Power Measures in Large Weighted Voting Games

Asymptotic Properties and Numerical Methods

Dissertation
zur Erlangung des Doktorgrades
des Fachbereichs Mathematik
der Universität Hamburg

vorgelegt von
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aus Tübingen

Hamburg 2004

Als Dissertation angenommen vom Fachbereich
Mathematik der Universität Hamburg

auf Grund der Gutachten von
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Hamburg, den 21.10.2003

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Dekan des Fachbereichs Mathematik

Acknowledgments

I want to express my gratitude to everyone who supported me so that I could complete this work. Firstly, I am grateful to the people that educated me, and to my parents in particular.

I am especially grateful to Moshé Machover and Claus Peter Ortlieb where the latter already acted as a supervisor to my diploma thesis. Without their support, it wouldn't have been possible to realize my PhD in applied mathematics as an external candidate. In that respect I also wish to thank the Department of Mathematics at Hamburg University, whose cooperative and encouraging attitude towards students has always been most impressing.

Some of the thesis has been written during my time as a research assistant at the Institute of SocioEconomics at Hamburg University. I thank Manfred Holler who supervised me on part of this research and for giving me the invaluable opportunity to frequently represent my research on conferences. I also profited a lot from the cooperation with Mathew Braham.

Since his profile is very close to mine, my coauthor Moshé Machover represented the strongest influence on this research on voting power. I am also very grateful to Abraham Neyman and Guillermo Owen for their kind help on Part I of this document.

I thank Maurice Koster for wonderful kinds of reasons.

A significant contribution was by all those people that kept me *from* working. Many thanks in that respect go to my dear old soulmates Volker Schirp and Susanne Vorreiter. I thank them for contributing to make my life less ordinary and I am especially grateful for their support in gloomy periods. Lucas Erbsman has always been a very special and important person - I thank him for his precious time. I am grateful to Isabel Guillén who always filled my office with laughter and energy and to whom I owe many wise decisions. Moreover, my PhD time was particularly enlightened by my dear friends Olaf Casimir, Barbara Dziersk, Melanie and Tillmann Esser, Daniel Friedrich, Christian Hahne, Ilona Isforth, Stefan Kock, Holger Strulik and Arnold van Meteren. I am also grateful for the stimulating support of Jochen Bigus. With greatest pleasure I look back to our female circle of friends at Hamburg University ('die Kreischziegen') which has given academic life a most cheerful aspect. In this respect I thank Silke Bender, Nicola Brandt, Heide Coenen, Daniela Felsch, Sandra Greiner, Isabel Guillén, Ria Steiger for their company. Hamburg, March 2004.

Contents

Chapter 1

Preliminaries

1.1 Introduction

This thesis deals with the asymptotic properties of simple weighted voting games when there are many 'small' voters. Institutions with a large number of participants are common in political and economic life; examples are markets, stock companies and executive boards, for example the EU Council of Ministers. The analysis of structural properties of collective decision-making rules has a well established history in game theory and social choice theory. Measurements of voting power have been proven to be useful as instruments to analyze collective decision-making rules which can be modelled as a simple (voting) game. This relates to any collective body that makes yes-or-no decisions by vote.

This thesis deals almost exclusively with *a priori* voting analysis. Contrary to *actual* (a posteriori) analysis, it models the voting system as an 'abstract shell', without taking into consideration voters' preferences, the range of issues over which a decision is taken or the degree of affinity between the voters. This abstraction seems to be necessary to focus on the legislature itself in a pure sense. Roth (1988, p. 9) puts it this way:

'Analyzing voting rules that are modelled as simple games abstracts away from the particular personalities and political interests present in particular voting environments, but this abstraction is what makes the analysis focus on the rules themselves rather than on other aspects of the political environment. This kind of analysis seems to be just what is needed to analyze the voting rules in a new constitution, for example, long before the specific issues to be voted on arise or the specific factions and personalities that will be involved can be identified.'

The theoretical and empirical literature on the field of voting power can roughly be divided into two fields: one studying individual and the other focussing on global measures. Individual power measures – such as the classical power measure proposed by Shapley & Shubik (1954) and Banzhaf (1965) – focus on the question to what 'extent' a given member is able to control the outcome of a collective decision. Global measures deal with global characteristics of decision rules, for example the ease with which the decision rule responds to fluctuations in the voters' wishes or the propensity to approve bills (which was introduced by Coleman (1971) as the power of a collectivity to act).

Power measures represent a useful instrument to shed light on the different aspects of voting scenarios, both in political as well as financial fields. However, the extent of common acceptance and applicability to real-life voting design is still modest. A major limiting factor is presumably that the computation of power measures is not straightforward – especially when the number of voters is large – such that specific software has to be written or installed in order to be able to evaluate voting systems at all. In this respect, the limit theorems developed for *weighted voting games* in this thesis clearly serve as a convenient approximation for large weighted voting games.

Weighted voting games play a central role, not only because they are very common in economic and political organizations but also because many voting systems can be equivalently represented as such.[1] In a weighted voting game each board member is assigned to a non-negative number as weight, and a certain positive number is fixed as quota. Many organizations have systems of governance by weighted voting, examples for economic organizations are the International Monetary Fund, the World Bank, stock companies, etc. In federal political bodies the weights are usually designed to reflect the number of inhabitants of each represented state; examples are the EU's Council of Ministers and the US Presidential Electoral College.

A widespread fallacy is that under a weighted voting decision rule the powers of the voters are proportional to their respective weights. A simple counterexample is a game of three voters, one being endowed with 2% of the total weight sum and the other 98% evenly split up among the other two voters. If the rule is that any coalition with a combined voting weight of more than 50% of the total weight sum is winning (simple majority rule) than any two

[1] Freixas & Zwicker (2003) give a combinatorial characterization of such games.

voters constitute a winning coalition. Hence each voter has exactly the same power despite an extremely skewed weight distribution. A further counterexample is the voting system of the EU's Council of Ministers between 1958 and 1972. All six countries were endowed with an even number of voting weights, except Luxembourg with a voting weight of one. Since the required threshold for a proposal to pass was an even number, Luxembourg's choice never made any difference. In modern game theoretic terms, Luxembourg was a dummy voter, being entitled with one vote for 310 000 inhabitants in contrast to Germany with only one vote for its 13 572 500 inhabitants.

Interestingly, however, both real-life and randomly generated weighted voting games with many voters provide much empirical evidence that the following general rule holds for the most measures of voting power prevalent in the literature: if the distribution of the weights is not too skewed (in other words, the ratio of the largest weight to the smallest is not very high), then the relative powers of the voters tend to approximate closely to their respective relative weights. Hence the latter serves as an approximation of the former which becomes rapidly more accurate with increasing size of the voting body. For example in the Treaty of Nice, subject to the EU's prospective enlargement to 27 members, the ratio of the normalized Banzhaf measure as well as the Shapley-Shubik index to the normalized voting weight lies within a range of 0.95 to 1.05. This suggests a *relative* irrelevance which seems to hold for prevalent power measures: with increasing number of voters the power ratio of any two voters converges to the ratio of the voting weights, irrespective of the specific power measure chosen.

Note that this observation is not concerned with *absolute* voting power. For instance, consider two voting systems with the same set of voters. Pick two voters, say a and b, and assume that in the first voting system they have the same voting power whereas in the second system a has more power than b but less in absolute terms compared to the first scenario. Then, in relative terms, a is better off in the second scenario compared to b, however, in absolute terms a is worse off. This fundamental difference becomes apparent when considering the classical power measures: by definition the Shapley-Shubik index sums up to one whereas the total of Banzhaf measures (generically) changes with different decision rules. Although one should expect that it is the absolute power that is of major interest, people concerned with real-life voting systems – in politics and economics – seem to care almost exclusively about relative power.

A fundamental mathematical basis of the relative irrelevance phenomenon contributes to overcome a second limiting factor towards the modest acceptance of power measures – that of arbitrariness: the large variety of individual power measures leads to ambiguity in evaluating voting systems. This widens the problem of choosing the 'right' power concept and designing an eligible voting system so as to fit a given power distribution, respectively.

Part I of the thesis provides proofs concerning the normalized irrelevance statement for the two most prominent measures of power, i.e. the Banzhaf measure and Shapley-Shubik index, as well as a generalized Banzhaf measure in weighted voting games with abstentions.

The assumption that the distribution of weights is not too skewed suggests that in games with a significant large number of voters any single voter will have negligible power - a setup literature refers to as 'non-atomic' games.

Part II of the thesis gives an analysis of what happens to global measures in a weighted voting game when the total weight's fraction of a finite number of voters (the atomic part) stays constant, whilst the remaining block of votes is broken up and distributed among an increasing number of 'small' voters (the non-atomic part). Literature refers to the limit scenario with infinitely many small voters whose weight is infinitely small as an 'oceanic game'. This scenario seems adequate to represent situations with such a large number of small voters that any of them has a negligible effect on the outcome. For example, a common scenario in stock companies is that each shareholder is entitled with a number of votes (voting weight) proportional to their relative capital contribution: usually a small group of 'major' voters owns a significant number of votes – reflecting their large proportion of ownership of the capital stock – accompanied by a large 'pool' of small voters where each of these 'minor' voters has a negligible effect on voting outcomes.

Part III regards computation of power measures in large weighted voting games and serves to enrich the results by empirical data. The evaluation of large voting systems struggles with the problem that the exact calculation is not feasible due to exponential time complexity. For weighted voting games literature provides methods of avoiding exponential time complexity, however, their applicability is limited: the methods invoke the use of arrays whose dimensions can cause substantial storage requirements. Part III extends the prevalent numerical methods to more general weighted voting games and shows how the storage schemes designed for sparse matrices can be used to significantly cut down the storage extent. Here, significantly means that this method allows the evaluation of weighted voting games which represents an impossible task with prevalent methods, given modern computer power. Furthermore, the approach to voting power in this thesis is probabilistic – other than axiomatic – which allows the application of powerful approximation tools of stochastics, primarily the central limit theorem. This part discusses and extends the prevalent approximation methods which are widespread in the literature.

1.2 Basic Definitions

Let N be a nonempty finite set to which we shall refer as *assembly*. The elements of N are called *voters* and we shall often identify them with the integers $1, 2, ..., n$, where $n = |N|$. A play of the voting game consists in a *division*,[2] in which each voter chooses one of two options (usually, 'yes' and 'no'). Any subset of $S \subseteq N$ is called a *coalition*. By referring to a *decision rule* we shall associate the following mathematical structure.

Definition 1.1 A *simple voting game* – briefly, SVG – is a collection $\mathcal{W}$ of subsets of N, satisfying the following three conditions:

1. $N \in \mathcal{W}$,
2. $\varnothing \notin \mathcal{W}$,
3. if $S \in \mathcal{W}$ and $S \subseteq T$ then $T \in \mathcal{W}$ (Monotonicity).

We shall refer to a coalition $S \subseteq N$ as *winning* or *losing*, according as $S \in \mathcal{W}$ or $S \notin \mathcal{W}$.
The following concept shall prove useful throughout the discussion.

Definition 1.2 Let $\mathcal{W}$ be an SVG with assembly N. A *characteristic* (or *coalitional*) *function* v is a map from the set of all coalitions $S \subseteq N$ to $\{0, 1\}$ such that

$$v(S) = \begin{cases} 1 & \text{if } S \in \mathcal{W}, \\ 0 & \text{otherwise.} \end{cases}$$

This thesis deals almost exclusively with a special class of SVGs called weighted voting games.

[2]Here we follow Felsenthal and Machover (1997, p. 335) in borrowing the term from English parliamentary usage to denote the *collective* act of a voting body, whereby each individual member casts a vote.

Definition 1.3 A *weighted voting game* – briefly, WVG –

$$[q; w_1, w_2, ..., w_n] \tag{1.1}$$

is given by an assignment of a non-negative real *weight* w_k to each voter $k \in N$, and a *relative Quota* $q \in (0, 1)$ such that

$$v(S) = \begin{cases} 1 & \text{if } \sum_{k \in S} w_k \geq q \sum_{k \in N} w_k, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

The blunt inequality $\geq$ in (1.2) may be replaced by the sharp inequality $>$. In this case we shall use the notation

$$< q; w_1, w_2, \ldots, w_n > . \tag{1.3}$$

Remarks 1.4
(i) Any particular WVG represented by (1.1) can be put into the form (1.3) by slightly adjusting the quota q and vice versa such that the definitions (1.1) and (1.3) are equivalent: they determine the same class of structure.
(ii) Definition (1.1) and (1.3) may be rewritten by replacing the relative quota by q multiplied by the total weight of N, i.e. $c := q \sum_{k \in N} w_k$. We shall refer to c as the *absolute* quota.
(iii) In Chapter 4 we shall modify (1.2) by introducing the concept of a *ternary* WVG in which any voter faces the option to abstain as a tertium quid besides ‘yes’ and ‘no’. When there is risk of confusion we shall refer to a WVG with a characteristic function as defined in (1.2) as a *binary* WVG.

Part I

L.S. Penrose's Limit Theorem

Chapter 2

Introduction to Part I

In his 1946 paper (p. 53), Lionel Penrose gave the first definition of a priori voting power. According to this definition, as slightly amended in his 1952 booklet (pp. 7–8), the voting power of voter a equals the probability ψ_a of a 'being able to influence a decision either way'. Here it is assumed a priori that all voters other than a vote independently of one another, each voting 'yes' and 'no' with equal probability; so that all divisions of those voters into 'yes' and 'no' camps are equiprobable. Then ψ_a is the probability of the event that those voters are so divided that a's vote will tip the balance: if a votes 'yes' the act in question will be adopted, and if s/he votes 'no' the act will be blocked.[1]

Penrose always assumes that decisions are subject to the simple majority rule, whereby each voter must vote either 'yes' or 'no' (so that no abstentions are admitted) and a proposed bill is adopted iff it receives over half of all votes. However, he allows the formation of blocs, so that a bloc-voter can have any positive integral number of votes. Thus the decision rules he considers are a special case of what is known in the voting-power literature as a 'weighted voting game' (WVG, see Definition 1.3).

Penrose confines his attention to the special case in which q equals or slightly exceeds $\frac{1}{2}$.[2] For such WVGs, he derives in his 1952 the following approxima-

[1] We have stated the a priori assumption more fully than Penrose, who merely says that the other voters are assumed to act 'at random'. The definition he had given in his 1946 took $\psi_a/2$ rather than ψ_a itself as a's voting power; the difference is of course inessential.
Penrose's measure ψ is often referred to in the literature as 'the [absolute] Banzhaf index' and denoted by 'β''. In using 'ψ' we are following Owen (1995).

[2] In fact, he seems to be thinking of (1.2) with $>$ instead of $\geq$, and $q = 1/2$; see Remark 1.4(i).

tion for the voting power ψ_a of voter a:

$$\psi_a \approx w_a \sqrt{\frac{2}{\pi \sum_{k \in N} {w_k}^2}}. \tag{2.1}$$

In deriving (2.1) he assumes that the number of voters is large, and w_a is small compared to the sum S of all weights.[3] Note that as w_a/S becomes vanishingly small, so do both sides of (2.1). Thus $\approx$ must be taken to mean that the *relative* error of the approximation tends to 0; in other words, the ratio between the two sides tends to 1.

Implicit in this approximation formula is a limit theorem about the behavior of the ratio between the voting powers of any two voters, a and b: if the number of voters increases indefinitely, while existing voters always keep their old weights and the relative quota is pegged at $\frac{1}{2}$, then (under suitable conditions),

$$\frac{\psi_a}{\psi_b} \to \frac{w_a}{w_b}. \tag{2.2}$$

Penrose does not present a rigorous proof of (2.1) or (2.2), but merely outlines an argument, which is presumably based on some version of the central limit theorem of probability theory.

Unfortunately, (2.1) and (2.2) do not always hold under the conditions assumed by Penrose. For example, let $0 < w' < w$, and for any positive integer n put

$$\mathcal{W}^{(n)} := [(w' + nw)/2; w', \underbrace{w, w, \ldots, w}_{n \text{ times}}], \tag{2.3}$$

where $(w' + nw)/2$ represents the absolute quota. Thus, voters $2, \ldots, n+1$ have the same weight, which is greater than that of voter 1; and a bill is adopted iff it receives at least (and hence in fact more than) half the total weight.[4] Clearly, for any fixed n the voting powers $\psi_k[\mathcal{W}^{(n)}]$, for $k = 2, \ldots, n+1$, are positive and equal to one another. But

$$\psi_1[\mathcal{W}^{(n)}] = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \psi_2[\mathcal{W}^{(n)}] & \text{if } n \text{ is even.} \end{cases} \tag{2.4}$$

Hence (2.2) does not hold in this case for $a = 1$ and $b > 1$.

[3] In stating (2.1) and the assumptions under which it is derived we are paraphrasing Penrose. For his own formulation see his 1952, p. 71f.

[4] See Definition 1.3 and Remark 1.4(ii).

Nevertheless, experience suggests that such counter-examples are atypical, contrived exceptions. Both real-life and randomly generated WVGs with many voters provide much empirical evidence that (2.2) holds in most cases, as a general rule: if the distribution of weights is not too skewed (in other words, the ratio of the largest weight to the smallest is not very high), then the relative powers of the voters tend to approximate closely to their respective relative weights. Moreover, this is the case not only for multi-voter WVGs with $q = \frac{1}{2}$, but also for those with any $q \in (0, 1)$.

By the *relative power* of voter a in a WVG $\mathcal{W}$ we mean here a's Banzhaf (briefly, Bz) index β, obtained by normalizing (or relativizing) the Penrose measure:

$$\beta_a[\mathcal{W}] := \frac{\psi_a[\mathcal{W}]}{\sum_{k \in N} \psi_k[\mathcal{W}]}. \tag{2.5}$$

(see Definition 3.6). Similarly, a's *relative weight* $\bar{w}_a$ in $\mathcal{W}$ is obtained by dividing a's weight by the total weight of all voters:

$$\bar{w}_a[\mathcal{W}] := \frac{w_a}{\sum_{k \in N} w_k}. \tag{2.6}$$

The typical tendency of the values of β to approximate to the respective relative weights in multi-voter WVGs is illustrated in Tables 13.1 and 13.2. The WVGs shown in these tables are taken from Felsenthal & Machover(2001). Both are decision rules designed for the so-called qualified majority voting (QMV) in the EU's Council of Ministers following its prospective enlargement to 27 member states. $\mathcal{N}_{27}$ (Table 13.1) is prescribed in the Treaty of Nice (2001);[5] Rule B (Table 13.2) is a 'benchmark' rule proposed in Felsenthal & Machover (2001).

In each of these tables, column (1) gives the weights of the voters. The absolute and relative quota are stated at the bottom of the table. Column (2) gives the respective relative weights $\bar{w}$ as percentages. Column (3) gives the relative voting powers as measured by the Bz index β, also in percentage terms. Column (4) gives the ratio of the Bz index to the respective relative weight. Note that all the figures in this column are quite close to 1. In Table 13.1 they are well within the range 1 ± 0.1. In Table 13.2 – where the quota is nearer half the total weight – the approximation is even better: the ratios are all well within the range 1 ± 0.01.

The same tendency is also apparent in Table 13.4, which is based on a WVG model of the Electoral College that elects the President of the US. The figures

[5] $\mathcal{N}_{27}$ is not stated in the treaty in this simple form, as a WVG; but it can be reduced to the form shown in Table 13.1. For details see Felsenthal & Machover (2001, Section 3).

for β are quite close to those for $\bar{w}$.

Moreover, a similar phenomenon is observable not only for the Bz index, but for also for some other indices of voting power, notably the Shapley–Shubik (briefly, S-S) index ϕ.[6] (For a definition of the S-S index see Section 3.1.) This typical behavior of ϕ is also illustrated in Tables 13.1 and 13.2. In these tables, column (5) gives the values of the S-S index in percentage terms and column (6) gives the ratio of these values to the respective relative weights. Note that all these ratios are well within the range 1 ± 0.05. The same tendency is evident also in Table 13.4: compare the figures for ϕ with those for $\bar{w}$.

This suggests a general problem: *under what conditions does the ratio of the voting powers of any two voters, as measured by a given index, converge to the ratio of their weights?*

In order to make this problem more precise, let us introduce the following framework.

Definition 2.1 Let
$$N^{(0)} \subsetneqq N^{(1)} \subsetneqq N^{(2)} \subsetneqq \cdots$$
be an infinite increasing chain of finite non-empty sets, and let
$$N = \bigcup_{n=0}^{\infty} N^{(n)}. \tag{2.7}$$
Let w be a weight function that assigns to each $a \in N$ a positive real number w_a as weight; and let q be a real $\in (0, 1)$.
For each $n \in \mathbb{N}$ let $\mathcal{W}^{(n)}$ be the WVG whose assembly is $N^{(n)}$ – each voter $a \in N^{(n)}$ being endowed with the pre-assigned weight w_a – and whose relative quota is q.
We shall then say that $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is a q-*chain* of WVGs.

Remark 2.2 In what follows, whenever we shall refer to a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$, we shall assume that the $N^{(n)}$, N and w are as specified in Definition 2.1: $N^{(n)}$ is the assembly of $\mathcal{W}^{(n)}$, N is given by (2.7), and w is the weight function.

[6] Thus, in multi-voter WVGs in which the distribution of weights is not extremely skewed, the respective values of β and ϕ tend, as a general rule, to be quite close to each other. This phenomenon has helped to foster the widespread fallacy that these two indices *always* behave alike, and so must have more or less the same meaning. This fallacy is criticized in Felsenthal & Machover (1998).

Definition 2.3 Let ξ be an index of voting power. We shall say that *Penrose's Limit Theorem (PLT) holds for the q-chain* $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ *with respect to the index* ξ *and* $a, b \in N$ if

$$\lim_{n\to\infty} \frac{\xi_a[\mathcal{W}^{(n)}]}{\xi_b[\mathcal{W}^{(n)}]} = \frac{w_a}{w_b}. \tag{2.8}$$

Remarks 2.4 (i) Note that $\xi_a[\mathcal{W}^{(n)}]/\xi_b[\mathcal{W}^{(n)}]$ in (2.8) is undefined if $a \notin N^{(n)}$ or $b \notin N^{(n)}$, but this does not matter because $a, b \in N^{(n)}$ for all sufficiently large n.

In preparation for what follows, we introduce two items of notation.

First, note that if $a \in N^{(n)}$ the *relative* weight of a in $\mathcal{W}^{(n)}$ – unlike a's absolute weight w_a – depends on n. We denote this relative weight by $\bar{w}_a^{(n)}$; thus

$$\bar{w}_a^{(n)} := \frac{w_a}{\sum_{k\in N^{(n)}} w_k}. \tag{2.9}$$

Second, for each $a \in N$ we put

$$N_a^{(n)} := \{k \in N^{(n)} : w_k = w_a\}. \tag{2.10}$$

The members of $N_a^{(n)}$ have the same weight as a, and we shall therefore refer to them as *replicas* of a.

In the analysis of Part I we shall distinguish between a *measure* and *index* of voting power. We shall use 'index' to indicate that a measure of voting power satisfies a normalization condition: the power measures of the voters sum up to 1.

In Chapter 3 we shall prove that PLT holds for the classical Shapley-Shubik and Banzhaf index under suitable conditions. In Chapter 4, Definition 2.1 and 2.3 will be extended to weighted voting games, in which voters have the option of abstaining.

Chapter 3

PLT for Binary WVGs

3.1 PLT for Replicative q-Chains and the S-S index

Definition 3.1 The *Shapley-Shubik index* – briefly, S-S index – is the function ϕ that assigns to any SVG $\mathcal{W}$ and any voter a of $\mathcal{W}$ a value $\phi_a[\mathcal{W}]$ given by

$$\phi_a[\mathcal{W}] := \sum_{S \subseteq N} \frac{(|S|-1)!(n-|S|)!}{n!}(v(S) - v(S) - \{a\}),$$

where v is the characteristic function of $\mathcal{W}$. (Here, N is the assembly of $\mathcal{W}$ with $n = |N|$.)

In this section we shall prove that PLT holds with respect to the S-S index for a special class of chains. The main special property of these chains is that any $a \in N$ is eventually (that is, for sufficiently large n) accompanied by sufficiently many replicas in $N^{(n)}$. Let us make this more precise.

Definition 3.2
(i) We shall say that the q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is *non-atomic* if

$$\lim_{n\to\infty} \max\{\bar{w}_a^{(n)} | \ a \in N^{(n)}\} = 0. \tag{3.1}$$

(ii) We say that the q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is *replicative* with respect to $a \in N$ if there is a positive constant C_a such that for all sufficiently large n

$$\sum_{k \in N_a^{(n)}} \bar{w}_k^{(n)} > C_a. \tag{3.2}$$

Remark 3.3 Condition (3.1) is essentially the one assumed by Penrose: the relative weight of each voter becomes negligibly small.
The second condition (3.2) ensures that, nevertheless, the total relative weight of voter a's replicas does *not* become negligibly small.

Our main result in this section is

Theorem 3.4 *If $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is a non-atomic q-chain then PLT holds with respect to the S-S index for those $a, b \in N$ for which $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is replicative.*

Proof We shall show that if $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ is replicative with respect to $a \in N$

$$\lim_{n\to\infty} \frac{\phi_a[\mathcal{W}^{(n)}]}{\bar{w}_a^{(n)}} = 1, \tag{3.3}$$

from which our theorem clearly follows.

To this end, we invoke a result of Neyman (1982, Theorem 9.8), according to which (3.1) implies that

$$\lim_{n\to\infty} \sum_{k\in N^{(n)}} \left|\phi_k[\mathcal{W}^{(n)}] - \bar{w}_k^{(n)}\right| = 0.$$

Now let $a \in N$. Then we have, *a fortiori*,

$$\lim_{n\to\infty} \sum_{k\in N_a^{(n)}} \left|\phi_k[\mathcal{W}^{(n)}] - \bar{w}_k^{(n)}\right| = 0,$$

which can be written as

$$\lim_{n\to\infty} \sum_{k\in N_a^{(n)}} \bar{w}_k^{(n)} \left|\frac{\phi_k[\mathcal{W}^{(n)}]}{\bar{w}_k^{(n)}} - 1\right| = 0. \tag{3.4}$$

However, all the $k \in N_a^{(n)}$ are replicas of a, so they all have the same value of ϕ and the same weight as a. Hence (3.4) can be written as follows:

$$\lim_{n\to\infty} \left|\frac{\phi_a[\mathcal{W}^{(n)}]}{\bar{w}_a^{(n)}} - 1\right| \sum_{k\in N_a^{(n)}} \bar{w}_k^{(n)} = 0.$$

It now follows from (3.2) that (3.3) holds – as claimed. □

Remark 3.5 In the definition of WVG, the blunt inequality $\geq$ in (1.2) can be replaced by a sharp inequality $>$ (see Remark 1.4(i)). The two definitions are equivalent, however, the relative quota q of a WVG in the blunt sense may not work for the sharp sense, but may need to be slightly adjusted (and vice versa). Consequently, the corresponding definitions of q-chain and non-atomic q-chain in the sharp sense do not yield the same classes as our present Definitions 2.1 and 3.2. Nevertheless, Theorem 3.4 applies to non-atomic q-chains in the sharp sense as well, because Neyman's result, on which our proof depends, also covers this case – see Neyman 1981, Lemma 3.2.

3.2 PLT for 1/2-Chains and the Banzhaf Index

Definition 3.6 The *Penrose* or *Banzhaf measure* – briefly, Bz measure – is the function ψ that assigns to any SVG $\mathcal{W}$ and any voter a of $\mathcal{W}$ a value $\psi_a[\mathcal{W}]$ given by

$$\psi_a[\mathcal{W}] := \frac{1}{2^{n-1}} \sum_{S \subseteq N} (v(S) - v(S - \{a\})), \tag{3.5}$$

where v is the characteristic function of $\mathcal{W}$.
The *Banzhaf index* – briefly, Bz index – is the function β defined by

$$\beta_a[\mathcal{W}] := \frac{\psi_a[\mathcal{W}]}{\sum_{k \in N} \psi_k[\mathcal{W}].} \tag{3.6}$$

(As usual, N is the assembly of $\mathcal{W}$ and $n = |N|$.)

Remark 3.7 The Penrose (Banzhaf) measure $\psi_a[\mathcal{W}]$ of voter a denotes the probability that a is *critical* under $\mathcal{W}$ – that is, that the other voters of $\mathcal{W}$ are so divided that a's vote can tip the balance – if every outcome is equally likely. For WVGs as given by (1.1) this implies that $\psi_a[\mathcal{W}]$ is equal to the probability that the combined weight sum of the voters other than a lies in the interval $\left[q\left(\sum_{k \in N} w_k\right) - w_a, q \sum_{k \in N} w_k\right)$, where every voter $k \in N$ is equally likely to vote 'yes' or 'no'.

Given a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ of WVGs, we associate with it the family of independent random variables $\{Y_k | \ k \in N\}$, indexed by N, such that for every $a \in N$,

$$\text{Prob}\,\{Y_a = w_a\} = \text{Prob}\,\{Y_a = 0\} = \frac{1}{2}. \tag{3.7}$$

We consider the chain

$$\mathfrak{Y} := \left\{ \{Y_k |\ k \in N^{(n)}\} |\ n \in \mathbb{N} \right\} \tag{3.8}$$

of (finite) sets of these random variables.
For any $a \in N$ let us put

$$S_{\neg a}^{(n)} := \left(\sum_{k \in N^{(n)}} Y_k \right) - Y_a, \quad \mu_{\neg a}^{(n)} := E\left[S^{(n)}\right]_{\neg a}, \quad \sigma_{\neg a}^{(n)} := \left(Var\left[S_{\neg a}^{(n)} \right] \right)^{\frac{1}{2}}.$$

And let $\bar{S}_{\neg a}^{(n)}$ be the 'standardized' form of $S_{\neg a}^{(n)}$, i.e.

$$\bar{S}_{\neg a}^{(n)} := \frac{S_{\neg a}^{(n)} - \mu_{\neg a}^{(n)}}{\sigma_{\neg a}^{(n)}}. \tag{3.9}$$

Using the definition of the Y_a it is easy to obtain the following explicit expression for $\mu_{\neg a}^{(n)}$ and $\sigma_{\neg a}^{(n)}$

$$\mu_{\neg a}^{(n)} = \frac{\left(\sum_{k \in N^{(n)}} w_k \right) - w_a}{2}, \tag{3.10}$$

$$\sigma_{\neg a}^{(n)} = \frac{\left[\left(\sum_{k \in N^{(n)}} w_k^2 \right) - w_a^2 \right]^{\frac{1}{2}}}{2}. \tag{3.11}$$

Definition 3.8 We shall say that the chain $\mathfrak{Y}$ satisfies the *special local central limit (SLCL) condition* if, for every $a \in N$,

$$\lim_{n \to \infty} \mathrm{Prob} \left\{ \bar{S}_{\neg a}^{(n)} \in \left[-\frac{w_a}{2\sigma_{\neg a}^{(n)}}, \frac{w_a}{2\sigma_{\neg a}^{(n)}} \right) \right\} \frac{\sigma_{\neg a}^{(n)}}{w_a} = \frac{1}{\sqrt{2\pi}}; \tag{3.12}$$

and for all $a, b \in N$,

$$\lim_{n \to \infty} \frac{\sigma_{\neg a}^{(n)}}{\sigma_{\neg b}^{(n)}} = 1. \tag{3.13}$$

Remark 3.9 The $\bar{S}_{\neg a}^{(n)}$ are evidently discrete random variables with mean 0. We shall be interested in cases where their standard deviations, $\sigma_{\neg a}^{(n)}$, tend to ∞ with n. Then equation (3.12) says that the average density of $\bar{S}_{\neg a}^{(n)}$ in a half-open interval around 0, whose length becomes vanishingly small, approaches the value of the standard normal density function φ at 0, namely $1/\sqrt{2\pi}$. This means that $\mathfrak{Y}$ obeys a special case (namely, at 0) of the local central limit theorem of probability theory.

The main result in this section is

Theorem 3.10 *Let $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ be a $\frac{1}{2}$-chain of WVGs. If its associated chain $\mathfrak{Y}$ satisfies the SLCL condition, then PLT holds for $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ with respect to the Bz index and any $a, b \in N$.*

Proof Let $a \in N$ and take n large enough so that $a \in N^{(n)}$. Then, by definition, the Penrose measure of a in $\mathcal{W}^{(n)}$ is given by

$$\psi_a[\mathcal{W}^{(n)}] = \mathrm{Prob}\left\{ S_{\neg a}^{(n)} \in \left[\frac{1}{2}(\sum_{k\in N^{(n)}} w_k) - w_a, \frac{1}{2}\sum_{k\in N^{(n)}} w_k \right) \right\}$$

(see Remark 3.7). Using (3.9) and (3.10), this can be re-written as

$$\psi_a[\mathcal{W}^{(n)}] = \mathrm{Prob}\left\{ \bar{S}_{\neg a}^{(n)} \in \left[-\frac{w_a}{2\sigma_{\neg a}^{(n)}}, \frac{w_a}{2\sigma_{\neg a}^{(n)}} \right) . \right\}$$

Invoking (3.12) we obtain

$$\lim_{n\to\infty} \psi_a[\mathcal{W}^{(n)}] \frac{\sigma_{\neg a}^{(n)}}{w_a} = \frac{1}{\sqrt{2\pi}}. \tag{3.14}$$

Hence by (3.13)

$$\lim_{n\to\infty} \frac{\psi_a[\mathcal{W}^{(n)}]}{\psi_b[\mathcal{W}^{(n)}]} = \frac{w_a}{w_b}.$$

Finally, using (3.6) we get

$$\lim_{n\to\infty} \frac{\beta_a[\mathcal{W}^{(n)}]}{\beta_b[\mathcal{W}^{(n)}]} = \frac{w_a}{w_b},$$

as claimed. □

Combining (3.14) and (3.11) we get:

Corollary 3.11 *Let $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ be a $\frac{1}{2}$-chain of WVGs. If* (3.12) *holds for the associated chain $\mathfrak{Y}$, then*

$$\psi_a[\mathcal{W}^{(n)}] \approx w_a \sqrt{\frac{2}{\pi\{(\sum_{k\in N^{(n)}} w_k^2) - w_a^2\}}}. \tag{3.15}$$

This is a slightly improved version of Penrose's approximation formula (2.1). Of course, if – as Penrose assumes – each individual weight w_a becomes relatively negligible, then the difference between the two approximations is likewise negligible.

Remark 3.12 Owen (1995, pp. 272, 297) gives an approximation formula for ψ as well as for ϕ in multi-voter WVGs. His approximations are based on an interval version of the central limit theorem (see Remark 3.14 (i)) and are stated without proof and without specifying the precise conditions under which they hold (proving the validity of these approximations is not straightforward – for a discussion see Chapter 12. Nevertheless, the numerical approximations he obtains for the Penrose measures ψ of the bloc-voters in the US Presidential Electoral College – shown in the last column of Table XII.4.1 of Owen (1995, p. 297) – are closer than ours, which are based on (3.15) above and shown in the last column of our Table 13.4. (The exact values of ψ, correct to six decimal figures, are shown in the penultimate column of Table 13.4.)

As an example of an application of Theorem 3.10, we prove the following:

Theorem 3.13 *Let $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ be a $\frac{1}{2}$-chain such that its weight function w assumes only finitely many values, all of them positive integers; and such that the greatest common divisor of those values w_a that occur infinitely often is* 1. *Then the associated chain $\mathfrak{Y}$ satisfies the SLCL condition. Hence PLT holds for $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ with respect to the Bz index and any $a, b \in N$. Also,* (3.15) *holds.*

Proof To prove that (3.12) holds for any $a \in N$, observe that, since all possible values of $S_{\neg a}^{(n)}$ are integers, all possible values of $\bar{S}_{\neg a}^{(n)}$ belong to a lattice whose span is $1/\sigma_{\neg a}^{(n)}$. In the half-open interval

$$\left[-\frac{w_a}{2\sigma_{\neg a}^{(n)}}, \frac{w_a}{2\sigma_{\neg a}^{(n)}}\right)$$

there are exactly w_a points of this lattice: say $x_i^{(n)}, i = 1, 2, \ldots, w_a$.
We invoke a well-known version of the local central limit theorem – see Petrov (1975, p. 189, Theorem 2); see also Remark 3.14(i). From this theorem it follows that if n is sufficiently large then for each $i = 1, 2, \ldots, w_a$ the product

$$\text{Prob}\left\{\bar{S}_{\neg a}^{(n)} = x_i^{(n)}\right\} \sigma_{\neg a}^{(n)} \tag{3.16}$$

is arbitrarily close to $\varphi(x_i^{(n)})$. Also, from (3.11) it is clear that $\lim_{n\to\infty} \sigma_{\neg a}^{(n)} = \infty$; thus for a sufficiently large n each of the $x_i^{(n)}$ is arbitrarily close to 0,

hence the product (3.16) is arbitrarily close to $\varphi(0) = (2\pi)^{-\frac{1}{2}}$. But the left-hand side of (3.12) is simply the arithmetic mean of the w_a products (3.16); so it also gets arbitrarily close to $(2\pi)^{-\frac{1}{2}}$, as required.
As for (3.13): we have just seen that as n increases, $\sigma_{\neg a}^{(n)}$ grows without bound. Clearly, the term w_a^2 in (3.11) becomes negligible. Therefore (3.13) holds. □

Remarks 3.14
(i) For the proof of Theorem 3.13 we invoke a version of the local central limit theorem as given in Petrov (1975, p.189 Theorem 2). This Theorem deals with a sequence of independent integer-valued random variables each having finite variance, such that the set of distinct distributions of these variables is finite. The key condition is that the greatest common divisor of the maximal spans of those distributions that occur infinitely often in the sequence is 1. For details see Petrov (1975, ibid.).
(ii) Note that the chain as given by (2.3) fails to satisfy the condition of Theorem 3.13. For positive integers $w' < w$, the gcd of the weights that occur infinitely often is $w > 1$.
(iii) Empirical and computational experience provides much evidence that Theorem 3.10 and Theorem 3.13 also hold for general $q \in (0,1)$. However, a general proof shows one major difficulty: application of Petrov's version of the local central limit theorem analogously to the $q = 1/2$ case yields

$$\psi_a[\mathcal{W}^{(n)}]\sigma_{\neg a}^{(n)} = w_a \varphi(m_{\neg a}^{(n)}) + \varepsilon_{\neg a}^{(n)}, \tag{3.17}$$

where $m_{\neg a}^{(n)}$ is a mean value and $\varepsilon_{\neg a}^{(n)}$ is the approximation error which tends to 0 with increasing n. For $q = 1/2$, the mean value is arbitrarily close to 0 for any sufficiently large n. However, for $q \neq 1/2$ the mean value tends to $\pm\infty$ such that $w_a \varphi(m_{\neg a}^{(n)})$ also tends to zero. Hence it has to be shown that the *relative* error of the approximation tends to 0.

Chapter 4

PLT for Ternary WVGs

4.1 Introduction

In real life decisions, the option to abstain is one that can undoubtedly influence the outcome of a vote. This is clearly evident in the most commonly used rule in decision-making bodies: the simple majority, whereby a resolution passes if, and only if, more voters vote for it than against it. Unless specified otherwise, this rule does not treat abstentions as tantamount to either 'yes' or 'no'. Certainly, there are many real-life decision rules that do no treat abstention as a distinct third option. For example, in the Council of Ministers of the European Union, abstention usually counts as a 'no', except when an issue to be vote upon is basic constitutional. In this case abstention counts as a 'yes'. However, these are exceptions since in most real-life situations abstention is a *tertium quid*. In the United Nations Security Council (UNSC) abstention plays a key role: an abstaining permanent member is usually not interpreted as a vetoer. Since Article 27 of the UN Charter requires a minimum of nine affirmative members abstention is not treated tantamount to 'yes' either.[1] In each of the two houses of the US Congress the rule is that for a proposal to pass a certain percentage[2] of the members *present* has to be achieved (provided that a quorum of half the membership is present).

[1] Probably the most famous example of a significant abstention effect occurred in 1950 when the USSR's boycott of the UNSC led to a resolution of sending UN forces to Korea. Although the USSR strongly opposed it, their absence – 'passive' abstention – did not prevent the passage of the motion.

[2] The voting rule depends on the nature of the issue at hand. In some cases it is simple majority, in some the needed affirmative share is two-thirds.

Somewhat surprisingly, the literature has only recently started to take any notice of it. The widely used instrument to analyze voting power is that of a SVG – as given by Definition 1.2 – which is *binary* in that they assume that each voter has just two options: 'yes' and 'no'. This shortcoming is even more surprising as social choice theory does not in general impose strict preference orderings, i.e. it allows for indifference over alternatives. In their 1998 Felsenthal & Machover criticize the 'misreporting' of some authors to squeeze rules into the SVG corset when abstention is a distinct third option.[3] To overcome this shortcoming Felsenthal & Machover (1997, 1998) propose a setup called a *ternary voting game* (TVG) by defining an appropriate generalization of a SVG: abstention is added as a third option alongside 'yes' and 'no'. This extends an earlier step in this direction taken by Fishburn (1973, pp. 53-55).

Whereas in their TVG setup an analogous definition of the Bz measure follows more or less naturally, the translation of the S-S index is less obvious. The authors construct it by means of an alternative representation of this index (Felsenthal & Machover 1996).

More recently, Braham & Steffen (2002) remarked that the simple majority rule is often specified as *counting only the votes of those voting* ('yes' or 'no') so that abstention can be seen as tantamount to 'non-participation'. From this they argue that in contrast to Felsenthal & Machover who treat 'abstain' as symmetric to 'yes' and 'no', abstentions are to be treated separately. They point out that the TVG structure assumes that voters can choose simultaneously between 'yes', 'no' and 'abstain', when in fact the 'counting the votes of those voting' implies a sequential choice structure: a voter first decides whether to vote at all, and then to vote 'yes' or 'no'. This approach suggests other generalizations of the Bz and S-S index than the ones proposed by Felsenthal and Machover.

In this chapter, we will develop a probabilistic characterization of power in games with abstentions (not necessarily weighted) which constitutes a unifying approach to power measures in TVGs based on different modelling of the nature of abstentions. This will be achieved by recourse to a probabilistic interpretation of voting power, such that it is expressed as an expected *contribution* of a voter to the outcome of the vote (i.e. the practical difference that a voter makes). This unifying characterization shows a guideline for choosing

[3] They offer the hypothesis that 'the misreporting is due to what philosophers of science have called *theory-laden* or *theory-biased* observation - a common occurrence, akin to optical illusion, whereby an observer's perception is unconsciously distorted so as to fit a preconception', p. 280, as well as 1997.

the appropriate power concept in games with abstentions. Furthermore, in WVGs this interpretation allows to apply the powerful tools of stochastics, primarily important for approximation purposes, and will eventually detect PLT in WVGs with abstention as a distinct third option.

The chapter is organized as follows. Section 4.2 gives a general probabilistic interpretation for voting games with abstentions. The prevalent concepts of the nature of abstention are introduced in Section 4.3. Section 4.4 discusses PLT in weighted voting games with abstentions.

4.2 Probabilistic Interpretation

Definition 4.1 [Felsenthal & Machover (1997, 1998)]
(i) A *tripartition* of a set N is a map T from N to $\{-1, 0, 1\}$. We denote by T^-, T^0 and T^+ the inverse images of $\{-1\}, \{0\}$ and $\{1\}$ respectively under T:

$$\begin{aligned} T^- &= \{k \in N|\ T(k) = -1\}, \\ T^0 &= \{k \in N|\ T(k) = 0\}, \\ T^+ &= \{k \in N|\ T(k) = 1\}. \end{aligned} \tag{4.1}$$

We define partial ordering $\leq$ among tripartitions: if T_1 and T_2 are two tripartitions of N, we define

$$T_1 \leq T_2 \ :\Leftrightarrow\ T_1(k) \leq T_2(k) \text{ for all } k \in N.$$

(ii) By a *ternary voting game* – briefly TVG – we mean a mapping W from the set $\{-1, 0, 1\}$ of all tripartitions of N to $\{-1, 1\}$, satisfying the following three conditions:

$$\begin{aligned} &(1) \qquad T^+ = N \Rightarrow W(T) = 1; \\ &(2) \qquad T^- = N \Rightarrow W(T) = -1; \\ &(3) \qquad \textit{Monotonicity}: T_1 \leq T_2 \Rightarrow WT_1 \leq WT_2. \end{aligned}$$

We call W the *outcome* of T (under W).
(iii) If $T(a) \geq 0$, we denote by '$T_{a\downarrow}$' the tripartition such that $T_{a\downarrow} = T(a) - 1$ but $T_{k\downarrow}(k) = T(k)$ for all other $k \in N$. If $W(T) = 1$ and $W(T_{a\downarrow}) = -1$ we say that a is *positively W-critical for T*.

Remarks 4.2
(i) A ternary division T is interpreted as a voting division which allows

abstentions. T^- and T^+ are interpreted as the sets of 'no' and 'yes' voters, T^0 as the set of abstainers respectively. $T(k)$ can be interpreted as a degree of support of voter k for the decision in question.

(ii) Whether an affirming coalition T^+ induces the passage of a proposal also depends on the abstainers T^0, i.e. there are decision rules with which it is possible that the same set of 'yes' voters T^+ is successful in one tripartition, but not strong enough to let the bill pass in another tripartition.

For our purpose it shall prove useful to work with a TVG model that is more 'one-sided' with respect to the outcome. In analogy to SVGs we define:

Definition 4.3
Let W be a TVG on N. A *ternary characteristic* or *tripartitional* function v is the map from the set of all tripartitions $\{-1, 0, 1\}^N$ to $\{1, 0\}$ such that, for any tripartition T,

$$v(T) = \begin{cases} 1 \text{ if } W(T) = 1, \\ \\ 0 \text{ otherwise.} \end{cases} \tag{4.2}$$

In SVGs the S-S index and Bz measure of a voter $a \in N$ is defined as the (weighted) sum of *contributions* $C_a(S) := v(S) - v(S - \{a\})$ voter a brings to each possible coalition $S \subset N$ in which s/he is a member (see Definition 3.1 and 3.6). The contributions are weighted with coalition-specific factors $f_a(S)$. Let ξ stand for either the Bz measure or S-S index of voter a. Then for all $a \in N$ the general form of ξ_a is given by

$$\xi_a = \sum_{S \subset N| \ a \in S} f_a(S) C_a(S), \tag{4.3}$$

where $f_a(S)$ equals $(|S| - 1)!(n - |S|)!/n!$ for the S-S index and $1/2^{n-1}$ for the Bz measure.

Analogously, we shall model power of voter $a \in N$ in a game with abstentions as the weighted sum of his or her *contributions* $C_a(T)$ to each possible tripartition. The contributions are weighted by a factor $f_a(T)$ which can be interpreted as the probability that the specific tripartition T forms. Formally, let ξ denote a measure of voting power in TVGs, then

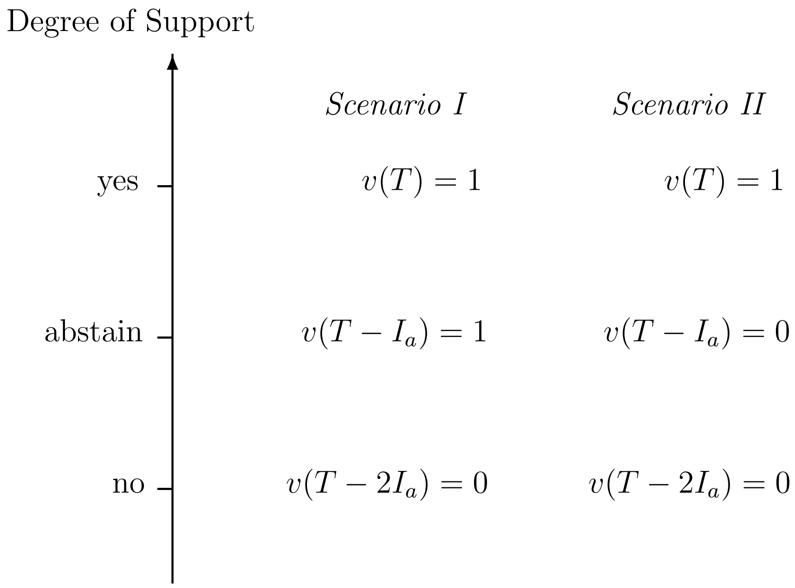
$$\xi_a = \sum_{T \in S_N} f_a(T) C_a(T). \tag{4.4}$$

Consider first the contribution term $C_a(T)$. For each $a \in N$ we define an *indicator function* of a, I_a, as a function on N such that $I_a(k) = 1$ for $k = a$ and zero elsewhere. Put

$$C_a(T) := \begin{cases} v(T) - v(T - 2I_a) \text{ if } a \in T^+, \\ \\ 0 \text{ otherwise.} \end{cases} \tag{4.5}$$

Hence $C_a(T) = 1$ iff voter a's choice makes a practical difference to the outcome by affirming the issue to vote upon instead of rejecting it. Note that it is not important at which level the change in the outcome occurs, i.e. whether ceteris paribus from a's switch from 'yes' to 'abstain' or from 'abstain' to 'no'. Figure 4.1 gives an illustration.

Figure 4.1: Contribution of a Voter



In Scenario I, the bill passes even with voter a switching from 'yes' to 'abstain'. But T^+ no longer has a majority when a votes 'no' instead of abstaining. In Scenario II, the change in the outcome occurs when voter a decides to abstain instead of voting 'yes'. If decreasing support of a has no effect on the outcome this implies $v(T) = v(T - I_a) = v(T - 2I_a)$ and voter a has a contribution of zero (i.e. makes no practical difference to the outcome).

Technically, C_a works as a filter. With it's binary values of either 0 or 1, it causes the sum in (4.4) to be taken only over the specific probabilities where a is considered to be crucial. With (4.5) it is therefore possible to read (4.4) as the probability that voter a is decisive as a 'yes' voter in a random tripartition X

$$\xi_a = Prob\{X \text{ wins, } X - 2I_a \text{ loses} |\ a \in T^+\}. \tag{4.6}$$

Equation (4.6) is a direct extension of the terms of a voter having power which Straffin uses for the SVG framework: the power of voter a is 'the probability that a bill passes if we assume a votes for it, but would fail if a voted against it' (Straffin 1994, p. 1136).

The next section discusses the tripartition specific factor $f_a(T)$ of expression (4.3). The probability that a tripartition T forms hinges on two settings: the nature of the abstention decision and the behavioral assumptions about the voters.

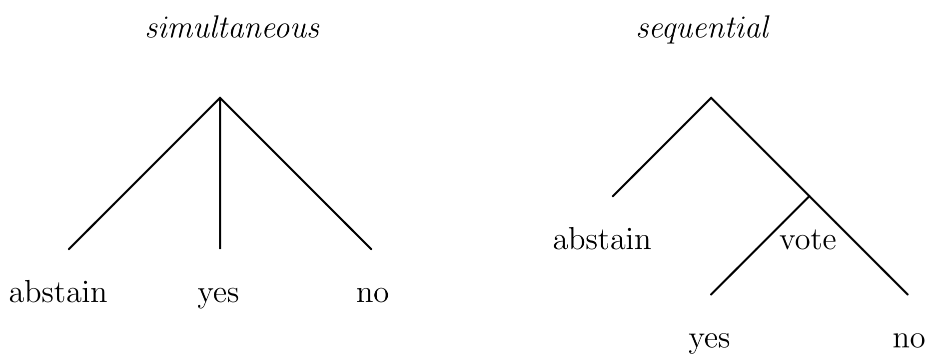
4.3 Nature of Abstention

In their 1997, Felsenthal & Machover treat abstention on a par with 'yes' and 'no' which implies that the voter decides simultaneously between the three options. In contrast Braham & Steffen (2002) propose a sequential structure: first the voter decides whether to vote or not and then to vote 'yes' or 'no'. Figure 4.2 illustrates the two approaches.

In the simultaneous approach, an abstaining voter can be thought of as being present in an assembly but indecisive about the issue to vote on. In this case the voter may feel neither affirmative nor negative about the proposal and thus chooses to declare 'I abstain' or casts an empty ballot. This is what Machover (2002) has called *active abstention*. In this case the voter is part of the quorum, even though his or her decision is neutral.

A different form of abstention takes place if the voter simply does not participate in the division. This may occur if the voter is prevented for any reason or if the issue to vote on is of minor interest to the voter, such that the costs of voting are higher than the expected pleasure of being on the winning side. Machover (2002) has termed this *abstention by default* and is reflected by the sequential approach proposed by Braham & Steffen (2002).

Figure 4.2: Nature of Abstention



In general, decision rules are blind to the distinction between the two kinds of abstentions. But in some cases active and default abstentions are specified. For example, in the US Congress active abstainers are counted for purpose of a quorum. So, if a quorum is not present because too many have abstained by default, no voting can take place at all.[4] But if a quorum is present and all present actively abstain the outcome according to the ordinary majority rule in the house of Representatives is presumably negative since the number of 'yes' voters is not greater than the number of 'no' voters.

Remark 4.4 If a quorum is required for a vote to take place, there is neither acceptance nor rejection of the proposal in case of absence of that quorum. Given abstention by default one could extend the binary outcomes of v in (4.2) by a third one, 'defer', representing a tie. In the present account we shall not discuss ties.

The common a priori assumption of the SVG setup of the voter voting independently is easily translated into the TVG framework. However, the spirit of a priori ignorance is less obvious when it comes to assigning probabilities to the single options in either approach of the nature of abstention. The route that Felsenthal & Machover have taken is to appeal Bernoulli's Principle of Insufficient Reason[5] to justify assigning a priori probabilities of 1/3 for each option. In their 1997 and 1998 they define a generalization of the

[4]This is a simplifying assumption as, in fact, there must be a motion to consider the quorum in order for a count to even take place. For details see Felsenthal & Machover (1998), Chapter 4.

[5]This principle claims that each of the alternatives should have equal probability if there is no known reason for assigning unequal ones (for more details see, for example, Felsenthal & Machover 1998).

Bz measure as follows:

Definition 4.5 Let W be a TVG with assembly N and let $a \in N$. We define the *Bz score* $\eta[W]$ by stipulating that $\eta_a[W]$ is the number of tripartitions of N for which a is positively W-critical.

We define the Bz *index of voting power* $\beta[W]$ by putting

$$\beta_a[W] := \frac{\eta_a[W]}{\sum_{k \in N} \eta_k[W]}. \tag{4.7}$$

We define the *Bz measure of voting power* $\psi[W]$ by putting

$$\psi_a[W] := \frac{\eta_a[W]}{3^{n-1}}. \tag{4.8}$$

Here, as usual, $n = |N|$ is the number of voters in W.

In terms of (4.4) they put

$$f_a(T) := 3^{n-1}. \tag{4.9}$$

However, the symmetry in TVGs is much less self-evident in comparison to the SVG setup. In the following we will therefore stick to a more general treatment as proposed as an alternative by the authors in their 1997 (p. 340). We assume that the a priori probability of any given voter abstaining is $t \in (0, 1)$ and s/he votes 'yes' and 'no' each with probability $(1 - t)/2$. Hence

$$f_a(T) = t^{|T^0|}((1 - t)/2)^{n-1-|T^0|}, \tag{4.10}$$

such that (4.9) is given by $t = 1/3$. Following the sequential approach of Braham & Steffen any voter first decides *whether* to vote or abstain with probability $1 - t$ and t respectively. In the second stage s/he decides *how* to vote, i.e. to choose either 'yes' or 'no' with probability 1/2 each. This provides

$$f_a(T) \quad = \quad t^{|T^0|}(1 - t)^{n-1-|T^0|}(1/2)^{n-1-|T^0|} \tag{4.11}$$

which equals (4.10). However, an a priori argument appealing Bernoulli's Principle of Insufficient Reason suggests $t = 1/2$ for the sequential approach.

With a tripartition specific factor as in (4.10) we will refer to the power measure as defined in (4.4) as the *generalized Bz measure* and denote it by ψ. Analogously, we shall refer to it's normalized form as *generalized Bz index*.

Remarks 4.6
(i) Braham & Steffen (2002) model voting games with abstentions (by default) as a whole bundle of SVGs in which each assembly consists of the non-abstaining voters. They express power as an expected value: power is the weighted sum of power in each single SVG. However, their concept is controversial in that an abstaining voter never exerts any power. Hence with expression (4.4) we only partly follow their concept of power – by a suitable tripartition specific factor $f_a(T)$ covering the sequential approach.

(ii) From (4.11) it is apparent that in the sequential approach the actual order of a voter's decision is not important. The order in Figure 4.2 is that which is observed, i.e. we see people either going to vote or not and then casting a 'yes' or 'no' ballot. However, the decision to vote 'yes' or 'no' may have been prior to the decision whether to abstain or participate in the vote.

4.4 PLT in WVGs with Abstentions

In WVGs with abstention as a tertium quid we shall consider the rule that the issue passes if the combined weight of affirming voters meets or exceeds some preset relative weight share *of those voting* (either 'yes' or 'no'). When there is no risk of confusion we shall stick to the same bracket notation of a WVG in the original (binary) setup as in (1.1) and (1.3).

Definition 4.7 A *ternary weighted voting game* – briefly, TVWG –

$$[q; w_1, w_2, ..., w_n]$$

is given by an assignment of a non-negative real w_k to each voter $k \in N$, and a relative Quota q such that for any tripartition T of N

$$v(T) = \begin{cases} 1 & \text{if } \sum_{k \in T^+} w_k \geq q \sum_{k \in N - T^0} w_k, \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

We shall use the notation

$$< q; w_1, w_2, ..., w_n >$$

for a TWVG when the blunt inequality $\geq$ in (4.12) is replaced by the sharp inequality $>$.

Remark 4.8
The rule given in (4.12) may be rewritten such that a bill is passed iff the

total weight of those voting for it is at least $\tilde{q} = q/(1-q)$ times the total weight of those voting against it.

Let the following random variables denote the decision of every voter $k \in N$, i.e.

$$Z_k = \begin{cases} 0 \text{ if } k \text{ abstains }, \\ w_k \text{ otherwise}, \end{cases} \tag{4.13}$$

$$\tag{4.14}$$

$$Y_k = \begin{cases} w_k \text{ if } k \text{ votes 'yes'}, \\ 0 \quad \text{if } k \text{ votes 'no'}. \end{cases} \tag{4.15}$$

Put

$$S_{\neg a} := \left(\sum_{k \in N} Y_k\right) - Y_a, \qquad W_{\neg a} := \left(\sum_{k \in N} Z_k\right) - Z_a. \tag{4.16}$$

Then (4.6) provides

$$\xi_a = Prob\{q(W_{\neg a} + w_a) - w_a \leq S_{\neg a} < q(W_{\neg a} + w_a)\}. \tag{4.17}$$

Note that in contrast to the SVG setup the majority quota $q(W_{\neg a} + w_a)$ is random (see also Remark 3.7).

Let V_k denote the random variable

$$V_k := Y_k - qZ_k$$

which takes the values

$$V_k = \begin{cases} 0 & & t, \\ (1-q)w_k & \text{with probability} & (1-t)/2, \\ -qw_k & & (1-t)/2. \end{cases} \tag{4.18}$$

Put

$$X_{\neg a} := \left(\sum_{k \in N} V_k\right) - V_a.$$

Subtracting $qW_{\neg a}$ in (4.17) provides

$$\xi_a = \text{Prob}\{(q-1)w_a \leq X_{\neg a} < qw_a\}. \tag{4.19}$$

Given a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$ of TWVGs, we associate with it the family $\{V_k |\ k \in N\}$ of independent random variables indexed by N. We consider the chain

$$\mathfrak{V} := \left\{\{V_k |\ k \in N^{(n)}\} |\ n \in \mathbb{N}\right\}. \tag{4.20}$$

For any $a \in N$ we put

$$X^{(n)}_{\neg a} := \left(\sum_{k \in N^{(n)}} Z_k \right) - Z_a, \quad \mu^{(n)}_{\neg a} := E\left[X^{(n)}_{\neg a}\right], \quad \sigma^{(n)}_{\neg a} := \left(Var\left[X^{(n)}_{\neg a}\right]\right)^{\frac{1}{2}}.$$

Let $\bar{X}^{(n)}_{\neg a}$ be the 'standardized' form of $X^{(n)}_{\neg a}$, i.e.

$$\bar{X}^{(n)}_{\neg a} := \frac{X^{(n)}_{\neg a} - \mu^{(n)}_{\neg a}}{\sigma^{(n)}_{\neg a}}. \tag{4.21}$$

From (4.18) we obtain the following explicit expressions for $\mu^{(n)}_{\neg a}$ and $\sigma^{(n)}_{\neg a}$

$$\mu^{(n)}_{\neg a} = (1-t)(1-2q)\frac{\left(\sum_{k \in N^{(n)}} w_k\right) - w_a}{2}, \tag{4.22}$$

$$\left(\sigma^{(n)}_{\neg a}\right)^2 = (1-t)\left[(1-q)^2 + q^2 - \frac{1-t}{2}(1-2q)^2\right]\frac{\left(\sum_{k \in N^{(n)}} w_k^2\right) - w_a^2}{2} \tag{4.23}$$

Definition 4.9 We shall say that the chain $\mathcal{V}$ satisfies the *special local central limit condition* if, for every $a \in N$,

$$\lim_{n \to \infty} \text{Prob}\left\{ \bar{X}^{(n)}_a \in \left[-\frac{w_a}{2\sigma^{(n)}_{\neg a}}, \frac{w_a}{2\sigma^{(n)}_{\neg a}} \right) \right\} \frac{\sigma^{(n)}_{\neg a}}{w_a} = \frac{1}{\sqrt{2\pi}}; \tag{4.24}$$

and for all $a, b \in N$

$$\lim_{n \to \infty} \frac{\sigma^{(n)}_{\neg a}}{\sigma^{(n)}_{\neg b}} = 1. \tag{4.25}$$

Remark 4.10 Definition 4.9 is the analogue of Definition 3.8 for ternary WVGs. Note that the $\bar{X}^{(n)}_{\neg a}$ are discrete random variables with mean 0. Again, we shall be interested in cases where their standard deviations, $\sigma^{(n)}_{\neg a}$, tend to ∞ with n. For an interpretation of (4.24) see Remark 3.9.

Theorem 4.11 *Let $\left\{W^{(n)}\right\}_{n=0}^{\infty}$ be a $\frac{1}{2}$-chain of TWVGs. If its associated chain $\mathcal{V}$ satisfies the SLCL condition, then PLT holds with respect to the generalized Bz index and any $a, b \in N$.*

Proof Let $a \in N$ and take n large enough so that $a \in N^{(n)}$. Then, by definition, the generalized Bz measure of a in $W^{(n)}$ is given by

$$\psi_a\left[W^{(n)}\right] = \text{Prob}\left\{ \bar{X}^{(n)}_a \in \left[-\frac{w_a}{2\sigma^{(n)}_{\neg a}}, \frac{w_a}{2\sigma^{(n)}_{\neg a}} \right) \right\}.$$

Invoking (4.24) we obtain

$$\lim_{n\to\infty} \psi_a \left[W^{(n)}\right] \frac{\sigma_{\neg a}^{(n)}}{w_a} = \frac{1}{\sqrt{2\pi}}. \tag{4.26}$$

Hence by (4.25)

$$\lim_{n\to\infty} \frac{\psi_a \left[W^{(n)}\right]}{\psi_b \left[W^{(n)}\right]} = \frac{w_a}{w_b}. \tag{4.27}$$

Finally, using (4.7) and (4.8) we obtain

$$\lim_{n\to\infty} \frac{\beta_a \left[W^{(n)}\right]}{\beta_b \left[W^{(n)}\right]} = \frac{w_a}{w_b}. \tag{4.28}$$

□

From (4.26) and (4.23) follows

Corollary 4.12 *If* (4.24) *holds, then*

$$\psi_a[\mathcal{W}^{(n)}] \approx w_a \sqrt{\frac{2}{(1-t)\pi\{(\sum_{k\in N^{(n)}} w_k^2) - w_a^2\}}}. \tag{4.29}$$

Table 13.3 illustrates the tendency of the generalized Bz index to approximate to the relative weights in large TWVGs. The numbers are taken from $\mathcal{N}_{27}$ in Table 13.1, however, for the sake of numerical comparability the rule is replaced by a ternary weighted voting rule according to (4.12). Column (1) and (2) give, the absolute and relative weights respectively. Column(3) gives the generalized Bz index for $t = 1/3$ in percentage terms and column (4) gives the ratio of these values to the respective relative weights which are within the range 1 ± 0.12. The values suggest a slower convergence in comparison with a binary weighted voting rule as in $\mathcal{N}_{27}$ from Table 13.1. Column (5) provides the exact values of ψ whereas the last column gives numerical approximations based on (4.29).

Theorem 4.13 *Let* $\left\{W^{(n)}\right\}_{n=0}^{\infty}$ *be a* $\frac{1}{2}$*-chain of TWVGs such that its weight function assumes only finitely many values, all of them positive integers; and such that the greatest common divisor of those values* w_a *that occur infinitely often is* 1*. Then the associated chain* $\mathcal{V}$ *satisfies the SLCL condition. Hence PLT holds with respect to the generalized Bz index and any* $a, b \in N$*. Also,* (4.29) *holds.*

Proof To show that (4.24) holds for any $a \in N$, observe that all possible values of $X_{\neg a}^{(n)}$ are integers multiplied by $1/2$ and therefore belong to a lattice whose span is $1/2$. Hence all possible values of $\bar{X}_a^{(n)}$ belong to a lattice whose span is $1/(2\sigma_{\neg a}^{(n)})$. In the half open interval $\left[-w_a/(2\sigma_{\neg a}^{(n)}), w_a/(2\sigma_{\neg a}^{(n)})\right)$ there are exactly $2w_a$ points of this lattice: say $x_i^{(n)}, i = 1, 2, ..., 2w_a$. We invoke Pevtrov's version of the local central limit theorem (1975, p. 189, Theorem 2; see also Remark 3.14 (i)). It follows that if n is sufficiently large then for each $i = 1, 2, ..., 2w_a$ the product

$$\text{Prob}\left\{\bar{X}_a^{(n)} = x_i^{(n)}\right\} 2\sigma_{\neg a}^{(n)} \tag{4.30}$$

is arbitrarily close to $\varphi(x_i^{(n)})$. From (4.23) it is clear that $\lim_{n\to\infty} \sigma_{\neg a}^{(n)} = \infty$; thus for a sufficiently large n each of the $x_i^{(n)}$ is arbitrarily close to 0. Hence the product (4.30) is arbitrarily close to $\varphi(0) = (2\pi)^{-1/2}$. The left-hand side of (4.24) is just the arithmetic mean of $2w_a$ many products (4.30) and hence tends to $(2\pi)^{-1/2}$ as required.
With $n \to \infty$ the term w_a^2 in (4.23) becomes negligible and (4.25) holds. □

Remark 4.14 The limit theorems in this Section are quite analogous those in the binary WVG setup of Section 3.2. Therefore Remarks (3.14) also apply here.

Chapter 5

Discussion of Part I

PLT may best be regarded not as a single theorem but – like the Central Limit Theorem of probability theory, with which it has some affinity – as an open-ended research programme covering many related results. The results of Chapter 3 and Chapter 4 is merely a modest contribution to this programme.

Empirical-computational evidence suggests that similar results hold for other classes of q-chains with respect to the S-S as well as other indices of voting power.

In fact, it seems to us likely that PLT holds *almost always*, in a sense that can be made precise, along the following lines.

Let $\mathbb{N}^+$ be the set of positive integers and consider the Cartesian product space

$$\mathfrak{W} = (0,1) \times \mathbb{N}^{+\mathbb{N}}.$$

Each member of $\mathfrak{W}$ is then an infinite sequence of the form $(q; w_0, w_1, \ldots)$ where $q \in (0,1)$ and the w_n are positive integers. Such a sequence gives rise to a q-chain $\{\mathcal{W}^{(n)}\}_{n=0}^{\infty}$, where $N^{(n)} = \{0, 1, \ldots, n\}$ for each $n \in \mathbb{N}$.

Further, we can regard $\mathfrak{W}$ as a product *probability* space by taking $(0,1)$ with the Lebesgue probability measure, and each copy of $\mathbb{N}^+$ with a reasonable probability distribution: say a geometric distribution ($\text{Prob}\{k\} = 2^{-k}$), or a Poisson distribution ($\text{Prob}\{k\} = \mathrm{e}^{-1}/(k-1)!$).

Or, instead of confining ourselves to integer weights, we can allow arbitrary positive real weights. To this end we can replace $\mathbb{N}^+$ by the set $\mathbb{R}^+$ of positive reals, with some reasonable probability measure on each copy – using, say, a

Gaussian density f on the positive half-line:

$$f(x) = \sqrt{\frac{2\mathrm{e}^{-x^2}}{\pi}}.$$

It now makes precise sense to talk about the probability that PLT holds, with respect to a given index, for the chain corresponding to a randomly chosen member of $\mathfrak{W}$.

Empirical evidence suggests that PLT holds with probability 1 with respect to S-S and the Bz index, as well as the generalized Bz index.

So far, the PLT results are concerned with a priori measuring of voting power which considers the voting body as an abstract shell in order to focus on the rules themselves without considering any further structures (see the introduction of this thesis for a more detailed explanation). However, computational experience provides much evidence that similar results can be derived not only for a larger class of a priori measures of voting power, but also for measures estimating the actual (a posteriori) voting power distribution which make use of empirical data on actual divisions (see for example Owen 1971, Straffin 1994).

Furthermore, the obvious and strong connection between measures of power in games, values and allocation schemes for cooperative games suggests that PLT plays a role in a broader context: the general cooperative games. Out of this reason the class of weighted voting games subject to a prior bias of the voters' regarding the bill voted upon lays embedded in the class of chance-constrained cooperative games (Charnes and Granot 1976 and 1977) and stochastic cooperative games (Borm and Suijs 2002). Furthermore, PLT is directly related to the area of cooperative games with a continuum of voters.

Literature provides a large variety of applications of power indices and cooperative solutions to real-world scenarios and also a large variety of characterizations of the most prevalent power indices and cooperative concepts like e.g. weighted Shapley values, random order values, etc. (for a survey see Owen 1995). The probabilistic structure of these concepts suggests a similar mathematical limit behavior and hence qualify for further heading for PLT statements.

Part II

Global Asymptotic Properties

Chapter 6

Introduction to Part II

Decision rules can be characterized in terms of the way in which voting power of individuals is distributed – as represented for example by the S-S index or Bz measure (see Definition 3.1 and 3.6) – or by some global value. This chapter is concerned with the latter, specifically one that was introduced by Coleman in his 1971 as the 'power of a collectivity to act'. Coleman defined this measure as the a priori probability that a committee representing this collectivity will be able to pass a random bill that comes before it. The measure is simply the cardinality of winning coalitions divided by all possible coalitions. Formally, for a given SVG $\mathcal{W}$ the *power of a collectivity to act* A is defined by

$$A[\mathcal{W}] := \frac{|\mathcal{W}|}{2^n}. \tag{6.1}$$

If we read $|\mathcal{W}|$ as the number of outcomes that lead to action, then A is defined as the *relative* number of voting outcomes leading to action. It reflects the ease of how the individual members' interests in a collective action can be translated into actual collective action. This ease is at a *minimum* if the collectivity operated under a decision rule in which each member has a veto – unanimity – since only the grand coalition (the total assembly) can initiate action, i.e. $A = 1/2^n$. If the committee operates under simple majority rule and has an odd number of members, then exactly half of the coalitions can initiate action (for an even number of members it is slightly less than one half). The power of the collectivity is at a *maximum* under what Rae (1969) has called a 'rule of individual initiative': where action can be initiated by a single individual, for example when s/he gives a fire alarm. In this case A is obtained by $A = 1 - (1/2^n)$. Unless n is very small A will be close to one.[1]

Following Felsenthal & Machover(1998, p. 62) we can think of Coleman's A as measuring the propensity of a committee to approve a random proposal, i.e. the *complaisance* of the rule $\mathcal{W}$. Felsenthal & Machover suggest that it may be more appropriate to think not in terms of complaisance but of the 'resistance' of a SVG. They introduce a *resistance coefficient R* which can be regarded as measuring the opposite of complaisance. It is defined by

$$\mathrm{R}[\mathcal{W}] := \frac{2^{n-1} - |\mathcal{W}|}{2^{n-1} - 1}. \tag{6.2}$$

R is a simple linear transformation of A and for large values of n, A approximates to $A \approx 1 - \mathrm{R}/2$. Although there is no substantial difference between A and R, Felsenthal & Machover define resistance in this way because it allows for easier comparisons of decision rules. For ordinary majority rule (with an odd number of voters) which gives equal a priori probabilities to positive and negative outcomes, $\mathrm{R} = 0$. For the unanimity rule, which is the most resistant, $\mathrm{R} = 1$.[2]

The interest in Coleman's A or Felsenthal and Machover's R is that they allow us to say something about the ability of a collectivity that uses voting to make its decisions not only to act, but as Coleman himself said, '... to act in accord with the aims or interests of some members, but often against the aims or interests of others. Thus for a collectivity of a given size, the greater the power of the collectivity to act, the more power it has to act against the interests of some of the members' (1971, p. 277).

Interest in such a global measure as A and R has recently emerged. Baldwin et. al. (2000), Felsenthal & Machover (2001), Leech (2002a) have all made use of A and R to evaluate the decision rules for the Council of Ministers (CM) of the EU prescribed by the Treaty of Nice for various scenarios of EU enlargement. All these studies suggest that for weighted voting games A falls (R increases) as the number of voters increases. Table 6.1 is taken from Felsenthal & Machover (2001) and gives the decision rules of the CM from 1958 to 1995. The greatest number of issues in EU parlance, except those concerned with the constitution of the EU itself, is decided by a rule

[1] This generally reflects the situation in which a public good, or a public bad, can be supplied by only a few members of a collectivity.

[2] It is easily verified that R is the unique coefficient that satisfies the following three conditions: (i) R is a linear function of $|\mathcal{W}|$; (ii) R achieves a maximal value of 1 when $|\mathcal{W}|$ has its least possible value (which happens for the unanimity rule, where $|\mathcal{W}| = 1$); (iii) R achieves a minimal value of -1 when $|\mathcal{W}|$ has its greatest possible value (which is achieved with the rule of individual initiative where $|\mathcal{W}| = 2^n - 1$).

Table 6.1: QMV weights and quota, first five periods

Country	1958	1973	1981	1986	1995
Germany	4	10	10	10	10
Italy	4	10	10	10	10
France	4	10	10	10	10
Neth'lnds	2	5	5	5	5
Belgium	2	5	5	5	5
Lux'mbrg	1	2	2	2	2
UK		10	10	10	10
Denmark		3	3	3	3
Ireland		3	3	3	3
Greece			5	5	5
Spain				8	8
Portugal				5	5
Sweden					4
Austria					4
Finland					3
Total	17	58	63	76	87
Quota	12	41	45	54	62
Quota%	70.59	70.69	71.43	71.05	71.26
min#	3	5	5	7	8
A	0.2188	0.1465	0.1367	0.0981	0.0778
R	0.5806	0.7098	0.7280	0.8041	0.8445

Note The '*Quota* %' row gives the quota as percentage of the total weight (relative quota). The 'min#' row gives the least number of members whose total weight equals or exceeds the (absolute) quota. A is the Coleman index, R is the resistance coefficient (see (6.1) and (6.2)).

known as *qualified majority voting* (QMV). Until now, the QMV has been a purely weighted decision rule in terms of Definition 1.3: each member state is assigned a weight and a proposed act is adopted if the combined weight of those affirming it achieves a fixed absolute quota, i.e. abstention is treated as a tantamount to 'no'.

Table 6.1 shows a dramatic decrease of A (increase of R) from 0.2188 in 1958 to 0.07788 in 1995. With each enlargement the quotient was kept in the range of 71 ± 0.5 %. These numbers suggest that if the CM keeps enlarging, while keeping the quota more or less constant, the EU tends to immobilism.

In this respect, Felsenthal & Machover (2001, p.456) have noted:

> A very important fact, which is apparently not widely realized, about weighted decision rules is that if the quota is pegged at a constant percentage of the sum of weights, and if the percentage is greater than 50%, then as the number of voters increases the resistance tends to grow ...

Felsenthal & Machover do not provide a proof of this claim, which has major practical importance, particularly for ex ante evaluation of committee design. If it is generally true, then it suggests that as a committee expands we may have to adjust the quota if we to avoid creating an undue bias in favor of the status quo.

However, there is an aspect which sheds another light on the pessimistic prognosis of the EU. It can be argued that the pessimistic view disregards the fact that the exercises to decide upon differ in an essential way. The distinction is made by the following characteristics:

1. *Preference aggregation:* A decision has to be made aggregating the *views* or *interests* of its members.
2. *Truth-tracking:* There is a *true ordering* of the alternatives, i.e. from 'best' to worst, and the task is to make the best ('truest') collective choice.

(for the terminology, see for example List & Goodin 2001). Examples matching the first category are questions like 'Shall the subsidies for rural economy be disposed?', 'Shall the EU participate in the Iraq war?', 'Shall the European flag be red or blue?'. Such a group process is typically concerned with choosing among alternative proposals for action on the relative merits of which the members hold different views.

However, truth-tracking asks for a decision from an epistemic point of view. Here, it makes sense to head for a decision in terms of 'optimality': the 'true' ordering of alternatives is, for example, an ordering from best to worst in terms of some ideal standard or criterion, such as the public interest or justice or efficiency, and the group is concerned to make the 'best' or 'correct' choice. Questions like whether it is economically advantageous for the EU to introduce the Euro, whether a relaxed dismissal protection leads to less unemployment or if the rate of drug addicts decreases with drug legalization are issues matching the second category: there is a truth, yet unknown by the members of the decision making body. The probability of a voter being correct is taken to quantify the *competence* of a voter, i.e. his or her ability to pick the correct choice. Taking competence into account, the question arises if the pessimistic prognosis still holds on epistemic grounds, especially with regards to Condorcet's jury theorem: This theorem shows that on a dichotomous choice, individuals who all have the same competence above 0.5, can make collective decisions under simple majority rule with a probability of being right (collective competence) that approaches 1 as either the size of the group or the individual competence goes up. Briefly, if the aim is 'tracking the truth' then the jury theorem says that the larger the decision body the better – opposite to the pessimistic prognosis of the CM.

This part of the thesis deals with global asymptotic properties of WVGs when there are many small voters. Section 6.1 sets up the probabilistic machinery that we will use throughout this part. Section 6.2 defines the general setup of the games, which are weighted voting games with a set of voters divided in two parts: a finite set of voters with a fixed share of the total weight (the atomic part) and many 'small' voters (the non-atomic part). Section 7.1 discusses the passage of the Coleman index A to the limit when the number of small voters grows to infinity. Here, we shall focus on A simply because it is technically easier to handle than R. These results are extended in Section 7.2 to TWVGs. Section 7.3 provides statements estimating the rate of convergence which turns out to be very high under specific smoothness conditions. This justifies the limit value to serve as an approximation for large WVGs (as the CM). To illustrate the results developed in the previous chapters, Section 7.4 discusses their application to the CM. Section 8.1 introduces the classical formulation of Condorcet's jury theorem. Section 8.2 proves a generalization of the latter which allows to discuss the results of Section 7.1 – 7.3 from an epistemic point of view. Finally, the Chapter 9 concludes.

6.1 Preliminaries

Our main tool borrowed from probability theory is a general version of the central limit theorem. We shall use the symbol Φ to denote the standard normal distribution. Let $\{X_k\}_{k=1}^{\infty}$ be a sequence of independent random variables, at least one of which has a non-degenerate distribution. Let the distribution of X_k be denoted by F_k, its expectation by $E[X_k] = \mu_k$ and assume its variance $Var\left[X_k\right] = \sigma_k^2$ to be finite. Further put

$$(s_n)^2 := Var\left[\sum_{k\leq n} X_k\right] = \sum_{k\leq n} \sigma_k^2$$

and

$$S_n := \frac{1}{s_n}\sum_{k\leq n} X_k - \mu_k.$$

Theorem 6.1 (Lindeberg-Feller) *In order that*

$$\lim_{n\to\infty} \max_{k\leq n} \frac{\sigma_k}{s_n} = 0 \tag{6.3}$$

and

$$\lim_{n\to\infty} \sup_x \left|\mathrm{Prob}\left\{S_n < x\right\} - \Phi(x)\right| = 0 \tag{6.4}$$

it is necessary and sufficient that the following condition (the Lindeberg condition)be satisfied:

$$\lim_{n\to\infty} L_n(\varepsilon) = 0 \tag{6.5}$$

with

$$\begin{aligned} L_n(\varepsilon) &:= s_n^{-1}\sum_{k\leq n} E\left[(X_k-\mu_k)^2; |X_k-\mu_k| \geq \varepsilon s_n\right] \qquad (6.6) \\ &= s_n^{-1}\sum_{k\leq n}\int_{\{|x-\mu_k|\geq\varepsilon\sqrt{s_n}\}} (x-\mu_k)^2 dF_k(x) \end{aligned}$$

for every fixed $\varepsilon > 0$.

For a proof see e.g. Petrov (1975), p.100-101. We put

$$Q^{(n)} := \sum_{k\leq n} w_k^2.$$

Lemma 6.2 *For each k, let the independent random variable X_k be given by*

$$X_k = C_k w_k,$$

where the C_k are real-valued random variables with the same non-degenerate distribution on a compact set $[a,b]$ for all $k\in\mathbb{N}$. Then $\{X_k\}_{k=1}^{\infty}$ satisfies the Lindeberg condition (6.5) *iff*

$$\lim_{n\to\infty} \frac{w_n}{\sqrt{Q^{(n)}}} = 0. \tag{6.7}$$

Proof For each k follows

$$E[X_k] = cw_k \tag{6.8}$$

$$Var[X_k] = d^2 w_k^2, \tag{6.9}$$

where c and d are reals independent of k, with $d > 0$ (since any C_k has a non-degenerate distribution). Hence

$$s_n = d\sqrt{Q^{(n)}}. \tag{6.10}$$

Now suppose the Lindeberg condition (6.5) is satisfied. Then by Theorem 6.1 we have (6.3), from which (6.7) follows at once in view of (6.9) and (6.10).

Conversely, suppose that (6.7) holds. We now show that

$$\lim_{n\to\infty} \max_{k\le n} \frac{w_k}{\sqrt{Q^{(n)}}} = 0. \tag{6.11}$$

For any $\varepsilon > 0$ fix n' so large that $w_k/\sqrt{Q^{(k)}} < \varepsilon$ for all $k > n'$. Thus, for all $n > n'$ we have

$$\frac{w_k}{\sqrt{Q^{(n)}}} \le \frac{w_k}{\sqrt{Q^{(k)}}} < \varepsilon \qquad \text{for } k = n' + 1, ..., n.$$

Thus (6.11) holds. Now observe that for every k, the integral in (6.6) follows as

$$\int_{|x-cw_k|>\varepsilon d\sqrt{Q^{(n)}}} (x - cw_k)^2 dF_k(x). \tag{6.12}$$

But from $|x - cw_k| = |y - c|w_k$ for all $y \in [\alpha, \beta]$ and (6.11) it follows that, for any given $\varepsilon > 0$, if n is sufficiently large, then

$$|y - c|w_k < \varepsilon d\sqrt{Q^{(n)}}$$

for all $y \in [\alpha, \beta]$ and all $k \le n$. That implies the integral (6.12) vanishes for all $k \le n$. Hence (6.5) holds. $\square$

6.2 General Setup

Consider a partition of the set of voters N into two camps: we will denote the set of *major voters* in N by L which is either given by $\{1,...,l\}$, where l is a natural number. Note that $l=0$ takes care of the case where L is empty by the general convention that $\{1,\ldots,0\}$ is empty. The set of *minor voters* in N is denoted by $M^{(\nu)}=\{l+1,...,l+m^{(\nu)}\}$.
We shall consider weighted voting situations as follows: there is a fixed quota c and a fixed set of major voters L, where each major voter is endowed with a fixed voting weight which sum up to w_L, the combined voting weight of L. There is also a fixed total combined voting weight α of the minor voters $M^{(\nu)}$ such that the total weight sum is a fixed constant

$$W := w_L + \alpha. \tag{6.13}$$

However, the population number of $M^{(\nu)}$ grows to infinity whereas the individual weight of any minor voter tends to zero. Hence $M^{(\nu)}$ represents the non-atomic part of the game. Since we admit the case $L=\emptyset$ the analysis covers both oceanic as well as purely non-atomic games.

Let $\{\Gamma^{(\nu)}\}_{\nu\in\mathbb{N}}$ be a sequence of WVGs, as follows

$$\Gamma^{(\nu)} = [c; w_1,...,w_l,\alpha_1^{(\nu)},...,\alpha_{m^{(\nu)}}^{(\nu)}]. \tag{6.14}$$

Put $Q^{(\nu)} := \sum_{k\leq m^{(\nu)}}\left[\alpha_k^{(\nu)}\right]^2$. Let $\{\Gamma^{(\nu)}\}_{\nu\in\mathbb{N}}$ evolve such that

$$\sum_{k\leq m^{(\nu)}} \alpha_k^{(\nu)} = \alpha, \quad \text{for each } \nu, \tag{6.15}$$

for a fixed $\alpha>0$, and

$$\lim_{\nu\to\infty} \alpha_{max}^{(\nu)}/\sqrt{Q^{(\nu)}} = 0, \tag{6.16}$$

where $\alpha_{\max}^{(\nu)} := \max_{k\leq m^{(\nu)}} \alpha_k^{(\nu)}$.

Remark 6.3
Note that (6.16) ensures

$$\alpha_{\max}^{(\nu)} \to 0, \quad \text{as } \nu\to\infty, \tag{6.17}$$

which implies $m^{(\nu)}\to\infty$. Hence for $L=\emptyset$, the sequence $\{\Gamma^{(\nu)}\}_{\nu\in\mathbb{N}}$ represents a (normalized) non-atomic q-chain as defined in Definition 2.1 and 3.2 (i). However, it can be shown that $Q^{(\nu)}$ tends to zero so that condition (6.16) is stricter than (6.17) (see Lemma 14.1 in the appendix).

Chapter 7

Complaisance of WVGs

7.1 Complaisance in Binary WVGs

Assumption 7.1 (independence) *We assume that the voters act independently of one another and choose each option with equal probability.*

We shall use the notation $w(B) := \sum_{k \in B} w_k$ for the sum of the weights of the 'yes' voters $B \subseteq L$ among the major voters. Note that Assumption 7.1 implies that each $B \subseteq L$ occurs with probability $1/2^l$. Analogously we shall regard $\alpha^{(\nu)}(S) = \sum_{k \in S} \alpha_k^{(\nu)}$ as the weight sum of the affirming minor voters S, where S is a random subset of $M^{(\nu)}$. Assumption 7.1 implies that any random $S \subseteq M^{(\nu)}$ arises with probability $1/2^{m^{(\nu)}}$.

From the general definition of a weighted voting game (1.3) and complaisance (6.1) we get

$$A[\Gamma^{(\nu)}] = \frac{1}{2^l} \sum_{B \subseteq L} \text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\}. \tag{7.1}$$

In binary WVGs Assumption 7.1 implies complete symmetry among the two options 'yes' and 'no'. We should therefore expect that in the limit the continuous 'ocean' of coin-tossing minor voters would be equally divided, i.e. exactly half the minor voters state 'yes', 'no' respectively. This suggests to focus on the games

$$\begin{aligned} \Gamma_0 &= [c - \frac{\alpha}{2}; w_1, ..., w_l], \\ \Gamma_0' &= < c - \frac{\alpha}{2}; w_1, ..., w_l >, \end{aligned} \tag{7.2}$$

which represent (binary) WVGs involving major voters only, with a quota c reduced by $\alpha/2$.

Put

$$\mathcal{R} := \{c \,|\, 0 < c < w_L + \alpha\} \tag{7.3}$$
$$\mathcal{J} := \{c \,|\, \alpha/2 < c < w_L + \alpha/2\}\,. \tag{7.4}$$

Theorem 7.2 *In the sequence of games described by* (6.14)– (6.16)*, we have*

$$\lim_{\nu\to\infty} A[\Gamma^{(\nu)}] = \frac{1}{2}A[\Gamma_0] + \frac{1}{2}A[\Gamma_0'] \qquad \text{if } c \in \mathcal{J}. \tag{7.5}$$

For other values of c we have

$$\lim_{\nu\to\infty} A[\Gamma^{(\nu)}] = \begin{cases} 1 & \text{if } c < \alpha/2, \\ 1 - 1/2^{l+1} & \text{if } c = \alpha/2, \\ 1/2^{l+1} & \text{if } c = w_L + \alpha/2, \\ 0 & \text{if } c > w_L + \alpha/2. \end{cases} \tag{7.6}$$

Figure 7.1 illustrates the result.

Figure 7.1: Complaisance in Binary WVGs

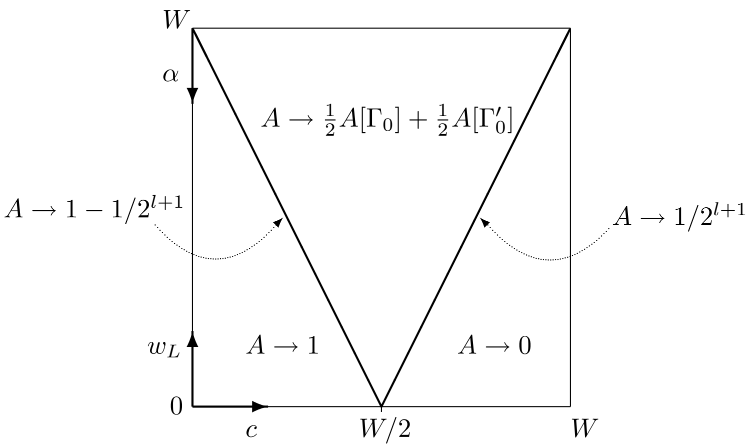


Figure 7.1 shows that the relation of the quota c to w_L is crucial for the limit value of A. At the interior region $\mathcal{J}$ the limiting value can be computed from

other WVGs: it is the arithmetical mean of complaisance of games involving just the major voters as defined in (7.2). Let the closure of $\mathcal{J}$ be denoted by $\bar{\mathcal{J}} := \{c \,|\, \alpha/2 \leq c \leq w_L + \alpha/2\}$. We see that in the domain $\mathcal{R} - \bar{\mathcal{J}}$ the influence of the major voters is 'destroyed': in the limit we have a combined voting weight of exactly $\alpha/2$ affirming minor voters such that $c < \alpha/2$ ensures that a proposal is always adopted. The same holds for the opposite. Even with all major voters affirming $w_L + \alpha/2 < c$ is too low to ever let pass a proposal.

For the proof of Theorem 7.2 we shall use the following Lemma.

Lemma 7.3 *Let $0 \leq z \leq \alpha$ and choose a subset $S \subseteq M^{(\nu)}$ at random. For the sequence of games (6.14) - (6.16) we have*

$$\lim_{n\to\infty} \mathrm{Prob}\{\alpha^{(\nu)}(S) \geq z\} = \begin{cases} 1 & \text{if } z < \alpha/2, \\ 1/2 & \text{if } z = \alpha/2, \\ 0 & \text{if } z > \alpha/2. \end{cases} \tag{7.7}$$

Proof We represent the vote of each minor voter $k+l \in M^{(\nu)}$ as the random variable

$$X_k^{(\nu)} = \begin{cases} \alpha_k^{(\nu)} & \text{if } k \text{ votes 'yes'} \\ 0 & \text{otherwise.} \end{cases} \tag{7.8}$$

Put

$$\mu := \sum_{k\leq m^{(\nu)}} E\left[X_k^{(\nu)}\right] = \frac{\alpha}{2}. \tag{7.9}$$

and

$$\left[s^{(\nu)}\right]^2 := \sum_{k\leq m^{(\nu)}} Var\left[X_k^{(\nu)}\right] = \frac{Q^{(\nu)}}{4}. \tag{7.10}$$

Theorem 6.1 provides

$$\begin{aligned} \lim_{\nu\to\infty} \mathrm{Prob}\{\alpha^{(\nu)}(S) < z\} &= \lim_{\nu\to\infty} \Phi(\frac{z-\mu}{s^{(\nu)}}) \\ &= \lim_{\nu\to\infty} \Phi(\frac{(z/\alpha - 1/2)\alpha}{s^{(\nu)}}). \end{aligned} \tag{7.11}$$

With increasing ν the standard deviation $s^{(\nu)}$ from (7.10) tends to zero. This implies that in (7.11) the sign of the term $(z/\alpha - 1/2)$ determines whether the argument of Φ converges to plus or minus infinity or is constantly zero and (7.7) follows with $\mathrm{Prob}\{\alpha^{(\nu)}(S) \geq z\} = 1 - \mathrm{Prob}\{\alpha(S)^{(\nu)} < z\}$. $\square$

Proof of Theorem 7.2 For the terms in (7.1) follows with Lemma 7.3

$$\operatorname{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\} \to \begin{cases} 0 & \text{if} \quad w(B) < c - \alpha/2, \\ 1/2 & \quad w(B) = c - \alpha/2, \\ 1 & \quad w(B) > c - \alpha/2. \end{cases} \tag{7.12}$$

For $c \in \mathcal{J}$ the games Γ_0 and Γ_0' are well defined, and for any $B \subseteq L$ for which the limit of $\operatorname{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\}$ is 1 we have that B is a winning coalition in both Γ_0 and Γ_0'. If the limit is $1/2$, the coalition B is winning in Γ_0 but not Γ_0'. This yields for (7.1)

$$\begin{aligned} 2^l A[\Gamma^{(\nu)}] &\to |\{B \subseteq L | w(B) > c - \alpha/2\}| + \frac{1}{2}|\{B \subseteq L | w(B) = c - \alpha/2\}| \\ &= \frac{1}{2}|\{B \subseteq L | w(B) > c - \alpha/2\}| + \frac{1}{2}|\{B \subseteq L | w(B) \geq c - \alpha/2\}| \\ &= \frac{1}{2} 2^l A[\Gamma_0] + \frac{1}{2} 2^l A[\Gamma_0'] \end{aligned}$$

and hence (7.5).

To see (7.6) note that from $c < \alpha/2$ follows that the third condition in (7.12) is fulfilled for any $B \subseteq L$ and hence $\operatorname{Prob}\left\{\alpha^{(\nu)}(S) \geq c + w(B)\right\} \to 1$ for all $B \subseteq L$.

The equality $c = \alpha/2$ implies $\operatorname{Prob}\left\{\alpha^{(\nu)}(S) \geq c + w(B)\right\} \to 1/2$ for $B = \emptyset$ and 1 otherwise which yields for $A[\Gamma^{(\nu)}]$ a limit value $1/2^l(1/2 + 2^l - 1)$.

If $c = w_L + \alpha/2$ then $\operatorname{Prob}\left\{\alpha^{(\nu)}(S) \geq c + w(B)\right\} \to 1/2$ for $B = L$ and 0 else.

Finally, from $c > w_L + \alpha/2$ follows that $\operatorname{Prob}\left\{\alpha^{(\nu)}(S) \geq c + w(B)\right\} \to 0$ for any $B \subseteq L$. □

7.2 Complaisance in Ternary WVGs

For the purpose of this section it is technically easier to work with the relative quota $q = c/W$ instead of the absolute quota c (see Remark 1.4(ii)).

Expression (4.12) is equivalent to

$$(1-q) \sum_{k \in T^+} w_k - q \sum_{k \in T^-} w_k \geq 0. \tag{7.13}$$

It shall prove useful to interpret a TWVG following (7.13) directly: any voter k is endowed with three voting weights $w_k^+ = (1-q)w_k, w_k^0 = 0$ and $w_k^- = -qw_k$. Voter k chooses either w_k^+ or w_k^- if s/he affirms or rejects the proposal, votes with w_k^0 if s/he abstains respectively. The proposal passes if the total weight sum in the ballot box exceeds or equals the absolute quota $c = 0$.

In a broader context, we can think of TWVGs as a subclass of

$$\Gamma := [c; \vec{w}_1, ..., \vec{w}_n], \tag{7.14}$$

where $n = |N|$ is the total number of voters and $\vec{w}_k = (w_k^+, w_k^0, w_k^-) \in \mathbb{R}^3$ for $k = 1, ..., n$ such that $w_k^+ \geq w_k^0 \geq w_k^-$. Read $w^+(T)$ as the weight sum of the affirming voters (analogously, $w^0(T)$ and $w^-(T)$ as the weight sum of the abstaining, rejecting voters respectively). The absolute quota is given by $c \in \mathbb{R}$ with $w^-(N) \leq c < w^+(N)$. Put $w(T) := w^+(T) + w^0(T) + w^-(T)$. In

Γ, a proposal passes iff

$$w(T) \geq c. \tag{7.15}$$

We denote by

$$\Gamma' =< c; \vec{w}_1, ..., \vec{w}_n > \tag{7.16}$$

the game in which the blunt equality in (7.15) is replaced by the sharp inequality.

Remark 7.4 The games (7.14) and (7.16) are a special case of weighted (j, k) games introduced by Freixas & Zwicker (2003). The authors define weighted voting for the context where voters cast ballots that choose from $j \geq 2$ levels of approval (which can be interpreted as ranging from total opposition to complete enthusiasm). In weighted (j, k) games each voter a is pre-assigned j real-valued weights $w_a^1 \geq w_a^2 \geq \ldots \geq w_a^j$. The output consists of k possible levels of collective support. In that sense TWVGs can be reformulated as weighted $(3, 2)$ games, following either (7.14) or (7.16), with abstention as a level of approval intermediate to 'yes' and 'no'.

In the following we will stick to the representation of a TWVG as

$$\Gamma = [0; \vec{w}_1, ..., \vec{w}_n],$$

and shall analyze the sequence

$$\Gamma^{(\nu)} = [0; \vec{w}_1, ..., \vec{w}_l, \vec{\alpha}_1^{(\nu)}, ..., \vec{\alpha}_{m^{(\nu)}}^{(\nu)}] \tag{7.17}$$

with

$$\vec{w}_k = (1-q, 0, -q)\ w_k$$

for the major voters $k = 1, ..., l$ and

$$\vec{\alpha}_k^{(\nu)} = (1-q, 0, -q)\,\alpha_k^{(\nu)}$$

for each minor voter $k + l \in M^{(\nu)}$.

Assumption 7.1 suggests that in the limit the continuous 'ocean' of minor voters would be equally divided in three parts, i.e. the respective weight effect in the ballot box of affirmers, rejecters and abstainers is $(1-q)\alpha/3, 0$ and $-q\alpha/3$. We should therefore expect for complaisance A a qualitative result comparable to the previous chapter, i.e. a categorization of different q/w_L-areas in which the influence of the major voter vanishes or the influence of the minors voters is only indirect by a reduction of the absolute quota. This points at a reduction of the quota consisting of the summed up weight contribution of each equal-sized camp of minor voters, i.e.

$$\tilde{c} := 1/3(1-q)\alpha - 1/3q\alpha = (1-2q)\alpha/3. \tag{7.18}$$

The analysis of the previous section suggests to focus on the games

$$\Gamma_0 := [-\tilde{c}; w_1, ..., w_l] \tag{7.19}$$

and

$$\Gamma_0' :=< -\tilde{c}; w_1, ..., w_l > . \tag{7.20}$$

In these games the voter set consists of the major voters only with a shift in the quota from 0 to $-\tilde{c}$. However, the direction of the quota adjustment and hence the profit of either the affirming or rejecting direction depends whether q is less or larger than 50 percent. With $q = 0.5$ there is no adjustment, i.e. $\tilde{c} = 0$.

Put

$$\begin{aligned} \mathcal{J} &:= \{q \,|\, -qw_L < -\tilde{c} < (1-q)w_L\} & (7.21)\\ &= \left\{q \,\Big|\, \frac{\alpha}{3w_L + 2\alpha} < q < \frac{3w_L+\alpha}{3w_L+2\alpha}\right\}. & (7.22)\end{aligned}$$

Theorem 7.5 *In the sequence of games described by* (6.14)– (6.16)*, we have*

$$\lim_{\nu\to\infty} A[\Gamma^{(\nu)}] = \frac{1}{2}A[\Gamma_0] + \frac{1}{2}A[\Gamma_0'], \quad \textit{if } q \in \mathcal{J}. \tag{7.23}$$

For $q \in (0,1] - \mathcal{J}$ *we have*

$$\lim_{\nu\to\infty} A[\Gamma^{(\nu)}] = \begin{cases} 1 & \textit{if} \quad q < \alpha/(3w_L+2\alpha),\\ 1-(1/2)(1/3^l) & \quad\quad q = \alpha/(3w_L+2\alpha),\\ (1/2)(1/3^l) & \quad\quad q = (3w_L+\alpha)/(3w_L+2\alpha),\\ 0 & \quad\quad q > (3w_L+\alpha)/(3w_L+2\alpha).\end{cases} \tag{7.24}$$

Figure 7.2 illustrates the result.

Figure 7.2: Complaisance in Ternary WVGs

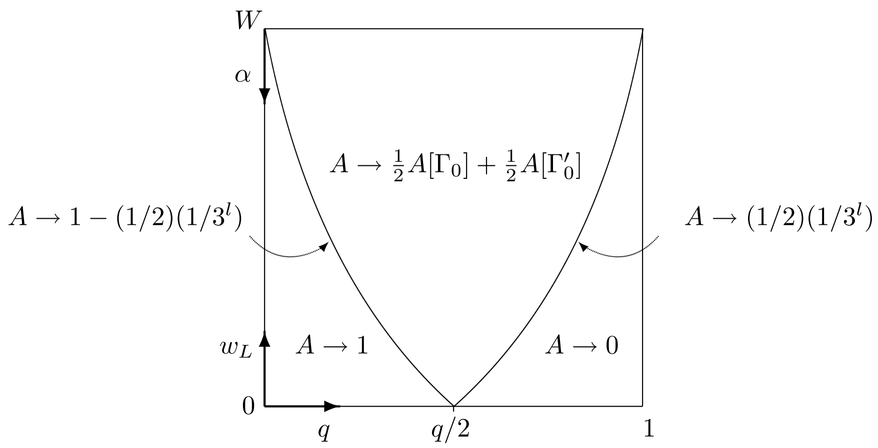


Figure 7.2 shows a qualitatively similar scenario to the results in Figure 7.1: there is an interior region $\mathcal{J}$ in which the major voters are not 'destroyed' and complaisance is given by the arithmetic mean of games involving just the major voters. However, including abstention as a distinct third option, the interior region $\mathcal{J}$ bends outwards. The curves describing the boundary points as indicated in the second and third row of (7.24) are now non-linear convex functions.

For the proof of Theorem 7.5 we will use the symbol T for tripartitions among the major voters only. Let $\mathbb{T}$ denote the set of all tripartitions among the major voters. Assumption 7.1 provides the probability for any $T \in \mathbb{T}$ to form by $1/3^l$. We shall mean by S any random tripartition among the minor voters (which occurs with probability $1/3^{m^{(\nu)}}$).

Proof of Theorem 7.5

Step 1: From (7.17) follows

$$A[\Gamma^{(\nu)}] = \frac{1}{3^l} \sum_{T \in \mathbb{T}} \mathrm{Prob}\{\alpha^{(\nu)}(S) \geq -w(T)\} \tag{7.25}$$

We represent the vote of any minor voter $k+l \in M^{(\nu)}$ by the random variable

$$X_k^{(\nu)} = \begin{cases} (1-q)\alpha_k^{(\nu)}, \\ 0, \\ -q\alpha_k^{(\nu)}, \end{cases} \tag{7.26}$$

where each value is taken with probability 1/3.
The first two moments of $\sum_{k \leq m^{(\nu)}} X_k^{(\nu)}$ are given by

$$\mu := \sum_{k \leq m^{(\nu)}} E\left[X_k^{(\nu)}\right] = 1/3(1-2q)\alpha, \tag{7.27}$$

which is equal to $\tilde{c}$ as defined in (7.18), and

$$\left[s^{(\nu)}\right]^2 := \sum_{k \leq m^{(\nu)}} Var\left[X_k^{(\nu)}\right] = \frac{2}{9}(q^2 - q + 1)Q^{(\nu)}. \tag{7.28}$$

Theorem 6.1 provides for each term in (7.25)

$$\begin{aligned} \lim_{\nu \to \infty} \mathrm{Prob}\left\{\alpha^{(\nu)}(S) \geq -w(T)\right\} &= 1 - \lim_{\nu \to \infty} \Phi\left(\frac{-w(T) - \mu}{s^{(\nu)}}\right) && (7.29) \\ &= 1 - \lim_{\nu \to \infty} \Phi\left(-\frac{w(T) + \tilde{c}}{s^{(\nu)}}\right) && (7.30) \\ &= \begin{cases} 0 & \text{if } w(T) < -\tilde{c}, \\ 1/2 & \text{if } w(T) = -\tilde{c}, \\ 1 & \text{otherwise.} \end{cases} && (7.31) \end{aligned}$$

Step 2: If $q \in \mathcal{J}$ the games Γ_0 and Γ_0' in (7.19) and (7.20) are well defined. For any $T \in \mathbb{T}$ for which the limit in (7.29) is 1 we have $w(T) > -\tilde{c}$ and hence T^+ is winning in both Γ_0 and Γ_0'. If the limit in (7.29) is 1/2 it follows that T^+ is winning in Γ_0 but losing in Γ_0'. Thus

$$3^l A[\Gamma^{(\nu)}] \to \frac{1}{2}|\{T | T^+ \text{ is winning in } \Gamma_0\}| + \frac{1}{2}|\{T | T^+ \text{ is winning in } \Gamma_0'\}|,$$

which proves (7.23).

To see (7.24) note that $q < \alpha/(2\alpha + 3w_L)$ is equivalent to $-\tilde{c} < -qw_L$. This implies $w(T) > -\tilde{c}$ for all $T \in \mathbb{T}$ and hence any probability term in (7.25) tends to 1.

Reformulation of $q = \alpha/(3w_L + 2\alpha)$ yields $-\tilde{c} = -qw_L$ which implies $w(T) = -\tilde{c}$ for the tripartition with $T^- = N$ and $w(T) > -\tilde{c}$ else.

From $q = (3w_L + \alpha)/(3w_L + 2\alpha)$ follows $-\tilde{c} = (1 - q)w_L$ and thus $w(T) = -\tilde{c}$ for the tripartition with $T^+ = N$ and $w(T) < -\tilde{c}$ else.

Finally $q > (3w_L + \alpha)/(3w_L + 2\alpha)$ implies $-\tilde{c} > (1 - q)w_L$and hence $w(T) < -\tilde{c}$ for every $T \in \mathbb{T}$. □

Remark 7.6 Assumption 7.1 is in the spirit of *a priori* voting analysis (see Section 1.1), however, the results of Theorem 7.2 and Theorem 7.5 can easily be adjusted to a broader context:

1. The crucial point in Assumption 7.1 is *independence* which allows us to use the Lindeberg-Feller Theorem 6.1. The results of Theorem 7.2 and Theorem 7.5 topologically stay the same by dropping the uniform distribution condition and replace it by any non-degenerate distribution on the voters option set. In particular, the results of Theorem 7.5 can easily be adjusted to the setup of TWVGs of Section 4.4 with a general probability to abstain $t \in (0, 1)$.

2. The proof techniques hinge on applying the independence assumption on the *minor voters* such that it is not necessary to specify voting behavior of the major voters.

7.3 Convergence Characteristics

The definitions of complaisance as in Sections 7.1 and 7.2 for binary and ternary WVGs show a common mathematical structure: from (7.1) and (7.25) follows that in both settings complaisance is a finite weighted sum of probabilities where each summand is of the form

$$\text{Prob}\left\{\alpha^{(\nu)}(T) \geq x\right\}, \qquad x \in \mathbb{R}. \tag{7.32}$$

In binary WVGs, T stands for a random bipartition among the minor voters, for a minor tripartition in ternary WVGs respectively. In both settings, the relation of x to the mean value μ of the random variable $\alpha^{(\nu)}(T)$ is crucial for the limit value of complaisance, i.e. for the sequence of voting games defined as in (6.14) - (6.16) we get

$$\text{Prob}\left\{\alpha^{(\nu)}(T) \geq x\right\} \to \begin{cases} 1 & \text{if } x < \mu, \\ 1/2 & \text{if } x = \mu, \\ 0 & \text{if } x > \mu. \end{cases} \tag{7.33}$$

(See (7.7) and (7.31)). For estimates of the rate of convergence for $\nu \to \infty$ it turns out that convergence happens to be fast if $x \neq \mu$ and if the distribution of the minor votes is reasonably smooth. This is due to the fact that under the smoothness condition the 'tails' of the sum of random variables display high convergence rates in areas of x unequal μ. In the following we will focus on weight distributions of the major voters and the quota c such that $x \neq \mu$ is always matched. In binary games, x takes the values $c - w(B)$ for some $B \subseteq L$ and the expected value of $\alpha^{(\nu)}(T)$ is given by $\alpha/2$ (see (7.1) and (7.9)). Hence for a 'tail' estimation we shall exclude the set P given by

$$P := \{c \,|\, c - w(B) = \alpha/2 \text{ for some } B \subseteq L\}\,.$$

Analogously, in TWVGs we have $x = -w(T)$ for some $T \in \mathbb{T}$ and the expected value of $\alpha^{(\nu)}(T)$ follows as $1/3(1-2q)\alpha$, as denoted by (7.25) and (7.27), and the set with $x = \mu$ is given by

$$P_{\mathbb{T}} := \{q \,|\, -w(T) = 1/3(1-2q)\alpha \text{ for some } T \in \mathbb{T}\}\,.$$

For a 'tail' estimation we shall use the following Lemma.

Lemma 7.7 *Let $\{Z_k\}_{k=1}^{\infty}$ be a sequence of independent real-valued random variables such that $|Z_k| \leq 1$ for all k. Let further $\{c_k\}_{k=1}^{\infty}$ be a sequence of real constants such that*

$$s^2 = \sum_{k=0}^{\infty} {c_k}^2 < \infty. \tag{7.34}$$

Then

$$\tilde{Z} = \sum_{k=0}^{\infty} c_k(Z_k - \mu_k), \tag{7.35}$$

with $\mu_k = E\left[X_k\right]$, satisfies

$$\text{Prob}\left\{\tilde{Z} > \delta s\right\} \leq \exp\left[-\frac{\delta^2}{2}\right], \quad \text{Prob}\left\{\tilde{Z} < -\delta s\right\} \leq \exp\left[-\frac{\delta^2}{2}\right] \tag{7.36}$$

for each number $\delta > 0$.

For a proof see Kemperman (1964).

Theorem 7.8 *In the sequence of games defined by* (6.14) – (6.16), *if* $c \notin P$ *we have for the binary case*

$$\left| A\left[\Gamma^{(\nu)}\right] - \lim_{i\to\infty} A\left[\Gamma^{(i)}\right] \right| = \mathcal{O}\left(\exp\left[-\left(\frac{\lambda}{\sqrt{m^{(\nu)}}\alpha_{\max}^{(\nu)}}\right)^2\right]\right), \tag{7.37}$$

where λ *is a positive constant.*

The same holds for ternary games if $q \notin P_{\mathbb{T}}$.

Proof For a fixed ν let

$$Z_k = \begin{cases} \alpha_k^{(\nu)}/\alpha_{\max}^{(\nu)} & \text{with probability } 1/2, \\ 0 & \text{otherwise,} \end{cases} \tag{7.38}$$

for each $k = 1, ..., m^{(\nu)}$. For $k > m^{(\nu)}$ set $Z_k \equiv 0$. For the constants c_k, put

$$c_k = \begin{cases} \alpha_{\max}^{(\nu)} & \text{for } k \leq m^{(\nu)}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.39}$$

From (7.34) follows

$$s = \sqrt{m^{(\nu)}}\alpha_{\max}^{(\nu)}.$$

This setting allows us to identify the $\tilde{Z}$ in (7.35) with $\alpha^{(\nu)}(T) - \mu$, where $\mu = \alpha/2$, and we get from (7.36)

$$\text{Prob}\left\{|\alpha^{(\nu)}(T) - \mu| > \delta s\right\} \leq \exp\left[-\frac{\delta^2}{2}\right],$$

for any positive number $\delta > 0$. Putting $\varepsilon := \delta s$ yields the reformulation

$$\text{Prob}\left\{|\alpha^{(\nu)}(T) - \mu| > \varepsilon\right\} \leq \exp\left[-\left(\frac{\varepsilon}{2\sqrt{m^{(\nu)}}\alpha_{\max}^{(\nu)}}\right)^2\right]. \tag{7.40}$$

Finally, from definition (7.1) follows that $A\left[\Gamma^{(\nu)}\right]$ is a weighted sum of finitely many terms where each term can be estimated by (7.40). For $c \notin P$ we have $\varepsilon > 0$ for each of those terms which proves (7.37).

For TWVGs the proof follows analogously with replacing (7.38) by

$$Z_k = \begin{cases} (1-q)\alpha_k^{(\nu)}/\alpha_{\max}^{(\nu)} & \text{with probability } 1/3, \\ 0 & \text{with probability } 1/3, \\ -q\alpha_k^{(\nu)}/\alpha_{\max}^{(\nu)} & \text{otherwise.} \end{cases}$$

and $\mu = 1/3(1-2q)\alpha$. $\square$

From Theorem 7.8 follows that the rate of converges hinges on $\sqrt{m^{(\nu)}}\alpha^{(\nu)}_{\max}$ (note that this term also reflects the ratio of $\alpha^{(\nu)}_{\max}$ to the mean minor weight $\alpha/m^{(\nu)}$). If the weight distribution among the minor voters is sufficiently smooth so that $\sqrt{m^{(\nu)}}\alpha^{(\nu)}_{\max}$ tends to zero sufficiently fast, we can expect high rates of convergence as indicated by (7.37). The following statement follows directly from (7.40).

Corollary 7.9 *Under the conditions of Theorem 7.8 and if*

$$\alpha^{(\nu)}_{\max} = \mathcal{O}\left(1/(\sqrt{m^{(\nu)}})^{1+\gamma}\right),$$

for any $\gamma \geq 0$, we have

$$\left| A\left[\Gamma^{(\nu)}\right] - \lim_{i\to\infty} A\left[\Gamma^{(i)}\right]\right| = \mathcal{O}\left(\exp\left[-\left(\lambda m^{(\nu)}\right)^{\gamma}\right]\right), \tag{7.41}$$

where λ is a positive constant.

The symmetric case $\alpha^{(\nu)}_{\max} = \alpha/m^{(\nu)}$ is covered by $\gamma = 1$ for which (7.41) states that the difference of $A\left[\Gamma^{(\nu)}\right]$ to the limit decreases exponentially in $m^{(\nu)}$.

We shall see that with an increasing number of members the scenario of the CM is qualitatively comparable to the 'one person one vote' situation as in the symmetric case, however, there is of course a distortion effect due to unequal weight distribution. The following theorem provides a statement which is more tailored to the discussion of the EU, considering the ratio $\alpha^{(\nu)}_{\max}/\alpha^{(\nu)}_{\min}$.

Theorem 7.10 *Let $\alpha_1, ..., \alpha_{m^{(\nu)}}$ be positive numbers totalling α, and T a random subset of $\{1, ..., m^{(\nu)}\}$. If every subset is equally probable then for any $\varepsilon > 0$,*

$$\mathrm{Prob}\left\{\alpha(T) > \alpha/2 + \varepsilon\right\} \leq \exp\left[-\frac{8m^{(\nu)}\varepsilon^2\theta}{\alpha^2(1+\theta)^2}\right],$$

where θ denotes $\alpha^{(\nu)}_{\max}/\alpha^{(\nu)}_{\min}$.

For a proof see Hoeffding (1963).

7.4 The EU Council of Ministers

This section applies the results of the previous sections to the evolution of the system of Qualified Majority Voting used by the Council of the European Union since its origin in 1958. The substantial characteristic of the process is that with each enlargement the maximal normalized voting weight decreases while the relative quota was kept more or less constant 71% as indicated by Table 6.1 and Table 13.1. This suggests to focus on games

$$\Gamma^{(\nu)} = [q; \alpha_1, ..., \alpha_\nu] \tag{7.42}$$

with an empty atomic part, i.e. we put the set of major voters $L = \emptyset$ and assume a fixed relative quota q. We shall identify the five scenarios from 1958 – 1995 and the QMV following its prospective enlargement to 27 as sequence elements $\Gamma^{(6)}, \Gamma^{(9)}, ..., \Gamma^{(15)}, \Gamma^{(27)}$, where the index denotes the size of the Council. Without loss of generality put $\alpha = 1$. The second row of Table 7.1 suggests that these games can be interpreted as elements of a sequence matching condition (6.16).

Table 7.1: Evolution of the CM

	$\Gamma^{(6)}$	$\Gamma^{(9)}$	$\Gamma^{(10)}$	$\Gamma^{(12)}$	$\Gamma^{(15)}$	$\Gamma^{(27)}$	...	
$\alpha_{\max}^{(\nu)}$	0.2353	0.1724	0.1587	0.1316	0.1149	0.0852	...	$\to 0$
$\alpha_{\max}^{(\nu)}/\sqrt{Q^{(\nu)}}$	0.5298	0.4603	0.4486	0.4131	0.3994	0.3627	...	$\to 0$

In the CM under QMV abstention is not a distinct third option but is treated as a tantamount to a 'no' vote. The limit scenario for non-atomic binary weighted voting games is depicted by the horizontal axis $w_L = 0$ in Figure 7.1. Theorem 7.2 provides for games with an empty set of major voters and $\alpha = 1$

$$\lim_{\nu\to\infty} A\left[\Gamma^{(\nu)}\right] = \begin{cases} 1 & \text{if } q < 1/2, \\ 1/2 & \text{if } q = 1/2, \\ 0 & \text{if } q > 1/2. \end{cases} \tag{7.43}$$

The Figures 7.3 and 7.4 at the end of this chapter illustrate the scenario. Figure 7.3 gives $A\left[\Gamma^{(\nu)}\right]$ for the six scenarios $\Gamma^{(6)}, \Gamma^{(9)}, ..., \Gamma^{(27)}$ as a function of $q \in (0,1)$. The points marked with ' $*$ ' are the corresponding realizations of q given by Table 6.1 and Table 13.1. The step function in the front indicates the limit values with increasing number of voters. As a reference scenario,

Figure 7.4 provides the same picture for the symmetric weight distribution 'one person one vote' with $\alpha_k = \alpha/\nu$ for all $k = 1, ..., \nu$ which are qualitatively similar to the setting of the CM.

In all scenarios we observe that convergence to the limit tends to be relatively quick for $q \neq 0.5$. The convergence rate is the higher, the closer q is located to the boundaries $q = 0$ and $q = 1$ as indicated by Theorem 7.10. This explains the rapid convergence of complaisance of the CM to zero by setting $\epsilon = q - 0.5 = 0.71 - 0.5 = 0.21$ and $m^{(\nu)} = \nu$.

Interestingly, Figures 7.3 and 7.4 identify another indicator for the dramatic decrease of A in Table 6.1: both display a high sensitivity in changes in q especially for low values of ν. It turns out that this sensitivity has a large impact. The dramatic effect in the first three columns of Table 6.1 had its origin partly in a slight increase in the quota from 70.59% in $\Gamma^{(6)}$ to 70.69% in $\Gamma^{(9)}$ and 71.43% in $\Gamma^{(10)}$. Row A_{71} in Table 7.2 gives complaisance of the voting systems if the quota q would have been kept at constant 0.71% which represents the arithmetic mean of the relative quota from 1958-1995. Analogously, R_{71} denotes 'resistance' as measured by R with a fixed 71% Quota. In this case the first scenario would have started already with a lower value $A\left[\Gamma^{(6)}\right]$ and hence the difference in comparison to the subsequent scenario would have been less significant. In fact, with a fixed quota $q = 0.71$ complaisance would have increased from 1973 to 1981.

Table 7.2: Complaisance of QMV with fixed 71%, first five periods

Country	1958	1973	1981	1986	1995
Quota %	70.59	70.69	71.43	71.05	71.26
A	0.2188	0.1465	0.1367	0.0981	0.0778
A_{71}	0.1562	0.1309	0.1367	0.0981	0.0778
R	0.5806	0.7098	0.7280	0.8041	0.8445
R_{71}	0.7097	0.7412	0.7280	0.8041	0.8445

Note The 'A_{71}' and 'R_{71}' row give complaisance and resistance as measured by (6.1) and (6.2) if the quota was kept at exact 71%.

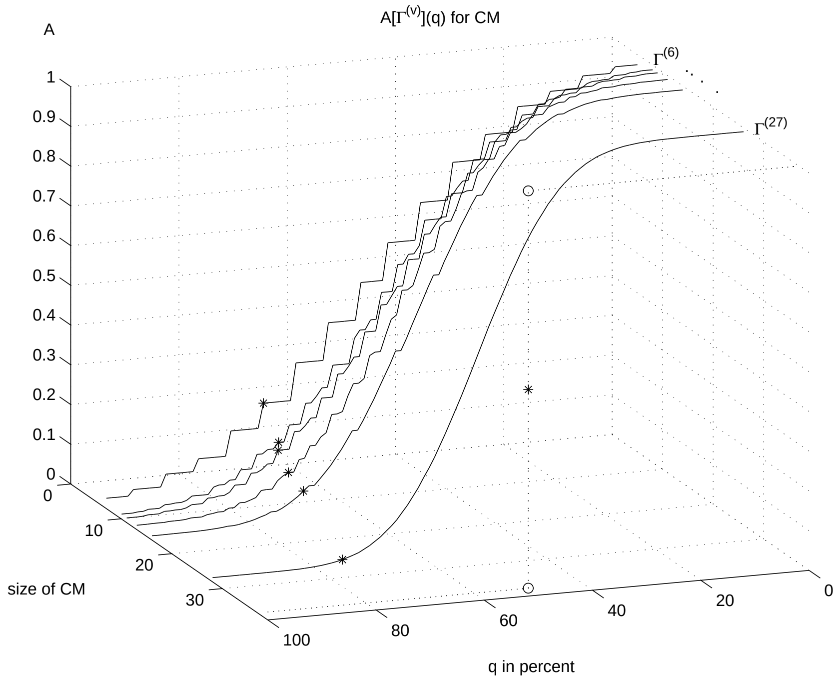


Figure 7.3: Complaisance in EU Council of Ministers

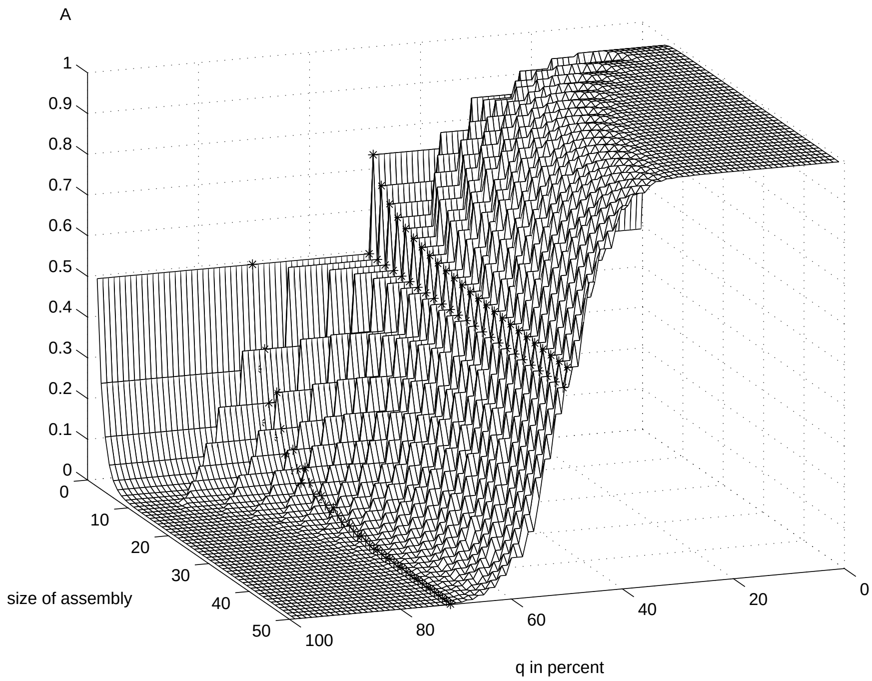


Figure 7.4: One Person One Vote

Chapter 8

In Search of the Truth

8.1 Introduction to Condorcet's Jury Theorem

The research in this chapter is rooted in a tradition which goes back to Condorcet (1785). Consider a group N confronting a dichotomous choice, while the members of the group are all assumed to possess more or less reliable perceptions of which alternatives 'ought' to be chosen. The fundamental premise is that there exists some procedure-independent fact of the matter as to what the best or right outcome is. We shall base the discussion on the following cover story: assume that in a jury trial, the probability that the defendant is guilty of the offense charged is $\theta \in [0, 1]$. Hence there is a truth independent of the jury, yet unknown to the members of the jury. We will assume that each member k possesses a more or less reliable perception about the truth. This degree of knowledge is modelled by $p_k \in (0, 1)$, the judgemental competence of the voter. It is the probability that the voter will make the correct choice (the 'better') of the two available to him or her. If the proposal to be voted upon is whether the defendant shall be convicted then p_k is the probability that k votes 'yes' if the defendant is guilty and 'no' if the person is innocent.

The jury's competence should be judged by the likelihood of the verdict being correct. For a given SVG Γ let $C[\Gamma]$ denote the probability that the decision rule in Γ leads to a correct choice. We shall refer to $C[\Gamma]$ as *jury competence* or *group judgemental accuracy*.[1] We further define $\mathcal{M}_n$ as the

[1] For the terminology, see for example Shapley & Grofman (1984).

simple majority game, i.e. the SVG whose winning coalitions are just those subsets of the voter set N with cardinality larger than $n/2$. Condorcet's jury theorem provides a statement for the jury competence of $\mathcal{M}_n$. Assume for simplicity n to be odd and put $m = (n+1)/2$.

Theorem 8.1 *(Condorcet jury theorem) (Condorcet, 1785; see also Grofman et al., 1983) Assume that the voters' choices are independent of one another and are homogenous, i.e. the probability that voter k's choice is correct is given by $p_k = p$ for all $k \in \{1, 2, , \ldots, n\}$. Then*

$$C[\mathcal{M}_n] = \sum_{h=m}^{n} \binom{n}{h} p^h (1-p)^{n-h}.$$

Moreover, if $1 > p > 1/2$, then $C[\mathcal{M}_n]$ is monotonically increasing in n and $\lim_{n\to\infty} C[\mathcal{M}_n] = 1$; if $0 < p < 1/2$, then $C[\mathcal{M}_n]$ is monotonically decreasing in n and $\lim_{n\to\infty} C[\mathcal{M}_n] = 0$; while if $p = 1/2$ then $C[\mathcal{M}_n] = 1/2$ for all odd n (and $\lim_{n\to\infty} C[\mathcal{M}_n] = 1/2$).

The result says that if each voter makes the correct choice with a given probability larger than 1/2, the correct option of being the majority winner converges to certainty monotonically as the number of voters tends to infinity. This result constitutes an important pro-democratic argument and has been extended in many ways by statisticians, economists, political scientists, etc. For example, Shapley & Grofman (1984) show that the group decision procedure that maximizes group judgemental accuracy is a weighted majority voting rule that assigns weights w_k equal to $\log p_k/(1-p_k)$. However, for our purpose to offset the results in context of Coleman's A of the previous sections, we shall prove a generalized statement weighted majority games when there are many small voters as defined by the setting (6.14) – (6.16).

8.2 Generalization of Condorcet's Jury Theorem

Assumption 8.2 (homogeneity) *There are exactly two alternatives, only one of which is correct (or equivalently, one of which is 'better' than the other) with probability $\theta \in [0,1]$. We fix an arbitrary real $p \in (0,1)$ and assume that each minor voter $k + l \in M^{(\nu)}$ acts independently and makes the correct choice (i.e. the 'better' choice) with probability p.*

For our purpose it will prove useful to decompose C into probabilities of avoiding errors of Type I and II in a statistical sense: Let C_I denote the probability of avoiding a Type I error or equivalently $(1 - C_I)$ the probability of a true hypothesis being rejected. In terms of the cover story, if the hypothesis is that the defendant is guilty, $(1 - C_I)$ is the probability that a guilty person will be found not guilty. Analogously, let C_{II} denote the probability of avoiding a Type II error, i.e. an innocent will be convicted. Group competence follows as

$$C = \theta C_I + (1 - \theta) C_{II}. \tag{8.1}$$

For the moment, put $\theta = 1$ (the defendant is guilty). We can then interpret $C = C_I$ as a more general term of complaisance of the jury if an affirmative group answer leads to conviction of the defendant. This is more general because the voters are now assumed to vote 'yes' with $p \in (0, 1)$ and complaisance as discussed in Section 7.1 follows by setting $p = 1/2$. We can now modify the results of Theorem 7.2 to the more general setting. For this purpose we redefine the games in (7.2) to

$$\begin{aligned} \Gamma_0 &:= [\,c - p\alpha; w_1, ..., w_l] \\ \Gamma'_0 &:= \; < c - p\alpha; w_1, ..., w_l >, \end{aligned} \tag{8.2}$$

as well as the inner area $\mathcal{J}$ from (7.4) to

$$\mathcal{J} := \{c \,|\, p\alpha < c < w_L + p\alpha\}. \tag{8.3}$$

Let $\mathcal{B}_n$ denote the unanimity SVG which consists of the grand coalition N only. Let $\mathcal{B}_n^*$ denote the *dual* of $\mathcal{B}_n$, representing what Rae (1969) has called a 'rule of individual initiative': any coalition except the $\emptyset$ is winning.

The next result is a statement about jury competence avoiding an error of Type I.

Theorem 8.3 *In a jury trial, assume that the minor voters' choices are independent of one another and are homogenous, i.e. the probability that voter k's choice is correct is given by* $p_k = p \in (0, 1)$ *for all* $k \leq m^{(\nu)}$. *For the sequence of voting games* (6.14) – (6.16) *we have*

$$\lim_{\nu \to \infty} C_I[\Gamma^{(\nu)}] = \frac{1}{2} C_I[\Gamma_0] + \frac{1}{2} C_I[\Gamma'_0] \quad \textit{if } c \in \mathcal{J}. \tag{8.4}$$

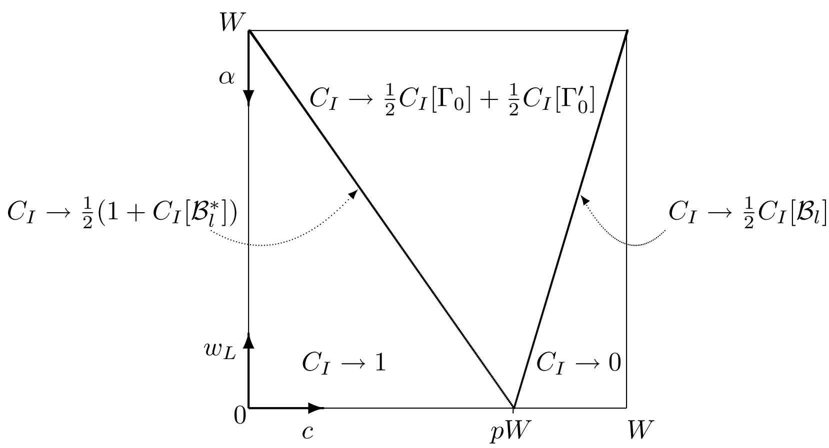
For $c \notin \mathcal{J}$ *follows*

$$\lim_{\nu \to \infty} C_I[\Gamma^{(\nu)}] = \begin{cases} 1 & \textit{if } c < p\alpha, \\ \frac{1}{2}(1 + C_I[\mathcal{B}_l^*]) & \textit{if } c = p\alpha, \\ \frac{1}{2} C_I[\mathcal{B}_l] & \textit{if } c = w_L + p\alpha, \\ 0 & \textit{if } c > w_L + p\alpha. \end{cases} \tag{8.5}$$

Remark 8.4 In Theorem 8.3 there is no need to specify the competence of the major voters. Their competence enters generally by means of the major collective competencies of the games Γ_0 and Γ'_0, the games $\mathcal{B}_l$ and $\mathcal{B}_l^*$ respectively.

The results for $p > 0.5$ are illustrated in Figure 8.1. The scenario is topologically equivalent to Figure 7.1. However, the introduction of $p > 0.5$ has led to a distortion effect of the inner area.

Figure 8.1: Limit scenario for C if $\theta = 1$



Proof of Theorem 8.3 Let the random variable Y denote a random coalition among the major voters. Setting $\theta = 1$, we get

$$C[\Gamma^{(\nu)}] = C_I[\Gamma^{(\nu)}] = \sum_{B \subseteq L} \text{Prob}\{Y = B\}\text{Prob}\{\alpha^{(\nu)}(S) \geq c - w(B)\}, \quad (8.6)$$

where S is a random subset of $M^{(\nu)}$. The following is almost analogous to the proof of Theorem 7.2 in that the mean value $\mu = \alpha/2$ has to be replaced by $\mu = p\alpha$. A suitable modification of Lemma 7.3 with respect to the mean value provides

$$\text{Prob}\left\{\alpha^{(\nu)}(S) \geq c - w(B)\right\} \to \begin{cases} 0 & w(B) < c - p\alpha, \\ 1/2 \quad \text{if} & w(B) = c - p\alpha, \\ 1 & w(B) > c - p\alpha. \end{cases} \tag{8.7}$$

For $c \in \mathcal{J}$ the WVGs in (8.2) are well-defined. If $w(B) > c - p\alpha$ then $B \subseteq L$ is winning in both Γ_0 and Γ_0', whereas if $w(B) = c - p\alpha$ then B is winning in Γ_0 only. With (8.7) this provides for (8.6)

$$\begin{aligned} & \lim_{\nu\to\infty} C_I[\Gamma^{(\nu)}] \\ = \; & \sum_{w(B)>c-p\alpha} \text{Prob}\{Y = B\} + \frac{1}{2} \sum_{w(B)=c-p\alpha} \text{Prob}\{Y = B\} \\ = \; & \frac{1}{2} \sum_{w(B)>c-p\alpha} \text{Prob}\{Y = B\} + \frac{1}{2} \sum_{w(B)\geq c-p\alpha} \text{Prob}\{Y = B\} \\ = \; & \frac{1}{2} C_I[\Gamma_0] + \frac{1}{2} C_I[\Gamma_0'], \end{aligned}$$

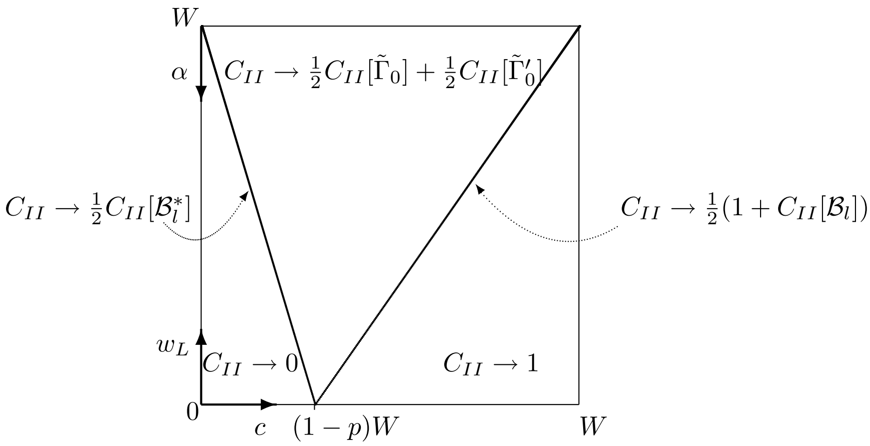
and hence (8.4).
Further, $c < p\alpha$ guarantees that $w(B) > c - p\alpha$ for all $B \subseteq L$. Similarly, $c > w_L + p\alpha$ ensures that $w(B) < c - \alpha$ for all $B \subseteq L$. With (8.7) this provides the first and last row of (8.5).
For $c = p\alpha$ then $w(B) = c - p\alpha$ for $B = \emptyset$ and $w(B) > c - p\alpha$ else. Hence for (8.6) follows with (8.7)

$$\begin{aligned} C_I[\Gamma^{(\nu)}] &= \sum_{B\neq\emptyset} \text{Prob}\{Y = B\} + \frac{1}{2}\text{Prob}\{Y = \emptyset\}, \\ &= \sum_{B\neq\emptyset} \text{Prob}\{Y = B\} + \frac{1}{2}(1 - \sum_{B\neq\emptyset} \text{Prob}\{Y = B\}) \end{aligned}$$

which proves the second row of (8.5).
To see the third row note that from $c = w_L + p\alpha$ follows that $w(B) = c - p\alpha$ for $B = L$ and $w(B) < c - p\alpha$ otherwise. □

For $\theta = 0$ we have that $C = C_{II}$ is the likelihood that an innocent defendant is found 'not guilty' in which case voting 'no' is the correct choice. This implies that the voters vote 'yes' with probability $(1 - p)$ leading to a distortion effect of the inner area opposite to the shift of Figure 8.1, as well as another adjustment of the quotas of the games played among the major voters only given by (8.2). Figure 8.2 illustrates this scenario – a precise statement is given in Theorem 8.5.

Figure 8.2: Limit scenario for C if $\theta = 0$



For $\tilde{\Gamma}_0$ and $\tilde{\Gamma}_0'$ see (8.10).

In summary, we get

Theorem 8.5 (generalized Condorcet jury theorem) *In a jury trial, let the probability that the defendant is guilty of the offense charged be given by $\theta \in [0,1]$. Assume that the minor voters' choices are independent of one another and are homogenous, i.e. the probability that voter k's choice is correct is given by $p_k = p \in (0,1)$ for all $k \leq m^{(\nu)}$. In the sequence of games described by* (6.14) - (6.16) *the jury competence follows as*

$$\lim_{\nu \to \infty} C[\Gamma^{(\nu)}] = \theta C_I + (1-\theta) C_{II}, \tag{8.8}$$

(i) where C_I is given by

$$C_I = \frac{1}{2} C_I[\Gamma_0] + \frac{1}{2} C_I[\Gamma_0'], \qquad \textit{if } c \in \mathcal{J}.$$

For other values of c we have

$$C_I = \begin{cases} 1 & \textit{if } c < p\alpha, \\ \frac{1}{2}(1 + C_I[\mathcal{B}_l^*]) & \textit{if } c = p\alpha, \\ \frac{1}{2} C_I[\mathcal{B}_l] & \textit{if } c = w_L + p\alpha, \\ 0 & \textit{if } c > w_L + p\alpha. \end{cases} \tag{8.9}$$

(ii) Put

$$\begin{aligned} \tilde{\Gamma}_0 &:= [c - (1-p)\alpha; w_1, w_2, ..., w_l], \\ \tilde{\Gamma}_0' &:= < c - (1-p)\alpha; w_1, w_2, ..., w_l > . \end{aligned} \tag{8.10}$$

For C_{II} follows

$$C_{II} = \frac{1}{2}C_{II}[\tilde{\Gamma}_0] + \frac{1}{2}C_{II}[\tilde{\Gamma}'_0], \qquad \textit{if } c \in \tilde{\mathcal{J}},$$

with

$$\tilde{\mathcal{J}} := \{c|(1-p)\alpha < c < w_L + (1-p)\alpha\}.$$

For other values of c we have

$$C_{II} = \begin{cases} 0 & \textit{if } c < (1-p)\alpha, \\ \frac{1}{2}C_{II}[\mathcal{B}_l^*] & \textit{if } c = (1-p)\alpha, \\ \frac{1}{2}(1 + C_{II}[\mathcal{B}_l]) & \textit{if } c = w_L + (1-p)\alpha, \\ 1 & \textit{if } c > w_L + (1-p)\alpha. \end{cases} \tag{8.11}$$

Statement (ii) follows analogously to the proof of Theorem 8.3 by replacing p by $(1-p)$.

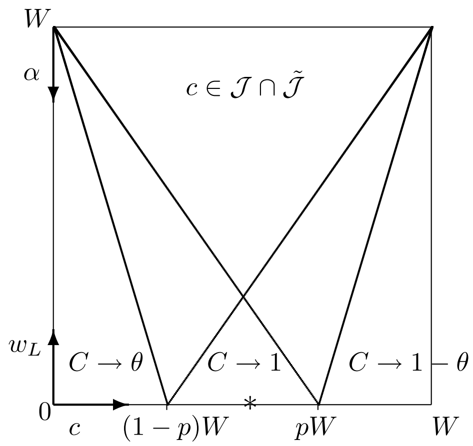
Remark 8.6 Theorem 8.5 does not require a specification of the competence of the major voters (see Remark 8.4).

The qualitative result of Theorem 8.5 is that for values of the quota c outside the closure of $\mathcal{J} \cup \tilde{\mathcal{J}}$ the jury competence is independent from the competence level of either the major or minors voters as indicated in Figure 8.3. The relative quotas in the area around $q = 50\%$ prove to be the best 'truth-tracker' since jury competence reaches infallibility, i.e. $C = 1$. This area enlarges with an increasing competence p of the minor voters due to a shift of the overlapping triangles in opposite direction. However, for any fixed $q \in (0, 1)$ increasing voting weight of the major voters w_L leads to areas in which the jury competence depends on the competence of major voters only: the limit value C can be computed from WVGs involving just the major voters in which the minor voters only have an implicit effect of manipulating the quota. This effect represents a shift of the quota c in 'truth direction': If the defendant is guilty then the threshold (quota) $c - p\alpha$ necessary for conviction decreases with increasing competence p of the minor voters (analogously, if the defendant is not guilty the likelihood of acquittal increases with increasing competence of the minor voter).
The result of Theorem 8.1 – the classical version of Condorcet – is indicated by the point marked with '$*$' on the horizontal axis $w_L = 0$.

The convergence behaves qualitatively similar to the convergence of complaisance A as discussed in Section 7.3: for statements measuring the rate of convergence of C_I and C_{II}, the mean value $\mu = \alpha/2$ has to be replaced by

$p\alpha$ or $(1-p)\alpha$ respectively. Thus the limit C serving as an approximation for finite real-world scenarios is justified in (c, w_L) areas with high convergence rates. In Section 7.3 we have seen that this is the case under suitable smoothness conditions with regards to allocation of the fixed weight block of the minor voters but also conditions ensuring that A (C respectively) consists of summands representing 'tails' of sum of random variables.

Figure 8.3: Generalized Jury Theorem



Chapter 9

Discussion of Part II

Our discussion has shown that the pessimistic prognosis of the EU's CM arose from the desire to fit group decisions into one class. The classification of problems into two main categories leads to another conclusion: if the group decision is the aggregation of individual views or interests then indeed a quota of above 50% implies a tendency to immobilism (with a significant convergence rate). This supports the pessimistic view.

It is tempting to criticize this pessimism because it is based on the a priori assumption of each 'yes' and 'no' choice being equally likely. This assumption does not match the observation of the real-world voting scenario of for example the CM: when it comes to voting, the affirmative votes usually represent a majority. However, this argument disregards two essential aspects of measuring voting power. First of all, it leaves the basis of an a priori analysis which considers a voting game as an abstract shell, ignoring factors such as an individual voter's prior bias for the issues voted upon, or affinities and disaffinities between voters. This abstraction is 'what makes the analysis focus on the rules themselves rather than on the other aspects of the political environment' (Roth 1988, p. 9, see also introduction to this thesis). This environment may change over time and hence the de facto (a posteriori) voting power can only be interpreted as a 'snap-shot'.[1]

The second essential aspect disregarded by this critique is that a vote held, for example, in the CM is usually the result of a foregoing bargaining process: before the formal vote is taken there is usually a whole series of shadow or

[1]For a more extended discussion of a priori versus actual (a posteriori) voting power and further references see e.g. Felsenthal & Machover (1998).

straw divisions - which comes to a halt when a majority can be expected.[2] In that sense, complaisance can be thought of as measuring the barrier that members of a committee have to overtake via negotiations and bargaining in order to approve a given proposal. A decreasing A (increasing resistance R) increases this barrier which is usually reflected by a long pre-vote period – a clear indicator of immobilism.

However, the generalization of Condorcet's jury theorem has shown that if the intention is to arrive at a collective decision in terms of a 'correct' judgement (truth-tracking), the prognosis of the CM's reliability depends on the competence level of its members. The variety of the issues to vote upon suggests a symmetric a priori setting of θ to $1/2$ (note that values for θ other than $1/2$ make sense if the jury is e.g. exclusively concerned with penal jurisdiction; in this case θ could represent a measure of the crime rate). Hence if we interpret the CM as a jury with more or less homogenous competence of its members $p > 0.5$, then the prognosis goes in the opposite direction: in the worst case the judgemental competence of the Council tends to 50% if $p \leq q$. For $p > q = 0.71$, however, the Council tends to infallibility with an increasing member set.

In both respects – a parliament acting as a decision committee or truth-tracking jury – the shift of q versus the 50 percent mark goes in the desired direction. In terms of the first category, it maintains a suitable level of complaisance, as measured by A, as the number of voters increases. If the CM is interpreted as a knowledge aggregation machine heading for the truth of a matter, then this shift increases the probability that the scenario is covered by the (triangle shaped) $C = 1$ area of Figure 8.3.

The classification of problems into two main categories has also an impact on the assessments of large decision making bodies with an atomic part. These are characterized by a small set of voters with a large voting weight and a large 'pool' of small voters – a typical scenario in, for example, shareholding. Likewise the 50% quota has an outstanding role: If the combined weight sum of the major voters is relatively small this quota prevents the tendency of complaisance to the extremes 0 and 1. With respect to truth tracking it increases the likelihood that the suitable WVG model, including competencies, matches the interior region of infallibility of Figure 8.3.

[2] Also, the CM seems to publish only positive outcomes, i.e. when acts have been adopted.

Part III

Numerical Methods for Large WVGs

Chapter 10

Introduction

The simplest method to compute power measures is to follow their definition directly. This suggests an algorithm which generates any possible division (tripartition) and determines the value of the characteristic function. The advantage of such an approach is the broad applicability to various concepts of cooperative game theory. However, a major disadvantage is that complexity of calculations grows exponentially with the number of voters. Even with a small assembly the necessary computation time easily exceeds the bounds of a possible realization: in case of n voters where each voter faces two options ('yes' or 'no'), we face an exponential complexity of order 2^n. This effect worsens with an enlarged option set, for example if the option to abstain is included in the model. Despite the technical advances and enormous progresses in making more powerful computers, this can not solve the fundamental nature of the problem at hand. For instance, yet it is still not possible to solve the States game in the US presidential Electoral College with 51 members. For a long time, enumerating all possible coalitions seemed insurmountable and methods of approximating the power measures were used instead. In their 1960, Mann & Shapley obtained approximations of the S-S-index for the Electoral College games (which then had 50 voters) using Monte Carlo methods. These numerical techniques can be loosely described as statistical simulation methods, where statistical simulation is defined in quite general terms to be any method that utilizes sequences of random numbers to perform the simulation. With respect to SVGs it implies the simple estimation of the power measures from a random sample of coalitions. Although the approximation errors can be reduced substantially with increasing sample size, however, the disadvantage is clearly that it is not exact.

The key computational idea that caused a breakthrough for the exact computation of WVGs is due to David G. Cantor.[1] He proposed to reformulate the definition of the S-S index such that it can be determined by means of the coefficients of specific polynomials, so called *generating functions*. The problem reduces then to carry out iterations whose computational effort increases polynomially (instead of exponentially) with increasing number of voters.

Unfortunately, this method is not without limitation due to another algorithmic complexity which is memory requirement (space complexity). This problem can be substantial, both in terms of integer size and array dimension. In turn, this widened the need for approximation methods. For example, due to space complexity Leech (2002b) uses approximations to evaluate the (weighted) rules of the International Monetary Fund (IMF). However, the prevalent numerical methods which are based on generating functions operate with matrices which typically match the characterization 'sparse', i.e. a large share of the entries is zero. The following chapter demonstrates how the data can be restructured by storage schemes for sparse matrices, resulting in a significant reduction in storage requirements. It allows for the evaluation of WVGs which otherwise show insurmountable storage requirement given modern computer power.

Chapter 11 is concerned with exact evaluation of WVGs, Chapter 12 discusses approximation methods. Section 11.1-11.3 briefly reviews the numerical methods and procedures that the prevalent literature provides for classical power measures. Section 11.1 characterizes the classical measures by a common structure which reveals the information necessary for computation. Section 11.2 and 11.3 introduce the classical method of generating functions to evaluate (binary) WVGs. Section 11.4 extends the prevalent numerical methods to TWVGs. Section 11.5 considers data schemes designed for sparse matrices.

[1] His suggestion was made to Mann & Shapley, following a lecture at Princeton university in October 1960.

Chapter 11

Exact Evaluation of WVGs

11.1 Common Structure of Classical Measures

The probabilistic approach of the individual and global measures considered in this thesis provides a unifying characterization which reveals the structural characteristics essential for computation. Consider first the global measures, complaisance A and jury competence C. Generally, in binary SVGs the common structure is given by

$$\sum_{S \subset N} fac(S)v(S) + const(S), \tag{11.1}$$

where $fac(S)$ is interpreted as the probability that coalition S forms and $const(S)$ is a constant, each depending on $S \subset N$. Table 11.1 specifies both terms.

The Bz measure ψ and S-S index ϕ in binary SVGs have the general form

$$\sum_{S \subset N} fac(S) \left(v(S) - v(S \backslash \{a\}) \right), \tag{11.2}$$

for each $a \in N$. The coalition specific factor $fac(S)$ is given by Table 11.2.

Technically, the term with the characteristic function in (11.1) and (11.2) works as a filter and leads to counting up selected probabilities $fac(S)$. Note that for complaisance and the Bz measure the factor $fac(S)$ is the same for all coalitions $S \subset N$ such that coalition size s represents superfluous

Table 11.1: Global Measures in Binary SVG Setup

	$fac(S)$	$const(S)$	coalition specific information
A	$1/2^n$	0	none
C	$\theta p^s(1-p)^{n-s}$ $-(1-\theta)(1-p)^s p^{n-s}$	$(1-\theta)(1-p)^s p^{n-s}$	coalition size s

Note The symbol s denotes the cardinality of $S \subset N$. Recall that θ and p are probabilities given by the Condorcet model. The third column gives the information necessary to evaluate $fac(S)$ and $const(S)$.

information. Therefore the corresponding algorithmic evaluation simplifies considerably as demonstrated in the next two sections. It turns out that computation reduces to determination of the number of coalitions with a combined voting weight in a specific range. The computation of jury competence and S-S index requires the additional coalition specific information of the number of affirming voters.

Table 11.2: Individual Measures in Binary SVG Setup

	$fac(S)$	coalition specific information
ψ_a	$1/2^{n-1}$	none
φ_a	$s!(n-s-1)!/n!$	number of affirming voters s

11.2 Complaisance and the Banzhaf Measure

Evaluating complaisance and the Bz measures requires the computation of winning coalitions, the number of winning coalitions in which a voter is critical respectively. Let d_h denote the number of coalitions whose members have combined voting weight h. Let c denote the absolute quota such that each coalition with weight sum $h \geq c$ is a winning coalition. Complaisance then follows as the number

$$A = \frac{1}{2^n} \sum_{h=c}^{w(N)} d_h. \tag{11.3}$$

Let $d_h^{\neg a}$ be the number of coalitions not containing a with sum of votes equal to h. Voter a is critical for a coalition with combined weight h iff $c - w_a \leq h < c$. Then the Bz measure can be reformulated as

$$\psi_a = \frac{1}{2^n} \sum_{h=c-w_a}^{c-1} d_h^{\neg a}. \tag{11.4}$$

Consider the *generating function*

$$f(x) = \prod_{k=1}^{n} (1 + x^{w_k}) \tag{11.5}$$

which is a polynomial in x of degree $w(N)$. Multiplication provides

$$f(x) = \sum_{h=0}^{w(N)} d_h x^h.$$

The coefficient of x^h is precisely d_h. Obviously, it is always true that $d_0 = d_{w(N)} = 1$ since there is only one coalition with combined voting weight zero, i.e. the empty set, and only one coalition with combined voting weight $w(N)$ which is the grand coalition.

Furthermore, successive multiplication of factors of the polynomial $f(x)$ omitting $k = a$ yields

$$f(x)/(1 + x^{w_a}) = \sum_{h=0}^{w(N)-w_a} d_h^{\neg a} x^h.$$

Hence the problem of computing A and ψ_a reduces to multiplication of factors of polynomial (11.5) and determination of the coefficients d_h and $d_h^{\neg a}$. Successive multiplication with one factor at a time provides

$$
\begin{aligned}
f(x) &= \prod_{k=1}^{n}(1+x^{w_k}) \\
&= [1+x^{w_1}]\prod_{k=2}^{n}(1+x^{w_k}) \\
&= \left[1+x^{w_1}+x^{w_2}+x^{w_1+w_2}\right]\prod_{k=3}^{n}(1+x^{w_k}) \\
&= \ldots \\
&= \sum_{h=0}^{w(N)} d_h x^h.
\end{aligned}
$$

Let the polynomial in the square brackets at stage $m \in \{0, ..., n\}$ be generally given by

$$
\left[d_0^{(m)} + d_1^{(m)}x + d_2^{(m)}x^2 + ... + d_{w(N)}^{(m)}x^{w(N)}\right].
$$

The coefficients of stage m can be updated from stage $m-1$ according to the iteration

$$
d_h^{(m)} = d_h^{(m-1)} + d_{h-w_m}^{(m-1)}. \tag{11.6}
$$

The last term is understood to be zero if either subscript is negative. At each iteration step m the coefficients are collected in an array $D^{(m)}$ of length $(w(N)+1)$, where the initial array $D^{(0)}$ is all zero except $d_0^{(0)} = 1$. This process generates a sequence of coefficient arrays $\left\{D^{(m)}\right\}_{m=0}^{n}$. After n iterations, this gives the required coefficients d_h and complaisance A can be computed by expression (11.3). The polynomial factors can be introduced in any order and the end result will be the same.

To compute the Bz measure of any voter a, the coefficients $d_h^{\neg a}$ can be obtained by dividing the generating function (11.5) by the factor $(1+x^{w_a})$. This means 'reversing' (11.6) once, i.e.

$$
d_h^{\neg a} = d_h - d_{h-w_a}.
$$

Example 11.1 Consider the game $\Gamma = [c; 1, 1, 2, 3]$. The generating function is given by

$$
f(x) = (1+x)(1+x)(1+x^2)(1+x^3).
$$

The evolution of iteration (11.6) is illustrated in Table 11.3. Zero coefficients are left out. The first column represents the initial array with $D^{(0)} =$

$[1, 0, ..., 0]$ The algorithmic evaluation is that at each iteration step any entry unequal zero is identified with $d_{h-w_m}^{(m-1)}$ and added to the entry $d_h^{(m-1)}$. For example, in the last iteration step any non-zero entry in column $m = 3$ is added to the entry $w_4 = 3$ rows below. The result is stored in array $m = 4$.

Table 11.3

h	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$m = 4$
0	1	1	1	1	1
1		1	2	2	2
2			1	2	2
3				2	3
4				1	3
5					2
6					2
7					1

◁

Time savings in operations and storage requirements are possible as follows:

1. There is no need to store the entire sequence $\left\{D^{(m)}\right\}_{m=0}^{n}$ since each array $D^{(m)}$ can simply be gained from updating $D^{(m-1)}$.

2. The array $D^{(m)}$ is symmetric according to the identity

$$d_h = d_{w(N)-h}.$$

   Hence only the upper half of $D^{(m)}$ needs to be determined and formulas (11.3) and (11.4) can be rewritten using only values $h \leq w(N)/2 + 1$.

3. The summation of the Bz measure (11.4) does only use values $d_h \leq c-1$ and hence only the first c rows of the array $D^{(m)}$ need to be computed.

11.3 Computation of Jury Competence and the S-S Index

For evaluating jury competence and the S-S index not only the combined voting weight of a coalition needs to be determined but also the number of coalition members. This can be performed by introducing an additional argument to the generating function according to

$$f(x,y) = \prod_{k=1}^{n}(1 + x^{w_k}y). \tag{11.7}$$

Successive multiplication of factors provides

$$f(x,y) = \sum_{h=0}^{w(N)}\sum_{k=0}^{n} d_{hk}x^h y^k,$$

where the coefficients d_{hk} of $x^h y^k$ denote the number of coalitions with k members and a sum of weights equal to h.

These coefficients can be stored in a $(w(N)+1)\times(n+1)$ matrix $D^{(n)}$, which is computed iteratively by successive multiplication of (11.7) implying the iteration rule

$$d_{hk}^{(m)} = d_{hk}^{(m-1)} + d_{h-w_m,k-1}^{(m-1)}, \quad \text{for } m = 1,2,\ldots,n. \tag{11.8}$$

The last term is understood to be zero if either subscript is negative. The initial matrix $D^{(0)}$ is all zero except $d_{00}^{(0)} = 1$. Again, the polynomial factors can be introduced in any order and the end result will be the same.

With these coefficients jury competence can be reformulated as

$$C = \theta\sum_{k=0}^{n} p^k(1-p)^{n-k}\sum_{h=c}^{w(N)} d_{hk} + (1-\theta)\sum_{k=0}^{n} p^{n-k}(1-p)^k\sum_{h=0}^{c-1} d_{hk}. \tag{11.9}$$

(Recall that $\theta, p \in (0,1)$ are exogenously given by the Condorcet jury model).

Let $d_{hk}^{\neg a}$ denote the number of ways in which k voters, other than a, can have a sum of weights equal to h. For the S-S index follows

$$\phi_a = \sum_{k=0}^{n-1}\frac{(k-1)!(n-k)!}{n!}\sum_{h=c-w_a}^{c-1} d_{hk}^{\neg a}, \tag{11.10}$$

where $d_{hk}^{\neg a}$ results from multiplying out the polynomial $f(x,y)$ omitting $k=a$, i.e.

$$f(x,y)/(1+x^{w_a}y) = \sum_{h=0}^{w(N)-w_a}\sum_{k=0}^{n-1} d_{hk}^{\neg a}x^h y^k.$$

The coefficients $d_{hk}^{\neg a}$ can be obtained from $D^{(n)}$ by 'reversing' (11.8) once, i.e.

$$d_{hk}^{\neg a} = d_{hk} - d_{h-w_a,k-1}.$$

Example 11.2 Consider again $\Gamma = [\,c\,;\,1,1,2,3\,]$. Figure 11.4 illustrates iteration (11.8). The algorithmic evaluation of (11.8) is such that at any iteration step m any non-zero entry is identified as $d_{h-w_m,k-1}^{(m-1)}$ and added to the matrix entry $d_{hk}^{(m-1)}$, i.e. one column to the right and w_m rows below. The result is stored in $D^{(m)}$. ◁

Figure 11.4

	$m=0$					$m=1$					$m=2$				
$h \backslash k$	0	1	2	3	4	0	1	2	3	4	0	1	2	3	4
0	1					1					1				
1							1					2			
2													1		
3															
4															
5															
6															
7															

	$m=3$					$m=4$				
$h \backslash k$	0	1	2	3	4	0	1	2	3	4
0	1					1				
1		2					2			
2		1	1				1	1		
3			2				1	2		
4				1				2	1	
5								1	1	
6									2	
7										1

Possible time and storage savings:

1. There is no need to store the entire sequence $\left\{D^{(m)}\right\}_{m=0}^{n}$ since $D^{(m)}$ can be gained from $D^{(m-1)}$ by simple superimposition.

2. Since $d_{hk}^{(m)} = 0$ for $k > m$ the iteration (11.8) need to be performed only for $k \leq m$.

3. The matrix $D^{(m)}$ is symmetric according to the identity

$$d_{hk} = d_{w(N)-h,n-k}.$$

Thus only values d_{hk} of $D^{(m)}$ have to be iteratively computed with $k \leq n/2$. Or, putting it differently, (11.9) and (11.10) can be rewritten using only values $k \leq n/2$.

11.4 Evaluating TWVGs

This section extends the classical numerical methods for binary WVGs which allow the evaluation of TWVGs. The methods lead to techniques for exact computation of the generalized Bz measure ψ as discussed Chapter 4 as well as complaisance A in TWVGs (see Section 7.2).
Let $\mathcal{T}$ denote the set of all tripartitions on N. In a tripartition $T \in \mathcal{T}$ let ℓ denote the number of abstainers such that $\ell = |T^0|$. With (4.4) and (4.10) the structure of the generalized Bz measure is given by

$$\psi_a = \sum_{T \in \mathcal{T}} fac(T) C_a(T), \tag{11.11}$$

where $fac(T) = t^\ell((1-t)/2)^{n-1-\ell}$.
Analogously, with either forms of abstentions – active abstention or abstention by default – complaisance in TWVGs takes the form

$$A = \sum_{T \in \mathcal{T}} fac(T) v(T), \tag{11.12}$$

with $fac(T) = t^\ell((1-t)/2)^{n-\ell}$.

Consider now the following generating function

$$\begin{aligned} f(x,y) &= \prod_{k=1}^{n} (1 + x^{w_k} + y^{w_k}), & (11.13)\\ &= \sum_{h=0}^{w(N)} \sum_{\ell=0}^{w(N)} d_{h\ell} x^h y^\ell, & (11.14) \end{aligned}$$

where any coefficient $d_{h\ell}$ can be interpreted as counting the number of tripartitions in which the camp of 'yes' voters have a combined vote h, whereas the votes of those abstaining count up to ℓ.

Successive multiplication of (11.13) gives rise to the iteration

$$(d_{h\ell})^{(m)} = (d_{h\ell})^{(m-1)} + (d_{h-w_m,\ell})^{(m-1)} + (d_{h,\ell-w_m})^{(m-1)}. \tag{11.15}$$

Terms with negative subscript are understood to be zero. This provides a sequence of $(w(N)+1) \times (w(N)+1)$ matrices $\left\{D^{(m)}\right\}_{m=0}^{n}$ where the initialization $D^{(0)}$ is all zero except $d_{00} = 1$.

Let $\lceil r \rceil$ denote the smallest integer larger than or equal to r. Given the coefficients of the matrix $D^{(m)}$ complaisance from (11.12) follows as

$$A = \sum_{\ell=0}^{w(N)} \sum_{h=\lceil q(w(N)-\ell) \rceil}^{w(N)-\ell} fac(\ell) d_{h\ell}. \tag{11.16}$$

For the generalized Bz measure let $d_{h\ell}^{\neg a}$ denote the number of tripartitions on $N \backslash \{a\}$ such that the affirming and abstaining voters have a combined voting weight h and ℓ respectively. Then (11.11) can be reformulated as

$$\psi_a = \sum_{\ell=0}^{w(N)} \sum_{h=\lceil q(w(N)-\ell) \rceil - w_a}^{\lceil q(w(N)-\ell) \rceil - 1} fac_a(T) d_{h\ell}^{\neg a}. \tag{11.17}$$

(Note that the coefficients $d_{h\ell}^{\neg a}$ can be computed from $D^{(n)}$ by one reverse iteration step, using (11.15) with $w_n = w_a$.)

Example 11.3 $\Gamma = [\,c\,;\, 1, 1, 2, 3\,]$. Consider Figure 11.5 at the end of this section. At every iteration step $m-1$ the algorithm adds any non-zero coefficient in $D^{(m-1)}$ located w_m to the right and w_m below and stores it in $D^{(m)}$. $\triangleleft$

Possible time and storage savings:

1. Similar to earlier cases, there is no need to store the entire sequence $\{D^{(m)}\}_{m=0}^{n}$ since $D^{(m)}$ can be gained from $D^{(m-1)}$ by simple superimposition.

2. Since the matrix $D^{(m)}$ is symmetric, for (11.16) and (11.17) only values $d_{h\ell}$ with $\ell \leq h$ have to be computed .

Remarks 11.4
(i) It is an easy task to modify the generating function (11.13) to specify the scenario counting. For example, besides the combined voting weight it is possible to determine the actual number of voters in a camp of equal choice. This suggests a generating function of the form

$$f(x, \bar{x}, y, \bar{y}) = \prod_{k=1}^{n} (1 + x^{w_k} \bar{x} + y^{w_k} \bar{y}). \tag{11.18}$$

Multiplication provides terms of the form $x^h \bar{x}^{\bar{h}} y^\ell \bar{y}^{\bar{\ell}}$ whose coefficients denote the number of tripartitions consisting of $\bar{h}$ 'yes' voters having a combined voting weight h, $\bar{\ell}$ abstainers with weight sum ℓ respectively.

(ii) The methodology in this section allows the evaluation of characteristics of weighted (j, k) games (see Remark 7.4) by means of a slight modification of either (11.13) or (11.18). If the corresponding weights are integer-valued, then the evaluation with respect to the weight sum of camps of equal approval level works with the following functional form

$$f(x_1, x_2, ..., x_j) = \prod_{k=1}^{n} (x_1^{w_k^1} + x_2^{w_k^2} + ... + x_j^{w_k^j}). \tag{11.19}$$

Note that TWVGs can be reformulated as $(3, 2)$ games. However, this additional effort is not necessary due to the special role of the abstention decision: it's affect on the threshold (that has to be achieved in order to let a bill pass) is taken into account by the double summation in (11.16) and (11.17) such that the generating function (11.13) does the trick.
Extension of each term $x_i^{w_k^i}$ to $x_i^{w_k^i} \bar{x}_i$ as in (11.18) provides the additional information of the number of voters in each camp of equal choice.

Figure 11.5

$m = 0$

$h\backslash k$	0	1	2	3	4	5	6	7
0	1							
1								
2								
3								
4								
5								
6								
7								

$m = 1$

$h\backslash k$	0	1	2	3	4	5	6	7
0	1	1						
1	1							
2								
3								
4								
5								
6								
7								

$m = 2$

$h\backslash k$	0	1	2	3	4	5	6	7
0	1	2	1					
1	2	2						
2	1							
3								
4								
5								
6								
7								

$m = 3$

$h\backslash k$	0	1	2	3	4	5	6	7
0	1	2	2	2	1			
1	2	2	2	2				
2	2	2	2					
3	2	2						
4	1							
5								
6								
7								

$m = 4$

$h\backslash k$	0	1	2	3	4	5	6	7
0	1	2	2	3	3	2	2	1
1	2	2	2	4	2	2	2	
2	2	2	2	2	2	2		
3	3	4	2	4	3			
4	3	2	2	3				
5	2	2	2					
6	2	2						
7	1							

11.5 Storage Schemes for Sparse Matrices

The coefficient matrices considered in Section 11.2- 11.4 feature a common property: a large share of the entries are zero, especially in early stages of iteration. This suggests the use of storage methods designed for sparse matrices. Briefly, a matrix is sparse if many of its coefficients are zero. This section extends the numerical methods discussed so far by integrating storage schemes that avoid the explicit storage of the zeros and can lead to enormous computational savings. The storage method in this section as well as further methods can be found in advanced literature on numerical mathematics of matrix operations (especially concerned with sparse matrices is e.g. Duff et al. (1986)).

The benefit from using methods designed for sparse matrices is twofold. First, the significantly reduced size of the arrays involved induces time saving. This is especially important in procedures which repeatedly call subroutines of evaluation.[1] Secondly, these methods allow the evaluation of large weighted voting systems which have seemed to cause insurmountable demands in storage. An example is given in Table 13.5. The corresponding $(w(N)+1) \times (w(N)+1)$ matrices $\left\{D^{(m)}\right\}_{m=0}^{n}$ as introduced in Section 11.4 have such large dimensions, $w(N) = 105\,412$, that they are not representable by MATLAB 6.5. However, the evaluation is possible with storage techniques for sparse matrices. The corresponding source codes can be found in Chapter 15.

The key idea is briefly explained. In order to avoid confusion with the term 'coefficient' as an entry of the matrix and the coefficient of the generating functions we shall use the term *entry* to refer to those matrix coefficients that are handled explicitly. A convenient way to store a sparse matrix is to store its set of non zero entries in the form of a set of triples $(d_{h\ell}, h, \ell)$, where $d_{h\ell}$ is associated with the entry in row h and column ℓ and the triples are held as three integer arrays. Since the discussion in this section is only concerned with matrix operations, we abstract from any meaning of h and ℓ and start counting them from one such that $h, \ell \geq 1$. The iteration processes of the previous sections demand a scheme that allows for easy inserting of new entries. This is performed by a data structure based on row-linked lists: the matrix is stored as a collection of rows, each in a linked list. The essence of such a listing scheme is that for each row h there is a pointer (head-pointer)

[1] For example, this is the case when dealing with the *inverse problem* of calculating voting weights so as to fit a given power distribution. A typical routine starts with an initial weight distribution which is repeatedly updated.

to the first entry and each entry is accompanied by another pointer (row-link) which points to the next entry of the hth row or is null if it is the last entry of row h.

As an example consider the matrix in Figure 11.6 (zero coefficients are left out). This matrix might have the representation as in Table 11.7.

Figure 11.6

$h\backslash\ell$	1	2	3	4	5	6	7	8
1	1							
2	2	1						
3	2	2	2					
4	2	1						
5	1							
6								
7								
8								

Here, the links have been constructed such that the rows are scanned in column order. The array $HEAD$ works as a head-pointer and the array $LINK$ points to the next row-element. The arrays COL and VAL store the column number and entry. To illustrate the use of this scheme take the fourth row of Figure 11.6. $HEAD(4) = 7$ says that the first entry in row 4 can be found in column 7 of Table 11.7. Further, $COL(7) = 1$ means that it is in the $(4, 1)$ position and has value $VAL(7) = 2$. From $LINK(7) = 8, COL(8) = 2, VAL(8) = 1$ follows that the next entry of row 4 can be found in column 8 of Table 11.7 and is in the $(4, 2)$ position with value 1. Since $LINK(8) = 0$ there are no further entries in row 4. The fact that the array $HEAD$ consists of only five elements indicates that the elements of row 5 to 8 of the matrix in Figure 11.6 are entirely zero.

It is an easy task to insert a new entry $d_{h\ell}$: the arrays COL and VAL are extended by ℓ and $d_{h\ell}$. $HEAD$ and $LINK$ have to be extended or modified such that they carry the information of the row index h as well as

Table 11.7

Subscripts	1	2	3	4	5	6	7	8	9
$HEAD$	1	2	4	7	9				
COL	1	1	1	1	2	3	1	2	1
VAL	1	2	1	2	2	2	2	1	1
$LINK$	0	3	0	5	6	0	8	0	0

the predecessor and successor of $d_{h\ell}$ in row h in column order. For example, for the insertion of d_{34} we set COL(10)=4, VAL(10)=d_{34}, $LINK(10) = 0$ (d_{34} is the last entry in row 3) and modify $LINK(6) = 10$ (the predecessor of d_{34} in row 3 is stored with subscript 6).

Note that at first sight it might seem that the newly introduced schemes are not so efficient, since in Table 11.7 there are many non-zero entries and more than in Figure 11.6. However, typically the number of zeros grows fast as the problem under consideration gets larger, and then the new method begins to pay off.

Chapter 12

Approximation Methods

The key idea of the prevalent approximation methods for large WVGs, binary and ternary, is to use the probabilistic interpretation in order to apply approximation tools of stochastics. In particular, the assumption that the voters vote independently (see Assumption 7.1) allows the interpretation of voting as a repeated independent random experiment (one voter after another) such that the central limit theorem can be applied. This suggests to use the normal distribution in order to approximate the Banzhaf measure, complaisance as well as jury competence. This also applies to the S-S index. Due to a result of Straffin, see for example his 1982, pp. 297–299, we know that the S-S index ϕ can be derived by a model in which the voters vote independently assuming that the affirming probability of any voter is uniformly distributed over $[0, 1]$.

Applying results of the CLT has proven to be a powerful approximation tool for large (binary) WVGs in a number of studies, see for example Owen (1975a, 1975b, 1995), Leech(1988, 1992, 2002b), Widgren (1994). Furthermore, the numerical extent is of only linear complexity since the methods basically require the computations of the first two moments of sums of independent random variables.

However, the literature treats this method heuristically, without specification of precise conditions proving the validity of the normal distribution as an approximation tool. Rigorous validation is not straightforward which is illustrated by the approximation formula of Penrose (2.1). In cases where the approximation is expected to hold, both the voting power of each voter and the term approximating it tend to 0 as the number of voters increases. In order to validate the approximation, it must be proved not only that the error term tends to 0 but that it does faster than the approximating term.

In other words, it must be proven that the relative error tends to 0. With respect to the Banzhaf measure, Corollary 3.15 and 4.12 give precise conditions proving the validity of the approximation based on the (local) CLT. Recall also that under the conditions of the PLT Theorems (as given by Theorem 3.4, Theorem 3.10 and Theorem 4.11), the normalized voting weight of any voter serves as a proxy for the corresponding S-S index and normalized Banzhaf measure.

In Part II we have seen that the uniform convergence condition (6.7) in Lemma 6.2 plays the key role in affirming the validity of using the normal distribution as approximations for global measures - which represents the main machinery in the proof of Theorem 7.2, Theorem 7.5 and Theorem 8.5. Finally, combining the convergence characteristics of the global measures (as discussed in Section 7.3) with the rates of convergence of the approximations provides an assessment of the quality of the approximation tools based on the integral form of the CLT as given by the Lindeberg-Feller Theorem 6.1.

Part IV

Appendix

Chapter 13

Tables

Table 13.1: QMV under $\mathcal{N}_{27}$

Country	(1) w	(2) $\bar{w}\,(\%)$	(3) $\beta\,(\%)$	(4) (3) : (2)	(5) $\phi\,(\%)$	(6) (5) : (2)
Germany	118	8.5199	7.7145	0.905	8.6799	1.019
UK	117	8.4477	7.7145	0.913	8.6670	1.026
France	117	8.4477	7.7145	0.913	8.6670	1.026
Italy	117	8.4477	7.7145	0.913	8.6670	1.026
Spain	108	7.7978	7.3732	0.946	7.9888	1.024
Poland	108	7.7978	7.3732	0.946	7.9888	1.024
Romania	56	4.0433	4.2771	1.058	3.9925	0.987
Netherlands	52	3.7545	3.9900	1.063	3.6866	0.982
Greece	48	3.4657	3.7092	1.070	3.3977	0.980
Czech Rep	48	3.4657	3.7092	1.070	3.3977	0.980
Belgium	48	3.4657	3.7092	1.070	3.3977	0.980
Hungary	48	3.4657	3.7092	1.070	3.3977	0.980
Portugal	48	3.4657	3.7092	1.070	3.3977	0.980
Sweden	40	2.8881	3.1126	1.078	2.8137	0.974
Bulgaria	40	2.8881	3.1126	1.078	2.8137	0.974
Austria	40	2.8881	3.1126	1.078	2.8137	0.974
Slovakia	28	2.0217	2.1984	1.087	1.9594	0.969
Denmark	28	2.0217	2.1984	1.087	1.9594	0.969
Finland	28	2.0217	2.1984	1.087	1.9594	0.969
Ireland	28	2.0217	2.1984	1.087	1.9594	0.969
Lithuania	28	2.0217	2.1984	1.087	1.9594	0.969
Latvia	16	1.1552	1.2603	1.091	1.1209	0.970
Slovenia	16	1.1552	1.2603	1.091	1.1209	0.970
Estonia	16	1.1552	1.2603	1.091	1.1209	0.970
Cyprus	16	1.1552	1.2603	1.091	1.1209	0.970
Luxembourg	16	1.1552	1.2603	1.091	1.1209	0.970
Malta	12	0.8664	0.9514	1.098	0.8310	0.959
Total	1 385	100.0001	100.0002		99.9997	

Quota: 1 034 = 74.66% of 1 385.

Note For explanations see Introduction to Part I.

Table 13.2: Rule B (benchmark QMV rule for enlarged CM)

Country	(1) w	(2) $\bar{w}$ (%)	(3) β (%)	(4) (3) : (2)	(5) ϕ (%)	(6) (5) : (2)
Germany	954	9.5381	9.6184	1.008	9.9894	1.047
UK	810	8.0984	8.1441	1.006	8.3359	1.029
France	809	8.0884	8.1338	1.006	8.3246	1.029
Italy	799	7.9884	8.0312	1.005	8.2122	1.028
Spain	661	6.6087	6.6219	1.002	6.6886	1.012
Poland	655	6.5487	6.5606	1.002	6.6232	1.011
Romania	499	4.9890	4.9813	0.998	4.9627	0.994
Netherlands	418	4.1792	4.1665	0.997	4.1221	0.982
Greece	342	3.4193	3.4050	0.996	3.3470	0.979
Czech Rep	338	3.3793	3.3649	0.996	3.3066	0.978
Belgium	337	3.3693	3.3549	0.996	3.2965	0.978
Hungary	335	3.3493	3.3349	0.996	3.2761	0.978
Portugal	333	3.3293	3.3149	0.996	3.2559	0.978
Sweden	313	3.1294	3.1151	0.995	3.0553	0.976
Bulgaria	302	3.0194	3.0051	0.995	2.9453	0.975
Austria	299	2.9894	2.9751	0.994	2.9153	0.975
Slovakia	245	2.4495	2.4365	0.995	2.3755	0.970
Denmark	243	2.4295	2.4166	0.995	2.3556	0.970
Finland	239	2.3895	2.3766	0.995	2.3157	0.969
Ireland	204	2.0396	2.0277	0.994	1.9706	0.966
Lithuania	203	2.0296	2.0176	0.994	1.9604	0.966
Latvia	164	1.6397	1.6299	0.994	1.5783	0.963
Slovenia	148	1.4797	1.4706	0.994	1.4223	0.961
Estonia	127	1.2697	1.2615	0.994	1.2175	0.959
Cyprus	91	0.9098	0.9042	0.994	0.8693	0.955
Luxembourg	69	0.6899	0.6854	0.993	0.6580	0.954
Malta	65	0.6499	0.6457	0.994	0.6200	0.954
Total	10 002	100.0000	100.0000		99.9999	

Quota: 6 000 = 59.99% of 10 002.

Note For explanations see Introduction to Part I.

Table 13.3: Rule C (benchmark QMV rule under $\mathcal{N}_{27}$ with abstention as a tertium quid)

Country	(1) w	(2) $\bar{w}$ (%)	(3) β (%)	(4) (3) : (2)	(5) ψ (%)	(6) ψ_{appr} (%)
Germany	118	8.5199	7.7542	0.910	7.4814	7.2218
UK	117	8.4477	7.7131	0.913	7.4417	7.1782
France	117	8.4477	7.7131	0.913	7.4417	7.1782
Italy	117	8.4477	7.7131	0.913	7.4417	7.1782
Spain	108	7.7978	7.3101	0.938	7.0529	6.7684
Poland	108	7.7978	7.3101	0.938	7.0529	6.7684
Romania	56	4.0433	4.2473	1.050	4.0978	3.8736
Netherlands	52	3.7545	3.9708	1.058	3.8311	3.6190
Greece	48	3.4657	3.6894	1.065	3.5596	3.3606
Czech Rep	48	3.4657	3.6894	1.065	3.5596	3.3606
Belgium	48	3.4657	3.6894	1.065	3.5596	3.3606
Hungary	48	3.4657	3.6894	1.065	3.5596	3.3606
Portugal	48	3.4657	3.6894	1.065	3.5596	3.3606
Sweden	40	2.8881	3.1128	1.078	3.0033	2.8324
Bulgaria	40	2.8881	3.1128	1.078	3.0033	2.8324
Austria	40	2.8881	3.1128	1.078	3.0033	2.8324
Slovakia	28	2.0217	2.2165	1.096	2.1385	2.0142
Denmark	28	2.0217	2.2165	1.096	2.1385	2.0142
Finland	28	2.0217	2.2165	1.096	2.1385	2.0142
Ireland	28	2.0217	2.2165	1.096	2.1385	2.0142
Lithuania	28	2.0217	2.2165	1.096	2.1385	2.0142
Latvia	16	1.1552	1.2862	1.113	1.2409	1.1675
Slovenia	16	1.1552	1.2862	1.113	1.2409	1.1675
Estonia	16	1.1552	1.2862	1.113	1.2409	1.1675
Cyprus	16	1.1552	1.2862	1.113	1.2409	1.1675
Luxembourg	16	1.1552	1.2862	1.113	1.2409	1.1675
Malta	12	0.8664	0.9692	1.119	0.9351	0.8795
Total	1 385	100.0001	100.0025			

Quota: 1 034 = 74.66% of 1 385.

Note For explanations see Section 4.4.

Table 13.4: US Presidential Electoral College (1970 Census)

	No.	w	$\bar{w}$ (%)	ϕ (%)	β (%)	ψ	ψ *approx*
	1	45	8.3643	8.8309	8.8816	0.379366	0.403527
	1	41	7.6208	7.9727	7.9513	0.339629	0.359921
	1	27	5.0186	5.0963	5.0457	0.215522	0.224441
	2	26	4.8327	4.8977	4.8499	0.207159	0.215510
	1	25	4.6468	4.6999	4.6553	0.198844	0.206653
	1	21	3.9033	3.9169	3.8865	0.166007	0.171900
	2	17	3.1599	3.1466	3.1308	0.133730	0.138057
	1	14	2.6022	2.5767	2.5708	0.109809	0.113150
	2	13	2.4164	2.3882	2.3852	0.101879	0.104923
	3	12	2.2305	2.2004	2.2000	0.093970	0.096728
	1	11	2.0446	2.0133	2.0152	0.086078	0.088564
	4	10	1.8587	1.8270	1.8308	0.078202	0.080426
	4	9	1.6729	1.6413	1.6468	0.070341	0.072314
	2	8	1.4870	1.4563	1.4631	0.062493	0.064224
	4	7	1.3011	1.2719	1.2796	0.054656	0.056153
	4	6	1.1152	1.0883	1.0964	0.046830	0.048100
	1	5	0.9294	0.9053	0.9133	0.039012	0.040061
	9	4	0.7435	0.7230	0.7305	0.031201	0.032034
	7	3	0.5576	0.5413	0.5477	0.023396	0.024017
Total	51	538	99.9998	100.0009	100.0005		

Quota: 270 = 50.19% of 538.

Note For the purpose of this table, the Electoral College is regarded as a WVG, in which each 'voter' is a bloc of Electors for a State, or for the District of Columbia. The number of Electors in each bloc is taken as the weight w of this bloc-voter. The first column, headed 'No.', shows the number of blocs with a given weight w. This way of modelling the Electoral College involves some over-simplification, because there may be more than two candidates, and since 1969 the Electors of Maine did not have to vote as a single bloc. (Since 1993, the same applies to Nebraska.) We use this model here for the sake of computational illustration, and for comparison with Table XII.4.1 of Owen (1995, p. 297), which is based on the same model. For further explanations, see Remark 3.12.

Table 13.5: Election of an executive director in the IMF with abstention as a tertium quid

	(1)	(2)	(3)	(4)	(5)	(6)
Country	w	$\bar{w}\,(\%)$	$\beta\,(\%)$	(3) : (2)	ψ	$\psi approx$
Armenia	1 170	1.1099	0.9148	0.824	0.016478	0.020336
Bosnia & Herzegovina	1 941	1.8413	1.5275	0.830	0.027514	0.033743
Bulgaria	6 652	6.3105	5.3210	0.843	0.095841	0.116010
Croatia	3 901	3.7007	3.0428	0.822	0.054808	0.067877
Cyprus	1 646	1.5615	1.2850	0.823	0.023145	0.028612
Georgia	1 753	1.6630	1.3824	0.831	0.024900	0.030473
Israel	9 532	9.0426	7.8489	0.868	0.141374	0.166853
Macedonia	939	0.8908	0.7403	0.831	0.013334	0.016320
Moldova	1 482	1.4059	1.1684	0.831	0.021045	0.025760
Netherlands	51 874	49.2107	55.5154	1.128	0.999944	0.996588
Romania	10 552	10.0102	8.7239	0.872	0.157135	0.185009
Ukraine	13 970	13.2528	12.5296	0.945	0.225683	0.246580
Total	105 412	99.9999	100.0000			

Quota: 52 707 = 50.00% of 105 412.

Note The voting weights are taken from the IMF Annual Report for 2002 and are designed to elect a director for the Executive Board. Currently, there are twenty four executive directors: five are appointed by the five member countries with the largest financial distribution and nineteen are elected by groups (at present, the country of the elected director of the group in Table 13.5 is the Netherlands). There is a minimum and a maximum percentage of the eligible votes that a nominee must receive in order to be elected (apparently, the duty of the maximum percentage is to prevent too great disparities in the voting strength among the directors). However, for the sake of computational illustration of both PLT for TWVGs and storage schemes designed for sparse matrices the ternary weighted voting rule as well as the 50% quota applies here (for a detailed account of the constitution of the IMF, see for example Gold 1972 or Leech 2002b). The tendency of the generalized Bz index β to approximate to the respective relative weights is weaker than in Table 13.3 due to a highly skewed weight distribution.

Chapter 14

Miscellaneous

Lemma 14.1 *Let $\alpha_1, ..., \alpha_{m^{(\nu)}}$ be positive numbers totalling α. If $\alpha_{max}^{(\nu)} \to 0$ as $\nu \to \infty$ then the sum of the squares tends to zero, i.e.*

$$\lim_{\nu\to\infty} \sum_{k \le m^{(\nu)}} (\alpha_k^{(\nu)})^2 = 0. \tag{14.1}$$

Proof Take $\epsilon \in (0,1)$. Then there is a ν^* such that $\alpha_{max}^{(\nu)} < \epsilon^2$ for all $\nu \ge \nu^*$. Take integers $t_1 \le t_2 \le ... \le t_{\lceil \epsilon^{-1} \rceil} = m^{(\nu)}$, such that for all $j \in \{1, 2, ..., \lceil \epsilon^{-1} \rceil\}$

$$\sum_{k=t_j+1}^{t_{j+1}} \alpha_k^{(\nu)} \le (\epsilon + 2\epsilon^2)\alpha. \tag{14.2}$$

Hence

$$\sum_{k=t_j+1}^{t_{j+1}} \left(\alpha_k^{(\nu)}\right)^2 \le \left(\epsilon + 2\epsilon^2\right)^2 \alpha^2$$

such that

$$\begin{aligned} \sum_{k \le m^{(\nu)}} (\alpha_k^{(\nu)})^2 &\le \sum_{j=1}^{\lceil \epsilon^{-1} \rceil} \sum_{k=t_j+1}^{t_{j+1}} \left(\alpha_k^{(\nu)}\right)^2 \le \sum_{j=1}^{\lceil \epsilon^{-1} \rceil} \left(\epsilon + 2\epsilon^2\right)^2 \\ &= \left\lceil \epsilon^{-1} \right\rceil \cdot \left(\epsilon + 2\epsilon^2\right)^2 \le \\ &\le \left(\frac{1}{\epsilon} + 1\right) \left(\epsilon^2 + 4\epsilon^3 + 4\epsilon^4\right) = \mathcal{O}\left(\epsilon\right). \end{aligned} \tag{14.3}$$

□

Chapter 15

Source Code

%POWERSPARSE: calculates coefficient matrix using row linked scheme,
calls module UPDATE and SWINGCHECK

```
w=[1170,1941,6652,3901,1646,1753,9532,939,1482,51874,10552,13970]
%w=[118,117,117,117,108,108,56,52,48,48,48,48,48,40,40,40,28,28,28,28,28,16,16,16,16,16,12];

sum(w) q=0.5; time=clock;

%******************* Initialisation ***************
n=length(w);
x=zeros(1,n);
w0=w;

%****************** Calculation of the Coefficient Matrix *********************

for a=1:1:n
    w=w0;
    if a>1 & w(a)==w(a-1)
        x(a)=x(a-1);
    else
        wa=w(a);
        w(a)=[];
        %*********** Calculate Coefficient Matrix with w reduced by wa
        HEAD=1;
        COL=1;
        VAL=1;
```

```
NEXT=0;

for m=1:1:n-1
   time=[time;clock];
   HEADold=HEAD;
   COLold=COL;
   VALold=VAL;
   NEXTold=NEXT;
   for i=1:1:length(HEADold)
       pos=HEADold(i);
       if ~(pos==0)      %row i has an entry in the scheme
          h=i+w(m);
          l=COLold(pos);
          dhl=VALold(pos);
          update


          h=i;
          l=COLold(pos)+w(m);
          update

          while ~ (NEXTold(pos)==0) %go through row i in col. order
             pos=NEXTold(pos);
             h=i+w(m);
             l=COLold(pos);
             dhl=VALold(pos);
             update

             h=i;
             l=COLold(pos)+w(m);
             update
          end
       end
    end
end

%************ Counting the Swings ****************************

  for i=1:1:length(HEAD)
     pos=HEAD(i);
     if ~(pos==0) %row i has an entry in the scheme
```

```
            h=i;
            l=COL(pos);
            dhl=VAL(pos);
            swingcheck

            while ~(NEXT(pos)==0) %go through row i in col. order
               pos=NEXT(pos);
               l=COL(pos);
               dhl=VAL(pos);
               swingcheck

            end
         end
      end


   %*********************
   end
%*****************
end
```

%UPDATE MODULE: Inserts triple (dhl,h,l) into row-linked scheme if position (h,l) does not yet exists as an entry, adds dhl to entry at position (h,l) otherwise

```
rows=length(HEAD);
if (h>rows) %row h enters the scheme with dhl as the only entry in row h and HEAD enlarges
   new=length(VAL)+1;
   if h-rows-1>=0
      FILL=zeros(1,h-rows-1);
      HEAD=[HEAD,FILL,new];
   else
      HEAD(h)=new;
   end
   NEXT=[NEXT,0];
   COL=[COL,l];
   VAL=[VAL,dhl];
else
   if HEAD(h)==0 %row h enters the scheme with dhl as the only entry in row h
```

```
      new=length(VAL)+1;
      HEAD(h)=new;
      NEXT=[NEXT,0];
      COL=[COL,l];
      VAL=[VAL,dhl];
   else
      if COL(HEAD(h))>l %dhl is the new first entry of row h in column order
         new=length(VAL)+1;
         NEXT=[NEXT,HEAD(h)];
         HEAD(h)=new;
         COL=[COL,l];
         VAL=[VAL,dhl];
      else %find entry in row h with column index smaller than or equal l
         predpos=HEAD(h);
         checkpos=NEXT(predpos);
         while ~(checkpos==0) & COL(checkpos)<=l
            predpos=checkpos;
            checkpos=NEXT(predpos);
         end
         if COL(predpos)==l %update existing entry at position (h,l)
            VAL(predpos)=VAL(predpos)+dhl;
         else %entry at position (h,COL(predpos)) is the predecessor of new entry dhl in column order
            new=length(VAL)+1;
            NEXT(new)=NEXT(predpos);
            NEXT(predpos)=new;
            VAL=[VAL,dhl];
            COL=[COL,l];
         end
      end
   end
end
```

```
%SWINGCHECK: Checks triple (h,l,dhl) for possible swing contribution; adds dhl to number of swings if h,l
```

```
Q=round(q*(sum(w0)-(l-1)));
if Q<q*(sum(w0)-(l-1))
   Q=Q+1;
```

```
end

if (h-1)>=(Q-wa) & (h-1)<Q
    x(a)=x(a)+dhl;
end
```

Chapter 16

Basic Notations and Abbreviations

Basic Notations

c absolute quota

p probability of picking the right choice in a dichotomous choice situation

q relative quota

t probability of abstaining

v characteristic function

w voting weight

A complaisance

C jury competence/ group judgemental accuracy

L major voters

M minor voters

N assembly

R resistance

$\mathcal{W}$ simple voting game

α combined voting weight of minor voters

β Banzhaf index

θ probability of a defendant being guilty

ϕ Shapley Shubik index

ψ Penrose/Banzhaf measure

List of Abbreviations

Bz – Banzhaf

CM – Council of Ministers

EU – European Union

IMF – International Monetary Fund

PI – Power index

S-S – Shapley-Shubik

SVG – simple voting game

TVG – ternary voting game

PLT – Penrose's Limit Theorem

WVG – weighted voting game

TWVG – ternary weighted voting game

UNSC – United Nations Security Council

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Abstract

This thesis deals with the asymptotic properties of weighted voting games (WVGs) when there are many ‘small’ voters. It comprises three main parts.

Part I is concerned with *Penrose's Limit Theorem* (PLT). This research goes back to a presumption of L.S. Penrose concerning an asymptotic property of some sequences of WVGs with an increasing number of voters: under certain conditions the ratio between the voting power of any two voters (according to various measures of voting powers) approaches the ratio of their weights. So far there has been no rigorous proof of PLT for any non-trivial class of cases and counterexamples to Penrose's claim can be constructed. Part I introduces the concept of *q-chains* of weighted voting games and considers the question whether for a given q-chain and a given power index the PLT holds true. It provides sufficient conditions for the two most prominent power indices - the Shapley-Shubik and the Banzhaf index. The main result with respect to the Shapley-Shubik index (Theorem 3.4) states that given a *non-atomic* q-chain PLT holds for those voters for which the chain is *replicative*. Also, the PLT is proved with respect to the Banzhaf index for an important class of WVG-sequences with quota 1/2 (Theorem 3.13). Finally, the thesis contains an analogue of the last mentioned result for weighted decision rules that admit abstention as a *tertium quid* (Theorem 4.13).

Part II is concerned with the asymptotic behaviour of some global quantities of WVGs. Here, the setup is such that there are two kinds of voters: a fixed (possibly empty) set of *major voters* with fixed weights (the atomic part), and a growing population of *minor voters* with weights converging uniformly to zero (the non-atomic part). The question under consideration is what happens when the number of minor voters tends to infinity. First, the analysed quantity is *complaisance* introduced by J.S. Coleman in 1971 as the 'power of a collectivity to act'. Here, the the decision making body in binary WVGs (Theorem 7.2) and ternary WVGs (Theorem 7.5) is considered as a 'preference-aggregating machine'. Second, decision-making is

assumed as 'truth-tracking' such that there is a 'right answer' but the voters only have partial information and imperfect competence for detecting the truth. The *Condorcet jury theorem* considers a quantity called the *collective competence*, i.e. the probability of the decision-making body to arrive at the correct decision. This part of the thesis extends the celebrated theorem to general q-chains (Theorem 8.5).

Part III develops numerical methods for computing the quantities considered in Part I and II. The standard approach of evaluating WVGs exactly is the method of *generating functions* known from combinatorics, however, it shows insurmountable demand in storage in large WVGs. Part III shows how to overcome this difficulty by using methods designed for sparse matrices (Section 11.5 and source code in Chapter 15). Chapter 12 establishes a foundation of the widespread but merely heuristically stated approximation methods for WVGs by proving the validity of the normal distribution as an approximation tool.

Zusammenfassung - German Abstract

Die Dissertation befaßt sich mit den asymptotischen Eigenschaften von gewichteten Abstimmungsspielen mit großer Anzahl von 'kleinen' Spielern. Die Arbeit besteht aus drei Hauptteilen.

Teil I handelt von *Penrose's Limit Theorem* (PLT). Die Arbeit greift eine Vermutung von L.S. Penrose auf, welche besagt, daß unter bestimmten Bedingungen der Quotient der Machtindizes zweier Spieler gegen den Quotient ihrer jeweiligen Abstimmungsgewichte konvergiert. Diese Vermutung gilt für zahlreiche Machtmaße. Sie war bisher unbewiesen, und es lassen sich Gegenbeispiele zu Penroses Vermutung finden. Teil I führt die Definition einer q-Kette ein und diskutiert die Frage, ob PLT für eine gegebene q-Kette und einen gegebenen Machtindex gilt. Die Arbeit entwickelt hinreichende Bedingungen für die beiden klassischen Machtindizes: den Schapley-Shubik und den Banzhaf Index. Das Haupttheorem bezüglich des Shapley-Shubik Index besagt, daß in *nicht-atomaren* Ketten PLT für Wähler gilt, für die die Kette *replikativ* ist (Theorem 3.4). Bezüglich des Banzhaf Index wird PLT für eine wichtige Klasse von Ketten mit Quote $q = 1/2$ bewiesen (Theorem 3.13). Weiterhin baut die Arbeit analog das zuletzt genannte Resultat für gewichtete Abstimmungsspiele aus, welche Enthaltung modellieren (Theorem 4.13).

Teil II handelt vom asymptotischen Verhalten einiger globaler Maße von gewichteten Abstimmungsspielen. Hier unterscheidet das Grundmodell zwei Arten von Spielern: eine gegebene (möglicherweise leere) Menge von *Hauptwählern* mit fixen Abstimmungsgewichten (der atomare Teil), sowie eine wachsende Population von 'kleinen' Wählern, deren Gewichte gleichmäßig zu null konvergieren (der nicht-atomare Teil). Die zentrale Frage ist was passiert, wenn die Anzahl der kleinen Spieler gegen unendlich geht. Zunächst untersucht die Arbeit ein Maß namens 'complaisance', welches 1971 von J.S.

Coleman als die 'Macht einer Kollektivität zu handeln' eingeführt wurde. Dieser Ansatz interpretiert das Entscheidungsorgan als eine 'Maschine', welche Präferenzen aggregiert (siehe Theorem 7.2 und 7.5 für binäre bzw. ternäre gewichtete Abstimmungsspiele). Anschließend wird die Beschlußfassung als 'Wahrheitsaufspürung' interpretiert: es gibt eine richtige Entscheidung, jedoch besitzen die Spieler nur einen Teil der Information und weisen imperfekte Entscheidungskompetenzen auf. Das berühmte *Condorcet Jury Theorem* betrachtet ein Maß der 'kollektiven Kompetenz', welches die Wahrscheinlichkeit eines Entscheidungsorgans darstellt, eine richtige Entscheidung zu treffen. Teil II der Arbeit erweitert das klassische Theorem zu allgemeinen q-Ketten (Theorem 8.5).
Teil III entwickelt numerische Methoden zur Errechnung der Maße, welche in Teil I und II diskutiert wurden. Die *Methode der erzeugenden Funktionen* der Kombinatorik ist Standard für die exakte Auswertung von gewichteten Abstimmunsspielen, welche bei großen Abstimmungsspielen jedoch einen unüberwindlichen Speicheraufwand verlangt. Teil III zeigt, wie Speichermethoden für dünn besetzte Matritzen dieses Problem lösen kann (Abschnitt 11.5, Quellcode in Kapitel 15). Kapitel 12 schafft eine Fundamentierung der verbreiteten, bisher jedoch nur heuristisch gehandhabten Approximationsmethoden für gewichtetete Abstimmungsspiele.

Curriculum Vitae

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L.S. Penrose's Limit Theorem: Proof of Some Special Cases (mit Moshé Machover), *Mathematical Social Sciences*, 2004, Vol. 47(1), 37-49.

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