Log Geometric Techniques For Open Invariants in Mirror Symmetry

Dissertation zur Erlangung des Doktorgrades
an der Fakultät für Mathematik, Informatik und Naturwissenschaften
Fachbereich Mathematik
der Universität Hamburg

vorgelegt von
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Tag der Disputation: 16 August 2016

Dissertation with the aim of achieving a doctoral degree
at the Faculty of Mathematics, Informatics
and Natural Sciences
Department of Mathematics
of the University of Hamburg

submitted by
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Date of disputation: 16 August 2016

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Eidesstattliche Versicherung

Declaration on oath

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Diessertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

I hereby declare, on oath, that I have written the present dissertation on my own and have not used other than the acknowledged resources and aids.

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Acknowledgements

First and foremost I would like to thank my advisor Bernd Siebert who was an excellent mentor for me. His support and contribution with his ideas on this thesis is priceless. I appreciate the time and effort he put into it. I am grateful for all his guidance and criticism.

I also thank Mark Gross for refereeing this thesis, for his support and encouragement during my PhD. I am so glad that I had the chance to struggle through the framework of the Gross-Siebert program on mirror symmetry. I am deeply grateful to the two founders of this program for the wonderful examples of people they provided me as great research mathematicians.

Special thanks also to Mohammed Abouzaid for suggesting the question on the Tate curve during a discussion with Bernd.

I would also like to thank all the members of the RTG 1670, especially our spokesperson Professor Christoph Schweigert for the funding sources that made my PhD work possible and for the very nice work environment in the RTG. I also appreciate the kindness and help of all the secretaries of the GRK; Ms Kortmann and Ms Harich during the first funding period and currently during the second period, Ms Mierswa Silva.

I thank all my teachers, especially to the geometry topology group at Middle East Technical University from where I had obtained my Bachelors degree. I also thank to all mathematicians with whom I had the opportunity to do various useful mathematical discussions, among some are Sean Keel, Travis Mandel, Mark McLean, Oliver Fabert, Kai Cielebak, Luis Diogo, Vivek Shende, Gätan Borot, Yıldıray Ozan, Janko Latschev and many other people whose names do not come to my mind at the moment.

I would like to thank Paul Seidel for organising a special semester on Homological Mirror Symmetry at the IAS school next term and providing me the opportunity to participate in it, which is a very nice motivation for me.

Thanks to Ludmil Katzarkov and Denis Auroux for organising wonderful conferences, especially in Miami and having provided me support to participate in them. Thanks also to Selman Akbulut and Turgut Önder for the Gökova Geometry Topology conferences which are also wonderful. Special thanks also to Özgür Ceyhan for his support and for providing me the opportunity to visit Luxembourg.

Thanks to my committee members Klaus Kröncke, Joerg Teschner, Chris Wendl and Ingenuin Gassa for the time they saved for me making the PhD defense possible.

I also thank my academic-brother Helge Ruddat for his encouragement and support during these years and my office mate Carsten Liese for being a really nice friend going through the Gross-Siebert program with me.

I thank my family and friends for all their love and encouragement. I am grateful to my father Yaşar (R.I.P.) and my mother Keziban for bringing me to the world. Thanks to my sister Handan who is a doctor of medicine and mother of two cute kids *Ceren* and *Selen* for showing me the virtue of hard work, my brother Erdinç for the amazing library of philosophy books he needed to leave behind while moving to England.

N. Hülya Argüz
Hamburg, July 2016

This thesis consists of two seperate parts. Each part has a seperate introduction and table of contents.

PART I: *Log-geometric invariants of degenerations with a view toward symplectic cohomology : the Tate curve*

This part is my own work.

PART II: *On the real locus in the Kato-Nakayama space of logarithmic spaces with a view toward toric degenerations*

This part is joint work with Bernd Siebert, which grew out from my MSc thesis (2013) "Lagrangian submanifolds in Kato-Nakayama spaces" that I pursued under his supervision. There I had studied the topology of Kato-Nakayama spaces. In particular, studied the Kato-Nakayama space over the central fiber of a toric degeneration of K3 surfaces and investigated the real locus Lagrangian in this set-up. Afterwards we wanted to introduce the framework of real log schemes and then to generalize these results to arbitrary dimensions. Bernd provided a draft for the first part on real log schemes, in which I filled in some details and proofs and merged it together with the results of my MSc thesis. He afterwards went through it and provided generalizations for many of the results. Moreover, all discussions involving gluing data are contributed by him.

LOG-GEOMETRIC INVARIANTS OF DEGENERATIONS WITH A VIEW TOWARD SYMPLECTIC COHOMOLOGY : THE TATE CURVE

HÜLYA ARGÜZ

Contents

Date: January 3, 2017.

Introduction.

Based on a proposal by Mohammed Abouzaid and Bernd Siebert, we suggest an algebraic geometric approach to the Fukaya category in terms of (log) Gromov-Witten theory. Our aim is to understand the symplectic Fukaya category which involves Lagrangians and holomorphic disks and does not appear to be amenable to the refined and computationally effective methods of algebraic geometry. Contrary to such expectations we suggest that a version of log Gromov-Witten theory applied on a certain algebraic-geometric degeneration of the symplectic manifold sometimes does this job.

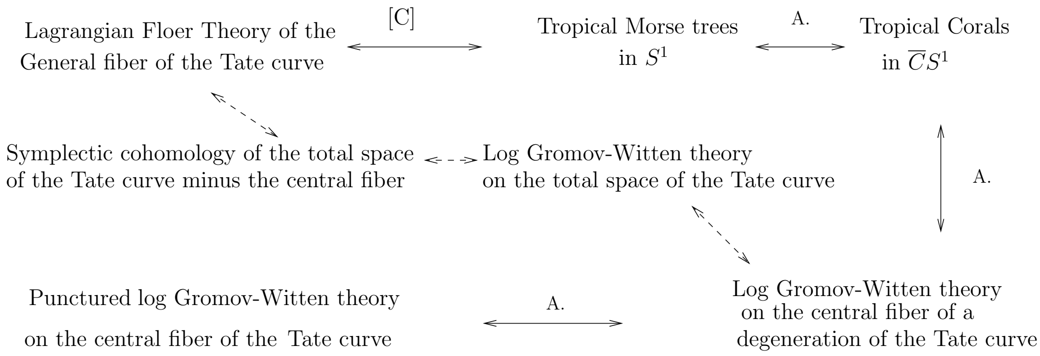
We work in the set up of toric degenerations ([GS3]). We suggest that the Lagrangian Floer theory of the general fiber of a toric degeneration is equivalent to the punctured log Gromov-Witten theory of the central fiber. We work out this correspondence on the easiest non-trivial example of a ($\mathbb{Z}$-quotient of) a toric degeneration, the Tate curve. The method in principle can however be applied to any variety with a toric degeneration in the sense of the Gross-Siebert program [GS2].

The general fiber of the Tate curve is an elliptic curve and the central fiber is topologically a pinched torus. To study the Lagrangian Floer theory of the elliptic curve we look at the tropical Morse category introduced by Abouzaid, Gross and Siebert. The idea is to approximate holomorphic disks between Lagrangians by structures arising from combinatorial objects, so called *tropical Morse trees*. The equivalence of the Tropical Morse category and the Fukaya category, for the elliptic curve has been worked out by M. Gross in [C]. Roughly speaking, a tropical Morse tree is the tropical analogue of the gradient flow tree in Morse theory. To be able to use enumerative techniques of traditional tropical geometry we introduce objects which we call *tropical corals* and show the correspondence between tropical Morse trees in S^1 and tropical corals in the truncated cone $\overline{C}S^1$ over S^1 ([A]).

We then define algebraic geometric objects which we call *log corals* whose tropicalizations give tropical corals. These are stable log maps with some non-complete components mapping to the central fiber of the "degenerate" Tate curve, a product of the affine line $\mathbb{A}^1$ with the central fiber of the Tate curve itself. We set up the counting problems for tropical corals as well as log corals on a defined stable range. Our main result is the following.

Theorem 0.1. *(5.11)*
Choosing incidence conditions in a certain stable range, the count of log corals is well-defined (and each log coral in the count is unobstructed). Moreover, the number of tropical corals agrees with the number of log corals.

Then we show that log corals in the central fiber of the "degeneration" of the Tate curve actually correspond to certain "punctured" stable log maps in the central fiber of the Tate curve itself. Hence, at the end we have the following correspondences where the ones with dashed lines are conjectural.



Thus, we arrive at a correspondence between the Lagrangian Floer theory of the general fiber of the Tate curve and the punctured log Gromov-Witten theory of its central fiber. We believe, once a suitable version of symplectic cohomology theory for the total space minus the central fiber of the Tate curve is set up, we can deform our log corals from the central fiber of the degenerate Tate curve to the general fiber of it (which is the Tate curve itself) and hence find the correct analytic versions for the cylinders one needs to approximate in symplectic cohomology. The Tate curve minus its central fiber is topologically the mapping torus of a Dehn twist. The relatively similar case of symplectic cohomology of Hamiltonian mapping tori have been considered in [F]. For cases similar as ours with non-Hamiltonian isotopies, there is no explicit description yet.

One of our main interests in an approach to define a version of symplectic cohomology in terms of log Gromov-Witten invariants is this: The mirror construction of [GS2], using a scattering procedure on an integral affine manifold, in which the scattering functions have enumerative meanings in terms of log Gromov-Witten invariants ([GPS]) produces a mirror family for a given log Calabi-Yau. It appears that the coordinate ring of the mirror family can be constructed directly by the symplectic cohomology ring SH^* as conjectured in the first preprint version of [GHK] and in the presentation of Bernd Siebert at the string math conference whose slides are available at [S], where the ring SH^* is suggested to be described in terms of punctured log Gromov-Witten invariants [ACGS]. Note that in [GHK], the structure coefficients that arise in the scattering procedure are constructed tropically. For a geometric interpretation one needs to study the corresponding punctured log Gromov-Witten invariants that correspond to the tropical invariants. Throughout this paper, we study the relation of these log geometric invariants to tropical geometry for the case of the Tate curve in detail and discuss the Floer theoretic perspectives.

Conventions. We work in the category of schemes of finite type over the complex number field $\mathbb{C}$, though in general one can work over any algebraically closed field $\Bbbk$ of characteristic 0. We fix

$$N = \mathbb{Z}^n, \ \ N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}, \ \ M = \mathrm{Hom}(N, \mathbb{Z}), \ \ M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$$

If Σ is a fan in $N_{\mathbb{R}}$ then $X(\Sigma)$ denotes the associated toric $\mathbb{C}$-variety with big torus $\mathrm{Int}X(\Sigma) \simeq \mathbb{G}(N) \subset X(\Sigma)$, whose complement is referred to as the *toric boundary* of $X(\Sigma)$.

For a subset $\Xi \subset N_{\mathbb{R}}$, $L(\Xi) \subset N_{\mathbb{R}}$ referred to as the *linear space associated to* Ξ, denotes the linear subspace spanned by differences $v - w$ for v, w in Ξ.

An *integral affine structure* on an n-dimensional manifold B is given by an open cover $\{U_i\}$ of B along with coordinate charts $\psi_i : U_i \to \mathbb{R}^n$, whose transition functions $\psi_i \circ \psi_j^{-1}$ lie in $\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$. An *integral affine manifold with singularities* is a topological manifold B, admitting an integral affine structure on a subset $B \setminus \Delta \subseteq B$ for some set $\Delta \subset B$ with the property that Δ is a locally finite union of codimension ≥ 2 submanifolds of B.

Given a monoid P, define the monoid ring

$$\mathbb{C}[P] := \bigoplus_{p \in P} \mathbb{C} z^p$$

where z^p is a symbol and multiplication is determined by $z^p \cdot z^{p'} = z^{p+p'}$.

1. The Tate curve and its degeneration

The Tate curve is a degeneration of elliptic curves; a family $T \to D$ where D can be viewed either as the unit disc

$$D = \{u \in \mathbb{C} \mid |u| < 1\}$$

or in the category of schemes,

$$D = \operatorname{Spec} \mathbb{C}[\![u]\!]$$

where $\mathbb{C}[\![u]\!]$ is the ring of formal power series in u with coefficients in $\mathbb{C}$. Throughout this paper, we will restrict our attention to the latter case and view the Tate curve as a curve over the complete discrete valuation ring

$$R := \mathbb{C}[\![u]\!]$$

The construction of the Tate curve is a special case of a construction of Mumford for degenerations of abelian varieties ([Mu]). For the details of the construction we also refer to ([C],8.4).

In this section, we will first build the *unfolded Tate curve* and its degeneration, where both are particular cases of a *toric degeneration* of a toric variety. We first review roughly how this construction works ([GS3],[NS]).

The initial data consists of an integral affine manifold $B \subseteq N_{\mathbb{R}}$, possibly with non-empty boundary ∂B, together with a *polyhedral decomposition* $\mathscr{P}$ of B, which is a covering $\mathscr{P} = \{\Xi\}$ of B by convex polyhedra such that

i. If $\Xi \in \mathscr{P}$ and $\Xi' \subset \Xi$ is a face, then $\Xi' \in \mathscr{P}$.

ii. If $\Xi, \Xi' \in \mathscr{P}$, then $\Xi \cap \Xi'$ is a common face of Ξ and Ξ'.

For each $\Xi \in \mathscr{P}$ let $C(\Xi)$ be the closure of the cone spanned by $\Xi \times \{1\}$ in $N_{\mathbb{R}} \times \mathbb{R}$:

$$C(\Xi) = \overline{\left\{a \cdot (n,1) \,\middle|\, a \geq 0, n \in \Xi\right\}} \tag{1.1}$$

Note that taking the closure here will be important in case Ξ is unbounded. We use the convex polyhedral cone $C(\Xi)$ to define the fan

$$\widetilde{\Sigma}_{\mathscr{P}} := \left\{\sigma \subset C(\Xi) \text{ face} \,\middle|\, \Xi \in \mathscr{P}\right\}$$

which we refer to as the fan associated to $\mathscr{P}$. By construction, the projection onto the second factor

$$N_{\mathbb{R}} \times \mathbb{R} \to (N_{\mathbb{R}} \times \mathbb{R})/N_{\mathbb{R}} = \mathbb{R}$$

defines a non-constant map of fans from the fan $\widetilde{\Sigma}_{\mathscr{P}}$ to the fan $\{0, \mathbb{R}_{\geq 0}\}$ of $\mathbb{A}^1$, hence a flat toric morphism

$$\pi : X \longrightarrow \mathbb{A}^1$$

where X is the toric variety associated to the fan $\widetilde{\Sigma}_{\mathscr{P}}$. Throughout this paper we will say "*X is obtained from a polyhedral decomposition $\mathscr{P}$ of B*" or "*$\pi : X \to \mathbb{A}^1$ is the degeneration associated to* $(\mathscr{P}, B)$" referring to this process of constructing $\pi : X \to \mathbb{A}^1$.

Note that $\pi : X \to \mathbb{A}^1$ is a degeneration of toric varieties. To describe the general fiber of π, we first identify $N_{\mathbb{R}}$ with $N_{\mathbb{R}} \times \{0\} \subset N_{\mathbb{R}} \times \mathbb{R}$ and define the *asymptotic fan* using Lemma 3.3 in [NS] as

$$\Sigma_{\mathscr{P}} = \big\{\sigma \cap (N_{\mathbb{R}} \times \{0\}) \,\big|\, \sigma \in \widetilde{\Sigma}_{\mathscr{P}}\big\}.$$

as a fan in $N_{\mathbb{R}}$. Let $X_{\mathscr{P}}$ be the toric variety associated to the fan $\Sigma_{\mathscr{P}}$. By Lemma 3.4 of [NS] we have

$$\pi^{-1}(\mathbb{A}^1 \setminus \{0\}) = X_{\mathscr{P}} \times (\mathbb{A}^1 \setminus \{0\})$$

and the closed fibers of π over $\mathbb{A}^1 \setminus \{0\}$ are all pairwise isomorphic. Thus, the toric variety

$$X_t := X_{\mathscr{P}}$$

is the general fiber of $\pi : X \to \mathbb{A}^1$. To describe the central fiber

$$\pi^{-1}(0) = X_0$$

observe that for $\Xi \in \mathscr{P}$ the set of adjacent $\Xi' \in \mathscr{P}$ define a fan Σ_Ξ by

$$\Sigma_\Xi = \big\{\mathbb{R}_{\geq 0} \cdot (\Xi' - \Xi) \subset N_{\mathbb{R}}/L(\Xi) \,\big|\, \Xi' \in \mathscr{P}, \Xi \subset \Xi'\big\}. \tag{1.2}$$

in $N_{\mathbb{R}}/L(\Xi)$, where $L(\Xi) \subset N_{\mathbb{R}}$ is the linear subspace associated to Ξ. Let X_Ξ be the toric variety associated to the fan Σ_Ξ. By Proposition 3.5 in [NS], there exist closed embeddings $X_\Xi \to \pi^{-1}(0)$, $\Xi \in \mathscr{P}$ compatible with morphisms $X_\Xi \to X_{\Xi'}$ for $\Xi' \subset \Xi$, inducing an isomorphism

$$\pi^{-1}(0) \simeq \varinjlim_{\Xi \in \mathscr{P}} X_\Xi$$

Now, to construct the unfolded Tate curve, we take $B = \mathbb{R}$ and endow it with a *b-periodic polyhedral decomposition* $\mathscr{P}_b$ defined as follows.

Definition 1.1. Let

$$\Xi_j := [jb, (j+1)b], \ j \in \mathbb{Z}$$

be a closed interval of integral length b in $\mathbb{R}$. Let

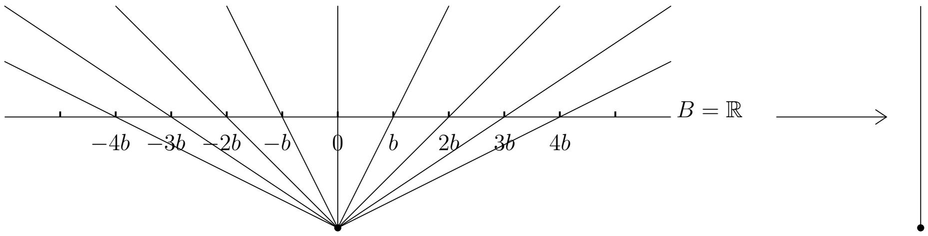
$$\Xi'_j := \Xi_j \cap \Xi_{j+1}$$

be a common face of two such intervals. The *integral b-periodic polyhedral decomposition $\mathscr{P}_b$ of* $\mathbb{R}$ is the covering

$$\mathscr{P}_b := \{\Xi_j\} \cup \{\Xi'_j\}$$

of $\mathbb{R}$, where $j \in \mathbb{Z}$. We refer to each Ξ_j and Ξ'_j as a *face* of $\mathscr{P}_b$. The *maximal cells* of $\mathscr{P}_b$ are the faces Ξ_j, $j \in \mathbb{Z}$ and the *vertices* of $\mathscr{P}_b$ are the 0-faces Ξ'_j, $j \in \mathbb{Z}$.

The following figure illustrates $\widetilde{\Sigma}_{\mathscr{P}_b}$ together with the map $\widetilde{\Sigma}_{\mathscr{P}_b} \to \{0, \mathbb{R}_{\geq 0}\}$.



Note that each face $\Xi_j \in \mathscr{P}_b$ is bounded and the cone $C(\Xi_j)$ is given as

$$C(\Xi_j) = \mathbb{R}_{\geq 0} \cdot (\Xi_j \times \{1\}).$$

The toric fan

$$\widetilde{\Sigma}_{\mathscr{P}_b} := \{\sigma \subset C(\Xi_j) \text{ face} \mid \Xi_j \in \mathscr{P}_b\} \tag{1.3}$$

associated to $\mathscr{P}_b$ has support in $(\mathbb{R} \times \mathbb{R}_{>0}) \cup \{(0,0)\}$. The projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{R} = \mathbb{R}$ onto the second factor defines a map of fans $\widetilde{\Sigma}_{\mathscr{P}_b} \to \{0, \mathbb{R}_{\geq 0}\}$ which induces the morphism

$$\pi : X \longrightarrow \mathbb{A}^1$$

referred to as the *unfolded Tate curve*.

Remark 1.2. The b-periodic polyhedral decomposition $\mathscr{P}_b$ of $\mathbb{R}$ is obtained by rescaling the 1-periodic polyhedral decomposition $\mathscr{P}_1$ of $\mathbb{R}$ by b. The degeneration associated to $(\mathscr{P}_b, \mathbb{R})$ is obtained from the degeneration associated to $(\mathscr{P}_1, \mathbb{R})$ by the base change $u \mapsto u^b$.

Since each $\Xi_j \in \mathscr{P}_b$ is bounded, the asymptotic fan $\Sigma_{\mathscr{P}_b}$ consists of the single point $(0,0) \in \mathbb{R}^2$. Therefore, the degeneration $\pi : X \to \mathbb{A}^1$ associated to $(\mathbb{R}, \mathscr{P}_b)$ satisfies

$$\pi^{-1}(\mathbb{A}^1 \setminus \{0\}) = \mathbb{G}_m \times (\mathbb{A}^1 \setminus \{0\}) \tag{1.4}$$

where $\mathbb{G}_m$ is the algebraic torus and hence the general fiber of $\pi : X \to \mathbb{A}^1$ is

$$X_t = \mathbb{G}_m$$

Define the central fiber X_0 of $\pi : X \to \mathbb{A}^1$ as follows. For each vertex $v \in \mathscr{P}_b$ let

$$\Sigma_v = \{\mathbb{R}_{\geq 0} \cdot (\Xi - v) \subset \mathbb{R} \mid \Xi \in \mathscr{P}, v \in \Xi\}$$

So, Σ_v is the fan of the toric variety

$$X_v = \mathbb{P}^1$$

Similarly, each closed interval $\Xi \in \mathscr{P}_b$ defines a fan Σ_Ξ for the toric variety X_Ξ which is a point of intersection of X_v and $X_{v'}$ where v and v' denote the vertices adjacent to Ξ. Hence, we obtain X_0 as an infinite chain of projective lines $\mathbb{P}^1$, glued pairwise together along an A_{b-1} singularity in X:

$$X_0 := \bigcup_{\infty} \mathbb{P}^1$$

Remark 1.3. Let $B \subseteq N_{\mathbb{R}}$ be an integral affine manifold, endowed with a polyhedral decomposition $\mathscr{P}$ and let $\Xi \in \mathscr{P}$ be a maximal cell. Let $\rho_1, \ldots, \rho_n$ be the edges of the cone $C(\Xi) = \mathbb{R}_{\geq 0} \cdot (\Xi \times \{1\})$ and let u_{ρ_i} be the generator of $\rho_i \cap (N \oplus \mathbb{Z})$, which is referred to as the *ray generator* of ρ_i for $i = 1, \ldots, n$. Then we use the notational convention

$$C(u_{\rho_1}, \ldots, u_{\rho_n}) := \{\lambda_1 u_{\rho_1} + \ldots, +\lambda_n u_{\rho_n} \mid \lambda_1, \ldots, \lambda_n \geq 0\} \subset N_{\mathbb{R}} \times \mathbb{R}$$

for $C(\Xi) = C(u_{\rho_1}, \ldots, u_{\rho_n})$.

Fix a b-periodic polyhedral decomposition $\mathscr{P}_b$ of $\mathbb{R}$ and let $\Xi := [a, a+b] \subset \mathbb{R}$ be a maximal cell of $\mathscr{P}_b$, so that

$$C(\Xi) = C((a,1),(a+b,1)) \tag{1.5}$$

The dual cone $C(\Xi)^\vee \subset M_{\mathbb{R}}$ is given by

$$C(\Xi)^\vee = C((1,-a),(-1,a+b)) \tag{1.6}$$

The generators of the associated monoid ring $\mathbb{C}[C(\Xi)^\vee \cap M]$ are

$$\left\{z^{(1,-a)}, z^{(-1,a+b)}, z^{(0,1)}\right\}$$

We have an isomorphism

$$\begin{aligned} \varphi : \mathbb{C}[C(\Xi)^\vee \cap M] &\longrightarrow \mathbb{C}[x,y,u]/(xy-u^b) \\ z^{(1,-a)} &\longmapsto x \\ z^{(-1,a+b)} &\longmapsto y \\ z^{(0,1)} &\longmapsto u \end{aligned}$$

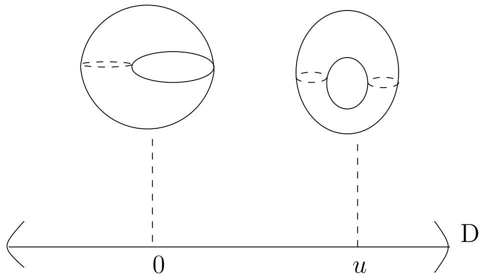
Hence, an affine cover for the total space of the unfolded Tate curve is given by a countable number of copies of

$$\operatorname{Spec} \mathbb{C}[x,y,u]/(xy-u^b) \tag{1.7}$$

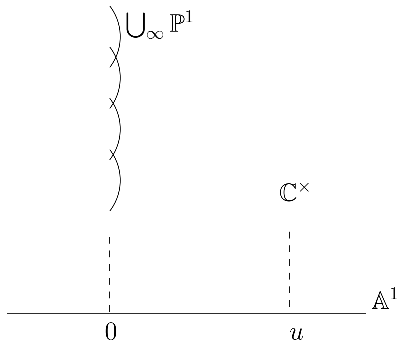
Remark 1.4. To define the Tate curve, observe that $\mathbb{Z}$ acts on the fan $\widetilde{\Sigma}_{\mathscr{P}_b}$ associated to the b-periodic integral polyhedral decomposition $\mathscr{P}_b$ of $\mathbb{R}$, by translation by b. This action induces a $\mathbb{Z}$-action on the unfolded Tate curve $\pi : X \to \mathbb{A}^1$, which is properly discontinuous in the analytic topology once restricting to the unit disc $D = \{u \in \mathbb{C} \mid |u| < 1\}$. Taking the quotient defines the *analytic Tate curve*

$$\Pi : T \to D$$

The fibre T_u of $\Pi : T \to D$ over $u \in D \setminus \{0\}$ is the elliptic curve $E := \mathbb{C}^*/(z \sim u^b \cdot z)$ viewed as a complex manifold. The central fiber T_0 is the nodal elliptic curve.



The analytic Tate curve $T \to D$



The unfolded Tate curve $X \to \mathbb{A}^1$

As we would like to stay on the category of schemes for technical reasons, we first define the formal scheme

$$\hat{X} = \lim X^k$$

where X^k is the k-th order thickening of $\pi^{-1}(0)$, that is, the subscheme of the unfolded Tate curve $\pi : X \to \mathrm{Spec}[u]$ defined by the equation $u^{k+1} = 0$. Then the quotient $\hat{X}/\mathbb{Z}$ makes sense as a formal scheme. There is a map of formal schemes

$$\hat{\mathfrak{T}} : \hat{X}/\mathbb{Z} \to \hat{\mathbb{A}}$$

induced by π. Here $\hat{\mathbb{A}} = \mathrm{Spf}\mathbb{C}[\![u]\!]$ is the ringed space consisting of the point 0 and the ring $\mathbb{C}[\![u]\!]$. Since, there is an ample line bundle $\mathcal{L}$ over $\hat{X}/\mathbb{Z}$ ([C], pg. 620), Grothendieck's Existence Theorem ([Gr], 5.4.5) ensures that $\hat{X}/\mathbb{Z}$ is obtained by the formal completion of the genuine scheme

$$\mathfrak{T} \to \mathrm{Spec}\,\mathbb{C}[[u]]$$

which we refer to as the *Tate curve.*

Note that the generic fibre of the Tate curve in this case, is an elliptic curve over $\mathbb{C}((u))$ and the central fiber is the nodal elliptic curve.

In section 6, we will see that the invariants on the Tate curve lift uniquely to the unfolded case, so we will disregard the $\mathbb{Z}$-quotient for computational convenience in the next sections.

Next we will construct a degeneration of the unfolded Tate curve. For this we first need the following definitions.

Definition 1.5. Let $B \subset N_{\mathbb{R}}$ be an integral affine manifold endowed with a polyhedral decomposition $\mathscr{P}$. For a cell $\Xi \in \mathscr{P}$, let $C(\Xi)$ be as in (1.5). If $\Xi_1 \subset \Xi_2$ taking cones we obtain $C(\Xi_1) \subseteq C(\Xi_2)$. Now, we can define the cone $C(B)$ over B as

$$C(B) = \bigcup_{\Xi \in \mathscr{P}} C(\Xi)$$

Note that $C(B) \subseteq N_{\mathbb{R}} \times \mathbb{R}$ admits an integral affine structure with a singularity at the origin in $N_{\mathbb{R}} \times \mathbb{R}$ ([GHKS] , 4.11). The *truncated cone* $\overline{C}B$ over B is the manifold with boundary with underlying topological space

$$\overline{C}B := \{(x,h) \in C(B) \mid h \geq 1\}$$

in the cone $C(B)$, endowed with the induced affine structure. It admits a polyhedral decomposition $\overline{C}\mathscr{P}$ with cells

$$\overline{C}\Xi := \{(x,h) \in C(\Xi) \mid h \geq 1, \Xi \in \mathscr{P}\}$$

The maximal cells of $\overline{C}\mathscr{P}$ are the cells $\overline{C}\Xi$ such that Ξ is a maximal cell of $\mathscr{P}$.

Now, our initial data to build the *degeneration of the unfolded Tate curve* is given by the tuple $(\overline{C}\mathbb{R}, \overline{C}\mathscr{P}_b)$. Here, $\overline{C}\mathbb{R}$ is the truncated cone over $\mathbb{R}$ endowed with the polyhedral decomposition $\overline{C}\mathscr{P}_b$ with maximal cells $\overline{C}\Xi$, where $\mathscr{P}_b$ is the b-periodic polyhedral decomposition of $\mathbb{R}$. Let $C(\overline{C}\Xi)$ be the cone over $\overline{C}\Xi$ The toric fan

$$\widetilde{\Sigma}_{\overline{C}\mathscr{P}_b} := \{\sigma \subset C(\overline{C}\Xi) \text{ face} \mid \overline{C}\Xi \in \overline{C}\mathscr{P}_b\}$$

has support in

$$\mathbb{R}_{\geq 0} \cdot (\mathbb{R} \times \mathbb{R}_{\geq 1} \times \{1\})$$

The projection map

$$\begin{aligned} (\mathrm{pr}_2, \mathrm{pr}_3) : \mathbb{R} \times \mathbb{R} \times \mathbb{R} &\longrightarrow \mathbb{R} \times \mathbb{R} \\ (x,y,z) &\longmapsto (x,y) \end{aligned}$$

onto the second and third factors defines a map of fans

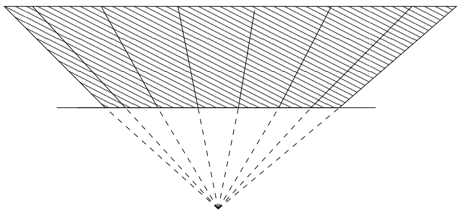
$$(\mathrm{pr}_2, \mathrm{pr}_3) : \widetilde{\Sigma}_{\overline{C}\mathscr{P}_b} \longrightarrow \{0, \mathbb{R}_{\geq 0}\} \times \{0, \mathbb{R}_{\geq 0}\}$$

(1.8)

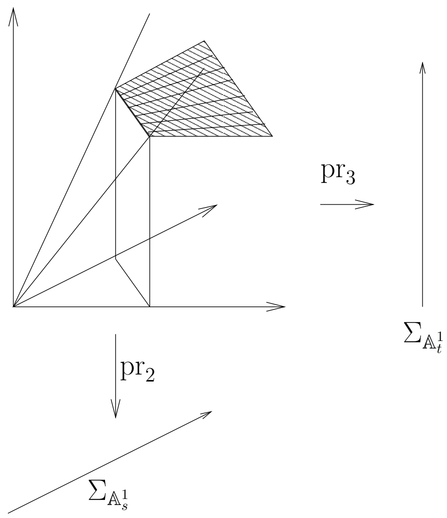
which induces the morphism

$$\tilde{\pi} : Y \to \operatorname{Spec} \mathbb{C}[s,t]$$

referred to as the *degeneration of the unfolded Tate curve*, where Y is the toric variety associated to the fan $\widetilde{\Sigma}_{\overline{C}\mathscr{P}_b}$. The following figure illustrates $\widetilde{\Sigma}_{\overline{C}\mathscr{P}_b}$ together with the projection map $(\mathrm{pr}_2, \mathrm{pr}_3)$.



The truncated cone $\overline{C}(\Xi)$



The cone $C(\overline{C}(\Xi))$ over the truncated cone

The central fiber Y_0 of the degeneration of the unfolded Tate curve $Y \to \operatorname{Spec} \mathbb{C}[s,t]$ over $t = 0$ is constructed as follows. For any cell $\overline{C}\Xi \in \overline{C}\mathscr{P}_b$, define $\Sigma_{\overline{C}\Xi}$ analogously to (1.2) and denote by $Y_{\overline{C}\Xi}$ the toric variety associated to $\Sigma_{\overline{C}\Xi}$. Then, for a vertex $v \in \mathscr{P}_b$, the truncated cone $\overline{C}v$ over v is the fan of the toric variety

$$Y_{\overline{C}v} = \mathbb{P}^1 \times \mathbb{A}^1$$

Recall that any maximal cell $\Xi \in \mathscr{P}_b$ is a closed interval of $\mathbb{R}$. The fan $\Sigma_{\overline{C}\Xi}$ associated to $\overline{C}\Xi$ is the fan of the toric variety

$$Y_{\Xi} = \{q\} \times \mathbb{A}^1$$

where $\{q\} \times \mathbb{A}^1$ is a component of the singular locus, given by the intersection of $Y_{\overline{C}v}$ and $Y_{\overline{C}v'}$ for the cells $\overline{C}v, \overline{C}v' \in \overline{C}\mathscr{P}_b$ adjacent to $\overline{C}\Xi$. Hence, Y_0 is obtained as the product of the affine line with the central fiber X_0 of the unfolded Tate curve. So, we have

$$Y_0 := \mathbb{A}^1_s \times \bigcup_{\infty} \mathbb{P}^1 = \mathbb{A}^1_s \times X_0$$

Now, let $\Xi := [a, a+b]$ be a maximal cell of $\mathscr{P}_b$ so that

$$C(\overline{C}(\Xi)) = C((a,1,1),(a+b,1,1),(a+b,1,0),(a,1,0))$$

Its dual $C(\overline{C}(\Xi))^{\vee}$ is given by

$$C(\overline{C}(\Xi))^{\vee} = C((0,1,-1),(-1,a+b,0),(0,0,1),(1,-a,0)) \tag{1.9}$$

We have an isomorphism

$$\begin{array}{rcl}\widetilde{\varphi}:\mathbb{C}[C(\overline{C}\Xi)^\vee\cap(M\oplus\mathbb{Z})] & \longrightarrow & \mathbb{C}[x,y,s,t]/(xy-(st)^b)\\ z^{(1,-a,0)} & \longmapsto & x\\ z^{(-1,a+b,0)} & \longmapsto & y\\ z^{(0,1,-1)} & \longmapsto & s\\ z^{(0,0,1)} & \longmapsto & t\end{array}$$

Hence, an affine cover for the total space Y of the degeneration of the unfolded Tate curve is given by a countable number of copies of

$$\operatorname{Spec}\mathbb{C}[x,y,s,t]/(xy-(st)^b \tag{1.10}$$

The main theorem of this section is the following.

Theorem 1.6. *Let $\pi:X\to\operatorname{Spec}\mathbb{C}[u]$ and $\tilde{\pi}:Y\to\operatorname{Spec}\mathbb{C}[s,t]$ be the degenerations associated to $(\mathscr{P}_b,\mathbb{R})$ and $(\overline{C}\mathscr{P}_b,\overline{C}\mathbb{R})$ respectively. Then, $\tilde{\pi}:Y\to\operatorname{Spec}\mathbb{C}[s,t]$ is obtained from $\pi:X\to\operatorname{Spec}\mathbb{C}[u]$ by the base change $u\mapsto st$.*

Proof. Let $\Xi:=[a,a+b]\subset\mathbb{R}$ be a maximal cell of $\mathscr{P}_b$ and let $\overline{C}\mathscr{P}_b$ be the corresponding maximal cell in $\overline{C}\Xi$. Then, we have the projection map

$$\begin{array}{rcl}(\mathrm{pr}_1,\mathrm{pr}_2):N_{\mathbb{R}}\times\mathbb{R}\times\mathbb{R} & \longrightarrow & N_{\mathbb{R}}\times\mathbb{R}\\ C(\overline{C}\Xi) & \longmapsto & C(\Xi)\end{array}$$

whose dual induces the embedding

$$\begin{array}{c}j:C(\Xi)^\vee\hookrightarrow C(\overline{C}\Xi)^\vee\\ (m_1,m_2)\mapsto(m_1,m_2,0)\end{array}$$

With Equation 1.6 and Equation 1.9, we obtain

$$C(\overline{C}\Xi)^\vee=j(C(\Xi)^\vee)+\mathbb{R}_{\geq0}(0,1,-1)+\mathbb{R}_{\geq0}(0,0,1)$$

Let

$$\phi_j:\mathbb{C}[C(\Xi)^\vee\cap M]\to\mathbb{C}[C(\overline{C}\Xi)^\vee\cap M\oplus\mathbb{Z}]$$

be the map induced by $j:C(\Xi)^\vee\hookrightarrow C(\overline{C}\Xi)^\vee$ on the level of monoid rings. Explicitly, we have

$$\begin{array}{rcl}\phi_j:\mathbb{C}[C(\Xi)^\vee\cap M] & \longrightarrow & \mathbb{C}[C(\overline{C}\Xi)^\vee\cap M\oplus\mathbb{Z}]\\ x:=z^{(1,-a)} & \longmapsto & z^{(1,-a,0)}=x\\ y:=z^{(-1,a+b)} & \longmapsto & z^{(-1,a+b,0)}=y\\ u:=z^{(0,1)} & \longmapsto & z^{(0,1,0)}=z^{(0,1,-1)}\cdot z^{(0,0,1)}=st\end{array}$$

Hence, we obtain

$$\begin{aligned}
\widetilde{\varphi} \circ \phi_j \circ \varphi : \mathbb{C}[x,y,u]/(xy-u^b) &\longrightarrow \mathbb{C}[x,y,s,t]/(xy-(st)^b) \\
x &\longmapsto x \\
y &\longmapsto y \\
u &\longmapsto st
\end{aligned}$$

where φ is the isomorphism defined in 1.7 and $\widetilde{\varphi}$ is the isomorphism defined in 1.10. Note that the projection $(\mathrm{pr}_1, \mathrm{pr}_2) : N_{\mathbb{R}} \times \mathbb{R} \times \mathbb{R} \to N_{\mathbb{R}} \times \mathbb{R}$ defines a map of fans from the fan $\widetilde{\Sigma}_{\mathscr{P}_b}$ defined in 1.3 and $\widetilde{\Sigma}_{\overline{C}\mathscr{P}_b}$ defined in 1.8. Hence, the compatibility of the gluing of affine patches follows by Theorem 1.13 in [Oda]. Therefore, we obtain $\tilde{\pi} : Y \to \operatorname{Spec} \mathbb{C}[s,t]$ from $\pi : X \to \operatorname{Spec} \mathbb{C}[u]$ by the base change $u \mapsto st$. $\square$

Remark 1.7. Consider the Tate curve

$$\mathfrak{T} \to \operatorname{Spec} \mathbb{C}[\![u]\!]$$

The base change $u \mapsto st$ induces the map

$$\mathbb{C}[\![u]\!] \to \mathbb{C}[s][\![t]\!]$$

where $\mathbb{C}[s][\![t]\!]$ is the ring of formal power series in t, admitting coefficients in the polynomial algebra $\mathbb{C}[s]$. So, we obtain the degeneration

$$\widetilde{\mathfrak{T}} \longrightarrow \mathbb{C}[s][\![t]\!]$$

of the Tate curve in the category of schemes. It will be important to have the s-variable not as a formal variable if one wants to consider deformation theory of the log geometric invariants on the central fiber over $t = 0$ of the degeneration of the Tate curve, which we introduce in the next sections.

Our final aim in this section is to investigate the charts for the log structure $\alpha_{Y_0} : \mathcal{M}_{Y_0} \to \mathcal{O}_{Y_0}$ on the central fiber $Y_0 \to \operatorname{Spec} \mathbb{C}[s]$ over $t = 0$ of the degeneration of the unfolded Tate curve. For the definition of a log structure and a chart for a log structure see A.11 and A.28. We will refer to the charts for the log structure on Y_0 often in the next sections, to study the log geometric invariants.

Recall that the total space of the degeneration of the unfolded Tate curve is the toric variety $Y \to \operatorname{Spec} \mathbb{C}[s,t]$ associated to the fan 1.8. We endow Y with the divisorial log structure $\alpha_Y : \mathcal{M}_Y \to \mathcal{O}_Y$ defined as in A.17. Here, we take the divisor

$$\tilde{D} := \tilde{\pi}^{-1}(st = 0) \subset Y$$

Following the discussion in the appendix A.29, we use the following toric charts for the log structure on Y throughout the text. The fan describing the toric variety containing Y consisted of the origin 0 and of cones $C(\sigma)$ over cells σ of $\overline{C}\mathscr{P}_b$, where $\mathscr{P}_b$ is a b-periodic polyhedral decomposition of $\mathbb{R}$. The origin yields a trivial chart for the complement of Y in the toric variety and is irrelevant for our considerations. The maximal cells of $\overline{C}\mathscr{P}_b$ are of the form $\overline{C}\Xi$ for Ξ an interval in $\mathbb{R}$ of length b, embedded in the lower boundary of $B = \overline{C}\mathbb{R}$. Thus (A.1) provides a covering system of charts of Y of the form

$$(C(\overline{C}\Xi))^\vee \cap (M \oplus \mathbb{Z}) \longrightarrow \Gamma(U_i, \mathcal{M}_Y), \tag{1.11}$$

where $U_i \subset Y$ is the open subset $\operatorname{Spec} \mathbb{C}[(C(\overline{C}\Xi))^\vee \cap (M \oplus \mathbb{Z})]$ of Y defined by $\overline{C}\Xi_i$.

Explicitly a chart for the log structure $\mathcal{M}_Y$ is given by the map

$$\begin{aligned} C(\overline{C}\Xi)^\vee \cap (M \oplus \mathbb{Z}) &\longrightarrow \mathbb{C}[x,y,s,t]/(xy - (st)^b) \\ (1,-a,0) &\longmapsto x \\ (-1,a+b,0) &\longmapsto y \\ (0,1,-1) &\longmapsto s \\ (0,0,1) &\longmapsto t \end{aligned} \tag{1.12}$$

So, by Discussion A.29, we obtain the following canonical description of the stalks of $\overline{\mathcal{M}}_Y$.

Proposition 1.8. *Let $\tau \subset B$ be a cell in the polyhedral decomposition $\overline{C}\mathscr{P}$ of B and $T_\tau \subset Y$ the torus of the corresponding toric stratum of Y. Then for $x \in T_\tau$, the map* (1.11) *induces a canonical isomorphism*

$$\overline{\mathcal{M}}_{Y,x} \simeq \big((C(\tau))^\vee \cap (M \oplus \mathbb{Z})\big)/\big((C(\tau))^\perp \cap (M \oplus \mathbb{Z})\big).$$

Since the log structure $\mathcal{M}_Y$ is fine, $\overline{\mathcal{M}}_Y$ is constant along open toric strata. The monoid $\overline{\mathcal{M}}_Y^{gp}$ is constructible and $\overline{\mathcal{M}}_Y \subset \overline{\mathcal{M}}_Y^{gp}$ is given by generization maps between stalks of strata $\overline{\mathcal{M}}_{Y,\eta}$, where η denotes the generic point of the irreducible component of a toric strata in Y (Proposition 1.1, [SS]). This holds also for the pull-back log structure $\mathcal{M}_{Y_0}$ on the central fiber Y_0 over $t = 0$. Indeed, after restricting the chart for $\mathcal{M}_Y$ to $t = 0$ we obtain a chart for the log structure $\mathcal{M}_{Y_0}$ on the central fiber Y_0 over $t = 0$.

Remark 1.9. To save notation let us write

$$C_{\mathbb{Z}} := \{p \in C \cap \mathbb{Z}^n \mid C \subset \mathbb{R}^n\}$$

for the integral points of a cone C in a finitely generated free abelian group. Then, the sections of the ghost sheaf

$$\overline{\mathcal{M}}_Y := \mathcal{M}_Y / \mathcal{O}_Y^\times$$

are in one-to-one correspondence with the points $p \in C(\overline{C}\Xi)^\vee_{\mathbb{Z}}$, since for each $p \in C(\overline{C}\Xi)^\vee_{\mathbb{Z}}$ the regular function

$$z^p \in \mathbb{C}[x,y,s,t]/(xy-(st)^b)$$

is invertible away from the toric boundary of the affine toric variety

$$Y_i = \operatorname{Spec}\mathbb{C}[x,y,s,t]/(xy-(st)^b) = \operatorname{Spec}\mathbb{C}[C(\overline{C}\Xi_i)^\vee_{\mathbb{Z}}]$$

corresponding to a cell $\overline{C}\Xi_i$ of $\overline{C}\mathscr{P}$. We use the following notational convention. Given an integral monoid P and a point $p \in P$ such that $z^p \in \mathbb{C}[P]$, we denote by s_p the corresponding section in $\mathcal{M}_Y$ and by $\overline{z^p}$ the section in $\overline{\mathcal{M}}_Y$ obtained as the image of s_p under the quotient map $\kappa : \mathcal{M}_Y \to \overline{\mathcal{M}}_Y$.

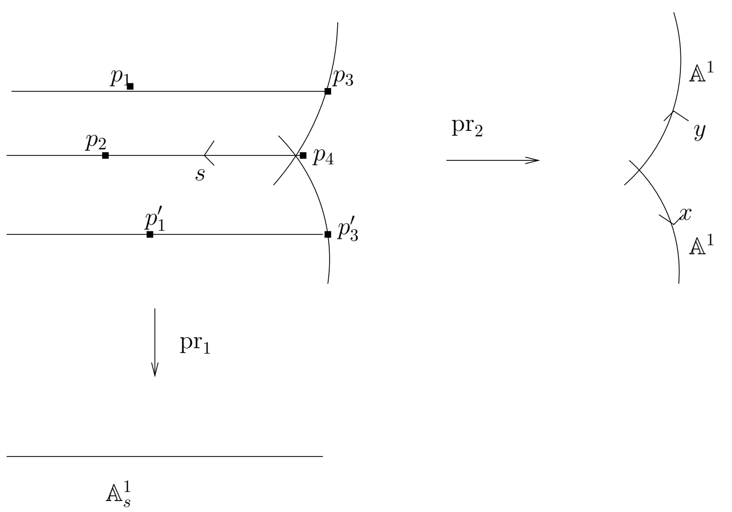
We will describe the stalks of the ghost sheaf $\overline{\mathcal{M}}_{Y_0}$ on Y_0 more explicitly in the remaining part of this section. First recall the description (1.10) of the affine cover for Y which induces by restricting to $t=0$ an affine cover of Y_0 that is given by a countable union of the open sets

$$\mathcal{U} = \operatorname{Spec}\mathbb{C}[x,y,s,t]/(xy)$$

The following figure illustrates $\mathcal{U} \subset Y_0 = \mathbb{A}^1_s \times X_0$ together with the projection maps

$$\begin{aligned} \mathrm{pr}_1 &: Y_0 \longrightarrow \mathbb{A}^1_s \\ \mathrm{pr}_2 &: Y_0 \longrightarrow \bigcup_{\infty} \mathbb{P}^1 \end{aligned}$$

onto the first and second coordinates.



We have four different types of points on $\mathcal{U}$ and the stalks of $\mathcal{M}_{Y_0}$ for each type are given as follows.

$$\begin{array}{lclcrcc}
p_1 & \in & \mathcal{U} \setminus \{(y=0) \cup (s=0)\} & \Longrightarrow & \overline{\mathcal{M}}_{Y_0,p_1} & = & \langle \bar{t} \rangle \\
p_1' & \in & \mathcal{U} \setminus \{(x=0) \cup (s=0)\} & \Longrightarrow & \overline{\mathcal{M}}_{Y_0,p_1'} & = & \langle \bar{t} \rangle \\
p_2 & = & x=y=0 & \Longrightarrow & \overline{\mathcal{M}}_{Y_0,p_2} & = & \langle \overline{x}, \overline{y}, \bar{t} \mid \overline{xy} = \bar{t}^b \rangle \\
p_3 & = & x=s=0 & \Longrightarrow & \overline{\mathcal{M}}_{Y_0,p_3} & = & \langle \overline{x}, \overline{s}, \bar{t} \mid \overline{x} = \overline{s}\bar{t}^b \rangle \\
p_3' & = & y=s=0 & \Longrightarrow & \overline{\mathcal{M}}_{Y_0,p_3'} & = & \langle \overline{y}, \overline{s}, \bar{t} \mid \overline{y} = \overline{s}\bar{t}^b \rangle \\
p_4 & = & x=y=s=0 & \Longrightarrow & \overline{\mathcal{M}}_{Y_0,p_4} & = & \langle \overline{x}, \overline{y}, \overline{s}, \bar{t} \mid \overline{xy} = (\overline{s}\bar{t})^b \rangle
\end{array}$$

Remark 1.10. In the above description of the stalks we use the following notational convention. When we write $\overline{\mathcal{M}}_{Y_0,p_i} = \langle \bar{t} \rangle$, for $i = 1, 2$, this means there is an isomorphism

$$\begin{array}{ccc}
\mathbb{N} & \longrightarrow & \overline{\mathcal{M}}_{Y_0,p_1} \\
1 & \longmapsto & \bar{t}
\end{array}$$

Analogously, we present a monoid with a set of generators G and relations R among elements of G by $\langle G \mid R \rangle$.

2. A tropical counting problem

2.1. **Tropical corals.** Let $\bar{\Gamma}$ be a finite, connected 1-dimensional simplicial complex. Denote the set of vertices of $\bar{\Gamma}$ by $V(\bar{\Gamma})$ and the subset of vertices of valency k in $V(\bar{\Gamma})$ by $V_k(\bar{\Gamma})$. The set of edges of $\bar{\Gamma}$ is denoted by $E(\bar{\Gamma})$. Consider the additional datum of a function

$$w_{\bar{\Gamma}} : E(\bar{\Gamma}) \to \mathbb{N} \setminus \{0\}$$

called the weight function on $E(\bar{\Gamma})$. The image of $e \in E(\bar{\Gamma})$ under w is referred to as the *weight of* e. The set of vertices adjacent to an edge $e \in E(\bar{\Gamma})$ is denoted by ∂e. A *bilateral graph* is the geometric realization of $\bar{\Gamma}$ such that:

(i) $\bar{\Gamma}$ has no divalent vertices.

(ii) There are sets of vertices

$$V^+(\bar{\Gamma}) := \{\mathrm{v}_1^+, \cdots, \mathrm{v}_l^+\}$$

referred to as *the set of positive vertices* and

$$V^-(\bar{\Gamma}) := \{\mathrm{v}_1^-, \cdots, \mathrm{v}_m^-\}$$

referred to as *the set of negative vertices* such that

$$V(\bar{\Gamma}) = V^+(\bar{\Gamma}) \ \amalg V^0(\bar{\Gamma}) \ \amalg \ V^-(\bar{\Gamma})$$

where $V^0(\bar{\Gamma})$ is referred to as *the set of interior vertices.*

(iii) All positive vertices are univalent:

$$V^+(\bar{\Gamma}) \subseteq V_1(\bar{\Gamma})$$

and the set of edges

$$E^+(\bar{\Gamma}) := \{e_1^+, \ldots, e_l^+ \mid \partial e_i^+ \cap v_i^+ \neq \emptyset\} \subset E(\bar{\Gamma})$$

is referred to as the set of *positive* edges of $\bar{\Gamma}$.

(iv) Let

$$V_k^-(\bar{\Gamma}) := V^-(\bar{\Gamma}) \cap V_k(\bar{\Gamma})$$

be the set of negative vertices of $\bar{\Gamma}$ with valency k. Then, the set of all univalent vertices of $\bar{\Gamma}$ is

$$V_1(\bar{\Gamma}) = V_1^-(\bar{\Gamma}) \amalg V^+(\bar{\Gamma})$$

For each $v \in V_1^-(\bar{\Gamma})$, let e_v be the edge adjacent to v. Throughout this section we omit the case where the cardinalities of both $V_1^-(\bar{\Gamma})$ and of $E(\bar{\Gamma})$ are one, as it can be treated easily in all the arguments we use. So, by connectivity of $\bar{\Gamma}$ for each $v \in V_1^-(\Gamma)$, there exist $v' \in V^0(\Gamma)$ such that

$$\partial e_v = \{v, v'\}$$

referred to as the *interior vertex associated to* v.

(v) The first Betti number of $\bar{\Gamma}$ is zero.

Remark 2.1. Note that condition (v) is necessary only if one is interested in rational curve counts on the algebraic side that we will discuss in the next sections, but it can be omitted to study more general cases.

Let $\bar{\Gamma}$ be a bilateral graph and let $|\bar{\Gamma}|$ be the geometric realization of $\bar{\Gamma}$. Define the non-compact geometric realization of $\bar{\Gamma}$ with positive vertices removed as

$$\Gamma := |\bar{\Gamma}| \setminus V^+(\bar{\Gamma}) \tag{2.1}$$

referred to as a *coral graph.* Note that Γ admits half-edges

$$E^+(\Gamma) := \{e_i \mid e_i = e_i^+ \setminus \{v_i^+\} \text{ where } e_i^+ \in E^+(\bar{\Gamma}) \text{ and } \partial e_i^+ = v_i^+ \in V^+(\bar{\Gamma})\}$$

referred to as the set of *positive edges* of Γ. A *k-labelled coral graph* denoted by $(\Gamma, \mathbb{E})$ is a coral graph Γ together with a choice of an ordered k-tuple of positive edges

$$\mathbb{E} = (E_1, \ldots, E_k) \subset E^+(\Gamma)$$

We sometimes say $(\Gamma, \mathbb{E})$ is a labelled coral graph if it is k-labelled for $k \in \mathbb{N} \setminus 0$.

The sets of vertices and edges of Γ are

$$\begin{aligned} V(\Gamma) &= V(\bar{\Gamma}) \setminus \{\mathrm{v}_1^+, \cdots, \mathrm{v}_l^+\} \\ E(\Gamma) &= \big(E(\bar{\Gamma}) \setminus E^+(\bar{\Gamma})\big) \cup E^+(\Gamma) \end{aligned}$$

The set

$$E^b(\Gamma) := E(\Gamma) \setminus E^+(\Gamma)$$

is referred to as the set of *bounded edges* of Γ.

The set

$$V^-(\Gamma) := V^-(\bar{\Gamma})$$

is referred to as the set of *negative vertices* of Γ.

Note that Γ is endowed with a weight function $w : E(\Gamma) \to \mathbb{N} \setminus \{0\}$ defined by

$$w = \begin{cases} w_{\bar{\Gamma}} & \text{on } E(\bar{\Gamma}) \setminus E^+(\bar{\Gamma}) \\ w_{\bar{\Gamma}}(e_i^+) & \text{on } e_i \in E^+(\Gamma), \text{ for } i = 1, \ldots, l \end{cases}$$

Definition 2.2. Let (Γ, w) be a coral graph Γ endowed with a weight function $w : E(\Gamma) \to \mathbb{N} \setminus \{0\}$. A *parameterized tropical coral* in $\overline{C}\mathbb{R}$ is a *proper* map

$$h : \Gamma \to \overline{C}\mathbb{R}$$

satisfying the following:

(i) For all $e \in E(\Gamma)$, the restriction $h|_e$ is an embedding and $h(E)$ is contained in an integral affine submanifold of $\overline{C}\mathbb{R}$.

(ii) For all $v \in V^0(\bar{\Gamma})$;

$$h(v) \in \overline{C}\mathbb{R} \setminus \partial\overline{C}\mathbb{R}$$

where $\partial\overline{C}\mathbb{R}$ denotes the boundary of the truncated cone $\overline{C}\mathbb{R}$. Moreover, the following balancing condition holds:

$$\sum_{j=1}^{k} w(e_j)u_j = 0$$

where

$$\{e_1, \ldots, e_k \in E(\Gamma) \mid v \in \partial e_j \text{ for each } j = 1, \ldots, k\}$$

is the set of edges adjacent to v, $w(e_i) \in \mathbb{N} \setminus \{0\}$ is the weight on e_j and $u_j \in N$ is the primitive integral vector emanating from $h(v)$ in the direction of $h(e_j)$ for $j = 1, \ldots, k$.

(iii) For all $v \in V^-(\Gamma)$;

$$h(v) \subset\in \overline{C}\mathbb{R}$$

and there exist $w_v \in \mathbb{N} \setminus \{0\}$ associated to v, referred to as the *weight on* v such that the following balancing condition holds:

$$w_v \cdot u_v + \sum_{j=1}^{n} w(e_j)u_j = 0$$

$$\{e_1, \ldots, e_k \in E(\Gamma) \mid v \subset \partial e_j \text{ for each } j = 1, \ldots, k\}$$

is the set of edges adjacent to v and $u_v \in N$ is the primitive integral vector emanating from $h(v)$ in the direction of the origin in $N_{\mathbb{R}}$.

(iv) For all $e \in E^+(\Gamma)$, the restriction to $h(e)$ of the projection map

$$\mathrm{pr}_2 : \overline{C}\mathbb{R} \to [1, \infty)$$

onto the second factor is proper.

An *isomorphism* of tropical corals $h : \Gamma \to \overline{C}\mathbb{R}$ and $h' : \Gamma' \to \overline{C}\mathbb{R}$ is a homeomorphism $\Phi : \Gamma \to \Gamma'$ respecting the weights of the edges and such that $h = h' \circ \Phi$. A *tropical coral* is an isomorphism class of parameterized tropical corals. A *k-labelled tropical coral* denoted by

$$(\Gamma, \mathbb{E}, h)$$

is a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ together with a choice of an ordered k-tuple of positive edges

$$\mathbb{E} = (e_1, \ldots, e_k) \text{ with } e_i \subset E^+(\Gamma) \text{ for } i = 1, \ldots, k$$

Remark 2.3. The definition of a tropical coral can be generalized to tropical corals in

$$(\overline{C}B, \overline{C}\mathscr{P})$$

for any integral affine manifold B endowed with a polyhedral decomposition $\overline{C}\mathscr{P}$. Indeed, to study the invariants of the Tate curve, we shall apply the quotient given by the $\mathbb{Z}$-action on $(\mathbb{R}, \mathscr{P}_b)$ and work over

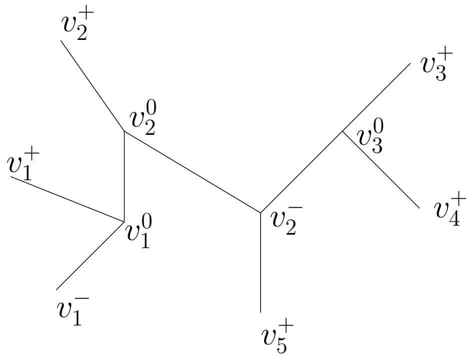
$$B := \overline{C}(\mathbb{R}/\mathbb{Z}) = \overline{C}S^1$$

Condition (iv) of the definition 2.2 will then ensure that there is no infinite wrapping of the unbounded edges of tropical corals in $\overline{C}S^1$. We ignore the $\mathbb{Z}$-quotient for the time being for computational purposes, since the tropical corals in $\overline{C}S^1$ lift to $\overline{C}\mathbb{R}$, as well as the corresponding log geometric invariants lift to the unfolded Tate curve as we will see in section 6.

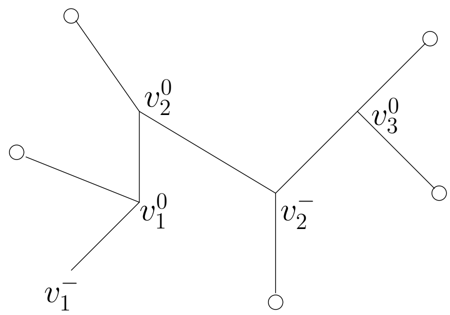
Example 2.4. The following figure illustrates a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ with

$$\begin{aligned} V^-(\Gamma) &= \{v_1^-, v_2^-\} \\ V^0(\Gamma) &= \{v_1^0, v_2^0, v_3^0\} \end{aligned}$$

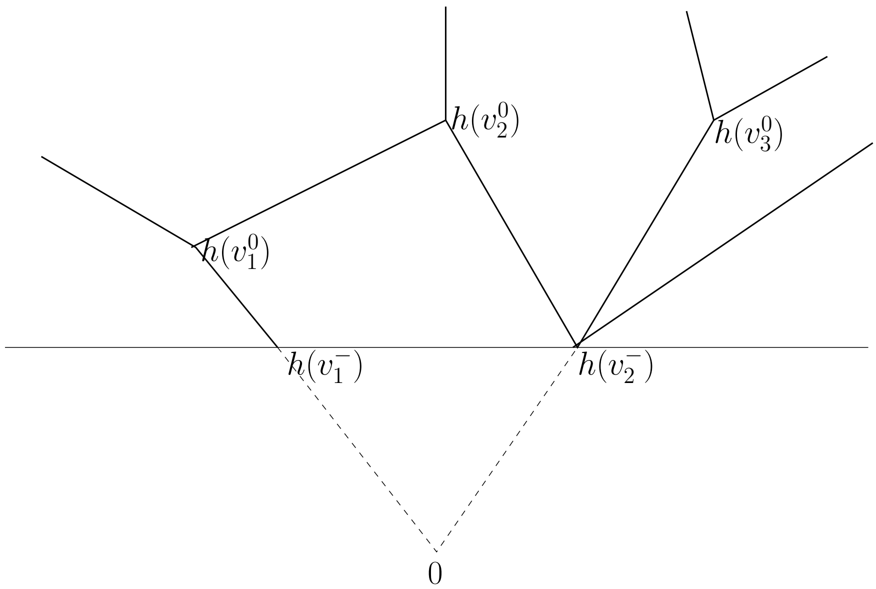
and $|E^+(\Gamma)| = 4$. The origin in $N_{\mathbb{R}}$ is labelled by 0.



A bilateral graph $\bar{\Gamma}$



The coral graph $\Gamma = \bar{\Gamma} \setminus V^+(\Gamma)$

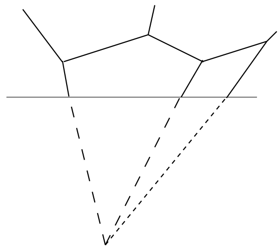


A tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$

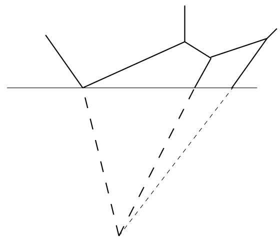
Definition 2.5. We call a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ *general* if the following conditions hold:

(i) All vertices $v \in V^0(\Gamma)$ are trivalent.
(ii) All vertices $v \in V^-(\Gamma)$ are univalent.

Otherwise it is called *degenerate*. In the following picture we illustrate a general and a degenerate tropical coral.



A general tropical coral



A degenerate tropical coral

2.2. **Incidences for tropical corals.**

Definition 2.6. Let $(\Gamma, \mathbb{E})$ be a labelled coral graph. Let

$$F(\Gamma) = \{(v, e) \,|\, e \in E(\Gamma) \text{ and } v \in \partial e\}$$

be the set of *flags* of Γ. The tuple (Γ, u) consisting of Γ and a map

$$\begin{aligned} u : F(\Gamma) \ \amalg \ V^-(\Gamma) &\longrightarrow N \\ F(\Gamma) \ni (v, e) &\longmapsto \quad u_{v,e} \\ V^-(\Gamma) \ni v &\longmapsto \quad u_v \end{aligned}$$

where u_v and $u_{v,e}$ are primitive integral vectors in N, is referred to as the *type of* Γ.

Definition 2.7. The *type of a tropical coral* $(\Gamma, \mathbb{E}, h)$ denoted by (Γ, u) by suppressing h and $\mathbb{E}$ in the notation, is the type of $(\Gamma, \mathbb{E})$ where the map $u : F(\Gamma) \ \amalg \ V^-(\Gamma) \to N$ is given by assigning to each $(u, v) \in F(\Gamma)$ the primitive integral vector $u_{v,e} \in N$ emanating from $h(v)$ in the direction of $h(e)$ and assigning to each $v \in V^-(\Gamma)$ the primitive integral vector $u_v \in N$ emanating from $h(v)$ in the direction of the origin.

We denote the set of tropical corals of type (Γ, u) by

$$\mathfrak{T}_{(\Gamma,u)} := \{h : \Gamma \to \overline{C}\mathbb{R} \text{ tropical coral} \mid h \text{ has type } (\Gamma, u)\}$$

To set up the counting problem we need to define *tropical incidence conditions*,

$$(\Delta, \lambda) \tag{2.2}$$

where Δ is referred to as the *degree* and λ is referred to as an *asymptotic constraint*, which we define in a moment.

Throughout the next sections we use the following conventions:

$$\begin{aligned} N_0 &:= \{n \in N \mid \operatorname{pr}_2(n) = 0\} \\ N_{>0} &:= \{n \in N \mid \operatorname{pr}_2(n) > 0\} \\ N_{<0} &:= \{n \in N \mid \operatorname{pr}_2(n) < 0\} \end{aligned}$$

where $N = \mathbb{Z}^2$. Moreover, given a set $\mathcal{A}$ we denote by $|\mathcal{A}|$ the cardinality of $\mathcal{A}$. Now we are ready to define the *degree* Δ of a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$, as a map $\Delta : N \setminus N_0 \to \mathbb{N}$.

Definition 2.8. The *degree of a type* (Γ, u) of a coral graph Γ, denoted by

$$\Delta := (\overline{\Delta}, \underline{\Delta})$$

is the map $\Delta : N \setminus N_0 \to \mathbb{N}$ of finite support given by

$$\Delta = \begin{cases} \overline{\Delta} & \text{on } N_{>0} \\ \underline{\Delta} & \text{on } N_{<0} \end{cases}$$

where

$$\overline{\Delta}(n) := \left| \, \left\{ (v, e) \mid e \in E^+(\Gamma) \text{ and } w(e) \cdot u_{(v,e)} = n \right\} \, \right|$$

where we consider $E^+(\Gamma)$ as a subset of $F(\Gamma)$, $w(e)$ is the weight on $e \in E(\Gamma)$ and $u_{(v,e)} := u((v, e))$. Similarly,

$$\underline{\Delta}(n) := \left| \, \left\{ v \in V^-(\Gamma) \, \middle| \, w_v \cdot u_v = n \right\} \, \right|$$

where $w_v \in \mathbb{N} \setminus \{0\}$ is the weight on $v \in V^-(\Gamma)$ and $u_v := u(v)$.

Definition 2.9. The *degree of a tropical coral* $h : \Gamma \to \overline{C}\mathbb{R}$, denoted by $\Delta := (\overline{\Delta}, \underline{\Delta})$ by suppressing h in the notation, is the degree of its type.

In other words, the degree of a tropical coral is the abstract set of directions of unbounded edges together with their weights, with repetitions allowed.

Remark 2.10. Note that

$$| \, \overline{\Delta} \, | := \sum_{n \in N_{>0}} \overline{\Delta}(n)$$

is equal to the cardinality of the set of unbounded edges of h and

$$| \, \underline{\Delta} \, | := \sum_{n \in N_{<0}} \underline{\Delta}(n)$$

is equal to the number of negative vertices of h.

Definition 2.11. An *asymptotic constraint of k-incidences* for a k-tuple of integral vectors $(u_1, \ldots, u_k) \subset N_{\mathbb{R}}$ is a k-tuple

$$\lambda = (\lambda_1, \ldots, \lambda_k) \in \prod_{i=1}^{k} N_{\mathbb{R}} / \mathbb{R} \cdot u_i$$

Let (Γ, u) be the type of a k-labelled coral graph $(\Gamma, \mathbb{E})$ where

$$\mathbb{E} = (e_1, \ldots, e_k) \text{ with } e_i \subset E^+(\Gamma) \; for \; i = 1, \ldots, k$$

Assume $\partial e_i = v_i \in V^0(\Gamma)$ and let

$$u_i := u((v_i, e_i)) \subset N$$

be the associated primitive integral vector to e_i. Recall from the definition of the degree of (Γ, u) that each $u_i \in N$ for $i = 1, \dots, k$ is determined by the degree Δ of (Γ, u).

Then, *an asymptotic constraint of k-incidences for the degree* Δ of (Γ, u) is an asymptotic constraint for the k-tuple of integral vectors $(u_1, \dots, u_k) \subset N_{\mathbb{R}}$ associated to $\mathbb{E} = (e_1, \dots, e_k)$.

An asymptotic constraint $\lambda = (\lambda_1, \dots, \lambda_k) \in \prod_{i=1}^k N_{\mathbb{R}}/\mathbb{R} \cdot u_i$ for a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$, is an asymptotic constraint for its type. Given an asymptotic constraint for $h : \Gamma \to \overline{C}\mathbb{R}$, we say *$h$ matches* λ if

$$q_i(h(e_i)) = \{\lambda_i\}$$

for all $i = 1, \dots, k$ under the quotient map

$$q_i : N_{\mathbb{R}} \longrightarrow N_{\mathbb{R}}/u_i \cdot \mathbb{R}$$

Let $\lambda = (\lambda_1, \dots, \lambda_k)$ be an asymptotic constraint for the type (Γ, u) of a tropical coral and assume the degree of the type (Γ, u) is $\Delta = (\overline{\Delta}, \underline{\Delta})$. Then, we call λ *general for* Δ if the following conditions are satisfied

(i) $k = \mid \overline{\Delta} \mid -1$.
(ii) Any tropical coral of degree $\Delta = (\overline{\Delta}, \underline{\Delta})$ with $k = |\overline{\Delta}| - 1$, matching λ is general.

We denote the set of tropical corals of type (Γ, u) matching an asymptotic constraint λ by

$$\mathfrak{T}_{(\Gamma,u)}(\lambda) := \{h : \Gamma \to \overline{C}\mathbb{R} \text{ tropical coral} \mid h \text{ has type } (\Gamma, u) \text{ and } h \text{ matches } \lambda\}$$

Next we will discuss how to set up suitable constraints to obtain the cardinality

$$|\mathfrak{T}_{(\Gamma,u)}(\lambda)|$$

as a finite number. For this we need to define a *stable range* of constraints and *good constraints* in this stable range.

The stable range of constraints is related to the issue of rescalings of tropical corals, which we discuss in a moment. First note that there is a length function on $E(\Gamma)$ defined as follows.

Definition 2.12. Let $e \in E(\Gamma)$ be an edge such that $h(e)$ has integral affine length L_e in $\overline{C}\mathbb{R}$, endowed with the polyhedral decomposition $\overline{C}\mathscr{P}_b$ for fixed $b \in \mathbb{N} \setminus \{0\}$ and let

$w(e)$ be the weight on e. Define the function $l : E(\Gamma) \to \mathbb{R}_{\geq 0}$ referred to as the *length function on* $E(\Gamma)$ by

$$l(e) := L_e \cdot \frac{b}{w(e)}$$

From the definition of the length function it follows that a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ rescales the length of each edge $e \in E(\Gamma)$ by $\frac{w(e)}{b}$.

Now, let $h : \Gamma \to \overline{C}\mathbb{R}$ be a tropical coral. The *rescaled coral*

$$s \cdot h : \Gamma \to \overline{C}\mathbb{R}$$

is the tropical coral obtained from $h : \Gamma \to \overline{C}\mathbb{R}$ by rescaling each L_e by $s \geq 1$. We refer to this process of obtaining $s \cdot h$ from h as *rescaling* h *by* s.

Assume h matches the asymptotic constraint $\lambda = (\lambda_1, \cdots, \lambda_k)$. Rescale each of the incidence conditions λ_i simultaneously with $s \in \mathbb{R}_{\geq 1}$ to obtain

$$s \cdot \lambda = (s \cdot \lambda_1, \cdots, s \cdot \lambda_k)$$

Then, the rescaled coral $s \cdot h$ matches $s \cdot \lambda$.

Remark 2.13. Note that if λ is a general constraint for a tropical coral h, then $s \cdot \lambda$ with $s \in \mathbb{R}_{\geq 1}$ is a general constraint for the rescaled coral $s \cdot h$.

2.3. **Extending a tropical coral to a tropical curve.** In this section we describe how to construct the extension $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ of a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$.

Construction 2.14. Let $h : \Gamma \to \overline{C}\mathbb{R}$ be a tropical coral and let $\widetilde{\Gamma}$ be the (geometric realization of the) graph

$$\widetilde{\Gamma} := \left(\Gamma \setminus V_1^-(\Gamma)\right) \bigcup E_{n>1}^-$$

where the set

$$E_{n>1}^- = \{e \text{ edge} \mid \partial e = v \text{ for } v \in V_{n>1}^-\}$$

consists of abstract half-edges e which are inserted at a negative vertex $v \in V_n^-$ of valency $n > 1$ and $\partial e = \{v\}$. We refer to each such e as the *edge inserted at* $v \in V_n^-$ for $n > 1$.

Let $E_1^-(\Gamma)$ the set of edges adjacent to univalent negative vertices of the coral graph Γ. Define

$$E_1^-(\widetilde{\Gamma}) := \{e_v \mid e_v = e_{v^-} \setminus v^-, \text{ for } e_{v^-} \in E_1^-(\Gamma)\}$$

the set of edges obtained by omitting $v^- \in V_1^-(\Gamma)$ from $e_{v^-} \in E_1^-(\Gamma)$. and refer to

$$E^-(\widetilde{\Gamma}) := E_1^-(\widetilde{\Gamma}) \bigcup E_{n>1}^-$$

as the *set of negative edges* of $\widetilde{\Gamma}$. The set

$$E^+(\widetilde{\Gamma}) := E^+(\Gamma)$$

is referred to as the *set of positive edges* of $\widetilde{\Gamma}$. and the set

$$E^{\infty}(\widetilde{\Gamma}) := E^{+}(\widetilde{\Gamma}) \bigcup E^{-}(\widetilde{\Gamma})$$

is referred to as the *set of unbounded edges* of $\widetilde{\Gamma}$.

Now, given a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$, we describe how to construct a map $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$. Let $u_v \in N$ and $\mathcal{U}_v \in N$ be the primitive integral vectors associated to edges $e_v \in E_1^{-}(\widetilde{\Gamma})$ and $E_v \in E_{n>1}^{-}(\widetilde{\Gamma})$ respectively, defined as follows.

For each $e_v \in E_1^{-}(\widetilde{\Gamma})$, let v be the interior vertex adjacent to v and let u_v be the primitive integral vector emanating from $h(v)$ in the direction of the origin in $N_{\mathbb{R}}$.

Let $E_v \in E_{n>1}^{-}$ be an edge inserted at v and let $\mathcal{U}_v$ be the primitive integral vector emanating from $h(v)$ in the direction of the origin in $N_{\mathbb{R}}$.

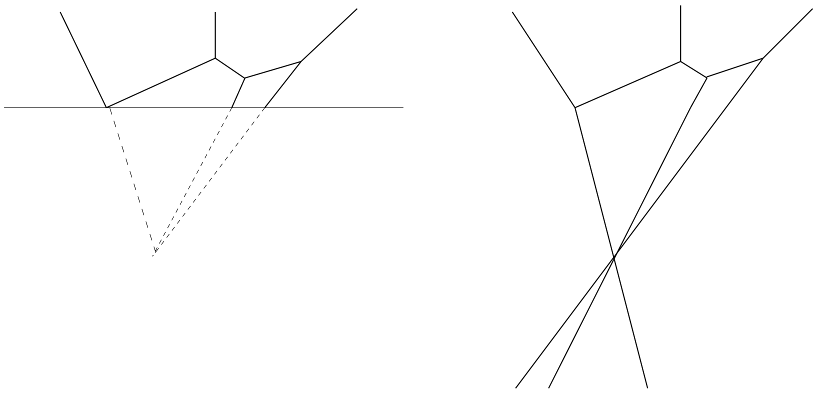
Now define $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ as

$$\widetilde{h} = \begin{cases} h & \text{on } \widetilde{\Gamma} \setminus E^{-}(\widetilde{\Gamma}) \\ u_v \cdot \mathbb{R}_{\geq 0} & \text{for } e_v \in E_1^{-}(\widetilde{\Gamma}) \\ \mathcal{U}_v \cdot \mathbb{R}_{\geq 0} & \text{for } E_v \in E_{n>1}^{-} \end{cases}$$

The map $\widetilde{h} : \widetilde{\Gamma} \longrightarrow \mathbb{R}^2$ is referred to as the tropical *extension* of h.

Note that the origin is not a vertex of $V(\widetilde{\Gamma})$ and hence the Betti number of $\widetilde{\Gamma}$ is the same as the Betti number of Γ which is zero.

Example 2.15. The following figure illustrates the tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ together with its extension $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$.



$h : \Gamma \to \overline{C}B$ $\qquad$ $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$

Remark 2.16. Let $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ be the tropical extension of a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$. Let $w : E(\Gamma) \to \mathbb{N} \setminus \{0\}$ be the weight function on the bilateral graph Γ.

Recall that for each edge $e_v \in E_1^{-}(\widetilde{\Gamma})$, we have $e_v = e_{v^-} \setminus v^-$ for $v^- \in V_1^{-}(\Gamma)$ and $e_{v^-} \in E_1^{-}(\Gamma)$. And for each edge $E_v \in E_{n>1}$, we have $v \in V^{-}(\Gamma) \setminus V_1^{-}(\Gamma)$ such that E_v is adjacent to v. Let $w_v \in \mathbb{N} \setminus \{0\}$ be as in Definition 2.2, (iii).

Then the weight function

$$\widetilde{w} : E(\widetilde{\Gamma}) \to \mathbb{N} \setminus \{0\}$$

on $E(\widetilde{\Gamma})$ is given by

$$\widetilde{w} := \begin{cases} w & \text{on } E(\widetilde{\Gamma}) \setminus E^-(\widetilde{\Gamma}) \\ w(e_{v^-}) & \text{for } E_v \in E_1^-(\widetilde{\Gamma}) \\ w_v & \text{for } E_v \in E_{n>1} \end{cases}$$

Let $h : \Gamma \to \overline{C}\mathbb{R}$ be a tropical coral of type

$$(\Gamma, u)$$

Then the tropical extension $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ is a particular type of a tropical curve defined as in ([NS], Definition 1.1). The of type of

$$(\widetilde{\Gamma}, \widetilde{u}) \tag{2.3}$$

is the graph $\widetilde{\Gamma}$ together with the map

$$\widetilde{u} : F(\widetilde{\Gamma}) \to N$$

given by

$$\widetilde{u} := \begin{cases} u & \text{on } F(\Gamma) \setminus V_1^-(\Gamma) \\ \mathcal{U}_v & \text{for each } E_v \in E_{n>1}^- \end{cases}$$

where we view $F(\Gamma) \setminus V_1^-(\Gamma)$ and $E_{n>1}^-$ as subsets of $F(\widetilde{\Gamma})$ and $\mathcal{U}_v$ is defined as in 2.14.

Note that, given a tropical extension $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ of type $(\widetilde{\Gamma}, \widetilde{u})$ of a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$, the restriction $\overline{\widetilde{h}} : \overline{\widetilde{\Gamma}} \to \overline{C}\mathbb{R}$ of $\widetilde{h} : \overline{\Gamma} \to \overline{C}\mathbb{R}$ to $\widetilde{h}^{-1}\overline{C}\mathbb{R}$ is clearly equal to $h : \Gamma \to \overline{C}\mathbb{R}$. The type (Γ, u) is determined by the restriction of $\widetilde{u}$ to $F(\overline{\widetilde{\Gamma}})$.

A tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ has degree

$$\Delta := (\overline{\Delta}, \underline{\Delta})$$

if and only if the tropical extension $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ of h has degree

$$\widetilde{\Delta} := \Delta \tag{2.4}$$

Finally, observe that $h : \Gamma \to \overline{C}\mathbb{R}$ matches the general constraint

$$\lambda = (\lambda_1, \ldots, \lambda_k) \in \prod_{i=1}^{k} N_{\mathbb{R}}/\mathbb{R} \cdot u_i$$

if and only if the tropical extension $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ matches the general constraint

$$\widetilde{\lambda} = (\lambda_1, \ldots, \lambda_k, 0, \ldots, 0) \in \prod_{i=1}^{k} N_{\mathbb{R}}/\mathbb{R} \cdot u_i \times \prod_{j=1}^{m} N_{\mathbb{R}}/u_j^- \cdot \mathbb{R} \tag{2.5}$$

where

$$m = \mid V^{-}(\Gamma) \mid$$

We refer to $(\widetilde{\Delta}, \widetilde{\lambda})$ as the tropical incidences on the extension and denote by

$$\mathcal{T}_{(\widetilde{\Gamma}, \widetilde{u})}(\widetilde{\lambda})$$

the set of tropical curves of type $(\widetilde{\Gamma}, \widetilde{u})$ matching a constraint $\widetilde{\lambda}$.

Lemma 2.17. *The map given by*

$$\begin{array}{ccc} \mathcal{T}_{(\Gamma,u)}(\lambda) & \longrightarrow & \mathcal{T}_{(\widetilde{\Gamma}, \widetilde{u})}(\widetilde{\lambda}) \\ h & \longmapsto & \widetilde{h} \end{array}$$

where $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ is the tropical extension of $h : \Gamma \to \overline{C}\mathbb{R}$, is injective.

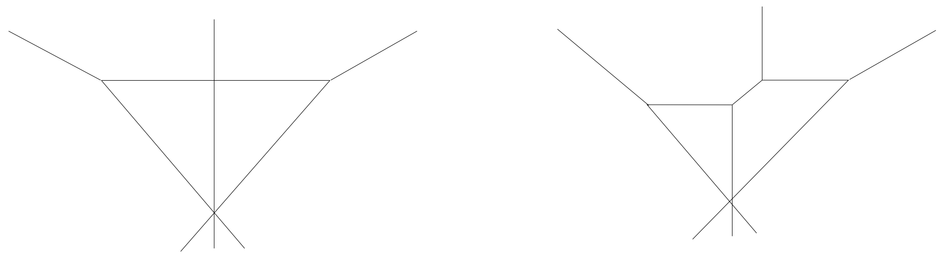
Proof. The result is an immediate consequence of the construction of tropical extension 2.14. □

2.4. **The count of tropical corals.** In this section we define the count of tropical corals of degree Δ matching a general asymptotic constraint λ.

Proposition 2.18. *For any map $\Delta \in \mathrm{Map}(N \setminus \{N_0\}, \mathbb{N})$ of finite support, there are only finitely many types of tropical corals of degree Δ.*

Proof. Let $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ be the extension of $h : \Gamma \to \overline{C}\mathbb{R}$. Then, $\widetilde{h}$ has degree $\widetilde{\Delta} = \Delta$. By Proposition 2.1 in [NS] there are only finitely many types of tropical curves of degree $\widetilde{\Delta}$. Since h is obtained by the restriction of $\widetilde{h}$, the result follows. □

Example 2.19. In the following figure we illustrate two tropical corals with the same degree but different types.



Two different types of tropical corals of the same degree

The number of types of tropical corals of same degree Δ can be enumerated by integral subdivisions of an integral polygon so that the tropical coral is realized as the dual graph of the subdivision, analogously to the case of tropical curves ([M],§4).

Our main aim in this section is to define a range of asymptotic constraints such that the count of tropical corals of a given degree matching an asymptotic constraint in this range is well-defined. More specifically, we would like to show that all tropical curves with a given degree and matching certain constraints are obtained as extensions of tropical corals of the same degree.

Now for any $\Delta \in \mathrm{Map}(N \setminus N_0, \mathbb{N})$ we first would like to establish the structure of the moduli space $\mathfrak{T}_{(\Gamma,u)}$ of fixed type matching degree Δ.

We first endow $\mathfrak{T}_{(\Gamma,u)}$ with an integral affine structure as follows. Let $(h : \Gamma \to \overline{C}B) \in \mathfrak{T}_{(\Gamma,u)}$ be a general tropical coral with m negative vertices and l unbounded edges. Since $h : \Gamma \to \overline{C}B$ is general from definition 2.5 it follows that the number of its bounded edges which are not connected to a negative vertex is equal to $l + m - 3$. Label these edges by by $\{e_1, \ldots, e_{l+m-3}\}$ and let $\rho(e_i)$ be the affine length of e_i. For an arbitrary vertex $v \in V(\Gamma) \setminus V^-(\Gamma)$ define

$$\begin{aligned} \Phi : \mathfrak{T}_{(\Gamma,u)} &\hookrightarrow N_{\mathbb{R}} \times \mathbb{R}^{l+m-3}_{\geq 0} \\ \Gamma &\mapsto (h(v), \rho(e_1), \ldots, \rho(e_{l+m-3})) \end{aligned} \tag{2.6}$$

The negative vertices are fixed by fixing the degree. Fixing one vertex that is not negative and the affine lengths of all bounded edges not adjacent to negative vertices in a tropical coral determines it uniquely. So, Φ is injective and determines an embedding $\mathfrak{T}_{(\Gamma,u)}$ into $N_{\mathbb{R}} \times \mathbb{R}^{m+l-3}_{\geq 0} \cong \mathbb{R}^{m+l-1}_{\geq 0}$. This induces a natural integral affine structure on $\mathfrak{T}_{(\Gamma,u)}$. Now, our aim is to prove the following main theorem of this section.

Theorem 2.20. *Let (Γ, u) be a general type of tropical corals of fixed degree Δ with l unbounded edges*

$$e_1, \ldots, e_l$$

with $\partial e_i = v_i$ for $v_i \in V(\Gamma)$ [1]*. Let u_i be the primitive integral vector in $N_{\mathbb{R}}$ emanating from v_i in the direction of $h(e_i)$. Assume $\mathfrak{T}_{(\Gamma,u)}$ is non-empty. Then for any sequence of indices $1 \leq i_1 < \ldots < i_k \leq l$ with $k \leq l-1$ the map*

$$\begin{aligned} \mathrm{ev}_{i_1,\ldots,i_k} : \mathfrak{T}_{(\Gamma,u)} &\longrightarrow \prod_{\mu=1}^{k} N_{\mathbb{R}}/\mathbb{R} \cdot u_{i_\mu} \\ h &\longmapsto \big([h(v_{i_1})], \ldots, [h(v_{i_k})]\big) \end{aligned} \tag{2.7}$$

is an integral affine submersion.

We will first generalize the concept of tropical corals and introduce *tropical bouquets* and discuss their main features. We afterwards will provide the proof of Theorem 2.20 for tropical bouquets, so the result in particular will hold for tropical corals.

[1]Not all v_i may be distinct, repetitions are allowed

Definition 2.21. Let $\Gamma_{m,l}$ be a coral graph with

$$\begin{aligned} V^-(\Gamma) &:= \{v_1^-, \ldots, v_m^-\} \\ E^+(\Gamma) &:= \{e_1, \ldots, e_l\} \end{aligned}$$

A *tropical bouquet* $h : \Gamma_{m,l} \to \mathbb{R}^2$ is a proper map such that

(i) h satisfies all conditions of the definition of a parameterized tropical coral except some of the unbounded edges $e \subset E^+(\Gamma)$ can be contained in $\mathbb{R}^2$ rather than in $\overline{C}\mathbb{R}$ and it is possible that the projection $\mathrm{pr}_2 : \mathbb{R}^2 \to \mathbb{R}$ onto the second factor is not proper on e.

(ii) Let 0 be the origin in $\mathbb{R}^2$, then

$$0 \notin h(e) \ \text{ for any } e \in E^+(\Gamma)$$

Note that tropical corals are particular types of tropical bouquets, in which the image of all unbounded edges lie in $\overline{C}\mathbb{R}$ and the projection of each unbounded edge onto the second factor is proper.

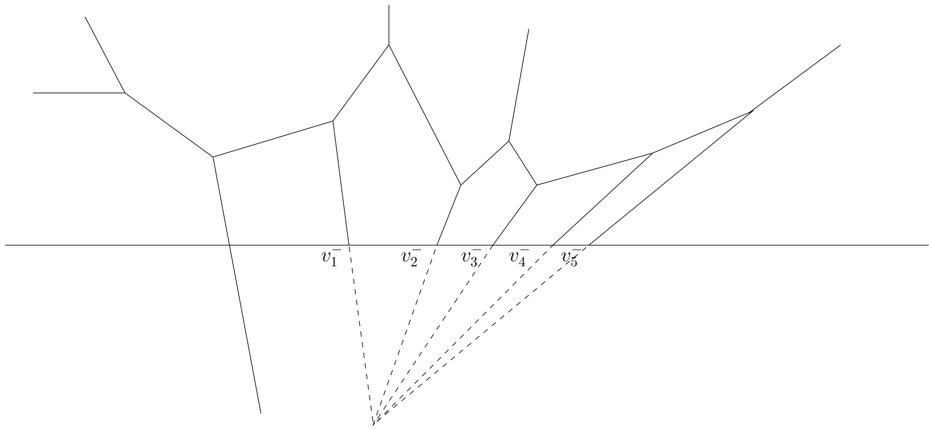
Definition 2.22. We call a tropical bouquet $h : \Gamma_{m,l} \to \mathbb{R}^2$ *general* if the following conditions hold:

(i) All vertices $v \in V^0(\Gamma_{m,l})$ are trivalent.

(ii) All vertices $v \in V^-(\Gamma_{m,l})$ are univalent.

Otherwise it is called *degenerate*.

Example 2.23. The following picture illustrates a general tropical bouquet $h : \Gamma_{5,6} \to \mathbb{R}^2$.



The *type* of a tropical bouquet $h : \Gamma_{m,l} \to \overline{C}\mathbb{R}^2$ is defined analogously to the type of a tropical coral and is denoted by $(\Gamma_{m,l}, u)$. The set of isomorphism classes of tropical bouquets of a given type $(\Gamma_{m,l}, u)$ is denoted by $\mathfrak{T}_{(\Gamma_{m,l},u)}$.

We describe the gluing process of two tropical bouquets as follows. Let

$$\begin{aligned}(h_1 : \Gamma^1_{m_1,l_1} \to \mathbb{R}^2) &\in \mathfrak{T}_{(\Gamma_{m_1,l_1},u_1)}\\ (h_2 : \Gamma^2_{m_2,l_2} \to \mathbb{R}^2) &\in \mathfrak{T}_{(\Gamma_{m_2,l_2},u_2)}\end{aligned}$$

and assume there exist edges

$$\begin{aligned}e_1 &\in E^+(\Gamma_{m_1,l_1}) \text{ such that } e_1 \text{ is adjacent to } v_1 \in V^0(\Gamma_{m_1,l_1})\\ e_2 &\in E^+(\Gamma_{m_2,l_2}) \text{ such that } e_2 \text{ is adjacent to } v_2 \in V^0(\Gamma_{m_2,l_2})\end{aligned}$$

Let

$$\begin{aligned}w_1 : E(\Gamma_{m_1,l_1}) &\to \mathbb{N} \setminus \{0\}\\ w_2 : E(\Gamma_{m_2,l_2}) &\to \mathbb{N} \setminus \{0\}\end{aligned}$$

be the weight functions and assume furthermore we have

$$w_1(e_1) = w_2(e_2)$$

Assume that

$$u_{e_1} = -u_{e_2}$$

where u_{e_i} denotes the primitive integral vector emanating from v_i in the direction of e_i. Let

$$N_{\mathbb{R}}/\mathbb{R}u := N_{\mathbb{R}}/\mathbb{R}u_{e_1} = N_{\mathbb{R}}/\mathbb{R}u_{e_2}$$

Then the maps

$$\begin{aligned}f_1 : \mathfrak{T}_{(\Gamma_{m_1,l_1},u_1)} &\to N_{\mathbb{R}}/\mathbb{R}u_{e_1}\\ h_1 &\mapsto [h_1(v_1)]\end{aligned}$$

$$\begin{aligned}f_2 : \mathfrak{T}_{(\Gamma_{m_2,l_2},u_2)} &\to N_{\mathbb{R}}/\mathbb{R}u_{e_2}\\ h_2 &\mapsto [h_2(v_2)]\end{aligned} \tag{2.8}$$

induces the map

$$\begin{aligned}f : \mathfrak{T}_{(\Gamma_{m_1,l_1},u_1)} \times \mathfrak{T}_{(\Gamma_{m_2,l_2},u_2)} &\to N_{\mathbb{R}}/\mathbb{R}u\\ (h_1, h_2) &\mapsto [h_1(v_1) - h_2(v_2)]\end{aligned}$$

Assume

$$f(h_1, h_2) = 0 \in N_{\mathbb{R}}/\mathbb{R}u$$

and

$$h(v_1) - h(v_2) = \lambda \cdot u_1 \text{ for } \lambda \in \mathbb{R}_{>0}$$

Then we can define a glued tropical bouquet

$$h_{12} : \Gamma_{m,l} \to \mathbb{R}^2$$

as follows. Define the vertex set $V(\Gamma_{m,l})$ as the disjoint union

$$V(\Gamma_{m,l}) := V(\Gamma_1) \amalg V(\Gamma_2)$$

and the edge set $E(\Gamma_{m,l})$ as

$$E(\Gamma_{m,l}) := E(\Gamma_1) \setminus \{e_1\} \amalg E(\Gamma_2) \setminus \{e_2\} \amalg \{e_{12}\}$$

where e_{12} is the edge such that

$$\partial e_{12} = \{v_1, v_2\}$$

Define E_{12} to be the line segment in $\overline{C}\mathbb{R}$ such that

$$\partial E_{12} = \{h(v_1), h(v_2)\}$$

Now, define the map $h_{12} : \Gamma_{m,l} \to \mathbb{R}^2$ by

$$h_{12} := \begin{cases} h_1 & \text{on } V(\Gamma^1_{m_1,l_1}) \cup E(\Gamma^1_{m_1,l_1}) \setminus \{e_1\} \\ E_{12} & \text{on } e_{12} \\ h_2 & \text{on } V(\Gamma^2_{m_2,l_2}) \cup E(\Gamma^2_{m_2,l_2}) \setminus \{e_2\} \end{cases}$$

We refer to $h_{12} : \Gamma_{m,l} \to \mathbb{R}^2$ as the *gluing* of h_1 and h_2 along the edges e_1 and e_2.

Lemma 2.24. *Any tropical bouquet* $(h_{12} : \Gamma_{m,l} \to \mathbb{R}^2) \in \mathcal{T}_{(\Gamma_{m,l},u)}$ *where* $m > 1$*, can be obtained by gluing two tropical bouquets*

$$h_1 \in \mathfrak{T}_{(\Gamma_{m_1,l_1},u_1)}$$
$$h_2 \in \mathfrak{T}_{(\Gamma_{m_2,l_2},u_2)}$$

such that

$$m = m_1 + m_2$$

and

$$l = (l_1 - 1) + (l_2 - 1) = l_1 + l_2 - 2$$

Proof. Choose two vertices

$$v_i, v_j \in V^-(\Gamma_{m,l})$$

Since $\Gamma_{m,l}$ is a tree by the definition of a coral graph, we have a path $\mathscr{P}_n$ given by a union of n bounded edges in $E(\Gamma_{m,l})$ connecting v_i to v_j. By connectedness of $h(\Gamma_{m,l})$ it follows that there exists at least one edge

$$e \in \mathscr{P}_n \text{ such that } 0 \notin L(h(e))$$

where O denotes the origin and $L(h(e))$ the affine line containing $h(e)$. Take a point

$$p \in e \setminus \partial e$$

Then

$$\Gamma_{m,l} \setminus \{p\} = \Gamma_{m_1,l_1} \ \amalg \ \Gamma_{m_2,l_2}$$

where Γ_{m_1,l_1} and Γ_{m_2,l_2} are two trees with non-compact edges

$$e_1 \in E^+(\Gamma_{m_1,l_1})$$
$$e_2 \in E^+(\Gamma_{m_2,l_2})$$

such that

$$e \setminus \{p\} = e_1 \ \amalg \ e_2$$

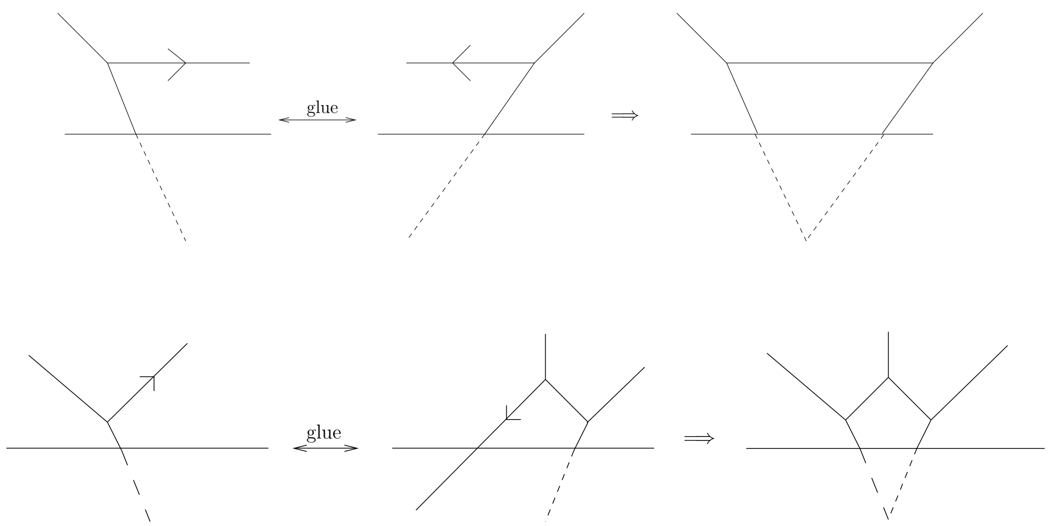
Define $h_1 : \Gamma_{m_1,l_1} \to \mathbb{R}^2$ and $h_2 : \Gamma_{m_2,l_2} \to \mathbb{R}^2$ by

$$h_1 \ := \ h\big|_{\Gamma_{m_1,l_1}}$$
$$h_2 \ := \ h\big|_{\Gamma_{m_2,l_2}}$$

Note that we apply an extension to $h_1(e_1)$ and $h_2(e_2)$ to half-lines in $\mathbb{R}^2$ and by abuse of notation denote the new maps again by h_1 and h_2.

Then, $h : \Gamma_{m,l} \to \mathbb{R}^2$ is obtained by gluing h_1 and h_2 along the edges e_1 and e_2. $\square$

Example 2.25. The following images illustrate tropical bouquets obtained by the gluing two tropical bouquets along their labelled edges.



Proof of Theorem 2.20: We will prove the theorem for tropical bouquets of general type $(\Gamma_{m,l}, u)$ of degree Δ. By Lemma 2.24, any such tropical bouquet $(h : \Gamma_{m,l} \to \mathbb{R}^2) \in \mathcal{T}_{(\Gamma_{m,l},u)}$ is obtained by gluing two tropical bouquets

$$h_1 \in \mathfrak{T}_{(\Gamma_{m_1,l_1},u_1)}$$
$$h_2 \in \mathfrak{T}_{(\Gamma_{m_2,l_2},u_2)}$$

along edges $e_1 \in E^+(\Gamma_{m_1,l_1}, u_1)$ with $\partial e_1 = v_1$ and $e_2 \in E^+(\Gamma_{m_2,l_2}, u_2)$ with $\partial e_2 = v_2$ so that

$$E(\Gamma_{m,l}, u) = E(\Gamma_{m_1,l_1}, u_1) \setminus \{e_1\} \amalg E(\Gamma_{m_2,l_2}, u_2) \setminus \{e_2\} \amalg e$$

with $\partial e = \{v_1, v_2\}$. Let

$$\begin{aligned} \{e_1\} \cup \{e_{i_r}, \ldots, e_{i_r} \mid r \leq l_1 - 1\} &\subseteq E^+(\Gamma_{m_1,l_1}, u_1) \\ \{e_2\} \cup \{e_{i_{r+1}}, \ldots, e_{i_k} \mid k - r \leq l_2 - 1\} &\subseteq E^+(\Gamma_{m_2,l_2}, u_2)\} \end{aligned}$$

Now, we will use induction on l. For $l = 1$, we need to have a unique negative vertex. Let v be the negative vertex and $e \in E(\Gamma_{m,l})$ be the edge with $\partial e = v$. Extend e, to obtain the tropical curve $\widetilde{h} : \widetilde{\Gamma_l^m} \to \mathbb{R}^2$. In this case the result follows from ([M] Proposition 2.14, [NS] Proposition 2.4).

Assume the theorem is true for any $2 \leq l_i < l$. Let u_i be the direction vector for e_i, that is the primitive integral vector emanating from $h_i(v_i)$ in the direction of $h_i(e_i)$ for $i = 1, 2$. Define the direction vectors u_{i_μ} for the unbounded edges e_{i_μ} analogously. Then, by the induction hypothesis we have submersions

$$\begin{aligned} \mathcal{T}_{(\Gamma_{m_1,l_1},u_1)} &\longrightarrow \prod_{\mu=1}^{r} N_{\mathbb{R}}/\mathbb{R} \cdot u_{i_\mu} \\ h_1 &\longmapsto ([h_1(v_{i_1})], \ldots, [h_1(v_{i_r})]) \\ \mathcal{T}_{(\Gamma_{m_2,l_2},u_2)} &\longrightarrow N_{\mathbb{R}}/\mathbb{R} \cdot u_2 \times \prod_{\mu=r+1}^{k} N_{\mathbb{R}}/\mathbb{R} \cdot u_{i_\mu} \\ h_2 &\longmapsto \big([h_2(v_2)], ([h_2(v_{i_{r+1}})], \ldots, [h_2(v_{i_k})])\big) \end{aligned}$$

Hence, we obtain a submersion

$$\begin{aligned} \mathcal{F} : \mathfrak{T}_{(\Gamma_{m_1,l_1},u_1)} \times_{N_{\mathbb{R}}/\mathbb{R}u} \mathfrak{T}_{(\Gamma_{m_2,l_2},u_2)} &\longrightarrow \prod_{\mu=1}^{r} N_{\mathbb{R}}/\mathbb{R} \cdot u_{i_\mu} \times \prod_{\mu=r+1}^{k-1} N_{\mathbb{R}}/\mathbb{R} \cdot u_{i_\mu} \\ (h_1, h_2) &\longmapsto \big([h_1(v_{i_1})], \ldots, [h_1(v_{i_r})], [h_2(v_{i_{r+1}})], \ldots, [h_2(v_{i_{k-1}})]\big) \end{aligned}$$

where

$$N_{\mathbb{R}}/\mathbb{R}u := N_{\mathbb{R}}/\mathbb{R}u_1 = N_{\mathbb{R}}/\mathbb{R}u_2$$

and the fibered coproduct is defined via the morphisms in Equation (2.8). Define

$$\begin{aligned} \mathcal{G} : \mathfrak{T}_{(\Gamma_{m_1,l_1},u_1)} \times_{N_{\mathbb{R}}/\mathbb{R}u} \mathfrak{T}_{(\Gamma_{m_2,l_2},u_2)} &\longrightarrow \mathbb{R} \\ (h_1, h_2) &\longrightarrow \lambda \end{aligned}$$

where $\lambda \in \mathbb{R}$ is defined by

$$h(v_2) - h(v_2) = \lambda \cdot u_{e_1}$$

Then, by the construction of gluing of bouquets we obtain

$$\mathfrak{T}_{(\Gamma_{m,l},u)} = \mathcal{G}^{-1}(\mathbb{R}_{>0}) \tag{2.9}$$

Hence, the inclusion $\mathcal{G}^{-1}(\mathbb{R}_{>0}) \subset \mathfrak{T}_{(\Gamma_{m_1,l_1},u_1)} \times_{N_{\mathbb{R}}/\mathbb{R}u} \mathfrak{T}_{(\Gamma_{m_2,l_2},u_2)}$ followed by the submersion $\mathcal{F}$ gives the desired submersion $\mathrm{ev}_{i_1,\ldots,i_k}$.

Corollary 2.26. *The set $\mathfrak{T}_{(\Gamma_{m,l},u)}$ of isomorphism classes of general tropical bouquets of a given type $(\Gamma_{m,l}, u)$ forms the interior of a convex polyhedron of dimension $l-1$ where l is the number of unbounded edges of Γ.*

Proof. The fact that $\mathfrak{T}_{(\Gamma_{m,l},u)}$ is a convex polytope follows by its description in equation (2.9) and the induction hypothesis.

By Theorem 2.20 we obtain a submersion

$$\mathcal{F}|_{\mathcal{G}^{-1}(\mathbb{R}_{>0})} : \mathcal{T}_{(\Gamma,u)} \longrightarrow \prod_{i=1}^{k} N_{\mathbb{R}}/\mathbb{R}\cdot u_i$$

where $1 \leq k \leq l-1$. Hence, for $k = l-1$ the result follows. □

Corollary 2.27. *The set $\mathfrak{T}_{(\Gamma,u)}$ of isomorphism classes of general tropical corals of a given type (Γ, u) forms the interior of a convex polyhedron of dimension $l-1$ where l is the number of unbounded edges of Γ.*

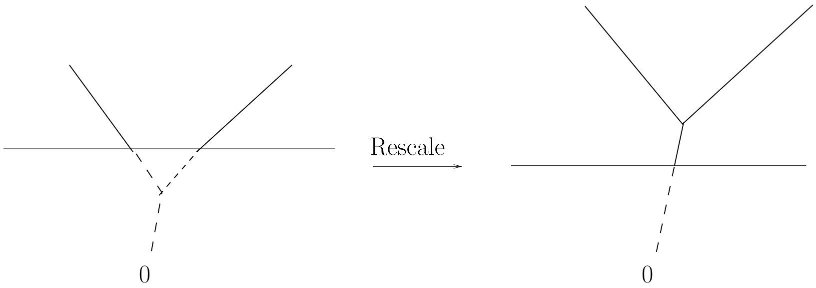
Proof. A tropical coral is a special type of a tropical bouquet in which all unbounded edges have positive direction vectors. Hence, the result follows. □

Remark 2.28. By Theorem 2.20 there is no dependence among the general asymptotic constraints and hence if there exist a general constraint λ for the degree of (Γ, u) such that the set $\mathfrak{T}_{(\Gamma,u)}(\lambda)$ of tropical corals of a type (Γ, u) matching λ is non-empty then the cardinality $|\mathfrak{T}_{(\Gamma,u)}(\lambda)|$ is equal to 1.

Remark 2.29. A non-general tropical coral can always be deformed into a general tropical coral analogously to the case of tropical curves ([M],§2). This is possible since non-general types of tropical corals are obtained by taking the limit of the lengths of some edges in general types of tropical corals to zero. It follows that the non-general tropical corals form a lower dimensional strata of the moduli space $\mathfrak{T}_{\Gamma,u}$ similar to ([M], Proposition 2.14). Hence, the types of the non-general corals form a nowhere dense subset in the space of constraints. This ensures the existence of general constraints.

Before setting up the tropical counting problem we define a range of constraints λ which will ensure that we get a well-defined count, independent of the constraints chosen within this range. We first need a couple of lemmas for this.

Let λ be an asymptotic constraint and assume the set of tropical corals of type (Γ, u) matching λ is empty. After rescaling λ by $s \in \mathbb{R}_{>0}$ it is possible to obtain a non-empts set of tropical corals of type (Γ, u) matching $s \cdot \lambda$ as illustrated in the following figure, where by O we denote the origin in $N_{\mathbb{R}}$.



To obtain a well-defined count we want to choose our constraints such that we also avoid the possibility of obtaining new tropical corals after rescaling.

Lemma 2.30. *Fix the degree* $\Delta \in \mathrm{Map}(N \setminus N_0, \mathbb{N})$. *Let* (Γ, u) *be the type of a tropical coral of degree* Δ. *Then* $\forall \lambda$ *such that* λ *is a general asymptotic constraint for* (Γ, u) *one of the following holds.*

(i) $\forall s \geq 1$, $\mathfrak{T}_{(\Gamma,u)}(s \cdot \lambda) = \emptyset$.

(ii) $\exists s_0 \geq 1$ *such that* $\forall s \geq s_0$, $|\mathfrak{T}_{(\Gamma,u)}(s \cdot \lambda) = 1|$

Proof. If there exists a tropical coral $h \in \mathcal{T}_{(\Gamma,u)}$ matching a general asymptotic constraint λ, then the rescaled coral $s \cdot h$ with $s \geq 1$ matches $s \cdot \lambda$. This operation clearly does not change the type. Moreover, since λ is general for h, then $s \cdot \lambda$ is general for $s \cdot h$. Hence, the result follows. $\square$

We need the following definition to show the count we define will be independent of the choice of the constraint.

Definition 2.31. Let $\Delta = (\overline{\Delta}, \underline{\Delta})$ be the degree of type (Γ, u) of a coral graph and

$$E^+(\Gamma) = \{e_1, \ldots, e_{k+1}\}$$

be the set of positive edges of Γ with $\partial e_i = v_i$ and $u((v_i, e_i)) = u_i$ for $i = 1, \ldots, k+1$. Define the cone

$$\mathcal{C}_\Delta := \{a_1 u_1 + \ldots + a_k u_k + a_{k+1} u_{k+1} \mid a_1, \ldots, a_{k+1} \in \mathbb{R}_{\geq 0}\} \subset N_{\mathbb{R}}$$

so that the image of $\mathcal{C}_\Delta$ in $N_{\mathbb{R}}/\mathbb{R} \cdot u_i \cong \mathbb{R}$ is a half-space for $u_i \in \partial \mathcal{C}_\Delta$ and otherwise it is all of $N_{\mathbb{R}}/\mathbb{R} u_i$. Then,

$$\lambda = (\lambda_1, \ldots, \lambda_k) \in \prod_{i=1}^{k} N_{\mathbb{R}}/\mathbb{R} \cdot u_i$$

is called a *good constraint for* Δ if in the former case

$$\lambda_i \in \mathrm{int}(\mathcal{C}_\Delta/\mathbb{R}\cdot u_i) \subset N_\mathbb{R}/\mathbb{R}\cdot u_i$$

Lemma 2.32. *Let λ be a good general constraint. Then any tropical curve $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ of degree Δ matching the constraint*

$$\widetilde{\lambda} = (\lambda, 0, \dots, 0)$$

where the last m entries are 0, is obtained as an extension of a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ of degree Δ matching λ after a possible rescaling, where Γ has m negative vertices.

Proof. It is enough to show that the images $\widetilde{h}(V(\widetilde{\Gamma}))$ of vertices of $\widetilde{\Gamma}$ lie inside the cone $\mathcal{C}_\Delta$ defined in 2.31, since then either

$$\widetilde{h}(V(\widetilde{\Gamma})) \subset \overline{C}\mathbb{R} \cap \mathcal{C}_\Delta$$

and the restriction of $\widetilde{h}$ is already a tropical coral, or by rescaling $\widetilde{h}$ with some $s \in \mathbb{R}_{\geq 1}$, we can ensure all vertices will be in $\overline{C}\mathbb{R} \cap \mathcal{C}_\Delta$.

Now assume there exist a vertex v of $\widetilde{\Gamma}$ such that $h(v) \notin \mathcal{C}_\Delta$. Then it follows that there exists at least one unbounded edge e of $\widetilde{\Gamma}$ with direction vector u_e emanating from $h(v)$ in the direction of $h(e)$ with $u_e \notin \mathcal{C}_\Delta$: If v is the only vertex of $\widetilde{\Gamma}$ with $h(v)$ not included in $\mathcal{C}_\Gamma$ then this is obvious by the balancing condition at v. If not, then take a longest path from v to a vertex v' of $\widetilde{\Gamma}$ such that $h(v') \notin \mathcal{C}_\Gamma$, which exists since Γ is connected. In this case v' must be adjacent to an unbounded edge e of $\widetilde{\Gamma}$ such that the direction vector $u_e \notin \mathcal{C}_\Delta$ by the balancing condition at v'. But the existence of such an edge e contradicts that $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ has degree Δ. Hence, the result follows. □

Remark 2.33. Recall general constraints exist as non-general types of tropical corals form a nowhere dense subset in the space of constraints (Remark 2.29). Note that good constraints form an open set inside the set of constraints. Hence, the existence of good general constraints follows.

To avoid the possibility of obtaining new tropical corals after rescaling that match a given constraint we will need the following definition.

Definition 2.34. Let $\Delta \in \mathrm{Map}(N \setminus N_0, \mathbb{N})$ be a map of finite support. Define the *stable range $\mathcal{S}$ of constraints for* Δ as the set of asymptotic constraints λ for Δ satisfying

(i) λ is a good general asymptotic constraint for Δ.
(ii) For any type (Γ, u) of degree Δ, if $\mathfrak{T}_{(\Gamma,u)}(\lambda) = \emptyset$, then $\mathfrak{T}_{(\Gamma,u)}(s\cdot\lambda) = \emptyset$, $\forall s \geq 1$.

Now we are ready to define the tropical count. Let $\Delta \in \mathrm{Map}(N \setminus N_0, \mathbb{N})$ be a map of finite support. Then there are only finitely many types of tropical corals

$$(\Gamma_1, u_1), \cdots, (\Gamma_n, u_n)$$

of degree Δ by Proposition 2.18.

Let $\mathcal{C}_\Delta$ be as in Definition 2.31, so that all all good general constraints lie inside the interior of $\mathcal{C}_\Delta$. Recall that good general constraints for Δ exist (2.33). Fix one good general constraint λ.

For each type of tropical coral (Γ_i, u_i), let $\mathcal{S}_i$ be the stable range defined as in Definition 2.34 for $i = 1, \cdots, n$. By Lemma 2.30, there exists $s_i \geq 1$ such that

$$s_i \cdot \lambda \subset \mathcal{S}_i$$

for $i = 1, \ldots, n$. Now define

$$\mathcal{S} := \bigcap_i \mathcal{S}_i$$

referred to as the stable range for Δ. Then $\mathcal{S}$ is non-empty, since by taking the maximum

$$s_0 = \max\{s_i \mid i = 1, \ldots, n\}$$

we ensure $\mathfrak{T}_{(\Gamma_i,u_i)}(s_0\lambda)$ is non-empty if it becomes non-empty after further rescaling by any positive real number, for all $i = 1, \ldots, n$. Hence, we have $s_0\lambda \in S$. So, either

$$\mathfrak{T}_{(\Gamma_i,u_i)}(s_0\lambda) = \emptyset$$

or

$$\left| \, \mathfrak{T}_{(\Gamma_i,u_i)}(s_0\lambda) \, \right| = 1$$

for $i = 1, \cdots, n$ by Remark 2.28. Assume we are in the latter case and the count is non-zero, so that $h_i : \Gamma_i \to \overline{C}\mathbb{R}$ for $i = 1, \ldots, k$ for $k \in \mathbb{N}$ is a list of general tropical corals of type (Γ_i, u_i) matching $s_0\lambda \in \mathcal{S}$. [2] Define the *tropical count* $N^{trop}_{\Delta,\lambda}$ as follows.

Let $V(\Gamma_i)\backslash V^-(\Gamma_i)$ be the set of all non-negative vertices of Γ_i, so that each $v \in V(\Gamma_i)$ is trivalent. For each $v \in V(\Gamma_i) \setminus V^-(\Gamma_i)$ define the multiplicity at v as follows. Choose two arbitrary edges e_1 and e_2 adjacent to v. Let u_1 and u_2 denote the primitive integral vectors emanating from v in the direction of e_1 and e_2 respectively. Let $w_1, w_2 \in \mathbb{N}\backslash\{0\}$ be the associated weights to e_1, e_2. Then we define the multiplicity of v as

$$\mathrm{Mult(v)} := w_1 \cdot w_2 \cdot |\det(u_1, u_2)|$$

The multiplicity of Γ (Definition 2.16,[M]) is defined to be the product

$$\mathrm{Mult(\Gamma_i)} := \prod_{v \in V_3(\Gamma_i)} \mathrm{Mult(v)}$$

[2]We fix $b \in \mathbb{N} \setminus \{0\}$ defining the polyhedral decomposition $\overline{C}\mathscr{P}_b$ of $\overline{C}\mathbb{R}$, sufficiently big such that all vertices of $h_i(\Gamma_i)$ are integral.

Then, the tropical count is defined by

$$N^{\mathrm{trop}}_{\Delta,\lambda} := \sum_{i=1}^{n} \frac{1}{\prod_j d_{ij}} \frac{1}{\prod_k e_{ik}} \cdot \mathrm{Mult}(\Gamma_i) \tag{2.10}$$

where d_{ij}'s are the weights of the unbounded edges and e_{ik}'s are the weights of the edges adjacent to a negative vertex of Γ_i.

Lemma 2.35. *Let λ be a general good constraint in the stable range $\mathcal{S}$ for Δ. Then the tropical count $N^{\mathrm{trop}}_{\Delta,\lambda}$ is independent of the choice of λ.*

Proof. Any tropical coral with m negative vertices of degree Δ matching λ has a unique extension which is a tropical curve of degree Δ matching $(\lambda, 0 \ldots, 0)$ where the last m entries are zero by Lemma 2.17.

Moreover, any tropical curve of degree Δ matching $(\lambda, 0 \ldots, 0)$ is obtained as the extension of a tropical coral of degree Δ matching λ after a possible rescaling by Lemma 2.32 since λ is a good constraint. By condition *ii* of Definition 2.34 we avoid the possibility of rescaling, hence any tropical curve is obtained as the extension of a tropical coral.

Therefore, the count of tropical curves of degree Δ matching $\widehat{\lambda} = (\lambda, 0 \ldots, 0)$ which are extensions of tropical corals is equal to the count of tropical corals of degree Δ matching λ which is given by equation (2.10) (Theorem 3.4 in [GPS]). Note that in the extension of a tropical coral, we omit the divalent vertices which correspond to negative vertices of the tropical coral. So, there is a one-to-one correspondence between the unbounded edges of an extension of a tropical coral that pass through the origin with the edges of the tropical coral that are adjacent to negative vertices with the corresponding weights preserved.

The cardinality of the set of tropical curves of degree Δ matching $\widehat{\lambda}$ is independent of the choice of the constraint by [GM]. Hence, the result follows. $\square$

3. A curve counting problem

3.1. **Log corals.** The central topic in this section is to define and count algebraic geometric objects that correspond to tropical corals. We assume familiarity with log Gromov-Witten theory.

Recall from [GS4], [AC] that Gromov-Witten theory has been generalised to the setting of logarithmic geometry. One works over a base log scheme $(S, \mathcal{M}_S)$. The scheme S in practice could be the spectrum of a discrete valuation ring with the log structure induced by the closed point (one-parameter degeneration), or it could be $\operatorname{Spec} \Bbbk$, for $\Bbbk$ an algebraically closed field of characteristic zero, endowed with the trivial log structure (absolute situation), or $\operatorname{Spec} \Bbbk$ endowed with *the standard log structure* (central fibre of one-parameter degeneration). The standard log structure up to isomorphism is given uniquely by a monoid Q with $Q^\times = \{0\}$ giving rise to the log structure

$$Q \oplus \Bbbk^\times \longrightarrow \Bbbk, \quad (q, a) \longmapsto \begin{cases} a, & q = 0 \\ 0, & q \neq 0. \end{cases}$$

on $\operatorname{Spec} \Bbbk$. We will restrict our attention to the latter case and take the log point endowed with the standard log structure as a base scheme. Throughout this paper we assume

$$\Bbbk = \mathbb{C} \text{ and } Q := \mathbb{N}$$

and denote the standard log point by

$$\operatorname{Spec} \mathbb{C}^\dagger := (\operatorname{Spec} \mathbb{C}, \mathbb{N} \oplus \mathbb{C}^\times).$$

One generalises the notion of a stable map to the log setting as follows. Consider an ordinary stable map with a number, say ℓ, of marked points. Thus we have a proper curve C with at most nodes as singularities, a regular map $f : C \to X$, a tuple

$$\mathbf{x} = (x_1, \ldots, x_\ell)$$

of closed points in the non-singular locus of C. Moreover the triple $(C, \mathbf{x}, f)$ is supposed to fulfill the stability condition of finiteness of the group of automorphisms of $(C, \mathbf{x})$ commuting with f. To promote such a stable map to a stable log map amounts to endow all spaces with (fine, saturated) log structures and lift all morphisms to morphisms of log spaces. Then $C \to \operatorname{Spec} \mathbb{C}$ is promoted to a smooth morphism of log spaces

$$\pi : C^\dagger \longrightarrow \operatorname{Spec} \mathbb{C}^\dagger.$$

and we have a log morphism (Definition A.34)

$$f : C^\dagger \longrightarrow X^\dagger.$$

where $X^\dagger$ denotes the log scheme $(X, \mathcal{M}_X)$, endowed with a log structure $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ and $C^\dagger$ denotes the log scheme C, endowed with a log structure $\alpha_C : \mathcal{M}_C \to \mathcal{O}_C$. Given a morphism of log spaces $f : C^\dagger \longrightarrow X^\dagger$, we denote by $\underline{f} : C \longrightarrow X$ the underlying morphism of schemes as well as the underlying morphism on topological spaces.

Throughout this paper we will assume that the arithmetic genus of the domain curve C is zero;

$$g(C) = 0$$

and thus will work on the Zariski site, rather than the étale site which would be needed for more general cases.

Let $x \in X$ be a closed point in and let $f_x^\flat : \mathcal{M}_{X,f(x)} \to \mathcal{M}_{C,x}$ be the morphism of monoids induced by $f : C^\dagger \longrightarrow X^\dagger$. Then, by the definition of a log morphism we obtain the following commutative diagram on the level of stalks

$$\begin{tikzcd} \mathcal{M}_{X,f(x)} \arrow[r, "f_x^\flat"] \arrow[d, "\alpha_{X,f(x)}"'] & \mathcal{M}_{C,x} \arrow[d, "\alpha_{C,x}"] \\ \mathcal{O}_{X,f(x)} \arrow[r, "f_x^\sharp"] & \mathcal{O}_{C,x} \end{tikzcd} \tag{3.1}$$

Let $\kappa : \mathcal{M}_X \xrightarrow{/\mathcal{O}_X^\times} \overline{\mathcal{M}}_X$ be the quotient homomorphism. By the commutativity of the above diagram there is a morphism induced by f on the level of ghost sheaves, denoted by

$$\overline{f}_x^\flat : \overline{\mathcal{M}}_{Y_0,f(x)} \to \overline{\mathcal{M}}_{C,x}$$

for a closed point $x \in C$. By abise of notation morphism

$$\overline{f}_x^\flat : \overline{\mathcal{M}}^{gp}_{Y_0,f(x)} \to \overline{\mathcal{M}}^{gp}_{C,x}$$

on group level is also denoted by $\overline{f}_x^\flat$.

We demand the morphism $f : C^\dagger \to \operatorname{Spec} \mathbb{C}^\dagger$ to be *log smooth*. This means locally on C and X, we have the following commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \operatorname{Spec} \mathbb{Z}[P] \arrow[d] \\ \operatorname{Spec} \mathbb{C} \arrow[r] & \operatorname{Spec} \mathbb{Z}[\mathbb{N}] \end{tikzcd} \tag{3.2}$$

such that

(i) The horizontal maps induce charts $P \to \mathcal{M}_C$ and $\mathbb{N} \to \mathbb{N} \oplus \mathbb{C}^\times$ for the log structures on C and $\operatorname{Spec} \mathbb{C}$ respectively.

(ii) The right vertical arrow is induced by a map of toric monoids $\mathbb{N} \to P$.

(iii) The induced morphism

$$C \to \operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec} \mathbb{Z}[P]$$

is a smooth morphism of schemes.

We furthermore demand that the regular points of C where π is not strict are exactly the marked points. We will recall the precise shape of such log structures on nodal curves instantly.

Remark 3.1. After a moment of thought one may conclude that an algebraic stack based on this notion of a stable log smooth map over the log point $(\operatorname{Spec} \Bbbk, Q \oplus \mathbb{C}^\times)$ can never be of finite type, because for a given stable log map one can always enlarge the monoid Q, for example by embedding Q into $Q \oplus \mathbb{N}^r$. To solve this issue, a basic insight in [GS4], [AC] is that there is a universal, minimal choice of Q. In this *basic monoid* there are just enough generators and relations to lift $f : C \to X$ to a morphism of log spaces while maintaining log smoothness of $C^\dagger \to \operatorname{Spec} \Bbbk^\dagger$. After the usual fixing of topological data (genus, homology class etc.) the corresponding stack of *basic* stable log maps turns out to be a proper Deligne-Mumford stack.

For the present paper this general theory is both a bit too general and still a bit too limited. It is too general because we will end up with a finite list of unobstructed stable log maps over the standard log point. In particular, there is always a distinguished morphism

$$(\operatorname{Spec} \mathbb{C}, \mathbb{N}) \to (\operatorname{Spec} \mathbb{C}, Q)$$

from any of our stable log maps to the corresponding log map with the basic monoid just coming from our degeneration situation. Therefore, throughout this paper we comfortably assume the basic monoid Q is given by the natural numbers $Q := \mathbb{N}$. Moreover, there is no need for working with higher dimensional moduli spaces or with virtual fundamental classes, as the moduli space of the stable log maps over $\operatorname{Spec} \mathbb{C}^\dagger$ form a proper Deligne-Mumford stack of finite type ([GS4], §3).

The general theory is also too restricted because we will have to admit non-complete domains. The presence of non-complete components requires an ad hoc treatment of compactness of our moduli space that is special to our situation.

In this section we define and discuss the properties of the special kind of stable log maps with non-complete components which we refer to as *log corals*. In the next sections we show that these types of stable log maps over the standard log point $\mathbb{C}^\dagger$ yield tropical corals. We first recall a couple of definitions from [GS4].

Definition 3.2. A *log smooth curve* over the standard log point

$$\operatorname{Spec} \mathbb{C}^\dagger := (\operatorname{Spec} \mathbb{C}, \mathbb{N} \oplus \mathbb{C})$$

consists of a fine saturated log scheme

$$C^\dagger := (C, \mathcal{M}_C)$$

with a log smooth, integral morphism

$$\pi : C^\dagger \to \operatorname{Spec} \mathbb{C}^\dagger$$

of relative dimension 1 such that every fibre of π is a reduced and connected curve.

Note that we do not demand C to be *proper* which is the case in ([GS4], Defn. 1.3).

If we specify a tuple of sections $\mathbf{x} := (x_1, \cdots, x_l)$ of π so that over the non-critical locus $U \subset C$ of π we have

$$\overline{\mathcal{M}}_C|_U \simeq \pi^* \overline{\mathcal{M}}_{\operatorname{Spec} \mathbb{C}^\dagger} \oplus \bigoplus_i x_{i*} \mathbb{N}_{\operatorname{Spec} \mathbb{C}}$$

then we call the log smooth curve $C^\dagger$ *marked* and denote it by

$$(C/\operatorname{Spec} \mathbb{C}^\dagger, \mathbf{x}, f).$$

Before going through more details we would first like to make a few remarks on the local structure of log smooth curves $\pi : C^\dagger \to \operatorname{Spec} \mathbb{C}^\dagger$ over the standard log point $\operatorname{Spec} \mathbb{C}^\dagger$. Let $0 \in \operatorname{Spec} \mathbb{C}$ be the closed point. Then we have an isomorphism

$$\begin{array}{rcl} \overline{\mathcal{M}}_{\operatorname{Spec} \mathbb{C}, 0} & \longrightarrow & \mathbb{N} \\ \bar{t}^a & \longmapsto & a \end{array}$$

Let $\sigma : \mathbb{N} \to \mathcal{M}_{\operatorname{Spec} \mathbb{C}, 0}$ be the chart for the log structure on $\operatorname{Spec} \mathbb{C}$ around $0 \in \operatorname{Spec} \mathbb{C}$, given by

$$\sigma(q) = \begin{cases} 1 & \text{if } q = 0 \\ 0 & \text{if } q > 0 \end{cases}$$

Now, we state the following crucial theorem, which is a special case of a theorem due to Kato ([Kf, p.222]), as we are restricting our attention to log smooth curves over the standard log point.

Theorem 3.3. *Locally C is isomorphic to one of the following log schemes V over* $\operatorname{Spec} \mathbb{C}$.

(i) $\operatorname{Spec}(\mathbb{C}[z])$ *with the log structure induced from the homomorphism*

$$\mathbb{N} \longrightarrow \mathcal{O}_V, \quad q \longmapsto \sigma(q).$$

(ii) $\operatorname{Spec}(\mathbb{C}[z])$ *with the log structure induced from the homomorphism*

$$\mathbb{N} \oplus \mathbb{N} \longrightarrow \mathcal{O}_V, \quad (q, a) \longmapsto z^a \sigma(q).$$

(iii) $\mathrm{Spec}(\mathbb{C}[z,w]/(zw-t))$ *with* $t \in \mathfrak{m}$*, where* $\mathfrak{m}$ *is the maximal ideal in* $\mathbb{C}$ *and with the log structure induced from the homomorphism*

$$\mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2 \longrightarrow \mathcal{O}_V, \quad \big(q,(a,b)\big) \longmapsto \sigma(q) z^a w^b.$$

Here $\mathbb{N} \to \mathbb{N}^2$ *is the diagonal embedding and* $\mathbb{N} \to \mathbb{N}$*,* $1 \mapsto \rho_q$ *is some homomorphism uniquely defined by* $C \to \mathrm{Spec}\,\mathbb{C}$*. Moreover,* $\rho_q \neq 0$*.*

In this list, the morphism $C^\dagger \to \mathrm{Spec}\,\mathbb{C}^\dagger$ *is represented by the canonical maps of charts* $\mathbb{N} \to \mathbb{N}$*,* $\mathbb{N} \to \mathbb{N} \oplus \mathbb{N}$ *and* $\mathbb{N} \to \mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2$*, respectively where we identify the domain* $\mathbb{N}$ *always with the first factor of the image.*

Remark 3.4. In Theorem 3.3 cases (i), (ii), (iii) correspond to neighbourhoods of general points, marked points and nodes of C respectively. Thus, we have the following table describing the local structure of $\overline{\mathcal{M}}_C$ for all possible types of points on C.

Points x on C	$\overline{\mathcal{M}}_{C,x}$
$x = \eta$ a generic point	$\mathbb{N}$
$x = p$ a marked point	$\mathbb{N}^2$
$x = q$ a node	$\mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2$

Note that the base moniod $\mathbb{N}$, together with the choice of $\rho_q \neq 0$ at each node $q \in C$, determines the log structure $\alpha_C : \mathcal{M}_C \to \mathcal{O}_C$.

Definition 3.5. Nodes and marked points of a log smooth curve $C^\dagger \to \mathrm{Spec}\,\mathbb{C}^\dagger$ are referred to as *special points* of C.

Throughout this paper we work with log maps to the central fiber over $t = 0$ of the degeneration of the unfolded Tate curve

$$Y_0 = \mathbb{A}^1_s \times \bigcup_{\infty} \mathbb{P}^1$$

where the charts for the log structure on $\mathcal{M}_{Y_0}$ are given in 1.12. The advantage on working with Y_0 is as follows. Since the total space Y is a toric variety, the log structure $\overline{\mathcal{M}}^{gp}_Y$ is globally generated. In particular, for every closed point $x \in Y_0$

$$\Gamma(Y_0, \overline{\mathcal{M}}^{gp}_{Y_0}) \twoheadrightarrow \overline{\mathcal{M}}^{gp}_{Y_0,x}$$

is surjective. Now we are ready to generalise the notion of a log map over $\mathrm{Spec}\,\mathbb{C}^\dagger$ ([GS4], 1.2) to the case where the domain includes some special non-complete components.

Definition 3.6. Let $(C/\mathrm{Spec}\,\mathbb{C}^\dagger, \mathbf{x}, f)$ be a marked log smooth curve and let

$$Y_0^\dagger = (Y_0, \mathcal{M}_{Y_0})$$

be the central fiber of the unfolded Tate curve over $t = 0$ endowed with the log structure induced by the embedding of Y_0 into the toric variety Y. A morphism of log schemes $f : C^\dagger \to Y_0$ is called a *log map* over $\operatorname{Spec} \mathbb{C}^\dagger$ if the following holds:

i.) The morphism f fits into the following commutative diagram

$$\begin{tikzcd} C^\dagger \arrow[r, "f"] \arrow[dr, "\pi"'] & (X, \mathcal{M}_{Y_0}) \arrow[d] \\ & \operatorname{Spec} \mathbb{C}^\dagger \end{tikzcd}$$

ii.) For each non-complete irreducible component $C' \subset C$, there is an isomorphism $C' \cong \mathbb{A}^1$.

iii.) The map $s \circ f\big|_{C'} : C' \to \mathbb{A}^1$ is dominant, where

$$s : Y_0 \to \mathbb{A}^1_s$$

is the projection map.

iv.) For each marked point $p_i \in C$

$$f(p_i) \in (Y_0)_{s=0}$$

where $(Y_0)_{s=0}$ is the fiber over $0 \in \mathbb{A}^1_s$ under the map $s : Y_0 \to \mathbb{A}^1$.

v.) The automorphism group $\operatorname{Aut}(C/\operatorname{Spec} \mathbb{C}^\dagger, \mathbf{x}, f)$ is finite.

Before proceeding, we will take a little time to discuss the log structure on the non-complete components of the domain of log maps in more detail.

Let $f : C^\dagger \to Y_0^\dagger$ be a stable log map. Let $C' \subset C$ be a non-complete component with generic point η. Denote the function field of $C' = \mathbb{A}^1$ by

$$\mathcal{O}_{C',\eta} = k(z)$$

Let $Z \subset Y_0$ be the smallest toric stratum containing $f(C')$. Then, we have one of the two following cases.

i) $\dim \overline{Z} = 2$: in this case we call f *transverse* at C'. A chart for the log structure $\mathcal{M}_{Y_0}$ around $f(\eta)$ is given by

$$\begin{aligned} \overline{\mathcal{M}}_{Y_0}|_{\mathcal{U}} &\to \mathcal{M}_{Y_0}|_{\mathcal{U}} \\ \bar{t} &\mapsto s_t \end{aligned} \tag{3.3}$$

for an open neighbourhood $\mathcal{U}$ of $f(\eta)$. The coordinates on $\mathcal{U}$ are given either by

$$\{x, t, s \mid x = (st)^b\}$$

or by

$$\{y, t, s \mid y = (st)^b\}$$

where t is fixed. Throughout this section without loss of generality we assume the former. The morphism

$$\begin{aligned} f_\eta^\flat : \mathcal{M}_{Y_0,f(\eta)} &\longrightarrow \mathcal{M}_{C,\eta} \\ s_x &\longmapsto \varphi_x(z) \\ s_s &\longmapsto \varphi_s(z) \end{aligned}$$

is only determined by $\varphi_x, \varphi_s \in k(z)\backslash\{0\}$ which are determined by $f_\eta^\sharp : \mathcal{O}_{Y_0,f(\eta)} \to \mathcal{O}_{C,\eta}$.

ii) $\dim \overline{Z} = 1$: in this case we call f *non-transverse* at C'. A chart for the log structure $\mathcal{M}_{Y_0}$ around $f(\eta)$ is given by

$$\begin{aligned} \overline{\mathcal{M}}_{Y_0}|_{\mathcal{U}} &\longrightarrow \mathcal{M}_{Y_0}|_{\mathcal{U}} \\ \overline{x} &\longmapsto s_x \\ \overline{y} &\longmapsto s^{-b}s_y \\ \overline{t} &\longmapsto s_t \end{aligned} \tag{3.4}$$

for an open neighbourhood $\mathcal{U}$ of $f(\eta)$, where $\overline{x} \cdot \overline{y} = \overline{t}^b$. The coordinates on $\mathcal{U}$ are given by $\{x, y, s, t \mid xy = (st)^b\}$. The morphism

$$\begin{aligned} f_\eta^\flat : \mathcal{M}_{Y_0,f(\eta)} &\longrightarrow \mathcal{M}_{C,\eta} \\ s_x &\longmapsto \varphi_x(z) \cdot t^\alpha \\ s_y &\longmapsto \varphi_y(z) \cdot t^\beta \\ s_s &\longmapsto \varphi_s(z) \end{aligned}$$

is determined by $\varphi_x, \varphi_y, \varphi_s \in k(z) \setminus \{0\}$ and $\alpha, \beta \in \mathbb{N} \setminus \{0\}$ such that $\alpha + \beta = b$ and $\varphi_x\varphi_y = \varphi_s^b$.

To set up the log counting problem in the next section, analogously as in the tropical counting problem, we restrict our attention to *general* log maps defined as follows.

Definition 3.7. A stable log map $f : C^\dagger \to {Y_0}^\dagger$ is called *general* if each non-complete component $C' \subset C$ has only one special point and each complete component contains at most three special points.

Note that for a general stable log map if a non-complete component $C' \subset C$ admits a marked point as the unique special point, then by the connectivity of C there can not exist any other component. Since this case can be treated easily in all arguments we use, we omit it and assume the unique special point on each non-complete component on a general coral is a node.

Let $f : C^\dagger \to {Y_0}^\dagger$ be a general log coral. Let $C' = \operatorname{Spec} \mathbb{C}[z] \subset C$ be a non-complete component with generic point η and

$$\mathcal{O}_{C',\eta} = \mathbb{C}(z)$$

Take charts for the log structure $\mathcal{M}_{Y_0}$ as in 1.12. Then, we have one of the following two cases.

(i) If f is transverse at C': We have a section

$$s_x \in \Gamma(\mathcal{U}, \mathcal{M}_{Y_0})$$

for an appropriate open subset $\mathcal{U} \subset Y_0$ of Y_0 with $f(C') \subset \mathcal{U}$ so that

$$\alpha_{Y_0}(s_x) = x \in \mathcal{O}^\times_{Y_0}(\mathcal{U}).$$

Let

$$f^{-1}\alpha_{Y_0} : f^{-1}\mathcal{M}_{Y_0} \to f^{-1}\mathcal{O}_{Y_0}$$

be the morphism between the inverse image sheaves of monoids on C and let

$$f^{-1}s_x \in \Gamma(f^{-1}(\mathcal{U}), f^{-1}\mathcal{M}_{Y_0}|_{C'})$$

be the section of $f^{-1}\mathcal{M}_{Y_0}|_{C'}$ induced by s_x. Then we obtain a rational function

$$\varphi_x \in k(z) \setminus \{0\}$$

on C' given by

$$\varphi_x = f^{-1}\alpha_{Y_0}(f^{-1}s_x). \tag{3.5}$$

ii.) If f is non-transverse at C': We have

$$s_{x^\beta/y^\alpha} \in \Gamma(\mathcal{U}, \mathcal{M}^{gp}_{Y_0})$$

for an appropriate open subset $\mathcal{U} \subset Y_0$ with $f(C') \subset \mathcal{U}$ and $\alpha, \beta \in \mathbb{N} \setminus \{0\}$ are as in 3.5. Then

$$f^b_\eta s_{x^\beta/y^\alpha} \in \Gamma(C', f^{-1}\mathcal{M}^{gp}_{Y_0}|_{C'})$$

is a section of $f^{-1}\mathcal{M}^{gp}_{Y_0}|_{C'}$. By 3.5,

$$f^b_\eta s_{x^\beta/y^\alpha} = \varphi^\beta_x/\varphi^\alpha_y$$

is a rational function.

Now, let $C' = \mathbb{P}^1 \setminus \{\infty\} \subset C$ be as above. Then, we have the following important condition, which will give us the tropical balancing condition on the negative vertices of a tropical coral as we will see in section 4.

Definition 3.8. Let $val_\infty : k(z) \to \mathbb{Z}$ be the valuation at $z = \infty$. We call f *asymptotically parallel* at C' if the following holds.

i) $val_\infty(\varphi_x) = 0$, if f is transverse at C'.
ii) $val_\infty(\varphi_x^\beta/\varphi_y^\alpha) = 0$, if f is non-transverse at C' .

Finally, we are ready to define *log corals.*

Definition 3.9. A log map $f : C^\dagger \to {Y_0}^\dagger$ over $\mathbb{C}^\dagger$ is called a *log coral* if f is *asymptotically parallel at* C' for each non-complete component $C' \subset C$.

We will sometimes replace Y_0 by an open subset of it containing $\operatorname{im} f$ or by the $\mathbb{Z}$-quotient of it. The definition of log corals then still makes sense with the obvious modifications.

Remark 3.10. Let $f : C^\dagger \to {Y_0}^\dagger$ be a stable log map and let $C' \subset C$ be a non-complete component. Assume $q = C' \cap \mathbb{P}^1$ is a nodal point of C' mapping to $(Y_0)_{s=0}$. If f is transverse at C', let $\varphi_x \in k(z) \setminus \{0\}$ be as in (3.5). Assume

$$p = f(q).$$

Then, we have

$$\varphi_x(q) = x(p)$$

which is the position of the intersection point $f(q) = f(C') \cap (Y_0)_{s=0}$ on $(Y_0)_{s=0}$.

If f is non-transverse at C', let $\widetilde{Y}_0$ be the weighted blow-up of Y_0 at $x = y = 0$ with weights $\alpha, \beta \in \mathbb{N} \setminus \{0\}$. By [AW] the count of stable log maps is invariant under blowing-up, thus generally we restrict our attention to log maps $f : C^\dagger \to {Y_0}^\dagger$ lifting to $\widetilde{f} : C^\dagger \to \widetilde{Y}_0^\dagger$. Let $\varphi_x^\beta/\varphi_y^\alpha \in k(z) \setminus \{0\}$ be as in 3.5. Assume

$$\widetilde{p} = \widetilde{f}(q).$$

Then, we have

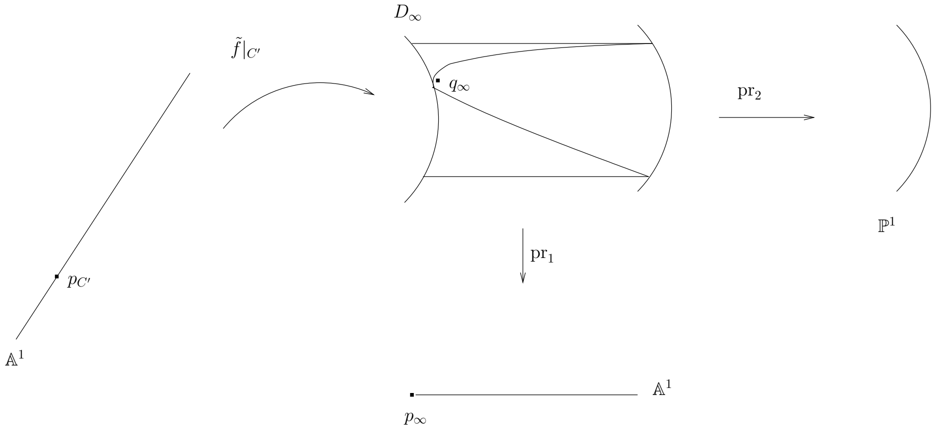
$$\varphi_x^\beta/\varphi_y^\alpha(q) = x^\beta/y^\alpha(\widetilde{p})$$

which is the position of the point of intersection of $\widetilde{f}(C')$ with the exceptional divisor on $\widetilde{Y}_0$.

Let $\mathbb{A}^1 \times \mathbb{P}^1 \subset Y_0$ be an irreducible component of Y_0, compactified to $\mathbb{P}^1 \times \mathbb{P}^1$ along $D_\infty = \mathbb{P}^1$ to which we refer to as *the divisor at infinity.* Let

$$q_\infty = \tilde{\varphi}(p_{C'}) \in D_\infty.$$

Then we refer to as q_∞ *the point at infinity.* See the following figure for an illustration of it.



Before proceeding in the next section with tropicalizations, we would like to make a couple of more remarks concerning the position of the point q_∞ on D_∞. Let $f : C^\dagger \to {Y_0}^\dagger$ be a log coral and $C' \subset C$ be a non-complete irreducible component of C. The restriction $f|_{C'}$ factors over an irreducible component $\mathbb{A}^1 \times \mathbb{P}^1 \subset Y_0$. Hence we have a morphism $f' : C' \to \mathbb{A}^1 \times \mathbb{P}^1$ fitting into the following diagram.

(3.6)
$$\begin{tikzcd} C' \arrow[r, "f|_{C'}"] \arrow[dr, dashed, "f'"'] & Y_0 \\ & \mathbb{A}^1 \times \mathbb{P}^1 \arrow[u, hook] \end{tikzcd}$$

Let

$$\mathrm{pr}_1 : \mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{A}^1$$
$$\mathrm{pr}_2 : \mathbb{A}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$$

be the projection maps and let

$$\begin{aligned} \varphi &:= \mathrm{pr}_1 \circ f' : C' \to \mathbb{A}^1 \\ \phi &:= \mathrm{pr}_2 \circ f' : C' \to \mathbb{P}^1 \end{aligned}$$

$$\begin{tikzcd} & & \mathbb{A}^1 \\ C' \arrow[r, "f'"] & \mathbb{A}^1 \times \mathbb{P}^1 \subset Y_0 \arrow[ur, "\mathrm{pr}_1"] \arrow[dr, "\mathrm{pr}_2"] & \\ & & \mathbb{P}^1 \end{tikzcd}$$

Note that the map $\varphi : \mathbb{A}^1 \to \mathbb{A}^1$ is identically equal with $s \circ f|_{C'}$ as in 3.6, so it is dominant. Thus, it extends uniquely to a morphism $\tilde{\varphi} : \mathbb{P}^1 \to \mathbb{P}^1$ where the domain of

$\tilde{\varphi}$ is the completion

$$\{p_{C'}\} \ \cup \ C' \cong \mathbb{P}^1$$

of C' along the point at infinity $p_{C'}$ on C' and

$$\tilde{\varphi}(z) := \begin{cases} \varphi(z) & \text{for } z \in C' \\ q_\infty & \text{for } z = p_{C'} \end{cases}$$

By the Hurwitz formula $\tilde{\varphi} : \mathbb{P}^1 \to \mathbb{P}^1$ is a covering totally branched at infinity. The branching order is defined as follows.

Definition 3.11. Let $f : C^\dagger \to Y_0^\dagger$ be a log coral with a non-complete component $C' \subset C$ and let $\varphi = s \circ f|_{C'}$ be defined as in (Defn. 3.6,(iii)). We say C' has *branching order* $w \in \mathbb{N} \setminus \{0\}$ if the degree of the covering $\tilde{\varphi} : \mathbb{P}^1 \to \mathbb{P}^1$ equals w.

We will describe the position of q_∞ on D_∞ in a moment. Recall φ_x is given as in (3.5) if f is transverse at C' and $\varphi_x^\beta/\varphi_y^\alpha$ if f is non-transverse at C'.

Definition 3.12. Let

$$\begin{array}{rcl} \upsilon : \operatorname{Spec} \mathbb{C}[w] \setminus \{0\} & \to & \operatorname{Spec} \mathbb{C}[z] \setminus \{0\} \\ w & \mapsto & \dfrac{1}{z} \end{array}$$

be the morphism which induces the map between fraction fields given by

$$\begin{array}{rcl} \upsilon^* : \mathbb{C}(z) & \to & \mathbb{C}(w) \\ r & \mapsto & r \circ \upsilon \end{array}$$

Let

$$\overline{\varphi_x} := \upsilon^*(\varphi_x) \ \text{ and } \ \overline{\varphi_x^\beta/\varphi_y^\alpha} := \upsilon^*(\varphi_x^\beta/\varphi_y^\alpha)$$

Then, the value over $w = 0$ given by

i.) $\overline{\varphi_x}(0)$ if f is transverse at C'

ii.) $\overline{\varphi_x^\beta/\varphi_y^\alpha}(0)$ if f is non-transverse at C'

gives the position of $q_\infty \subset D_\infty$ which we refer to as the *virtual position at infinity.*

3.2. **The set up of the counting problem.** In this section we set up the counting problem for log corals $f : C^\dagger \to {Y_0}^\dagger$. The count depends on the notions of coral degree Δ, degeneration order λ and log constraint ρ that we will define in a moment.

Recall the charts for $\mathcal{M}_C$ are described using Theorem 3.3. For generic points we have the isomorphism

$$\begin{array}{r} \overline{\mathcal{M}}^{gp}_{C,\eta} \to \mathbb{Z} \\ \bar{t}^a \mapsto a \end{array}$$

and for marked points p of C have the isomorphism

$$\begin{aligned}\overline{\mathcal{M}}^{gp}_{C,p} &\to \mathbb{Z}\oplus\mathbb{Z}\\ (\bar{t}^a,\bar{z}^b) &\mapsto (a,b)\end{aligned}$$

and throughout this section we denote by $\mathrm{pr}_i : \overline{\mathcal{M}}^{gp}_{C,p} \to \mathbb{Z}$ the projection to the i-th factor for $i = 1, 2$.

Explicitly, we take the following charts for $\mathcal{M}_C$ around generic and marked points.

(i) For a generic point η of an irreducible component of C;

$$\begin{array}{ccc}\overline{\mathcal{M}}_{C,\eta} & \longrightarrow & \mathcal{M}_{C,\eta}\\ \bar{t} & \longmapsto & s_t\end{array}$$

(ii) For a marked point p of C;

$$\begin{array}{ccc}\overline{\mathcal{M}}_{C,p} & \longrightarrow & \mathcal{M}_{C,p}\\ \bar{t} & \longmapsto & s_t\\ \bar{z} & \longmapsto & s_z\end{array}$$

Throughout this section we assume the charts for the log structure $\mathcal{M}_{Y_0}$ on the central fiber Y_0 over $t = 0$ of the unfolded Tate curve are given as in 1.12.

We first discuss how to assign an element of N to marked points and to non-complete irreducible components of the domain C of a log coral $f : C^\dagger \to Y_0^\dagger$. This will lead us to the definition of the coral degree Δ and degeneration order λ.

Let $f : C^\dagger \to Y_0^\dagger$ be a log coral with marked points $(p_1, \ldots, p_{l+1})$. Following the general scheme in [GS4], for each marked point p_i we now define an element of N recording the logarithmic contact order of p_i with Y_0 as follows. Let $\Xi \in \mathscr{P}$ be the interval such that the corresponding toric chart

$$(C(\overline{C}\Xi))^\vee \cap (M\oplus\mathbb{Z}) \longrightarrow \Gamma(U_\Xi, \mathcal{M}_Y)$$

covers $f(p_i)$. Composing with $\overline{f^\flat}$ and restricting to p_i thus yields a homomorphism of monoids

$$(C(\overline{C}\Xi))^\vee \cap (M\oplus\mathbb{Z}) \longrightarrow \overline{\mathcal{M}}_{Y,f(p_i)} \xrightarrow{\overline{f^\flat_{p_i}}} \overline{\mathcal{M}}_{C,p_i} = \mathbb{N}\oplus\mathbb{N}.$$

The two factors of $\mathbb{N}\oplus\mathbb{N}$ in $\overline{\mathcal{M}}_{C,p_i}$ are generated by the smoothing parameter s_t and by the equation defining p_i in C, respectively. Taking the composition with the projection to the second factor thus defines a homomorphism

$$u_i : (C(\overline{C}\Xi))^\vee \cap (M\oplus\mathbb{Z}) \longrightarrow \mathbb{N},$$

that is, an element of $C(\overline{C}\Xi) = ((C(\overline{C}))^\vee)^\vee$. Since by definition this homomorphism maps $(0,1) \in M\oplus\mathbb{Z}$ to 0, it lies in $(0,1)^\perp = N\oplus\{0\} \subset N\oplus\mathbb{Z}$. Moreover, identifying

$N \oplus \{0\}$ with N, the intersection of $C(\overline{C}\Xi)$ with $N_{\mathbb{R}} \oplus \{0\}$ agrees with the asymptotic cone of $\overline{C}\Xi$. We have thus defined, for each marked point $p_i \in C$, an element $u_i \in N$ contained in the asymptotic cone of the relevant cell of $\overline{C}\mathscr{P}$.

Now, given a log coral $f : C^\dagger \to Y_0^\dagger$, again following [GS4], each unbounded component $C' \simeq \mathbb{A}^1$ of C determines an integral point in $B = \overline{C}\mathbb{R}$ as follows. Let $\Xi \in \mathscr{P}$ be the interval with the corresponding log chart containing $f(\eta)$. The composition of the chart with the restriction to the stalk at η defines a homomorphism

$$\phi_\eta : (C(\overline{C}\Xi))^\vee \cap (M \oplus \mathbb{Z}) \longrightarrow \overline{\mathcal{M}}_{Y,f(\eta)} \xrightarrow{\overline{f_\eta^\flat}} \overline{\mathcal{M}}_{C,\eta} = \mathbb{N}.$$

Since f is a log morphism relative to the standard log point, $\phi_\eta(0,1) = 1$. Thus viewed as an element of $N \oplus \mathbb{Z}$, we have $\phi_\eta \in N \oplus \{1\}$. Define

$$u_\eta = \phi_\eta - (0,1) \tag{3.7}$$

as an element of $N = N \oplus \{0\} \subset N \oplus \mathbb{Z}$. Moreover, $\phi_\eta(s) = 0$ for C' does not lie in Y_0. This means $\phi_\eta \in C(\Xi \times \{1\}) = C(\partial B)$, and in turn u_η maps to an element of $\mathbb{N} \setminus \{0\}$ under the projection $N = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ to the height.

To define the coral degree for C', we need to multiply u_η with the contact order at infinity. This contact order is simply defined by pulling back $s \in \mathcal{O}(Y)$ to C' and taking the negative of the valuation at ∞, the missing point of $C' \simeq \mathbb{A}^1 \subset \mathbb{P}^1$:

$$w_\eta = -\operatorname{val}_\infty(f^\sharp(s)) \tag{3.8}$$

Note that $w_\eta \in \mathbb{N} \setminus \{0\}$ since $f^\sharp(s)$ is a non-constant regular function on $C' \simeq \mathbb{A}^1$.

Now, under these conventions we can define the coral degree as follows.

Definition 3.13. The *coral degree* (*C-degree*) with $\ell+1$ positive and m negative entries is a tuple

$$\Delta := (\overline{\Delta}^{\ell+1}, \underline{\Delta}^m) \subset N^{l+1} \times N^m$$

of elements in the lattice N with

$$\operatorname{pr}_2(\overline{\Delta}_i) > 0$$
$$\operatorname{pr}_2(\underline{\Delta}_j) < 0$$

for all i, j where by $\operatorname{pr}_2 : N = \mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}$ we denote the projection onto the second factor.

Definition 3.14. A log coral $f : C^\dagger \to Y_0^\dagger$ with $\ell + 1$ marked points

$$\{p_0, p_1, \ldots, p_\ell\}$$

and m non-complete components

$$C'_1, \cdots, C'_m$$

is said to be of coral degree $\Delta := (\overline{\Delta}^{\ell+1}, \underline{\Delta}^m)$ if the following conditions hold

(i) For each marked point p_i of C, the composition

$$\mathrm{pr}_2 \circ (\overline{f}^\flat_{f(p_i)})^{gp} : \overline{\mathcal{M}}^{gp}_{Y_0,f(p_i)} \to \overline{\mathcal{M}}^{gp}_{C,p_i} \to \mathbb{Z}$$

equals $\overline{\Delta}_i$.

(ii) For each non-complete component $C' \subset C$ with generic point η_j, the morphism $w_j \cdot u_j$ defined by Equations (3.7) and (3.8) equals $\underline{\Delta}_j$.

Example 3.15. Let $f : C^\dagger \to {Y_0}^\dagger$ be a log coral. Let η be the generic point of a non-complete component $C' \subset C$ and let $p \in C$ be a marked point. Take charts for the log structures $\mathcal{M}_C$ around p and η as in 3.2.

If f is transverse at C', then the map $\bar{f}^\flat_{f(\eta)} : \overline{\mathcal{M}}_{Y_0,f(\eta)} \longrightarrow \overline{\mathcal{M}}_{C,\eta}$ is given by $\bar{t} \longmapsto \bar{s}_t$. Assume f is non-transverse at C' and let $\bar{f}^\flat_{f(\eta)}$ be given by

$$\begin{aligned} \bar{f}^\flat_{f(\eta)} : \overline{\mathcal{M}}_{Y_0,f(\eta)} &\longrightarrow \overline{\mathcal{M}}_{C,\eta} \\ \bar{x} &\longmapsto \bar{s}_t^\alpha \\ \bar{y} &\longmapsto \bar{s}_t^\beta \\ \bar{t} &\longmapsto \bar{s}_t^b \end{aligned} \tag{3.9}$$

where $\alpha, \beta \in \mathbb{N} \setminus \{0\}$ such that $\alpha + \beta = b$. Then fixing the coral degree of f, fixes α and β.

Therefore, the map $\bar{f}^\flat_{f(\eta)}$ is determined by the degree at generic points η of a non-complete component of C.

Now, assume p is a marked point and take a chart for $\mathcal{M}_{Y_0}$ around $f(p)$ as follows.

$$\begin{aligned} \overline{\mathcal{M}}_{Y_0,f(p)} &\longrightarrow \mathcal{M}_{Y_0,f(p)} \\ \overline{x} &\longmapsto s_x \\ \overline{y} &\longmapsto s_y \\ \overline{s} &\longmapsto s_s \\ \bar{t} &\longmapsto s_t \end{aligned}$$

where

$$\overline{xy} = (\overline{s}\bar{t})^b \text{ in } \overline{\mathcal{M}}_{X,f(p)}$$

is induced by the equation

$$s_x s_y = (s_s s_t)^b \text{ in } \mathcal{M}_{Y_0}$$

Let $\bar{f}^{\flat}_{f(p)} : \overline{\mathcal{M}}_{Y_0,f(p)} \to \overline{\mathcal{M}}_{C,p}$ be given as

$$\begin{array}{rcl} \bar{f}^{\flat}_{f(p)} : \overline{\mathcal{M}}_{Y_0,f(p)} & \longrightarrow & \overline{\mathcal{M}}_{C,p} \\ \bar{x} & \longmapsto & \bar{t}^{f} \cdot \bar{z}^{e} \\ \bar{y} & \longmapsto & \bar{t}^{h} \cdot \bar{z}^{g} \\ \bar{s} & \longmapsto & \bar{t}^{d} \cdot \bar{z}^{c} \\ \bar{t} & \longmapsto & \bar{t} \end{array}$$

where $f, h, d \in \mathbb{N} \setminus \{0\}$ and $e, g, c \in \mathbb{N}$. Fixing the coral degree of f fixes the z-powers e, g, c. To obtain a finite number of possibilities for the morphism $\bar{f}^{\flat} : f^{-1}\overline{\mathcal{M}}_{Y_0,f(p)} \to \overline{\mathcal{M}}_{C,p}$ on the level of ghost sheaves we need more constraints also regarding the t-powers, which we define in the following definition.

Definition 3.16. Let $f : C^{\dagger} \to {Y_0}^{\dagger}$ be a log coral of coral degree $\Delta := (\overline{\Delta}^{\ell+1}, \underline{\Delta}^{m})$. Let $\{p_0, \ldots, p_\ell\} \subset C$ be the set of marked points on C, such that $p_j \in C_j$ for a complete component C_j of C. Let η_j denote the generic point of C_j and let

$$\lambda := (\lambda_1, \cdots, \lambda_\ell) \in \prod_{i=1}^{\ell} N/(N \cap \mathbb{R}\overline{\Delta}_j)$$

We say $f : C^{\dagger} \to Y_0^{\dagger}$ has *degeneration order* λ if the image of

$$(\bar{f}_{\eta_j} : \overline{\mathcal{M}}_{Y_0,f(\eta_j)} \to \overline{\mathcal{M}}_{C,\eta_j} \cong \mathbb{N}) \in N$$

under the quotient map

$$N \to N/(N \cap \mathbb{R}\overline{\Delta}_j)$$

is equal to λ_j for all $j = 1, \ldots, \ell$.

Note that the degeneration order describes a constraint on all but one marked points.

So far we have defined incidence conditions on the level of ghost sheaves which we will see all have tropical analogues. We will observe in the section 4 that fixing the coral degree of a log coral $f : C^{\dagger} \to {Y_0}^{\dagger}$ corresponds to fixing the degree of the corresponding tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ and fixing the degeneration order of f corresponds to fixing asymptotic constraints for h. Therefore, given a scheme theoretic morphism $\underline{f} : C \to Y_0$, the finiteness of the possibilities for the morphisms on the level of ghost sheaves $\overline{f}^{\flat} : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ follows from the fact that the number of tropical corals with a fixed degree matching asymptotic constraints on all but one of its unbounded edges is finite which we had seen in section 2.4.

Next we describe an additional constraint referred to as a *log constraint*, which does not have a tropical analogue, but will be required to ensure the finiteness of the lifts of the morphism $\bar{f}^{\flat} : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ on the level of ghost sheaves to a log morphism

$f : C^\dagger \to {Y_0}^\dagger$. For this we first need the following definition concerning the positions of marked points. Recall that we require the marked points on C to map to the toric boundary of a toric stratum in Y_0 under f. This is analogous to the situation treated in [GPS].

Definition 3.17. Let $f : C^\dagger \to {Y_0}^\dagger$ be a log coral with marked points $\{p_0, \ldots, p_\ell\} \subset C$, such that $p_j \in C_j$ for a complete component $C_j \subset C$. Let φ_x and $\varphi_x^\beta/\varphi_y^\alpha$ be defined analogously to (3.5) and (**??**) respectively. Recall by (Defn. 3.6, (iii)) $f(p)$ lies inside the big torus orbit of an irreducible component the toric boundary $(Y_0)_{s=0}$ for any marked point p of the domain.

i.) If f does not contract $C_j \cong \mathbb{P}^1$: Then

$$\varphi_x(p_j) \in \mathbb{C}^\times$$

is the position of $f(p_j)$ in the big torus orbit $\mathbb{C}^\times \subset C_j$.

ii.) If f contracts $C_j \cong \mathbb{P}^1$ to a nodal intersection point p of two complete components: Let E be the exceptional divisor on the (weighted) blow-up of Y_0 as in Remark 3.10. Then,

$$\varphi_x^\beta/\varphi_y^\alpha(p_j) \in \mathbb{C}^\times$$

is the position of $f(p_j)$ in the big torus orbit $\mathbb{C}^\times \subset E$.

We refer to $\varphi_x(p_j)$ in case (i) and to $\varphi_x^\beta/\varphi_y^\alpha(p_j)$ in case (ii) as the *marked point position*. Note that we will discuss a more intrinsic definition for the log constraints in the following section (5.3), which are defined in a moment in terms of marked point positions. Once we have the intrinsic definition it will be more explicit why $\varphi_x(p_j)$ and $\varphi_x^\beta/\varphi_y^\alpha(p_j)$ are indeed elements of $\mathbb{C}^\times$.

Now, let ℓ, m be positive integers. A tuple

$$\rho = (\overline{\rho}^\ell, \underline{\rho}^m) \in (\mathbb{C}^\times)^\ell \times (\mathbb{C}^\times)^m$$

is called a *log constraint of order* (ℓ, m).

Definition 3.18. A log coral $f : C^\dagger \to Y_0^\dagger$ matches a log constraint $\rho = (\overline{\rho}^\ell, \underline{\rho}^m)$ if

i.) C has $\ell + 1$ marked points $\{p_0, \ldots, p_\ell\}$ and the marked point position of p_j defined in Definition 3.17 equals $\overline{\rho_j}$, for $j = 1, \ldots, \ell$.

ii.) C has m unbounded components $\{C_1', \ldots, C_m'\}$ and the virtual position at infinity defined in Definition 3.12 for C_i' is equals $\underline{\rho_i}$, for $i = 1, \ldots, m$.

Definition 3.19. Let $\mathfrak{L}_{\Delta,\lambda,\rho}$ be the set of all log corals $f : C^\dagger \to Y_0^\dagger$ of coral degree Δ, degeneration order λ and matching a log constraint ρ. Define $N^{log}_{\Delta,\lambda,\rho}$ to be the cardinality

$$N^{log}_{\Delta,\lambda,\rho} := |\mathfrak{L}_{\Delta,\lambda,\rho}|$$

If $N^{log}_{\Delta,\lambda,\rho}$ is finite this defines a number giving us a count of log corals. In 5.11 we will see that, indeed

$$N^{log}_{\Delta,\lambda,\rho} = N^{trop}_{\Delta,\lambda}$$

where $N^{trop}_{\Delta,\lambda}$ is the tropical count defined in 2.4 for λ a good asymptotic constraint on the stable range. On the log side λ is a suitably chosen degeneration order which under tropicalization will correspond to good asymptotic constraints on the stable range.

Remark 3.20. A priori, $\mathfrak{L}_{\Delta,\lambda,\rho}$ has the structure of a stack, and it is a closed substack of a larger algebraic stack of log maps with some non-complete components. In the present case we are in the comfortable situation that due to unobstructedness of deformations, $\mathfrak{L}_{\Delta,\lambda,\rho}$ turns out to be a reduced scheme over $\mathbb{C}$ of finite length, hence really does not carry more information than the underlying set.

4. Tropicalizations

In this section we will define the tropicalization of a log scheme and explain how a log coral $f : C^\dagger \to Y_0^\dagger$ induces a map

$$f^{\mathrm{trop}} : \mathrm{Trop}(C^\dagger) \to \mathrm{Trop}(Y_0^\dagger)$$

between the associated tropical spaces, referred to as the *tropicalization of* f. We will see that actually the data $(\underline{f}, \overline{f}^\flat)$ given by the scheme-theoretic map $\underline{f} : C \to Y_0$ together with a morphism $\overline{f}^\flat : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ between sheaves of monoids on the ghost sheaf level is enough to determine the tropicalization of f.

The main result we need to establish in this section is that the tropicalization of a log coral is a tropical coral. The substantial point in this statement is that f being asymptotically parallel implies the balancing condition also on the vertices mapping to the lower boundary of the truncated cone $\overline{C}\mathbb{R}$. We assume $\overline{C}\mathbb{R}$ is endowed with a polyhedral decomposition $\overline{C}\mathscr{P}_b$ as in Definition 1.5 for a fixed $b \in \mathbb{N} \setminus \{0\}$. We often suppress $\overline{C}\mathscr{P}_b$ in the notation when referring to $\overline{C}\mathbb{R}$.

We first review how to define the tropicalization $\mathrm{Trop}(X)$ of a log scheme $(X, \mathcal{M}_X)$ ([GS4], Appendix B.). Then, we discuss how to obtain a morphism $f^{\mathrm{trop}} : \mathrm{Trop}(C) \to \mathrm{Trop}(Y_0)$ given a log coral $f : C^\dagger \to Y_0^\dagger$, which will give us the tropical coral obtained by tropicalizing f.

Definition 4.1. Let $X^\dagger$ be a log scheme endowed with the Zariski topology. The tropical space associated to $X^\dagger$ denoted by $\mathrm{Trop}(X)$ referred to as the *tropicalization of* $X^\dagger$ is defined as

$$\mathrm{Trop}(X) := \left(\coprod_{x \in \underline{X}} \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}) \right) \Big/ \sim$$

where the disjoint union is over all scheme-theoretic points x of $\underline{X}$ and the equivalence relation is generated by the identifications of faces given by dualizing generization maps $\overline{\mathcal{M}}_{X,x} \to \overline{\mathcal{M}}_{X,x'}$ when x is a specialization of x'. One then obtains for each x a map

$$i_x : \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}) \to \mathrm{Trop}(X).$$

which is injective since $\overline{\mathcal{M}}_X$ is fine in the Zariski topology.

Now, given a log coral $f : C^\dagger \to {Y_0}^\dagger$ and a closed point $x \in \underline{C}$, the map $\bar{f}_x^\flat : \overline{\mathcal{M}}_{Y_0,f(x)} \to \overline{\mathcal{M}}_{C,x}$ induces a morphism $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}, \mathbb{R}_{\geq 0}) \to \mathrm{Hom}(\overline{\mathcal{M}}_{Y,f(x)}, \mathbb{R}_{\geq 0})$. Hence we obtain the morphism

$$f^{\mathrm{trop}} : \mathrm{Trop}(C) \to \mathrm{Trop}(Y_0)$$

referred to as the tropicalization of f which is compatible with the equivalence relations defining $\mathrm{Trop}(C)$ and $\mathrm{Trop}(Y_0)$. We will establish the one-to-one correspondence between the data

$$(\underline{f}, \overline{f}^\flat)$$

and a tropical coral of a fixed degree matching a general asymptotic constraint in detail in this section. In the following sections we will see that there will be exactly N^{trop} log corals $f : C^\dagger \to Y_0^\dagger$ with given tropicalization, where N^{trop} is the tropical count defined in section 2.4. This will show that the tropical and log counts match.

We restrict to the case of current interest that Y_0 is the fibre over $t = 0$ of the degeneration of the unfolded Tate curve $Y \to \operatorname{Spec} \mathbb{C}[s, t]$ defined by an integral polyhedral decomposition $\overline{C}\mathscr{P}_b$ of (part of) $N_\mathbb{R}$ with $N \simeq \mathbb{Z}^2$ ([GS3] and [NS]). Thus the cones of the fan defining Y are $C(\overline{C}\Xi) \subset N_\mathbb{R} \oplus \mathbb{R}$, and the morphism to $\mathbb{A}^1$ is defined by projecting this fan to the last coordinate.

Given a log coral $f : C^\dagger \to Y_0^\dagger$ and a closed point $x \in C$ of the domain if $f(x)$ lies in the big cell of the toric stratum of Y_0 defined by $\overline{C}\Xi \in \overline{C}\mathscr{P}_b$, then

$$\overline{\mathcal{M}}_{Y_0,f(x)} = C(\overline{C}\Xi)^\vee \cap (M \oplus \mathbb{Z})/C(\overline{C}\Xi)^\perp \cap (M \oplus \mathbb{Z})$$

Dualizing yields

$$\overline{\mathcal{M}}^\vee_{Y_0,f(x)} \subset C(\overline{C}\Xi) \cap (N \oplus \mathbb{Z}),$$

with $M = \mathrm{Hom}(N, \mathbb{Z})$. To save notation let us write $C_\mathbb{Z}$ for the integral points of a cone C in a finitely generated free abelian group.

For $\overline{\mathcal{M}}^\vee_{C,x}$ we have the three possibilities $\mathbb{N}$, $\mathbb{N} \oplus \mathbb{N}$ and $\mathbb{N} \oplus_\mathbb{N} \mathbb{N}^2$, for x a non-special or generic point, a marked point or a node, respectively by Theorem 3.3.

For any (scheme-theoretic) point $x \in C$, the map of log structures on the ghost sheaf level is given by a homomorphism of toric monoids[3] $\overline{f}^b_x : \overline{\mathcal{M}}_{Y_0,f(x)} \to \overline{\mathcal{M}}_{C,x}$. Dualizing yields

$$(\overline{f}^b_x)^\vee : \overline{\mathcal{M}}^\vee_{C,x} \longrightarrow \overline{\mathcal{M}}^\vee_{Y_0,f(x)}. \tag{4.1}$$

We claim that the collection of all these data is conveniently encoded in a tropical coral. Given the data $(\underline{f}, \overline{f}^\flat)$ of a scheme-theoretic map $\underline{f} : C \to Y_0$ and a morphism $\overline{f}^\flat : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ on the level of ghos sheaves we construct a tropical coral associated to this data, referred to as the tropicalization of f (or the tropicalization of $(\underline{f}, \overline{f}^\flat)$) as follows.

[3]For higher genus of C we need to work with the étale topology on C here, but since our interest in this paper is with rational curves exclusively, the Zariski topology suffces.

Construction 4.2. (1) *(Vertices)* If $x = \eta$ is a generic point, (4.1) defines a map

$$V_\eta : \mathbb{N} \longrightarrow C(\overline{C}\Xi)_{\mathbb{Z}}.$$

Tracing the map to the standard log point shows that $V_\eta(1)$ projects to $b \in \mathbb{Z}$ under the projection $C(\overline{C}\Xi)_{\mathbb{Z}} \to \mathbb{Z}$ to the last coordinate. Thus V_η may be viewed as an integral point in $b \cdot \overline{C}\Xi \subset N_{\mathbb{R}} \times \{b\}$, the copy of $N_{\mathbb{R}}$ at height b in $N_{\mathbb{R}} \times \mathbb{R}$.

(2) *(Edges)* If $x = q$ is a nodal point of C with local equation $zw = t^{e_q}$, we have a homomorphism

$$(\overline{f}^b_q)^\vee : (\mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2)^\vee \longrightarrow C(\overline{C}\Xi)_{\mathbb{Z}}.$$

The domain of this map is the set of integral points of $C([0, e_q])$, the cone over an interval of integral length e_q. If η_1, η_2 are the generic points of the two irreducible components of C containing q (necessarily different since genus$(C) = 0$), then the two extremal generators $(0, 1)$ and $(e_q, 1)$ of this cone map to V_{η_1} and V_{η_2}, respectively (see [GS4], discussion 1.8 for details). Thus $(\overline{f}^b_q)^\vee$ is completely determined by V_η of the adjacent irreducible components and e_q. This data can be conveniently encoded by an edge connecting V_{η_1} and V_{η_2} in $N_{\mathbb{R}} \times \{b\}$, with associated weight $w_q = e_q$. It can also be shown that there is an integral vector $u_q \in N$ with

$$V_{\eta_2} - V_{\eta_1} = e_q u_q.$$

(3) *(Marked points)* If $x = p$ is a marked point, the homomorphism is

$$(\overline{f}^b_q)^\vee : \mathbb{N} \oplus \mathbb{N} \longrightarrow C(\overline{C}\Xi)_{\mathbb{Z}}.$$

The morphism to the standard log point generates one of the two factors of $\mathbb{N}\oplus\mathbb{N}$, say the first one. Then the second factor is generated by a local coordinate of C at p. Then $(\overline{f}^b_q)^\vee$ is a map with domain the integral points of the two-dimensional cone $\mathbb{R}^2_{\geq 0}$. The first generator $(1, 0)$ maps to V_η, with η the generic point of the irreducible component containing p. The second generator maps to some element $u_p \in C(\overline{C}\Xi) \cap (N \times \{0\})$. Thus u_p can be viewed as an element of the asymptotic cone of $\overline{C}\Xi$. We represent this element tropically by an unbounded edge emanating from V_η in direction u_p with weight equal to the divisibility of u_q (the largest natural number w_p with $w_p^{-1} u_q \in N$. We refer to $w_p \in \mathbb{N} \setminus \{0\}$ as the *contact order* of the marked point p.

(4) *(Balancing)* For η the generic point of a complete irreducible component of C the adjacent edges fulfill the balancing condition

$$\sum_{x \in \mathrm{cl}(\eta)} w_x u_x = 0.$$

Here x runs over the set of special points p, q in the closure of η in C.

Remark 4.3. In [GS4], Proposition 1.15, a modified balancing condition is stated for general log schemes. It is shown that for a stable log map $f : C^\dagger \to Y^\dagger$

$$\sum_{x \in \mathrm{cl}(\eta)} w_x u_x + \sum_{x \in \mathrm{cl}(\eta)} \tau_x = 0.$$

where x runs over the set of special points p, q in the closure of η in C. Here, given an irreducible component $D \subset C$ with normalization $g : \tilde{D} \to C$ and $\mathcal{M} := f^*\mathcal{M}_X$, the map τ_x is given by the composition

$$\tau_x : \Gamma(\tilde{D}, g^*\overline{\mathcal{M}}) \to \mathrm{Pic}\tilde{D} \xrightarrow{deg} \mathbb{Z}$$

which is the morphism mapping a section $m \in \Gamma(\tilde{D}, g^*\overline{\mathcal{M}})$ to the degree of the corresponding $\mathcal{O}^\times_{\tilde{D}}$ torsor $L_m \in \mathcal{M}|_{\tilde{D}}$. Note that any section $m \in \Gamma(\tilde{D}, g^*\overline{\mathcal{M}})$ gives rise to an $\mathcal{O}^\times_{\tilde{D}}$ torsor

$$L_m := \kappa^{-1}(m) \subset g^*\mathcal{M}$$

for $\kappa : \mathcal{M} \xrightarrow{/\mathcal{O}^\times} \overline{\mathcal{M}}$ the quotient homomorphism. To adopt this result to our case we need to show that τ_x vanishes identically. This actually holds in all toric situations. Let $z^m \in K(Y)$ be the rational function corresponding to $m \in \Gamma(\tilde{D}, g^*\overline{\mathcal{M}})$. Clearly, the sum of the orders of poles and zeroes of z^m is zero. Hence the global section of L_m induced by z^m has degree zero.

Let now $f : (C, \mathcal{M}_C) \to (Y_0, \mathcal{M}_{Y_0})$ be a log coral. Thus Y_0 is the central fibre of the degeneration of the unfolded Tate curve and C has some non-compact components, isomorphic to $\mathbb{A}^1$. The associated tropical curve then has one vertex in the interior of $\overline{C}(B)$ for each irreducible component mapping to $s^{-1}(0) \subset Y_0$ and one vertex in the lower boundary $\partial\overline{C}B$ for each $\mathbb{A}^1$-component. Each marked point of C necessarily maps to $s^{-1}(0)$ and gives rise to an unbounded edge (a positive end of the associated tropical coral). The balancing condition at the interior vertices follows from the balancing result of [GS4]. What is not obvious from the established picture is the balancing condition at the vertices on $\partial\overline{C}B$. We show this at the end of this section. First let is describe the tropicalizations of the domain C and of Y_0 in more detail.

The tropicalization $\mathrm{Trop}(C)$ of the domain of a log coral will be given by the *dual graph of* C constructed as follows.

Construction 4.4. Let $f : C^\dagger \to Y_0^\dagger$ be a map log coral. Assume $g(C) = 0$. Then, the *dual graph* of C is constructed as follows.

i.) Define the set of vertices $V(\mathcal{G})$ so that there is a vertex $v_j \in V(\mathcal{G})$ for each irreducible component $C_j \subset C$.

ii.) Define the set of bounded edges $E^b(\mathcal{G})$ so that an edge $e_{ij} \in E^b(\mathcal{G})$ connecting two vertices v_i and v_j exists if and only if the corresponding irreducible components $C_i \subset C$ and $C_j \subset C$ intersect.

iii.) Define the set of unbounded edges denoted by $E^+(\mathcal{G})$ so that for each edge $e_j \in E^+(\mathcal{G})$ adjacent to a vertex $v_j \in V(\mathcal{G})$ there exist a marked point $p_j \in C_j \subset C$ on the irreducible component C_j corresponding to v_j.

We refer to a vertex $v \in V(\mathcal{G})$ a negative vertex if $v \in \partial\overline{C}\mathbb{R}$ and in this case the corresponding component $C_v \subset C$ is non-complete.

Proposition 4.5. *Let $f : C^\dagger \to Y_0^\dagger$ be a log coral. Then, the tropicalization* $\mathrm{Trop}(C)$ *of the domain $C^\dagger$ is the cone $C(\mathcal{G})$ over the dual graph $\mathcal{G}$ of C.*

Proof. Recall from Theorem 3.3 that there are three possibilities of closed points $x \in C$, with the stalk $\overline{\mathcal{M}}_{C,x}$ is either $\mathbb{N}$, $\mathbb{N} \oplus \mathbb{N}$ or $\mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2$, for x a generic point, a marked point or a node, respectively. Then, the vertices of $\mathcal{G}$ correspond to generic points η, unbounded edges (or the flags of the graph $\mathcal{G}$) correspond to the marked points and bounded edges correspond to nodes. So, item (i) is a direct consequence of the Definition 4.1. □

Note that there is a length function $\ell : E^b(\mathcal{G}) \to \mathbb{Z}$ defined as follows. Let $e_q \in E^b(\mathcal{G})$ be a bounded edge such that e_q corresponds to the node $q \in C$ with

$$\overline{\mathcal{M}}_{C,q} = \mathbb{N} \oplus_{\mathbb{N}} \mathbb{N}^2 = S_e$$

where S_e is the monoid obtained by the push-out of the diagonal map $\mathbb{N} \to \mathbb{N}^2$ and the the homothety $\mathbb{N} \xrightarrow{\cdot e} \mathbb{N}$ for $e \in \mathbb{N} \setminus \{0\}$. Define

$$\begin{aligned} \ell : E^b(\mathcal{G}) &\to \mathbb{Z} \\ e_q &\longmapsto e \end{aligned}$$

Now, given a log coral $f : C^\dagger \to Y_0^\dagger$ to describe the tropicalization of Y_0 we first define the tropicalization of Y.

Proposition 4.6. *Let $Y \to \operatorname{Spec}\mathbb{C}[s,t]$ be the degeneration of the unfolded Tate curve. And let Y_0 be the central fiber over $t = 0$. Then,*

(i) *The tropicalization* $\mathrm{Trop}(Y)$ *of Y is the cone $C(\overline{C}\mathbb{R})$ over the truncated cone $\overline{C}\mathbb{R}$ endowed with the polyhedral decomposition $\overline{C}\mathscr{P}_b$.*

(ii) $\mathrm{Trop}(Y) = \mathrm{Trop}(Y_0)$

Proof. Recall from section 1 that Y is the toric variety associated to $(\overline{C}\mathscr{P}_b, \overline{C}\mathbb{R})$. Since the log structure on Y is fine and constant along any open toric strata it follows from Definition 4.1 that $\mathrm{Trop}(Y)$ is the cone over the fan associated to Y. Note that this holds for any toric variety, hence in particular for Y. This proves (i). To show (ii), observe that for the subvariety $Y_0 \subset Y$ the following holds. For any closed point $x \in Y$, we have $\overline{x} \cap Y_0 \neq \emptyset$. Therefore, (ii) follows. $\square$

Lemma 4.7. *Let $f : C^\dagger \to {Y_0}^\dagger$ be a log coral with associated tropicalization map $f^{\mathrm{trop}} : C(\mathcal{G}) \to C(\overline{C}\mathbb{R})$. Then, there are height functions $h_C : C(\mathcal{G}) \to \mathbb{R}_{\geq 0}$ and $h_{Y_0} : C(\overline{C}\mathbb{R}) \to \mathbb{R}_{\geq 0}$ fitting into the following commutative diagram.*

$$\begin{tikzcd} C(\mathcal{G}) \arrow[r, "f^{\mathrm{trop}}"] \arrow[dr, "h_C"'] & C(\overline{C}\mathbb{R}) \arrow[d, "h_{Y_0}"] \\ & \mathbb{R}_{\geq 0} \end{tikzcd}$$

Proof. Let $\operatorname{Spec}\mathbb{C}^\dagger := (\operatorname{Spec}\mathbb{C}, \mathbb{N}\oplus\mathbb{C})$ be the standard log pont. By Defiintion 4.1, it follows that the tropicalization of $\operatorname{Spec}\mathbb{C}^\dagger$ is

$$\mathrm{Trop}(\operatorname{Spec}\mathbb{C}^\dagger) = \mathbb{R}_{\geq 0}$$

Hence, the result follows by the functoriality of the tropicalization map and the commutativity if the diagram in Definition 3.6. $\square$

By restricting $f^{\mathrm{trop}} : C(\mathcal{G}) \to C(\overline{C}\mathbb{R})$ to $h_C^{-1}(1)$, we obtain a map $h : \Gamma \to \overline{C}\mathbb{R}$ which is a tropical coral.

We will discuss in more detail the correspondence between scheme theoretic maps $\underline{f} : C \to Y_0$ together with a morphism on the level of ghost sheaves $\overline{f}^\flat : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ and tropical corals $h : \Gamma \to \overline{C}\mathbb{R}$.

Now, we are ready to prove the main theorem of this section.

Proposition 4.8. *The tropicalization of a log coral is a tropical coral.*

Proof. It remains to prove the balancing condition at the lower boundary of $\overline{C}B$. Let $f : C^\dagger \to Y_0^\dagger$ be a log map such that C has an unbounded irreducible component $C' \subset C$ with generic point η and let $v \in \overline{C}B$ be the vertex corresponding to C'. Define the quotient map

$$\pi_v : N \to N \;/\; \mathbb{R}\cdot v \cap N$$

So, π_v gives an element $m_v \in \mathrm{Hom}(N, \mathbb{Z}) = M$. Define

$$x_v := z^{m_v} \in \mathbb{C}[M]$$

Observe that $f^*(x_v) = f^*(x)$ if f is transverse at C' and $f^*(x_v) = f^*(x^\beta)/f^*(y^\alpha)$ if f is non-transverse at C' where x and α, β are as in (3.5) and 3.5. In any case, we have

$$\sum_{z\in C'} val_z(f^*x_v) = \sum_{z\in\mathbb{P}^1\setminus\{\infty\}} val_z(f^*x_v) = 0$$

since from the asymptotically parallel condition we know $val_\infty(f^*x_v) = 0$ and the sum of orders of poles and zeroes of the rational function f^*x_v on $\mathbb{P}^1$ is zero.

Let $z_1, \dots, z_k \in C'$ be the special points and let the corresponding edges emanating from V have direction vectors (with weights) $\xi_1, \dots, \xi_k$. Note that if z_i is a marked point then $\xi_i = u_i$ and if z_i is a node then $z_i = u_q$ where u_i, u_q are defined as in $(2), (3)$ in the construction of the tropical data associated to a log coral in 4.2.

Then, $\pi_v(\sum_{i=1}^k(\xi_i)) = \sum_{z\in C'} val_z(f^*x_v) = 0$ and hence the tropical balancing condition is satisfied.

□

Remark 4.9. Let $f : C^\dagger \to {Y_0}^\dagger$ be a log coral with tropicalization $h : \Gamma \to \overline{C}\mathbb{R}$. Then it follows from the definition of the incidence conditions on tropical and log corals that under the tropicalization map, the coral degree of $f : C^\dagger \to {Y_0}^\dagger$ corresponds to the degree of $h : \Gamma \to \overline{C}\mathbb{R}$, while the degeneration order of $f : C^\dagger \to {Y_0}^\dagger$ corresponds to the asymptotic constraints on $h : \Gamma \to \overline{C}\mathbb{R}$. Note also that, we fix the degeneration order of a log coral in a way so that the corresponding tropical incidence conditions are in the stable range, hence the tropicalization defines a general tropical coral.

Furthermore, the contact order at a marked point (4.2,(3)) corresponds to the weight on the corresponding unbounded edge while the branching order of a non-complete component $C' \subset C$ (Defn. 3.11) of a general log coral corresponds to the weight associated to the edge adjacent to the vertex $v_{C'} \in \overline{C}\mathbb{R}$ corresponding to C'.

5. The log count agrees with the tropical count

In the last section 4 we have seen that a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ encodes the data of a morphism $\overline{f}^{\flat} : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ on the level of ghost sheaves on C and Y_0. In this section we discuss the number of lifts of the morphism $\overline{f}^{\flat}$ to a log coral $f : C^{\dagger} \to Y_0^{\dagger}$. We refer to this number as $N^{\log}$. We show that the tropical count defined in section 2.4 equals $N^{\log}$.

For this, we will discuss the *extensions of log corals* which will allow us to reduce the situation to the case treated in [NS]. We will exhibit the bijection between the log corals $f : C^{\dagger} \to Y_0^{\dagger}$ and their extensions

$$f_e : \overline{C}^{\dagger} \to \overline{U_0}^{\dagger}$$

for a suitably chosen open subset $U_0 \subset Y_0$ and completions $\overline{C}$ of C and $\overline{U_0}$ of U_0 and matching certain incidence conditions. We will first describe the construction of the compactified target $\overline{U_0}$.

Let Δ be a degree of a tropical coral and let λ be an asymptotic constraint. Let $I \subset \mathbb{R}$ be a closed interval in $\mathbb{R}$, chosen big enough such that the images of all tropical corals $h_i : \Gamma_i \to \overline{C}\mathbb{R}$ for $i = 1, \ldots, k$ matching (Δ, λ) have support in $\overline{C}I \subset \overline{C}\mathbb{R}$.

Now, let $\mathscr{P}_b$ be the b-periodic polyhedral decomposition of $\mathbb{R}$ as in section 1 for $b \in \mathbb{N} \setminus \{0\}$ fixed. Define a polyhedral decomposition

$$\mathscr{P}' \subset \mathscr{P}_b$$

such that cells $\sigma' \in \mathscr{P}$ are obtained by $\sigma' = \sigma \cap I$ for a cell $\sigma \in \mathscr{P}$.

Let $\overline{C}\mathscr{P}'$ be the polyhedral decomposition of $\overline{C}I$ induced by $\mathscr{P}'$. Consider the toric degeneration associated to $(\overline{C}\mathscr{P}', \overline{C}I)$. Let $U_0 \subset Y_0$ be the fiber over $t = 0$ of this degeneration, which by construction is an open subset of Y_0 such that the image of any log coral $f : C^{\dagger} \to Y_0^{\dagger}$ whose tropicalization is h lies inside U_0.

To define the completion $\overline{U_0}$ of U_0 we extend the polyhedral decomposition $\overline{C}\mathscr{P}'$ of $\overline{C}I$ to a polyhedral decomposition $\tilde{\mathscr{P}}$ of $\mathbb{R}^2$ by applying the following steps:

1.) Add the horizontal line $l := \mathbb{R} \times \{1\}$ at height 1 to $|\overline{C}\mathscr{P}'|$.
2.) Extend each half line meeting l to a line which passes through the origin. This defines a polyhedral decomposition of $\mathbb{R}^2$ denoted by $\tilde{\mathscr{P}}$.
3.) Add 2-cells bounded by the 1-cells obtained by items 1.) and 2.), to achieve $\tilde{\mathscr{P}} = N_{\mathbb{R}} = \mathbb{R}^2$.

The new polyhedral decomposition $\tilde{\mathscr{P}}$ is called *the extended polyhedral decomposition* of $\overline{C}\mathscr{P}$. Then $\overline{U_0}$ is the fibre over $t = 0$ of the toric degeneration $\overline{U} \to \operatorname{Spec} \mathbb{C}[s, t]$ associated to $(N_{\mathbb{R}}, \tilde{P})$. We endow $\overline{U}$ with the divisorial log structure $\mathcal{M}_{(\overline{U}, \overline{U_0})}$. Then $\overline{U_0}$ and $U_0 \subset \overline{U}$ carry log structures obtained by the pull-back of $\mathcal{M}_{(\overline{U}, \overline{U_0})}$.

Note that $\overline{U_0}$ is a complete variety containing U_0 as an open subset. Topologically, we have

$$\overline{U_0} = \Big(\bigcup_n \mathbb{P}^1 \times \mathbb{P}^1\Big) \amalg_{\cup_n \mathbb{P}^1} Z_0$$

where Z_0 is the complete toric variety described by the $\mathbb{Z}_2$ symmetric fan at the origin whose one cells are given by m lines passing through the origin in $\mathbb{R}^2$, where m is equal to the number of negative vertices of the coral graph Γ.

We define the extension $f_e : \overline{C}^\dagger \to \overline{U_0}^\dagger$ of a log coral f as follows.

Definition 5.1. Let $f : C^\dagger \to Y_0^\dagger$ be a log coral with tropicalization $h : \Gamma \to \overline{C}\mathbb{R}$. Let $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ be the tropical extension of $h : \Gamma \to \overline{C}\mathbb{R}$ constructed as in 2.14. We call a stable log map $f_e : \overline{C}^\dagger \to \overline{U_0}^\dagger$ a *log extension* of f if $f_e|_C = f$ and the tropicalization of f_e is $\widetilde{h}$.

It follows from the definition of a log extension $f_e : \overline{C}^\dagger \to \overline{U_0}^\dagger$ of a log coral $f : C^\dagger \to Y_0^\dagger$ that the domain $\overline{C}$ is obtained from C as follows.

1.) For each non-complete component $C' \subset C$, take the completion $\overline{C'}$ of C' at the point at infinity $q_{C'}$.
2.) Attach transversally a copy of $\mathbb{P}^1$ for each non-complete component $C' \subset C$ such that
$$q_{C'} = \overline{C'} \cap \mathbb{P}^1$$
is the nodal intersection point.

Note that the log extension of a log coral is a particular type of a stable log map in the sense of [GS4]. Our next aim is to set up a counting problem for stable log maps $f : \tilde{C} \to \overline{U_0}$ that extend log corals uniquely. So, let $\tilde{C}$ be a proper log smooth curve over $\operatorname{Spec}\mathbb{C}^\dagger$, which is the case treated in [GS4]. We also assume $g(\overline{C}) = 0$ throughout this section. To set up the counting problem for stable log maps $f : \overline{C} \to \overline{U_0}$ that extend log corals we need to fix the data given by $(\widetilde{\Delta}, \widetilde{\lambda}, \widetilde{\rho})$ analogously to section 3.2, which we define in a moment.

The log degree $\widetilde{\Delta}$ of a stable log map is defined as follows, analogously as in Definition 3.13 of the coral degree of a log coral.

Definition 5.2. The *log degree* with $\ell + 1$ positive and m negative entries is a tuple elements in N given by

$$\tilde{\Delta} := (\overline{\tilde{\Delta}}^{\ell+1}, \underline{\tilde{\Delta}}^{m}) \subset N^{\ell+1} \times N^m$$

with

$$\operatorname{pr}_2(\overline{\tilde{\Delta}}_i) > 0$$
$$\operatorname{pr}_2(\underline{\tilde{\Delta}}_j) < 0$$

A log map $f : \tilde{C}^\dagger \to \overline{U_0}^\dagger$ with $(\ell+1)+m$ marked points $x_0, \dots, x_\ell, p'_1, \dots, p'_m$ for some $\ell, m \in \mathbb{N} \setminus \{0\}$ is said to be of log degree $\tilde{\Delta} := (\overline{\tilde{\Delta}}^{\ell+1}, \underline{\tilde{\Delta}}^m)$ if the following conditions hold

(i) For $i = 1, \dots, \ell+1$, the composition

$$\mathrm{pr}_2 \circ (\overline{f}^\flat_{f(x_i)})^{gp} : \overline{\mathcal{M}}^{gp}_{Y_0, f(x_i)} \to \overline{\mathcal{M}}^{gp}_{C, x_i} \to \mathbb{Z}$$

equals $\overline{\tilde{\Delta}}_i$.

(ii) For $j = 1, \dots, m$ the composition

$$\mathrm{pr}_2 \circ (\overline{f}^\flat_{f(p_j)}) gp : \overline{\mathcal{M}}^{gp}_{Y_0, f(p_j)} \to \overline{\mathcal{M}}^{gp}_{C, p_j} \to \mathbb{Z}$$

equals $\underline{\tilde{\Delta}}_j$.

The degeneration order $\widetilde{\lambda}$ for a stable log map is also defined analogously to 3.16. As the definitions follow straight forward, we leave the details to the reader.

Note that the coral degree of a log coral defined in 3.13 immediately determines the log degree of its extension. So, Δ defines $\tilde{\Delta}$ and the same holds for the degeneration order: the degeneration order λ of a log coral determines the extended degeneration order $\widetilde{\lambda}$ of its extension.

Now we will discuss how to define the extended log constraints intrinsically from the constraint of a tropical coral.

Discussion 5.3. We discuss an equivalent, but more intrinsic definition to the log constraint ρ defined in 3.18.

Let $u \in N$ be an asymptotic direction (one of the entries of the log-coral degree, which corresponds to one of the entries of the degree of the tropicalization) and λ a degeneration order. Let $\tilde{\lambda} \in N$ be a lift of λ under the quotient homomorphism $N \to N/\mathbb{Z} \cdot u$. Then, the pair (u, λ) defines the two-plane

$$H_{u,\lambda} = \mathbb{R} \cdot (u, 0) + \mathbb{R} \cdot (\tilde{\lambda}, 1) \subset N_{\mathbb{R}} \oplus \mathbb{R}.$$

Let $m_{u,\lambda}$ span the rank one subgroup $H^\perp_{u,\lambda} \cap (M \oplus \mathbb{Z})$ of $M \oplus \mathbb{Z}$. To fix signs we ask $m_{u,\lambda}$ to evaluate positively on $(1, 0, 0)$. Denote by

$$s_{u,\lambda} \in \Gamma(Y_0, \mathcal{M}^{\mathrm{gp}}_{Y_0})$$

the corresponding section induced by the rational function $z^{m_{u,\lambda}}$ on Y.

Thus for any log coral $f : C^\dagger \to Y_0^\dagger$ we obtain a section

$$f^\flat(s_{u,\lambda}) \in \Gamma(C, \mathcal{M}^{\mathrm{gp}}_C)$$

The point of the log coral to have degree u and degeneration order λ at a marked point p is that $f_p^\flat(s_{u,\lambda})$ lies in $\mathcal{O}_{C,p}^\times$. It thus makes sense to evaluate at p to obtain a non-zero complex number

$$f_p^\flat(s_{u,\lambda})(p) \in \mathbb{C}^\times.$$

Similarly, when we consider a non-complete component $C' \subset C$ we take the two-plane spanned by $(u, 0)$ and $(v, 1)$ where v is the tropical vertex on ∂B defining the non-complete component. The monomial thus obtained is denoted $s_{u,v}$. Now a log coral $f : (C, \mathcal{M}_C) \to (Y, \mathcal{M}_Y)$ with an unbounded component $C' \subset C$ with generic point corresponding to v is asymptotically parallel if and only if $f^\flat(s_{u,v})$ extends over the missing point ∞ of $C' \simeq \mathbb{P}^1 \setminus \{\infty\}$. In this case we can thus obtain a well-defined complex number as

$$f_p^\flat(s_{u,v})(\infty) \in \mathbb{C}^\times.$$

Indeed, in the notation of Definition 3.18, $f_p^\flat(s_{u,\lambda}) = \varphi_x$ if f is transverse at C' and $f_p^\flat(s_{u,\lambda}) = \varphi_{x^\beta/y^\alpha}$ if f is non-transverse at C'.

Definition 5.4. We call a tuple

$$\tilde{\rho} = (\overline{\tilde{\rho}}^\ell, \underline{\tilde{\rho}}^m) \subset (\mathbb{C}^\times)^\ell \times (\mathbb{C}^\times)^m$$

a *log constraint for a stable log map*, of order (ℓ, m). A stable log map $f : \tilde{C}^\dagger \to \overline{U_0}^\dagger$ with $\ell + 1 + m$ marked points for $\ell, m \in \mathbb{N} \setminus \{0\}$ as in Definition 5.2 matches a log constraint $\tilde{\rho} = (\overline{\tilde{\rho}}^\ell, \underline{\tilde{\rho}}^m)$ if the following holds. $\tilde{C}$ has $\ell + 1 + m$ marked points $\{x_0, \ldots, x_\ell, p_1, \ldots, p_m\}$ with

i.) $f_{x_j}^\flat(s_{u,\lambda})(x_j) = \overline{\tilde{\rho}_j}$, for $j = 1, \ldots, \ell$, that is for all but one marked points.
ii.) $f_{p_j}^\flat(s_{u,v})(\infty) = \underline{\tilde{\rho}_j}$, for $j = 1, \ldots, m$.

Definition 5.5. We refer to the incidence conditions $(\widetilde{\Delta}, \widetilde{\lambda}, \widetilde{\rho})$ as the *extended log incidences.*

Lemma 5.6. *Let $\tilde{f} : \overline{C}^\dagger \to \overline{U}_0^\dagger$ be a stable log map of degree Δ, degeneration order λ and log constraint ρ. Then the projection from the fibre product $\tilde{f} \times_{\overline{U}_0} U_0 \to U_0$ composed with the open embedding $U_0 \to Y_0$ is a log coral of the same degree Δ, degeneration order λ and log constraint ρ.*

Proof. The result is an immediate consequence of the definition of extended log incidences and the fact that $f^\flat(s_{u,p})$ is a non-zero regular function on the whole added component, hence constant. Therefore, the extended log incidences on the additional marked points of the log extension is immediately encoded on the non-complete components of the log coral. □

Our main result in this section is the following theorem.

Theorem 5.7. *Let Δ, λ, ρ be degree, degeneration order and log constraint respectively. Denote by $\widetilde{\mathcal{L}}_{\Delta,\lambda,\rho}$ and $\mathcal{L}_{\Delta,\lambda,\rho}$ the sets of isomorphism classes of stable log maps and of log corals of degree Δ, degeneration order λ and log constraint ρ, respectively. Then the forgetful map*

$$\widetilde{\mathfrak{L}}_{\Delta,\lambda,\rho} \to \mathfrak{L}_{\Delta,\lambda,\rho}$$

defined in Lemma 5.6, is bijective.

Before proceeding with the proof of this theorem we have a couple of remarks and the following lemma, which will be useful.

We say the data given by the tuple $(\underline{f}, \overline{f}^\flat)$ consisting of a scheme-theoretic morphism $\underline{f} : C \to Y_0$ and a morphism $\overline{f}^\flat : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ *lifts to a log coral* if we can define a stable log map $f : C^\dagger \to {Y_0}^\dagger$ with underlying scheme-theoretic map $\underline{f}$ and compatible with $\overline{f}^\flat$ such that the pull-back log structure $f^*\mathcal{M}_{Y_0} \to \mathcal{O}_C$ factors over a log structure $\mathcal{M}_C \to \mathcal{O}_C$ that is smooth over $(\operatorname{Spec}\mathbb{C}, \mathbb{N} \oplus \mathbb{C}^\times)$. That is, we have the following commutative diagram

$$\begin{tikzcd} f^*\mathcal{M}_{Y_0} \arrow[r] \arrow[dr, dashed] & \mathcal{O}_C \\ & \mathcal{M}_C \arrow[u, dashed] \end{tikzcd}$$

where $C^\dagger$ is a log smooth curve over $(\operatorname{Spec}\mathbb{C}, \mathbb{N} \oplus \mathbb{C}^\times)$.

Theorem 5.8. *Let $\underline{f} : C \to Y_0$ be a scheme-theoretic morphism and $\overline{f}^\flat : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ be a morphism on the level of ghost sheaves which corresponds to a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ under tropicalization as discussed in section 4. Then, $\overline{f}^\flat : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{C}$ lifts to a log coral $f : C^\dagger \to {Y_0}^\dagger$.*

Proof. Since Y_0 is toric, $\overline{\mathcal{M}}^{gp}$ is globally generated. Thus, for each closed point $x \in Y_0$ there is a surjection $\Gamma(Y_0, \overline{\mathcal{M}}^{gp}_{Y_0}) \to \overline{\mathcal{M}}^{gp}_{Y_0,x}$ fitting into the following commutative diagram

$$\begin{tikzcd} \Gamma(Y_0, \overline{\mathcal{M}}^{gp}_{Y_0}) \arrow[r] & \overline{\mathcal{M}}^{gp}_{Y_0,x} \\ \Gamma(Y_0, \mathcal{M}^{gp}_{Y_0}) \arrow[r] \arrow[u] & \mathcal{M}^{gp}_{Y_0,x} \arrow[u] \end{tikzcd}$$

where the vertical arrows are given by morphisms induced by the quotient homomorphism

$$\kappa : \mathcal{M}_{Y_0} \xrightarrow{/\mathcal{O}^\times_{Y_0}} \overline{\mathcal{M}}_{Y_0}$$

By abuse of notation we denote the induced morphisms $\Gamma(Y_0, \mathcal{M}_{Y_0}^{gp}) \to \Gamma(Y_0, \overline{\mathcal{M}}_{Y_0}^{gp})$ and $\mathcal{M}_{Y_0,x}^{gp} \to \overline{\mathcal{M}}_{Y_0,x}^{gp}$ also by κ. The lift of $\overline{f}^{\flat} : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ is uniquely determined by $f^{\flat} : f^{-1}\mathcal{M}_{Y_0} \to \mathcal{M}_C$ it is enough to define the morphism between the torsors

$$\varphi_{\overline{m}} : \mathcal{L}_{\overline{m}} \to \mathcal{L}_{f^{\flat}(\overline{m})}$$

where the torsor $\mathcal{L}_{\overline{m}}$ associated to a section $\overline{m} \in \Gamma(C, f^{-1}\overline{\mathcal{M}}_{Y_0}^{gp})$ is defined as

$$\mathcal{L}_{\overline{m}} := \kappa^{-1}(\overline{m}) \subset f^{-1}\mathcal{M}_{Y_0}$$

and $\mathcal{L}_{f^{\flat}(\overline{m})}$ is defined analogously. Any of the torsors $\mathcal{L}_m \subset f^{-1}\mathcal{M}_{Y_0}$ are trivial since we are in a purely toric situation. The triviality of $\mathcal{L}_{\overline{m}}$ in a toric situation follows since the degree of $\mathcal{L}_{\overline{m}} := \kappa^{-1}(\overline{m})$ is by definition the sum of poles and zeroes of the corresponding rational function $z^{\overline{m}}$ on the total space $Y \to \mathbb{A}^2$ of the degeneration of the unfolded Tate curve. Thus, the degree of $\mathcal{L}_{\overline{m}}$ is clearly zero.

Therefore a necessary and sufficient condition for $\varphi_{\overline{m}} : \mathcal{L}_{\overline{m}} \to \mathcal{L}_{f^{\flat}(\overline{m})}$ to exist is $\mathcal{L}_{f^{\flat}(\overline{m})}$ to be trivial. Since the genus of C is zero, the torsor $\mathcal{L}_{f^{\flat}(\overline{m})}$ is trivial if and only if the degree of $\mathcal{L}_{f^{\flat}(\overline{m})}$ restricted to each irreducible component $C_\eta \subset C$ vanishes. That is,

$$\deg_{C_\eta} \mathcal{L}_{f^{\flat}(\overline{m})} = 0$$

Assume C_η is a complete component of the domain admitting nodal points $q_1, \ldots, q_n$ and marked points $p_1, \ldots, p_m$. Let the corresponding morphisms u_{q_i}, u_{p_j} be defined as in section 4 for each $i = 1, \ldots, n$ and $j = 1, \ldots, m$. By Remark 4.3 in [GS4] we have

$$\deg_{C_\eta} \mathcal{L}_{f^{\flat}(\overline{m})} = \sum_i \langle u_{p_i}, f^{\flat}(\overline{m})\rangle + \sum_j \langle u_{q_i}, f^{\flat}(m)\rangle$$

which vanishes by the balancing condition 4.3. If $C_\eta \subset C$ is a non-complete component we again have the vanishing of $\deg_{C_\eta} \mathcal{L}_{f^{\flat}(\overline{m})}$ as a consequence of the balancing as shown in the proof of Proposition 4.8.

Therefore, $\mathcal{L}_{f^{\flat}(\overline{m})}$ is trivial and hence a lift $f^{\flat} : f^{-1}\mathcal{M}_{Y_0} \to \mathcal{M}_C$ of $\overline{f}^{\flat} : f^{-1}\overline{\mathcal{M}}_{Y_0} \to \overline{\mathcal{M}}_C$ exists. $\square$

Note that from the proof of the above theorem it follows that any morphism on the level of ghost sheaves lifts to a stable log map if its tropicalization is a tropical curve with first Betti number zero.

Proof of Theorem 5.7

Let

$$(f : C^{\dagger} \to Y_0^{\dagger}) \in \mathcal{L}_{\Delta,\lambda,\rho}$$

be a log coral and with support in

$$U_0 \subset Y_0.$$

We have to show that f has exactly one extension

$$\tilde{f} : \overline{C}^\dagger \to \overline{U}_0^\dagger$$

to a stable log map in $\widetilde{\mathcal{L}}_{\Delta,\lambda,\rho}$.

The tropicalization

$$h : \Gamma \to B = \overline{C}\mathbb{R} \subset N_\mathbb{R}$$

of f has a unique extension

$$\tilde{h} : \tilde{\Gamma} \to N_\mathbb{R}$$

to a tropical curve of degree Δ and asymptotic constraint λ as discussed in section 2.3. By functoriality of tropicalizations, the tropical curve $\tilde{h}$ is the tropicalization of any extension of the log coral f to a stable log map $\tilde{f}$ with the same log degree and degeneration order.

If Γ has m negative vertices, the extension of the tropical coral adds m vertices, each mapping to $0 \in N_\mathbb{R}$, while the vertices of the tropical coral on ∂B turn into divalent vertices of the tropical curve. Thus for the domain $\overline{C}$ of the extension $\tilde{f}$, there is no choice other than to replace each non-complete component of C, an $\mathbb{A}^1$, by a chain of two copies of $\mathbb{P}^1$'s.

Denote by $D \subset \overline{C}$ one of the added components. We can determine its image $\tilde{f}(D)$ by consideration of the log constraints as follows. Recall from the discussion of log constraints above (5.5) that each unbounded component of the log coral determines a monomial $z^{m_{u,\lambda}}$ on Y, hence a section

$$s_{u,\lambda} \in \Gamma(Y_0, \mathcal{M}_{Y_0}^{\mathrm{gp}})$$

with the property that $f^\flat(s_{u,\lambda})$ is invertible at ∞. Moreover, the corresponding log constraint ρ is the value

$$f^\flat(s_{u,p})(\infty) \in \mathbb{C}^\times.$$

For the extended stable log map $\tilde{f}$, the torsor for the corresponding section $\overline{\tilde{f}^\flat}(\overline{s}_{u,\lambda})$ of $f^*\overline{\mathcal{M}}_{\overline{U}_0}$ is trivial by the proof of Theorem 5.8. Indeed, since $m_{u,\lambda}$ is orthogonal to the 2-plane in $N_\mathbb{R} \oplus \mathbb{R}$ spanned by the leg of $\tilde{h}$ in direction u, as a rational function $z^{m_{u,\lambda}}$ is defined on the added component $Z_0 \subset \overline{U}_0$ and restricts to the constant function on $f(D)$, with value the log constraint $\rho \in \mathbb{C}^\times$. From the tropical picture and the discussion in [NS],§5.1, we see that $\tilde{f}|_D : \mathbb{P}^1 \to Z_0$ is a "divalent line". This means that after a toric blow up (making $f(D)$ disjoint from zero-dimensional strata of Z_0) and removing the toric prime divisors from Z_0 disjoint from $\tilde{f}(D)$, the restriction $\tilde{f}|_D$ is a cover of a line

$$\{a\} \times \mathbb{P}^1 \subset \mathbb{C}^\times \times \mathbb{P}^1$$

branched at most over the two intersection points with the toric boundary. The covering degree agrees with the weight w of the edges adjacent to the vertex of $\tilde{\Gamma}$ corresponding to D, hence is determined completely by the tropical extension $\tilde{h}$. Now the point is that $z^{m_{u,\lambda}}$ restricts to the projection to the first factor $\mathbb{C}^\times \times \mathbb{P}^1 \to \mathbb{C}^\times$. Hence the position $a \in \mathbb{C}^\times$ of the line agrees with the value of

$$f^\flat(s_{u,\lambda}) = f^\sharp(z^{m_{u,\lambda}})$$

along D which in turn agrees with the log constraint ρ of either the log coral or the stable log map. Conversely, $z^{m_{u,\lambda}} = \rho$ defines a rational curve in Z_0, and the w-fold branched cover defines the extension $\tilde{f}$ on D.

We have now found a unique extension of $\underline{f} : \underline{C} \to \underline{U}_0$ to a stable map $\underline{\tilde{f}} : \underline{\overline{C}} \to \underline{Y}_0$. It remains to extend the log structure. First, note that the extension $\overline{C}$ of C admits a log structure $\mathcal{M}_{\overline{C}}$ obtained from the log structure $\mathcal{M}_C$ on C such that $\mathcal{M}_{\overline{C}}$ is unique up to isomorphism and

$$\overline{C}^\dagger = (\overline{C}, \mathcal{M}_{\overline{C}})$$

is a log smooth curve over the standard log point $\operatorname{Spec}\mathbb{C}^\dagger$. This follows immediately as the extension $\overline{C}$ is obtained by adding only one node and one transversally attached $\mathbb{P}^1$ component with a unique marked point on each non-complete component of C.

Each added component $D \subset \overline{C}$ maps into the interior of Z_0. Hence the extension of the log structure away from the added node $q \in \overline{C}$ is uniquely determined by strictness, as in the torically transverse case of ([NS], Prop. 7.1). For the extension at q observe that the stalk of $\mathcal{M}^{\mathrm{gp}}_{\overline{U}_0}$ at $f(q)$ can be generated by two toric generators s_u, s_v fulfilling the equation

$$s_u \cdot s_v = s_t^b$$

Explicitly, since all vertices of the tropical coral h map to integral points (see footnote in §2.4), we can write $u = (a, 1)$ with $a \in \mathbb{Z}$. Then

$$m_{u,\lambda} = (-1, a, 0)$$

and the quotient

$$(M \oplus \mathbb{Z})/\mathbb{Z} \cdot m_{u,\lambda}$$

can be generated by the images of $(0, -1, 1)$ and $(0, 1, 0)$. Let

$$\begin{aligned} u &= z^{(0,-1,1)} = s^{-1} \\ v &= z^{(0,1,0)} \end{aligned}$$

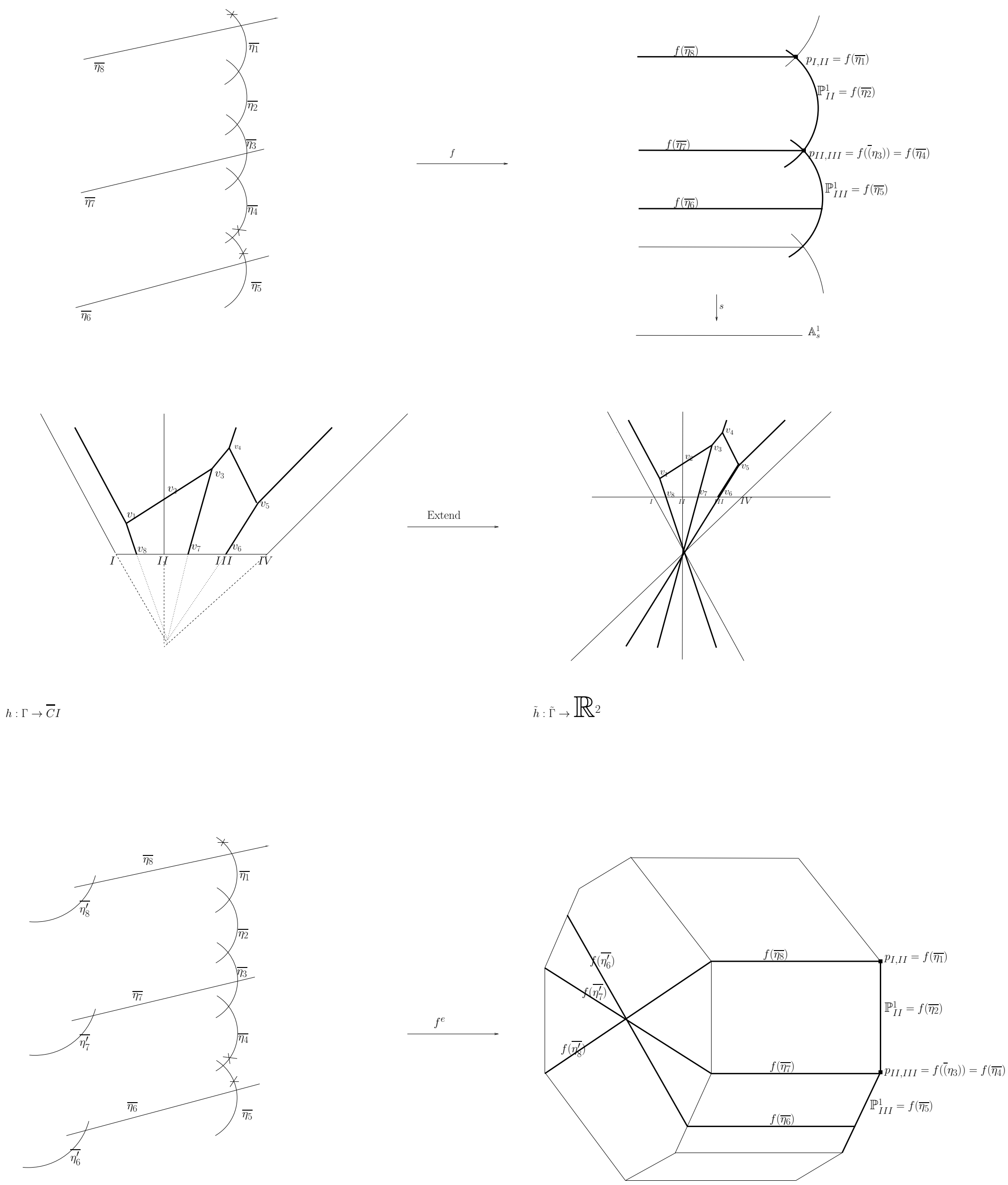
be the corresponding rational functions on $\overline{U}$ and s_u, s_v the corresponding global sections of $\mathcal{M}^{\mathrm{gp}}_{\overline{U}_0}$. If $\sigma \in \tilde{\mathscr{P}}$ is the minimal cell containing $(a, 1)$ and $(0, 0)$, then $(0, -1, 1)$

and $(0,1,0)$ lie in $(C(\sigma))^\vee$. Thus

$$s_u, s_v \in \mathcal{M}_{\overline{U}_0, f(q)}$$

and they generate $\mathcal{M}^{gp}_{\overline{U}_0,f(q)}$ up to powers of $s_{u,\lambda}$. Now $s_{u,\lambda}$ becomes invertible upon pull-back by $\tilde{f}$ and $\tilde{f}^\flat(s_{u,\lambda})$ is therefore already given by the scheme-theoretic discussion above. Thus to extend $f:(C,\mathcal{M}_C)\to(Y_0,\mathcal{M}_{Y_0})$ to D it remains to specify the value of $\tilde{f}^\flat_q$ on s_u and s_v in a way compatible with the structure maps to $\mathcal{O}_{\overline{U}_0,f(q}$ and with the equation $s_u s_v = s_t^b$. But this situation is now completely analogous to the torically transverse case, treated in the proof of Proposition 7.1 of [NS]. The discussion shows that $\tilde{f}^\flat_q$ exists uniquely as a map of log structures over the trivial log point $\operatorname{Spec}\mathbb{C}$, but that working over the standard log point $\operatorname{Spec}\mathbb{C}^\dagger$ brings in a choice of a w-th root of unity. But there is also an action of the Galois group $\mathbb{Z}/m$ of the w-fold cyclic branched cover $\underline{\tilde{f}}|_D : D\to\tilde{f}(D)$ which acts simply transitively on the set of such choices. Thus the extension is unique up to isomorphism, finishing the proof.

Example 5.9. The following figure illustrates a map $f : C^\dagger \to Y_0^\dagger$ and its log extension $f_e : \overline{C}^\dagger \to \overline{U_0}^\dagger$ together with their tropicalizations $h : \Gamma \to \overline{C}\mathbb{R}$ and $\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2$ respectively.



Our final goal is to show that the tropical count of tropical corals and the log count of log corals match. The strategy is to use a correspondence theorem for log extensions and tropical extensions, by reducing the situation to the torically transverse situation in a toric degeneration of a complete toric variety already treated in [NS]. Toric transversality ([NS], Definition 4.1) means that the image of $f : C \to Y_0$ is disjoint from toric strata of codimension larger than one. The associated tropical curve then can be read off as the dual intersection complex, and it maps to the 1-skeleton of the polyhedral decomposition defining the considered degeneration.

Let $f_e : (\overline{C}, \mathcal{M}_{\overline{C}}) \to (\overline{U_0}, \mathcal{M}_{\overline{U_0}})$ be the log extension of a given log coral $f : C^\dagger \to {Y_0}^\dagger$, matching log incidences $(\tilde{\Delta}, \tilde{\lambda}, \tilde{\rho})$. Assume f_e has tropicalization

$$(\widetilde{h} : \widetilde{\Gamma} \to \mathbb{R}^2) \in \mathcal{T}_{(\Gamma,u)}(\lambda)$$

of degree Δ, where $\mathbb{R}^2$ is endowed with the extended polyhedral decomposition $\tilde{\mathscr{P}}$ defined in 5. Now, to ensure that the image of $\widetilde{h}$ remains inside the 1-skeleton of the polyhedral decomposition $\tilde{\mathscr{P}}$ refine $\tilde{\mathscr{P}}$ accordingly. To obtain an appropriate refinement we simply add $\widetilde{\Gamma}$ to the 1-skeleton of $\tilde{\mathscr{P}}$. We do this refinement to $\tilde{\mathscr{P}}$ simultaneously for all tropical curves $\widetilde{h}_i : \widetilde{\Gamma}_i \to \mathbb{R}^2$ of degree Δ matching λ and of type (Γ, u). Note that by section 2.4, there are finitely many such tropical curves. This way obtain a new polyhedral decomposition $\tilde{\mathscr{P}}'$ of $\mathbb{R}^2$. Denote by W_0 the fiber over $t = 0$ of the toric degeneration associated to $\tilde{\mathscr{P}}'$. So, W_0 carries the natural log structure $\mathcal{M}_{W_0}$ obtained by the pull-back of the divisorail log structure on the total space given by the divisor $W_0 \subset W$.

We first construct a log smooth degeneration of $(\overline{U}_0, \mathcal{M}_{U_0})$ to

$$(W_0, \mathcal{M}_{W_0})$$

This statement follows from the following more general result on refinements of polyhedral decompositions and their associated toric degenerations of toric varieties.

Lemma 5.10. *Let $\mathscr{P}$ be an integral polyhedral decomposition of $N_{\mathbb{R}}$ and $\mathscr{P}'$ an integral refinement. Denote by $\pi : X \to \mathbb{A}^1$ and $\pi' : X' \to \mathbb{A}^1$ be the associated toric degenerations of toric varieties. Assume that $\mathscr{P}$ and $\mathscr{P}'$ are regular, that is, they support strictly convex piecewise affine functions. Then there is a two-parameter degeneration*

$$\tilde{\pi} : \tilde{X} \longrightarrow \mathbb{A}^2$$

with restrictions to $\mathbb{A}^1 \times \mathbb{G}_m$ and to $\mathbb{G}_m \times \mathbb{A}^1$ equal to $\pi \times \mathrm{id}_{\mathbb{G}_m}$ and to $\mathrm{id}_{\mathbb{G}_m} \times \pi'$, respectively.

Proof. Let φ, φ' be strictly convex, piecewise affine functions with corner loci (or non-differentiability loci) $\mathscr{P}$, $\mathscr{P}'$, respectively. We can assume φ, φ' are defined over the rational numbers and hence, after rescaling, that they are defined over the integers. Denote by $\Phi : N_{\mathbb{R}} \oplus \mathbb{R}_{>0} \to \mathbb{R}$ the homogenization of φ:

$$\Phi : N_{\mathbb{R}} \longrightarrow \mathbb{R}, \quad \Phi(x, \lambda) = \lambda \cdot \varphi\big(\frac{x}{\lambda}\big).$$

Note that Φ is the restriction of a linear function on the cone $C\sigma$ for any $\sigma \in \mathscr{P}$. Now the fan $\Sigma = C\mathscr{P}$ can be defined by the corner locus of Φ, that is, the maximal elements of Σ are the domains of linearity of φ. Similarly, we define the homogenization Φ' of φ'.

To define a two-parameter degeneration we use the fan $\tilde{\Sigma}$ in $N_{\mathbb{R}} \oplus \mathbb{R} \oplus \mathbb{R}$ defined by the corner locus of the following piecewise linear function:

$$\Psi(x, \lambda, \mu) := \Phi(x, \lambda) + \Phi'(x, \mu).$$

Now if

$$(x, \lambda, \mu) \in |\tilde{\Sigma}|$$

then Ψ is linear on

$$(x, \lambda, \mu) + \{0\} \times \mathbb{R}^2_{\geq 0}.$$

Thus the projection

$$N_{\mathbb{R}} \oplus \mathbb{R}^2 \to \mathbb{R}^2$$

induces a map of $\tilde{\Sigma}$ to the fan of $\mathbb{A}^2$. Define $\tilde{X}$ as the toric variety defined by $\tilde{\Sigma}$ and

$$\tilde{\pi} : \tilde{X} \to \mathbb{A}^2$$

as the toric morphism defined by the map of fans just described. The restriction of $\tilde{\pi}$ to $\mathbb{A}^1 \times \mathbb{G}_m$ is described by the intersection of $\tilde{\Sigma}$ with $N_{\mathbb{R}} \times \mathbb{R} \times \{0\}$. This intersection is the corner locus of

$$\Psi|_{N_{\mathbb{R}} \times \mathbb{R} \times \{0\}}.$$

But

$$\Psi(x, \lambda, 0) = \Phi(x, \lambda),$$

so this corner locus defines $C(\mathscr{P})$. Thus

$$\tilde{\pi}|_{\mathbb{A}^1 \times \mathbb{G}_m} = \pi \times \mathrm{id}_{\mathbb{G}_m}.$$

Analogously we conclude

$$\tilde{\pi}|_{\mathbb{G}_m \times \mathbb{A}^1} = \mathrm{id}_{\mathbb{G}_m} \times \pi'$$

□

With the construction of Lemma 5.10 we are now in position to show that our moduli space of log corals (or of their extensions to stable log maps in $\overline{U}_0$) has cardinality equal to the count of tropical corals. Given log degree Δ, degeneration order λ and log constraint ρ define

$$N_{\Delta,\lambda,\rho} := |\mathcal{L}_{\Delta,\lambda,\rho}|.$$

where $\mathcal{L}_{\Delta,\lambda,\rho}$ is as in Theorem 5.7.

Theorem 5.11. $N_{\Delta,\lambda,\rho} = N^{\mathrm{trop}}_{\Delta,\lambda,\rho}$

Proof. Note first that the polyhedral decomposition for the $\overline{U} \to \mathbb{A}^1$ is regular by inspection. Recall the polyhedral decomposition $\tilde{\mathscr{P}}'$ containing all extended tropical curves of degree Δ and asymptotic constraint λ in its 1-skeleton, with vertices mapping to vertices. By further refinement and rescaling we may assume that $\tilde{\mathscr{P}}'$ is integral and regular. Lemma 5.10 now provides a two-parameter degeneration

$$\tilde{\pi} : \tilde{X} \longrightarrow \mathbb{A}^2$$

isomorphic to $\pi : \overline{U} \to \mathbb{A}^1$ over $1 \times \mathbb{A}^1$ and to the degeneration $\pi' : X' \to \mathbb{A}^1$ defined by $\tilde{\mathscr{P}}'$ over $\mathbb{A}^1 \times \{1\}$. For the latter degeneration, the log constraints ρ translate into point constraints along with multiplicities along toric divisors on the general fibre of π'. This is the situation treated in the refinement [GPS] of [NS]. Hence all stable log curves in the central fibre are torically transverse and the relevant moduli space of stable log curves with incidence conditions consists of $N^{\mathrm{trop}}_{\Delta,\lambda}$ reduced (and unobstructed) points. By unobstructedness of deformations, the same statement is then true for stable log maps to a general fibre of π', or of $\tilde{\pi}$.

By deformation invariance of log Gromov-Witten invariants we obtain the same virtual count by looking at the central fibre $(\overline{U_0}, \mathcal{M}_{\overline{U_0}})$ of π, as a log space over the standard log point. But in fact, by the same arguments as in [NS] it can be easily shown that this moduli problem is unobstructed, and hence the moduli space $\tilde{\mathcal{L}}_{\Delta,\lambda,\rho}$ of extended stable log maps is a finite set of cardinality equal to the log Gromov-Witten invariant. Finally, in Theorem 5.7 we have established a bijection between $\tilde{\mathcal{L}}_{\Delta,\lambda,\rho}$ and $\mathcal{L}_{\Delta,\lambda,\rho}$, giving the claimed result. $\square$

6. The $\mathbb{Z}$-quotient

In this section we show the correspondence between log corals $f : C^\dagger \to {Y_0}^\dagger$ where Y_0 is the central fiber of degeneration of the "unfolded" Tate curve and the log corals $\tilde{f} : (\tilde{C}, \mathcal{M}_{\tilde{C}}) \to (\tilde{T}_0, \mathcal{M}_{\tilde{T}_0})$, where $\tilde{T}_0$ is the central fiber of the degeneration of the Tate curve obtained by taking the $\mathbb{Z}$-quotient $\pi : Y_0 \xrightarrow{/\mathbb{Z}} \tilde{T}_0$ as explained in the first section. Given a log coral $f : C^\dagger \to {Y_0}^\dagger$, one naturally obtains a log coral

$$\tilde{f} = \pi \circ f : (\tilde{C}, \mathcal{M}_{\tilde{C}}) \to (\tilde{T}_0, \mathcal{M}_{\tilde{T}_0})$$

?n the following Theorem, we show that given a log coral $\tilde{f} : (\tilde{C}, \mathcal{M}_{\tilde{C}}) \to (\tilde{T}_0, \mathcal{M}_{\tilde{T}_0})$ there is a lift of it to a log coral $f : C^\dagger \to {Y_0}^\dagger$.

Theorem 6.1. *Let $\tilde{f} : (\tilde{C}, \mathcal{M}_{\tilde{C}}) \to (\tilde{T}_0, \mathcal{M}_{\tilde{T}_0})$ be a log coral fitting into the following Cartesian diagram*

(6.1)
$$\begin{tikzcd} \tilde{C} \times_{\tilde{T}_0} Y_0 \arrow[r, "/\mathbb{Z}"'] \arrow[d, "f"] & \tilde{C} \arrow[d, "\tilde{f}"] \\ Y_0 \arrow[r, "/\mathbb{Z}"'] & \tilde{T}_0 \end{tikzcd}$$

The fiber product $\tilde{C} \times_{\tilde{T}_0} Y_0$ is isomorphic to a disjoint union of copies of $\tilde{C}$, that is

$$\tilde{C} \times_{\tilde{T}_0} Y_0 = \coprod_{\mathbb{Z}} \tilde{C}$$

and $\tilde{f} = \coprod_{n \in \mathbb{Z}} \Phi_n \circ f$ where $\Phi_n : Y_0 \to Y_0$ is the $\mathbb{Z}$-action on Y_0 induced by the $\mathbb{Z}$-action on the Mumford fan defined as in section 1.

Proof. It is enough to show that each connected component $C' \subset \tilde{C} \times_{\tilde{T}_0} Y_0$ is isomorphic to $\tilde{C}$. This will follow from the fact that any étale proper map from a connected curve to a nodal rational curve is an isomorphism.

The map $\pi : \tilde{C} \times_{\tilde{T}_0} Y_0 \to \tilde{C}$ is étale since $Y_0 \to \tilde{T}_0$ is étale, hence it is a local isomorphism in the étale topology.

Claim: C' has finitely many irreducible components.

To prove the claim it is enough to show that $\pi^{-1}(\tilde{C})$ has finitely many irreducible components, since $\tilde{C}$ has finitely many irreducible components by the definition of a log coral.

Let $\Gamma_{C'}$ and $\Gamma_{\tilde{C}}$ be the dual graphs of C' and $\tilde{C}$ respectively. That is, the graph obtained by replacing each irreducible component by a vertex, each node by a bounded edge and each marked point by an unbounded edge. The map $\pi|_{C'} : C' \to \tilde{C}$ induces a map

$$\Gamma_\pi : \Gamma_{C'} \to \Gamma_{\tilde{C}}$$

which is a local isomorphism. So, for every vertex $v \in \Gamma_{C'}$, the restriction $\Gamma_\pi|_{Star(v)}$ is an isomorphism onto $Star(\Gamma_\pi(v))$ where for any vertex v, $Star(v)$ denotes the subgraph consisting of the vertex v, edges containing v and their vertices. Since $\Gamma_{C'}$ is connected and $\Gamma_{\tilde{C}}$ is a tree, Γ_π is a covering map. As any covering map factors through the universal covering and the fundamental group $\pi_1(\Gamma_C) = 0$, it follows that Γ_π is an isomorphism. Hence, the number of vertices of $\Gamma_{C'}$ is finite and the claim follows. This shows that the map $\pi|_{C'} : C' \to \tilde{C}$ is proper.

By the Hurwitz formula we have $\pi^* w_{\tilde{C}} = w_{C'}$ where $w_{\tilde{C}}$ and $w_{C'}$ denote the canonical bundles over C and C' respectively. Applying the Riemann Roch Theorem for nodal curves, we obtain

$$2g - 2 = \deg w_{C'} = d \cdot \deg(w_{\tilde{C}}) = -2d$$

where $g = g(C') = 0$ as each component of C' is a copy of $\mathbb{P}^1$ and the dual graph is a tree. Hence, $d = 1$. Therefore, $\pi|_{C'} : C' \to \tilde{C}$ is an isomorphism. $\square$

It follows that a log coral $\tilde{f} : (\tilde{C}, \mathcal{M}_{\tilde{C}}) \to (\tilde{T}_0, \mathcal{M}_{\tilde{T}_0})$ lifts uniquely to a log coral $\tilde{f}|_{C'} : (C', \mathcal{M}_{C'}) \to {Y_0}^\dagger$, once fixing the connected component $C' \subset \tilde{C} \times_{\tilde{T}_0 Y_0}$. The idea here is similar to the standard fact in covering theory, that a lift of a path to the covering space is determined uniquely once fixing an initial point. We will explain this in more detail in the rest of this section.

Let us now discuss how to translate our results on the (degenerate) unfolded Tate curve to its $\mathbb{Z}$-quotient, hence to the degenerate Tate curve $\tilde{T}_0$. Note that as a scheme, $\tilde{T}_0$ is the product of a nodal elliptic curve E with $\mathbb{A}^1$.

Let us first set up the counting problem on the tropical side. Denote the correspnding affine manifold by $\overline{B} = B/\mathbb{Z}$, topologically

$$S^1 \times \mathbb{R}_{\geq 0}.$$

Each direction of positive or negative ends (the directions associated to positive edges and to edges adjacent to negative vertices of a coral graph) that enter the definition of the degree of a tropical coral is now only defined up to the $\mathbb{Z}$-action. Thus specifying a degree Δ on $\overline{B}$ amounts to choosing an equivalence class under the $\mathbb{Z}$-action of the corresponding data on B. If for given degree Δ and asymptotic constraint ρ on $\overline{B}$, we look at tropical corals with a fixed number of positive and negative ends, but each direction and asymptotic constraint restricted only to an equivalence class under the $\mathbb{Z}$-action, there will always be infinitely many such tropical corals or none. Indeed, given any tropical coral fulfilling the given constraints on its ends, the composition with the action of any integer on B will produce another one. These tropical corals related bvy the $\mathbb{Z}$-action induce equal tropical objects on $\overline{B}$. Thus we should rather restrict to $\mathbb{Z}$-equivalence classes of tropical corals. A simple way to break the $\mathbb{Z}$-symmetry is to choose one representant of the $\mathbb{Z}$-equivalence class of directions of one of the ends, say the first one.

Another source of infinity in the count is more fundamental and is part of the nature of the problem. It is related to the fact that the counting problem in symplectic cohomology produces an infinite sum with single terms weighted by the symplectic area of the pseudo-holomorphic curve with boundary. The logarithmic analogue runs as follows. For a log coral $f : C^\dagger \to Y_0^\dagger$ look at the underlying scheme theoretic morphisms and take the composition

$$\underline{C} \to \underline{Y}_0 \to \underline{X}_0 = E \times \mathbb{A}^1$$

This morphism extends uniquely to a morphism

$$\underline{\tilde{C}} \to E \times \mathbb{P}^1$$

from a complete curve $\underline{\tilde{C}}$ to $E \times \mathbb{P}^1$. The algebraic-geometric analogue of the symplectic area is the degree of the composition

$$\underline{\tilde{C}} \to E \times \mathbb{P}^1 \to E.$$

Since "degree" is already taken for something else, let us call the degree of $\underline{\tilde{C}} \to E$ the *log-area* of the log coral. For a fixed log-area A, the moduli space of log corals of given degree Δ, degeneration order λ and log constraint ρ is then again finite. Denote the corresponding cardinality by

$$N_{\Delta,\lambda,\rho}(A).$$

We may then define the count of log corals on $(X_0, \mathcal{M}_{X_0})$ as the formal power series

$$N_{\Delta,\lambda,\rho}(X_0, \mathcal{M}_{X_0}) = \sum_{A \in \mathbb{N}} N_{\Delta,\lambda,\rho}(A) q^A \in \mathbb{C}[\![q]\!]. \tag{6.2}$$

The tropical analogue of the area is given by a tropical intersection number ([AR]) as follows. For each vertex $(a \cdot b, 1)$ of $\mathscr{P}_b$ we have the line

$$L_a = \mathbb{R} \cdot (a, 1) \subset N_\mathbb{R}$$

through the origin. Each tropical coral has a well-defined intersection number with L_a. Define the *tropical area* of a tropical coral as the sum of the intersection numbers of its extension to a tropical curve in $N_\mathbb{R}$ with L_a, for all $a \in \mathbb{Z}$. The tropical intersection number bounds the number of crossings of the tropical coral with unbounded 1-cells of the polyhedral decomposition of B. Hence, with the $\mathbb{Z}$-action moded out as before, we also obtain a finite tropical count

$$N^{\mathrm{trop}}_{\Delta,\lambda}(A)$$

of tropical corals in $\overline{CB}$ of fixed tropical area A for tropical corals of fixed degree Δ and matching ageneral asymptotic constraint λ. We then have the following tropical analogue of (6.2)

$$N^{\mathrm{trop}}_{\Delta,\lambda}(\overline{CB}) = \sum_{A\in\mathbb{N}} N^{\mathrm{trop}}_{\Delta,\lambda}(A)q^A \in \mathbb{C}[\![q]\!].$$

where $N^{\mathrm{trop}}_{\Delta,\lambda}(\overline{CB})$ denotes the count of tropical corals in $\overline{CB}$ with degree Δ, matching the asymptotic constraint λ. Our results on the unfolded Tate curve readily give

$$N^{\mathrm{trop}}_{\Delta,\lambda}(A) = N_{\Delta,\lambda,\rho}(A)$$

for any $A \in \mathbb{N}$, Δ, λ and ρ. Summing over $A \in \mathbb{N}$ we obtain our main result for the Tate curve:

Theorem 6.2. $N_{\Delta,\lambda,\rho}(X_0, \mathcal{M}_{X_0}) = N^{\mathrm{trop}}_{\Delta,\lambda}(\overline{CB})$.

Hence, we have the equivalence of the tropical count of corals in $\overline{CB}$ and log corals in X_0.

7. The punctured invariants of the central fiber of the Tate curve

In this section we will see how the log corals on the central fiber of the "degeneration" of the Tate curve describe punctured invariants on the central fiber of the Tate curve itself. For the theory of punctured log Gromov-Witten invariants we refer to [ACGS], for which a brief summary can also be found in [GS6]. Here we just give the definitions of punctured invariants by simply adopting it to our situation.

As discussed in section 6, since log corals on the Tate curve lift to the unfolded Tate curve, for convenience we again work on the unfolded case and show the correspondence between log corals $f : C^\dagger \to {Y_0}^\dagger$ and $f^\circ : (C^\circ, \mathcal{M}_{C^\circ}) \to (X_0, \mathcal{M}_{X_0})$. Recall from section 1 that Y_0 is the central fiber of the degeneration of the unfolded Tate curve given by

$$Y_0 = \mathbb{A}^1 \times \bigcup_{\infty} \mathbb{P}^1 = \mathbb{A}^1 \times X_0$$

and the fiber over $0 \in \mathbb{A}^1_s$ under the dominant map $s : Y_0 \to \mathbb{A}^1_s$, denoted by $(Y_0)_{s=0}$ is equal to

$$(Y_0)_{s=0} = X_0$$

where X_0 is the central fiber of the unfolded Tate curve.

Throughout this section we will restrict our attention to general log corals. The following proposition states an important property for such corals.

Proposition 7.1. *Let $f : C^\dagger \to {Y_0}^\dagger$ be a general log coral and let $C' \subset C$ be a non-complete component. Then, the map $\mathrm{pr}_2 \circ \tilde{f}_{|C'} : C' \cong \mathbb{A}^1 \to \mathbb{P}^1$ defined as in 3.6 is constant.*

Proof. Let $\tilde{f}(z) : \mathbb{P}^1 \to \mathbb{P}^1$ be the extension of $\mathrm{pr}_2 \circ \tilde{f}_{|C'} : C' \cong \mathbb{A}^1 \to \mathbb{P}^1$ as in 3.6. If f is non-transverse at C', the result follows trivially. Assume f is transverse at C', then

$$\left(\tilde{f}_{|C'}(C') \ \cap \ (Y_0)_{s=0}\right) \subset \mathbb{P}^1 \setminus \{0, \infty\}$$

where by $\{0, \infty\}$ we denote the lower dimensional toric strata in $\mathbb{P}^1$. Furthermore, by the asymptotically parallel condition the point at infinity intersects the divisor at infinity at a point

$$\tilde{f}_{C'}(p_{C'}) = q_\infty \in \mathbb{C}^\times \subset \mathbb{P}^1$$

Hence we have $\tilde{f}(C' \cup p_{C'}) \in \mathbb{P}^1 \setminus \{0\}$ affine. Since a morphism from a complete variety to an affine variety is constant, $\tilde{\phi}(z)$ is constant, hence the result follows. □

Thus, the non-complete components $C' \subset C$ do not carry any additional information except at the position of the node q_∞. Thus, we can trade the whole component $\tilde{f}_{|C'}$ by $f(q) = f(C') \cap (Y_0)_{s=0}$ which will carry us to $X_0 = (Y_0)_{s=0}$. On the tropical side this corresponds to restricting the tropicalization $h : \Gamma \to \overline{C}\mathbb{R}$ of $f : C^\dagger \to {Y_0}^\dagger$ to $h^{-1}(\overline{C}\mathbb{R} \setminus \partial(\overline{C}\mathbb{R}))$.

Definition 7.2. Let $h : \Gamma \to \overline{C}\mathbb{R}$ be a tropical coral. The restriction

$$h^\circ := h|_{h^{-1}(\overline{C}\mathbb{R}\setminus\partial(\overline{C}\mathbb{R}))} : \Gamma^\circ \to \overline{C}\mathbb{R} \setminus \partial(\overline{C}\mathbb{R})$$

is called the *restricted tropical coral.*

Note that omitting the vertices corresponding to the non-complete components of a log coral trades each non-complete component $C' \subset C$ with a marked point on the $\mathbb{P}^1$-component adjacent to C' on the scheme-theoretic level. Let $f : C^\dagger \to {Y_0}^\dagger$ be a log coral and let $C_k \subset C$ for $k = 1, \ldots, m$ be the non-complete components of C such that

$$C = \coprod_{k=1}^{m} C'_k \ \amalg_{p_k} \ \tilde{C}$$

where for each $k = 1, \ldots, m$ there exists a single point $p_k \in C'_k \bigcap \tilde{C}$.

To investigate the punctured log Gromow-Witten invariants

$$\tilde{f} : (\tilde{C}, \mathcal{M}^\circ_{\tilde{C}}) \to (X_0, \mathcal{M}_{X_0}))$$

we first need to define a suitable log structure $\alpha^\circ : \mathcal{M}^\circ_{\tilde{C}} \to \mathcal{O}_{\tilde{C}}$, which we will do by using two log structures on $\tilde{C}$ that we are already familiar with.

First note that since $C^\dagger \to (\operatorname{Spec}\mathbb{C}, \mathbb{N}\oplus\mathbb{C}^\times)$ is a smooth log curve over the standard log point $(\operatorname{Spec}\mathbb{C}, \mathbb{N} \oplus \mathbb{C}^\times)$ so is $(\tilde{C}, \mathcal{M}_{\tilde{C}})$. Consider each marked point $p_k \in \tilde{C}$ as a smooth section $p_k : \operatorname{Spec}\mathbb{C} \to \tilde{C}$. Around each p_k, the log structure $\mathcal{M}_{\tilde{C}}$ on $\tilde{C}$ is given as in (3.3, (iii)). Away from points p_k, define $\alpha_{\tilde{C}} : \mathcal{M}_{\tilde{C}} \to \mathcal{O}_{\tilde{C}}$ as the restriction of the log structure $\alpha : \mathcal{M}_C \to \mathcal{O}_C$ on C to $\tilde{C}$.

We have a second familiar log structure on $\tilde{C}$ described by the divisorial log structure. Observe that the images $\operatorname{im}(p_k) \subseteq \tilde{C}$ under the maps $p_k : \operatorname{Spec}\mathbb{C} \to \tilde{C}$ define a Cartier divisor in $\tilde{C}$ and hence associated to $\tilde{C}$ there is the divisorial log structure which we denote by $\alpha_P : P \to \mathcal{O}_{\tilde{C}}$ where P denotes the sheaf of monoids on $\tilde{C}$ which is given by the regular functions on $\tilde{C}$ vanishing at $D := \bigcup_k p_k \in \tilde{C}$.

Now, we can define $\mathcal{M}^\circ_{\tilde{C}}$ as follows. Let $\mathcal{M}^\circ_{\tilde{C}} \subseteq \mathcal{M}_{\tilde{C}} \oplus_{\mathcal{O}^\times_{\tilde{C}}} P^{gp}$ be the sheaf of monoids on $\tilde{C}$ such that a section s of $\mathcal{M}_{\tilde{C}} \oplus_{\mathcal{O}^\times_{\tilde{C}}} P^{gp}$ over an étale open set $U \to \tilde{C}$ is a section of $\mathcal{M}^\circ_{\tilde{C}}$ if for all $x \in U$ and representatives $(t_x, t'_x) \in \mathcal{M}_{\tilde{C},x} \oplus_{\mathcal{O}^\times_{\tilde{C}}} P^{gp}_x$ of $s_x \in (\mathcal{M}_{\tilde{C}} \oplus_{\mathcal{O}^\times_{\tilde{C}}} P^{gp})_x$, we have $t'_x \in P_x$ unless $\alpha_{\tilde{C}}(t_x) = 0 \in {\mathcal{O}_{C,x}}^\times$. By this construction we obtain $\mathcal{M}^\circ_{\tilde{C}}$ as the largest subsheaf of $\mathcal{M}_{\tilde{C}} \oplus_{\mathcal{O}^\times_{\tilde{C}}} P^{gp}$ to which the sum of homomorphisms $\alpha_{\tilde{C}} : \mathcal{M}_{\tilde{C}} \to \mathcal{O}_{\tilde{C}}$ and $\alpha_P : P \to \mathcal{O}_C$ extends. The morphism

$\alpha^\circ : \mathcal{M}^\circ_{\tilde{C}} \to \mathcal{O}_{\tilde{C}}$ is then given by

$$\begin{array}{rcl} \alpha^\circ_{\tilde{C}} : \mathcal{M}^\circ_{\tilde{C}} & \to & \mathcal{O}_{\tilde{C}} \\ (t,t') & \mapsto & \alpha_{\tilde{C}}(t)\cdot \alpha_P(t') \end{array}$$

where the product is interpreted to be zero if $\alpha_{\tilde{C}}(t) = 0$, even if $\alpha_P(t')$ is undefined.

Remark 7.3. Punctured curves can allow negative contact orders. Given a log morphism $f : (\tilde{C}, \mathcal{M}^\circ_{\tilde{C}}) \to (X_0, \mathcal{M}_{X_0})$ we obtain a composed map

$$u_{p_k} : \overline{\mathcal{M}}_{X_0,f(p_k)} \to \overline{\mathcal{M}^\circ}_{\tilde{C},p_k} \subseteq \mathbb{N}\oplus\mathbb{Z} \xrightarrow{\mathrm{pr}_2} \mathbb{Z}$$

This is clearly analogous to the case in the non-punctured situation, but now it is possible that the image of u_p does not lie in $\mathbb{N}$.

We define incidence conditions for punctured maps $\tilde{f} : (\tilde{C}, \mathcal{M}^\circ_{\tilde{C}}) \to (X_0, \mathcal{M}_{X_0})$ analogously to the case of extended log corals as in 5.5. We define $\Delta^\circ := \tilde{\Delta}$, $\lambda^\circ := \tilde{\lambda}$. A slight difference occurs in the definition of the log constraint $\tilde{\rho} := (\overline{\tilde{\rho}}, \underline{\tilde{\rho}})$. We define $\rho^\circ := (\overline{\rho^\circ}, \underline{\rho^\circ})$ such that $\overline{\rho^\circ} := \overline{\tilde{\rho}}$. However, one needs to pay attention to the definition of $\underline{\rho^\circ}$ as the marked points $p_j \in C^e_j$ on extended log corals in the punctured situation are traded by marked points p_k on the $\mathbb{P}^1$-component connected to the non-complete component $C'_k \subset C$. Therefore the log constraint $\underline{\rho^\circ}$ is the marked point position of $p_k \in \mathbb{P}^1$ which is identical with the virtual position of the point at infinity $q^k_{C'}$ of C'_k by 7.1.

Denote the set of stable log maps $\tilde{f} : (\tilde{C}, \mathcal{M}^\circ_{\tilde{C}}) \to (X_0, \mathcal{M}_{X_0})$ matching $(\Delta^\circ, \lambda^\circ, \rho^\circ)$ by $\mathfrak{L}^\circ{}_{\Delta^\circ,\lambda^\circ,\rho^\circ}$.

Lemma 7.4. *The forgetful map*

$$\begin{array}{rl} \mathfrak{L}_{\Delta,\lambda,\rho} & \longrightarrow \mathfrak{L}^\circ_{\Delta^\circ,\lambda^\circ,\rho^\circ} \\ f & \quad\mapsto \tilde{f} \end{array}$$

is a bijection.

Proof. We only need to show given a punctured map $\tilde{f} : (\tilde{C}, \mathcal{M}^\circ_{\tilde{C}}) \to (X_0, \mathcal{M}_{X_0})$, we can extend it to a log coral $f : C^\dagger \to {Y_0}^\dagger$. As a scheme theoretic morphism define

$$f = \begin{cases} \tilde{f} & \text{on } \tilde{C} \hookrightarrow C \\ \tilde{f}(p_k) & \text{on each } C_k \cong \mathbb{A}^1 \subset C \setminus \tilde{C} \end{cases}$$

which naturally lifts to a log morphism ([ACGS]). □

8. Floer theoretic perspectives

The *Tropical Morse Category*, introduced by Abouzaid, Gross and Siebert provides an approach to the homological mirror symmetry conjecture. The idea is to relate both the Fukaya category on one side and the category of coherent sheaves on the other side to the tropical Morse category. For the case of the Elliptic curve these relations are studied in detail in ([C], §8).

The structure coefficients in the Tropical Morse category are given by tropical Morse trees which we will review in a moment. Then we will describe how to associate a tropical coral of a given type to a tropical Morse tree. We obtain a one-to-one correspondence between tropical Morse trees and the types of tropical corals. Hence, each tropical Morse tree gives an $n-1$ dimensional moduli space of tropical corals $h : \Gamma \to \overline{C}\mathbb{R}$ (2.26) where n denotes the number of positive edges of Γ.

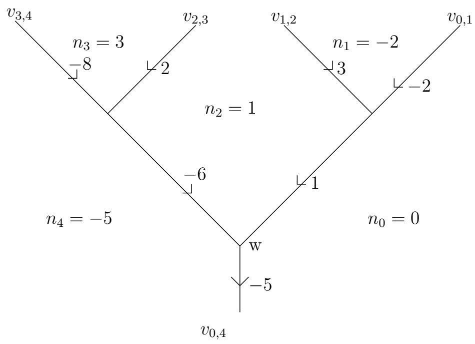
Definition 8.1. A metric *ribbon tree* $\mathcal{S}$ is a connected tree with a finite number of vertices and edges, with no bivalent vertices, with the additional data of a cyclic ordering of edges at22 each vertex and a length assigned to each edge in $(0, \infty)$.

We refer to the external vertices of a ribbon tree $\mathcal{S}$ as *leaves*. We call $\mathcal{S}$ *rooted* if there is one external distinguished vertex referred to as the *root vertex*. We assume in a rooted Ribbon tree all edges are directed towards the root vertex. We refer to the root vertex as the *outgoing vertex* and all other external vertices as *incoming vertices*. We refer to edges adjacent to a vertex v as *incoming edge* if the assigned direction points towards v and *outgoing edge* if the direction points outwards. The cyclic ordering on the ribbon tree $\mathcal{S}$ specifies an embedding $\mathcal{S} \hookrightarrow \mathbb{R}^2$ up to isotopy. This given a Ribbon tree with n external vertices we obtain a decomposition of $\mathbb{R}^2$ into n regions.

Definition 8.2. Let S be a rooted ribbon tree with $d+1$ external vertices giving a decomposition of $\mathbb{R}^2$ into $d+1$ regions. Choose $n_0, \ldots, n_d \in \mathbb{Z}$ to label each region respecting the cyclic orientation. Then assign to each edge e lying between regions labelled by n_i and n_j the integer $n_e := n_j - n_i$ respecting the cyclic order. We refer to the ribbon tree whose edges are labelled with the assigned integers as *decorated.*

We also label each external vertex of the ribbon tree adjacent to the edge lying between regions labelled by n_i and n_j as v_{ij}.

Example 8.3. The following figure illustrates a ribbon graph with 4 incoming leaves, giving a decomposition of the plane into 5 regions assigned with numbers $n_0, \ldots, n_4$. The integers $n_e = n_i - n_j$ are written on each edge e.



Definition 8.4. Let B be an integral affine manifold and $\mathcal{S}$ be a ribbon tree with $d+1$ external vertices. Let $n_0, \ldots, n_d \in \mathbb{Z}$ be the set of integers describing a decoration on $\mathcal{S}$ as in 8.2 so that to each edge e we assign $n_e \in \mathbb{Z}$. Identify each edge e of $\mathcal{S}$ with $[0,1]$ with coordinate s and the orientation on e pointing from 0 to 1. A *tropical Morse tree* is a map $\phi : B \to \mathcal{S}$ satisfying the following:

i.)

$$\phi(v) := \begin{cases} p_{i,i+1} & \text{if v is the i}-\text{th external incoming vertex} \\ p_{0,d} & \text{if v is the root vertex} \end{cases}$$

where $\phi(v) := p_{i,i+1}$ for

$$p_{i,i+1} \in B\Big(\frac{1}{n_{i+1} - n_i}\mathbb{Z}\Big)$$

and

$$p_{0,d} \in B\Big(\frac{1}{n_d - 0}\mathbb{Z}\Big)$$

ii.) For an edge e of $\mathcal{S}$, $\phi(e)$ is either an affine line segment not necessarily integral or a point in B.

iii.) For each edge e, there is a section $v_e \in \Gamma(e, (\phi\big|_e)^* TB)$ satisfying

1) $v_e(v) = 0$ for each external vertex v adjacent to the edge e.
2) For each edge $e \cong [0,1]$ and $s \in [0,1]$, we have $v_e(s)$ is tangent to $\phi(e)$ at $\phi(s)$, pointing in the same direction as the orientation on $\phi(e)$ induced by that on e. By identifying $(\phi\big|_e)^* TB$ with the trivial bundle over e using the affine structure, we have

$$\frac{d}{ds} v_e(s) = n_e \phi_* \frac{\partial}{\partial_s}$$

3) If v is an internal vertex of $\mathcal{S}$ with incoming edges $e_1 \ldots, e_p$ and outgoing edge e_{out}, then

$$v_{e_{out}}(v) = \sum_{i=1}^{p} v_{e_i}(v)$$

Remark 8.5. In the original definition of tropical Morse trees ([C], Definition 8.19) there is an extra assumption to ensure the existence of a map from the moduli space of tropical Morse trees to the moduli space of ribbon graphs,which we skip as it is inessential for our purposes.

We refer to v_e as the *velocity* and to n_e as the *acceleration*. By the assumption that $v_e(v) = 0$ on each external edge we can classify all contracted edges. Namely we contract an edge if we are in one of the two cases

1. An edge e such that e is an external edge adjacent to an incoming vertex and $n_e < 0$
2. The edge e is adjacent to the root vertex and $n_e > 0$

The incoming external leaves with negative acceleration are being contracted to ensure the velocity increases starting from zero along the adjacent edges. The edge adjacent to the root vertex gets contracted if the acceleration is positive as we want the velocity to be zero again once arriving at the root vertex.

Now we describe how to assign a polygon to a tropical Morse tree. The general construction is described in ([C], 8.4.4). We restrict our attention to the case $B = S^1$ and $X(B) = E$ where E denotes the elliptic curve and show that to every tropical Morse tree $\phi : \mathcal{S} \to B$ where $B = \mathbb{R}/d\mathbb{Z}$ for d the number of leaves of $\mathcal{S}$ there corresponds a piecewise linear disk bounded by the Lagrangian sections of the elliptic curve E.

Identify E with TB/Λ where $\Lambda \subseteq TB$ is the sheaf of integral tangent vectors. We denote the coordinate on $B = \mathbb{R}/d\mathbb{Z}$ by y and the fiberwise coordinates by x, so that coordinates on $E = TB/\Lambda$ are given by $\{x, y\}$. Define a section $B \to E$ via

$$\sigma_n(y) = (y, -n \cdot y)$$

and set $L_n := \sigma_n(B)$. Let e be an edge of $\mathcal{S}$ labelled by $n_j - n_i$ for $j > i$. Define

$$\begin{aligned} R_e : e \times [0,1] &\to E \\ (s,t) &\mapsto \sigma_{n_i}(\phi(s)) - t \cdot v_e(s) \end{aligned} \tag{8.1}$$

where $t \cdot v_e(s)$ is viewed as a tangent vector at $\phi(s)$. Let v_{in} and v_{out} denote the vertices of e such that the assigned direction on e is from v_{out} to v_{in}. Then $R_e(v_{in} \times \{1\}) \subseteq L_{n_j}$ implies $R_e(e \times \{1\}) \subseteq L_{n_j}$ (for details see [C], pg 631).

Remark 8.6. For convenience we work with the lifts of a tropical Morse trees $\phi : \mathcal{R} \to S^1 = \mathbb{R}/d\mathbb{Z}$ to $\phi : \mathcal{R} \to \mathbb{R}$, which by abuse of notation we also denote by ϕ.

Example 8.7. The following figure is a tropical Morse tree from the ribbon graph we had seen in example 8.3 onto $\mathbb{R}$. We illustrate the image on three copies of $\mathbb{R}$ which are identified along vertical projection. The reason we use two copies is that the image is not injective and we go back and forth in the direction assigned with arrows. We choose

$$p_{01} = 8 = \frac{y}{3} \in \frac{1}{3}\mathbb{Z}$$
$$p_{12} = 12 = \frac{z}{2} \in \frac{1}{2}\mathbb{Z}$$
$$p_{23} = 0 \in \frac{1}{2}\mathbb{Z}$$
$$p_{34} = 6 = \frac{x}{8} \in \frac{1}{2}\mathbb{Z}$$

Now for the image $\phi(w) \in \mathbb{R}$ we have a 1-parameter family of possibilities, which will be the reason of the associated polygon to be non-convex. We want the following equalities to be satisfied to ensure balancing: Let *right side velocities* be the sum of velocities assigned to the the images of edges with acceleration $n_1 - n_0, n_2 - n_1, n_2 - n_0$ and let *left side velocities* be the sum of velocities assigned to the the images of edges with acceleration $n_4 - n_3, n_3 - n_2, n_4 - n_2$ and let

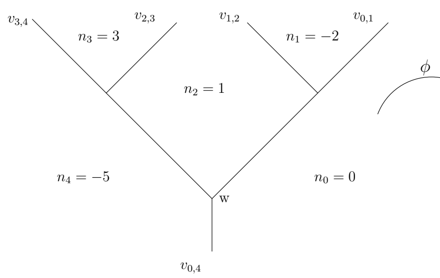
i The right side velocities should be positive;
$2 \cdot \frac{x}{8} - 6 \cdot (\phi(w) - \frac{x}{8}) > 0$

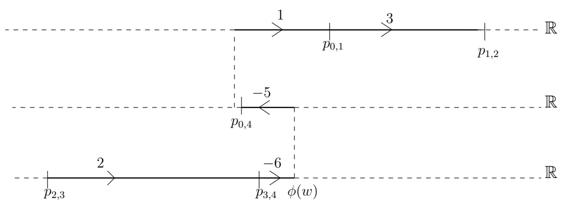
ii The left side velocities should be positive;
$\phi(w) - \frac{z}{2} + 3 \cdot (\frac{z}{2} - \frac{y}{3}) > 0$

where we had chosen x, y, z as above. Hence we need to have;

$$x - 6 \cdot (\phi(w)) > 0 \implies 8 > \phi(w)$$
$$\phi(w) + z - y > 0 \implies \phi(w) > 0$$

We choose $\phi(w) = 7$. Then by the balancing condition we get $p_{04} = \frac{28}{5} \in \frac{1}{5}\mathbb{Z}$.

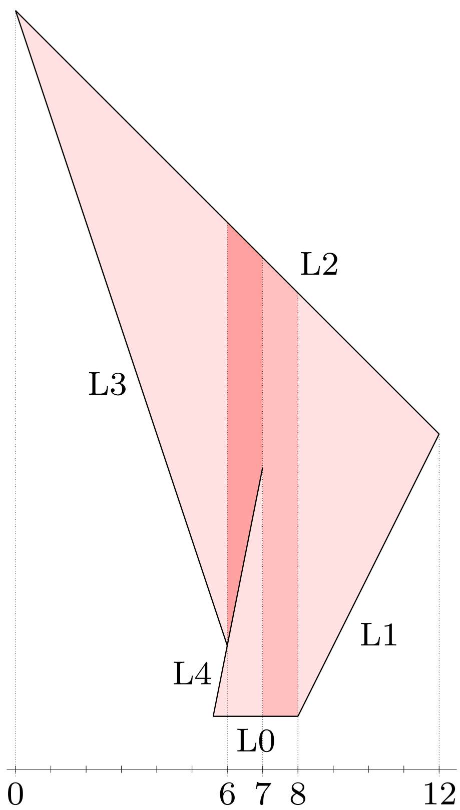




The image of ϕ in $\mathbb{R}$

A tropical Morse tree $\phi : \mathcal{S} \to \mathbb{R}$

The associated polygon is illustrated in the following picture. Note that in this situation it is not convex.



We will in a moment explain how to associate a tropical coral $h : \Gamma \to \overline{C}\mathbb{R}$ to a given tropical Morse tree $\phi : \mathcal{R} \to \mathbb{R}$.

Mark the position of each $p_{ij} = \phi(v_{ij})$ on the boundary $\partial\overline{C}\mathbb{R}$, as well as the position of the image of the root vertex $\phi(w) \in \partial\overline{C}\mathbb{R}$. Then take the inverse image of each p_{ij} and of $\phi(w)$ under the radial projection map

$$\pi_{u_{ij}} : \overline{C}\mathbb{R} \to \overline{C}\mathbb{R}/\mathbb{R} \cdot u_{ij}$$

where u_{ij} denotes the vector $\langle p_{ij}, 1\rangle$ as well as

$$\pi_{u_w} : \overline{C}\mathbb{R} \to \overline{C}\mathbb{R}/\mathbb{R} \cdot u_w$$

where u_w is the vector $\langle \phi(w), 1\rangle$. We denote the rays obtained on the truncated cone $\overline{C}\mathbb{R}$ by

$$R_{p_{ij}} := \pi_{u_{ij}}^{-1}(p_{ij}) \text{ and } R_{\phi(w)} := \pi_{u_w}^{-1}(\phi(w))$$

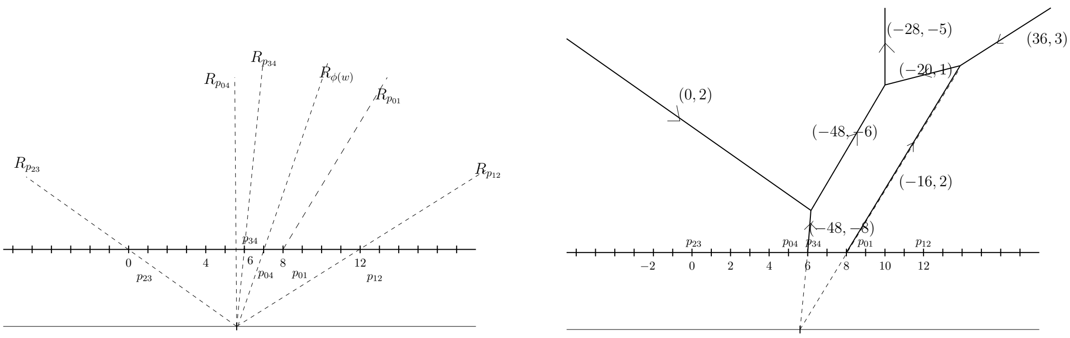
Then we assign a direction vector to each ray given by

$$u = \begin{cases} (n_j - n_i) \cdot \langle p_{ij}, 1\rangle & \text{for each } R_{p_{ij}} \\ (n_k - n_0) \cdot \langle \phi(w), 1\rangle & \text{for } R_{\phi(w)} \end{cases}$$

The rays $R_{p_{ij}}$ will form all unbounded edges together with the edges connected to negative vertices. Note that to the assigned direction vector of the rays there is an orientation we determine compatibly with the orientation of the polygon we had discussed. The idea is to patch these rays together (in cases they correspond to the edges adjacent to negative vertices we just take the restriction onto a finite part of them), by adding additional bounded segments to ensure the balancing at each vertex holds to form the associated tropical coral.

We start by choosing a point p_w such that $p_w \in R_{\phi(w)}$ is far up from the boundary $\partial\overline{C}\mathbb{R}$. This will at the end correspond to an interior vertex of the associated tropical coral. Paying attention to the cyclic order encoded in the tropical Morse tree, we patch the rays together with some possible additional segments ensuring balancing. This procedure will become clear once illustrated in a couple of examples.

Example 8.8. We illustrate the tropical coral associated to the tropical Morse tree of example 8.7.



Let us step by step explain how we draw it. Take a point $p_w \in R_{\phi(w)}$ which will correspond to an interior vertex of the tropical coral. To determine the edges adjacent to p_w, observe that there are three edges of the ribbon graph adjacent to $w \in \mathcal{R}$. Denote each edge with assigned acceleration $n_j - n_i$ by e_{ij} and assign a velocity vector to each edge by balancing condition;

$$u_{e_{02}} + (-2)\cdot(p_{01},1) + 3\cdot(p_{12},1) = 0 \implies u_{e_{02}} = (-20,1)$$

$$u_{e_{24}} + (-1)\cdot 2\cdot(p_{23},1) + (-8)\cdot(p_{34},1) = 0 \implies u_{e_{24}} = (-48,-6) \tag{8.2}$$

As $w \in \mathcal{R}$ is adjacent to e_{02}, e_{24} and the edge connected to the root vertex v_{04}, in the corresponding tropical coral p_w will be adjacent to the edges with degree vector $u_{e_{02}}, u_{e_{24}}$ and $(n_4 - n_0)\cdot(p_{04},1)$. Patch these together paying attention to the orientation. Note that the orientation on the tropical coral should be compatible with the flow on the tropical Morse tree where everything flows towards the root vertex. Continue this procedure and patch all edges to obtain the tropical coral.

Appendix A. A glance at logarithmic geometry

We give basic definitions and a few results we had used throughout the paper on logarithmic geometry which is a pretty broad area of study ([Kk], [Kf], [O]).

Monoids play a crucial role in the theory of logarithmic geometry. We first review quickly some basics on monoids. For a detailed study of monoids in log geometry see [O].

Definition A.1. A *monoid* is a set $\mathcal{M}$ with an associative binary operation with a unit. The monoid operation is usually written additively, in which case we will denote the identity by 0. All monoids we will consider will be commutative monoids.

The basic example of a monoid is the set of natural numbers $\mathbb{N}$ together with the operation addition.

Definition A.2. A *homomorphism of monoids* is a function $\beta : \mathscr{P} \to \mathscr{Q}$ between monoids such that $\beta(0) = (0)$ and $\beta(p + p') = \beta(p) + \beta(p')$.

Definition A.3. The *Grothendieck group of a monoid* $\mathscr{P}$ is the abelian group generated by a monoid $\mathscr{P}$, denoted by $\mathscr{P}^{gp}$ defined as follows.

$$\mathscr{P}^{gp} \;:=\; \{\mathscr{P} \times \mathscr{P}/\sim\}$$

with the eqquivalence relation compatible with the monoid operation defined as $(x, y) \sim (x', y')$ if and only if there exists an element $p \in \mathscr{P}$ such that $pxy' = pyx'$.

Note that the there is a natural map from $\mathscr{P}$ to its associated Grothendieck group $\mathscr{P}^{gp}$ sending an element $p \in \mathscr{P}$ to the equivalance class $(p, 1)$ denoted by $p/1$. So, for any abelian group G and a monoid $\mathscr{P}$, we have $\mathrm{Hom}_{Mon}(\mathscr{P}, G) = \mathrm{Hom}_{Ab}(\mathscr{P}^{gp}, G)$. Considering the basic example of a monoid $\mathbb{N}$ we have $\mathbb{N}^{gp} = \mathbb{Z}$. In general, $\mathscr{P}^{gp}$ will be the smallest group containing $\mathscr{P}$.

Definition A.4. A monoid $\mathscr{P}$ is called *integral* if the map $\mathscr{P} \to \mathscr{P}^{gp}$ is injective.

For a monoid to be integral is equivalent to say that the cancellation law holds. That is a monoid $\mathscr{P}$ is injective if $p + q = p' + q \Rightarrow p = p'$.

Definition A.5. A monoid $\mathscr{P}$ is called *fine* if it is finitely generated and integral. $\mathscr{P}$ is called *saturated* if it is integral and whenever $p \in \mathscr{P}^{gp}$ such that $\mathrm{m}p \in \mathscr{P}$ then $p \in \mathscr{P}$. $\mathscr{P}$ is called *toric* if it is fine, saturated and $\mathscr{P}^{gp}$ is torsion free.

Example A.6. Consider the semigroup generated by 2 and 3 in $\mathbb{N}$. This gives the monoid $\mathscr{P} = \{0, 2, 3, 4, ...\}$. Then we have $1 \in \mathscr{P}^{gp} = \mathbb{Z}$, but $1 \notin \mathscr{P}$. So, $\mathscr{P}$ is not saturated. The associated $\mathbb{C}$-algebra to $\mathscr{P}$ is $\mathcal{R} = \mathbb{C}[x^2, x^3]$. Then the corresponding variety will be $\mathrm{Spec}\mathcal{R} = V(y^2 - x^3) \in \mathbb{A}^2$ which is clearly not normal. We see that in order to get a normal variety at the end, one must start with a saturated monoid.

We have the following lemma due to Gordan.

Lemma A.7. *Let L be a finitely generated free $\mathbb{Z}$ module. Let $L_{\mathbb{R}} = L \otimes_{\mathbb{Z}} \mathbb{R}$. Let σ be a strongly convex rational polyhedral cone in $L_{\mathbb{R}}$. This means σ is a cone with apex at the origin generated by a finite number of vectors in the lattice L and it contains no line through the origin. Then $P := L \cap \sigma$ is a toric monoid. σ is generated by P and $\dim \sigma = \mathrm{rank} P^{gp}$.*

The standard way to obtain a toric variety X is to start with a cone σ as in A.7 generated by a toric monoid $\mathscr{P}$. The dual cone $\tilde{\sigma}$ defines a fan for X. Using Laurent monomials pass to the associated finitely generated monomial algebra $R[\mathscr{P}]$. Then $\mathcal{U}_\sigma := \mathrm{Spec}(R[\mathscr{P}])$ is an affine toric variety. Note that $\mathcal{U}_\sigma$ contains a copy of $(\mathbb{C}^\times)^n$. If we start with a fan, that is a collection of cones σ we can glue together corresponding the copies of $(\mathbb{C}^\times)^n$ for each cone to obtain a general toric variety. The closed points of a toric variety X obtained from a toric monoid $\mathscr{P}$ correspond to semigroup homomorphisms via the natural isomorphism $\mathrm{Specm} R[\mathscr{P}] = \mathrm{Hom}_{sg}(\mathscr{P}, \mathbb{C})$.

Example A.8. Let $\mathscr{P} = \{< p_1, p_2, p_3, p_4 > \ \mid p_i \in \mathbb{N} \text{ for } i = 1, ..., 4 \text{ and } p_1 + p_3 = p_2 + p_4\ \}$. Clearly, $\mathscr{P}$ is fine and saturated. Furthermore, $\mathscr{P}^{gp} = \mathbb{Z}^3$ which is torsion free. Hence, $\mathscr{P}$ is toric and $R[\mathscr{P}] \cong \mathbb{C}[x, y, w, t]/(xy - wt)$. So, the associated toric variety is the complex 3-dimensional conifold $xy - zt$. Recall that generally for a toric variety X_σ where σ is the cone defined by a monoid $\mathscr{P}$, we have $\mathbb{Z}^{\dim_{\mathbb{C}} X} \cong \mathscr{P}^{gp}$.

Remark A.9. For a toric variety X_σ where σ is the cone defined by a monoid $\mathscr{P}$, we have $\mathbb{Z}^{\dim_{\mathbb{C}} X} \cong \mathscr{P}^{gp}$.

Now we begin with the basic definitions and examples of log structures.

Definition A.10. Let X be an analytic space with the usual analytic topology or generally a scheme so that the underlying space is endowed with the étale topology. A *pre log structure* on X is a sheaf of monoids $\mathcal{M}$ on X together with a homomorphism of monoids $\beta : \mathcal{M} \longrightarrow \mathcal{O}_X$ where we consider the structure sheaf $\mathcal{O}_X$ as a monoid with respect to multiplication.

Definition A.11. A pre log structure on X with the morphism of monoids $\alpha : \mathcal{M} \longrightarrow (\mathcal{O}_X, \cdot)$ is called a *log structure* if α induces an isomorphism

$$\alpha\big|_{\alpha^{-1}(\mathcal{O}_X^\times)} : \alpha^{-1}(\mathcal{O}_X^\times) \longrightarrow \mathcal{O}_X^\times$$

We will call a scheme X associated with a log structure $\alpha_X : \mathcal{M}_x \longrightarrow \mathcal{O}_X$ a log scheme and denote it by $(X, \mathcal{M}_X)$ or by $X^\dagger$.

Remark A.12. If X is a space carrying a log structure $\alpha : \mathcal{M}_X \to \mathcal{O}_X^\times$ and a section $s_m \in \mathcal{M}_X$ is mapped to an invertible section f of the structure sheaf $\mathcal{O}_X^\times$ this implies that s_m is invertible and $s_m^{-1} = f^{-1}$.

Definition A.13. Given a log scheme $(X, \mathcal{M}_X)$ we define $\overline{\mathcal{M}}_X := \mathcal{M}_X \big/ \mathcal{O}_X^\times$. We call $\overline{\mathcal{M}}_X$ the *ghost sheaf* of $(X, \mathcal{M}_X)$.

The ghost sheaf $\overline{\mathcal{M}}_X$ of $(X, \mathcal{M}_X)$ carries very important geometric information. We will mention briefly about this in ...

Now we immediately start with the basic examples.

Example A.14. *The trivial log structure.*
Let X be a scheme and

$$\mathcal{M}_X := \mathcal{O}_X^\times$$
$$\alpha_X : \mathcal{O}_X^\times \longrightarrow \mathcal{O}_X$$

be the inclusion. Clearly, this defines a log structure on X, called the trivial log structure.

Example A.15. *The standard log point.*
Let

$$X := \operatorname{Spec} \mathbb{C}$$
$$\mathcal{M}_X := \mathbb{C}^\times \oplus \mathbb{N}$$

Define $\alpha_X : \mathcal{M}_X \to \mathbb{C}$ as follows.

$$\alpha_X(x, n) := \begin{cases} x & \text{if } n = 0 \\ 0 & \text{if } n \neq 0 \end{cases}$$

A point $\operatorname{Spec} \mathbb{C}$ together with this log structure is called the standard log point.

Example A.16. *The polar log point.*
Let

$$X := \operatorname{Spec} \mathbb{C}$$
$$\mathcal{M}_X := S^1 \times \mathbb{R}_{\geq 0}$$

Here, $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers with the monoid structure given by multiplication and S^1 is the set of complex numbers of absolute value 1 again with multiplication. Define

$$\alpha : S^1 \times \mathbb{R}_{\geq 0} \to \mathbb{C}$$
$$(e^{i\phi} r) \to r e^{i\phi}$$

This is by definition a prelog structure on X. Observe that $\alpha\big|_{\alpha^{-1}(\mathbb{C}^\times)}$ is an isomorphism. We have $\alpha^{-1}(\mathbb{C}^\times) = \mathbb{R}_{>0} \times S^1$, and

$$\alpha\big|_{\alpha^{-1}(\mathbb{C}^\times)} : \mathbb{C}^\times \to \mathbb{R}_{>0} \times S^1$$
$$z \to (|z|, arg(z))$$

Thus, α gives a log structure on $\operatorname{Spec}\mathbb{C}$ and together with this log structure $\operatorname{Spec}\mathbb{C}$ is called the polar log point.

Definition A.17. *The divisorial log structure.*
Let $D \subset X$ be a divisor. Let $j : X \setminus D \to X$ the embedding of the complement. Take the sheaf of monoids on X defined as follows.

$$\mathcal{M}_{(X,D)} := j_*(\mathcal{O}^\times_{X\setminus D}) \cap \mathcal{O}_X$$

So, $\mathcal{M}_{(X,D)}$ is the sheaf of regular functions on X with zeroes in $D \subset X$, namely the regular functions on X which are units on $X \setminus D$.

We have $\alpha_X : \mathcal{M}_{(X,D)} \hookrightarrow \mathcal{O}_X$. Hence we get, $\alpha^{-1}(\mathcal{O}^\times_X) = \mathcal{O}^\times_X \subset \mathcal{M}_{(X,D)}$ which shows $\alpha\big|_{\mathcal{O}^\times_X} : \alpha^{-1}(\mathcal{O}^\times_X) \longrightarrow \mathcal{O}^\times_X$ is an isomorphism.

Remark A.18. There is a short exact sequence $0 \longrightarrow \mathcal{O}^\times_X \longrightarrow \mathcal{M}^{gp}_X \longrightarrow \mathcal{M}^{gp}_X \big/ \mathcal{O}^\times_X \longrightarrow 0$. If we have a log scheme $(X, \mathcal{M}_{(X,D)})$ equipped with the divisorial log structure then $\mathcal{M}^{gp}_X \big/ \mathcal{O}^\times_X$ will be equal to the sheaf of non-effective Cartier divisors on X with support in $X \setminus D$.

Remark A.19. If X is a scheme with log structure $\mathcal{M}_X$ then from the definition A.11 it follows that the only invertible section of $\overline{\mathcal{M}}_X := \mathcal{M}_X \big/ \mathcal{O}^\times_X$ is the identity.

Example A.20. Let $X = \mathbb{A}^2$ be the affine plane with coordinates x, y and let

$$D = (xy = 0) \subset X$$

Consider the divisorial log structure $M_{(X,D)} \hookrightarrow \mathcal{O}_X$ on X. By definition $M_{(X,D)}$ is the sheaf of regular functions on X with zeroes on $D := \{xy = 0\}$. The global sections of the ghost sheaf $\overline{\mathcal{M}}_{(X,D)}$ are generated by

$$\overline{\mathcal{M}}_{(X,D)} = \langle \overline{x}, \overline{y} \rangle$$

We have the following isomorphisms on stalk level

i.) For $p \in \{x = 0, y \neq 0\}$:

$$\overline{\mathcal{M}}_{(X,D),p} \longrightarrow \mathbb{N}$$
$$\overline{x}^a \mapsto a$$

ii.) For $p \in \{x \neq 0, y = 0\}$:

$$\begin{aligned} \overline{\mathcal{M}}_{(X,D),p} &\longrightarrow \mathbb{N} \\ \overline{y}^b &\mapsto b \end{aligned}$$

iii.) For $p = (0,0)$:

$$\begin{aligned} \overline{\mathcal{M}}_{(X,D),p} &\longrightarrow \mathbb{N}^2 \\ \overline{x}^a\overline{y}^b &\mapsto (a,b) \end{aligned}$$

Definition A.21. *Log structure associated to a prelog structure.*

Let $\alpha : \mathscr{P} \longrightarrow (\mathcal{O}_X, \cdot)$ be a prelog structure on X. One can force a log structure on X as follows. Take the monoid

$$\mathscr{P}^a := \mathscr{P} \oplus \mathcal{O}_X^\times \Big/ \{(p, \alpha(p)^{-1}) \,|\, p \in \alpha^{-1}(\mathcal{O}_X^\times)\}$$

We will define

$$\alpha^a(p,h) = h \cdot \alpha(p)$$

Let us check that the pair $(\alpha^a, \mathscr{P}^a)$ gives a log structure on X. For this, we need to show that the map $\alpha^a\big|_{(\alpha^a)^{-1}(\mathcal{O}_X^\times)}$ is an isomorphism.

Clearly, $\alpha^a\big|_{(\alpha^a)^{-1}(\mathcal{O}_X^\times)} : (\alpha^a)^{-1}(\mathcal{O}_X^\times) \to \mathcal{O}_X^\times$ is surjective, since for any $a \in \mathcal{O}_X^\times$ we have $\alpha^a(1,a) = a$.

Let $\overline{(x,a)} \in \ker \alpha^a\big|_{(\alpha^a)^{-1}}$ we will show $\overline{(x,a)} = (1,1)$. Here we use multiplicative notation and denote the identity elements of the monoids $\mathcal{O}_X$ and $\mathscr{P}$ by 1 and the identity element of $\mathscr{P}^a$ by $(1,1)$.

$\alpha^a(\overline{(x,a)}) = 1 \Rightarrow \alpha(x) \cdot a = 1 \Rightarrow (x,a) = (x, \alpha(x)^{-1})$ and $x \in \alpha^{-1}(\mathcal{O}_X^\times)$. Note that under the equivalence relation on $\mathscr{P}^a$, two sections (x,a) and (y,b) in $\mathscr{P}^a$ are equal if there are local sections $\alpha(p)$ and $\alpha(q)$ of $\mathcal{O}_X^\times$ such that $(x,a) \cdot (q, \alpha(q)^{-1}) = (y,b) \cdot (p, \alpha(p)^{-1})$.

So, we get $(x,a) \sim (1,1)$. Thus, $(\alpha^a, \mathscr{P}^a)$ is a log structure.

Remark A.22. From the definition of $\mathscr{P}^a$ it follows that $\mathscr{P}/\alpha^{-1}(\mathcal{O}_X^\times)$ is isomorphic to $\mathscr{P}^a/\mathcal{O}_X^\times$

Example A.23. Let us give a simple example to show how this construction works. Let $\alpha : \mathbb{N}^2 \to \mathbb{N}$ be the map defined as $\alpha(x,y) = x$. In this example we take additive monoids $\mathbb{N}^2$ and $\mathbb{N}$ with identity 0. Clearly, $\alpha\big|_{\alpha^{-1}(\mathbb{N}^\times)}$ is not an isomorphism . Here, $\mathbb{N}^\times = 0$ and all elements $\{(0,0),(0,1),(0,2),(0,3)\cdots\}$ map to 0 under α. We define a new monoid $\mathcal{M}^a$ as follows.

$$\mathcal{M}^a := \mathbb{N}^2 \oplus \mathbb{N}^\times \Big/ \{(x,y), \alpha(x,y)^{-1} \,|\, (x,y) \in \alpha^{-1}(\mathbb{N}^\times)\}$$

We define the map $\alpha^a : \mathcal{M}^a \to \mathbb{N}$ as $\alpha^a(x,y,a) = a + \alpha(x,y)$. Now it is easy to check $\alpha^a\big|_{(\alpha^a)^{-1}(\mathbb{N}^\times)}$ is an isomorphism. If $\alpha^a(x,y,a) = a + \alpha(x,y) = 0$ we have $a = 0$ and $\alpha(x,y) = 0$. These are elements of the set $\{(0,0,0),(0,1,0),(0,2,0)\cdots\}$. Notice that all elements of this set are identified under the equivalence relation to $(0,0,0)$. Hence we have $(\alpha^a)^{-1}(0) = (0,0,0)$.

Definition A.24. *Induced log structures.* Let Y be an algebraic space with a log structure $\alpha_Y : \mathcal{M}_Y \longrightarrow \mathcal{O}_Y$. Let $f : X \to Y$ be a morphism of algebraic spaces. Consider the map $f^{-1}\mathcal{M}_Y \to f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ which defines a pre log structure on X. The log structure associated to this pre log structure is called the induced log structure or the pull back log structure on X and is usually denoted by $\mathcal{M}_X = f^{-1}\mathcal{M}_Y$.

In a moment we will define log charts, first introduced by Kato in [Kk]. First we have the following definitions.

Definition A.25. A log structure $\mathcal{M}$ on a scheme X is called *coherent* if étale locally on X there exists a finitely generated monoid $\mathscr{P}$ and a homomorphism $\mathscr{P}_X \to \mathcal{O}_X$ whose associated log structure is isomorphic to $\mathcal{M}$. Here $\mathscr{P}_X$ denotes the constant sheaf corresponding to $\mathscr{P}$. $\mathcal{M}$ is called *integral* if $\mathcal{M}$ is a sheaf of integral monoids. If $\mathcal{M}$ is both coherent and integral then it is called *fine* .

Remark A.26. If $\mathcal{M}$ is coherent (resp. integral) the stalk $\mathcal{M}_{\mathcal{O}^\times_{X,\overline{x}},\overline{x}}$ is finitely generated (resp. integral).

Definition A.27. For a scheme X with a fine log structure $\mathcal{M}$ a *chart for* $\mathcal{M}$ is a homomorphism $\mathscr{P}_X \to \mathcal{M}$ for a finitely generated integral monoid $\mathscr{P}$ which induces $\mathscr{P}^a \cong \mathcal{M}$ over an étale open subset of X. Recall that as $\mathscr{P}_X$ we denote the constant sheaf P on X.

Definition A.28. For morphism $f : (X,\mathcal{M}) \to (Y,\mathcal{N})$ of schemes with fine log structures a *chart for* f is a triple $(\mathscr{P}_X \to \mathcal{M}, \mathcal{Q}_Y \to \mathcal{N}, \mathcal{Q} \to \mathscr{P})$ where $\mathscr{P}_X \to \mathcal{M}$, $\mathcal{Q}_Y \to \mathcal{N}$ are charts of $\mathcal{M}$ and $\mathcal{N}$ respectively and $\mathcal{Q} \to \mathscr{P}$ is a homomorphism for which the following diagram commutes.

$$\begin{tikzcd} \mathcal{Q}_X \arrow[r, "h"] \arrow[d] & \mathscr{P}_X \arrow[d] \\ f^{-1}\mathcal{N} \arrow[r] & \mathcal{M} \end{tikzcd}$$

A chart of f also exists étale locally. For a fine log scheme X with log structure $\mathcal{M}$ and a chart $\beta : \mathscr{P} \to \Gamma(X,\mathcal{M})$ we have a natural map $\mathscr{P} \overset{\beta}{\to} \mathcal{M} \overset{\alpha}{\to} \mathcal{O}_X$. Let

$\phi = \alpha \circ \beta$ be the map of monoids $\mathscr{P} \to \mathcal{O}_X$. Since $\mathcal{M}_X$ is the log structure associated to this pre log structure by A.22 we have $\mathscr{P}/\phi^{-1}(\mathcal{O}_X^\times)$ is isomorphic to $\mathcal{M}/\mathcal{O}_X^\times = \overline{\mathcal{M}}$. Therefore, for a chart $\mathscr{P}$ of a log structure $\mathcal{M}$ we have a surjective map $\mathscr{P} \to \overline{\mathcal{M}}$. This is surjective on the stalks. So, for a geometric point $\overline{x} \in \mathcal{U}$ for some open set $\mathcal{U} \in X$ we have the following commutative diagram.

$$\begin{tikzcd}
\mathscr{P} = \Gamma(\mathcal{U}, \mathscr{P}) \arrow[r, "h"] \arrow[d] & \Gamma(\mathcal{U}, \overline{\mathcal{M}}_{\mathcal{U}}) \arrow[d] \\
\mathscr{P} = \mathscr{P}_{\overline{x}} \arrow[r] & \overline{\mathcal{M}}_{\mathcal{U},\overline{x}}
\end{tikzcd}$$

where $\mathcal{M}_{\mathcal{U}}$ denotes the pull back of the sheaf $\mathcal{M}$ to $\mathcal{U}$. Observe that this implies $\Gamma(\mathcal{U}, \overline{\mathcal{M}}_{\mathcal{U}}) \to \overline{\mathcal{M}}_{\mathcal{U},\overline{x}}$ is surjective.

Discussion A.29. Let

$$N \simeq \mathbb{Z}^n$$

be a finitely generated free abelian group and

$$M = \operatorname{Hom}(N, \mathbb{Z})$$

its dual. Let X be a toric variety with fan Σ in the vector space $N_{\mathbb{R}}$, endowed with the *toric log structure* $\mathcal{M}_X$. That is; $\mathcal{M}_X = \mathcal{M}_{X,D}$ is the divisoral log structure with $D \subset X$ the toric boundary divisor. For $\sigma \in \Sigma$ we have the affine toric patch

$$U_\sigma = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

of X. The toric log structure $\mathcal{M}_X$ is locally generated by the monomial functions on this open subset, that is, the canonical map

$$\sigma^\vee \cap M \longrightarrow \mathbb{C}[\sigma^\vee \cap M], \quad m \longmapsto z^m \tag{A.1}$$

is a chart for the log structure on U_σ (see Example A.31). The minimal toric stratum of U_σ is the algebraic torus

$$T_\sigma = \operatorname{Spec} \mathbb{C}[M/(\sigma^\perp \cap M)].$$

Now the following conditions are equivalent for $m \in \sigma^\vee \cap M$:

(1) z^m is invertible on U_σ;
(2) z^m is invertible on the smallest dimensional toric stratum of U_σ;
(3) z^m is invertible on any toric prime divisor containing T_σ;
(4) For any $n \in N \cap \sigma$ it holds $< n, m >= 0$;
(5) $m \in \sigma^\perp$.

We thus obtain the following statement.

Proposition A.30. *For any $x \in T_\sigma$ the toric chart* (A.1) *induces a canonical isomorphism*

$$\sigma^\vee \cap M/\sigma^\perp \cap M \xrightarrow{\sigma} \overline{\mathcal{M}}_{X,x}.$$

□

Example A.31. (3.19,[G]) Let X be the affine toric variety

$$X = \operatorname{Spec} \mathbb{C}[\sigma^\vee \cap M]$$

endowed with the toric log structure $\mathcal{M}_X$. Then $\mathcal{M}_{(X,D)}$ is fine and a chart for $\mathcal{M}_X$ is given by A.1 as in the Discussion A.29. Denote by $P := \sigma^\vee \cap M$ and let P_X be corresponding constant sheaf on $X = \operatorname{Spec} \mathbb{C}[P]$. The map $P \to \mathbb{C}[P]$ induces a morphism $P_X \to \mathcal{O}_X$ which factors as

$$P \longrightarrow \mathcal{M}_{(X)} \hookrightarrow \mathcal{O}_X$$

via $p \mapsto z^p \in \mathcal{M}_{(X)}$ since z^p is a regular function on X invertible on the big torus orbit, on the complement of the toric boundary divisor D. This gives a map

$$P \oplus \mathcal{O}_X^\times \to \mathcal{M}_{(X)}$$

whose kernel on an open subset $U \subseteq X$, consists precisely of the pairs (p, z^{-p}) such that z^{-p} is invertible on U.

Moreover, for a geometric point $x \in X$, a function defined in a neighbourhood U, which is invertible on $U \setminus D$ is of the form $h \cdot z^p$,for $p \in P$ and $h \in \mathcal{O}_X^\times(U)$

Thus, we have a map $P \longrightarrow \mathcal{M}_{(X)}$ whose associated log structure by construction is isomorphic to $\mathcal{M}_{(X)}$. This shows that A.1 is a chart for the toric log structure $\mathcal{M}_X$ on X.

Example A.32. Let $X = {\operatorname{Spec} \mathbb{C}[x,y,w,t]} \big/ {(xy - wt)}$. A fan for X is given by the dual cone $\tilde{\sigma} = \mathbb{R}_{\geq 0}(0,1,0) + \mathbb{R}_{\geq 0}(-1,0,1) + \mathbb{R}_{\geq 0}(0,-1,1) + \mathbb{R}_{\geq 0}(1,0,0)$. Then we have the four toric invariant divisors

$$D_1 = (y = w = 0),\ D_2 = (x = w = 0),\ D_3 = (x = t = 0),\ D_4 = (y = t = 0)$$

Let D be the canonical divisor. That is $D = -D_1 - D_2 - D_3 - D_4$. Note that sections $\Gamma(X, \mathcal{M}_{X,D})$ of the log structure $\mathcal{M}_{X,D}$ are $s_x,\ s_y,\ ,s_w,\ s_t$ where s_x denotes functions of the form x^a for $a \in \mathbb{N}$ and $s_y,\ ,s_w,\ s_t$ are defined analogously. Clearly these are regular functions having zeroes in D. We have $\mathscr{P} =< e_1, \cdots e_4 | e_1 + e_2 = e_3 + e_4 >$. Define a chart $\mathscr{P} \to \Gamma(X, \mathcal{M}_{X,D})$ as follows.

$$\begin{aligned} \mathscr{P} &\xrightarrow{\alpha} \Gamma(X, \mathcal{M}_{X,D}) \\ e_1 &\to x \\ e_2 &\to y \\ e_3 &\to w \\ e_4 &\to t \end{aligned}$$

Remark A.33. Generally a toric variety that is not neccessarily affine is a fine log scheme. Note that a toric variety is obtained by gluing its affine patches. So, we can define a log structure on a toric variety that will restrict to the canonical log structure on each affine patch. This log structure is fine.

Note also that if we consider a toric variety X with the divisorial log structure $\mathcal{M}_{(X,D)}$ and if we take D to be an arbitrary divisor but not the canonical divisor, then the log structure is not necessarily fine.

Our final few words on the brief introduction to log geometry will be concerned on log morphisms.

Definition A.34. If X and Y are log schemes with sheafs of monoids $\mathcal{M}_X$ and $\mathcal{M}_X$, then we define a morphism $(f, f^\flat)$ from $(X, \mathcal{M})$ to $(Y, \mathcal{M}_Y)$ so that $f : X \longrightarrow Y$ is a morphism of the underlying schemes and $f^\flat : f^{-1}\mathcal{M}_Y \longrightarrow \mathcal{M}_X$ is a homomorphism of sheafs of monoids where $f^{-1}\mathcal{M}_Y$ denotes the inverse image of the sheaf $\mathcal{M}_Y$ so that the following diagram commutes

$$\begin{tikzcd} f^{-1}\mathcal{M}_Y \arrow[r, "f^\flat"] \arrow[d, "\alpha_Y"'] & \mathcal{M}_X \arrow[d, "\alpha_X"'] \\ f^{-1}\mathcal{O}_Y \arrow[r, "f^\sharp"'] & \mathcal{O}_X \end{tikzcd}$$

We call $f := (f, f^\flat)$ a *morphism of log schemes.*

Recall that if we have a morphism $f : X \to Y$ of analytic spaces where Y is equipped with a log structure $\mathcal{M}_Y$ there is an induced log structure on X denoted by $f^*\mathcal{M}_Y$.

Definition A.35. A morphism $f : X \to Y$ between log schemes $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ is called *strict* if $\mathcal{M}_X$ is isomorphic to the induced log structure $f^*\mathcal{M}_Y$ from Y.

Let $(X, \mathcal{M}_X)$ be a fine log scheme with a chart $\mathscr{P} \to \Gamma(X, \mathcal{M}_X)$. Then $X \to \mathrm{Spec}(\mathbb{C}[\mathscr{P}])$ is a strict morphism of analytic spaces.

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ON THE REAL LOCUS IN THE KATO-NAKAYAMA SPACE OF LOGARITHMIC SPACES WITH A VIEW TOWARD TORIC DEGENERATIONS

Contents

Introduction.

We study real structures in the toric degenerations introduced by Gross and the second author in the context of mirror symmetry [GS1], [GS3]. A *toric degeneration* in this sense is a degeneration of algebraic varieties $\delta : \mathfrak{X} \to T = \operatorname{Spec} R$ with R a discrete valuation ring and with central fibre $X_0 = \delta^{-1}(0)$ a union of toric varieties, glued pairwise along toric divisors. Here $0 \in \operatorname{Spec} R$ is the closed point. We also require that δ is toroidal at the zero-dimensional toric strata, that is, étale locally near these points, δ is given by a monomial equation in an affine toric variety. For an introductory survey of toric degenerations see [GS4].

Probably the most remarkable aspect of toric degenerations is that they can be produced *canonically* from the central fibre X_0 and some residual information on the family $\mathfrak{X}$, captured by what is called a *log structure*. While the reconstruction is done

Date: December 21, 2016.

by an inductive procedure involving a *wall structure* [GS3], and is typically impossible to carry through in practice, many features of the family are already contained in the log structure. A simple characterization of the nature of the log structure in the present situation has been given in [GS1], Theorem 3.27. It says that if at a general point of the singular locus of X_0 where two irreducible components meet, $\mathfrak{X}$ is given as $xy = f \cdot t^e$ with $t \in R$ generating the maximal ideal, the log structure captures $e \in \mathbb{N}$ and the restriction of f to $x = y = 0$.

An important feature of the log structure for the present paper is that the topology of the degeneration can be read off canonically. Indeed, just as for any logarithmic space over the complex numbers, to X_0 there is a canonically and functorially associated topological space X_0^{KN}, its *Kato-Nakayama space* or *Betti realization* [KN]. It comes with a continuous map to the analytic space X_0^{an} associated to X_0. Moreover, in the present case, there is a map $X_0^{\mathrm{KN}} \to S^1$, coming from functoriality and the fact that the closed point in T as a divisor also comes with a log structure, with S^1 its Kato-Nakayama space. Now it follows from the main result of [NO] that the map $X_0^{\mathrm{KN}} \to S^1$ is homeomorphic to the preimageunder δ of a small circle about 0 in T. See the discussion at the beginning of §4.2 for details. In particular, by restricting to the fibre over, say $1 \in S^1$, we obtain a topological space $X_0^{\mathrm{KN}}(1)$ homeomorphic to a general fiber $\mathcal{X}_t$ of an analytic model $\mathcal{X}$ of the degeneration $\mathfrak{X}$.

Our primary interest in this paper are real structures in X_0 and their lift to X_0^{KN}. The main reason for being interested in real structures in this context is that the real locus produces natural Lagrangian submanifolds on any complex projective manifold defined over $\mathbb{R}$. Thus assuming the analytic model $\mathcal{X}$ is defined over $\mathbb{R}$, it comes with a natural family of degenerating Lagrangian submanifolds. Again we can study these Lagrangians by means of their analogues in X_0^{KN}. Note that if X_0 is defined over $\mathbb{R}$ and the functions f on the double locus defining the log structure are as well, then the canonical family $\mathfrak{X}$ is already defined over $\mathbb{R}$, see [GS3], Theorem 5.2.

Once we have a real Lagrangian $L \subset \mathcal{X}_t$, a holomorphic disc with boundary on L glues with its complex conjugate to a rational curve $C \subset \mathcal{X}_t$ with a real involution. Real rational curves are amenable to techniques of algebraic geometry and notably of log Gromov-Witten theory of the central fibre X_0. Thus real Lagrangians provide an algebraic-geometric path to open Gromov-Witten invariants and the Fukaya category. See [So],[PSW] and [FOOO] for previous work in this direction without degenerations.

In Section 1 we introduce the straightforward notion of a real structure on a log space along with basic properties. Our main example is the central fibre of a degeneration defined over $\mathbb{R}$, with its natural log structure. In Section 2 we recall the definition of the Kato-Nakayama space X^{KN} over a log scheme X as a topological space along with some properties needed in later sections. We then show that the real involution on a real log scheme lifts canonically to its Kato-Nakayama space (Section 3). Section 4 is devoted to the toric degeneration setup. We describe the Kato-Nakayama space as glued from standard pieces, torus bundles over the momentum polytopes of the irreducible components of the central fibre $X_0 \subset \mathfrak{X}$, and in terms of global monodromy data. Under the presence of a real structure we give a similar description for the real locus. For real structures inducing the standard real structure on each toric irreducible component of X_0, the real locus in the Kato-Nakayama space of X_0 is a branched cover of the union B of momentum polyhedra, the integral affine manifold of half the real dimension of a general fibre governing the inductive construction of $\mathfrak{X}$. For a concrete example we study the case of a toric degeneration of quartic $K3$ surfaces, reproducing a result of Castaño-Bernard and Matessi [CBM] on the topology of the real locus of an SYZ-fibration with compatible real involution in our setup.

Conventions. We work in the category of log schemes of finite type over $\mathbb{C}$ with log structures in the étale topology, but use the analytic topology from Section 2 on. Similar discussions are of course possible in the categories of algebraic log stacks over $\mathbb{C}$ or of complex analytic log spaces. Throughout this paper we assume basic familiarity with log geometry at the level of [Kf]. For more details we encourage the reader to also look at [Kk], [O]. The structure homomorphism of a log space $(X, \mathcal{M}_X)$ is denoted $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$, or just α if X is understood. The standard log point $(\operatorname{Spec}\mathbb{C}, \mathbb{N}\oplus\mathbb{C}^\times)$ is denoted by $O^\dagger$.

For $a = re^{i\varphi} \in \mathbb{C} \setminus \{0\}$ we denote by $\arg(a) = \varphi \in \mathbb{R}/2\pi i\mathbb{Z}$ and by $\operatorname{Arg}(a) = e^{i\varphi} = a/|a|$.

1. Real structures in log geometry

Recall that for a scheme $\bar{X}$ defined over $\mathbb{R}$ the Galois group $G(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$ acts on the assoicated complex scheme $X = \bar{X} \times_{\operatorname{Spec}\mathbb{R}} \operatorname{Spec}\mathbb{C}$ by means of the universal property of the cartesian product

$$\begin{array}{ccc} X & \longrightarrow & \bar{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{R}. \end{array}$$

The generator of the Galois action thus acts on X as an involution of schemes over $\mathbb{R}$ making the following diagram commutative

(1.1)
$$\begin{array}{ccc} X & \xrightarrow{\iota} & X \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{\text{conj}} & \operatorname{Spec}\mathbb{C}. \end{array}$$

Here conj denotes the $\mathbb{R}$-linear automorphism of $\operatorname{Spec}\mathbb{C}$ defined by complex conjugation.

Conversely, a *real structure* on a complex scheme X is an involution $\iota : X \to X$ of schemes over $\mathbb{R}$ fitting into the commutative diagram (1.1). It is not hard to see that if X is separated then X is defined over $\mathbb{R}$ with ι the generator of the Galois action ([Hr], II Ex.4.7). A pair (X, ι) is called a *real scheme*. By abuse of notation we usually omit ι when talking about real schemes.

Definition 1.1. Let $(X, \mathcal{M}_X)$ be a log scheme over $\mathbb{C}$ with a real structure $\iota_X : X \to X$ on the underlying scheme. Then a *real structure* on $(X, \mathcal{M}_X)$ *(lifting* ι_X*)* is an involution

$$\tilde{\iota}_X = (\iota_X, \iota_X^\flat) : (X, \mathcal{M}_X) \longrightarrow (X, \mathcal{M}_X)$$

of log schemes over $\mathbb{R}$ with underlying scheme-theoretic morphism ι_X. The data consisting of $(X, \mathcal{M}_X)$ and the involutions ι_X, $\iota_X^\flat$ is called a *real log scheme.*

In talking about real log schemes the involutions ι_X, $\iota_X^\flat$ are usually omitted from the notation. We also sometimes use the notation ι_X for the involution of the log space $(X, \mathcal{M}_X)$ and in this case write $\underline{\iota}_X$ if we want to emphasize we mean the underlying morphism of schemes.

Definition 1.2. Let $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$ be real log schemes. A morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of real log schemes is called *real* if the following diagram is commutative.

$$\begin{array}{ccc} f^{-1}\iota_Y^{-1}\mathcal{M}_Y & \xrightarrow{f^{-1}\iota_Y^\flat} & f^{-1}\mathcal{M}_Y \\ {\scriptstyle \iota_X^{-1}f^\flat}\downarrow & & \downarrow{\scriptstyle f^\flat} \\ \iota_X^{-1}\mathcal{M}_X & \xrightarrow{\iota_X^\flat} & \mathcal{M}_X. \end{array}$$

Here the left-hand vertical arrow uses the identification $\iota_Y \circ f = f \circ \iota_X$.

Remark 1.3. For a real morphism of real log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ the following diagram commutes.

$$\begin{tikzcd}
& f^{-1}\iota_Y^{-1}\mathcal{O}_Y \arrow[rr] \arrow[dd] & & f^{-1}\mathcal{O}_Y \arrow[dd] \\
f^{-1}\iota_Y^{-1}\mathcal{M}_Y \arrow[ur] \arrow[rr] \arrow[dd] & & f^{-1}\mathcal{M}_Y \arrow[ur] \arrow[dd] & \\
& \iota_X^{-1}\mathcal{O}_X \arrow[rr] & & \mathcal{O}_X \\
\iota_X^{-1}\mathcal{M}_X \arrow[ur] \arrow[rr] & & \mathcal{M}_X \arrow[ur] &
\end{tikzcd}$$

In fact, commutativity on the (1) bottom, (2) top, (3) right (4) left (5) back and (6) front faces follows from the assumptions that (1) $(X, \mathcal{M}_X)$ is a real log scheme, (2) $(Y, \mathcal{M}_Y)$ is a real log scheme, (3) f is a morphism of log schemes, (4) ι_X^{-1} applied to the right face plus the identity $f \circ \iota_X = \iota_Y \circ f$, (5) f induces a real morphism on the underlying schemes and (6) f is a real morphism of real log structures.

Given a real log scheme $(X, \mathcal{M}_X)$ with $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ the structure homomorphism, for any geometric point $\bar{x} \to X$ we have a commutative diagram

$$\begin{tikzcd}
\mathcal{M}_{X,\bar{x}} \arrow[r, "\iota^\flat"] \arrow[d, "\alpha_{\bar{x}}"'] & \mathcal{M}_{X,\iota(\bar{x})} \arrow[r, "\iota^\flat"] \arrow[d, "\alpha_{\iota(\bar{x})}"'] & \mathcal{M}_{X,\bar{x}} \arrow[d, "\alpha_{\bar{x}}"'] \\
\mathcal{O}_{X,\bar{x}} \arrow[r, "\iota^\sharp"] & \mathcal{O}_{X,\iota(\bar{x})} \arrow[r, "\iota^\sharp"] & \mathcal{O}_{X,\bar{x}}.
\end{tikzcd}$$

The compositions of the maps in the two horizontal sequences are the identity on $\mathcal{M}_{X,\bar{x}}$ and on $\mathcal{O}_{X,\bar{x}}$, respectively. For the next result recall that if X is a pure-dimensional scheme and $D \subset X$ is a closed subset of codimension one, then the subsheaf $\mathcal{M}_{(X,D)} \subset \mathcal{O}_X$ of regular functions with zeros contained in D defines the *divisorial log structure* on X associated to D.

Proposition 1.4. *Let X be a pure-dimensional scheme, $D \subset X$ a closed subset of codimension one and let $\mathcal{M}_X = \mathcal{M}_{(X,D)}$ be the associated divisorial log structure. Then a real structure ι on X lifts to $\mathcal{M}_X$ iff $\iota(D) = D$. Moreover, in this case the lift $\iota^\flat$ is uniquely determined as the restriction of $\iota^\sharp$ to $\mathcal{M}_{(X,D)} \subset \mathcal{O}_X$.*

Proof. Let $\iota : X \to X$ be a real structure on X with $\iota(D) = D$. Then $\iota(X \setminus D) = X \setminus D$ and hence $\iota^\sharp$ restricts to an isomorphism $\varphi : \iota^{-1}\mathcal{O}^\times_{X\setminus D} \to \mathcal{O}^\times_{X\setminus D}$. By definition of $\mathcal{M}_{(X,D)}$, φ induces an isomorphism $\iota^\flat : \iota^{-1}\mathcal{M}_{(X,D)} \to \mathcal{M}_{(X,D)}$. Hence we get a real structure $(\iota, \iota^\flat) : (X, \mathcal{M}_{(X,D)}) \to (X, \mathcal{M}_{(X,D)})$ on $(X, \mathcal{M}_{(X,D)})$ lifting ι.

Conversely, let the real structure $\iota : X \to X$ lift to $(\iota, \iota^\flat) : (X, \mathcal{M}_{(X,D)}) \longrightarrow (X, \mathcal{M}_{(X,D)})$. In other words, there exists a morphism $\iota^\flat : \iota^{-1}\mathcal{M}_{(X,D)} \longrightarrow \mathcal{M}_{(X,D)}$

making the following diagram commute.

$$
\begin{tikzcd}
\iota^{-1}\mathcal{M}_{(X,D)} \arrow[r, "\iota^{-1}\alpha"] \arrow[d, "\iota^\flat"'] & \iota^{-1}\mathcal{O}_X \arrow[d, "\iota^\sharp"] \\
\mathcal{M}_{(X,D)} \arrow[r, "\alpha"] & \mathcal{O}_X
\end{tikzcd}
\tag{1.2}
$$

Let $D = \bigcup_\mu D_\mu$ be the decomposition into irreducible components. Since $\iota^2 = \mathrm{id}_X$ it suffices to show $\iota(D) \subset D$, or $\iota(D_\mu) \subset D$ for every μ. Fix μ and let $U \subset X$ be an affine open subscheme with $U \cap D_\mu \neq \emptyset$. Let $f \in \mathcal{O}_X(U) \setminus \{0\}$ be such that $D \subset V(f)$. Then $U \cap D_\mu \subset U \cap D \subset V(f)$. Write $V(f) = (D_\mu \cap U) \cup E$ with $E \subset V(f)$ the union of the irreducible components of $V(f)$ different from D_μ. Replacing U by $U \setminus E$ we may assume $V(f) = U \cap D_\mu$. Note that U may not be affine anymore, but this is not important from now on.

Taking sections of Diagram (1.2) over $\iota^{-1}(U)$ shows that $f \circ \iota = \iota^\sharp(f)$ lies in $\mathcal{M}_{(X,D)}\big(\iota^{-1}(U)\big) \subset \mathcal{O}_X\big(\iota^{-1}(U)\big)$. By the definition of $\mathcal{M}_{(X,D)}$ this implies $V(f\circ\iota) \subset D$. But also

$$V(f \circ \iota) = \iota^{-1}\big(V(f)\big) = \iota^{-1}(U \cap D_\mu) = \iota(U \cap D_\mu).$$

Taken together this shows that $\iota(U \cap D_\mu) \subset D$. Since U is open with $U \cap D_\mu \neq \emptyset$ we obtain the desired inclusion $\iota(D_\mu) \subset D$. □

Proposition 1.5. *Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be strict and assume that the morphism $\underline{f}$ of the underlying schemes is compatible with real structures ι_X on X and ι_Y on Y. Then for any real structure $\iota_Y^\flat$ on $\mathcal{M}_Y$ lifting ι_Y there exists a unique real structure $\iota_X^\flat$ on $\mathcal{M}_X$ lifting ι_X and compatible with f.*

Proof. By strictness we can assume the log structure $\mathcal{M}_X$ on X is the pull-back log structure $f^*\mathcal{M}_Y = f^{-1}\mathcal{M}_Y \oplus_{f^{-1}\mathcal{O}_Y^\times} \mathcal{O}_X^\times$. Hence,

$$\iota_X^{-1}\mathcal{M}_X = \iota_X^{-1} f^{-1}\mathcal{M}_Y \oplus_{\iota_X^{-1} f^{-1}\mathcal{O}_Y^\times} \iota_X^{-1}\mathcal{O}_X^\times = f^{-1}\iota_Y^{-1}\mathcal{M}_Y \oplus_{f^{-1}\iota_Y^{-1}\mathcal{O}_Y^\times} \iota_X^{-1}\mathcal{O}_X^\times.$$

Now for a lift $\iota_X^\flat : \iota_X^{-1}\mathcal{M}_X \to \mathcal{M}_X$ of $\iota_X^\sharp$ compatible with f, the composition

$$\varphi : f^{-1}\iota_Y^{-1}\mathcal{M}_Y \longrightarrow \iota_X^{-1}\mathcal{M}_X \xrightarrow{\iota_X^\flat} \mathcal{M}_X = f^*\mathcal{M}_Y$$

factors over $f^{-1}\iota_Y^\flat : f^{-1}\iota_Y^{-1}\mathcal{M}_Y \to f^{-1}\mathcal{M}_Y$ and is hence determined by f and $\iota_Y^\flat$. Similarly, the composition

$$\psi : \iota_X^{-1}\mathcal{O}_X^\times \longrightarrow \iota_X^{-1}\mathcal{M}_X \xrightarrow{\iota_X^\flat} \mathcal{M}_X = f^*\mathcal{M}_Y$$

factors over $\iota_X^\sharp : \iota_X^{-1}\mathcal{O}_X^\times \to \mathcal{O}_X^\times$ and thus is known by assumption. Since $\underline{f}$ is a real morphism of real schemes, φ and ψ agree on $f^{-1}\iota_Y^{-1}\mathcal{O}_Y^\times$. Hence the unique existence of $\iota_X^\flat$ with the requested properties follows from the universal property of the fibered sum. □

Explicit computations are most easily done in charts adapted to the real structure.

Definition 1.6. Let $(X, \mathcal{M}_X)$ be a log scheme with a real structure $(\iota_X, \iota_X^\flat)$. A chart $\beta : P \to \Gamma(U, \mathcal{M}_X)$ for $(X, \mathcal{M}_X)$ is called a *real chart* if (1) $\iota_X(U) = U$ and (2) there exists an involution $\iota_P : P \to P$ such that for all $p \in P$ it holds $\beta\big(\iota_P(p)\big) = \iota_X^\flat\big(\beta(p)\big)$.

Example 1.7. An involution ι_P of a toric monoid P induces an antiholomorphic involution on $\mathbb{C}[P]$ by mapping $\sum_p a_p z^p$ to $\sum_p \overline{a_p} z^{\iota_P(p)}$. The induced real structure on the toric variety $X_P = \operatorname{Spec} \mathbb{C}[P]$ permutes the irreducible components of the toric divisor $D_P \subset X_P$ and hence, by Proposition 1.4 induces a real structure on $(X_P, \mathcal{M}_{(X_P, D_P)})$. We claim the canonical toric chart

$$\beta : P \longrightarrow \Gamma(X_P, \mathcal{M}_{(X_P,D_P)}), \quad p \longmapsto z^p$$

is a real chart. Indeed, for any $p \in P$ we have $\beta(\iota_P(p)) = z^{\iota(p)} = \iota^\sharp_{X_P}(z^p) = \iota^\flat_{X_P}(z^p)$, the last equality due to Proposition 1.4.

Real charts may not exist, a necessary condition being that X has a cover by affine open sets that are invariant under the real involution ι_X. This is the only obstruction:

Lemma 1.8. *Let $(X, \mathcal{M}_X)$ be a real log scheme with involution ι_X. Let $U \subset X$ be a ι_X-invariant open set supporting a chart $\beta : P \to \Gamma(U, \mathcal{M}_X)$. Then there also exists a real chart $\beta' : P' \to \Gamma(U, \mathcal{M}_X)$ for $\mathcal{M}_X$ on U.*

Proof. We claim that

$$\tilde{\beta} : P \oplus P \longrightarrow \Gamma(U, X), \quad \tilde{\beta}(p, p') = \beta(p) \cdot \iota_X^\flat\big(\beta(p')\big)$$

is a real chart. Since $\tilde{\beta}$ restricts to β on the first summand of $\tilde{P}$, this is still a chart for $\mathcal{M}_X$ on U. For the involution on the monoid $\tilde{P} = P \oplus P$ we take $\iota_{\tilde{P}}(p, p') = (p', p)$. Then indeed for any $(p, p') \in \tilde{P}$ we have

$$\tilde{\beta}\big(\iota_{\tilde{P}}(p, p')\big) = \tilde{\beta}(p', p) = \beta(p') \cdot \iota_X^\flat(\beta(p)) = \iota_X^\flat\big(\iota_X^\flat(\beta(p')) \cdot \beta(p)\big) = \iota_X^\flat\big(\tilde{\beta}(p, p')\big),$$

verifying the condition for a real chart. □

Note that if X is a separated scheme, real charts always exist at any point x in the fixed locus of ι_X. In fact, take any chart defined in a neighbourhood U of X, restrict to $U \cap \iota_X(U)$, still an affine open set by separatedness, and apply Lemma 1.8.

Proposition 1.9. *Cartesian products exist in the category of real log schemes.*

Proof. Let $(X, \mathcal{M}_X)$, $(S, \mathcal{M}_S)$, $(T, \mathcal{M}_T)$ be real log schemes endowed with morphisms $f : (X, \mathcal{M}_X) \to (T, \mathcal{M}_T)$ and $g : (S, \mathcal{M}_S) \to (T, \mathcal{M}_T)$. Then the fibre product in the category of log schemes $(S \times_T X, \mathcal{M}_{S\times_T X})$ fits into the following cartesian diagram.

$$\begin{tikzcd} (S \times_T X, \mathcal{M}_{S\times_T X}) \arrow[r, "p_X"] \arrow[d, "p_S"] & (X, \mathcal{M}_X) \arrow[d, "f"] \\ (S, \mathcal{M}_S) \arrow[r, "g"] & (T, \mathcal{M}_T) \end{tikzcd} \tag{1.3}$$

The log structure on the fiber product $S \times_T X$ is given by $\mathcal{M}_{S\times_T X} = p_X^*\mathcal{M}_X \oplus_{p_T^*\mathcal{M}_T} p_S^*\mathcal{M}_S$. By the universal property of the fibered coproduct the existence of real structures on $(X, \mathcal{M}_X), (S, \mathcal{M}_S)$ and $(T, \mathcal{M}_T)$ ensures the existence of a real structure on $(S \times_T X, \mathcal{M}_{S\times_T X})$. ☐

Note that in general the amalgamated sum of fine log structures $p_X^*\mathcal{M}_X \oplus_{p_T^*\mathcal{M}_T} p_Y^*\mathcal{M}_Y$ is only coherent, but not even integral. To take the fibred product in the category of fine log schemes requires the further step of integralizing $(S\times_T X, \mathcal{M}_{S\times_T X})$. Given a monoid P with integralization P_{int} and a chart $U \to \operatorname{Spec}\mathbb{Z}[P]$ for a log scheme $(U, \mathcal{M}_U)$, the integralization of $(U, \mathcal{M}_U)$ is the closed subscheme $U \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[P^{\text{int}}]$ of U with the log structure defined by the chart $U \to \operatorname{Spec}\mathbb{Z}[P] \to \operatorname{Spec}\mathbb{Z}[P^{\text{int}}]$. A similar additional step is needed for staying in the category of saturated log schemes. Fortunately, we are only interested in the case that g is strict, and in this case the fibre product in all categories agree. See [O], Ch.III, §2.2.1, for details.

Example 1.10. Let S be the spectrum of a discrete valuation ring with residue field $\mathbb{C}$ and $\delta : \mathfrak{X} \to S$ be a flat morphism. Let $0 \in S$ be the closed point, $X_0 = \delta^{-1}(0)$ and consider δ as a morphism of log schemes with divisorial log structures $\delta : (\mathfrak{X}, \mathcal{M}_{(\mathfrak{X},X_0)}) \to (S, \mathcal{M}_{(S,0)})$. If δ commutes with real structures on $\mathfrak{X}$ and S, then by Proposition 1.4, the morphism δ is naturally a real morphism of real log schemes. Taking the base change by the strict morphism $(\operatorname{Spec}\mathbb{C}, \mathbb{N} \oplus \mathbb{C}^\times) \to (S, \mathcal{M}_{(S,0)})$, Proposition 1.9 leads to a real log scheme $(X_0, \mathcal{M}_{X_0})$ over the standard log point $O^\dagger = (\operatorname{Spec}\mathbb{C}, \mathbb{N} \oplus \mathbb{C}^\times)$.

2. The Kato-Nakayama space of a log space

2.1. **Generalities on Kato-Nakayama spaces.** For the rest of the paper we work in the analytic topology. If R is a finitely generated $\mathbb{C}$-algebra we write $\operatorname{Specan} R$ for the analytic space associated to the complex scheme $\operatorname{Spec} R$.

To any log scheme $(X, \mathcal{M}_X)$ over $\mathbb{C}$, Kato and Nakayama in [KN] have introduced a topological space $(X, \mathcal{M}_X)^{\text{KN}}$, its *Kato-Nakayama space* or *Betti-realization*. We review this definition and its basic properties first before discussing the additional properties coming from a real structure. Denote by $\Pi^\dagger = (\operatorname{Specan}\mathbb{C}, \mathcal{M}_\Pi)$ the *polar log point*, with log structure

$$\alpha_\Pi : \mathcal{M}_{\Pi,0} = \mathbb{R}_{\geq 0} \times U(1) \longrightarrow \mathbb{C}, \quad (r, e^{i\varphi}) \longmapsto r \cdot e^{i\varphi}.$$

There is an obvious map $\Pi^\dagger \to \operatorname{Specan}\mathbb{C}$ making $\Pi^\dagger$ into a log space over $\mathbb{C}$. Note $\overline{\mathcal{M}}_{\Pi,0} = U(1)$, so this log structure is not fine. As a set define

$$(X, \mathcal{M}_X)^{\mathrm{KN}} := \operatorname{Hom}\big(\Pi^\dagger, (X, \mathcal{M}_X)\big),$$

the set of morphisms of complex analytic log spaces $\Pi^\dagger \to (X, \mathcal{M}_X)$. Note that a log morphism $f : \Pi^\dagger \to (X, \mathcal{M}_X)$ is given by its set-theoretic image, a point $x = \varphi(0) \in X$, and a monoid homomorphism $f^\flat : \mathcal{M}_{X,x} \to \mathbb{R}_{\geq 0} \times U(1)$. Forgetting the monoid homomorphism thus defines a map

$$\pi : (X, \mathcal{M}_X)^{\mathrm{KN}} \longrightarrow X. \tag{2.1}$$

We endow $(X, \mathcal{M}_X)^{\mathrm{KN}}$ with the following topology. A local section $\sigma \in \Gamma(U, \mathcal{M}_X^{\mathrm{gp}})$, $U \subset X$ open, defines a map

$$\mathrm{ev}_\sigma : \pi^{-1}(U) \longrightarrow \mathbb{R}_{\geq 0} \times U(1), \quad f \longmapsto f^\flat \circ \sigma. \tag{2.2}$$

As a subbasis of open sets on $(X, \mathcal{M}_X)^{\mathrm{KN}}$ we take $\mathrm{ev}_\sigma^{-1}(V)$, for any $U \subset X$ open, $\sigma \in \Gamma(U, \mathcal{M}_X^{\mathrm{gp}})$ and $V \subset \mathbb{R}_{\geq 0} \times U(1)$ open. The forgetful map π is then clearly continuous.

If the log structure is understood, we sometimes write X^{KN} instead of $(X, \mathcal{M}_X)^{\mathrm{KN}}$ for brevity.

Remark 2.1. The following more explicit set-theoretic description of $(X, \mathcal{M}_X)^{\mathrm{KN}}$ is sometimes useful. A log morphism $f : \Pi^\dagger \to (X, \mathcal{M}_X)$ with $f(0) = x$ is equivalent to a choice of monoid homomorphism $f^\flat$ fitting into the commutative diagram

$$\begin{tikzcd} \mathcal{M}_{X,x} \arrow[r, "f^\flat = (\rho,\theta)"] \arrow[d, "\alpha_{X,x}"'] & \mathbb{R}_{\geq 0} \times U(1) \arrow[d, "\alpha_\Pi"] \\ \mathcal{O}_{X,x} \arrow[r, "\mathrm{ev}_x"'] & \mathbb{C} \end{tikzcd}$$

Here $\alpha_{X,x}$ is the stalk of the structure morphism $\alpha_X : \mathcal{M}_X \to \mathcal{O}_X$ of X and ev_x takes the value of a function at the point x. This diagram implies that the first component ρ of $f^\flat$ is determined by x and the structure homomorphism by

$$\rho(\sigma) = \big|\big(\alpha_{X,x}(\sigma)\big)(x)\big|.$$

Thus giving f is equivalent to selecting the point $x \in X$ and a homomorphism $\theta : \mathcal{M}_{X,x} \to U(1)$ with the property that for any $\sigma \in \mathcal{M}_{X,x}$ it holds

$$(\alpha_{X,x}(\sigma))(x) = \big|(\alpha_{X,x}(\sigma))(x)\big| \cdot \theta(\sigma).$$

Since both sides vanish unless $\sigma \in \mathcal{O}_{X,x}^\times \subset \mathcal{M}_{X,x}$, this last property needs to be checked only on invertible elements. Note also that a homomorphism $\mathcal{M}_{X,x} \to U(1)$ extends to $\mathcal{M}_{X,x}^{\mathrm{gp}}$ since $U(1)$ is an abelian group. Summarizing, we have a canonical identification

$$(X, \mathcal{M}_X)^{\mathrm{KN}} = \Big\{(x,\theta) \in \textstyle\prod_x \mathrm{Hom}(\mathcal{M}_{X,x}^{\mathrm{gp}}, U(1)) \,\Big|\, \forall h \in \mathcal{O}_{X,x}^\times : \theta(h) = \frac{h(x)}{|h(x)|}\Big\} \tag{2.3}$$

In this description we adopt the occasional abuse of notation of viewing $\mathcal{O}_{X,x}^\times$ as a submonoid of $\mathcal{M}_{X,x}$ by means of the structure homomorphism $\mathcal{M}_{X,x} \to \mathcal{O}_{X,x}$. From (2.3), for $s \in \mathcal{M}_{X,x}$ any point $f \in X^{\mathrm{KN}}$ over $x \in X$ defines an element $\theta(s) \in U(1)$. We refer to this element of $U(1)$ as the *phase* of s at f. If $s \in \mathcal{O}_{X,x}^\times$ then the phase of any point of X^{KN} over x agrees with $\mathrm{Arg}(s) = e^{i\arg(s)}$.

Next we give an explicit description of $(X, \mathcal{M}_X)^{\mathrm{KN}}$ assuming the log structure has a chart with a fine monoid. For a fine monoid P, we have $P^{\mathrm{gp}} \simeq T \oplus \mathbb{Z}^r$ with T finite. Thus the set $\mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ is in bijection with $|T|$ copies of the real torus $U(1)^r$ by means of choosing generators. This identification is compatible with the topology on $\mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ defined by the subbasis of topology consisting of the sets

$$V_p := \big\{\varphi \in \mathrm{Hom}(P^{\mathrm{gp}}, U(1)) \,\big|\, \varphi(p) \in V\big\}, \tag{2.4}$$

for all $V \subset U(1)$ open and $p \in P^{\mathrm{gp}}$.

Proposition 2.2. *Let P be a fine monoid and let X be an analytic space endowed with the log structure defined by a holomorphic map $g : X \to \mathrm{Specan}\,\mathbb{C}[P]$. Then there is a canonical closed embedding of $(X, \mathcal{M}_X)^{\mathrm{KN}}$ into $X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ with image*

$$\Big\{(x,\lambda) \in X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1)) \,\Big|\, \forall p \in P :\, g^\sharp(z^p)_x \in \mathcal{O}_{X,x}^\times \Rightarrow \lambda(p) = \mathrm{Arg}(g^\sharp(z^p)(x))\Big\}.$$

Proof. Denote by $\beta : P \to \Gamma(X, \mathcal{M}_X)$ the chart given by g and by $\beta_x : P \to \mathcal{M}_{X,x}$ the induced map to the stalk at $x \in X$. Recall the description (2.3) of $(X, \mathcal{M}_X)^{\mathrm{KN}}$ by pairs (x, θ) with $x \in X$ and $\theta : \mathcal{M}_{X,x}^{\mathrm{gp}} \to U(1)$ a group homomorphism extending $h \mapsto h(x)/|h(x)|$ for $h \in \mathcal{O}_{X,x}^\times$. With this description, the canonical map in the statement is

$$\Psi : (X, \mathcal{M}_X)^{\mathrm{KN}} \longrightarrow X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1)), \quad (x,\theta) \longmapsto \big(x, \theta \circ \beta_x^{\mathrm{gp}}\big).$$

Here $\beta_x^{\mathrm{gp}} : P^{\mathrm{gp}} \to \mathcal{M}_{X,x}^{\mathrm{gp}}$ is the map induced by β_x on the associated groups.

To prove continuity of Ψ, let $p \in P^{\mathrm{gp}}$ and $V \subset U(1)$ be open. Then Ψ^{-1} of $X \times V_p$ with $V_p \subset \mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ the basic open set from (2.4), equals $\mathrm{ev}^{-1}_{\beta^{\mathrm{gp}}(p)}(\mathbb{R}_{\geq 0} \times V)$, with

ev_σ defined in (2.2). By the definition of the topology, $\Psi^{-1}(X \times V_p) \subset (X, \mathcal{M}_X)^{\mathrm{KN}}$ is thus open. Continuity of the first factor π of Ψ being trivial, this shows that Ψ is continuous.

We next check that $\mathrm{im}(\Psi)$ is contained in the stated closed subset of $X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1))$. Let $(x, \theta) \in (X, \mathcal{M}_X)^{\mathrm{KN}}$. For $p \in P$ the required equation $g^\sharp(z^p)(x) = \lambda(p) \cdot \big|g^\sharp(z^p)(x)\big|$ for $\lambda = \theta \circ \beta_x^{\mathrm{gp}}$ is non-trivial only if $h := g^\sharp(z^p) \in \mathcal{O}_{X,x}^\times$. In this case, $\beta_x(p)$ maps to h under the structure homomorphism $\mathcal{M}_{X,x} \to \mathcal{O}_{X,x}$ and hence

$$\big(\theta \circ \beta_x(p)\big)(x) = \frac{h(x)}{\big|h(x)\big|} = \frac{g^\sharp(z^p)(x)}{\big|g^\sharp(z^p)(x)\big|},$$

verifying the required equality.

Conversely, assume $(x, \lambda) \in X \times \mathrm{Hom}(P^{\mathrm{gp}}, U(1))$ fulfills

$$g^\sharp(z^p)(x) = \lambda(p) \cdot \big|g^\sharp(z^p)(x)\big|. \tag{2.5}$$

Denote by $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ the structure homomorphism. Then $\mathcal{M}_{X,x}$ fits into the cocartesian diagram of monoids

$$\begin{tikzcd} \beta_x^{-1}(\mathcal{O}_{X,x}^\times) \arrow[r] \arrow[d, "\alpha_x \circ \beta_x"'] & P \arrow[d, "\beta_x"] \\ \mathcal{O}_{X,x}^\times \arrow[r] & \mathcal{M}_{X,x} \end{tikzcd} .$$

Consider the pair of homomorphisms $\mathrm{Arg} \circ \mathrm{ev}_x : \mathcal{O}_{X,x}^\times \to U(1)$, and $\lambda : P \to U(1)$, with ev_x evaluation at x. In view of $(\alpha \circ \beta)(p) = g^\sharp(z^p)$, Equation (2.5) says precisely that the compositions of these two maps with the maps from $\beta_x^{-1}(\mathcal{O}_{X,x}^\times)$ agree. By the universal property of fibred sums we thus obtain a homorphism $\mathcal{M}_{X,x} \to U(1)$. Define $\theta : \mathcal{M}_{X,x}^{\mathrm{gp}} \to U(1)$ as the induced map on associated groups. For $h \in \mathcal{O}_{X,x}^\times$ it holds $\theta(h) = h(x)/|h(x)|$ and hence $(x, \theta) \in (X, \mathcal{M}_X)^{\mathrm{KN}}$. It is now not hard to see that the map $(x, \lambda) \mapsto (x, \theta)$ is inverse to Ψ and continuous as well. □

2.2. **Examples of Kato-Nakayama spaces.** We next discuss a few examples of Kato-Nakayama spaces, geared towards toric degenerations. Unless otherwise stated, N denotes a finitely generated free abelian group, $M = \mathrm{Hom}(N, \mathbb{Z})$ its dual and $N_\mathbb{R}$, $M_\mathbb{R}$ are the associated real vector spaces. If $\sigma \subset N_\mathbb{R}$ is a cone then the set of monoid homomorphisms $\sigma^\vee = \mathrm{Hom}(\sigma, \mathbb{R}_{\geq 0}) \subset M_\mathbb{R}$ denotes its dual cone. A *lattice polyhedron* is the intersection of rational half-spaces in $M_\mathbb{R}$ with an integral point on each minimal face.

The basic example is a canonical description of the Kato-Nakayama space of a toric variety defined by a momentum polytope. We use a rather liberal definition of a momentum map, not making any reference to a symplectic structure. Let $\Xi \subset M_\mathbb{R}$

be a full-dimensional, convex lattice polyhedron. Let X be the associated complex toric variety. A basic fact of toric geometry states that the fan of X agrees with the normal fan Σ_Ξ of Ξ. From this description, X is covered by affine toric varieties $\operatorname{Specan}\mathbb{C}[\sigma^\vee \cap M]$, for $\sigma \subset \Sigma_\Xi$. Since the patching is monomial, it preserves the real structure of each affine patch. Hence the real locus $\operatorname{Hom}(\sigma^\vee, \mathbb{R}) \subset \operatorname{Hom}(\sigma^\vee, \mathbb{C})$ of each affine patch glues to the real locus $X_\mathbb{R} \subset X$. Unlike in the definition of $\sigma^\vee$, here $\mathbb{R}$ and $\mathbb{C}$ are multiplicative monoids. Moreover, inside the real locus of each affine patch there is the distinguished subset

$$\sigma = \operatorname{Hom}(\sigma^\vee, \mathbb{R}_{\geq 0}) \subset \operatorname{Hom}(\sigma^\vee, \mathbb{R}),$$

with "Hom" referring to homomorphisms of monoid. These also patch via monomial maps to give the *positive real locus* $X_{\geq 0} \subset X_\mathbb{R}$.

Having introduced the positive real locus $X_{\geq 0} \subset X_\mathbb{R}$ we are in position to define abstract momentum maps.

Definition 2.3. Let X be the complex toric variety defined by a full-dimensional lattice polyhedron $\Xi \subset M_\mathbb{R}$. Then a continuous map

$$\mu : X \longrightarrow \Xi$$

is called an *(abstract) momentum map* if the following holds.

(1) μ is invariant under the action of $\operatorname{Hom}(M, U(1))$ on X.
(2) The restriction of μ maps $X_{\geq 0}$ homeomorphically to Ξ, thus defining a section $s_0 : \Xi \to X$ of μ with image $X_{\geq 0}$.
(3) The map

$$\operatorname{Hom}(M, U(1)) \times \Xi \longrightarrow X, \quad (\lambda, x) \longmapsto \lambda \cdot s_0(x) \tag{2.6}$$

induces a homeomorphism $\operatorname{Hom}(M, U(1)) \times \operatorname{Int}(\Xi) \simeq X \setminus D$, where $D \subset X$ is the toric boundary divisor.

Projective toric varieties have a momentum map, see e.g. [Fu], §4.2. For an affine toric variety $\operatorname{Specan}\mathbb{C}[P]$, momentum maps also exist. One natural construction discussed in detail in [NO], §1, is a simple formula in terms of generators of the toric monoid P ([NO], Definition 1.2). Some work is however needed to show that if $P = \sigma^\vee \cap M$, then the image of this momentum map is the cone $\sigma^\vee$ spanned by P. We give here another, easier but somewhat ad hoc construction of a momentum map.

Proposition 2.4. *An affine toric variety $X = \operatorname{Specan}\mathbb{C}[\sigma^\vee \cap M]$ has a momentum map with image the defining rational polyhedral cone $\sigma^\vee \subset M_\mathbb{R}$.*

Proof. If the minimal toric stratum $Z \subset X$ is of dimension $r > 0$, we can decompose $\sigma^\vee \simeq C + \mathbb{R}^r$ and acordingly $X \simeq \overline{X} \times (\mathbb{C}^*)^r$ with $\overline{X}$ a toric variety with a zero-dimensional toric stratum. The product of a momentum map $\overline{X} \to C$ with the momentum map

$$(\mathbb{C}^*)^r \longrightarrow \mathbb{R}^r, \quad (z_1, \dots, z_r) \longmapsto \big(\log|z_1|, \dots, \log|z_r|\big)$$

is then a momentum map for X. We may therefore assume that X has a zero-dimensional toric stratum, or equivalently that $\sigma^\vee$ is strictly convex.

Now embed X into a projective toric variety $\tilde{X}$ and let $\mu : \tilde{X} \to \Xi$ be a momentum map mapping the zero-dimensional toric stratum of X to the origin. Then the cone in $M_\mathbb{R}$ spanned by Ξ equals $\sigma^\vee$. By replacing Ξ with its intersection with an appropriate affine hyperplane we may assume that Ξ is the convex hull of 0 and a disjoint facet $\omega \subset \Xi$. Then $X = \mu^{-1}(\Xi \setminus \omega)$. To construct a momentum map for X with image $\sigma^\vee$ let $q : M_\mathbb{R} \to \mathbb{R}$ be the quotient by T_ω. Then $q(\Xi)$ is an interval $[0, a]$ with $a > 0$. Now $f(x) = x/(a - x)$ maps the half-open interval $[0, a)$ to $\mathbb{R}_{\geq 0}$. A momentum map for X with image $\sigma^\vee$ is then defined by

$$z \longmapsto (f \circ q)\big(\mu(z)\big) \cdot \mu(z).$$

□

Our next result concerns the announced canonical description of the Kato-Nakayama space of a toric variety with a momentum map.

Proposition 2.5. *Let X be a complex toric variety with a momentum map $\mu : X \to \Xi \subset M_\mathbb{R}$ and let $\mathcal{M}_X$ be the toric log structure on X. Then the map* (2.6) *factors through a canonical homeomorphism*

$$\Xi \times \mathrm{Hom}(M, U(1)) \longrightarrow (X, \mathcal{M}_X)^{\mathrm{KN}}.$$

Proof. The toric variety X is covered by open affine sets of the form $\operatorname{Specan} \mathbb{C}[P]$ with $P^{\mathrm{gp}} = M$, and these are charts for the log structure. Thus the local description of X^{KN} in Proposition 2.2 as a closed subset globalizes to define a closed embedding

$$\iota : X^{\mathrm{KN}} \longrightarrow X \times \mathrm{Hom}(M, U(1)).$$

With $s_0 : \Xi \to X$ the section of μ with image $X_{\geq 0} \subset X$, consider the continuous map

$$\Phi : \Xi \times \mathrm{Hom}(M, U(1)) \longrightarrow X \times \mathrm{Hom}(M, U(1)), \quad (a, \lambda) \longmapsto \big(\lambda \cdot s_0(a), \lambda\big).$$

Here $\lambda \in \mathrm{Hom}(M, U(1))$ acts on X as an element of the algebraic torus $\mathrm{Hom}(M, \mathbb{C}^\times)$. The map Φ has the continuous left-inverse $\pi \times \mathrm{id}$. Thus to finish the proof it remains to show $\mathrm{im}(\Phi) = \mathrm{im}(\iota)$.

Indeed, according to Proposition 2.2, $(x,\lambda) \in X \times \operatorname{Hom}(M, U(1))$ lies in $\iota(X^{\mathrm{KN}})$ iff for all $m \in M$ with z^m defined at x it holds $z^m(x) = \lambda(m) \cdot |z^m(x)|$. But this equation holds if and only if $x = \lambda \cdot \sigma(a)$ for $a = \mu(x)$ since $\sigma(a) \in X_{\geq 0}$ implies

$$z^m(\lambda \cdot \sigma(a)) = \lambda(m) \cdot z^m(\sigma(a)) = \lambda(m) \cdot |z^m(x)|.$$

Thus $(x,\lambda) \in X^{\mathrm{KN}}$ iff $(x,\lambda) = (\lambda \cdot \sigma(\mu(x)), \lambda)$, that is, iff $(x,\lambda) \in \operatorname{im}(\Phi)$. □

Remark 2.6. The left-hand side in the statement of Proposition 2.5 can also be written $T^*_\Xi/\check{\Lambda}$ where $\check{\Lambda} \subset T_\Xi$ is the local system of integral cotangent vectors. Indeed, for any $y \in \Xi$ we have the sequence of canonical isomorphisms

$$T^*_{\Xi,y}/\check{\Lambda}_y \longrightarrow \operatorname{Hom}(M,\mathbb{R})/\operatorname{Hom}(M,\mathbb{Z}) = \operatorname{Hom}(M,\mathbb{R}/\mathbb{Z}) = \operatorname{Hom}(M,U(1)).$$

Example 2.7. Let $X = \mathbb{A}^1 = \operatorname{Specan}\mathbb{C}[\mathbb{N}]$ be endowed with the divisorial log structure $\mathcal{M}_{(X,\{0\})}$. By Proposition 2.4 there exists a momentum map $\mu : X \to \mathbb{R}_{\geq 0}$. Explicitly, in the present case one may simply take $\mu(z) = |z|$ where z is the toric coordinate. According to Proposition 2.5, $X^{\mathrm{KN}} \cong \mathbb{R}_{\geq 0} \times S^1$ canonically. The map $\pi : X^{\mathrm{KN}} \to X$ is a homeomorphism onto the image over $\mathbb{A}^1 \setminus \{0\}$ and has fibre $S^1 = \operatorname{Hom}(\mathcal{M}_{(X,\{0\})}, U(1)) = \operatorname{Hom}(\mathbb{N}, U(1))$ over 0. Thus X^{KN} is homeomorphic to the *oriented real blow up* of $\mathbb{A}^1$ at 0.

Example 2.8. More generally, Let $(X, \mathcal{M}_{(X,D)})$ be the divisorial log structure on a complex scheme X with a normal crossings divisor $D \subset X$. Then the Kato-Nakayama space X^{KN} of X can be identified with the oriented real blow up of X along D. At a point $x \in X$ the map $X^{\mathrm{KN}} \to X$ has fibre $(S^1)^k$ with k the number of irreducible components of D containing x.

Example 2.9. Let $X = \mathbb{P}^2$ with the toric log structure. There exists a momentum map $\mu : \mathbb{P}^2 \to \Xi$ with $\Xi = \operatorname{conv}\{(0,0),(1,0),(0,1)\} \subset M_\mathbb{R}$ the 2-simplex and $M = \mathbb{Z}^2$. The momentum map exhibits the algebraic torus $(\mathbb{C}^\times)^2 \subset \mathbb{P}^2$ as a trivial $(S^1)^2$-bundle over $\operatorname{Int}\Xi$. Intrinsically, the 2-torus fibres of μ over $\operatorname{Int}(\Xi)$ are $\operatorname{Hom}(M, U(1))$. Over a face $\tau \subset \Xi$, the 2-torus fibre collapses via the quotient map given by restriction,

$$\operatorname{Hom}(M, U(1)) \longrightarrow \operatorname{Hom}(M \cap T_\tau, U(1)),$$

where $T_\tau \subset M_\mathbb{R}$ is the tangent space of τ. The quotient yields an S^1 over the interior of an edge of Ξ and a point over a vertex.

Now going over to the Kato-Nakayama space simply restores the collapsed directions, thus yielding the trivial product $\Xi \times (S^1)^2$. The fibre of $X^{\mathrm{KN}} \to X$ over the interior of a toric stratum given by the face $\tau \subset \Xi$ are the fibres of $\operatorname{Hom}(M, U(1)) \to \operatorname{Hom}(M \cap T_\tau, U(1))$.

An analogous discussion holds for all toric varieties with a momentum map.

We finish this section with an instructive non-toric example that features a non-fine log structure. It discusses the most simple non-toric example of a toric degeneration, the subject of Section 4.

Example 2.10. Let $X = \operatorname{Specan} \mathbb{C}[x, y, w^{\pm 1}, t]/(xy - t(w + 1))$, considered as a holomorphic family of complex surfaces $\delta : X \to \mathbb{C}$ via projection by t. For fixed $t \neq 0$ we can eliminate w to arrive at $\delta^{-1}(t) \simeq \mathbb{C}^2$. For $t = 0$ we have $\pi^{-1}(0) = \mathbb{C}^2 \amalg_{\mathbb{C}} \mathbb{C}^2$, two copies of the affine plane with coordinates x, w and y, w, respectively, glued semi-normally along the line $x = y = 0$. Denote $X_0 = \delta^{-1}(0)$, let $\mathcal{M}_X = \mathcal{M}_{(X,X_0)}$ be the log structure defined by the family and $\mathcal{M}_{X_0}$ its restriction to the fibre over 0. Then $(X_0, \mathcal{M}_{X_0})$ comes with a log morphism f to the standard log point $O^\dagger = (\mathrm{pt}, \mathbb{C}^\times \oplus \mathbb{N})$. We want to discuss the Kato-Nakayama space of $(X_0, \mathcal{M}_{X_0})$ together with the map to S^1, the Kato-Nakayama space of $O^\dagger$.

First note that X has an A_1-singularity at the point p_0 with coordinates $x = y = t = 0$, $w = -1$. Any Cartier divisor at p_0 with support contained in X_0 is defined by a power of t. Hence $\overline{\mathcal{M}}_{X_0,p_0} = \mathbb{N}$, while at a general point p of the double locus $(X_0)_{\mathrm{sing}} \simeq \mathbb{C}$, the central fibre is a normal crossings divisor in a smooth space and $\overline{\mathcal{M}}_{X_0,p} = \mathbb{N}^2$. In particular, $\overline{\mathcal{M}}_{X_0}$ is not a fine sheaf at p_0. On the other hand $(X_0, \mathcal{M}_{X_0})$ is a typical example of Ogus' notion of *relative coherence*. In this category, the main result of [NO] still says that X^{KN} is homeomorphic relative $(\mathbb{C}, \mathcal{M}_{\mathbb{C}})^{\mathrm{KN}} = S^1 \times \mathbb{R}_{\geq 0}$ to $X_0^{\mathrm{KN}} \times \mathbb{R}_{\geq 0}$. In particular, the fibre of $f^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to S^1 = (O^\dagger)^{\mathrm{KN}}$ over $e^{i\phi} \in S^1$ is homeomorphic to $\mathbb{C}^2$. We want to verify this statement explicitly.

As a matter of notation we write s_x, s_y, s_t for the sections of $\mathcal{M}_X$ or of $\mathcal{M}_{X_0}$ defined by the monomial functions indicated in the subscripts. We also use s_t to denote the generator of the log structure $\mathcal{M}_{O^\dagger}$ of $O^\dagger$.

Since s_t generates $\mathcal{M}_{X_0,p_0}$ as a log structure, according to (2.3) the fibre of $\pi : X_0^{\mathrm{KN}} \to X_0$ over p_0 is a copy of $U(1)$, by mapping $\theta \in \operatorname{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,p_0}, U(1))$ to its value on s_t. The projection to $S^1 = (O^\dagger)^{\mathrm{KN}}$ can then be viewed as the identity.

On the complement of p_0 the log structure is fine, but there is no global chart. We rather need two charts, defined on the open sets

$$\begin{aligned} U &= X_0 \setminus (x = y = 0) = \operatorname{Specan} \big(\mathbb{C}[x^{\pm 1}, w^{\pm 1}] \times \mathbb{C}[y^{\pm 1}, w^{\pm 1}]\big) \\ V &= X_0 \setminus (w = -1) = \operatorname{Specan} \mathbb{C}[x, y, w^{\pm 1}]_{w+1}, \end{aligned}$$

respectively. The charts are as follows:

$$\begin{aligned} \varphi : \mathbb{N} &\longrightarrow \Gamma(U, \mathcal{M}_{X_0}), \quad \varphi(1) = s_t. \\ \psi : \mathbb{N}^2 &\longrightarrow \Gamma(V, \mathcal{M}_{X_0}), \quad \varphi(a, b) = s_x^a \cdot s_y^b. \end{aligned}$$

Proposition 2.2 now exhibits U^{KN}, V^{KN} as closed subsets of $U \times U(1)$ and $V \times U(1)^2$, respectively. In each case, the projections to the $U(1)$-factors are defined by evaluation of $\theta \in \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,x}, U(1))$ on monomials. We write these $U(1)$-valued functions defined on open subsets of X_0^{KN} by $\theta_t, \theta_x, \theta_y, \theta_w$ according to the corresponding monomial. Since $f : (X_0, \mathcal{M}_{X_0}) \to O^\dagger$ is strict over U, we have $U^{\mathrm{KN}} = U \times U(1)$ with $f^{\mathrm{KN}} = \theta_t$ the projection to $U(1)$. For V^{KN}, over the double locus $x = y = 0$ the fibre of the projection $V^{\mathrm{KN}} \to V$ is all of $U(1)^2$, while for $x \neq 0$ the value of θ_x is determined by $\arg x$. An analogous statement holds for $y \neq 0$.

To patch the descriptions of X_0^{KN} over the two charts amounts to understanding the map $V^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$, the image telling the value of $\theta \in \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,x}, U(1))$ on s_t. Over $V = X_0 \setminus (w = -1)$ we have the equation $s_t = (w+1)^{-1} s_x s_y$. Thus, say over $x \neq 0$, we had the description of V^{KN} by the value of $\theta \in \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,x}, U(1))$ on s_y. Then

$$\theta_t = \frac{\mathrm{Arg}(x)}{\mathrm{Arg}(w+1)} \cdot \theta_y. \tag{2.7}$$

Thus the identification with U^{KN} is twisted both by the phases of x and of $w+1$. A similar description holds for $y \neq 0$.

For $t = \tau e^{i\phi} \neq 0$ denote by $X_0^{\mathrm{KN}}(e^{i\phi})$ the fibre over $e^{i\phi} \in U(1) = (O^\dagger)^{\mathrm{KN}}$ and similarly $U^{\mathrm{KN}}(e^{i\phi})$, $V^{\mathrm{KN}}(e^{i\phi})$. It is now not hard to construct a homeomorphism between $U^{\mathrm{KN}}(e^{i\phi}) \cup V^{\mathrm{KN}}(e^{i\phi})$ and $\delta^{-1}(t) \simeq \mathbb{C}^2 \setminus \{0\}$. For example, there exists a unique such homeomorphism that on $(x = 0) \subset U^{\mathrm{KN}}(e^{i\phi})$ restricts to

$$(\mathbb{C}^*)^2 \ni (y = se^{i\psi}, w) \longmapsto \left(\frac{(w+1) \cdot \tau e^{i(\phi - \psi)}}{s + |(w+1)\tau|^{1/2}}, (s + |(w+1)\tau|^{1/2}) e^{i\psi} \right) \in \mathbb{C}^2,$$

and to a similar map with the roles of x and y swapped on $(y = 0) \subset U^{\mathrm{KN}}(e^{i\phi})$. This form of the homeomorphism comes from considering the degeneration $xy = (w+1)t$ as a family of normal crossing degenerations of curves parametrized by $w = \mathrm{const}$. Details are left to the reader.

Example 2.11. An alternative and possibly more useful way to discuss the Kato-Nakayama space of the degeneration $xy = (w+1)t$ in Example 2.10, is in terms of closed strata and the momentum maps of the irreducible components $Y_1 = (y = 0)$, $Y_2 = (x = 0)$ of X_0 and of their intersection $Z = Y_1 \cap Y_2$. Endow Y_1, Y_2, Z with the log structures making the inclusions into X_0 strict. Away from p_0 we then have global charts defined by s_t, s_x for Y_1, by s_t, s_y for Y_2 and by s_x, s_y for Z. By functoriality, the fibre of $\pi : X_0^{\mathrm{KN}} \to X_0$ over these closed strata Y_1, Y_2, Z agrees with Y_1^{KN}, Y_2^{KN}, Z^{KN}, respectively. Therefore, we can compute X_0^{KN} as the fibred sum

$$X_0^{\mathrm{KN}} = Y_1^{\mathrm{KN}} \amalg_{Z^{\mathrm{KN}}} Y_2^{\mathrm{KN}}.$$

Away from the singular point $p_0 \in X_0$ of the log structure, Y_1^{KN} is the Kato-Nakayama space of Y_1 as a toric variety times an additional S^1-factor coming from s_t, and similarly for Y_2. Since each Y_i has a momentum map μ_i with image the half-plane $\mathbb{R}_{\geq 0} \times \mathbb{R}$, Proposition 2.5 gives a description of Y_i^{KN} as $\mathbb{R}_{\geq 0} \times \mathbb{R} \times U(1)^3/\sim$ with the $U(1)$-factors telling the phases of w, s_t and of s_x (for $i = 1$) or of s_y (for $i = 2$), respectively. We assume that the momentum map maps p_0 to $(0,0) \in \mathbb{R}_{\geq 0} \times \mathbb{R}$. Note that $w \neq 0$, so the phase of w is already determined uniquely at any point of X_0. The indicated quotient takes care of the special point p_0 by collapsing a $U(1)$ over $(x, y, w) = (0, 0, -1)$ as follows. Restricting the projection

$$\mathbb{R}_{\geq 0} \times \mathbb{R} \times U(1)^3 \longrightarrow Y_1$$

to $x = 0$ yields a $U(1)^2$-bundle over $\mathbb{C}^*$, the w-plane. The two $U(1)$-factors record the phases of s_t and s_x, respectively. Now the quotient collapses the second $U(1)$-factor over $w = -1$, reflecting the fact that only s_t survives in $\mathcal{M}_{X_0,p_0}$.

Again by functoriality, the restriction of either Y_i^{KN} to $\{0\} \times \mathbb{R}$ yields Z^{KN}. Using s_x, s_y as generators for $\mathcal{M}_Z$ over $Z \setminus \{p_0\} = \mathbb{C}^* \setminus \{-1\}$ we see

$$Z^{\mathrm{KN}} \;=\; \mathbb{R} \times U(1)^3/\sim \;=\; \mathbb{C}^* \times U(1)^2/\sim$$

Now the three $U(1)$-factors tell the phases θ_w, θ_x, θ_u of w, s_x, s_y. The equivalence relation collapses the $U(1)$-subgroup

$$\big\{(\theta_x, \theta_y) \in U(1)^2 \,\big|\, \theta_x \cdot \theta_y = 1\big\}$$

over $-1 \in \mathbb{C}^*$.

Thus over the circle $|w| = a$ inside the double locus $x = y = 0$, X_0^{KN} is a trivial $U(1)^2$-bundle as long as $a \neq 1$, hence a 3-torus. This 3-torus fibres as a trivial bundle of 2-tori over $(O^\dagger)^{\mathrm{KN}} = S^1$. If $a = 1$, one of the $U(1)$-factors collapses to a point over $w = -1$, leading to a trivial family of pinched 2-tori over S^1.

A nontrival torus fibration arises if we consider a neighbourhood of the double locus. This is most easily understood by viewing X_0^{KN} as a torus fibration over $\mathbb{R}^2$ by taking the union of the momentum maps

$$\mu : X_0 \longrightarrow \mathbb{R}^2, \quad \mu|_{Y_1} = \mu_1, \quad \mu|_{Y_2} = -\mu_2.$$

Denote by $X_0^{\mathrm{KN}}(\theta)$ the fibre of $X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = S^1$ over $\theta \in S^1$. Write $\mu^{\mathrm{KN}} = \pi \circ \mu :$ $X_0^{\mathrm{KN}} \to \mathbb{R}^2$ and $\mu^{\mathrm{KN}}(\theta)$ for the restriction to $X_0^{\mathrm{KN}}(\theta)$. For any $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ the fibre $(\mu^{\mathrm{KN}})^{-1}(a, b)$ is a 3-torus trivially fibred by 2-tori over $(O^\dagger)^{\mathrm{KN}} = S^1$. We also have trivial torus bundles over the half-spaces $(\mathbb{R}_{\geq 0} \times \mathbb{R}) \setminus \{(0, 0)\}$ and $(\mathbb{R}_{\leq 0} \times \mathbb{R}) \setminus \{(0, 0)\}$ as well as over $\mathbb{R} \times (\mathbb{R} \setminus \{0\})$. However, the torus bundle is non-trivial over any loop about $(0,0)$. The reason is that the equation $xy = t(w+1)$ gives the identification of torus fibrations over the two half planes via

$$\theta_y = \theta_x^{-1} \cdot \mathrm{Arg}(w+1) \cdot \theta_t.$$

Now $\mathrm{Arg}(w+1)$ restricted to the circle $|w| = a$ with $a < 1$ is homotopic to a constant map, while for $a > 1$ this restriction has winding number 1. This means that for $\theta \in S^1$, the topological monodromy of the 2-torus fibration $\mu^{\mathrm{KN}}(\theta) : X_0^{\mathrm{KN}} \to \mathbb{R}^2$ along a counterclockwise loop about $(0,0) \in \mathbb{R}^2$ is a (negative) Dehn-twist. Thus $\mu^{\mathrm{KN}}(\theta)$ is homeomorphic to a neighbourhood of an I_1-singular fibre (a nodal elliptic curve) of an elliptic fibration of complex surfaces.

3. The Kato-Nakayama space of a real log space

Let us now combine the topics of Sections 1 and 2 and consider the additional structure on the Kato-Nakayama space of a log space induced by a real structure. Throughout this section we identify $\mathbb{Z}/2\mathbb{Z}$ with the multiplicative group with two elements ± 1.

The conjugation involution on $\mathbb{C}$ lifts to the log structure of the polar log point $\mathcal{M}_\Pi = \mathbb{R}_{\geq 0} \times U(1)$ by putting $\iota_\Pi^\flat(r, e^{i\varphi}) = (r, e^{-i\varphi})$.

Definition 3.1. Let $(X, \mathcal{M}_X)$ be a real log space with $(\iota_X, \iota_X^\flat) : (X, \mathcal{M}_X) \to (X, \mathcal{M}_X)$ its real involution. We call the map

$$\iota_X^{\mathrm{KN}} : X^{\mathrm{KN}} \longrightarrow X^{\mathrm{KN}}, \quad \big(f : \Pi^\dagger \to (X, \mathcal{M}_X)\big) \longrightarrow \iota_X^\flat \circ f \circ \iota_\Pi^\flat.$$

the *lifted real involution*.

Proposition 3.2. *The lifted real involution ι_X^{KN} is continuous and is compatible with the underlying real involution ι_X of X under the projection $\pi : X^{\mathrm{KN}} \to X$.*

Proof. Both statements are immediate from the definitions. □

By the definition and proposition we thus see that the real locus of X has a canonical lift to X^{KN}.

Definition 3.3. Let $(X, \mathcal{M}_X)$ be a real log space and $\iota_X^{\mathrm{KN}} : X^{\mathrm{KN}} \to X^{\mathrm{KN}}$ the lifted real involution. We call the fixed point set of ι_X^{KN} the *real locus of* X^{KN}, denoted $X_\mathbb{R}^{\mathrm{KN}} \subset X^{\mathrm{KN}}$.

To describe the real locus in toric degenerations, one main interest in this paper is the study of the real locus $X_\mathbb{R}^{\mathrm{KN}} \subset X^{\mathrm{KN}}$. We first discuss the fibres of the restriction $\pi_\mathbb{R} : X_\mathbb{R}^{\mathrm{KN}} \to X_\mathbb{R}$ of the projection $\pi : X^{\mathrm{KN}} \to X$. If $x \in X_\mathbb{R}$ then $\iota_X^\flat$ induces an involution on $\mathcal{M}_{X,x}$ and on the quotient $\overline{\mathcal{M}}_{X,x} = \mathcal{M}_{X,x}/\mathcal{O}_{X,x}^\times$. If the involution on $\overline{\mathcal{M}}_{X,x}$ is trivial, $\pi_{\mathbb{R}}^{-1}(x)$ is easy to describe. Recall from (2.3) that $\pi^{-1}(x)$ can be identified with the set of homomorphisms $\theta : \mathcal{M}_{X,x}^{\mathrm{gp}} \to U(1)$ given on invertible functions $h \in \mathcal{O}_{X,x}^{\times}$ by $\theta(h) = h(x)/|h(x)|$.

Proposition 3.4. *Let $(X, \mathcal{M}_X)$ be a real log space and $x \in X_{\mathbb{R}}$.*

(1) *In the description (2.3) of $\pi^{-1}(x)$, an element $\theta \in \mathrm{Hom}(\mathcal{M}_{X,x}^{\mathrm{gp}}, U(1))$ lies in $X_{\mathbb{R}}^{\mathrm{KN}}$ if and only if $\theta \circ \iota_{X,x}^{\flat} = \overline{\theta}$, the complex conjugation of θ.*

(2) *If $\iota_X^{\flat}$ induces a trivial action on $\overline{\mathcal{M}}_{X,x}$, then $\pi_{\mathbb{R}}^{-1}(x)$ is canonically a torsor for the group $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}, \mathbb{Z}/2\mathbb{Z})$.*

Proof. 1) Let $\tilde{x} \in \pi^{-1}(x)$, given by a log morphism $f : \Pi^{\dagger} \to (X, \mathcal{M}_X)$ with image x. Then $\tilde{x} \in X_{\mathbb{R}}^{\mathrm{KN}}$ if and only if $f \circ \iota_{\Pi} = \iota_X \circ f$. Now writing $\tilde{x} = (x, \theta)$ as in (2.3), we have $f \circ \iota_{\Pi} = (x, \overline{\theta})$ and $\iota_X \circ f = (x, \theta \circ \iota_{X,x}^{\flat})$. Comparing the two equations yields the statement.

2) Denote by $\kappa : \mathcal{M}_X \to \overline{\mathcal{M}}_X$ the quotient homomorphism. We define the action of $\sigma \in \mathrm{Hom}(\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}, \mathbb{Z}/2\mathbb{Z})$ on $\pi^{-1}(x)$ by

$$\big(\sigma \cdot \theta\big)(s) = \sigma(\kappa_x(s)) \cdot \theta(s)$$

for $\theta \in \mathrm{Hom}(\mathcal{M}_{X,x}^{\mathrm{gp}}, U(1))$. In this definition we take $\sigma(\kappa_x(s)) \in \mathcal{O}_{X,x}^{\times}$ by means of the identifcation $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$. Now by (1) together with the additional hypotheses, θ defines a point in $X_{\mathbb{R}}^{\mathrm{KN}}$ iff $\theta = \overline{\theta}$. This is the case iff θ takes values ± 1. This condition is preserved by the action of $\mathrm{Hom}(\overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}, \mathbb{Z}/2\mathbb{Z})$. Conversely, if θ_1, θ_2 define elements in $\pi_{\mathbb{R}}^{-1}(x) \subset X_{\mathbb{R}}^{\mathrm{KN}}$ then they both take values in $\mathbb{Z}/2\mathbb{Z} \subset U(1)$ and in any case they agree on $\mathcal{O}_{X,x}^{\times}$. Thus $\theta_1 \circ \theta_2^{-1}$ factors over the quotient map $\kappa_x : \mathcal{M}_{X,x}^{\mathrm{gp}} \to \overline{\mathcal{M}}_{X,x}^{\mathrm{gp}}$ to define a homomorphism $\sigma : \overline{\mathcal{M}}_{X,x}^{\mathrm{gp}} \to \mathbb{Z}/2\mathbb{Z}$. Then $\theta_1 = \sigma \cdot \theta_2$, showing that the action is simply transitive. $\square$

Concretely, in the fine saturated case, Proposition 3.4,(2) says that if the stalk of $\overline{\mathcal{M}}_X^{\mathrm{gp}}$ at $x \in X_{\mathbb{R}}$ has rank r, and $\iota_x^{\flat}$ induces a trivial action on $\overline{\mathcal{M}}_{X,x}$, then $\pi^{-1}(x)$ consists of 2^r points. This seems to contradict the expected smoothness of $X_{\mathbb{R}}^{\mathrm{KN}}$ in log smooth situations, but we will see in the toric situation how this process can sometimes merely separate sheets of a branched cover. The reason is that the real picture interacts nicely with the momentum map description of X^{KN}.

Proposition 3.5. *Let $(X, \mathcal{M}_X)$ be a toric variety with its toric log structure and $\mu : X \to \Xi \subset M_{\mathbb{R}}$ a momentum map. Let ι_X be the unique real structure on $(X, \mathcal{M}_X)$ lifting the standard real structure according to Proposition 1.4. Then there is a canonical decomposition*

$$X_{\mathbb{R}}^{\mathrm{KN}} \simeq \Xi \times \mathrm{Hom}(M, \mathbb{Z}/2\mathbb{Z}),$$

with the projection to Ξ giving the composition $\mu \circ \pi_{\mathbb{R}} : X_{\mathbb{R}}^{\mathrm{KN}} \to \Xi$.

Proof. Recall the section $\sigma : \Xi \to X$ of the momentum map with image $X_{\geq 0} \subset X_{\mathbb{R}}$. For $x \in \Xi$, Proposition 2.5 identifies $\pi^{-1}(\mu^{-1})(x) \subset X^{\mathrm{KN}}$ with pairs $(\lambda \cdot \sigma(x), \lambda) \in X \times \mathrm{Hom}(M, U(1))$. The action of ι_X^{KN} on this fibre is

$$(\lambda \cdot \sigma(x), \lambda) \longmapsto (\overline{\lambda} \cdot \sigma(x), \overline{\lambda}).$$

Thus $(\lambda \cdot \sigma(x), \lambda)$ gives a point in $X_{\mathbb{R}}^{\mathrm{KN}}$ if and only if $\lambda = \overline{\lambda}$. This is the case iff λ takes values in $\mathbb{R} \cap U(1) = \{\pm 1\}$, giving the result. □

Without the assumption of a trivial action on the ghost sheaf $\overline{\mathcal{M}}_{X,x}$, the fibre of $X_{\mathbb{R}}^{\mathrm{KN}} \to X_{\mathbb{R}}$ can be non-discrete.

Example 3.6. Let X be a complex variety with a real structure ι_X and a ι_X-invariant simple normal crossings divisor D with two irreducible components D_1, D_2. Assume there is a real point $x \in D_1 \cap D_2$ and ι_X exchanges the two branches of D at x. Denote by $\iota_X^\flat$ the induced real structure on $\mathcal{M}_X = \mathcal{M}_{(X,D)}$ according to Proposition 1.4. Then $P = \overline{\mathcal{M}}_{X,x} = \mathbb{N}^2$ and $\iota_{X,x}^\flat(a, b) = (b, a)$. The action extends to an involution ι_M of $M = P^{\mathrm{gp}} = \mathbb{Z}^2$. In the present case there is a subspace $M' \subset M$ with $M' \oplus \iota_M(M') = M$, e.g. $M' = \mathbb{Z} \cdot (1, 0)$. Then $\theta : M \to U(1)$ can be prescribed arbitrarily on M' and extended uniquely to M by enforcing $\theta \circ \iota_M = \overline{\theta}$. Thus in the present case $\pi_{\mathbb{R}}^{-1}(x) = \mathrm{Hom}(\mathbb{Z}, U(1)) = S^1$.

In the general case, say with $\overline{\mathcal{M}}_{X,x}$ a fine monoid, we can write $\mathcal{M}_{X,x}^{\mathrm{gp}} = M \oplus \mathcal{O}_{X,x}^\times$ with M a finitely generated abelian group and such that $\iota_{X,x}^\flat$ acts by an involution ι_M on M and by $\iota_{X,x}^\sharp$ on $\mathcal{O}_{X,x}^\times$. Then $\pi^{-1}(x) = \mathrm{Hom}(M, U(1))$ is a disjoint union of tori, one copy of $\mathrm{Hom}(M/T, U(1))$ for each element of the torsion subgroup $T \subset M$. The fibres $\pi_{\mathbb{R}}^{-1}(x)$ for $x \in X_{\mathbb{R}}$ are the preimage of the diagonal torus of the map

$$\mathrm{Hom}(M, U(1)) \longrightarrow \mathrm{Hom}(M, U(1)) \times \mathrm{Hom}(M, U(1)), \quad \theta \longmapsto (\theta \circ \iota_M, \overline{\theta}).$$

4. The case of toric degenerations

4.1. **Toric degenerations and their intersection complex.** We now focus attention to toric degenerations, as first introduced in [GS1], Definition 4.1. As already stated in the introduction, a *toric degeneration* in this sense is a proper flat map of normal connected schemes $\delta : \mathfrak{X} \to \operatorname{Spec} R$ with R a discrete valuation ring and such that the central fibre X_0 is a reduced union of toric varieties; the toric irreducible components of X_0 are glued pairwise along toric strata in such a way that the dual intersecting complex is a closed topological manifold, of the same dimension n as the fibres of δ. In particular, the notion of toric strata of X_0 makes sense. It is then also required that near each zero-dimensional toric stratum of X_0, étale locally δ is isomorphic to a monomial map of toric varieties. Since R is a discrete valuation ring this amounts to describing $\mathfrak{X}$ étale locally as $\operatorname{Spec}\mathbb{C}[P]$ with P a toric monoid and f by one monomial $t = z^{\rho_P}$, $\rho_P \in P$. This last formulation then holds locally outside a closed subset $Z \subset X_0$ of codimension 2 and not containing any zero-dimensional toric strata. For the precise list of conditions we refer to [GS1], Definition 4.1. Under these conditions it turns out that the generic fibre $\mathfrak{X}_\eta$ is a Calabi-Yau variety.

We refer to [GS3], §1 for a more thorough review of toric degenerations as described here. Various generalizations of toric degenerations have also been considered, notably including dual intersection complexes that are non-compact or have non-empty boundary [CPS], higher dimensional base spaces [GHK],[GHKS] and log singular loci containing zero-dimensional toric strata [GHK]. While much of the following discussion holds in these more general setups, to keep the presentation simple we restrict ourselves to the original Calabi-Yau case.

Asuming that X_0 is projective, let $\mathscr{P} = \{\sigma\}$ be the set of momentum polytopes of the toric strata and $\mathscr{P}_{\max} \subset \mathscr{P}$ the maximal elements under inclusion. For $\tau \in \mathscr{P}$ we denote by $X_\tau \subset X_0$ the correspoinding toric stratum. View $B = \bigcup_{\sigma\in\mathscr{P}_{\max}} \sigma$ as a cell complex with attaching maps defined by the intersection patterns of the toric strata. The barycentric subdivision of $(B, \mathscr{P})$ is then canonically isomorphic to the barycentric subdivision of the dual intersection complex of X_0, as simplicial complexes. Thus B is a topological manifold. There is a generalized momentum map $\mu : X_0 \to B$ that restricts to the toric momentum map $X_\tau \to \tau$ on each toric stratum of X_0 ([RS], Proposition 3.1). Unlike in [GS1], for simplicity of notation we assume that no irreducible component of X_0 self-intersects. On the level of the cell complex $(B, \mathscr{P})$ this means that for any $\tau \in \mathscr{P}$ the map $\tau \to B$ is injective. We call $(B, \mathscr{P})$ the *intersection complex* or *cone picture* of the polarized central fibre X_0.

The log structure $\mathcal{M}_{X_0}$ on X_0 induced from the degeneration can be conveniently described as follows. At a general point of $(X_0)_{\mathrm{sing}}$, exactly two irreducible components X_σ, $X_{\sigma'} \subset X_0$ intersect. At such a point there is a local description of $\mathfrak{X}$ of the form

$$uv = f(z_1, \ldots, z_{n-1}) \cdot t^\kappa, \tag{4.1}$$

with t a generator of the maximal ideal of R, $z_1, \ldots, z_{n-1}$ toric coordinates for the maximal torus of $X_\sigma \cap X_{\sigma'}$ and u, v restricting either to 0 or to a monomial on X_σ, $X_{\sigma'}$. One of the main results of [GS1] is the statement that the restriction of the function f is well-defined after choosing u, v and that this restriction classifies $\mathcal{M}_{X_0}$. The zero locus of f in X_ρ specifies the locus where the log structure $\mathcal{M}_{X_0}$ is not fine. Thus $\mathcal{M}_{X_0}$ is fine outside a closed subset $Z \subset (X_0)_{\mathrm{sing}}$ of codimension two, a union of hypersurfaces on the irreducible components X_ρ of X_0, $\dim\rho = n-1$. Conversely, there is a sheaf on X_0 with support on $(X_0)_{\mathrm{sing}}$ which is an invertibe $\mathcal{O}_{X_0}$-module on the open dense subset where X_0 is normal crossings, with sections classifiying log structures arising from a local embedding into a toric degeneration (see [GS1], Theorem 3.22 and Definition 4.21). Thus the moduli space of log structures on X_0 that look like coming from a toric degeneration can be explicitly described by an open subset of $\Gamma((X_0)_{\mathrm{sing}}, \mathcal{F})$ for some coherent sheaf $\mathcal{F}$ on $(X_0)_{\mathrm{sing}}$.

The ghost sheaf $\overline{\mathcal{M}}_{X_0}$ can be read off from a multivalued piecewise affine function φ on $\mathscr{P}$. This function is uniquely described by one integer κ_ρ on each codimension one cell $\rho \in \mathscr{P}$. If $uv = f \cdot t^\kappa$ is the local description of $\mathfrak{X}$ at a general point of X_ρ, then $\kappa_\rho = \kappa$. Each codimension two cell τ imposes a linear condition on the κ_ρ for all $\rho \supset \tau$ assuring the existence of a local single-valued representative of φ in a neighbourhood of τ (see [GHKS], Example 1.11). Note that the local representative φ is only defined up to a linear function. Thus globally φ can be viewed as a *multi-valued piecewise linear function*, a section of the sheaf of pieceweise linear functions modulo linear functions. We write $(B, \mathscr{P}, \varphi)$ for the complete tuple of discrete data associated to a toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$, still refereed to as the intersection complex (now *polarized* by φ).

The interpretation of the cells of $\mathscr{P}$ as momentum polyhedra endows B with the structure of an *integral affine manifold* on the interiors of the maximal cells, that is, a manifold with coordinate changes in $\mathrm{Aff}(\mathbb{Z}^n) = \mathbb{Z}^n \rtimes \mathrm{GL}(n, \mathbb{Z})$. On such manifolds it makes sense to talk about integral points as the preimage of $\mathbb{Z}^n$ under any chart, and they come with a local system Λ of integral tangent vectors. An important insight is that the log structure on X_0 provides a canonical extension of this affine structure over the complement in B of the amoeba image $\mathcal{A} := \mu(Z)$ of the log singular locus $Z \subset (X_0)_{\mathrm{sing}}$. In the local description at a codimension one cell $\rho = \sigma \cap \sigma'$, the affine structure of the adjacent maximal cells σ, σ' already agree on their common face ρ. So the extension at $x \in \operatorname{Int} \rho \setminus \mathcal{A}$ only requires the identification of $\xi \in \Lambda_{\sigma,x}$ with $\xi' \in \Lambda_{\sigma',x}$, each complementary to $\Lambda_{\rho,x}$. In a local description $uv = f \cdot t^{\kappa_\rho}$ we have $u|_{X_\sigma} = z^m$, $v|_{X_{\sigma'}} = z^{m'}$ by the assumption on u, v to be monomial on one of the adjacent components X_σ, $X_{\sigma'}$. One then takes $\xi = m$, $\xi' = -m' + m_x$. Here $m_x \in \Lambda_\rho$ is defined by the homotopy class of $f|_{\mu^{-1}(x)}$, see [RS], Construction 2.2 for details. This defines the integral affine structure on $B \setminus \mathcal{A}$ away from codimension two cells.

Lemma 4.1. *The integral affine structure on the interiors of the maximal cells $\sigma \in \mathscr{P}$ and on* $\operatorname{Int} \rho \setminus \mathcal{A}$ *for all codimension one cells ρ extends uniquely to $B \setminus \mathcal{A}$.*

Proof. Uniqueness is clear because the extension is already given on an open and dense subset.

At a vertex $v \in B$ we have $\mu^{-1}(v) = X_v$, a zero-dimensional toric stratum. Let $U \to \operatorname{Specan} \mathbb{C}[P]$ with $P = K \cap \mathbb{Z}^{n+1}$ and $t = z^{\rho_P}$, $\rho_P \in P$, be a toric chart for

$\delta : \mathfrak{X} \to \operatorname{Specan} R$ at X_v. Here K is an $(n+1)$-dimensional rational polyhedral cone, not denoted $\sigma^\vee$ to avoid confusion with the cells of B. There is then a local identification of μ with the composition

$$\mu_v : \operatorname{Specan} \mathbb{C}[P] \xrightarrow{\mu_P} K \longrightarrow \mathbb{R}^{n+1}/\mathbb{R} \cdot \rho_P$$

of the momentum map for $\operatorname{Specan} \mathbb{C}[P]$ with the projection from the cone K along the line through ρ_P. Since $\rho_P \in \operatorname{Int} K$, this map projects ∂K to a complete fan Σ_v in $\mathbb{R}^{n+1}/\mathbb{R} \cdot \rho_P$. The irreducible components of X_0 containing X_v have affine toric charts given by the facets of K. Thus this fan describes X_0 at X_v as a gluing of affine toric varieties. Now any momentum map μ of a toric variety provides an integral affine structure on the image with $R^1\mu_*\underline{\mathbb{Z}}$ the sheaf of integral tangent vectors on the interior. In the present case, this argument shows first that the restriction of $R^1\mu_{v*}\underline{\mathbb{Z}}$ to the interior of each maximal cone $K' \in \Sigma_v$ can be canonically identified with the sheaf of integral tangent vectors Λ on the interiors of maximal cells of B. Second, the argument shows that $R^1\mu_{v*}\underline{\mathbb{Z}}$ restricted to $\operatorname{Int} K'$ can be identified with the (trivial) local system coming from the integral affine structure provided by $\mathbb{Z}^{n+1}/\mathbb{Z} \cdot \rho_P$. The fan thus provides an extension of the sheaf Λ over a neighbourhood of σ and hence also of the integral affine structure. A possible translational part in the local monodromy does not arise by the given gluing along lower dimensional cells.

For any $\tau \in \mathscr{P}$, the extension at the vertices of $\mathscr{P}$ provides also the extension on any connected component of $\tau \setminus \mathcal{A}$ containing a vertex. If $\mathcal{A} \cap \tau$ has connected components not containing a vertex, one can in any case show the existence of a toric model with fan $\partial K/\mathbb{R} \cdot \rho_P$ of not necessarily strictly convex rational polyhedral cones. The argument given at a vertex then works analogously. □

4.2. **The Kato-Nakayama space of a toric degeneration.** Throughout the following discussion we fix $(X_0, \mathcal{M}_{X_0})$ the central fibre of a toric degeneration with log singular locus $Z \subset (X_0)_{\text{sing}}$, $(B, \mathscr{P})$ its intersection complex and $\mu : X_0 \to B$ a momentum map. The main result of the section gives a canonical description of the Kato-Nakayama space of $(X_0, \mathcal{M}_{X_0})$ as a torus bundle over B, away from the amoeba image $\mathcal{A} = \mu(Z) \subset B$. We denote by

$$\mu^{\text{KN}} : X_0^{\text{KN}} \xrightarrow{\pi} X_0 \xrightarrow{\mu} B$$

the composition of the projection of the Kato-Nakayama space with the momentum map and write $Z^{\text{KN}} = \pi^{-1}(Z) \subset X_0^{\text{KN}}$. We also fix once and for all a generator t of the maximal ideal of R and accordingly identify the closed point in $\operatorname{Spec} R$ with the induced log structure with the standard log point $O^\dagger$. Thus we have a log morphism $\delta : (X_0, \mathcal{M}_{X_0}) \to O^\dagger$, inducing a continuous map $\delta^{\text{KN}} : X_0^{\text{KN}} \to (O^\dagger)^{\text{KN}} = U(1)$.

Our interest in X_0^{KN} comes from the fact that it captures the topology of an analytic family inducing the given log structure on X_0, for a large class of spaces. This statement is based on a result of Nakayama and Ogus, which involves the following generalization of the notion of a fine log structure. A log space $(X, \mathcal{M}_X)$ is *relatively coherent* if locally in X the log structure $\mathcal{M}_X$ is isomorphic to a sheaf of faces of a fine log structure.

Theorem 4.2. *(*[NO]*, Theorem 5.1) Let* $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ *be a proper, separated, exact and relatively smooth morphism of log analytic spaces, with* $(Y, \mathcal{M}_Y)$ *fine and* $(X, \mathcal{M}_X)$ *relatively coherent. Then* $f^{\mathrm{KN}} : (X, \mathcal{M}_X)^{\mathrm{KN}} \to (Y, \mathcal{M}_Y)^{\mathrm{KN}}$ *is a topological fibre bundle with fibres oriented manifolds with boundary.*

Being a topological fibre bundle says that f^{KN} is locally in Y homeomorphic to the projection from a product. The technical heart of the proof is a local product decomposition for maps of real cones induced by exact homomorphisms of fine monoids ([NO], Theorem 0.2). From this result it follows easily that f^{KN} is a topological submersion, that is, locally in X a projection of a product ([NO], Theorem 3.7). In a final step one applies a result of Siebenmann ([Si], Corollary 6.14) to conclude the fibre bundle property.

We can verify the hypothesis of Theorem 4.2 for analytic smoothings of $(X_0, \mathcal{M}_{X_0})$ for the case of *simple singularities.* The notion of simple singularities has been introduced in [GS1] as an indecomposability condition on the local affine monodromy around the singular locus $\Delta \subset B$ of the affine structure on the dual intersection complex of $(X_0, \mathcal{M}_{X_0})$. It implies local rigidity of the singular locus of the log structure as needed in the smoothing algorithm ([GS3], Definition 1.26), but unlike local rigidity, being simple imposes conditions in all codimensions.

Proposition 4.3. *Let* $(X_0, \mathcal{M}_{X_0})$ *be the central fibre of a troic degeneration with simple singularities. Then* $(X_0, \mathcal{M}_{X_0}) \to \operatorname{Spec} O^\dagger$ *as well as any analytic family* $\mathcal{X} \to D = \{z \in \mathbb{C} \,\big|\, |z| < 1\}$ *with* X_0 *as central fibre and inducing the given log structure* $\mathcal{M}_{X_0}$*, fulfills the conditions of Theorem 4.2. In particular,* $\delta^{\mathrm{KN}} : (X_0, \mathcal{M}_{X_0}) \to U(1)$ *and* $(\mathcal{X}, \mathcal{M}_{\mathcal{X}, X_0})^{\mathrm{KN}} \to (D, \mathcal{M}_{D,0})^{\mathrm{KN}}$ *are topological fibre bundles with fibres closed manifolds.*

Proof. Under the assumptions of simple singularities, [GS2], Theorem 2.11 and Corollary 2.17 show that $\delta : (X_0, \mathcal{M}_{X_0}) \to O^\dagger$ as well as any analytic family inducing the given log structure on X_0 away from codimension three are relatively coherent. The log structure on the parameter space being generated by one element, exactness is trivial. Moreover, δ is *vertical* as a log morphism (the image of $\delta^\flat$ is not contained in any proper face), and hence the fibres have no boundary according to [NO], Theorem 5.1. $\square$

The preceding discussion motivates the study of $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to U(1)$. First we show that the log singular locus can be dealt with by taking closures, even stratawise. For $\tau \in \mathscr{P}$ denote by $X_\tau^{\mathrm{KN}} = \pi^{-1}(X_\tau) \subset X_0^{\mathrm{KN}}$ and by $Z_\tau^{\mathrm{KN}} = Z^{\mathrm{KN}} \cap X_\tau^{\mathrm{KN}}$.

Lemma 4.4. *On each toric stratum $X_\tau \subset X_0$, the preimage $Z_\tau^{\mathrm{KN}} \subset X_\tau^{\mathrm{KN}}$ of the log singular locus is a nowhere dense closed subset.*

Proof. It suffices to prove the statement over an irreducible component $X_\sigma \subset X_0$, $\sigma \subset B$ a maximal cell. Let $x \in Z \cap X_\sigma$ and $X_\tau \subset X_\sigma$ the minimal toric stratum containing x. Since Z does not contain zero-dimensional toric strata, x is not the generic point $\eta \in X_\tau$. We claim that the generization map $\chi_{\eta x} : \overline{\mathcal{M}}_{X_0,x} \to \overline{\mathcal{M}}_{X_0,\eta}$ is injective. In fact, $\mathcal{M}_{X_0}$ is locally the divisorial log structure for a toric degeneration. Hence the stalks of $\overline{\mathcal{M}}_{X_0}$ are canonically a submonoid of $\mathbb{N}^r$ with r the number of irreducible components of X_0 containing X_τ, say $X_{\sigma_1}, \ldots, X_{\sigma_r}$. An element $a \in \mathbb{N}^r$ lies in $\overline{\mathcal{M}}_{X_0,x}$ iff $\sum a_i X_{\sigma_i}$ is locally at x a Cartier divisor, in a local description as the central fibre of a toric degeneration. In any case, both $\overline{\mathcal{M}}_{X_0,x}$ and $\overline{\mathcal{M}}_{X_0,\eta}$ are submonoids of the same $\mathbb{N}^r$, showing the claimed injectivity of $\chi_{\eta x}$.

Now take a chart $\overline{\mathcal{M}}_{X_0,\eta} \to \Gamma(U, \mathcal{M}_{X_0})$ with U a Zariski-open neighbourhood of η in $X_0 \setminus Z$. Then Proposition 2.2 yields a canonical homeomorphism $\pi^{-1}(U) = U \times \mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0,\eta}, U(1))$. By the definition of the topology on X_0^{KN}, the composition

$$U \times \mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0,\eta}, U(1)) \longrightarrow \mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0,\eta}, U(1)) \longrightarrow \mathrm{Hom}(\overline{\mathcal{M}}^{\mathrm{gp}}_{X_0,x}, U(1))$$

of the projection with pull-back by the generization map $\chi_{\eta x}^*$ is continuous. By injectivity of $\chi_{\eta x}$ this composition is surjective. Since $x \in \mathrm{cl}(U)$ we conclude that $\pi^{-1}(x)$ is contained in the closure of $\pi^{-1}(U)$, showing the desired density. □

For $\sigma \in \mathscr{P}_{\max}$ denote by $\mathcal{M}_{X_\sigma}$ the toric log structure for the irreducible component $X_\sigma \subset X_0$ and by X_σ^{KN} its Kato-Nakayama space. By [GS1], Lemma 5.13, there is a canonical isomorphism

$$\mathcal{M}^{\mathrm{gp}}_{X_0}|_{X_\sigma \setminus Z} \simeq \mathcal{M}^{\mathrm{gp}}_{X_\sigma}|_{X_\sigma \setminus Z} \oplus \mathbb{Z}, \tag{4.2}$$

the $\mathbb{Z}$-factor generated by the generator t of $\mathfrak{m}_R$ chosen above.

Lemma 4.5. *For $\sigma \in \mathscr{P}_{\max}$ denote by $\Lambda_\sigma = \Gamma(\mathrm{Int}\,\sigma, \Lambda)$ the group of integral tangent vector fields on σ. Then there is a canonical continuous surjection*

$$\Phi_\sigma : \sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1)) \longrightarrow (\mu^{\mathrm{KN}})^{-1}(\sigma) \subset X_0^{\mathrm{KN}},$$

which is a homeomorphism onto the image over the complement of the log singular locus $Z \subset X_0$.

With respect to the product decomposition

$$\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1)) = X_\sigma^{\mathrm{KN}} \times U(1),$$

of the domain of Φ*, the restrictions of* $\pi : X_0^{\mathrm{KN}} \to X_0$ *and* $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to U(1)$ *to* $(\mu^{\mathrm{KN}})^{-1}(\sigma)$ *are given by the projection to* X_σ^{KN} *followed by* $X_\sigma^{\mathrm{KN}} \to X_\sigma$ *and by the projection to* $U(1)$*, respectively.*

Proof. By Lemma 4.4 we may establish the result away from Z and then extend Φ_σ by continuity. For any $x \in X_\sigma \setminus Z$, the isomorphism (4.2) establishes a canonical bijection

$$\mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_0,x}, U(1)) \longrightarrow \mathrm{Hom}(\mathcal{M}^{\mathrm{gp}}_{X_\sigma,x}, U(1)) \times \mathrm{Hom}(\mathbb{Z}, U(1)).$$

This bijection is compatible with the fibrewise description of the Kato-Nakayama spaces of X_0 and of X_σ in (2.3), respectively, as well as with the definition of the topology. Now varying $x \in X_\sigma \setminus Z$, Proposition 2.5 turns the first factor into the complement of a closed, nowhere dense subset (to become $\Phi_\sigma^{-1}(Z^{\mathrm{KN}})$) in $\sigma \times \mathrm{Hom}(\Lambda_\sigma, U(1))$. The inverse of this description of $(\mu^{\mathrm{KN}})^{-1}(\sigma)$ over the complement $X_\sigma^{\mathrm{KN}} \setminus Z^{\mathrm{KN}}$ defines the map Φ_σ over $X_0^{\mathrm{KN}} \setminus Z^{\mathrm{KN}}$.

The statements in the second paragraph are immediate from the definitions. □

Proposition 4.6. *Away from the amoeba image* $\mathcal{A} \subset B$ *of the log singular locus* $Z \subset (X_0)_{\mathrm{sing}}$*, the projection* $\mu^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to B$ *is a bundle of real* $(n+1)$*-tori. Similarly, over* $B \setminus \mathcal{A}$ *the restriction of* μ^{KN} *to a fibre of* $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$ *is a bundle of* n*-tori.*

Proof. For $\sigma \in \mathscr{P}_{\max}$ denote by $T_\sigma = \mathrm{Hom}(\Lambda_\sigma, U(1)) \times U(1)$ the $(n+1)$-torus fibre of μ^{KN} over σ in the description of Lemma 4.5. For $x \in B \setminus \mathcal{A}$ let $\tau \in \mathscr{P}$ be the unique cell with $x \in \mathrm{Int}\,\tau$. Let $n = \dim B$ and $k = \dim \tau$. Then in an open contractible neighbourhood $U \subset B \setminus \mathcal{A}$ of x, the polyhedral decomposition $\mathscr{P}$ looks like the product of $\Lambda_\tau \otimes_{\mathbb{Z}} \mathbb{R}$ with an $n-k$-dimensional complete fan Σ_τ in the vector space with lattice $\Lambda_x/\Lambda_{\tau,x}$. Over each maximal cell σ containing τ, we have the canonical homeomorphism of $(\mu^{\mathrm{KN}})^{-1}(\sigma)$ with $\sigma \times \mathrm{Hom}(\Lambda_\sigma, U(1)) \times U(1)$ provided by Lemma 4.5. Thus for any pair of maximal cells $\sigma, \sigma' \supset \tau$ we obtain a homeomorphism of torus bundles

$$\Phi_{\sigma'\sigma} : (U \cap \sigma \cap \sigma') \times T_\sigma \longrightarrow (U \cap \sigma \cap \sigma') \times T_{\sigma'}.$$

We only claim a fibre-preserving homeomorphism of total spaces here, $\Phi_{\sigma,\sigma'}$ does in general not preserve the torus actions. In any case, these homeomorphisms are compatible over triple intersections, hence provide homeomorphisms of torus bundes also for maximal cells intersecting in higher codimension. This way we have described $\pi^{-1}(U)$ as the gluing of trivial torus bundles over a decomposition of U into closed subsets, a clutching construction.

To prove local triviality from this description of $\pi^{-1}(U)$, replace U by a smaller neighbourhood of x that is star-like with respect to a point $y \in \mathrm{Int}(\sigma)$ for some maximal cell $\sigma \ni x$. By perturbing y slightly, we may assume that the rays emanating from y intersect each codimension one cell ρ with $\rho \cap U \neq \emptyset$ transversaly. To obtain a fibre-preserving homeomorphism $\pi^{-1}(U) \simeq U \times T_\sigma$, connect any other point $y \in U$ with x by a straight line segment γ. Then γ passes through finitely many maximal cells σ'. At each change of maximal cell apply the relevant $\Phi_{\sigma'\sigma''}$ to obtain the identification of the fibre over y with T_σ. □

Remark 4.7. Let us describe explicitly the homeomorphism of torus bundles $\Phi_{\sigma'\sigma}$ in the proof of Proposition 4.6, locally around some $x \in B \setminus \mathcal{A}$. We restrict to the basic case $\sigma \cap \sigma' = \rho$ of codimension one. Let f_ρ be the function defining the log structure along X_ρ according to (4.1). Then there is first a strata-preserving isomorphism of $X_\rho \subset X_\sigma$ with $X_\rho \subset X_{\sigma'}$. This isomorphism is given by *(closed) gluing data*, see [GS1], Definition 2.3 and Definition 2.10. In the present case, gluing data are homomorphisms $\Lambda_\rho \to \mathbb{C}^\times$ fulfilling a cocyle condition in codimension two. The Kato-Nakayama space has an additional $U(1)$-factor coming from the deformation parameter t. This additional factor gets contracted in X_0 along X_ρ, but not in X_0^{KN}. Thus over X_ρ, the Kato-Nakayama space is a $U(1)^2$-fibration. One factor captures the phase of the deformation parameter t, the other the phase of the monomial u (or v) describing X_ρ as a divisor in X_σ and $X_{\sigma'}$, respectively. In these coordinates for X_0^{KN} over σ and σ', the gluing $\Phi_{\sigma'\sigma}$ is determined by taking the argument of (4.1):

$$\arg(u) + \arg(v) = \kappa_\rho \cdot \arg(t) + \arg(f). \tag{4.3}$$

Unless f is monomial, this equation is not compatible with the fibrewise action of $\mathrm{Hom}\big(\Lambda_x, U(1)\big) \times U(1)$ suggested by Lemma 4.5. In the case of nontrivial *(open) gluing data*, this definition of $\Phi_{\sigma'\sigma}$ has to be corrected by the scaling factors in $\mathbb{C}^\times$ by which u, v differ from toric monomials in X_σ, $X_{\sigma'}$, respectively.

By Proposition 4.6 we may now view the subset $(\mu^{\mathrm{KN}})^{-1}(B \setminus \mathcal{A}) \subset X_0^{\mathrm{KN}}$ as a torus bundle over $B \setminus \mathcal{A}$. For $U \subset B \setminus A$ any subset, we write $X_0|_U = (\mu^{\mathrm{KN}})^{-1}(U)$, viewed as a topological torus bundle over U. Generally, topological r-torus bundles are fibre bundles with structure group $\mathrm{Homeo}(T^r)$, the group of homeomorphisms of the r-torus. There is the obvious subgroup $U(1)^r \rtimes \mathrm{GL}(r, \mathbb{Z})$ of homeomorphisms that lift to affine transformations on the universal covering $\mathbb{R}^{r+1} \to T^r = \mathbb{R}^r/\mathbb{Z}^r$. In higher dimensions (certainly for $r \geq 5$), there exist exotic homeomorphisms that are not isotopic to a linear one ([Ht], Theorem 4.1). However, in the present situation such exotic transition maps do not occur, and we can even find a system of local trivializations with transition maps induced by locally constant affine transformations.

Lemma 4.8. *The torus bundle $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}$ has a distinguished atlas of local trivializations with transition maps in $U(1)^{n+1} \rtimes \mathrm{GL}(n+1,\mathbb{Z})$.*

Proof. It suffices to consider the attaching maps between the trivial pieces $(\mu^{\mathrm{KN}})^{-1}(\sigma) = \sigma\times\mathrm{Hom}(\Lambda_\sigma\oplus\mathbb{Z}, U(1))$ of Lemma 4.5 for maximal cells σ,σ' with $\rho=\sigma\cap\sigma'$ of codimension one. Let $V\subset\rho$ be a connected component of $\rho\setminus\mathcal{A}$. In Remark 4.7 we saw that the transition maps over V are given by the equation $\arg(v) = -\arg(u)+\arg(f)+\kappa_\rho\cdot\arg(t)$. Now $\mu^{\mathrm{KN}}|_V$ factors over the Kato-Nakayama space of X_ρ, which can be trivialized as $V\times\mathrm{Hom}(\Lambda_\rho, U(1))\times U(1)^2$. The last factor is given by $(\mathrm{Arg}(u),\mathrm{Arg}(t))$, say, and the transition map transforms this trivialization into the description with $(\mathrm{Arg}(v),\mathrm{Arg}(t))$. Thus this transition is the identity on the first $n-1$ coordinates given by Λ_ρ and on $\mathrm{Arg}(t)$, while on the last coordinate it is given by $\mathrm{Arg}(u)^{-1}$ times the phase of the algebraic function $f\cdot t^{\kappa_\rho}$. The homotopy class of this map is given by the winding numbers of a generating set of closed loops in $\pi_1(T^{n-1}\times T^1)=\mathbb{Z}^n$. These winding numbers define a monomial function z^m on $V\times\mathrm{Hom}(\Lambda_\rho\oplus\mathbb{Z}, U(1))$ with $z^{-m}\cdot f$ homotopic to a constant map. The transition function is therefore isotopic to $(\mathrm{id}_{T^{n-1}}, \mathrm{Arg}(z^m\cdot t^{\kappa_\rho}\cdot u^{-1}), \mathrm{id}_{U(1)})$, fibrewise a linear transformation of $T^{n+1}=T^n\times U(1)$ with coordinates $\mathrm{Arg}(z)$ for T^n and $\mathrm{Arg}(u)$ for $U(1)$, respectively.

The translational factor of $U(1)^{n+1}$ arises because non-trivial gluing data change the meaning of monomials on the maximal cells by constants. See the discussion after Corollary 4.8 below for some comments on gluing data. $\square$

The topological classification of torus bundles with transition functions taking values in $U(1)^r\rtimes\mathrm{GL}(r,\mathbb{Z})$ works in analogy with the Lagrangian fibration case discussed in [Du]. Let $\mu: X\to B$ be such a torus bundle of relative dimension r. Then $\Lambda = R^1\mu_*\underline{\mathbb{Z}}$ is a local system with fibres $\Lambda_x = H^1(\mu^{-1}(x),\mathbb{Z})\simeq\mathbb{Z}^r$. The torus $\mathrm{Hom}(\Lambda_x, U(1))\simeq U(1)^r$ acts fibrewise by translation on any trivialization $\mu^{-1}(U)\simeq U\times T^r$, and this action does not depend on the trivialization. Hence $X\to B$ can be viewed as a $\mathcal{H}om(\Lambda,\underline{U}(1))$-torsor. In particular, if $\mu: X\to B$ has a section, then $X\simeq\mathcal{H}om(\Lambda,\underline{U}(1))\simeq\check{\Lambda}\otimes U(1)$ is isomorphic to the trivial torsor. Here we write $\check{\Lambda}=\mathcal{H}om(\Lambda,\underline{\mathbb{Z}})$ and view the locally constant sheaf $\mathcal{H}om(\Lambda,\underline{U}(1))$ as a topological space via its espace étalé. The cohomology group $H^1(X,\check{\Lambda}\otimes U(1))$ classifies isomorphism classes of $\check{\Lambda}\otimes U(1)$-torsors by the usual Čech description. Moreover, from the exact sequence

$$0\longrightarrow\check{\Lambda}\longrightarrow\check{\Lambda}\otimes_{\mathbb{Z}}\mathbb{R}\longrightarrow\check{\Lambda}\otimes_{\mathbb{Z}}U(1)\longrightarrow 0,$$

exhibiting $\check{\Lambda}\otimes_{\mathbb{Z}}U(1)$ fibrewise as a quotient of a vector space modulo a lattice, we obtain the long exact cohomology sequence

$$\ldots\longrightarrow H^1(B,\check{\Lambda})\longrightarrow H^1(B,\check{\Lambda}\otimes\mathbb{R})\longrightarrow H^1(B,\check{\Lambda}\otimes U(1))\xrightarrow{\delta} H^2(B,\check{\Lambda})\longrightarrow\ldots$$

The image of the class $[X] \in H^1(B, \check\Lambda \otimes U(1))$ of $X \to B$ as a $\check\Lambda \otimes U(1)$-torsor under the connecting homomorphism, defines the obstruction $\delta[X] \in H^2(B, \check\Lambda)$ to the existence of a continuous section of $\mu : X \to B$. The proof works as in the symplectic case discussed in [Du], p.696f.

For a local system Λ on a topological space M with fibres $\mathbb{Z}^r$ we write $\operatorname{Hom}(\Lambda, U(1))$ for the associated torus bundle. As a set, $\operatorname{Hom}(\Lambda, U(1)) = \coprod_{x\in M} \operatorname{Hom}(\Lambda_x, U(1))$, and the topology is generated by sets, for $m \in \Gamma(U, \Lambda)$ with $U \subset X$ open and $V \subset U(1)$ open,

$$V_m = \big\{(x, \varphi) \,\big|\, x \in U,\ \varphi \in \operatorname{Hom}(\Lambda_x, U(1)),\ \varphi(m) \in V\big\}.$$

With this notation, we summarize the general discussion on torus bundles with locally constant transition functions in the following proposition.

Proposition 4.9. *Let M be a topological manifold and $\mu : X \to M$ a fibre bundle with locally constant transition functions with values in $U(1)^r \rtimes \operatorname{GL}(r, \mathbb{Z})$. Then up to isomorphism, $X \to M$ is given uniquely by the local system $\Lambda = R^1\mu_*\underline{\mathbb{Z}}$ with fibres $\mathbb{Z}^r$ and a class $[X] \in H^1(M, \check\Lambda \otimes U(1))$.*

Moreover, a continuous section of μ exists if and only if $\delta([X]) \in H^2(M, \check\Lambda)$ vanishes. In this case, $X \simeq \operatorname{Hom}(\Lambda, U(1))$, as a torus bundle with locally constant transition functions in $U(1)^r \rtimes \operatorname{GL}(r, \mathbb{Z})$. □

For the Kato-Nakayama space $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}$, the governing bundle $R^1\mu_*\underline{\mathbb{Z}}$ is identified as follows. Recall that the multivalued piecewise affine function φ encoded in the $\kappa_\rho \in \mathbb{N}$ defines an integral affine manifold $\mathbb{B}_\varphi$ with an integral affine action by $(\mathbb{R}, +)$, making $\mathbb{B}_\varphi$ a torsor over $B = \mathbb{B}_\varphi/\mathbb{R}$ ([GHKS], Construction 1.14). This torsor comes with a canonical piecewise affine section locally representing φ. The pull-back of $\Lambda_{\mathbb{B}_\varphi}$ under this section defines an extension

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \mathcal{P} \longrightarrow \Lambda \longrightarrow 0 \tag{4.4}$$

of Λ by the constant sheaf $\underline{\mathbb{Z}}$ on $B \setminus \mathcal{A}$. The extension class of this sequence equals $c_1(\varphi) \in \operatorname{Ext}^1(\Lambda, \underline{\mathbb{Z}}) = H^1(B \setminus \mathcal{A}, \check\Lambda)$, called the first Chern class of φ from its mirror dual interpretation (see [GHKS], Equation (1.5)).

Remark 4.10. Much as in the discussion of the radiance obstruction in [GS1], §1.1, the first Chern classs $c_1(\varphi)$ can be interpreted as an element in group cohomology $H^1(\pi_1(B \setminus \mathcal{A}, x), \check\Lambda_x)$. Here $\check\Lambda_x \simeq \mathbb{Z}^n$ is a $\pi_1(B \setminus \mathcal{A}, x)$-module by means of parallel transport in $\check\Lambda$ along closed loops based at some fixed $x \in B \setminus \mathcal{A}$. As an element in group cohomology, $c_1(\varphi)$ is given by a twisted homomorphism $\lambda : \pi_1(B \setminus \mathcal{A}, x) \to \check\Lambda_x$, $\gamma \mapsto \lambda_\gamma$, determining the monodromy of $\mathcal{P}$ around a closed loop γ based at x as follows:

$$\Lambda_x \oplus \mathbb{Z} \longrightarrow \Lambda_x \oplus \mathbb{Z}, \quad (v, a) \longmapsto \big(T_\gamma \cdot v, \lambda_\gamma \cdot v + a\big).$$

Here $T_\gamma \in \mathrm{GL}(\Lambda_x)$ is the monodromy of Λ along γ and we have chosen an isomorphism $\mathcal{P}_x \simeq \Lambda_x \oplus \mathbb{Z}$. Being a twisted homomorphism means that for a composition $\gamma_1\gamma_2$ of two loops based at x,

$$\lambda_{\gamma_1\gamma_2} = \lambda_{\gamma_2} \circ T_{\gamma_1} + \lambda_{\gamma_1}.$$

Here we use the convention that $\gamma_1\gamma_2$ runs through γ_2 first, and hence $T_{\gamma_1\gamma_2} = T_{\gamma_2} \circ T_{\gamma_1}$. This interpretation is also compatible with the fact that under discrete Legendre duality, the roles of $c_1(\varphi)$ and the radiance obstruction swap ([GS1], Proposition 1.50,3).

For each point $x \in B$ there is a chart for the log structure on X_0 with monoid $\mathbb{C}[\mathcal{P}_x^+]$, where $\mathcal{P}_x^+ \subset \mathcal{P}_x$ is a certain submonoid of positive elements with $(\mathcal{P}_x^+)^{\mathrm{gp}} = \mathcal{P}_x$ ([GHKS], §2.2 and [GS3], Construction 2.7). Hence from the local description of $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}$ in Lemma 4.5 and in Proposition 4.6, the following result is immediate:

Lemma 4.11. *Writing $\breve{\mu}$ for the restriction of $\mu^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to B$ to $B \setminus \mathcal{A}$, there exists a canonical isomorphisms of local systems $R^1\breve{\mu}_*\underline{\mathbb{Z}} = \mathcal{P}$.* □

In view of this lemma, an immediate corollary from Proposition 4.9 is a complete topological description of $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}}$ over large open sets.

Corollary 4.12. *Denote by $\tilde{\mathcal{A}} \subset B$ a closed subset containing $\mathcal{A}$ and such that $B \setminus \tilde{\mathcal{A}}$ retracts to a one-dimensional cell complex. Then as a topological torus bundle, $X_0^{\mathrm{KN}}|_{B\setminus\tilde{\mathcal{A}}}$ is isomorphic to* $\mathrm{Hom}(\mathcal{P}, \underline{U}(1))$.

Proof. By Lemma 4.8 we can treat $X_0^{\mathrm{KN}}|_{B\setminus\tilde{\mathcal{A}}}$ as a torus bundles with locally constant transition functions in $U(1)^{n+1} \rtimes \mathrm{GL}(n+1,\mathbb{Z})$. By Proposition 4.9 the obstruction to the existence of a continuous section then lies in $H^2(B \setminus \tilde{\mathcal{A}}, \mathcal{P}^\vee)$. This cohomology group vanishes by the assumption on the existence of a retraction. □

We should emphasize that in this corollary, we have first used Lemma 4.8 to reduce to the case of locally constant transition functions. As discussed in Remark 4.7, the transition functions for $X_0^{\mathrm{KN}}|_{B\setminus\mathcal{A}} \to B \setminus \mathcal{A}$ between the canonical charts coming from toric geometry are not locally constant, and hence Corollary 4.8 makes a purely topological statement.

The remainder of this subsection derives a more canonical description of X_0^{KN} over a somewhat smaller set by controlling the gluing data. The various charts for $(X_0, \mathcal{M}_{X_0})$ are related by parallel transport inside $\mathcal{P}$, but the monomials may be rescaled due to non-trivial *gluing data*. Gluing data have already been introduced in [GS1], but §1.2 in [GS3] or §5.2 in [GHKS] may contain more palatable accounts. Multiples of the deformation parameter t are well-defined on all charts, hence define a constant subsheaf with fibres $\mathbb{Z} \oplus \mathbb{C}^\times$. Monomials therefore define a refinement of (4.4):

$$0 \longrightarrow \underline{\mathbb{Z}} \oplus \underline{\mathbb{C}}^\times \longrightarrow \tilde{\mathcal{P}} \longrightarrow \Lambda \longrightarrow 0. \tag{4.5}$$

The extension class $\big(c_1(\varphi), s\big) \in \operatorname{Ext}^1(\Lambda, \underline{\mathbb{Z}} \oplus \underline{\mathbb{C}}^\times) = H^1(B \setminus \mathcal{A}, \check{\Lambda}) \oplus H^1(B \setminus \mathcal{A}, \check{\Lambda} \otimes \mathbb{C}^\times)$ has as second component (the restriction to $B \setminus \mathcal{A}$ of) the gluing data s, as discussed in [GHKS], Remark 5.16.[1] Furthermore, dividing out $\mathbb{R}_{>0} \subset \mathbb{C}^\times$ defines an extension $\widehat{\mathcal{P}}$ of Λ by $\underline{\mathbb{Z}} \oplus \underline{U}(1)$ with class $\big(c_1(\varphi), \operatorname{Arg}(s)\big) \in \operatorname{Ext}^1(\Lambda, \underline{\mathbb{Z}} \oplus \underline{U}(1)) = H^1(B \setminus \mathcal{A}, \check{\Lambda}) \oplus H^1(B \setminus \mathcal{A}, \check{\Lambda} \otimes U(1))$. Taking this latter extension and the extension of Λ by $\underline{\mathbb{Z}}$ from (4.4) as two columns, we obtain the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
1 \arrow[r] & \underline{U}(1) \arrow[r] \arrow[d, "="] & \underline{\mathbb{Z}} \oplus \underline{U}(1) \arrow[r] \arrow[d] & \underline{\mathbb{Z}} \arrow[r] \arrow[d] & 0 \\
1 \arrow[r] & \underline{U}(1) \arrow[r] & \widehat{\mathcal{P}} \arrow[r] \arrow[d] & \mathcal{P} \arrow[r] \arrow[d] & 0 \\
 & & \Lambda \arrow[r, "="] \arrow[d] & \Lambda \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd} \tag{4.6}$$

Note that the extension of $\underline{\mathbb{Z}}$ by $\underline{U}(1)$ in the second row is trivial by construction. The middle row now defines an extension of $\mathcal{P}$ by $\underline{U}(1)$.

We can use the extension $\widehat{\mathcal{P}}$ to give a *canonical* description of X_0^{KN} on a large subset of $B \setminus \mathcal{A}$, assuming the open gluing data *normalizes* the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ ([GS1], Definition 4.23). Being normalized means that if $f_{\rho,v}$ is the slab function describing the log structure near a zero-dimensional toric stratum $x \in X_0$ along a codimension one stratum $X_\rho \subset X_0$ with $x \in X_\rho$, then $f_{\rho,v}(x) = 1$. By the discussion after Definition 4.23 in [GS1], there always exist open gluing data normalizing a given toric log Calabi-Yau space, so this assumption imposes no restriction. Note

[1]The discussion in [GHKS] is on the complement of a part $\tilde{\Delta} \subset B$ of the codimension two skeleton of the barycentric subdivision. There is a retraction of $\mathcal{A}$ to a subset of $\tilde{\Delta}$. However, the discussion on monomials works on any cell $\tau \in \mathscr{P}$ not contained in $\mathcal{A}$ and with $\mathfrak{X} \to \operatorname{Spec} R$ locally toroidal at some point of X_τ.

that while previously we had viewed slab functions only as functions on (analytically) open subsets of the big torus of X_ρ, hence as Laurent polynomials, in the setup of [GS1] or [GS3] they extend as regular functions to the zero-dimensional toric stratum they take reference to.

Proposition 4.13. *Denote by* $B' = \bigcup_{\sigma\in\mathscr{P}_{\max}} \operatorname{Int}\sigma \cup \big\{v \in \mathscr{P}^{[0]}\big\}$ *the subset of* B *covered by the interiors of maximal cells and the vertices of* $\mathscr{P}$*. Assume that the toric log Calabi-Yau space* $(X_0, \mathcal{M}_{X_0})$ *is normalized with respect to open gluing data* s*. Denote by* $\widehat{\mathcal{P}}$ *the extension of* Λ *by* $\mathbb{Z}\oplus U(1)$ *in* (4.6)*. Write* $\operatorname{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))^\circ \subset \operatorname{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))$ *for the space of fibrewise homomorphisms restricting to the identity on* $\underline{U}(1) \subset \widehat{\mathcal{P}}$*. Then there is a canonical homeomorphism*

$$\operatorname{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))^\circ|_{B'} \xrightarrow{\simeq} X_0^{\mathrm{KN}}\big|_{B'}$$

of topological fibre bundles over B'*.*

Moreover, the class in $H^1(B', \mathcal{P}^\vee \otimes U(1))$ *defining* $X_0^{\mathrm{KN}}|_{B'}$ *as a topological torus bundle with locally constant transition functions in* $U(1)^{n+1} \rtimes \mathrm{GL}(n+1,\mathbb{Z})$ *according to Lemma 4.8, agrees with the class* $\big(c_1(\varphi), \operatorname{Arg}(s)\big) \in \operatorname{Ext}^1(\Lambda, \underline{\mathbb{Z}}\oplus\underline{U}(1))$ *of the extension* $\widehat{\mathcal{P}}$ *of* $\mathcal{P}$ *by* $\underline{U}(1)$ *in* (4.6)*.*

Proof. From its origin in the bundle $\tilde{\mathcal{P}}$ of monomials, we obtain a canonical description of $\widehat{\mathcal{P}}$ over B' as follows. Over a maximal cell σ, we have a canonical isomorphism of $\widehat{\mathcal{P}}|_\sigma$ with the trivial bundle with fibre $\Lambda_\sigma \oplus \mathbb{Z} \oplus U(1)$. Then if $\sigma, \sigma' \in \mathscr{P}_{\max}$ and $v \in \sigma\cap\sigma'$ is a vertex, the open gluing data s define a multiplicative function $s_{v,\sigma} : \Lambda_\sigma \to \mathbb{C}^\times$, and similarly for σ'. Now glue the trivial bundles on σ, σ' by means of $\operatorname{Arg}\big(s_{v,\sigma'}\cdot s_{v,\sigma}^{-1}\big)$ on the $U(1)$-factor. The gluing on the discrete part $\Lambda_\sigma \oplus \mathbb{Z}$ is governed by a local representative of the MPL function φ, to yield $\mathcal{P}$. Accordingly, we obtain a description of $\operatorname{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))^\circ$ by gluing trivial pieces $\sigma \times \operatorname{Hom}(\Lambda_\sigma\oplus\mathbb{Z}, U(1))$.

Now the point is that if f is normalized, the gluing of X_0^{KN} from the same canonical trivial pieces in Lemma 4.5 is given by exactly the same procedure over vertices. Indeed, given a codimenson one cell ρ and a vertex $v \in \rho$, in the formula $\arg(u) + \arg(v) = \arg(f_{\rho,v}) + \kappa_\rho \cdot \arg(t)$ the term involving $f_{\rho,v}$ disappears due to the normalization condition.

The statement on the extension class is immediate by exhibiting the class in $H^1(\mathcal{P}^\vee \otimes U(1))$ of $\operatorname{Hom}(\widehat{\mathcal{P}}, \underline{U}(1))^\circ|_{B'}$ as a torus bundle according to Proposition 4.9 using the description of $\widehat{\mathcal{P}}$ by gluing constant sheaves over maximal cells just given. □

To describe the fibres of $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$, we need another extension. For $\phi \in U(1)$ denote by $\Psi_\phi : \mathbb{Z}\oplus U(1) \to U(1)$ the homomorphism mapping $(1,1)$ to

ϕ and inducing the identity on $U(1)$. We have a morphism of extensions

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\mathbb{Z}} \oplus \underline{U}(1) & \longrightarrow & \widehat{\mathcal{P}} & \longrightarrow & \Lambda & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \Psi_\phi} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & \underline{U}(1) & \longrightarrow & \widehat{\mathcal{P}}_\phi & \longrightarrow & \Lambda & \longrightarrow & 0, \end{array}$$

with the lower row having extension class $\Psi_\phi(s) \in \mathrm{Ext}^1(\Lambda, \underline{U}(1)) = H^1(B\backslash\mathcal{A}, \check{\Lambda}\otimes\underline{U}(1))$.

Corollary 4.14. *With the same assumptions as in Proposition 4.13, the fibre of* δ^{KN} : $X_0^{\mathrm{KN}}|_{B'} \to (\operatorname{Spec} O^\dagger)^{\mathrm{KN}} = U(1)$ *over* $\phi \in U(1)$ *is isomorphic to the n-torus bundle with local system* Λ *and with extension class* $\Psi_\phi(s) \in H^1(B', \check{\Lambda} \otimes \underline{U}(1))$.

Proof. By Lemma 4.5, the restriction of δ^{KN} to the canonical piece $\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1))$ is given by composing with the inclusion $\mathbb{Z} \to \Lambda_\sigma \oplus \mathbb{Z}$. The statement follows by tracing through the gluing descriptions of $\widehat{\mathcal{P}}_\phi$ and of the extension class defined by $(\delta^{\mathrm{KN}})^{-1}(\phi)$. □

Remark 4.15. Let us emphasize the role of the normalization condition here. The canonical description over maximal cells in Lemma 4.5 is based on toric monomials. To extend this description over a point $x \in B \setminus \mathcal{A}$ in a codimension one cell ρ, we need the gluing equation (4.3) to be monomial along the fibres of the momentum map. This condition means that the restriction of f to a fibre of the momentum map $X_\rho \to \rho$ is monomial. This is a strong condition that in case the Newton polyhedron of f is full-dimensional fails everywhere except at the zero-dimensional toric strata of X_ρ. The normalization condition then says that the non-trivial gluing of the torus fibres over the vertices only comes from the gluing data, hence is entirely determined by the extension class of $\widehat{\mathcal{P}}$.

4.3. **Study of the real locus.** We now turn to toric degenerations with a real structure, or rather to the corresponding toric log Calabi-Yau space ([GS1], Definition 4.3) that arise as central fibre $(X_0, \mathcal{M}_{X_0})$ of a toric degeneration, as discussed in §4.1. We call a toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ *standard real* if it has a real structure inducing the standard real structure on its toric irreducible components and which is compatible with the standard real structure on the standard log point. Since the morphism $\delta : (X_0, \mathcal{M}_{X_0}) \to O^\dagger$ is strict at the generic points of the irreducible components of X_0, and since any section of $\mathcal{M}_{X_0}$ that is supported on higher codimensional strata is trivial (constant 1), there is at most one such real structure on $(X_0, \mathcal{M}_{X_0})$. This real structure is called the *standard real structure.*

Standard real structures appear to be the only class of real structures on toric log-Calabi-Yau spaces that exist in great generality. While other real structures, for example those lifting an involution on B, should be extremely interesting in more specific situations, we therefore restrict the following discussion to standard real structures.

Proposition 4.16. *Let $(X_0, \mathcal{M}_{X_0})$ be a polarized toric log Calabi-Yau space ([GS1], Definition 4.3) with intersection complex $(B, \mathscr{P})$. Then there is a standard real structure on $(X_0, \mathcal{M}_{X_0})$ if and only if the following hold:*

1. *There exist open gluing data $s = (s_{\omega\tau})$ with $X_0 \simeq X_0(B, \mathscr{P}, s)$ such that $s_{\omega\tau}$ takes values in $\mathbb{R}^\times$ rather than in $\mathbb{C}^\times$.*
2. *The slab functions $f_{\rho,v} \in \mathbb{C}[\Lambda_\rho]$ describing the log structure $\mathcal{M}_{X_0}$ for gluing data as in (1) are defined over $\mathbb{R}$, that is $f_{\rho,v} \in \mathbb{R}[\Lambda_\rho]$ for any $\rho \in \mathscr{P}$ of codimension one and $v \in \rho$ a vertex.*

Proof. The proof is by revision of the arguments in [GS1].

1) If $s = (s_{\omega\tau})_{\omega,\tau}$ are open gluing data taking values in $\mathbb{R}^\times \subset \mathbb{C}^\times$, the construction of $X_0(B, \mathscr{P}, s)$ by gluing affine toric varieties in [GS1], Definition 2.28 readily shows that the real structures on the irreducible components induce a real structure on $X_0(B, \mathscr{P}, s)$.

Conversely, given $(X_0, \mathcal{M}_{X_0})$ with a standard real structure, Theorem 4.14 in [GS1] constructs open gluing data s and an isomorphism $X_0 \simeq X_0(B, \mathscr{P}, s)$. The construction has two steps. First, X_0 being glued from toric varieties, there exist *closed* gluing data $\overline{s}$ inducing this gluing. If X_0 admits a standard real structure, $\overline{s}$ automatically takes real values. In a second step one shows that the closed gluing data are the image of open gluing data as in [GS1], Lemma 2.29 and Proposition 2.32,2. This step uses a chart for the log structure at a zero-dimensional toric stratum $x \in X_0$. In view of the given real structure on $(X_0, \mathcal{M}_{X_0})$, this chart can be taken real (Lemma 1.8). With this choice of chart, the construction of open gluing data in the proof of [GS1], Theorem 4.14, indeed produces real open gluing data.

2) The relation between the slab functions $f_{\rho,v}$ and charts for the log structure is given in [GS1], Theorem 3.22. At a zero-dimensional toric stratum $x \in X_0$ the description in terms of open gluing data yields an isomorphism of an open affine neighbourhood in $\operatorname{Spec} X_0$ with $\operatorname{Spec} \mathbb{C}[P]/(z^{\rho_P})$, with $P = \overline{\mathcal{M}}_{X,x}$ and $\rho_P \in P$ corresponding to the deformation parameter t. The facets of P are in one-to-one correspondence with the irreducible components of X_0 containing x. Now charts for the log structure on this open subset are of the form

$$P \longrightarrow \mathbb{C}[P]/(z^{\rho_P}), \quad p \longmapsto h_p \cdot z^p$$

with h_p an invertible function on $V(p)$, the closure of the open subset $(z^p \neq 0) \subset \operatorname{Spec}\mathbb{C}[P]$. The equation describing this chart in terms of functions on codimension one strata writes the slab function as a quotient of piecewise multiplicative functions g_v defined in terms of h_p. This equation, with the slab function written $\xi_\omega(h)$ in [GS1], shows that describing a real chart via real open gluing data yields real slab functions $f_{\rho,v}$.

Conversely, given real open gluing data and real slab functions, the real structure on $\operatorname{Spec}\mathbb{C}[P]$ induces the involution $\iota^\flat_{X_0}$ defining a standard real structure on $(X_0, \mathcal{M}_{X_0})$. □

In the case of positive and simple singularities, a polarized toric log Calabi-Yau space with given intersection complex $(B, \mathscr{P}, \varphi)$ is defined uniquely up to isomorphism by so-called *lifted gluing data* $s \in H^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^\times)$ ([GS1], Theorem 5.4).[2] Lifted gluing data both contain moduli of open gluing data and moduli of the log structure given by the slab functions. In terms of lifted gluing data the existence of a standard real structure has a simple cohomological formulation.

Corollary 4.17. *Assuming $(B, \mathscr{P})$ positive and simple, then the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ defined by lifted gluing data $s \in H^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^\times)$ is standard real if and only if s lies in the image of*

$$H^1(B, \iota_*\check{\Lambda} \otimes \mathbb{R}^\times) \longrightarrow H^1(B, \iota_*\check{\Lambda} \otimes \mathbb{C}^\times).$$

Proof. This follows again by inspection of the corresponding results in [GS1], here Theorems 5.2 and 5.4. □

Remark 4.18. It is worthwhile pointing out that real structures on $(X_0, \mathcal{M}_{X_0})$ are compatible with the smoothing algorithm of [GS3] in the following way. Assume that $(X_0, \mathcal{M}_{X_0})$ is a toric log Calabi-Yau space for which the smoothing algorithm of [GS3] works, for example with associated intersection complex $(B, \mathscr{P})$ positive and simple. Assume that $(X_0, \mathcal{M}_{X_0})$ has a real structure, not necessarily standard. The real involution then induces a possibly non-trivial involution on the intersection complex $(B, \mathscr{P})$. But in any case, $(X_0, \mathcal{M}_{X_0})$ has a description by open gluing data $s = (s_{\omega\tau})$ and slab functions $f_{\rho,v}$ with the real involution lifting to an action on these data. By the strong uniqueness of the smoothing algorithm it is then not hard to see that the real involution extends to the constructed family $\mathfrak{X} \to \operatorname{Spec}\mathbb{C}[\![t]\!]$.

Note also that by Proposition 4.16, the locally rigid case with standard real structure is already covered in [GS3], Theorem 5.2.

[2]The theorem makes also the converse statement using the dual intersection complexes; working polarized as we do here, imposes a codimension one constraint, see the definition of $A_{\mathbb{P}}$ in [GHKS], §A.2.

Let us now assume we have a standard real structure on $(X_0, \mathcal{M}_{X_0})$. We wish to understand the topology of the real locus $X_{0,\mathbb{R}}^{\mathrm{KN}} \subset X_0^{\mathrm{KN}}$, the fixed locus of the lifted real involution of X_0^{KN} from Definition 3.1. First, since $(O^\dagger)_{\mathbb{R}}^{\mathrm{KN}} = \{\pm 1\}$, the real locus of $X_{0,\mathbb{R}}^{\mathrm{KN}}$ decomposes into two parts, the preimages of ± 1 under $\delta^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to (O^\dagger)^{\mathrm{KN}} = U(1)$. Denote by $X_{0,\mathbb{R}}^{\mathrm{KN}}(\pm 1)$ these two fibres.

Proposition 4.19. *The restriction of $\mu^{\mathrm{KN}} : X_0^{\mathrm{KN}} \to B$ to the real locus exhibits $X_{0,\mathbb{R}}^{\mathrm{KN}}$ as a surjection with finite fibres. Over $B \setminus \mathcal{A}$, this map is a topological covering map with fibres of cardinality 2^{n+1}.*

Proof. Let $\sigma \in \mathscr{P}$ be a maximal cell. In the canonical identification Φ_σ of Lemma 4.5, the standard real involution on $(\mu^{\mathrm{KN}})^{-1}(\sigma) \subset X_0^{\mathrm{KN}}$ lifts to the involution of $\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1))$ that acts by the identity on σ and by multiplication by -1 in $\Lambda_\sigma \oplus \mathbb{Z}$. The fixed point set of this involution over each point in σ is the set of two-torsion points $(\pm 1, \ldots, \pm 1)$ of $U(1)^{n+1}$. In particular, away from $\mathcal{A} \subset B$, the projection $X_{0,\mathbb{R}}^{\mathrm{KN}} \to B$ is a 2^{n+1}-fold unbranched cover.

In any case, $\Phi_\sigma\big(\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, \{\pm 1\})\big)$ is a closed subset in X_0^{KN} containing $X_{\sigma,\mathbb{R}} \setminus Z$ and projecting with fibres of cardinality at most 2^{n+1} to σ. The statement on finiteness of all fibres then follows if Z is nowhere dense in $X_{0,\mathbb{R}}^{\mathrm{KN}}$. This statement follows as in Lemma 4.4 noting that the generization maps between stalks of $\mathcal{M}_{X_0}$ at real points are compatible with the real involution. $\square$

We thus see that $X_{0,\mathbb{R}}^{\mathrm{KN}}$ can be understood by studying (a) the unbranched covering over $B \setminus \mathcal{A}$ and (b) the behaviour near the log singular locus by means of the canonical uniformization map Φ_σ of Lemma 4.5. Sometimes, e.g. in dimension two, the unbranched cover together with the fact that $X_{0,\mathbb{R}}^{\mathrm{KN}}$ is a topological manifold, determines $X_{0,\mathbb{R}}^{\mathrm{KN}}$ completely.

For the unbranched cover, Lemma 4.5 together with the gluing equation (4.3) in Remark 4.7 provide a full description of $X_{0,\mathbb{R}}^{\mathrm{KN}}$. Note also that the gluing equation involves the term $\kappa_\rho \cdot \arg(t)$, which for κ_ρ odd and $\mathrm{Arg}(t) = -1$ leads to a difference in the identification of branches over neighboring maximal cells.

We formulate this discussion as a proposition, omitting the obvious proof.

Proposition 4.20. *Let $(X_0, \mathcal{M}_{X_0})$ be endowed with a standard real structure, described by real open gluing data and real slab functions, following Proposition 4.16. Then the description of X_0^{KN} in Remark 4.7 as glued from trivial pieces $\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1)) \setminus \Phi_\sigma^{-1}(Z^{\mathrm{KN}})$ via Equation (4.3), exhibits the real locus $X_{0,\mathbb{R}}^{\mathrm{KN}} \setminus Z^{\mathrm{KN}}$ as glued from $\big(\sigma \times \mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, \{\pm 1\})\big) \setminus \Phi_\sigma^{-1}(Z^{\mathrm{KN}})$. In particular, the sign of the function f describing the gluing over a connected component of $\rho \setminus \mathcal{A}$, $\dim \rho = n - 1$, influences* *the identification of branches suggested by the identification of integral tangent vectors through affine parallel transport.* □

As emphasized, in general the specific choice of slab functions changes the topology of $X_{0,\mathbb{R}}^{\mathrm{KN}}$, and hence has to be studied case by case. Assuming without loss of generality that the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ is normalized by the open gluing data, we can however give a neat global description over the large subset $B' \subset B$ considered in Proposition 4.13. In the simple case, there is a retraction of $B \setminus \mathcal{A}$ to B' and this result is strong enough to understand the unbranched cover over $B \setminus \mathcal{A}$ completely. In the general case, this result can be complemented by separate studies along the interior of codimension one cells to gain a complete understanding of the part of the real locus covering $B \setminus \mathcal{A}$.

As a preparation, we need to discuss the effect of the real involution on Diagram (4.6), and in particular on the middle vertical part, the exact sequence

$$0 \longrightarrow \underline{\mathbb{Z}} \oplus \underline{U}(1) \longrightarrow \widehat{\mathcal{P}} \longrightarrow \Lambda \longrightarrow 0.$$

The action on the discrete part $\underline{\mathbb{Z}}$ and Λ is induced by the action on the cohomology of the torus fibres, which is multiplication by -1. Similarly, we can act by multiplication with -1 on each entry of the sequence defining $\mathcal{P}$, forming the next to rightmost column in (4.6). For the extension by $U(1)$, however, taking the pushout with complex conjugation on $U(1)$, maps the extension class $s \in \mathrm{Ext}^1(\Lambda, \underline{U}(1))$ to its complex conjugate $\overline{s}$. Thus only if this class is real, reflected in a real choice of open gluing data (Proposition 4.16), there is an involution on $\widehat{\mathcal{P}}$ inducing multiplication by -1 on $\underline{\mathbb{Z}}$ and Λ and the conjugation on $U(1)$. Note also that the extension class is real if and only if it lies in the image of $\mathrm{Ext}^1(\Lambda, \underline{\mathbb{Z}} \oplus \{\pm 1\})$ under the inclusion $\{\pm 1\} \to U(1)$. In this case, the extension of Λ by $\underline{\mathbb{Z}} \oplus U(1)$ is obtained by pushout from an extension by $\underline{\mathbb{Z}} \oplus \{\pm 1\}$. We now assume such an involution $\iota_{\widehat{\mathcal{P}}}$ of $\widehat{\mathcal{P}}$ exists.

Proposition 4.21. *Assume that the toric log Calabi-Yau space $(X_0, \mathcal{M}_{X_0})$ is given by real open gluing data and real normalized slab functions. Then in the canonical description of X_0^{KN} over $B' \subset B$ given in Proposition 4.13, the real locus is given by*

$$\mathrm{Hom}(\widehat{\mathcal{P}}, \{\pm 1\})^0|_{B'} \subset \mathrm{Hom}(\widehat{\mathcal{P}}, U(1))^0|_{B'}.$$

Proof. Recall the trivialization with fibres $\Lambda_\sigma \oplus \mathbb{Z} \oplus U(1)$ of $\widehat{\mathcal{P}}$ over the interior of a maximal $\sigma \in \mathscr{P}$ used in the proof of Proposition 4.13. In this trivialization, the involution $\iota_{\widehat{\mathcal{P}}}$ acts by -1 on $\Lambda_\sigma \oplus \mathbb{Z}$ and by conjugation on $U(1)$. Taking homomorphisms to $U(1)$ and restricting to those homomorphisms inducing the identity on the $U(1)$-factor, identifies the fibres of X_0^{KN} over $\mathrm{Int}\,\sigma$ with $\mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, U(1))$. The fixed point locus of the induced action of $\iota_{\widehat{\mathcal{P}}}$ is then the set of homomorphisms to the two-torsion points of $U(1)$, that is, $\mathrm{Hom}(\Lambda_\sigma \oplus \mathbb{Z}, \{\pm 1\})$, as claimed. □

Remark 4.22. The topology of the 2^{n+1}-fold cover of B' can also be described in terms of the monodromy representation as follows. Analogously to the discussion for $\mathcal{P}$ in Remark 4.10, the monodromy representation of $\widehat{\mathcal{P}}$ is given by viewing $(c_1(\varphi), s) \in \mathrm{Ext}^1(\Lambda, \underline{\mathbb{Z}} \oplus \underline{U}(1))$ as a pair of twisted homomorphisms,

$$(\lambda, \theta) : \pi_1(B', x) \longrightarrow \mathrm{Hom}(\Lambda_x, \mathbb{Z} \oplus U(1)).$$

Explicitly, for a closed loop γ at x, the action of $(\lambda, \theta)(\gamma) = (\lambda_\gamma, \theta_\gamma)$ on the fibre of $\widehat{\mathcal{P}}_x \simeq \Lambda_x \oplus \mathbb{Z} \oplus U(1)$ is

$$\Lambda_x \oplus \mathbb{Z} \oplus U(1) \ni (v, a, \beta) \longmapsto (T_\gamma \cdot v, \lambda_\gamma \cdot v + a, \theta_\gamma(v) \cdot \beta).$$

Here $T_\gamma \in \mathrm{GL}(\Lambda_x)$ is from parallel transport in Λ. If the open gluing data s are real, θ takes values in $\{\pm 1\} \subset U(1)$. Thus in the real case, (λ, θ) is a twisted homomorphism with values in $\mathrm{Hom}(\Lambda_x, \mathbb{Z} \oplus \{\pm 1\})$.

In view of Proposition 4.21, the monodromy representation of $X_{0,\mathbb{R}}^{\mathrm{KN}}$ over B' is given by the induced action on $\mathrm{Hom}\big(\Lambda_x \oplus \mathbb{Z} \oplus \{\pm 1\}, \{\pm 1\}\big)^\circ = \mathrm{Hom}(\Lambda_x, \{\pm 1\}) \oplus \mathrm{Hom}(\mathbb{Z}, \{\pm 1\})$. Note that the last summand in $\Lambda_x \oplus \mathbb{Z} \oplus \{\pm 1\}$ does not contribute to the right-hand side, since we restricted to those homomorphisms inducing the identity on $0 \oplus 0 \oplus \{\pm 1\}$. The action of a closed loop γ on $\mathrm{Hom}(\Lambda_x, \{\pm 1\}) \oplus \mathrm{Hom}(\mathbb{Z}, \{\pm 1\})$ is now readily computed as

$$(\varphi, \mu) \longmapsto \big(\varphi \circ T_\gamma + \mu \circ \lambda_\gamma + \theta_\gamma, \mu\big), \tag{4.7}$$

Here we wrote the group structure on $\{\pm 1\}$ additively. This formula gives an explicit description of $X_{0,\mathbb{R}}^{\mathrm{KN}}$ over B' in terms of a permutation representation of $\pi_1(B', x)$ on the set $\check{\Lambda}_x / 2\check{\Lambda}_x \oplus \{\pm 1\}$ of cardinality 2^{n+1}.

In this description, the map to the real part $(O^\dagger)_{\mathbb{R}}^{\mathrm{KN}} = \{\pm 1\}$ of the Kato-Nakayama space $U(1)$ of the standard log point, is induced by the inclusion $\mathbb{Z} \oplus \{\pm 1\} \to \Lambda_x \oplus \mathbb{Z} \oplus \{\pm 1\}$. Thus to describe the fibres over $\{\pm 1\} \subset (O^\dagger)^{\mathrm{KN}}$ in $X_{0,\mathbb{R}}^{\mathrm{KN}}$ simply amounts to restricting to $\mu = \pm 1$ in (4.7). In particular, $c_1(\varphi)$ only becomes relevant for the fibre over -1. This fact can also be seen from the gluing description of (4.3), where κ_ρ is the only place for $c_1(\varphi)$ to enter.

5. Examples

5.1. **A toric degeneration of quartic K3 surfaces.** As a first application of the general results in this paper, we look at an example of a toric degeneration of real quartic K3 surfaces.

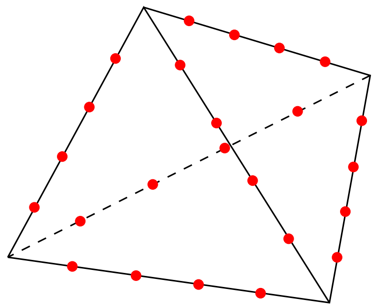

FIGURE 5.1. $(B, \mathscr{P})$ for a quartic K3 surface

Consider $(B, \mathscr{P})$ the polyhedral affine manifold that as an integral cell complex is the boundary of a 3-simplex, with four focus-focus singularities on each edge and with the complete fan at each vertex the fan of $\mathbb{P}^2$ (Figure 5.1). There are four maximal cells, each isomorphic to the standard simplex in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$ and $(0,1)$. The edges have integral length 1 and are identified pairwise to yield the boundary of a tetrahedron. On each of the six edges there are four singular points of the afffine structure, with monodromy conjugate to $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. We use standard ("vanilla") gluing data, that is, $s_{\omega\tau} = 1$ for all $\omega, \tau \in \mathscr{P}$, $\omega \subset \tau$. Then X_0 is isomorphic as a scheme to $Z_0Z_1Z_2Z_3 = 0$ in $\mathbb{P}^3$, a union of four copies of $\mathbb{P}^2$. As the MPL-function defining the ghost sheaf $\overline{\mathcal{M}}_{X_0}$ we take the function with kink $\kappa_\rho = 1$ on each of the edges. The moduli space of toric log Calabi-Yau structures on X_0 with the given $\overline{\mathcal{M}}_{X_0}$ is described by the space of global sections of an invertible sheaf $\mathcal{LS}$ on the double locus $(X_0)_{\mathrm{sing}}$. This line bundle has degree 4 on each of the six $\mathbb{P}^1$-components. The section is explicitly described by the 12 slab functions $f_{\rho,v}$. For each edge $\rho \in \mathscr{P}$ there are two slab functions, related by the equation $f_{\rho,v}(x) = x^4 f_{\rho,v'}(x^{-1})$ for x the toric coordinate on $\mathbb{P}^1$ (see e.g. [GS3], Equation 1.11). Explicitly, restricting to the normalized case, we have $f_{\rho,v} = 1 + a_1x + a_2x^2 + a_3x^3 + x^4$, the highest and lowest coefficients being 1 due to the normalization condition at the two zero-dimensional toric strata of the projective line $X_\rho \subset X_0$. The other coefficients $a_i \in \mathbb{C}$ are free, to give a total of $6 \cdot 3 = 18$ parameters. Taking into account the additional deformation parameter t, this number is in agreement with the 19 dimensions of projective smoothings of X_0. See the appendix of [GHKS] for a discussion of projectivity in this context.

This example does not have simple singularities, but it is locally rigid in the sense of [GS3], Definition 1.26. Thus the smoothing algorithm of [GS3] works, yielding a one-parameter smoothing of X_0, one for each choice of slab functions. According to Proposition 4.16 this smoothing is real if and only if all slab functions are real, that is, all coefficients $a_i \in \mathbb{R}$. To obtain 4 focus-focus singularities on each edge of $\mathscr{P}$

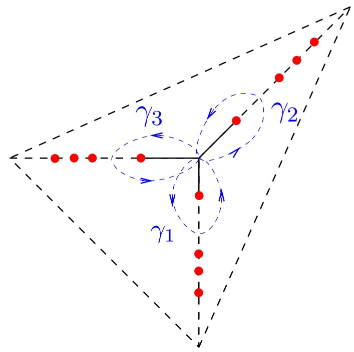

FIGURE 5.2. Chart at a vertex of $(B, \mathscr{P})$ from Figure 5.1

as drawn in Figure 5.1, we need to choose the a_i in such a way that the 4 zeroes of $f_{\rho,v} = 1 + a_1x + a_2x^2 + a_3x^3 + x^4$ have pairwise different absolute values. These are then also all real. This condition is open in the Euclidean topology, but the closure is a proper subset of $\mathbb{R}^3$, the space of tuples (a_1, a_2, a_3). The precise choice does not matter for the following discussion and we assume such a choice has been made on each edge.

Proposition 5.1. *Let $(X_0, \mathcal{M}_{X_0})$ be the union of four copies of $\mathbb{P}^2$ with the real log structure as described. Denote by $\mathcal{A} \subset B$ the pairwise different images of the 24 singular points of the log structure (the zero loci of the slab functons).*

Then the fibre $X^{\mathrm{KN}}_{0,\mathbb{R}}(1)$ of $\delta^{\mathrm{KN}}_{\mathbb{R}} : X^{\mathrm{KN}}_{0,\mathbb{R}} \to \{\pm 1\} \subset U(1)$ has two connected components, one mapping homeomorphically to B, the other a branched covering of degree 3, unbranched over $B \setminus \mathcal{A}$ and with a simple branch point over each point of $\mathcal{A}$. In particular, the latter component is a closed orientable surface of genus 10.

Proof. In the present case of vanilla gluing data, the positive real sections $\sigma \times \{1\} \subset \sigma \times \mathrm{Hom}(\Lambda_\sigma, \{\pm 1\})$ of the pieces over maximal cells (Lemma 4.5) are compatible to yield a section of $X^{\mathrm{KN}}_{0,\mathbb{R}}(1) \to B$. At each point of $\mathcal{A}$ there are local analytic coordinates x, y, w, defined over $\mathbb{R}$, with $(X_0, \mathcal{M}_{X_0})$ isomorphic to the central fibre of the degeneration $xy = t(w + 1)$ discussed in Example 2.11. In this example, the real locus of the Kato-Nakayama space has three connected components, with two being sections and one a two-fold branched cover with one branch point.

To finish the proof we have to study the global monodromy representation $\pi_1(B \setminus \mathcal{A}) \to S_4$ into the permutations of a fibre and show it has at most two irreducible subrepresentations. We compute a part of the affine monodromy representation and then use Remark 4.22 and notably Equation (4.7) to obtain the induced monodromy representation in S_4.

Figure 5.2 depicts a chart at a vertex with its three adjacent maximal cells. The chart gives the affine coordinates in the union of the three triangles minus the dotted lines. The locations of the 12 singular points on the outer dotted lines are irrelevant in this chart and are hence omitted. We look at the part of the fundamental group spanned by the three loops γ_1, γ_2, γ_3. Each encircles one focus-focus singularities on an edge containing the vertex and hence the affine monodromy is conjugate to $\left(\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right)$. In particular, the translational part vanishes. Concretely, in standard coordinates of $\mathbb{R}^2$, the monodromy matrices T_i along γ_i are

$$T_1 = \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \quad T_2 = \begin{pmatrix}2 & -1\\ 1 & 0\end{pmatrix}, \quad T_3 = \begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}. \tag{5.1}$$

Now while the γ_i are not loops inside B' as treated in Remark 4.22, it is not hard to see that (4.7) still applies in the present case. We have $\mu = 1$ since we look at $X^{\mathrm{KN}}_{0,\mathbb{R}}(1)$ and $\theta_{\gamma_i} = 1$ also for the translational parts. Thus (4.7) says that the branches transform according to the linear part of the affine monodromy. Now indeed a slab function $f_{\rho,v}$ changes signs locally along the real locus over an edge whenever crossing a focus-focus singularity. For $X^{\mathrm{KN}}_{0,\mathbb{R}}(1)$ this means that the two branches given by $\mathrm{Hom}(\Lambda_\rho, \{\pm 1\}) \subset \mathrm{Hom}(\Lambda_\sigma, \{\pm 1\})$ have trivial monodromy around any focus-focus singularity on ρ, while the two other branches swap.

It thus remains to compute the action of T_i on the two-torsion points of $\mathbb{Z}^2/2\mathbb{Z}^2$. These are the four vectors

$$v_0 = \begin{pmatrix}0\\0\end{pmatrix}, \quad v_1 = \begin{pmatrix}1\\0\end{pmatrix}, \quad v_2 = \begin{pmatrix}0\\1\end{pmatrix}, \quad v_3 = \begin{pmatrix}1\\1\end{pmatrix}.$$

Here v_0 is the point in the positive real locus, yielding the section of $X^{\mathrm{KN}}_{0,\mathbb{R}}(1) \to B$. The permutation of the indices of the other three vectors yield the three transpositions (12), (13), (23) for T_1, T_2 and T_3, respectively. These transpositions act transitively on $\{1,2,3\}$, showing connectedness of the cover of degree 3. This component is a genus 10 surface by the Riemann Hurwitz formula. □

5.2. **Toric degenerations of K3 surfaces for simple singularities.** As a second, related family of examples we consider toric degenerations of K3 surfaces such that the associated intersection complex $(B, \mathscr{P})$ has simple singularities. In this case the possible topologies of $X^{\mathrm{KN}}_{0,\mathbb{R}}$ are determined by Proposition 4.21. Interestingly, for the fibre over $1 \in (O^\dagger)^{\mathrm{KN}} = U(1)$, the question becomes a purely group-theoretic one. In fact, according to (4.7), for $\mu = 1$ the translational part λ of the affine monodromy representation does not enter in the computation. Moreover, by a classical result of Livné and Moishezon, the linear part of the monodromy representation for an affine structure on S^2 with 24 focus-focus singularities is unique up to equivalence [Mo], p.179. The result says that there exists a set of standard generators $\gamma_1, \dots, \gamma_{24}$ of $\pi_1(S^2 \setminus 24 \text{ points}, x)$, closed loops pairwise only intersecting at x and with composition

$\gamma_1 \cdot \ldots \cdot \gamma_{24}$ homotopic to the constant loop, such that the monodromy representation takes the form

$$T_{\gamma_i} = \begin{cases} T_3, & i \text{ odd} \\ T_1, & i \text{ even,} \end{cases}$$

with T_1, T_3 as in (5.1). As in §5.1, the corresponding monodromy of the four elements in $\check{\Lambda}_x/2\check{\Lambda}_x \simeq \mathbb{Z}^2/2\mathbb{Z}^2$ are (12) and (23), respectively. Thus the computation only depends on the choice of the twisted homomorphism $\theta \in H^1(B', \check{\Lambda} \otimes \{\pm 1\})$. Now each θ_γ acts by translation on the fibre $\mathbb{Z}^2/2\mathbb{Z}^2$. If θ_γ is nontrivial, the permutation is a double transposition. But any double transposition together with (12) and (23) acts transitively on the 4-element set. Thus $X_{0,\mathbb{R}}^{\mathrm{KN}}$ is connected as soon as $\theta \neq 0$; otherwise we have two connected components as in Proposition 5.1.

Proposition 5.2. *Let $(X_0, \mathcal{M}_{X_0})$ be a toric log K3 surface with intersection complex $(B, \mathscr{P})$ having simple singularities and endowed with a standard real structure. Denote by $\theta \in H^1(B, \check{\Lambda} \otimes \{\pm 1\})$ be the argument of the associated lifted real gluing data according to Corollary 4.17. Then $X_{0,\mathbb{R}}^{\mathrm{KN}}(1)$ has a connected component mapping homeomorphically to $B \simeq S^2$ if and only if $\theta = 0$, and is otherwise connected.* □

For $X_{0,\mathbb{R}}^{\mathrm{KN}}(-1)$, the translational part of the affine monodromy enters in (4.7). The action is also by translation, hence lead to a double transposition if non-trivial. A similar analysis then shows that if $X_{0,\mathbb{R}}^{\mathrm{KN}}(-1)$ is not connected, one connected component maps homeomorphically to $B \simeq S^2$.

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