# Bang-bang control of parabolic equations

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und Prof. Dr. Daniel Wachsmuth

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Prof. Dr. Bernd Siebert
Leiter des Fachbereichs Mathematik

# Contents

# Introduction

In this thesis, a class of optimal control problems governed by the heat equation is considered. The task is – roughly speaking – to track a desired *given state* $y_d$ by an *optimal state* $y = y(u)$. This state is realized as the solution of the heat equation, the *optimal control* $u$ being its right-hand side. This control is sought for. More precisely, we want to minimize the tracking-type functional

$$J(u) := \frac{1}{2}\|y(u) - y_d\|_Y^2 + \frac{\alpha}{2}\|u\|_U^2$$

where the optimal control has to lie in a set of *admissible controls* $U_{\text{ad}} \subset U$, for some properly chosen time-dependent function spaces $U$ and $Y$. We assume control constraints of box type, i.e., $a \leq u \leq b$ should hold almost everywhere in space and time for fixed bounds $a$ and $b$.

Via the parameter $\alpha \geq 0$ the possibility to weight the influence of (the norm of) the control is given. In many applications, this norm is interpreted as a measure for the control costs. For example, heating processes can be modeled in the above setting. The control might be a temperature source, and since temperature and energy costs are proportional, minimizing the temperature by fixing a nonzero $\alpha$ is thus meaningful from a modeling viewpoint.

However, if the control costs are negligible or even not meaningful at all, one might be interested in the *limit problem* $\alpha = 0$.

For example, in biochemical processes, the control might be the concentration of an activator of a reaction in a substrate with concentration $y$, which one wants to get close to a certain desired one. Here, control costs in terms of concentrations seem to be not meaningful in general.

Let us also mention as a second example the optimal control approach to inverse problems. Consider the heat equation $y_t - \nabla \cdot D\nabla y = f$ with a fixed function $f$ and a matrix-valued diffusion coefficient $D$ which might depend on space and time and is unknown. A solution $y^\delta$, for example from a measurement, is known, and the task is now to identify $D$. Writing

$y = y(D)$ for the solution of the equation in dependence of the diffusion matrix, one can tackle this problem in the framework of the control problem from above: $D$ is the control, $y(D)$ the state and $y_d = y^\delta$ the solution one wants to get close to. In this parameter identification problem, setting $\alpha = 0$ is again a natural choice.

Apart from the modeling question of selecting $\alpha$ positive or zero, the nonzero choice has mathematical benefits: The functional to minimize has by choosing $\alpha > 0$ a unique solution which fulfills a projection equation. This equation can be used to numerically solve the problem by a fixed-point iteration or the more efficient semismooth Newton method.

The limit case differs from the case $\alpha > 0$. The projection equation does not hold anymore and one is confronted with a loss of regularity in the function space where $u$ lives in. We therefor call the problem in the case of $\alpha > 0$ the *regular problem.*

The optimal control in the limit case is often discontinuous, but has a special structure: It takes values only on the bounds $a$ and $b$ of the set of admissible controls $U_{\mathrm{ad}}$. Such controls are called *bang-bang controls.*

In order to numerically solve the limit case, a famous idea from the theory of inverse problems can be applied, since the limit problem can be interpreted as an inverse problem with convex constraints. The idea of the so-called *Tikhonov regularization* consists in solving the regular case $\alpha > 0$ as an approximation of the limit problem. By the convergence of $u_\alpha$ to $u_0$ when $\alpha$ tends to zero, this method is justified. In this step, an error is introduced, the so-called *regularization error* $\|u_\alpha - u_0\|_U$.

As a next step, the control problem with $\alpha > 0$ is discretized in space and time (parameters $h$ and $k$, respectively) to solve it on a computer. We thereby introduce a second error, the *discretization error* $\|u_\alpha - u_\alpha^{kh}\|_U$.

The total error consists thus of two ingredients: The regularization and the discretization error. If *a-priori error estimates* are at hand for both, one can derive a coupling rule for $\alpha$, $k$ and $h$ for an efficient numerical solving.

It is the aim of this thesis to establish a numerical analysis with error estimates as described above.

In **chapter one**, the class of optimal control problems depending on $\alpha \geq 0$ mentioned above is introduced in detail.

The functional analytic setting is provided, existence and uniqueness of the state equation and the optimal control problem are discussed, as well as regularity issues. A necessary and also sufficient condition to characterize the solution of the optimal control problem is established, which is a key ingredient in the later analysis.

We then analyze the error introduced by the Tikhonov regularization. We first recall some well-known results from the general theory of inverse problems. After that, we show that under additional conditions assumed to hold for the limit problem, better results for the rate of convergence can be given. Here we present some new convergence rates, which improve known results. We show that the additional conditions required to obtain the improved convergence rates are not only sufficient but in some situations even necessary.

Finally, for bang-bang solutions a second sufficient condition is introduced, from which one can derive the same convergence rates. Almost-necessity of the condition and the relation to the previously used one are analyzed. With this second condition, an error bound on the time derivative of the control with respect to $\alpha$ is derived, which will be useful later to improve convergence rates for the discrete regularized solutions.

Having error estimates for the regularization error at hand, in **chapter two** an appropriate discretization of the optimal control problem is set up.

Therefor, we first consider a finite element discretization of the state equation (the heat equation) and an *adjoint equation*. The particular choice used here, a Petrov–Galerkin scheme, was recently proposed by Hinze, Vierling and the author in [DHV15]. We recall the results of the semi-discretization in time carried out there and enlarge the analysis to a full-discretization in time and space. Stability and error estimates are derived in different norms.

After that, we formulate and analyze the discretization of the optimal control problem. Here, the *Variational Discretization concept* introduced by Hinze in [Hin05] is used. At first, estimates for the error between regularized control and discrete regularized control are shown, which are not robust if $\alpha$ tends to zero and lead to non-optimal estimates for the total error. They are however of independent interest and later used to derive refined estimates.

We also show that although the state is approximated only roughly, a projection of the state available without further effort converges with a higher order.

We then derive robust estimates, which lead to better estimates for the total error if the limit problem is sufficiently regular.

Finally, using an estimate derived at the end of the first chapter, we improve these robust estimates further.

In the **third chapter**, we report and comment on some numerical calculations to support the analytical findings. We start with the non-robust error estimates, i.e., $\alpha > 0$ fixed, and consider the asymptotic behavior of the time ($k \to 0$) and space ($h \to 0$) discretization.

After that, we look at the regularization error in dependence of a problem specific parameter $\kappa$. We observe in the control the improved estimates from the first chapter.

Finally, we couple regularization and discretization parameters to approximate the limit problem, i.e., $\alpha = 0$. We observe the behavior predicted by the theorems from the second chapter.

# 1 The continuous optimal control problem

In this chapter the class of optimal control problems depending on $\alpha \geq 0$ is introduced, which we are interested in.

The functional analytic setting is provided in detail with the associated time-dependent function spaces. Thereafter, existence and uniqueness of the state equation and the optimal control problem is discussed, as well as regularity issues. A necessary and also sufficient condition for the solution is established, which is a key ingredient in the later analysis.

Here, we mention the monographs [Eva98], [Hin+09], [Wlo87], [LM72], [Trö05] and [GGZ74] as background references for the theory of optimal control with PDE constraints, partial differential equations, and the time-dependent function spaces. For functional analytic issues, we refer to the books of [Alt02] and [Bre10].

We then analyze the error introduced by the Tikhonov regularization. We first recall some well-known results from the general theory of inverse problems with convex constraints, mainly taken from [EHN00], see also [Neu86]. Afterwards, we show that under additional conditions assumed to hold for the limit problem, better results for the rate of convergence are given. Here we present some new convergence rates, which improve known results from the elliptic case (using the Laplace equation instead of the heat equation), where several ideas have been taken from.

We show that the additional conditions required to obtain the improved convergence rates are not only sufficient but even necessary, at least in some situations.

Finally, for bang-bang solutions a second sufficient condition is introduced, from which one can derive the same convergence rates. Almost-necessity of the condition and the relation to the previously used one are analyzed. With this second condition, an error bound on the time derivative of the control with respect to $\alpha$ is derived, which will be useful later to improve convergence rates for the discrete regularized solutions.

## 1.1 Problem setting

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be a spatial domain which is assumed to be bounded and convex with a polygonal boundary $\partial\Omega$. Furthermore, a fixed time interval $I := (0, T) \subset \mathbb{R}$, $0 < T < \infty$, a desired state $y_d \in L^2(I, L^2(\Omega))$, a non-negative real constant $0 \le \alpha \in \mathbb{R}$, and an initial value $y_0 \in L^2(\Omega)$ are prescribed.

With the Gelfand triple

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$$

we consider the following optimal control problem

$$\begin{aligned} &\min_{y \in Y, u \in U_{\mathrm{ad}}} J(y, u) \quad \text{with} \quad J(y, u) := \frac{1}{2}\|y - y_d\|^2_{L^2(I, L^2(\Omega))} + \frac{\alpha}{2}\|u\|^2_U, \\ &\text{s.t. } y = S(Bu, y_0) \end{aligned} \tag{$\mathbb{P}$}$$

where $U_{\mathrm{ad}} \subset U$ is the set of admissible controls, $B$ a properly chosen control operator (see below), and

$$Y := W(I) := \left\{ v \in L^2(I, H_0^1(\Omega)) \;\middle|\; v_t \in L^2(I, H^{-1}(\Omega)) \right\}$$

is the state space. We use the notation $v_t$ and $\partial_t v$ for weak time derivatives.

The operator

$$S : L^2(I, H^{-1}(\Omega)) \times L^2(\Omega) \to W(I), \quad (f, g) \mapsto y := S(f, g), \tag{1.1}$$

denotes the weak solution operator associated with the heat equation, i.e., the linear parabolic problem

$$\begin{aligned} \partial_t y - \Delta y &= f \quad \text{in } I \times \Omega\,, \\ y &= 0 \quad \text{in } I \times \partial\Omega\,, \\ y(0) &= g \quad \text{in } \Omega\,. \end{aligned}$$

The weak solution is defined as follows. For $(f, g) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$ the function $y \in W(I)$ satisfies with $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{H^{-1}(\Omega) H_0^1(\Omega)}$ the two equations

$$y(0) = g \tag{1.2a}$$

$$\int_0^T \left\langle \partial_t y(t), v(t) \right\rangle + a(y(t), v(t))\, dt = \int_0^T \left\langle f(t), v(t) \right\rangle dt \qquad \forall\, v \in L^2(I, H_0^1(\Omega)). \tag{1.2b}$$

Note that by the embedding $W(I) \hookrightarrow C([0,T], L^2(\Omega))$, see, e.g., [Eva98, Theorem 5.9.3], the first relation is meaningful.
In the preceding equation, the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is given by

$$a(y,v) := \int_\Omega \nabla y(x) \nabla v(x)\; dx.$$

We show below that (1.2) yields an operator, the operator $S$ mentioned above.

For the admissible set $U_{\text{ad}}$, the control space $U$, and the control operator $B$ we consider two situations.

1. (Distributed controls) With a control region $\Omega_U := I \times \Omega$, $D := 1$, $U := L^2(\Omega_U, \mathbb{R}^D)$, and fixed bounds $a, b \in L^\infty(\Omega_U, \mathbb{R}^D)$ with $a \le b$ almost everywhere (a.e.) in $I \times \Omega$, we consider the closed and convex set

   $$U_{\text{ad}} := \{u \in U \mid a(t,x) \le u(t,x) \le b(t,x) \text{ a.e. in } I \times \Omega\}.$$

   The control operator $B : U \to L^2(I, H^{-1}(\Omega))$ is given by $B := \text{Id}$, i.e., the identity mapping induced by the standard Sobolev embedding $L^2(\Omega) \hookrightarrow H^{-1}(\Omega)$.

2. (Located controls) Given $D \in \mathbb{N}_{>0}$, bounds $a, b \in L^\infty(I, \mathbb{R}^D)$ with $a(t) \le b(t)$ (by components), we consider with $\Omega_U := I$ the control space $U := L^2(\Omega_U, \mathbb{R}^D)$ and its closed and convex subset

   $$U_{\text{ad}} := \{u \in U \mid \forall\, i \in \{1, \dots, D\} : a_i(t) \le u_i(t) \le b_i(t) \text{ a.e. in } I\}.$$

   With $D$ fixed functionals $g_i \in H^{-1}(\Omega)$ the linear and continuous control operator $B$ is given by

   $$B : L^2(I, \mathbb{R}^D) \to L^2(I, H^{-1}(\Omega)), \quad u \mapsto \left( t \mapsto \sum_{i=1}^{D} u_i(t) g_i \right). \tag{1.3}$$

   For later use we observe that the adjoint operator $B^*$ is given by

   $$B^* : L^2(I, H_0^1(\Omega)) \to L^2(I, \mathbb{R}^D),$$
   $$(B^* q)(t) = \Big( \langle g_1, q(t) \rangle_{H^{-1}(\Omega) H_0^1(\Omega)} \;,\; \dots \;,\; \langle g_D, q(t) \rangle_{H^{-1}(\Omega) H_0^1(\Omega)} \Big)^T. \tag{1.4}$$

If furthermore $g_i \in L^2(\Omega)$ for all $1 \leq i \leq D$ holds, we can consider $B$ as an operator

$$B : L^2(I, \mathbb{R}^D) \to L^2(I, L^2(\Omega))$$

and get the adjoint operator $B^* : L^2(I, L^2(\Omega)) \to L^2(I, \mathbb{R}^D)$ as

$$(B^* q)(t) = \left( (g_1, q(t))_{L^2(\Omega)} \ , \ \ldots \ , \ (g_D, q(t))_{L^2(\Omega)} \right)^T . \tag{1.5}$$

Note that the adjoint operator $B^*$ (and also the operator itself) is preserving time regularity, i.e.,

$$B^* : H^k(I, X) \to H^k(I, \mathbb{R}^D) \text{ for } k \geq 0 \tag{1.6}$$

where $X$ is a subspace of $L^2(\Omega)$ depending on the regularity of the $g_i$ (as noticed just before), e.g., $X = L^2(\Omega)$ or $X = H_0^1(\Omega)$.

## 1.2 Existence and uniqueness

**Lemma 1** (Properties of the solution operator $S$)**.**

1. *For every $(f, g) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$ a unique state $y \in W(I)$ satisfying* (1.2) *exists. Thus the operator $S$ from* (1.1) *exists. Furthermore the state fulfills*

$$\|y\|_{W(I)} \leq C \left( \|f\|_{L^2(I, H^{-1}(\Omega))} + \|g\|_{L^2(\Omega)} \right) . \tag{1.7}$$

2. *Consider the bilinear form $A : W(I) \times W(I) \to \mathbb{R}$ given by*

$$A(y, v) := \int_0^T - \left\langle v_t, y \right\rangle + a(y, v)\, dt + \left\langle y(T), v(T) \right\rangle \tag{1.8}$$

*with $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{H^{-1}(\Omega) H_0^1(\Omega)}$. Then for $y \in W(I)$, equation* (1.2) *is equivalent to*

$$A(y, v) = \int_0^T \left\langle f, v \right\rangle dt + (g, v(0))_{L^2(\Omega)} \quad \forall\ v \in W(I). \tag{1.9}$$

*Furthermore, $y$ is the only function in $W(I)$ fulfilling equation* (1.9)*.*

*Proof.* The first part is a standard result, see, e.g., [Eva98, Theorem 7.1.3, 7.1.4] in combination with [Hin+09, Theorem 1.33] or [Hin+09, Theorem 1.35, 1.37].

For the second part, we first note that (1.2) can be rewritten as one equation if the test space is minimized, i.e.,

$$\int_0^T \langle \partial_t y(t), v(t) \rangle_{H^{-1}(\Omega) H_0^1(\Omega)} + a(y(t), v(t))\, dt + (y(0), v(0)_{L^2(\Omega)} \\ = \int_0^T \langle f(t), v(t) \rangle_{H^{-1}(\Omega) H_0^1(\Omega)}\, dt + (g, v(0))_{L^2(\Omega)} \quad \forall\, v \in W(I).$$

From this the claim follows with integration by parts of functions in $W(I)$, see [Hin+09, Theorem 1.32] or [GGZ74, Satz IV.1.17]. □

Note that in (1.8) we have due to the Gelfand triple

$$\langle y(T), v(T) \rangle_{H^{-1}(\Omega) H_0^1(\Omega)} = (y(T), v(T))_{L^2(\Omega)}.$$

The reason why we use the left expression is given in the discussion after (2.16).

An advantage of the formulation (1.9) in comparison to (1.2) is the fact that the weak time derivative $y_t$ of $y$ is *not* part of the equation. Later in discretizations of this equation, it offers the possibility to consider states which do not possess a weak time derivative.

We can now establish the existence of a solution to problem $(\mathbb{P})$.

**Lemma 2** (Unique solution of the o.c.p.)**.**
*The optimal control problem* $(\mathbb{P})$ *admits a unique solution* $(\bar{y}, \bar{u}) \in Y \times U$*, which can be characterized by the first order necessary and sufficient optimality condition*

$$\bar{u} \in U_{\mathrm{ad}}, \quad (\alpha \bar{u} + B^* \bar{p}, u - \bar{u})_U \geq 0 \quad \forall\ u \in U_{\mathrm{ad}} \tag{1.10}$$

*where* $B^*$ *denotes the adjoint operator of* $B$*, and the so-called optimal adjoint state* $\bar{p} \in W(I)$ *is the unique weak solution to the adjoint problem*

$$\begin{aligned} -\partial_t \bar{p} - \Delta \bar{p} &= h \quad \text{in } I \times \Omega\,, \\ \bar{p} &= 0 \quad \text{on } I \times \partial\Omega\,, \\ \bar{p}(T) &= 0 \quad \text{on } \Omega \end{aligned}$$

*with $h := \bar{y} - y_d$. This weak solution is defined and uniquely determined by the equation*

$$
\begin{aligned}
A(v,\bar{p}) &\overset{(1.8)}{=} \int_0^T -\langle \bar{p}_t, v\rangle_{H^{-1}(\Omega)H_0^1(\Omega)} + a(v,\bar{p})\,dt + (v(T),\bar{p}(T))_{L^2(\Omega)} \\
&= \int_0^T \langle h, v\rangle_{H^{-1}(\Omega)H_0^1(\Omega)}\,dt \quad \forall\, v \in W(I),
\end{aligned}
\tag{1.11}
$$

*which has by integration by parts the equivalent formulation*

$$
\begin{aligned}
\int_0^T \langle v_t, \bar{p}\rangle_{H^{-1}(\Omega)H_0^1(\Omega)} + a(v,\bar{p})\,dt + (v(0),\bar{p}(0))_{L^2(\Omega)} \\
= \int_0^T \langle h, v\rangle_{H^{-1}(\Omega)H_0^1(\Omega)}\,dt \quad \forall\, v \in W(I).
\end{aligned}
$$

*Proof.* This follows from standard results, see, e.g., [Hin+09, Theorem 1.46, p. 66] or [Trö05, Satz 2.14]. Note that the theorem remains valid even in the case $\alpha = 0$ since in our setting $U_{\text{ad}}$ is bounded and the cost functional is strictly convex. Note further that $\bar{p} = \tilde{S}^*(\bar{y} - y_d)$ where $\tilde{S} : L^2(I, H^{-1}(\Omega)) \to L^2(I, L^2(\Omega))$ is the operator $f \mapsto S(f, 0) \in W(I) \hookrightarrow L^2(I, L^2(\Omega))$, which is the solution operator from above in combination with a canonical embedding. □

As a consequence of the fact that $U_{\text{ad}}$ is a closed and convex set in a Hilbert space we have the following lemma.

**Lemma 3.** *In the case $\alpha > 0$ the variational inequality (1.10) is equivalent to*

$$
\bar{u} = P_{U_{\text{ad}}}\left(-\frac{1}{\alpha} B^* \bar{p}\right) \tag{1.12}
$$

*where $P_{U_{\text{ad}}} : U \to U_{\text{ad}}$ is the orthogonal projection.*

*Proof.* See [Hin+09, Corollary 1.2, p. 70] with $\gamma = \frac{1}{\alpha}$. □

The orthogonal projection in (1.12) can be made explicit in our setting.

**Lemma 4.** *Let us for $a, b \in \mathbb{R}$ with $a \le b$ consider the projection of a real number $x \in \mathbb{R}$ into the interval $[a, b]$, i.e., $P_{[a,b]}(x) := \max\{a, \min\{x, b\}\}$.*

1. *In the case of distributed controls there holds for* $v \in L^2(I, L^2(\Omega))$

$$P_{U_{\text{ad}}}(v)(t,x) = P_{[a(t,x),b(t,x)]}(v(t,x)) \quad a.e.$$

2. *In the case of located controls we have for* $v \in L^2(I, \mathbb{R}^D)$

$$P_{U_{\text{ad}}}(v)(t) = \big(P_{[a_i(t),b_i(t)]}(v_i(t))\big)_{i=1}^{D} \quad a.e.$$

*Proof.* Since $L^2(I, L^2(\Omega)) \cong L^2(I \times \Omega)$ and $L^2(I, \mathbb{R}^D) \cong L^2(I, \mathbb{R})^D$ hold, the claim is a consequence of the fact that the projection can be characterized by an inequality (see, e.g., [Hin+09, Lemma 1.10, p. 67]), which by a Lebesgue point argument holds pointwise, see, e.g., [Trö05, S. 54]. □

We now derive an explicit characterization of the optimal control.

**Lemma 5.** *If* $\alpha > 0$*, then for almost all* $(t,x) \in I \times \Omega$ *there holds for the optimal control*

$$\bar{u}(t,x) = \begin{cases} a(t,x) & \text{if } B^*\bar{p}(t,x) + \alpha a(t,x) > 0, \\ -\alpha^{-1}B^*\bar{p}(t,x) & \text{if } B^*\bar{p}(t,x) + \alpha \bar{u}(t,x) = 0, \\ b(t,x) & \text{if } B^*\bar{p}(t,x) + \alpha b(t,x) < 0, \end{cases} \tag{1.13a}$$

*in the case of distributed controls, and for every* $1 \le i \le D$ *and almost all* $t \in I$ *there holds*

$$\bar{u}_i(t) = \begin{cases} a_i(t) & \text{if } (B^*\bar{p})_i(t) + \alpha a_i(t) > 0, \\ -\alpha^{-1}(B^*\bar{p})_i(t) & \text{if } (B^*\bar{p})_i(t) + \alpha \bar{u}_i(t) = 0, \\ b_i(t) & \text{if } (B^*\bar{p})_i(t) + \alpha b_i(t) < 0, \end{cases} \tag{1.13b}$$

*in the case of located controls.*

*Suppose* $\alpha = 0$ *is given. Then the optimal control fulfills a.e. in the case of distributed controls*

$$\bar{u}(t,x) = \begin{cases} a(t,x) & \text{if } B^*\bar{p}(t,x) > 0, \\ b(t,x) & \text{if } B^*\bar{p}(t,x) < 0, \end{cases} \tag{1.14a}$$

*and in the case of located controls*

$$\bar{u}_i(t) = \begin{cases} a_i(t) & \text{if } (B^*\bar{p})_i(t) > 0, \\ b_i(t) & \text{if } (B^*\bar{p})_i(t) < 0. \end{cases} \tag{1.14b}$$

*Proof.* We only consider distributed controls. The case of located controls follows by obvious modifications.

Let us first note that the variational inequality (1.10) is for $\alpha \geq 0$ equivalent to the following pointwise one.

$$\forall'(t,x) \in I \times \Omega \ \forall\ v \in [a(t,x), b(t,x)] : \\ (\alpha\bar{u}(t,x) + B^*\bar{p}(t,x), v - \bar{u}(t,x))_{\mathbb{R}} \geq 0. \quad (1.15)$$

This can be shown via a Lebesgue point argument, see the proof of [Trö05, Lemma 2.26]. By cases, one immediately derives (1.13) and (1.14) from (1.15). □

As a consequence we have in the case of distributed controls and $\alpha = 0$: If $B^*\bar{p}$ vanishes only on a subset of $I \times \Omega$ with Lebesgue measure zero, the optimal control $\bar{u}$ only takes values on the bounds $a, b$ of the admissible set $U_{\text{ad}}$. In this case $\bar{u}$ is called a *bang-bang solution*, accordingly defined in the case of located controls.

## 1.3 On regularity

In this section we recall some regularity results concerning the weak solution $y$ of the state equation (1.2) and the weak solution $p$ of the adjoint state equation (1.11). Afterwards we pose an assumption on the regularity of the data of problem ($\mathbb{P}$), from which we can derive more regularity of the optimal solution triple $(\bar{u}, \bar{y}, \bar{p})$. This regularity is needed for the convergence rates in the numerical realization of the problem.

We use here and in what follows the notation

$$\|\cdot\| := \|\cdot\|_{L^2(\Omega)} \quad \text{and} \quad \|\cdot\|_I := \|\cdot\|_{L^2(I,L^2(\Omega))},$$

and similarily we write for scalar products

$$(\cdot,\cdot) := (\cdot,\cdot)_{L^2(\Omega)} \quad \text{and} \quad (\cdot,\cdot)_I := (\cdot,\cdot)_{L^2(I,L^2(\Omega))}.$$

Let us start with the following standard result.

**Lemma 6** (More regularity). *For $f, h \in L^2(I, L^2(\Omega))$ and $g \in H_0^1(\Omega)$ the solutions $y$ of (1.2) and $p$ of (1.11) satisfy*

$$y, p \in L^2(I, H^2(\Omega) \cap H_0^1(\Omega)) \bigcap H^1(I, L^2(\Omega)) \hookrightarrow C([0,T], H_0^1(\Omega)). \quad (1.16)$$

*Furthermore, with some constant $C > 0$ there holds*

$$\|y\|_I + \|\partial_t y\|_I + \|\Delta y\|_I + \max_{t\in[0,T]} \|y(t)\|_{H^1(\Omega)} \le C\left(\|f\|_I + \|g\|_{H^1(\Omega)}\right),$$

*and*

$$\|\partial_t p\|_I + \|\Delta p\|_I + \max_{t\in[0,T]} \|p(t)\|_{H^1(\Omega)} \le C\|h\|_I.$$

*Proof.* See [Eva98, Theorems 7.1.5 and 5.9.4]. □

**Remark 7.** *As an immediate consequence we get: The optimal adjoint state $\bar{p}$ has the regularity* (1.16)*. If we assume $y_0 \in H_0^1(\Omega)$ and in the case of located controls $g_i \in L^2(\Omega)$, the same holds true for the optimal state $\bar{y}$.*

In order to achieve second order convergence in time we need more regularity, i.e., at least two weak time derivatives.

**Lemma 8** (High regularity)**.** *Let $f, h \in H^1(I, L^2(\Omega))$, $f(0), h(T) \in H_0^1(\Omega)$, and $g \in H_0^1(\Omega)$ with $\Delta g \in H_0^1(\Omega)$. Then the weak solutions $y$ of* (1.9) *and $p$ of* (1.11) *satisfy*

$$y, p \in H^1(I, H^2(\Omega) \cap H_0^1(\Omega)) \bigcap H^2(I, L^2(\Omega)) \hookrightarrow C^1(\bar{I}, H_0^1(\Omega)). \tag{1.17}$$

*With some constant $C > 0$ we have the a priori estimates*

$$\begin{aligned}&\|\partial_t^2 y\|_I + \|\partial_t \Delta y\|_I + \max_{t\in[0,T]} \|\nabla \partial_t y(t)\| \\ &\qquad \le C\left(\|f\|_{H^1(I,L^2(\Omega))} + \|f(0)\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)} + \|\Delta g\|_{H^1(\Omega)}\right),\end{aligned}$$

*and*

$$\|\partial_t^2 p\|_I + \|\partial_t \Delta p\|_I + \max_{t\in[0,T]} \|\nabla \partial_t p(t)\| \le C\left(\|h\|_{H^1(I,L^2(\Omega))} + \|h(T)\|_{H^1(\Omega)}\right).$$

*Proof.* This can be found in [MV11, Proposition 2.1]. □

We will also make use of three weak time derivatives, which the optimal adjoint state possesses in the case $\alpha > 0$. To this end, we need the following Lemma.

**Lemma 9.** *Let $p$ be the weak solution of* (1.11) *for a right-hand side $h$ with regularity $h \in H^2(I, L^2(\Omega)) \bigcap H^1(I, H^2(\Omega) \cap H_0^1(\Omega))$ and $\Delta h(T) \in H_0^1(\Omega)$. Then $p$ fulfills*

$$p \in H^3(I, L^2(\Omega)) \bigcap H^2(I, H^2(\Omega) \cap H_0^1(\Omega)) \hookrightarrow C^2(\bar{I}, H_0^1(\Omega)), \tag{1.18}$$

*and with some constant $C > 0$ the estimate*

$$\|\partial_t^3 p\|_I + \|\partial_t^2 \Delta p\|_I + \max_{t\in[0,T]} \|\nabla \partial_t^2 p(t)\| \leq C \left(\|\partial_t^2 h\|_I + \|\nabla(\partial_t h(T) + \Delta h(T))\|\right) \tag{1.19}$$

*holds true.*

*Proof.* This follows along the lines of the proof of [SV13, Lemma 2], making use of Lemma 6 and Lemma 8. □

To derive high regularity for the optimal state $\bar{y}$ and the adjoint state $\bar{p}$, we have to assume more regularity on the data than stated at the beginning of this chapter.

**Assumption 10.** *Let $y_d \in H^2(I, L^2(\Omega)) \bigcap H^1(I, H^2(\Omega) \cap H_0^1(\Omega))$ with $\Delta y_d(T) \in H_0^1(\Omega)$ and $y_0 \in H_0^1(\Omega)$. Furthermore, we expect $\Delta y_0 \in H_0^1(\Omega)$. In the case of distributed controls, we assume $a$, $b \in H^1(I, L^2(\Omega)) \bigcap C(\bar{I}, H_0^1(\Omega) \cap C(\bar{\Omega}))$. In the case of located controls, we assume $g_i \in H_0^1(\Omega)$, $i = 1, \dots, D$, and $a, b \in W^{1,\infty}(I, \mathbb{R}^D)$.*

In view of the relation (1.12), the following lemma is useful to derive regularity for the optimal control $\bar{u}$. It is sometimes called *Stampacchia's lemma*, since its core can be traced back to [Sta64, Lemme 1.1].

**Lemma 11** (Stability of the projection)**.** *With $a, b \in U$ where $U$ is a Hilbert space specified below, consider the orthogonal projection onto the set $U_{\text{ad}} := \{u \in U \mid a \leq u \leq b \text{ a.e.}\}$ and one of the following situations with $k \in \{0, 1\}$, $1 \leq p \leq \infty$.*

1. *$U := L^2(\Omega)$ and $V := W^{k,p}(\Omega)$ or $V := H_0^1(\Omega)$,*
2. *$U := L^2(I, \mathbb{R}^D)$ and $V := W^{k,p}(I, \mathbb{R}^D)$,*
3. *$U := L^2(I, L^2(\Omega))$ and $V := H^k(I, L^2(\Omega))$ or $V := L^2(I, H^k(\Omega))$,*
4. *$U := L^2(I, L^2(\Omega))$ and $V := C(\bar{I}, H_0^1(\Omega))$,*

*5.* $U := L^2(I, L^2(\Omega))$ *and* $V := C(\bar{I}, C(\bar{\Omega})) \cong C(\bar{I} \times \bar{\Omega})$*.*

*Then there holds the following stability result. If* $a$*,* $b$*, and* $v$ *are in* $V$*, so is* $P_{U_{\rm ad}}(v)$ *and the inequality*

$$\|P_{U_{\rm ad}}(v)\|_V \le C\left(\|a\|_V + \|b\|_V + \|v\|_V\right) \tag{1.20}$$

*is fulfilled with a constant* $C > 0$*.*

*Furthermore in the situations 1, 2, and 3, the projection* $P_{U_{\rm ad}} : V \to V$ *is Lipschitz continuous if* $k = 0$ *and continuous if both* $k = 1$ *and* $1 \le p < \infty$*.*

*Proof.* Let us first note that by Lemma 4 the projection $Pu := P_{U_{\rm ad}}u$ can be written as

$$Pu = a + [(u-b)^- + b - a]^+ \tag{1.21}$$

where $u^+$ is (almost everywhere) the positive part of the function $u$, i.e.,

$$\cdot^+ : U \to U, \quad u \mapsto u^+, \quad \text{with} \qquad u^+(x) = \begin{cases} u(x) & u(x) > 0 \\ 0 & u(x) \le 0 \end{cases}. \tag{1.22}$$

Accordingly, $u^-$ is the negative part.

Situation 1.
The fact that $P$ is $V$-preserving and the representation

$$Dv^+ = \begin{cases} Dv & v > 0 \\ 0 & v \le 0 \end{cases} \tag{1.23}$$

for the weak derivative $Dv^+$ of $v^+$ are classic results, see, e.g., [Sta64, Lemme 1.1] and [Zie89, Corollary 2.1.8, p. 47]. From this representation, estimate (1.20) immediately follows from (1.21) and (1.22).

It remains to prove continuity. Since (pointwise) Lipschitz continuity is obvious for $u^+$, it immediately carries over to $P : L^p \to L^p$ for $1 \le p \le \infty$ by (1.21).

Let now $V = H^1(\Omega)$ or $V = H^1_0(\Omega)$ and a sequence $(v_n)$ with $V \ni v_n \to v \in V$ be given. We have to show $\|Dv_n^+ - Dv^+\|_{L^2(\Omega)} \to 0$.

With the help of (1.23) we get

$$\begin{aligned} &\|Dv_n^+ - Dv^+\|^2_{L^2(\Omega)} \\ &\le \|Dv_n - Dv\|^2_{L^2(\Omega)} + \int_{\{v_n>0,\ v\le 0\}} |Dv_n(x)|^2 dx + \int_{\{v_n\le 0,\ v>0\}} |Dv(x)|^2 dx \\ &=: I + II + III. \end{aligned} \tag{1.24}$$

By construction of $(v_n)$, the first term vanishes if $n$ goes to infinity. Thus it remains to estimate the terms $II$ and $III$.

From $(v_n)$ we select a subsequence converging almost everywhere to $v$, denoted again by $v_n$, see, e.g. [Alt02, Lemma 1.18, p. 52].

Consider now term $III$. We will show that the Lebesgue measure of the sets $E_n := \{v_n \leq 0,\ v > 0\}$ vanishes if $n$ approaches infinity. As a consequence, term $III$ itself goes to zero by [Alt02, Lemma A 1.16, p. 82].

Let us fix an $\epsilon > 0$. Continuity from above of the image measure guarantees

$$\exists\ \epsilon_2 > 0 : \text{meas}(\{0 < v < \epsilon_2\}) < \frac{\epsilon}{2}.$$

By Egorov's theorem, see, e.g. [Alt02, A 1.17, p. 83], we conclude the existence of a set $\tilde{E}_\epsilon$ such that $\text{meas}(\Omega \backslash \tilde{E}_\epsilon) < \frac{\epsilon}{2}$ and $v_n \to v$ uniformly on the set $\tilde{E}_\epsilon$.

Therefore, we can choose an $N(\epsilon) \in \mathbb{N}$ such that there holds

$$\forall\ n \in \mathbb{N} : n > N(\epsilon) \Rightarrow \|v_n - v\|_{L^\infty(\tilde{E}_\epsilon)} < \frac{\epsilon_2}{2}.$$

With the subset $E_n^1 := \{v_n \leq 0,\ 0 < v < \epsilon_2\}$ of $E_n$ we conclude for $n > N(\epsilon)$

$$\begin{aligned} \text{meas}(E_n) &\leq \text{meas}(\Omega \backslash \tilde{E}_\epsilon) + \text{meas}(\tilde{E}_\epsilon \cap E_n) < \frac{\epsilon}{2} + \text{meas}(\tilde{E}_\epsilon \cap E_n^1) \\ &\leq \frac{\epsilon}{2} + \text{meas}(\{0 < v < \epsilon_2\}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

Thus $\text{meas}(E_n) \to 0$ holds for $n \to \infty$.

Consider now term $II$. We estimate for a measurable subset $X$ of $\Omega$

$$\int_X |Dv_n|^2 \leq C \left( \int_X |Dv_n - Dv|^2 + \int_X |Dv|^2 \right),$$

where the first addend vanishes by assumption if $n$ tends to infinity. Taking $X := \{v_n > 0,\ v \leq 0\}$, we only have to estimate $\int_{E_n} |Dv|^2$, now with $E_n := \{v_n > 0,\ v < 0\}$, since $Dv = 0$ a.e. on $\{v = 0\}$ by [EG92, Theorem 4(iv), p. 130]. Estimating $\int_{E_n} |Dv|^2$ can be done analogously as for $III$.

Note finally that we have shown: Every sequence $V \ni v_n \to v \in V$ possesses a subsequence with $Dv_n^+ \to Dv^+$. Thus by contradiction the convergence is valid for the whole sequence.

For $V := W^{k,p}(\Omega)$, the above proof has to be changed at obvious places.

Situation 2.
By the isomorphism $W^{k,p}(I,\mathbb{R}^D) \cong W^{k,p}(I,\mathbb{R})^D$, the claim is an immediate consequence of Situation 1, since for the projection there holds $P_{C_1\times C_2}(x_1,x_2) = P_{C_1}(x_1) \times P_{C_2}(x_2)$ where $C_i$ denote nonempty closed convex subsets of two Hilbert spaces $H_i$ and $x_i \in H_i$.

Situation 3.
Since the isomorphisms

$$H^k(I,L^2(\Omega)) \cong \big\{ f \in L^2(I\times\Omega) \;\big|\; D_t^k f \in L^2(I\times\Omega) \big\} \quad \text{and}$$
$$L^2(I,H^k(\Omega)) \cong \big\{ f \in L^2(I\times\Omega) \;\big|\; D_x^k f \in L^2(I\times\Omega) \big\}$$

are known to hold (see [LM72, Vol. II, p.5]), this case reduces to Situation 1.

Situation 4.
Well-definedness follows from the a.e. equality $v^+(t) = (v(t))^+$ in $H^1(\Omega)$ (compare Lemma 4) and continuity in Situation 1.

Situation 5.
Since (pointwise) Lipschitz continuity is obvious for $u^+$, it immediately carries over to $P : C(\bar{I}\times\bar{\Omega}) \to C(\bar{I}\times\bar{\Omega})$ by (1.21). □

We can now derive regularity for the triple $(\bar{u},\bar{y},\bar{p})$ from the Assumption 10. The result is an extension of [MV11, Proposition 2.3] and [SV13, Lemma 2].

**Lemma 12** (Regularity of problem $(\mathbb{P})$, $\alpha > 0$)**.** *Let Assumption 10 hold and let $\alpha > 0$. For the unique solution $(\bar{y},\bar{u})$ of $(\mathbb{P})$ and the corresponding adjoint state $\bar{p}$ there holds*

- $\bar{p} \in H^3(I,L^2(\Omega)) \bigcap H^2(I,H^2(\Omega)\cap H_0^1(\Omega)) \hookrightarrow C^2(\bar{I},H_0^1(\Omega))$,
- $\bar{y} \in H^2(I,L^2(\Omega)) \bigcap H^1(I,H^2(\Omega)\cap H_0^1(\Omega)) \hookrightarrow C^1(\bar{I},H_0^1(\Omega))$, *and*
- $\bar{u} \in W^{1,\infty}(I,\mathbb{R}^D)$ *in the case of located controls or*
- $\bar{u} \in H^1(I,L^2(\Omega)) \cap C(\bar{I},H_0^1(\Omega)) \cap C(\bar{I}\times\bar{\Omega})$ *in the case of distributed controls.*

*With some constant $C > 0$ independent of $\alpha$, we have the a priori estimates*

$$\|\partial_t^2 \bar{y}\|_I + \|\partial_t \Delta \bar{y}\|_I + \max_{t\in[0,T]} \|\nabla \partial_t \bar{y}(t)\|$$
$$\le d_1(\bar{u}) := C\left(\|B\bar{u}\|_{H^1(I,L^2(\Omega))} + \|\nabla B\bar{u}(0)\| + \|\nabla \Delta y_0\|\right),$$
$$\|\partial_t^2 \bar{p}\|_I + \|\partial_t \Delta \bar{p}\|_I + \max_{t\in[0,T]} \|\nabla \partial_t \bar{p}(t)\|$$
$$\le d_0(\bar{u}) := C\left(\|y_d\|_{H^1(I,L^2(\Omega))} + \|\nabla y_d(T)\| + \|B\bar{u}\|_I + \|\nabla y_0\|\right), \text{ and}$$
$$\|\partial_t^3 \bar{p}\|_I + \|\partial_t^2 \Delta \bar{p}\|_I + \max_{t\in[0,T]} \|\nabla \partial_t^2 \bar{p}(t)\|$$
$$\le d_1^+(\bar{u}) := d_1(\bar{u}) +$$
$$C\left(\|\partial_t^2 y_d\|_I + \|\nabla \partial_t y_d(T)\| + \|\nabla \Delta y_d(T)\| + \|\nabla B\bar{u}(T)\|\right). \quad (1.25)$$

*Proof.* From Lemma 6 we conclude that the optimal state $\bar{y}$ – for the present – has regularity $H^1(I, L^2(\Omega))$, and $\bar{y}(T) \in H_0^1(\Omega)$. Thus, by Lemma 8 the optimal adjoint state $\bar{p}$ has regularity

$$\bar{p} \in H^2(I, L^2(\Omega)) \cap C(\bar{I}, H_0^1(\Omega)) \cap C(\bar{I} \times \bar{\Omega}), \quad (1.26)$$

and the a priori estimate with $\|\partial_t^2 \bar{p}\|_I$ is valid.

In the case of located controls, we by (1.6) conclude $B^*\bar{p} \in H^2(I, \mathbb{R}^D) \hookrightarrow W^{1,\infty}(I, \mathbb{R}^D)$. Finally, from Lemma 11, 2., and the projection formula (1.12) we get the regularity for $\bar{u}$.

In the case of distributed controls, by Assumption 10 and Lemma 11, 3.-5., the regularity of $\bar{p}$ as given above in (1.26) is almost preserved when switching from $\bar{p}$ to $\bar{u}$ but the term $H^2(I, L^2(\Omega))$ has to be replaced by $H^1(I, L^2(\Omega))$.

Using Lemma 8 again, we obtain the regularity for $\bar{y}$ (note: $Bu(0) \in H_0^1(\Omega)$) and the estimate with $\|\partial_t^2 \bar{y}\|_I$.

With this estimate, the estimate $\|\nabla(\partial_t \bar{y}(T) + \Delta \bar{y}(T))\| \le C\|\nabla B\bar{u}(T)\|$, which holds since $B\bar{u} \in C(\bar{I}, H_0^1(\Omega))$ as we just saw, and Lemma 9, we conclude the existence of three weak time derivatives for $\bar{p}$ and the estimate with $\|\partial_t^3 \bar{p}\|_I$. □

**Remark 13** (Regularity in the case $\alpha = 0$)**.** *In the case $\alpha = 0$, we by inspecting the proof of Lemma 12 can only derive less regularity. The adjoint $\bar{p}$ now has the regularity given in Lemma 8, for $\bar{y}$ we can only conclude the regularity from Lemma 6, compare Remark 7.*

*Since* (1.12) *does not hold, we can not derive regularity for* $\bar{u}$ *from that of* $\bar{p}$ *as above. We only know from the definition of* $U_{\text{ad}}$ *that* $\bar{u} \in L^\infty(\Omega_U, \mathbb{R}^D)$, *but might be discontinuous as we will see later.*

## 1.4 Tikhonov regularization

In this section, we collect some results concerning the convergence of the Tikhonov regularized solution to the limit problem. Furthermore, convergence rates will be given. We start with results which are well-known from inverse problem theory and can be directly applied to our situation. Afterwards we state more refined results, where we benefit from and extend recent results for elliptic optimal control problems.

For this section, it is useful to rewrite problem ($\mathbb{P}$) in the reduced form

$$\min_{u \in U_{\text{ad}}} J_\alpha(u) \quad \text{with} \quad J_\alpha(u) := \frac{1}{2}\|Tu - z\|_H^2 + \frac{\alpha}{2}\|u\|_U^2 \qquad (\mathbb{P}_\alpha)$$

with $H := L^2(I, L^2(\Omega))$, fixed data $z := y_d - S(0, y_0)$ and the linear and continuous control-to-state operator $T : U \to H$, $Tu := S(Bu, 0)$.

We declare the notation $(\bar{u}_\alpha, \bar{y}_\alpha, \bar{p}_\alpha)$ for the unique solution of problem ($\mathbb{P}_\alpha$), which coincides with $(\bar{u}, \bar{y}, \bar{p})$ from Lemma 2.

The limit problem, i.e.,

$$\min_{u \in U_{\text{ad}}} J_0(u) = \min_{u \in U_{\text{ad}}} \frac{1}{2}\|Tu - z\|_H^2, \qquad (\mathbb{P}_0)$$

can be interpreted as an inverse problem with convex constraints (given by $U_{\text{ad}}$), which is the starting point of the analysis below.

Let us finally mention that all results of this section hold in a more general setting: One can replace the operator $T = S(B\cdot, 0) : U \to H$ by an arbitrarily chosen continuous linear operator $T$, mapping from some $L^2$ space to some Hilbert space, if the Hilbert space adjoint $T^*$ maps continuously to $L^\infty$, at least on the range of $T$. Lemma 12 and (1.5) show that this property is fulfilled for problem ($\mathbb{P}_\alpha$).

### 1.4.1 Convergence results from general inverse problem theory

From Lemma 2 we already know that a unique solution $\bar{u}_0$ of ($\mathbb{P}_0$) exists. As a consequence, we get convergence $\bar{u}_\alpha \to \bar{u}_0$ if $\alpha \to 0$ and even a first convergence rate for the error of the optimal state $\bar{y}_\alpha$, as the following Theorem shows. This Theorem is a collection of classic result from the theory of inverse problems with convex constraints, see, e.g., [EHN00, Chapter 5.4] or [Neu86].

**Theorem 14.** *For the solutions* $(\bar{u}_\alpha, \bar{y}_\alpha)$ *of* $(\mathbb{P}_\alpha)$ *and* $(\bar{u}_0, \bar{y}_0)$ *of* $(\mathbb{P}_0)$, *there holds*

1. *The optimal control and the optimal state depend continuously on* $\alpha$. *More precisely, the inequality*
$$\|\bar{y}_{\alpha'} - \bar{y}_\alpha\|_H^2 + \alpha'\|\bar{u}_{\alpha'} - \bar{u}_\alpha\|_U^2 \le (\alpha - \alpha')(\bar{u}_\alpha, \bar{u}_{\alpha'} - \bar{u}_\alpha)_U \tag{1.27}$$
*holds for all* $\alpha \ge 0$ *and all* $\alpha' \ge 0$.

2. *The regularized solutions converge to the unregularized one, i.e.,*
$$\|\bar{u}_\alpha - \bar{u}_0\|_U \to 0 \quad \text{if } \alpha \to 0. \tag{1.28}$$

3. *The optimal state satisfies the rate of convergence*
$$\|\bar{y}_\alpha - \bar{y}_0\|_H = o(\sqrt{\alpha}). \tag{1.29}$$

4. *The optimal control* $\bar{u}_\alpha$ *and the optimal state* $\bar{y}_\alpha$ *depend Lipschitz continuously on the data* $z$. *More precisely, consider two solutions* $\bar{u}_\alpha$ *and* $\bar{u}'_\alpha$ *of* $(\mathbb{P}_\alpha)$ *for data* $z$ *and* $z'$, *respectively. Then there holds*
$$\sqrt{\alpha}\|\bar{u}_\alpha - \bar{u}'_\alpha\|_U + \|\bar{y}_\alpha - \bar{y}'_\alpha\|_H \le \|z - z'\|_H \tag{1.30}$$

*Proof.* 1. From the definition of $\bar{u}_\alpha$ and $\bar{u}_0$ we infer
$$\alpha\|\bar{u}_\alpha\|_U^2 \le \|T\bar{u}_\alpha - z\|_H^2 - \|T\bar{u}_0 - z\|_H^2 + \alpha\|\bar{u}_\alpha\|_U^2 \le \alpha\|\bar{u}_0\|_U^2,$$
thus
$$\forall\ \alpha \ge 0 : \|\bar{u}_\alpha\|_U \le \|\bar{u}_0\|_U. \tag{1.31}$$
Let us repeat (1.10) in the new notation:
$$\bar{u}_\alpha \in U_{\mathrm{ad}}, \quad (\alpha\bar{u}_\alpha + B^*\bar{p}_\alpha, u - \bar{u}_\alpha)_U \ge 0 \quad \forall\ u \in U_{\mathrm{ad}}.$$
We now consider for some $\alpha, \alpha' \ge 0$ this inequality once with $(\alpha, u) := (\alpha, \bar{u}_{\alpha'})$, and once with $(\alpha, u) := (\alpha', \bar{u}_\alpha)$. Adding both, we obtain

$$(\alpha\bar{u}_\alpha - \alpha'\bar{u}_{\alpha'} + B^*(\bar{p}_\alpha - \bar{p}_{\alpha'}), \bar{u}_{\alpha'} - \bar{u}_\alpha)_U \geq 0. \tag{1.32}$$

We rewrite this inequality as

$$\forall\alpha \geq 0\ \forall\alpha' \geq 0 : \|\bar{y}_{\alpha'} - \bar{y}_\alpha\|_H^2 + \alpha'\|\bar{u}_{\alpha'} - \bar{u}_\alpha\|_U^2 \leq (\alpha - \alpha')(\bar{u}_\alpha, \bar{u}_{\alpha'} - \bar{u}_\alpha)_U,$$

which gives the desired continuity.

2. and 3. Taking $\alpha' = 0$ we can estimate

$$\|\bar{y}_0 - \bar{y}_\alpha\|_H^2 \leq \alpha\|\bar{u}_\alpha\|_U\|\bar{u}_0 - \bar{u}_\alpha\|_U. \tag{1.33}$$

Thus with the help of (1.31), we conclude

$$\lim_{\alpha\to 0} \bar{y}_\alpha = \bar{y}_0. \tag{1.34}$$

Let $(\alpha_n)$ be a sequence with $\alpha_n \to 0$ for $n \to \infty$. From (1.31) we get the existence of an element $\tilde{u}_0$ and a subsequence of $(\alpha_n)$, again denoted by $(\alpha_n)$, with $\bar{u}_{\alpha_n} \rightharpoonup \tilde{u}_0$. Since $U_{\text{ad}}$ is closed and convex, it is weakly sequentially closed. We thus have $\tilde{u}_0 \in U_{\text{ad}}$. From weak continuity of $T$ we conclude with the help of (1.34): $T\tilde{u}_0 = T\bar{u}_0$. Since $\bar{u}_0$ is the unique solution to problem $(\mathbb{P}_0)$, we conclude $\tilde{u}_0 = \bar{u}_0$, thus $\bar{u}_\alpha \rightharpoonup \bar{u}_0$ as $\alpha \to 0$. Strong convergence now follows from (1.31), as shows

$$\|\bar{u}_\alpha - \bar{u}_0\|_U^2 = \|\bar{u}_\alpha\|_U^2 + \|\bar{u}_0\|_U^2 - 2(\bar{u}_\alpha, \bar{u}_0)_U \leq 2(\bar{u}_0 - \bar{u}_\alpha, \bar{u}_0)_U \overset{\alpha\to 0}{\longrightarrow} 0.$$

We have thus shown

$$\lim_{\alpha\to 0} \bar{u}_\alpha = \bar{u}_0, \tag{1.35}$$

which, together with (1.33), gives $\|\bar{y}_\alpha - \bar{y}_0\|_H = o(\sqrt{\alpha})$.

4. Let us now consider the Lipschitz continuity with respect to the data $z$. We get in the same way as (1.32) the inequality

$$(\alpha(\bar{u}_\alpha - \bar{u}'_\alpha) + B^*(\bar{p}_\alpha - \bar{p}'_\alpha), \bar{u}'_\alpha - \bar{u}_\alpha)_U \geq 0,$$

which can be rewritten as

$$\|\bar{y}'_\alpha - \bar{y}_\alpha\|_H^2 + \alpha\|\bar{u}'_\alpha - \bar{u}_\alpha\|_U^2 \leq (z' - z, \bar{y}'_\alpha - \bar{y}_\alpha)_H.$$

This gives the desired estimate of the Lipschitz continuity at once. $\square$

### 1.4.2 Refined convergence rates under additional assumptions

We now consider better regularization error estimates which are tailored to optimal control problems rather then general inverse problems. We use ideas from recent results for elliptic optimal control problems to derive estimates for the parabolic case. We improve known results with the help of an $L^1$-norm estimate, which usefulness is revisited also later in the analysis of the discretization error.

From now onwards we assume

$$a \leq 0 \leq b \tag{1.36}$$

in a pointwise almost everywhere sense where $a$ and $b$ are the bounds of the admissable set $U_{\text{ad}}$. For $(\mathbb{P}_0)$, the problem we finally want to solve, this assumption can always be met by a simple transformation of the variables.

To prove better rates of convergence with respect to $\alpha$, we rely on the following assumption.

**Assumption 15.** *Let distributed controls be given. There exist a set $A \subset \Omega_U$, a function $w \in H$ with $T^*w \in L^\infty(\Omega_U, \mathbb{R}^D)$, and constants $\kappa > 0$ and $C \geq 0$, such that there holds the inclusion $\{x \in \Omega_U \mid B^*\bar{p}_0(x) = 0\} \subset A^c$ for the complement $A^c = \Omega_U \backslash A$ of $A$ and in addition*

1. *(source condition)*

$$\chi_{A^c}\bar{u}_0 = \chi_{A^c} P_{U_{\text{ad}}}(T^*w). \tag{1.37}$$

2. *(($\bar{p}_0$-)measure condition)*

$$\forall\, \epsilon > 0: \quad \text{meas}(\{x \in A \mid 0 \leq |B^*\bar{p}_0(x)| \leq \epsilon\}) \leq C\epsilon^\kappa \tag{1.38}$$

*with the convention that $\kappa := \infty$ if the left-hand side of* (1.38) *is zero for some $\epsilon > 0$.*

*In the case of located controls, the preceding conditions have to be fulfilled by each of the $D$ components of $\bar{u}_0$ – add a subscript index $i \in \{1, \ldots, D\}$ to $B^*\bar{p}_0$, $U_{\text{ad}}$, $T^*$, and $\bar{u}_0$.*

Source conditions of the form $\bar{u}_0 = P_{U_{\text{ad}}}(T^*w)$ are well known in the theory of inverse problems with convex constraints, see [Neu86] and [EHN00].

However, since they are usually posed almost everywhere, thus globally, they are unlikely to hold in the optimal control setting. For example, the condition $\bar{u}_0 = P_{U_{\text{ad}}}(T^*w)$ together with continuous bounds $a$ and $b$ implies in our parabolic situation (compare (1.26) in Lemma 12) that $\bar{u}_0$ is continuous, too. However, discontinuous controls in the case $\alpha = 0$ are often observed, see, e.g., the test examples in the numerics chapter later. Therefore a localized variant of the general source condition is more useful.

Similar measure conditions were previously used for control problems with elliptic PDEs, starting with the analysis in [WW11a] and [DH12]. In the latter paper, Deckelnick and Hinze used the measure condition to derive a-priori error estimates for discretization errors of $(\mathbb{P}_0)$.

A condition related to the measure condition was also used to establish stability results for bang-bang control problems with autonomous ODEs, see [Fel03, Assumption 2]. There, a condition on the gradient of $\bar{p}_0$ is imposed, thus no measure enters the formulation. The measure condition can be interpreted as a weakening of this gradient condition, as was shown in [DH12, Lemma 3.2].

In all above-mentioned references, the measure condition was assumed to hold globally, i.e., Assumption 15 holds with $\text{meas}(A^c) = 0$. Together with (1.14) one immediately observes that this implies bang-bang controls.

The combination of both conditions in Assumption 15 turned out to be very useful in the context of elliptic optimal control problems, see [WW11b; WW13]. Although we are actually interested in the investigation of bang-bang controls, we use this more general condition due to the low additional effort.

Key ingredient in our analysis of the regularization error and also of the discretization error considered later is the following lemma, which is extracted from the proof of [WW11b, Theorem 3.20]. For its origins see also the discussion at the end of the bibliography.

**Lemma 16.** *Let Assumption 15.2 hold. For the solution $\bar{u}_0$ of $(\mathbb{P}_0)$, there holds with some constant $C > 0$ independent of $\alpha$ and $u$*

$$C\|u - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} \leq (B^*\bar{p}_0, u - \bar{u}_0)_U \quad \forall\, u \in U_{\text{ad}}. \tag{1.39}$$

*Proof.* Let us consider distributed controls first.

For $\epsilon > 0$, we define $B_\epsilon := \{x \in A \mid |B^*\bar{p}_0| \geq \epsilon\}$. Using the (pointwise) optimality condition (1.15) and Assumption 15.2, we conclude for some

$u \in U_{\mathrm{ad}}$

$$\begin{aligned}\int_{\Omega_U} (B^*\bar{p}_0, u - \bar{u}_0)_{\mathbb{R}} = \int_{\Omega_U} |B^*\bar{p}_0||u - \bar{u}_0| &\geq \epsilon \|u - \bar{u}_0\|_{L^1(B_\epsilon)} \\ &\geq \epsilon \|u - \bar{u}_0\|_{L^1(A)} - \epsilon \|u - \bar{u}_0\|_{L^1(A \backslash B_\epsilon)} \\ &\geq \epsilon \|u - \bar{u}_0\|_{L^1(A)} - \epsilon \|u - \bar{u}_0\|_{L^\infty(\Omega_U)} \operatorname{meas}(A \backslash B_\epsilon) \\ &\geq \epsilon \|u - \bar{u}_0\|_{L^1(A)} - c\epsilon^{\kappa+1} \|u - \bar{u}_0\|_{L^\infty(\Omega_U)}\end{aligned}$$

where without loss of generality $c > 1$.

Setting $\epsilon := c^{-2/\kappa} \|u - \bar{u}_0\|_{L^1(A)}^{1/\kappa} \|u - \bar{u}_0\|_{L^\infty(\Omega_U)}^{-1/\kappa}$ yields

$$\int_{\Omega_U} (B^*\bar{p}_0, u - \bar{u}_0)_{\mathbb{R}} \geq \tilde{C} \|u - \bar{u}_0\|_{L^1(A)}^{1+1/\kappa},$$

since $\|u - \bar{u}_0\|_{L^\infty(\Omega_U)} \leq C$ for some $C = C(a, b) > 0$ independent of $u$ by the definition of $U_{\mathrm{ad}}$.

In the case of located controls, observe first that (1.39) is valid for each component $\bar{u}_0^i$, $1 \leq i \leq D$, of $\bar{u}_0$. This can be shown as above. From this and the estimate

$$\begin{aligned}\left( \int_A \left( \sum_{i=1}^{D} |u - \bar{u}_0^i|^D \right)^{1/D} \right)^{1+1/\kappa} &\leq \left( C \sum \int |u - \bar{u}_0^i| \right)^{1+1/\kappa} \\ &\leq C \sum \left( \int |u - \bar{u}_0^i| \right)^{1+1/\kappa},\end{aligned}$$

the claim follows at once. □

With the previous Lemma, we can now improve the inequality (1.27) (setting there $\alpha := 0$) from general inverse problem theory, since the error in the control in the $L^1$ norm now appears on the left-hand side with a factor C>0 independent of $\alpha$. This is in contrast to the error in the $L^2$ norm.

**Lemma 17.** *Let Assumption 15.2 hold (with possibly* $\operatorname{meas}(A) = 0$*). Then there holds for some $C > 0$ independent of $\alpha$*

$$\begin{aligned}\|\bar{y}_\alpha - \bar{y}_0\|_H^2 + C \|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A, \mathbb{R}^D)}^{1+1/\kappa} + \alpha \|\bar{u}_\alpha - \bar{u}_0\|_U^2 \\ \leq \alpha (\bar{u}_0, \bar{u}_0 - \bar{u}_\alpha)_U \quad \forall\ \alpha > 0.\end{aligned}$$

*Proof.* Adding the necessary condition for $\bar{u}_\alpha$ (1.10) with $u := \bar{u}_0$, i.e.,

$$0 \leq (\alpha \bar{u}_\alpha + B^* \bar{p}_\alpha, \bar{u}_0 - \bar{u}_\alpha)_U \,,$$

to the estimate (1.39) of Lemma 16 with $u := \bar{u}_\alpha$, we get

$$\begin{aligned} C\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} &\leq (B^*(\bar{p}_\alpha - \bar{p}_0), \bar{u}_0 - \bar{u}_\alpha)_U + \alpha(\bar{u}_\alpha, \bar{u}_0 - \bar{u}_\alpha)_U \\ &\leq -\|\bar{y}_\alpha - \bar{y}_0\|_H^2 + \alpha(\bar{u}_\alpha - \bar{u}_0, \bar{u}_0 - \bar{u}_\alpha)_U \\ &\qquad + \alpha(\bar{u}_0, \bar{u}_0 - \bar{u}_\alpha)_U \\ &\leq -\|\bar{y}_\alpha - \bar{y}_0\|_H^2 - \alpha\|\bar{u}_\alpha - \bar{u}_0\|_U^2 + \alpha(\bar{u}_0, \bar{u}_0 - \bar{u}_\alpha)_U. \end{aligned}$$

□

The following Lemma is extracted from the proof of [WW11b, Lemma 3.9]. It shows how the source condition (Assumption 15.1) is taken into account to reduce the error estimate to the set $A$.

**Lemma 18.** *Let Assumption 15.1 (source condition) be satisfied. Then there holds with a constant $C > 0$*

$$(\bar{u}_0, \bar{u}_0 - u)_U \leq C(\|T(u - \bar{u}_0)\|_H + \|u - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}) \quad \forall\, u \in U_{\mathrm{ad}}.$$

*Proof.* The source condition is equivalent to

$$0 \leq (\chi_{A^c}(\bar{u}_0 - T^* w), u - \bar{u}_0)_U \quad \forall\, u \in U_{\mathrm{ad}}.$$

Using this representation, we can estimate

$$\begin{aligned} (\bar{u}_0, \bar{u}_0 - u)_U &\leq (\chi_{A^c} T^* w, \bar{u}_0 - u)_U + (\chi_A \bar{u}_0, \bar{u}_0 - u)_U \\ &\leq (w, T(\bar{u}_0 - u))_H + (-T^* w + \bar{u}_0, \chi_A\,(\bar{u}_0 - u))_U \,. \end{aligned}$$

Since $T^* w \in L^\infty(\Omega_U, \mathbb{R}^D)$, we get the claim. □

Using this Lemma, we can now state regularization error estimates. We consider different situations with respect to the fulfillment of parts of Assumption 15.

**Theorem 19.** *For the regularization error there holds with positive constants $c$ and $C$ indepent of $\alpha > 0$ the following.*

1. *The error in the optimal state fulfills the rate of convergence*

$$\|\bar{y}_\alpha - \bar{y}_0\|_H = o(\sqrt{\alpha}).$$

2. *Let Assumption 15.1 be satisfied with* $\text{meas}(A) = 0$ *(source condition holds a.e. on the domain). Then the optimal control converges with the rate*

$$\|\bar{u}_\alpha - \bar{u}_0\|_U \leq C\sqrt{\alpha}, \tag{1.40}$$

*and the optimal state converges with the improved rate*

$$\|\bar{y}_\alpha - \bar{y}_0\|_H \leq C\alpha. \tag{1.41}$$

3. *Let Assumption 15.2 be satisfied with* $\text{meas}(A^c) = 0$ *(measure condition holds a.e. on the domain). Then the estimates*

$$\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(\Omega_U,\mathbb{R}^D)} \leq C\alpha^\kappa \tag{1.42}$$

$$\|\bar{u}_\alpha - \bar{u}_0\|_U \leq C\alpha^{\kappa/2} \tag{1.43}$$

$$\|\bar{y}_\alpha - \bar{y}_0\|_H \leq C\alpha^{(\kappa+1)/2} \tag{1.44}$$

*hold true. If* $\kappa > 1$ *holds and in addition*

$$T^* : \text{range}(T) \to L^\infty(\Omega_U, \mathbb{R}^D) \qquad \textit{exists and is continuous,} \tag{1.45}$$

*we can improve* (1.44) *to*

$$\|\bar{y}_\alpha - \bar{y}_0\|_H \leq C\alpha^\kappa. \tag{1.46}$$

4. *Let Assumption 15 be satisfied with* $\text{meas}(A) \cdot \text{meas}(A^c) > 0$ *(source and measure condition on parts of the domain). Then the following estimates hold true.*

$$\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)} \leq C\alpha^{\min(\kappa,\, \frac{2}{1+1/\kappa})} \tag{1.47}$$

$$\|\bar{u}_\alpha - \bar{u}_0\|_U \leq C\alpha^{\min(\kappa,\, 1)/2} \tag{1.48}$$

$$\|\bar{y}_\alpha - \bar{y}_0\|_H \leq C\alpha^{\min((\kappa+1)/2,\, 1)} \tag{1.49}$$

*Proof.* 1. The estimate is just a repetition of (1.29).

3. Let us recall the estimates of Lemma 17, i.e.,

$$\|\bar{y}_\alpha - \bar{y}_0\|_H^2 + C\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} + \alpha\|\bar{u}_\alpha - \bar{u}_0\|_U^2 \leq \alpha(\bar{u}_0, \bar{u}_0 - \bar{u}_\alpha)_U. \quad (1.50)$$

By Young's inequality we can estimate

$$\alpha\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)} \leq C\alpha^{\kappa+1} + C\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa}. \quad (1.51)$$

If $A = \Omega_U$ up to a set of measure zero, we can combine both estimates, since $\bar{u}_0 \in U_{\text{ad}} \subset L^\infty$, and move the second summand of the just mentioned estimate to the left. This yields the claim since

$$\frac{\kappa + 1}{1 + 1/\kappa} = \kappa.$$

The improved estimate (1.46) can be obtained with the help of (1.42) as follows

$$\begin{aligned}\|\bar{y}_\alpha - \bar{y}_0\|_H^2 &= (T^*(\bar{y}_\alpha - \bar{y}_0), \bar{u}_\alpha - \bar{u}_0)_U \leq C\|T^*(\bar{y}_\alpha - \bar{y}_0)\|_{L^\infty}\|\bar{u}_\alpha - \bar{u}_0\|_{L^1} \\ &\leq C\|\bar{y}_\alpha - \bar{y}_0\|_H\|\bar{u}_\alpha - \bar{u}_0\|_{L^1} \leq C\|\bar{y}_\alpha - \bar{y}_0\|_H\, \alpha^\kappa.\end{aligned}$$

2.+4. We combine (1.50) with the estimate of Lemma 18 (with $u := \bar{u}_\alpha$), invoke Cauchy's inequality and get

$$\begin{aligned}&\|\bar{y}_\alpha - \bar{y}_0\|_H^2 + C\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} + \alpha\|\bar{u}_\alpha - \bar{u}_0\|_U^2 \\ &\quad\leq \alpha(\bar{u}_0, \bar{u}_0 - \bar{u}_\alpha)_U \leq C\alpha(\|\bar{y}_\alpha - \bar{y}_0\|_H + \|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}) \\ &\qquad\leq C\alpha^2 + \frac{1}{2}\|\bar{y}_\alpha - \bar{y}_0\|_H^2 + C\alpha\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}.\end{aligned}$$

We now move the second addend to the left.

If $\text{meas}(A) = 0$ (case 2.), we are done. Otherwise (case 4.) we continue estimating, making use of (1.51), to get

$$\|\bar{y}_\alpha - \bar{y}_0\|_H^2 + C\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} + \alpha\|\bar{u}_\alpha - \bar{u}_0\|_U^2 \leq C\alpha^{\min(2,\kappa+1)},$$

from which the claim follows. □

Some remarks on the previous theorem are in order.

Let us compare the first with the other cases, where Assumption 15 is taken (partially) into account. In all cases, we get an improved convergence rate for the optimal state.

The second estimate replicates a well known estimate from the theory of inverse problems with convex constraints, see, e.g., [Neu86] and [EHN00, Theorem 5.19]. However, as indicated in the discussion after Assumption 15, this situation is unlikely to hold in the context of optimal control problems.

Concerning the "min"-functions in the estimates, we note that the left argument is chosen if $\kappa < 1$, the right one if $\kappa > 1$. In the case $\kappa = 1$, both expressions coincide. Thus the worse part of Assumption 15 with respect to the items 2. and 3. dominates the convergence behavior of the regularization errors.

As mentioned after Assumption 15, case 3. implies bang-bang controls.

By Lemma 12 and Remark 13 we can immediately see that the assumption (1.45) on $T^*$ is fulfilled for our parabolic problem. We even more get the estimate

$$\|\bar{p}_\alpha - \bar{p}_0\|_{L^\infty(\Omega \times I)} \le C\|\bar{y}_\alpha - \bar{y}_0\|_H \le C\alpha^\kappa$$

for the optimal adjoint state.

Let us finally mention that the cases 3. and 4. unify (with respect to $\kappa$) and improve (for $\kappa < 1$) recently obtained regularization estimates from [WW11b, section 3.3].

### 1.4.3 Necessity of the additional assumptions

Let us now consider the question of necessity of Assumption 15 to obtain convergence rates, thus a converse of Theorem 19.

We first show that a convergence rate $\|\bar{y}_\alpha - \bar{y}_0\|_H \le C\alpha$ implies the source condition (1.37) to hold at least on $\{x \in \Omega_U \mid B^*\bar{p}_0(x) = 0\}$.

The following Theorem is mainly taken from [WW13, Theorem 4]. It resembles a necessity result known from inverse problem theory, see, e.g., [EHN00, Theorem 5.19] or [Neu86]. However, in inverse problems, the condition $T\bar{u}_0 = z$ is typically assumed.

**Theorem 20.** *If we assume a convergence rate $\|\bar{y}_\alpha - \bar{y}_0\|_H = \mathcal{O}(\alpha)$, then there exists a function $w \in H$ such that $\bar{u}_0 = P_{U_{\mathrm{ad}}}(T^*w)$ holds pointwise* *a.e. on* $K := \{x \in \Omega_U \mid B^*\bar{p}_0(x) = 0\}$*. Thus* (1.37) *holds on* $K$ *instead of* $A^c$*.*

*If even* $\|\bar{y}_\alpha - \bar{y}_0\|_H = o(\alpha)$*, then* $\bar{u}_0$ *vanishes on* $K$*.*

*Proof.* Let us first define a test function $\hat{u} \in U_{\mathrm{ad}}$ for $x \in \Omega_U$ by

$$\hat{u}(x) \begin{cases} = a(x) & \text{if } B^*\bar{p}_0(x) > 0, \\ \in [a(x), b(x)] & \text{if } B^*\bar{p}_0(x) = 0, \\ = b(x) & \text{if } B^*\bar{p}_0(x) < 0. \end{cases}$$

We consider the necessary condition (1.10) for $\bar{u}_\alpha$, i.e.,

$$(\alpha\bar{u}_\alpha + B^*\bar{p}_\alpha, u - \bar{u}_\alpha)_U \geq 0 \quad \forall\, u \in U_{\mathrm{ad}},$$

for the special case $u = \hat{u}$ and add the necessary condition for $\bar{u}_0$, evaluated at $u = \bar{u}_\alpha$. We obtain

$$0 \leq \left(\alpha\bar{u}_\alpha + B^*\left(\bar{p}_\alpha - \bar{p}_0\right), \hat{u} - \bar{u}_\alpha\right)_U + \left(B^*\bar{p}_0, \hat{u} - \bar{u}_\alpha + \bar{u}_\alpha - \bar{u}_0\right)_U .$$

By construction of $\hat{u}$ and the representation of $\bar{u}_0$ from Lemma 5, we conclude that the second scalar product vanishes. Thus we end up with

$$0 \leq \left(\alpha\bar{u}_\alpha + T^*T\left(\bar{u}_\alpha - \bar{u}_0\right), \hat{u} - \bar{u}_\alpha\right)_U ,$$

and dividing the expression by $\alpha$ and taking the limit we get with the help of (1.28) the inequality

$$0 \leq (T^*\dot{y}_0 + \bar{u}_0, \hat{u} - \bar{u}_0)_U$$

for any weak subsequential limit $\dot{y}_0$ of $\frac{1}{\alpha}(\bar{y}_\alpha - \bar{y}_0)$, which exists due to the assumption of the Theorem. The first assertion is now a direct consequence of the construction of $\hat{u}$.

The second assertion follows from the equality $\dot{y}_0 = 0$ in case of $\|\bar{y}_\alpha - \bar{y}_0\|_H = o(\alpha)$. □

**Remark 21.** *In the assumptions of the previous Theorem, one can replace the norm* $\|\bar{y}_\alpha - \bar{y}_0\|_H$ *by the norm* $\|B^*(\bar{p}_\alpha - \bar{p}_0)\|_{L^2(K,\mathbb{R}^D)}$*, which follows from the proof.*

We next show that if (1.45) and $\kappa > 1$ hold true, convergence as in Theorem 19.3 implies the measure condition (1.38).

**Theorem 22.** *Let us assume* $\{x \in \Omega_U \mid B^*\bar{p}_0(x) = 0\} \subset A^c$ *for some given set* $A \subset \Omega_U$. *Let us further assume that*

$$\exists\ \sigma > 0\ \forall'\ x \in \Omega_U : \quad a \leq -\sigma < 0 < \sigma \leq b. \tag{1.52}$$

*If* $\kappa > 1$ *and convergence rates* $\|\bar{u}_\alpha - \bar{u}_0\|^p_{L^p(A,\mathbb{R}^D)} + \|B^*(\bar{p}_\alpha - \bar{p}_0)\|_{L^\infty(A,\mathbb{R}^D)} \leq C\alpha^\kappa$ *are known to hold for some real* $p \geq 1$, *then the measure condition* (1.38) *from Assumption 15 is fulfilled.*

*Proof.* We consider distributed controls only. The case of located controls is obtained in the same way by considering the $D$ component functions of the involved functions.

Let us split the set $A$ into the subsets

$$\begin{aligned}
A_0 &:= \{x \in A \mid -B^*\bar{p}_0 < 0 \text{ and } \alpha a \geq -B^*\bar{p}_\alpha\}, \\
A_1 &:= \{x \in A \mid -B^*\bar{p}_0 < 0 \text{ and } \alpha a < -B^*\bar{p}_\alpha < \alpha b\}, \\
A_2 &:= \{x \in A \mid -B^*\bar{p}_0 < 0 < \alpha b \leq -B^*\bar{p}_\alpha\}, \\
A_3 &:= \{x \in A \mid -B^*\bar{p}_0 > 0 \text{ and } \alpha a < -B^*\bar{p}_\alpha < \alpha b\}, \\
A_4 &:= \{x \in A \mid -B^*\bar{p}_0 > 0 > \alpha a \geq -B^*\bar{p}_\alpha\}, \quad \text{and} \\
A_5 &:= \{x \in A \mid -B^*\bar{p}_0 > 0 \text{ and } \alpha b \leq -B^*\bar{p}_\alpha\}.
\end{aligned}$$

Thus $A = A_0 \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$, and from Lemma 5 we infer

$$\begin{aligned}
\int_A |\bar{u}_0 - \bar{u}_\alpha|^p &= \int_{A_1} |a + \alpha^{-1}B^*\bar{p}_\alpha|^p + \int_{A_3} |b + \alpha^{-1}B^*\bar{p}_\alpha|^p + \int_{A_2 \cup A_4} |a - b|^p \\
&\geq \int_{A_1} |a + \alpha^{-1}B^*\bar{p}_\alpha|^p + \int_{A_3} |b + \alpha^{-1}B^*\bar{p}_\alpha|^p \\
&\geq (\frac{\sigma}{2})^p \operatorname{meas}(\left\{x \in A \;\middle|\; |B^*\bar{p}_\alpha| \leq \frac{\sigma}{2}\alpha\right\}).
\end{aligned} \tag{1.53}$$

Thus from $\|\bar{u}_\alpha - \bar{u}_0\|^p_{L^p(A)} \leq C\alpha^\kappa$ and (1.53) we conclude

$$\operatorname{meas}(\{x \in A \mid |B^*\bar{p}_\alpha| \leq C_1\alpha\}) \leq C_2\alpha^\kappa.$$

Since $\kappa > 1$ and $\|B^*(\bar{p}_\alpha - \bar{p}_0)\|_{L^\infty(A)} \leq C\alpha^\kappa$, we get for some arbitrarily chosen $x \in A$ with $|B^*\bar{p}_0(x)| \leq \alpha C_1/2$ the estimate

$$|B^*\bar{p}_\alpha(x)| \leq |B^*\bar{p}_0(x)| + |B^*(\bar{p}_\alpha - \bar{p}_0)(x)| \leq \frac{C_1}{2}(\alpha + \alpha^{\kappa-\epsilon}) \leq C_1\alpha$$

for some sufficiently small $\epsilon = \epsilon(C_1, \kappa) > 0$. Consequently, we have

$$\text{meas}(\left\{ x \in A \;\middle|\; |B^*\bar{p}_0| \leq \frac{C_1}{2}\alpha \right\}) \leq C_2\alpha^\kappa.$$

□

Concerning the previous Theorem, let us mention the related result [WW13, Theorem 8]. It has the same implication, but assumes (1.43) and (1.44), which imply the prerequisites of Theorem 22 in case of (1.45).

For the case $\kappa \leq 1$, it is an open question whether the previous Theorem (and likewise [WW13, Theorem 8]) is valid.

Let us also note that the assumption $a \leq -\sigma < 0 < \sigma \leq b$ in the previous Theorem can be replaced by the weaker one

$$a \leq -\sigma < 0 \text{ on } B^*\bar{p}_0 > 0 \quad \text{and} \quad 0 < \sigma \leq b \text{ on } B^*\bar{p}_0 < 0,$$

as an inspection of the previous proof shows.

### 1.4.4 On the time derivative of the regularized control for bang-bang solutions

In this subsection, we consider bang-bang solutions, i.e.,

$$\text{meas}(\{x \in \Omega_U \mid B^*\bar{p}_0(x) = 0\}) = 0. \tag{1.54}$$

We introduce a second measure condition. This new condition implies the same convergence results as in Theorem 19.3, thus it can replace the $\bar{p}_0$-measure condition (1.38) from Assumption 15.

We then show that the new condition is almost necessary to obtain these convergence rates.

Finally, it turns out that the new and the old measure condition coincide if the limit problem is of certain regularity.

The reason to introduce this new condition is that it leads to an improved bound on the decay of smoothness in the derivative of the optimal control when $\alpha$ tends to zero. This bound will be useful later to derive improved convergence rates for the discretization errors.

**Definition 23** ($\bar{p}_\alpha$-measure condition)**.** *Let distributed controls be given. If for the set*

$$I_\alpha := \{x \in \Omega_U \mid \alpha a < -B^*\bar{p}_\alpha < \alpha b\} \tag{1.55}$$

*the condition*

$$\exists\ \bar{\alpha} > 0\ \forall\ 0 < \alpha < \bar{\alpha} : \quad \operatorname{meas}(I_\alpha) \leq C\alpha^\kappa \tag{1.56}$$

*holds true (with the convention that* $\kappa := \infty$ *if the measure in* (1.56) *is zero for all* $0 < \alpha < \bar{\alpha}$*), we say that the* $\bar{p}_\alpha$-measure condition *is fulfilled.*

*In the case of located controls, the modifications mentioned in Assumption 15 have to be applied.*

The equality in the estimate (1.53) from the proof of Theorem 22 shows that if the $\bar{p}_\alpha$-measure condition holds and we assume the additional condition $\operatorname{meas}(A_2 \cup A_4) \leq C\alpha^\kappa$ (with $A_i$ as in that proof), we get the convergence rate $\|\bar{u}_\alpha - \bar{u}_0\|^p_{L^p(A,\mathbb{R}^D)} \leq C\alpha^\kappa$ for each $1 \leq p < \infty$.

Interestingly, this additional condition is not needed to obtain convergence in the control, as we will now show.

**Theorem 24.** *If the* $\bar{p}_\alpha$*-measure condition* (1.56) *and the conditions* (1.52) *and* (1.45) *are fulfilled, the convergence rates*

$$\|\bar{u}_\alpha - \bar{u}_0\|_{L^1(\Omega_U,\mathbb{R}^D)} \leq C\alpha^\kappa \quad \text{and} \quad \|\bar{y}_\alpha - \bar{y}_0\|_I \leq C\alpha^{(\kappa+1)/2} \tag{1.57}$$

*hold true. If* $\kappa > 1$*, we have the improved estimate*

$$\|\bar{y}_\alpha - \bar{y}_0\|_I \leq C\alpha^\kappa. \tag{1.58}$$

*Proof.* We consider distributed controls only. The case of located controls is obtained in the same way by considering the $D$ component functions of the involved functions.

Let $u \in U_{\text{ad}}$ be arbitrarily chosen. For the active set $I^c_\alpha$ of $\bar{p}_\alpha$, which is the complement of the inactive set $I_\alpha$ defined in (1.55), we have by Lemma 5, making use of (1.52), the estimate

$$(B^*\bar{p}_\alpha, u - \bar{u}_\alpha)_{I^c_\alpha} = \int_{I^c_\alpha} |B^*\bar{p}_\alpha||u - \bar{u}_\alpha| \geq \sigma\alpha\|u - \bar{u}_\alpha\|_{L^1(I^c_\alpha)}. \tag{1.59}$$

Invoking the $\bar{p}_\alpha$-measure condition (1.56), we get on the inactive set itself the estimate

$$|(B^*\bar{p}_\alpha, u - \bar{u}_\alpha)_{I_\alpha}| \leq C\alpha\|u - \bar{u}_\alpha\|_{L^1(I_\alpha)} \leq CC_{ab}\alpha^{\kappa+1} \tag{1.60}$$

with $C_{ab} = \max(\|a\|_\infty, \|b\|_\infty)$. Consequently, with $L^1 := L^1(\Omega_U)$ we get

$$\begin{aligned}\sigma\alpha\|u-\bar{u}_\alpha\|_{L^1} - \alpha^{\kappa+1} &\overset{(1.56)}{\leq} \sigma\alpha\|u-\bar{u}_\alpha\|_{L^1} - \sigma\alpha\|u-\bar{u}_\alpha\|_{L^1(I_\alpha)} \\ &= \sigma\alpha\|u-\bar{u}_\alpha\|_{L^1(I_\alpha^c)} \\ &\overset{(1.59)}{\leq} (B^*\bar{p}_\alpha, u-\bar{u}_\alpha)_{I_\alpha^c} \\ &= (B^*\bar{p}_\alpha, u-\bar{u}_\alpha) - (B^*\bar{p}_\alpha, u-\bar{u}_\alpha)_{I_\alpha} \\ &\overset{(1.60)}{\leq} (B^*\bar{p}_\alpha, u-\bar{u}_\alpha) + C\alpha^{\kappa+1}.\end{aligned} \tag{1.61}$$

Rearranging terms, we conclude

$$\sigma\alpha\|u-\bar{u}_\alpha\|_{L^1} \leq (B^*\bar{p}_\alpha, u-\bar{u}_\alpha) + C\alpha^{\kappa+1}. \tag{1.62}$$

Taking $u := \bar{u}_0$ in the previous equation and adding the necessary condition (1.10) for $\bar{u}_0$ for the special case $u := \bar{u}_\alpha$, i.e.,

$$(-B^*\bar{p}_0, \bar{u}_0 - \bar{u}_\alpha) \geq 0, \tag{1.63}$$

we get the estimate

$$\begin{aligned}\sigma\alpha\|\bar{u}_0-\bar{u}_\alpha\|_{L^1} &\leq (B^*(\bar{p}_\alpha - \bar{p}_0), \bar{u}_0-\bar{u}_\alpha) + C\alpha^{\kappa+1} \\ &= -\|\bar{y}_\alpha - \bar{y}_0\|_I^2 + C\alpha^{\kappa+1},\end{aligned} \tag{1.64}$$

from which the claim follows.

The improved estimate can be established as in the proof of Theorem 19.

□

The $\bar{p}_\alpha$-measure condition (1.56) is slightly stronger than what actually is necessary in order to obtain convergence rates.

**Corollary 25.** *Let us assume* $\{x \in \Omega_U \mid B^*\bar{p}_0(x) = 0\} \subset A^c$ *for some given set* $A \subset \Omega_U$. *Let us further assume that* (1.52) *is valid.*

*If the convergence rate* $\|\bar{u}_\alpha - \bar{u}_0\|^p_{L^p(A,\mathbb{R}^D)} \leq C\alpha^\kappa$ *is known to hold for some real* $p \geq 1$ *and some real* $\kappa > 0$, *then the measure condition*

$$\mathrm{meas}(\{x \in A \mid \alpha(a+\epsilon) \leq -B^*\bar{p}_\alpha(x) \leq \alpha(b-\epsilon)\}) \leq \frac{C}{\epsilon^p}\alpha^\kappa \tag{1.65}$$

*is fulfilled for each* $0 < \epsilon < \sigma$.

*Proof.* This follows from the proof of Theorem 22. □

If the limit problem is of certain regularity, the $\bar{p}_\alpha$-measure condition is not stronger than the $\bar{p}_0$-measure condition, and, as we show afterwards, both conditions coincide.

**Lemma 26.** *Let Assumption 15 hold with* $\operatorname{meas}(A^c) = 0$ *($\bar{p}_0$-measure condition is valid a.e. on $\Omega_U$). Let furthermore $\kappa \geq 1$ and* (1.45) *be valid. Then the $\bar{p}_\alpha$-measure condition* (1.56) *is fulfilled.*

*Proof.* Since the set $I_\alpha$ from (1.55) fulfills $I_\alpha \subset \{x \in \Omega_U \mid |B^*\bar{p}_\alpha| \leq C\alpha\}$ with $C = \max(\|a\|_\infty, \|b\|_\infty)$, we conclude with (1.45) and Theorem 19 that if $x \in I_\alpha$ and $\kappa \geq 1$, we have

$$|B^*\bar{p}_0| \leq |B^*\bar{p}_\alpha| + |B^*(\bar{p}_0 - \bar{p}_\alpha)| \leq C\alpha.$$

With the $\bar{p}_0$-measure condition (1.38) we obtain the estimate

$$\operatorname{meas}(I_\alpha) \leq \operatorname{meas}(\{x \in \Omega_U \mid |B^*\bar{p}_0| \leq C\alpha\}) \leq C\alpha^\kappa,$$

which concludes the proof. □

**Corollary 27.** *Let a bang-bang solution be given, i.e.,* (1.54) *holds true. In the case of $\kappa > 1$,* (1.45)*, and* (1.52)*, both measure conditions are equivalent.*

*Proof.* One direction of the claim, namely "$\bar{p}_0$-m.c. $\Rightarrow \bar{p}_\alpha$-m.c.", has already been shown in Lemma 26.

For the other direction, we know from Theorem 24 that the convergence rates (1.57) hold, which by (1.45) and Theorem 22 imply the $\bar{p}_0$-measure condition. □

Let us now consider located controls. Since $\bar{p}_\alpha \in C^1(\bar{I}, L^2(\Omega))$ for $\alpha \geq 0$ by Lemma 12 and Remark 13, we conclude

$$\|\partial_t B^*\bar{p}_\alpha\|_{L^\infty(I,\mathbb{R}^D)} \leq C\|\partial_t\bar{p}_\alpha\|_{L^\infty(I,L^2(\Omega))} \leq C + C\|\bar{u}_\alpha\|_U \leq C$$

with a constant $C > 0$ independent of $\alpha$ due to the definition of $U_{\text{ad}}$. With this estimate, the projection formula (1.12) and (the proof of) Lemma 11 we obtain with $L := L^\infty(\Omega_U, \mathbb{R}^D)$ the bound

$$\|\partial_t\bar{u}_\alpha\|_L \leq \frac{1}{\alpha}\|\partial_t B^*\bar{p}_\alpha\|_L + \|\partial_t a\|_L + \|\partial_t b\|_L \leq C\frac{1}{\alpha}, \tag{1.66}$$

if $\alpha > 0$ is sufficiently small.

If the $\bar{p}_\alpha$-measure condition (1.56) is valid, this decay of smoothness in terms of $\alpha$ can be relaxed in weaker norms, as the following Lemma shows.

**Lemma 28** (Smoothness decay in the derivative)**.** *Let the $\bar{p}_\alpha$-measure condition* (1.56) *be fulfilled and located controls be given. Then there holds with $C_{ab} = \|\partial_t a\|_{L^\infty(\Omega_U,\mathbb{R}^D)} + \|\partial_t b\|_{L^\infty(\Omega_U,\mathbb{R}^D)}$ for sufficiently small $\alpha > 0$ and $1 \leq p < \infty$ the inequality*

$$\|\partial_t \bar{u}_\alpha\|_{L^p(\Omega_U,\mathbb{R}^D)} \leq C \max(C_{ab}, \alpha^{\kappa/p-1}) \tag{1.67}$$

*with a constant $C > 0$ independent of $\alpha$. Note that $C_{ab} = 0$ in the case of constant control bounds $a$ and $b$.*

*Proof.* We invoke (1.56) and (1.66) to get the estimate

$$\begin{aligned} \|\partial_t \bar{u}_\alpha\|^p_{L^p(\Omega_U,\mathbb{R}^D)} &\leq \operatorname{meas}(I_\alpha)\|\partial_t \bar{u}_\alpha\|^p_{L^\infty(\Omega_U,\mathbb{R}^D)} + \operatorname{meas}(\Omega_U) C^p_{ab} \\ &\leq C \max(\alpha^{\kappa-p}, C^p_{ab}) \end{aligned}$$

with the set $I_\alpha$ from (1.55). □

# 2 The discretized problem

For the numerical treatment of problem ($\mathbb{P}$) we introduce finite element discretizations of the state equation and the adjoint equation. In a first step we only discretize in time. We use piecewise linear continuous Ansatz functions and piecewise constant (discontinuous) test functions for the discretization of the adjoint equation. This yields a semidiscrete Crank–Nicolson scheme. For the state equation we switch Ansatz and test space. The spatial discretization is obtained in a second step by usual conforming finite elements. We carefully seperate the discretization errors into the influences of time and space, respectively. Stability and error estimates are derived in different norms.

After that, we formulate and analyze the variational discretization of the optimal control problem. At first, estimates for the error between regularized control and discrete regularized control are shown, which are not robust if $\alpha$ tends to zero and lead to non-optimal estimates for the total error. We then derive robust estimates, which lead to better estimates for the total error if the limit problem is sufficiently regular. Finally, we improve these robust estimates further for bang-bang controls.

## 2.1 Time discretization of the state and adjoint equation

Let us as a first step consider a time discretization. Since the space variables are not touched, we remain in an infinite dimensional but semidiscrete setting.

Large parts of this section rely on recent results obtained in [DHV15], which itself is founded on the paper [MV11].

Consider a partition $0 = t_0 < t_1 < \cdots < t_M = T$ of the time interval $\bar{I}$. With $I_m = [t_{m-1}, t_m)$ we have $[0,T) = \bigcup_{m=1}^{M} I_m$. Furthermore, let $t_m^* = \frac{t_{m-1}+t_m}{2}$ for $m = 1, \ldots, M$ denote the interval midpoints. By $0 =:$

$t_0^* < t_1^* < \cdots < t_M^* < t_{M+1}^* := T$ we get a second partition of $\bar{I}$, the so-called *dual partition*, namely $[0,T) = \bigcup_{m=1}^{M+1} I_m^*$, with $I_m^* = [t_{m-1}^*, t_m^*)$. The grid width of the first mentioned (primal) partition is given by the parameters $k_m = t_m - t_{m-1}$ and

$$k = \max_{1 \leq m \leq M} k_m.$$

Here and in what follows we assume $k < 1$. We also denote by $k$ (in a slight abuse of notation) the grid itself.

For the $L^2$ stability of the operator $\pi_{P_k^*}$ given in Lemma 33, we need the following condition on sequences of time grids.

**Assumption 29.** *There exist constants* $0 < \kappa_1 \leq \kappa_2 < \infty$ *independent of* $k$ *such that*

$$\kappa_1 \leq \frac{k_m}{k_{m+1}} \leq \kappa_2$$

*holds for all* $m = 1, 2, \ldots, M-1$.

Furthermore, for the analysis of the pointwise-in-time stability and for error estimates, a second condition on sequences of time grids has to be presumed.

**Assumption 30.** *There exists a constant* $\mu > 0$ *independent of* $k$ *such that*

$$k \leq \mu \min_{m=1,2,\ldots,M} k_m.$$

On these partitions of the time interval, we define the Ansatz and test spaces of the Petrov–Galerkin schemes. These schemes will replace the continuous-in-time weak formulations of the state equation and the adjoint equation, i.e., (1.9) and (1.11), respectively. To this end, we define at first for an arbitrary Banach space $X$ the semidiscrete function spaces

$$P_k(X) := \left\{ v \in C([0,T], X) \;\middle|\; v\big|_{I_m} \in \mathcal{P}_1(I_m, X) \right\} \hookrightarrow H^1(I, X), \quad (2.1a)$$

$$P_k^*(X) := \left\{ v \in C([0,T], X) \;\middle|\; v\big|_{I_m^*} \in \mathcal{P}_1(I_m^*, X) \right\} \hookrightarrow H^1(I, X), \quad (2.1b)$$

and

$$Y_k(X) := \left\{ v : [0,T] \to X^* \;\middle|\; v\big|_{I_m} \in \mathcal{P}_0(I_m, X) \right\}. \quad (2.1c)$$

Here, $\mathcal{P}_i(J, X)$, $J \subset \bar{I}$, $i \in \{0, 1\}$, is the set of polynomial functions in time of degree at most $i$ on the interval $J$ with values in $X$. We note that functions in $P_k(X)$ can be uniquely determined by $M+1$ elements from $X$. The same holds true for functions $v \in Y_k(X)$ but with $v(T)$ only uniquely determined in $X^*$ by definition of the space. The reason for this is given in the discussion below (2.16). Furthermore, for each function $v \in Y_k(X)$ we have $[v] \in L^2(I, X)$ where $[.]$ denotes the equivalence class with respect to the almost-everywhere relation.

### 2.1.1 Interpolation operators

In the sequel, we will frequently use the following interpolation operators.

1. (Orthogonal projection) $\mathcal{P}_{Y_k(X)} : L^2(I, X) \to Y_k(X)$

$$\mathcal{P}_{Y_k(X)} v\big|_{I_m} := \frac{1}{k_m} \int_{t_{m-1}}^{t_m} v \, dt, \; m = 1, \ldots, M, \; \mathcal{P}_{Y_k(X)} v(T) := 0 \quad (2.2)$$

2. (Midpoint interpolation) $\Pi_{Y_k(X)} : C([0, T], X) \to Y_k(X)$

$$\Pi_{Y_k(X)} v\big|_{I_m} := v\left(t_m^*\right), \; m = 1, \ldots, M, \; \Pi_{Y_k(X)} v(T) := v(T). \quad (2.3)$$

3. (Piecewise linear interpolation on the dual grid) $\pi_{P_k^*(X)} : C([0, T], X) \cup Y_k(X) \to P_k^*(X)$

$$\begin{aligned} \pi_{P_k^*(X)} v\Big|_{I_1^* \cup I_2^*} &:= v(t_1^*) + \frac{t - t_1^*}{t_2^* - t_1^*} \left(v(t_2^*) - v(t_1^*)\right), \\ \pi_{P_k^*(X)} v\Big|_{I_m^*} &:= v(t_{m-1}^*) + \frac{t - t_{m-1}^*}{t_m^* - t_{m-1}^*} \big(v(t_m^*) - v(t_{m-1}^*)\big) \\ &\qquad \text{for } m = 3, \ldots, M-1, \\ \pi_{P_k^*(X)} v\Big|_{I_M^* \cup I_{M+1}^*} &:= v(t_{M-1}^*) + \frac{t - t_{M-1}^*}{t_M^* - t_{M-1}^*} \big(v(t_M^*) - v(t_{M-1}^*)\big). \end{aligned} \quad (2.4)$$

The interpolation operators are obviously linear mappings. Furthermore, they are bounded, and we have error estimates, as the following lemma shows.

In addition to the notation introduced at the beginning of section 1.3, adding a subscript $I_m$ to a norm will indicate an $L^2(I_m, L^2(\Omega))$ norm in the following. Inner products are treated in the same way.

Note that in all of the following results $C$ denotes a generic, strict positive real constant that does not depend on quantities which appear to the right or below of it.

**Lemma 31.** *For the midpoint interpolation and the orthogonal projection there holds continuity in the sense*

$$\|\Pi_{Y_k(X)} v\|_{L^2(I,X)} \leq C\sqrt{T}\,\|v\|_{C([0,T],X)} \quad \forall\, v \in C([0,T],X),$$

*and*

$$\|\mathcal{P}_{Y_k(X)} v\|_{L^2(I,X)} \leq \|v\|_{L^2(I,X)} \quad \forall\, v \in L^2(I,X).$$

*Let $y \in H^1(I_m, X) \hookrightarrow C(\bar{I}_m, X)$. Then the error estimates*

$$\|y - \Pi_{Y_k(X)} y\|_{L^2(I_m,X)} + \|y - \mathcal{P}_{Y_k(X)} y\|_{L^2(I_m,X)} \leq C k_m \|\partial_t y\|_{L^2(I_m,X)} \tag{2.5}$$

*and*

$$\|y - \Pi_{Y_k(X)} y\|_{L^\infty(I_m,X)} + \|y - \mathcal{P}_{Y_k(X)} y\|_{L^\infty(I_m,X)} \leq C\sqrt{k_m}\|\partial_t y\|_{L^2(I_m,X)} \tag{2.6}$$

*hold true.*

*Proof.* The proof follows from direct calculations. □

By squaring, summing up over time, and taking the square root, the preceding error estimates remain valid if all indices $m$ are removed, especially $y \in H^1(I,X)$ is assumed.

The following lemma, see also the proof of [DHV15, Corollary 4.3], provides a link between the orthogonal and the midpoint interpolation.

**Lemma 32.** *Let $y$ be a function with $y \in H^2(I,X)$. For the error between the orthogonal projection and the midpoint interpolation, defined in* (2.2) *and* (2.3)*, respectively, there holds*

$$\|\Pi_{Y_k(X)} y - \mathcal{P}_{Y_k(X)} y\|_{L^2(I_m,X)} \leq k_m^2 \|\partial_t^2 y\|_{L^2(I_m,X)}. \tag{2.7}$$

*Proof.* Let $\|\cdot\| := \|\cdot\|_X$ for this proof and suppose $w \in C^2(I_m, X) \cap H^2(I_m, X)$. With a Taylor expansion of $w$ at $t_m^*$ we obtain

$$\begin{aligned}\left\|\int_{t_{m-1}}^{t_m} w(t) - w(t_m^*) dt\right\|^2 &= \left\|\int_{t_{m-1}}^{t_m} (t - t_m^*)\partial_t w(t_m^*) + \int_{t_m^*}^{t} (t-s)\partial_t^2 w(s) ds dt\right\|^2 \\ &\leq k_m \int_{t_{m-1}}^{t_m} \left\|\int_{t_m^*}^{t} (t-s)\partial_t^2 w(s) ds\right\|^2 dt \leq k_m^4 \int_{t_{m-1}}^{t_m} \int_{t_m^*}^{t} \left\|\partial_t^2 w(s)\right\|^2 ds dt \\ &\leq k_m^5 \int_{t_{m-1}}^{t_m} \left\|\partial_t^2 w(s)\right\|^2 ds, \quad (2.8)\end{aligned}$$

where we have used the Cauchy-Schwarz inequality twice. With this inequality, we conclude

$$\begin{aligned}&\|\Pi_{Y_k(X)} w - \mathcal{P}_{Y_k(X)} w\|^2_{L^2(I_m, X)} \\ &\qquad = k_m \left\|\frac{1}{k_m}\int_{t_{m-1}}^{t_m} w(t) - w(t_m^*) dt\right\|^2 \leq k_m^4 \|\partial_t^2 w\|^2_{L^2(I_m, X)},\end{aligned}$$

which is (2.7) for $w$ instead of $y$. The result now follows by density of the space $C^2(I_m, X) \cap H^2(I_m, X)$ in $H^2(I_m, X)$. □

**Lemma 33.** *Let Assumption 29 be met. The interpolation operator* $\pi_{P_k^*(X)}$ *defined in* (2.4) *is stable in the sense*

$$\|\pi_{P_k^*(X)} w_k\|_{L^2(I,X)} \leq C \|w_k\|_{L^2(I,X)} \quad \forall\, w_k \in Y_k(X)$$

*where* $C > 0$ *is a constant independent of* $k$*, and fulfills the error estimate*

$$\|w - \pi_{P_k^*(X)} w\|_{L^2(I,X)} \leq C k^2 \|\partial_t^2 w\|_{L^2(I,X)} \quad \forall\, w \in H^2(I, X).$$

*Proof.* See [MV11, Lemma 5.6] for a similar result. □

### 2.1.2 Schemes, stability, and error estimates

The first step in the semidiscretization of problem ($\mathbb{P}$) consists in defining a discrete adjoint function $p_k$ as a counterpart of $p$ given by (1.11). Here we use more regular ansatz functions compared to the state discretization introduced below, namely functions in $P_k$. The reason will become clear in the later analysis of the error in the control.

In this subsection, we now consider the concrete case $X := H_0^1(\Omega)$ if not otherwise stated and abbreviate

$$P_k := P_k(H_0^1(\Omega)), \quad P_k^* := P_k^*(H_0^1(\Omega)), \quad \text{and } Y_k := Y_k(H_0^1(\Omega)). \tag{2.9}$$

We extend the bilinear form $A$ of (1.8) in its first argument to $W(I) \cup Y_k$, thus consider the operator

$$A : W(I) \cup Y_k \times W(I) \to \mathbb{R}, \quad A \text{ given by (1.8)}. \tag{2.10}$$

**Definition 34** (Time-discrete adjoint equation)**.** *For $h \in L^2(I, H^{-1}(\Omega))$ find $p_k \in P_k$ such that*

$$A(\tilde{y}, p_k) = \int_0^T \langle h(t), \tilde{y}(t) \rangle_{H^{-1}(\Omega) H_0^1(\Omega)} \, dt \quad \forall \, \tilde{y} \in Y_k. \tag{2.11}$$

This problem admits a unique solution $p_k \in P_k$. This follows from the subsequent considerations. We have a unique decomposition of a function $p_k \in P_k$ via

$$p_k(t) = \sum_{i=0}^{M} p_i \Lambda_i(t)$$

with coefficients $p_i \in H_0^1(\Omega)$ and $\Lambda_i \in P_k(\mathbb{R})$ being the usual hat functions defined by $\Lambda_i(t_j) = \delta_{ij}$ for $i, j \in \{0, 1, \ldots, M\}$. Using this representation, the coefficients $p_i$ are determined by the following backward in time Crank–Nicolson scheme

$$\begin{cases} p_M = 0\,, \\ \hline (p_{i-1} - p_i, \phi) + \frac{1}{2} k_i \left( \nabla(p_i + p_{i-1}), \nabla \phi \right) = \left\langle \int_{I_i} h(t) \, dt, \phi \right\rangle_{H^{-1}(\Omega) H_0^1(\Omega)} \\ \qquad \forall \, i \in \{M, M-1, \ldots, 1\} \quad \forall \, \phi \in H_0^1(\Omega). \end{cases} \tag{2.12}$$

**Remark 35.** *Note that if the data has the regularity $h \in L^2(I, L^2(\Omega))$, we have from* (2.12) *and elliptic regularity theory that $\Delta p_k\big|_{I_m} \in \mathcal{P}_1(I_m, L^2(\Omega))$ for all $m = 1, \ldots, M$.*

We start the analysis of this scheme by giving a stability result, which is a variant of [MV11, Lemma 4.2]. For a second stability result assuming nonsmooth data, see also Corollary 41.

**Lemma 36.** *Let $p_k \in P_k$ solve (2.11) with $h \in L^2(I, L^2(\Omega))$. Then there exists a constant $C > 0$ independent of $k$ such that*

$$\|p_k\|_{H^1(I,L^2(\Omega))} + \|\nabla p_k\|_{C(\bar{I},L^2(\Omega))} + \|\mathcal{P}_{Y_k} \Delta p_k\|_I \le C\|h\|_I.$$

*If furthermore $h \in L^2(I, H_0^1(\Omega))$ holds, we have*

$$\|\partial_t \nabla p_k\|_I + \|\Delta p_k\|_{C(\bar{I},L^2(\Omega))} \le C\|\nabla h\|_I.$$

*Proof.* For a fixed $m \in \{1, \ldots, M\}$ we define $\tilde{y} \in Y_k$ by $\tilde{y}\big|_{I_m} := -\partial_t p_k\big|_{I_m}$ and zero elsewhere. Testing with $\tilde{y}$ in (2.11) we obtain using integration by parts in the space $W(I)$

$$\begin{aligned}
A(\tilde{y}, p_k) &= \frac{1}{2}\|\nabla p_k(t_{m-1})\|^2_{L^2(\Omega)} + \|\partial_t p_k\|^2_{I_m} \\
&= \int_{I_m} \langle h, \tilde{y}\rangle_{H^{-1}(\Omega)H_0^1(\Omega)} + \frac{1}{2}\|\nabla p_k(t_m)\|^2_{L^2(\Omega)} \\
&\le \|h\|_{I_m}\|\partial_t p_k\|_{I_m} + \frac{1}{2}\|\nabla p_k(t_m)\|^2_{L^2(\Omega)} \\
&\le \frac{1}{2}\left(\|h\|^2_{I_m} + \|\partial_t p_k\|^2_{I_m} + \|\nabla p_k(t_m)\|^2_{L^2(\Omega)}\right),
\end{aligned}$$

where we used the Cauchy-Schwarz inequality and Cauchy's inequality. Rearranging terms yields

$$\|\nabla p_k(t_{m-1})\|^2_{L^2(\Omega)} + \|\partial_t p_k\|^2_{I_m} \le \|h\|^2_{I_m} + \|\nabla p_k(t_m)\|^2_{L^2(\Omega)} \quad \forall\, m = 1, \ldots, M.$$

Since $p_k\big|_{I_m} \in \mathcal{P}_1(I_m, H_0^1(\Omega))$ and $p_k(t_M) = 0$, we arrive at

$$\|\partial_t p_k\|_I + \|\nabla p_k\|_{C(\bar{I},L^2(\Omega))} \le C\|h\|_I. \tag{2.13}$$

The first estimate except for the last summand now follows by the fundamental theorem of calculus for $H^1(I, X)$ functions, see, e.g., [Eva98, Theorem 5.9.2].

From Remark 35 we know $\Delta p_k\big|_{I_m} \in \mathcal{P}_1(I_m, L^2(\Omega))$. Therefore, using integration by parts in space, we derive from (2.11) the equation

$$(-\partial_t p_k, \tilde{y})_I - (\Delta p_k, \tilde{y})_I + \langle \tilde{y}(T), p_k(T)\rangle_{H^{-1}(\Omega)H_0^1(\Omega)} = (h, \tilde{y})_I \quad \forall\, \tilde{y} \in Y_k. \tag{2.14}$$

Since no space derivatives of $\tilde{y}$ appear in (2.14) anymore, we can extend the equation by density to the space

$$\hat{Y}_k := \left\{ v : [0,T] \to H^{-1}(\Omega) \;\middle|\; v\big|_{I_m} \in \mathcal{P}_0(I_m, L^2(\Omega)) \right\} .$$

We test (2.14) with the function $\hat{y} := -\mathcal{P}_{Y_k}\Delta p_k \in \hat{Y}_k$ and get since $p_k(T) = 0$

$$(-\partial_t p_k, -\mathcal{P}_{Y_k}\Delta p_k)_I - (\Delta p_k, -\mathcal{P}_{Y_k}\Delta p_k)_I = (h, -\mathcal{P}_{Y_k}\Delta p_k)_I,$$

from which by orthogonality, i.e.,

$$(\Delta p_k - \mathcal{P}_{Y_k}\Delta p_k, \mathcal{P}_{Y_k}\Delta p_k)_I = 0,$$

and (2.13) the estimate $\|\mathcal{P}_{Y_k}\Delta p_k\|_I \le C\|h\|_I$ follows.

Let us now assume that $h \in L^2(I, H_0^1(\Omega))$. With some $m \in \{1, \dots, M\}$ fixed, we test (2.14) with a function $\hat{y} \in \hat{Y}_k$, given by $\hat{y}\big|_{I_m} := \partial_t \Delta p_k\big|_{I_m}$, and zero elsewhere. We obtain

$$\begin{aligned}
&\|\partial_t \nabla p_k\|_{I_m}^2 + \frac{1}{2}\|\Delta p_k(t_{i-1})\|^2 - \frac{1}{2}\|\Delta p_k(t_i)\|^2 \\
&\quad = (\nabla h, \partial_t \nabla p_k)_{I_m} \le \frac{1}{2}\left(\|\nabla h\|_{I_m}^2 + \|\partial_t \nabla p_k\|_{I_m}^2\right) \quad \forall\, m = 1, \dots, M.
\end{aligned}$$

From this, the second estimate follows as above. □

**Lemma 37.** *Let $p \in H^1(I, H^2(\Omega) \cap H_0^1(\Omega)) \bigcap H^2(I, L^2(\Omega)))$ solve (1.11) for some $h$, which is, e.g., the case (compare Lemma 8) if $h \in H^1(I, L^2(\Omega))$, $h(T) \in H_0^1(\Omega)$. Let furthermore $p_k \in P_k$ solve (2.11) for the same $h$. Then there holds*

$$\|p_k - p\|_I \le Ck^2 \left(\|\partial_t^2 p\|_I + \|\partial_t \Delta p\|_I\right).$$

*Proof.* See [MV11, Lemma 6.3] for a similar result. □

Let us consider the discretization of the state equation (1.9).

**Definition 38** (Time-discrete state equation)**.**
*For $(f, g) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$ find $y_k \in Y_k$, such that*

$$A(y_k, v_k) = \int_0^T \langle f(t), v_k(t)\rangle_{H^{-1}(\Omega)H_0^1(\Omega)}\, dt + (g, v_k(0)) \quad \forall\, v_k \in P_k. \quad (2.15)$$

In view of (1.9), this is a Petrov–Galerkin discretization of the state and by decomposing $y_k \in Y_k$ as

$$y_k = \sum_{i=1}^{M} y_i \chi_{I_i} + y_{M+1}\chi_T$$

with $y_i \in H_0^1(\Omega)$ for $i = 1, \dots, M$, $y_{M+1} \in H^{-1}(\Omega)$, we end up with the following scheme, which has to hold for arbitrary $\phi \in H_0^1(\Omega)$ with $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_{H^{-1}(\Omega) H_0^1(\Omega)}$.

$$\left\{\begin{array}{l}
(\phi, y_1 - g) + \frac{1}{2} k_1 (\nabla y_1, \nabla \phi) = \left\langle \int_{I_1} \frac{-(t - t_1)}{k_1} f(t)\, dt, \phi \right\rangle \\
\hline
(\phi, y_{i+1} - y_i) + \frac{1}{2} k_i (\nabla y_i, \nabla \phi) + \frac{1}{2} k_{i+1} (\nabla y_{i+1}, \nabla \phi) \\
= \left\langle \int_{I_i} \frac{t - t_{i-1}}{k_i} f(t)\, dt, \phi \right\rangle + \left\langle \int_{I_{i+1}} -\frac{t - t_{i+1}}{k_{i+1}} f(t)\, dt, \phi \right\rangle \quad \forall\, 1 \le i \le M-1 \\
\hline
\left\langle y_{M+1} - y_M, \phi \right\rangle + \frac{1}{2} k_M (\nabla y_M, \nabla \phi) = \left\langle \int_{I_M} \frac{t - t_{M-1}}{k_M} f(t)\, dt, \phi \right\rangle
\end{array}\right. \tag{2.16}$$

These equations can be solved subsequently from above to below yielding unique coefficients $y_1, y_2, \dots, y_{M+1}$ and therefore finally $y_k \in Y_k$. This follows from the fact that each of the first $M$ equations is a uniquely solvable elliptic equation (by the Lax-Milgram lemma) and the last equation determines $y_{M+1} \in H^{-1}(\Omega)$ uniquely from $y_M$.

Note that the first equation can be interpreted as a (Rannacher) start-up step, see [Ran84], for the Crank–Nicolson scheme given by the next equations except the last one.

**Remark 39.** *Note that if the data has the regularity $f \in L^2(I, L^2(\Omega))$, we have from* (2.16) *and elliptic regularity theory that $\Delta y_k\big|_{I_m} \in \mathcal{P}_0(I_m, L^2(\Omega))$ for all $m = 1, \dots, M$. Finally we get $y_{M+1} \in L^2(\Omega)$.*

As a first step in the analysis of this scheme, we consider the stability for different norms depending on the assumed smoothness of the data.

**Lemma 40.** *Let $y_k \in Y_k$ be the solution of (2.15) for data $(f, g) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$. Then there holds with a constant $C > 0$ independent of $k$ the stability estimate*

$$\|y_k\|_I \leq C \left( \|f\|_{L^2(I,H^{-1}(\Omega))} + \|g\| \right).$$

*If furthermore $f \in L^2(I, L^2(\Omega))$ is fulfilled, we have*

$$\|\nabla y_k\|_I \leq C \left( \|f\|_I + \|g\| \right).$$

*If even $f \in L^2(I, H_0^1(\Omega))$ and $g \in H_0^1(\Omega)$, there holds*

$$\|\Delta y_k\|_I \leq C \left( \|\nabla f\|_I + \|\nabla g\| \right).$$

*If $f \in Y_k$ and $g \in H_0^1(\Omega)$, we have*

$$\|\Delta y_k\|_I \leq C \left( \|f\|_I + \|\nabla g\| \right).$$

*Proof.* Let $p_k := p_k(y_k) \in P_k$ be the solution of (2.11) with right-hand side $y_k$. Using $y_k$ as a test function, too, we get using integration by parts in the space $W(I)$

$$\begin{aligned}
\|y_k\|_I^2 &= A(y_k, p_k) \\
&= \int_0^T \langle f, p_k \rangle_{H^{-1}(\Omega) H_0^1(\Omega)} \, dt + (g, p_k(0)) \\
&\leq C \left( \|f\|_{L^2(I,H^{-1}(\Omega))} \|p_k\|_{L^2(I,H_0^1(\Omega))} + \|g\| \|p_k(0)\| \right) \\
&\leq C \left( \|f\|_{L^2(I,H^{-1}(\Omega))} + \|g\| \right) \|y_k\|_I.
\end{aligned}$$

Note that Lemma 36 was used in the last step.

For the second assertion, we now assume $f \in L^2(I, L^2(\Omega))$. From Remark 39 we know $\Delta y_k\big|_{I_m} \in \mathcal{P}_0(I_m, L^2(\Omega))$ and $y_k(T) \in L^2(\Omega)$. Therefore, using integration by parts in space, we derive from (2.15) the equation

$$\begin{aligned}
(-\partial_t v_k, y_k)_I - (v_k, \Delta y_k)_I + (y_k(T), v_k(T)) \\
= (f, v_k)_I + (g, v_k(0)) \quad \forall\, v_k \in P_k. \quad (2.17)
\end{aligned}$$

Since no space derivatives of $v_k$ appear in (2.17) anymore, we can extend the equation by density to the space

$$\hat{P}_k := \left\{ v \in C([0,T], L^2(\Omega)) \;\middle|\; v\big|_{I_m} \in \mathcal{P}_1(I_m, L^2(\Omega)) \right\}.$$

Let $v_k := v_k(y_k) \in P_k$ be the solution of (2.11) with right-hand side $y_k$. Since by Remark 35 we have $\Delta v_k \in \hat{P}_k$, we get by (2.14), integration by parts in space, and (2.17) the estimate

$$\begin{aligned}
\|\nabla y_k\|_I^2 &= (-\Delta y_k, y_k)_I \\
&= (-\partial_t v_k, -\Delta y_k)_I - (\Delta v_k, -\Delta y_k)_I \\
&= (-\partial_t(-\Delta v_k), y_k)_I - (-\Delta v_k, \Delta y_k)_I \\
&= (f, -\Delta v_k)_I + (g, -\Delta v_k(0)) \\
&\leq C(\|f\|_I + \|g\|)\|\nabla y_k\|_I,
\end{aligned}$$

with the help of the second part of Lemma 36.

The third assertion can be derived like the preceding one, using $v_k := v_k(-\Delta y_k)$, i.e., $v_k$ is the solution of (2.11) with right-hand side $-\Delta y_k$. We get

$$\begin{aligned}
\|\Delta y_k\|_I^2 &= (-\partial_t v_k, -\Delta y_k)_I - (\Delta v_k, -\Delta y_k)_I \\
&= (-\partial_t(-\Delta v_k), y_k)_I - (-\Delta v_k, \Delta y_k)_I \\
&= (f, -\Delta v_k)_I + (g, -\Delta v_k(0)) \\
&= (\nabla f, \nabla v_k)_I + (\nabla g, \nabla v_k(0))_I \\
&\leq C(\|\nabla f\|_I + \|\nabla g\|)\|\Delta y_k\|_I,
\end{aligned}$$

again with the help of Lemma 36.

If $f \in Y_k$, we use the first part of the previous estimate to get with the help of orthogonality and the first estimate from Lemma 36

$$\begin{aligned}
\|\Delta y_k\|_I^2 &= (f, -\Delta v_k)_I + (g, -\Delta v_k(0)) \\
&= (f, -\mathcal{P}_{Y_k}\Delta v_k)_I + (\nabla g, \nabla v_k(0)) \\
&\leq C\|\Delta y_k\|_I(\|f\|_I + \|\nabla g\|),
\end{aligned}$$

from which the fourth assertion follows. □

With the help of this lemma we can establish stability of the discrete adjoint solution given data with minimal smoothness.

**Corollary 41.** *Let $p_k \in P_k$ solve* (2.11) *with $h \in L^2(I, H^{-1}(\Omega))$. Then there exists a constant $C > 0$ independent of $k$ such that*

$$\|p_k\|_I \leq C\|h\|_{L^2(I,H^{-1}(\Omega))}.$$

*Proof.* Let $y_k := y_k(p_k, 0)$ be the solution of (2.15) with right-hand side $p_k$ and initial datum zero. We obtain

$$\begin{aligned}\|p_k\|_I^2 = A(y_k, p_k) = \int_I \langle h, y_k\rangle_{H^{-1}(\Omega)H_0^1(\Omega)} \\ \leq \|h\|_{L^2(I,H^{-1}(\Omega))}\|y_k\|_{L^2(I,H_0^1(\Omega))} \leq C\|h\|_{L^2(I,H^{-1}(\Omega))}\|\nabla y_k\|_I \\ \leq C\|h\|_{L^2(I,H^{-1}(\Omega))}\|p_k\|_I,\end{aligned} \tag{2.18}$$

where the Poincaré inequality in space and Lemma 40 were used. □

Although we consider nonconforming discretization schemes, the following important property still holds due to the dense embedding $W(0,T) \overset{d}{\hookrightarrow} L^2(I, H_0^1(\Omega))$.

**Remark 42.** *Let $y$ be the solution of* (1.9) *for some* $(f, g) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$ *and $y_k \in Y_k$ be the solution of* (2.15) *for the same* $(f, g)$. *Consider also the solution $p$ of* (1.11) *for some* $h \in L^2(I, H^{-1}(\Omega))$ *and let $p_k \in P_k$ solve* (2.11) *for the same $h$.*

*Then for the bilinear form $A$ defined in* (2.10), *we have the property of* Galerkin orthogonality, *which reads*

$$\begin{aligned} A(\tilde{y}, p - p_k) = 0 \quad \forall\ \tilde{y} \in Y_k, \\ A(y - y_k, \tilde{p}) = 0 \quad \forall\ \tilde{p} \in P_k. \end{aligned} \tag{2.19}$$

We now consider the error of the time discretization and establish - as a byproduct - a superconvergence result, which will be useful in the later analysis. The error estimate is an adaptation of [MV11, Lemma 5.2], whereas the superconvergence result is a slightly improved variant of [DHV15, Corollary 4.3].

**Lemma 43.** *Let $(f, g)$ fulfill the requirements of Lemma 6, $y \in W(I)$ be the solution of* (1.9), *$y_k \in Y_k$ be the solution of* (2.15). *Then for the error we have the estimate*

$$\|y - y_k\|_I \leq Ck\left(\|\partial_t y\|_I + \|\Delta y\|_I\right) \leq Ck\left(\|f\|_I + \|g\|_{H^1(\Omega)}\right). \tag{2.20}$$

*If furthermore $(f, g)$ has the regularity of Lemma 8, there holds the superconvergence result*

$$\begin{aligned}\|y_k - \mathcal{P}_{Y_k} y\|_I \leq Ck^2\|\partial_t \Delta y\|_I \\ \leq Ck^2\left(\|f\|_{H^1(I,L^2(\Omega))} + \|f(0)\|_{H^1(\Omega)} + \|g\|_{H^1(\Omega)} + \|\Delta g\|_{H^1(\Omega)}\right).\end{aligned} \tag{2.21}$$

*Proof.* Let $p_k := p_k(y_k - \mathcal{P}_{Y_k} y) \in P_k$ solve (2.11) with $h := y_k - \mathcal{P}_{Y_k} y$. With the help of Galerkin orthogonality (2.19), the definition of $\mathcal{P}_{Y_k}$, and integration by parts in space we derive

$$\begin{aligned}\|h\|_I^2 &= A(h, p_k) = A(y - \mathcal{P}_{Y_k} y, p_k)\\ &= -(\partial_t p_k, y - \mathcal{P}_{Y_k} y)_I + (\nabla p_k, \nabla(y - \mathcal{P}_{Y_k} y))_I\\ &= (\nabla p_k, \nabla(y - \mathcal{P}_{Y_k} y))_I = -(p_k, \Delta(y - \mathcal{P}_{Y_k} y))_I\\ &= -(p_k - \mathcal{P}_{Y_k} p_k, \Delta y - \mathcal{P}_{Y_k} \Delta y)_I\\ &\leq Ck^2 \|\partial_t p_k\|_I \|\partial_t \Delta y\|_I \leq Ck^2 \|h\|_I \|\partial_t \Delta y\|_I.\end{aligned} \tag{2.22}$$

In the last steps, we used the estimate (2.5) and Lemma 36. Invoking Lemma 8, we get (2.21).

To prove (2.20), we use the first part of (2.22) to get

$$\begin{aligned}\|h\|_I^2 &= (\nabla p_k, \nabla(y - \mathcal{P}_{Y_k} y))_I = (\nabla p_k - \mathcal{P}_{Y_k} \nabla p_k, \nabla(y - \mathcal{P}_{Y_k} y))_I\\ &= (\nabla p_k - \mathcal{P}_{Y_k} \nabla p_k, \nabla y)_I = -(p_k - \mathcal{P}_{Y_k} p_k, \Delta y)_I\\ &\leq Ck \|\partial_t p_k\|_I \|\Delta y\|_I \leq Ck \|h\|_I \|\Delta y\|_I.\end{aligned} \tag{2.23}$$

Making use of the splitting

$$\|y - y_k\|_I \leq \|y - \mathcal{P}_{Y_k} y\|_I + \|\mathcal{P}_{Y_k} y - y_k\|_I,$$

the estimate (2.20) is now a consequence of (2.23), (2.5) and Lemma 6. □

As a consequence of this result, for Petrov–Galerkin approximations $y_k \in Y_k$ of solutions $y \in W(I)$ of (1.9) we can only expect $\mathcal{O}(k)$ convergence. Since $y_k$ is piecewise constant in time, this is of course no surprise.

To obtain $\mathcal{O}(k^2)$ convergence for the control approximations and even for discretized states in problem $(\mathbb{P})$, we rely on the following superconvergence result for the midpoint interpolation $\Pi_{Y_k}$, which was given above for the orthogonal projection $\mathcal{P}_{Y_k}$ in (2.21). Note that the result can also be found in [MV11, Lemma 5.3], but with another proof.

**Corollary 44.** *Let $y, y_k$ solve* (1.9) *and* (2.15), *respectively, with data $f$ and $g$. Assume that $y$ has the regularity $y \in H^1\left(I, H^2(\Omega) \cap H_0^1(\Omega)\right) \bigcap H^2\left(I, L^2(\Omega)\right)$, which, e.g., is fulfilled if $f$ and $g$ satisfy the regularity requirements of Lemma 8. Then there holds*

$$\|y_k - \Pi_{Y_k} y\|_I \leq Ck^2 \left(\|\partial_t^2 y\|_I + \|\partial_t \Delta y\|_I\right). \tag{2.24}$$

*Proof.* With the result of Lemma 32 at hand, the claim is an immediate consequence of (2.21). □

Let us now consider stability and error estimates of time-discrete states in $L^\infty(I, L^2(\Omega))$.

**Lemma 45.** *Let Assumption 30 hold and $y \in W(I)$ and $y_k \in Y_k$ be the solution of (1.9) and (2.15), respectively, both for data $(f, g) \in L^2(I, L^2(\Omega)) \times H_0^1(\Omega)$. Then there holds the stability estimate*

$$\|y_k\|_{L^\infty(I,L^2(\Omega))} \leq C(\sqrt{k}+1)\left(\|f\|_I + \|g\|_{H^1(\Omega)}\right). \tag{2.25}$$

*For the error we have the estimate*

$$\|y - y_k\|_{L^\infty(I,L^2(\Omega))} \leq C\sqrt{k}\left(\|\partial_t y\|_I + \|\Delta y\|_I\right). \tag{2.26}$$

*Proof.* In view of Lemma 6, the stability estimate (2.25) is an immediate consequence of (2.26). Thus it remains to show (2.26). Making use of the properties (2.5) and (2.6) of the midpoint interpolation from Lemma 31, and

$$\|y\|_{I_m} = \sqrt{k_m}\,\|y\|_{L^\infty(I_m,L^2(\Omega))} \quad \forall\, y \in \mathcal{P}_0(I_m, L^2(\Omega)), \tag{2.27}$$

we get

$$\begin{aligned}
&\|y - y_k\|_{L^\infty(I_m,L^2(\Omega))}\\
&\leq \|y - \Pi_{Y_k} y\|_{L^\infty(I_m,L^2(\Omega))} + \|\Pi_{Y_k} y - y_k\|_{L^\infty(I_m,L^2(\Omega))}\\
&\leq \|y - \Pi_{Y_k} y\|_{L^\infty(I_m,L^2(\Omega))} + k_m^{-1/2}\|\Pi_{Y_k} y - y_k\|_{I_m}\\
&\leq \|y - \Pi_{Y_k} y\|_{L^\infty(I_m,L^2(\Omega))} + k_m^{-1/2}\left(\|\Pi_{Y_k} y - y\|_{I_m} + \|y - y_k\|_{I_m}\right)\\
&\leq C\left(\sqrt{k_m}\|\partial_t y\|_{I_m} + k_m^{-1/2}\left(k_m\|\partial_t y\|_{I_m} + \|y - y_k\|_{I_m}\right)\right).
\end{aligned} \tag{2.28}$$

With the help of (2.20) from Lemma 40 and Assumption 30, we are done. □

A better convergence rate of the error is possible if the continuous state $y$ has more regularity, as the next Lemma shows.

**Lemma 46.** *Let Assumption 30 be fulfilled and let $y$ and $y_k$ be as in Corollary 44. For the error of the state, we have the improved estimate*

$$\|y - y_k\|_{L^\infty(I,L^2(\Omega))} \le Ck\left(\|\partial_t y\|_{L^\infty(I,L^2(\Omega))} + \|\partial_t \Delta y\|_I\right). \tag{2.29}$$

*Proof.* With (2.21) and (2.27) we deduce similarly as in (2.28)

$$\begin{aligned}
&\|y - y_k\|_{L^\infty(I_m,L^2(\Omega))} \\
&\le \|y - \mathcal{P}_{Y_k} y\|_{L^\infty(I_m,L^2(\Omega))} + \|\mathcal{P}_{Y_k} y - y_k\|_{L^\infty(I_m,L^2(\Omega))} \\
&\le \|y - \mathcal{P}_{Y_k} y\|_{L^\infty(I_m,L^2(\Omega))} + k_m^{-1/2}\|\mathcal{P}_{Y_k} y - y_k\|_{I_m} \\
&\le C\left(\|y - \mathcal{P}_{Y_k} y\|_{L^\infty(I_m,L^2(\Omega))} + k_m^{-1/2} k^2 \|\partial_t \Delta y\|_I\right) \\
&\le C\left(k_m \|\partial_t y\|_{L^\infty(I_m,L^2(\Omega))} + k_m^{-1/2} k^2 \|\partial_t \Delta y\|_I\right).
\end{aligned}$$

□

We have seen above in Lemma 43 that if the state is discretized piecewise constant in time, we can only expect first order convergence. The following Lemma shows that a projected version of the discretized state converges with order two to its continuous counterpart, if both depend on the same given data. This will be used later to derive a similar result about the optimal state and a projection of its discrete analogon.

**Lemma 47.** *Let Assumption 29 be fulfilled and $y$ and $y_k$ be given as in Corollary 44. Then there holds*

$$\|\pi_{P_k^*} y_k - y\|_I \le Ck^2\left(\|\partial_t^2 y\|_I + \|\partial_t \Delta y\|_I\right).$$

*Proof.* Making use of the splitting

$$\begin{aligned}
\|\pi_{P_k^*} y_k - y\|_I &= \|\pi_{P_k^*}(y_k - \Pi_{Y_k} y)\|_I + \|\pi_{P_k^*} \Pi_{Y_k} y - y\|_I \\
&= \|\pi_{P_k^*}(y_k - \Pi_{Y_k} y)\|_I + \|\pi_{P_k^*} y - y\|_I,
\end{aligned}$$

the claim is an immediate consequence of Lemma 33 and Corollary 44. □

One essential ingredient of our convergence analysis is given by the following result, a slightly improved version of [DHV15, Lemma 4.9].

**Lemma 48.** *Let $y$ and $y_k$ be as in Corollary 44, and let $p_k(h) \in P_k$ denote the solution to* (2.11) *with right-hand side $h$. Then we have*

$$\|p_k(y_k - y)\|_{C(\bar{I},L^2(\Omega))} \leq Ck^2 \|\partial_t \Delta y\|_I.$$

*Proof.* By definition of the orthogonal projection we have

$$(y_k - y, \tilde{y})_I = (y_k - \mathcal{P}_{Y_k} y, \tilde{y})_I \quad \forall\, \tilde{y} \in Y_k,$$

and since $p_k$ solves (2.11) one immediately obtains

$$p_k(y_k - y) = p_k(y_k - \mathcal{P}_{Y_k} y).$$

Hence by Lemma 36 and (2.21) we get

$$\begin{aligned}\|p_k(y_k - y)\|_{C(\bar{I},L^2(\Omega))} &= \|p_k(y_k - \mathcal{P}_{Y_k} y)\|_{C(\bar{I},L^2(\Omega))} \\ &\leq C\|y_k - \mathcal{P}_{Y_k} y\|_I \leq Ck^2 \|\partial_t \Delta y\|_I,\end{aligned}$$

which is the claim. □

## 2.2 Space and time discretization of state and adjoint equation

Using continuous piecewise linear functions in space, we can derive fully discretized variants of the state and adjoint equation.

We consider a regular triangulation $\mathcal{T}_h$ of $\Omega$ with mesh size

$$h := \max_{T \in \mathcal{T}_h} \operatorname{diam}(T),$$

see, e.g., [BS08, Definition (4.4.13)], and $N = N(h)$ triangles. We assume that $h < 1$. We also denote by $h$ (in a slight abuse of notation) the grid itself.

With the space

$$X_h := \left\{ \phi_h \in C^0(\bar{\Omega}) \;\middle|\; \phi_h\big|_T \in \mathcal{P}_1(T,\mathbb{R}) \quad \forall\, T \in \mathcal{T}_h \right\} \tag{2.30}$$

we define $X_{h0} := X_h \cap H_0^1(\Omega)$ to discretize $H_0^1(\Omega)$.

For the space grid we make use of a standard grid assumption, as we did for the time grid. This assumption is also referred to as quasi-uniformity.

**Assumption 49.** *There exists a constant $\mu > 0$ independent of $h$ such that*

$$h \leq \mu \min_{T \in \mathcal{T}_h} \operatorname{diam}(T).$$

With this assumption, the inverse estimate

$$\|\nabla \phi_h\| \leq C h^{-1} \|\phi_h\| \quad \forall\, \phi_h \in X_h \tag{2.31}$$

is known to hold, see, e.g., [BS08, (4.5.12)].

Furthermore, the grid assumption guarantees that for every $x_h \in X_{h0}$ the discrete Sobolev inequality

$$\|x_h\|_{L^\infty(\Omega)} \leq C \ell(h) \|\nabla x_h\| \tag{2.32}$$

is valid where

$$\ell(h) = \begin{cases} |\log h|^{\frac{1}{2}} & \text{if } d = 2, \\ h^{-\frac{1}{2}} & \text{if } d = 3. \end{cases} \tag{2.33}$$

For a proof of this result, we refer for $d = 2$ to [Tho06, Lemma 6.4] or [Xu89, Theorem 3.4]. From the inverse inequality

$$\|x_h\|_{L^\infty(\Omega)} \leq C h^{-1/2} \|x_h\|_{L^6(\Omega)},$$

which can be found in [BS08, (4.5.12)], the case $d = 3$ follows with the help of the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$.

## 2.2.1 Interpolation operators

**Lemma 50** (Ritz projection)**.** *By $R_h : H_0^1(\Omega) \to X_{h0}$ we denote the Ritz projection which is defined by the relation*

$$(\nabla R_h f, \nabla \phi_h) = (\nabla f, \nabla \phi_h) \quad \forall\, \phi_h \in X_{h0}. \tag{2.34}$$

*It is stable in $H_0^1(\Omega)$ with*

$$\|\nabla R_h f\| \leq \|\nabla f\| \quad \forall\, f \in H_0^1(\Omega)$$

*and it fulfills the error estimate*

$$\|f - R_h f\| + h\|\nabla(f - R_h f)\| \leq C h^s \|f\|_{H^s(\Omega)} \qquad \forall\, f \in H^s(\Omega) \cap H_0^1(\Omega),\ 1 \leq s \leq 2.$$

*If furthermore Assumption 49 is satisfied, the Ritz projection has the almost maximum-norm stability property*

$$\|R_h f\|_{L^\infty(\Omega)} \leq C|\log h|\|f\|_{L^\infty(\Omega)},$$

*from which the error estimate*

$$\|R_h f - f\|_{L^\infty(\Omega)} \leq C|\log h| h^2 \|f\|_{W^{2,\infty}(\Omega)}$$

*follows.*

*Proof.* The $L^2$ results are well known, see, e.g., [Tho06, (1.24), Lemma 1.1, and chap. 19], and so are the $L^\infty$ results for $d = 2$, see, e.g., [Tho06, Theorem 1.4 and the discussion afterwards] or [BS08, chap. 8.5]. The case $d = 3$ was recently established in [LV16, Theorem 12]. □

We now extend the Ritz projection to a time-dependent operator.

**Lemma 51** (Time-extended Ritz projection)**.** *The time-extended Ritz projection $R_h : L^2(I, H_0^1(\Omega)) \to L^2(I, X_{h0})$ is defined by the relation*

$$(\nabla R_h f, \nabla \phi_h)_I = (\nabla f, \nabla \phi_h)_I \quad \forall\ \phi_h \in L^2(I, X_{h0}). \tag{2.35}$$

*It is stable in $L^p(I, H_0^1(\Omega))$ with*

$$\|\nabla R_h f\|_{L^p(I,L^2(\Omega))} \leq \|\nabla f\|_{L^p(I,L^2(\Omega))} \quad \forall\ f \in L^p(I, H_0^1(\Omega)),\ 2 \leq p \leq \infty, \tag{2.36}$$

*and it fulfills the error estimate*

$$\|f - R_h f\|_{L^p(I,L^2(\Omega))} + h\|\nabla(f - R_h f)\|_{L^p(I,L^2(\Omega))} \leq Ch^s \|f\|_{L^p(I,H^s(\Omega))}$$
$$\forall\ f \in L^p(I, H^s(\Omega) \cap H_0^1(\Omega)),\ 1 \leq s \leq 2 \leq p \leq \infty. \tag{2.37}$$

*If furthermore Assumption 49 is satisfied, the Ritz projection has the almost maximum-norm stability property with respect to space*

$$\|R_h f\|_{L^p(I,L^\infty(\Omega))} \leq C|\log h|\|f\|_{L^p(I,L^\infty(\Omega))} \quad \forall\ 2 \leq p \leq \infty, \tag{2.38}$$

*from which one can derive the error estimate*

$$\|R_h f - f\|_{L^p(I,L^\infty(\Omega))} \leq C|\log h| h^2 \|f\|_{L^p(I,W^{2,\infty}(\Omega))} \quad \forall\ 2 \leq p \leq \infty. \tag{2.39}$$

*Proof.* By a Lebesgue point argument, one can show that (2.35) is equivalent to "(2.34) holds for almost all $t \in I$". Therefore, invoking Lemma 50, one can immediately derive the claim. □

For the error analysis of the fully discrete adjoint equation, we need the following time projection, mapping time-differentiable function to piecewise linear ones.

**Definition 52** (Piecewise linear projection). *The operator $P^t : H^1(I, X) \to P_k(X)$ is defined by the relation*

$$P^t w(t_m) = w(t_m) \quad \forall\, 0 \leq m \leq M. \tag{2.40}$$

Note that since $P_k(X)$ is a space of piecewise linear functions, the operator $P^t$ is just the interpolation in time. Therefore, we have stability in the sense

$$\|P^t w\|_{L^\infty(I,X)} \leq C\|w\|_{L^\infty(I,X)} \leq C\|w\|_{H^1(I,X)}, \tag{2.41}$$

and by standard techniques one can show the error estimate

$$\|w - P^t w\|_{H^{s,l}(I,X)} \leq C k^{r-s} \|w\|_{H^{r,l}(I,X)} \tag{2.42}$$

where $0 \leq s \leq 1 \leq r \leq 2 \leq l \leq \infty$, and $w$ is supposed to have the regularity on the right-hand side. See, e.g., [AM89, Lemma 2.2] for a similar result.

If $X$ is a Hilbert space with inner product $(.,.)_X$, there is another possibility to define $P^t$, which will be useful in the later analysis, namely

$$\begin{cases} P^t w(T) = w(T), \\ \int_I ((P^t w)_t, p^k_t)_X \, dt = \int_I (w_t, p^k_t)_X \, dt \quad \forall\, p^k \in P_k(X). \end{cases} \tag{2.43}$$

To see the equivalence of both definitions, we define some $p^k \in P_k(X)$ by

$$p^k(t) := \begin{cases} (t - t_m)\,\phi & \text{if } t_m < t \leq T, \\ 0 & \text{if } t \leq t_m \end{cases}$$

where $\phi \in X$ is arbitrary and $m \in \{0, 1, \dots, M-1\}$. Plugging $p^k$ into the second equation of (2.43) and making use of the first one, we conclude (2.40).

To see the converse, note that (2.40) implies $\int_{I_m} (P^t w)_t \, dt = \int_{I_m} w_t \, dt$ for all $1 \leq m \leq M$ by the fundamental theorem of calculus. Since the derivatives of functions in $P_k(X)$ are piecewise constant in $X$, we get (2.43).

### 2.2.2 Schemes, stability, and error estimates

We now define fully discrete ansatz and test spaces, directly derived from their semidiscrete counterparts from (2.1), namely

$$P_{kh} := P_k(X_{h0}), \quad P^*_{kh} := P^*_{kh}(X_{h0}), \quad \text{and } Y_{kh} := Y_k(X_{h0}). \tag{2.44}$$

These spaces lead to fully discrete state and adjoint equations, naturally derived from the semidiscrete counterparts given by Definition 38 and 34, respectively.

**Definition 53** (Fully discrete adjoint equation)**.** *For $h \in L^2(I, H^{-1}(\Omega))$ find $p_{kh} \in P_{kh}$ such that*

$$A(\tilde{y}, p_{kh}) = \int_0^T \langle h(t), \tilde{y}(t)\rangle_{H^{-1}(\Omega)H^1_0(\Omega)}\, dt \quad \forall\ \tilde{y} \in Y_{kh}. \tag{2.45}$$

**Definition 54** (Fully discrete state equation)**.** *For $(f, g) \in L^2(I, H^{-1}(\Omega))\times L^2(\Omega)$ find $y_{kh} \in Y_{kh}$, such that*

$$A(y_{kh}, v_{kh}) = \int_0^T \langle f(t), v_{kh}(t)\rangle_{H^{-1}(\Omega)H^1_0(\Omega)}\, dt + (g, v_{kh}(0)) \quad \forall\ v_{kh} \in P_{kh}. \tag{2.46}$$

Existence and uniqueness of these two schemes follow as in the semidiscrete case discussed above.

**Remark 55.** *Note that in view of Remark 39 and Remark 35, in the fully discrete setting we can only conclude $\Delta y_{kh}\big|_{I_m} \in \mathcal{P}_0(I_m, H^{-1}(\Omega))$ and $\Delta p_{kh}\big|_{I_m} \in \mathcal{P}_1(I_m, H^{-1}(\Omega))$ for all $m = 1, \dots, M$, even if $f \in L^2(I, L^2(\Omega))$. The reason is the fact that $X_{h0} \not\subset H^2(\Omega)$.*

As in the semidiscrete case, we start the analysis of the fully discrete schemes with some stability results.

**Lemma 56.** *Let $p_{kh} \in P_{kh}$ solve (2.45) with $h \in L^2(I, L^2(\Omega))$. Then there exists a constant $C > 0$ independent of $k$ and $h$ such that*

$$\|p_{kh}\|_{H^1(I,L^2(\Omega))} + \|\nabla p_{kh}\|_{C(\bar{I},L^2(\Omega))} \le C\|h\|_I.$$

*Proof.* Following the proof of Lemma 36 with the obvious modifications gives the claim. □

Similar to Remark 42, we have the following.

**Remark 57.** *Let $y$ be the solution of* (1.9) *for some* $(f, g) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$ *and let* $y_{kh} \in Y_{kh}$ *be the solution of* (2.46) *for the same* $(f, g)$. *Consider also the solution $p$ of* (1.11) *for some* $h \in L^2(I, H^{-1}(\Omega))$ *and let* $p_{kh} \in P_{kh}$ *solve* (2.45) *for the same $h$.*

*Then for the bilinear form $A$ defined in* (2.10)*, we have the property of* Galerkin orthogonality *which reads*

$$\begin{aligned} A(\tilde{y}, p - p_{kh}) &= 0 \quad \forall\ \tilde{y} \in Y_{kh}, \\ A(y - y_{kh}, \tilde{p}) &= 0 \quad \forall\ \tilde{p} \in P_{kh}. \end{aligned} \tag{2.47}$$

*In addition to Remark 42, we have*

$$\begin{aligned} A(\tilde{y}, p - p_k) &= 0 \quad \forall\ \tilde{y} \in Y_{kh}, \\ A(y - y_k, \tilde{p}) &= 0 \quad \forall\ \tilde{p} \in P_{kh}. \end{aligned} \tag{2.48}$$

The next Lemma is used in the proof of the next but one Lemma. It is a variant of [MV11, Lemma 5.4, Corollary 5.5].

**Lemma 58.** *Let* $y_k = y_k(f, g) \in Y_k$ *and* $\tilde{y}_k = y_k(\mathcal{P}_{Y_k} f, g) \in Y_k$ *be the solutions of* (2.15) *for some* $(f, g) \in L^2(I, L^2(\Omega)) \times L^2(\Omega)$. *By* $y_{kh} = y_{kh}(f, g) \in Y_{kh}$ *and* $\tilde{y}_{kh} = y_{kh}(\mathcal{P}_{Y_k} f, g) \in Y_{kh}$ *we denote the solutions of* (2.46).

*Then it holds with a constant $C > 0$ independent of $k$ and $h$*

$$\|y_k - \tilde{y}_k\|_I + \|y_{kh} - \tilde{y}_{kh}\|_I \leq Ck\|f\|_I.$$

*If in addition* $f \in H^1(I, L^2(\Omega))$ *holds, we have*

$$\|y_k - \tilde{y}_k\|_I + \|y_{kh} - \tilde{y}_{kh}\|_I \leq Ck^2\|\partial_t f\|_I.$$

*Proof.* We substract the defining equation (2.15) for $y_k$ from the corresponding one for $\tilde{y}_k$, using $v_k \in P_k$ as a test function defined by $\partial_t v_k := y_k - \tilde{y}_k$ and $v_k(T) := 0$. Using orthogonality, we get

$$\|y_k - \tilde{y}_k\|_I^2 - (\nabla v_k, \partial_t \nabla v_k)_I = (v_k, f - \mathcal{P}_{Y_k} f)_I = (v_k - \mathcal{P}_{Y_k} v_k, f - \mathcal{P}_{Y_k} f).$$

Using integration by parts, we conclude

$$\begin{aligned} \|y_k - \tilde{y}_k\|_I^2 + \frac{1}{2}\|\nabla v_k(0)\|^2 &\leq C\|v_k - \mathcal{P}_{Y_k} v_k\|_I \|f - \mathcal{P}_{Y_k} f\|_I \\ &\leq Ck\|\partial_t v_k\|_I \|f - \mathcal{P}_{Y_k} f\|_I \leq Ck\|y_k - \tilde{y}_k\|_I \|f - \mathcal{P}_{Y_k} f\|_I, \end{aligned}$$

where (2.5) was used. Depending on the regularity of $f$, one can now invoke (2.5) again or make use of the stability of $\mathcal{P}_{Y_k}$ given in Lemma 31.

In the same way the estimate for $y_{kh}$ can be derived. □

We now use Galerkin orthogonality and the Ritz projection to establish stability of the gradient of a fully discrete state $y_{kh}$. Note that we can not argue as in the proof of Lemma 40. We also provide an error estimate.

**Lemma 59.** *Let $y$ be the solution of (1.9) for some $(f, g) \in L^2(I, H^{-1}(\Omega)) \times L^2(\Omega)$ and let $y_{kh} \in Y_{kh}$ be the solution of (2.46) for the same $(f, g)$. Then with a constant $C > 0$ independent of $k$ and $h$, it holds*

$$\|y_{kh}\|_I \leq C \left( \|f\|_{L^2(I,H^{-1}(\Omega))} + \|g\| \right).$$

*If furthermore Assumption 49 is satisfied as well as $f \in L^2(I, L^2(\Omega))$, we have*

$$\|\nabla y_{kh}\|_I \leq C \left( \|f\|_I + \|g\| \right).$$

*If in addition, without requiring Assumption 49, the regularity $g \in H_0^1(\Omega)$ is fulfilled, we have the error estimate*

$$\|y - y_{kh}\|_I \leq C(h^2 + k) \left( \|f\|_I + \|\nabla g\| \right). \tag{2.49}$$

*Proof.* Following the proof of Lemma 40 using Lemma 56 leads to the first estimate.

For the second estimate, let $p_{kh} = p_{kh}(\tilde{h})$ be the solution of (2.45) with right-hand side $\tilde{h} \in Y_{kh}$ given by

$$\tilde{h} := \begin{cases} y_{kh} - R_h y_k & \text{if } 0 \leq t < T, \\ 0 & \text{if } t = T \end{cases}$$

where $y_k = y_k(f, g)$ is the solution of (2.15). We then get by Galerkin orthogonality, the definition of the Ritz projection (2.35), and Lemma 56

$$\begin{aligned} \|\tilde{h}\|_I^2 &= A(\tilde{h}, p_{kh}) \\ &= -(\partial_t p_{kh}, y_k - R_h y_k)_I + (\nabla(y_k - R_h y_k), \nabla p_{kh})_I \\ &\leq \|\partial_t p_{kh}\|_I \|y_k - R_h y_k\|_I \leq C \|\tilde{h}\|_I \|y_k - R_h y_k\|_I. \end{aligned} \tag{2.50}$$

Together with the error estimate of the Ritz projection (2.37) we obtain

$$\|y_{kh} - R_h y_k\|_I \leq Ch \|\nabla y_k\|_I.$$

Using the inverse estimate (2.31), with the help of (2.36) we get

$$\begin{aligned}\|\nabla y_{kh}\|_I &\le \|\nabla(y_{kh} - R_h y_k)\|_I + \|\nabla R_h y_k\|_I \\ &\le Ch^{-1}\|y_{kh} - R_h y_k\|_I + \|\nabla R_h y_k\|_I \\ &\le Ch^{-1}h\|\nabla y_k\|_I + C\|\nabla y_k\|_I.\end{aligned}$$

Using Lemma 40, we conclude the second estimate.

For the error estimate, consider the solutions $\tilde{y}_k$ and $\tilde{y}_{kh}$ from Lemma 58. We split the error into four parts

$$\begin{aligned}\|y - y_{kh}\|_I &\le C\|y - y_k\|_I + \|y_k - \tilde{y}_k\|_I + \|\tilde{y}_k - \tilde{y}_{kh}\|_I + \|\tilde{y}_{kh} - y_{kh}\|_I \\ &= I + II + III + IV,\end{aligned}$$

We can estimate $I$ by Lemma 43, and the summands $II$ and $IV$ can be treated by Lemma 58.

Thus it remains to estimate summand $III$. We use (2.50) together with (2.37) and the last estimate of Lemma 40 to get

$$\begin{aligned}\|\tilde{y}_{kh} - \tilde{y}_k\|_I &\le \|\tilde{y}_{kh} - R_h\tilde{y}_k\|_I + \|R_h\tilde{y}_k - \tilde{y}_k\|_I \\ &\le Ch^2\|\Delta\tilde{y}_k\|_I \le Ch^2(\|f\|_I + \|\nabla g\|). \quad (2.51)\end{aligned}$$

□

For completeness, let us also mention the stability of the solution of the fully discrete adjoint equation for nonsmooth data.

**Corollary 60.** *Let Assumption 49 be satisfied. Let $p_{kh} \in P_{kh}$ solve* (2.45) *with $h \in L^2(I, H^{-1}(\Omega))$. Then there exists a constant $C > 0$ independent of $k$ and $h$ such that*

$$\|p_{kh}\|_I \le C\|h\|_{L^2(I,H^{-1}(\Omega))}.$$

*Proof.* The proof can be established as the corresponding one in Corollary 41 making use of Lemma 59. □

In the previous corollary, the Assumption 49 is redundant and stability can be established even pointwise in time.

**Lemma 61.** *Let $p_{kh} \in P_{kh}$ solve* (2.45) *with $h \in L^2(I, H^{-1}(\Omega))$. Then there exists a constant $C > 0$ independent of $k$ and $h$ such that*

$$\|p_{kh}\|_{L^\infty(I,L^2(\Omega))} \le C\|h\|_{L^2(I,H^{-1}(\Omega))}.$$

*Proof.* To prove this, combine [AM89, Theorem 3.1] with the formula (2.6) loc. cit. □

Let us now consider the error of the fully discrete adjoint state. We begin with an $L^2(I, L^2(\Omega))$ norm result.

**Lemma 62.** *Let $p$ solve* (1.11) *for some $h$ such that $p$ has the regularity $p \in H^1\left(I, H^2(\Omega) \cap H_0^1(\Omega)\right) \bigcap H^2\left(I, L^2(\Omega)\right)$, which is fulfilled, e.g., if $h$ satisfies the regularity requirements of Lemma 8. Let furthermore $p_{kh} \in P_{kh}$ solve* (2.45) *for the same $h$. Then it holds*

$$\|p_{kh} - p\|_I \leq C(k^2 + h^2)(\|p_{tt}\|_I + \|\Delta p_t\|_I).$$

*Proof.* We start with the splitting

$$\|p - p_{kh}\|_I \leq \|p - P^t R_h p\|_I + \|P^t R_h p - p_{kh}\|_I,$$

and estimate

$$\begin{aligned}\|p - P^t R_h p\|_I &= \|p - P^t p\|_I + \|P^t p - P^t R_h p\|_I \\ &\leq C(k^2\|p_{tt}\|_I + h^2\|\Delta p_t\|_I),\end{aligned}$$

where (2.42), (2.41), and (2.37) were used. We set

$$e_{kh} := P^t R_h p - p_{kh} \in P_{kh},$$

and observe that $e_{kh}(T) = 0$ by (2.40). Consider the solutions $y = y(e_{kh}, 0)$ and $y_{kh} = y_{kh}(e_{kh}, 0)$ of the state equation (1.9) and the fully discrete state equation (2.46), respectively, with right-hand side $e_{kh}$ and initial value zero. By Galerkin orthogonality, the definition of the Ritz projection (2.35), and (2.43) we conclude

$$\begin{aligned}\|e_{kh}\|_I^2 &= A(e_{kh}, y_{kh}) = A(P^t R_h p - p, y_{kh}) \\ &= -(\partial_t(P^t R_h p - p), y_{kh})_I + (\nabla(P^t R_h p - p), \nabla y_{kh})_I \\ &= -(R_h p_t - p_t, y_{kh})_I - (\Delta(P^t p - p), y_{kh} - y)_I - (P^t p - p, \Delta y)_I \\ &\leq C(h^2\|\Delta p_t\|_I\|y_{kh}\|_I + k\|\Delta p_t\|_I(k + h^2)\|e_{kh}\|_I + k^2\|p_{tt}\|_I\|\Delta y\|_I),\end{aligned}$$

where in the last step the error estimates for the Ritz projection (2.37), the projection $P^t$ given by (2.42), and (2.49) were used.

Consequently, since $k < 1$ was assumed at the beginning of the chapter, we arrive at

$$\|e_{kh}\|_I \leq C(h^2 + k^2)(\|\Delta p_t\|_I + \|p_{tt}\|_I).$$

□

**Remark 63.** *Since the underlying scheme of the fully discrete adjoint equation is the Crank–Nicolson scheme, the convergence rate $h^2 + k^2$ is a well known result. It can also be found in [AM89, Corollary 3.4], where more smoothness on $p$ is assumed than we just did above. This difference is crucial for $\bar{p}$ if $\alpha = 0$.*

We next give a superconvergence result which is the key ingredient to establish a pointwise-in-time error estimate for the fully discrete adjoint state.

**Lemma 64.** *Let the assumptions of Lemma 62 be fulfilled. Then there holds*

$$\|\partial_t(P^t R_h p - p_{kh})\|_I + \|\nabla(P^t R_h p - p_{kh})\|_{C(\bar{I},L^2(\Omega))} \leq C\left(h^2 + k\right)\|\Delta p_t\|_I.$$

*If furthermore $p$ has the additional regularity $p \in H^2(I, H^2(\Omega) \cap H^1_0(\Omega))$, the superconvergence property*

$$\|\partial_t(P^t R_h p - p_{kh})\|_I + \|\nabla(P^t R_h p - p_{kh})\|_{C(\bar{I},L^2(\Omega))} \leq C\left(h^2\|\Delta p_t\|_I + k^2\|\Delta p_{tt}\|_I\right)$$

*is valid.*

*Proof.* From Galerkin orthogonality we conclude

$$A(p - P^t R_h p + P^t R_h p - p_{kh}, y_{kh}) = 0.$$

Setting $p_1 := p - P^t R_h p$ and $p_2 := P^t R_h p - p_{kh} \in P_{kh}$, we rewrite this as

$$-(\partial_t(p_1 + p_2), y_{kh})_I + (\nabla(p_1 + p_2), \nabla y_{kh})_I = 0.$$

Seperating $p_1$ and $p_2$, we obtain

$$-(\partial_t p_2, y_{kh})_I + (\nabla p_2, \nabla y_{kh})_I = (\partial_t p_1, y_{kh})_I - (\nabla p_1, \nabla y_{kh})_I.$$

Plugging in $y_{kh} \in Y_{kh}$ defined by $y_{kh}\big|_{I_m} := -\partial_t p_2\big|_{I_m}$ for some fixed interval $I_m$ and zero elsewhere, applying integration by parts, and making use of the relations (2.35) and (2.43) gives

$$\begin{aligned} &\|\partial_t p_2\|_{I_m}^2 + \frac{1}{2}\|\nabla p_2(t_{i-1})\|^2 \\ &= \frac{1}{2}\|\nabla p_2(t_i)\|^2 - (\partial_t p_1, \partial_t p_2)_{I_m} + (\nabla p_1, \nabla \partial_t p_2)_{I_m} \\ &= \frac{1}{2}\|\nabla p_2(t_i)\|^2 - (\partial_t (p - R_h p), \partial_t p_2)_{I_m} - (\nabla(p - P^t p), \nabla \partial_t p_2)_{I_m} \\ &= \frac{1}{2}\|\nabla p_2(t_i)\|^2 - (\partial_t (p - R_h p) + \Delta(p - P^t p), \partial_t p_2)_{I_m}. \end{aligned}$$

Using the Cauchy-Schwarz and Cauchy's inequality, we end up with

$$\begin{aligned} &\frac{1}{2}\left(\|\partial_t p_2\|_{I_m}^2 + \|\nabla p_2(t_{i-1})\|^2\right) \\ &\qquad\qquad \leq \frac{1}{2}\left(\|\nabla p_2(t_i)\|^2 + \|p_t - R_h p_t\|_{I_m}^2 + \|\Delta p - P^t \Delta p\|_{I_m}^2\right). \end{aligned}$$

We recall $p_2(T) = 0$, and since $P_{kh}$ is piecewise linear in time, we conclude

$$\begin{aligned} \|\partial_t p_2\|_I + \|\nabla p_2\|_{C(\bar{I}, L^2(\Omega))} &\leq \left(\|p_t - R_h p_t\|_I + \|\Delta p - P^t \Delta p\|_I\right) \\ &\leq C\left(h^2\|\Delta p_t\|_I + k\|\Delta p_t\|_I\right), \end{aligned}$$

where (2.42) and (2.37) were used.

If $p$ fulfills the additional regularity, one can replace the term "$k\|\Delta p_t\|_I$" by "$k^2\|\Delta p_{tt}\|_I$". □

**Lemma 65.** *Let the assumptions of Lemma 62 be fulfilled. Then it holds*

$$\|p - p_{kh}\|_{L^\infty(I, L^2(\Omega))} \leq C(h^2 + k)\left(\|\Delta p_t\|_I + \|p_t\|_{L^\infty(I, L^2(\Omega))}\right).$$

*If in addition $p \in H^2(I, H^2(\Omega) \cap H_0^1(\Omega))$ and $p_{tt} \in L^\infty(I, L^2(\Omega))$ is known to hold, the improved estimate*

$$\begin{aligned} \|p - p_{kh}\|_{L^\infty(I, L^2(\Omega))} \leq C(h^2 + k^2)&\left(\|\Delta p_t\|_I + \|p_t\|_{L^\infty(I, L^2(\Omega))}\right) \\ &+ Ck^2\left(\|\Delta p_{tt}\|_I + \|p_{tt}\|_{L^\infty(I, L^2(\Omega))}\right) \end{aligned}$$

*is valid.*

*Proof.* We split the error into two parts and use Lemma 64 to conclude with $L := L^\infty(I, L^2(\Omega))$

$$\begin{aligned}\|p - p_{kh}\|_L &\le \|P^t R_h p - p_{kh}\|_L + \|p - P^t R_h p\|_L \\ &\le \|\partial_t(P^t R_h p - p_{kh})\|_I + \|P^t p - P^t R_h p\|_L + \|p - P^t p\|_L \\ &\le C\left((h^2 + k)\|\Delta p_t\|_I + \|p - R_h p\|_L + \|p - P^t p\|_L\right),\end{aligned}$$

where stability of $P^t$ given by (2.41) and the fact that $(P^t R_h p - p_{kh})(T) = 0$ were used. With the error estimate of the Ritz projection (2.37) and the error estimate of the time projection (2.42), we finally get

$$\begin{aligned}&\|p - p_{kh}\|_{L^\infty(I,L^2(\Omega))} \\ &\le C\left((h^2 + k)\|\Delta p_t\|_I + h^2\|\Delta p\|_{L^\infty(I,L^2(\Omega))} + k\|p_t\|_{L^\infty(I,L^2(\Omega))}\right).\end{aligned}$$

If the additional regularity holds, the modifications are obvious. □

Let us now establish the analog of Lemma 48 and the superconvergence result of Lemma 43, i.e., (2.21) for fully discretized objects.

**Lemma 66.** *Let $y \in Y$ and $y_{kh} \in Y_{kh}$ solve* (1.9) *and* (2.46)*, respectively, with data $(f, g)$ as in Lemma 8. By $p_{kh}(h) \in P_{kh}$ we denote the solution to* (2.45) *with right-hand side $h$. Then it holds*

$$\|y_{kh} - \mathcal{P}_{Y_k} y\|_I + \|p_{kh}(y_{kh} - y)\|_{C(\bar{I},L^2(\Omega))} \le C(k^2 F_1(f, g) + h^2 F_2(f, g))$$

*with*

$$F_2(f, g) := \|f\|_I + \|g\|_{H^1(\Omega)}$$

*and*

$$F_1(f, g) := F_2(f, g) + \|\partial_t f\|_I + \|f(0)\|_{H^1(\Omega)} + \|\Delta g\|_{H^1(\Omega)}.$$

*Proof.* We first observe that the estimate (2.21) in combination with (2.51) and Lemma 58 yields the inequality

$$\|y_{kh} - \mathcal{P}_{Y_k} y\|_I \le C(k^2 F_1(f, g) + h^2 F_2(f, g)).$$

From this, we get analogously to the proof of Lemma 48

$$\begin{aligned}\|p_{kh}(y_{kh} - y)\|_{C(\bar{I},L^2(\Omega))} &= \|p_{kh}(y_{kh} - \mathcal{P}_{Y_k} y)\|_{C(\bar{I},L^2(\Omega))} \\ &\le C\|y_{kh} - \mathcal{P}_{Y_k} y\|_I,\end{aligned}$$

where Lemma 56 was used. Combining both estimates proves the claim. □

## 2.3 Discretization of the optimal control problem

With the results of the previous sections, we are now able to introduce the discretized optimal control problem which reads

$$\min_{y_{kh} \in Y_{kh}, u \in U_{\mathrm{ad}}} J(y_{kh}, u) = \min \frac{1}{2}\|y_{kh} - y_d\|_I^2 + \frac{\alpha}{2}\|u\|_U^2, \quad \text{s.t. } y_{kh} = S_{kh}(Bu, y_0) \tag{$\mathbb{P}_{kh}$}$$

where $\alpha$, $B$, $y_0$, $y_d$, and $U_{\mathrm{ad}}$ are taken as in ($\mathbb{P}$) and $S_{kh}$ is the solution operator associated to the fully discrete state equation (2.46). Recall that the space $Y_{kh}$ was introduced in (2.44).

For every $\alpha > 0$, this problem admits a unique solution triple ($\bar{u}_{kh}$, $\bar{y}_{kh}$, $\bar{p}_{kh}$) where $\bar{y}_{kh} = S_{kh}(B\bar{u}_{kh}, y_0)$ and $\bar{p}_{kh}$ denotes the discrete adjoint state which is the solution of the fully discrete adjoint equation (2.45) with right-hand side $h := \bar{y}_{kh} - y_d$. The first order necessary and sufficient optimality condition for problem ($\mathbb{P}_{kh}$) is given by

$$\bar{u}_{kh} \in U_{\mathrm{ad}}, \quad (\alpha\bar{u}_{kh} + B^*\bar{p}_{kh}, u - \bar{u}_{kh})_U \geq 0 \quad \forall\, u \in U_{\mathrm{ad}}, \tag{2.52}$$

which can be rewritten as

$$\bar{u}_{kh} = P_{U_{\mathrm{ad}}}\left(-\frac{1}{\alpha}B^*\bar{p}_{kh}\right). \tag{2.53}$$

The before mentioned facts can be proven in the same way as for the continuous problem ($\mathbb{P}$).

Note that the control space $U$ is not discretized in the formulation ($\mathbb{P}_{kh}$). In the numerical treatment, the relation (2.53) is instead exploited to get a discrete control. This approach is called *Variational Discretization* and was introduced by Hinze in [Hin05], see also [Hin+09, Chapter 3.2.5] for further details.

**Remark 67.** *In the case $\alpha = 0$, problem ($\mathbb{P}_{kh}$) has at least one solution, but only $\bar{y}_{kh}$ and $\bar{p}_{kh}$ are unique, whereas an associated optimal control is in general non-unique. The reason is that $f \mapsto S_{kh}(f, y_0)$ is not injective in contrast to $f \mapsto S(f, y_0)$.*

### 2.3.1 Error estimates for the regularized problem

In what follows, we use the notation $y_{kh}(v) := S_{kh}(Bv, y_0)$ with $v \in U_{\mathrm{ad}}$, and $p_{kh}(h)$ is an abbreviation of the solution to (2.45) with right-hand side

$h \in L^2(I, H^{-1}(\Omega))$. Furthermore, $y(v)$ and $p(h)$ denote the continuous counterparts. Note that therefore we have $\bar{y} = y(\bar{u})$, $\bar{y}_{kh} = y_{kh}(\bar{u}_{kh})$, $\bar{p} = p(\bar{y} - y_d)$, and $\bar{p}_{kh} = p_{kh}(\bar{y}_{kh} - y_d)$.

The following Lemma provides a first step towards an error estimate with respect to the control and state discretization. It is the fully discrete variant of [DHV15, Lemma 5.1].

**Lemma 68.** *Let $\bar{u}$ and $\bar{u}_{kh}$ solve $(\mathbb{P})$ and $(\mathbb{P}_{kh})$, respectively, both for the same $\alpha \geq 0$. Then there holds*

$$\alpha\|\bar{u}_{kh} - \bar{u}\|_U^2 + \|\bar{y}_{kh} - y_{kh}(\bar{u})\|_I^2 \\ \leq \Big(B^*\Big(p_{kh}(\bar{y} - y_d) - \bar{p} + p_{kh}(y_{kh}(\bar{u}) - \bar{y})\Big), \bar{u} - \bar{u}_{kh}\Big)_U .$$

*Proof.* Inserting $\bar{u}_{kh}$ into (1.10) and $\bar{u}$ into (2.52) and adding up the resulting inequalities yields

$$\Big(\alpha(\bar{u}_{kh} - \bar{u}) + B^*(\bar{p}_{kh} - \bar{p}), \bar{u}_{kh} - \bar{u}\Big)_U \leq 0.$$

After some simple manipulations we obtain

$$\alpha\|\bar{u}_{kh} - \bar{u}\|_U^2 \leq \Big(B^*\Big(p_{kh}(\bar{y} - y_d) - \bar{p} + p_{kh}(y_{kh}(\bar{u})) - p_{kh}(\bar{y})\Big), \bar{u} - \bar{u}_{kh}\Big)_U \\ + \Big(B^*\Big(\bar{p}_{kh} - p_{kh}(y_{kh}(\bar{u}) - y_d)\Big), \bar{u} - \bar{u}_{kh}\Big)_U ,$$

and since the last line equals $-\|\bar{y}_{kh} - y_{kh}(\bar{u})\|_I^2$, we end up with the desired estimate by moving this term to the left. □

We can now prove an error estimate, which resembles the standard estimate for variational discretized controls. It is build upon [DHV15, Theorem 5.2]. Since we are interested in the limit behavior $\alpha \to 0$, we try to give a precise dependence of the right-hand side on $\alpha$. Note the splitting in terms of the quantities $d_0$ and $d_1$. In contrast to $d_0$, the term $d_1$ is *not* bounded if $\alpha \to 0$.

**Theorem 69.** *Let $\bar{u}$ and $\bar{u}_{kh}$ solve $(\mathbb{P})$ and $(\mathbb{P}_{kh})$, respectively, both for the same $\alpha \geq 0$. Then there exists a constant $\alpha_{\max} > 0$ independent of $k$*

*and h, so that for all $0 \le \alpha \le \alpha_{\max}$ (with the convention "$1/0 = \infty = d_1$" in the case of $\alpha = 0$) the estimate*

$$\begin{aligned}\sqrt{\alpha}\|\bar{u}_{kh} - \bar{u}\|_U &+ \|\bar{y}_{kh} - y_{kh}(\bar{u})\|_I \\ &\le C \min\left(\frac{k^2+h^2}{\sqrt{\alpha}} d_0, (k+h)\sqrt{\|\bar{u}_{kh} - \bar{u}\|_U \sqrt{d_0}}\right) \\ &\quad + C\min\left(k^2 d_1, k d_0\right) + C h^2 d_0 \\ &\le C \max(d_0+1, \sqrt{d_0}) \min\left(\frac{k^2}{\alpha} + \frac{h^2}{\sqrt{\alpha}}, k+h\right)\end{aligned} \tag{2.54}$$

*is satisfied. Here, the constants*

$$d_0 = d_0(\bar{u}) = \|y_d\|_{H^1(I,L^2(\Omega))} + \|\nabla y_d(T)\| + \|B\bar{u}\|_I + \|\nabla y_0\|$$

*and*

$$d_1 = d_1(\bar{u}) = \|B\bar{u}\|_{H^1(I,L^2(\Omega))} + \|\nabla B\bar{u}(0)\| + \|\nabla \Delta y_0\|$$

*are from the estimates* (1.25) *in Lemma 12.*

*Proof.* We split the right-hand side of the estimate from Lemma 68 and get with the Cauchy-Schwarz inequality

$$\begin{aligned}&\alpha\|\bar{u}_{kh} - \bar{u}\|_U^2 + \|\bar{y}_{kh} - y_{kh}(\bar{u})\|_I^2 \\ &\le \|p_{kh}(\bar{y}-y_d)-\bar{p}\|_I\|\bar{u}-\bar{u}_{kh}\|_U + \left(B^*\left(p_{kh}\left(y_{kh}\left(\bar{u}\right) - \bar{y}\right)\right), \bar{u} - \bar{u}_{kh}\right)_U = I + II.\end{aligned}$$

With the help of Lemma 62 and Lemma 12, we conclude

$$\|p_{kh}(\bar{y} - y_d) - \bar{p}\|_I \le C(k^2+h^2)(\|\bar{p}_{tt}\|_I + \|\Delta \bar{p}_t\|_I) \le C(k^2+h^2)d_0.$$

Now we use Cauchy's inequality to obtain

$$I \le \frac{C}{\alpha}\|p_{kh}(\bar{y} - y_d) - \bar{p}\|_I^2 + \frac{\alpha}{2}\|\bar{u} - \bar{u}_{kh}\|_U^2.$$

Here, the second addend can be moved to the left. Both estimates can be summarized as

$$\sqrt{I} \le C \min\left(\frac{k^2+h^2}{\sqrt{\alpha}} d_0, (k+h)\sqrt{\|\bar{u}_{kh} - \bar{u}\|_U \sqrt{d_0}}\right).$$

The addend $II$ can be estimated as

$$II = (y_{kh}(\bar{u}) - \bar{y}, y_{kh}(\bar{u}) - \bar{y}_{kh})_I \leq \frac{1}{2}(\|y_{kh}(\bar{u}) - \bar{y}\|_I^2 + \|y_{kh}(\bar{u}) - \bar{y}_{kh}\|_I^2).$$

We move the second term to the left. Note that in the previous estimate $\bar{y}$ can be replaced by $\mathcal{P}_{Y_k}\bar{y}$ by definition of $\mathcal{P}_{Y_k}$. We thus can invoke either the error estimate of the state equation (2.49) from Lemma 59 or the superconvergence result from Lemma 66. In conclusion, we have

$$\sqrt{II} \leq C \min\left((k + h^2)d_0, k^2 d_1 + h^2 d_0\right) = \min\left(kd_0, k^2 d_1\right) + h^2 d_0.$$

Together with the estimate for $\sqrt{I}$, we obtain the first inequality of the claim.

For the second inequality, we first note that with the help of the projection formula (1.12), Lemma 11, and Lemma 6 one immediately derives the estimate

$$\begin{aligned} &\|\bar{u}\|_{H^1(I,\tilde{U})} + \|B\bar{u}(0)\|_{H^1(\Omega)} \\ &\leq \frac{C}{\alpha}(\|\bar{p}\|_{H^1(I,L^2(\Omega))} + \|\bar{p}(0)\|_{H^1(\Omega)}) + C(a) + C(b) \\ &\leq \frac{C}{\alpha}(\|y_d\|_I + \|\bar{u}\|_U + \|y_0\|_{H^1(\Omega)}) + C(a) + C(b) \end{aligned} \tag{2.55}$$

where $\tilde{U} \in \{\mathbb{R}^D, L^2(\Omega)\}$, depending on whether located or distributed controls are given, and $C(x) = \|x\|_{H^1(I,\tilde{U})} + \|x(0)\|_X$ with $X = H^1(\Omega)$ (distributed controls) or $X = \mathbb{R}^D$ (located controls). This term is bounded due to Assumption 10.

Since there exists an $\alpha_{\max} > 0$, depending only on the data $a$, $b$, $y_0$, $y_d$, such that

$$\forall\ 0 \leq \alpha \leq \alpha_{\max} : \qquad d_1 + d_1^+ \leq C\frac{1}{\alpha}(d_0 + 1) \tag{2.56}$$

holds with $d_1^+ := d_1^+(\bar{u})$ from the estimates (1.25) in Lemma 12, and since $\sqrt{\|\bar{u}_{kh} - \bar{u}\|_U}$ is bounded independently of $\alpha$ due to the definition of $U_{\mathrm{ad}}$, we get the claim. □

From the proof of the previous Theorem, one can immediately derive a first robust (with respect to $\alpha \to 0$) error bound for the optimal state.

**Corollary 70.** *Let $\bar{u}$ and $\bar{u}_{kh}$ solve* ($\mathbb{P}$) *and* ($\mathbb{P}_{kh}$)*, respectively, both for the same arbitrarily chosen $\alpha \geq 0$. Then there holds with a constant $C > 0$ independent of $\alpha$ that*

$$\|\bar{y} - \bar{y}_{kh}\|_I \leq C(k+h)\max(d_0 + 1, \sqrt{d_0})$$

*with $d_0$ as in Theorem 69.*

*Proof.* Combining

$$\|\bar{y} - \bar{y}_{kh}\|_I \leq \|y_{kh}(\bar{u}) - \bar{y}_{kh}\|_I + \|\bar{y} - y_{kh}(\bar{u})\|_I$$

with the previous Theorem and (2.49) from Lemma 59 proves the claim. □

Now, from the above Theorem we derive further non-robust estimates for the discrete state and adjoint state. Finally, we prove second order convergence for $\pi_{P_k^*}\bar{y}_{kh}$, i.e., the piecewise linear interpolation on the dual grid of the optimal state. This function is obtained for free from $\bar{y}_{kh}$, since $\bar{y}_{kh}$ only has to be evaluated on the dual time grid. Compare [DHV15, Theorem 5.3] for the convergence of the interpolation in the semidiscrete case.

**Corollary 71.** *Let $\bar{u}$ and $\bar{u}_{kh}$ denote the solutions to* ($\mathbb{P}$) *and* ($\mathbb{P}_{kh}$)*, respectively, both for the same sufficiently small $\alpha > 0$ (in the sense of Theorem 69). With $d_0$ and $d_1$ as in Theorem 69 and*

$$d_1^+ := d_1^+(\bar{u}) = d_1(\bar{u}) + C(\|\partial_t^2 y_d\|_I + \|\nabla\partial_t y_d(T)\| + \|\nabla\Delta y_d(T)\| + \|\nabla B\bar{u}(T)\|$$

*from the estimates* (1.25) *in Lemma 12, the estimates*

$$\begin{aligned}
\|\bar{u} - \bar{u}_{kh}\|_U &\leq C(\frac{k^2 d_1}{\sqrt{\alpha}} + \frac{k^2 + h^2}{\alpha} d_0) \leq C(\frac{k^2}{\alpha^{3/2}} + \frac{h^2}{\alpha})(d_0 + 1),\\
\|\bar{y} - \bar{y}_{kh}\|_I &\leq C(k + \frac{k^2}{\alpha} + \frac{h^2}{\sqrt{\alpha}})(d_0 + 1), \quad \text{and}\\
\alpha\|\bar{u} - \bar{u}_{kh}\|_{L^\infty(I,\tilde{U})} &+ \|\bar{p} - \bar{p}_{kh}\|_{L^\infty(I,L^2(\Omega))} + \|\bar{y} - \pi_{P_k^*}\bar{y}_{kh}\|_I\\
&\leq C(k^2 d_1^+ + \frac{k^2 + h^2}{\sqrt{\alpha}} d_0) \leq C(\frac{k^2}{\alpha} + \frac{h^2}{\sqrt{\alpha}})(d_0 + 1)
\end{aligned}$$

*hold with $\tilde{U} \in \{\mathbb{R}^D, L^2(\Omega)\}$ depending on whether located or distributed controls are given.*

*Proof.* The first estimate for the optimal control and the estimate for the optimal state follow from Theorem 69. For the latter, we argue as in the proof of Corollary 70.

For the optimal adjoint state, we split the error into three parts to obtain with $L := L^\infty(I, L^2(\Omega))$

$$\begin{aligned}&\|\bar p - \bar p_{kh}\|_L \\ &\le \|\bar p - p_{kh}(\bar y - y_d)\|_L + \|p_{kh}(\mathcal{P}_{Y_k}\bar y - y_{kh}(\bar u))\|_L + \|p_{kh}(y_{kh}(\bar u) - \bar y_{kh})\|_L.\end{aligned}$$

With the second error estimate from Lemma 65, the regularity given in Lemma 12, and the estimate from Lemma 66, we conclude

$$\|\bar p - p_{kh}(\bar y - y_d)\|_L + \|p_{kh}(\mathcal{P}_{Y_k}\bar y - y_{kh}(\bar u))\|_L \le C(h^2 d_0 + k^2 d_1^+),$$

since $d_1 \le d_1^+$.

Stability from Lemma 56 combined with Theorem 69 gives the estimate

$$\|p_{kh}(y_{kh}(\bar u) - \bar y_{kh})\|_L \le C\frac{k^2 + h^2}{\sqrt{\alpha}} d_0 + Ck^2 d_1 + Ch^2 d_0.$$

From this, we get

$$\|\bar p - \bar p_{kh}\|_L \le C\frac{k^2 + h^2}{\sqrt{\alpha}} d_0 + Ck^2 d_1^+.$$

The projection formulae (1.12) and (2.53), Lipschitz continuity of the projection given in Lemma 11, and stability of $B^*$ yield

$$\|\bar u - \bar u_{kh}\|_{L^\infty(I,\tilde U)} \le C\frac{1}{\alpha}\|\bar p - \bar p_{kh}\|_L.$$

Together which with the just established estimate this yields the pointwise-in-time error estimate for the optimal control.

Let us now treat the error $\|\bar y - \pi_{P_k^*}\bar y_{kh}\|_I$. We split the norm into three parts:

$$\begin{aligned}&\|\bar y - \pi_{P_k^*}\bar y_{kh}\|_I \\ &\qquad \le \|\bar y - \pi_{P_k^*} y_k(\bar u)\|_I + \|\pi_{P_k^*}(y_k(\bar u) - y_{kh}(\bar u))\|_I + \|\pi_{P_k^*}(y_{kh}(\bar u) - \bar y_{kh})\|_I.\end{aligned}$$

For the first term we use the superconvergence Lemma 47 and regularity from Lemma 12 to get the estimate

$$\|\bar y - \pi_{P_k^*} y_k(\bar u)\|_I \le Ck^2 d_1. \tag{2.57}$$

The second addend can be estimated using stability of the interpolation operator given by Lemma 33, the connection between semidiscrete and fully discrete state given by the estimate (2.51) and Lemma 58. Altogether, this yields

$$\|\pi_{P_k^*}(y_k(\bar{u}) - y_{kh}(\bar{u}))\|_I \le C\|y_k(\bar{u}) - y_{kh}(\bar{u})\|_I \le C(k^2 d_1 + h^2 d_0).$$

Finally, for the third term, we use again Lemma 33 and Theorem 69 to obtain the estimate

$$\|\pi_{P_k^*}(y_{kh}(\bar{u}) - \bar{y}_{kh})\|_I \le C(k^2 d_1 + \frac{k^2 + h^2}{\sqrt{\alpha}} d_0).$$

Collecting all estimates leads to

$$\|\bar{y} - \pi_{P_k^*}\bar{y}_{kh}\|_I \le C(k^2 d_1 + \frac{k^2 + h^2}{\sqrt{\alpha}} d_0).$$

Using the inequality (2.56), we can finally reduce the non-robust constants $d_1$ and $d_1^+$ to the robust one $d_0$. □

Let us comment on the estimates of Theorem 69 and Corollary 71. These estimates show that if $\alpha > 0$ is fixed, we have convergence rates $h^2 + k^2$ except for the state error. Invoking the regularization error, one obtains estimates for the total error between the limit problem and the discrete regularized one. From this, a coupling rule for the parameters $\alpha$, $k$ and $h$ can be derived.

As an example, consider the error in the projected state for the special case $\kappa = 1$. With the help of Theorem 19.3 (or 4), and Corollary 71 we get with the inequality (2.56) the estimate

$$\begin{aligned}\|\bar{y}_0 - \pi_{P_k^*}(\bar{y}_{kh})\|_I &\le \|\bar{y}_0 - \bar{y}_\alpha\|_I + \|\bar{y}_\alpha - \pi_{P_k^*}(\bar{y}_{kh})\|_I \\ &\le C(\alpha + k^2 d_1^+ + \frac{k^2 + h^2}{\sqrt{\alpha}} d_0) \le C(\alpha + \frac{k^2}{\alpha} + \frac{h^2}{\sqrt{\alpha}})(d_0 + 1), \quad (2.58)\end{aligned}$$

which implies $\|\bar{y}_0 - \pi_{P_k^*}(\bar{y}_{kh})\|_I \le Ck = Ch^{4/3}$ when setting $\alpha = k = h^{4/3}$.

However, if the decay estimate $d_1^+ \le \frac{C}{\alpha}$, i.e., (2.56), can be improved, we can get a better convergence rate (with respect to $k$) for the total error. In subsection 1.4.4 we saw that this is indeed possible in the bang-bang case.

Unfortunately, space convergence of order $h^2$ is not achievable in the above mentioned estimates if $\alpha$ tends to zero due to $\alpha$ in the denominator. To overcome this, we establish other estimates in the next subsection. The question of improving the decay estimate (2.56) is discussed in the next but one subsection using the estimates of the next subsection.

### 2.3.2 Robust error estimates

All the previous estimates (except Corollary 70) are not robust for $\alpha \to 0$, since $\alpha$ appears always in a denominator on the right-hand side. Especially, convergence of order $h^2$ is not achievable as discussed at the end of the previous subsection. With some refined analysis, however, one can show estimates which are robust with respect to $\alpha \to 0$.

A key ingredient is Lemma 16, which was also very important for the derivation of the regularization error.

Recall the notation from the beginning of subsection 2.3.1.

**Theorem 72.** *Let Assumption 15 be fulfilled so that either* (1.42) *or* (1.47) *from Theorem 19 holds. We denote the valid convergence rate for the control by* $\alpha^{\omega_1}$. *Then, either* (1.44) *or* (1.49) *is fulfilled. We abbreviate the corresponding convergence rate by* $\alpha^{\omega_2}$.

*Let* $\bar{u}_0$ *be the solution of* $(\mathbb{P}_0)$ *with associated state* $\bar{y}_0$. *For some* $\alpha \geq 0$ *let in addition* $\bar{u}_d := \bar{u}_{\alpha,kh} \in U_{\mathrm{ad}}$ *be a (compare Remark 67) or (if* $\alpha > 0$*) the solution of* $(\mathbb{P}_{kh})$ *with associated discrete state* $\bar{y}_d$ *and adjoint state* $\bar{p}_d$. *Recall that we fix* $D := 1$ *in the case of distributed controls. Then there holds*

$$\begin{aligned}
&\|\bar{u}_0 - \bar{u}_d\|_{L^1(A,\mathbb{R}^D)} \leq C(\alpha^{\omega_1} \\
&+ \|B^*(p_{kh} - p)(y(\bar{u}_d) - y_d)\|^{\kappa}_{L^\infty(A,\mathbb{R}^D)} + \|B^*(p_{kh} - p)(y(\bar{u}_d) - y_d)\|^{\frac{1}{1+1/\kappa}}_{L^1(A^c,\mathbb{R}^D)} \\
&+ \|B^* p_{kh}(y_{kh}(\bar{u}_d) - y(\bar{u}_d))\|^{\kappa}_{L^\infty(A,\mathbb{R}^D)} + \|B^* p_{kh}(y_{kh}(\bar{u}_d) - y(\bar{u}_d))\|^{\frac{1}{1+1/\kappa}}_{L^1(A^c,\mathbb{R}^D)})
\end{aligned} \tag{2.59}$$

*for the error in the control and*

$$\begin{aligned}\|\bar y_0 - \bar y_d\|_I &\le C(\alpha^{\omega_2}\\ &+ \|B^*(p_{kh}-p)(y(\bar u_d)-y_d)\|_{L^\infty(A,\mathbb{R}^D)}^{\frac{1+\kappa}{2}} + \|B^*(p_{kh}-p)(y(\bar u_d)-y_d)\|_{L^1(A^c,\mathbb{R}^D)}^{1/2}\\ &+ \|B^* p_{kh}(y_{kh}(\bar u_d)-y(\bar u_d))\|_{L^\infty(A,\mathbb{R}^D)}^{\frac{1+\kappa}{2}} + \|B^* p_{kh}(y_{kh}(\bar u_d)-y(\bar u_d))\|_{L^1(A^c,\mathbb{R}^D)}^{1/2}\\ &+ \|y_{kh}(\bar u_d)-y(\bar u_d)\|_I) \end{aligned} \tag{2.60}$$

*for the error in the state.*

*Proof.* To the estimate (1.39) from Lemma 16 with $u := \bar u_d$, i.e.,

$$C\|\bar u_d - \bar u_0\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} \le (-B^*\bar p_0, \bar u_0 - \bar u_d)_U, \tag{2.61}$$

we add the necessary condition (2.52) for $\bar u_d$ with $u := \bar u_0$, which can be rewritten as

$$\alpha\|\bar u_0 - \bar u_d\|_U^2 \le (\alpha \bar u_0 + B^*\bar p_d, \bar u_0 - \bar u_d)_U\,. \tag{2.62}$$

We end up with

$$\begin{aligned}&\|\bar u_0 - \bar u_d\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} + \alpha\|\bar u_0 - \bar u_d\|_U^2 + \|y(\bar u_0) - y(\bar u_d)\|_I^2\\ &\quad\le C\Big( - B^* p(y(\bar u_d)-y_d) + B^* p_{kh}(y_{kh}(\bar u_d)-y_d) + \alpha\bar u_0, \bar u_0 - \bar u_d\Big)_U\\ &\quad\le C\Big(\underbrace{B^*(p_{kh}-p)(y(\bar u_d)-y_d)}_{I} + \underbrace{B^* p_{kh}(y_{kh}(\bar u_d)-y(\bar u_d))}_{II}\\ &\qquad + \underbrace{\alpha\bar u_0}_{III}, \bar u_0 - \bar u_d\Big)_U. \end{aligned} \tag{2.63}$$

We now use Lemma 18, Cauchy's and Young's inequality to estimate $III$ as

$$\begin{aligned}\alpha(\bar u_0, \bar u_0 - \bar u_d)_U &\le \alpha C\left(\|T(\bar u_d - \bar u_0)\|_H + \|\bar u_d - \bar u_0\|_{L^1(A,\mathbb{R}^D)}\right)\\ &\le C\alpha^2 + \frac{1}{4}\|T(\bar u_d - \bar u_0)\|_H^2 + C\alpha^{1+\kappa} + \frac{1}{4}\|\bar u_d - \bar u_0\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa}.\end{aligned}$$

The $\alpha$-free terms can now be moved to the left, since $\|T(\bar u_d - \bar u_0)\|_H = \|y(\bar u_d) - y(\bar u_0)\|_I$. Note that $C\alpha^2$ can be omitted if $A = \Omega$, compare (1.51). Thus only the term $C\alpha^{2\,\omega_2}$ remains on the right-hand side.

For $I$ and $II$, we proceed with the help of Young's inequality to obtain

$$
\begin{aligned}
&(\sim, \bar{u}_0 - \bar{u}_d)_U \\
&= (\sim, \bar{u}_0 - \bar{u}_d)_{L^2(A,\mathbb{R}^D)} + (\sim, \bar{u}_0 - \bar{u}_d)_{L^2(A^c,\mathbb{R}^D)} \\
&\leq C\|\sim\|^{1+\kappa}_{L^\infty(A,\mathbb{R}^D)} + \frac{1}{4}\|\bar{u}_0 - \bar{u}_d\|^{1+1/\kappa}_{L^1(A,\mathbb{R}^D)} + \|\sim\|_{L^1(A^c,\mathbb{R}^D)}\|b-a\|_{L^\infty(A^c,\mathbb{R}^D)}
\end{aligned}
$$

and move the second addend to the left.

Finally, we end up with

$$
\begin{aligned}
&\|\bar{u}_0 - \bar{u}_d\|^{1+1/\kappa}_{L^1(A,\mathbb{R}^D)} + \alpha\|\bar{u}_0 - \bar{u}_d\|^2_U + \|y(\bar{u}_0) - y(\bar{u}_d)\|^2_I \leq C(\alpha^{2\,\omega_2} \\
&+\|B^*(p_{kh}-p)(y(\bar{u}_d)-y_d)\|^{1+\kappa}_{L^\infty(A,\mathbb{R}^D)} + \|B^*(p_{kh}-p)(y(\bar{u}_d)-y_d)\|_{L^1(A^c,\mathbb{R}^D)} \\
&+\|B^*p_{kh}(y_{kh}(\bar{u}_d)-y(\bar{u}_d))\|^{1+\kappa}_{L^\infty(A,\mathbb{R}^D)} + \|B^*p_{kh}(y_{kh}(\bar{u}_d)-y(\bar{u}_d))\|_{L^1(A^c,\mathbb{R}^D)}.
\end{aligned}
$$

From this we conclude the claim for the optimal control.

The just established estimate together with the decomposition

$$
\|\bar{y}_0 - \bar{y}_d\|_I \leq \|y_{kh}(\bar{u}_d) - y(\bar{u}_d)\|_I + \|y(\bar{u}_d) - y(\bar{u}_0)\|_I
$$

yields the claim for the optimal state. □

**Remark 73.** *The error estimate* (2.59) *in the previous Theorem for* $\alpha > 0$ *is also valid if* $\bar{u}_0$ *is replaced by* $\bar{u}_\alpha$*, i.e., the solution of* $(\mathbb{P})$ *for some* $\alpha > 0$*, since by Theorem 19 we can estimate*

$$
\begin{aligned}
\|\bar{u}_\alpha - \bar{u}_d\|_{L^1(A,\mathbb{R}^D)} &\leq \|\bar{u}_\alpha - \bar{u}_0\|_{L^1(A,\mathbb{R}^D)} + \|\bar{u}_0 - \bar{u}_d\|_{L^1(A,\mathbb{R}^D)} \\
&\leq C\alpha^{\omega_1} + \|\bar{u}_0 - \bar{u}_d\|_{L^1(A,\mathbb{R}^D)}.
\end{aligned} \tag{2.64}
$$

*Likewise, in* (2.60) *the state* $\bar{y}_0$ *can be replaced by* $\bar{y}_\alpha$.

*We will make use of this fact in the proof of the next Theorem.*

**Remark 74.** *In the previous Theorem, both total errors turned out to be of two ingredients: A discretization error and a regularization error, the latter given by the terms* $\alpha^{\omega_1}$ *and* $\alpha^{\omega_2}$*, respectively, which are precisely the regularization errors from Theorem 19. However, there is one exception: The bang-bang case with* $\kappa > 1$*. Here, we expect from* (1.46) *a regularization error of order* $\alpha^\kappa$*, which is weakend to* $\alpha^{(\kappa+1)/2}$ *in* (2.60)*. Fortunately, with some more effort, this setback can be overcome if all norms in* (2.60) *(without taking into account their exponents) show the same asymptotic behavior. This is the case for our discretization from above. Thus, we can improve the theorem in this special case, see the next theorem.*

In combination with the error estimates for the state and adjoint state equations previously derived, we can now prove a first error estimate between solutions of $(\mathbb{P}_{kh})$ and $(\mathbb{P}_0)$, which is robust if $\alpha$ tends to zero. In view of the numerical verification, we restrict ourselves now to the situation $A = \Omega_U$ and located controls.

**Theorem 75.** *Let the assumptions of Theorem 72 be fulfilled. Further, we assume located controls and $A = \Omega_U$ (measure condition on the whole domain).*

*Then there hold the estimates*

$$\|\bar{u}_0 - \bar{u}_d\|_U^2 + \|\bar{u}_0 - \bar{u}_d\|_{L^1(\Omega_U,\mathbb{R}^D)} \leq C\left(\alpha + h^2 + k\right)^\kappa \left(1 + d_0(\bar{u}_d)^\kappa\right) \tag{2.65}$$

*for the error in the control, for the auxiliary error*

$$\|\bar{y}_d - y_{kh}(\bar{u}_\alpha)\|_I^2 \leq C(h^2 + k) d_0(\bar{u}_\alpha)\Big(\alpha^\kappa + (h^2 + k)^\kappa d_0(\bar{u}_d)^\kappa\Big) \tag{2.66}$$

*where by $\bar{u}_\alpha$ we denote the solution of* $(\mathbb{P})$*, and*

$$\|\bar{y}_d - \bar{y}_0\|_I \leq C\left(\alpha^{\frac{1+\kappa}{2}} + (h^2 + k)^{\min(1,\frac{1+\kappa}{2})}\right)\left(1 + d_0(\bar{u}_d)^{\min(1,\frac{1+\kappa}{2})}\right) \tag{2.67}$$

*for the error in the state.*

*If $\kappa > 1$, we have the improved convergence rate*

$$\|\bar{y}_d - \bar{y}_0\|_I \leq C(\alpha^\kappa + h^2 + k)(1 + \max\Big(d_0(\bar{u}_d)^\kappa, d_0(\bar{u}_\alpha)\Big)), \tag{2.68}$$

*thus observe the regularization error* (1.46).

*Proof.* Combining Theorem 72 with the adjoint error estimate in Lemma 65, the adjoint stability from Lemma 56, the error estimate (2.49) in Lemma 59, and the regularity given in Lemma 12 and Remark 13, we achieve (2.65) and (2.67) except for the $U$ error in the control. This error can be derived from the corresponding $L^1$ error by the estimate

$$\begin{aligned}\|\bar{u}_0 - \bar{u}_d\|_U^2 &\leq \|\bar{u}_0 - \bar{u}_d\|_{L^\infty(\Omega_U,\mathbb{R}^D)}\|\bar{u}_0 - \bar{u}_d\|_{L^1(\Omega_U,\mathbb{R}^D)} \\ &\leq \|b - a\|_{L^\infty(\Omega_U,\mathbb{R}^D)}\|\bar{u}_0 - \bar{u}_d\|_{L^1(\Omega_U,\mathbb{R}^D)},\end{aligned} \tag{2.69}$$

which follows immediately from standard $L^p$ interpolation, see, e.g., [AF03, Theorem 2.11], and the definition of $U_{\mathrm{ad}}$.

Let us now tackle the improved state convergence, thereby proving the estimate (2.66). We split the error into three parts and obtain with the help of (1.46) and the error estimate (2.49) from Lemma 59

$$\begin{aligned}
&\|\bar{y}_d - \bar{y}_0\|_I^2 \\
&\le C\Big(\|\bar{y}_d - y_{kh}(\bar{u}_\alpha)\|_I^2 + \|y_{kh}(\bar{u}_\alpha) - y(\bar{u}_\alpha)\|_I^2 + \|y(\bar{u}_\alpha) - y(\bar{u}_0)\|_I^2\Big) \\
&\le C\Big(\|\bar{y}_d - y_{kh}(\bar{u}_\alpha)\|_I^2 + (h^2+k)^2 d_0^2(\bar{u}_\alpha) + \alpha^{2\kappa}\Big),
\end{aligned}$$

where we also used (1.25) from Lemma 12.

For the remaining term, we invoke Lemma 68 in combination with (2.59) and Remark 73 and setting $L := L^\infty(I, L^2(\Omega))$ we obtain with the stability of $B^*$ for located controls

$$\begin{aligned}
&\|\bar{y}_d - y_{kh}(\bar{u}_\alpha)\|_I^2 \\
&\le C\Big(\|p_{kh}(\bar{y}_\alpha - y_d) - \bar{p}_\alpha\|_L + \|p_{kh}(y_{kh}(\bar{u}_\alpha) - \bar{y}_\alpha)\|_L\Big)\|\bar{u}_\alpha - \bar{u}_d\|_{L^1(\Omega_U, \mathbb{R}^D)} \\
&\le C\Big(\|p_{kh}(\bar{y}_\alpha - y_d) - \bar{p}_\alpha\|_L + \|p_{kh}(y_{kh}(\bar{u}_\alpha) - \bar{y}_\alpha)\|_L\Big)\cdot \\
&\qquad \Big(\alpha^\kappa + \|(p_{kh} - p)(y(\bar{u}_d) - y_d)\|_L^\kappa + \|p_{kh}(y_{kh}(\bar{u}_d) - y(\bar{u}_d))\|_L^\kappa\Big).
\end{aligned} \tag{2.70}$$

Invoking again Lemma 65, Lemma 56, estimate (2.49) from Lemma 59, and Lemma 12, we get

$$\|\bar{y}_d - y_{kh}(\bar{u}_\alpha)\|_I^2 \le C(h^2+k)d_0(\bar{u}_\alpha)\Big(\alpha^\kappa + (h^2+k)^\kappa d_0^\kappa(\bar{u}_d)\Big),$$

which is the auxiliary estimate (2.66) of the statement.

If $\kappa > 1$, we can use the Cauchy-Schwarz inequality to get from it the estimate

$$\|\bar{y}_d - y_{kh}(\bar{u}_\alpha)\|_I^2 \le C\Big((h^2+k)^2 d_0^2(\bar{u}_\alpha) + \alpha^{2\kappa} + (h^2+k)^{1+\kappa} d_0(\bar{u}_\alpha) d_0^\kappa(\bar{u}_d)\Big).$$

Since $\kappa > 1$, collecting all estimates yields the inequality

$$\|\bar{y}_d - \bar{y}_0\|_I^2 \le C(\alpha^{2\kappa} + (h^2+k)^2 \max\Big(d_0^{2\kappa}(\bar{u}_d), d_0^2(\bar{u}_\alpha)\Big)),$$

from which we finally get (2.68). □

**Corollary 76.** *Let the assumptions of the previous theorem hold. For the adjoint state we have the error estimate*

$$\|\bar{p}_0 - \bar{p}_d\|_{L^\infty(I,L^2(\Omega))} \le C(\alpha^{\max(\frac{1+\kappa}{2},\kappa)} + (k+h^2)^{\min(1,\frac{1+\kappa}{2})} C(\bar{u}_d, \bar{u}_\alpha))$$

*with* $C(\bar{u}_d, \bar{u}_\alpha) = \max(1, d_0(\bar{u}_d), d_0(\bar{u}_\alpha))^{\max(1,\frac{1+\kappa}{2})}$.

*Proof.* Inspecting the proof of Corollary 71, we get the estimate

$$\|\bar{p}_\alpha - \bar{p}_d\|_{L^\infty(I,L^2(\Omega))} \le C((k+h^2)d_0(\bar{u}_\alpha) + \|y_{kh}(\bar{u}_\alpha) - \bar{y}_d\|_I).$$

The last addend can be estimated with the auxiliary estimate (2.66) from the previous theorem and Cauchy's inequality. We obtain

$$\|\bar{p}_\alpha - \bar{p}_d\|_{L^\infty(I,L^2(\Omega))} \le C(\alpha^{\max(\frac{1+\kappa}{2},\kappa)} + (k+h^2)^{\min(1,\frac{1+\kappa}{2})} C(\bar{u}_d, \bar{u}_\alpha)).$$

Invoking the regularization errors (1.44) and (1.46) proves the claim. □

### 2.3.3 Improved estimates for bang-bang controls

As motivated at the end of subsection 2.3.1, improving the decay estimate (2.56) with the help of the results of subsection 1.4.4 leads to improved (non-robust) error estimates. However, the convergence rate $h^2$ is not achievable in these estimates, but the robust estimates from Theorem 72 overcome this problem. On the other hand, in Theorem 72 we have $\bar{u}_d$ on the right-hand side instead of $\bar{u}_\alpha$, so that the results of subsection 1.4.4 can not be directly applied. Therefor, we have to estimate some additional terms in combination with Theorem 72 to finally get the desired improved estimates.

**Theorem 77.** *Let the assumptions of Theorem 72 be fulfilled. Further, we assume located controls and* $A = \Omega_U$ *up to a set of measure zero (measure condition on the whole domain). If* $\kappa < 1$*, we additionally require the* $\bar{p}_\alpha$*-measure condition* (1.56). *(For* $\kappa \ge 1$*, this condition is automatically met by Lemma 26.)*

*Then, for* $\alpha > 0$ *sufficiently small,* $d_0 := d_0(\bar{u}_\alpha)$ *given as in Theorem 69, and* $C_{ab}$ *defined in Lemma 28 it holds*

$$\|\bar{u}_0 - \bar{u}_d\|_U^2 + \|\bar{u}_0 - \bar{u}_d\|_{L^1(\Omega_U,\mathbb{R}^D)} \le C\left(\alpha + h^2 + k^2 \max(1, C_{ab}, \alpha^{\kappa/2-1})\right)^\kappa (1 + d_0^\kappa)$$

*for the error in the control.*

*Proof.* Let us recall the estimate (2.63) from the proof of Theorem 72, i.e.,

$$\begin{aligned}
&\|\bar u_0 - \bar u_d\|^{1+1/\kappa}_{L^1(A,\mathbb{R}^D)} + \alpha\|\bar u_0 - \bar u_d\|^2_U + \|y(\bar u_0) - y(\bar u_d)\|^2_I \\
&\le C\Big( - B^* p(y(\bar u_d) - y_d) + B^* p_{kh}(y_{kh}(\bar u_d) - y_d) + \alpha \bar u_0, \bar u_0 - \bar u_d\Big)_U,
\end{aligned}$$

which we rearrange as follows:

$$\begin{aligned}
&\|\bar u_0 - \bar u_d\|^{1+1/\kappa}_{L^1(A,\mathbb{R}^D)} + \alpha\|\bar u_0 - \bar u_d\|^2_U + \|y_{kh}(\bar u_0) - y_{kh}(\bar u_d)\|^2_I \\
&\le C\Big( \underbrace{-B^* p(y(\bar u_0) - y(\bar u_\alpha))}_{I} \underbrace{-B^* p(y(\bar u_\alpha) - y_d) + B^* p_{kh}(y_{kh}(\bar u_\alpha) - y_d)}_{IIa} \\
&\qquad \underbrace{+\alpha \bar u_0}_{IIb} \underbrace{+B^* p_{kh}(y_{kh}(\bar u_0) - y_{kh}(\bar u_\alpha))}_{III}, \bar u_0 - \bar u_d\Big)_U.
\end{aligned} \tag{2.71}$$

For term $III$, we use Cauchy's inequality to get

$$\begin{aligned}
&(y_{kh}(\bar u_0) - y_{kh}(\bar u_\alpha)), y_{kh}(\bar u_0) - y_{kh}(\bar u_d))_I \\
&\qquad \le C\|y_{kh}(\bar u_0) - y_{kh}(\bar u_\alpha)\|^2_I + \frac{1}{16}\|y_{kh}(\bar u_0) - y_{kh}(\bar u_d)\|^2_I,
\end{aligned}$$

and move the latter addend to the left-hand side of (2.71). We split the former addend with the help of (2.49) from Lemma 59 and the regularization errors (1.43) and (1.44) to obtain with the help of Young's inequality

$$\begin{aligned}
\|y_{kh}(\bar u_0) - y_{kh}(\bar u_\alpha)\|^2_I &\le C(\|(\hat y_{kh} - \hat y)(\bar u_0 - \bar u_\alpha)\|_I + \|y(\bar u_0) - y(\bar u_\alpha)\|_I)^2 \\
&\le C((k + h^2)\alpha^{\kappa/2} + \alpha^{\frac{1+\kappa}{2}})^2 \\
&\le C(k + h^2)^{2(\kappa+1)} + C\alpha^{1+\kappa}
\end{aligned} \tag{2.72}$$

where $\hat y_{kh}$ and $\hat y$ denote the solution operators for the state equation with initial value zero.

For $IIb$, we invoke again Young's inequality and the inclusion $\bar u_0 \in U_{\text{ad}} \subset L^\infty$ to get the estimate

$$\alpha(\bar u_0, \bar u_0 - \bar u_d)_U \le C\alpha\|\bar u_d - \bar u_0\|_{L^1(\Omega_U,\mathbb{R}^D)} \le C\alpha^{\kappa+1} + \frac{1}{16}\|\bar u_d - \bar u_0\|^{1+1/\kappa}_{L^1(\Omega_U,\mathbb{R}^D)}.$$

We now move the second summand to the left of (2.71) since $A = \Omega_U$ up to a set of measure zero.

The addend $IIa$ can be rewritten and estimated with again the help of Young's inequality to get

$$\begin{aligned}
&\Big(-B^*p(y(\bar u_\alpha)-y_d)+B^*p_{kh}(y_{kh}(\bar u_\alpha)-y_d),\bar u_0-\bar u_d\Big)_U\\
&\le C\Big(B^*(p_{kh}-p)(y(\bar u_\alpha)-y_d)+B^*p_{kh}(y_{kh}(\bar u_\alpha)-y(\bar u_\alpha)),\bar u_0-\bar u_d\Big)_U\\
&\le C\|B^*(p_{kh}-p)(y(\bar u_\alpha)-y_d)+B^*p_{kh}(y_{kh}(\bar u_\alpha)-y(\bar u_\alpha))\|^{1+\kappa}_{L^\infty(\Omega_U,\mathbb{R}^D)}\\
&\qquad+\frac{1}{16}\|\bar u_0-\bar u_d\|^{1+1/\kappa}_{L^1(\Omega_U,\mathbb{R}^D)}.
\end{aligned}$$

The last addend can now be moved to the left of (2.71).

For summand $I$, we add an additional term to get

$$\begin{aligned}
&\Big(-B^*p(y(\bar u_0)-y(\bar u_\alpha)),\bar u_0-\bar u_d\Big)_U\\
&\quad=\Big(B^*(p_{kh}-p)(y(\bar u_0)-y(\bar u_\alpha))-B^*p_{kh}(y(\bar u_0)-y(\bar u_\alpha)),\bar u_0-\bar u_d\Big)_U.
\end{aligned}$$

We estimate the second addend with the help of the regularization error (1.44) as

$$\Big(y(\bar u_0)-y(\bar u_\alpha),y_{kh}(\bar u_0)-y_{kh}(\bar u_d)\Big)_I\le C\alpha^{1+\kappa}+\frac{1}{16}\|y_{kh}(\bar u_0)-y_{kh}(\bar u_d)\|_I^2,$$

and move the second addend to the left of (2.71). For the remaining addend, we use again the above mentioned results and the estimate (2.69) to obtain

$$\begin{aligned}
&\Big(B^*(p_{kh}-p)(y(\bar u_0)-y(\bar u_\alpha)),\bar u_0-\bar u_d\Big)_U\\
&=\Big(y(\bar u_0)-y(\bar u_\alpha),(\hat y_{kh}-\hat y)(\bar u_0-\bar u_d)\Big)\\
&\le C\|y(\bar u_0)-y(\bar u_\alpha)\|_I^2+C\|(\hat y_{kh}-\hat y)(\bar u_0-\bar u_d)\|_I^2\\
&\le C\alpha^{1+\kappa}+C(k+h^2)^2\|\bar u_0-\bar u_d\|_U^2\\
&\le C\alpha^{1+\kappa}+C(k+h^2)^2\|\bar u_0-\bar u_d\|_{L^1(\Omega_U,\mathbb{R}^D)}\\
&\le C\alpha^{1+\kappa}+C(k+h^2)^{2(\kappa+1)}+\frac{1}{16}\|\bar u_0-\bar u_d\|^{1+1/\kappa}_{L^1(\Omega_U,\mathbb{R}^D)}
\end{aligned}$$

and move the last term to the left of (2.71).

Collecting all previous estimates, we with $L := L^\infty(I, L^2(\Omega))$ obtain

$$\begin{aligned}
&\|\bar{u}_0 - \bar{u}_d\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} + \alpha\|\bar{u}_0 - \bar{u}_d\|_U^2 + \|y_{kh}(\bar{u}_0) - y_{kh}(\bar{u}_d)\|_I^2 \\
&\leq C\Big(\alpha^{\kappa+1} + (k + h^2)^{2(\kappa+1)} + \|(p_{kh} - p)(y(\bar{u}_\alpha) - y_d)\|_L^{1+\kappa} \\
&\quad + \|p_{kh}(y_{kh}(\bar{u}_\alpha) - \mathcal{P}_{Y_k} y(\bar{u}_\alpha))\|_L^{1+\kappa}\Big).
\end{aligned}$$

Note that we introduced the orthogonal projection $\mathcal{P}_{Y_k}$ in the last addend, which is possible due to the definition of the fully discrete adjoint equation (2.45). Furthermore, we used stability of $B^*$ for located controls.

We combine the previous estimate with the (improved) adjoint error estimate from Lemma 65, the adjoint stability from Lemma 56, and the superconvergence result from Lemma 66, making use of the regularity given in Lemma 12, to get

$$\begin{aligned}
&\|\bar{u}_0 - \bar{u}_d\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} + \alpha\|\bar{u}_0 - \bar{u}_d\|_U^2 + \|y_{kh}(\bar{u}_0) - y_{kh}(\bar{u}_d)\|_I^2 \\
&\leq C\Big(\alpha + h^2 d_0 + k^2(1 + d_1^+(\bar{u}_\alpha))\Big)^{1+\kappa}.
\end{aligned} \tag{2.73}$$

With the help of the estimate given in Lemma 28 for $p = 2$, i.e.,

$$\|\partial_t \bar{u}_\alpha\|_{L^2(\Omega_U,\mathbb{R}^D)} \leq C \max(C_{ab}, \alpha^{\kappa/2-1}),$$

we conclude that for $\alpha > 0$ sufficiently small it holds

$$d_1^+(\bar{u}_\alpha) \leq C + C \max(C_{ab}, \alpha^{\kappa/2-1}). \tag{2.74}$$

In conclusion, we get

$$\begin{aligned}
&\|\bar{u}_0 - \bar{u}_d\|_{L^1(A,\mathbb{R}^D)}^{1+1/\kappa} + \alpha\|\bar{u}_0 - \bar{u}_d\|_U^2 + \|y_{kh}(\bar{u}_0) - y_{kh}(\bar{u}_d)\|_I^2 \\
&\leq C\Big(\alpha + h^2 d_0 + k^2 \max(1, C_{ab}, \alpha^{\kappa/2-1})\Big)^{1+\kappa}.
\end{aligned}$$

Finally, recall that the $U$ error in the control can be derived from the corresponding $L^1$ error using the estimate (2.69). □

From the previous theorem we get coupling rules for $\alpha$ and $k$, always with $\alpha = h^2$, and convergence rates, which are depicted in the following table.

Note that in any case we get a better rate than $k^\kappa$ proven in Theorem 75.

| $\alpha =$ | $\|\bar{u}_d - \bar{u}_0\|_{L^1(\Omega_U,\mathbb{R}^D)} \leq C \ldots$ | if |
|---|---|---|
| $k^{4/(4-\kappa)}$ | $\alpha^\kappa = h^{2\kappa} = k^{4\kappa/(4-\kappa)}$ | $\kappa < 2$ |
| $k^2$ | $\alpha^\kappa = h^{2\kappa} = k^{2\kappa}$ | $\kappa \geq 2$ |

Table 2.1: Coupling and convergence implied by Theorem 77.

**Corollary 78.** *Let the assumptions of the previous Theorem hold. For the adjoint and the projected state we have the error estimate*

$$\begin{aligned}&\|\bar{p}_0 - \bar{p}_d\|_{L^\infty(I,L^2(\Omega))} + \|\bar{y}_0 - \pi_{P_k^*}\bar{y}_d\|_I \\ &\qquad \leq C\alpha^{\max(\frac{\kappa+1}{2},\kappa)} + C\Big(h^2 d_0 + k^2 \max(1, C_{ab}, \alpha^{\kappa/2-1})\Big)^{\min(1,\frac{\kappa+1}{2})}.\end{aligned}$$

*Proof.* Inspecting the proof of Corollary 71, we obtain the estimate

$$\|\bar{p}_\alpha - \bar{p}_d\|_{L^\infty(I,L^2(\Omega))} + \|\bar{y}_\alpha - \pi_{P_k^*}\bar{y}_d\|_I \leq C(k^2 d_1^+ + h^2 d_0 + \|y_{kh}(\bar{u}_\alpha) - \bar{y}_d\|_I).$$

To estimate the last addend, let us first combine the estimate (2.73) from the proof of Theorem 77 with Remark 73 to get

$$\|\bar{u}_\alpha - \bar{u}_d\|_{L^1(A,\mathbb{R}^D)} \leq C\Big(\alpha + h^2 d_0 + k^2(1 + d_1^+(\bar{u}_\alpha))\Big)^\kappa.$$

With this estimate, we now follow the proof of Theorem 75 from the entry point (2.70) onwards. We obtain

$$\begin{aligned}&\|\bar{p}_\alpha - \bar{p}_d\|^2_{L^\infty(I,L^2(\Omega))} + \|\bar{y}_\alpha - \pi_{P_k^*}\bar{y}_d\|^2_I \\ &\quad \leq C\left(\left(h^2 d_0 + k^2 d_1^+\right)^2 + \left(h^2 d_0 + k^2 d_1^+\right)\left(\alpha + h^2 d_0 + k^2\left(1 + d_1^+\right)\right)^\kappa\right).\end{aligned}$$

With Young's inequality, the regularization error (1.44), property (1.45), and the decay estimate (2.74), we finally get the claim. □

# 3 Numerics

We will now consider some test examples in order to finally validate numerically the results of the previous chapters.

As we have previously said, we solve numerically the regularized problem $(\mathbb{P}_{kh})$ for some $\alpha > 0$ as an approximation of the limit problem $(\mathbb{P}_0)$. Thus, we have the influence of two errors: The regularization error in dependence of the parameter $\alpha > 0$ and the discretization error due to space and time approximation. The second error depends on the fineness of the space and time grid, respectively, thus on the parameters $h$ and $k$.

We first consider the time discretization error for fixed positive $h$ and $\alpha$ by taking $k \to 0$. Here, we mainly recall the discussion of [DHV15]. In addition to the semidiscrete error analysis in [DHV15], the discussion is now founded on the fully discrete estimates of the previous chapter. Therefore, the numerical behavior of the error is added if $h \to 0$, again for fixed $\alpha > 0$, but now of course with fixed $k$ instead of $h$.

Second, we investigate the regularization error for fixed small discretization parameters $k$ and $h$ in dependence of the parameter $\kappa$ from the measure condition (1.38) if $\alpha \to 0$.

As a third step, we couple regularization and discretization parameters as proposed by Theorem 77 and Table 2.1.

In all examples we make use of the fact that instead of the linear control operator $B$, given by (1.3), we can also use an *affine linear* control operator

$$\tilde{B} : U \to L^2(I, H^{-1}(\Omega))\,, \quad u \mapsto g_0 + Bu \tag{3.1}$$

where $g_0$ is a fixed function. If we assume that $g_0$ is an element of the space $H^1(I, L^2(\Omega))$ with $g_0(0) \in H_0^1(\Omega)$ and $g_0(T) \in H_0^1(\Omega)$, the preceding theory remains valid since $g_0$ can be interpreted as a modification of $y_d$.

## 3.1 The discretization error for fixed $\alpha > 0$

The following first example is taken from [DHV15, Section 6.2]. It is an example for $(\mathbb{P})$ with $\alpha > 0$ fixed. Here, we denote the optimal solution triple by $(\bar{u}, \bar{y}, \bar{p})$.

With a space-time domain $\Omega \times I := (0,1)^2 \times (0, 0.5)$, we consider one located control function $\bar{u}$, i.e., $D := 1$, and a constant $a := 2$, not to be confused with the lower bound $a_1$ of the admissible set $U_{\mathrm{ad}}$ defined below. This constant $a$ influences the number of switching points between the active and inactive set. Furthermore, we define the functions

$$\begin{aligned} g_1(x_1, x_2) &:= \sin(\pi x_1)\sin(\pi x_2)\,, \\ w_a(t, x_1, x_2) &:= \cos\left(\frac{t}{T}\,2\pi a\right) \cdot g_1(x_1, x_2)\,, \\ \bar{y}(t, x_1, x_2) &:= w_a(t, x_1, x_2)\,, \quad \text{and} \\ \bar{p}(t, x_1, x_2) &:= w_a(t, x_1, x_2) - w_a(T, x_1, x_2)\,. \end{aligned} \tag{3.2}$$

Consequently, the initial value of the optimal state $\bar{y}$ is

$$y_0(x_1, x_2) = \bar{y}(0, x_1, x_2) = g_1(x_1, x_2)\,,$$

and for the other problem data we obtain

$$g_0 = g_1 2\pi \left(-\frac{a}{T}\sin\left(\frac{t}{T}\,2\pi a\right) + \pi\cos\left(\frac{t}{T}\,2\pi a\right)\right) - B\bar{u}\,, \tag{3.3}$$

$$y_d = g_1\left(\cos\left(\frac{t}{T}\,2\pi a\right)(1 - 2\pi^2) - \frac{2\pi a}{T}\sin\left(\frac{t}{T}\,2\pi a\right) + 2\pi^2\cos(2\pi a)\right)\,,$$

and the optimal control

$$\bar{u} = P_{U_{\mathrm{ad}}}\left(-\frac{1}{4\alpha}\cos\left(\frac{t}{T}\,2\pi a\right) + \frac{1}{4\alpha}\right)\,.$$

Here, we use the fact that the adjoint operator of $B$ is given by

$$(B^* z)(t) = \int_\Omega z(t, x_1, x_2) \cdot g_1(x_1, x_2)\, dx_1 dx_2\,,$$

compare (1.4). Note that we consider the adjoint of $B$, not of $\tilde{B}$.

Finally, we choose the regularization parameter $\alpha := 1$ and define the bounds of the admissible set $U_{\text{ad}}$ as $a_1 := 0.2$ and $b_1 := 0.4$.

Note that this example fulfills Assumption 10.

We solve $(\mathbb{P}_{kh})$ numerically with the above data using a fixed-point iteration for equation (2.53). Each fixed-point iteration is initialized with the starting value $u_{kh}^{(0)} := a_1$ which is the lower bound of the admissible set. As a stopping criterion for the fixed-point iteration, we require for the discrete adjoint states belonging to the current and the last iterate that

$$\|B^* \left(p_{kh}^{(i)} - p_{kh}^{(i-1)}\right)\|_{L^\infty(\Omega\times I)} < t_0$$

where $t_0 := 10^{-5}$ is a prescribed threshold.

### 3.1.1 Error in time ($k \to 0$, $h$ and $\alpha > 0$ fixed)

We discretize in space with a fixed number of nodes $\text{Nh} = (2^7+1)^2 = 16\,641$. We examine the behavior of the temporal convergence by considering a sequence of meshes with $\text{Nk} = (2^\ell + 1)$ nodes at refinement levels $\ell = 1, 2, 3, 4, 5, 6, 7, 8$.

Table 3.1 shows the behavior of several errors in time between the exact control $\bar{u}$ and its computed discretized counterpart $u_{kh}$, obtained by the fixed-point iteration. Furthermore, the *experimental order of convergence* (EOC) is given. The table indicates an error behavior of $\mathcal{O}(k^2)$ for the $L^2$ error in the control, which is in accordance with Theorem 69. Furthermore, the error of the adjoint, see Table 3.4, shows the same behavior as expected by Corollary 71. Here, we note that the EOC deteriorates in our numerical example if the temporal error reaches the size of the spatial error, which in the numerical investigations is fixed through the choice of Nh given above. See, e.g., the last lines in Table 3.1, Table 3.4, and Table 3.6.

Since the state is discretized piecewise constant in time, the order of convergence is only one. This is depicted in Table 3.2.

A better and second order convergent approximation of the state is given by the projection $\pi_{P_k^*} y_{kh}$ of the computed discrete state $y_{kh}$, see Corollary 71 and for the corresponding numerical results see Table 3.3. This better approximation of the state can be obtained without further numerical effort: One only has to interpret the vector containing the values of $y_k$ on each interval $I_m$ as a vector of linearly-in-time linked values on the gridpoints of the dual grid $t_1^* < \cdots < t_M^*$.

Figure 3.1 illustrates the convergence of $u_{kh}$ to $\bar{u}$. Note that the intersection points between the inactive set $\mathcal{I}_{kh} := \{t \in I \mid a < u_{kh}(t) < b\}$ and the active set $\mathcal{A}_{kh} := I \backslash \mathcal{I}_{kh}$ need not coincide with the time grid points since we use variational discretization for the control.

Let us further note that the number of fixed-point iterations does not depend on the fineness of the time grid size. In our example, two iterations suffice to reach the above mentioned threshold $t_0$.

| $\ell$ | $\|\bar{u} - u_{kh}\|$ $L^1(I,\mathbb{R})$ | $\|\bar{u} - u_{kh}\|$ $L^2(I,\mathbb{R})$ | $\|\bar{u} - u_{kh}\|$ $L^\infty(I,\mathbb{R})$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.04925427 | 0.09237138 | 0.20000000 | / | / | / |
| 2 | 0.00256632 | 0.01106114 | 0.07336869 | 4.26 | 3.06 | 1.45 |
| 3 | 0.00403215 | 0.01144324 | 0.04704583 | -0.65 | -0.05 | 0.64 |
| 4 | 0.00069342 | 0.00204495 | 0.00893696 | 2.54 | 2.48 | 2.40 |
| 5 | 0.00016762 | 0.00050729 | 0.00249463 | 2.05 | 2.01 | 1.84 |
| 6 | 0.00003989 | 0.00011939 | 0.00064497 | 2.07 | 2.09 | 1.95 |
| 7 | 0.00000948 | 0.00003227 | 0.00020672 | 2.07 | 1.89 | 1.64 |
| 8 | 0.00000764 | 0.00002142 | 0.00009457 | 0.31 | 0.59 | 1.13 |

Table 3.1: First example: Errors and EOC in the control ($\alpha > 0$, $k \to 0$).

| $\ell$ | $\|\bar{y} - y_{kh}\|$ $L^1(I,L^1(\Omega))$ | $\|\bar{y} - y_{kh}\|$ $L^2(I,L^2(\Omega))$ | $\|\bar{y} - y_{kh}\|$ $L^\infty(I,L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.19644927 | 0.41294081 | 2.24551425 | / | / | / |
| 2 | 0.12998104 | 0.25395823 | 1.25550373 | 0.60 | 0.70 | 0.84 |
| 3 | 0.05657200 | 0.11245327 | 0.66590819 | 1.20 | 1.18 | 0.91 |
| 4 | 0.02614960 | 0.05648390 | 0.38823773 | 1.11 | 0.99 | 0.78 |
| 5 | 0.01277718 | 0.02830060 | 0.19379413 | 1.03 | 1.00 | 1.00 |
| 6 | 0.00634467 | 0.01413902 | 0.09325101 | 1.01 | 1.00 | 1.06 |
| 7 | 0.00316732 | 0.00702903 | 0.04324651 | 1.00 | 1.01 | 1.11 |
| 8 | 0.00158309 | 0.00343000 | 0.01843334 | 1.00 | 1.04 | 1.23 |

Table 3.2: First example: Errors and EOC in the state ($\alpha > 0$, $k \to 0$).

| $\ell$ | $\lVert\bar{y} - \pi_{P_k^*} y_{kh}\rVert$ $L^1(I, L^1(\Omega))$ | $\lVert\bar{y} - \pi_{P_k^*} y_{kh}\rVert$ $L^2(I, L^2(\Omega))$ | $\lVert\bar{y} - \pi_{P_k^*} y_{kh}\rVert$ $L^\infty(I, L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.19734452 | 0.42154165 | 2.65669891 | / | / | / |
| 2 | 0.13173168 | 0.25800727 | 1.39668789 | 0.58 | 0.71 | 0.93 |
| 3 | 0.03422500 | 0.07418402 | 0.40783930 | 1.94 | 1.80 | 1.78 |
| 4 | 0.01080693 | 0.02168391 | 0.15176831 | 1.66 | 1.77 | 1.43 |
| 5 | 0.00282859 | 0.00567595 | 0.04685968 | 1.93 | 1.93 | 1.70 |
| 6 | 0.00071212 | 0.00143268 | 0.01229008 | 1.99 | 1.99 | 1.93 |
| 7 | 0.00017551 | 0.00035509 | 0.00311453 | 2.02 | 2.01 | 1.98 |
| 8 | 0.00004104 | 0.00008530 | 0.00078765 | 2.10 | 2.06 | 1.98 |

Table 3.3: First example: Errors and EOC in the projected state ($\alpha > 0$, $k \to 0$).

### 3.1.2 Error in space ($h \to 0$, $k$ and $\alpha > 0$ fixed)

Let us now examine the behavior of the spatial convergence by considering a sequence of meshes with Nh $= (2^\ell + 1)^2$ nodes at refinement levels $\ell = 1, 2, 3, 4, 5, 6$. In time, we discretize with a fixed number of nodes Nk $= (2^{13} + 1) = 8\,193$. All other parameters remain unchanged.

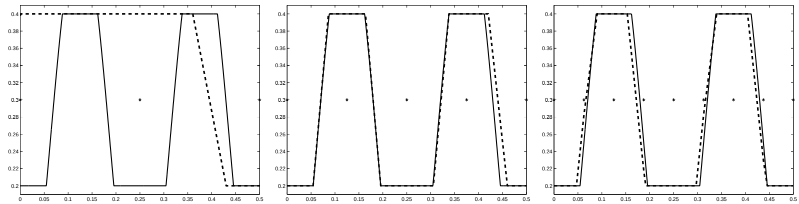
From the Tables 3.5, 3.6, 3.7, and 3.8, we observe a convergence rate of $\mathcal{O}(h^2)$ for the quantities from above, where a $\mathcal{O}(k^2)$ behavior was observed. This second order convergence in space also holds for the optimal state, which is in accordance with the theory, see again Corollary 71.

For the convergence in the optimal control, see also Figure 3.2.

## 3.2 Behavior of the regularization error

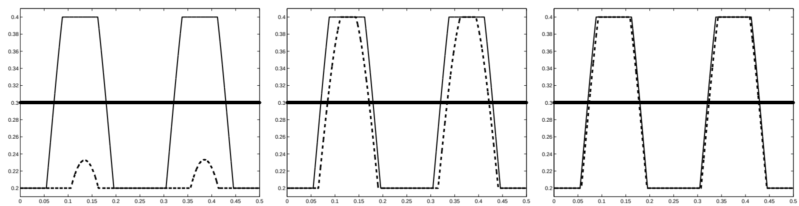
We now want to validate the improved convergence rates for the regularization error given in Theorem 19.3.

Here, we report only on the errors in the optimal control since we observed no or only poor convergence in the error of the optimal state and adjoint state, respectively. This might be due to the fact that the influence of the space- and time-discretization error is much larger than that of the regularization error in higher dimensions. This phenomenon was also observed for elliptic problems, compare [WW11a].



(a) $\ell = 1$ (b) $\ell = 2$ (c) $\ell = 3$

Figure 3.1: First example: Optimal control $\bar{u}$ (solid) and computed counterpart $u_{kh}$ (dashed) over time at refinement level $\ell$ ($\alpha > 0$, $k \to 0$). For $\ell \geq 4$, a difference is not visible any more.



(a) $\ell = 1$ (b) $\ell = 2$ (c) $\ell = 3$

Figure 3.2: First example: Optimal control $\bar{u}$ (solid) and computed counterpart $u_{kh}$ (dashed) over time at refinement level $\ell$ ($\alpha > 0$, $h \to 0$). For $\ell \geq 4$, a difference is not visible any more.

| $\ell$ | $\|\bar{p}-p_{kh}\|$ $L^1(I,L^1(\Omega))$ | $\|\bar{p}-p_{kh}\|$ $L^2(I,L^2(\Omega))$ | $\|\bar{p}-p_{kh}\|$ $L^\infty(I,L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.20659855 | 0.46853028 | 2.86360259 | / | / | / |
| 2 | 0.03491931 | 0.08118048 | 0.56829981 | 2.56 | 2.53 | 2.33 |
| 3 | 0.01994220 | 0.04100552 | 0.20495644 | 0.81 | 0.99 | 1.47 |
| 4 | 0.00440890 | 0.00895349 | 0.05815307 | 2.18 | 2.20 | 1.82 |
| 5 | 0.00105993 | 0.00215639 | 0.01668075 | 2.06 | 2.05 | 1.80 |
| 6 | 0.00026116 | 0.00053258 | 0.00447036 | 2.02 | 2.02 | 1.90 |
| 7 | 0.00006984 | 0.00014824 | 0.00116014 | 1.90 | 1.85 | 1.95 |
| 8 | 0.00004199 | 0.00008530 | 0.00046798 | 0.73 | 0.80 | 1.31 |

Table 3.4: First example: Errors and EOC in the adjoint state ($\alpha > 0$, $k \to 0$).

| $\ell$ | $\|\bar{u}-u_{kh}\|$ $L^1(I,\mathbb{R})$ | $\|\bar{u}-u_{kh}\|$ $L^2(I,\mathbb{R})$ | $\|\bar{u}-u_{kh}\|$ $L^\infty(I,\mathbb{R})$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.04086721 | 0.08276876 | 0.20000000 | / | / | / |
| 2 | 0.00945532 | 0.02308470 | 0.08258124 | 2.11 | 1.84 | 1.28 |
| 3 | 0.00210310 | 0.00583995 | 0.02197847 | 2.17 | 1.98 | 1.91 |
| 4 | 0.00051500 | 0.00146824 | 0.00557569 | 2.03 | 1.99 | 1.98 |
| 5 | 0.00012813 | 0.00036763 | 0.00139896 | 2.01 | 2.00 | 1.99 |
| 6 | 0.00003200 | 0.00009195 | 0.00035003 | 2.00 | 2.00 | 2.00 |

Table 3.5: First example: Errors and EOC in the control ($\alpha > 0$, $h \to 0$).

As a second example, we consider the limit problem ($\mathbb{P}_0$) and choose the optimal control to be the lower bound of the admissible set, i.e., $\bar{u} := a_1 := -0.2$ for some fixed $\kappa > 0$. For the upper bound we set $b_1 := 0.2$. The optimal adjoint state is chosen as

$$\bar{p}(t,x_1,x_2) := (T-t)^{1/\kappa} g_1(x_1,x_2)\,,$$

from which we derive

$$-\partial_t \bar{p} - \Delta\bar{p} = \frac{1}{\kappa}(T-t)^{1/\kappa-1} g_1 - (T-t)^{1/\kappa}\Delta g_1 = \bar{y} - y_d\,.$$

From this relation, keeping $\bar{y}$ as defined in (3.2) we get $y_d$. We define $g_0$ as

| $\ell$ | $\|\bar{y} - y_{kh}\|$ $L^1(I, L^1(\Omega))$ | $\|\bar{y} - y_{kh}\|$ $L^2(I, L^2(\Omega))$ | $\|\bar{y} - y_{kh}\|$ $L^\infty(I, L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.01983293 | 0.06277033 | 0.26858776 | / | / | / |
| 2 | 0.00843691 | 0.01822315 | 0.07296899 | 1.23 | 1.78 | 1.88 |
| 3 | 0.00242357 | 0.00478896 | 0.01962996 | 1.80 | 1.93 | 1.89 |
| 4 | 0.00062702 | 0.00121437 | 0.00528421 | 1.95 | 1.98 | 1.89 |
| 5 | 0.00015867 | 0.00031824 | 0.00161243 | 1.98 | 1.93 | 1.71 |
| 6 | 0.00005127 | 0.00012228 | 0.00069019 | 1.63 | 1.38 | 1.22 |

Table 3.6: First example: Errors and EOC in the state ($\alpha > 0$, $h \to 0$).

| $\ell$ | $\|\bar{y} - \pi_{P_k^*} y_{kh}\|$ $L^1(I, L^1(\Omega))$ | $\|\bar{y} - \pi_{P_k^*} y_{kh}\|$ $L^2(I, L^2(\Omega))$ | $\|\bar{y} - \pi_{P_k^*} y_{kh}\|$ $L^\infty(I, L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.01983293 | 0.06277028 | 0.26830125 | / | / | / |
| 2 | 0.00843691 | 0.01822290 | 0.07258642 | 1.23 | 1.78 | 1.89 |
| 3 | 0.00242354 | 0.00478800 | 0.01924801 | 1.80 | 1.93 | 1.91 |
| 4 | 0.00062688 | 0.00121058 | 0.00490340 | 1.95 | 1.98 | 1.97 |
| 5 | 0.00015806 | 0.00030345 | 0.00123143 | 1.99 | 2.00 | 1.99 |
| 6 | 0.00003959 | 0.00007589 | 0.00030816 | 2.00 | 2.00 | 2.00 |

Table 3.7: First example: Errors and EOC in the projected state ($\alpha > 0$, $h \to 0$).

in (3.3) but with $\bar{u} = a_1$. All other data remain unchanged with respect to the preceding section. Thus besides $\bar{u}$, only $\bar{p}$, $y_d$, and $g_0$ are altered.

This example fulfills the measure condition (1.38) of Assumption 15 with $\text{meas}(A^c) = 0$ and exponent $\kappa$ from above.

We solve the regularized problem $(\mathbb{P}_{kh})$ again using a fixed-point iteration procedure. To this end, we consider a fixed fine space-time mesh with $\text{Nh} = (2^5 + 1)^2$ nodes in space and $\text{Nk} = (2^{11} + 1)$ nodes in time. The regularization parameter $\alpha = 2^{-\ell}$ is considered for $\ell = 1, 2, 3, 4, 5, 6$.

The problem is solved for different values of $\kappa$, namely $\kappa = 0.3, 0.5, 1, 2$. Note however, that Assumption 10 is only fulfilled if $\kappa \leq 1$.

Let us remark that the convergence of the fixed-point iteration does not depend on the starting value.

| $\ell$ | $\|\bar{p}-p_{kh}\|$ $L^1(I,L^1(\Omega))$ | $\|\bar{p}-p_{kh}\|$ $L^2(I,L^2(\Omega))$ | $\|\bar{p}-p_{kh}\|$ $L^\infty(I,L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.07287573 | 0.24360487 | 1.09944996 | / | / | / |
| 2 | 0.03553656 | 0.07951728 | 0.34753670 | 1.04 | 1.62 | 1.66 |
| 3 | 0.01030205 | 0.02112693 | 0.09132732 | 1.79 | 1.91 | 1.93 |
| 4 | 0.00267154 | 0.00535947 | 0.02309948 | 1.95 | 1.98 | 1.98 |
| 5 | 0.00067405 | 0.00134469 | 0.00579137 | 1.99 | 1.99 | 2.00 |
| 6 | 0.00016890 | 0.00033647 | 0.00144891 | 2.00 | 2.00 | 2.00 |

Table 3.8: First example: Errors and EOC in the adjoint state ($\alpha > 0$, $h \to 0$).

As one can see from the Tables 3.9, 3.10, 3.11, and 3.12, the improved convergence rates of Theorem 19 for the optimal control, more precisely (1.42) and (1.43), can be observed numerically. It seems that they cannot be improved any further.

Let us also comment on the convergence in the $L^\infty$ norm of $\bar{u}$ depicted in Table 3.12. This phenomenon is due to the simplicity of our test example. If $\alpha$ is taken sufficiently small with fixed $k$ and $h$, the regularized *numerical* solution coincides with the lower bound of the admissible set, which is the solution of the limit problem. Compare Figure 3.3 to see this. This behavior can be observed also in the $L^1$ and $L^2$ norm and for other values of $\kappa$.

| $\ell$ | $\|\bar{u}-u_{kh}\|$ $L^1(I,\mathbb{R})$ | $\|\bar{u}-u_{kh}\|$ $L^2(I,\mathbb{R})$ | $\|\bar{u}-u_{kh}\|$ $L^\infty(I,\mathbb{R})$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.09417668 | 0.13354708 | 0.19999938 | / | / | / |
| 2 | 0.08837777 | 0.12648809 | 0.19999875 | 0.09 | 0.08 | 0.00 |
| 3 | 0.07681662 | 0.11533688 | 0.19999751 | 0.20 | 0.13 | 0.00 |
| 4 | 0.06212895 | 0.10353644 | 0.19999505 | 0.31 | 0.16 | 0.00 |
| 5 | 0.05008158 | 0.09264117 | 0.19999018 | 0.31 | 0.16 | 0.00 |
| 6 | 0.04011694 | 0.08237596 | 0.19998064 | 0.32 | 0.17 | 0.00 |

Table 3.9: Second example: Errors and EOC in the control ($\kappa = 0.3$, $\alpha \to 0$, $h$, $k$ fixed).

| $\ell$ | $\Vert\bar{u}-u_{kh}\Vert$ $L^1(I,\mathbb{R})$ | $\Vert\bar{u}-u_{kh}\Vert$ $L^2(I,\mathbb{R})$ | $\Vert\bar{u}-u_{kh}\Vert$ $L^\infty(I,\mathbb{R})$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.07912861 | 0.11494852 | 0.19999937 | / | / | / |
| 2 | 0.05957289 | 0.09753159 | 0.19999875 | 0.41 | 0.24 | 0.00 |
| 3 | 0.04204449 | 0.08187630 | 0.19999757 | 0.50 | 0.25 | 0.00 |
| 4 | 0.02963509 | 0.06865675 | 0.19999536 | 0.50 | 0.25 | 0.00 |
| 5 | 0.02084162 | 0.05749818 | 0.19999143 | 0.51 | 0.26 | 0.00 |
| 6 | 0.01463170 | 0.04811089 | 0.19998479 | 0.51 | 0.26 | 0.00 |

Table 3.10: Second example: Errors and EOC in the control ($\kappa = 0.5$, $\alpha \to 0$, $h$, $k$ fixed).

| $\ell$ | $\Vert\bar{u}-u_{kh}\Vert$ $L^1(I,\mathbb{R})$ | $\Vert\bar{u}-u_{kh}\Vert$ $L^2(I,\mathbb{R})$ | $\Vert\bar{u}-u_{kh}\Vert$ $L^\infty(I,\mathbb{R})$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.04006495 | 0.07304858 | 0.19993848 | / | / | / |
| 2 | 0.02000722 | 0.05160925 | 0.19987712 | 1.00 | 0.50 | 0.00 |
| 3 | 0.00998774 | 0.03646496 | 0.19975470 | 1.00 | 0.50 | 0.00 |
| 4 | 0.00498724 | 0.02576440 | 0.19951038 | 1.00 | 0.50 | 0.00 |
| 5 | 0.00249053 | 0.01820019 | 0.19902435 | 1.00 | 0.50 | 0.00 |
| 6 | 0.00123906 | 0.01282180 | 0.19804869 | 1.01 | 0.51 | 0.01 |

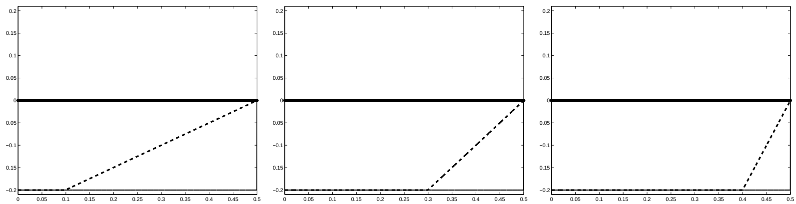
Table 3.11: Second example: Errors and EOC in the control ($\kappa = 1$, $\alpha \to 0$, $h$, $k$ fixed).

## 3.3 Coupling regularization and discretization parameters

We now couple the regularization parameter $\alpha$ with the discretization parameters $h$ and $k$ in a way which allows for optimal convergence.

For the limit problem ($\mathbb{P}_0$), we consider a third test example which is a bang-bang problem with $\operatorname{meas}(A^c) = 0$ and $\kappa = 1$ in Assumption 15. We choose an optimal adjoint state

$$\bar{p} := \frac{-T}{2\pi a} \sin\left(\frac{t}{T} 2\pi a\right) g_1 \,,$$



(a) $\ell = 1$ (b) $\ell = 2$ (c) $\ell = 3$

Figure 3.3: Second example: Optimal control $\bar{u}$ (solid) and computed counterpart $u_{kh}$ (dashed) over time after level $\ell$ ($\kappa = 1$, $\alpha \to 0$, $h$, $k$ fixed).

| $\ell$ | $\|\bar{u} - u_{kh}\|$ $L^1(I, \mathbb{R})$ | $\|\bar{u} - u_{kh}\|$ $L^2(I, \mathbb{R})$ | $\|\bar{u} - u_{kh}\|$ $L^\infty(I, \mathbb{R})$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.01081546 | 0.03305084 | 0.19723389 | / | / | / |
| 2 | 0.00279478 | 0.01690248 | 0.19446840 | 1.95 | 0.97 | 0.02 |
| 3 | 0.00074507 | 0.00878066 | 0.18893681 | 1.91 | 0.94 | 0.04 |
| 4 | 0.00020543 | 0.00463711 | 0.17787362 | 1.86 | 0.92 | 0.09 |
| 5 | 0.00005823 | 0.00246523 | 0.15574724 | 1.82 | 0.91 | 0.19 |
| 6 | 0.00001564 | 0.00125068 | 0.11149448 | 1.90 | 0.98 | 0.48 |

Table 3.12: Second example: Errors and EOC in the control ($\kappa = 2$, $\alpha \to 0$, $h$, $k$ fixed).

which is nonzero almost everywhere, and since

$$-\partial_t \bar{p} - \Delta \bar{p} = \cos\left(\frac{t}{T} 2\pi a\right) g_1 - \frac{T}{2\pi a} \sin\left(\frac{t}{T} 2\pi a\right) 2\pi^2 g_1 = \bar{y} - y_d\,,$$

we get the function $y_d$ by taking $\bar{y}$ as in (3.2). From the relation (1.14) we conclude that the optimal control is given by

$$\bar{u} = \begin{cases} a_1 & \text{if } B^*\bar{p} > 0, \\ b_1 & \text{if } B^*\bar{p} < 0 \end{cases}$$

where $a_1$, $b_1$, and all other data are taken from the first example. Note that $B^*\bar{p}(t) = (g_1, \bar{p}(t))_{L^2(\Omega)}$ and $(g_1, g_1)_{L^2(\Omega)} = 0.25$.

Since $\kappa = 1$ in this example, we conclude with Theorem 77, Corollary 78, and the second line of Table 2.1 the estimate

$$\|\bar{u}_0 - \bar{u}_d\|_U^2 + \|\bar{u}_0 - \bar{u}_d\|_{L^1(A,\mathbb{R})} + \|\bar{p}_0 - \bar{p}_d\|_{L^\infty(I,L^2(\Omega))} + \|\bar{y}_0 - \pi_{P_k^*}\bar{y}_d\|_I \le C(\alpha + h^2 + k^{4/3}). \quad (3.4)$$

Consequently, we set Nh $= (2^\ell + 1)^2$, Nk $= (2^{3/2\ell+1} + 1)$, and $\alpha = 2^{-2\ell}$ with $\ell = 1, 2, 3, 4, 5, 6$, to obtain second order convergence with respect to $h$ in (3.4).

The results are given in Tables 3.13, 3.14, 3.15, and 3.16. We also refer to Figure 3.4.

As one can see from the tables, the coupling shows the expected behavior for the error in the optimal control, projected state, and adjoint state.

Note that for the state $\bar{y}$, we observe convergence of order 3/2, which means by the coupling from above ($k = h^{3/2}$) first order convergence in $k$. Thus, it is in accordance with our expectation.

| $\ell$ | $\|\bar{u} - u_{kh}\|$ $L^1(I,\mathbb{R})$ | $\|\bar{u} - u_{kh}\|$ $L^2(I,\mathbb{R})$ | $\|\bar{u} - u_{kh}\|$ $L^\infty(I,\mathbb{R})$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.05208333 | 0.10206207 | 0.20000000 | / | / | / |
| 2 | 0.05156250 | 0.10155048 | 0.20000000 | 0.01 | 0.01 | 0.00 |
| 3 | 0.01551730 | 0.05249039 | 0.20000000 | 1.73 | 0.95 | 0.00 |
| 4 | 0.00395214 | 0.02696386 | 0.20000000 | 1.97 | 0.96 | 0.00 |
| 5 | 0.00100074 | 0.01375946 | 0.20000000 | 1.98 | 0.97 | -0.00 |
| 6 | 0.00026290 | 0.00704586 | 0.20000000 | 1.93 | 0.97 | 0.00 |

Table 3.13: Third example: Errors and h-EOC in the control ($\alpha = k^{4/3} = h^2$).

## 3.4 Final remarks

Let us mention that the convergence of the fixed-point iteration is in general guaranteed only for values of $\alpha$ not too small. This is an immediate consequence of Banach's fixed-point theorem in combination with (2.53).

| $\ell$ | $\Vert\bar{y} - y_{kh}\Vert$ $L^1(I, L^1(\Omega))$ | $\Vert\bar{y} - y_{kh}\Vert$ $L^2(I, L^2(\Omega))$ | $\Vert\bar{y} - y_{kh}\Vert$ $L^\infty(I, L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.04168338 | 0.14344433 | 0.77006182 | / | / | / |
| 2 | 0.02298795 | 0.05061771 | 0.24946457 | 0.86 | 1.50 | 1.63 |
| 3 | 0.00877452 | 0.01795226 | 0.08863801 | 1.39 | 1.50 | 1.49 |
| 4 | 0.00314952 | 0.00624197 | 0.02943581 | 1.48 | 1.52 | 1.59 |
| 5 | 0.00111871 | 0.00218973 | 0.00994956 | 1.49 | 1.51 | 1.56 |
| 6 | 0.00039580 | 0.00077075 | 0.00339060 | 1.50 | 1.51 | 1.55 |

Table 3.14: Third example: Errors and h-EOC in the state ($\alpha = k^{4/3} = h^2$).

| $\ell$ | $\Vert\bar{y} - \pi_{P_k^*} y_{kh}\Vert$ $L^1(I, L^1(\Omega))$ | $\Vert\bar{y} - \pi_{P_k^*} y_{kh}\Vert$ $L^2(I, L^2(\Omega))$ | $\Vert\bar{y} - \pi_{P_k^*} y_{kh}\Vert$ $L^\infty(I, L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.03984472 | 0.12699052 | 0.67616861 | / | / | / |
| 2 | 0.01063414 | 0.02423705 | 0.15855276 | 1.91 | 2.39 | 2.09 |
| 3 | 0.00235558 | 0.00482756 | 0.02588151 | 2.17 | 2.33 | 2.61 |
| 4 | 0.00059757 | 0.00116777 | 0.00526572 | 1.98 | 2.05 | 2.30 |
| 5 | 0.00015345 | 0.00029551 | 0.00128779 | 1.96 | 1.98 | 2.03 |
| 6 | 0.00003968 | 0.00007581 | 0.00032323 | 1.95 | 1.96 | 1.99 |

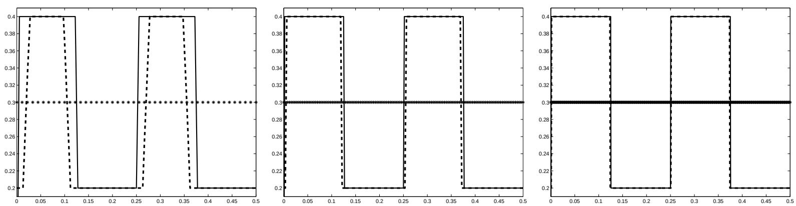
Table 3.15: Third example: Errors and h-EOC in the projected state ($\alpha = k^{4/3} = h^2$).

In the numerical examples we considered, no convergence problems occurred, even for very small values of $\alpha$. This might be due to the fact that we consider controls which "live" in one space dimension only. For higher dimensions, the situation is more delicate. There, the application of semismooth Newton methods has turned out to be fruitful, see [HV12] for its numerical analysis in the case of variational discretization of elliptic optimal control problems.

For a discretization of ($\mathbb{P}$) in the regular case ($\alpha > 0$), let us finally mention a discontinuous Galerkin approach analyzed recently in [SV13], based on the results in [MV08a] and [MV08b].

| $\ell$ | $\Vert\bar{p}-p_{kh}\Vert$ $L^1(I,L^1(\Omega))$ | $\Vert\bar{p}-p_{kh}\Vert$ $L^2(I,L^2(\Omega))$ | $\Vert\bar{p}-p_{kh}\Vert$ $L^\infty(I,L^\infty(\Omega))$ | EOC $L^1$ | EOC $L^2$ | EOC $L^\infty$ |
|---|---|---|---|---|---|---|
| 1 | 0.00175355 | 0.00559389 | 0.02497779 | / | / | / |
| 2 | 0.00052886 | 0.00120225 | 0.00578048 | 1.73 | 2.22 | 2.11 |
| 3 | 0.00012807 | 0.00026289 | 0.00128201 | 2.05 | 2.19 | 2.17 |
| 4 | 0.00003156 | 0.00006214 | 0.00028508 | 2.02 | 2.08 | 2.17 |
| 5 | 0.00000786 | 0.00001530 | 0.00006829 | 2.01 | 2.02 | 2.06 |
| 6 | 0.00000195 | 0.00000377 | 0.00001649 | 2.01 | 2.02 | 2.05 |

Table 3.16: Third example: Errors and h-EOC in the adjoint state ($\alpha = k^{4/3} = h^2$).



(a) $\ell = 3$ (b) $\ell = 4$ (c) $\ell = 5$

Figure 3.4: Third example: Optimal control $\bar{u}$ (solid) and computed counterpart $u_{kh}$ (dashed) over time after level $\ell$ ($\alpha = k^{4/3} = h^2$).

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Let us comment on literature related to this thesis. We restrict ourselves to more or less recent papers which are concerned with discretizations of bang-bang optimal control problems (ocp) and which present error estimates.

We try to formulate the cited results in the notation of this thesis for easy comparison.

Let us start with the numerical treatment of bang-bang ocps governed by *ordinary* differential equations (ODEs), which has attracted some interest in recent years.

A related ODE problem to $(\mathbb{P})$ with $\alpha = 0$ reads

$$\begin{aligned} &\min_{y\in Y, u\in U_{\mathrm{ad}}} J(y,u) \quad \text{with} \quad J(y,u) := \frac{1}{2}\int_0^T y(t)^T W(t) y(t)dt, \\ &\text{s.t. } \dot{y} = B(t)u(t) \quad \forall' \, t \in I := [0,T], \\ &y(0) = y_0, \end{aligned} \tag{OQ}$$

with $U_{\mathrm{ad}} \subset U := L^2(I,\mathbb{R}^m)$ defined by pointwise box constrains, a state space $Y = H^1(I,\mathbb{R}^n)$, and Lipschitz continuous functions $W : [0,T] \to \mathbb{R}^{n\times n}$ and $B : [0,T] \to \mathbb{R}^{n\times m}$. It is also assumed that the matrices $W(t)$ are symmetric and positive semidefinite. This problem has at least one solution.

In this context, the variational inequality (1.10) is called *minimum principle* and one can establish, analogously to our parabolic setting, an adjoint ODE for some quantity $p$ with right-hand side $y$ and $p(T) = 0$. Note that $y_d$ from $(\mathbb{P})$ is zero in (OQ). An explicit characterization for the optimal control holds true, depending on the values of the so-called switching-function $\sigma(t) := B(t)^T p(t)$, analogously to the located case of (1.14). It is now assumed that

> (A1) There exists a solution $(u^*, y^*)$ of (OQ) such that each of the components of $\sigma$ has finitely many zeros, all collected in $\Sigma = \{s_1, \dots, s_l\}$ with $0, T \notin \Sigma$,

and for the set of active indices $\mathcal{I}(s_j) := \{1 \le i \le m \mid \sigma_i(s_j) = 0\}$ of the components of the switching function we demand the property

> (A2) There exist $\bar{\sigma} > 0$, $\bar{\tau} > 0$, such that
>
> $$\forall\, j \in \{1,\dots,l\}\; \forall\, i \in \mathcal{I}(s_j)\; \forall\, \tau \in [s_j - \bar{\tau}, s_j + \bar{\tau}] : \quad |\sigma_i(\tau)| \ge \bar{\sigma}|\tau - s_j|$$
>
> and $\sigma_i$ changes sign in $s_j$, i.e., $\sigma_i(s_j - \bar{\tau})\sigma_i(s_j + \bar{\tau}) < 0$.

Both assumptions (A1) and (A2) ensure uniqueness of the optimal control $u^*$ and imply the measure condition (1.38) on the whole interval $[0, T]$ with $\kappa = 1$.

In [Alt+12], the Euler discretization is used to discretize this problem directly (as no regularization is used) and $\mathcal{O}(\sqrt{h})$ convergence is shown for any discrete optimal control in $L^1$ and for the discrete state and the discrete adjoint state in $L^\infty$ where $h$ is the mesh size in time. Concerning the non-uniqueness of $u_{0,h}$, recall Remark 67. With an implicit method, the same convergence order is achieved in [AS14a].

In [Alt+12], a stronger condition than (A2) is introduced, too, namely

> (A3) The function $B$ is differentiable with Lipschitz continuous derivative and there exists $\bar{\sigma} > 0$ such that
>
> $$\min_{1 \le j \le l} \min_{i \in \mathcal{I}(s_j)} (|\dot{\sigma}_i(s_j)|) \ge 2\bar{\sigma}.$$

In the elliptic context, let us mention the gradient condition [DH12, Lemma 3.2] related to (A3). It implies the measure condition (1.38) in the case of $\kappa = 1$.

With condition (A3), an improvement of the convergence of the above mentioned quantities from $\mathcal{O}(\sqrt{h})$ to $\mathcal{O}(h)$ is shown. These results have later been carried over to an implicit scheme in [AS14b].

Discretization combined with regularization provides an interesting alternative to the direct solution of the limit problem since the regularized problems possess more regularity. We already saw this in the parabolic context, but this also holds in the ODE case.

In [AS11], the problem (OQ) is regularized by adding a term $\frac{\alpha}{2}\|u\|^2_{L^2(I,\mathbb{R}^m)}$, i.e., by an $L^2$ regularization as in our problem $(\mathbb{P})$. Then, the projection formula (1.12) is available in the ODE context, too.

A result similar to Lemma 16 with $\kappa = 1$, extracted from [Fel03, Lemma 3.3], is used to derive linear convergence of the regularization error in the control in the $L^1$ norm and in the state in the $W^{1,1}$ norm. This motivated us to formulate and use Lemma 16 in the PDE context.

Without assuming (A2), convergence of the state in the $L^2$ norm of order $\mathcal{O}(\sqrt{\alpha})$ can be shown, which corresponds in our context to the unconditional convergence (1.29). With (A1) and (A2) and the Euler discretization, they obtain $\|u_{\alpha,h} - u_0\|_{L^1} \le C\frac{h}{\alpha} + C\alpha \le \sqrt{h}$ with the coupling $\alpha = \sqrt{h}$.

Note that non-robust estimates as in Corollary 71 have been used to derive this result.

Replacing (A2) by (A3), this estimate is improved in [Sey13] to $\|u_{\alpha,h} - u_0\|_{L^1} \leq Ch$ with the coupling $\alpha = h$. For state and adjoint state, a convergence rate of $\mathcal{O}(h)$ in $L^\infty$ is also shown.

In [Sey15], a refinement of condition (A2) is introduced, namely

(A2k) There is a smallest natural number $k \in \mathbb{N}$ for which there exist constants $\bar{\sigma}, \bar{\tau} > 0$ such that

$$\forall\, j \in \{1, \ldots, l\}\; \forall\, i \in \mathcal{I}(s_j)\; \forall\, \tau \in [s_j - \bar{\tau}, s_j + \bar{\tau}] : \quad |\sigma_i(\tau)| \geq \bar{\sigma}|\tau - s_j|^k. \tag{3.5}$$

Note that this condition implies the measure condition (1.38) with $\kappa = 1/k$. With the condition (A2k), a variant of Lemma 16 is established, and convergence, e.g., in the control $\|u_\alpha - u_0\|_{L^1} \leq C\alpha^{1/k}$ is shown. This is in accordance with (1.42). With the coupling $\alpha = h$ an error $\|u_{\alpha,h} - u_0\|_{L^1} \leq Ch^{1/(k+1)}$ in the control is shown for the Euler discretization. Errors for state and adjoint state are also derived.

In [Sey14], much of the above discussed material is collected in one reference (in German language). Moreover, the discrete regularization assuming (A2k) is additionally analyzed for an implicit discretization. The convergence rates are the same as for the explicit Euler discretization. This also holds true if (A2k) is replaced by (A3), and hence, e.g., linear convergence in the control in the $L^1$ norm can be achieved.

Let us now consider *elliptic* problems. Here, the heat equation in our setting is replaced by the Poisson equation with, e.g., homogeneous Dirichlet boundary values and a control acting as right-hand side.

In [DH12], variational discretization with piecewise linear and continuous discretizations of state and adjoint state is applied to the limit problem $\alpha = 0$. This yields linear convergence for state and adjoint state in $L^2$ and $L^\infty$, respectively, without any further assumptions, compare Corollary 70. Note that similar to Remark 67, the discrete control $u_{0,h}$ is non-unique. Discrete state and adjoint state, however, are unique.

Better rates are obtainable, e.g., $\mathcal{O}(h^2)$ in the above mentioned quantities if $\kappa = 1$, if the measure condition (1.38) holds a.e. on the domain, and if $p$ is sufficiently regular.

A convergence estimate for discrete controls in $L^1(A)$ is also derived. These estimates are comparable to Theorem 72 in the special case of $\alpha = 0$.

We note that with our proof technique one can improve the rates in [DH12] with respect to $\kappa$.

Key ingredient to prove the error estimates is the estimate

$$\|u_0 - u_{0,h}\|_{L^1(A)} \leq C\|p_0 - p_{0,h}\|^{\kappa}_{L^\infty}. \tag{3.6}$$

In [GY11], this estimate is generalized (with respect to $\alpha$) to

$$\|u_\alpha - u_{\alpha,h}\|_{L^1(\Omega)} \leq C\|p_\alpha - p_{\alpha,h}\|_{L^\infty}$$

assuming a measure condition stronger than the $\bar{p}_\alpha$-measure condition (1.56) in the special case of $\kappa = 1$ on the whole domain. However, for the total error $\|u_0 - u_{\alpha,h}\|_{L^1(A)}$, which is not considered in [GY11], this strengthening does not improve the estimates.

In [WW11a], first estimates for the regularization error assuming the measure condition (1.38) are derived together with non-robust finite element estimates. Let us also mention that a-posteriori error estimates as well as an additional $L^1$ term in the cost functional are considered. The $L^1$ term is also included in [WW11b] and [WW13].

In [WW11b], improved regularization error estimates are obtained, and the measure condition is generalized to the Assumption 15. As noted above, in Theorem 19 we further improved these estimates.

It is also discussed how the condition (1.45), which is fulfilled in our situation, can be weakened. A weakening of the source condition (1.37) to so-called *power type source conditions* is also derived. Additionally, noise of level $\delta$ in the desired state (i.e., $\|y_d - y_d^\delta\| \leq \delta$) and a parameter choice rule $\alpha(\delta)$ are discussed.

The paper [WW13] discusses necessity and sufficiency of conditions for convergence rates in the regularization, similar to Theorem 20 and Theorem 22. See also the remarks near these theorems.

In [Wac13], a parameter choice rule $\alpha(h)$ is developed to select $\alpha$ adaptively depending on a-posteriori available quantities. The rule selects $\alpha(h) \sim h^2$ for an example with $\kappa < 1$. This is theoretically justified by the a-priori *robust* estimates derived in [Wac14]. We note that with our proof technique, one can improve these rates with respect to $\kappa$.

Let us finally refer to the discussions after Assumption 15 and at the end of the numerics chapter for further literature.

# Abstract

In this thesis, a class of optimal control problems governed by the heat equation is considered. The task is to minimize the tracking-type functional

$$J(u) := \frac{1}{2}\|y(u) - y_d\|_Y^2 + \frac{\alpha}{2}\|u\|_U^2$$

in the *limit case* $\alpha = 0$.

The optimal control in the limit case is often discontinuous, but has a special structure: It takes values only on the bounds $a$ and $b$ of the set of admissible controls $U_{\text{ad}} = \{u \in U \mid a \le u \le b\}$. Such controls are called *bang-bang controls*.

To stabilize the limit problem, the case $\alpha > 0$ is considered, which is a *Tikhonov regularization* and introduces a *regularization error*.

As a next step, the control problem with $\alpha > 0$ is discretized in space and time. We thereby introduce a second error, the *discretization error*.

If *a-priori error estimates* are at hand for both errors, one can derive a coupling rule for discretization and regularization parameters for an efficient numerical solving. It is the aim of this thesis to establish such a numerical analysis.

In **chapter one**, the class of optimal control problems is introduced. Existence, uniqueness and regularity are discussed. We then analyze the Tikhonov regularization error. We first recall some well-known results. After that, we show that under additional conditions, better results for the rate of convergence can be given. For bang-bang solutions, a second sufficient condition is introduced. With it, an error bound on the time derivative of the control with respect to $\alpha$ is derived, which will be useful later to improve convergence rates for the discrete regularized solutions.

Having estimates for the regularization error at hand, in **chapter two** an appropriate discretization of the optimal control problem is set up. Therefor, we first consider finite element discretizations of the state and adjoint equation. Stability and error estimates are derived in different norms. After that, we formulate and analyze the variational discretization of the optimal control problem. At first, estimates for the error between regularized control and discrete regularized control are shown, which are not robust if $\alpha$ tends to zero and lead to non-optimal estimates for the total error. We then derive robust estimates, which lead to better estimates for the total error if the limit problem is sufficiently regular. Finally, we improve these robust estimates further for bang-bang controls.

In the **third chapter**, we report and comment on some numerical calculations to support the analytical findings.

# Zusammenfassung


Diese Arbeit beschäftigt sich mit einer Klasse von Optimalsteuerungsproblemen mit der Wärmeleitungsgleichung. Ziel ist die Minimierung eines Tracking-Type-Funktionals

$$J(u) := \frac{1}{2}\|y(u) - y_d\|_Y^2 + \frac{\alpha}{2}\|u\|_U^2$$

im *Grenzfall* $\alpha = 0$.

Die optimale Steuerung im Grenzfall ist oft unstetig, hat aber eine spezielle Struktur: Sie nimmt nur Werte an auf den Schranken $a$ und $b$ der Menge zulässiger Steuerungen $U_{\mathrm{ad}} = \{u \in U \mid a \leq u \leq b\}$. Solche Steuerungen werden *Bang-Bang-Steuerungen* genannt.

Zur Stabilisierung des Grenzproblems wird der Fall $\alpha > 0$ betrachtet, eine *Tichonow-Regularisierung*, die einen *Regularisierungsfehler* einführt.

Als nächster Schritt wird das Kontrollproblem mit $\alpha > 0$ in Zeit und Ort diskretisiert. Dadurch wird ein zweiter Fehler, der *Diskretisierungsfehler*, eingeführt.

Sind *A-priori-Fehlerschätzer* etabliert, kann eine Kopplungsregel zwischen Diskretisierungs- und Regularisierungsparametern hergeleitet werden zur effizienten numerischen Lösung. Ziel dieser Arbeit ist eine solche numerische Analyse.

Im **ersten Kapitel** wird die Klasse von Optimalsteuerungsproblemen eingeführt. Existenz, Eindeutigkeit und Regularität werden diskutiert. Danach analysieren wir den Tichonow-Regularisierungsfehler. Wir wiederholen zunächst bekannte Resultate. Danach zeigen wir unter zusätzlichen Bedingungen bessere Ergebnisse für die Konvergenzrate. Für Bang-Bang-Steuerungen wird eine zweite hinreichende Bedingung eingeführt. Mit ihr wird eine Fehlerschranke für die Zeitableitung der Steuerung bezüglich $\alpha$ hergeleitet. Damit können später Konvergenzraten für die diskreten regularisierten Lösungen verbessert werden.

Im **zweiten Kapitel** wird eine geeignete Diskretisierung des Kontrollproblems betrachtet. Zuerst werden Finite-Elemente-Diskretisierungen von Zustands- und Adjungierten-Gleichung eingeführt. Stabilität und Fehlerabschätzungen in verschiedenen Normen werden hergeleitet. Danach formulieren und analysieren wir die variationelle Diskretisierung des Kontrollproblems. Zuerst werden Fehlerabschätzungen zwischen regularisierter Kontrolle und diskreter regularisierter Kontrolle gezeigt, welche nicht robust sind, falls $\alpha$ gegen Null strebt, und zu nicht-optimalen Abschätzungen für den Gesamtfehler führen. Danach leiten wir robuste Abschätzungen her, die zu besseren Abschätzungen für den Gesamtfehler führen, falls das Grenzproblem hinreichend regulär ist. Am Ende verbessern wir diese robusten Abschätzungen weiter für Bang-Bang-Steuerungen.

Im **dritten Kapitel** berichten wir von einigen numerischen Berechnungen, die die analytischen Ergebnisse bestätigen.