.02$ |
| $spread$ | $-3.03^{**}$ | $-2.35^{**}$ | $-2.51$ |
| $surplus$ | $1.62$ | $1.73^{*}$ | $1.44$ |
| $M2$ | $-6.40^{***}$ | $-3.06^{***}$ | $-6.34^{***}$ |
| $spend$ | $-5.47^{***}$ | $-3.29^{***}$ | $-5.66^{***}$ |
| $debt$ | $-6.52^{***}$ | $-4.6^{***}$ | $-4.31^{***}$ |
| $output$ | $-5.56^{***}$ | $-2.46^{**}$ | $-4.85^{***}$ |

can be traced back to the high persistence of the time series and the low power of unit root tests.

### 2.7.2 TVP-VAR

Figure 15 shows the time-varying coefficients for a TVP-VAR estimated on output, inflation, surplus, interest rate with two lags.

### 2.7.3 Multivariate Modelling

We consider a multivariate setting with two explanatory variables - inflation and surplus - while innovations are Gaussian distributed. This model can be formulated as

$$i_t = \alpha_0 + \alpha_{1,S_t}\tilde{y}_t + \alpha_{2,S_t}\pi_t + \alpha_{3,S_t}s_t + \varepsilon_{S_t}, \varepsilon_t \sim \mathcal{N}\left(0, \sigma^2_{i,S_t}\right), \tag{82}$$

$$s_t = \beta_0 + \beta_{1,S_t}G_t + \beta_{2,S_t}I_t + \beta_{3,S_t}i_t + \eta_{S_t}, \eta_t \sim \mathcal{N}\left(0, \sigma^2_{s,S_t}\right), \tag{83}$$

where $S_t$ is the state in time $t$ and we assume two states. The state-dependent variance of the innovations is $\sigma^2$, while the $\alpha$'s and $\beta$'s are the coefficients of the explanatory variables. Further, there exists a transition matrix $P$ that describes the likelihood of state changes,

$$\mathbb{P}_t = \begin{bmatrix} p_{11,t} & p_{12,t} \\ p_{21,t} & p_{22,t} \end{bmatrix}, \tag{84}$$

where $p_{ij}$ gives the probability from changing from state $i$ to state $j$ at time $t$.

Our estimation yields several interesting results. First, let us discuss the parameter estimates of the augmented Taylor-type interest rate rule. We find the following

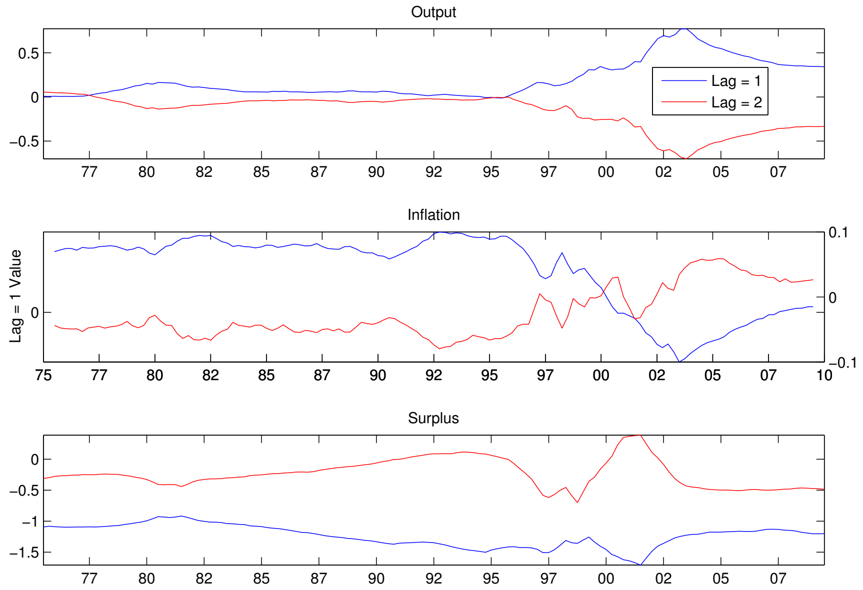


**Figure 15:** Coefficients for output, inflation, and surplus for the interest rate from the TVP-VAR.

coefficients

$$b_t = \begin{bmatrix} \underset{(0.01)}{-0.23} \\ \underset{(0.01)}{0.23} \end{bmatrix} i_t + \eta_t, \eta_t \sim \mathcal{N}\left(0, \begin{bmatrix} \underset{(1.08)}{5.92} \\ \underset{(0.11)}{0.63} \end{bmatrix}\right), \tag{85}$$

$$i_t = 2.12 + 0.24\tilde{y}_t + \underset{(0.11)}{0.88}\pi_t + \begin{bmatrix} \underset{(0.11)}{-2.6} \\ \underset{(0.13)}{1.9} \end{bmatrix} \tilde{b}_t + \varepsilon_t, \varepsilon_t \sim \mathcal{N}\left(0, \begin{bmatrix} \underset{(7.38)}{42.04} \\ \underset{(0.48)}{2.58} \end{bmatrix}\right), \tag{86}$$

where standard errors are shown in parenthesis indicating that all coefficients are estimated significantly.

### 2.7.4 Equation System

The system of log-linear equations for $\{c, i, \pi, y, Z, n, w, b, g, \tau, u^g, u^\tau, MC\}$ is

$$
\begin{aligned}
1 &: \quad c_t = c_{t+1} - \frac{1}{\sigma}\left[i_t - \pi_{t+1} - \rho\right], \\
2 &: \quad \pi_t = \beta\pi_{t+1} + \kappa MC_t, \\
3 &: \quad yy_t = cc_t + gg_t, \\
4 &: \quad y_t = Z_t + (1-\alpha)\, n_t, \\
5 &: \quad \varphi N_t + \sigma C_t = w_t - \frac{\tau}{1-\tau}\tau_t, \\
6 &: \quad BB_t + WN\tau\left[\tau_t + W_t + N_t\right] = iB\left[i_{t-1} + B_{t-1}\right] + GG_t, \\
7 &: \quad \hat{\imath}_t = \rho + \phi_\pi \hat{\pi}_t + \phi_{B,S}\hat{B}_t, \\
8 &: \quad \ln Z_t = \rho_Z \ln Z_{t-1} + \varepsilon_{Z,t}, \\
9 &: \quad \hat{G}_t = -\kappa_g \hat{Y}_t - \kappa_B \hat{B}_t + u_t^g, \\
10 &: \quad \hat{\tau}_t = -\zeta_g \hat{Y}_t - \zeta_B \hat{B}_t + u_t^\tau, \\
11 &: \quad u_t^g = \rho_g u_{t-1}^g + \varepsilon_t^g, \\
12 &: \quad u_t^\tau = \rho_\tau u_{t-1}^\tau + \varepsilon_t^\tau, \\
13 &: \quad MC_t = W_t - Z_t - \alpha N_t.
\end{aligned}
$$

# CHAPTER III

# DELAYS IN PUBLIC GOODS

## 3.1 Introduction

During the Great Recession governments around the world used fiscal policy measures to counter the large adverse effects on real activity. For example, in the United States the 2009 American Recovery and Reinvestment Act (ARRA, for short) provided 550 Billion U. S. Dollar of government spending to foster economic growth and create new jobs (see, for example, Bernstein and Romer (2009)). Furthermore, it is not just the total amount of spending provided, it is the composition of the spending program that matters (see, for example, Feltenstein and Ha (1995), Turnovsky and Fisher (1995), and Devarajan et al. (1996)). The ARRA spend roughly 30 percent, i.e. 160 Billion U. S. Dollar, on investment into public capital.[1] Along this line, government spending for public capital is a non-negligible share (approx. 20 percent) of total spending. In Europe, Gemmell et al. (2012) point out that the second largest budgetary position is cohesion policy, giving grants to underdeveloped regions. Further, public capital does not just play an important role for developed countries but also is crucial for the growth in developing countries. In those countries, public capital is understood to remove the bottlenecks for economic growth, such as insufficient levels of infrastructure, energy production and distribution, education, health care, and communication systems.

Technically, it is a shared believe that investment into public capital is superior to wasteful government (consumption) expenditures. Public capital investments do not just trigger Keynesian demand-side effects over the short-run, as government consumption does, but also creates supply-side effects. Those effects are likely to raise growth through its effect on private capital's marginal productivity.

However, a common factor in developed and developing countries is that investments into public capital are associated with delays (put differently, time lags). Public infrastructure programs, in particular, are subject to large implementation delays

[1] For South Korea, the share of infrastructure spending in the 2008 stimulus package was 43 percent (4.5 Billion U. S. Dollar), while the 2009 stimulus package in Australia spend roughly 36 percent (15 Billion U. S. Dollar) on infrastructure.

due to the required planning, bidding, contracting, and construction process and often require coordination of different regional governments. In the developing country environment, Pritchett (2000) argues that government expenditures are not necessarily productivity enhancing. Due to corruption, inefficiency, and misallocation, there is a questionmark when infrastructure projects are completed and what their performance will be. Examples are the famous "white elephant" projects of unfinished infrastructure programs (roads, bridges, airports, etc.). While the existence of delays is widely acknowledged and an essential part of policy reforms, they are ignored in almost all macroeconomic models.[2]

This paper builds a stylized growth model with delays (i.e., implementation lags) in government investment projects. We contrast the canonical modelling assumption of direct implementation of those projects and discuss the effects of delays on macroeconomic variables.

Technically, we develop a stochastic endogenous growth model in continuous time with public capital. The key difference to the existing literature is the implementation of uncertainty in the public capital accumulation process; by assuming that it is subject to Poisson uncertainty. The government continuously spends but the completion of the public investment project is unknown in advance. We build on the model by Turnovsky (1997) who builds on Barro (1990) and, in addition, introduce implementation lags (driven by Poisson shocks) and depreciation rates for the public and the private capital stock.

Our paper has several contributions. First, we provide the numerical solution for the model with Poisson uncertainty and the (nested) Turnovsky (1997) model using the waveform relaxation algorithm provided by Posch and Trimborn (2013). We consider the Turnovsky (1997) model to be a baseline model as it is does not feature delays and is, therefore, purely deterministic. Further, it contains all basic features commonly assumed in the literature on fiscal policy in growth models. Then, we calibrate the model to match stylized facts of the U. S. economy and discuss the optimal policy functions. We find that the deterministic and the stochastic policy functions are substantially different. The introduced uncertainty about the accumulation of public capital therefore has sizable effects on the behavior of risk-averse agents.

Second, we use the model to discuss the effects of three policy reforms implemented by the U. S. government. We find that an increase in government expenditures (the

[2]For example, Power and King Jr. (2009) in the Wall Street Journal present anecdotal evidence about the delay in spending parts of the ARRA expenditures (governed by the Department of Energy).

ARRA program) raises the growth rate while an increase in the income tax rate (the Omnibus Reconciliation Act) reduces the growth rate. Finally, we consider a policy reform exclusive to our model, namely a reallocation of government expenditures towards projects not associated with implementation lags (the "New Deal", or military buildups). We find that such a policy increases the growth rate. While the effects are smaller compared to the increase in government spending, the main advantage of this policy reform is that it does not generate additional costs. Policy reforms in the Turnovsky (1997) model intuitively have larger effects because, without lags, the public capital stock growths faster and agents do not face uncertainty. However, we find that the Turnovsky (1997) model overshoots the empirically observed values for key macoroeconomic variables significantly.

Finally, our paper also sheds some light on the reasons for different development dynamics across countries. We find that countries like China or India, with long lags and more spending associated with implementation lags, have higher growth rates due to an uncertainty effect on agents' behavior. They tend to consume less and save more, therefore, increasing the accumulation of private and public capital stocks.

The paper is organized as follows. The next section discusses the related literature and section 3 develops our model. Section 4 provides the numerical solution for our model, discusses the resulting policy functions and transitional dynamics. Section 5 evaluates three types of policy reforms, while section 6 briefly concludes.

## *3.2 Literature Review*

The empirical literature on the effects of government expenditures and growth dates back to Ratner (1983). Using annual data for the U. S. he finds an elasticity of output to public capital of 0.06. Aschauer (1989) controlling for capital utilization finds a value of 0.39. Bom and Ligthart (2009) perform a meta-analysis of the literature and estimate an average value of 0.184 for the elasticity.

Easterly and Rebelo (1993) and Canning and Pedroni (2004) also find that public capital has positive effects on long-run growth. The papers by Easterly and Levine (1997), Canning (1999), and Canning and Bennathan (2000) find that infrastructure indicators have a significant effect on growth. Micro level studies (for example, Reinikka and Svensson (1999)) find large effects of public capital on growth.

A different stream in the literature tries to identify the factors of growth. Temple (1999) and Rodrik (2003), for example, find three main factors: geography, institutions, and trade. All of them are directly, or indirectly, linked to government expenditures. Geographical deficiencies can, at least partially, be removed by investment into infrastructure (roads, harbors, etc.). Intuitively, the quality of institutions depends on the expenditures into the number and the skill of public administration officers as well as on the judicial authority. Lastly, trade costs are related to the quality of public services (customs, transportation links, etc.).

The empirical literature focusing on the non-linear effects of debt on growth goes back to the work by Cohen (1997) showing that a higher debt rescheduling probability reduces growth. Elbadawi et al. (1997) find a critical threshold of 97 percent debt-to-GDP ratio. Patillo et al. (2002) and Patillo et al. (2004) consider total external debt and find evidence for an inverted-U shaped relationship.

Then, the often cited paper by Reinhart and Rogoff (2010) relies on statistical methods and finds a critical debt-to-GDP threshold of 90 percent. Papers estimating a threshold of roughly 90 percent are Kumar and Woo (2010), Cecchetti et al. (2011), and Padoan et al. (2012). For Europe, Checherita and Rother (2010) and Baum et al. (2012) find similar results. Caner et al. (2010) and Elmeskov and Sutherland (2012) report a threshold of roughly 80 percent.

The seminal contribution in the theoretical literature is due to Barro (1990). He introduces productive public capital into an endogenous growth model. He finds that expenditures into public capital instead of wasteful consumption increases the growth rate. While Barro (1990) treats public capital as a flow variable, Futagami et al. (1993) treat it as a stock variable. A key finding in this paper is the existence of transitional dynamics not present in the Barro (1990) model. Those dynamics imply that the welfare maximizing tax rate is smaller than the growth rate maximizing tax rate.

Various other authors extended the Barro (1990) model along different dimensions.[3] Barro and Sala-i-Martin (1992) focus on rivalry and excludability in the provision of public goods while Turnovsky (1996) and Fisher and Turnovsky (1998) analyze the effects of congestion.

Brauninger (2005) and Yakita (2008) introduce government debt. Brauninger (2005) shows that there exists a threshold for the effect of the deficit ratio for the growth ratio. Yakita (2008) relying on the stock approach, shows that the threshold is a function of the initial level of public capital.[4] Figure 16 shows the time series of government deficit for the United States. With the exception of the early 1950's

[3] For models with elastic labor supply see Tanzi and Zee (1993), Milesi-Feretti and Roubini (1994), and Turnovsky (2000).

[4] Except debt, expenditures can also be financed via taxes, inflation, or budget reallocations. See, for example, Christie and Rioja (2012), Gemmell et al. (2012), and Denaux (2007).

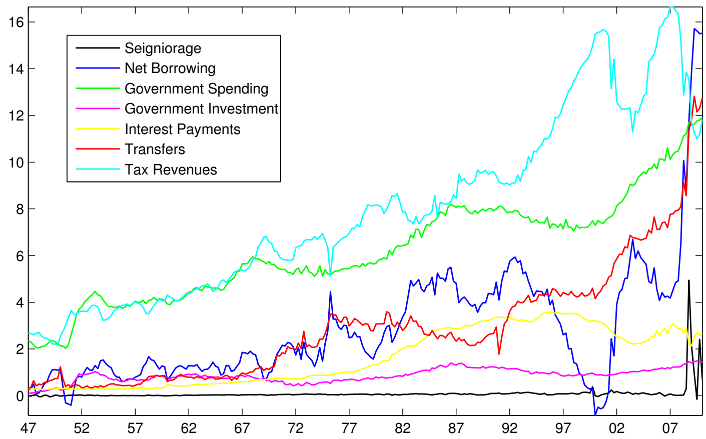


**Figure 16:** History of fiscal policy in the United States. Vertical axis measures Billion of real U. S. Dollar, horizontal axis measures quarters.

and the late Clinton and early Bush years the U. S. government was running deficits in every given period. Especially, the build up during the Reagan years and the almost exponential trend starting in 2007 have increased the awareness of the fiscal challenges in the future. In the words of former Chairman Bernanke (2010): "Amid all of the uncertainty surrounding the long-term economic and budgetary outlook, one certainty is that both current and future Congresses and Presidents will have to make some very tough decisions to put the budget back on a sustainable trajectory."

In theory, there are various channels through which government debt affects the economy in the short- and, more importantly, in the long-run. The main transmission channels work through higher long-term interest rates, increased uncertainty about the fiscal sustainability, (expectations about) higher tax rates crowding-out private investment, therefore, harming capital accumulation. Moreover, various empirical studies find that debt has non-linear effects on the growth rate, see, for example, Checherita and Rother (2010) and Reinhart and Rogoff (2010). Those studies point towards a Laffer-curve behavior of the growth rate in the debt level. Put differently, below a certain threshold (often found to be around a 90 percent debt-to-GDP ratio) additional debt has positive effects on the growth rate which is reversed after reaching the threshold.

Along this line, the role of financing decisions for growth effects of government expenditures goes back to Turnovsky (1996). In his model the optimal mix of consumption taxes, income taxes, and debt depends on the level of infrastructure relative to the social optimum. Ghosh and Mourmouras (2004) consider the implications of the golden rule of public finance and find that the rule, on the one hand, is an effective device to restrict the composition of government expenditures and, on the other hand, find that an increase in government consumption lowers welfare. Similarly, Greiner and Semmler (2000) find that the growth effects crucially depend on the current budgetary regime. Christie and Rioja (2012) build a two-sector endogenous growth model and find that the impact of government expenditures on infrastructure, education, and healthcare on the growth rate depends on the way the spending is financed and on the initial budgetary position.

Further, Feltenstein and Ha (1995), Turnovsky and Fisher (1995), and Devarajan et al. (1996) stress that the composition of government expenditures matters. Turnovsky and Fisher (1995) focus on the different growth effects of government consumption vs. infrastructure expenditures. Feltenstein and Ha (1995) show that public capital has different effects across sectors while Devarajan et al. (1996) show that the condition for higher spending leading to higher growth depends on the share of government expenditures allocated to the different components of government expenditures.

Saint Paul (1992) and Mourmouras and Lee (1999) consider Non-Ricardian agents à la Blanchard (1985) - Yaari (1965).[5] While Mourmouras and Lee (1999) do not consider debt, Saint Paul (1992) shows that higher debt always decreases the growth rate and harms future generations.

Zhu (1992) applies the Ramsey approach to optimal taxation in a stochastic growth model. He finds that the government can make use of state contingent capital income taxes, replacing state contingent bonds, to ensure optimal quantity allocations. Further, it is shown that the result of a zero capital tax rate does not hold in a stochastic growth model. Finally, the paper shows that consumption and investment to output ratios move inversely to the expenditure to output ratio, while the employment rate and the optimal labor income tax rate move with the expenditure to output ratio. Chamley (2001) shows that the famous zero capital taxation result depends on the assumptions of long horizon and perfect markets. Due to borrowing constraints, agents are not able to insure against idiosyncratic risks, cannot smooth consumption, and, hence, have a motive for precautionary savings. It is shown that a tax is efficient if and only if savings are positively or negatively correlated with consumption. Benavie et al. (1996) use a stochastic growth model with capital adjustment costs. They show that an increase in the income tax rate reduces growth and increase the volatility of the growth rate. Increasing government spending has

[5] Other papers considering Non-Ricardian agents include Boldrin (1992), Jones and Manuelli (1992), and Rivas (2003).

no effect on mean or variance of the growth rate.

## *3.3 The Model*

In the following section we describe our model environment and provide the solution to the optimal control problems. We start with the centralized model, where a social planner maximizes welfare and proceed by introducing government debt and solving the decentralized version of our model. In the following, we will abstract from money as we focus on the real effects of fiscal policy. Further, we model the economy as a representative worker-entrepreneur with infinite planning horizon and perfect foresight.

### 3.3.1 The Centralized Model

*3.3.1.1 The Framework*

The model builds mainly on the work by Futagami et al. (1993) and Turnovsky (1997) who build on Barro (1990). In contrast to those papers, we introduce uncertainty into the accumulation process of public capital. We assume that the point in time at which the government investment project is finalized is not known for sure. Put differently, the increase in public capital stock, i.e., the implementation shock, occurs at a random point in time. This assumption pays tribute to the various kinds of lags associated with government investment programs.

There is a social planner seeking to maximize the lifetime utility of a representative agent given by

$$\mathbb{E}_t \int_0^{\infty} e^{-\rho t} u\left(C_t\right) dt, \tag{87}$$

where $C_t$ denotes consumption and $\rho > 0$ is the discount factor. The mathematical expectation operator is denoted by $\mathbb{E}_t$ and $u\left(C_t\right)$ is the agent's utility function. For the utility function we assume the following specification

$$u\left(C_t\right) = \frac{C_t^{1-\gamma}}{1-\gamma}, \tag{88}$$

where $\gamma > 0$ is the elasticity of substitution.

Output is produced using a Cobb-Douglas type production technology

$$Y_t = A K_{G,t}^{\alpha} K_t^{1-\alpha} \equiv A \left(\frac{K_{G,t}}{K_t}\right)^{\alpha} k_t, \tag{89}$$

where $K_G$ is the public capital stock and $k$ is the private capital stock of an individual firm. Further, $0 < \alpha < 1$ is the elasticity of public capital and $A > 0$ gives the level of technology. Then, the economy wide aggregate stock is given by $K = Nk$, where $N$ gives the number of individual firms. Along this line, $Y = Ny$, but we assume $N = 1$ for the remainder as in Turnovsky (1997).

The resource constraint is given by

$$Y_t = C_t + I_t + G_t, \tag{90}$$

where private capital stock accumulation is driven by private investment

$$dK_t = (I_t - \delta_K K_t)dt, \tag{91}$$

where $\delta_K \geq 0$ denotes the rate of physical capital depreciation of the private capital stock. Further, the accumulation process for public capital is

$$dK_{G,t} = \left[\theta G_t - \delta_G K_{G,t}\right] dt + \left[(1-\theta)\, G_{t-}\right] dN_t, \tag{92}$$

where $\delta_G \geq 0$ is the depreciation rate of the public capital stock. Further, $dN_t$ is a Poisson process counting the number of implementations with $\lambda \geq 0$ the arrival rate of the implementation shocks, so $\mathbb{E}_t dK_{G,t} = \left[(\theta + (1-\theta)\lambda)G_t\right] dt$.[6] The parameter $0 \leq \theta \leq 1$ is thought of to categorize public investment projects into projects with and without implementation lags. Our new approach is that some infrastructure projects, such as a bridge or a road, require running costs long before completion, however, they only contribute to public capital by the time they have been successfully implemented. Typically, this completion date is random. From the planner's perspective, we may thus view investment into public capital as risky.

Assume that government investment is a fixed share, $0 \leq g \leq 1$, of total output[7]

$$G_t = gY_t. \tag{93}$$

Finally, the budget constraint is given by

$$\begin{aligned} dK_t &= \left[Y_t - C_t - G_t - \delta_K K_t\right] dt, & (94) \\ &= \left[(1-g)\, A K_{G,t}^{\alpha} K_t^{1-\alpha} - C_t - \delta_K K_t\right] dt. & (95) \end{aligned}$$

*3.3.1.2 Solution*

In the following section we derive the solution for the centralized model. For this purpose, we apply the methods of dynamic stochastic programming.

[6] We define $G_{t-} \equiv \lim_{s \to t} G_s$ for $s < t$ is the left-limit at $t$. Intuitively, this variable represents the value of government expenditures the instant before a successful implementation of a new project.

[7] In order for the equilibrium to be sustainable, government spending itself needs to be pinned to an index of growth, such as output.

Consider the control problem maximizing utility (87) subject to (92) and (95). Choosing an admissible control $C_t \in U_C$ and defining $V(K_t, K_{G,t})$ as the value function, Bellman's principle gives

$$\rho V(K_t, K_{G,t}) = \max_{C_t \in U_C} \left\{ u(C_t) + \frac{1}{dt}\mathbb{E}_t dV(K_t, K_{G,t}) \right\}, \tag{96}$$

subject to

$$\begin{aligned} dK_t &= \left[(1-g) A K_{G,t}^{\alpha} K_t^{1-\alpha} - C_t - \delta_K K_t\right] dt, & (97)\\ dK_{G,t} &= \left[\theta g A K_{G,t}^{\alpha} K_t^{1-\alpha} - \delta_G K_{G,t}\right] dt + \left[(1-\theta) g A K_{G,t-}^{\alpha} K_{t-}^{1-\alpha}\right] dN_t, & (98)\\ (K_0, K_{G,0}) &\in \mathbb{R}_+^2. & (99) \end{aligned}$$

Here, $\mathbf{K} = (K_t, K_{G,t}) \in U_{\mathbf{K}}$ is the vector of state variables and $U_{\mathbf{K}} \subseteq \mathbb{R}^2$ is the state space. The control variable is the rate of consumption, $C_t \in U_C$, and $U_C \subseteq \mathbb{R}$ is the admissible control region.

The first-order condition for this problem is given by

$$u_C(C_t) = V_K(K_t, K_{G,t}), \tag{100}$$

which makes the control a function of the state variables for all $t \in [0, \infty)$.

Then, after some algebra (see the technical appendix), we may summarize the equilibrium dynamics as

$$\begin{aligned} dq_t &= \left\{ \begin{array}{c} \left[\left[\rho - \theta g \alpha A z_t^{\alpha-1} + \delta_G\right] q_t - \alpha A (1-g) z_t^{\alpha-1}\right] \\ - \left[\left[\rho - (1-g)(1-\alpha) A z_t^{\alpha} + \delta_K\right] q_t - q_t^2 \left[\theta g A (1-\alpha) z_t^{\alpha}\right]\right] \\ +\lambda \left[q_t \tilde{c}_t^{-\gamma} + ((1-\theta)(1-\alpha) g A z_t^{\alpha}) q_t^2 \tilde{q}_t \tilde{c}_t^{-\gamma} \right. \\ \left. - \left(1 + (1-\theta) \alpha g A z_t^{\alpha-1}\right) q_t \tilde{q}_t \tilde{c}_t^{-\gamma}\right] \end{array} \right\} dt \\ &\quad + \left[\tilde{q}_t q_t - q_t\right] dN_t, & (101)\\ dz_t &= \left[\theta g A z_t^{\alpha} - \delta_G z_t - (1-g) A z_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt \\ &\quad + \left[(1-\theta) g A z_t^{\alpha}\right] dN_t, & (102)\\ dc_t &= c_t \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{\theta g A (1-\alpha)}{\gamma} q_t z_t^{\alpha} - (1-g) A z_t^{\alpha} \\ + \frac{\lambda}{\gamma}\left[\tilde{c}_t^{-\gamma} + ((1-\theta)(1-\alpha) g A z_t^{\alpha}) q_t \tilde{q}_t \tilde{c}_t^{-\gamma} - 1\right] + c_t \end{array} \right\} dt \\ &\quad + \left[\tilde{c}_t c_t - c_t\right] dN_t, & (103) \end{aligned}$$

where we defined

$$z_t \equiv \frac{K_{G,t}}{K_t}, \quad q_t \equiv \frac{V_{K_G}}{V_K}, \quad \text{and} \quad c_t \equiv \frac{C_t}{K_t}, \tag{104}$$

as the public-to-private capital stock ratio, the ratio of their respective shadow prices (costate variables), and the consumption to private capital ratio, respectively, and

$$\begin{aligned} \tilde{c}_t &= \frac{c\left(K_t, K_{G,t}\left(1 + gA(K_{G,t}/K_t)^{\alpha-1}\right)\right)}{c(K_t, K_{G,t})}, & (105)\\ \tilde{q}_t &= \frac{q\left(K_t, K_{G,t}\left(1 + gA(K_{G,t}/K_t)^{\alpha-1}\right)\right)}{q(K_t, K_{G,t})}, & (106) \end{aligned}$$

defines the optimal response of the consumption to private capital ratio and the ration of the shadow prices an instant after the successful implementation. Note that $\tilde{c}-1$ denotes the percentage change in the consumption to private capital ratio, i.e., the percentage change of consumption, after successful implementation.

Our solution nests two special cases: If we set $\theta = 1$ it resembles the Turnovsky (1997) model (set $\delta_K = \delta_G = 0$),

$$
\begin{aligned}
dq_t &= \left\{ \begin{array}{c} \delta_G q_t + \left[(1-g)(1-\alpha) z_t q_t - \alpha(1-g)\right] A z_t^{\alpha-1} - \delta_K q_t \\ +\left[(1-\alpha) z_t q_t - \alpha\right] q_t g A z_t^{\alpha-1} \end{array} \right\} dt, && (107)\\
dz_t &= \left[ g A z_t^{\alpha} - \delta_G z_t - (1-g) A z_t^{\alpha+1} + c_t z_t + \delta_K z_t \right] dt, && (108)\\
dc_t &= c_t \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{gA(1-\alpha)}{\gamma} q_t z_t^{\alpha} \\ -(1-g) A z_t^{\alpha} + c_t \end{array} \right\} dt, && (109)
\end{aligned}
$$

whereas for $\theta = 0$ it resembles the pure implementation lags model,

$$
\begin{aligned}
dq_t &= \left\{ \begin{array}{c} \delta_G q_t + \left[(1-g)(1-\alpha) z_t q_t - \alpha(1-g)\right] A z_t^{\alpha-1} - \delta_K q_t \\ +\lambda \left[ (1-\tilde{q}_t) q_t + \tilde{q}_t \left[(1-\alpha) z_t q_t - \alpha\right] q_t g A z_t^{\alpha-1} \right] \tilde{c}_t^{-\gamma} \end{array} \right\} dt \\
&\quad + \left[\tilde{q}_t q_t - q_t\right] dN_t, && (110)\\
dz_t &= \left[ -(1-g) A z_t^{\alpha+1} - \delta_G z_t + c_t z_t + \delta_K z_t \right] dt + \left[ g A z_t^{\alpha} \right] dN_t, && (111)\\
dc_t &= c_t \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K \\ +\frac{\lambda}{\gamma} \left[ \tilde{c}_t^{-\gamma} + \left((1-\alpha) g A z_t^{\alpha}\right) q_t \tilde{q}_t \tilde{c}_t^{-\gamma} - 1 \right] - (1-g) A z_t^{\alpha} + c_t \end{array} \right\} dt \\
&\quad + \left[\tilde{c}_t c_t - c_t\right] dN_t. && (112)
\end{aligned}
$$

Intuitively, the Turnovsky (1997) model is purely deterministic and no shocks occur. It can be considered a benchmark case in which there is no uncertainty about public capital accumulation. In contrast, the pure implementation lags model does not have a deterministic part in the process for public capital accumulation. In this version, all government expenditures are associated with implementation lags. Hence, uncertainty is largest in this version of the model.

### 3.3.2 The Decentralized Model

*3.3.2.1 The Framework*

A representative agent maximizes utility

$$\mathbb{E}_t \int_0^{\infty} u(C_t)\, e^{-\rho t} dt, \tag{113}$$

subject to the flow budget constraint

$$dK_t + dB_t = \left[ (1-\tau_y) \left[ (r_{K,t} - \delta_K) K_t + r_{B,t} B_t \right] - (1+\tau_c) C_t - T_t \right] dt, \tag{114}$$

where $\tau_y \geq 0$ is the income tax rate and $\tau_c \geq 0$ is the consumption tax rate. Bond holdings are denoted by $B_t$ and the private physical capital stock is $K_t$. Lump-sum taxes are denoted by $T_t$. Public capital and thus $r_{K,t}$ and $r_{B,t}$ are taken as given and considered independent from own actions (agents are atomistic), thus private capital accumulation follows

$$dK_t = [I_t - \delta_K K_t] \, dt, \tag{115}$$

where $I_t$ denotes private gross investment into the physical capital good.

The government invests an amount $G$ per period into the accumulation of public capital. A share $\theta$ of this investment directly increases the public capital stock, while the share $1-\theta$ only works with a lag. Using $G_t = gY_t$ gives

$$dK_{G,t} = \left[\theta g A K_{G,t}^{\alpha} K_t^{1-\alpha} - \delta_G K_{G,t}\right] dt + \left[(1-\theta) g A K_{G,t-}^{\alpha} K_{t-}^{1-\alpha}\right] dN_t. \tag{116}$$

In the decentralized version government spending are financed using bonds (purchased by the representative agent) and collecting tax revenues (income $\tau_y$ and consumption taxes $\tau_c$). Government debt follows the accumulation process

$$dB_t = [G_t + (1-\tau_y)\, r_t B_t - \tau_y Y_t - \tau_c C_t - T] \, dt. \tag{117}$$

Using the agent's budget constraint

$$dB_t = [(1-\tau_y)\,[Y_t + r_t B_t] - (1+\tau_c)\, C_t - I_t - T_t] \, dt, \tag{118}$$

and the government's budget constraint gives the aggregate resource constraint in the decentralized economy

$$Y_t = C_t + I_t + G_t. \tag{119}$$

#### 3.3.2.2 *Solution*

Consider the control problem maximizing utility (113) subject to (114) and (115). Choosing the admissible controls $C_t \in U_C$ and $I_t \in U_I$, while defining $V(K_t, B_t)$ as the value function, Bellman's principle gives

$$\rho V(K_t, B_t) = \max_{C_t \in U_C, I_t \in U_I} \left\{ u(C_t) + \frac{1}{dt} \mathbb{E}_t dV(K_t, B_t) \right\}, \tag{120}$$

subject to

$$\begin{aligned} dK_t &= [I_t - \delta_K K_t] \, dt, && (121)\\ dK_t + dB_t &= [(1-\tau_y)\,[(r_{K,t} - \delta_K) K_t + r_{B,t} B_t] - (1+\tau_c)\, C_t - T_t] \, dt, && (122)\\ (K_0, B_0) &\in \mathbb{R}_+^2. && (123) \end{aligned}$$

Here, $\mathbf{K} = (K_t, B_t) \in U_{\mathbf{K}}$ is the vector of state variables and $U_{\mathbf{K}} \subseteq \mathbb{R}^2$ is the state space. The control variables are the rate of consumption, $C_t \in U_C$, and investment, $I_t \in U_I$ .The admissible control regions are $U_C \subseteq \mathbb{R}$ and $U_I \subseteq \mathbb{R}$.

The first-order conditions for this problem are given by

$$\begin{aligned} u_C(C_t) &= (1+\tau_c) V_B(K_t, B_t), \quad (124) \\ V_K(K_t, B_t) &= V_B(K_t, B_t), \quad (125) \end{aligned}$$

which makes each control a function of the state variables for all $t \in [0, \infty)$.

Then, after some algebra (see the technical appendix), we arrive at the Euler equation

$$dC_t = -\frac{C_t}{\gamma}(\rho - [(1-\tau_y)(r_{K,t} - \delta_K)])dt. \quad (126)$$

In equilibrium, it holds

$$r_{K,t} = (1-\alpha) A z_t^{\alpha}, \quad (127)$$

where $z_t = K_{G,t}/K_t$ and, via Ito's lemma,

$$dz_t = \left[\theta g A z_t^{\alpha} - \delta_G z_t - (1-g) A z_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt + \left[(1-\theta) g A z_t^{\alpha}\right] dN_t. \quad (128)$$

Defining consumption in per capital terms, $c_t = C_t/K_t$, and using the aggregate private capital accumulation equation (95), gives

$$dc_t = c_t \left[ \frac{(1-\tau_y)((1-\alpha) A z_t^{\alpha} - \delta_K) - \rho}{\gamma} - (1-g) A z_t^{\alpha} + c_t + \delta_K \right], \quad (129)$$

together with eq. (128) summarizing equilibrium dynamics. This system resembles the decentralized version in Turnovsky (1997), setting $\delta_K = \delta_G = 0$.

## 3.4 Numerical Solution

### 3.4.1 Calibration

Before we discuss the numerical solution, we present the calibration (annual rates) to be used in the remainder of the paper. Table 5 presents our calibration.

We set the rate of time preference to 0.03 which equals an annual interest rate of about 3 percent. The elasticity of the production function w.r.t. the government capital stock is set to 0.184. Based on the meta-analysis by Bom and Ligthart (2009) who find this value to be an average values across studies. However, this analysis is performed on a cross-country data set. For the United States the estimated value range from -0.144 to 0.48. Hence, the value can be considered a fair average of the empirically found values.

**Table 5:** Calibration.

| Parameter | Value | Objective |
|---|---|---|
| $\alpha$ Elasticity of gov. capital | 0.184 | Bom and Lighthart (2009) |
| $\rho$ Time preference | 0.03 | Free parameter |
| $\gamma$ El. of substitution | 3.5 | Free parameter |
| $A$ Technology level | 0.5 | Free parameter |
| $g$ Gov. Spending rate | 0.05 | NIPA Data |
| $\delta_K$ Depreciation rate, Private | 0.05 | BEA Data |
| $\delta_G$ Depreciation rate, Public | 0.02 | BEA Data |
| $\theta$ Share of riskless investment | 0.32 | Own calculations |
| $\lambda$ Arrival rate | 0.2 | Own calculations |
| $\tau_y$ Income tax rate | 0.35 | U. S. Data |

The private capital depreciation rate is set to its commonly assumed value of 0.05. The public capital stock depreciates at the (annual) rate of 0.02 in line with the estimates of the U. S. Bureau of Economic Analysis (BEA, for short) national accounts and the value used by Baxter and King (1993). We further assume that the government spends 5 percent of output for investment projects. Finally, we use the remaining two free parameters to calibrate the model to stylized facts of the U. S. economy. We target the long-run (consumption) growth rate of 2 to 3 percent (annually), the ratio of consumption to private capital of about 0.65, a ratio of capital to output of about 3, and a ratio of public-to-private capital of about 0.25. In order to match those numbers, we set the technology level to 0.5 and the elasticity of substitution is calibrated to be 3.5 allowing for a reasonable degree of risk aversion.

There are still two parameters to bet set. First, we start with the share of government investment without a time lag, $\theta$, i.e. the share of "riskless" investment. Here our calibration strategy makes use of the NIPA tables of the BEA. Using table 3.9.5 we are able to disaggregate total government expenditures (in Billion of U. S. Dollar) into consumption and investment. Further, the data allows to classify four subcategories of government investment: structures, equipment, software, and research & development (R&D, for short). We assume that investment into software and equipment increase the government's capital stock without a lag (that is, within one year). On the flipside, investment into structures and R&D do feature a lag. Figure 17

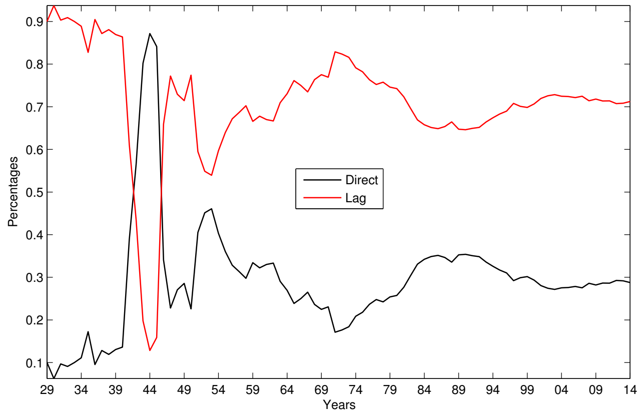


**Figure 17:** Time series of the share of investment with (red) and without (black) lag.

presents the calculated time series for the share of investment with (black line) and without lag (red line) over time from 1929 to 2014. After World War II (and after the "New Deal" program) we observe that the share of investment without a lag is fairly stable varying between 0.2 and 0.4. The average over our sample gives a value of $\theta$ of 0.32, i.e. 32 percent of total government investment expenditures work without a lag. This implies a share of 68 percent of government investment expenditures that will only increase the government capital stock after some time. To be clear, the spending for those projects occurs today, but the effect on the public capital stock occurs later.

Second, we need to calibrate the arrival rate of government investment projects with a lag, $\lambda$, i.e. the arrival rate of the Poisson shocks. The literature on the so-called "time overrun" of investment projects is sparse. Sovacool et al. (2014) present results for the time and cost overrun of electricity projects in a world-wide sample. They report average time overruns between 0 and 43 months with standard deviations from 0 to 58 months. Those numbers give an upper bound for the duration of large infrastructure projects of six years and a value of $\lambda$ of approximately 0.2. To contrast those numbers, we also estimate a vector error correction model (see the technical appendix for details) and obtain first significant effects of a shock in investment (with lag) on output after five years. Therefore, our baseline calibration is to set $\lambda$ to 0.2, implying the arrival of an infrastructure project every five years (on average).

Given this calibration, we are able to match our targets, although the ratio of public-to-private capital is significantly lower than observed empirically (see table 6). The empirical value for $z$, the ratio of public-to-private capital, is 0.32 the historical average from 1947 to 2007. The ratio of consumption-to-output is 0.64 in the data

**Table 6:** The table compares the empirical values with the expected values. Those are obtained simulating the model for 500 years and averaging over 500 simulations.

| | Decentralized Model | Turnovsky (1997) Model ($\theta = 1$) | Data |
|---|---|---|---|
| $z$ | 0.14 | 0.32 | 0.32 |
| $C/Y$ | 0.70 | 0.72 | 0.64 |
| $K/Y$ | 2.89 | 2.47 | 3 |
| $C_G$ | 1.71 | 2.26 | 1.83 |

(NIPA table 1.1.6) and it is 0.7 in the model.[8] The private-capital-to-output ratio commonly used in the literature is 3 while our model generates a value of 2.89. According to Mehra and Prescott (1985) the historical consumption growth rate ($C_G$) in the United States is 1.83 percent. and the model generates a value of 1.71.

### 3.4.2 Policy Functions

In this section we discuss the numerical solution to the centralized and the decentralized model. In order to provide this solution, we use the waveform relaxation algorithm suggested by Posch and Trimborn (2013). It allows us to numerically compute the transition process in dynamic equilibrium models with Poisson shocks. Technically, the system of SDEs is transformed into a system of retarded functional differential equations. Then, a waveform relaxation algorithm is used which involves to provide an initial guess of the policy function and, then, to solve the system of deterministic ordinary difference equations using existing methods.

Figure 18 compares the centralized and the decentralized policy functions. In blue dashed, we plot the deterministic policy function for the centralized model. At the same time, this policy function also represents the solution to the Turnovsky (1997) model, i.e. the model without implementation lags. The red line shows the stochastic policy function in the centralized model, and the black line shows the policy function for the decentralized model with implementation lags and where tax rates are set to zero.

First, we observe that the deterministic and the stochastic policy functions are quite different for the given calibration. This proofs that the uncertainty about the accumulation of public capital does have sizable effects on agents' behavior. We can infer that consumption is lower for any given public-to-private-capital ratio. This

[8] To obtain the numbers for the model, we simulate the model for 500 years and average the numbers over 500 simulations.

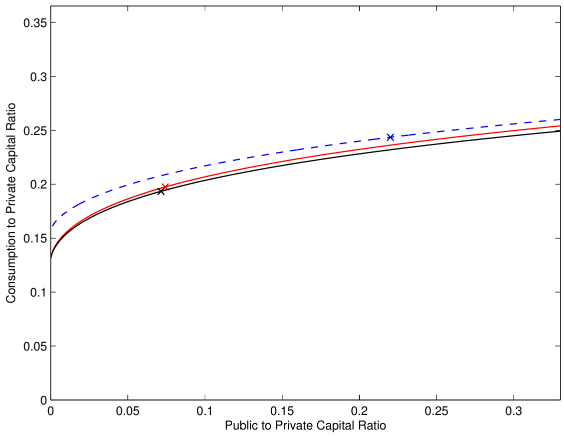


**Figure 18:** In this figure we show the policy function of consumption-to-private-capital ratio and the (conditional) steady state values of public-to-private-capital ratio and consumption-to-private-capital ratio: (i) centralized with implementation lags - solid red, (ii) decentralized with implementation lags - solid black, vs. (iii) centralized with no implementation lags (Turnovsky (1997)) - dashed blue.

result is intuitive, as public capital accumulation in an environment with lags is expected at a lower rate. Precisely, this rate is $\theta + (1-\theta)\lambda = 0.46$ instead of 1 because roughly two thirds of the investment into public capital is assumed to be governed by implementation lags and on average only one fifth of the cumulative public investment into infrastructure will translate into public capital over the course of the next year.[9]

Second, we find that the decentralized policy function is strictly below the centralized policy function for all public-to-private-capital ratios. Since the solution in the centralized model is a first-best solution this result shows that the decentralized model (here, without taxes) can not replicate the first best solution.

In figure 19 we provide the numerical solution to the above considered cases assuming $\lambda = 1$. This allows us to identify the pure effect of the implementation lags.

In this solution, effects are solely a risk adjustment due to the precautionary savings motive. If roughly two thirds of the public investment expenditure feature implementation lags of about one year on average, consumption would be about 13 percent lower (comparing the steady state values) and less public capital relative to private capital will be accumulated, the public-to-private-capital ratio decreases by 61 percent (again comparing the steady state values).

[9]Recall that $\lambda = 0.2$ can be interpreted as on average one implementation within five years.

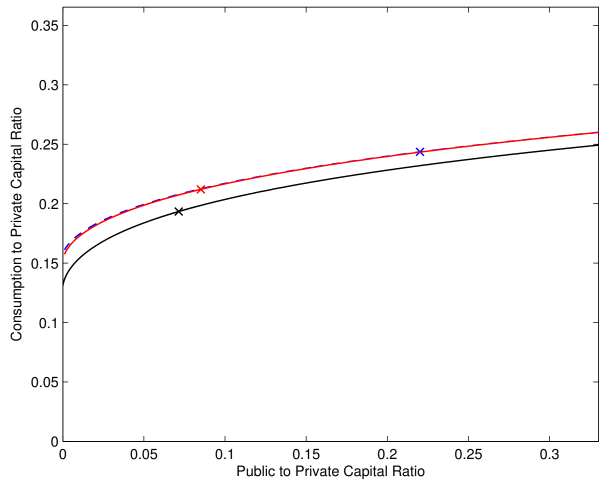


**Figure 19:** In this figure we show the policy functions of consumption-to-private-capital ratio and the (conditional) steady state values of public-to-private-capital ratio and consumption-to-private-capital ratio for $\lambda = 1$: (i) centralized with implementation lags - solid red, (ii) decentralized with implementation lags - solid black, vs. (iii) centralized with no implementation lags (Turnovsky (1997)) - dashed blue.

In general, there are three channels through which uncertainty affects the outcome. First, the income (or precautionary savings) effect leads, ceteris paribus, to higher savings because more uncertainty implies a higher probability of lower consumption tomorrow. Second, the intertemporal substitution effect lowers the marginal propensity to save for risk-avers agents. Finally, consumption is affected by the value of the intertemporal elasticity of substitution and the curvature of the production technology (as it affects effective risk aversion).

### 3.4.3 The Role of Income Taxes

For a low income tax rate (0.1), we observe that the decentralized policy function is close to the deterministic centralized one (see figure 20 for $\tau_y = 0.1$ and figure 21 for $\tau_y = 0.4$). Intuitively, since there is no jump in the consumption Euler equation in neither of those two models, we can expect them to be close to each other. As we can see, the decentralized steady state is to the north-east of the deterministic centralized steady state. In this environment, there is more consumption and more capital accumulation. The higher tax rate allows the government to increase spending by more as in the low tax rate scenario. Therefore, the reallocation from private to public capital is larger. As a consequence, the increased spending and its positive effect towards production (due to more public capital being used in the production

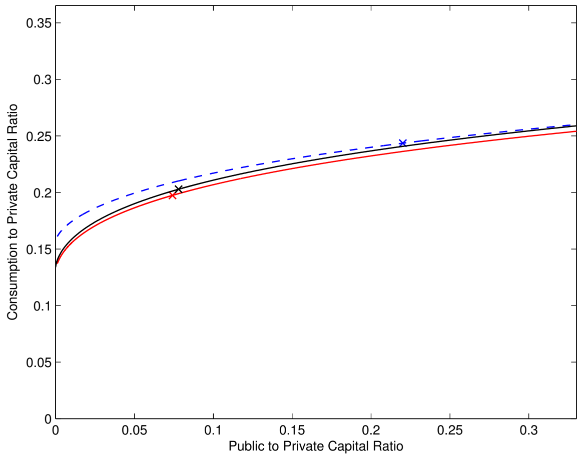


**Figure 20:** In this figure we show the policy functions of consumption-to-private-capital ratio and the (conditional) steady state value of public-to-private-capital ratio and consumption-to-private-capital ratio: (i) centralized with implementation lags - solird red, (ii) decentralized with implementation lags - solid black, vs. (iii) centralized no implementation lags (Turnovsky (1997)) - dashed blue for a low income tax rate of $\tau_y = 0.1$.

process) outweighs the negative effect of income taxes through the demand side.

Ceteris paribus, there is less consumption and less capital accumulation in this version of the model because agents need to pay income taxes. Lower consumption also implies lower output (demand-side effect) and less private capital will be accumulated as a consequence of rational agents' maximizing behavior. Governments do collect the taxes and issue bonds and are able to spend more. This positive effect is stronger to counterbalance the former, negative effect. To some extend, this is due to the implementation lags in government capital as the higher government spending does not directly increase the government capital stock which, potentially, could outweigh the adverse effect towards the private capital stock in the production function.

### 3.4.4 Transitional Dynamics

The dynamics of the centralized and the decentralized model are presented in figure 22.

We start by discussing the transitional dynamics in the centralized model. Upon arrival of the shock to the public capital stock, $z$, the relation of shadow prices as well as consumption - via the Euler equation - jump. This response can be seen in the top

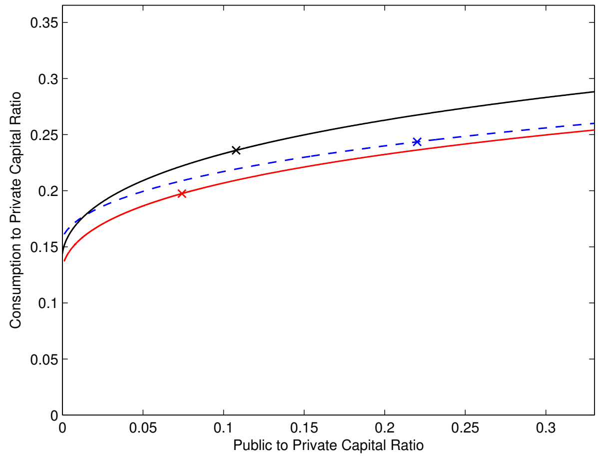


**Figure 21:** In this figure we show the policy functions of consumption-to-private-capital ratio and the (conditional) steady state value of public-to-private-capital ratio and consumption-to-private-capital ratio: (i) centralized with implementation lags - solird red, (ii) decentralized with implementation lags - solid black, vs. (iii) centralized no implementation lags (Turnovsky (1997)) - dashed blue for a low income tax rate of $\tau_y = 0.4$.

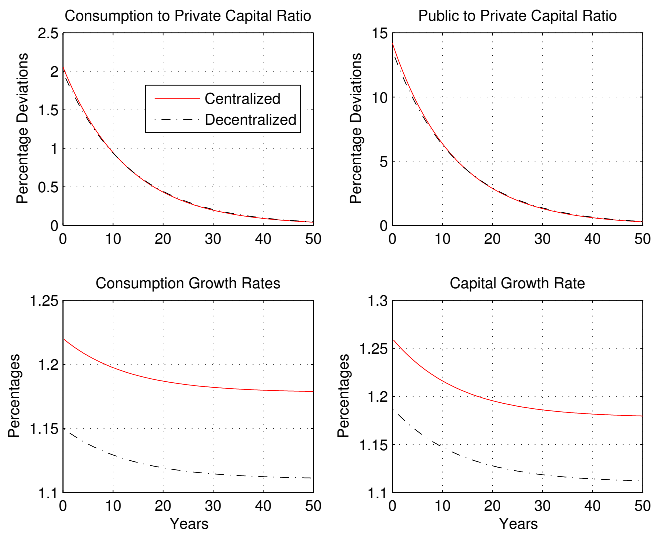


**Figure 22:** In this figure we show (clockwise from top to bottom) the transitional dynamics of consumption-to-private-capital ratio, public-to-private-capital ratio, consumption growth rates, and the capital growth rate after a successful implementation of public capital.

left and top right panel of figure 22. With more public capital more output can be produced. This increase in output is consumed and invested into accumulating new private capital. Private investment therefore increases the private capital stock over time. Similarly, the rise in output increases - automatically - government spending (cf. eq. (93)). As a consequence, the government capital stock increases over time. The dynamics of private and public capital stock over time then lead to a decrease in the public-to-private-capital ratio as shown in the top right panel of figure 22. Over time, the model converges to the new steady state with a higher consumption-to-private-capital and a higher public-to-private-capital ratio; driven by the decrease in $z$. Further, figure 22 also plots the response of the decentralized model (black dashed lines). Although the Euler equation in the decentralized model does not contain a jump term, the consumption-to-private-capital ratio jumps because rational agents anticipate that $z$ will jump. Hence, it is optimal for $c$ to jump as well.

### 3.4.5 Time Series Simulation

In this section we present the realization of a simulation of the centralized model over 60 years (see figure 23). In this realization, we observe ten shocks to the public capital stock (top left panel). On average we should observe (60/5 =) twelve shocks which implies that we find less public capital accumulation as agents expected. This again demonstrates the role played by the uncertainty introduced by implementation lags. Agents expected two more realizations of public capital projects such that there is less public capital accumulation as they expected.

Further, we observe the time series for the public-to-private capital ratio and the consumption-to-private-capital ratio (top left panels) and find the jumps as we expect from the equation system (eqs. (128) and (129)) summarizing equilibrium dynamics. The jumps in public capital can be inferred from the bottom right panel plotting the level of the public capital stock. Here, we observe the increase generated by the arrival of the public investment projects that are affected with lags. In contrast, the level of private capital is less affected by the jumps. Both levels show that our model generates exponential growth in line with Kaldor's facts. The average time between two jumps of five years can hardly be seen in this simulation. We observe periods with shocks every year (the three shocks after 35 years) and periods without any shock (from year 24 to year 35). We find that the jumps lead to strong increases in the growth rates by about one to two percentage points.

As we have already mentioned, the Turnovsky (1997) model is nested in our model, if we set $\theta = 1$, and can be considered to be a benchmark case. We also plot the solution to this model in figure 23 by a red dashed line. We observe substantial

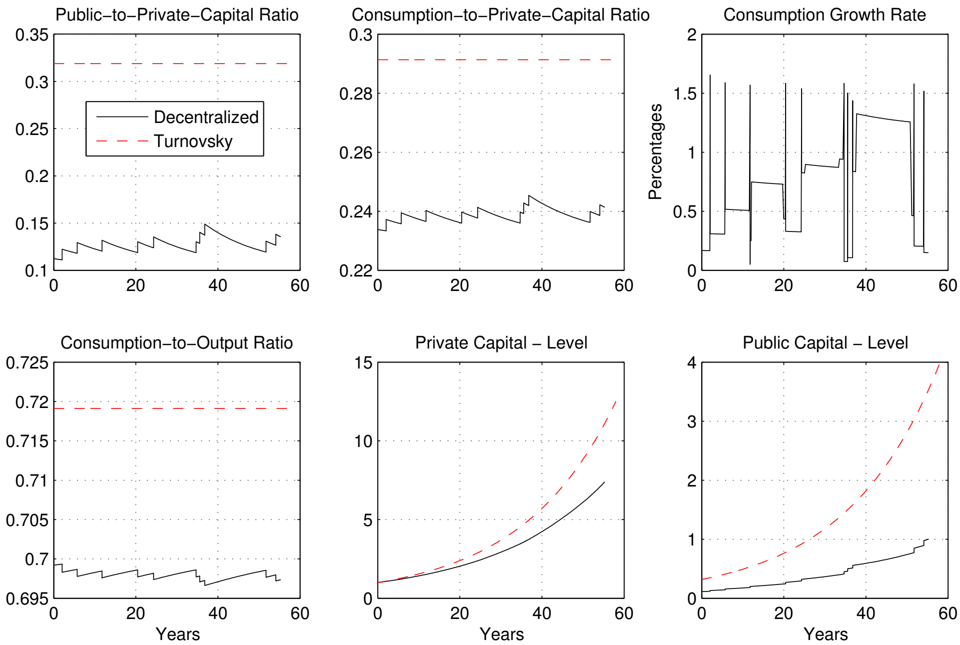


**Figure 23:** The figure shows one realization of a simulation of the decentralized model over 500 years (only 50 years plotted, plotted in black). In addition, we plot the solution for the Turnovsky (1997) model (red dashed line).

differences across the two models. Recall that there are no jumps in the Turnovksy (1997) model as it is purely deterministic. The public-to-private-capital ratio and the consumption-to-private-capital ratio are larger as in our decentralized model. In the absence of uncertainty about the accumulation process of public capital (also compare the difference in the policy functions) and the faster accumulation of public capital raise these ratios. Furthermore, since the full amount of public expenditures will increase the public capital stock without any implementation lags, we observe a faster accumulation of public and private capital.

### 3.4.6 Robustness

In this section we provide a robustness check of our results for the centralized model to variations in the two most important parameters $\lambda$ and $\theta$ (see figure 24).

The top left panel plots the steady state values of the consumption-to-private-capital ratio. If $\theta \to 0$, relatively more government investments feature lags. This implies more uncertainty about the path of government capital and, therefore, output. Hence, this creates more incentives for risk averse agents to increase private investment financed via higher savings through lower consumption. If $\lambda \to 0$ shocks to the government capital stock occur less frequent. On average, agents have to wait

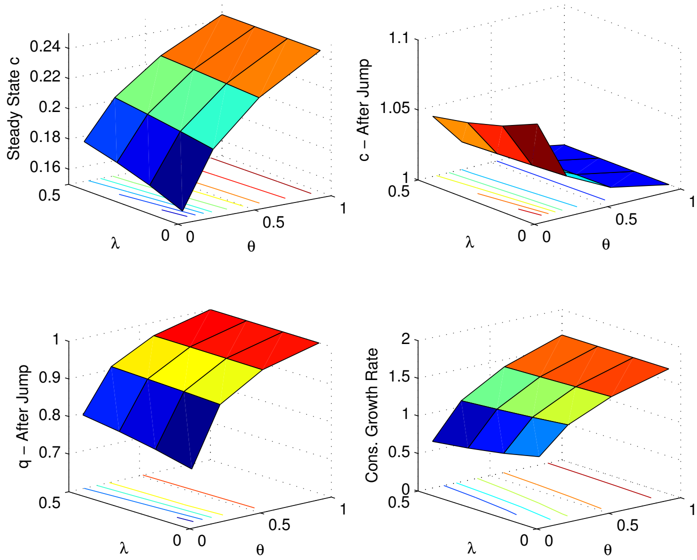


**Figure 24:** In this figure we show (clockwise from top to bottom) the (conditional) steady state of consumption to private capital ratio, the jump term of consumption $\tilde{c}$, the jump term of the ratio of the costate variables $\tilde{q}$, and the consumption growth rate for different values of $\lambda$ and $\theta$.)

longer until government expenditures have positive effects along the supply side. This implies less government capital accumulation forcing agents to increase private capital by lowering consumption.

The top right and the bottom left panel plot the values of the consumption-to-private-capital ratio, $c$, and the ratio of the shadow prices, $q$, after the jump. We observe a larger effect of a jump from lower steady states. If $\theta \to 0$, a jump is associated with a much larger share of total government expenditures. As a consequence, the effects on the economy are larger. Letting $\lambda \to 0$ again reduces the frequency of arrivals of government investment shocks. This creates more uncertainty about the government capital accumulation process and, with risk averse agents, optimal plans are less depend on government capital. As a consequence, agents are more surprised by the jumps, having larger effects on output and consumption.

Finally, the bottom right panel shows the plot for the consumption growth rate. If $\theta \to 0$, more investment with lags will result in a lower consumption growth rate. Larger parts of the government expenditures are associated with lags. This generates more uncertainty and less government capital accumulation. Hence, the consumption growth rate will be lower. Further, we find some degree of non-linearity of the effects of varying the arrival rate of shocks for different values of $\theta$. For small $\theta$, less frequent shocks, i.e. $\lambda \to 0$, from a smaller steady state (for consumption) will have larger effects and, therefore, a larger consumption growth rate in steady state. However, for large $\theta$, lags are of lesser importance and the effect of varying $\lambda$ are smaller. More frequent shocks though lead to a slightly higher consumption growth rate due to faster government capital accumulation.

Assuming that developing countries like China or India have a low $\lambda$, i.e. long lags, and a low $\theta$, i.e. more spending towards investment with lags, have higher consumption growth rates than developed countries like the United States or Germany with short lags. The reason is that lags have effects on the optimal consumption/investment decision of risk averse private agents. Longer lags essentially increase the uncertainty about the path of government capital as an input factor for production. Risk averse agents prefer to "insure" against the dependence on the uncertain government investment and consume less and save (invest) more.

## 3.5 Policy Experiments

In this section we want to discuss three historical policy experiments. We start by deriving the analytical long-run effects of fiscal policy, i.e. varying spending and taxes. We do so to provide some intuition for the following results based on the numerical solution of the model. Then, we consider an increase in the government spending rate, $g$, that is calibrated to match the increase in government expenditures generated by the ARRA program. The second reform is an increase in the share of investment without a lag, i.e. a reallocation of government expenditures. This reallocation could, for example, be triggered by military buildups (e.g. Korean War or the Carter-Reagan buildup). Finally, we consider an increase in the income tax rate that is in line with the increase in the tax rate due to the Omnibus Budget Reconciliation Act in 1993 under the Clinton administration.

### 3.5.1 Long-Run Fiscal Policy Effects

In this section we want to analytically compute the long-run steady state effects of fiscal policies in the decentralized model. To be precise, we consider changes in the income tax rate, $\tau_y$, and the government spending rate, $g$. To do so, we consider the steady state of the decentralized model given by (assuming $\delta_K = \delta_G = 0$)

$$0 = \frac{(1-\tau_y)\left((1-\alpha) A\hat{z}^{\alpha}\right) - \rho}{\gamma} - (1-g) A\hat{z}^{\alpha} + \hat{c}, \tag{130}$$

$$0 = \theta g A\hat{z}^{\alpha} - (1-g) A\hat{z}^{\alpha+1} + \hat{c}\hat{z}, \tag{131}$$

where the hat denotes steady state values. In the background of the model, government debt and the consumption tax (effectively a lump-sum tax) adjust such that the government's budget constraint holds.

After some algebra (see the technical appendix) the steady state effects of a change in the income tax rate on $\hat{z}, \hat{c}$, and the steady state growth rate, $\hat{\phi}$, are

$$\frac{d\hat{z}}{d\tau_y} = \frac{\hat{z}^2}{\gamma\left[\frac{(1-\tau_y)}{\gamma}\alpha\hat{z} + \theta g\right]} > 0, \tag{132}$$

$$\frac{d\hat{c}}{d\tau_y} = \frac{A\hat{z}^{\alpha}\left[(1-g)\alpha\hat{z} + \theta g(1-\alpha)\right]}{\gamma\left[\frac{(1-\tau_y)}{\gamma}\alpha\hat{z} + \theta g\right]} > 0, \tag{133}$$

$$\frac{d\hat{\phi}}{d\tau_y} = -\frac{gA(1-\alpha)\hat{z}^{\alpha}}{\gamma\left[\frac{(1-\tau_y)}{\gamma}\alpha\hat{z} + \theta g\right]} < 0. \tag{134}$$

Consider a ceteris paribus decrease in the income tax rate. Lower income taxes will increase the incentives to save by increasing the net rate of return to private capital. As a consequence, agents consume less and save more. This results in a lower consumption-to-capital ratio. Further, because agents hold more private capital, the growth rate of private capital increases. Since agents acquire more private capital relative to public capital, the public-to-private capital rate decreases in the new long-run steady state.

Further, we can compare the long-run effects for different values of $\theta$. If we assume that $\theta = 1$, we replicate the version established by Turnovsky (1997) without any lags. In the model where we assume that $0 \leq \theta < 1$, we observe that the effects are smaller, as government expenditures - at least to some extend - will only increase the public capital stock, i.e. be productive, after some time. Hence, as the government spends but the spending is not productive, the effects of policy changes are smaller in this environment.

The effects of a change in the government spending rate are described by

$$\frac{d\hat{z}}{dg} = \frac{\hat{z}}{(1-\alpha)\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha\hat{z}+g\right]} > 0, \tag{135}$$

$$\frac{d\hat{c}}{dg} = \frac{A\hat{z}^{\alpha}\left[\frac{\alpha-g}{(1-\alpha)} - \frac{(1-\tau_y)\alpha(\theta+\hat{z})}{\theta\gamma}\right]}{\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha\hat{z}+g\right]}, \tag{136}$$

$$\frac{d\hat{\phi}}{dg} = \frac{(1-\tau_y)\,\alpha A\hat{z}^{\alpha}}{\gamma\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha\hat{z}+g\right]} > 0. \tag{137}$$

Increasing the government spending rate leads to an increase in the capital growth rate. This increase also implies that the growth rate in the entire economy increases. In contrast, the effect on consumption is ambiguous. It depends on the sign of $\alpha - g$. Again, a model comparison across the parameter $\theta$ yields the result that implementation lags lead to smaller effects of fiscal policy as well as lower growth rates.

### 3.5.2 Government Expenditures

Our first policy experiments is an increase in government expenditures, $g$. We calibrate the increase according to the ARRA program. According to Baker and Deutsch (2009), this program increased government expenditures by 2.6 percentage points. Figure 25 shows the transitional dynamics from increasing government expenditures without the realization of Poisson shocks in order to isolate the pure policy effect.

As we have concluded in the previous section, an increase in government spending increases the public-to-private capital ratio ($z$) as more public capital is accumulated (even with lags) and less output is available for private capital accumulation. The latter also implies that less output is available for consumption. However, the increase in government spending is not wasted but transformed into a higher public capital stock. This serves as an input factor in the production technology and, therefore, increases output. The increase in output then allows agents to consume more and we observe an increase in the consumption-to-private-capital ratio. Finally, it is

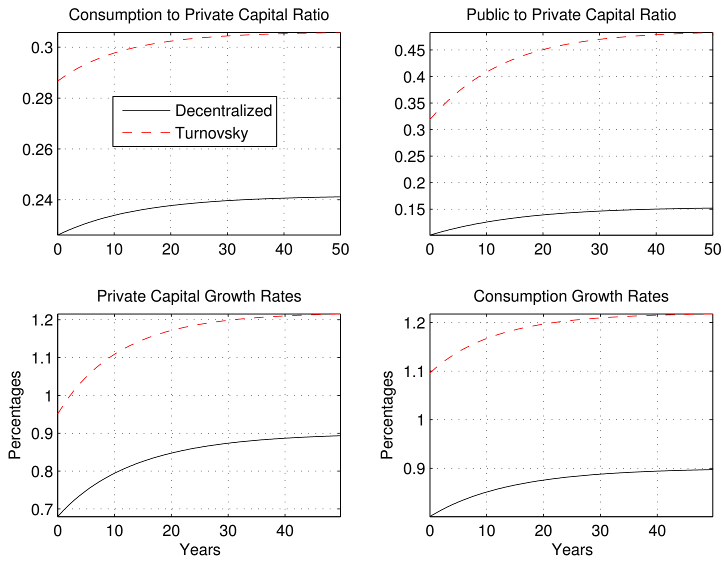


**Figure 25:** The figure plots the transitional dynamics for the decentralized (solid black) and the Turnovsky (1997) model (red dashed) after increasing government spending, $g$, without any Poisson shocks.

worth stressing that the convergence process shows a high degree of persistence and is completed roughly 50 years after the implementation of the policy.

The steady state effects of this policy reform are presented in table 7. We find that the increase in government expenditures raises the (consumption) growth rate by 0.21 percentage points from 1.71 to 1.92 (12 percent). At the same time we see a small drop in the consumption-to-output ratio (0.7 to 0.68, 3 percent) and the private-capital-to-output ratio (2.89 to 2.7, 7 percent) driven by a larger increase in output.

Figure 25 also plots the transitional dynamics for the Turnovsky (1997) model after the implementation of the reform. Quantitatively we obtain the same effects. However, the steady state effects are larger compared to the decentralized model. The public-to-private capital ratio increases from 0.32 to 0.48 (50 percent) in the Turnovsky (1997) version of our model. This increase is about twice as large as in our model. Similarly, the increase of the consumption-to-private-capital ratio is larger in the Turnovsky (1997) model (0.0191) compared to the decentralized model (0.0151), as shown in table 8. The intuition for the larger effects in the Turnovsky (1997) model is straightforward: since it is purely deterministic, it features a faster accumulation of the public capital stock. Increasing government expenditures directly translates into an even faster capital accumulation. With implementation lags, the increase - to some extend (32 percent, to be precise) - will only become effective after some (random) time. However, we infer that the model performs worse in matching the observed values compared to the model with delays. Most importantly, it predicts a much higher (consumption) growth rate of 2.43 percent.

**Table 7:** The table presents the values of key variables before and after the policy reforms. For the numbers we simulate the model for 500 years and compute the average over 500 simulations. $\Delta g$ is the increase in government spending from 0.05 to 0.076, $\Delta\theta$ is the reallocation reform, $\theta$ increases from 0.32 to 0.5, and $\Delta\tau_y$ is an increase in the income tax rate from 0.35 to 0.4.

| | Data | Initial | $\Delta g$ | $\Delta\theta$ | $\Delta\tau_y$ |
|---|---|---|---|---|---|
| $z$ | 0.32 | 0.14 | 0.21 | 0.19 | 0.15 |
| $C/Y$ | 0.64 | 0.70 | 0.68 | 0.71 | 0.71 |
| $K/Y$ | 3 | 2.89 | 2.70 | 2.74 | 2.88 |
| $C_G$ | 1.83 | 1.71 | 1.92 | 1.87 | 1.58 |
| $std\,(C_G)$ | 0.03 | 0.0041 | 0.0045 | 0.0041 | 0.0038 |
| $std(C/Y)$ | 0.04 | 0.0007 | 0.0013 | 0.0011 | 0.0006 |

**Table 8:** The table presents the values of key variables before and after the policy reforms in the Turnovsky (1997) model. For the numbers we simulate the model for 500 years and compute the average over 500 simulations. $\Delta g$ is the increase in government spending from 0.05 to 0.076 and $\Delta\tau_y$ is an increase in the income tax rate from 0.35 to 0.4.

| | Data | Initial | $\Delta g$ | $\Delta\tau_y$ |
|---|---|---|---|---|
| $z$ | 0.32 | 0.32 | 0.48 | 0.34 |
| $C/Y$ | 0.64 | 0.72 | 0.70 | 0.73 |
| $K/Y$ | 3 | 2.47 | 2.29 | 2.44 |
| $C_G$ | 1.83 | 2.18 | 2.43 | 2.02 |

So far, we considered the pure effect of implementing the policy reform. Figure 26 presents the simulated time series of key variables over 60 years with (black) and without the policy reform (red dashed) conditional on the same realization of Poisson shocks. First, the increase in government expenditures, $g$, increases the jump size of the Poisson shocks. This is visible, for example, in the time series for the public capital stock (bottom right panel). Given the exponential growth property of our model, the initial small differences accumulate over time and lead to sizable and increasing differences in the public capital stock over time. This raises output in the simulation with policy reform relative to the simulation without the policy reform.

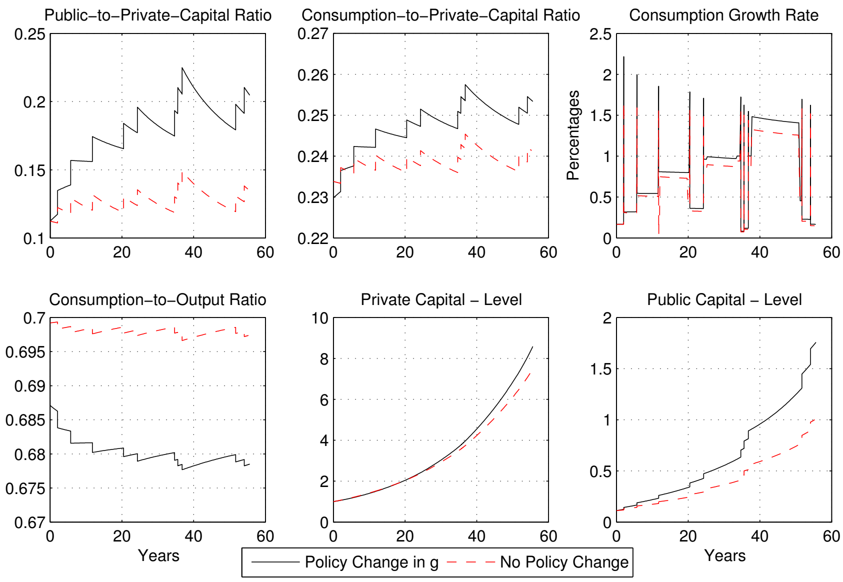


**Figure 26:** This figure plots one realization (the same across models) with (black line) and without (red dashed line) increasing government spending, $g$.

This increase in output than leads to a higher path for the private capital stock and leads to a higher public-to-private-capital ratio (from 0.14 to 0.21) and a higher consumption-to-private-capital ratio. Finally, we want to comment on the dynamics of government debt: We find that this policy, while increasing the growth rate, does increase the debt-to-GDP ratio (exponentially), while the ratio decreases without the policy reform.

### 3.5.3 Reallocating Government Expenditures

The composition of government expenditures plays a key role in the overall effectiveness of fiscal policy (see, for example, Feltenstein and Ha (1995), Turnovsky and Fisher (1995), and Devarajan et al. (1996)). Our model adds another dimension to this discussion. Policy has control over the share of spending going into investment projects with and without lags (parameter $\theta$ in the model). As we will show, this policy reform creates large effects on macroeconomic variables. Following the work by Ramey and Shapiro (1998) we study a policy reform related to reallocating government expenditures driven by military buildups. Considering figure 17 we will evaluate the effects of the Korean War (1950-1953). During this time the share of government expenditures without lags increased to a value of about 50 percent. Therefore, we consider the effects from increasing $\theta$ from its steady state value of 0.32 to a value of 0.5. A different motivation for this type of policy reform is the "New Deal" program

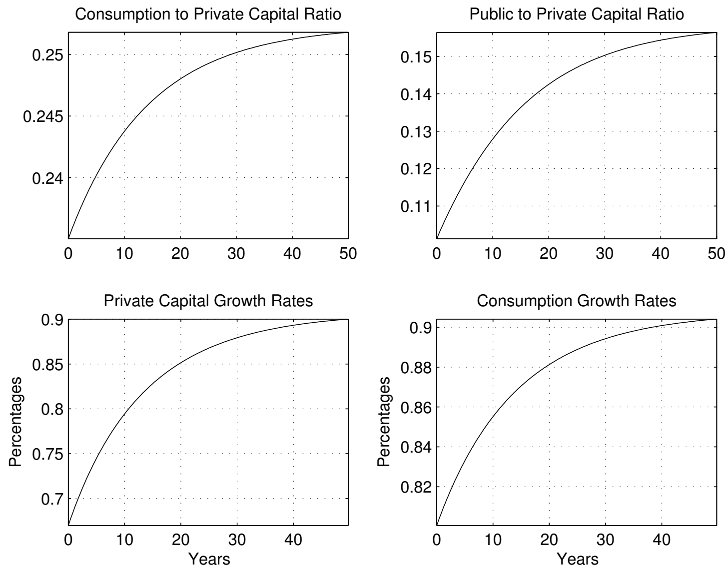


**Figure 27:** The figure plots the transitional dynamics for the decentralized model after reallocating government expenditures, i.e. increasing $\theta$, without any Poisson shocks.

under President Roosevelt. A part of this policy program is the "Public Works Administration" that mainly used government expenditures to invest into infrastructure. Consistently, we can understand this policy as a desire of the government to avoid expenditures with delays. This could be motivated by the need of a fast response due to a recession or political economy issues, like elections.

Figure 27 shows the transitional dynamics for this policy reform. This increase implies that after the reform not 32 percent but 50 percent of government spending are not affected by implementation lags. Put differently, an additional amount of 18 percentage points of government spending now directly increases the public capital stock. This lowers the uncertainty about the path of the public capital stock and lowers the incentives for precautionary actions and increases the information agents have. The increase in $\theta$ increases the public-to-private-capital ratio because more of the government spending are directly transformed into public capital. More public capital increases output and the consumption-to-private-capital ratio increases. With more output being produced also more private capital will be accumulated due to higher savings (investments).

Again, we also want to discuss the differences generated by this policy reform over time. For this purpose, figure 28 plots the simulated time series of key variables over 60 years with (black) and without the policy reform (red dashed) conditional on the same realization of Poisson shocks that we have used in the previous section. The increase in $\theta$ is an increase in the deterministic part of the public capital accumulation. Hence, a larger share of total government expenditures will be free of implementation lags, reducing uncertainty about the path of public capital, and directly increasing the public capital stock. Therefore, it is not surprising that the quantitative effects are similar to the increase in government expenditures.

In this simulation there are two counteracting effects at work. First, the increase in the deterministic part of the public capital stock accumulation process increases output, consumption, and allows a faster public and private (due to higher savings) capital stock accumulation. Therefore, we obtain an increase in the public-to-private-capital ratio (0.14 to 0.19 or 36 percent, cf. table 7) and the consumption-to-private-capital ratio (0.24 to 0.26, 8 percent).

However, as we can infer from the bottom right panel, the jump size of the Poisson shocks will be smaller, because less government expenditures are allocated to those type of projects. Hence, if those projects are realized, their effect on the public capital stock, ceteris paribus, is smaller. Therefore, the difference in the simulated path of public capital with and without the reform is smaller compared to the difference for the increase in government spending (here the relative shares stayed constant). Accordingly, we find that the increase in the (consumption) growth rate is smaller compared to the increase in government spending (0.17 vs. 0.21). Further, in this model the debt-to-GDP ratio decreases over time, while we found an increase in the previous policy reform. In conclusion, this policy reform has positive real effects that, however, are smaller compared to the government spending reform. The advantage of this reform is that it does not create additional costs, as it is simply a reallocation of existing expenditures. However, we should not overstress this result as it relates to military buildups. Because our model does not distinguish between different effects of different types of public capital along the supply-side, it can not be recommended to focus exclusively on projects without lags. Intuitively, investments into software will have different supply-side effects than investments into roads, harbors, or bridges.

### 3.5.4 Income Tax Policy

Our last policy experiment considers an increase in the income tax rate. Under President Clinton, the Omnibus Budget Reconciliation Act in 1993 increased the income tax rate by five percentage points. We study the effects of this policy reform using our model.

Figure 29 plots the convergence path towards the new steady state after the implementation of the reform. The increase in the income tax rate has the conventional

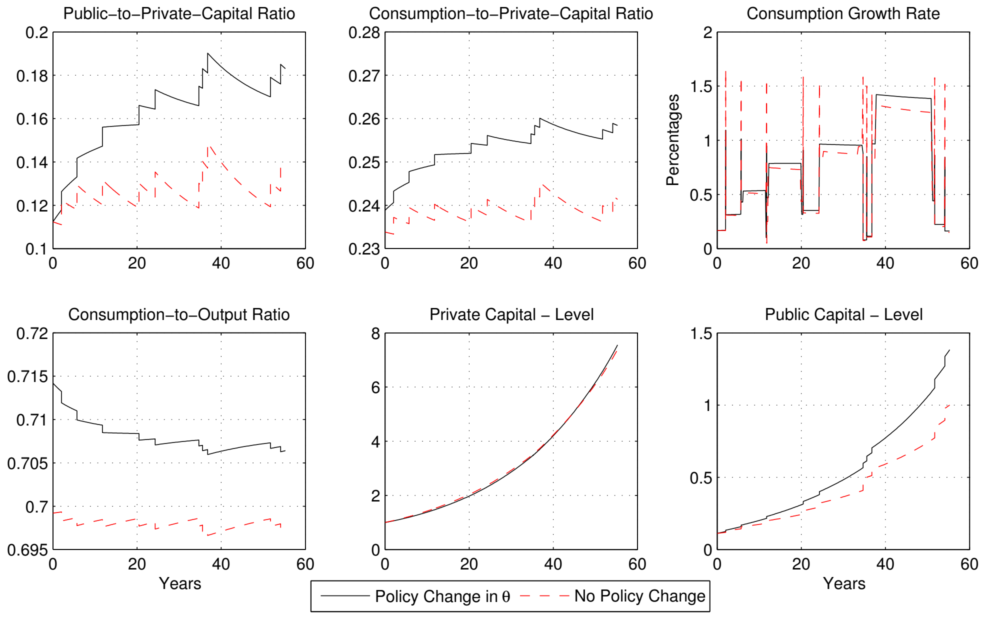


**Figure 28:** This figure plots one realization (the same across models) with (black line) and without (red dashed line) reallocating government expenditures, increasing $\theta$.

effects: the reduction of the effective wage rate lowers disposable income and, therefore, lowers consumption and investment. At the same time, a higher income tax rate reduces the incentives to save because the net rate of return to private capital is decreased. Hence, agents want to consume more and save (invest) less. Therefore, the consumption-to-private capital ratio increases and is higher in the new steady state compared to the initial steady state. Since agents save less, private capital accumulates slower, drops and then converges to the new steady state driven by increasing output.

Again, figure 29 plots transitional dynamics in the Turnovsky (1997) model. Here, we do observe smaller effects as in the decentralized model. The consumption-to-private-capital ratio increases only by 0.0032, while it increases by 0.0101 in the decentralized model. Similarly, the public-to-private-capital ratio increases by 0.0211 vs. 0.0308 in the decentralized model. Again, we observe that the Turnovsky (1997) model overshoots the values for the (consumption) growth rate.

Figure 30 plots the simulated time series of key variables over 60 years with (black) and without the policy reform (red dashed) conditional on the same realization of Poisson shocks that we have used in the previous sections. As we have discussed, we do observe a small increase in the public-to-private-capital ratio (from 0.14 to 0.15 or 7 percent, cf. table 7) and the consumption-to-private-capital ratio (from

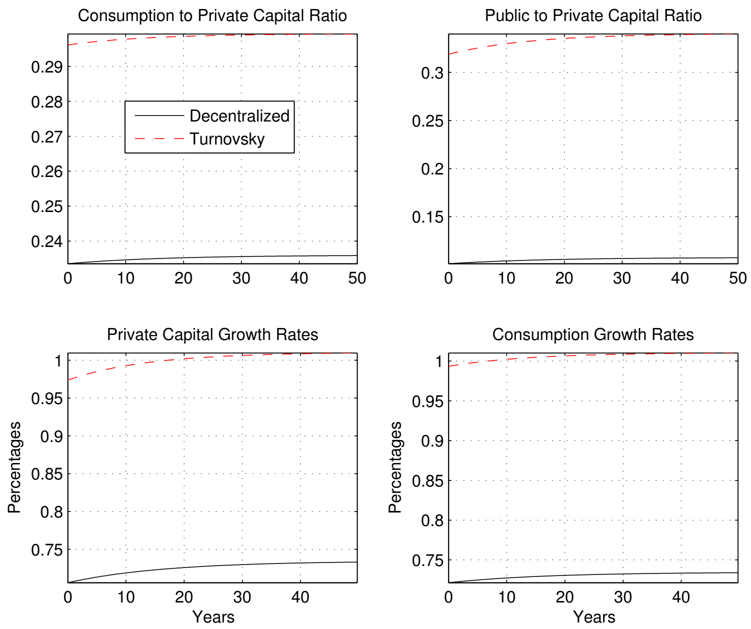


**Figure 29:** The figure plots the transitional dynamics for the decentralized and the Turnovsky (1997) model after increasing the income tax, $\tau_y$, without any Poisson shocks.

0.2376 to 0.2477, 4 percent). Further, we explained that private and public capital accumulate at a smaller pace and, hence, the (consumption) growth rate drops from 1.71 to 1.58 percent (down 8 percent). We find that the model generates a lower debt-to-GDP ratio compared to the scenario without the policy reform. Intuitively, higher tax income will allow to finance more spendings without accumulating debt and to reduce the debt burden.

In conclusion, the increase in income taxes lowers the accumulation of both capital stocks and leads to a reduction in the (consumption) growth rate.

## 3.6 Conclusion

Public investment projects whether in developed or developing countries are associated with implementation delays. They are subject to large implementation delays due to the required planning, bidding, contracting, and construction process and often require coordination of different regional governments. We add to the literature on fiscal policy in endogenous growth models by building a stochastic endogenous growth model in continuous time with public capital. In our model, implementation lags generate uncertainty in the public capital accumulation process: the government continuously spends but the completion of the public investment project is unknown.

Our paper has several contributions. First, we provide a numerical solution of this - and the nested Turnovsky (1997) - model under Poisson uncertainty calibrated

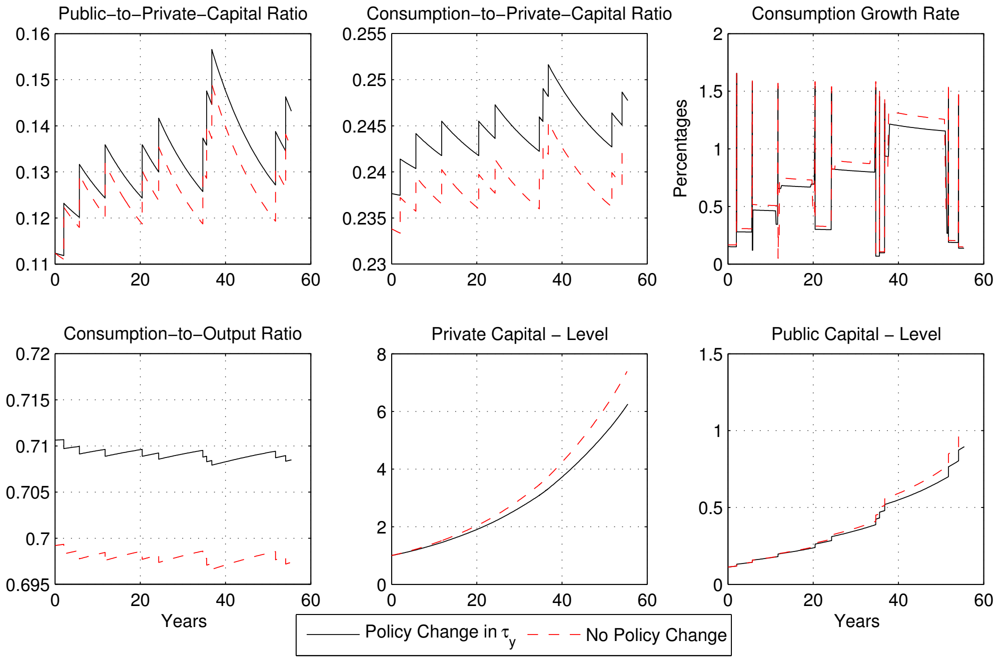


**Figure 30:** This figure plots one realization (the same across models) with (black line) and without (red dashed line) increasing the income tax rate, $\tau_y$.

on the U.S. economy. We find that the implementation lags in the accumulation of public capital have sizable effects on agents' behavior. We show that the deterministic and the stochastic policy functions are substantially different such that consumption is always lower for any given public-to-private-capital ratio. Second, we evaluate the effects of three historical policy reforms. We find that an increase in government expenditures raises the growth rate while an increase in the income tax rate reduces the growth rate. We then consider a policy reform exclusive to our model, namely a reallocation of government expenditures towards projects not associated with implementation lags. We find that such a policy increase the growth rate. While the effects are smaller compared to the increase in government spending, the main advantage of this policy reform is that it does not generate additional costs.

Finally, our robustness check allows us to comment on different development dynamics across countries. Countries like China or India, with long lags and more spending associated with implementation lags, have higher growth rates due to an uncertainty effect on agents' behavior generated by implementation lags. Agents consume less and save more, therefore, increasing the accumulation of private and public capital.

We identify two main directions for future research. First, it will be interesting to introduce default (risk) on government bonds. This will affect agents' decisions and one could further link the Poisson arrival rate to debt dynamics. There could be a threshold for the debt-to-GDP ratio (e.g., to model the fiscal cliff in the United States) above which spendings are cut and public investment projects take even more time to be completed. Second, our model is suitable to address issues related to development economics, such as poverty traps. A model version with multiple equilibria and different values of the Poisson arrival rate could offer interesting insights in the different development processes of countries. We also plan to add a welfare analysis of the considered policy reforms.

## *3.7 Technical Appendix*

### 3.7.1 Implementation Lags

Bellman equation (BE, for short) is

$$\rho V\left(K,K_G\right)=\max_C\left\{u\left(C\right)+\frac{1}{dt}\mathbb{E}_t dV\left(K,K_G\right)\right\},$$

subject to

$$\begin{aligned} dK_t &= \left[\left(1-g\right)AK_{G,t}^{\alpha}K_t^{1-\alpha}-C_t-\delta_K K_t\right]dt, \\ dK_{G,t} &= \left[gAK_{G,t}^{\alpha}K_t^{1-\alpha}\right]dN_t. \end{aligned}$$

**Step 1: FOC**

We start by computing the derivative $dV$, dropping time indices

$$\begin{aligned} dV\left(K,K_G\right) &= V_K\left(K,K_G\right)\left[\left(1-g\right)AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right]dt+V_{K_G}\left(K,K_G\right)0dt \\ &\quad+\left[V\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V\left(K,K_G\right)\right]dN. \end{aligned}$$

Such that the BE is given by

$$\begin{aligned} \rho V\left(K,K_G\right) &= \max_C\left\{u\left(C\right)+\frac{1}{dt}\mathbb{E}_t dV\left(K,K_G\right)\right\}, \\ &= \max_C\left\{\begin{array}{c} u\left(C\right)+V_K\left(K,K_G\right)\left[\left(1-g\right)AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right] \\ +\lambda\left[V\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V\left(K,K_G\right)\right] \end{array}\right\}. \end{aligned}$$

The FOC is

$$u_C\left(C\right)=V_K\left(K,K_G\right).$$

**Step 2: Evolution of Co-States**

Consider the problem

$$\rho V\left(K,K_G\right)=\begin{array}{c} u\left(C\left(K\right)\right)+V_K\left(K,K_G\right)\left[\left(1-g\right)AK_G^{\alpha}K^{1-\alpha}-C\left(K\right)-\delta_K K\right] \\ +\lambda\left[V\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V\left(K,K_G\right)\right] \end{array},$$

1. take the derivative w.r.t. $K$

$$\begin{aligned} \rho V_K\left(K,K_G\right) &= V_K\left(K,K_G\right)\left[\left(1-g\right)\left(1-\alpha\right)AK_G^{\alpha}K^{-\alpha}-\delta_K\right] \\ &\quad+V_{KK}\left(K,K_G\right)\left[\left(1-g\right)AK_G^{\alpha}K^{1-\alpha}-C\left(K\right)-\delta_K K\right] \\ &\quad+\lambda\left[\begin{array}{c} V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left(\left(1-\alpha\right)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ -V_K\left(K,K_G\right) \end{array}\right]. \end{aligned}$$

Then,

$$\begin{aligned} dV_K(K,K_G) &= V_{KK}(K,K_G)\left[(1-g)AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right]dt \\ &\quad+\left[V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G)\right]dN. \end{aligned}$$

Combining

$$\begin{aligned} dV_K(K,K_G) &= \left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]V_K(K,K_G)\,dt \\ &\quad-\lambda\left[\begin{array}{c} V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left((1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G) \end{array}\right]dt \\ &\quad+\left[V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G)\right]dN. \end{aligned}$$

Hence,

$$\begin{aligned} dV_K(K,K_G) &= \left[\begin{array}{c} \left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]V_K(K,K_G) \\ -\lambda\left[\begin{array}{c} V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left((1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ -V_K(K,K_G) \end{array}\right] \end{array}\right]dt \\ &\quad+\left[V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G)\right]dN. \end{aligned}$$

2. Take the derivative w.r.t. $K_G$

$$\begin{aligned} \rho V_{K_G}(K,K_G) &= V_K(K,K_G)(1-g)\alpha AK_G^{\alpha-1}K^{1-\alpha}dt \\ &\quad+V_{KK_G}(K,K_G)\left[(1-g)AK_G^{\alpha}K^{1-\alpha}-C(K,K_G)-\delta_K K\right]dt \\ &\quad+\lambda\left[\left(1+\alpha gAK_G^{\alpha-1}K^{1-\alpha}\right)V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]. \end{aligned}$$

Then,

$$\begin{aligned} dV_{K_G}(K,K_G) &= V_{KK_G}(K,K_G)\left[(1-g)AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right]dt \\ &\quad+\left[V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]dN. \end{aligned}$$

Combining

$$\begin{aligned} dV_{K_G}(K,K_G) &= \rho V_{K_G}(K,K_G)\,dt \\ &\quad-V_K(K,K_G)\left[\alpha A(1-g)K_G^{\alpha-1}K^{1-\alpha}\right]dt \\ &\quad-\lambda\left[\left(1+\alpha gAK_G^{\alpha-1}K^{1-\alpha}\right)V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]dt \\ &\quad+\left[V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]dN. \end{aligned}$$

Hence,

$$\begin{aligned} dV_{K_G}(K,K_G) &= \left\{\begin{array}{c} \rho V_{K_G}(K,K_G)-V_K(K,K_G)\left[\alpha A(1-g)K_G^{\alpha-1}K^{1-\alpha}\right] \\ -\lambda\left[\left(1+\alpha gAK_G^{\alpha-1}K^{1-\alpha}\right)V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right] \end{array}\right. \\ &\quad+\left[V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]dN. \end{aligned}$$

**Step 3: Insert FOCs**

$$
\begin{aligned}
du_C(C) &= dV_K(K, K_G) \\
&= \left[ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) AK_G^{\alpha} K^{-\alpha} + \delta_K\right] V_K(K, K_G) \\ -\lambda \left[ \begin{array}{c} V_K\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left((1-\alpha) gAK_G^{\alpha}K^{-\alpha}\right) V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right) - V_K(K, K_G) \end{array} \right] \end{array} \right] dt \\
&\quad + \left[V_K\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right) - V_K(K, K_G)\right] dN.
\end{aligned}
$$

Moreover, it holds

$$
\begin{aligned}
du_C(C) &= \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) AK_G^{\alpha} K^{-\alpha} + \delta_K\right] u_C(C) \\ -\lambda \left[ u_C\left(\tilde{C}C\right) + \left((1-\alpha) gAK_G^{\alpha}K^{-\alpha}\right) V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right) - u_C(C) \right] \end{array} \right\} \\
&\quad + \left[u_C\left(\tilde{C}C\right) - u_C(C)\right] dN,
\end{aligned}
$$

where $\tilde{C}$ is the optimal consumption jump term defined as

$$
\tilde{C} = \frac{C\left(K, K_G\left(1 + gA(K_G/K)^{\alpha-1}\right)\right)}{C(K, K_G)},
$$

and $\tilde{C} - 1$ denotes the percentage change in consumption after implementation of the public good.

**Step 4: CVF to Keynes-Ramsey Rule**

Let

$$
f(u_C(C)) = C,
$$

apply CVF to $f(u_C(C))$ using

$$
\begin{aligned}
f\left(u_C\left(\tilde{C}C\right)\right) &= \tilde{C}C, \\
f_C(u_C(C)) &= \frac{df(u_C(C))}{du_C(C)} = \frac{dC}{du_C(C)} = \frac{1}{u_{CC}(C)}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
df(u_C(C)) &= f_C(u_C(C)) \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) AK_G^{\alpha} K^{-\alpha} + \delta_K\right] u_C(C) \\ -\lambda \left[ \begin{array}{c} u_C\left(\tilde{C}C\right) - u_C(C) \\ +\left((1-\alpha) gAK_G^{\alpha}K^{-\alpha}\right) V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right) \end{array} \right] \end{array} \right\} dt \\
&\quad + \left[f\left(u_C\left(\tilde{C}C\right)\right) - f(u_C(C))\right] dN.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
dC &= \frac{1}{u_{CC}(C)}\left\{\begin{array}{c}\left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]u_C(C)\\ -\lambda\left[\begin{array}{c}u_C\left(\tilde{C}C\right)+\left((1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)\\ -u_C(C)\end{array}\right]\end{array}\right\}dt\\
&\quad+\left[\tilde{C}C-C\right]dN,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{u_{CC}(C)}{u_C(C)}dC &= \left\{\begin{array}{c}\left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]\\ -\lambda\left[\frac{u_C\left(\tilde{C}C\right)}{u_C(C)}+\left((1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right]\end{array}\right\}dt\\
&\quad+\frac{u_{CC}(C)}{u_C(C)}\left[\tilde{C}C-C\right]dN.
\end{aligned}
$$

Assume the following CRRA preferences ($\gamma\to 1:\ln C$)

$$
\begin{aligned}
u(C) &= \frac{C^{1-\gamma}}{1-\gamma},\\
u_C(C) &= C^{-\gamma},\\
u_{CC}(C) &= -\gamma C^{-(\gamma+1)}.
\end{aligned}
$$

Inserting

$$
\begin{aligned}
\frac{-\gamma C^{-(\gamma+1)}}{C^{-\gamma}}dC &= \left\{\begin{array}{c}\left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]\\ -\lambda\left[\tilde{C}^{-\gamma}+\left((1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right]\end{array}\right\}dt\\
&\quad+\frac{-\gamma C^{-(\gamma+1)}}{C^{-\gamma}}\left[\tilde{C}C-C\right]dN.
\end{aligned}
$$

Gives

$$
\begin{aligned}
-\frac{\gamma}{C}dC &= \left\{\begin{array}{c}\left[\rho-(1-g)(1-\alpha)A\left(\frac{K_G}{K}\right)^{\alpha}+\delta_K\right]\\ -\lambda\left[\tilde{C}^{-\gamma}+\left((1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right]\end{array}\right\}dt\\
&\quad-\frac{\gamma}{C}\left[\tilde{C}C-C\right]dN.
\end{aligned}
$$

The system is now given by

$$
\begin{aligned}
-\frac{\gamma}{C_t}dC_t &= \left\{\begin{array}{c}\left[\rho-(1-g)(1-\alpha)A\left(\frac{K_{G,t}}{K_t}\right)^{\alpha}+\delta_K\right]\\ -\lambda\left[\tilde{C}^{-\gamma}+\left((1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right]\end{array}\right\}dt\\
&\quad-\frac{\gamma}{C_t}\left[\tilde{C}_tC_t-C_t\right]dN_t,\\
dK_t &= \left[(1-g)AK_{G,t}^{\alpha}K_t^{1-\alpha}-C_t-\delta_KK_t\right]dt,\\
dK_{G,t} &= \left[gAK_{G,t}^{\alpha}K_t^{1-\alpha}\right]dN_t.
\end{aligned}
$$

#### 3.7.1.1 Process for $z_t$

We want to write the system in one state variable. Hence, we define

$$z_t = \frac{K_{G,t}}{K_t},$$

where

$$\begin{aligned} dK_t &= \left[(1-g)\, AK_{G,t}^{\alpha} K_t^{1-\alpha} - C_t - \delta_K K_t\right] dt, \\ dK_{G,t} &= \left[gAK_{G,t}^{\alpha} K_t^{1-\alpha}\right] dN_t. \end{aligned}$$

Ito's Lemma for Poisson processes gives (dropping time indices)

$$\begin{aligned} dz &= \left[\frac{1}{K}0 - \frac{K_G}{K^2}\left[(1-g)\, AK_G^{\alpha} K^{1-\alpha} - C - \delta_K K\right]\right] dt \\ &\quad + \left[\frac{K_G + gAK_G^{\alpha} K^{1-\alpha}}{K} - \frac{K_G}{K}\right] dN, \end{aligned}$$

simplify

$$\begin{aligned} dz &= \left[-\frac{K_G}{K^2}(1-g)\, AK_G^{\alpha} K^{1-\alpha} + \frac{K_G}{K^2} C + \frac{K_G}{K^2}\delta_K K\right] dt \\ &\quad + \left[\frac{K_G + gAK_G^{\alpha} K^{1-\alpha}}{K} - \frac{K_G}{K}\right] dN, \end{aligned}$$

using the definition for $z$ gives

$$dz = \left[-\frac{z}{K}(1-g)\, AK_G^{\alpha} K^{1-\alpha} + \frac{z}{K} C + \delta_K z\right] dt + \left[\frac{K_G}{K} + g\frac{AK_G^{\alpha} K^{1-\alpha}}{K} - z\right] dN,$$

further

$$dz = \left[-z(1-g)\, A\left(\frac{K_G}{K}\right)^{\alpha} + z\frac{C}{K} + \delta_K z\right] dt + \left[z + gA\left(\frac{K_G}{K}\right)^{\alpha} - z\right] dN,$$

then,

$$dz = \left[-z(1-g)\, Az^{\alpha} + z\frac{C}{K} + \delta_K z\right] dt + \left[gAz^{\alpha}\right] dN.$$

Finally,

$$dz = \left[z\frac{C}{K} - (1-g)\, Az^{\alpha+1} + \delta_K z\right] dt + \left[gAz^{\alpha}\right] dN.$$

Then, the reduced system is

$$\begin{aligned} -\frac{\gamma}{C_t} dC_t &= \left[\begin{array}{c} \left[\rho - (1-g)(1-\alpha)\, A\left(\frac{K_{G,t}}{K_t}\right)^{\alpha} + \delta_K\right] \\ -\lambda\left[\tilde{C}^{-\gamma} + \left((1-\alpha)\, gAK_G^{\alpha} K^{-\alpha}\right)\frac{V_{K_G}\left(K, K_G + gAK_G^{\alpha} K^{1-\alpha}\right)}{u_C(C)} - 1\right] \end{array}\right] dt \\ &\quad -\frac{\gamma}{C_t}\left[\tilde{C}_t C_t - C_t\right] dN_t, \\ dz_t &= \left[z_t\frac{C_t}{K_t} - (1-g)\, Az_t^{\alpha+1} + \delta_K z\right] dt + \left[gAz_t^{\alpha}\right] dN_t. \end{aligned}$$

We can further simplify

$$\begin{aligned}
-\frac{\gamma}{C_t}dC_t &= \left[\begin{array}{c} \rho-(1-g)(1-\alpha)Az_t^{\alpha}+\delta_K \\ -\lambda\left[\tilde{C}^{-\gamma}+((1-\alpha)gAK_G^{\alpha}K^{-\alpha})\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right]\end{array}\right]dt \\
&\quad -\gamma\left[\frac{\tilde{C}_tC_t}{C_t}-1\right]dN_t, \\
dz_t &= \left[z_t\frac{C_t}{K_t}-(1-g)Az_t^{\alpha+1}+\delta_K z\right]dt+\left[gAz_t^{\alpha}\right]dN_t.
\end{aligned}$$

*3.7.1.2 Process for $c_t$*

Here, the problem is $C/K$. Therefore, we define $c = C/K$ as per capital (private) consumption. Then, we need

$$\begin{aligned}
dC_t &= -\frac{C_t}{\gamma}\left\{\begin{array}{c} \rho-(1-g)(1-\alpha)Az_t^{\alpha}+\delta_K \\ -\lambda\left[\tilde{C}^{-\gamma}+((1-\alpha)gAK_G^{\alpha}K^{-\alpha})\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right]\end{array}\right\}dt \\
&\quad +\left[\tilde{C}_tC_t-C_t\right]dN_t, \\
dz_t &= \left[z_t\frac{C_t}{K_t}-(1-g)Az_t^{\alpha+1}+\delta_K z_t\right]dt+\left[gAz_t^{\alpha}\right]dN_t, \\
dK_t &= \left[(1-g)AK_{G,t}^{\alpha}K_t^{1-\alpha}-C_t-\delta_K K_t\right]dt.
\end{aligned}$$

Ito's Lemma for Poisson processes yields

$$\begin{aligned}
dc &= \left[\begin{array}{c} -\frac{C}{\gamma}\frac{1}{K}\left\{\begin{array}{c} \left[\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K\right] \\ -\lambda\left[\tilde{C}^{-\gamma}+((1-\alpha)gAK_G^{\alpha}K^{-\alpha})\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right]\end{array}\right\} \\ -\frac{C}{K^2}\left[(1-g)AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right]\end{array}\right]dt \\
&\quad +\left[\frac{C+C\left[\frac{\tilde{C}C}{C}-1\right]}{K}-\frac{C}{K}\right]dN.
\end{aligned}$$

Simplify

$$\begin{aligned}
dc &= \left[\begin{array}{c} \frac{-C}{\gamma K}\left[\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K\right] \\ +\frac{C}{\gamma K}\lambda\left[\tilde{C}^{-\gamma}+((1-\alpha)gAK_G^{\alpha}K^{-\alpha})\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right] \\ -\frac{C}{K^2}(1-g)AK_G^{\alpha}K^{1-\alpha}+\frac{C^2}{K^2}+\frac{C}{K^2}\delta_K K\end{array}\right]dt \\
&\quad +\left[\frac{C+C\left[\frac{\tilde{C}C}{C}-1\right]}{K}-\frac{C}{K}\right]dN.
\end{aligned}$$

Using the definition of $c$

$$dc = \left[ \begin{array}{c} -\frac{c}{\gamma}\left[\rho - (1-g)(1-\alpha) Az^{\alpha} + \delta_K\right] \\ +c\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma} + ((1-\alpha) gAz^{\alpha}) \frac{V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)} - 1\right] \\ -c(1-g) Az^{\alpha} + c^2 + \delta_K c \end{array} \right] dt$$
$$+ \left[c + c\frac{\tilde{c}c}{c} - c - c\right] dN.$$

Further

$$dc = \left[ \begin{array}{c} -c\frac{\rho}{\gamma} + c\frac{(1-g)(1-\alpha)}{\gamma} Az^{\alpha} - c\frac{\delta_K}{\gamma} \\ +c\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma} + ((1-\alpha) gAz^{\alpha}) \frac{V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)} - 1\right] \\ -c(1-g) Az^{\alpha} + c^2 + \delta_K c \end{array} \right] dt$$
$$+ [\tilde{c}c - c] dN.$$

Then,

$$dc = c \left[ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} Az^{\alpha} - \frac{\delta_K}{\gamma} - (1-g) Az^{\alpha} - \frac{\rho}{\gamma} \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma} + ((1-\alpha) gAz^{\alpha}) \frac{V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)} - 1\right] + c + \delta_K \end{array} \right] dt$$
$$+ [\tilde{c}c - c] dN,$$

and

$$dc = c \left[ \begin{array}{c} (1-g) Az^{\alpha}\left[\frac{(1-\alpha)}{\gamma} - 1\right] - \frac{\rho}{\gamma} \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma} + ((1-\alpha) gAz^{\alpha}) \frac{V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)} - 1\right] + c + \delta_K - \frac{\delta_K}{\gamma} \end{array} \right] dt$$
$$+ [\tilde{c}c - c] dN,$$

then

$$dc = c \left[ \begin{array}{c} (1-g) Az^{\alpha}\left[\frac{(1-\alpha)-\gamma}{\gamma}\right] - \frac{\rho}{\gamma} \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma} + ((1-\alpha) gAz^{\alpha}) \frac{V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K(K, K_G)} \frac{V_K\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)} - 1\right] \\ +c + \frac{(\gamma-1)}{\gamma}\delta_K \end{array} \right] dt$$
$$+ [\tilde{c}c - c] dN,$$

hence,

$$dc = c \left[ \begin{array}{c} (1-g) Az^{\alpha}\left[\frac{(1-\alpha)-\gamma}{\gamma}\right] - \frac{\rho}{\gamma} \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma} + ((1-\alpha) gAz^{\alpha}) \frac{V_{K_G}\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K\left(K, K_G + gAK_G^{\alpha}K^{1-\alpha}\right)} \tilde{c}^{-\gamma} - 1\right] \\ +c + \frac{(\gamma-1)}{\gamma}\delta_K \end{array} \right] dt$$
$$+ [\tilde{c}c - c] dN.$$

Finally,

$$
\begin{aligned}
dc_t &= c_t \left[ \begin{array}{c} \left[\frac{1-\alpha-\gamma}{\gamma}\right](1-g)\,Az_t^{\alpha} - \frac{\rho}{\gamma} \\ +\frac{\lambda}{\gamma}\left[\tilde{c}_t^{-\gamma} + ((1-\alpha)\,gAz_t^{\alpha})\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}\tilde{c}_t^{-\gamma} - 1\right] \\ +c_t + \frac{(\gamma-1)}{\gamma}\delta_K \end{array} \right] dt \\
&\quad + \left[\tilde{c}_t c_t - c_t\right] dN_t, \\
dz_t &= \left[z_t c_t - (1-g)\,Az_t^{\alpha+1} + \delta_K z_t\right] dt + \left[gAz_t^{\alpha}\right] dN.
\end{aligned}
$$

*3.7.1.3 Process for $q_t$*

Now, define

$$
q = \frac{V_{K_G}}{V_K},
$$

and use

$$
\begin{aligned}
dV_K &= \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha)\,AK_G^{\alpha}K^{-\alpha} + \delta_K\right] V_K(K,K_G) \\ -\lambda \left[ \begin{array}{c} V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ +((1-\alpha)\,gAz^{\alpha})\,V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - V_K(K,K_G) \end{array} \right] \end{array} \right\} dt \\
&\quad + \left[V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - V_K(K,K_G)\right] dN, \\
dV_{K_G} &= \left\{ \begin{array}{c} \rho V_{K_G}(K,K_G) - V_K(K,K_G)\left[\alpha A(1-g)\,K_G^{\alpha-1}K^{1-\alpha}\right] \\ -\lambda \left[ \left(1+\alpha gAz^{\alpha-1}\right) V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - V_{K_G}(K,K_G) \right] \end{array} \right\} dt \\
&\quad + \left[V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - V_{K_G}(K,K_G)\right] dN.
\end{aligned}
$$

Ito's Lemma for Poisson processes yields

$$
\begin{aligned}
dq &= \frac{1}{V_K}\left[ \begin{array}{c} \rho V_{K_G}(K,K_G) - V_K(K,K_G)\left[\alpha A(1-g)\,z^{\alpha-1}\right] \\ -\lambda\left[\left(1+\alpha gAz^{\alpha-1}\right)V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - V_{K_G}(K,K_G)\right] \end{array} \right] dt \\
&\quad - \frac{V_{K_G}}{(V_K)^2}\left[ \begin{array}{c} \left[\rho - (1-g)(1-\alpha)\,Az^{\alpha} + \delta_K\right] V_K(K,K_G) \\ -\lambda \left[ \begin{array}{c} V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ +((1-\alpha)\,gAz^{\alpha})\,V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - V_K(K,K_G) \end{array} \right] \end{array} \right] dt \\
&\quad + \left[\frac{V_{K_G} + \left[V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - V_{K_G}(K,K_G)\right]}{V_K + \left[V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - V_K(K,K_G)\right]} - \frac{V_{K_G}}{V_K}\right] dN.
\end{aligned}
$$

Simplify to arrive at

$$
\begin{aligned}
dq &= \left[\begin{array}{c} \rho\frac{V_{K_G}(K,K_G)}{V_K} - \left[\alpha A\left(1-g\right)z^{\alpha-1}\right] \\ -\lambda\left[\left(1+\alpha gAz^{\alpha-1}\right)\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K} - \frac{V_{K_G}(K,K_G)}{V_K}\right] \end{array}\right] dt \\
&\quad - \left[\left[\rho-\left(1-g\right)\left(1-\alpha\right)Az^{\alpha}+\delta_K\right]\frac{V_{K_G}}{\left(V_K\right)^2}V_K\left(K,K_G\right)\right] dt \\
&\quad + \lambda\left[\begin{array}{c} \frac{V_{K_G}}{(V_K)^2}V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left(\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}}{(V_K)^2}V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ -\frac{V_{K_G}}{(V_K)^2}V_K\left(K,K_G\right) \end{array}\right] dt \\
&\quad + \left[\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)} - \frac{V_{K_G}}{V_K}\right] dN.
\end{aligned}
$$

Using $q$

$$
\begin{aligned}
dq &= \left[\rho q-\alpha A\left(1-g\right)z^{\alpha-1}\right] - \lambda\left[\left(1+\alpha gAz^{\alpha-1}\right)\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K} - q\right] dt \\
&\quad - \left[\left[\rho-\left(1-g\right)\left(1-\alpha\right)Az^{\alpha}+\delta_K\right]q\right] dt \\
&\quad + \lambda\left[\begin{array}{c} \frac{V_{K_G}}{(V_K)^2}V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left(\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}}{(V_K)^2}V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - q \end{array}\right] dt \\
&\quad + \left[\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)} - q\right] dN.
\end{aligned}
$$

Then,

$$
\begin{aligned}
dq &= \left[\rho q-\alpha A\left(1-g\right)z^{\alpha-1}\right] dt - \lambda\left[\left(1+\alpha gAz^{\alpha-1}\right)\frac{V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K} - q\right] dt \\
&\quad - \left[\left[\rho-\left(1-g\right)\left(1-\alpha\right)Az^{\alpha}+\delta_K\right]q\right] dt \\
&\quad + \lambda\left[\begin{array}{c} \frac{V_{K_G}}{(V_K)^2}V_K\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left(\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}}{(V_K)^2}V_{K_G}\left(K,K_G+gAK_G^{\alpha}K^{1-\alpha}\right) - q \end{array}\right] dt + \left[\tilde{q}q-q\right] dN.
\end{aligned}
$$

The system now reads

$$
\begin{aligned}
dq &= \left[\rho q-\alpha A\left(1-g\right)z^{\alpha-1}\right] dt - \lambda\left[\begin{array}{c} \left(1+\alpha gAz^{\alpha-1}\right)q\tilde{q}\tilde{c}^{-\gamma} \\ -q\tilde{c}^{-\gamma} - \left(\left(1-\alpha\right)gAz^{\alpha}\right)q^2\tilde{q}\tilde{c}^{-\gamma} \end{array}\right] dt \\
&\quad - \left[\left[\rho-\left(1-g\right)\left(1-\alpha\right)Az^{\alpha}+\delta_K\right]q\right] dt + \left[\tilde{q}q-q\right] dN_t.
\end{aligned}
$$

The final system is

$$dc_t = \left\{ \begin{array}{c} c_t \left[ \left[ \frac{1-\alpha-\gamma}{\gamma} \right] (1-g) A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{\lambda}{\gamma} \left[ \tilde{c}_t^{-\gamma} + ((1-\alpha) g A z_t^{\alpha}) q_t \tilde{q}_t \tilde{c}_t^{-\gamma} - 1 \right] \right] \\ +c_t \left[ c_t + \frac{(\gamma-1)}{\gamma} \delta_K \right] \end{array} \right\} dt \tag{138}$$
$$+ \left[ \tilde{c}_t c_t - c_t \right] dN_t,$$

$$dz_t = \left[ z_t c_t - (1-g) A z_t^{\alpha+1} + \delta_K z_t \right] dt + \left[ g A z_t^{\alpha} \right] dN_t, \tag{139}$$

$$dq_t = \left\{ \begin{array}{c} \left[ \rho q_t - \alpha A (1-g) z_t^{\alpha-1} \right] + \lambda \left[ \begin{array}{c} q_t \tilde{c}_t^{-\gamma} - \left( 1 + \alpha g A z_t^{\alpha-1} \right) q_t \tilde{q}_t \tilde{c}_t^{-\gamma} \\ + ((1-\alpha) g A z_t^{\alpha}) q_t^2 \tilde{q}_t \tilde{c}_t^{-\gamma} \end{array} \right] \\ - \left[ \left[ \rho - (1-g)(1-\alpha) A z_t^{\alpha} + \delta_K \right] q_t \right] \end{array} \right\} dt \tag{140}$$
$$+ \left[ \tilde{q}_t q_t - q_t \right] dN_t,$$

where $z = K_G/K$ and $c = C/K$. Further

$$\tilde{c} = \frac{c\left(K, K_G\left(1 + gA(K_G/K)^{\alpha-1}\right)\right)}{c\left(K, K_G\right)},$$
$$\tilde{q} = \frac{q\left(K, K_G\left(1 + gA(K_G/K)^{\alpha-1}\right)\right)}{q\left(K, K_G\right)}.$$

### 3.7.2 Turnovsky Model

Bellman equation (BE, for short) is

$$\rho V(K, K_G) = \max_C \left\{ u(C) + \frac{1}{dt} dV(K, K_G) \right\},$$

subject to

$$dK_t = \left[ (1-g) A K_{G,t}^{\alpha} K_t^{1-\alpha} - C_t - \delta_K K_t \right] dt,$$
$$dK_{G,t} = \left[ g A K_{G,t}^{\alpha} K_t^{1-\alpha} \right] dt.$$

**Step 1: FOC**

We start by computing the derivative $dV$, dropping time indices

$$dV(K, K_G) = V_K(K, K_G) \left[ (1-g) A K_G^{\alpha} K^{1-\alpha} - C - \delta_K K \right] dt + V_{K_G}(K, K_G) \left[ g A K_G^{\alpha} K^{1-\alpha} \right] dt.$$

Such that the BE is given by

$$\begin{aligned} \rho V(K, K_G) &= \max_C \left\{ u(C) + \frac{1}{dt} dV(K, K_G) \right\}, \\ &= \max_C \left\{ \begin{array}{c} u(C) + V_K(K, K_G) \left[ (1-g) A K_G^{\alpha} K^{1-\alpha} - C - \delta_K K \right] \\ + V_{K_G}(K, K_G) \left[ g A K_G^{\alpha} K^{1-\alpha} \right] \end{array} \right\}. \end{aligned}$$

The FOC is

$$u_C(C) = V_K(K, K_G).$$

**Step 2: Evolution of Co-States**

Consider the problem

$$\rho V(K, K_G) = \begin{array}{c} u(C(K, K_G)) + V_K(K, K_G)\left[(1-g) A K_G^{\alpha} K^{1-\alpha} - C(K, K_G) - \delta_K K\right] \\ +V_{K_G}(K, K_G)\left[g A K_G^{\alpha} K^{1-\alpha}\right] \end{array},$$

1. Take the derivative w.r.t. $K$

$$\begin{aligned} \rho V_K(K, K_G) &= V_K(K, K_G)\left[(1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} - \delta_K\right] \\ &\quad + V_{KK}(K, K_G)\left[(1-g) A K_G^{\alpha} K^{1-\alpha} - C(K, K_G) - \delta_K K\right] \\ &\quad + V_{K_G}(K, K_G)\left[g A (1-\alpha) K_G^{\alpha} K^{-\alpha}\right] + V_{K_G, K}(K, K_G)\left[g A K_G^{\alpha} K^{1-\alpha}\right]. \end{aligned}$$

Then,

$$\begin{aligned} dV_K(K, K_G) &= V_{KK}(K, K_G)\left[(1-g) A K_G^{\alpha} K^{1-\alpha} - C - \delta_K K\right] dt \\ &\quad + V_{K_G, K}(K, K_G)\left[g A K_G^{\alpha} K^{1-\alpha}\right] dt. \end{aligned}$$

Combining

$$\begin{aligned} dV_K(K, K_G) &= \left[\rho - (1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} + \delta_K\right] V_K(K, K_G)\, dt \\ &\quad - V_{K_G}(K, K_G)\left[g A (1-\alpha) K_G^{\alpha} K^{-\alpha}\right] dt. \end{aligned}$$

Hence,

$$dV_K(K, K_G) = \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} + \delta_K\right] V_K(K, K_G) \\ -V_{K_G}(K, K_G)\left[g A (1-\alpha) K_G^{\alpha} K^{-\alpha}\right] \end{array} \right\} dt.$$

2. Take the derivative w.r.t. $K_G$

$$\begin{aligned} \rho V_{K_G}(K, K_G) &= V_K(K, K_G)(1-g)\alpha A K_G^{\alpha-1} K^{1-\alpha} \\ &\quad + V_{KK_G}(K, K_G)\left[(1-g) A K_G^{\alpha} K^{1-\alpha} - C(K, K_G) - \delta_K K\right] \\ &\quad + V_{K_G}(K, K_G)\left[g A \alpha K_G^{\alpha-1} K^{1-\alpha}\right] + V_{K_G, K_G}(K, K_G)\left[g A K_G^{\alpha} K^{1-\alpha}\right]. \end{aligned}$$

Then,

$$\begin{aligned} dV_{K_G}(K, K_G) &= V_{KK_G}(K, K_G)\left[(1-g) A K_G^{\alpha} K^{1-\alpha} - C - \delta_K K\right] dt \\ &\quad + V_{K_G, K_G}(K, K_G)\left[g A K_G^{\alpha} K^{1-\alpha}\right] dt. \end{aligned}$$

Combining

$$\begin{aligned} dV_{K_G}(K, K_G) &= \left[\rho - g\alpha A K_G^{\alpha-1} K^{1-\alpha}\right] V_{K_G}(K, K_G)\, dt \\ &\quad - V_K(K, K_G)\left[\alpha A (1-g) K_G^{\alpha-1} K^{1-\alpha}\right] dt. \end{aligned}$$

**Step 3: Insert FOCs**

$$\begin{aligned}
du_C(C) &= dV_K(K, K_G), \\
&= \left\{ \begin{array}{c} [\rho - (1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} + \delta_K] V_K(K, K_G) \\ -V_{K_G}(K, K_G) [gA(1-\alpha) K_G^{\alpha} K^{-\alpha}] \end{array} \right\} dt.
\end{aligned}$$

Put differently,

$$\begin{aligned}
dV_K(K, K_G) &= \left\{ \begin{array}{c} [\rho - (1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} + \delta_K] V_K(K, K_G) \\ -V_{K_G}(K, K_G) [gA(1-\alpha) K_G^{\alpha} K^{-\alpha}] \end{array} \right\} dt, \\
dV_{K_G}(K, K_G) &= \left\{ \begin{array}{c} \left[\rho - g\alpha A K_G^{\alpha-1} K^{1-\alpha}\right] V_{K_G}(K, K_G) \\ -V_K(K, K_G) \left[\alpha A(1-g) K_G^{\alpha-1} K^{1-\alpha}\right] \end{array} \right\} dt, \\
dK_t &= \left[(1-g) A K_{G,t}^{\alpha} K_t^{1-\alpha} - C_t - \delta_K K_t\right] dt, \\
dK_{G,t} &= \left[gA K_{G,t}^{\alpha} K_t^{1-\alpha}\right] dt.
\end{aligned}$$

*3.7.2.1 Process for $z_t$*

Define

$$z_t = \frac{K_{G,t}}{K_t},$$

where

$$\begin{aligned}
dK_t &= \left[(1-g) A K_{G,t}^{\alpha} K_t^{1-\alpha} - C_t - \delta_K K_t\right] dt, \\
dK_{G,t} &= \left[gA K_{G,t}^{\alpha} K_t^{1-\alpha}\right] dt.
\end{aligned}$$

Computing the differential for $z_t$ gives (dropping time indices)

$$\begin{aligned}
dz &= \left[\frac{1}{K} gA K_G^{\alpha} K^{1-\alpha} - \frac{K_G}{K^2} \left[(1-g) A K_G^{\alpha} K^{1-\alpha} - C - \delta_K K\right]\right] dt, \\
&= \left[gA K_G^{\alpha} K^{-\alpha} - \frac{K_G}{K}(1-g) A K_G^{\alpha} K^{-\alpha} + C\frac{K_G}{K^2} + \frac{K_G}{K^2}\delta_K K\right] dt, \\
&= \left[gA\left(\frac{K_G}{K}\right)^{\alpha} - \frac{K_G}{K}(1-g) A \left(\frac{K_G}{K}\right)^{\alpha} + \frac{C}{K}\frac{K_G}{K} + \frac{K_G}{K}\delta_K\right] dt, \\
&= \left[gAz^{\alpha} - z(1-g) A z^{\alpha} + \frac{C}{K} z + \delta_K z\right] dt, \\
&= \left[gAz^{\alpha} - (1-g) A z^{\alpha+1} + \frac{C}{K} z + \delta_K z\right] dt,
\end{aligned}$$

such that

$$dz_t = \left[gAz_t^{\alpha} - (1-g) A z_t^{\alpha+1} + \frac{C_t}{K_t} z_t + \delta_K z_t\right] dt.$$

Then, the new system is given by

$$\begin{aligned}
dV_K &= \{[\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K]V_K(K,K_G)-V_{K_G}(K,K_G)[gA(1-\alpha)z^{\alpha}]\}\,dt,\\
dV_{K_G} &= \left\{\left[\rho-g\alpha Az^{\alpha-1}\right]V_{K_G}(K,K_G)-V_K(K,K_G)\left[\alpha A(1-g)z^{\alpha-1}\right]\right\}dt,\\
dz_t &= \left[gAz_t^{\alpha}-(1-g)Az_t^{\alpha+1}+\frac{C_t}{K_t}z_t+\delta_K z_t\right]dt.
\end{aligned}$$

*3.7.2.2 Process for $q_t$*

Define

$$q=\frac{V_{K_G}}{V_K},$$

and use

$$\begin{aligned}
dV_K &= \{[\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K]V_K(K,K_G)-V_{K_G}(K,K_G)[gA(1-\alpha)z^{\alpha}]\}\,dt,\\
dV_{K_G} &= \left\{\left[\rho-g\alpha Az^{\alpha-1}\right]V_{K_G}(K,K_G)-V_K(K,K_G)\left[\alpha A(1-g)z^{\alpha-1}\right]\right\}dt.
\end{aligned}$$

Computing the differential for $q$ gives

$$\begin{aligned}
dq &= \frac{1}{V_K}\left[\left[\rho-g\alpha Az^{\alpha-1}\right]V_{K_G}(K,K_G)-V_K(K,K_G)\left[\alpha A(1-g)z^{\alpha-1}\right]\right]dt\\
&\quad-\frac{V_{K_G}}{(V_K)^2}[[\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K]V_K(K,K_G)-V_{K_G}(K,K_G)[gA(1-\alpha)z^{\alpha}]]\,dt.
\end{aligned}$$

Simplify

$$\begin{aligned}
dq &= \left[\left[\rho-g\alpha Az^{\alpha-1}\right]\frac{V_{K_G}}{V_K}-\left[\alpha A(1-g)z^{\alpha-1}\right]\right]dt\\
&\quad-\left[[\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K]\frac{V_{K_G}}{(V_K)^2}V_K-\frac{V_{K_G}}{(V_K)^2}V_{K_G}[gA(1-\alpha)z^{\alpha}]\right]dt,
\end{aligned}$$

Using $q$

$$\begin{aligned}
dq &= \left[\left[\rho-g\alpha Az^{\alpha-1}\right]q-\alpha A(1-g)z^{\alpha-1}\right]dt\\
&\quad-\left[[\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K]q-q^2[gA(1-\alpha)z^{\alpha}]\right]dt,
\end{aligned}$$

Then,

$$\begin{aligned}
dq &= \rho q-g\alpha qAz^{\alpha-1}-\alpha A(1-g)z^{\alpha-1}\\
&\quad-\rho q+(1-g)(1-\alpha)qAz^{\alpha}-\delta_K q+q^2gA(1-\alpha)z^{\alpha},
\end{aligned}$$

Hence,

$$\begin{aligned}
dq &= -g\alpha qAz^{\alpha-1}-\alpha A(1-g)z^{\alpha-1}+(1-g)(1-\alpha)qAz^{\alpha}-\delta_K q+q^2gA(1-\alpha)z^{\alpha},\\
&= -g\alpha qAz^{\alpha-1}-\alpha Az^{\alpha-1}+\alpha Agz^{\alpha-1}+qAz^{\alpha}-gqAz^{\alpha}-\alpha qAz^{\alpha}+\alpha gqAz^{\alpha}-\delta_K q\\
&\quad+q^2gAz^{\alpha}-\alpha q^2gAz^{\alpha},
\end{aligned}$$

The system now is,

$$\begin{aligned}
dq &= \left[(1-\alpha) zq - \alpha\right] Az^{\alpha-1}\left[1-g+qg\right] - \delta_K q, \\
dz_t &= \left[gAz_t^{\alpha} - (1-g) Az_t^{\alpha+1} + \frac{C_t}{K_t} z_t + \delta_K z_t\right] dt, \\
dV_K &= \left\{\left[\rho - (1-g)(1-\alpha) Az^{\alpha} + \delta_K\right] V_K(K, K_G) - V_{K_G}(K, K_G)\left[gA(1-\alpha) z^{\alpha}\right]\right\} dt.
\end{aligned}$$

*3.7.2.3 Process for $c_t$*

As an intermediate step compute $dC_t$. Let

$$f(u_C(C)) = C,$$

apply CVF to $f(u_C(C))$ using

$$f_C(u_C(C)) = \frac{df(u_C(C))}{du_C(C)} = \frac{dC}{du_C(C)} = \frac{1}{u_{CC}(C)}.$$

Then,

$$\begin{aligned}
du_C(C) &= \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) AK_G^{\alpha}K^{-\alpha} + \delta_K\right] u_C(C) \\ -V_{K_G}(K, K_G)\left[gA(1-\alpha) K_G^{\alpha}K^{-\alpha}\right] \end{array} \right\} dt. \\
df(u_C(C)) &= f_C(u_C(C)) \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) AK_G^{\alpha}K^{-\alpha} + \delta_K\right] u_C(C) \\ -V_{K_G}(K, K_G)\left[gA(1-\alpha) K_G^{\alpha}K^{-\alpha}\right] \end{array} \right\} dt.
\end{aligned}$$

Moreover,

$$dC = \frac{1}{u_{CC}(C)} \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) AK_G^{\alpha}K^{-\alpha} + \delta_K\right] u_C(C) \\ -V_{K_G}(K, K_G)\left[gA(1-\alpha) K_G^{\alpha}K^{-\alpha}\right] \end{array} \right\} dt,$$

and

$$\frac{u_{CC}(C)}{u_C(C)} dC = \left\{\rho - (1-g)(1-\alpha) AK_G^{\alpha}K^{-\alpha} + \delta_K - \frac{V_{K_G}(K, K_G)}{u_C(C)}\left[gA(1-\alpha) K_G^{\alpha}K^{-\alpha}\right]\right\} dt.$$

Assume the following CRRA preferences ($\gamma \to 1 : \ln C$)

$$\begin{aligned}
u(C) &= \frac{C^{1-\gamma}}{1-\gamma}, \\
u_C(C) &= C^{-\gamma}, \\
u_{CC}(C) &= -\gamma C^{-(\gamma+1)}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{-\gamma}{C}dC &= \left\{ \begin{array}{c} \rho - (1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} + \delta_K \\ -\frac{V_{K_G}(K,K_G)}{C^{-\gamma}} \left[ gA(1-\alpha) K_G^{\alpha} K^{-\alpha} \right] \end{array} \right\} dt, \\
dC &= \frac{-C}{\gamma} \left\{ \begin{array}{c} \rho - (1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} + \delta_K \\ -\frac{V_{K_G}(K,K_G)}{C^{-\gamma}} \left[ gA(1-\alpha) K_G^{\alpha} K^{-\alpha} \right] \end{array} \right\} dt, \\
dC_t &= C_t \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A K_{G,t}^{\alpha} K_t^{-\alpha} - \frac{(\rho+\delta_K)}{\gamma} \\ +\frac{V_{K_G}(K,K_G)}{\gamma C_t^{-\gamma}} \left[ gA(1-\alpha) K_{G,t}^{\alpha} K_t^{-\alpha} \right] \end{array} \right\} dt.
\end{aligned}$$

Define

$$c_t = \frac{C_t}{K_t},$$

and apply Ito's lemma

$$\begin{aligned}
dc &= \frac{1}{K} C \left\{ \frac{(1-g)(1-\alpha)}{\gamma} A K_G^{\alpha} K^{-\alpha} - \frac{(\rho+\delta_K)}{\gamma} + \frac{V_{K_G}(K,K_G)}{\gamma C^{-\gamma}} \left[ gA(1-\alpha) K_G^{\alpha} K^{-\alpha} \right] \right\} dt \\
&\quad -\frac{C}{K^2} \left[ (1-g) A K_G^{\alpha} K^{1-\alpha} - C - \delta_K K \right] dt.
\end{aligned}$$

Then,

$$\begin{aligned}
dc &= \frac{C}{K} \left\{ \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{\rho}{\gamma} - \frac{\delta_K}{\gamma} + \frac{V_{K_G}(K,K_G)}{\gamma C^{-\gamma}} \left[ gA(1-\alpha) z^{\alpha} \right] \right\} dt \\
&\quad -\frac{C}{K} \left[ (1-g) A z^{\alpha} - \frac{C}{K} - \delta_K \right] dt.
\end{aligned}$$

Hence,

$$\begin{aligned}
dc &= c \left\{ \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{(\rho+\delta_K)}{\gamma} + C^{\gamma} \frac{V_{K_G}(K,K_G)}{\gamma} \left[ gA(1-\alpha) z^{\alpha} \right] \right\} dt \\
&\quad -c \left[ (1-g) A z^{\alpha} - c - \delta_K \right] dt,
\end{aligned}$$

and the system reads

$$\begin{aligned}
dc &= c \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{(\rho+\delta_K)}{\gamma} + \frac{V_{K_G}(K,K_G)}{\gamma C^{-\gamma}} \left[ gA(1-\alpha) z^{\alpha} \right] \\ -\left[ (1-g) A z^{\alpha} - c - \delta_K \right] \end{array} \right\} dt, \\
&= c \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{(\rho+\delta_K)}{\gamma} + \frac{V_{K_G}(K,K_G)}{\gamma u_C(C)} \left[ gA(1-\alpha) z^{\alpha} \right] \\ -\left[ (1-g) A z^{\alpha} - c - \delta_K \right] \end{array} \right\} dt, \\
&= c \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{(\rho+\delta_K)}{\gamma} + \frac{V_{K_G}(K,K_G)}{\gamma V_K(K,K_G)} \left[ gA(1-\alpha) z^{\alpha} \right] \\ -\left[ (1-g) A z^{\alpha} - c - \delta_K \right] \end{array} \right\} dt.
\end{aligned}$$

Use $q$

$$dc = c \left\{ \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{\rho}{\gamma} - \frac{\delta_K}{\gamma} + \frac{q}{\gamma} \left[ gA(1-\alpha) z^{\alpha} \right] - \left[ (1-g) A z^{\alpha} - c - \delta_K \right] \right\} dt,$$

Finally,

$$
\begin{aligned}
dc_t &= c_t \left\{ \frac{(1-g)(1-\alpha)}{\gamma} A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{gA(1-\alpha)}{\gamma} q_t z_t^{\alpha} - (1-g) A z_t^{\alpha} + c_t \right\} dt, && (141)\\
dq_t &= \left[(1-\alpha) z_t q_t - \alpha\right] A z_t^{\alpha-1} \left[1-g+q_t g\right] - \delta_K q_t, && (142)\\
dz_t &= \left[gAz_t^{\alpha} - (1-g) A z_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt, && (143)
\end{aligned}
$$

which is system (6) in Turnovsky (iff $\delta_K = 0$).

*3.7.2.4 Steady State*

Using ($dc_t = dz_t = dq_t = 0$)

$$
\begin{aligned}
0 &= c\left\{ \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{gA(1-\alpha)}{\gamma} q z^{\alpha} - (1-g) A z^{\alpha} + c \right\},\\
0 &= \left[(1-\alpha) z q - \alpha\right] A z^{\alpha-1} \left[1-g+qg\right] - \delta_K q,\\
0 &= gAz^{\alpha} - (1-g) A z^{\alpha+1} + cz + \delta_K z,
\end{aligned}
$$

gives

$$
\begin{aligned}
0 &= \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{gA(1-\alpha)}{\gamma} q z^{\alpha} - (1-g) A z^{\alpha} + c,\\
c &= (1-g) A z^{\alpha} - gAz^{\alpha-1} - \delta_K,
\end{aligned}
$$

Steady state growth rate

$$
\begin{aligned}
Y &= AK_G^{\alpha} K^{1-\alpha},\\
\phi_g &= \frac{dK_G}{K_G} = \frac{gAK_G^{\alpha}K^{1-\alpha}}{K_G} = gAz^{\alpha-1},\\
\phi_k &= \frac{dK}{K} = \frac{Y}{K} - \frac{dK_G}{K} - \frac{C}{K} = (1-g) A z^{\alpha} - c,\\
\phi &= \phi_g = \phi_k = gAz^{\alpha-1} = (1-g) A z^{\alpha} - c,\\
&= \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{gA(1-\alpha)}{\gamma} q z^{\alpha}.
\end{aligned}
$$

For the first-best equilibrium, take the derivative of BE w.r.t. $g$

$$
\begin{aligned}
dV(K, K_G) &= V_K(K, K_G)\left[(1-g) AK_G^{\alpha}K^{1-\alpha} - C - \delta_K K\right] dt\\
&\quad + V_{K_G}(K, K_G)\left[gAK_G^{\alpha}K^{1-\alpha}\right] dt,\\
V_K AK_G^{\alpha}K^{1-\alpha} &= V_{K_G} AK_G^{\alpha}K^{1-\alpha},\\
q &= 1,
\end{aligned}
$$

and, hence,

$$
\begin{aligned}
0 &= \left[(1-\alpha)\, zq - \alpha\right] Az^{\alpha-1} \left[1-g+g\right] - \delta_K, \\
\frac{\delta_K}{\left[1-g+g\right](1-\alpha)\, A} &= z^{\alpha} - \frac{\alpha}{(1-\alpha)} z^{\alpha-1}.
\end{aligned}
$$

Assuming $\delta_K = 0$ gives

$$
\begin{aligned}
z^{\alpha} - \frac{\alpha}{(1-\alpha)} z^{\alpha-1} &= 0, \\
z^{\alpha} &= \frac{\alpha}{(1-\alpha)} z^{\alpha-1}, \\
z &= \frac{\alpha}{(1-\alpha)}.
\end{aligned}
$$

For $c$

$$
\begin{aligned}
\frac{(1-g)(1-\alpha)}{\gamma} Az^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{gA(1-\alpha)}{\gamma} z^{\alpha} - (1-g)\, Az^{\alpha} + c &= 0, \\
(1-g)(1-\alpha)\, Az^{\alpha} - \rho + (\gamma-1)\,\delta_K + gA(1-\alpha)\, z^{\alpha} - \gamma(1-g)\, Az^{\alpha} + \gamma c &= 0, \\
(1-\alpha)\, Az^{\alpha} - g(1-\alpha)\, Az^{\alpha} - \rho + (\gamma-1)\,\delta_K + gA(1-\alpha)\, z^{\alpha} - \gamma(1-g)\, Az^{\alpha} + \gamma c &= 0, \\
(1-\alpha)\, Az^{\alpha} - \rho + (\gamma-1)\,\delta_K - \gamma(1-g)\, Az^{\alpha} + \gamma c &= 0, \\
\rho + \gamma(1-g)\, Az^{\alpha} - (\gamma-1)\,\delta_K - (1-\alpha)\, Az^{\alpha} &= \gamma c, \\
\frac{\rho + Az^{\alpha}\left[\gamma(1-g) - (1-\alpha)\right]}{\gamma} &= c,
\end{aligned}
$$

Then,

$$
\begin{aligned}
\phi &= \phi_g = \phi_k = gAz^{\alpha-1} = (1-g)\, Az^{\alpha} - c, \\
gAz^{\alpha-1} &= (1-g)\, Az^{\alpha} - c, \\
gA\frac{1}{z} &= (1-g)\, A - \frac{c}{z^{\alpha}}, \\
gA\frac{1}{\frac{\alpha}{(1-\alpha)}} &= (1-g)\, A - \frac{c}{z^{\alpha}}, \\
g(1-\alpha) &= \alpha(1-g) - \frac{\alpha c}{Az^{\alpha}}, \\
g - g\alpha &= \alpha - \alpha g - \frac{\alpha c}{Az^{\alpha}}, \\
g &= \alpha - \frac{\alpha c}{Az^{\alpha}}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
c &= \frac{\rho + Az^{\alpha}\left[\gamma\left(1-g\right)-\left(1-\alpha\right)\right]}{\gamma}, \\
&= \frac{\rho + Az^{\alpha}\left[\gamma\left(1-\alpha+\frac{\alpha c}{Az^{\alpha}}\right)-\left(1-\alpha\right)\right]}{\gamma}, \\
&= \frac{\rho + Az^{\alpha}\left[\gamma-\alpha\gamma+\gamma\frac{\alpha c}{Az^{\alpha}}-1+\alpha\right]}{\gamma}, \\
&= \frac{\rho + Az^{\alpha}\left[\gamma-\alpha\gamma-1+\alpha\right]+\gamma\alpha c}{\gamma}, \\
&= \frac{\rho + Az^{\alpha}\left[\gamma-\alpha\gamma-1+\alpha\right]}{\gamma}+\alpha c, \\
&= \frac{\rho + Az^{\alpha}\left[\gamma-\alpha\gamma-1+\alpha\right]}{\left(1-\alpha\right)\gamma}, \\
c &= \frac{\rho - \left(1-\gamma\right)\alpha Az^{\alpha}\left[1-\alpha\right]}{\left(1-\alpha\right)\gamma}.
\end{aligned}
$$

And,

$$
\begin{aligned}
\phi &= \phi_g = \phi_k = gAz^{\alpha-1} = \left(1-g\right)Az^{\alpha}-c, \\
&= \left(1-g\right)Az^{\alpha}-c, \\
&= \left(1-g\right)Az^{\alpha}-c, \\
&= \left(1-\alpha+\frac{\alpha c}{Az^{\alpha}}\right)Az^{\alpha}-c, \\
&= \left(1-\alpha\right)Az^{\alpha}+\alpha c-c, \\
&= \left(1-\alpha\right)Az^{\alpha}-\left(1-\alpha\right)c, \\
\frac{\phi}{\left(1-\alpha\right)} &= Az^{\alpha}-c, \\
&= Az^{\alpha}-\frac{\rho-\left(1-\gamma\right)\alpha Az^{\alpha}\left[1-\alpha\right]}{\left(1-\alpha\right)\gamma}, \\
&= Az^{\alpha}-\frac{\rho}{\left(1-\alpha\right)\gamma}+\frac{\left(1-\gamma\right)\alpha A}{\gamma}z^{\alpha}, \\
&= \left[A+\frac{\left(1-\gamma\right)\alpha A}{\gamma}\right]z^{\alpha}-\frac{\rho}{\left(1-\alpha\right)\gamma}, \\
&= \left[\frac{\gamma A+\left(1-\gamma\right)\alpha A}{\gamma}\right]z^{\alpha}-\frac{\rho}{\left(1-\alpha\right)\gamma}, \\
\phi &= \left[\gamma A+\left(1-\gamma\right)\alpha A\right]\frac{\left(1-\alpha\right)}{\gamma}z^{\alpha}-\frac{\rho}{\gamma}, \\
&= \frac{\left[\gamma+\alpha-\gamma\alpha\right]\left(1-\alpha\right)Az^{\alpha}-\rho}{\gamma}, \\
&= \frac{\left[\gamma\left(1-\alpha\right)+\alpha\right]\left(1-\alpha\right)Az^{\alpha}-\rho}{\gamma},
\end{aligned}
$$

$$
\begin{aligned}
\phi &= \phi_g = \phi_k = gAz^{\alpha-1} = (1-g)\,Az^{\alpha} - c, \\
&= \frac{(1-g)\,(1-\alpha)}{\gamma}Az^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{gA\,(1-\alpha)}{\gamma}qz^{\alpha}, \\
\phi &= \frac{(1-g)\,(1-\alpha)}{\gamma}Az^{\alpha} - \frac{\rho}{\gamma} + \frac{gA\,(1-\alpha)}{\gamma}z^{\alpha}, \\
&= \left[\frac{(1-g)\,(1-\alpha)}{\gamma} + \frac{g\,(1-\alpha)}{\gamma}\right] Az^{\alpha} - \frac{\rho}{\gamma}, \\
&= \left[\frac{(1-\alpha)}{\gamma}\right] Az^{\alpha} - \frac{\rho}{\gamma}, \\
&= \frac{(1-\alpha)\,Az^{\alpha} - \rho}{\gamma}.
\end{aligned}
$$

To sum up, the steady state values are derived from

$$
\begin{aligned}
0 &= \frac{(1-g)\,(1-\alpha)}{\gamma}A\tilde{z}^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{gA\,(1-\alpha)}{\gamma}\tilde{q}\tilde{z}^{\alpha} - (1-g)\,A\tilde{z}^{\alpha} + \tilde{c}, \\
0 &= [(1-\alpha)\,\tilde{z}\tilde{q} - \alpha]\,A\tilde{z}^{\alpha-1}\,[1-g+qg] - \delta_K\tilde{q}, \\
0 &= gA\tilde{z}^{\alpha} - (1-g)\,A\tilde{z}^{\alpha+1} + cz + \delta_K\tilde{z},
\end{aligned}
$$

assuming $\delta_K = 0$

$$
\begin{aligned}
\tilde{c} &= (1-g)\,A\tilde{z}^{\alpha} - gA\tilde{z}^{\alpha-1}, \\
\tilde{z}\tilde{q} &= \frac{\alpha}{(1-\alpha)}, \\
0 &= \frac{(1-g)\,(1-\alpha)}{\gamma}A\tilde{z}^{\alpha} - \frac{\rho}{\gamma} + \frac{gA\,(1-\alpha)}{\gamma}q\tilde{z}^{\alpha} - (1-g)\,A\tilde{z}^{\alpha} + \tilde{c},
\end{aligned}
$$

with the growth rates

$$
\begin{aligned}
\phi_g &= gA\tilde{z}^{\alpha-1}, \\
\phi_k &= (1-g)\,A\tilde{z}^{\alpha} - \tilde{c}, \\
\phi &= \phi_g = \phi_k = gA\tilde{z}^{\alpha-1} = (1-g)\,A\tilde{z}^{\alpha} - \tilde{c}, \\
&= \frac{(1-g)\,(1-\alpha)}{\gamma}A\tilde{z}^{\alpha} - \frac{\rho}{\gamma} + \frac{gA\,(1-\alpha)}{\gamma}\tilde{q}\tilde{z}^{\alpha}.
\end{aligned}
$$

The first-best equilibrium is characterized by the following steady state values (denoted by a $\hat{}$) - assuming $\delta_K = 0$ -

$$\begin{aligned}
\hat{q} &= 1, \\
\hat{z} &= \frac{\alpha}{(1-\alpha)}, \\
\hat{c} &= \frac{\rho - (1-\gamma)\,\alpha A \hat{z}^{\alpha}\,[1-\alpha]}{(1-\alpha)\,\gamma}, \\
\hat{g} &= \alpha - \frac{\alpha c}{A\hat{z}^{\alpha}}, \\
\hat{\phi} &= \frac{(1-\alpha)\,A\hat{z}^{\alpha} - \rho}{\gamma}.
\end{aligned}$$

#### *3.7.2.5 Long-Run Fiscal Effects*

Take

$$\begin{aligned}
\tilde{c} &= (1-g)\,A\tilde{z}^{\alpha} - gA\tilde{z}^{\alpha-1}, \\
\tilde{z}\tilde{q} &= \frac{\alpha}{(1-\alpha)}, \\
0 &= \frac{(1-g)(1-\alpha)}{\gamma}A\tilde{z}^{\alpha} - \frac{\rho}{\gamma} + \frac{gA(1-\alpha)}{\gamma}q\tilde{z}^{\alpha} - (1-g)\,A\tilde{z}^{\alpha} + \tilde{c}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{(1-g)(1-\alpha)}{\gamma}A\tilde{z}^{\alpha} - \frac{\rho}{\gamma} + \frac{gA(1-\alpha)}{\gamma}q\tilde{z}^{\alpha} - gA\tilde{z}^{\alpha-1} &= 0, \\
\frac{(1-g)(1-\alpha)}{\gamma}Az - \frac{\rho}{\gamma\tilde{z}^{\alpha-1}} + \frac{gA(1-\alpha)}{\gamma}qz - gA &= 0, \\
\frac{(1-g)(1-\alpha)}{\gamma}Az - \frac{\rho}{\gamma\tilde{z}^{\alpha-1}} + \frac{gA(1-\alpha)}{\gamma}\frac{\alpha}{(1-\alpha)} - gA &= 0, \\
\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha} - \frac{\rho}{\gamma} + \frac{gA(1-\alpha)}{\gamma}\frac{\alpha}{(1-\alpha)}\tilde{z}^{\alpha-1} - gA\tilde{z}^{\alpha-1} &= 0, \\
\frac{(1-g)(1-\alpha)}{\gamma}z^{\alpha} - \frac{\rho}{A\gamma} + \frac{g}{\gamma}\alpha\tilde{z}^{\alpha-1} - g\tilde{z}^{\alpha-1} &= 0.
\end{aligned}$$

Take derivative

$$\begin{aligned}
0 = {} & \frac{(1-g)(1-\alpha)}{\gamma}\alpha z^{\alpha-1}\frac{dz}{dg} + \frac{g}{\gamma}\alpha(\alpha-1)\,z^{\alpha-2}\frac{dz}{dg} \\
& -g(\alpha-1)\,z^{\alpha-2}\frac{dz}{dg} + \frac{-(1-\alpha)}{\gamma}z^{\alpha} + \frac{1}{\gamma}\alpha\tilde{z}^{\alpha-1} - \tilde{z}^{\alpha-1}.
\end{aligned}$$

Then,

$$
\begin{aligned}
\frac{dz}{dg}\left[\frac{(1-g)(1-\alpha)}{\gamma}\alpha z^{\alpha-1}+\frac{g}{\gamma}\alpha(\alpha-1)z^{\alpha-2}-g(\alpha-1)z^{\alpha-2}\right] &= \tilde{z}^{\alpha-1}-\frac{1}{\gamma}\alpha\tilde{z}^{\alpha-1} \\
&\quad +\frac{(1-\alpha)}{\gamma}z^{\alpha}, \\
\frac{dz}{dg}\left[\frac{(1-g)(1-\alpha)}{\gamma}\alpha+\frac{g}{\gamma}\alpha(\alpha-1)\frac{1}{z}-g(\alpha-1)\frac{1}{z}\right] &= 1-\frac{1}{\gamma}\alpha+\frac{(1-\alpha)}{\gamma}z, \\
\frac{dz}{dg}\left[\frac{(1-g)(1-\alpha)}{\gamma}\alpha z+\frac{g}{\gamma}\alpha(\alpha-1)-g(\alpha-1)\right] &= z-\frac{1}{\gamma}\alpha z+\frac{(1-\alpha)}{\gamma}z^{2}, \\
\frac{dz}{dg}\left[\frac{(1-g)(1-\alpha)}{\gamma}\alpha z+\frac{g}{\gamma}\alpha(\alpha-1)-g(\alpha-1)\right] &= z\left[1+\frac{1}{\gamma}\left[(1-\alpha)z-\alpha\right]\right], \\
(1-\alpha)\frac{dz}{dg}\left[\frac{(1-g)}{\gamma}\alpha z-\frac{g}{\gamma}\alpha+g\right] &= \frac{z\left[1+\frac{1}{\gamma}\left[(1-\alpha)z-\alpha\right]\right]}{1}, \\
\frac{dz}{dg}\left[\frac{(1-g)}{\gamma}\alpha z-\frac{g}{\gamma}\alpha+g\right] &= \frac{z\left[1+\frac{1}{\gamma}\left[(1-\alpha)z-\alpha\right]\right]}{(1-\alpha)}, \\
\frac{dz}{dg}\left[g+\frac{1}{\gamma}\left[(1-g)\alpha z-g\alpha\right]\right] &= \frac{z\left[1+\frac{1}{\gamma}\left[(1-\alpha)z-\alpha\right]\right]}{(1-\alpha)}, \\
\frac{z\left[1+\frac{1}{\gamma}\left[(1-\alpha)z-\alpha\right]\right]}{(1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)z-g\right]\right]} &= \frac{dz}{dg}.
\end{aligned}
$$

Take derivative

$$
\begin{aligned}
\tilde{c} &= (1-g)\,A\tilde{z}^{\alpha} - gA\tilde{z}^{\alpha-1}, \\
\frac{dc}{dg} &= -A\tilde{z}^{\alpha} - A\tilde{z}^{\alpha-1} + (1-g)\,A\alpha\tilde{z}^{\alpha-1}\frac{dz}{dg} \\
&\quad -gA\,(\alpha-1)\,\tilde{z}^{\alpha-2}\frac{dz}{dg}, \\
&= \frac{dz}{dg}\left[(1-g)\,A\alpha\tilde{z}^{\alpha-1} - gA\,(\alpha-1)\,\tilde{z}^{\alpha-2}\right] \\
&\quad -A\tilde{z}^{\alpha} - A\tilde{z}^{\alpha-1}, \\
&= \frac{z\left[1+\frac{1}{\gamma}\left[(1-\alpha)\,z-\alpha\right]\right]}{(1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]}\left[\begin{array}{c} (1-g)\,A\alpha\tilde{z}^{\alpha-1} \\ -gA\,(\alpha-1)\,\tilde{z}^{\alpha-2} \end{array}\right] \\
&\quad -A\tilde{z}^{\alpha} - A\tilde{z}^{\alpha-1}, \\
(1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]\frac{dc}{dg} &= \tilde{z}^{\alpha}\left[1+\frac{1}{\gamma}\left[(1-\alpha)\,z-\alpha\right]\right]\left[(1-g)\,A\alpha - gA\,(\alpha-1)\,\frac{1}{z}\right] \\
&\quad -A\tilde{z}^{\alpha}\left[1+\frac{1}{z}\right](1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right], \\
&= A\tilde{z}^{\alpha}\left[1+\frac{1}{\gamma}\left[(1-\alpha)\,z-\alpha\right]\right]\left[(1-g)\,\alpha + g\,(1-\alpha)\,\frac{1}{z}\right] \\
&\quad -A\tilde{z}^{\alpha}\left[1+\frac{1}{z}\right](1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right], \\
\frac{(1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]}{A\tilde{z}^{\alpha}}\frac{dc}{dg} &= \left[1+\frac{1}{\gamma}(1-\alpha)\,z-\frac{1}{\gamma}\alpha\right]\left[(1-g)\,\alpha + g\,(1-\alpha)\,\frac{1}{z}\right] \\
&\quad -(1-\alpha)\left[1+\frac{1}{z}\right]\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right], \\
&= \frac{1}{\gamma}\left[\gamma+(1-\alpha)\,z-\alpha\right]\left[\alpha-g\alpha+g\frac{1}{z}-g\alpha\frac{1}{z}\right] \\
&\quad -(1-\alpha)\,\frac{1}{\gamma}\left[1+\frac{1}{z}\right]\left[\gamma g+\alpha z-\alpha gz-\alpha g\right], \\
\frac{\gamma\,(1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]}{A\tilde{z}^{\alpha}}\frac{dc}{dg} &= \left[\gamma+(1-\alpha)\,z-\alpha\right]\left[\alpha-g\alpha+g\frac{1}{z}-g\alpha\frac{1}{z}\right] \\
&\quad -(1-\alpha)\left[1+\frac{1}{z}\right]\left[\gamma g+\alpha z-\alpha gz-\alpha g\right], \\
&= \gamma\alpha-\gamma g\alpha+\gamma g\frac{1}{z}-\gamma g\alpha\frac{1}{z}+(1-\alpha)\,z\alpha \\
&\quad -(1-\alpha)\,zg\alpha+(1-\alpha)\,zg\frac{1}{z}-(1-\alpha)\,zg\alpha\frac{1}{z} \\
&\quad -\alpha\alpha+\alpha g\alpha-\alpha g\frac{1}{z}+\alpha g\alpha\frac{1}{z} \\
&\quad -(1-\alpha)\left[\begin{array}{c} \gamma g+\alpha z-\alpha gz-\alpha g+\gamma g\frac{1}{z} \\ +\alpha\frac{1}{z}z-\alpha gz\frac{1}{z}-\alpha g\frac{1}{z} \end{array}\right].
\end{aligned}
$$

Then,

$$\begin{aligned}
&= \gamma\alpha - \gamma g\alpha + \gamma g\frac{1}{z} - \gamma g\alpha\frac{1}{z} + (1-\alpha)\, z\alpha \\
&\quad - (1-\alpha)\, zg\alpha + (1-\alpha)\, zg\frac{1}{z} - (1-\alpha)\, zg\alpha\frac{1}{z} - \alpha\alpha + \alpha g\alpha - \alpha g\frac{1}{z} + \alpha g\alpha\frac{1}{z} \\
&\quad - (1-\alpha)\left[\gamma g + \alpha z - \alpha g z - \alpha g + \gamma g\frac{1}{z} + \alpha\frac{1}{z}z - \alpha g z\frac{1}{z} - \alpha g\frac{1}{z}\right], \\
&= \gamma\alpha - \gamma g\alpha + \gamma g\frac{1}{z} - \gamma g\alpha\frac{1}{z} + (1-\alpha)\, z\alpha \\
&\quad - (1-\alpha)\, zg\alpha + (1-\alpha)\, g - (1-\alpha)\, g\alpha - \alpha\alpha + \alpha g\alpha - \alpha g\frac{1}{z} + \alpha g\alpha\frac{1}{z} \\
&\quad - (1-\alpha)\left[\gamma g + \alpha z - \alpha g z - \alpha g + \gamma g\frac{1}{z} + \alpha - \alpha g - \alpha g\frac{1}{z}\right], \\
&= \gamma\alpha - \gamma g\alpha + \gamma g\frac{1}{z} - \gamma g\alpha\frac{1}{z} + (1-\alpha)\, z\alpha \\
&\quad - (1-\alpha)\, zg\alpha + (1-\alpha)\, g - (1-\alpha)\, g\alpha - \alpha\alpha + \alpha g\alpha - \alpha g\frac{1}{z} + \alpha g\alpha\frac{1}{z} \\
&\quad -\gamma g - \alpha z + \alpha g z + \alpha g - \gamma g\frac{1}{z} - \alpha + \alpha g + \alpha g\frac{1}{z} + \alpha\gamma g + \alpha\alpha z - \alpha\alpha g z \\
&\quad -\alpha\alpha g + \alpha\gamma g\frac{1}{z} + \alpha\alpha - \alpha\alpha g - \alpha\alpha g\frac{1}{z}, \\
&= \gamma\alpha - \gamma g\alpha + \gamma g\frac{1}{z} - \gamma g\alpha\frac{1}{z} + z\alpha - \alpha\alpha z - zg\alpha + \alpha z g\alpha + g - \alpha g - g\alpha \\
&\quad +\alpha g\alpha - \alpha\alpha + \alpha g\alpha - \alpha g\frac{1}{z} + \alpha g\alpha\frac{1}{z} \\
&\quad -\gamma g - \alpha z + \alpha g z + \alpha g - \gamma g\frac{1}{z} - \alpha + \alpha g + \alpha g\frac{1}{z} + \alpha\gamma g + \alpha\alpha z - \alpha\alpha g z \\
&\quad -\alpha\alpha g + \alpha\gamma g\frac{1}{z} + \alpha\alpha - \alpha\alpha g - \alpha\alpha g\frac{1}{z}, \\
&= \gamma\alpha - \alpha + g - \gamma g, \\
&= -(1-\gamma)\,\alpha + g\,(1-\gamma), \\
&= (g-\alpha)(1-\gamma), \\
\frac{dc}{dg} &= \frac{A\tilde{z}^{\alpha}(g-\alpha)(1-\gamma)}{\gamma(1-\alpha)\left[g + \frac{\alpha}{\gamma}\left[(1-g)\, z - g\right]\right]}.
\end{aligned}$$

For the steady state growth rate, use

$$\begin{aligned}
\phi &= (1-g)\,Az^{\alpha}-c,\\
\frac{d\phi}{dg} &= -Az^{\alpha}+(1-g)\,A\alpha z^{\alpha-1}\frac{dz}{dg}-\frac{dc}{dg},\\
&= -Az^{\alpha}+(1-g)\,A\alpha z^{\alpha-1}\frac{z\left[1+\frac{1}{\gamma}\left[(1-\alpha)\,z-\alpha\right]\right]}{(1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]}\\
&\quad -\frac{A\tilde{z}^{\alpha}\,(g-\alpha)\,(1-\gamma)}{\gamma\,(1-\alpha)\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]},\\
\frac{\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]}{Az^{\alpha}}\frac{d\phi}{dg} &= -\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]+\frac{(1-g)\,\alpha\left[1+\frac{1}{\gamma}\left[(1-\alpha)\,z-\alpha\right]\right]}{(1-\alpha)}\\
&\quad -\frac{(g-\alpha)\,(1-\gamma)}{\gamma\,(1-\alpha)},\\
&= -g-\frac{\alpha}{\gamma}z+zg\frac{\alpha}{\gamma}+\frac{\alpha}{\gamma}g+\frac{\frac{1}{\gamma}\,(1-g)\,\alpha\left[\gamma+z-\alpha z-\alpha\right]}{(1-\alpha)}\\
&\quad -\frac{(g-\gamma g-\alpha+\alpha\gamma)}{\gamma\,(1-\alpha)},\\
\frac{\gamma\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]}{Az^{\alpha}}\frac{d\phi}{dg} &= -g\gamma-\alpha z+zg\alpha+\alpha g+\frac{(1-g)\,\alpha\left[\gamma+z-\alpha z-\alpha\right]}{(1-\alpha)}\\
&\quad -\frac{(g-\gamma g-\alpha+\alpha\gamma)}{(1-\alpha)},\\
&= \frac{\alpha-g+\alpha g-\alpha\alpha}{(1-\alpha)},\\
&= \frac{(1-\alpha)\,(\alpha-g)}{(1-\alpha)},\\
\frac{d\phi}{dg} &= \frac{(\alpha-g)\,Az^{\alpha}}{\gamma\left[g+\frac{\alpha}{\gamma}\left[(1-g)\,z-g\right]\right]}.
\end{aligned}$$

### 3.7.3 Turnovsky with Lags (Full Model)

The Bellman equation is

$$\rho V\left(K,K_G\right)=\max_{C}\left\{u\left(C\right)+\frac{1}{dt}\mathbb{E}_t dV\left(K,K_G\right)\right\},$$

subject to

$$\begin{aligned}
dK_t &= \left[(1-g)\,AK_{G,t}^{\alpha}K_t^{1-\alpha}-C_t-\delta_K K_t\right]dt,\\
dK_{G,t} &= \left[\theta gAK_{G,t}^{\alpha}K_t^{1-\alpha}-\delta_G K_{G,t}\right]dt+\left[(1-\theta)\,gAK_{G,t}^{\alpha}K_t^{1-\alpha}\right]dN_t.
\end{aligned}$$

**Step 1: FOC**

We start by computing the derivative $dV$, dropping time indices

$$\begin{aligned}
dV\left(K,K_G\right) &= V_K\left(K,K_G\right)\left[(1-g)\,AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right]dt \\
&\quad +V_{K_G}\left(K,K_G\right)\left[\theta g AK_G^{\alpha}K^{1-\alpha}-\delta_G K_G\right]dt \\
&\quad +\left[V\left(K,K_G+(1-\theta)\,gAK_G^{\alpha}K^{1-\alpha}\right)-V\left(K,K_G\right)\right]dN.
\end{aligned}$$

Such that the BE is given by

$$\begin{aligned}
\rho V\left(K,K_G\right) &= \max_C\left\{u\left(C\right)+\frac{1}{dt}\mathbb{E}_t dV\left(K,K_G\right)\right\}, \\
&= \max_C\left\{\begin{array}{c}
u\left(C\right)+V_K\left(K,K_G\right)\left[(1-g)\,AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right] \\
+V_{K_G}\left(K,K_G\right)\left[\theta g AK_G^{\alpha}K^{1-\alpha}-\delta_G K_G\right] \\
+\lambda\left[V\left(K,K_G+(1-\theta)\,gAK_G^{\alpha}K^{1-\alpha}\right)-V\left(K,K_G\right)\right]
\end{array}\right\}.
\end{aligned}$$

The FOC is

$$u_C\left(C\right)=V_K\left(K,K_G\right).$$

**Step 2: Evolution of Co-States**

Consider the problem

$$\rho V\left(K,K_G\right)=\begin{array}{c}
u\left(C\left(K,K_G\right)\right)+V_K\left(K,K_G\right)\left[(1-g)\,AK_G^{\alpha}K^{1-\alpha}-C\left(K,K_G\right)-\delta_K K\right] \\
+V_{K_G}\left(K,K_G\right)\left[\theta g AK_G^{\alpha}K^{1-\alpha}-\delta_G K_G\right] \\
+\lambda\left[V\left(K,K_G+(1-\theta)\,gAK_G^{\alpha}K^{1-\alpha}\right)-V\left(K,K_G\right)\right]
\end{array},$$

1. Take the derivative w.r.t. $K$

$$\begin{aligned}
\rho V_K\left(K,K_G\right) &= V_K\left(K,K_G\right)\left[(1-g)\,(1-\alpha)\,AK_G^{\alpha}K^{-\alpha}-\delta_K\right] \\
&\quad +V_{KK}\left(K,K_G\right)\left[(1-g)\,AK_G^{\alpha}K^{1-\alpha}-C\left(K,K_G\right)-\delta_K K\right] \\
&\quad +V_{K_G}\left(K,K_G\right)\left[\theta g A\,(1-\alpha)\,K_G^{\alpha}K^{-\alpha}\right]+V_{K_G K}\left(K,K_G\right)\left[\theta g AK_G^{\alpha}K^{1-\alpha}-\delta_G K_G\right] \\
&\quad +\lambda\left[\begin{array}{c}
V_K\left(K,K_G+(1-\theta)\,gAK_G^{\alpha}K^{1-\alpha}\right) \\
+\left((1-\theta)\,(1-\alpha)\,gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+(1-\theta)\,gAK_G^{\alpha}K^{1-\alpha}\right) \\
-V_K\left(K,K_G\right)
\end{array}\right].
\end{aligned}$$

Then,

$$\begin{aligned}
dV_K\left(K,K_G\right) &= V_{KK}\left(K,K_G\right)\left[(1-g)\,AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right]dt \\
&\quad +V_{K_G K}\left(K,K_G\right)\left[\theta g AK_G^{\alpha}K^{1-\alpha}-\delta_G K_G\right]dt \\
&\quad +\left[V_K\left(K,K_G+(1-\theta)\,gAK_G^{\alpha}K^{1-\alpha}\right)-V_K\left(K,K_G\right)\right]dN.
\end{aligned}$$

Combining

$$\begin{aligned}
dV_K(K,K_G) &= \left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]V_K(K,K_G)\,dt \\
&\quad -V_{K_G}(K,K_G)\left[\theta gA(1-\alpha)K_G^{\alpha}K^{-\alpha}\right]dt \\
&\quad -\lambda\left[\begin{array}{c} V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G) \\ +\left((1-\theta)(1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)\end{array}\right]dt \\
&\quad +\left[V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G)\right]dN.
\end{aligned}$$

Hence,

$$\begin{aligned}
dV_K(K,K_G) &= \left\{\begin{array}{c} \left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]V_K(K,K_G) \\ -V_{K_G}(K,K_G)\left[\theta gA(1-\alpha)K_G^{\alpha}K^{-\alpha}\right] \\ -\lambda\left[\begin{array}{c} V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)+ \\ \left((1-\theta)(1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right) \\ -V_K(K,K_G)\end{array}\right]\end{array}\right\} \\
&\quad +\left[V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G)\right]dN_t.
\end{aligned}$$

2. Take the derivative w.r.t. $K_G$

$$\begin{aligned}
\rho V_{K_G}(K,K_G) &= V_K(K,K_G)(1-g)\alpha AK_G^{\alpha-1}K^{1-\alpha} \\
&\quad +V_{KK_G}(K,K_G)\left[(1-g)AK_G^{\alpha}K^{1-\alpha}-C(K,K_G)-\delta_K K\right] \\
&\quad +V_{K_G}(K,K_G)\left[\theta gA\alpha K_G^{\alpha-1}K^{1-\alpha}-\delta_G\right] \\
&\quad +V_{K_GK_G}(K,K_G)\left[\theta gAK_G^{\alpha}K^{1-\alpha}-\delta_G K_G\right] \\
&\quad +\lambda\left[\begin{array}{c} \left(1+(1-\theta)\alpha gAK_G^{\alpha-1}K^{1-\alpha}\right)V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right) \\ -V_{K_G}(K,K_G)\end{array}\right].
\end{aligned}$$

Then,

$$\begin{aligned}
dV_{K_G}(K,K_G) &= V_{KK_G}(K,K_G)\left[(1-g)AK_G^{\alpha}K^{1-\alpha}-C-\delta_K K\right]dt \\
&\quad +V_{K_GK_G}(K,K_G)\left[\theta gAK_G^{\alpha}K^{1-\alpha}-\delta_G K_G\right]dt \\
&\quad +\left[V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]dN.
\end{aligned}$$

Combining

$$\begin{aligned}
dV_{K_G}(K,K_G) &= \left[\rho-\theta g\alpha AK_G^{\alpha-1}K^{1-\alpha}+\delta_G\right]V_{K_G}(K,K_G)\,dt \\
&\quad -V_K(K,K_G)\left[\alpha A(1-g)K_G^{\alpha-1}K^{1-\alpha}\right]dt \\
&\quad -\lambda\left[\begin{array}{c} \left(1+(1-\theta)\alpha gAK_G^{\alpha-1}K^{1-\alpha}\right)V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right) \\ -V_{K_G}(K,K_G)\end{array}\right]dt \\
&\quad +\left[V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]dN.
\end{aligned}$$

**Step 3: Insert FOCs**

$$\begin{aligned}
du_C(C) &= dV_K(K,K_G) \\
&= \left\{ \begin{array}{c} \left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]V_K(K,K_G) \\ -V_{K_G}(K,K_G)\left[\theta gA(1-\alpha)K_G^{\alpha}K^{-\alpha}\right] \\ -\lambda\left[\begin{array}{c} V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left((1-\theta)(1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right) \\ -V_K(K,K_G) \end{array}\right] \end{array} \right\} dt \\
&\quad + \left[V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G)\right]dN_t.
\end{aligned}$$

Put differently,

$$\begin{aligned}
dV_K(K,K_G) &= \left\{ \begin{array}{c} \left[\rho-(1-g)(1-\alpha)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]V_K(K,K_G) \\ -V_{K_G}(K,K_G)\left[\theta gA(1-\alpha)K_G^{\alpha}K^{-\alpha}\right] \\ -\lambda\left[\begin{array}{c} V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)+ \\ \left((1-\theta)(1-\alpha)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right) \\ -V_K(K,K_G) \end{array}\right] \end{array} \right\} dt \\
&\quad + \left[V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G)\right]dN_t, \\
dV_{K_G}(K,K_G) &= \left\{ \begin{array}{c} \left[\rho-\theta g\alpha AK_G^{\alpha-1}K^{1-\alpha}+\delta_G\right]V_{K_G}(K,K_G) \\ -V_K(K,K_G)\left[\alpha A(1-g)K_G^{\alpha-1}K^{1-\alpha}\right] \\ -\lambda\left[\begin{array}{c} \left(1+(1-\theta)\alpha gAK_G^{\alpha-1}K^{1-\alpha}\right)V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right) \\ -V_{K_G}(K,K_G) \end{array}\right] \end{array} \right\} dt \\
&\quad + \left[V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]dN, \\
dK_t &= \left[(1-g)AK_{G,t}^{\alpha}K_t^{1-\alpha}-C_t-\delta_K K_t\right]dt, \\
dK_{G,t} &= \left[\theta gAK_{G,t}^{\alpha}K_t^{1-\alpha}-\delta_G K_{G,t}\right]dt+\left[(1-\theta)gAK_{G,t}^{\alpha}K_t^{1-\alpha}\right]dN_t.
\end{aligned}$$

*3.7.3.1 Process for $z_t$*

Define

$$z_t=\frac{K_{G,t}}{K_t},$$

where

$$\begin{aligned}
dK_t &= \left[(1-g)AK_{G,t}^{\alpha}K_t^{1-\alpha}-C_t-\delta_K K_t\right]dt, \\
dK_{G,t} &= \left[\theta gAK_{G,t}^{\alpha}K_t^{1-\alpha}-\delta_G K_{G,t}\right]dt+\left[(1-\theta)gAK_{G,t}^{\alpha}K_t^{1-\alpha}\right]dN_t.
\end{aligned}$$

Ito's Lemma for Poisson processes yields for $z_t$ gives (dropping time indices)

$$\begin{aligned}
dz &= \left[\frac{1}{K}\left[\theta g A K_G^{\alpha} K^{1-\alpha} - \delta_G K_G\right] - \frac{K_G}{K^2}\left[(1-g) A K_G^{\alpha} K^{1-\alpha} - C - \delta_K K\right]\right] dt \\
&\quad + \left[\frac{K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}}{K} - \frac{K_G}{K}\right] dN, \\
&= \left[\theta g A K_G^{\alpha} K^{-\alpha} - \delta_G \frac{K_G}{K} - \frac{K_G}{K}(1-g) A K_G^{\alpha} K^{-\alpha} + C\frac{K_G}{K^2} + \frac{K_G}{K^2}\delta_K K\right] dt \\
&\quad + \left[\frac{K_G}{K} + (1-\theta) g A K_G^{\alpha} K^{-\alpha} - \frac{K_G}{K}\right] dN, \\
&= \left[\theta g A \left(\frac{K_G}{K}\right)^{\alpha} - \delta_G \frac{K_G}{K} - \frac{K_G}{K}(1-g) A \left(\frac{K_G}{K}\right)^{\alpha} + \frac{C}{K}\frac{K_G}{K} + \frac{K_G}{K}\delta_K\right] dt \\
&\quad + \left[\frac{K_G}{K} + (1-\theta) g A \left(\frac{K_G}{K}\right)^{\alpha} - \frac{K_G}{K}\right] dN, \\
&= \left[\theta g A z^{\alpha} - \delta_G z - z(1-g) A z^{\alpha} + \frac{C}{K} z + \delta_K z\right] dt + \left[z + (1-\theta) g A z^{\alpha} - z\right] dN, \\
&= \left[\theta g A z^{\alpha} - \delta_G z - (1-g) A z^{\alpha+1} + \frac{C}{K} z + \delta_K z\right] dt + \left[(1-\theta) g A z^{\alpha}\right] dN,
\end{aligned}$$

such that

$$dz = \left[\theta g A z^{\alpha} - \delta_G z - (1-g) A z^{\alpha+1} + \frac{C}{K} z + \delta_K z\right] dt + \left[(1-\theta) g A z^{\alpha}\right] dN.$$

Then, the new system is given by

$$\begin{aligned}
dV_K &= \left\{\begin{array}{c} \left[\rho - (1-g)(1-\alpha) A z^{\alpha} + \delta_K\right] V_K(K, K_G) \\ -V_{K_G}(K, K_G)\left[\theta g A (1-\alpha) z^{\alpha}\right] \\ -\lambda \left[\begin{array}{c} V_K\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) \\ +\left((1-\theta)(1-\alpha) g A K_G^{\alpha} K^{-\alpha}\right) V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) \\ -V_K(K, K_G) \end{array}\right] \end{array}\right\} dt \\
&\quad + \left[V_K\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) - V_K(K, K_G)\right] dN, \\
dV_{K_G} &= \left\{\begin{array}{c} \left[\rho - \theta g \alpha A z^{\alpha-1} + \delta_G\right] V_{K_G}(K, K_G) - V_K(K, K_G)\left[\alpha A (1-g) z^{\alpha-1}\right] \\ -\lambda \left[\begin{array}{c} \left(1 + (1-\theta)\alpha g A K_G^{\alpha-1} K^{1-\alpha}\right) V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) \\ -V_{K_G}(K, K_G) \end{array}\right] \end{array}\right\} dt \\
&\quad + \left[V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) - V_{K_G}(K, K_G)\right] dN, \\
dz &= \left[\theta g A z^{\alpha} - \delta_G z - (1-g) A z^{\alpha+1} + \frac{C}{K} z + \delta_K z\right] dt + \left[(1-\theta) g A z^{\alpha}\right] dN.
\end{aligned}$$

#### *3.7.3.2 Process for $q_t$*

Define

$$q = \frac{V_{K_G}}{V_K},$$

and use

$$
\begin{aligned}
dV_K &= \left\{ \begin{array}{c} [\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K]V_K(K,K_G)-V_{K_G}(K,K_G)[\theta gA(1-\alpha)z^{\alpha}] \\ -\lambda\left[\begin{array}{c} V_K(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}) \\ +((1-\theta)(1-\alpha)gAK_G^{\alpha}K^{-\alpha})V_{K_G}(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}) \\ -V_K(K,K_G) \end{array}\right] \end{array} \right\} dt \\
&\quad +\left[V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_K(K,K_G)\right]dN, \\
dV_{K_G} &= \left\{ \begin{array}{c} [\rho-\theta g\alpha Az^{\alpha-1}+\delta_G]V_{K_G}(K,K_G)-V_K(K,K_G)[\alpha A(1-g)z^{\alpha-1}] \\ -\lambda\left[\begin{array}{c} (1+(1-\theta)\alpha gAz^{\alpha-1})V_{K_G}(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}) \\ -V_{K_G}(K,K_G) \end{array}\right] \end{array} \right\} dt \\
&\quad +\left[V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)-V_{K_G}(K,K_G)\right]dN.
\end{aligned}
$$

Ito's Lemma for Poisson processes yields for $dq$

$$
\begin{aligned}
dq &= \frac{1}{V_K}\left[ \begin{array}{c} [\rho-\theta g\alpha Az^{\alpha-1}+\delta_G]V_{K_G}(K,K_G)-V_K(K,K_G)[\alpha A(1-g)z^{\alpha-1}] \\ -\lambda[(1+(1-\theta)\alpha gAz^{\alpha-1})V_{K_G}(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha})-V_{K_G}(K,K_G)] \end{array}\right] dt \\
&\quad -\frac{V_{K_G}}{(V_K)^2}\left\{ \begin{array}{c} [\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K]V_K(K,K_G)-V_{K_G}(K,K_G)[\theta gA(1-\alpha)z^{\alpha}] \\ -\lambda\left[\begin{array}{c} V_K(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}) \\ +((1-\theta)(1-\alpha)gAK_G^{\alpha}K^{-\alpha})V_{K_G}(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}) \\ -V_K(K,K_G) \end{array}\right] \end{array} \right\} \\
&\quad +\left[\frac{V_{K_G}+[V_{K_G}(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha})-V_{K_G}(K,K_G)]}{V_K+[V_K(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha})-V_K(K,K_G)]}-\frac{V_{K_G}}{V_K}\right]dN.
\end{aligned}
$$

Simplify

$$
\begin{aligned}
dq &= \left\{ \begin{array}{c} [\rho-\theta g\alpha Az^{\alpha-1}+\delta_G]\frac{V_{K_G}}{V_K}-[\alpha A(1-g)z^{\alpha-1}] \\ -\lambda\left[(1+(1-\theta)\alpha gAz^{\alpha-1})\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K}-\frac{V_{K_G}(K,K_G)}{V_K}\right] \end{array} \right\} dt \\
&\quad -\left[[\rho-(1-g)(1-\alpha)Az^{\alpha}+\delta_K]\frac{V_{K_G}}{(V_K)^2}V_K-\frac{V_{K_G}}{(V_K)^2}V_{K_G}[\theta gA(1-\alpha)z^{\alpha}]\right]dt \\
&\quad +\lambda\left[ \begin{array}{c} \frac{V_{K_G}}{(V_K)^2}V_K(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha})-\frac{V_{K_G}}{(V_K)^2}V_K(K,K_G) \\ +((1-\theta)(1-\alpha)gAK_G^{\alpha}K^{-\alpha})\frac{V_{K_G}}{(V_K)^2}V_{K_G}(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}) \end{array}\right] dt \\
&\quad +\left[\frac{V_{K_G}(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha})}{V_K(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha})}-\frac{V_{K_G}}{V_K}\right]dN.
\end{aligned}
$$

Using $q$

$$
\begin{aligned}
dq &= \left[\left[\rho - \theta g\alpha A z^{\alpha-1} + \delta_G\right] q - \alpha A (1-g) z^{\alpha-1}\right] dt \\
&\quad -\lambda \left[\left(1 + (1-\theta)\alpha g A z^{\alpha-1}\right) \frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{V_K} - q\right] dt \\
&\quad - \left[\left[\rho - (1-g)(1-\alpha) A z^{\alpha} + \delta_K\right] q - q^2 \left[\theta g A (1-\alpha) z^{\alpha}\right]\right] dt \\
&\quad +\lambda \left[\begin{array}{c} \frac{V_{K_G}}{(V_K)^2} V_K\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) \\ + \left((1-\theta)(1-\alpha) g A K_G^{\alpha} K^{-\alpha}\right) \frac{V_{K_G}}{(V_K)^2} V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) - q \end{array}\right] dt \\
&\quad + \left[\frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{V_K\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)} - q\right] dN.
\end{aligned}
$$

Then,

$$
\begin{aligned}
dq &= \left[\left[\rho - \theta g\alpha A z^{\alpha-1} + \delta_G\right] q - \alpha A (1-g) z^{\alpha-1}\right] dt \\
&\quad -\lambda \left[\left(1 + (1-\theta)\alpha g A z^{\alpha-1}\right) \frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{V_K} - q\right] dt \\
&\quad - \left[\left[\rho - (1-g)(1-\alpha) A z^{\alpha} + \delta_K\right] q - q^2 \left[\theta g A (1-\alpha) z^{\alpha}\right]\right] dt \\
&\quad +\lambda \left[\begin{array}{c} \frac{V_{K_G}}{(V_K)^2} V_K\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) \\ + \left((1-\theta)(1-\alpha) g A z^{\alpha}\right) \frac{V_{K_G}}{(V_K)^2} V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) - q \end{array}\right] dt \\
&\quad + \left[\tilde{q} q - q\right] dN.
\end{aligned}
$$

The system now is,

$$
\begin{aligned}
dq &= \left[\left[\rho - \theta g\alpha A z^{\alpha-1} + \delta_G\right] q - \alpha A (1-g) z^{\alpha-1}\right] dt \\
&\quad -\lambda \left[\left(1 + (1-\theta)\alpha g A z^{\alpha-1}\right) \frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{V_K} - q\right] dt \\
&\quad - \left[\left[\rho - (1-g)(1-\alpha) A z^{\alpha} + \delta_K\right] q - q^2 \left[\theta g A (1-\alpha) z^{\alpha}\right]\right] dt \\
&\quad +\lambda \left[\begin{array}{c} \frac{V_{K_G}}{(V_K)^2} V_K\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) \\ + \left((1-\theta)(1-\alpha) g A z^{\alpha}\right) \frac{V_{K_G}}{(V_K)^2} V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) - q \end{array}\right] dt \\
&\quad + \left[\tilde{q} q - q\right] dN, \\
dz_t &= \left[\theta g A z_t^{\alpha} - \delta_G z_t - (1-g) A z_t^{\alpha+1} + \frac{C_t}{K_t} z_t + \delta_K z_t\right] dt + \left[(1-\theta) g A z_t^{\alpha}\right] dN_t, \\
dV_{K_G} &= \left\{\begin{array}{c} \left[\rho - \theta g\alpha A z^{\alpha-1} + \delta_G\right] V_{K_G}(z) - V_K(z)\left[\alpha A (1-g) z^{\alpha-1}\right] \\ -\lambda \left[\begin{array}{c} \left(1 + (1-\theta)\alpha g A z^{\alpha-1}\right) V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) \\ -V_{K_G}(K, K_G) \end{array}\right] \end{array}\right\} dt \\
&\quad + \left[V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right) - V_{K_G}(K, K_G)\right] dN.
\end{aligned}
$$

*3.7.3.3 Process for $c_t$*

As an intermediate step compute $dC_t$. Let

$$f\left(u_C\left(C\right)\right)=C,$$

apply CVF to $f\left(u_C\left(C\right)\right)$ using

$$f_C\left(u_C\left(C\right)\right)=\frac{df\left(u_C\left(C\right)\right)}{du_C\left(C\right)}=\frac{dC}{du_C\left(C\right)}=\frac{1}{u_{CC}\left(C\right)}.$$

Then,

$$\begin{aligned}du_C\left(C\right) &= \left\{\begin{array}{c}\left[\rho-\left(1-g\right)\left(1-\alpha\right)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]u_C\left(C\right)\\ -V_{K_G}\left(K,K_G\right)\left[\theta gA\left(1-\alpha\right)K_G^{\alpha}K^{-\alpha}\right]\\ -\lambda\left[\begin{array}{c}u_C\left(\tilde{C}C\right)-u_C\left(C\right)\\ +\left(\left(1-\theta\right)\left(1-\alpha\right)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+\left(1-\theta\right)gAK_G^{\alpha}K^{1-\alpha}\right)\end{array}\right]\end{array}\right\}dt\\ &\quad+\left[u_C\left(\tilde{C}C\right)-u_C\left(C\right)\right]dN,\\ dC &= f_C\left(u_C\left(C\right)\right)\left\{\begin{array}{c}\left[\rho-\left(1-g\right)\left(1-\alpha\right)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]u_C\left(C\right)\\ -V_{K_G}\left(K,K_G\right)\left[\theta gA\left(1-\alpha\right)K_G^{\alpha}K^{-\alpha}\right]\\ -\lambda\left[\begin{array}{c}u_C\left(\tilde{C}C\right)-u_C\left(C\right)+\\ \left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)V_{K_G}\left(K,K_G+\left(1-\theta\right)gAK_G^{\alpha}K^{1-\alpha}\right)\end{array}\right]\end{array}\right\}\\ &\quad+\left[f\left(u_C\left(\tilde{C}C\right)\right)-f\left(u_C\left(C\right)\right)\right]dN.\end{aligned}$$

Moreover,

$$\begin{aligned}dC &= \frac{1}{u_{CC}\left(C\right)}\left\{\begin{array}{c}\left[\rho-\left(1-g\right)\left(1-\alpha\right)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]u_C\left(C\right)\\ -V_{K_G}\left(K,K_G\right)\left[\theta gA\left(1-\alpha\right)K_G^{\alpha}K^{-\alpha}\right]\\ -\lambda\left[\begin{array}{c}u_C\left(\tilde{C}C\right)-u_C\left(C\right)\\ +\left(\left(1-\theta\right)\left(1-\alpha\right)gAK_G^{\alpha}K^{-\alpha}\right)V_{K_G}\left(K,K_G+\left(1-\theta\right)gAK_G^{\alpha}K^{1-\alpha}\right)\end{array}\right]\end{array}\right\}\\ &\quad+\left[\tilde{C}C-C\right]dN,\end{aligned}$$

and

$$\begin{aligned}\frac{u_{CC}\left(C\right)}{u_C\left(C\right)}dC &= \left\{\begin{array}{c}\left[\rho-\left(1-g\right)\left(1-\alpha\right)AK_G^{\alpha}K^{-\alpha}+\delta_K\right]-\frac{V_{K_G}\left(K,K_G\right)}{u_C\left(C\right)}\left[\theta gA\left(1-\alpha\right)K_G^{\alpha}K^{-\alpha}\right]\\ -\lambda\left[\frac{u_C\left(\tilde{C}C\right)}{u_C\left(C\right)}+\left(\left(1-\theta\right)\left(1-\alpha\right)gAK_G^{\alpha}K^{-\alpha}\right)\frac{V_{K_G}\left(K,K_G+\left(1-\theta\right)gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C\left(C\right)}-1\right]\end{array}\right\}\\ &\quad+\frac{u_{CC}\left(C\right)}{u_C\left(C\right)}\left[\tilde{C}C-C\right]dN.\end{aligned}$$

Assume the following CRRA preferences ($\gamma \to 1 : \ln C$)

$$\begin{aligned} u(C) &= \frac{C^{1-\gamma}}{1-\gamma}, \\ u_C(C) &= C^{-\gamma}, \\ u_{CC}(C) &= -\gamma C^{-(\gamma+1)}. \end{aligned}$$

Then,

$$\begin{aligned}
\frac{-\gamma}{C} dC &= \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} + \delta_K\right] - \frac{V_{K_G}(K,K_G)}{C^{-\gamma}} \left[\theta g A (1-\alpha) K_G^{\alpha} K^{-\alpha}\right] \\ -\lambda \left[\tilde{C}^{-\gamma} + ((1-\theta)(1-\alpha) g A z^{\alpha}) \frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{u_C(C)} - 1\right] \end{array} \right\} dt \\
&\quad - \frac{\gamma}{C}\left[\tilde{C}C - C\right] dN, \\
dC &= \frac{-C}{\gamma} \left\{ \begin{array}{c} \left[\rho - (1-g)(1-\alpha) A K_G^{\alpha} K^{-\alpha} + \delta_K\right] - \frac{V_{K_G}(K,K_G)}{C^{-\gamma}} \left[\theta g A (1-\alpha) K_G^{\alpha} K^{-\alpha}\right] \\ -\lambda \left[\tilde{C}^{-\gamma} + ((1-\theta)(1-\alpha) g A z^{\alpha}) \frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{u_C(C)} - 1\right] \end{array} \right\} dt \\
&\quad + \left[\tilde{C}C - C\right] dN, \\
&= C \left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma} A K_G^{\alpha} K^{-\alpha} - \frac{\rho}{\gamma} - \frac{\delta_K}{\gamma}\right] + \frac{V_{K_G}(K,K_G)}{\gamma C^{-\gamma}} \left[\theta g A (1-\alpha) K_G^{\alpha} K^{-\alpha}\right] \\ + \frac{\lambda}{\gamma} \left[\tilde{C}^{-\gamma} + ((1-\theta)(1-\alpha) g A z^{\alpha}) \frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{u_C(C)} - 1\right] \end{array} \right\} dt \\
&\quad + \left[\tilde{C}C - C\right] dN, \\
&= C_t \left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma} A K_{G,t}^{\alpha} K_t^{-\alpha} - \frac{\rho}{\gamma} - \frac{\delta_K}{\gamma}\right] + \frac{V_{K_G}(K,K_G)}{\gamma C_t^{-\gamma}} \left[\theta g A (1-\alpha) K_{G,t}^{\alpha} K_t^{-\alpha}\right] \\ + \frac{\lambda}{\gamma} \left[\tilde{C}^{-\gamma} + ((1-\theta)(1-\alpha) g A z^{\alpha}) \frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{u_C(C)} - 1\right] \end{array} \right\} dt \\
&\quad + \left[\tilde{C}C - C\right] dN.
\end{aligned}$$

Define

$$c_t = \frac{C_t}{K_t},$$

and apply Ito's lemma

$$\begin{aligned}
dc &= \frac{C}{K} \left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma} A K_G^{\alpha} K^{-\alpha} - \frac{\rho}{\gamma} - \frac{\delta_K}{\gamma}\right] + \frac{V_{K_G}(K,K_G)}{\gamma C^{-\gamma}} \left[\theta g A (1-\alpha) K_G^{\alpha} K^{-\alpha}\right] \\ + \frac{\lambda}{\gamma} \left[\tilde{C}^{-\gamma} + ((1-\theta)(1-\alpha) g A z^{\alpha}) \frac{V_{K_G}\left(K, K_G + (1-\theta) g A K_G^{\alpha} K^{1-\alpha}\right)}{u_C(C)} - 1\right] \end{array} \right\} dt \\
&\quad - \frac{C}{K^2}\left[(1-g) A K_G^{\alpha} K^{1-\alpha} - C - \delta_K K\right] dt \\
&\quad + \left[\frac{C + \left[\tilde{C}C - C\right]}{K} - \frac{C}{K}\right] dN.
\end{aligned}$$

Then,

$$
\begin{aligned}
dc &= \frac{C}{K}\left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}-\frac{\delta_K}{\gamma}\right]+\frac{V_{K_G}(z)}{\gamma C^{-\gamma}}\left[\theta gA\left(1-\alpha\right)z^{\alpha}\right] \\ +\frac{\lambda}{\gamma}\left[\tilde{C}^{-\gamma}+\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right] \end{array}\right\}dt \\
&\quad -\frac{C}{K}\left[\left(1-g\right)Az^{\alpha}-\frac{C}{K}-\delta_K\right]dt+\left[\frac{\tilde{C}C}{K}-\frac{C}{K}\right]dN.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
dc &= c\left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}-\frac{\delta_K}{\gamma}\right]+C^{\gamma}\frac{V_{K_G}(z)}{\gamma}\left[\theta gA\left(1-\alpha\right)z^{\alpha}\right] \\ +\frac{\lambda}{\gamma}\left[\tilde{C}^{-\gamma}+\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right] \end{array}\right\}dt \\
&\quad -c\left[\left(1-g\right)Az^{\alpha}-c-\delta_K\right]dt+\left[\tilde{c}c-c\right]dN.
\end{aligned}
$$

Then,

$$
\begin{aligned}
dc &= c\left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}-\frac{\delta_K}{\gamma}\right]+\frac{V_{K_G}(z)}{\gamma C^{-\gamma}}\left[\theta gA\left(1-\alpha\right)z^{\alpha}\right] \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma}+\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right] \\ -\left[\left(1-g\right)Az^{\alpha}-c-\delta_K\right] \end{array}\right\}dt \\
&\quad +\left[\tilde{c}c-c\right]dN, \\
&= c\left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}-\frac{\delta_K}{\gamma}\right]+\frac{V_{K_G}(z)}{\gamma u_C(C)}\left[\theta gA\left(1-\alpha\right)z^{\alpha}\right] \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma}+\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right] \\ -\left[\left(1-g\right)Az^{\alpha}-c-\delta_K\right] \end{array}\right\}dt \\
&\quad +\left[\tilde{c}c-c\right]dN, \\
&= c\left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}-\frac{\delta_K}{\gamma}\right]+\frac{V_{K_G}(z)}{\gamma V_K(z)}\left[\theta gA\left(1-\alpha\right)z^{\alpha}\right] \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma}+\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right] \\ -\left[\left(1-g\right)Az^{\alpha}-c-\delta_K\right] \end{array}\right\}dt \\
&\quad +\left[\tilde{c}c-c\right]dN.
\end{aligned}
$$

Use $q$

$$
\begin{aligned}
dc &= c\left\{ \begin{array}{c} \left[\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}-\frac{\delta_K}{\gamma}\right]+\frac{q}{\gamma}\left[\theta gA\left(1-\alpha\right)z^{\alpha}\right]-\left[\left(1-g\right)Az^{\alpha}-c-\delta_K\right] \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma}+\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right] \end{array}\right\}dt \\
&\quad +\left[\tilde{c}c-c\right]dN.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
dq &= \left[\left[\rho-\theta g\alpha Az^{\alpha-1}+\delta_G\right]q-\alpha A\left(1-g\right)z^{\alpha-1}\right]dt \\
&\quad -\lambda\left[\left(1+\left(1-\theta\right)\alpha gAz^{\alpha-1}\right)\frac{V_{K_G}\left(K,K_G+\left(1-\theta\right)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K}-q\right]dt \\
&\quad -\left[\left[\rho-\left(1-g\right)\left(1-\alpha\right)Az^{\alpha}+\delta_K\right]q-q^2\left[gA\left(1-\alpha\right)z^{\alpha}\right]\right]dt \\
&\quad +\lambda\left[\begin{array}{c}\frac{V_{K_G}}{\left(V_K\right)^2}V_K\left(K,K_G+\left(1-\theta\right)gAK_G^{\alpha}K^{1-\alpha}\right) \\ +\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}}{\left(V_K\right)^2}V_{K_G}\left(K,K_G+\left(1-\theta\right)gAK_G^{\alpha}K^{1-\alpha}\right)-q\end{array}\right]dt \\
&\quad +\left[\tilde{q}q-q\right]dN, \\
dz &= \left[\theta gAz^{\alpha}-\delta_G z-\left(1-g\right)Az^{\alpha+1}+cz+\delta_K z\right]dt+\left[\left(1-\theta\right)gAz^{\alpha}\right]dN, \\
dc &= c\left\{\begin{array}{c}\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}+\frac{(\gamma-1)}{\gamma}\delta_K+\frac{\theta gA(1-\alpha)}{\gamma}qz^{\alpha}-\left(1-g\right)Az^{\alpha}+c \\ +\frac{\lambda}{\gamma}\left[\tilde{c}^{-\gamma}+\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{u_C(C)}-1\right]\end{array}\right\}dt \\
&\quad +\left[\tilde{c}c-c\right]dN.
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
dq &= \left[\left[\rho-\theta g\alpha Az^{\alpha-1}+\delta_G\right]q-\alpha A\left(1-g\right)z^{\alpha-1}\right]dt \\
&\quad -\lambda\left[\left(1+\left(1-\theta\right)\alpha gAz^{\alpha-1}\right)\frac{V_{K_G}\left(K,K_G+\left(1-\theta\right)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K}-q\right]dt \\
&\quad +\lambda\left[\begin{array}{c}q\frac{V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K} \\ +\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)q\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K}-q\end{array}\right]dt \\
&\quad -\left[\left[\rho-\left(1-g\right)\left(1-\alpha\right)Az^{\alpha}+\delta_K\right]q-q^2\left[\theta gA\left(1-\alpha\right)z^{\alpha}\right]\right]dt+\left[\tilde{q}q-q\right]dN.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
dq &= \left[\left[\rho-\theta g\alpha Az^{\alpha-1}+\delta_G\right]q-\alpha A\left(1-g\right)z^{\alpha-1}\right]dt \\
&\quad +\lambda\left[\begin{array}{c}q\frac{V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K} \\ +\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)q\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K}\frac{V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}-q \\ -\left(1+\left(1-\theta\right)\alpha gAz^{\alpha-1}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K}\frac{V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}+q\end{array}\right] \\
&\quad -\left[\left[\rho-\left(1-g\right)\left(1-\alpha\right)Az^{\alpha}+\delta_K\right]q-q^2\left[\theta gA\left(1-\alpha\right)z^{\alpha}\right]\right]dt+\left[\tilde{q}q-q\right]dN, \\
dz &= \left[\theta gAz^{\alpha}-\delta_G z-\left(1-g\right)Az^{\alpha+1}+cz+\delta_K z\right]dt+\left[\left(1-\theta\right)gAz^{\alpha}\right]dN, \\
dc &= c\left\{\begin{array}{c}\frac{(1-g)(1-\alpha)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}+\frac{(\gamma-1)}{\gamma}\delta_K+\frac{\theta gA(1-\alpha)}{\gamma}qz^{\alpha}-\left(1-g\right)Az^{\alpha}+c \\ +\frac{\lambda}{\gamma}\left[\begin{array}{c}\tilde{c}^{-\gamma}-1 \\ +\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)\frac{V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K}\frac{V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}{V_K\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right)}\end{array}\right]\end{array}\right. \\
&\quad +\left[\tilde{c}c-c\right]dN.
\end{aligned}
$$

The final system is

$$
\begin{aligned}
dq_t &= \left\{ \begin{array}{c} \left[\left[\rho - \theta g \alpha A z_t^{\alpha-1} + \delta_G\right] q_t - \alpha A \left(1-g\right) z_t^{\alpha-1}\right] \\ -\left[\left[\rho - \left(1-g\right)\left(1-\alpha\right) A z_t^{\alpha} + \delta_K\right] q_t - q_t^2 \left[\theta g A \left(1-\alpha\right) z_t^{\alpha}\right]\right] \\ +\lambda \left[q_t \tilde{c}_t^{-\gamma} + \left(\left(1-\theta\right)\left(1-\alpha\right) g A z_t^{\alpha}\right) q_t^2 \tilde{q}_t \tilde{c}_t^{-\gamma} - \left(1 + \left(1-\theta\right) \alpha g A z_t^{\alpha-1}\right) q_t \tilde{q}_t \tilde{c}_t^{-\gamma}\right] \end{array} \right\} dt \\
&\quad + \left[\tilde{q}_t q_t - q_t\right] dN_t, \qquad (144) \\
dz_t &= \left[\theta g A z_t^{\alpha} - \delta_G z_t - \left(1-g\right) A z_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt + \left[\left(1-\theta\right) g A z_t^{\alpha}\right] dN_t, \qquad (145) \\
dc_t &= c_t \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma} \delta_K + \frac{\theta g A (1-\alpha)}{\gamma} q_t z_t^{\alpha} \\ +\frac{\lambda}{\gamma} \left[\tilde{c}_t^{-\gamma} + \left(\left(1-\theta\right)\left(1-\alpha\right) g A z_t^{\alpha}\right) q_t \tilde{q}_t \tilde{c}_t^{-\gamma} - 1\right] - \left(1-g\right) A z_t^{\alpha} + c_t \end{array} \right\} dt \\
&\quad + \left[\tilde{c}_t c_t - c_t\right] dN_t, \qquad (146)
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{c} &= \frac{c\left(K, K_G\left(1 + gA(K_G/K)^{\alpha-1}\right)\right)}{c\left(K, K_G\right)}, \\
\tilde{q} &= \frac{q\left(K, K_G\left(1 + gA(K_G/K)^{\alpha-1}\right)\right)}{q\left(K, K_G\right)},
\end{aligned}
$$

Note that for $\theta = 1$ it resembles the Turnovsky model (141) to (143),

$$
\begin{aligned}
dq_t &= \left\{\delta_G q_t + \left[\left(1-g\right)\left(1-\alpha\right) z_t q_t - \alpha\left(1-g\right)\right] A z_t^{\alpha-1} - \delta_K q_t + \left[\left(1-\alpha\right) z_t q_t - \alpha\right] q_t g A z_t^{\alpha-1}\right\} dt, \\
dz_t &= \left[g A z_t^{\alpha} - \delta_G z_t - \left(1-g\right) A z_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt, \\
dc_t &= c_t \left\{ \frac{(1-g)(1-\alpha)}{\gamma} A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma} \delta_K + \frac{g A (1-\alpha)}{\gamma} q_t z_t^{\alpha} - \left(1-g\right) A z_t^{\alpha} + c_t \right\} dt,
\end{aligned}
$$

whereas for $\theta = 0$ it resembles the pure implementation lags model (138) to (140),

$$
\begin{aligned}
dq_t &= \left\{ \begin{array}{c} \delta_G q_t + \left[\left(1-g\right)\left(1-\alpha\right) z_t q_t - \alpha\left(1-g\right)\right] A z_t^{\alpha-1} - \delta_K q_t \\ +\lambda \left[(1-\tilde{q}_t) q_t + \tilde{q}_t \left[\left(1-\alpha\right) z_t q_t - \alpha\right] q_t g A z_t^{\alpha-1}\right] \tilde{c}_t^{-\gamma} \end{array} \right\} dt \\
&\quad + \left[\tilde{q}_t q_t - q_t\right] dN_t, \\
dz_t &= \left[-\left(1-g\right) A z_t^{\alpha+1} - \delta_G z_t + c_t z_t + \delta_K z_t\right] dt + \left[g A z_t^{\alpha}\right] dN_t, \\
dc_t &= c_t \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma} \delta_K \\ +\frac{\lambda}{\gamma} \left[\tilde{c}_t^{-\gamma} + \left(\left(1-\alpha\right) g A z_t^{\alpha}\right) q_t \tilde{q}_t \tilde{c}_t^{-\gamma} - 1\right] - \left(1-g\right) A z_t^{\alpha} + c_t \end{array} \right\} dt \\
&\quad + \left[\tilde{c}_t c_t - c_t\right] dN_t.
\end{aligned}
$$

*3.7.3.4 Steady State*

Use the system

$$
\begin{aligned}
dq_t &= \left\{ \begin{array}{c} \left[\left[\rho - \theta g\alpha A z_t^{\alpha-1} + \delta_G\right] q_t - \alpha A\left(1-g\right) z_t^{\alpha-1}\right] \\ -\left[\left[\rho - \left(1-g\right)\left(1-\alpha\right) A z_t^{\alpha} + \delta_K\right] q_t - q_t^2\left[\theta g A\left(1-\alpha\right) z_t^{\alpha}\right]\right] \\ +\lambda\left[q_t\tilde{c}_t^{-\gamma} + \left(\left(1-\theta\right)\left(1-\alpha\right) g A z_t^{\alpha}\right) q_t^2 \tilde{q}_t \tilde{c}_t^{-\gamma} - \left(1 + \left(1-\theta\right)\alpha g A z_t^{\alpha-1}\right) q_t \tilde{q}_t \tilde{c}_t^{-\gamma}\right] \end{array} \right\} dt \\
&\quad + \left[\tilde{q}_t q_t - q_t\right] dN_t, \\
dz_t &= \left[\theta g A z_t^{\alpha} - \delta_G z_t - \left(1-g\right) A z_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt + \left[\left(1-\theta\right) g A z_t^{\alpha}\right] dN_t, \\
dc_t &= c_t \left\{ \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z_t^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{\theta g A(1-\alpha)}{\gamma} q_t z_t^{\alpha} \\ +\frac{\lambda}{\gamma}\left[\tilde{c}_t^{-\gamma} + \left(\left(1-\theta\right)\left(1-\alpha\right) g A z_t^{\alpha}\right) q_t \tilde{q}_t \tilde{c}_t^{-\gamma} - 1\right] - \left(1-g\right) A z_t^{\alpha} + c_t \end{array} \right\} dt \\
&\quad + \left[\tilde{c}_t c_t - c_t\right] dN_t,
\end{aligned}
$$

and set $(dc_t = dz_t = dq_t = 0)$, gives

$$
\begin{aligned}
0 &= \begin{array}{c} \left[\left[\rho - \theta g\alpha A z^{\alpha-1} + \delta_G\right] q - \alpha A\left(1-g\right) z^{\alpha-1}\right] \\ -\left[\left[\rho - \left(1-g\right)\left(1-\alpha\right) A z^{\alpha} + \delta_K\right] q - q^2\left[\theta g A\left(1-\alpha\right) z^{\alpha}\right]\right] \\ +\lambda\left[q + \left(\left(1-\theta\right)\left(1-\alpha\right) g A z^{\alpha}\right) q^2 - \left(1 + \left(1-\theta\right)\alpha g A z^{\alpha-1}\right) q\right] \end{array}, \\
0 &= \theta g A z^{\alpha} - \delta_G z - \left(1-g\right) A z^{\alpha+1} + cz + \delta_K z, \\
0 &= \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{\rho}{\gamma} + \frac{(\gamma-1)}{\gamma}\delta_K + \frac{\theta g A(1-\alpha)}{\gamma} q z^{\alpha} \\ +\frac{\lambda}{\gamma}\left[1 + \left(\left(1-\theta\right)\left(1-\alpha\right) g A z^{\alpha}\right) q - 1\right] - \left(1-g\right) A z^{\alpha} + c \end{array}.
\end{aligned}
$$

Setting $\delta_K = \delta_G = 0$ yields

$$
\begin{aligned}
0 &= \begin{array}{c} \left[\left[\rho - \theta g\alpha A z^{\alpha-1}\right] q - \alpha A\left(1-g\right) z^{\alpha-1}\right] \\ -\left[\left[\rho - \left(1-g\right)\left(1-\alpha\right) A z^{\alpha}\right] q - q^2\left[\theta g A\left(1-\alpha\right) z^{\alpha}\right]\right] \\ +\lambda\left[q + \left(\left(1-\theta\right)\left(1-\alpha\right) g A z^{\alpha}\right) q^2 - \left(1 + \left(1-\theta\right)\alpha g A z^{\alpha-1}\right) q\right] \end{array}, \\
0 &= \theta g A z^{\alpha} - \left(1-g\right) A z^{\alpha+1} + cz, \\
0 &= \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{\rho}{\gamma} + + \frac{\theta g A(1-\alpha)}{\gamma} q z^{\alpha} \\ +\frac{\lambda}{\gamma}\left[1 + \left(\left(1-\theta\right)\left(1-\alpha\right) g A z^{\alpha}\right) q - 1\right] - \left(1-g\right) A z^{\alpha} + c \end{array}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
0 &= \begin{array}{c} -q\theta g\alpha A z^{\alpha-1} - \alpha A\left(1-g\right) z^{\alpha-1} \\ +\left(1-g\right)\left(1-\alpha\right) q A z^{\alpha} + q^2\theta g A\left(1-\alpha\right) z^{\alpha} \\ +\lambda\left(\left(1-\theta\right)\left(1-\alpha\right) g A z^{\alpha}\right) q^2 - \lambda\left(1-\theta\right)\alpha g A z^{\alpha-1} q \end{array}, \\
c &= \left(1-g\right) A z^{\alpha} - \theta g A z^{\alpha-1}, \\
0 &= \begin{array}{c} \frac{(1-g)(1-\alpha)}{\gamma} A z^{\alpha} - \frac{\rho}{\gamma} + + \frac{\theta g A(1-\alpha)}{\gamma} q z^{\alpha} \\ +\frac{\lambda}{\gamma} + \frac{\lambda}{\gamma}\left(\left(1-\theta\right)\left(1-\alpha\right) g A z^{\alpha}\right) q - \frac{\lambda}{\gamma} - \left(1-g\right) A z^{\alpha} + c \end{array}.
\end{aligned}
$$

First-best

$$\begin{aligned}
\rho V(K,K_G) &= \max_C \left\{ \begin{array}{c} u(C) + V_K(K,K_G)\left[(1-g)AK_G^{\alpha}K^{1-\alpha} - C - \delta_K K\right] \\ +V_{K_G}(K,K_G)\left[\theta g A K_G^{\alpha}K^{1-\alpha} - \delta_G K_G\right] \\ +\lambda\left[V\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right) - V(K,K_G)\right] \end{array} \right\}, \\
0 &= -V_K(K,K_G)AK_G^{\alpha}K^{1-\alpha} + V_{K_G}(K,K_G)\theta A K_G^{\alpha}K^{1-\alpha} \\
&\quad +\lambda(1-\theta)AK_G^{\alpha}K^{1-\alpha}V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right), \\
V_K(K,K_G)AK_G^{\alpha}K^{1-\alpha} &= V_{K_G}(K,K_G)\theta A K_G^{\alpha}K^{1-\alpha} \\
&\quad +\lambda(1-\theta)AK_G^{\alpha}K^{1-\alpha}V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right), \\
V_K(K,K_G) &= V_{K_G}(K,K_G)\theta \\
&\quad +\lambda(1-\theta)V_{K_G}\left(K,K_G+(1-\theta)gAK_G^{\alpha}K^{1-\alpha}\right), \\
1 &= q\theta + \lambda(1-\theta)\tilde{q}q, \\
1 &= q\left(\theta + \lambda(1-\theta)\tilde{q}\right), \\
q &= \frac{1}{\theta + \lambda(1-\theta)\tilde{q}}.
\end{aligned}$$

Steady State growth rate and optimal government spending rate

$$\begin{aligned}
\phi &= \theta g A z^{\alpha-1} = (1-g)Az^{\alpha} - c, \\
\theta g &= (1-g)z - \frac{c}{Az^{\alpha-1}}, \\
\theta g + gz &= z - \frac{c}{Az^{\alpha-1}}, \\
g &= \frac{z - \frac{c}{Az^{\alpha-1}}}{\theta + z}.
\end{aligned}$$

Then, in equilibrium

$$\begin{aligned}
q &= \frac{1}{\theta + \lambda(1-\theta)\tilde{q}}, \\
q &= \frac{1}{\theta + (1-\theta)\lambda}.
\end{aligned}$$

### 3.7.3.5 Long-Run Fiscal Effects

Start by

$$
\begin{aligned}
0 &= \begin{array}{c} -q\theta g\alpha Az^{\alpha-1}-\alpha A\left(1-g\right)z^{\alpha-1}+\left(1-g\right)\left(1-\alpha\right)qAz^{\alpha}+q^{2}\theta gA\left(1-\alpha\right)z^{\alpha} \\ +\lambda\left(\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}\right)q^{2}-\lambda\left(1-\theta\right)\alpha gAz^{\alpha-1}q \end{array}, \\
\theta g\alpha Az^{\alpha-1} &= \lambda\left(1-\theta\right)\left(1-\alpha\right)gAz^{\alpha}q-\lambda\left(1-\theta\right)\alpha gAz^{\alpha-1}-\alpha A\left(1-g\right)\frac{z^{\alpha-1}}{q} \\
&\quad +\left(1-g\right)\left(1-\alpha\right)Az^{\alpha}+q\theta gA\left(1-\alpha\right)z^{\alpha}, \\
\theta g\alpha z^{-1} &= \lambda\left(1-\theta\right)\left(1-\alpha\right)gq-\frac{\lambda\left(1-\theta\right)\alpha g}{z}-\frac{\alpha\left(1-g\right)}{zq}+\left(1-g\right)\left(1-\alpha\right)+q\theta g\left(1-\alpha\right), \\
\frac{\alpha\left(1-g\right)}{zq} &= \lambda\left(1-\theta\right)\left(1-\alpha\right)gq+q\theta g\left(1-\alpha\right)-\frac{\lambda\left(1-\theta\right)\alpha g}{z}-\theta g\alpha z^{-1}+\left(1-g\right)\left(1-\alpha\right), \\
\frac{\alpha\left(1-g\right)}{zq} &= q\left[\lambda\left(1-\theta\right)\left(1-\alpha\right)g+\theta g\left(1-\alpha\right)\right]-\frac{\lambda\left(1-\theta\right)\alpha g}{z}-\theta g\alpha z^{-1}+\left(1-g\right)\left(1-\alpha\right), \\
qz &= \frac{\alpha\left(1-g\right)}{q\left[\lambda\left(1-\theta\right)\left(1-\alpha\right)g+\theta g\left(1-\alpha\right)\right]-\frac{\lambda\left(1-\theta\right)\alpha g}{z}-\theta g\alpha z^{-1}+\left(1-g\right)\left(1-\alpha\right)}, \\
0 &= \alpha\left(1-g\right)qz-\left[\begin{array}{c} q\left[\lambda\left(1-\theta\right)\left(1-\alpha\right)g+\theta g\left(1-\alpha\right)\right] \\ -\frac{\lambda\left(1-\theta\right)\alpha g}{z}-\theta g\alpha z^{-1}+\left(1-g\right)\left(1-\alpha\right) \end{array}\right], \\
0 &= q^{2}z\left[\lambda\left(1-\theta\right)\left(1-\alpha\right)g+\theta g\left(1-\alpha\right)\right]-\lambda\left(1-\theta\right)\alpha gq-q\theta g\alpha \\
&\quad +\left(1-g\right)\left(1-\alpha\right)qz-\alpha\left(1-g\right).
\end{aligned}
$$

Further,

$$
\begin{aligned}
\frac{\theta g\alpha}{z}+\frac{\alpha\left(1-g\right)}{zq}+\left(1-g\right)\left(1-\alpha\right) &= \lambda\left(1-\theta\right)\left(1-\alpha\right)gq-\frac{\lambda\left(1-\theta\right)\alpha g}{z}+q\theta g\left(1-\alpha\right), \\
c &= \left(1-g\right)Az^{\alpha}-\theta gAz^{\alpha-1}, \\
0 &= \frac{\left(1-g\right)\left(1-\alpha\right)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}+\frac{\theta gA\left(1-\alpha\right)}{\gamma}qz^{\alpha} \\
&\quad -\left(1-g\right)Az^{\alpha}+c+\frac{\lambda}{\gamma}\left(1-\theta\right)\left(1-\alpha\right)gAqz^{\alpha}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
0 &= \frac{\left(1-g\right)\left(1-\alpha\right)}{\gamma}Az^{\alpha}-\frac{\rho}{\gamma}+\frac{\theta gA\left(1-\alpha\right)}{\gamma}qz^{\alpha}-\theta gAz^{\alpha-1}+\frac{\lambda}{\gamma}\left(1-\theta\right)\left(1-\alpha\right)gAqz^{\alpha}, \\
0 &= \frac{\left(1-g\right)\left(1-\alpha\right)}{\gamma z}-\frac{\rho}{Az^{\alpha-1}\gamma}+\frac{\theta g\left(1-\alpha\right)}{\gamma}qz-\theta g+\frac{\lambda}{\gamma}\left(1-\theta\right)\left(1-\alpha\right)gqz, \\
0 &= \frac{\left(1-g\right)\left(1-\alpha\right)}{\gamma z}-\frac{\rho}{Az^{\alpha-1}\gamma}-\theta g+qz\left[\frac{\theta g\left(1-\alpha\right)}{\gamma}+\frac{\lambda}{\gamma}\left(1-\theta\right)\left(1-\alpha\right)g\right].
\end{aligned}
$$

Derivative

$$
\begin{aligned}
0 &= -\frac{(1-\alpha)}{\gamma z} - \frac{(1-g)(1-\alpha)}{\gamma} z^{-2}\frac{dz}{dg} - \frac{\rho}{A\gamma}(1-\alpha) z^{-\alpha}\frac{dz}{dg} - g + qz\frac{\theta(1-\alpha)}{\gamma} \\
&\quad + \frac{\lambda}{\gamma}(1-\theta)(1-\alpha) qz \\
&\quad + \frac{\theta g(1-\alpha)}{\gamma} z\frac{dq}{dg} + \frac{\theta g(1-\alpha)}{\gamma} q\frac{dz}{dg} + \frac{\lambda}{\gamma}(1-\theta)(1-\alpha) gq\frac{dz}{dg} \\
&\quad + \frac{\lambda}{\gamma}(1-\theta)(1-\alpha) gz\frac{dq}{dg}, \\
0 &= q^2 z\left[\lambda(1-\theta)(1-\alpha) + \theta(1-\alpha)\right] - \lambda(1-\theta)\alpha q - q\theta\alpha - (1-\alpha) qz + \alpha \\
&\quad + \frac{dq}{dg}\left[2qz\left[\lambda(1-\theta)(1-\alpha) g + \theta g(1-\alpha)\right] - \lambda(1-\theta)\alpha g - \theta g\alpha + (1-g)(1-\alpha) z\right] \\
&\quad + \frac{dz}{dg}\left[q^2\left[\lambda(1-\theta)(1-\alpha) g + \theta g(1-\alpha)\right] - (1-\alpha) q\right], \\
&= -q^2 z\left[\lambda(1-\theta)(1-\alpha) + \theta(1-\alpha)\right] + \lambda(1-\theta)\alpha q + q\theta\alpha + (1-\alpha) qz - \alpha \\
&\quad - \frac{dz}{dg}\left[q^2\left[\lambda(1-\theta)(1-\alpha) g + \theta g(1-\alpha)\right] - (1-\alpha) q\right] \\
&\quad - \frac{dq}{dg}\left[2qz\left[\lambda(1-\theta)(1-\alpha) g + \theta g(1-\alpha)\right] - \lambda(1-\theta)\alpha g - \theta g\alpha + (1-g)(1-\alpha) z\right], \\
&= \lambda(1-\theta)\alpha q + q\theta\alpha + (1-\alpha) qz - \alpha - q^2 z\left[\lambda(1-\theta)(1-\alpha) + \theta(1-\alpha)\right] \\
&\quad - \frac{dz}{dg}\left[q^2\left[\lambda(1-\theta)(1-\alpha) g + \theta g(1-\alpha)\right] - (1-\alpha) q\right].
\end{aligned}
$$

Combine

$$
\begin{aligned}
0 &= -\frac{(1-\alpha)}{\gamma z} - \frac{(1-g)(1-\alpha)}{\gamma} z^{-2}\frac{dz}{dg} - \frac{\rho}{A\gamma}(1-\alpha) z^{-\alpha}\frac{dz}{dg} - g + qz\frac{\theta(1-\alpha)}{\gamma} \\
&\quad + \frac{\lambda}{\gamma}(1-\theta)(1-\alpha) qz \\
&\quad + \frac{\theta g(1-\alpha)}{\gamma} z\frac{dq}{dg} + \frac{\theta g(1-\alpha)}{\gamma} q\frac{dz}{dg} + \frac{\lambda}{\gamma}(1-\theta)(1-\alpha) gq\frac{dz}{dg} \\
&\quad + \frac{\lambda}{\gamma}(1-\theta)(1-\alpha) gz\frac{dq}{dg}, \\
0 &= \frac{dz}{dg}\left[\frac{\lambda}{\gamma}(1-\theta)(1-\alpha) gq + \frac{\theta g(1-\alpha)}{\gamma} q - \frac{(1-g)(1-\alpha)}{\gamma} z^{-2} - \frac{\rho}{A\gamma}(1-\alpha) z^{-\alpha}\right] \\
&\quad + \frac{dq}{dg}\left[\frac{\lambda}{\gamma}(1-\theta)(1-\alpha) gz + \frac{\theta g(1-\alpha)}{\gamma} z\right] \\
&\quad - \frac{(1-\alpha)}{\gamma z} - g + qz\frac{\theta(1-\alpha)}{\gamma} + \frac{\lambda}{\gamma}(1-\theta)(1-\alpha) qz.
\end{aligned}
$$

### 3.7.4 Decentralized

A representative agent maximizes utility

$$\mathbb{E}_t \int_0^{\infty} u\left(C_t\right) e^{-\rho t} dt,$$

subject to the flow budget constraint

$$dK_t + dB_t = \left[(1-\tau_y)\left[(r_{K,t}-\delta_K)K_t + r_{B,t}B_t\right] - (1+\tau_c)\, C_t - T_t\right] dt,$$

where $\tau_y$ is the income tax rate and $\tau_c$ is the consumption tax rate. Bond holdings are denoted by $B_t$ and the private physical capital stock is $K_t$. Lump-sum taxes are denoted by $T_t$. Public capital and thus $r_{K,t}$ and $r_{B,t}$ are taken as given and considered independent from own actions (agents are atomistic), thus private capital accumulation follows

$$dK_t = \left[I_t - \delta_K K_t\right] dt,$$

where $I_t$ denotes private gross investment into the physical capital good.

#### *3.7.4.1 Solution*

BE is

$$\rho V\left(K_t, B_t\right) = \max_{(C,I)\in(U_C\times U_I)} \left\{ u\left(C\right) + \frac{1}{dt}\mathbb{E}_t dV\left(K_t, B_t\right) \right\},$$

subject to

$$\begin{aligned} dK_t &= [I_t - \delta_K K_t]dt. \\ dB_t &= \left[(1-\tau_y)\left[(r_{K,t}-\delta_K)K_t + r_{B,t}B_t\right] - (1+\tau_c)\, C_t - T_t\right] dt - dK_t. \end{aligned}$$

Re-write

$$\begin{aligned} dK_t &= [I_t - \delta_K K_t]dt, \\ dB_t &= \left[(1-\tau_y)\left[(r_{K,t}-\delta_K)K_t + r_{B,t}B_t\right] - (1+\tau_c)\, C_t - T_t\right] dt - [I_t - \delta_K K_t]dt. \end{aligned}$$

**Step 1: FOC**

We start by computing the derivative $dV$, dropping time indices

$$\begin{aligned} dV\left(K,B\right) &= V_K\left(K,B\right)[I_t - \delta_K K_t]dt - V_B(K,B)[I_t - \delta_K K_t]dt \\ &\quad + V_B\left(K,B\right)\left[(1-\tau_y)\left[(r_{K,t}-\delta_K)K_t + r_{B,t}B_t\right] - (1+\tau_c)\, C_t - T_t\right] dt. \end{aligned}$$

Such that the BE is given by

$$\begin{aligned} \rho V\left(K,B\right) &= \max_{(C,I)} \left\{ u\left(C\right) + \frac{1}{dt}\mathbb{E}_t dV\left(K,B\right) \right\}, \\ &= \max_{(C,I)} \left\{ \begin{array}{c} u\left(C\right) + V_K\left(K,B\right)[I_t - \delta_K K_t] - V_B(K,B)[I_t - \delta_K K_t] \\ + V_B\left(K,B\right)\left[(1-\tau_y)\left[(r_{K,t}-\delta_K)K_t + r_{B,t}B_t\right] - (1+\tau_c)\, C_t - T_t\right] \end{array} \right\}. \end{aligned}$$

The FOC w.r.t. $C$ and $I$ are

$$u_C(C) - (1+\tau_c) V_B(K,B) = 0,$$

and

$$V_K(K,B) = V_B(K,B).$$

**Step 2: Evolution of Co-States**

Consider the problem

$$\begin{aligned} \rho V(K,B) &= u(C) + V_K(K,B)[I_t - \delta_K K_t] - V_B(K,B)[I_t - \delta_K K_t] \\ &\quad + V_B(K,B)[(1-\tau_y)[(r_{K,t} - \delta_K)K_t + r_{B,t}B_t] - (1+\tau_c)C_t - T_t], \end{aligned}$$

take the derivative w.r.t. $K$

$$\begin{aligned} \rho V_K(K,B) &= V_{KK}(K,B)[I_t - \delta_K K_t] - V_K(K,B)\delta_K - V_{BK}[I_t - \delta_K K_t] + V_B(K,B)\delta_K \\ &\quad + V_{BK}(K,B)[(1-\tau_y)[(r_{K,t} - \delta_K)K_t + r_{B,t}B_t] - (1+\tau_c)C_t - T_t] \\ &\quad + V_B(K,B)[(1-\tau_y)(r_{K,t} - \delta_K)]. \end{aligned}$$

Then, we compute the differential of $V_K(K,B)$ via change of variable formula:

$$\begin{aligned} dV_K(K,B) &= V_{KK}(K,B)[I - \delta_K K]dt - V_{BK}(K,B)[I - \delta_K K_t]dt \\ &\quad + V_{BK}(K,B)[(1-\tau_y)[(r_{K,t} - \delta_K)K_t + r_{B,t}B_t] - (1+\tau_c)C_t - T_t]\,dt. \end{aligned}$$

Combining and using that $V_K(K,B) = V_B(K,B)$ and thus $V_{KK} = V_{BK}$

$$dV_K(K,B) = (\rho - [(1-\tau_y)(r_{K,t} - \delta_K)])V_K(K,B).$$

**Step 3: Euler equation**

$$\begin{aligned} u_C(C) &= (1+\tau_c) V_B(K,B), \\ du_C(C) &= (1+\tau_c)\, dV_B(K,B), \\ du_C(C) &= (1+\tau_c)\, dV_K(K,B), \\ du_C(C) &= (1+\tau_c)(\rho - [(1-\tau_y)(r_{K,t} - \delta_K)])V_K(K,B), \\ dC &= -\frac{C}{\gamma}(1+\tau_c)(\rho - [(1-\tau_y)(r_{K,t} - \delta_K)])\frac{V_K(K,B)}{u_C(C)}, \\ dC &= -\frac{C}{\gamma}(\rho - [(1-\tau_y)(r_{K,t} - \delta_K)]). \end{aligned}$$

*Public Sector*

The government invests an amount $G$ per period into the accumulation of public capital. A share $\theta$ of this investment directly increases the public capital stock, while the share $1-\theta$ only works with a lag. Using $G_t = gY_t$ gives

$$dK_{G,t} = \left[\theta g A K_{G,t}^{\alpha} K_t^{1-\alpha} - \delta_G K_{G,t}\right] dt + (1-\theta) g A K_{G,t-}^{\alpha} K_{t-}^{1-\alpha} dN_t.$$

In the decentralized version government spendings are financed using bonds (purchased by the representative agent) and collecting tax revenues (income $\tau_y$ and consumption taxes $\tau_c$). Government debt follows the accumulation process

$$dB_t = G_t + (1-\tau_y)\, r_t B_t - \tau_y Y_t - \tau_c C_t - T.$$

Using the agent's budget constraint

$$dB_t = (1-\tau_y)\left[Y_t + r_t B_t\right] - (1+\tau_c)\, C_t - I_t - T_t,$$

and the government's budget constraint gives the aggregate resource constraint in the decentralized economy

$$Y_t = C_t + I_t + G_t.$$

*Equilibrium Dynamics*

It holds,

$$r_{K,t} = (1-\alpha)\, A K_G^{\alpha} K^{-\alpha},$$

using

$$z_t = \frac{K_{G,t}}{K_t},$$

gives

$$r_{K,t} = (1-\alpha)\, A z^{\alpha}.$$

As in the centralized version,

$$dz_t = \left[\theta g A z_t^{\alpha} - \delta_G z_t - (1-g)\, A z_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt + \left[(1-\theta)\, g A z_t^{\alpha}\right] dN_t.$$

Therefore,

$$\begin{aligned} dC_t &= -\frac{C_t}{\gamma}(\rho - [(1-\tau_y)\left((1-\alpha)\, A z_t^{\alpha} - \delta_K\right)]), \\ dz_t &= \left[\theta g A z_t^{\alpha} - \delta_G z_t - (1-g)\, A z_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt + \left[(1-\theta)\, g A z_t^{\alpha}\right] dN_t. \end{aligned}$$

We want to express consumption in per capital terms. Therefore,

$$c_t = \frac{C_t}{K_t},$$

and apply Ito's Lemma using the aggregate capital accumulation process

$$dK_t = \left[(1-g)\, AK_{G,t}^{\alpha} K_t^{1-\alpha} - C_t - \delta_K K_t\right] dt,$$

because we are now dealing with aggregate values, gives

$$\begin{aligned}
dc_t &= \frac{1}{K}\left[-\frac{C}{\gamma}(\rho - [(1-\tau_y)(r_{K,t}-\delta_K)])\right] - \frac{C}{K^2}\left[(1-g)\, AK_G^{\alpha}K^{1-\alpha} - C - \delta_K K\right], \\
&= \frac{c}{\gamma}([(1-\tau_y)(r_{K,t}-\delta_K)] - \rho) - c\left[(1-g)\, AK_G^{\alpha}K^{-\alpha} - \frac{C}{K} - \delta_K\right], \\
&= \frac{c}{\gamma}([(1-\tau_y)(r_{K,t}-\delta_K)] - \rho) - c\left[(1-g)\, Az^{\alpha} - c - \delta_K\right], \\
&= c\left[\frac{(1-\tau_y)((1-\alpha)\, Az^{\alpha} - \delta_K) - \rho}{\gamma} - (1-g)\, Az^{\alpha} + c + \delta_K\right].
\end{aligned}$$

Finally, the system is given by (cf. Turnovsky system 6)

$$\begin{aligned}
dc_t &= c_t\left[\frac{(1-\tau_y)((1-\alpha)\, Az_t^{\alpha} - \delta_K) - \rho}{\gamma} - (1-g)\, Az_t^{\alpha} + c_t + \delta_K\right], \\
dz_t &= \left[\theta g Az_t^{\alpha} - \delta_G z_t - (1-g)\, Az_t^{\alpha+1} + c_t z_t + \delta_K z_t\right] dt + \left[(1-\theta)\, gAz_t^{\alpha}\right] dN_t.
\end{aligned}$$

#### *3.7.4.2 Steady State*

In steady state,

$$\begin{aligned}
0 &= \frac{(1-\tau_y)((1-\alpha)\, Az^{\alpha} - \delta_K) - \rho}{\gamma} - (1-g)\, Az^{\alpha} + c + \delta_K, \\
0 &= \theta g Az^{\alpha} - \delta_G z - (1-g)\, Az^{\alpha+1} + cz + \delta_K z.
\end{aligned}$$

such that

$$c = (1-g)\, Az^{\alpha} - \theta g Az^{\alpha-1} + \delta_G - \delta_K,$$

assuming $\delta_G = \delta_K = 0$ gives

$$c = (1-g)\, Az^{\alpha} - \theta g Az^{\alpha-1}.$$

Further,

$$\frac{(1-\tau_y)(1-\alpha)\, Az^{\alpha} - \rho}{\gamma} - (1-g)\, Az^{\alpha} + c = 0.$$

The capital growth rates are

$$\begin{aligned}
\phi_g &= gAz^{\alpha-1}, \\
\phi_k &= (1-g)\, Az^{\alpha} - c.
\end{aligned}$$

*3.7.4.3 Long-Run Fiscal Effects*

The effects of taxes and spending are

$$\begin{aligned}
\frac{(1-\tau_y)(1-\alpha)Az^{\alpha}-\rho}{\gamma}-\theta gAz^{\alpha-1} &= 0,\\
-\frac{(1-\alpha)Az^{\alpha}}{\gamma}+\frac{(1-\tau_y)(1-\alpha)A}{\gamma}\alpha z^{\alpha-1}\frac{dz}{d\tau_y}-\theta gA(\alpha-1)z^{\alpha-2}\frac{dz}{d\tau_y} &= 0,\\
\frac{dz}{d\tau_y}\left[\frac{(1-\tau_y)(1-\alpha)}{\gamma}\alpha z+\theta g(1-\alpha)\right] &= \frac{(1-\alpha)z^2}{\gamma},\\
\frac{dz}{d\tau_y}\left[\frac{(1-\tau_y)}{\gamma}\alpha z+\theta g\right] &= \frac{z^2}{\gamma},\\
\frac{dz}{d\tau_y} &= \frac{z^2}{\gamma\left[\frac{(1-\tau_y)}{\gamma}\alpha z+\theta g\right]}.
\end{aligned}$$

Then,

$$\begin{aligned}
\frac{(1-\tau_y)(1-\alpha)Az^{\alpha}-\rho}{\gamma}-\theta gAz^{\alpha-1} &= 0,\\
\frac{(1-\tau_y)(1-\alpha)A}{\gamma}\alpha z^{\alpha-1}\frac{dz}{dg}-\theta Az^{\alpha-1}-\theta gA(\alpha-1)z^{\alpha-2}\frac{dz}{dg} &= 0,\\
\frac{(1-\tau_y)(1-\alpha)A}{\gamma}\alpha z\frac{dz}{dg}-\theta Az-\theta gA(\alpha-1)\frac{dz}{dg} &= 0,\\
\frac{dz}{dg}\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right] &= \frac{z}{(1-\alpha)},\\
\frac{dz}{dg} &= \frac{z}{(1-\alpha)\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}.
\end{aligned}$$

Using,

$$\begin{aligned}
c &= (1-g)Az^{\alpha}-\theta gAz^{\alpha-1},\\
\frac{(1-\tau_y)(1-\alpha)Az^{\alpha}-\rho}{\gamma}-(1-g)Az^{\alpha}+c &= 0.
\end{aligned}$$

gives

$$\begin{aligned}
-\frac{(1-\alpha)Az^{\alpha}}{\gamma}+\frac{(1-\tau_y)(1-\alpha)A\alpha z^{\alpha-1}}{\gamma}\frac{dz}{d\tau_y}-(1-g)\alpha Az^{\alpha-1}\frac{dz}{d\tau_y}+\frac{dc}{d\tau_y} &= 0,\\
(1-g)A\alpha z^{\alpha-1}\frac{dz}{d\tau_y}-\theta gA(\alpha-1)z^{\alpha-2}\frac{dz}{d\tau_y}-\frac{dc}{d\tau_y} &= 0,\\
\frac{z^2\left[(1-g)A\alpha z^{\alpha-1}-\theta gA(\alpha-1)z^{\alpha-2}\right]}{\gamma\left[\frac{(1-\tau_y)}{\gamma}\alpha z+\theta g\right]} &= \frac{dc}{d\tau_y},\\
\frac{Az^{\alpha}\left[(1-g)\alpha z+\theta g(1-\alpha)\right]}{\gamma\left[\frac{(1-\tau_y)}{\gamma}\alpha z+\theta g\right]} &= \frac{dc}{d\tau_y}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{dc}{dg} &= -Az^{\alpha}-\theta Az^{\alpha-1}+(1-g)\,A\alpha z^{\alpha-1}\frac{dz}{dg}-\theta g\,(\alpha-1)\,Az^{\alpha-2}\frac{dz}{dg},\\
&= \frac{dz}{dg}\left[(1-g)\,A\alpha z^{\alpha-1}-\theta g\,(\alpha-1)\,Az^{\alpha-2}\right]-Az^{\alpha}-\theta Az^{\alpha-1},\\
&= \frac{Az^{\alpha}}{(1-\alpha)\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}\left[(1-g)\,\alpha-\theta g\,(\alpha-1)\,z^{-1}\right]-Az^{\alpha}-\theta Az^{\alpha-1},\\
\frac{\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}{Az^{\alpha}}\frac{dc}{dg} &= \frac{(1-g)}{(1-\alpha)}\alpha+\frac{\theta g}{z}-\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]\left[1+\frac{\theta}{z}\right],\\
&= \frac{(1-g)}{(1-\alpha)}\alpha+\frac{\theta g}{z}-\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]\left[1+\frac{\theta}{z}\right],\\
&\quad \frac{(1-\tau_y)}{\theta\gamma}\alpha z+\frac{(1-\tau_y)}{\theta\gamma}\alpha z\frac{\theta}{z}+g+g\frac{\theta}{z},\\
\frac{\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}{Az^{\alpha}}\frac{dc}{dg} &= \frac{(1-g)}{(1-\alpha)}\alpha+\frac{\theta g}{z}-\frac{(1-\tau_y)}{\theta\gamma}\alpha z-\frac{(1-\tau_y)}{\theta\gamma}\alpha\theta-g-g\frac{\theta}{z},\\
&= -\frac{(1-\tau_y)\,\alpha\,(\theta+z)}{\theta\gamma}+\frac{(1-g)}{(1-\alpha)}\alpha+\frac{\theta g}{z}-g-\frac{g\theta}{z},\\
&= -\frac{(1-\tau_y)\,\alpha\,(\theta+z)}{\theta\gamma}+\frac{(1-g)}{(1-\alpha)}\alpha-g,\\
&= -\frac{(1-\tau_y)\,\alpha\,(\theta+z)}{\theta\gamma}+\frac{(1-g)\,\alpha-(1-\alpha)\,g}{(1-\alpha)},\\
&= -\frac{(1-\tau_y)\,\alpha\,(\theta+z)}{\theta\gamma}+\frac{\alpha-g\alpha-g+\alpha g}{(1-\alpha)},\\
&= \frac{\alpha-g}{(1-\alpha)}-\frac{(1-\tau_y)\,\alpha\,(\theta+z)}{\theta\gamma},\\
\frac{dc}{dg} &= \frac{Az^{\alpha}\left[\frac{\alpha-g}{(1-\alpha)}-\frac{(1-\tau_y)\alpha(\theta+z)}{\theta\gamma}\right]}{\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}.
\end{aligned}$$

Use

$$\phi = gAz^{\alpha-1},$$

take derivative

$$\begin{aligned}
\frac{d\phi}{d\tau_y} &= gA\,(\alpha-1)\,z^{\alpha-2}\frac{dz}{d\tau_y},\\
&= gA\,(\alpha-1)\,z^{\alpha-2}\frac{z^2}{\gamma\left[\frac{(1-\tau_y)}{\gamma}\alpha z+\theta g\right]},\\
&= -\frac{gA\,(1-\alpha)\,z^{\alpha}}{\gamma\left[\frac{(1-\tau_y)}{\gamma}\alpha z+\theta g\right]}.
\end{aligned}$$

Further,

$$\begin{aligned}
\phi &= (1-g)\,Az^{\alpha}-c,\\
\frac{d\phi}{dg} &= -Az^{\alpha}+(1-g)\,A\alpha z^{\alpha-1}\frac{dz}{dg}-\frac{dc}{dg},\\
&= -Az^{\alpha}+(1-g)\,\alpha\frac{Azz^{\alpha-1}}{(1-\alpha)\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}\\
&\quad -\frac{Az^{\alpha}\left[\frac{\alpha-g}{(1-\alpha)}-\frac{(1-\tau_y)\alpha(\theta+z)}{\theta\gamma}\right]}{\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]},\\
&= -Az^{\alpha}+(1-g)\,\alpha\frac{Az^{\alpha}}{(1-\alpha)\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}\\
&\quad -\frac{Az^{\alpha}\left[\frac{\alpha-g}{(1-\alpha)}-\frac{(1-\tau_y)\alpha(\theta+z)}{\theta\gamma}\right]}{\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]},\\
\frac{\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}{Az^{\alpha}}\frac{d\phi}{dg} &= -\frac{(1-\tau_y)}{\theta\gamma}\alpha z+\frac{\alpha g-g}{(1-\alpha)}+\frac{\alpha-\alpha g}{(1-\alpha)}+\frac{g-\alpha}{(1-\alpha)}+\frac{(1-\tau_y)\,\alpha\,(\theta+z)}{\theta\gamma},\\
&= -\frac{(1-\tau_y)}{\theta\gamma}\alpha z+\frac{(1-\tau_y)\,\alpha\,(\theta+z)}{\theta\gamma},\\
\frac{\theta\gamma\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}{Az^{\alpha}}\frac{d\phi}{dg} &= -(1-\tau_y)\,\alpha z+(1-\tau_y)\,\alpha\,(\theta+z)\,,\\
&= -\alpha z+\alpha z\tau_y+\alpha\theta+\alpha z-\tau_y\alpha\theta-\tau_y\alpha z,\\
&= \alpha\theta-\tau_y\alpha\theta,\\
&= (1-\tau_y)\,\alpha\theta,\\
\frac{d\phi}{dg} &= \frac{(1-\tau_y)\,\alpha\theta Az^{\alpha}}{\theta\gamma\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]},\\
\frac{d\phi}{dg} &= \frac{(1-\tau_y)\,\alpha Az^{\alpha}}{\gamma\left[\frac{(1-\tau_y)}{\theta\gamma}\alpha z+g\right]}.
\end{aligned}$$

#### *3.7.4.4 Optimal Tax Rate*

Which tax rate replicates the first-best equilibrium? Assuming a constant rax rate, we require the following to hold:

The decentralized solution

$$\begin{aligned}
0 &= \frac{(1-\tau_y)\left((1-\alpha)\,Az^{\alpha}\right)-\rho}{\gamma}-(1-g)\,Az^{\alpha}+c,\\
0 &= \theta gAz^{\alpha}-(1-g)\,Az^{\alpha+1}+cz.
\end{aligned}$$

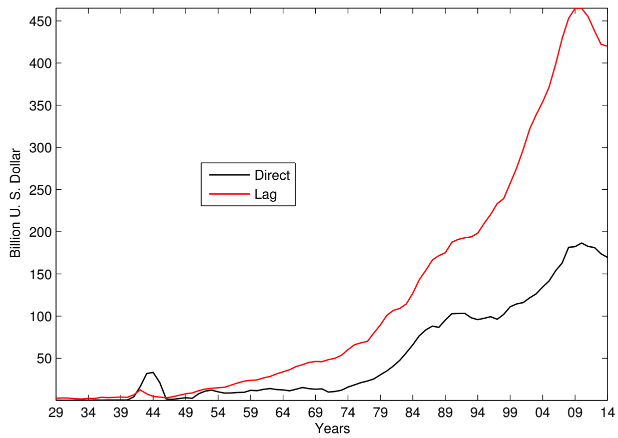


**Figure 31:** Time series of investment with (Lag) and without (Direct) time lags.

has to be equal to the centralized solution (for $\theta = 1$)

$$\begin{aligned}\tilde{c} &= (1-g)\, A\tilde{z}^{\alpha} - gA\tilde{z}^{\alpha-1},\\ \tilde{z}\tilde{q} &= \frac{\alpha}{(1-\alpha)},\\ 0 &= \frac{(1-\alpha)\, A\tilde{z}^{\alpha}\left[(1-g)+gq\right]}{\gamma} - \frac{\rho}{\gamma} - (1-g)\, A\tilde{z}^{\alpha} + \tilde{c}.\end{aligned}$$

This holds only true, iff

$$\begin{aligned}(1-\tau_y)(1-\alpha)\, Az^{\alpha} &= (1-\alpha)\, A\tilde{z}^{\alpha}\left[(1-g)+gq\right],\\ (1-\tau_y) &= (1-g)+gq,\\ \tau_y &= g\,(1-q)\,.\end{aligned}$$

### 3.7.5 Additional Graphics

#### *3.7.5.1 Government Investment*

Figure 31 plots the time series of government investment with and without lag over time (1929 to 2014).

#### *3.7.5.2 VECM*

We consider is a six-dimensional system

$$y_t = \left(I_t^L, I_t^W, G_t, I_t, C_t, Y_t\right)', \tag{147}$$

where output is denoted by $Y_t$, $C_t$ is consumption, $I_t$ is private investment, $I_t^L$ is public investment with lag, $I_t^W$ is public investment without lag, and $G_t$ is government consumption.

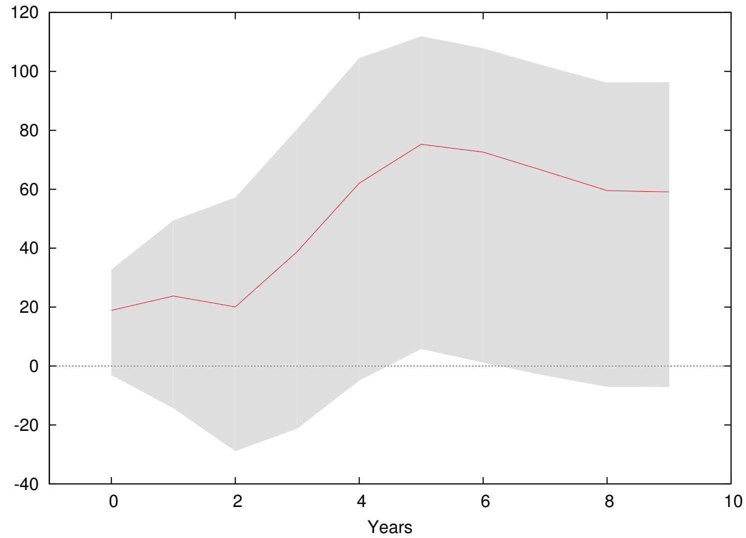


**Figure 32:** Impulse responses of GDP to a one percent standard shock in investment with lag. 15% confidence bands are shown in grey.

The cointegrating rank of this model is 1 (according to the Johansen cointegration test). Then, the VECM($p$) model is given by

$$\Delta y_t = \mu + \Pi y_{t-1} + \sum_{i=1}^{p-1} \Gamma_i \Delta y_{t-i} + u_t, \tag{148}$$

where $\Pi = \alpha\beta'$ with $\alpha$ and $\beta$ being $(K \times r)$ matrices, where $K$ is the dimension of the system and $r$ is the rank of $\Pi$. Further, $\Gamma_i$ are parameter matrices of size $(K \times K)$ and $\mu$ is an unrestricted constant. Finally, $u_t \sim (0, \Sigma_u)$ is white noise. We find that four lags are optimal. The reaction of GDP to a standard shock in investment with a lag is shown in Figure 32.

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# CHAPTER IV

# CURRICULUM VITAE

### Contact Information


- University of Hamburg
- Department of Economics
- Chair of Economics, Methods in Economics
- Von-Melle-Platz 5
- 20146 Hamburg, Germany
- Email: Dennis.Wesselbaum@Wiso.Uni-Hamburg.de


### Appointments

- Researcher, University of Hamburg, Chair of Economics, Methods in Economics, 09/2012 - 09/2015.
- Research Assistant, Department of Money and Macroeconomics, Chair in Monetary and Fiscal Policy, Professor Ester Faia, 10/2011 - 09/2012.
- Researcher, The Kiel Institute for the World Economy, Research areas "Monetary Policy under Market Imperfections" and "Reforming the Welfare Society", 03/2010 - 07/2011.

### Professional Activities

- Research Visit, Johann-Wolfgang Goethe University, Frankfurt am Main, Germany, August - September 2014.
- Research Visit, Federal Reserve Bank of Chicago, United States, September - November 2010.
- Visit, Faculty of Economics, University of Pavia, Italy, December 2009 and July 2010.
- Visit, Department of Economics, University of Louvain la Neuve, Belgium, January 2010.
- Fellow, Euro Area Business Cycle Network (EABCN), since 2009.

- Fellow, German Physical Society (DPG), since 2005.

**Education**

- Ph.D.
  - University of Hamburg, 09/2012 - 09/2015.
  - Dissertation on "Fiscal Policy in General Equilibrium."
- Diploma in Economics (M.Sc. equivalent)
  - University of Kiel, GPA 1.3 (3.8 U. S. scale), 10/2005 - 03/2010.
  - Specialization: Macro Labor Economics and Monetary Policy.
- Abitur
  - Friedrich-List-Gymnasium, Lübeck, GPA 1.5.

**Research**

- *Publications*
  - "Sectoral Labor Market Effects of Fiscal Spending." *Structural Change and Economic Dynamics*, forthcoming.
  - "The Intensive Margin Puzzle and Labor Market Adjustment Costs." *Macroeconomic Dynamics*, forthcoming.
  - "What drives Endogenous Growth in the United States?" *The B.E. Journal of Macroeconomics (Contributions)*, 15(1): 183-221, 2015.
  - "Bubbles over the U. S. Business Cycle: A Macroeconometric Approach." Joint with Marc Luik. *Journal of Macroeconomics*, 40: 27-41, 2014.
  - "Evaluating Labor Market Reforms: A Normative Analysis." Joint with Céline Poilly. *Journal of Macroeconomics*, 39: 156-170, 2014.
  - "On Time-dependent Business Cycle Distributions of Labor Market Variables." *European Economics Letters*, 3(1): 57-61, 2014.
  - "Staggered Wages, Sticky Prices, and Labor Market Dynamics in Matching Models." Joint with Janett Neugebauer. *Applied Economics Quarterly*, 60(3): 159-177, 2014.
  - "Labour Market Dynamics in Australia." *The Australian Economic Review*, 47(2): 1-16, 2014.
  - "Firing Costs in a Business Cycle Model with Endogenous Separations." *Journal of Economic Studies*, forthcoming.

- "Firing Tax vs. Severance Payment - An Unequal Comparison." *Journal of Economic Studies*, 41(5): 721-736, 2014.
- "Reciprocity and Matching Frictions." *International Review of Economics*, 60(3): 247-268, 2013.
- "Gender-specific Differences in Labor Market Adjustment Patterns: Evidence from the United States." *The Social Science Journal*, 50(3): 381-385, 2013.
- "Sector-Specific Productivity Shocks in a Matching Model." *Economic Modelling*, 28(6): 2674-2682, 2011.
- "Extensive vs. Intensive Margin in Germany and the United States: Any Differences?" Joint with Christian Merkl. *Applied Economics Letters*, 18: 805-808, 2011.
- "Kernfusion - Die vernachlässigte Alternative." *Kiel Policy Brief*, No. 30, June 2011 and *ITER Newsline*, No. 186, July 2011.
- "Evaluating the Federal Reserve's Policy." *Kiel Policy Brief*, No. 23, January 2011.
- "Estimating the Impact of Fiscal Stimulus Package." Joint with Björn van Roye. In Klodt, Henning and Lehment, Harmen, eds.: *The Crisis and Beyond*, The Kiel Institute for the World Economy, November 2009.

• *Working Papers*

- "Procyclical Debt as Automatic Stabilizer", Banque de France Working Paper, No. 444, September 2013. Submitted.
- "Cheap Talk in a New Keynesian Model". Submitted.
- "Fiscal and Monetary Policy Interactions in New Zealand". Submitted.
- "Stochastic Volatility in the U.S. Labor Market". MPRA Working Paper, No. 43054. Submitted.
- "How Large are Firing Costs? A Cross-Country Study". Submitted.
- "Price Bargaining and the Business Cycle", Kiel Working Paper, No. 1629, June 2010. Submitted.
- "On the Introduction of Firing Costs", joint with Steffen Ahrens. Kiel Working Paper, No. 1559, October 2009.
- "Capital, Endogenous Separations, and the Business Cycle", joint with Björn van Roye. Kiel Working Paper, No. 1561, October 2009.
- "A Game Theoretical View on Efficiency Wage Theories", MPRA Paper, No. 18026, University Library Munich, October 2009.

**Research Grants**

- EES Research Exchange Grant, 2010.
- EES Research Exchange Grant, 2009.

**Presentations**

- *2014*
  - 1st International Research Conference on "Macroeconomic Policies and Financial Stability Issues in Emerging Markets", Central Bank of the Republic of Azerbaijan, Baku, October.
  - Money, Macro, and Finance (MMF) Research Group, 46th Annual Conference, University of Durham, September.
  - 4th SEEK Conference on "Public Finance and Income Distribution in Europe", ZEW, Mannheim, May.
  - 18th Conference Theory and Methods in Macroeconomics (T2M), University of Lausanne, February.
  - Doctoral Seminar, University of Hamburg, Hamburg, January and December.
- *2013*
  - 4th Conference on "Recent Developments in Macroeconomics", ZEW, Mannheim, July.
  - Seminar, Banque de France, Paris, July.
  - 19th International Conference on Computing in Economics and Finance, Society for Computational Economics, Vancouver, July.
  - Doctoral Seminar, University of Hamburg, Hamburg, May.
- *2011*
  - IAB/LASER Workshop "Increasing Labor Market Flexibility - Boon or Bane?", Nürnberg, March.
  - 4th RGS Doctoral Conference in Economics, TU Dortmund, February.
  - Macro Reading Group, Kiel Institute for the World Economy, Kiel.
- *2010*
  - Staff Seminar, Federal Reserve Bank of Chicago, September.
  - Zeuthen Workshop on Macroeconomics, University of Copenhagen, March.
  - Macro Reading Group, Kiel Institute for the World Economy, Kiel.

- *2009*
  - Staff Seminar, University of Pavia, December.
  - Macro Reading Group, Kiel Institute for the World Economy, Kiel (4x).

**Workshops and Internships**

- Session chair on "Government Debt", 4th SEEK Conference on "Public Finance and Income Distribution in Europe", ZEW, Mannheim, May 2014.
- Session chair on "Business Cycle Dynamics II", 4th Conference on "Recent Developments in Macroeconomics", ZEW, Mannheim, July 2013.
- Session chair on "Fiscal Policy III", 18th International Conference on Computing in Economics and Finance, Vancouver, July 2013.
- CFS Symposium "Global perspective on the financial crisis", Frankfurt, September 2011.
- MONFISPOL, Final Conference, Frankfurt, September 2011.
- Session Organizer "Avoiding Currency Wars and Ensuring Balanced Global Recovery", Global Economic Symposium 2011.
- Zeuthen Lectures on Macroeconomics, "Unemployment Fluctuations and Stabilization Policies: A New Keynesian Perspective", Prof. Jordi Galí, University of Copenhagen, March 2010.
- 8th Workshop on "Macroeconomic Dynamics: Theory and Applications", University of Pavia, Pavia, December 2009.
- Advanced Studies Program, The Kiel Institute for the World Economy. Course "Monetary Policy: Theory and Practice", by Larry Christiano, 2009.
- 1st EES Workshop on "The Labor Market and the Business Cycle", the Kiel Institute for the World Economy, Kiel, March 2009.
- Workshop on "Corporate Banking", Dresdner Bank, Kiel, January 2009.
- Student assistant in the research areas "Reforming the Welfare Society" (Alessio J. G. Brown) and "Monetary Policy under Market Imperfections" (Christian Merkl), at the Kiel Institute for the World Economy, Kiel, November 2008 - March 2010.
- Four-week internship, Corporate Auditing, Draeger Group, Lübeck, August 2007.
- Two-week internship, Research Unit, Draeger Group, Lübeck, April 2001.

## Teaching

- Tutorial, Mathematics II, Summer term 2015, University of Hamburg.
- Tutorial, Mathematics I, Winter term 2012/2013, 2013/2014, and 2014/2015, University of Hamburg.
- Supervisor, Bachelor Thesis (2x), University of Hamburg.
- Discussant, Advanced Studies Program, Kiel Institute for the World Economy, Kiel, February 2011.
- Seminar on "Fiscal Policy in the Great Recession." Summer term 2010, University of Kiel.

## Honors and Awards

- Outstanding Paper Award 2015, Emerald Literati Network (Journal of Economic Studies).
- Erich Schneider Memorial Price for the best Diploma Thesis in Economics, Kiel University, 2009.

## Referee Service

- Economics Letters, Economic Modelling (5x), International Journal of Manpower, Journal of Economic Studies (3x), Journal of Macroeconomics.

## Languages and Computer Skills

- Languages: German (native), English (fluent), French (basics), Italian (beginner).
- Computer Skills: Matlab, Gretl, EViews, Stata, WinIdea, Mathcad, Scientific Workplace, Latex.

# CHAPTER V

# ERKLÄRUNGEN

## 5.1 *Eidesstattliche Versicherung*

Ich, Dennis Wesselbaum, versichere an Eides statt, dass ich die Dissertation mit dem Titel:

"Fiscal Policy in General Equilibrium"

selbst und bei einer Zusammenarbeit mit anderen Wissenschaftlerinnen oder Wissenschaftlern gemäß den beigefügten Darlegungen nach §6 Abs. 3 der Promotionsordnung der Fakultät Wirtschafts- und Sozialwissenschaften vom 24. August 2010 verfasst habe. Andere als die angegebenen Hilfsmittel habe ich nicht benutzt.

Dennis Wesselbaum Date

## 5.2 *Erklärung zur Promotionsberatung*

Hiermit erkläre ich, Dennis Wesselbaum, dass ich keine kommerzielle Promotionsberatung in Anspruch genommen habe. Die Arbeit wurde nicht schon einmal in einem früheren Promotionsverfahren angenommen oder als ungenügend beurteilt.

Dennis Wesselbaum Date

## 5.3 *Selbstdeklaration*

Gemäß der geltenden Promotionsordnung ist bei einer kumulativen Dissertation eine Selbstdeklaration abzugeben, die meinen Beitrag zu den drei Kapiteln angibt.

Kapitel 1 „Procyclical Debt as Automatic Stabilizer“ und das Kapitel 2 „Cheap Talk in a New Keynesian Model“ sind Artikel, die ich als alleiniger Autor geschrieben habe. Dies beinhaltet die Ideenfindung, Literaturrecherche, Modelentwicklung und –lösung, Simulation und Interpretation der Ergebnisse (das „Schreiben“ des Artikels).

Kapitel 3 „Delays in Public Goods“ ist eine Zusammenarbeit mit Prof. Dr. Olaf Posch von der Universität Hamburg und Assoc. Prof. Santanu Chatterjee, Ph.D.

von der University of Georgia. Dieses Projekt basiert auf einem von mir entwickelten Research Proposal, welches dann zusammen weiterentwickelt wurde. Die Modelentwicklung und –lösung sowie die numerische Lösung habe ich eigenständig vorgenommen; allerdings in Absprache und unter der Ergebniskontrolle beider Co-Autoren. Die Literaturrecherche und die Formulierung der Ergebnisse habe ich dann in Absprache mit beiden Co-Autoren vorgenommen. Prozentual liegt mein Anteil an der Arbeit (Kapitel 3) bei ca. 90 Prozent.

Dennis Wesselbaum

Date