Threshold Results for Cycles

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CHAPTER 1

Introduction

This thesis splits into two parts concerning two problems in extremal combinatorics. In the first part we analyse thresholds for Ramsey-type properties in random discrete structures (see Theorem 5 and Theorem 8). In the second part we consider a generalisation of Dirac's theorem on Hamiltonian cycles to hypergraphs (see Theorem 12). For the basic notation which is not defined here we refer to the textbooks by Diestel [**10**], Bollobás [**4**], and Bondy and Murty [**5**].

§1.1. Sharp Thresholds

1.1.1. Thresholds for Random Graphs. In Part 1 we consider a question on random discrete structures, in particular, in Chapter 3 on random graphs. Random graph theory has its origin in the 1940s. In one of the first applications of random graphs Erdős [**12**] proved the existence of a certain combinatorial object for which no constructive proof is known until now.

Throughout the years the systematic study of random graphs grew into a field within graph theory on its own. While random graphs were initially used as a tool to prove existence results, Erdős and Rényi studied in a series of papers starting in 1959 [**13**] random graphs as objects themselves. Their paper [**14**] from 1960 is considered one of the most important ones on random graphs. There they investigate the so-called *evolution of the random graph*, i.e. they analyse how the structure of a random graph changes if the density increases, e.g. they contribute to answer questions such as "when is the random graph connected". In general the most common question is "for which density does the random graph with high probability satisfy a given property". For an overview about classic results we refer to the books by Bollobás [**3**] and by Janson, Łuczak, and Ruciński [**32**]. Nowadays random graph theory is an established branch of discrete mathematics lying at the focal point of graph theory, combinatorics, and probability theory.

In this thesis we use the binomial random graph model $G(n,p)$ which considers graphs with vertex set $[n] = \{1,\dots,n\}$ where each edge appears independently with probability p. Formally $G(n,p)$ can be constructed by the following random procedure. Let Ω_n be the set of all graphs with vertex set $[n]$, let $\mathcal{P}(\Omega_n)$ be the powerset of Ω_n, and let $\mathbb{P}_{G(n,p)}$ be a probability measure such that for each graph $G \in \Omega_n$ holds $\mathbb{P}_{G(n,p)}(\{G\}) = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}$. Then $G(n,p)$ is a short hand notation for the probability space $(\Omega_n, \mathcal{P}(\Omega_n), \mathbb{P}_{G(n,p)})$.

In this thesis we are interested in large graphs, which means we consider the behaviour of the random graph for $n \to \infty$. Moreover, in our case p depends on n and mostly in interesting cases $\lim_{n\to\infty} p(n) = 0$ holds. Let $\mathcal{A}_n \subseteq \Omega_n$ be a family of graphs with vertex set $[n]$ and let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$. We then call $\mathcal{A}$ a graph property, e.g. for $\mathcal{A}_n = \{G \in \Omega_n \colon G \text{ is connected}\}$ we obtain the property that a graph is connected (up to isomorphisms). In the following we will suppress the index n in $\mathcal{A}_n$ if it is clear which n is meant. We say that $G(n,p)$ satisfies $\mathcal{A}$ *asymptotically almost surely* (a.a.s.) if $\lim_{n\to\infty} \mathbb{P}(G(n,p) \in \mathcal{A}) = 1$, where we use the standard notation $\mathbb{P}(G(n,p) \in \mathcal{A})$ for the the term $\mathbb{P}_{G(n,p)}(\mathcal{A}_n)$.

A key concept in this area is the so-called *threshold function*, that is for a given property $\mathcal{A}$ a function $\hat{p} = \hat{p}(n)$ such that

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \in \mathcal{A}) = \begin{cases} 0, & \text{if } p = o(\hat{p})\,, \\ 1, & \text{if } p = \omega(\hat{p})\,. \end{cases}$$

This threshold function determines the critical value of p where the probability that $G(n,p)$ satisfies $\mathcal{A}$ “jumps” from zero to one. Note that a threshold function is not unique as any multiplication with a constant would also yield a threshold function. However, we talk about the threshold and often mean the order of magnitude of a threshold which is unique. Note also that the definition of threshold consists of two statements. We refer to the statement "$\lim_{n\to\infty} \mathbb{P}(G(n,p) \in \mathcal{A}) = 0$ if $p = o(\hat{p})$" as the 0-statement and to "$\lim_{n\to\infty} \mathbb{P}(G(n,p) \in \mathcal{A}) = 1$ if $p = \omega(\hat{p})$" as the 1-statement.

As it turns out, most "natural" graph properties have a threshold, for example it is easy to show that monotone graph properties have a threshold (see, e.g. [**22**]). Consequently, this leads to the question whether this result is sharp or in other words: Is it possible to improve $p = o(\hat{p})$ and $p = \omega(\hat{p})$?

In this sense we define a threshold to be *semi-sharp* if there are constants $C_1 \geqslant C_0 > 0$ such that

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \in \mathcal{A}) = \begin{cases} 0, & \text{if } p \leqslant C_0\hat{p}\,, \\ 1, & \text{if } p \geqslant C_1\hat{p} \end{cases}$$

and *sharp* if for all $\varepsilon > 0$

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \in \mathcal{A}) = \begin{cases} 0, & \text{if } p \leqslant (1-\varepsilon)\hat{p}\,, \\ 1, & \text{if } p \geqslant (1+\varepsilon)\hat{p}\,. \end{cases}$$

If a threshold is not sharp we call it *coarse*. For example the threshold for the property "$G(n,p)$ is connected" is a sharp threshold of order $\frac{\log n}{n}$, while for the property "$G(n,p)$ contains a triangle" it is $\frac{1}{n}$, coarse, and not even semi-sharp.

Concerning sharp thresholds only few results are known. The most important work in this area was done by Friedgut [**19**] who basically characterised the graph properties that do not have a sharp threshold as the ones that can be approximated by local properties. In Part 1, Chapter 3 we will investigate the sharpness of the threshold for some Ramsey-type properties of graphs. In Part 1, Chapter 4 we study the threshold for monochromatic Schur triples in colourings of random subsets of the integers.

1.1.2. Thresholds for Ramsey-type Properties. Ramsey theory is a branch of extremal combinatorics which started in 1930 with Ramsey's theorem [**42**]. The finite version states that for all $k, r \in \mathbb{N}$ there is $n_0 \in \mathbb{N}$ such that for all vertex colourings of the complete graph K_n with r colours, where $n \geqslant n_0$, there exists a monochromatic K_k. There are also some other versions including an infinite version, a version with edge colourings instead of vertex colourings or versions with more general graphs then cliques. We define the following common short notation $G \to (F)_r^e$ that means: For each edge colouring of G with r colours there is a monochromatic copy of F. When we talk about vertex colourings we use v instead of e, however, in this thesis we are mainly interested in edge colourings.

A common theme in recent years was the transfer of different results to a sparse random setting, for example this was done for the classical theorems of Ramsey, Turán and Szemerédi (see, e.g. [**8**, **25**, **46**, **52**]).

Here we are interested in the threshold behaviour for Ramsey properties of random graphs, in this thesis especially in the sharpness of a threshold for the case of edge colourings. For the case of *vertex colourings* we refer to [**23**], where it was shown that for strongly strictly balanced graphs F, i.e.

$$\forall F' \subsetneq F \text{ with } v(F') > 1 \colon \frac{e(F)}{v(F)-1} < \frac{e(F)-e(F')}{v(F)-v(F')},$$

the threshold for the property $G(n,p) \to (F)_r^v$ is sharp and it has order of magnitude $n^{-1/m_1(F)}$, where

$$m_1(F) = \max_{F' \subseteq F, v(F') \geqslant 2} \{e(F')/(v(F')-1)\}.$$

Heuristically a probability $p = \Theta\big(n^{-1/m_1(F)}\big)$ yields that in average a fixed vertex v should be contained in a constant number of copies of F. Then for $p = \omega(\hat{p})$ we expect many copies of F in the whole graph and consequently that somewhere there should exist a monochromatic copy of F. On the other side if $p = o(\hat{p})$

we expect only few copies of F which should make it possible to find an F-free colouring of $G(n,p)$.

For the case of *edge colourings* the order of magnitude changes. For star forests and an arbitrary number of colours r and $d = \Delta(F)$ the threshold coincides with the threshold for appearance of a vertex of degree $r(d-1)+1$ which is of order $n^{-1-\frac{1}{r(d-1)+1}}$ (see [**14**]). For other graphs which are not a star forest the threshold depends on the m_2-density

$$m_2(F) = \max\{d_2(F')\colon F' \subseteq F \text{ and } e(F') \geqslant 1\}\,,$$

where

$$d_2(F') = \begin{cases} \frac{e(F')-1}{v(F')-2}, & \text{if } v(F') > 2, \\ 1, & \text{if } F' = K_2\,. \end{cases} \tag{1}$$

In [**46**] Rödl and Ruciński proved the following semi-sharp behaviour of the threshold (parts of the theorem had been shown before, see also [**17**], [**40**] and [**45**]).

Theorem 1 (Rödl & Ruciński [**46**]). *For all $r \geqslant 2$, for all graphs F that are not a star forest the function $\hat{p} = \hat{p}(n) = n^{-1/m_2(F)}$ is the threshold for the property $G(n,p) \to (F)^e_r$. In fact, there exist constants $C_1 \geqslant C_0 > 0$ such that*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (F)^e_r) = \begin{cases} 0, & \text{if } p \leqslant C_0 n^{-1/m_2(F)}, \\ 1, & \text{if } p \geqslant C_1 n^{-1/m_2(F)}. \end{cases} \qquad \square$$

Note that $p = \Theta(n^{-1/m_2(F)})$ yields a similar behaviour as $p = \Theta(n^{-1/m_1(F)})$ in the context of vertex colourings. We expect for each fixed edge e a constant number of copies of F containing e.

Overall, the order of magnitude for edge colourings is known for all graphs F and for all number of colourings $r \geqslant 2$. In contrast to this the question whether the threshold is sharp seems to be much more complicated compared to the vertex colouring case and only few results were proved.

1.1.3. Sharp Thresholds for Ramsey Properties of Random Graphs. In the before mentioned paper [**23**] Friedgut and Krivelevich showed that the threshold for most trees and an arbitrary number of colours $r \geqslant 2$ is sharp.

THEOREM 2 (Friedgut & Krivelevich [**23**]). *For all $r \geqslant 2$, for every tree T which is not a star and in case of $r = 2$ not P_4 (a path of three edges), there exist constants $C_1 \geqslant C_0 > 0$ and a function $p(n) = c(n)n^{-1/m_1(F)}$ with $C_0 \leqslant c(n) \leqslant C_1$ such that*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (T)^e_r) = \begin{cases} 0, & \text{if } p \leqslant (1-\varepsilon)p(n), \\ 1, & \text{if } p \geqslant (1+\varepsilon)p(n). \end{cases} \qquad \square$$

In another involved paper Friedgut et al. showed that the threshold for a triangle K_3 (the complete graph on three vertices) and two colours is sharp.

THEOREM 3 (Friedgut, Rödl, Ruciński & Tetali [**24**]). *There exist positive constants c_0 and c_1 and a function $c(n)$ with $c_0 < c(n) < c_1$ such that for all $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (K_3)^e_2) = \begin{cases} 0, & \text{if } p \leqslant (1-\varepsilon)c(n)n^{-1/2}\,, \\ 1, & \text{if } p \geqslant (1+\varepsilon)c(n)n^{-1/2}\,. \end{cases} \qquad \square$$

Recently Friedgut, Hàn, Person and Schacht [**21**] developed in a paper on arithmetic progressions in random subsets of the integers a method that works to prove the sharpness for all bipartite graphs and two colours. We use this method to show the following extension of [**24**] for arbitrary cycles. In particular, we obtain a shorter proof of their theorem.

THEOREM 4 (Schacht & Sch.). *For a cycle C_k of length $k \geqslant 3$ there exist positive constants c_0 and c_1 and a function $c(n)$ with $c_0 < c(n) < c_1$ such that for all $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (C_k)^e_2) = \begin{cases} 0, & \text{if } p \leqslant (1-\varepsilon)c(n)n^{-(k-2)/(k-1)}, \\ 1, & \text{if } p \geqslant (1+\varepsilon)c(n)n^{-(k-2)/(k-1)}. \end{cases}$$

In fact, our proof also works for a more general class of graphs. We say a graph F is *nearly bipartite* if $e(F) \geqslant 2$ and there is a bipartite graph F' and some edge e such that $F = F' + e = (V(F'), E(F') \cup \{e\})$. Related to the definition of m_2-density we call a graph *strictly balanced* if $d_2(F') < d_2(F)$ for all $F' \subsetneq F$. Note that all cycles are nearly bipartite since removing one edge yields a bipartite graph, and they are strictly balanced. There are also other strictly balanced and nearly bipartite graphs, for example there exist such graphs which result from a cycle by adding some cords. Our main theorem which also implicates Theorem 4 is the following.

THEOREM 5 (Schacht & Sch.). *For all strictly balanced and nearly bipartite graphs F there exist positive constants c_0 and c_1 and a function $c(n)$ with $c_0 < c(n) < c_1$ such that for all $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}(G(n,p) \to (F)^e_2) = \begin{cases} 0, & \text{if } p \leqslant (1-\varepsilon)c(n)n^{-1/m_2(F)}, \\ 1, & \text{if } p \geqslant (1+\varepsilon)c(n)n^{-1/m_2(F)}. \end{cases}$$

We will give the proof of Theorem 5 in Chapter 3. Similar to the proofs in [**24**] and [**21**] the proof starts with a result by Friedgut and Bourgain given in [**19**] which yields a characterisation of graph properties with a coarse threshold (remember that coarse equals not sharp) as properties that can be approximated by local properties. A second main tool that we use is the recent hypergraph container theorem by Saxton and Thomason and by Balogh, Morris, and Samotij ([**51**], [**1**]). We will present these main tools as well as some concentration results of probability theory in Chapter 2.

1.1.4. Sharp Threshold for Monochromatic Schur Triples. In Ramsey theory we are not only interested in graphs but also in other discrete structures. For instance we can ask for integer value solutions of linear equations. Here one prime example is the following theorem by van der Waerden.

Theorem 6 (van der Waerden 1927 [**58**]). *For all integers $r \geqslant 1$ and $k \geqslant 1$ the following holds. For any partition $E_1 \cup \ldots \cup E_r = \mathbb{N}$ of the natural numbers there exists some $j \in [r]$ such that E_j contains an arithmetic progression of length k, that means there exist $a, \lambda \in \mathbb{N}$ with $\lambda > 0$ such that $a + i\lambda \in E_j$ for all $i = 0, \ldots, k-1$.* □

We can consider arithmetic progressions as solutions of special linear equations, for example a triple (x, y, z) with $y = x + \lambda$ and $z = x + 2\lambda$ is a solution of the equation $x + z = 2y$. The same can be done for longer arithmetic progressions, in general an arithmetic progression of length k is a solution $\vec{x} = (x_1, \ldots, x_k)$ of the equation system $x_i + x_{i+2} = 2x_{i+1}$ for all $i = 1, \ldots, k-2$. Clearly this can be expressed as a set of solutions of a homogeneous linear equation system given by a matrix A and more generally we can ask for solutions of such an equation system for arbitrary integer value matrices A.

We are also mainly interested in "non degenerated" solutions such that $x_i \neq x_j$ for all $i \neq j$, which leads to the following definition. A matrix A is called *irredundant* if there exists a solution $\vec{x} = (x_1, \ldots, x_k)$ of $A\vec{x} = 0$ with $x_i \neq x_j$ for all $i \neq j$. These solutions are called *proper solutions*.

Rado [**41**] found a characterisation when there are monochromatic solutions of $A\vec{x} = 0$. A Matrix A is called *partition regular* if for arbitrary $r \in \mathbb{N}$ and sufficiently large n every partition of $[n]$ into r classes has a partition class that contains a proper solution of $A\vec{x} = 0$.

All of these definitions and concepts can be done for $[n]$ as well as for $\mathbb{Z}_n$, the quotient ring $\mathbb{Z}/n\mathbb{Z}$. A related question is the following where we consider arbitrary subsets of $[n]$ respectively of $\mathbb{Z}_n$ of linear size instead of partitions. Which matrices A satisfy that for all $\varepsilon > 0$ and sufficiently large n every subset of $[n]$ respectively $\mathbb{Z}_n$ of size at least εn contains a proper solution of $A\vec{x} = 0$? Matrices that satisfy this condition are called *density regular* and it turns out that for example the matrix corresponding to arithmetic progressions of length k is density regular.

Another natural linear equation is given by the matrix $A_{ST} = \begin{pmatrix} 1 & 1 & -1 \end{pmatrix}$ which characterises triples (x, y, z) such that $x + y = z$. We call the set

$$\{(x, y, z) \in [n]^3 \colon y + z = x \quad \text{or} \quad x + z = y \quad \text{or} \quad x + y = z\}$$

(and analogue for $\mathbb{Z}_n$ instead of $[n]$) the set of Schur triples and Schur proved in [**54**] that for all $r \in \mathbb{N}$ for every partition of $[n]$ into r classes there exist monochromatic proper Schur triples providing n is sufficiently large. In other words: On the one hand he showed that A_{ST} is partition regular. On the other hand it is clear that A_{ST} is not density regular since for example the set of all odd numbers up to n does not contain a Schur triple. Note that in this thesis a triple (x, y, z) is called a Schur triple if one element is the sum of the other two, independently of the position of the elements in the triple.

Similar to questions in graph theory, theorems about solutions of linear equations had been transferred to a random setting that means to $[n]_p$ or $\mathbb{Z}_{n,p}$, where for some finite set Γ we denote by $[\Gamma]_p$ the binomial random subset such that each element of Γ is contained in $[\Gamma]_p$ independently with probability p, and we use the short hand notation $\mathbb{Z}_{n,p} = [\mathbb{Z}_n]_p$. As noted before we are interested in the threshold behaviour which in some sense is determined by the “densest subset” of a solution, similar to the m_1-density in case of vertex colourings for graphs or the m_2-density in the case of edge colourings.

For an $\ell \times k$ matrix A and a partition $W \cup \overline{W} = [k]$ of the columns of A, we denote by A_W the matrix obtained from A by restricting to the columns indexed by W. Then we define the density m_A of A by

$$m_A = \max_{W \cup \overline{W} = [k], |W| \geqslant 2} \frac{|W| - 1}{|W| - 1 + \operatorname{rank}(A_{\overline{W}}) - \operatorname{rank}(A)},$$

where we use $\operatorname{rank}(A_{\overline{W}}) = 0$ for $\overline{W} = \varnothing$.

We define the arrow notation also for random subsets of integers, that means for a subset $X \subseteq [n]$ or $X \subseteq \mathbb{Z}_n$ we write $X \to (A)_r$ if for any partition of X into r classes there is a proper solution of $A\vec{x} = 0$ contained in one partition class.

In [**47**] Rödl and Ruciński showed that irredundant, density regular matrices have a semi-sharp threshold for the property $[n]_p \to (A)_r$ which is of order n^{-1/m_A}. Friedgut, Rödl and Schacht verified the same semi-sharp threshold of order n^{-1/m_A} for all irredundant, partition regular integer matrices (see [**25**]). The special case for Schur triples and two colours was shown in [**26**] before, where as short hand notation we write ST instead of A_{ST}.

THEOREM 7 (Graham, Rödl & Ruciński [**26**]). *There exist constants $C_1 \geqslant C_0 > 0$ such that the following holds.*

$$\lim_{n\to\infty} \mathbb{P}([n]_p \to (\mathrm{ST})_2) = \begin{cases} 0, & \text{if } p \leqslant C_0 n^{-1/2}\,, \\ 1, & \text{if } p \geqslant C_1 n^{-1/2}\,. \end{cases} \qquad \square$$

Recently Friedgut, Hàn, Person and Schacht [**21**] verified the first sharp threshold result in this area by showing that in case of two colours and arbitrary $k \in \mathbb{N}$ the threshold for $\mathbb{Z}_{n,p} \to (A_k)_2$ is sharp. The method used there also gives the, in the graph section mentioned, sharpness concerning bipartite graphs.

We adapted the proof for the case of two colours and Schur triples, a non density regular matrix. The main result in this section is the following.

THEOREM 8. *There exist positive constants c_0 and c_1 and a function $c(n)$ with $c_0 < c(n) < c_1$ such that for all $\varepsilon > 0$ we have*

$$\lim_{n\to\infty} \mathbb{P}(\mathbb{Z}_{n,p} \to (\mathrm{ST})_2) = \begin{cases} 0, & \text{if } p \leqslant (1-\varepsilon)c(n)n^{-1/2}, \\ 1, & \text{if } p \geqslant (1+\varepsilon)c(n)n^{-1/2}. \end{cases} \qquad \square$$

This proof also uses some of the main tools of Theorem 5 from Chapter 2. The details of the proof (which follows the same method as the proof in [**21**]) will be presented in the second half of Part 1, in Chapter 4.

§1.2. Forcing Hamiltonian Cycles in Hypergraphs

In Part 2 of this thesis we consider minimum degree conditions that guarantee the existence of Hamiltonian cycles in hypergraphs. A fundamental result in the theory of Hamiltonian cycles is Dirac's theorem from 1952.

Theorem 9 (Dirac [**11**]). *Every graph G on $n \geqslant 3$ vertices with minimum degree $\delta(G) \geqslant n/2$ contains a Hamiltonian cycle.* □

Here we want to investigate minimum degree conditions for hypergraphs. As there are several ways to generalise the notion of minimum degree or of cycle, we start with the following definitions. For a k-uniform hypergraph $\mathcal{H} = (V, E)$ and $1 \leqslant s \leqslant k-1$ let $S \in \binom{V}{s}$. We denote by $\deg(S)$ the number of edges that contain S and by $N(S)$ the neighbourhood of S, i.e. sets $T \in \binom{V}{k-s}$ such that $T \mathbin{\dot\cup} S \in E$, consequently $|N(S)| = \deg(S)$. We call

$$\delta_s(\mathcal{H}) = \min\{\deg S \colon S \in \binom{V}{s}\}$$

the s-minimum degree of $\mathcal{H}$. Note that an i-minimum degree condition yields also some useful information about j-minimum degree conditions if $j < i$ since every j-set is contained in some i-set. In general the opposite is not true.

Let $k \in \mathbb{N}$ and $1 \leqslant \ell \leqslant k-1$. An ℓ-cycle is a k-uniform hypergraph $\mathcal{C}_\ell$ if a cyclic ordering of its vertices exists such that every edge of $\mathcal{C}_\ell$ consists of k consecutive vertices, the intersection of two consecutive edges (order given by the contained vertices) is precisely ℓ, and every vertex is contained in at least one edge. Note that if $2\ell < k$ there are vertices which are contained in exactly one edge and vertices that are contained in exactly two edges, while for $\ell = k-1$ each vertex is contained in exactly k edges.

These definitions allow for different generalisations of Dirac's theorem depending on the choice of the minimum degree and the value of ℓ and a lot of work was accomplished over the last 20 years (see e.g. [**43**] and the references therein). The starting point in this area is a conjecture by Katona and Kierstead [**33**] that for all

$k \geqslant 3$ each k-uniform hypergraph $\mathcal{H}$ on n vertices with $\delta_{k-1}(\mathcal{H}) \geqslant (1/2 + o(1))n$ contains a Hamiltonian $(k-1)$-cycle. This conjecture was verified by Rödl, Ruciński and Szemerédi ([**48**], [**49**]) by introducing a new method, the so-called *absorbing method*. In the absorbing method one tries to find a special large cycle that contains almost all vertices of $\mathcal{H}$. Afterwards the cycle can be extended by local changes to a Hamiltonian cycle. The difficulty in proving derives from the necessary preparation for the properties of the almost spanning cycle, which allow to conclude the missing vertices. We will also use this method in our proof and explain the details in Part 2.

The same question can be asked for 1-cycles instead of $(k-1)$-cycles. What is the necessary degree condition? Kühn and Osthus [**39**] gave the answer for 3-uniform hypergraphs. Is $\delta_2(\mathcal{H}) \geqslant (1/4 + o(1))n$ then $\mathcal{H}$ contains a Hamiltonian 1-cycle. In a next step Hàn and Schacht gave in general an asymptotic version for arbitrary $k \in \mathbb{N}$ and ℓ-cycles with $1 \leqslant \ell < k/2$ in [**27**] (see also [**34**]). In this context asymptotic means that we can choose $\gamma > 0$ in the next theorem arbitrary small such that for sufficiently large n the degree condition is sufficient.

THEOREM 10 (H. Hàn & Schacht [**27**]). *For all integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$ and every $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in (k-\ell)\mathbb{N}$ and*

$$\delta_{k-1}(\mathcal{H}) \geqslant \left(\frac{1}{2(k-\ell)} + \gamma\right) n$$

contains a Hamiltonian ℓ-cycle. □

The case $\ell = k-2$ is solved in [**38**] by Kühn, Mycroft and Osthus. Recently J. Han and Zhao [**28**] improved the last theorem to a sharp version, i.e. they managed to remove γ. To get a feeling about the improvement from an asymptotic to a sharp version it is useful to look at the so-called *extremal case*.

Consider an extremal example (i.e. one that maximises the number of edges) of a k-uniform hypergraph $\mathcal{H}$ on n vertices that does not contain a Hamiltonian

ℓ-cycle. It is known that such hypergraph looks as follows (for simplicity assume $n \in 2(k-\ell)\mathbb{N}$). Take a set A of size $\frac{n}{2(k-\ell)} - 1$, a set B of size $n - \frac{n}{2(k-\ell)} + 1$, and let a hyperedge e be contained in $E(\mathcal{H})$ if and only if e contains at least one vertex from A. Then on one hand $\mathcal{H}$ does not contain a Hamiltonian ℓ-cycle since each edge has to contain at least one vertex from A. For $2\ell < k$ each vertex in an ℓ-cycle is contained in at most two edges and consequently it is not possible to find $(k-\ell)n$ edges that form a cycle such that each vertex contains an element from A. On the other hand this hypergraph satisfies $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-\ell)} - \frac{1}{n})n$, i.e. it is not possible to improve the degree condition to $\delta_{k-1}(\mathcal{H}) \geqslant (\frac{1}{2(k-\ell)} - \gamma)n$ for some $\gamma > 0$ and arbitrary hypergraphs.

The proof of the sharp result is split into two cases. Either a hypergraph is in some sense close to the extremal example shown in the last paragraph, which yields some structure to work with, or it is not close, in which case it is possible to use the asymptotic result by Hàn and Schacht. In Chapter 5, for this we define the notion of (ℓ, ξ)-extremal graphs that are graphs which contain a set B of size $\lfloor \frac{2(k-\ell)-1}{2((k-\ell)} n \rfloor$ with $e(B) \leqslant \xi \binom{n}{k}$. We can think of "$\mathcal{H}$ is close to the extremal example" as "$\mathcal{H}$ is (ℓ, ξ)-extremal for some small $\xi > 0$".

One of the natural questions in this field is to find s-minimum degree conditions for smaller s, so the next step is to deal with $s = k-2$. Buß, H. Hàn and Schacht proved the 3-uniform case, which was later extended to the sharp version by J. Han and Zhao [**29**].

THEOREM 11 (Buß, H. Hàn & Schacht [**6**]). *For all $\gamma > 0$ there exists an n_0 such that every 3-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in 2\mathbb{N}$ and*

$$\delta_1(\mathcal{H}) \geqslant \left(\frac{7}{16} + \gamma\right) n$$

contains a Hamiltonian 1-cycle. $\square$

The extremal example for this condition remains the same and one can calculate to which $(k-2)$-minimum degree condition this leads. A quick estimation yields the following for a set $S \subseteq B$ of size $k-2$.

$$\deg(S) = |A||B| + \binom{|A|}{2} = \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - o(1)\right)\binom{n}{2}.$$

It turns out that the corresponding minimum degree condition is sufficient. Our main result is the following asymptotic version.

Theorem 12 (Bastos, Mota, Schacht, Schnitzer & Sch.). *For all integers $k \geqslant 4$ and $1 \leqslant \ell < k/2$ and every $\gamma > 0$ there exists an n_0 such that every k-uniform hypergraph $\mathcal{H} = (V, E)$ on $|V| = n \geqslant n_0$ vertices with $n \in (k-\ell)\mathbb{N}$ and*

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} + \gamma\right)\binom{n}{2}$$

contains a Hamiltonian ℓ-cycle.

In fact we show the following version for non (ℓ, ξ)-extremal hypergraphs. Theorem 12 follows directly from Theorem 13 since for given ℓ, k, and γ we can choose ξ sufficiently small and n sufficiently large such that the degree condition in Theorem 12 prevents the graph to be (ℓ, ξ)-extremal.

Theorem 13 (Bastos, Mota, Schacht, Schnitzer & Sch.). *For any $0 < \xi < 1$ and all integers $k \geqslant 4$ and $1 \leqslant \ell < k/2$, there exists $\gamma > 0$ such that the following holds for sufficiently large n. Suppose $\mathcal{H}$ is a k-uniform hypergraph on n vertices with $n \in (k-\ell)\mathbb{N}$ such that $\mathcal{H}$ is not (ℓ, ξ)-extremal and*

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - \gamma\right)\binom{n}{2}.$$

Then $\mathcal{H}$ contains a Hamiltonian ℓ-cycle.

Moreover, a proof for the sharp version is in preparation at the moment. However, this will not be covered in this thesis. The proof of Theorem 13 will be presented in Part 2.

Part 1

Sharp Thresholds

CHAPTER 2

Main Tools

Large parts of Chapter 2 and almost all of Chapter 3 are based on the Article [**53**], joint work with Mathias Schacht. Chapter 4 follows the same proof strategy and uses the ideas from [**21**] as well as from [**53**].

In this chapter we present the main tools which are similarly used in both parts about sharp thresholds, in Chapter 3 about nearly bipartite graphs and in Chapter 4 about Schur triples. These tools are mainly Friedgut's and Bourgain's criterion to characterise coarse thresholds and the recently developed hypergraph container lemma by Balogh, Morris, and Samotij and by Saxton and Thomason. Furthermore, we will recall some standard probabilistic estimates.

§2.1. Friedgut's Criterion for Coarse Thresholds

In [**19**] Friedgut characterised graph properties with a coarse threshold as those properties that can be approximated by local ones. We use the following version from [**20**, Theorem 2.4], where for a graph B and $n \geqslant v(B)$ we define $\Psi_{B,n}$ as the set of all injective embeddings of B into the complete graph K_n.

Theorem 14. *Let $\mathcal{A}$ be a monotone graph property with a coarse threshold. Then there exist $p = p(n)$, constants $\frac{1}{3} > \alpha > 0$, $\varepsilon > 0$, $\tau > 0$, and a graph B satisfying*

(*i*) $\alpha < \mathbb{P}(G(n,p) \in \mathcal{A}) < 1 - 3\alpha$ *and*

(*ii*) $\mathbb{P}(B \subseteq G(n,p)) > \tau$

such that for every graph property $\mathcal{G}$ with a.a.s. $G(n,p) \in \mathcal{G}$ there exist infinitely many $n \in \mathbb{N}$ and for each such n a graph $Z \in \mathcal{G}$ on n vertices such that the following holds.

(1) $\mathbb{P}(Z \cup h(B) \in \mathcal{A}) > 1 - \alpha$ *, where* $h \in \Psi_{B,n}$ *is chosen uniformly at random,*

(2) $\mathbb{P}(Z \cup G(n, \varepsilon p) \in \mathcal{A}) < 1 - 2\alpha$*,*

where the random graph $G(n, \varepsilon p)$ *and* Z *have the same vertex set.* □

Note that $\mathbb{P}(\cdot)$ in (i), (ii), (1), and (2) concern different probability spaces. While in (i) and (ii) it concerns the random graph $G(n, p)$, we consider h chosen uniformly at random in (1) and the random graph $G(n, \varepsilon p)$ in (2).

Roughly the theorem can be read as: If $\mathcal{A}$ has a coarse threshold and p is in range of the threshold, then there exists a small graph B (the "booster") and for infinitely many n a typical graph Z on n vertices with $Z \notin \mathcal{A}$ such that the following holds. Adding a random copy of B to Z increases the probability to maintain property $\mathcal{A}$ more than adding εpn^2 edges to Z. This is remarkable because B has at most K^2 edges which is much less then εpn^2 in typical applications. All in all we can conclude that property $\mathcal{A}$ depends on the local property that the small graph B is contained in $G(n, p)$.

Also in [**19**] (see the appendix there) Bourgain gave a similar characterisation for a more general setting than graphs, but with weaker conclusions. In particular, for $\mathbb{Z}_n$ the theorem says.

THEOREM 15. *There exist functions* $\delta(C, \tau)$ *and* $K(C, \tau)$ *such that the following holds. Let* $p = o(1)$*, let* $\mathcal{A}$ *be a monotone family of subsets of* $\mathbb{Z}_n$ *with*

$$\tau < \mu(p, \mathcal{A}) := \mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A}) < 1 - \tau \,, \tag{2}$$

and assume also $p \cdot \frac{d\mu(p,\mathcal{A})}{dp} \leqslant C$*. Then there exist some* $B \subset \mathbb{Z}_n$ *with* $|B| \leqslant K$ *such that*

$$\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A} | B \subseteq \mathbb{Z}_{n,p}) \geqslant \mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A}) + \delta \,. \tag{3}$$

□

In [**20**] it was observed that the proof of the last theorem in fact also yields the following stronger version with more than one B.

THEOREM 16. *There exist functions $\delta(C,\tau)$, $K(C,\tau)$ and $\eta(C,\tau)$ such that the following holds. Let $p = o(1)$, let $\mathcal{A}$ be a monotone family of subsets of $\mathbb{Z}_n$ with*

$$\tau < \mu(p,\mathcal{A}) := \mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A}) < 1 - \tau\,, \tag{4}$$

and assume also $p \cdot \frac{d\mu(p,\mathcal{A})}{dp} \leqslant C$. Then there exist a family $\mathcal{B}$ of subsets of $\mathbb{Z}_n$ such that

$$\mathbb{P}(B \subseteq \mathbb{Z}_{n,p} \textit{ for some } B \in \mathcal{B}) \geqslant \eta$$

and for all $B \in \mathcal{B}$ holds $|B| \leqslant K$ and

$$\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A} | B \subseteq \mathbb{Z}_{n,p}) \geqslant \mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A}) + \delta\,. \tag{5}$$

□

We will use the last theorem in Chapter 4 about Schur triples.

§2.2. Hypergraph Containers

We shall also use a recent result concerning independent sets in hypergraphs, which was obtained independently by Saxton and Thomason [**51**] and Balogh, Morris, and Samotij [**1**]. Here we will use the version from [**51**].

Let $\mathcal{H}$ be an ℓ-uniform hypergraph on $m = |V(\mathcal{H})|$ vertices. For a subset $\sigma \subset V(\mathcal{H})$ we define its *degree* by

$$d(\sigma) = |\{e \in E(\mathcal{H})\colon \sigma \subseteq e\}|\,.$$

For a vertex $v \in V$ and an integer j with $2 \leqslant j \leqslant \ell$ we consider the maximum degree over all j-element sets σ containing v

$$d^{(j)}(v) = \max\{d(\sigma)\colon v \in \sigma \subset V(\mathcal{H}) \text{ and } |\sigma| = j\}\,.$$

We denote by $d = \ell|E(\mathcal{H})|/m > 0$ the average degree of $\mathcal{H}$ and, following the notation of [**51**], for $\tau > 0$ and $j = 2, \ldots, \ell$ we set

$$\delta_j = \frac{1}{\tau^{j-1} m d} \sum_{v \in V(\mathcal{H})} d^{(j)}(v)$$

and

$$\delta(\mathcal{H},\tau) = 2^{\binom{\ell}{2}-1}\sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}}\delta_j\,.$$

We write $\mathcal{P}(X)$ for the power set of X and denote by $\mathcal{P}^s(X) = \mathcal{P}(X)\times\cdots\times\mathcal{P}(X)$ the s-fold cross product of $\mathcal{P}(X)$. Then the hypergraph container theorem in the version by Saxton and Thomason reads as follows.

Theorem 17 (Saxton & Thomason [**51**]). *Let $\mathcal{H}$ be an ℓ-uniform hypergraph on the vertex set $[m]$ and let $0 < \varepsilon < \frac{1}{2}$. Suppose that for $\tau > 0$ we have both $\delta(\mathcal{H},\tau) \leqslant \varepsilon/12\ell!$ and $\tau \leqslant 1/144\ell!^2\ell$. Then there exist a constant $c = c(\ell)$ and a collection $\mathcal{J} \subset \mathcal{P}([m])$ such that the following holds*

(*a*) *for every independent set I in $\mathcal{H}$ there exists $T = (T_1,\dots,T_s) \in \mathcal{P}^s(I)$ with $|T_i| \leqslant c\tau m$, $s \leqslant c\log(1/\varepsilon)$ and there exists a $J = J(T) \in \mathcal{J}$ only depending on T such that $I \subseteq J(T) \in \mathcal{J}$,*

(*b*) *$e(\mathcal{H}[J]) \leqslant \varepsilon e(\mathcal{H})$ for all $J \in \mathcal{J}$ and*

(*c*) *$\log|\mathcal{J}| \leqslant c\tau\log(1/\tau)\log(1/\varepsilon)m$.* □

§2.3. Probabilistic Inequalities

We will frequently use the following standard probabilistic estimates (see for example [**32**]).

Lemma 18 (Markov's inequality). *Let X be a non-negative random variable and $a > 0$, then*

$$\mathbb{P}(X \geqslant a) \leqslant \frac{\mathbb{E}[X]}{a}\,.$$ □

Lemma 19 (Chebyshev's inequality). *Let X be a random variable with finite expectation and finite non-zero variance and let $t > 0$, then*

$$\mathbb{P}\big(|X - \mathbb{E}[X]| \geqslant t\big) \leqslant \frac{\mathrm{Var}(X)}{t^2}\,.$$ □

LEMMA 20 (Chernoff's inequality). *Let X be a binomial distributed random variable and $t > 0$, then*

$$\mathbb{P}\big(|X - \mathbb{E}[X]| \geqslant t\big) \leqslant 2\exp\left(-\frac{t^2}{2(\mathbb{E}[X] + t/3)}\right) .$$

Similarly, for $\varepsilon \leqslant 3/2$ holds

$$\mathbb{P}\big(X \geqslant \varepsilon\,\mathbb{E}[X]\big) \leqslant \exp\left(-\frac{\varepsilon^2\,\mathbb{E}[X]}{3}\right) . \qquad \square$$

For Janson's inequality we refer to [**30**] (see also [**31**]).

LEMMA 21 (Janson's inequality). *Let Γ_p be a random subset of a finite set Γ and let $\mathcal{S}$ be a family of subsets of Γ. Let $\mathbb{1}_A$ be the characteristic function for the event $A \subseteq \Gamma_p$ and $X = \sum_{A\in\mathcal{S}} \mathbb{1}_A$ be the number of elements of $\mathcal{S}$ that are contained in Γ_p. Then*

$$\mathbb{P}(X = 0) \leqslant \exp\left(-\frac{\mathbb{E}[X]^2}{2\Delta}\right) ,$$

where

$$\Delta = \sum_{A\neq B\in\mathcal{S},A\cap B\neq\varnothing} \mathbb{E}[\mathbb{1}_A\mathbb{1}_B] . \qquad \square$$

CHAPTER 3

Nearly Bipartite Graphs

This chapter is based on [**53**], joint work with Mathias Schacht. Here we will prove the first main result, Theorem 5. We start with a section about the main lemmas, concepts and ideas of the proof. Afterwards we present the proof of Theorem 5 in Section 3.2. The proof of the two main lemmas are finally presented in Section 3.3 and Section 3.4, respectively.

The proof of Theorem 5 refines ideas from the work in [**21**] and also uses Friedgut's criterion for coarse thresholds [**19**] and the recent *hypergraph container* theorem of Balogh, Morris, and Samotij [**1**] and Saxton and Thomason [**51**]. In Section 3.1 we will reformulate Friedgut's criterion and in addition we will state the two main technical lemmas, Lemmas 23 and 24, which we will need in the proof of the main result. Section 3.2 is devoted to the proof of Theorem 5 based on these tools. In Section 3.3 and Section 3.4 we then prove Lemmas 23 and 24, respectively. We close with a few remarks concerning possible generalisations of Theorem 5 and related open questions.

§3.1. Main Lemmas and Ideas of the Proof

In this section we give an overview of the proof as well as an introduction of the necessary concepts for the proof of the main result. In particular we will present two probabilistic lemmas.

For definiteness we may assume that the vertex sets of K_n and $G(n,p)$ coincides with $[n]$. We use the following notation: For a graph B and $n \geqslant v(B)$ we define $\Psi_{B,n}$ as the set of all injective embeddings of B into the complete graph K_n. So $\Psi_{B,n}$ corresponds to the unlabelled copies of B in K_n and, clearly, $|\Psi_{B,n}| = \Theta(n^{v(B)})$.

The starting point of the proof is the Rödl-Ruciński theorem, Theorem 1, which establishes that $n^{-1/m_2(F)}$ is the threshold for the property $G(n,p) \to (F)_2^e$ for most graphs F. In view of Theorem 5 we restrict our discussion below to two colours and to strictly balanced and nearly bipartite graphs F. In particular, we have $m_2(F) > 1$ for every strictly balanced and nearly bipartite graph F since every nearly bipartite graph is required to have at least two edges by definition and we defined $d_2(K_2) = 1$. The assumptions of Theorem 5 are never met by forests F and for sharp thresholds of Ramsey properties of trees we refer to [**23**]. Consequently in the following we can exclude all forests (some forests exhibit a slightly different behaviour in this context see [**32**, Theorem 8.1] for details).

We will strengthen Theorem 1 and show that these thresholds are sharp. For that we will appeal to Friedgut's criterion (Theorem 14) for coarse thresholds and to a recent structural result on independent sets in hypergraphs (see Section 2.2) which play a crucial rôle in our proof. In Section 3.1.2 we introduce two somewhat technical probabilistic lemmas needed for the proof of Theorem 5. Section 3.1.3 establishes the connection between independent sets in hypergraphs and colourings of the edges of the random graph without monochromatic copies of the given graph F considered in our setting.

3.1.1. Friedgut's Criterion for Coarse Thresholds. Below we reformulate Theorem 14 suited for our application.

Corollary 22. *Let F be a strictly balanced and nearly bipartite graph. Assume that the property $G \to (F)_2^e$ does not have a sharp threshold. Then there exists a function $p(n) = c(n)n^{-1/m_2(F)}$ with $C_0 < c(n) < C_1$ for some $C_0, C_1 > 0$, there are constants $\frac{1}{3} > \alpha > 0$ and $\varepsilon > 0$, and there is a graph B with $B \nrightarrow (F)_2^e$ such that for infinitely many $n \in \mathbb{N}$ and for every family of graphs $\mathcal{G}$ on n vertices with a.a.s. $G(n,p) \in \mathcal{G}$ there exists a $Z \in \mathcal{G}$ such that the following hold*

(1) $\mathbb{P}(Z \cup h(B) \to (F)_2^e) > 1 - \alpha$, with $h \in \Psi_{B,n}$ chosen uniformly at random,

(2) $\mathbb{P}(Z \cup G(n, \varepsilon p) \to (F)_2^e) < 1 - 2\alpha$.

Corollary 22 is just a reformulation of Theorem 14 in our context. We give the details below.

Proof of Corollary 22. Note that conclusions (1) and (2) of Corollary 22 are identical to (1) and (2) of Theorem 14 for the monotone graph property $\mathcal{A} = \{G \colon G \to (F)_2^e\}$. Owing to Theorem 1 we infer that because of (*i*) in Theorem 14 the probability $p(n)$ must satisfy $p(n) = c(n)n^{-1/m_2(F)}$ where $C_0 < c(n) < C_1$ for constants C_0, C_1 given by Theorem 1. It is only left to show that $B \nrightarrow (F)_2^e$ is a consequence of (*ii*) of Theorem 14.

Recall that it was shown in [**44**, Theorem 6] that if $B \to (F)_2^e$ then $m(B) = \frac{e(B)}{v(B)} > m_2(F)$. Thus a standard application of Markov's inequality yields $\mathbb{P}(H \subseteq G(n,p)) = o(1)$ for every H with $H \to (F)_2^e$ and $p = \Theta(n^{-1/m_2(F)})$. Consequently the graph B provided by Theorem 14 must satisfy $B \nrightarrow (F)_2^e$, due to (*ii*) of Theorem 14.

$\square$

3.1.2. Main Probabilistic Lemmas. In this section we define an auxiliary hypergraph $\mathcal{H}$. The hypergraph $\mathcal{H}$ to which we will apply Theorem 17 depends on the graph $Z \in \mathcal{G}$ which will be provided by Friedgut's criterion (Corollary 22) applied for the strictly balanced, nearly bipartite graph F. For the verification of the assumptions of Theorem 17 we will restrict the family $\mathcal{G}$ containing Z. Recall that $\mathcal{G}$ can be chosen to be any graph property which is satisfied a.a.s. by $G(n,p)$ for every p with $p = \Theta(n^{-1/m_2(F)})$. In what follows we discuss the restrictions for the family $\mathcal{G}$ (see Lemmas 23 and 24 below) and for that we introduce the required notation.

Let Z and B be two subgraphs of the complete graph K_n. We say $z \in E(Z)$ *focuses* on $b \in E(B)$ if there exists a copy of F in $Z \cup B$ which contains z and b. We set

$$M(Z,B) = \{z \in E(Z) \colon \text{there is } b \in E(B) \text{ such that } z \text{ focuses on } b\}. \qquad (6)$$

The pair (Z, B) is called *interactive* if $E(Z) \cap E(B) = \varnothing$, $Z \nrightarrow (F)^e_2$, and $B \nrightarrow (F)^e_2$, but $Z \cup B \to (F)^e_2$. For a collection $\Xi \subset \Psi_{B,n}$ of embeddings of B into K_n the pair (Z, Ξ) is called *interactive* if $(Z, h(B))$ is interactive for all $h \in \Xi$. Furthermore, a pair (Z, Ξ) is *regular* if for all $h \in \Xi$ every $z \in E(Z)$ focuses on at most one $b \in E(h(B))$. We call $h \in \Psi_{B,n}$ *regular* w.r.t. Z if $(Z, \{h\})$ is regular. The hypergraphs $\mathcal{H}$ considered here are defined in terms of regular pairs (Z, Ξ).

For a pair (Z, Ξ) with $Z \subseteq K_n$ and $\Xi \subseteq \Psi_{B,n}$ we define the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi)$ with vertex set

$$V(\mathcal{H}) = E(Z)$$

and edge set

$$E(\mathcal{H}) = \{M(Z, h(B)) \colon h \in \Xi\}.$$

For our presentation it will be useful to consider orderings of the edges of the involved graphs and "order consistent" embeddings. For that we fix an arbitrary ordering of $E(K_n)$ and an ordering of $E(B)$. For an interactive and regular pair (Z, Ξ) and $h \in \Xi$ we say that $z \in M(Z, h(B)) = \{e_1, \ldots, e_\ell\}$ with $e_1 < e_2 < \cdots < e_\ell$ has *index* i if $z = e_i$. Furthermore, we call (Z, Ξ) and $\mathcal{H}(Z, \Xi)$ *index consistent* if for all $z \in E(Z)$ and all $h, h' \in \Xi$ with $z \in M(Z, h(B)) \cap M(Z, h'(B))$ the indices of z in $M(Z, h(B))$ and in $M(Z, h'(B))$ are the same. Let $b_1 < \cdots < b_{e(B)}$ be the ordering of the edges of B. Then the *profile* of $M(Z, h(B))$ is the function $\pi \colon [|M(Z, h(B))|] \to [e(B)]$ defined by $\pi(i) = j$ if and only if e_i focuses on $h(b_j)$. Since the pair (Z, Ξ) is regular, for each edge of $\mathcal{H}$ each e_i focuses on at most one $h(b_j)$ and, hence, the profile is well defined. We say (Z, Ξ) has profile π if all edges $M(Z, h(B))$ for $h \in \Xi$ have profile π. Note that in this case all sets $M(Z, h(B))$ have the same cardinality and $|M(Z, h(B))|$ is called the *length* of the profile π.

Having established this notation we now state the following technical lemma which gives one part of the graph property $\mathcal{G}$ for the application of Corollary 22.

Moreover, we shall also apply Theorem 17 which results in useful properties of the hypergraph $\mathcal{H}(Z, \Xi)$ for $Z \in \mathcal{G}$ and some appropriately chosen $\Xi \subseteq \Psi_{B,n}$.

Lemma 23. *For all constants $C_1 > C_0 > 0$, $\frac{1}{3} > \alpha > 0$ and graphs F and B, where F is strictly balanced, there exist $\alpha', \beta, \gamma > 0$ and $L \in \mathbb{N}$ such that for every $p = c(n)n^{-1/m_2(F)}$ with $C_0 \leqslant c(n) \leqslant C_1$ a.a.s. $Z \in G(n,p)$ satisfies the following. If*

$$\mathbb{P}(Z \cup h(B) \to (F)_2^e) > 1 - \alpha$$

then there exists $\Xi_{B,n} \subseteq \Psi_{B,n}$ with $|\Xi_{B,n}| \geqslant \alpha' n^2$ and $Z \cup h(B) \to (F)_2^e$ for all $h \in \Xi_{B,n}$ such that the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi_{B,n})$ is index consistent for some profile π of length $\ell \leqslant L$ and there is a family $\mathcal{C}$ of subsets of $V(\mathcal{H})$ satisfying

(1) $\log |\mathcal{C}| \leqslant e(Z)^{1-\gamma}$,
(2) $|C| \geqslant \beta e(Z)$ *for all* $C \in \mathcal{C}$ *and*
(3) every hitting set A of $\mathcal{H}$ contains a $C \in \mathcal{C}$, i.e. for every $A \subseteq V(\mathcal{H})$ with $e \cap A \neq \varnothing$ for all $e \in E(\mathcal{H})$ there exists $C \in \mathcal{C}$ with $C \subseteq A$.

Note that in contrast to the assumptions of Theorem 5 for Lemma 23 it is not required that the given graph F is nearly bipartite. However, for the proof of Theorem 5 we need another restriction on the family $\mathcal{G}$ (in Corollary 22) which is satisfied a.a.s. by $G(n,p)$ and makes use of the near-bipartiteness of F. For a nearly bipartite graph $F = F' + e$ we consider those pairs of vertices in K_n which complete a copy of the bipartite subgraph F' in a given subgraph of $G(n,p)$ to a full copy of F in K_n. Hence, for a graph $G \subseteq K_n$ we define the *basegraph* $\mathrm{Base}_F(G) \subseteq K_n$ with edge set

$$\big\{\{x,y\} \colon \exists F' \subseteq G \text{ such that } F' + \{x,y\} \text{ forms a copy of } F\big\}.$$

We require that for every relatively dense subgraph G' of $G(n,p)$ the basegraph spans many copies of F itself. More precisely, for a graph G on n vertices and a nearly bipartite graph $F = F' + e$ and $\lambda, \eta > 0$ we say G has the property $T(\lambda, \eta, F)$

if for every subgraph $G' \subset G$ with $e(G') \geqslant \lambda e(G)$ we have that the basegraph $\mathrm{Base}_F(G')$ contains at least $\eta n^{v(F)}$ copies of F.

Lemma 24 gives the second restriction for the family $\mathcal{G}$ for our application of Corollary 22. Assuming that there is no copy of F in the bigger colour class of Z, Lemma 24 will be helpful to find a copy of F in the intersection of $Z \cap G(n, \varepsilon p)$ with the other colour class.

LEMMA 24. *For all $\lambda > 0$, $C_1 > C_0 > 0$ and every strictly balanced and nearly bipartite graph F there exists $\eta > 0$ such that for $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ the random graph $G(n,p)$ a.a.s. satisfies $T(\lambda, \eta, F)$.*

3.1.3. Colourings and Hitting Sets. In this section we establish the connection between hitting sets of the hypergraph $\mathcal{H}(Z, \Xi)$ and F-free colourings of Z.

Recall that the definition of an interactive pair (Z, Ξ) says that for every embedding $h \in \Xi \subseteq \Psi_{B,n}$ the graphs Z and $h(B)$ are edge disjoint and $Z \nrightarrow (F)^e_2$ and $B \nrightarrow (F)^e_2$ but $Z \cup h(B) \rightarrow (F)^e_2$. Let $b_1, \ldots, b_K$ be an enumeration of $E(B)$ and fix an F-free colouring $\sigma\colon E(B) \rightarrow \{\text{red,blue}\}$. We copy this colouring for every $h \in \Xi$ by setting $\sigma_h\colon E(h(B)) \rightarrow \{\text{red,blue}\}$ with $\sigma_h(h(b_i)) = \sigma(b_i)$ for all $i = 1, \ldots, K$. Furthermore, let φ be an arbitrary F-free colouring of Z.

Since $Z \cup h(B) \rightarrow (F)^e_2$, the joint colouring of $Z \cup h(B)$ given by φ and σ_h yields a monochromatic copy of F and this copy must contain edges of both graphs, of Z and of $h(B)$. Thus each edge $M(Z, h(B))$ of the hypergraph $\mathcal{H}(Z, \Xi)$ contains an $e \in E(Z)$ which focuses on some $h(b)$ with $b \in E(B)$, where we have $\varphi(e) = \sigma_h(h(b)) = \sigma(b)$. We say such an edge $e \in E(Z)$ (resp. vertex $e \in V(\mathcal{H})$) is *activated* by φ, σ, and h. We define the set of activated vertices by

$$A^\sigma_\varphi = A^\sigma_\varphi(Z, \Xi) = \bigcup_{h \in \Xi} \{e \in E(Z)\colon e \text{ is activated by } \sigma,\ \varphi \text{ and } h\} \subseteq V(\mathcal{H})\,. \tag{7}$$

Note that by definition for an interactive pair (Z, Ξ) every edge $M(Z, h(B))$ of $\mathcal{H}(Z, \Xi)$ contains an activated vertex and, hence, the set of activated vertices A^σ_φ

is a hitting set of $\mathcal{H}(Z,\Xi)$. In what follows we will use different colourings φ of Z but we will always restrict to the same colouring σ of B.

Suppose that in addition we have a fixed ordering of $E(Z)$ and $E(B) = \{b_1, \ldots, b_K\}$. Further suppose that the interactive pair (Z,Ξ) is also index consistent with profile π of length ℓ. In particular, the hypergraph $\mathcal{H}(Z,\Xi)$ is ℓ-uniform.

It also follows from the definitions that for $z \in A^\sigma_\varphi \cap A^\sigma_{\varphi'}$ for two colourings φ and φ' we have $\varphi(z) = \varphi'(z)$. In fact, for $z \in A^\sigma_\varphi$ there exists an $h \in \Xi$ such that z is activated by σ, φ and h. Let i be the index of z in $M(Z, h(B))$, then z focuses on $h(b_{\pi(i)})$ and, therefore, $\varphi(z) = \sigma(b_{\pi(i)})$. Repeating the same argument for φ', we obtain from index consistency that $\varphi'(z) = \sigma(b_{\pi(i)}) = \varphi(z)$. We summarise these observations in the following fact.

Fact 25. *Let (Z,Ξ) be an interactive, regular and index consistent pair with profile π and let σ be an F-free colouring of $E(B)$ and φ be an F-free colouring of $E(Z)$. Then*

(A1) *$A^\sigma_\varphi(Z,\Xi)$ is a hitting set of $\mathcal{H}(Z,\Xi)$ and*

(A2) *for all F-free colourings φ' of $E(Z)$ and for all $z \in A^\sigma_\varphi \cap A^\sigma_{\varphi'}$ we have $\varphi(z) = \varphi'(z)$.* □

Now we are prepared to give the proof of the main theorem based on the lemmas and theorems of this section.

§3.2. Proof of the Main Theorem

The starting point of the proof is Friedgut's criterion (see Corollary 22) applied to the contradictory assumption, that the Ramsey property $G \to (F)^e_2$ for a given strictly balanced and nearly bipartite graph F has a coarse threshold. For that we define a family of graphs $\mathcal{G}$ having "useful" properties and Lemma 23 and Lemma 24 show that a.a.s. $G(n,p)$ displays these properties. Then Friedgut's criterion asserts for infinitely many $n \in \mathbb{N}$ the existence of an n-vertex graph $Z \in \mathcal{G}$, a graph B (called *booster*), constants $\frac{1}{3} > \alpha > 0$, $\varepsilon > 0$ and a family of embeddings $\Psi'_{B,n} \subseteq \Psi_{B,n}$ with $Z \cup h(B) \rightarrow (F)^e_2$ for all $h \in \Psi'_{B,n}$ and $|\Psi'_{B,n}| \geqslant (1-\alpha)|\Psi_{B,n}|$, but $\mathbb{P}(Z \cup G(n, \varepsilon p) \rightarrow (F)^e_2) < 1 - 2\alpha$. The goal is to find a contradiction to the last fact by showing $\mathbb{P}(Z \cup G(n, \varepsilon p) \rightarrow (F)^e_2) = 1 - o(1)$.

Let Φ be the set of all F-free colourings of Z. We have to show that for any $\varphi \in \Phi$ the probability to extend φ to an F-free colouring of $Z \cup G(n, \varepsilon p)$ is very small. We are able to show that this probability is of order $\exp(-\Omega(pn^2))$. Now we would like to use a union bound for all $\varphi \in \Phi$. However, we have only little control over $|\Phi|$ and the trivial upper bound $2^{\Theta(pn^2)}$ is too large to combine it with the bound from above $\exp(-\Omega(pn^2))$ to obtain for $\mathbb{P}(Z \cup G(n, \varepsilon p) \nrightarrow (F)^e_2)$ a bound of order $o(1)$ by the union bound.

Instead we shall find a partition of Φ into $2^{o(pn^2)}$ classes such that two colourings from the same partition class always agree on a large subset of Z. These subsets are called *cores*. Then we will show that the colouring of φ restricted to the associated core implies that φ is only with probability at most $\exp(-\Omega(pn^2))$ extendible to an F-free colouring of $Z \cup G(n, \varepsilon p)$. This allows us to use a union bound over all partition classes to get the desired upper bound on $\mathbb{P}(Z \cup G(n, \varepsilon p) \nrightarrow (F)^e_2)$ of order $o(1)$.

For the definition of the cores we will appeal to the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi)$ which was defined in Section 3.1.2. Recall that $V(\mathcal{H}) = e(Z)$ and hyperedges of $\mathcal{H}$ correspond to embeddings of B in K_n, which are given by a carefully chosen subset $\Xi \subseteq \Psi'_{B,n}$. In fact, we shall select $\Xi \subseteq \Psi'_{B,n}$ in such a way that we can apply the structural result on independent sets of hypergraphs by Saxton and Thomason [**51**] to $\mathcal{H}$ (see Lemma 23). In fact, the cores then correspond to the complements of the almost independent sets from $\mathcal{J}$ given by the Saxton-Thomason theorem (Theorem 17). This yields a small family $\mathcal{C}$ of subsets of $V(\mathcal{H})$, that means of size $2^{o(pn^2)}$, such that the elements $C \in \mathcal{C}$ are not too small and every hitting set of $\mathcal{H}$ contains at least one element from $\mathcal{C}$.

We then associate every F-free colouring φ of Z with a hitting set A_φ^σ of $\mathcal{H}$ (for some F-free colouring σ of B, see part (A1) of Fact 25) and thus we can associate to each such colouring φ a core $C \in \mathcal{C}$ contained in A_φ^σ. This allows us to define the desired partition of the set of colourings Φ using the "small" family of cores $\mathcal{C}$. Finally, we use the union bound to estimate the probability that there is an F-free colouring of Z that can be extended to an F-free colouring of $Z \cup G(n, \varepsilon p)$ by $o(1)$, which contradicts $\mathbb{P}(Z \cup G(n, \varepsilon p) \to (F)_2^e) < 1 - 2\alpha$. Below we give the details of this proof.

Proof of Theorem 5. Let $F = F' + \{a_1, a_2\}$ be a strictly balanced, nearly bipartite graph with F' being bipartite and assume for a contradiction that the property $G \to (F)_2^e$ does not have a sharp threshold.

We apply Corollary 22 and obtain a function $p(n) = c(n)n^{-1/m_2(F)}$ with $C_0 < c(n) < C_1$ for some $C_1 > C_0 > 0$, constants $\frac{1}{3} > \alpha > 0$, $\varepsilon > 0$ and a graph B with $B \nrightarrow (F)_2^e$.

For these parameters we apply Lemma 23 and obtain constants $\alpha', \beta, \gamma > 0$ and $L \in \mathbb{N}$. Set $\lambda = \beta/2$ and apply Lemma 24, which yields $\eta > 0$. Then let $\mathcal{G}_n$ be the family of graphs G on n vertices that satisfy the conclusions of Lemma 23 and Lemma 24 for the chosen parameters and $\frac{1}{4}pn^2 \leqslant e(G) \leqslant pn^2$. Since these properties hold a.a.s. in $G(n,p)$, it follows from Corollary 22, that there are infinitely many $n \in \mathbb{N}$ for which there is some $Z \in \mathcal{G}_n$ satisfying

(R1) $\mathbb{P}(Z \cup h(B) \to (F)_2^e) > 1 - \alpha$, with $h \in \Psi_{B,n}$ chosen uniformly at random,
(R2) $\mathbb{P}(Z \cup G(n, \varepsilon p) \to (F)_2^e) < 1 - 2\alpha$

as well as by Lemma 24

(T) Z has the property $T(\lambda, \eta, F)$

and

(Z) $\frac{1}{4}pn^2 \leqslant e(Z) \leqslant pn^2$.

Owing to $Z \in \mathcal{G}_n$ and (R1) we can use Lemma 23 to find some $\Xi_{B,n} \subseteq \Psi_{B,n}$ of size at least $\alpha' n^2$ with $Z \cup h(B) \to (F)_2^e$ for all $h \in \Xi_{B,n}$ such that the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi_{B,n})$ is index consistent with a profile π of length $\ell \leqslant L$ and such that there is a family $\mathcal{C}$ of subsets of $V(\mathcal{H})$ with

(C1) $\log |\mathcal{C}| \leqslant e(Z)^{1-\gamma}$,

(C2) $|C| \geqslant \beta e(Z)$ for all $C \in \mathcal{C}$ and

(C3) every hitting set A of $\mathcal{H}$ contains a set $C \in \mathcal{C}$.

Our proof is by contradiction and we shall establish such a contradiction to the assertion (R2).

Let Φ be the set of all F-free edge colourings of $E(Z)$ and pick an arbitrary F-free colouring σ of B. We want to split Φ into "few" classes. For this we use the correspondence between any colouring $\varphi \in \Phi$ and the hitting set $A_\varphi^\sigma = A_\varphi^\sigma(Z, \Xi_{B,n})$ of $\mathcal{H}$ given by part (A1) of Fact 25. Moreover, for $C \in \mathcal{C}$ we define

$$\Phi_C = \{\varphi \in \Phi \colon C \subseteq A_\varphi^\sigma\} .$$

Then $\Phi = \bigcup_{C \in \mathcal{C}} \Phi_C$ (not necessarily disjoint) since by (C3) for every $\varphi \in \Phi$ the hitting set A_φ^σ contains some $C \in \mathcal{C}$ and hence $\varphi \in \Phi_C$.

Part (A2) of Fact 25 asserts that $\varphi(z) = \varphi'(z)$ for all $z \in A_\varphi^\sigma \cap A_{\varphi'}^\sigma$ and colourings φ, $\varphi' \in \Phi$. In other words, all colourings in Φ_C agree on C and, hence, there exists a monochromatic subset $R_C \subseteq C$, say coloured red, of size at least $|C|/2 \geqslant \beta e(Z)/2 = \lambda e(Z)$ (see (C2) and the choice of λ). For the desired contradiction we add $G(n, \varepsilon p)$ to Z. We have to show that

$$\mathbb{P}(Z \cup G(n, \varepsilon p) \nrightarrow (F)_2^e) = o(1) .$$

For this purpose we find for all F-free colourings φ of Z an upper bound for the probability that φ is extendible to an F-free colouring of $Z \cup G(n, \varepsilon p)$. For φ we use only the colouring on the associated core $C \subseteq A_\varphi^\sigma$, instead of the colouring on all edges of Z. In this way we can deal with all embeddings $\varphi \in \Phi_C$ at once since they coincide on C.

Since the red colour class R_C contains at least $\lambda e(Z)$ edges it follows from Property (T), that there are at least $\eta n^{v(F)}$ copies of F in the basegraph $\mathrm{Base}_F(R_C)$ of R_C w.r.t. F. In an F-free colouring of $Z \cup G(n, \varepsilon p)$ all edges in

$$U_C = G(n, \varepsilon p) \cap \mathrm{Base}_F(R_C)$$

have to be coloured blue since every edge in $\mathrm{Base}_F(R_C)$ completes a red copy of F' in R_C to a copy of F. Consequently, φ cannot be extended to an F-free colouring of $Z \cup G(n, \varepsilon p)$ if U_C spans a copy of F. However, since $\mathrm{Base}_F(R_C)$ contains $\Omega(n^{v(F)})$ copies of F and $p = \Omega(n^{-1/m_2(F)})$ it follows from Janson's inequality [**30**] (see also [**31**]) that it is very unlikely that U_C is F-free. In fact, a standard application of Janson's inequality asserts that there exists some $\gamma' = \gamma'(\varepsilon, \eta, C_0, C_1, F)$ such that

$$\mathbb{P}(F \nsubseteq G(n, \varepsilon p) \cap \mathrm{Base}_F(R_C)) = \mathbb{P}(F \nsubseteq U_C) \leqslant \exp\left(-\gamma' n^{2-\frac{1}{m_2(F)}}\right). \tag{8}$$

We then deduce the desired contradiction to (R2) by

$$\begin{aligned}
\mathbb{P}(Z \cup G(n, \varepsilon p) \nrightarrow (F)^e_2) &\leqslant |\mathcal{C}| \cdot \max_{C \in \mathcal{C}} \mathbb{P}(\exists \varphi \in \Phi_C \colon \varphi \text{ is extendible to } U_C) \\
&\overset{(C1)}{\leqslant} \exp\left(e(Z)^{1-\gamma}\right) \cdot \max_{C \in \mathcal{C}} \mathbb{P}(F \nsubseteq U_C) \\
&\overset{(Z)}{\leqslant} \exp\left((pn^2)^{1-\gamma}\right) \cdot \max_{C \in \mathcal{C}} \mathbb{P}(F \nsubseteq U_C) \\
&\overset{(8)}{\leqslant} \exp\left((C_1 n^{2-\frac{1}{m_2(F)}})^{1-\gamma}\right) \cdot \exp\left(-\gamma' n^{2-\frac{1}{m_2(F)}}\right) \\
&< \alpha\,,
\end{aligned}$$

for sufficiently large n, since $\gamma > 0$ and C_1, γ, and γ' are constants independent of n. This concludes the proof of Theorem 5. □

§3.3. Proof of Lemma 23

The key tool to prove Lemma 23 is the *container theorem* (see Section 2.2). We shall apply Theorem 17 to the hypergraph $\mathcal{H}(Z, \Xi_{B,n})$. In order to satisfy the assumptions of Theorem 17 we may enforce some properties on the typical graph Z

and the family of embeddings $\Xi_{B,n}$. Firstly in Section 3.3.1 we will formulate some properties on Z that hold a.a.s. for $G(n,p)$ and which will turn out to be useful for locating a suitable family of embeddings $\Xi_{B,n} \subseteq \Psi_{B,n}$ (see Section 3.3.2). In Section 3.3.3 we finally check that for those choices the assumptions of Theorem 17 are satisfied by the hypergraph $\mathcal{H}(Z, \Xi_{B,n})$.

3.3.1. Some Typical Properties of $G(n,p)$. Corollary 22 yields a family of embeddings of B into K_n. We restrict ourselves to regular embeddings with foresight to the later parts of the proof. Actually we want that for every edge $e \in E(Z)$ and every embedding h there is at most one $b \in E(B)$ such that e focuses on $h(b)$. In addition there should be exactly one copy of F that contains e and $h(b)$ if e focuses on $h(b)$. There are three ways such that this fails.

Definition 26. *Let F, B, Z be graphs with $Z \subseteq K_n$. An embedding $h \in \Psi_{B,n}$ is bad (with respect to F and Z) if one of the following holds*

- (B1) *either there is a copy F_1 of F in $Z \cup h(B)$ that contains at least one edge of $E(Z) \setminus E(h(B))$ and at least two edges of $E(h(B))$,*
- (B2) *or there are distinct copies F_1 and F_2 of F in $Z \cup h(B)$ and edges e, $f_1 \neq f_2$ with $e \in E(Z) \setminus E(h(B))$ and $e \in E(F_1) \cap E(F_2)$, $f_1, f_2 \in E(h(B))$ such that $f_1 \in E(F_1)$ and $f_2 \in E(F_2)$*
- (B3) *or there are distinct copies F_1 and F_2 of F in $Z \cup h(B)$ and edges e, f with $e \in E(Z) \setminus E(h(B))$ and $e \in E(F_1) \cap E(F_2)$, $f \in E(h(B))$ and $f \in E(F_1) \cap E(F_2)$.*

Note that (B3) would be a special case of (B2) if we did not require $f_1 \neq f_2$ there. However, for the later discussion it is better to distinguish these cases, and the idea of excluding embeddings h because of (B3) will be used in the proof of Lemma 23 (see Lemma 36).

Fact 27. *For F, B and Z let $\Xi_{B,n} \subseteq \Psi_{B,n}$ be a family of embeddings such that properties* (B1) *and* (B2) *fail for every $h \in \Xi_{B,n}$. Then clearly the pair $(Z, \Xi_{B,n})$ is regular.* □

We shall show that for the random graph $Z = G(n,p)$ only a few embeddings $h \in \Psi_{B,n}$ are bad (see (Z5) in Definition 28 and Lemma 29 below), which enables us to focus on regular pairs $(Z, \Xi_{B,n})$. Moreover, we shall restrict to typical graphs Z, which render a few more somewhat technical properties such as containing roughly the expected number of some special subgraphs. We discuss those properties below.

Let $\mathcal{F}_-$ be the family of spanning subgraphs of F obtained by removing some edge and for a graph G we denote by $\mathcal{F}_-(G)$ the copies of the members of $\mathcal{F}_-$ in G. Furthermore, for an edge $e \in E(G)$ let $\mathcal{F}_-(G,e)$ be those copies in $\mathcal{F}_-(G)$ that contain e. For e_1, $e_2 \in \binom{V(G)}{2}$ let $\mathcal{P}(G, e_1, e_2)$ be the set of pairs (F_1, F_2) of two edge disjoint subgraphs of G such that

- F_1 and F_2 are copies of (possibly different) spanning subgraphs of F, each of which obtained from F by removing two edges,
- the intersection $V(F_1) \cap V(F_2) = \{x_1, x_2, \ldots, x_s\}$ contains at least two vertices, and
- $F_1 + \{x_1, x_2\} + e_1$ and $F_2 + \{x_1, x_2\} + e_2$ are isomorphic to F.

For $s \geqslant 2$ let $\mathcal{P}_s(G, e_1, e_2) \subseteq \mathcal{P}(G, e_1, e_2)$ be the set of pairs as in $\mathcal{P}(G, e_1, e_2)$ such that F_1 and F_2 intersect in exactly s vertices. Note that for $i = 1, 2$ by definition $e_i \neq \{x_1, x_2\}$ and e_i is not required to be an edge of G.

These concepts lead to the following definition of "good" graphs Z, where we impose that the sizes of the introduced families defined above are close to the respective expectation in $G(n,p)$. Then Lemma 29 states that a.a.s. $G(n,p)$ is indeed good for the right choice of parameters.

Definition 28. *For graphs F and B and constants $D > 0$, $\zeta > 0$, $\delta > 0$ and $p \in (0,1)$ we consider the set of graphs $\mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ on n vertices that is given by $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ if and only if*

(Z1) $\frac{1}{4}pn^2 \leqslant e(Z) \leqslant pn^2$,

(Z2) $|\mathcal{F}_-(Z)| \leqslant Dn^2$,

(Z3) $|\mathcal{F}_-(Z,e)| \leqslant \frac{D}{p}$ *for all* $e \in E(Z)$,

(Z4) $|\mathcal{P}(Z,e_1,e_2)| \leqslant \frac{D}{pn^\delta}$ *for all but at most* $\frac{Dpn^2}{n^\delta}$ *pairs of distinct edges* $e_1, e_2 \in E(Z)$ *and*

(Z5) $|\{h \in \Psi_{B,n} \colon h$ *is bad w.r.t.* F *and* $Z\}| \leqslant \frac{|\Psi_{B,n}|}{n^\zeta}$.

The following Lemma shows that a.a.s. $G(n,p) \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ for D sufficiently large and ζ and δ sufficiently small (in fact, our choice of δ will imply $pn^\delta \to 0$).

Lemma 29. *For every strictly balanced graph F, for every graph B, and for all constants $C_1 \geqslant C_0 > 0$ there are constants $D > 0$, $\zeta > 0$, and δ with $0 < \delta \leqslant \min\left\{\frac{1}{m_2(F)}, 1 - \frac{1}{m_2(F)}\right\}$ such that for $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ a.a.s. $G(n,p) \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$.*

We will split the proof into two parts: First we consider (Z1)-(Z4) which deals with subgraphs of Z (Lemma 30), and then we deal with the bad embeddings considered in (Z5) (Lemma 32).

Lemma 30. *For constants $C_1 \geqslant C_0 > 0$, a strictly balanced graph F, and p and n with $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ the following holds. There exist constants $D > 0$ and δ with $0 < \delta < \min\left\{\frac{1}{m_2(F)}, 1 - \frac{1}{m_2(F)}\right\}$ such that a.a.s. $G(n,p)$ satisfies the properties* (Z1), (Z2), (Z3), *and* (Z4) *with the parameters p, D, and δ and for the graph F.*

For the proof of Lemma 30 we note that property (Z1) follows directly from the concentration of the binomial distribution and (Z2) follows from (Z1) and (Z3).

The proof of (Z3) will make use of Spencer's extension lemma (Theorem 31 stated below). Finally, (Z4) follows from a standard second moment argument. Below we introduce the necessary notation for the statement of Theorem 31.

For a graph H and an ordered proper subset $R = (x_1, \dots, x_r)$ of $V(H)$ the pair (R, H) is called *rooted graph* with *roots* R. For an induced subgraph $H' = H[S]$ of H with $\{x_1, \dots, x_r\} \subsetneq S$ we say (R, H') is a rooted subgraph of (R, H). We define the density of a rooted graph (R, H) by

$$\mathrm{dens}(R, H) = \frac{e(H) - e(H[R])}{v(H) - |R|}\,.$$

Let $V(H) \setminus \{x_1, \dots, x_r\} = \{y_1, \dots, y_\nu\}$ for some $\nu \geqslant 1$. For a graph G with some marked vertices $(x'_1, \dots, x'_r)$ an ordered tuple $(y'_1, \dots, y'_\nu)$ is called an (R, H)-*extension of* $(x'_1, , \dots, x'_r)$ if

- the y'_i are distinct from each other and from the x'_j,
- $\{x'_i, y'_j\} \in E(G)$ whenever $\{x_i, y_j\} \in E(H)$ and
- $\{y'_i, y'_j\} \in E(G)$ whenever $\{y_i, y_j\} \in E(H)$.

The number of (R, H)-extensions $(y'_1, \dots, y'_\nu)$ is denoted by $N(x'_1, \dots, x'_r)$. Finally, we define $\mathrm{mad}(R, H)$ as the maximal average degree of a rooted graph (R, H) by

$$\mathrm{mad}(R, H) = \max\{\mathrm{dens}(R, H') \colon (R, H') \text{ is rooted subgraph of } (R, H)\}\,.$$

THEOREM 31 ([**55**, Theorem 3]). *Let (R, H) be an arbitrary rooted graph and let $\varepsilon > 0$. Then there exist t such that if $p \geqslant n^{-1/\mathrm{mad}(R,H)}(\log n)^{1/t}$ then a.a.s. in $G(n, p)$*

$$(1 - \varepsilon)\mathbb{E}[N(\boldsymbol{x}')] < N(\boldsymbol{x}') < (1 + \varepsilon)\mathbb{E}[N(\boldsymbol{x}')]$$

for all $\boldsymbol{x}' = (x'_1, \dots, x'_r)$ chosen from $[n]$. $\square$

PROOF OF LEMMA 30. (Z1) This follows from an application of Chernoff's inequality, Lemma 20.

(Z2) As already mentioned this property follows from (Z1) and (Z3). However, here is a standard direct proof based on the subgraph containment threshold in random graphs.

For $F_- \in \mathcal{F}_-$ let X be the random variable that counts the number of copies of F_- contained in $G(n,p)$. Using that $p = \Theta(n^{-1/m_2(F)})$ combined with the balancedness of F yields

$$\mathbb{E}[X] = \Theta\left(n^{v(F)}p^{e(F)-1}\right) = \Theta(n^2)\,.$$

Moreover, by the definition of the 2-density the expected number of copies of every non-trivial subgraph of $F_- \subset F$ is of order $\Omega(pn^2)$ and tends to infinity for $n \to \infty$. Consequently, X converges to $\mathbb{E}[X]$ in probability (see, e.g. [**32**, Remark 3.7]) and we have $\mathbb{P}(X \geqslant 2\mathbb{E}[X]) \to 0$ for $n \to \infty$. Summing over all $F_- \in \mathcal{F}_-$ yields the claim.

(Z3) Consider a graph $F_- \in \mathcal{F}_-$ and remove some edge $\{x_1, x_2\}$ from F_- and call the resulting graph F_{-2}. For $e \in \binom{[n]}{2}$ let X_e be the random variable that counts the number of copies of F_{-2} that build a copy of F_- by adding e and let X be the random variable that counts the number of copies of F_{-2} contained in $G(n,p)$.

Now we can use Spencer's extension lemma (Theorem 31). We consider the rooted graph $((x_1,x_2),F_-)$. Let $\hat{F}$ be an induced subgraph of F_- such that $((x_1,x_2),\hat{F})$ is a rooted subgraph of $((x_1,x_2),F_-)$ which maximizes the density $\mathrm{dens}((x_1,x_2),\hat{F})$. Since the graph $F \supsetneq F_- \supseteq \hat{F}$ is strictly balanced we have

$$m_2(F) > d_2(\hat{F}) = \frac{e(\hat{F})-1}{v(\hat{F})-2} = \mathrm{dens}((x_1,x_2),\hat{F}) = \mathrm{mad}((x_1,x_2),F_-)\,.$$

Consequently, Theorem 31 applied with $\varepsilon = 1$ implies a.a.s.

$$N(x_1',x_2') \leqslant 2\mathbb{E}(X_e) = O(p^{e(F)-2}n^{v(F)-2})$$

for every $x_1' \neq x_2' \in [n]$. Owing to $p = \Theta(n^{-1/m_2(F)})$ and the (strict) balancedness of F we have that $p^{e(F)}n^{v(F)} = \Theta(pn^2)$ and, consequently, for sufficiently large D the claim follows by summing over all choices of $F_- \in \mathcal{F}_-$ and $\{x_1, x_2\} \in E(F_-)$.

(Z4) We show that this property holds a.a.s. for

$$\delta = \frac{1}{6}\min\left\{\tfrac{1}{m_2(F)}, 1 - \tfrac{1}{m_2(F)}\right\} \tag{9}$$

and some $D > 0$ independent of n. In the proof below we distinguish several cases. In the first case we only look at configurations from $\mathcal{P}_2(G(n,p), e_1, e_2)$. Afterwards we consider configurations from $\mathcal{P}_s(G(n,p), e_1, e_2)$ for $s > 2$.

Case 1 : $\mathbf{s = 2}$. For two pairs $e_1 \neq e_2 \in \binom{[n]}{2}$ let X_{e_1,e_2} be the random variable given by $|\mathcal{P}_2(G(n,p), e_1, e_2)|$ and denote by v_1 and u_1 the elements of e_1 and by v_2 and u_2 the elements of e_2. We want to use Chebyshev's Inequality to obtain the claimed bound for most pairs. Consequently, we estimate the expectation and variance of X_{e_1,e_2}. We distinguish between the cases $e_1 \cap e_2 = \varnothing$ and $|e_1 \cap e_2| = 1$.

First let $e_1 \cap e_2 = \varnothing$. Since $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ and F is strictly balanced we have $n^{v(F)}p^{e(F)} = \Theta(pn^2)$ and

$$n^{v(F)-2}p^{e(F)-1} \leqslant C_1^{e(F)-1}\,. \tag{10}$$

For $F_0 \subseteq F$ with $v(F_0) \geqslant 2$ it follows from F being strictly balanced that there is some $d > 0$ only depending on F and C_0 such that

$$n^{v(F_0)}p^{e(F_0)} \geqslant dpn^2\,. \tag{11}$$

The expectation of X_{e_1,e_2} is

$$\mathbb{E}[X_{e_1,e_2}] \leqslant e(F)^4 n^{2v(F)-6}p^{2e(F)-4} \overset{(10)}{\leqslant} e(F)^4 C_1^{2e(F)-2} n^{-2}p^{-2} \tag{12}$$

and $\mathbb{E}[X_{e_1,e_2}] \to 0$ for n tending to infinity since $p = \Theta(n^{-1/m_2(F)})$ and $m_2(F) > 1$.

Now we estimate the variance of X_{e_1,e_2}. We will show

$$\mathrm{Var}(X_{e_1,e_2}) \leqslant \frac{c}{n^2p^2}\left(1 + \frac{1}{np^2}\right)$$

for some constant $c > 0$ depending only on F, C_0 and C_1. For this purpose let (F_a, F_b) and (F_c, F_d) be two different pairs of graphs that contribute to the number $|\mathcal{P}_2(G(n,p), e_1, e_2)|$ with

$$V(F_a) \cap V(F_b) = \{x_1, x_2\}, \quad V(F_c) \cap V(F_d) = \{y_1, y_2\}$$

and

$$V(F_a) \cap V(F_c) \supseteq \{u_1, v_1\} = e_1, \quad V(F_b) \cap V(F_d) \supseteq \{u_2, v_2\} = e_2 \,.$$

Recall that e_1 and e_2 are by definition of $\mathcal{P}_2(G(n,p), e_1, e_2)$ not necessarily contained in $G(n,p)$ and they are not contained as edges in any of the subgraphs F_a, F_b, F_c, and F_d (where $s = 2$ is used). We denote by $\mathcal{P}^2_{e_1,e_2}$ the family of isomorphism types of possible pairs $((F_a, F_b), (F_c, F_d))$ such that the conditions above are satisfied. If it is clear from the context we will sometimes drop the subscripts e_1 and e_2 to further ease the notation.

For $Q = ((F_a, F_b), (F_c, F_d)) \in \mathcal{P}^2_{e_1,e_2}$ let $\mathcal{S}_Q$ be the set of subsets of $[n]$ of size $v(F_a \cup F_b \cup F_c \cup F_d)$ that contain u_1, v_1, u_2, and v_2. For $S \in \mathcal{S}_Q$ let 1_S be the indicator random variable for the event "there exists a copy of Q in $G(n,p)$ on the vertex set S". Then

$$\mathrm{Var}(X_{e_1,e_2}) \leqslant \mathbb{E}[X_{e_1,e_2}] + \sum_{Q \in \mathcal{P}^2_{e_1,e_2}} \sum_{S \in \mathcal{S}_Q} \mathbb{P}(1_S = 1) \tag{13}$$

For the estimation of the term $\sum_{Q \in \mathcal{P}^2} \sum_{S \in \mathcal{S}_Q} \mathbb{P}(1_S = 1)$ we use the following notation. For $\alpha, \beta \in \{a, b, c, d\}$ and $\square \in \{\cup, \cap\}$ we set

$$v_{\alpha \square \beta} = v(F_\alpha \square F_\beta) \quad \text{and} \quad e_{\alpha \square \beta} = e(F_\alpha \square F_\beta) \,,$$

where $F_\alpha \cap F_\beta$ and $F_\alpha \cup F_\beta$ denotes the normal union and intersection of two graphs. Moreover, we can extend this to longer expressions of unions and intersections, like $v_{(\alpha \cap \beta) \cup \gamma}$, and we will make use of this short hand notation in the calculations below. We also set

$$v_{\alpha \smallsetminus \beta} = v_\alpha - v_{\alpha \cap \beta} \qquad \text{and} \qquad e_{\alpha \smallsetminus \beta} = e_\alpha - e_{\alpha \cap \beta} \,. \tag{14}$$

Note that $e_{\alpha \smallsetminus \beta}$ denotes the number of edges exclusively contained in F_α, which does not necessarily coincide with $e(F_\alpha - V(F_\beta))$. We estimate $\sum_{Q \in \mathcal{P}^2} \sum_{S \in \mathcal{S}_Q} \mathbb{P}(1_S = 1)$ by counting the number of choices for the vertices of the desired configuration and determine the number of needed edges. Recalling that every $Q \in \mathcal{P}^2_{e_1,e_2}$ corresponds to $((F_a, F_b), (F_c, F_d))$ we count those by first choosing (F_a, F_b), then F_c and then F_d and deal with the vertices and edges that are counted several times by looking at the intersections between the different copies of F.

$$\begin{aligned}
&\sum_{Q \in \mathcal{P}^2} \sum_{S \in \mathcal{S}_Q} \mathbb{P}(1_S = 1) \\
&\quad \leqslant \sum_{Q \in \mathcal{P}^2} (4v(F))! \cdot n^{2v(F)-6} p^{2e(F)-4} \cdot n^{v_{c \smallsetminus (a \cup b)}} p^{e_{c \smallsetminus (a \cup b)}} \cdot n^{v_{d \smallsetminus (a \cup b \cup c)}} p^{e_{d \smallsetminus (a \cup b \cup c)}} \qquad (15)\\
&\quad \overset{(14)}{=} (4v(F))! \sum_{Q \in \mathcal{P}^2} n^{4v(F)-6} p^{4e(F)-8} \cdot n^{-v_{c \cap (a \cup b)}} p^{-e_{c \cap (a \cup b)}} \cdot n^{-v_{d \cap (a \cup b \cup c)}} p^{-e_{d \cap (a \cup b \cup c)}} \\
&\quad = (4v(F))! \sum_{Q \in \mathcal{P}^2} n^2 p^{-4} (n^{v(F)-2} p^{e(F)-1})^4 n^{-v_{c \cap (a \cup b)}} p^{-e_{c \cap (a \cup b)}} n^{-v_{d \cap (a \cup b \cup c)}} p^{-e_{d \cap (a \cup b \cup c)}} \\
&\quad \overset{(10)}{\leqslant} C \sum_{Q \in \mathcal{P}^2} n^2 p^{-4} \cdot n^{-v_{c \cap (a \cup b)}} p^{-e_{c \cap (a \cup b)}} \cdot n^{-v_{d \cap (a \cup b \cup c)}} p^{-e_{d \cap (a \cup b \cup c)}}\,, \qquad (16)
\end{aligned}$$

where $C > 0$ is a constant depending only on F and C_1. For the estimation of

$$f_Q(n,p) := n^2 p^{-4} \cdot n^{-v_{c \cap (a \cup b)}} p^{-e_{c \cap (a \cup b)}} \cdot n^{-v_{(a \cup b \cup c) \cap d}} p^{-e_{(a \cup b \cup c) \cap d}} \qquad (17)$$

we distinguish several cases depending on the structure of Q.

First we consider terms in (16) with $\{x_1, x_2\} \subseteq V(F_c)$. Since $\{x_1, x_2, v_1, u_1\} \subseteq V(F_a \cap F_c)$ and $F_a \cap F_c \subseteq F_a \subset F$ we also know that $F_0 := (F_a \cap F_c) + \{x_1, x_2\} + e_1 \subseteq F$. Therefore,

$$\frac{1}{n^{v_{a \cap c}} p^{e_{a \cap c}}} = \frac{p^2}{n^{v(F_0)} p^{e(F_0)}} \overset{(11)}{\leqslant} \frac{p^2}{dpn^2} = \frac{p}{dn^2}\,.$$

Similarly, $(F_b \cap F_c) + \{x_1, x_2\} \subseteq F$ and $((F_a \cup F_b \cup F_c) \cap F_d) + \{y_1, y_2\} + e_2 \subseteq F$. The same argument yields

$$\frac{1}{n^{v_{b\cap c}} p^{e_{b\cap c}}} \leqslant \frac{1}{dn^2} \qquad \text{and} \qquad \frac{1}{n^{v_{(a\cup b\cup c)\cap d}} p^{e_{(a\cup b\cup c)\cap d}}} \leqslant \frac{p}{dn^2} \,.$$

Applying these bounds and the facts that $v_{a\cap b\cap c} \leqslant 2$ and $e_{a\cap b\cap c} = 0$ to (17) yields

$$\begin{aligned} f_Q(n,p) &= n^2 p^{-4} \cdot n^{-v_{a\cap c}} p^{-e_{a\cap c}} \cdot n^{-v_{b\cap c}} p^{-e_{b\cap c}} \cdot n^{v_{a\cap b\cap c}} \cdot n^{-v_{(a\cup b\cup c)\cap d}} p^{-e_{(a\cup b\cup c)\cap d}} \\ &\leqslant n^2 p^{-4} \cdot \frac{p}{dn^2} \cdot \frac{1}{dn^2} \cdot n^2 \cdot \frac{p}{dn^2} = \frac{1}{d^3 p^2 n^2} \,. \end{aligned} \tag{18}$$

By symmetry we obtain the same estimate in the case that $\{x_1, x_2\} \subseteq V(F_d)$ and in the remaining case we may assume

(I) $|V(F_c) \cap \{x_1, x_2\}| \leqslant 1$ and $|V(F_d) \cap \{x_1, x_2\}| \leqslant 1$.

Next we consider those terms in (16) with (I) and $v_{b\cap c} \geqslant 2$. By (I) we have $v_{a\cap b\cap c} \leqslant 1$. We proceed in a similar way as above. This time we use that $(F_a \cap F_c) + e_1 \subseteq F$ and similarly that $((F_a \cup F_b \cup F_c) \cap F_d) + e_2 + \{y_1, y_2\} \subseteq F$ and, therefore,

$$\frac{1}{n^{v_{a\cap c}} p^{e_{a\cap c}}} \overset{(11)}{\leqslant} \frac{1}{dn^2} \qquad \text{and} \qquad \frac{1}{n^{v_{(a\cup b\cup c)\cap d}} p^{e_{(a\cup b\cup c)\cap d}}} \overset{(11)}{\leqslant} \frac{p}{dn^2} \,.$$

Moreover, since we assume $v_{b\cap c} \geqslant 2$ we can apply (11) with $F_0 = F_b \cap F_c$

$$\frac{1}{n^{v_{b\cap c}} p^{e_{b\cap c}}} \leqslant \frac{1}{dpn^2} \,.$$

Combining these bounds with (17) and $v_{a\cap b\cap c} \leqslant 1$ and $e_{a\cap b\cap c} = 0$ yields

$$\begin{aligned} f_Q(n,p) &\leqslant n^2 p^{-4} \cdot n^{-v_{a\cap c}} p^{-e_{a\cap c}} \cdot n^{-v_{b\cap c}} p^{-e_{b\cap c}} \cdot n \cdot n^{-v_{(a\cup b\cup c)\cap d}} p^{-e_{(a\cup b\cup c)\cap d}} \\ &\leqslant n^2 p^{-4} \cdot \frac{1}{dn^2} \cdot \frac{1}{dpn^2} \cdot n \cdot \frac{p}{dn^2} = \frac{1}{d^3 p^4 n^3} \,. \end{aligned} \tag{19}$$

Next we consider the subcase of (I) when

$$v_{b\cap c} = 1 \qquad \text{and} \qquad V(F_c) \cap \{x_1, x_2\} = \varnothing \,.$$

Then we have $e_{b\cap c} = 0$ and $v_{a\cap b\cap c} = 0$. Since $(F_a \cap F_c) + e_1 \subseteq F$ and $((F_a \cup F_b \cup F_c) \cap F_d) + e_2 + \{y_1, y_2\} \subseteq F$ we get

$$\frac{1}{n^{v_{a\cap c}} p^{e_{a\cap c}}} \overset{(11)}{\leqslant} \frac{1}{dn^2} \qquad \text{and} \qquad \frac{1}{n^{v_{(a\cup b\cup c)\cap d}} p^{e_{(a\cup b\cup c)\cap d}}} \overset{(11)}{\leqslant} \frac{p}{dn^2}\,.$$

Consequently, in this case we have

$$\begin{aligned} f_Q(n,p) &= n^2 p^{-4} \cdot n^{-v_{a\cap c} - v_{b\cap c} + v_{a\cap b\cap c} - v_{(a\cup b\cup c)\cap d}} p^{-e_{a\cap c} - e_{b\cap c} + e_{a\cap b\cap c} - e_{(a\cup b\cup c)\cap d}} \\ &\leqslant n^2 p^{-4} \cdot n^{-v_{a\cap c}} p^{-e_{a\cap c}} \cdot n^{-1} \cdot n^{-v_{(a\cup b\cup c)\cap d}} p^{-e_{(a\cup b\cup c)\cap d}} \\ &\leqslant n^2 p^{-4} \cdot \frac{1}{dn^2} \cdot n^{-1} \cdot \frac{p}{dn^2} = \frac{1}{d^2 p^3 n^3}\,. \end{aligned} \tag{20}$$

For the last remaining cases we consider summands in (16) with (I) and

(A1) either $v_{b\cap c} = 1$ and $V(F_c) \cap \{x_1, x_2\} \neq \varnothing$ (and, hence, $V(F_b) \cap V(F_c) \subsetneq \{x_1, x_2\}$),

(A2) or $v_{b\cap c} = 0$.

In both cases together with (I) we get

$$v_{b\cap(a\cup c)\cap d} = |\{x_1, x_2\} \cap V(F_d)| \leqslant 1\,. \tag{21}$$

Based on (21) we treat both subcases in same way. We consider $((F_a \cup F_b) \cap F_c) + e_1 \subseteq F$, $(F_b \cap F_d) + e_2 \subseteq F$ and $((F_a \cup F_c) \cap F_d) + \{y_1, y_2\} \subseteq F$ and get

$$\frac{1}{n^{v_{(a\cup b)\cap c}} p^{e_{(a\cup b)\cap c}}} \overset{(11)}{\leqslant} \frac{1}{dn^2}\,,\ \frac{1}{n^{v_{b\cap d}} p^{e_{b\cap d}}} \overset{(11)}{\leqslant} \frac{1}{dn^2} \quad \text{and} \quad \frac{1}{n^{v_{(a\cup c)\cap d}} p^{e_{(a\cup c)\cap d}}} \overset{(11)}{\leqslant} \frac{1}{dn^2}\,,$$

which leads to

$$\begin{aligned} f_Q(n,p) &= n^2 p^{-4} \cdot n^{-v_{(a\cup b)\cap c} - v_{b\cap d} - v_{(a\cup c)\cap d} + v_{b\cap(a\cup c)\cap d}} \cdot p^{-e_{(a\cup b)\cap c} - e_{b\cap d} - e_{(a\cup c)\cap d} + e_{b\cap(a\cup c)\cap d}} \\ &\overset{(21)}{\leqslant} n^2 p^{-4} \cdot n^{-v_{(a\cup b)\cap c}} p^{-e_{(a\cup b)\cap c}} \cdot n^{-v_{b\cap d}} p^{-e_{b\cap d}} \cdot n^{-v_{(a\cup c)\cap d}} p^{-e_{(a\cup c)\cap d}} \cdot n \\ &\leqslant n^2 p^{-4} \cdot \left(\frac{1}{dn^2}\right)^3 \cdot n = \frac{1}{d^3 p^4 n^3}\,. \end{aligned} \tag{22}$$

Using the bounds from (18), (19), (20) and (22) and $pn \to \infty$ for $n \to \infty$ we summarize that there are constants $c', c > 0$ only depending on F, C_0 and C_1 such that for sufficiently large n

$$f_Q(n,p) \leqslant c' \left(\frac{1}{p^2 n^2} + \frac{1}{p^4 n^3} \right) .$$

Since the sum in (16) has finitely many summands, together with (13) and (16) it follows that

$$\mathrm{Var}(X_{e_1,e_2}) \leqslant \frac{c}{p^2 n^2} \left(1 + \frac{1}{p^2 n} \right) . \tag{23}$$

Recall that we want to show that there are at most $Dpn^2 n^{-\delta}$ pairs of edges e_1, e_2 in $G(n,p)$ so that $X_{e_1,e_2} > Dp^{-1} n^{-\delta}$ for some constant $D > 0$ independent of n and $\delta > 0$ chosen in (9). For this purpose we use Markov's Inequality and Chebyshev's Inequality. Let $t = p^{-1} n^{-\delta}$, then Chebyshev's Inequality tells us

$$\mathbb{P}(X_{e_1,e_2} \geqslant \mathbb{E}[X_{e_1,e_2}] + t) \leqslant \frac{\mathrm{Var}(X_{e_1,e_2})}{t^2} .$$

Let X be the number of pairs $(e_1, e_2) \in \binom{E(Z)}{2}$ with $X_{e_1,e_2} \geqslant 2p^{-1} n^{-\delta}$ and $e_1 \cap e_2 = \varnothing$. Since $\mathbb{E}[X_{e_1,e_2}] \leqslant t$ we have

$$\mathbb{E}[X] \leqslant \binom{pn^2}{2} \mathbb{P}(X_{e_1,e_2} \geqslant \mathbb{E}[X_{e_1,e_2}] + t) \tag{24}$$

$$\leqslant \frac{p^2 n^4}{2} \cdot \frac{cp^2 n^{2\delta}}{p^2 n^2} \left(1 + \frac{1}{np^2} \right) = \frac{1}{2} cp^2 n^{2+2\delta} \left(1 + \frac{1}{np^2} \right) . \tag{25}$$

We distinguish the cases $n^{-1} p^{-2} > 1$ and $n^{-1} p^{-2} \leqslant 1$. For $n^{-1} p^{-2} > 1$ we have for sufficiently large n

$$\mathbb{E}[X] \leqslant \frac{cp^2 n^{2+2\delta}}{np^2} \leqslant cn^{1+2\delta} \leqslant pn^{2-2\delta} ,$$

where the last inequality follows from our choice of $\delta < \frac{1}{4}(1 - \frac{1}{m_2(F)})$.

For the case $n^{-1} p^{-2} \leqslant 1$ we have for sufficiently large n

$$\mathbb{E}[X] \leqslant cp^2 n^{2+2\delta} \leqslant pn^{2-2\delta}$$

where the last inequality follows by the choice of $\delta < \frac{1}{4m_2(F)}$. Consequently, $\mathbb{E}[X] \leqslant pn^{2-2\delta}$ and by Markov's Inequality

$$\mathbb{P}(X > pn^{2-\delta}) \leqslant \frac{\mathbb{E}[X]}{pn^{2-\delta}} \leqslant n^{-\delta}$$

thus a.a.s. $X \leqslant pn^{2-\delta}$. For sufficiently large n this finishes the case $e_1 \cap e_2 = \varnothing$.

It remains the case when $|e_1 \cap e_2| = 1$. Now let $e_1, e_2 \in \binom{[n]}{2}$ with $|e_1 \cap e_2| = 1$. We repeat essentially the same calculations of the first case $e_1 \cap e_2 = \varnothing$ with the following differences.

- For the expectation of X_{e_1,e_2} in (12) we get
$$\mathbb{E}[X_{e_1,e_2}] = O\left(\frac{1}{np^2}\right).$$
- For the variance we will show
$$\operatorname{Var}(X_{e_1,e_2}) \leqslant \frac{c}{np^2}\left(1 + \frac{1}{np^2}\right).$$
In the calculation of the variance there is essentially one difference compared to the case $e_1 \cap e_2 = \varnothing$. In (15) we get
$$v_{a\cup b} - |\{x_1, x_2\} \cup \{v_1, u_1\} \cup \{v_2, u_2\}| \leqslant 2v(F) - 5$$
instead of $2v(F) - 6$ which leads to an additional n factor. This n factor carries over to
$$f_Q(n,p) := n^3p^{-4} \cdot n^{-v_{c\cap(a\cup b)}}p^{-e_{c\cap(a\cup b)}} \cdot n^{-v_{(a\cup b\cup c)\cap d}}p^{-e_{(a\cup b\cup c)\cap d}} \tag{26}$$
in (17).

  For the following case distinction we repeat in the case $\{x_1, x_2\} \subseteq V(F_c)$ the calculation, but keep the additional n factor. Consequently we get in (18)
$$f_Q(n,p) = O\left(\frac{1}{p^2n}\right).$$
Similarly we get with the additional n factor in (19)
$$f_Q(n,p) = O\left(\frac{1}{p^4n^2}\right).$$

The case $v_{b\cap c} = 1$ and $V(F_c) \cap \{x_1, x_2\} = \varnothing$ disappears since F_b and F_c intersect at least in $e_1 \cap e_2 \subseteq \{x_1, x_2\}$. For the same reason the case $v_{b\cap c} = 0$ disappears. For the last remaining case in (22) we get again the same bound with an additional factor of n

$$f_Q(n,p) = O\left(\frac{1}{p^4 n^2}\right) .$$

Consequently

$$\mathrm{Var}(X_{e_1,e_2}) \leqslant \frac{c}{np^2}\left(1 + \frac{1}{np^2}\right) .$$

- The expectation still satisfies $\mathbb{E}[X_{e_1,e_2}] \leqslant t$ for the same choice of $t = p^{-1}n^{-\delta}$. This follows since $\mathbb{E}[X_{e_1,e_2}] = O(\frac{1}{np^2})$, $t = \frac{1}{pn^\delta}$ and $\delta < 1 - \frac{1}{m_2(F)}$.
- Let X' be the number of pairs $(e_1, e_2) \in \binom{E(Z)}{2}$ satisfying $X_{e_1,e_2} \geqslant 2p^{-1}n^{-\delta}$ and $|e_1 \cap e_2| = 1$. We know by the condition $|e_1 \cap e_2| = 1$ that $X' \leqslant 2p^2n^3$, thus we get with X' instead of X in (24) a factor of $2p^2n^3$ instead of $\binom{pn^2}{2}$ which results in a factor of n^{-1} compared to the first case. Consequently the n^{-1} factor cancels with the n factor above which leads to the same order of magnitude in (25). Then the rest of the proof is the same as in the first case.

Setting $D' \geqslant 2$ sufficiently large such that $2p^{-1}n^{-\delta} \leqslant \frac{D'pn^2}{n^\delta}$ then yields

$$|\mathcal{P}_2(Z, e_1, e_2)| \leqslant \frac{D'}{pn^\delta} \tag{27}$$

for all but at most $\frac{D'pn^2}{n^\delta}$ pairs of edges $e_1, e_2 \in E(Z)$.

Case 2 : s > 2. We consider for $s > 2$ configurations from $\mathcal{P}_s(G(n,p), e_1, e_2)$, so for two pairs $e_1 \neq e_2 \in \binom{[n]}{2}$ let Y_{e_1,e_2} be the random variable given by $|\mathcal{P}_s(G(n,p), e_1, e_2)|$. Here it is sufficient to use Markov's inequality instead of Chebyshev's inequality which will allow us to avoid the calculation of the variance, but we still have to distinguish the cases $e_1 \cap e_2 = \varnothing$ and $|e_1 \cap e_2| = 1$.

For the first case let $e_1 \cap e_2 = \varnothing$. The expectation of Y_{e_1,e_2} is

$$\mathbb{E}[Y_{e_1,e_2}] \leqslant e(F)^4 n^{2v(F)-4-s} v(F)^s p^{2e(F)-4} \overset{(10)}{\leqslant} e(F)^4 v(F)^s C_1^{2e(F)-2} n^{-s} p^{-2} \leqslant C' n^{-3} p^{-2}$$

with $C' = e(F)^4 v(F)^s C_1^{2e(F)-2}$. We use Markov's inequality and get

$$\mathbb{P}\left(Y_{e_1,e_2} \geqslant \frac{1}{pn^\delta}\right) \leqslant C' n^{-3} p^{-2} \cdot pn^\delta = C' p^{-1} n^{-3+\delta}\,.$$

Let Y be the number of pairs $e_1, e_2 \in E(Z)$ with $e_1 \cap e_2 = \varnothing$ and $Y_{e_1,e_2} \geqslant p^{-1}n^{-\delta}$. Then

$$\mathbb{E}[Y] \leqslant \binom{pn^2}{2} C' n^{-3+\delta} p^{-1} \leqslant \frac{C' pn^{1+\delta}}{2}$$

and a second use of Markov's inequality yields

$$\mathbb{P}(Y \geqslant pn^{2-\delta}) \leqslant \frac{C' pn^{1+\delta}}{2pn^{2-\delta}} = o(1)$$

where the last inequality follows from our choice $\delta < 1/2$ and for sufficiently large n.

We repeat the same proof for the case $|e_1 \cap e_2| = 1$ with the following differences.

- $\mathbb{E}[Y_{e_1,e_2}] \leqslant C'' n^{-2} p^{-2}$ for some $C'' > 0$.
- $\mathbb{P}\left(Y_{e_1,e_2} \geqslant \frac{1}{pn^\delta}\right) \leqslant C'' p^{-1} n^{-2+\delta}$.
- $\mathbb{E}[Y] \leqslant 2p^2 n^3 C'' p^{-1} n^{-2+\delta} \leqslant 2C'' pn^{1+\delta}$.
- $\mathbb{P}(Y \geqslant pn^{2-\delta}) \leqslant \frac{2C'' pn^{1+\delta}}{pn^{2-\delta}} = o(1)$.

Consequently for all $s \geqslant 3$ we have $|\mathcal{P}_s(G(n,p), e_1, e_2)| \leqslant p^{-1} n^{-\delta}$ for all but at most $pn^{2-\delta}$ pairs of edges $e_1, e_2 \in E(Z)$. Together with (27) this concludes the proof of (Z4) and finishes the proof of Lemma 30. $\square$

The next lemma concerns property (Z5), which bounds the number of bad embeddings as defined in Definition 26.

Lemma 32. *For all graphs B and all strictly balanced graphs F, for all $C_1 \geqslant C_0 > 0$ and for $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ there exists $\zeta > 0$ such that a.a.s. $G(n,p)$ satisfies* (Z5).

Proof of Lemma 32. We shall show that there exist a $\xi > 0$ such that for any given $h \in \Psi_{B,n}$ we have for sufficiently large n

$$\mathbb{P}(h \text{ is bad w.r.t. } F \text{ and } G(n,p)) \leqslant n^{-\xi}\,.$$

Then the lemma follows from Markov's inequality with $\zeta = \xi/2$.

Let $h \in \Psi_{B,n}$ be fixed. We first consider the case that h is bad w.r.t. F and $G(n,p)$ because of (B1). Since F is strictly balanced, for all proper subgraphs $F_0 \subsetneq F$ with $e(F_0) \geqslant 2$ we have

$$\begin{aligned} p^{e(F_0)}n^{v(F_0)} &= pn^2 \cdot p^{e(F_0)-1}n^{v(F_0)-2} \\ &\geqslant pn^2 \cdot C_0^{e(F_0)-1} n^{-\frac{1}{m_2(F)}(e(F_0)-1)+v(F_0)-2} \\ &= pn^2 \cdot C_0^{e(F_0)-1} n^{(e(F_0)-1)\left(\frac{v(F_0)-2}{e(F_0)-1}-\frac{1}{m_2(F)}\right)} \\ &= pn^2 \cdot C_0^{e(F_0)-1} n^{(e(F_0)-1)\left(\frac{1}{d_2(F_0)}-\frac{1}{d_2(F)}\right)} \\ &\geqslant pn^2 \cdot n^{\xi'} \end{aligned} \tag{28}$$

for some $\xi' > 0$. We bound the probability for h being bad because of case (B1) by estimating the number of configurations leading to this event. In this case F_0 stands for the part of F that is contained in $h(B)$ and hence consists of at least two edges. Using again $n^{v(F)-2}p^{e(F)-1} \leqslant C_1^{e(F)-1}$ yields

$$\begin{aligned} \mathbb{P}(h \text{ is bad by (B1)}) &\leqslant \sum_{F_0 \subsetneq F, e(F_0)\geqslant 2} v(B)^{v(F_0)} n^{v(F)-v(F_0)} p^{e(F)-e(F_0)} \\ &\overset{(28)}{\leqslant} \sum_{F_0 \subsetneq F, e(F_0)\geqslant 2} v(B)^{v(F_0)} C_1^{e(F)-1} n^{-\xi'} \leqslant n^{-\xi_1} \end{aligned}$$

for some $\xi_1 > 0$ and sufficiently large n.

When we address the case (B2) we can assume that h is not bad because of case (B1). Hence, it suffices to consider copies F_1 and F_2 of F each intersecting $h(B)$ in precisely one edge and $F_0 := F_1 \cap F_2$ having no edge in $h(B)$. Again we will use $n^{v(F)-2}p^{e(F)-1} \leqslant C_1^{e(F)-1}$ and that $n^{v(F_0)}p^{e(F_0)} \geqslant dpn^2$ for $F_0 \subsetneq F$ with $e(F_0) \geqslant 1$ for some $d > 0$ only depending on F and C_0 (see (11)). Note that two fixed edges of $h(B)$ determine at least three vertices of $F_1 \cup F_2$.

$$\begin{aligned}
\mathbb{P}(h \text{ is bad by (B2) and not by (B1)}) &\leqslant \sum_{\substack{F_0 \subsetneq F \\ e(F_0) \geqslant 1}} v(B)^4 n^{2v(F)-v(F_0)-3} p^{2e(F)-e(F_0)-2} \\
&\leqslant \sum_{F_0} v(B)^4 C_1^{2e(F)-2} \frac{n}{p^{e(F_0)} n^{v(F_0)}} \\
&\leqslant \sum_{F_0} v(B)^4 C_1^{2e(F)-2} \frac{1}{dpn} \\
&\leqslant \sum_{F_0} v(B)^4 C_1^{2e(F)-3} d^{-1} n^{-(1-\frac{1}{m_2(F)})} \leqslant n^{-\xi_2}
\end{aligned}$$

for some $\xi_2 > 0$ since $m_2(F) > 1$.

For case (B3) we assume that h is not bad because of case (B1) or case (B2). Again we bound the probability by the expected number of options to obtain a configuration as in (B3). In this case F_0 stands for the intersection of two different copies of F and includes at least two edges, e and f from (B3), where f is also contained in $h(B)$.

$$\begin{aligned}
&\mathbb{P}(h \text{ is bad by (B3) and not by (B1) or (B2)}) \\
&\qquad \leqslant \sum_{\substack{F_0 \subsetneq F \\ e(F_0) \geqslant 2}} v(B)^2 n^{2v(F)-v(F_0)-2} p^{2e(F)-e(F_0)-1} \\
&\qquad \leqslant \sum_{F_0} v(B)^2 C_1^{2e(F)-2} \cdot pn^2 \cdot \frac{1}{p^{e(F_0)} n^{v(F_0)}} \\
&\qquad \overset{(28)}{\leqslant} n^{-\xi_3}
\end{aligned}$$

for some $\xi_3 > 0$ and, hence, $\mathbb{P}(h \text{ is bad}) \leqslant n^{-\xi}$ for any $0 < \xi < \min\{\xi_1, \xi_2, \xi_3\}$ and sufficiently large n. □

3.3.2. Restricting Embeddings of B. In this section we focus on restricting the family $\Psi_{B,n}$ of all embeddings B in K_n to a suitable subset $\Xi_{B,n}$ so that we can apply Theorem 17 for the proof of Lemma 23. In particular, our choice of $\Xi_{B,n}$ will ensure conditions on the maximum degree and maximum pair degree of $\mathcal{H} = \mathcal{H}(Z, \Xi_{B,n})$. For the control of the pair degree of $\mathcal{H}$ the following definition will be useful.

Definition 33. *For a pair of edges* $e_1, e_2 \in E(Z)$ *and an embedding* $h \in \Xi_{B,n} \subseteq \Psi_{B,n}$ *we write* $e_1 \approx_h e_2$ *if* e_1 *and* e_2 *both focus on* $h(B)$. *Moreover, if* e_1 *and* e_2 *focus jointly on only one edge of* $h(B)$, *then we write* $e_1 \sim_h e_2$. *We denote by* $c_{\Xi_{B,n}}(e_1, e_2)$ *the number of* $h \in \Xi_{B,n}$ *such that* $e_1 \approx_h e_2$.

In the next definition and lemma we define the properties of the desired family of embeddings.

Definition 34. *Let* F, B *be graphs and let* $\alpha > 0$. *We call a family* $\Xi_{B,n} \subseteq \Psi_{B,n}$ *of embeddings of* B *into* K_n α*-normal if the following conditions are satisfied.*

(N1) $|\Xi_{B,n}| \geqslant \alpha n^2$ *and*

(N2) $|V(h(B)) \cap V(h'(B))| \leqslant 1$ *for all* $h \neq h' \in \Xi_{B,n}$.

Lemma 35. *Let* F *and* B *be graphs. For all constants* $\frac{1}{3} > \alpha > 0$, $D > 0$, $1 > \zeta > 0$, $\min\{\frac{1}{m_2(F)}, 1 - \frac{1}{m_2(F)}\} > \delta > 0$, *and* $C_1 > C_0 > 0$ *there exists* $n_0 \in \mathbb{N}$ *such that for all* $n \geqslant n_0$ *and* $C_0 n^{-1/m_2(F)} \leqslant p \leqslant C_1 n^{-1/m_2(F)}$ *the following holds. If* $Z \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$ *and*

$$\mathbb{P}(Z \cup h(B) \to (F)^e_2) > 1 - \alpha$$

where $h \in \Psi_{B,n}$ *chosen uniformly at random then there exists* $\Xi^0_{B,n} \subseteq \Psi_{B,n}$ *such that*

(Ξ*1*) $\Xi^0_{B,n}$ *is* $\widetilde{\alpha}$*-normal for* $\widetilde{\alpha} = \widetilde{\alpha}(B) = \frac{1}{13 v(B)^4 v(B)!} > 0$,

(Ξ*2*) $Z \cup h(B) \to (F)^e_2$ *for all* $h \in \Xi^0_{B,n}$,

(Ξ*3*) *for all pairs* $\{e_1, e_2\} \in \binom{E(Z)}{2}$ *we have* $c_{\Xi^0_{B,n}}(e_1, e_2) \leqslant \frac{1}{p n^{\delta/2}}$,

(Ξ*4*) h *is not bad w.r.t.* F *and* Z *for all* $h \in \Xi^0_{B,n}$ *(see Definition 26), and*

(Ξ*5*) *for all* $h \in \Xi^0_{B,n}$ *we have* $E(h(B)) \cap E(Z) = \varnothing$.

A family $\Xi^0_{B,n}$ *is* $(\widetilde{\alpha}, Z)$*-normal if it satisfies conditions* (Ξ*1*), (Ξ*2*), (Ξ*3*), (Ξ*4*), *and* (Ξ*5*) *for a given* $Z \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$.

Proof of Lemma 35. Given F, B and the constants as above we set

$$\widetilde{\alpha} = \frac{1}{13v(B)^4 v(B)!}\,.$$

Let $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ and suppose $\mathbb{P}(Z \cup h(B) \to (F)^e_2) > 1 - \alpha$.

For the construction of $\Xi^0_{B,n}$ we start with the family $\Psi_{B,n}$ and remove embeddings that do not satisfy property (Ξ*2*), embeddings that do not satisfy property (Ξ*4*) and embeddings that will later lead to problems for (Ξ*3*). After that we choose at random $2\widetilde{\alpha}n^2$ embeddings which will induce property (Ξ*3*) and show that after deleting the embeddings that intersect in more than one vertex we keep $C\widetilde{\alpha}n^2$ of them with $C > 1$. Afterwards we remove embeddings not satisfying (Ξ*5*). Since $e(Z) = \Theta(pn^2)$ we keep at least $(C\widetilde{\alpha} - o(1))n^2 > \widetilde{\alpha}n^2$ embeddings h, which finishes the proof.

Since $\mathbb{P}(Z \cup h(B) \to (F)^e_2) > 1 - \alpha > 2/3$ there is a family $\Psi^1_{B,n} \subseteq \Psi_{B,n}$ of embeddings of B of size $\frac{2}{3}|\Psi_{B,n}|$ such that $Z \cup h(B) \to (F)^e_2$ for all $h \in \Psi^1_{B,n}$, i.e. $\Psi^1_{B,n}$ satisfies (Ξ*2*).

Moreover, since $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ there are at most $n^{-\zeta}|\Psi_{B,n}|$ embeddings that are bad w.r.t. F and Z. We remove those bad embeddings from $\Psi^1_{B,n}$. In this way for sufficiently large n we obtain a family $\Psi^2_{B,n} \subseteq \Psi^1_{B,n}$ of size at least $\frac{1}{2}|\Psi_{B,n}|$ that contains no bad embedding and, therefore, $\Psi^2_{B,n}$ satisfies (Ξ*4*).

Since $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ there are at most $\frac{Dpn^2}{n^\delta}$ pairs of distinct edges $e_1, e_2 \in E(Z)$ such that $|\mathcal{P}(Z,e_1,e_2)| > \frac{D}{pn^\delta}$. For those pairs of edges e_1, e_2 we delete all embeddings $h \in \Psi^2_{B,n}$ with $e_1 \sim_h e_2$. Since $|\mathcal{F}_-(Z,e)| \leqslant \frac{D}{p}$ for all $e \in E(Z)$ for $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ we delete at most

$$\frac{Dpn^2}{n^\delta} \cdot \frac{D}{p} v(F)^2 n^{v(B)-2} = \frac{D^2 v(F)^2 n^{v(B)}}{n^\delta} = o(|\Psi_{B,n}|)$$

embeddings from $\Psi^2_{B,n}$. So we get for sufficiently large n a family $\Psi^3_{B,n} \subseteq \Psi^2_{B,n}$ of size at least $\frac{1}{3}|\Psi_{B,n}|$ such that for all distinct e_1, $e_2 \in E(Z)$ we have

(F1) if $e_1 \sim_h e_2$ for some $h \in \Psi^3_{B,n}$, then $|\mathcal{P}(Z,e_1,e_2)| \leqslant \frac{D}{pn^\delta}$.

Next we will select a subset $\Psi^4_{B,n} \subseteq \Psi^3_{B,n}$, which allows us to bound $c_{\Psi^4_{B,n}}(e_1, e_2)$ for every pair of edges of Z. For this purpose for

$$\varepsilon = 2\widetilde{\alpha} = \frac{2}{13v(B)^4 v(B)!}$$

we select with repetition εn^2 times an element of $\Psi^3_{B,n}$, where we assume for simplicity that εn^2 is an integer. For every selection S we define a family of embeddings $\Psi_S \subseteq \Psi^3_{B,n}$ by taking all embeddings that were chosen at least once in S. We will show that the random selection S a.a.s. satisfies that $c_{\Psi_S}(e_1, e_2) \leqslant \frac{1}{pn^{\delta/2}}$ for all $e_1, e_2 \in E(Z)$ and that with probability less than $\frac{1}{2}$ there are more than $\frac{\varepsilon}{2} n^2$ embeddings that share at least two vertices with some other embedding in the selection.

First we show that a.a.s. $c_{\Psi_S}(e_1, e_2) \leqslant \frac{1}{pn^{\delta/2}}$ for all $e_1, e_2 \in E(Z)$. Since there are no bad embeddings w.r.t. F and Z in $\Psi^3_{B,n}$ we know that if e focuses on $h(B)$ then e focuses on exactly one edge in $E(h(B))$ (see property (B1) in Definition 26). Hence, for $e_1 \approx_h e_2$ we may consider the following two cases. Either $e_1 \sim_h e_2$ or e_1 and e_2 focus on two different edges in $h(B)$.

For the first case we shall use (F1) and $|\Psi^3_{B,n}| \geqslant \frac{1}{3}\binom{n}{v(B)}$ to bound the probability that $e_1 \sim_{h_i} e_2$. In fact,

$$\begin{aligned}
\mathbb{P}(e_1 \sim_{h_i} e_2) &\leqslant \frac{D}{pn^{\delta}} \cdot v(F)^2 \cdot \frac{v(B)^2 \cdot (n-2) \cdots (n - v(B) + 1)}{|\Psi^3_{B,n}|} \\
&\leqslant \frac{3Dv(F)^2 v(B)^2 v(B)!}{pn^{2+\delta}} .
\end{aligned}$$

In the second case we shall use (Z3) of Definition 28 for the upper bound on $|\mathcal{F}_-(Z, e)|$. This and the fact that two edges fix at least three vertices yield

$$\begin{aligned}
\mathbb{P}(e_1 \approx_{h_i} e_2 \text{ and not } e_1 \sim_{h_i} e_2) &\leqslant \frac{D^2}{p^2} \cdot v(F)^4 \cdot \frac{v(B)^3 \cdot (n-3) \cdots (n - v(B) + 1)}{|\Psi^3_{B,n}|} \\
&\leqslant \frac{3D^2 v(F)^4 v(B)^3 v(B)!}{p^2 n^3} .
\end{aligned}$$

Consequently

$$\mathbb{P}(e_1 \approx_{h_i} e_2) \leqslant 3Dv(F)^2 v(B)^2 v(B)! \left(\frac{1}{pn^{2+\delta}} + \frac{Dv(F)^2 v(B)}{p^2 n^3} \right) . \tag{29}$$

Since $\delta < 1 - \frac{1}{m_2(F)}$ we infer $n^\delta < C_0 n^{1-\frac{1}{m_2(F)}} < pn$ for sufficiently large n. Therefore the right hand side of (29) is of order $\Theta(\frac{1}{pn^{2+\delta}})$ and we can bound

$$\mathbb{P}(e_1 \approx_{h_i} e_2) \leqslant \frac{D_0}{pn^{2+\delta}} .$$

where $D_0 = 4Dv(F)^2 v(B)^2 v(B)!$. For the expected number of connections we get

$$\mathbb{E}[c_{\Psi_S}(e_1, e_2)] \leqslant \sum_{i=1}^{\varepsilon n^2} \mathbb{P}(e_1 \approx_{h_i} e_2) \leqslant \frac{\varepsilon D_0}{pn^\delta} .$$

Consequently, Chernoff's Inequality yields

$$\mathbb{P}\left(c_{\Psi_S}(e_1, e_2) \geqslant \frac{3}{2} \cdot \frac{\varepsilon D_0}{pn^\delta} \right) \leqslant \exp\left(-\frac{1}{12} \cdot \frac{\varepsilon D_0}{pn^\delta} \right) .$$

Note that $\frac{1}{pn^\delta} > n^\beta$ for some $\beta > 0$ since $\delta < \frac{1}{m_2(F)}$, hence, we can apply the union bound for all pairs of edges $e_1, e_2 \in E(Z)$ and get that a.a.s.

$$c_{\Psi_S}(e_1, e_2) \leqslant \frac{3\varepsilon D_0}{2pn^\delta} \leqslant \frac{1}{pn^{\delta/2}} .$$

Finally we verify that most pairs of selected embeddings intersect in at most one vertex. In fact, for $i = 1, \ldots, \varepsilon n^2$ let 1_{h_i} be the indicator random variable for the event "there is some $j \in [\varepsilon n^2] \setminus \{i\}$ such that $v(h_i(B) \cap h_j(B)) \geqslant 2$" and set $Y = \sum_{i=1}^{\varepsilon n^2} 1_{h_i}$. Then

$$\mathbb{E}[1_{h_1}] \leqslant (\varepsilon n^2 - 1) \frac{\binom{v(B)}{2} \cdot v(B)(v(B)-1) \cdot (n-2) \cdots (n - v(B) + 1)}{|\Psi^3_{B,n}|} \leqslant D_1 \varepsilon$$

for some constant $D_1 = D_1(B)$ with $0 < D_1 < \frac{3}{2} v(B)^4 v(B)!$ independent of ε. Hence,

$$\mathbb{E}[Y] \leqslant \varepsilon n^2 D_1 \varepsilon = D_1 \varepsilon^2 n^2$$

and by Markov's Inequality we get

$$\mathbb{P}(Y > 2\mathbb{E}[Y]) \leqslant \frac{1}{2} ,$$

so there is a selection S of εn^2 embeddings such that $Y \leqslant 2D_1\varepsilon^2 n^2$ and $c_{\Psi_S}(e_1, e_2) \leqslant \frac{1}{pn^{\delta/2}}$ for all pairs of edges. For this choice of S we can simply delete all those embeddings h_i that intersect with some other embedding h_j in at least two vertices. We call the remaining family $\Psi^4_{B,n}$. Using $D_1 \leqslant 3v(B)^4 v(B)!/2$ and the definition $\varepsilon = 2\widetilde{\alpha} = \frac{2}{13v(B)^4 v(B)!}$ yields

$$|\Psi^4_{B,n}| \geqslant \varepsilon n^2 - 2D_1\varepsilon^2 n^2 \geqslant C\widetilde{\alpha} n^2$$

for some $C > 1$ and, hence, $\Psi^4_{B,n}$ satisfies (Ξ*1*)–(Ξ*4*).

To achieve (Ξ*5*) we make use of $e(Z) \leqslant pn^2$ (see (Z1) of Definition 28). Since no two embeddings from $\Psi^4_{B,n}$ share an edge, we may remove all embeddings from $\Psi^4_{B,n}$ which share at least one edge with Z and this results in the desired family $\Xi^0_{B,n} \subseteq \Psi^4_{B,n}$ of size at least $\widetilde{\alpha} n^2$, which finishes the proof. $\square$

For Lemma 23 we have to show that there is a family of embeddings $\Xi_{B,n}$ such that the hypergraph $\mathcal{H}(Z, \Xi_{B,n})$ is index consistent with a profile π. Lemma 36 will ensure this.

Lemma 36. *For all constants $1 > \widetilde{\alpha} > 0$ and $D > 0$, for all graphs F and B with F being strictly balanced and with $E(B) = \{e_1, \ldots, e_K\}$, there exist $\alpha' > 0$ and $L \in \mathbb{N}$ such that every graph Z on n vertices with a fixed ordering of its edge set and the property*

(Z) $|\mathcal{F}_{-}(Z)| \leqslant Dn^2$

satisfies the following.

For every $(\widetilde{\alpha}, Z)$-normal family $\Xi^0_{B,n}$ there is an (α', Z)-normal family $\Xi_{B,n} \subset \Xi^0_{B,n}$ and there is a profile π of length at most L such that $(Z, \Xi_{B,n})$ is index consistent with profile π.

Below we consider Z and B to be fixed graphs and for a simpler notation we set

$$M_h = M(Z, h(B))$$

for $h \in \Psi_{B,n}$ (see (6) for the definition of $M(Z, h(B))$). Note that it is rather unlikely that M_h and $M_{h'}$ of $\mathcal{H}$ are equal for distinct $h, h' \in \Xi^0_{B,n}$ and, hence, Lemma 36 follows by a simple averaging argument. We will use Lemma 36 for $Z \in \mathcal{G}_{B,F,n,p}(D, \zeta, \delta)$ which satisfies (Z) by (Z2) from Definition 28.

Proof of Lemma 36. Let $1 > \widetilde{\alpha} > 0$, $D > 0$, F and B be given. We define

$$L = (e(F) - 1)\frac{2}{\widetilde{\alpha}} v(F)^2 D \qquad \text{and} \qquad \alpha' = \frac{\widetilde{\alpha}}{2L(KL)^L}\,.$$

Given some Z satisfying (Z) and an $(\widetilde{\alpha}, Z)$-normal family $\Xi^0_{B,n} \subseteq \Psi_{B,n}$ we will restrict $\Xi^0_{B,n}$ to the promised set $\Xi_{B,n}$ with the desired properties.

Note that the family $\Xi_{B,n} \subseteq \Xi^0_{B,n}$ inherits the properties (Ξ*2*)–(Ξ*5*) from the $(\widetilde{\alpha}, Z)$-normality of $\Xi^0_{B,n}$ since they are independent of $\widetilde{\alpha}$. Consequently, to establish that $\Xi_{B,n}$ is indeed (α', Z)-normal, we only have to focus on (Ξ*1*). Since again property (N2) of Definition 34 is inherited from the normality of $\Xi^0_{B,n}$, it suffices to show that $|\Xi_{B,n}| \geqslant \alpha' n^2$.

Because of (Z) we know that Z contains at most Dn^2 copies of some $F' \subseteq F$ with $e(F') = e(F) - 1$. Also due to $\Xi^0_{B,n}$ being $(\widetilde{\alpha}, Z)$-normal (see (Ξ*4*)) there are no bad embeddings w.r.t. F and Z in $\Xi^0_{B,n}$ and thus by Fact 27 the pair $(Z, \Xi^0_{B,n})$ is regular. In particular, for every $h \in \Xi^0_{B,n}$ we have that every edge $e \in M_h$ focuses on exactly one $b \in E(h(B))$. Furthermore, since every $h \in \Xi^0_{B,n}$ also does not satisfy (B3) of Definition 26, each $e \in M_h$ focuses on one $b \in E(h(B))$ in only one way, i.e. there is only one copy of F in $Z \cup h(B)$ containing b and e. Therefore, $\ell_h = |M_h|$ is a multiple of $e(F)-1$ and each M_h gives rise to $\ell_h/(e(F)-1)$ copies of graphs F' in Z, where each such F' is obtained from F by removing some edge. Clearly, each such $(e(F) - 1)$-element subset of M_h might be completed to a copy of F in at most $\binom{v(F)}{2} - e(F) + 1 < v(F)^2$ ways.

Applying the upper bound on the number of copies of F with one edge removed from (Z) yields

$$\sum_{h \in \Xi^0_{B,n}} \frac{\ell_h}{e(F) - 1} \leqslant v(F)^2 \cdot Dn^2\,.$$

So there are at most $\widetilde{\alpha}n^2/2$ embeddings $h \in \Xi^0_{B,n}$ with $\ell_h > L$, and, consequently, at least $\widetilde{\alpha}n^2/2$ embeddings $h \in \Xi^0_{B,n}$ with $\ell_h \leqslant L$. Since there are at most K^ℓ different profiles of length ℓ, there must be a profile π of length $\ell \leqslant L$ and a subset $\Xi'_{B,n} \subseteq \Xi^0_{B,n}$ with

$$|\Xi'_{B,n}| \geqslant \frac{1}{LK^L} \cdot \frac{\widetilde{\alpha}}{2} n^2$$

such that $(Z, \Xi'_{B,n})$ has profile π.

Next we apply another averaging argument to achieve index consistency. We consider some partition $Z_1 \cup \ldots \cup Z_\ell$ of Z into ℓ classes chosen uniformly at random. Recall that we ordered the edges of Z. For $h \in \Xi'_{B,n}$ consider $M_h = (z_1, \ldots, z_\ell)$ with the inherited ordering of Z. We include h in $\Xi_{B,n}$ if $z_i \in Z_i$ for all $i = 1, \ldots, \ell$. Clearly $\mathbb{P}(h \in \Xi_{B,n}) = \frac{1}{\ell^\ell}$ and $\mathbb{E}[|\Xi_{B,n}|] = \frac{|\Xi'_{B,n}|}{\ell^\ell}$, which means there is an $\Xi_{B,n} \subseteq \Xi'_{B,n}$ with

$$|\Xi_{B,n}| \geqslant |\Xi'_{B,n}|/\ell^\ell \geqslant \frac{1}{L^L} \frac{\widetilde{\alpha}n^2}{2LK^L} = \alpha' n^2 \,.$$

Now let $h, h' \in \Xi_{B,n}$ and let $z \in M_h \cap M_{h'}$. Since $z \in Z_j$ for some partition class Z_j we know that z has index j in both M_h and $M_{h'}$. Therefore $(Z, \Xi_{B,n})$ is index consistent which finishes the proof. □

3.3.3. Proof of Lemma 23. Finally we prove Lemma 23. The previous lemmas will be utilised to show that the hypergraph $\mathcal{H}(Z, \Xi)$ satisfies the conditions of Theorem 17 of Saxton and Thomason about independent sets in hypergraphs.

Proof of Lemma 23. Let constants $C_1 > C_0 > 0$, $\frac{1}{3} > \alpha > 0$ and graphs F and B with F being strictly balanced be given.

First we fix all constants used in the proof. For the given graphs F and B and the given constants C_1 and C_0 Lemma 29 yields constants $D > 0$, $\zeta > 0$, and δ with $0 < \delta < \min\left\{\frac{1}{m_2(F)}, 1 - \frac{1}{m_2(F)}\right\}$. Similarly Lemma 36 applied to F, B, D and

$$\widetilde{\alpha} = \frac{1}{13v(B)^4 v(B)!}$$

yields α' and L. Fixing an auxiliary constant

$$k = \binom{L}{e(F)-1}\binom{v(F)}{2}$$

allows us to set

$$\beta = \frac{\alpha'}{Dkv(F)^2} \quad \text{and} \quad \gamma = \frac{\delta}{10L}\,. \tag{30}$$

We shall show that α', β, γ, and L defined this way have the desired property. For that let $p = p(n) = c(n)n^{-1/m_2(F)}$ for some $c(n)$ satisfying $C_0 \leqslant c(n) \leqslant C_1$. We shall show that $G(n,p)$ a.a.s. satisfies the property of Lemma 23. Hence, in view of Lemma 29 we may assume that the graphs Z considered in Lemma 23 are from the set $\mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$. Moreover, let n be sufficiently large, so that Lemma 35 applied with F, B, α, D, ζ, δ, C_1 and C_0 holds for n.

Now let $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ such that for $h \in \Psi_{B,n}$ chosen uniformly at random we have

$$\mathbb{P}(Z \cup h(B) \to (F)^e_2) > 1 - \alpha\,.$$

Then Lemma 35 yields an $(\tilde{\alpha}, Z)$-normal family of embeddings $\Xi^0_{B,n} \subseteq \Psi_{B,n}$, i.e. the family $\Xi^0_{B,n}$ satisfies properties $(\,\Xi\mathit{1})$–$(\,\Xi\mathit{5})$ of Lemma 35 for the parameters chosen above.

Since $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ it satisfies property (Z2) of Definition 28 and, hence, Z satisfies in particular assumption (Z) of Lemma 36. Consequently, Lemma 36 yields an (α', Z)-normal family $\Xi_{B,n} \subseteq \Xi^0_{B,n}$ and a profile π of length $\ell \leqslant L$ such that the pair $(Z, \Xi_{B,n})$ is index consistent for π.

Next we consider the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi_{B,n})$ defined by

$$V(\mathcal{H}) = E(Z) \quad \text{and} \quad E(\mathcal{H}) = \{M(Z, h(B))\colon h \in \Xi_{B,n}\}\,,$$

where

$$M(Z, h(B)) = \{z \in E(Z)\colon \text{there is } b \in E(h(B)) \text{ such that } z \text{ focuses on } b\}\,.$$

Clearly, $\mathcal{H}$ is an ℓ-uniform hypergraph on $m = e(Z)$ vertices. Below we show that $\mathcal{H}$ satisfies the assumptions of Theorem 17 for

$$\varepsilon = \tfrac{1}{4} \qquad \text{and} \qquad \tau = n^{-\frac{\delta}{4(\ell-1)}}\,.$$

Since $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ it displays properties (Z1)–(Z5) of Definition 28. In particular, the property (Z1) guarantees

$$\frac{1}{4}pn^2 \leqslant e(Z) = m \leqslant pn^2 < n^2\,. \tag{31}$$

Now we bound $e(\mathcal{H})$. Since $\Xi_{B,n}$ is α'-normal, it follows from (N1) and (N2) of Definition 34 that $\alpha' n^2 \leqslant |\Xi_{B,n}| \leqslant n^2$ and, consequently, we have $e(\mathcal{H}) \leqslant n^2$. On the other hand, for any hyperedge M_h of size ℓ there are at most $\binom{\ell}{e(F)-1}$ different copies of some $F' \subseteq F$ with $e(F') = e(F) - 1$ in M_h and each such copy can be extended to F by at most $\binom{v(F)}{2}$ different boosters since all boosters are edge disjoint. Consequently, M_h could be the hyperedge for at most $\binom{\ell}{e(F)-1}\binom{v(F)}{2} \leqslant k$ different embeddings $h \in \Xi_{B,n}$ and, therefore, we have

$$\frac{\alpha' n^2}{k} \leqslant e(\mathcal{H}) \leqslant n^2\,. \tag{32}$$

Hence, for the average degree of $\mathcal{H}$ we obtain

$$d(\mathcal{H}) = \ell \cdot \frac{e(\mathcal{H})}{v(\mathcal{H})} \geqslant \ell \cdot \frac{\alpha' n^2}{k} \cdot \frac{1}{pn^2} = \frac{\ell\alpha'}{kp}\,.$$

We denote by $\Delta_1(\mathcal{H}) = \max_{v \in V(\mathcal{H})} |\{e \in E(\mathcal{H}) : e \text{ contains } v\}|$ the maximum vertex degree and by $\Delta_2(\mathcal{H}) = \max_{(v,v') \in \binom{V(\mathcal{H})}{2}} |\{e \in E(\mathcal{H}) : e \text{ contains } v \text{ and } v'\}|$ the maximum codegree of $\mathcal{H}$ and below we will bound $\Delta_1(\mathcal{H})$ and $\Delta_2(\mathcal{H})$.

We start with $\Delta_1(\mathcal{H})$. Suppose $e \in M(Z, h(B))$ for some $h \in \Xi_{B,n}$. Since $\Xi_{B,n}$ contains no bad embeddings w.r.t. F and Z and $E(h(B)) \cap E(Z) = \varnothing$ there exists a unique copy $F_- \in \mathcal{F}_-(Z,e)$ with $e \in E(F_-)$ and $f \in h(B)$ such that $F_- + f$ forms a copy of F. Moreover, since every two distinct embeddings h, $h' \in \Xi_{B,n}$ intersect in at most one vertex the degree of e in $\mathcal{H}$ is bounded by $|\mathcal{F}_-(Z,e)| \cdot \binom{v(F)}{2}$.

Consequently, it follows from property (Z3) given by $Z \in \mathcal{G}_{B,F,n,p}(D,\zeta,\delta)$ that

$$\Delta_1(\mathcal{H}) \leqslant \frac{D}{p} \cdot \binom{v(F)}{2}.$$

For $\Delta_2(\mathcal{H})$ we have to look at pairs of edges of Z. Two edges $e_1, e_2 \in E(Z)$ are both contained in $M(Z, h(B))$ if and only if $e_1 \approx_h e_2$. By (Ξ*3*) we know $c_{\Xi_{B,n}}(e_1, e_2) \leqslant \frac{1}{pn^{\delta/2}}$, so

$$\Delta_2(\mathcal{H}) \leqslant \frac{1}{pn^{\frac{\delta}{2}}}.$$

Note that $pn^{\delta/2} \to 0$ for $n \to \infty$ since $\delta \leqslant \frac{1}{m_2(F)}$.

In order to verify the assumptions of Theorem 17 we estimate $\delta(\mathcal{H}, \tau)$ for ε and τ defined above. Indeed we have

$$\begin{aligned}
\delta(\mathcal{H}, \tau) &= 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \frac{1}{\tau^{j-1} m d(\mathcal{H})} \sum_{v \in V(\mathcal{H})} d^{(j)}(v) \\
&\leqslant 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \frac{1}{\tau^{j-1} m d(\mathcal{H})} \cdot m \cdot \Delta_2(\mathcal{H}) \\
&\leqslant 2^{\binom{\ell}{2}-1} \sum_{j=2}^{\ell} \frac{1}{\tau^{\ell-1} d(\mathcal{H})} \cdot \Delta_2(\mathcal{H}) \\
&\leqslant 2^{\binom{\ell}{2}-1} \cdot \ell \cdot n^{\frac{\delta}{4}} \cdot \frac{kp}{\ell \alpha'} \cdot \frac{1}{pn^{\frac{\delta}{2}}} \\
&= 2^{\binom{\ell}{2}-1} \cdot \frac{k}{\alpha'} \cdot \frac{1}{n^{\frac{\delta}{4}}} \\
&\leqslant \frac{\varepsilon}{12\ell!},
\end{aligned}$$

where the last inequality holds for sufficiently large n.

By Theorem 17 there exist some constant $c = c(\ell)$ and a family $\mathcal{J} \subset \mathcal{P}(V(\mathcal{H}))$ satisfying (*a*), (*b*) and (*c*) from Theorem 17. We define

$$\mathcal{C} = \{C \subset V(\mathcal{H}) \colon C = V(\mathcal{H}) \smallsetminus J \text{ for one } J \in \mathcal{J}\}.$$

Below we show that $\mathcal{C}$ has the desired properties (1), (2) and (3) of Lemma 23.

(1) follows from (c) since $|\mathcal{C}| = |\mathcal{J}|$ and

$$\log|\mathcal{J}| \leqslant c\tau\log(1/\tau)\log(1/\varepsilon)m \leqslant m \cdot n^{-\frac{\delta}{4(\ell-1)}} c\log(1/\tau)\log(1/\varepsilon) \leqslant m^{1-\gamma},$$

where the last inequality follows for sufficiently large n from

$$m^{\gamma} \overset{(31)}{<} n^{2\gamma} \overset{(30)}{\leqslant} n^{\frac{\delta}{5\ell}},$$

since $c = c(\ell)$ and $\log(1/\varepsilon)$ are constants independent of n and $\log(1/\tau) < \log n$.

(2) follows from (b). Assume for a contradiction that there is $C \in \mathcal{C}$ with $|C| < \beta m$ and let $J = V \setminus C \in \mathcal{J}$. Then we count the number of hyperedges of $\mathcal{H}$.

$$\begin{aligned}
e(\mathcal{H}) &\leqslant e(\mathcal{H}[V \setminus C]) + |C| \cdot \Delta_1(\mathcal{H}) \\
&< e(\mathcal{H}[J]) + \beta m \cdot \frac{D}{p}\binom{v(F)}{2} \\
&\overset{(31)}{\leqslant} \varepsilon e(\mathcal{H}) + \beta D\binom{v(F)}{2} n^2 \\
&\overset{(32)}{\leqslant} \varepsilon e(\mathcal{H}) + \frac{\beta D k}{\alpha'}\binom{v(F)}{2} e(\mathcal{H}) \\
&= \left(\varepsilon + \frac{\beta D k}{\alpha'}\binom{v(F)}{2}\right) e(\mathcal{H}) \\
&\overset{(30)}{<} e(\mathcal{H})
\end{aligned}$$

with a contradiction, so $|C| \geqslant \beta m$ for all $C \in \mathcal{C}$.

(3) For a hitting set A of $\mathcal{H}$ consider the independent set $I = V \setminus A$. Hence by (a) of Theorem 17 there exists $J \in \mathcal{J}$ such that $I \subseteq J$ and, therefore, we have $A \supseteq V \setminus J = C$ which is an element of $\mathcal{C}$. $\square$

§3.4. Proof of Lemma 24

The proof of Lemma 24 follows the proof in [**24**, Lemma 2.3] and is based on an application of the regularity method for subgraphs of sparse random graphs which we introduce first.

Let $\varepsilon > 0$, $p \in (0,1]$ and $H = (V, E)$ be a graph. For X, $Y \subset V$ non-empty and disjoint let
$$d_{H,p}(X,Y) = \frac{e(X,Y)}{p|X||Y|}$$
and we say (X, Y) is (ε, p)-regular if
$$|d_{H,p}(X,Y) - d_{H,p}(X',Y')| < \varepsilon$$
for all subsets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geqslant \varepsilon|X|$ and $|Y'| \geqslant \varepsilon|Y|$. We will use the sparse regularity lemma in the following form (see, e.g. [**35**]).

Lemma 37. *For all $\varepsilon > 0$ and t_0 there exists an integer T_0 such that for every function $p = p(n) \gg 1/n$ a.a.s. $G \in G(n,p)$ has the following property. Every subgraph $H = (V, E)$ of G with $|V| = n$ vertices admits a partition $V = V_1 \cup \ldots \cup V_t$ satisfying*

- *(i) $t_0 \leqslant t \leqslant T_0$,*
- *(ii) $|V_1| \leqslant \cdots \leqslant |V_t| \leqslant |V_1| + 1$ and*
- *(iii) all but at most εt^2 pairs (V_i, V_j) with $i \neq j$ are (ε, p)-regular.* □

For a partition $\mathcal{P}$ as in the last lemma we call the graph $R = R(\mathcal{P}, d, \varepsilon)$ with vertex set $V(R) = \{V_1, \ldots, V_t\}$ and edges
$$\{V_i, V_j\} \in E(R) \iff (V_i, V_j) \text{ is } (\varepsilon, p)\text{-regular with } d_{H,p}(V_i, V_j) \geqslant d$$
the reduced graph w.r.t. $\mathcal{P}$, d, and ε.

The next lemma is a counting lemma for subgraphs of random graphs from [**1**, **9**, **51**]. For the proof of Lemma 24 we only need this (and the following lemma) for fixed bipartite graphs. However, we state those auxiliary lemmas in its general form.

Lemma 38. *For every graph F with vertex set $V(F) = [\ell]$ and $d > 0$ there exist $\varepsilon > 0$ and $\xi > 0$ such that for every $\eta > 0$ there exists $C > 0$ such that for $p > Cn^{-1/m_2(F)}$ a.a.s. $G \in G(n,p)$ satisfies the following.*

Let $H = (V_1 \cup \ldots \cup V_\ell, E_H)$ be an ℓ-partite (not necessarily induced) subgraph of G with vertex classes of size at least ηn and with the property that for every edge $\{i,j\} \in E(F)$ the pair (V_i, V_j) in H is (ε, p)-regular with density $d_{H,p}(V_i, V_j) \geqslant d$. Then the number of partite copies of F in H is at least

$$\xi p^{e(F)} \prod_{i=1}^{\ell} |V_i|\,,$$

where a partite copy is a graph homomorphism $\varphi\colon F \to H$ with $\varphi(i) \in V_i$. □

The next lemma bounds the number of edges between large sets of vertices of $G(n,p)$ as well as the number of copies of some bipartite graphs $F^\star$ with two vertices from a prescribed set W.

Lemma 39. *Let $F^\star$ be a graph with two vertices $a_1, a_2 \in V(F^\star)$ with $a_1 a_2 \notin E(F^\star)$. For all $(\log n)/n \leqslant p = p(n) < 1$ the random graph $G \in G(n,p)$ satisfies a.a.s. the following properties.*

(A1) *For all disjoint subsets U, $W \subseteq V(G)$ with $|U|, |W| \geqslant n/\log\log n$ we have*

$$p|U|^2/3 < e_G(U) < p|U|^2 \quad \text{and} \quad p|U||W|/2 < e_G(U,W) < 2p|U||W|\,.$$

(A2) *For all subsets $W \subset V(G)$ there exists a set of edges $E_0 \subseteq E(G)$ with $|E_0| = n\log n$ such that there are at most $2p^{e(F^\star)} n^{v(F^\star)-2} |W|^2$ many copies $\varphi(F^\star)$ of $F^\star$ in the graph $(V(G), E(G) \smallsetminus E_0)$ with $V(\varphi(F^\star)) \cap W = \{\varphi(a_1), \varphi(a_2)\}$.*

The proof of (A1) follows directly from Chernoff's inequality and the proof of (A2) is based on the so-called *deletion method* in form of the following lemma.

Lemma 40. [**32**, Lemma 2.51] *Let Γ be a set, $S \subseteq [\Gamma]^s$ and $0 < p < 1$. Then for every $k > 0$ with probability at least $1 - \exp(-\frac{k}{2s})$ there exists a set $E_0 \subset \Gamma_p$ of size k such that $\Gamma_p \smallsetminus E_0$ contains at most 2μ sets from S where μ is the expected number of sets from S contained in Γ_p.* □

Proof of Lemma 39. Since part (A1) follows from Chernoff's inequality, we will only focus on property (A2), which is a direct consequence of Lemma 40.

In fact, let V be a set of n vertices, $W \subset V$ and a graph $F^\star$ with two fixed vertices $a_1, a_2 \in V(F^\star)$ not forming an edge in $F^\star$. We use Lemma 40 with $\Gamma = \binom{V}{2}$, $s = e(F^\star)$,

$$S = \left\{\text{copies } \varphi(F^\star) \text{ of } F^\star \text{ in } (V, \Gamma) \text{ with } V(\varphi(F^\star)) \cap W = \{\varphi(a_1), \varphi(a_2)\}\right\},$$

p, and $k = n \log n$. In particular, $\Gamma_p = G(n,p)$ in our setup here. With probability at least $1 - \exp\left(-\frac{n \log n}{2e(F^\star)}\right)$ there exists a set $E_0 \subseteq E(G(n,p))$ of size at most $n \log n$ such that there are at most

$$2\mu \leqslant 2p^{e(F^\star)} n^{v(F^\star)-2} |W|^2$$

many copies $\varphi(F^\star)$ with $V(\varphi(F^\star)) \cap W = \{\varphi(a_1), \varphi(a_2)\}$ in $(V, E(G(n,p)) \smallsetminus E_0)$. The lemma then follows from the union bound applied for all 2^n possible choices $W \subset V$. □

Finally, we can prove Lemma 24. Let F be a strictly balanced and nearly bipartite graph. Let G be a typical graph (with respect to the properties of Lemmas 37–39) in $G(n,p)$ and let H be a subgraph of G with $|E(H)| \geqslant \lambda |E(G)|$. First we apply the sparse regularity lemma (Lemma 37) to H. Since H is relatively dense in $G(n,p)$ we infer that the corresponding reduced graph R (for suitable chosen parameters) has many, i.e. $\Omega(|V(R)|^2)$ edges. So we can find many large complete bipartite graphs in R. We conclude that there is some partition class $V_i \in V(R)$ contained in many complete bipartite graphs.

We analyse the graph $G_0 = \mathrm{Base}_H(F)[V_i]$ on the vertex set V_i with edges being those pairs in $\binom{V_i}{2}$ that complete a copy of the bipartite graph $F' \subseteq F' + e = F$ in H to a copy of F. We say that G_0 is (ϱ, d)-dense if for all $W \subseteq V(G_0)$ with $|W| \geqslant \varrho |V_i|$ we have $e_{G_0}(W) \geqslant d\binom{|W|}{2}$. It is well known that sufficiently large (ϱ, d)-dense graphs contain any fixed subgraph (see e.g. [**46**]).

Lemma 41. *For all $d > 0$ and F there exist ϱ, $c_0 > 0$ and $n_0 \in \mathbb{N}$ such that every (ϱ, d)-dense graph G_0 with $v(G_0) = n \geqslant n_0$ contains at least $c_0 n^{v(F)}$ copies of F.* $\square$

To show the (ϱ, d)-denseness of G_0 we consider $W \subseteq V_i$ with $|W| \geqslant \varrho|V_i|$. Then by Lemma 38 we will find many copies of F' in H where the missing edge has to be in $\binom{W}{2}$. Together with an upper bound for the number of graphs that are combinations of two different copies of F' ((A2) of Lemma 39) we ensure that not too many copies of F' are completed to F by the same pair in W. Thus there are many edges in $\mathrm{Base}_H(F)[W]$ and G_0 is (ϱ, d)-dense.

Proof of Lemma 24. Let $\lambda > 0, C_1 > C_0 > 0$ and let F be a strictly balanced nearly bipartite graph such that $F = F' + \{a_1, a_2\}$, where F' is bipartite with partition classes $A = \{a_1, \ldots, a_a\}$ and $B = \{b_1, \ldots, b_b\}$.

The Sparse Counting Lemma (Lemma 38) applied with F' and $d_{\mathrm{CL}} = \lambda/4$ yields constants $\varepsilon_{\mathrm{CL}} > 0$ and $\xi_{\mathrm{CL}} > 0$. Since we don't know whether the given constant C_0 is at least 1 or not, we find it convenient to fix an auxiliary constant

$$C_0' = \min\{1, C_0^{e(F)-1}\}\,. \tag{33}$$

Furthermore, we set

$$d = \frac{(\lambda/6)^{2(a-1)b} \cdot \xi_{\mathrm{CL}}^2 \cdot C_0^{2(e(F)-1)} \cdot C_0'}{64 \cdot a^{2a} b^{2b} \cdot (v(F)+1)^{v(F)} \cdot C_1^{2(e(F)-1)}}\,. \tag{34}$$

Next we appeal to Lemma 41. For F and for this choice of d this lemma yields constants ϱ, $c_0 > 0$ and $n_0 \in \mathbb{N}$. Furthermore, set

$$\varepsilon = \min\left\{\frac{\varrho\varepsilon_{\mathrm{CL}}}{4}, \frac{\lambda}{48}\right\} \qquad \text{and} \qquad t_0 = \frac{48ab}{\lambda}\,. \tag{35}$$

Lemma 37 applied with ε and t_0 yields $T_0 \in \mathbb{N}$ and Lemma 38 applied with $\eta_{\mathrm{CL}} = \varrho/(2T_0)$ yields C_{CL}. Finally, we fix the promised

$$\eta = c_0 T_0^{-v(F)}$$

and let $C_0 n^{-1/m_2(F)} \leqslant p = p(n) \leqslant C_1 n^{-1/m_2(F)}$. For later reference we note that due to the balancedness of F we have

$$p^{e(F)} n^{v(F)} \leqslant C_1^{e(F)-1} p n^2 \tag{36}$$

and owing to the choice of C_0' in (33) we have

$$p^{e(F_1)} n^{v(F_1)} \geqslant C_0' p n^2 \tag{37}$$

for every subgraph $F_1 \subseteq F$ with $e(F_1) \geqslant 1$. Moreover, since we applied Lemma 38 for $F' \subsetneq F$, the strict balancedness of F implies $m_2(F) > m_2(F')$. Consequently, for sufficiently large n we have

$$C_{\mathrm{CL}} n^{-1/m_2(F')} \leqslant C_0 n^{-1/m_2(F)} \leqslant p\,.$$

Since we have to show that $G(n,p)$ a.a.s. satisfies $T(\lambda,\eta,F)$ we can assume that n is arbitrarily large. Consider any $G \in G(n,p)$ that satisfies the properties of Lemma 37 and Lemma 38, as well as property (A1) and property (A2) of Lemma 39 for all bipartite graphs $F^\star$ such that $F^\star$ is the union of two different copies $\varphi_1(F')$ and $\varphi_2(F')$ of F' with $\{\varphi_1(a_1), \varphi_1(a_2)\} = \{\varphi_2(a_1), \varphi_2(a_2)\}$. In other words, for the rest of the proof we consider a fixed graph G to which we can apply the Lemmas 37–39 and we will show that such a G satisfies $T(\lambda,\eta,F)$. For that let $H \subseteq G$ with

$$e(H) \geqslant \lambda e(G) > \frac{1}{3}\lambda p n^2$$

where the second inequality follows from property (A1) of Lemma 39.

Lemma 37 applied to H yields a partition $\mathcal{P}$ of the vertices $V = V_1 \cupdot \ldots \cupdot V_t$ with at least $(1-\varepsilon)\binom{t}{2}$ many (ε, p)-regular pairs for some t with $t_0 \leqslant t \leqslant T_0$. We assume w.l.o.g. that t divides n. We infer that there are at least $\frac{\lambda}{6}\binom{t}{2}$ regular pairs with edge density at least $\frac{\lambda}{4}p$ since otherwise we could bound the number of edges of H by

$$\begin{aligned} e(H) &\leqslant \frac{\lambda}{6}\binom{t}{2} \cdot 2p\left(\frac{n}{t}\right)^2 + \binom{t}{2} \cdot \frac{\lambda}{4}p\left(\frac{n}{t}\right)^2 + \varepsilon\binom{t}{2} \cdot 2p\left(\frac{n}{t}\right)^2 + t \cdot p\left(\frac{n}{t}\right)^2 \\ &\leqslant \frac{1}{2}pn^2\left(\frac{\lambda}{3} + \frac{\lambda}{4} + 2\varepsilon + \frac{2}{t}\right) \\ &\overset{(35)}{\leqslant} \frac{1}{3}\lambda pn^2\,, \end{aligned}$$

which would contradict the derived lower bound $e(H) > \frac{1}{3}\lambda pn^2$.

Let $R = R(\mathcal{P}, d_{\mathrm{CL}}, \varepsilon)$ be the reduced graph w.r.t. the partition $\mathcal{P}$ and relative density $d_{\mathrm{CL}} = \frac{\lambda}{4}$. In particular R has exactly $t \geqslant t_0$ vertices and at least $\frac{\lambda}{6}\binom{t}{2}$ edges. It follows from the theorem of Kővári, Sós and Turán [**37**] (see, e.g. [**15**, Lemma 1]) that there are at least γt^{a+b-1} copies of the complete bipartite graph $K_{a-1,b}$ in R where[1]

$$\gamma = \gamma(F, \lambda) = \frac{1}{2}\frac{1}{(a-1)^{a-1}b^b}\left(\frac{\lambda}{6}\right)^{(a-1)b}. \tag{38}$$

Hence, there is a partition class V_{a_0} of $\mathcal{P}$ such that V_{a_0} is contained in at least γt^{a+b-2} copies of $K_{a-1,b}$ in R where V_{a_0} is always contained in partition class A of $K_{a-1,b}$ for these copies.

Our goal is to show that the graph G_0 induced by $\mathrm{Base}_F(H)$ on V_{a_0} is (ϱ, d)-dense, which due to our choice of c_0 and η above leads to $c_0(n/t)^{v(F)} > \eta n^{v(F)}$ copies of F in G_0 (see Lemma 41). So let $W \subseteq V_{a_0}$ with $|W| \geqslant \varrho|V_{a_0}|$ and fix some partition $W = W_1 \cupdot W_2$ with $|W_1| = |W_2| = |W|/2$ (for simplicity, we may assume

[1]Strictly speaking, in [**37**] no such lower bound on the number of copies of complete graphs in dense large graphs is given. However, the proof from [**37**] combined with standard convexity arguments gives the bound stated here and such an argument can be found for example in [**15**, Lemma 1].

that $|W|$ is even). Note that for any j for which (V_{a_0}, V_j) is (ε, p)-regular we still have that (W_1, V_j) and (W_2, V_j) are $(2\varepsilon/\varrho, p)$-regular.

We will ensure many copies of F' with $a_1 \in W_1$ and $a_2 \in W_2$ which force edges in $G_0 = \mathrm{Base}_F(H)[V_{a_0}]$. However, we have to make sure that not too many copies force the same edge in G_0. For this purpose we delete some edges by (A2) of Lemma 39 to restrict the number of graphs $F^\star$ that are unions of two different copies of F' that force the same edge in G_0.

Let $\varphi_1(F')$ and $\varphi_2(F')$ be two copies of F' satisfying $\varphi_1(\{a_1, a_2\}) = \varphi_2(\{a_1, a_2\})$ and let $F^\star = \varphi_1(F') \cup \varphi_2(F')$. We find by (A2) of Lemma 39 at most $n \log n$ edges $E_{F^\star}$ such that there are at most

$$2p^{e(F^\star)} n^{v(F^\star)-2} |W|^2 \tag{39}$$

copies of $F^\star$ in $(V(H), E(H) \setminus E_{F^\star})$ with $\varphi_1(a_1), \varphi_1(a_2) \in W_1 \cup W_2$. We repeat this argument for all possible graphs $F^\star$ that can be created this way and we denote by $\mathcal{F}^\star$ the family of those graphs. Since there are at most $2(a+1)^{a-2}(b+1)^b$ such graphs $F^\star$, in total we delete at most

$$2(a+1)^{a-2}(b+1)^b n \log n = o(pn^2)$$

edges of H, i.e. for $H' = H - \bigcup_{F^\star \in \mathcal{F}^\star} E_{F^\star}$ we have

$$e(H') \geqslant (1 - o(1)) e(H).$$

In particular, for sufficiently large n the density and the regularity of the pairs in the partition $\mathcal{P}$ is not affected much and (δ, p)-regular pairs in H are still $(2\delta, p)$-regular in H'.

Lemma 38 yields many copies of F' in H'. In fact, since $m_2(F') < m_2(F)$ we get

$$p \geqslant C_0 n^{-\frac{1}{m_2(F)}} > C_{\mathrm{CL}} n^{-\frac{1}{m_2(F')}}.$$

For any copy of $K_{a-1,b}$ in the reduced graph R that contains V_{a_0} among the $a-1$ classes of the bipartition of $K_{a-1,b}$ Lemma 38 applied with $\varepsilon_{\mathrm{CL}} \geqslant 4\varepsilon/\varrho$ (see (35)) yields at least

$$\xi_{\mathrm{CL}} p^{e(F)-1} \left(\frac{n}{t}\right)^{v(F)-2} |W_1||W_2| = \frac{1}{4} \xi_{\mathrm{CL}} p^{e(F)-1} \left(\frac{n}{t}\right)^{v(F)-2} |W|^2$$

partite copies of F' in H' with $a_1 \in W_1$ and $a_2 \in W_2$. Repeating this for the γt^{a+b-2} different copies of $K_{a-1,b}$ in R that contain V_{a_0} in the described way, in total we obtain at least

$$\begin{aligned}\gamma t^{v(F)-2} \cdot \frac{1}{4} \xi_{\mathrm{CL}} p^{e(F)-1} \left(\frac{n}{t}\right)^{v(F)-2} |W|^2 &= \frac{\gamma \xi_{\mathrm{CL}}}{4} \cdot p^{e(F)-1} n^{v(F)-2} |W|^2 \\ &\geqslant \frac{\gamma \xi_{\mathrm{CL}}}{4} \cdot C_0^{e(F)-1} |W|^2 \qquad (40)\end{aligned}$$

copies of F' in H' with $a_1 \in W_1$ and $a_2 \in W_2$. For a pair of vertices $e \in \binom{W}{2}$ we define

$$x_e = |\{\varphi(F') \text{ copy of } F' \text{ in } H' \colon e = \{\varphi(a_1), \varphi(a_2)\}\}| \ .$$

By (40) we know that

$$\sum_{e \in \binom{W}{2}} x_e \geqslant \frac{\gamma \xi_{\mathrm{CL}}}{4} \cdot C_0^{e(F)-1} |W|^2 \,. \qquad (41)$$

Let $\mathcal{W}_{>0} = \left\{e \in \binom{W}{2} : x_e \neq 0\right\}$ and $N = |\mathcal{W}_{>0}|$. Since this N corresponds to the number of edges in $\mathrm{Base}_{H'}(F)[W] \subseteq \mathrm{Base}_H(F)[W]$ we shall show that $N \geqslant d\binom{|W|}{2}$. For this purpose we use (41) and an upper bound for $\sum_{e \in \binom{W}{2}} x_e^2$ that follows from (39). In fact,

$$\sum_{e \in \binom{W}{2}} x_e^2 \overset{(39)}{\leqslant} |\mathcal{F}^\star| \cdot 2 p^{e(\hat{F})} n^{v(\hat{F})-2} |W|^2 \qquad (42)$$

where $\hat{F}$ is a graph in $\mathcal{F}^\star$ that maximises the value of $p^{e(F^\star)} n^{v(F^\star)-2}$ for $F^\star \in \mathcal{F}^\star$. We will show that $p^{e(\hat{F})} n^{v(\hat{F})-2}$ is bounded by a constant only depending on C_0, C_1 and F. In fact, for $F^\star = \varphi_1(F') \cup \varphi_2(F') \in \mathcal{F}^\star$ let $F_0 = \varphi_1(F') \cap \varphi_2(F')$ and

$e = \{\varphi_1(a_1), \varphi_1(a_2)\}$. In particular, $F_0 + e \subseteq F$ and we have

$$p^{e(F^\star)} n^{v(F^\star)-2} = \frac{p^{e(F^\star+e)} n^{v(F^\star+e)}}{pn^2} = \frac{\left(p^{e(F)} n^{v(F)}\right)^2}{p^{e(F_0+e)} n^{v(F_0+e)} \cdot pn^2} \overset{(36)}{\leqslant} \frac{C_1^{2e(F)-2} pn^2}{p^{e(F_0+e)} n^{v(F_0+e)}}$$
$$\overset{(37)}{\leqslant} \frac{C_1^{2e(F)-2}}{C_0'} \,.$$

Combining (42) with the simple upper bound $|\mathcal{F}^\star| \leqslant (v(F)+1)^{v(F)}$ and the last inequality yields

$$\sum_{e \in \binom{W}{2}} x_e^2 \leqslant 2(v(F)+1)^{v(F)} \frac{C_1^{2(e(F)-1)}}{C_0'} |W|^2 \,. \tag{43}$$

Finally, we establish the (ϱ, d)-denseness of G_0. In fact, from the Cauchy-Schwarz inequality we know

$$\left(\sum_{e \in \binom{W}{2}} x_e\right)^2 = \left(\sum_{e \in \mathcal{W}_{>0}} x_e\right)^2 \leqslant N \cdot \sum_{e \in \mathcal{W}_{>0}} x_e^2 = N \cdot \sum_{e \in \binom{W}{2}} x_e^2 \,.$$

and, consequently,

$$\begin{aligned} N \quad &\geqslant \quad \frac{\left(\sum_{e \in \binom{W}{2}} x_e\right)^2}{\sum_{e \in \binom{W}{2}} x_e^2} \\ &\overset{(41),(43)}{\geqslant} \frac{\left(\gamma \xi_{\mathrm{CL}} C_0^{e(F)-1} |W|^2/4\right)^2}{2(v(F)+1)^{v(F)} C_1^{2(e(F)-1)} |W|^2 / C_0'} \\ &> \quad \frac{\gamma^2 \xi_{\mathrm{CL}}^2 C_0^{2(e(F)-1)} C_0'}{16(v(F)+1)^{v(F)} C_1^{2(e(F)-1)}} \cdot \binom{|W|}{2} \\ &\overset{(34),(38)}{\geqslant} d \cdot \binom{|W|}{2} \,. \end{aligned}$$

Recalling that $W \subseteq V_{a_0}$ with $|W| \geqslant \varrho |V_{a_0}|$ was arbitrary, implies that G_0 is (ϱ, d)-dense which finishes the proof. $\square$

§3.5. Open Problems

3.5.1. Ramsey Properties of Nearly Partite Hypergraphs. Instead of nearly bipartite graphs one may consider *nearly k-partite k-uniform hypergraphs*, i.e. k-uniform hypergraphs with vertex partition $V_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_k$ and the property that at most one hyperedge is contained in V_1 and the remaining hyperedges contain exactly one vertex from each vertex class. Again one may require additional *balancedness* assumptions (similar as in Theorem 5). However, for the proof of a lemma corresponding to Lemma 24 one would need a sparse version of the so-called weak regularity lemma for hypergraphs and a corresponding embedding/counting lemma for subhypergraphs of random hypergraphs (see, e.g. [**9**, Section 5.1]). For the more relaxed version of nearly partite, which would allow the additional hyperedge to span across more than one vertex class, one would likely need sparse analogues of the strong hypergraph regularity method for subhypergraphs of random hypergraphs.

3.5.2. Ramsey Properties for More Colours and General Graphs. It would be very interesting to extend Theorem 5 to more general graphs F. The class of nearly bipartite graphs contains the triangle K_3 and an extension for all cliques would be desirable. The main obstacle seems to establish a suitable analogue of Lemma 24 for this case.

Another limitation is the restriction to two colours only. The Rödl-Ruciński theorem [**46**] applies, up to very few exceptions (see, e.g. [**32**, Section 8.1]), to arbitrary graphs and any number of colours $r \geqslant 2$. However, besides for the case of trees (see [**23**]), all known sharpness results address only the two-colour case and extending these results to more than two colours appears an interesting open problem in the area.

Finally, we mention that due to Friedgut's criterion the $c = c(n)$ in Theorem 5 is bounded by constants, but it may depend on n. It seems plausible, that a strengthening of Theorem 5 for some constant c independent of n also holds. However, this would likely require a very different approach to these problems.

CHAPTER 4

Schur Triples

Here we will prove Theorem 8. The proof builds on similar ideas as in [**21**, **53**].

§4.1. Main Lemmas

The proof of Theorem 8 has a very similar structure as the proof of Theorem 5 in Chapter 3. We start with a reformulation of Friedgut's and Bourgain's Criterion.

4.1.1. Friedgut's Criterion for Coarse Thresholds. We want to apply Bourgain's Criterion, Theorem 16. For that we need symmetric properties of the given ground set. To achieve the symmetry we switch from subsets of $[n]$ to subsets of $\mathbb{Z}_n$.

As a starting point for the application of Bourgain's Criterion we also need the result by Graham, Rödl & Ruciński that the corresponding threshold is semi-sharp. Theorem 7 concerns random subsets of $[n]$, however, the proof given in [**26**] also works similarly for random subsets of $\mathbb{Z}_n$. The 1-statement follows directly from the $[n]$-case. The 0-statement requires only slight changes in the calculations and we obtain the following Lemma

Lemma 42. *There exist constants $C_1 \geqslant C_0 > 0$ such that the following holds.*

$$\lim_{n\to\infty} \mathbb{P}(\mathbb{Z}_{n,p} \to (\mathrm{ST})_2) = \begin{cases} 0, & \text{if } p \leqslant C_0 n^{-1/2}\,, \\ 1, & \text{if } p \geqslant C_1 n^{-1/2}\,. \end{cases} \qquad \square$$

The starting point of the proof of Theorem 8 is again the criterion of properties with a coarse threshold by Friedgut and Bourgain. Since we deal with a monotone property in random subsets of $\mathbb{Z}_n$, this time we will use the version by Bourgain, Theorem 16 from Section 2.1.

We do not want that the booster set B already satisfies $B \to (\mathrm{ST})_2$. Since B is small the following lemma will guarantee that $B \nrightarrow (\mathrm{ST})_2$.

Lemma 43. *Let $\mathcal{B}$ be a family of subsets of $\mathbb{Z}_n$ such that every $B \in \mathcal{B}$ satisfies $|B| \leqslant \log n$ and $B \to (\mathrm{ST})_2$. Then for every function $p = p(n) = \Theta(n^{-1/2})$ holds $\mathbb{P}(B \subseteq \mathbb{Z}_{n,p}$ for some $B \in \mathcal{B}) = o(1)$.*

The proof is very similar to the proof of Theorem 2 in [**26**] and we only sketch the main differences. First in [**26**] the proof is given for random subsets of $[n]$ instead of random subsets of $\mathbb{Z}_n$. In the proof we only have to estimate the expectation of special configurations. Since all of those expectations have the same order of magnitude in $[n]$ and in $\mathbb{Z}_n$ the proof can be adapted to this setting. Secondly we want to prove the lemma for every $p = \Theta(n^{-1/2})$, where in [**26**] $p = cn^{-1/2}$ for a sufficiently small constant c. However, for the expectations mentioned above we do not need c to be small since in the proof this is only used for large configurations which are excluded here by $|B| \leqslant \log n$. This concludes the discussion of Lemma 43.

Recall, that in the graph case we used random embeddings of B, while in [**21**] for arithmetic progressions in $\mathbb{Z}_{n,p}$ random shifts of a booster B were used. Note that in both cases the property $B \nrightarrow (F)_2^e$ respectively $B \nrightarrow (AP_k)_2$ also hold with B replaced by $h(B)$ respectively B replaced by $B + x$ for some embedding h respectively some shift $x \in Z$.

For Schur triples we cannot use shifts of B since it can happen that $B \nrightarrow (\mathrm{ST})_2$ as well as $B + x \to (\mathrm{ST})_2$. Instead we take scalings of B, that means for $q \in \mathbb{Z}_n$ we take $qB = \{qb \colon b \in B\}$. It turns out that scalings preserve the property $B \nrightarrow (\mathrm{ST})_2$ at least for q coprime to n. Let $Q_n^\star = \{q \in \mathbb{Z}_n \colon \gcd(q, n) = 1\}$ be the elements of $\mathbb{Z}_n$ that are coprime to n. Then $|Q_n^\star|$ is given by Euler's totient function and it is known that for $n \geqslant 2$

$$|Q_n^\star| \geqslant \frac{n}{e^{\gamma^\star} \operatorname{loglog} n + \frac{3}{\operatorname{loglog} n}}$$

holds where $\gamma^\star \approx 0.577$ is Euler's constant [**50**, Theorem 15], $e^{\gamma^\star} \approx 1.78$. Since we do not want to distinct cases for different order of magnitude of $|Q_n|$ later in the proof, we fix for each $n \geqslant 2$ a subset $Q_n \subseteq Q_n^\star$ of size

$$|Q_n| = \left\lceil \frac{n}{e^{\gamma^\star} \operatorname{loglog} n + \frac{3}{\operatorname{loglog} n}} \right\rceil ,$$

in particular, we have for sufficiently large n

$$\frac{n}{2 \operatorname{loglog} n} \leqslant |Q_n| \leqslant \frac{n}{\operatorname{loglog} n} . \tag{44}$$

For $q \in Q_n$ the function $\psi_q \colon \mathbb{Z}_n \to \mathbb{Z}_n, n \mapsto qn$ is a bijection and for each Schur triple (x, y, z) also $(\psi_q(x), \psi_q(y), \psi_q(z))$ is a Schur triple. Similarly the preimage of any Schur triple is a Schur triple. With these preparations we can adapt Friedgut's and Bourgain's criterion, in this case Theorem 16, to our setting to obtain the following lemma.

Lemma 44. *Assume that the property $\{Z \subseteq \mathbb{Z}_n \colon Z \to (\mathrm{ST})_2\}$ does not have a sharp threshold. Then there exist constants $K \in \mathbb{N}$, $\alpha, \varepsilon, \mu > 0$ and $C_1 \geqslant C_0 > 0$ and a function $c \colon \mathbb{N} \to \mathbb{R}$ with $C_0 < c(n) < C_1$ such that for $p(n) = c(n)n^{-1/2}$ and for infinitely many $n \in \mathbb{N}$ the following holds.*

There is a subset $B_n \subset \mathbb{Z}_n$ with $|B_n| \leqslant K$ and $B_n \nrightarrow (\mathrm{ST})_2$ such that for every family $\mathcal{Z}$ of subsets from $\mathbb{Z}_n$ with $\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{Z}) \geqslant 1 - \mu$ there exists a $Z \in \mathcal{Z}$ such that

(1) $\mathbb{P}(Z \cup qB_n \to (\mathrm{ST})_2) > \alpha$, with $q \in Q_n$ chosen uniformly at random,
(2) $\mathbb{P}(Z \cup \mathbb{Z}_{n,\varepsilon p}) \to (\mathrm{ST})_2) < \frac{\alpha}{2}$.

Proof. Let $\mathcal{A} = \{Z \subseteq \mathbb{Z}_n \colon Z \to (\mathrm{ST})_2\}$, then Lemma 42 yields that a threshold function of $\mathcal{A}$ has order of magnitude $\Theta(n^{-1/2})$. We assume that this threshold is not sharp. It follows that there are constants $C, \tau > 0$, $C_1 \geqslant C_0 > 0$ and a function $c : \mathbb{N} \to \mathbb{R}$ with $C_0 \leqslant c(n) \leqslant C_1$ such that for $p = p(n) = c(n)n^{-1/2}$ there are infinitely many $n \in \mathbb{N}$ with

$$\tau < \mu(p, \mathcal{A}) := \mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A}) < 1 - \tau$$

and $p \cdot \frac{d\mu(p,\mathcal{A})}{dp} \leqslant C$, which are exactly the assumptions of Theorem 16. Consequently there are $\delta(C,\tau), \eta(C,\tau) > 0$ and $K(C,\tau) \in \mathbb{N}$ and for each of the infinitely many n there exists a family $\mathcal{B}_n$ of subsets from $\mathbb{Z}_n$ with the properties as in Theorem 16. Finally let $\bar{\mathcal{A}} = \mathcal{P}(\mathbb{Z}_n) \smallsetminus \mathcal{A} = \{Z \subseteq \mathbb{Z}_n \colon Z \nrightarrow (\mathrm{ST})_2\}$ be the complement of $\mathcal{A}$ and set

$$\mu < \frac{\delta\, \mathbb{P}(\mathbb{Z}_{n,p} \in \bar{\mathcal{A}})}{8}\,. \tag{45}$$

Since we assume that the threshold of $\mathcal{A}$ is not sharp there exists a sufficiently small $\varepsilon > 0$ and $\alpha > 0$ with $0 < \alpha < \delta/2$ such that the following holds.

$$\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{Z}' | \mathbb{Z}_{n,p} \in \bar{\mathcal{A}}) \geqslant 1 - \delta/4$$

where $\mathcal{Z}' \subseteq \bar{\mathcal{A}}$ is the family of sets $Z \in \bar{\mathcal{A}}$ with

$$\mathbb{P}(Z \cup \mathbb{Z}_{n,\varepsilon p} \rightarrow (\mathrm{ST})_2) < \alpha/2\,,$$

otherwise this would yield a contradiction to $p \cdot \frac{d\mu(p,\mathcal{A})}{dp} \leqslant C$.

We know from Theorem 16 that $\mathbb{P}(B_n \subseteq \mathbb{Z}_{n,p}$ for some $B_n \in \mathcal{B}_n) \geqslant \eta$ and from Lemma 43 that a.a.s. all $B_n \subset \mathbb{Z}_n$ of size at most K satisfy $B_n \nrightarrow (\mathrm{ST})_2$. Consequently for each n there exists a $B_n \in \mathcal{B}_n$ with $|B_n| \leqslant K$, $B_n \nrightarrow (\mathrm{ST})_2$ and

$$\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A} | B \subseteq \mathbb{Z}_{n,p}) \geqslant \mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A}) + \delta\,.$$

By symmetry for all $q \in Q_n$ also qB_n satisfies these three properties: $|qB_n| \leqslant K$ follows directly since ψ_q is bijective. For $qB_n \nrightarrow (\mathrm{ST})_2$ we use that ψ_q yields a bijection between the Schur triples: Any ST-free colouring of B_n can be transferred via ψ_q to an ST-free colouring of qB_n and vice versa since there is always a multiplicative inverse to q in $\mathbb{Z}_n$ which is also coprime to n. The third property follows from the same argument as the second combined with the fact that for any $A \subseteq \mathbb{Z}_n$ the probability $\mathbb{P}(A \subseteq \mathbb{Z}_{n,p})$ only depends on the size of A. Consequently for all $q \in Q_n$ holds

$$\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A} | qB_n \subseteq \mathbb{Z}_{n,p}) \geqslant \mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{A}) + \delta\,. \tag{46}$$

Let $\mathcal{Z}''$ be the family of subsets $Z \subseteq \mathbb{Z}_n$ such that

$$\mathbb{P}(Z \cup qB_n \in \mathcal{A}) > \frac{\delta}{2} > \alpha\,,$$

where here the probability comes from choosing q uniformly at random from Q_n. Then it follows from (46) that $\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{Z}'' | \mathbb{Z}_{n,p} \in \bar{\mathcal{A}}) \geqslant \delta/2$.

By the choice of μ in (45) it follows for any property $\mathcal{Z}$ with $\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{Z}) \geqslant 1-\mu$ that

$$\mathbb{P}(\mathbb{Z}_{n,p} \notin \mathcal{Z} \cap \mathcal{Z}' \cap \mathcal{Z}'' | \mathbb{Z}_{n,p} \in \bar{\mathcal{A}}) \leqslant \frac{\delta}{8} + \frac{\delta}{4} + \left(1 - \frac{\delta}{2}\right) \leqslant 1 - \frac{\delta}{8}$$

and consequently

$$\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{Z} \cap \mathcal{Z}' \cap \mathcal{Z}'') \geqslant \frac{\delta}{8} \cdot \mathbb{P}(\mathbb{Z}_{n,p} \in \bar{\mathcal{A}}) \geqslant \frac{\delta\tau}{8}\,.$$

Consequently $\mathcal{Z} \cap \mathcal{Z}' \cap \mathcal{Z}'' \neq \varnothing$ and any Z in this intersection satisfies the desired properties. □

4.1.2. Main Probabilistic Lemmas. Similarly as in Chapter 3 we will apply the hypergraph container theorem, Theorem 17, to a hypergraph $\mathcal{H}$ that depends on $B, Z, \Xi \subseteq \mathbb{Z}_n$, where B, Z are directly given by Bourgain's lemma (Lemma 44) and Ξ stands for a suitable family of scalings of B and will be a subfamily of the family of scalings that is given by Lemma 44.

The main work in the proof is to make suitable choices for $\mathcal{Z}$ and Ξ. We can guarantee that $Z \in \mathcal{Z}$ satisfies properties that hold for $\mathbb{Z}_{n,p}$ with probability at least $1 - \mu$ for μ given by Lemma 44. On the other hand we can also restrict Ξ as long as the remaining family has size $\Theta(\frac{n}{\log\log n})$. In this way we can ensure that Ξ only contains typical scalings.

Next we translate the definitions from Chapter 3 to the setting in $\mathbb{Z}_n$ with Schur triples. For given $B, Z \subseteq \mathbb{Z}_n$ we say that $z \in Z$ *focuses* on $b \in B$ if there exists $x \in Z$ such that (z, x, b) is a Schur triple. The set of focusing elements we call

$$M(Z, B) = \{z \in Z \colon \text{there is } b \in B \text{ s.t. } z \text{ focuses on } b\}\,.$$

The pair (Z, B) is called *interactive* if $Z \cap B = \varnothing$, $Z \nrightarrow (\mathrm{ST})_2$, $B \nrightarrow (\mathrm{ST})_2$, but $Z \cup B \to (\mathrm{ST})_2$. For a set Ξ of scalings of B we call the pair (Z, Ξ) *interactive* if (Z, qB) is interactive for every $q \in \Xi$.

We call (Z, Ξ) *regular* if for all $q \in \Xi$ every $z \in Z$ focuses on at most one $qb \in qB$. Compared to Chapter 3 we need a new definition since we cannot exclude the case that there is some $z \in Z$ that focuses on two elements of qB if there exist $b, b' \in B$ with $b + b' = 0$. We say (Z, Ξ) is *almost regular* if for all $q \in \Xi$ every $z \in Z$ focuses on at most two $qb, qb' \in qB$ and if in the case that z focuses on exactly two $qb \neq qb' \in qB$ then there exists $x \in Z$ such that $x - z = qb$ and $z - x = qb'$.

For a booster B and an almost regular and interactive pair (Z, Ξ) we define a hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi)$ with vertex set

$$V(\mathcal{H}) = Z$$

and edge set

$$E(\mathcal{H}) = \{M(Z, qB) \colon q \in \Xi\}.$$

For an interactive and almost regular pair (Z, Ξ) and $q \in \Xi$ we say that $z \in M(Z, qB) = \{z_1, \ldots, z_\ell\}$ with $z_1 < z_2 < \cdots < z_\ell$ has *index* i if $z = z_i$ (for that we fix the canonical ordering on $\mathbb{Z}_n$). Furthermore, we call (Z, Ξ) and $\mathcal{H}(Z, \Xi)$ *index consistent* if for all $z \in Z$ and all $q, q' \in \Xi$ with $z \in M(Z, qB) \cap M(Z, q'B)$ the indices of z in $M(Z, qB)$ and in $M(Z, q'B)$ are the same. Let $b_1 < \cdots < b_{|B|}$ be the natural ordering of the elements of B induced by the fixed one of $\mathbb{Z}_n$. Then the *profile* of $M(Z, qB)$ is the function $\pi \colon [|M(Z, qB)|] \to [|B|]^2$ defined by $\pi(i) = (j, j)$ if z_i focuses only on qb_j and by $\pi(i) = (j, k)$, if z_i focuses on qb_j as well as on qb_k in a way such that there exist $x \in Z$ with $z_i - x = qb_j$ and $x - z_i = qb_k$. Note that in this case it follows that $x \in M(Z, qB)$ and $\pi(x) = (k, j)$. Since the pair (Z, Ξ) is almost regular, for each edge of $\mathcal{H}$ each z_i focuses on at most two qb_j and, hence, the profile is well defined. We say (Z, Ξ) has profile π if all edges $M(Z, qB)$ for $q \in \Xi$ have profile π. Note that in this case all sets $M(Z, qB)$ have the same cardinality and $|M(Z, qB)|$ is called the *length* of the profile π.

Recall that we fixed for all $n \in \mathbb{N}$ a family Q_n of coprimes to n. With these definitions at hand we can formulate our main technical lemma which will similarly to Chapter 3 yield the desired family of cores.

Lemma 45. *For all constants $C_1 > C_0 > 0$, $\alpha, \mu > 0$, $K \in \mathbb{N}$ and any sequence $(B_m)_{m\in\mathbb{N}}$ of subsets $B_m \subset \mathbb{Z}_m$ with $|B_m| \leqslant K$ for all m, there exist $\alpha', \beta, \gamma > 0$ and $L, n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and every $p = c(n)n^{-1/2}$ with $C_0 \leqslant c(n) \leqslant C_1$ with probability at least $1 - \frac{\mu}{2}$ we have that $Z \in \mathbb{Z}_{n,p}$ satisfies the following. If*

$$\mathbb{P}(Z \cup qB_n \to (\mathrm{ST})_2) > \alpha$$

for $q \in Q_n$ chosen uniformly at random, then there exists $\Xi_n \subseteq Q_n$ with $|\Xi_n| \geqslant \frac{\alpha' n}{\log\log n}$ and $Z \cup qB_n \to (\mathrm{ST})_2$ for all $q \in \Xi_n$ such that the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi_n)$ is almost regular and index consistent for some profile π of length $\ell \leqslant L$ and there is a family $\mathcal{C}$ of subsets of $V(\mathcal{H})$ satisfying

(1) $\log|\mathcal{C}| \leqslant |Z|^{1-\gamma}$,
(2) $|C| \geqslant \beta|Z|$ for all $C \in \mathcal{C}$, and
(3) every hitting set A of $\mathcal{H}$ contains some $C \in \mathcal{C}$, i.e. for every $A \subseteq V(\mathcal{H})$ with $e \cap A \neq \varnothing$ for all $e \in E(\mathcal{H})$ there exists $C \in \mathcal{C}$ with $C \subseteq A$.

The main part of the proof (which will be given in Section 4.3) deals with the preparation of Z and Ξ such that the hypergraph container theorem, Theorem 17, can be applied to $\mathcal{H}(Z, \Xi)$.

The second probabilistic lemma is a preparation step for the second round when we will add εpn elements to Z, where $p = \Theta(n^{-1/2})$. For $S \subseteq \mathbb{Z}_n$ we define the *base set* of S by

$$\mathrm{Base}(S) = \{z \in \mathbb{Z}_n \colon \exists s, s' \in S \text{ s.t. } (s, s', z) \text{ is a Schur triple}\}.$$

We can consider $\mathrm{Base}(S)$ as the set of elements that complete a pair in S to a Schur triple. The next lemma shows that a.a.s. a random subset of size $\Omega(n^{1/2})$

contains at least a constant fraction of the expected number of Schur triples in its base set.

LEMMA 46. *For all* $\lambda > 0$, $C_1 > C_0 > 0$ *there exists* $\eta > 0$ *such that for* $C_0 n^{-1/2} \leqslant p \leqslant C_1 n^{-1/2}$ *the following a.a.s. holds. For every subset* $S \subseteq \mathbb{Z}_{n,p}$ *of size at least* λpn *there are at least* ηn^2 *Schur triples in* $\mathrm{Base}(S)$*.*

The proof follows the proof in [**21**]and is given in section 4.4.

4.1.3. Colourings and Hitting Sets. In the following we define the hitting set of $\mathcal{H}$ depending on a colouring of B and one of Z that establishes a connection between ST-free colourings of $Z \cup qB$ and the hypergraph container theorem, Theorem 17.

For $B, Z \subseteq \mathbb{Z}_n$ and a family Ξ of scalings of B let (Z, Ξ) be an interactive pair, i.e. $Z \nrightarrow (\mathrm{ST})_2$, $B \nrightarrow (\mathrm{ST})_2$ but $Z \cup qB \rightarrow (\mathrm{ST})_2$ for all $q \in \Xi$. In particular there exists an ST-free colouring $\sigma\colon B \rightarrow \{\text{red,blue}\}$ of B. We copy σ for all $q \in \Xi$ to qB by setting $\sigma_q\colon qB \rightarrow \{\text{red,blue}\}$ with $\sigma_q(qb) = \sigma(b)$. Furthermore, let φ be an arbitrary ST-free colouring of Z.

As (Z, Ξ) is interactive, for each $q \in \Xi$ for every 2-colouring of $Z \cup qB$ there exists a monochromatic Schur triple in $Z \cup qB$, in particular, for the colouring defined by σ_q and φ (later in the proof we will ensure that $Z \cap qB = \varnothing$ for all elements q in the chosen family Ξ and we assume this for now).

Since σ and φ both are ST-free it follows that there is a monochromatic Schur triple in $Z \cup qB$ that uses elements from Z as well as elements from qB. We call the elements from Z that are contained in such a Schur triple *activated* by σ, φ, and q and define for $\mathcal{H}$ the set of activated vertices by

$$A^\sigma_\varphi = A^\sigma_\varphi(Z, \Xi) = \bigcup_{q \in \Xi} \big\{ z \in Z\colon z \text{ is activated by } \sigma, \varphi \text{ and } q \big\} \subseteq V(\mathcal{H})\,.$$

In Chapter 3 we looked for an interactive, regular and index consistent pair (Z, Ξ) at A^σ_φ to obtain that $\varphi(z) = \varphi'(z)$ for all $z \in A^\sigma_\varphi \cap A^\sigma_{\varphi'}$ and ST-free colourings φ, φ' of Z. For Schur triples, however, we cannot exclude the following configuration where there are $x \neq z \in Z$ and $qb \neq qb' \in qB$ such that $z - x = qb$ and $x - z = qb'$. If we try to follow the same way as before it could happen that $\varphi(x) = \varphi(z) = \sigma(b) = \text{red}$ while $\varphi'(x) = \varphi'(z) = \sigma(b') = \text{blue}$ which would contradict $\varphi(z) = \varphi'(z)$ for all $z \in A^\sigma_\varphi \cap A^\sigma_{\varphi'}$.

To solve this issue we define $\widetilde{A}^\sigma_\varphi \subseteq A^\sigma_\varphi$ in the following way. Let (Z, Ξ) be an interactive, almost regular, and index consistent pair with profile π (compared to Chapter 3 we changed regular to almost regular). Remember that by index consistency each $z \in Z$ focuses for all $q \in \Xi$ on qb_j for exactly the same indices j, in fact, on at most two different qb_j and in the case that there are two different $qb_j \neq qb_k$ then there exists $x \in Z$ such that $z - x = qb_j$ and $x - z = qb_k$. It follows that such elements which focuses on two different $qb_j \neq qb_k$ appear in pairs (x, z) with $z - x = qb_j$ and $x - z = qb_k$.

The idea is to include in $\widetilde{A}^\sigma_\varphi \subseteq A^\sigma_\varphi$ all elements from A^σ_φ which focuses on exactly one qb_j, and to make choices for each pair (x, z) of elements which focuses on $qb_j \neq qb_k$ depending on the profile of x and z and the colour of qb_j and qb_k. The index consistency allows to make the definition of $\widetilde{A}^\sigma_\varphi$ by considering only one $q \in \Xi$. For given φ, σ and $q \in \Xi$ let $z \in A^\sigma_\varphi$ and let $i(z)$ be the index of z in $M(Z, qB)$, then

(A1) if there is only one $qb_j \in qB$ such that z focuses on qb_j let $z \in \widetilde{A}^\sigma_\varphi$,

(A2) if there are $qb_j \neq qb_k \in qB$ with $\sigma(b_j) = \sigma(b_k)$ such that z focuses on qb_j as well as on qb_k then let $z \in \widetilde{A}^\sigma_\varphi$,

(A3) if there are $qb_j \neq qb_k \in qB$ with $\sigma(b_j) \neq \sigma(b_k)$ and $j < k$ such that $\pi(i(z)) = (j, k)$ then let $z \in \widetilde{A}^\sigma_\varphi$,

(A4) if there are $qb_j \neq qb_k \in qB$ with $\sigma(b_j) \neq \sigma(b_k)$ and $j < k$ such that $\pi(i(z)) = (k, j)$ then let $z \notin \widetilde{A}^\sigma_\varphi$.

Note that in Case (A4), when $z \notin \widetilde{A}^{\sigma}_{\varphi}$, then there is some $x \in Z$ with index $i(x)$ such that $z - x = qb_k$ and $x - z = qb_j$. Consequently by definition $x \in A^{\sigma}_{\varphi}$ as well as $\pi(i(x)) = (j,k)$ and by (A3) holds $x \in \widetilde{A}^{\sigma}_{\varphi}$.

Consequently the definition ensures that $\widetilde{A}^{\sigma}_{\varphi}$ contains at least one element from all pairs (x,z) of elements which focuses on two different elements of qB. Since A^{σ}_{φ} is a hitting set of $\mathcal{H}(Z,\Xi)$ it follows that also $\widetilde{A}^{\sigma}_{\varphi}$ is a hitting set of $\mathcal{H}(Z,\Xi)$.

It remains to show that for any two ST-free colourings φ, φ' of Z and for $x \in \widetilde{A}^{\sigma}_{\varphi} \cap \widetilde{A}^{\sigma}_{\varphi'}$ holds $\varphi(x) = \varphi'(x)$. This follows since $\varphi(x)$ is determined by the elements of qB on which x focuses (the profile) as well as by the colouring σ on B. Since we have index consistency the focusing is the same for all $q \in \Xi$. The colouring of σ clearly does not depend on φ. Consequently the colour of x can be determined by the profile of the index of x in $M(Z,qB)$ and by σ which both do not depend on φ and we obtain $\varphi(x) = \varphi'(x)$. We summarise these consequences in the following fact.

Fact 47. *Let (Z,Ξ) be an interactive, almost regular and index consistent pair with profile π, let σ be an* ST*-free colouring of B and φ, φ' be* ST*-free colourings of Z. Then*

(A1) *$\widetilde{A}^{\sigma}_{\varphi}(Z,\Xi)$ is a hitting set of $\mathcal{H}(Z,\Xi)$ and*

(A2) *for all $x \in \widetilde{A}^{\sigma}_{\varphi} \cap \widetilde{A}^{\sigma}_{\varphi'}$ holds $\varphi(x) = \varphi'(x)$.* □

§4.2. Proof of Theorem 8

We adapt the proof of Chapter 3 to Schur triples. As starting point we apply Bourgain's criterion (see Lemma 44) to the contradictory assumption that $\mathbb{Z}_{n,p} \to (\mathrm{ST})_2$ has a coarse threshold. At this point we also have to define a family $\mathcal{Z}$ of subsets of $\mathbb{Z}_n$ with typical properties, which in our case are the properties of Lemma 45 and Lemma 46 that are guaranteed to hold simultaneously with probability at least $1 - \mu$ for sufficiently large n.

Lemma 44 yields constants $\alpha, \varepsilon, K, \mu, C_1$, and C_0 and for infinitely many n and $p = c(n)n^{-1/2}$ with $C_0 \leqslant c(n) \leqslant C_1$ a set $Z \subseteq \mathbb{Z}_n$ of size at most K and a small booster set $B_n \subseteq \mathbb{Z}_n$ such that

(B1) $\mathbb{P}(Z \cup qB_n \to (\mathrm{ST})_2) > \alpha$, with $q \in Q_n$ chosen uniformly at random,

(B2) $\mathbb{P}(Z \cup \mathbb{Z}_{n,\varepsilon p}) \to (\mathrm{ST})_2) < \frac{\alpha}{2}$.

We will find a contradiction to (B2).

Let Φ be the family of all ST-free colourings of Z and consider an arbitrary $\varphi \in \Phi$. We shall show that the probability that φ extends to an ST-free colouring of $Z \cup \mathbb{Z}_{n,\varepsilon p}$ is at most $\exp\big(-\Omega(pn)\big)$. This part uses in particular Lemma 46 and Janson's inequality.

To obtain the contradiction to (B2) we want to use a union bound over all ST-free colourings, but there are in total $2^{\Theta(pn)}$ ST-free colourings. This problem is solved by Lemma 45 which yields a family $\mathcal{C}$ of *cores* C with $|\mathcal{C}| = 2^{o(pn)}$ such that we can partition Φ in $|\mathcal{C}|$ classes and all colourings in the same class agree on the corresponding C.

The proof of Lemma 45 basically applies the hypergraph container theorem (Theorem 17) to the hypergraph $\mathcal{H}(Z, \Xi)$, which works for carefully chosen Z and Ξ (this will be done in Section 4.3). Afterwards the hitting set $\widetilde{A}^{\sigma}_{\varphi}$ establishes a connection between the cores and the colourings in Φ.

Finally the union bound yields that with probability at most $o(1)$ there is an ST-free colouring $\varphi \in \Phi$ that extends to an ST-free colouring of $Z \cup \mathbb{Z}_{n,\varepsilon p}$ which contradicts (B2). Below we show the details of the proof.

Proof of Theorem 8. Assume for a contradiction that the property $\mathbb{Z}_n \to (\mathrm{ST})_2$ does not have a sharp threshold. We apply Lemma 44 and obtain constants $K \in \mathbb{N}$, $\alpha, \varepsilon, \mu > 0$, $C_1 \geqslant C_0 > 0$ and a function $p(n) = c(n)n^{-1/2}$ with $C_0 \leqslant c(n) \leqslant C_1$ such that for infinitely many $n \in \mathbb{N}$ there is $B_n \subseteq \mathbb{Z}_n$ of size at most K and $B_n \nrightarrow (\mathrm{ST})_2$. Let $I \subseteq \mathbb{N}$ be the set of these $n \in \mathbb{N}$. Let $(B_n)_{n \in \mathbb{N}}$

be a sequence of subsets $B_n \subseteq \mathbb{Z}_n$ that is formed by B_n for $n \in I$ and $B_n = \varnothing$ for $n \notin I$.

Next we apply Lemma 45 to get $\alpha', \beta, \gamma > 0$ and $L \in \mathbb{N}$. Set $\lambda = \beta/4$, then Lemma 46 yields $\eta > 0$.

For $n \in \mathbb{N}$ let $\mathcal{Z}_n$ be the family of subsets of $\mathbb{Z}_n$ that satisfy $pn/2 \leqslant |Z| \leqslant 2pn$ and that satisfy the conclusions of Lemma 45 and of Lemma 46 for the chosen parameters. Since the properties in Lemma 45 hold for sufficiently large n with probability at least $1-\frac{\mu}{2}$ and the property in Lemma 46 as well as $pn/2 \leqslant |Z| \leqslant 2pn$ hold a.a.s. Lemma 44 yields some $Z \subseteq \mathcal{Z}_n$ such that

(B1) $\mathbb{P}(Z \cup qB_n \to (\mathrm{ST})_2) > \alpha$, with $q \in Q_n$ chosen uniformly at random,

(B2) $\mathbb{P}(Z \cup \mathbb{Z}_{n,\varepsilon p}) \to (\mathrm{ST})_2) < \frac{\alpha}{2}$,

and such that the conclusion of Lemma 46 holds. By (B1) we can apply Lemma 45 to obtain a profile π of length $\ell \leqslant L$ and $\Xi_n \subseteq Q_n$ with $|\Xi_n| \geqslant \frac{\alpha' n}{\log\log n}$ and $Z \cup qB_n \to (\mathrm{ST})_2$ for all $q \in \Xi_n$ such that the hypergraph $\mathcal{H} = \mathcal{H}(Z, \Xi_n)$ is almost regular and index consistent for π and such that there is a family $\mathcal{C}$ of subsets of $V(\mathcal{H})$ satisfying

(C1) $\log |\mathcal{C}| \leqslant |Z|^{1-\gamma}$,

(C2) $|C| \geqslant \beta |Z|$ for all $C \in \mathcal{C}$, and

(C3) every hitting set A of $\mathcal{H}$ contains some $C \in \mathcal{C}$.

We shall establish a contradiction to the assertion (B2). Let Φ be the set of all ST-free edge colourings of Z and pick an arbitrary ST-free colouring σ of B_n. We want to split Φ into "few" classes, for that we use the correspondence between any colouring $\varphi \in \Phi$ and the hitting set $\widetilde{A}^{\sigma}_{\varphi} = \widetilde{A}^{\sigma}_{\varphi}(Z, \Xi_{B,n})$ of $\mathcal{H}$ given by Fact 47. Moreover, for $C \in \mathcal{C}$ we define

$$\Phi_C = \{\varphi \in \Phi \colon C \subseteq \widetilde{A}^{\sigma}_{\varphi}\}\,.$$

Then $\Phi = \bigcup_{C \in \mathcal{C}} \Phi_C$, where this is not necessarily a disjoint union, since by (C3) for every $\varphi \in \Phi$ the hitting set $\widetilde{A}^{\sigma}_{\varphi}$ contains some $C \in \mathcal{C}$ and hence $\varphi \in \Phi_C$. Fact 47 also guarantees that $\varphi(z) = \varphi'(z)$ for all $z \in \widetilde{A}^{\sigma}_{\varphi} \cap \widetilde{A}^{\sigma}_{\varphi'}$ and arbitrary colourings $\varphi, \varphi' \in \Phi$. That means that all colourings in Φ_C agree on C and, consequently, there exists a monochromatic subset $R_C \subseteq C$ of size at least $|C|/2 \geqslant \beta|Z|/2 = \lambda pn$. For the desired contradiction we add $\mathbb{Z}_{n,\varepsilon p}$ to Z. If we show that

$$\mathbb{P}(Z \cup \mathbb{Z}_{n,\varepsilon p} \nrightarrow (\mathrm{ST})_2) = o(1)$$

this will contradict (B2).

We shall find for all ST-free colourings φ of Z an upper bound for the probability that φ is extendible to an ST-free colouring of $Z \cup \mathbb{Z}_{n,\varepsilon p}$. Instead of using the complete colouring given by φ we only need parts of the colouring on the associated core, on $R_C \subseteq C \subseteq \widetilde{A}^{\sigma}_{\varphi}$ such that we can deal with all colourings from Φ_C at once.

We know that $|R_C| \geqslant \lambda pn$ and therefore by the conclusion of Lemma 46 there are at least ηn^2 Schur triples in the corresponding base set $\mathrm{Base}(R_C)$ of R_C. Since all edges in R_C are coloured with one colour, lets say red, all elements in $U_C = \mathrm{Base}(R_C) \cap \mathbb{Z}_{n,\varepsilon p}$ have to be coloured with blue to prevent a red Schur triple. Consequently φ cannot be extended to a ST-free colouring of $Z \cup \mathbb{Z}_{n,\varepsilon p}$ if there is a Schur triple in U_C.

However, $\mathrm{Base}(R_C)$ contains ηn^2 Schur triples and $p = \Theta(n^{-1/2})$ and so we can use Janson's inequality [**30**] (see also [**31**]) to show that it is unlikely that U_C is ST-free.

In fact Janson's inequality implies that there exists some $\gamma' = \gamma'(\varepsilon, \eta, C_0, C_1)$ such that

$$\mathbb{P}\Big(\ \mathrm{Base}(R_C) \cap \mathbb{Z}_{n,\varepsilon p} \text{ is ST-free }\Big) = \mathbb{P}\Big(U_C \text{ is ST-free }\Big) \leqslant \exp\Big(-\gamma' n^{-1/2}\Big)\,. \quad (47)$$

By combining (C1) and (47) we deduce the desired contradiction:

$$\begin{aligned}
\mathbb{P}(Z \cup \mathbb{Z}_{n,\varepsilon p} \nrightarrow (\mathrm{ST})_2) &\leqslant |\mathcal{C}| \cdot \max_{C \in \mathcal{C}} \mathbb{P}\big(\exists \varphi \in \Phi_C \colon \varphi \text{ is extendible to } U_C\big) \\
&\overset{(C1)}{\leqslant} \exp\left(|Z|^{1-\gamma}\right) \cdot \max_{C \in \mathcal{C}} \mathbb{P}\big(U_C \text{ is ST-free }\big) \\
&\overset{(Z1)}{\leqslant} \exp\left((2pn)^{1-\gamma}\right) \cdot \max_{C \in \mathcal{C}} \mathbb{P}\big(U_C \text{ is ST-free }\big) \\
&\overset{(47)}{\leqslant} \exp\left((2C_1 n^{1/2})^{1-\gamma}\right) \cdot \exp\left(-\gamma' n^{1/2}\right) \\
&< \alpha\,,
\end{aligned}$$

for sufficiently large n, since $\gamma > 0$ and C_1, γ, and γ' are constants independent of n. This concludes the proof of Theorem 8. □

§4.3. Proof of Lemma 45

In this section we will give the proof of our main probabilistic lemma, Lemma 45, which includes the preparation step to choose the family $\mathcal{Z}$ of typical sets $Z \subseteq \mathbb{Z}_n$ and a convenient subset $\Xi \subseteq Q_n$ of scalings of the booster. Afterwards we apply the hypergraph container theorem to obtain the family of cores.

4.3.1. Some Typical Properties of $\mathbb{Z}_{n,p}$. We need properties that hold with high probability for $\mathbb{Z}_{n,p}$ and which give us some control over the hypergraph $\mathcal{H}(Z, \Xi)$. In particular we want to bound the number of pairs that build with some element of a booster a Schur triple and we want to get some information on $d(\mathcal{H})$, $\Delta_1(\mathcal{H})$, and $\Delta_2(\mathcal{H})$. For that we define the following for $Z \subseteq \mathbb{Z}_n$ and $z \in \mathbb{Z}_n$. Let

$$\mathcal{P}(n, z, Z) = \{(x, y) \in Z^2 \colon x \neq y \text{ and } (x, y, z) \text{ is a ST}\}\,, \tag{48}$$

be the pairs in Z^2 that form a Schur triple with z and let for $B \subseteq \mathbb{Z}_n$ and $q \in Q_n$

$$\mathcal{S}(n, z, B, q) = \{x \in \mathbb{Z}_n \colon \exists b \in B \text{ s.t. } (x, qb, z) \text{ is a ST}\} \text{ and}$$

$$\mathcal{S}(n, z, B) = \bigcup_{q \in Q_n} \mathcal{S}(n, z, B, q) \tag{49}$$

be the elements of $\mathbb{Z}_n$ that complete a Schur triple with $z \in Z$ and some $qb \in qB$ for fixed $q \in Q_n$ respectively an arbitrary $q \in Q_n$. Further, for $z' \in \mathbb{Z}_n$ let

$$\mathcal{S}_2(n,z,z',B,q) = \{(x,y) \in \mathbb{Z}_n^2 \colon x \neq y, \exists b \neq b' \in B \colon (z,qb,x) \text{ and } (z',qb',y) \text{ are ST}\}$$

$$\mathcal{S}_2(n,z,z',B) = \bigcup_{q \in Q_n} \mathcal{S}_2(n,z,z',B,q)\,. \tag{50}$$

be the pairs in $\mathbb{Z}_n^2$ that complete $z \in Z$ and $qb \in qB$ to a Schur triple as well as z' and $qb' \in qB$ for fixed $q \in Q_n$ respectively an arbitrary $q \in Q_n$.

We will use $|\mathcal{P}(n,z,Z)|$ to bound the number of Schur triples that can use elements of a given booster, while $\mathcal{S}(n,z,B)$ yields some information on $\Delta_1(\mathcal{H})$ and $\mathcal{S}_2(n,z,z',B)$ on $\Delta_2(\mathcal{H})$. Finally we define when a scaling of a booster is untypical.

Definition 48. *We say $q \in Q_n$ is bad w.r.t. B and Z if there are either*

(B1) *$b \in B$, $z \in Z$ such that (z,z,qb) is a Schur triple, or*
(B2) *$b \in B$, $z \in Z$, and, $x \neq x' \in Z \smallsetminus \{z\}$ such that (z,x,qb) and (z,x',qb) are Schur triples, or*
(B3) *$b \neq b' \in B$ and $z \in Z$, $x \neq x' \in Z \smallsetminus \{z\}$ such that (z,x,qb) and (z,x',qb') are Schur triples, or*
(B4) *$b \neq b' \in B$ and $x \neq z \in Z$ such that (z,x,qb) and (z,x,qb') are Schur triples but not simultaneously $x - z = qb$ and $z - x = qb'$.*

Note that any $z \in Z$ can focus onto two booster elements $qb \neq qb' \in qB$ for q not bad only in the case if there is an element $x \in Z$ with $x - z = qb$ and $z - x = qb'$ (all other cases are excluded by (B3) and (B4)). In this case $qb + qb' = 0$ and consequently $b + b' = 0$ follows. We deal with this case in another way later in the proof. In the following we define which properties a typical subset $Z \subseteq \mathbb{Z}_n$ should have.

Definition 49. *For $K \in \mathbb{N}$ and a subset $B_n \subset \mathbb{Z}_n$ with $|B_n| \leqslant K$, for $p \in (0,1)$, and for constants $D \in \mathbb{N}$, $\alpha, \zeta > 0$ we consider the set $\mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$ of subsets from $\mathbb{Z}_n$ that is given by $Z \in \mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$ if and only if*

(Z1) $\frac{1}{2}pn \leqslant |Z| \leqslant 2pn$, $0 \notin Z$,
(Z2) $|\mathcal{P}(n, z, Z)| \leqslant D$ *for all but at most* $\frac{\alpha}{6K^2}|Q_n|$ *many* $z \in \bigcup_{q \in Q_n} qB_n$,
(Z3) $|\mathcal{S}(n, z, B_n) \cap Z| \leqslant 8pK|Q_n|$ *for all* $z \in \mathbb{Z}_n$,
(Z4) $|\mathcal{S}^2(n, z, z', B_n) \cap Z^2| \leqslant n^{1/4}$ *for all pairs* $(z, z') \in \mathbb{Z}_n^2$ *with* $z \neq z'$,
(Z5) $|\{q \in Q_n : q$ *is bad w.r.t.* B_n *and* $Z\}| \leqslant n^{1-\zeta}$.

In the following lemma we show that $\mathbb{Z}_{n,p}$ with probability close to one satisfies (Z1)–(Z5).

Lemma 50. *For all constants $K \in \mathbb{N}$, $\alpha, \mu > 0$, and $C_1 \geqslant C_0 > 0$ and for all sequences $(B_m)_{m \in \mathbb{N}}$ of subsets from $\mathbb{Z}_m$ with $|B_m| \leqslant K$ there are constants $\zeta > 0$ and $D, n_0 \in \mathbb{N}$ such that for $n \geqslant n_0$ and $C_0 n^{-1/2} \leqslant p \leqslant C_1 n^{-1/2}$ holds $\mathbb{P}(\mathbb{Z}_{n,p} \in \mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)) \geqslant 1 - \frac{\mu}{2}$.*

We remark that in fact all properties but (Z2) hold a.a.s.

Proof. We have to show that properties (Z1)–(Z5) with Z replaced by $\mathbb{Z}_{n,p}$ hold with probability at least $1 - \mu/2$. (Z1) and (Z3)–(Z5) even hold a.a.s.

(Z1) $\frac{1}{2}pn \leqslant |Z| \leqslant 2pn$ follows a.a.s. directly from Chernoff's inequality, Lemma 20, and $0 \notin Z$ a.a.s. since $\mathbb{P}(0 \in \mathbb{Z}_{n,p}) = p$.

(Z2) holds with probability at least $1 - \mu/3$ for any $D \geqslant \frac{54 C_1^2 K^3}{\alpha \mu}$. For fixed $z \in \bigcup_{q \in Q_n} qB_n$ we estimate the expectation of $X_z = |\mathcal{P}(n, z, \mathbb{Z}_{n,p})|$ and use Markov's inequality. There are at most $3n$ possibilities to find a pair $x, y \in \mathbb{Z}_n^2$ such that (x, y, z) is a Schur triple, but $(x, y) \in \mathcal{P}(n, z, \mathbb{Z}_{n,p})$ only if x and y both are contained in $\mathbb{Z}_{n,p}$. We get

$$\mathbb{E}[X_z] \leqslant 3np^2 \leqslant 3C_1^2 \,,$$

and Markov's inequality, Theorem 18, yields

$$\mathbb{P}(X_z \geqslant D+1) \leqslant \frac{3C_1^2}{D+1}\,.$$

Now we can calculate the expected number of $z \in \bigcup_{q\in Q_n} qB_n$ with $X_z > D$ by

$$\mathbb{E}\Big[\big|\{z \in \bigcup_{q\in Q_n} qB_n \colon X_z > D\}\big|\Big] \leqslant \frac{3C_1^2}{D+1}\cdot K|Q_n|$$

and using Markov's inequality a second time gives

$$\mathbb{P}\left(\big|\{z \in \bigcup_{q\in Q_n} qB_n \colon X_z > D\}\big| \geqslant \frac{3C_1^2 K|Q_n|}{(D+1)}\cdot\frac{3}{\mu}\right) \leqslant \frac{\mu}{3}\,.$$

Consequently with probability at least $1-\frac{\mu}{3}$ we have $X_z < D$ for all but at most $\frac{9C_1^2K|Q_n|}{(D+1)\mu} \leqslant \frac{\alpha}{6K^2}|Q_n|$ many $z \in \bigcup_{q\in Q_n} qB_n$.

(Z3) follows a.a.s. from Chernoff's inequality. For $z \in \mathbb{Z}_n$ the expectation of $|\mathcal{S}(n,z,B_n) \cap Z|$ clearly has size $p|\mathcal{S}(n,z,B_n)|$. We show $|\mathcal{S}(n,z,B_n)| = \Theta(|Q_n|)$. For any $q \in Q_n$ there is at least one $b \in B_n$ and at least one $x \in \mathbb{Z}_n \smallsetminus \{z, qb\}$ such that (z, qb, x) is a Schur triple. For example $x = z + qb$ works since we may assume $z \neq 0$ and $qb \neq 0$. In this way we get $x \in \mathcal{S}(n,z,B_n)$, where each element from $\mathbb{Z}_n$ is reached at most $6K$ times (three choices for the orientation of (z, qb, x), two choices which element is multiplied by q, each element can be used by at most K different q) which yields the lower bound. On the other hand each $q \in Q_n$ contributes at most $4K$ different elements to $\mathcal{S}(n,z,B_n)$ (for each b at most 3 different x and maybe qb itself). Consequently we get

$$\frac{|Q_n|}{6K} \leqslant |\mathcal{S}(n,z,B_n)| \leqslant 4K|Q_n|\,,$$

in particular,

$$p|\mathcal{S}(n,z,B_n)| = \Theta(n^{-1/2}|Q_n|) = \Theta\left(\frac{n^{1/2}}{\operatorname{loglog} n}\right)\,.$$

Now Chernoff's inequality together with a simple union bound yields the statement.

(Z4) is done in a similar way as (Z3). For two fixed $z \neq z' \in \mathbb{Z}_n$ we will estimate the size of the family of pairs $(x, y) \in \mathbb{Z}_n^2$ such that $x \neq y$ and for each (x, y) there are $b \neq b' \in B$ with (z, qb, x) and $(z'qb', y)$ are Schur triples. We call such pairs connecting pairs and we will show that such a family has size $O(\frac{n}{\operatorname{loglog} n})$. Then the expected value of $|\mathcal{S}^2(n, z, z', B_n) \cap Z^2|$ is at most $O\big(p^2 \frac{n}{\operatorname{loglog} n}\big) = O\big(\frac{1}{\operatorname{loglog} n}\big)$.

Then we want to use Chernoff's inequality and a union bound again, but in contrast to (Z3) here the probabilities $\mathbb{P}((x, y) \in Z^2)$ and $\mathbb{P}((x', y') \in Z^2)$ for two connecting pairs are not independent if the pairs intersect. We solve this issue by splitting the family of connecting pairs into a constant number of families such that in each family all pairs are disjoint.

Let $z \neq z' \in \mathbb{Z}_n$. First we estimate the number of pairs $(x, y) \in \mathbb{Z}_n^2$ that connect z and z'. For any $q \in Q_n$ there are at most $9K^2$ possible connecting pairs for (z, z') that use the booster qB. Consequently we get $|\mathcal{S}^2(n, z, z', B_n)| = O(\frac{n}{\operatorname{loglog} n})$.

Now consider the auxiliary graph G where the connecting pairs are the vertices of G and $\{(x, y), (x'y')\}$ is an edge of G if and only if (x, y) and (x', y') are not disjoint. Consequently $v(G) = |\mathcal{S}^2(n, z, z', B_n)| = O(\frac{n}{\operatorname{loglog} n})$ and $\Delta(G) < 18K^2$. Since in general for the chromatic number $\chi(G) \leqslant \Delta(G) + 1$ holds there is a partition of the vertices into $m \leqslant 18K^2$ independent sets. Each independent set corresponds to a family of disjoint connecting pairs. In this way we get a partition of $\mathcal{S}^2(n, z, z', B_n)$ into m families $\mathcal{S}_1^2, \dots, \mathcal{S}_m^2$ of connecting pairs. For each family $\mathcal{S}_i^2$ we get

$$\mathbb{E}\big[|\mathcal{S}_i^2 \cap Z^2|\big] \leqslant p^2 |\mathcal{S}_i^2| = O\left(\frac{1}{\operatorname{loglog} n}\right)$$

and Chernoff's inequality yields

$$\mathbb{P}\left(|\mathcal{S}_i^2 \cap Z^2| \geqslant \frac{n^{1/4}}{m}\right) \leqslant \exp\big(-\Omega(n^{1/4})\big).$$

Clearly $|\mathcal{S}^2(n, z, z', B_n) \cap Z^2| \leqslant \sum_{i=1}^m |\mathcal{S}_i^2 \cap C|$ and consequently

$$\mathbb{P}(|\mathcal{S}^2(n, z, z', B_n) \cap Z^2| \geqslant n^{1/4}) \leqslant \exp\big(-\Omega(n^{1/4})\big).$$

Now we can use a union bound over all pairs $z, z' \in \mathbb{Z}_n$ which yields the claim.

For (Z5) we will show that there exists $\xi > 0$ such that for all $q \in Q_n$ and sufficiently large n

$$\mathbb{P}(q \text{ is bad w.r.t. } B_n \text{ and } \mathbb{Z}_{n,p}) \leqslant n^{-\xi}\,.$$

Then (Z5) follows from Markov's inequality with $\zeta = \xi/2$.

For fixed $q \in Q_n$ and $b \in B$ there are at most two different $z \in Z$ such that (z, z, qb) is a Schur triple. Each z appears with probability p and a union bound yields

$$\mathbb{P}(q \text{ is bad w.r.t. } B_n \text{ and } \mathbb{Z}_{n,p} \text{ because of (B1)}) \leqslant K \cdot 2 \cdot p \leqslant n^{-\xi_1}$$

for some $\xi_1 < 1/2$ and sufficiently large n.

For fixed $q \in Q_n$ we count configurations with $b \in B$, $z \in Z$, and $x \neq x' \in Z \smallsetminus \{z\}$ such that (z, x, qb) and (z, x', qb) are Schur triples. Then each configuration appears with probability p^3 and consequently

$$\mathbb{P}(q \text{ is bad w.r.t. } B_n \text{ and } \mathbb{Z}_{n,p} \text{ because of (B2)}) \leqslant K \cdot n \cdot 3 \cdot 3 \cdot p^3 \leqslant n^{-\xi_2}$$

for some $\xi_2 < 1/2$ and sufficiently large n.

For the case that a fixed $q \in Q_n$ is bad because of (B3) or (B4) we count configurations with $b \neq b' \in B$ and $z, x, x' \in Z$ such that (z, x, qb) and (z, x', qb') are Schur triples and distinguish between the cases $x \neq x'$ and $x = x'$. First let $x \neq x'$, then each configuration appears with probability p^3 and consequently

$$\mathbb{P}(q \text{ is bad w.r.t. } B_n \text{ and } \mathbb{Z}_{n,p} \text{ because of (B3)}) \leqslant K^2 \cdot n \cdot 3 \cdot 3 \cdot p^3 \leqslant n^{-\xi_3}$$

for $\xi_3 < 1/2$ and sufficiently large n.

For the last case assume $x = x'$. For each b, b', q the assumptions that (z, x, qb) and (z, x', qb') are Schur triples lead to a system of two linear equations (depending on the orientation of the Schur triples) which x and z have to satisfy. There are nine cases, but using symmetries between x and z and between qb and qb' we can restrict to the following four cases.

- Case A: $qb = x + z$ and $qb' = x + z$,
- Case B: $qb = x - z$ and $qb' = x - z$,
- Case C: $qb = x - z$ and $qb' = z - x$,
- Case D: $qb = z - x$ and $qb' = x + z$,

Case A and B do not have a solution since $qb \neq qb'$ for $b \neq b'$ and $q \in Q_n$. Solutions of case C do not lead to configurations that are bad in sense of (B4) by the formulation of (B4). It is left to deal with case D.

Let $qb = z - x$ and $qb' = x + z$, hence, the equations $2x = qb' - qb$ and $2z = qb' + qb$ hold. Consequently there are at most two possible solutions for x which determine the value of z. Each such x appears with probability p (and similarly each such y) and there are at most K^2 choices for b, b' so we conclude

$$\mathbb{P}(q \text{ is bad w.r.t. } B_n \text{ and } \mathbb{Z}_{n,p} \text{ because of (B4)}) \leqslant K^2 \cdot 2 \cdot p \leqslant n^{-\xi_4}$$

for some $\xi_4 < 1$. With $\xi = \min\{\xi_1, \xi_2, \xi_3, \xi_4\}$ and $\zeta = \xi/2$ Markov's inequality yields (Z5). $\square$

4.3.2. Restricting Embeddings of B. In addition to the properties of Z we also need some properties for $q \in Q_n$, that is why we restrict the family of scalings Q_n. In that sense we define normal scalings.

Definition 51. *For constants $\tilde{\alpha} > 0$, $D \in \mathbb{N}$ and $Z, B_n \subseteq \mathbb{Z}_n$ a set $\Xi_n^0 \subseteq \mathbb{Z}_n$ is called $(\tilde{\alpha}, B_n, D, Z)$-normal if the following properties are satisfied*

(Ξ*1*) $|\Xi_n^0| \geqslant \frac{\tilde{\alpha} n}{\log\log n}$,

(Ξ*2*) $Z \cup qB_n \to (\mathrm{ST})_2$ *for all* $q \in \Xi_n^0$,

(Ξ*3*) *q is not bad w.r.t. B_n and Z for all* $q \in \Xi_n^0$,

(Ξ*4*) $qB_n \cap q'B_n = \varnothing$ *for all* $q \neq q' \in \Xi_n^0$,

(Ξ*5*) *for all $q \in \Xi_n^0$ we have* $qB_n \cap Z = \varnothing$,

(Ξ*6*) $|\mathcal{P}(n, z, Z)| \leqslant D$ *for all* $z \in \bigcup_{q \in \Xi_n^0} qB_n$,

(Ξ*7*) *for all $q \in \Xi_n^0$ there are no $b, b' \in B_n$ and $z \in Z$ such that (z, qb, qb') is a Schur Triple.*

Now we will show that in the setting given by Bourgain's criterion, Lemma 44, we can find an $(\widetilde{\alpha}, B_n, D, Z)$-normal family of scalings.

LEMMA 52. *For all constants $\alpha, \zeta > 0$, $K, D \in \mathbb{N}$, and $C_1 > C_0 > 0$ and for a sequence $(B_m)_{m\in\mathbb{N}}$ of subsets $B_m \subseteq \mathbb{Z}_m$ with $|B_m| \leqslant K$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ and $C_0 n^{-1/2} \leqslant p \leqslant C_1 n^{-1/2}$ the following holds. If $Z \in \mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$ and*

$$\mathbb{P}(Z \cup qB_n \to (\mathrm{ST})_2) > \alpha \tag{51}$$

where $q \in Q_n$ chosen uniformly at random, then there exists $\Xi_n^0 \subseteq Q_n$ such that Ξ_n^0 is $(\widetilde{\alpha}, B_n, D, Z)$-normal with $\widetilde{\alpha} = \alpha/(50K^6)$.

PROOF. Given $(B_m)_{m\in\mathbb{N}}$ and the constants as above set $\widetilde{\alpha} = \frac{\alpha}{50K^6}$ and let n be sufficiently large. Let $C_0 n^{-1/2} \leqslant p \leqslant C_1 n^{-1/2}$, let $Z \in \mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$, and let (51) hold. We will obtain the desired set Ξ_n^0 by starting with Q_n and removing step by step elements.

We start with the family of coprimes Q_n which has size $|Q_n| \geqslant \frac{n}{2\log\log n}$. By assumption (51) there is a family $Q_n^1 \subseteq Q_n$ of size at least $\alpha|Q_n|$ such that for all $q \in Q_n^1$ we have $Z \cup qB_n \to (\mathrm{ST})_2$.

Next we remove from Q_n^1 all elements q such that q is bad w.r.t. B_n and Z. Since $Z \in \mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$ the family Y_n of bad elements w.r.t. B_n and Z has size at most $n^{1-\zeta}$ and we conclude that for sufficiently large n and $Q_n^2 = Q_n^1 \smallsetminus Y_n$ we have $|Q_n^2| \geqslant \frac{\alpha}{2}|Q_n|$.

Now we want to ensure ($\Xi 4$). Since $q \in Q_n$ is coprime w.r.t. n, the function $\psi_q \colon \mathbb{Z}_n \to \mathbb{Z}_n, n \mapsto qn$ is bijective, and thus for every $z \in \mathbb{Z}_n$ there are at most K many $q \in Q_n$ such that $qb = z$ for some $b \in B$. In fact, these are the unique q_i with $q_i b_i = z$ if we denote by b_i the elements from B_n. In a first step we pick greedily any $q \in Q_n^2$ and remove all $q' \in Q_n^2$ with $qB_n \cap q'B_n \neq \varnothing$. By the argument above in this way we pick at least one and remove at most $K^2 - 1$ elements from Q_n^2.

Repeating the same procedure as often as possible yields a set $Q_n^3 \subseteq Q_n^2$ with $|Q_n^3| \geqslant \frac{1}{K^2}|Q_n^2| \geqslant \frac{\alpha}{2K^2}|Q_n|$ such that $qB_n \cap q'B_n = \varnothing$ for all $q, q' \in Q_n^3$.

To obtain property ($\Xi 5$) we remove all elements q from Q_n^3 with $qB_n \cap Z \neq \varnothing$. By ($\Xi 4$) we remove at most $|Z|$ many elements from Q_n^3 and because of $Z \in \mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$ we know $|Z| \leqslant 2pn = \Theta(n^{1/2})$. On the other hand $|Q_n^3| = \Theta(\frac{n}{\log\log n})$ and we obtain a set $Q_n^4 \subseteq Q_n^3$ of size at least $\frac{\alpha}{3K^2}|Q_n|$ that satisfies ($\Xi 5$).

In the next step we remove all $q \in Q_n^4$ that contain some $z \in qB$ with $|\mathcal{P}(n, z, Z)| > D$. By $Z \in \mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$ (we use (Z2)) there are at most $\frac{\alpha}{6K^2}|Q_n|$ such z and by ($\Xi 4$) each z belongs to exactly one q. Consequently we remove at most $\frac{\alpha}{6K^2}|Q_n|$ many $q \in Q_n^4$ and obtain a set Q_n^5 of size at least $\frac{\alpha}{6K^2}|Q_n|$ that satisfies ($\Xi 6$).

To ensure ($\Xi 7$) we do a preparation step first. For $q \in Q_n$ let

$$A_q = \{a \in \mathbb{Z}_n \smallsetminus \{0\} \colon \exists b, b' \in B_n \text{ s.t. } a = qb - qb' \text{ or } a = qb + qb'\}$$

be the set of distances between elements of qB_n and sums of two elements of qB_n. Pick an arbitrary $q \in Q_n^5$ and remove from Q_n^5 all $q' \in Q_n^5$ with $A_{q'} \cap A_q \neq \varnothing$. We count the number of $q' \in Q_n^5$ that will be removed in this way. For each of less than $2K^2$ many $a \in A_q$ and for each of at most K^2 many combinations $b_1, b_2 \in B$ there is at most one $q' \in Q_n$ such that $a = q'(b_1 - b_2)$ (and the same for all $a = q'(b_1 + b_2)$). It follows that less than $4K^4$ different q' are removed from Q_n^5 and consequently we can repeat this procedure to get a set $Q_n^6 \subseteq Q_n^5$ of size at least $\frac{1}{4K^4}|Q_n^5| \geqslant \frac{\alpha}{24K^6}|Q_n|$ such that $A_q \cap A_{q'} = \varnothing$ for all $q, q' \in Q_n^6$.

Now for any $z \in Z$ and $b, b' \in B$ there is at most one $q \in Q_n^6$ such that z completes qb and qb' to a Schur triple: For $z \neq 0$ this is ensured by $A_q \cap A_{q'} = \varnothing$, and $Z \in \mathcal{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$ yields $0 \notin Z$. Consequently we can obtain ($\Xi 7$) by removing at most $|Z| = o(|Q_n|)$ many $q \in Q_n^6$.

The resulting set has the desired properties of the set Ξ_n^0, because $|\Xi_n^0| \geqslant \frac{\alpha}{25K^6}|Q_n| \geqslant \frac{\widetilde{\alpha}n}{\log\log n}$ for sufficiently large n and (Ξ*2*)–(Ξ*7*) are inherited in each step.

$\square$

Since there is only a bounded number of profiles we can use a simple averaging argument to achieve index consistency for a given family of scalings.

LEMMA 53. *For all constants $\widetilde{\alpha} > 0$ and $K, D \in \mathbb{N}$, for all sequences $(B_m)_{m\in\mathbb{N}}$ of subsets $B_m \subseteq \mathbb{Z}_m$ with $|B_m| \leqslant K$ for all $m \in \mathbb{N}$ there exist $\alpha' > 0$ and $L, n_0 \in \mathbb{N}$ such that for all $n \geqslant n_0$ every subset $Z \subset \mathbb{Z}_n$ satisfies the following.*

For every $(\widetilde{\alpha}, B_n, D, Z)$-normal family Ξ_n^0 there is an (α', B_n, D, Z)-normal family $\Xi_n \subset \Xi_n^0$ and there is a profile π of length at most L such that (Z, Ξ_n) is almost regular and index consistent with profile π.

PROOF. Let $\widetilde{\alpha}$, D and $(B_m)_{m\in\mathbb{N}}$ be given. We define

$$L = 2DK \qquad \text{and} \qquad \alpha' = \frac{\widetilde{\alpha}}{L^{L+1}K^{2L}}$$

and let n_0 be sufficiently large. Given some $Z \subseteq \mathbb{Z}_n$ and an $(\widetilde{\alpha}, B_n, D, Z)$-normal set $\Xi_n^0 \subseteq Q_n$ we restrict Ξ_n^0 to Ξ_n with the desired properties.

Properties (Ξ*2*)–(Ξ*7*) from Lemma 52 are inherited from Ξ_n^0 to Ξ_n since restricting does not destroy this properties that are independent of $\widetilde{\alpha}$. We only have to ensure (Ξ*1*) that means we have to keep a sufficiently large proportion of the elements from Ξ_n^0.

Consider the pair (Z, Ξ_n^0). Since (Ξ*3*) and (Ξ*7*) the pair (Z, Ξ_n^0) is by definition almost regular and, in particular, each $z \in Z$ focuses for each $q \in \Xi_n^0$ uniquely on at most two $qb \in qB$. If $z \in Z$ focuses for $q \in \Xi_n^0$ on exactly two $qb_j, qb_k \in qB$ then there is $x \in Z$ such that $z - x = qb_j$ and $x - z = qb_k$.

By the regularity we also know that each $q \in \Xi_n^0$ has a profile of length ℓ_q and by (Ξ*6*) this length is at most $\ell_q \leqslant 2DK = L$ since for each of at most K

elements $qb \in qB$ there are at most L pairs that form a Schur triple with qb. Only the elements of the pairs can be contained in M_q.

By the pigeonhole principle there is some $\ell \leqslant L$ such that at least $\frac{1}{L}|\Xi_n^0|$ many $q \in \Xi_n^0$ have a profile of length exactly ℓ. Since there are at most $(K^2)^\ell$ different profiles of length exactly ℓ there is a profile π of length ℓ and there are at least $\frac{1}{LK^{2L}}|\Xi_n^0|$ different $q \in \Xi_n^0$ which have all the same profile π. Let Ξ_n' be the set of these q.

Next we apply another averaging argument to achieve index consistency. We consider some partition $Z_1 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} Z_\ell$ of Z into ℓ classes chosen uniformly at random. For $q \in \Xi_n'$ consider $M_q = (z_1, \ldots, z_\ell)$ with the natural ordering of Z. We include q in Ξ_n if $z_i \in Z_i$ for all $i = 1, \ldots, \ell$. Clearly $\mathbb{P}(q \in \Xi_n) = \frac{1}{\ell^\ell}$ and $\mathbb{E}[|\Xi_n|] = \frac{|\Xi_n'|}{\ell^\ell}$, which means there is an $\Xi_n \subseteq \Xi_n'$ with

$$|\Xi_n| \geqslant \frac{|\Xi_n'|}{\ell^\ell} \geqslant \frac{1}{L^L} \cdot \frac{|\Xi_n^0|}{LK^{2L}} \geqslant \frac{\widetilde{\alpha} n}{L^{L+1} K^{2L} \operatorname{loglog} n} = \frac{\alpha' n}{\operatorname{loglog} n}\,.$$

Now let $q, q' \in \Xi_n$ and let $z \in M_q \cap M_{q'}$. Since $z \in Z_j$ for some partition class Z_j we know that z has index j in both M_q and $M_{q'}$. Therefore (Z, Ξ_n) is index consistent which finishes the proof. $\square$

4.3.3. Proof of Lemma 45. Finally in this section we combine the preparatory lemmas for a typical set Z and a normal family of scalings Ξ to apply the hypergraph container theorem to $\mathcal{H}(Z, \Xi)$.

Proof of Lemma 45. First, we fix all constants used in the proof. Let constants $C_1 > C_0 > 0$, $\alpha, \mu > 0$, $K \in \mathbb{N}$ and a sequence $(B_m)_{m \in \mathbb{N}}$ of subsets $B_m \subseteq \mathbb{Z}_m$ with $|B_m| \leqslant K$ for all m be given. Lemma 50 applied with the constants above yields $\zeta > 0$ and $D, n_{50} \in \mathbb{N}$. Similarly Lemma 53 applied with

$$\widetilde{\alpha} = \frac{\alpha}{50K^6}$$

yields $\alpha' > 0$ and $L, n_{53} \in \mathbb{N}$. We set

$$\beta = \frac{\alpha'}{96C_1^2 K \binom{\ell}{2}} \quad \text{and} \quad \gamma = \frac{1}{10L} \tag{52}$$

and show that α', β, γ and L defined in this way have the desired properties. Further, let $p = p(n) = c(n)n^{-1/2}$ for some $c(n)$ satisfying $C_0 \leqslant c(n) \leqslant C_1$. We shall show that $\mathbb{Z}_{n,p}$ satisfies with probability at least $1 - \frac{\mu}{2}$ the properties of Lemma 45, hence, by Lemma 50 we can assume that for sufficiently large n, in particular $n \geqslant \max\{n_{50}, n_{53}\}$, the set Z considered in Lemma 45 is from the set $\mathscr{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$. Moreover, let n be sufficiently large such that Lemma 52 applied with α, ζ, D, K, C_0, C_1, and $(B_m)_{m\in\mathbb{N}}$ holds for n.

Now let $Z \in \mathscr{Z}_{n,p,B_n}(\alpha, \zeta, D, K)$ such that for $q \in Q_n$ chosen uniformly at random we have

$$\mathbb{P}(Z \cup qB_n \to (\mathrm{ST})_2) > \alpha\,.$$

then Lemma 52 yields an $(\widetilde{\alpha}, B_n, D, Z)$-normal family $\Xi_n^0 \subseteq Q_n$ with $\widetilde{\alpha} = \alpha/(50K^6)$. By the choice of α' and L Lemma 53 yields an (α', B_n, D, Z)-normal family $\Xi_n \subseteq \Xi_n^0$ and a profile π of length $\ell \leqslant L$ such that (Z, Ξ_n) is almost regular and index consistent with profile π. Removing embeddings from Ξ_n does not destroy any of the properties (Ξ*2*)–(Ξ*7*), regularity, index consistency or a profile, so we can assume w.l.o.g. that $|\Xi_n| \leqslant \lceil \frac{\alpha' n}{\log\log n} \rceil$.

Consequently now we work with a set $Z \subseteq \mathbb{Z}_n$, $\Xi_n \subseteq Q_n$, and a profile π such that (Z1)–(Z5) from Lemma 50 are satisfied as well as (Ξ*1*)–(Ξ*7*) from Lemma 52 and (Z, Ξ_n) is index consistent with profile π.

We consider the hypergraphs $\mathcal{H} = \mathcal{H}(Z, \Xi_n)$ defined by

$$V(\mathcal{H}) = Z \quad \text{and} \quad E(\mathcal{H}) = \{M(Z, qB_n)\colon q \in \Xi_n\}\,,$$

where

$$M_q = M(Z, qB_n) = \{z \in E(Z)\colon \exists b \in B_n \text{ s.t. } z \text{ focuses on } qb\}\,.$$

Since (Z, Ξ_n) is index consistent with a profile of length ℓ we know that $\mathcal{H}$ is an ℓ-uniform hypergraph on $|Z|$ vertices. Our goal is to show that $\mathcal{H}$ satisfies the assumptions of the hypergraph container theorem for

$$\varepsilon = \frac{1}{4} \quad \text{and} \quad \tau = n^{-\frac{1}{8(\ell-1)}}\,.$$

For that we estimate $v(\mathcal{H}) = |Z|$, $e(\mathcal{H})$, $d(\mathcal{H})$, $\Delta_2(\mathcal{H})$, and finally $\delta(\mathcal{H}, \tau)$. To conclude the properties of Lemma 45 we will also need a bound on $\Delta_1(\mathcal{H})$.

By (Z1) we know

$$\frac{1}{2}pn \leqslant v(\mathcal{H}) = |Z| \leqslant 2pn\,. \tag{53}$$

For a bound on $e(\mathcal{H})$ remember that Ξ_n is (α', B_n, D, Z)-normal and by (Ξ*1*) $\frac{\alpha' n}{\operatorname{loglog} n} \leqslant |\Xi_n| \leqslant \lceil \frac{\alpha' n}{\operatorname{loglog} n} \rceil$.

(Z, Ξ_n) is almost regular and Ξ_n does not contain bad scalings w.r.t. B and Z, so each hyperedge M_q of length ℓ consists of $\ell/2$ pairs x_i, y_i, $i = 1 \ldots, \ell/2$, such that x_i and y_i focus onto the same element qb_i respectively onto the same elements qb and qb'. There are at most $\binom{\ell}{2}$ pairs x_i, y_i in M_q and each pair can be extended to a Schur triple by at most three different $q \in \Xi_n$, since the boosters qB_n and $q'B_n$ are disjoint by (Ξ*4*). Consequently M_q can be the hyperedge for at most $3\binom{\ell}{2}$ different $q \in \Xi_n$ and we get that

$$\frac{\alpha' n}{3\binom{\ell}{2} \operatorname{loglog} n} \leqslant e(\mathcal{H}) \leqslant \left\lceil \frac{\alpha' n}{\operatorname{loglog} n} \right\rceil\,. \tag{54}$$

For the average degree we get

$$d(\mathcal{H}) = \ell \cdot \frac{e(\mathcal{H})}{v(\mathcal{H})} \geqslant \ell \cdot \frac{\alpha' n}{3\binom{\ell}{2} \operatorname{loglog} n} \cdot \frac{1}{2pn} \geqslant \frac{\alpha'}{3\ell} \cdot \frac{1}{p \operatorname{loglog} n}\,.$$

Next we look at $\Delta_1(\mathcal{H})$, that means we count for $z \in Z$ the number of $q \in \Xi_n$ such that z focuses on qB_n. Since $qB_n \cap q'B_n = \varnothing$ an upper bound follows from (Z3) and we get

$$\Delta_1(\mathcal{H}) = \max_{z \in Z} \left\{ |\mathcal{S}(n, z, B_n) \cap Z| \right\} \leqslant \max_{z \in Z} \left\{ 8Kp|Q_n| \right\} \overset{(44)}{\leqslant} \frac{8Kpn}{\operatorname{loglog} n}\,. \tag{55}$$

For $\Delta_2(\mathcal{H})$ we consider pairs $z, z' \in Z$ that focus on the same qB_n. By definition of $\mathcal{S}_2$ and since $qB_n \cap q'B_n = \varnothing$ for $q \neq q'$, an upper bound of $\Delta_2(\mathcal{H})$ is given by

$\max_{z,z' \in Z^2}\{\mathcal{S}_2(n,z,z',B_n)\}$. Then Property (Z4) yields that

$$\Delta_2(\mathcal{H}) \leqslant \max_{z,z' \in Z^2}\{\mathcal{S}_2(n,z,z',B_n)\} \leqslant n^{1/4}\,.$$

Now we are prepared to estimate $\delta(\mathcal{H},\tau)$ to check the assumptions of Theorem 17. We have

$$\begin{aligned}
\delta(\mathcal{H},\tau) &= 2^{\binom{\ell}{2}-1}\sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \frac{1}{\tau^{j-1} m d(\mathcal{H})} \sum_{v\in V(\mathcal{H})} d^{(j)}(v) \\
&\leqslant 2^{\binom{\ell}{2}-1}\sum_{j=2}^{\ell} 2^{-\binom{j-1}{2}} \frac{1}{\tau^{j-1} m d(\mathcal{H})} \cdot m \cdot \Delta_2(\mathcal{H}) \\
&\leqslant 2^{\binom{\ell}{2}-1}\sum_{j=2}^{\ell} \frac{1}{\tau^{\ell-1} d(\mathcal{H})} \cdot \Delta_2(\mathcal{H}) \\
&\leqslant 2^{\binom{\ell}{2}-1} \cdot \ell \cdot n^{1/8} \cdot \frac{3\ell}{\alpha'} p \operatorname{loglog} n \cdot n^{1/4} \\
&\leqslant 2^{\binom{\ell}{2}-1} \cdot \frac{3\ell^2 C_1}{\alpha'} \cdot n^{-1/8} \operatorname{loglog} n \\
&\leqslant \frac{\varepsilon}{12\ell!}
\end{aligned}$$

for sufficiently large n.

By the hypergraph container theorem, Theorem 17, there exist some constant $c = c(\ell)$ and a family $\mathcal{J} \subset \mathcal{P}(V(\mathcal{H}))$ satisfying (a), (b), and (c) from Theorem 17. We define

$$\mathcal{C} = \{C \subset V(\mathcal{H}) \colon C = V(\mathcal{H}) \setminus J \text{ for one } J \in \mathcal{J}\}\,.$$

We claim that $\mathcal{C}$ has the desired properties (1), (2), and (3) of Lemma 45.

(1) follows from (c) since $|\mathcal{C}| = |\mathcal{J}|$ and

$$\log|\mathcal{J}| \leqslant c\tau \log(1/\tau)\log(1/\varepsilon) v(\mathcal{H}) \leqslant |Z| \cdot n^{-\frac{1}{8(\ell-1)}} c \log(1/\tau)\log(1/\varepsilon) \leqslant |Z|^{1-\gamma}\,,$$

where the last inequality follows for sufficiently large n from

$$|Z|^{\gamma} \leqslant n^{\gamma} \overset{(52)}{\leqslant} n^{\frac{1}{10L}} \leqslant n^{\frac{1}{10\ell}}\,,$$

since $c = c(\ell)$ and $\log(1/\varepsilon)$ are constants independent of n and $\log(1/\tau) < \log n$.

(2) follows from (b). Assume for a contradiction that there is $C \in \mathcal{C}$ with $|C| < \beta|Z|$ and let $J = V \setminus C \in \mathcal{J}$. Then counting the number of hyperedges of $\mathcal{H}$ will yield a contradiction:

$$\begin{aligned} e(\mathcal{H}) &\leqslant e(\mathcal{H}[V \setminus C]) + |C| \cdot \Delta_1(\mathcal{H}) \\ &\overset{(55)}{\leqslant} e(\mathcal{H}[J]) + \beta|Z| \cdot \frac{8Kpn}{\operatorname{loglog} n} \\ &\overset{(53)}{\leqslant} \varepsilon e(\mathcal{H}) + \beta 16 C_1^2 K \frac{n}{\operatorname{loglog} n} \\ &\overset{(54)}{\leqslant} \varepsilon e(\mathcal{H}) + \frac{\beta 48 C_1^2 K \binom{\ell}{2}}{\alpha'} e(\mathcal{H}) \\ &= \left(\varepsilon + \frac{\beta 48 C_1^2 K \binom{\ell}{2}}{\alpha'}\right) e(\mathcal{H}) \\ &\overset{(52)}{<} e(\mathcal{H}), \end{aligned}$$

so $|C| \geqslant \beta|Z|$ for all $C \in \mathcal{C}$.

(3) For a hitting set A of $\mathcal{H}$ consider the independent set $I = V \setminus A$. Hence by (a) of Theorem 17 there exists $J \in \mathcal{J}$ such that $I \subseteq J$ and, therefore, $A \supseteq V \setminus J = C$ which is an element of $\mathcal{C}$. $\square$

§4.4. Proof of Lemma 46

Let $Y \subseteq \mathbb{Z}_n$. In the following we consider the single row matrix

$$A = \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$$

that is chosen in a way such that a solution $\vec{v} = (x_1, x_2, y_1, y_2, z_1, z_2) \in Y^6$ to $A\vec{v} = 0$ corresponds to a Schur triple in $\operatorname{Base}(Y)$: If $\vec{v}$ is such a solution, then obviously $(x_2 - x_1) + (y_2 - y_1) = (z_2 - z_1)$. On the other hand $x_2 - x_1$, $y_2 - y_1$, and $z_2 - z_1$ are elements from $\operatorname{Base}(Y)$ and they form a Schur triple (which could be degenerated). Most of these solutions, however, will lead to non degenerated Schur triples.

FACT 54. *Let $n \geqslant 7$ and the single row matrix $A = \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$ be given. Consider the system $A\vec{v} = 0$ with $\vec{v} \in \mathbb{Z}_n^6$. Then A is irredundant, density regular and has density $m(A) = \frac{5}{4}$.*

PROOF. Remember that in the introduction we defined A as *irredundant* if there exists a solution $\vec{v} = (v_1, \ldots, v_k)$ of $A\vec{v} = 0$ with $v_i \neq v_j$ for all $i \neq j$ (note that in [**16**] another definition of irredundant is used that is equivalent for sufficiently large n). These solutions are called proper solutions. For $n \geqslant 7$ obviously there is a proper solution of $A\vec{v} = 0$ in $\mathbb{Z}_n$, so A is irredundant.

Clearly $(1, 1, 1, 1, 1, 1)^T$ is a solution (but not a proper solution) to $A\vec{v} = 0$ In [**16**, Theorem 2] it is shown that in this case (A has full rank and is irredundant) A is also density regular.

We defined the density $m(A)$ by

$$m_A = \max_{W \cup \overline{W} = [k], |W| \geqslant 2} \frac{|W| - 1}{|W| - 1 + \operatorname{rank}(A_{\overline{W}}) - \operatorname{rank}(A)}$$

and for $\overline{W} = \varnothing$ also $\operatorname{rank}_{\overline{W}} = 0$. For $\overline{W} \neq \varnothing$ obviously $\operatorname{rank}_{\overline{W}} = 1$. It follows that in our case

$$m(A) = \max\left\{1\,,\ \frac{6-1}{6-1+0-1}\right\} = \frac{5}{4}\,.$$

□

The next theorem follows from [**52**] and it is true even more generally for arbitrary irredundant, density regular matrices and $p \geqslant Cn^{-1/m(A)}$ for some $C \geqslant 0$. Although the theorem (Theorem 2.4.) presented there only guarantees one proper solution, the proof of the main lemma yields at least a constant fraction of the expected number of proper solutions (see Lemma 3.4.).

THEOREM 55 ([**52**]). *For $A = \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}$ and for all $\lambda > 0$ there exist $C > 0, \xi > 0$ such that for every sequence $p = p(n) \geqslant Cn^{-4/5}$ the following a.a.s. holds. Every subset of $\mathbb{Z}_{n,p}$ of size at least λpn contains at least $\xi p^6 n^5$ proper solutions of $A\vec{v} = 0$.*

We conclude this part with the proof of Lemma 46 which uses the ideas of the corresponding proof for arithmetic progressions in [**21**].

PROOF OF LEMMA 46. Let $\lambda > 0$, $C_1 > C_0 > 0$ be given and consider the matrix

$$A = \begin{pmatrix} -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}.$$

Owing to Fact 54 for $n \geqslant 7$ the matrix A has rank 1, is irredundant, and density regular with $m(A) = \frac{5}{4}$. Theorem 55 yields constants $C, \xi > 0$. We define the auxiliary constant $c = \max\{12, C_1^2\}$ and set $\eta = \frac{\xi^2 C_0^6}{432c^3}$. Let $C_0 n^{-1/2} \leqslant p \leqslant C_1 n^{-1/2}$.

Since we want to show that a.a.s. for every subset $S \subseteq \mathbb{Z}_{n,p}$ of size at least λpn there are at least ηn^2 Schur triples in $\mathrm{Base}(S)$ we can assume that n is sufficiently large, in particular $p \geqslant C_0 n^{-1/2} \geqslant Cn^{-4/5} = Cn^{-1/m(A)}$.

For $X \subseteq \mathbb{Z}_n$ let

$$\mathrm{ST}(X) = \{(x, y, z) \in X^3 \colon x, y \text{ and } z \text{ form a Schur triple}\}$$

be the set of (possibly degenerated) Schur triples that are contained in X. Note that every Schur triple in X appears at most 6 times in $\mathrm{ST}(X)$ and each non degenerated Schur triple exactly 6 times. For a Schur triple $(x, y, z) \in \mathrm{ST}(\mathbb{Z}_n)$ and $Y \subseteq \mathbb{Z}_n$ we denote by $\deg_Y(x, y, z)$ the number of six tuples $(x_1, x_2, y_1, y_2, z_1, z_2) \in Y^6$ with pairwise distinct entries such that $x = x_2 - x_1$, $y = y_2 - y_1$ and $z = z_1 - z_2$. Since (x, y, z) is a Schur triple all proper solutions to $A\vec{v} = 0$ are six tuples that contribute to $U(Y) = \sum_{(x,y,z)\in\mathrm{ST}(\mathbb{Z}_n)} \deg_Y(x, y, z)$.

We are also interested in $W(Y) = \sum_{(x,y,z)\in\mathrm{ST}(\mathbb{Z}_n)} \binom{\deg_Y(x,y,z)}{2}$ which can be bounded from above for all $Y \subseteq \mathbb{Z}_{n,p}$ simultaneously by $W(\mathbb{Z}_{n,p})$. In the following we will estimate the expectation and variance of $W(\mathbb{Z}_{n,p})$. For that we count for each triple $(x, y, z) \in \mathrm{ST}(\mathbb{Z}_n)$ (which are at most 6 times the number of Schur triples in $\mathbb{Z}_n$) the number of pairs of six tuples $(x_1, x_2, y_1, y_2, z_1, z_2) \in \mathbb{Z}_{n,p}^6$ and $(x_1', x_2', y_1', y_2', z_1', z_2') \in \mathbb{Z}_{n,p}^6$ each with pairwise distinct entries such that both contribute to $\deg_{\mathbb{Z}_{n,p}}(x, y, z)$. If we count the number of possible choices for these tuples we see that there are at most n choices for x_1. Any choice of x_1 determines the value of x_2 since $x_2 = x + x_1$, any choice of y_1 determines y_2, and z_1 determines z_2. So for the first six tuple there are at most $n(n-2)(n-4) < n^3$ possible configurations in $\mathbb{Z}_n$ and since all entries are pairwise distinct each configuration appears with probability p^6 in $\mathbb{Z}_{n,p}$.

Now for fixed $x, y, z, x_1, x_2, y_1, y_2, z_1, z_2$ we count the number of possible configurations for the second six tuple. First consider the two elements x_1' and x_2'. Similar to the calculations for the first six tuple we see that as long as x_1' and x_2' are distinct from the elements of the first six tuple this will yield a factor of at most np^2 for $\mathbb{E}[W(\mathbb{Z}_{n,p})]$. If they are not distinct from the first six tuple then there are either only six choices for x_1' or only six choices for x_2'. In both cases the remaining element is uniquely determined and we conclude that the resulting factor for $\mathbb{E}[W(\mathbb{Z}_{n,p})]$ for the number of possible choices of (x_1', x_2') is at most the constant 12. The same argument works for the pair y_1', y_2' and for the pair z_1', z_2'. Summarising we get

$$\mathbb{E}[W(\mathbb{Z}_{n,p})] \leqslant |\operatorname{ST}(\mathbb{Z}_n)| \cdot (np^2)^3 \max\{12, np^2\}^3 \leqslant 6n^2 \cdot c^3 n^3 p^6 = O(n^2)\,.$$

The estimation of the variance uses similar calculations. For $(x, y, z) \in \operatorname{ST}(\mathbb{Z}_n)$ we are looking for four six tuples $(x_1^i, x_2^i, y_1^i, y_2^i, z_1^i, z_2^i) \in \mathbb{Z}_{n,p}^6$, $i \in [4]$, each with pairwise distinct entries such that all of them contribute to $\deg_{\mathbb{Z}_{n,p}}(x, y, z)$. For the first two six tuples we get exactly the same calculation as for $\mathbb{E}[W(\mathbb{Z}_{n,p})]$. For the third six tuple we repeat the calculation as for the second one with the following difference: If the pair x_1^3, x_2^3 is not disjoint from the elements chosen for the first two six tuples, then we get instead of 6 at most 12 choices for x_1^3 respectively x_2^3 (and similarly for the pairs y_1^3, y_2^3 and z_1^3, z_2^3). Consequently the factor for $\operatorname{Var}(W(\mathbb{Z}_{n,p}))$ is equal to $\max\{24, np^2\}^3$. Similarly the factor for the fourth six tuple is $\max\{36, np^2\}^3$. As $np^2 = \Theta(1)$ we get in total

$$\operatorname{Var}(W(\mathbb{Z}_{n,p})) = \Theta(\mathbb{E}[W(\mathbb{Z}_{n,p})]) = O(n^2)\,.$$

Then Chebyshev's inequality, Lemma 19, yields that a.a.s.

$$W(\mathbb{Z}_{n,p}) \leqslant 2\mathbb{E}[W(\mathbb{Z}_{n,p})] \leqslant 12c^3n^5p^6\,.$$

Now let an arbitrary $S \subseteq \mathbb{Z}_{n,p}$ of size at least λpn be given. Since Theorem 55 as well as $W(\mathbb{Z}_{n,p}) \leqslant 12c^3n^5p^6$ hold a.a.s. for $\mathbb{Z}_{n,p}$ we can assume that in S there are at least ξn^5p^6 proper solutions to $A\vec{v} = 0$ and that $W(S) \leqslant W(\mathbb{Z}_{n,p}) \leqslant 12c^3n^5p^6$. We conclude that

$$U(S) = \sum_{(x,y,z)\in \mathrm{ST}(\mathbb{Z}_n)} \deg_S(x,y,z) \geqslant \xi n^5p^6$$

and

$$W(S) = \sum_{(x,y,z)\in \mathrm{ST}(\mathbb{Z}_n)} \binom{\deg_S(x,y,z)}{2} \leqslant 12c^3n^5p^6\,.$$

Note that in $U(S)$ and $W(S)$ the term $(x,y,z) \in \mathrm{ST}(\mathbb{Z}_n)$ can be replaced by $(x,y,z) \in \mathrm{ST}(\mathrm{Base}(S))$ since by definition triples in $\mathrm{ST}(\mathbb{Z}_n) \smallsetminus \mathrm{ST}(\mathrm{Base}(S))$ satisfy $\deg_S(x,y,z) = 0$. Finally the Cauchy-Schwarz inequality yields

$$\binom{\xi n^5p^6}{2} \leqslant \binom{U(S)}{2} \leqslant |\,\mathrm{ST}(\mathrm{Base}(S))| \cdot W(S) \leqslant |\,\mathrm{ST}(\mathrm{Base}(S))| \cdot 12c^3n^5p^6$$

and consequently

$$|\,\mathrm{ST}(\mathrm{Base}(S))| \geqslant \frac{\xi^2n^{10}p^{12}}{3\cdot 12c^3n^5p^6} = \frac{\xi^2}{36c^3}n^5p^6 \geqslant \frac{\xi^2C_0^6}{36c^3}n^2 = 12\eta n^2\,.$$

In $\mathrm{ST}(\mathrm{Base}(S))$ we count each Schur triple of $\mathrm{Base}(S)$ at most six times, consequently there are at least $2\eta n^2$ Schur triples contained in $\mathrm{Base}(S)$. On the other hand there are only $\Theta(n)$ degenerated Schur triples contained in $\mathbb{Z}_n$ and consequently for sufficiently large n there are at least ηn^2 non degenerated Schur triples contained in $\mathrm{Base}(Y)$ which finishes the proof. $\square$

Part 2

Hamiltonian Cycles in Hypergraphs

CHAPTER 5

Hamiltonian Cycles

In this chapter we will prove Theorem 13. This chapter is based on [**2**], joint work with Bastos, Mota, Schacht, and Schnitzer.

§5.1. Main Lemmas

5.1.1. Outline of the Proof of Theorem 13. The proof follows the *Absorbing Method* introduced by Rödl, Ruciński, and Szemerédi in [**48**]. For this, we derive the following lemmas. The Absorbing Lemma (Lemma 58), the Reservoir Lemma (Lemma 57), and the Path-Tiling Lemma (Lemma 65).

We call an ℓ-path $\mathcal{A} \subseteq \mathcal{H}$ a β*-absorbing path* for a k-uniform hypergraph $\mathcal{H}$ if for every subset $U \subset V(\mathcal{H})$ of size at most βn there exists an ℓ-path $\mathcal{Q}$ such that $V(\mathcal{Q}) = V(\mathcal{A}) \cup U$ and $\mathcal{Q}$ have the same ends as $\mathcal{A}$, for some $\beta > 0$. The Absorbing Lemma (Lemma 58) ensures the existence of a β-absorbing path $\mathcal{A}$. This reduces the problem of finding a Hamiltonian ℓ-cycle to that of finding an almost spanning ℓ-cycle that contains $\mathcal{A}$.

To obtain an almost spanning ℓ-cycle, we first find a bounded number (independent of $|V(\mathcal{H})|$) of ℓ-paths covering almost all vertices of $V(\mathcal{H}) \smallsetminus \mathcal{A}$ and then connect them using only vertices from a small set, a so-called *reservoir set* that we fix beforehand. The Reservoir Lemma (Lemma 57) shows that it is possible to find this reservoir set R such that any bounded number of disjoint ℓ-paths can be connected to an ℓ-cycle, only using vertices from R.

We can choose the sizes of $\mathcal{A}$ and R small enough, so that the remaining hypergraph satisfies almost the same degree condition as $\mathcal{H}$. Then the Path-Tiling Lemma (Lemma 65) ensures the existence of a collection of ℓ-paths covering almost all vertices of $V(\mathcal{H}) \setminus (\mathcal{A} \cup R)$. This is the only point in the proof where we use the exact value of the degree condition and the non-extremality of $\mathcal{H}$. (In fact, a proof for the corresponding version of the Path-Tiling Lemma for a direct proof of Theorem 12, which allows us to utilize a slightly larger degree condition, is a bit simpler.)

As mentioned before, the paths from the Path-Tiling Lemma and $\mathcal{A}$ can be connected by using vertices from R to an almost spanning ℓ-cycle containing $\mathcal{A}$. Since this ℓ-cycle contains almost all vertices of $\mathcal{H}$, the absorbing property of $\mathcal{A}$ allows us to absorb the leftover vertices, i.e. vertices that are not contained in any of the ℓ-paths and vertices that were not used to connect the ℓ-paths. The resulting ℓ-cycle is the desired Hamiltonian ℓ-cycle.

5.1.2. Connecting. In order to construct an almost spanning ℓ-cycle of a k-uniform hypergraph $\mathcal{H}$, we first find some ℓ-paths and connect them at their ends. Formally, given an ℓ-path $\mathcal{P} = v_1 \cdots v_t$ in $\mathcal{H}$, we define the *ends* of $\mathcal{P}$ as the sets $\{v_1, \ldots, v_\ell\}$ and $\{v_{t-\ell+1}, \ldots, v_t\}$. For a collection of $2m$ mutually disjoint sets of ℓ vertices X_i, Y_i we say that a set of ℓ-paths $\mathcal{T}_1, \ldots, \mathcal{T}_m$ *connects* $(X_i, Y_i)_{i\in[m]}$ if all paths are vertex-disjoint and X_i and Y_i are the ends of $\mathcal{T}_i$, for all $i \in [m]$. The connections for a given collection of disjoint ℓ-paths are given by the following lemma. In addition the lemma allows to restrict the edges used for the connection to a given "well-connected" subset R of vertices.

Lemma 56 (Connecting Lemma). *Let $\eta > 0$ and let $k \geqslant 4$, $1 \leqslant \ell < k/2$, and $m \geqslant 1$ be integers. Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph and $R \subset V$ with $|R| = r \geqslant 32km/\eta^3$. For every collection of $2m$ mutually disjoint sets X_i, $Y_i \in \binom{V}{\ell}$ and $V' = \bigcup_{i\in[m]}(X_i \cup Y_i) \cup R$ the following holds.*

If $\left|N(K) \cap \binom{R}{2}\right| \geqslant \eta\binom{r}{2}$ for all $K \in \binom{V'}{k-2}$, then there exist ℓ-paths $\mathcal{T}_1, \ldots, \mathcal{T}_m$ of size at most 4 connecting $(X_i, Y_i)_{i\in[m]}$, which contain vertices from V' only.

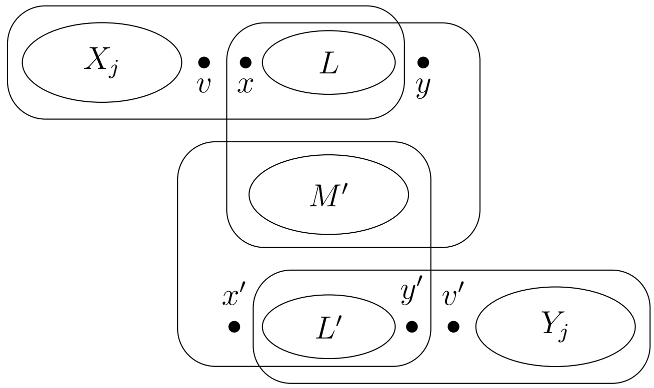

FIGURE 1. The path connecting (X_j, Y_j).

PROOF. Given $\eta > 0$ and integers $k \geqslant 4$, $1 \leqslant \ell < k/2$ and $m \geqslant 1$, let $\mathcal{H} = (V, E)$, $R \subseteq V$, X_i, Y_i for $i \in [m]$, and V' satisfy the assumptions of the lemma.

Suppose we have constructed ℓ-paths $\mathcal{T}_1, \ldots, \mathcal{T}_{j-1}$ connecting for some $j \leqslant m$ the pairs $(X_i, Y_i)_{i \in [j-1]}$ using only vertices from V'. We want to construct a path $\mathcal{T}_j$ with ends X_j and Y_j. We define $F_j = \bigcup_{i \in [m]}(X_i \cup Y_i) \cup \bigcup_{i \in [j-1]} V(\mathcal{T}_i)$ as the set of *forbidden vertices for* $\mathcal{T}_j$.

If $k - 2 \geqslant 2\ell = |X_j \cup Y_j|$, fix a set Z of size $k - 2 - 2\ell$ from $R \smallsetminus F_j$. Since $|R| = r \geqslant 32km/\eta^3$, we know that

$$\left| N(X_j \cup Y_j \cup Z) \cap \binom{R}{2} \right| \geqslant \eta \binom{r}{2} > \binom{r}{2} - \binom{r - 4km}{2} \geqslant \binom{r}{2} - \binom{|R \smallsetminus F_j|}{2}.$$

Hence, there exists a hyperedge $X_j \cup Y_j \cup Z'$ with $Z' \subseteq R \smallsetminus F_j$, which realizes the path $\mathcal{T}_j$.

It is left to consider the case that $2\ell = k - 1$. See Figure 1 for a drawing of the path we will construct in this case. For a set $A \subseteq V$, let $N_A(S) = N(S) \cap \binom{A}{k-s}$.

OBSERVATION. *For any $Z \in \{X_j, Y_j\}$ and $L \in \binom{R \smallsetminus F_j}{\ell - 1}$, there are at least $\eta r/4$ many vertices $z \in R \smallsetminus (F_j \cup L)$ with $|N_{R \smallsetminus F_j}(Z \cup L \cup \{z\})| \geqslant \eta r/4$.*

To see the observation note that we can consider $N_{R \smallsetminus F_j}(Z \cup L)$ as the edge set of a 2-graph with vertex set $R \smallsetminus (F_j \cup L)$. Since $r \geqslant 32km/\eta^3$, it follows from the degree condition of $\mathcal{H}$ into the set R that this graph has edge density at least $\eta/2$ and the observation follows.

Let $L \in \binom{R \smallsetminus F_j}{\ell-1}$ and let $x, y \in R \smallsetminus (F_j \cup L)$ be distinct. We say that (x, L, y) is an *extendable triple* in $R \smallsetminus F_j$ if

$$|N_{R \smallsetminus F_j}(X_j \cup L \cup \{x\})| \geqslant \eta r/4 \quad \text{and} \quad |N_{R \smallsetminus F_j}(Y_j \cup L \cup \{y\})| \geqslant \eta r/4.$$

The observation yields at least $(\eta r/4)(\eta r/4 - 1) > (\eta r/8)^2$ extendable triples (x, L, y) for any fixed $L \in \binom{R \smallsetminus F_j}{\ell-1}$.

Given $S \in \binom{R \smallsetminus F_j}{\ell-2}$ and an extendable triple (x, L, y) disjoint from S, $S \cup L \cup \{x, y\}$ is a $(k-2)$-element set. Consequently, the minimum degree condition of the lemma yields at least $\eta\binom{r}{2}$ pairs $M \in \binom{R}{2}$ such that $S \cup M \cup L \cup \{x, y\}$ is an edge of $\mathcal{H}$. Moreover, similarly as in the proof of the observation at least $(\eta/2)\binom{|R \smallsetminus F_j|}{2}$ of these pairs avoid F_j. Since this is true for every extendable triple and there are at least $\binom{|R \smallsetminus F_j|}{\ell-1}(\eta r/8)^2$ extendable triples, there exists an $M \in \binom{R \smallsetminus F_j}{2}$ that, together with S, forms an edge of $\mathcal{H}$ with at least $(\eta/2)(\eta r/8)^2\binom{|R \smallsetminus F_j|}{\ell-1}$ extendable triples. Since $r \geqslant 32km/\eta^3$, this is more than the number of triples that any single extendable triple can intersect with, so there exist two completely disjoint extendable triples (x, L, y) and (x', L', y') that form an edge of $\mathcal{H}$ together with $M' = M \cup S$.

By the definition of extendable triples we have

$$\left|N_{R \smallsetminus F_j}(X_j \cup L \cup \{x\})\right| \geqslant \eta r/4 > k + 1 = \left|M' \cup L' \cup \{x', y', y\}\right|$$

and

$$\left|N_{R \smallsetminus F_j}(Y_j \cup L' \cup \{y'\})\right| \geqslant \eta r/4 > k + 2 = \left|M' \cup L \cup \{x, y, x'\}\right| + 1.$$

Consequently there are $v, v' \in R \smallsetminus F_j$ such that the hyperedges

$$\{X_j \cup L \cup \{v, x\}\}, \quad \{M' \cup L \cup \{x, y\}\}, \quad \{M' \cup L' \cup \{x', y'\}\}, \quad \text{and} \quad \{Y_j \cup L' \cup \{y', v'\}\}$$

are edges of $\mathcal{H}$, which form a path of size 4 connecting (X_j, Y_j). $\square$

In the main proof we will connect ℓ-paths to an almost spanning ℓ-cycle. The Reservoir Lemma (stated below) ensures the existence of a small set R such that we can connect an arbitrary collection of at most $2m$ many ℓ-sets, only using vertices of R.

LEMMA 57 (Reservoir Lemma). *Let $\eta, \varepsilon > 0$ and let $k \geqslant 4$, $1 \leqslant \ell < k/2$, and $m \geqslant 1$ be integers. Then for every sufficiently large k-uniform hypergraph $\mathcal{H} = (V, E)$ on n vertices with $\delta_{k-2}(\mathcal{H}) \geqslant \eta\binom{n}{2}$ there is a set $R \subset V$ with $|R| \leqslant \varepsilon n$ such that the following holds.*

For every collection X_i, Y_i for $i \in [j]$ of $2j$ mutually disjoint sets of ℓ vertices, where $j \leqslant m$, there exist ℓ-paths $\mathcal{T}_1, \ldots, \mathcal{T}_j$ of size at most 4 connecting $(X_i, Y_i)_{i\in[j]}$ that, moreover, contain vertices from $\bigcup_{i\in[j]}(X_i \cup Y_i) \cup R$ only.

Lemma 57 is a consequence of Lemma 56, since one can show that with high probability a suitably sized random subset $R \subseteq V$ inherits an appropriately scaled minimum degree condition from $\mathcal{H}$. As a consequence such a set satisfies the assumptions of Lemma 56 (with $\eta/2$) and the lemma yields the conclusion of Lemma 57 (see, e.g. [**6**, Lemma 6] for a very similar argument).

5.1.3. Absorption. Given a k-uniform hypergraph $\mathcal{H}$ and $U \subset V$ with $|U| \in (k-\ell)\mathbb{N}$, we say that an ℓ-path $\mathcal{A}$ *absorbs* U if there exists an ℓ-path $\mathcal{Q}$ with the same ends as $\mathcal{A}$ and $V(\mathcal{Q}) = V(\mathcal{A}) \cup U$. At the end of the main proof we will absorb all vertices outside of an almost spanning ℓ-cycle to obtain a Hamiltonian ℓ-cycle using an *absorbing path* $\mathcal{A}$, i.e. a path that can absorb any set U of small linear size. The existence of such a path $\mathcal{A}$ is given by the following lemma.

LEMMA 58 (Absorbing Lemma). *For every η, $\zeta > 0$ and all integers $k \geqslant 4$ and $1 \leqslant \ell < k/2$ there exists $\varepsilon > 0$ such that the following holds for sufficiently large n. Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph on n vertices that satisfies $\delta_{k-2}(\mathcal{H}) \geqslant \eta\binom{n}{2}$. Then there is an ℓ-path $\mathcal{A}$ with $|V(\mathcal{A})| \leqslant \zeta n$ such that for all subsets $U \subset V \smallsetminus V(\mathcal{A})$ of size at most εn with $|U| \in (k-\ell)\mathbb{N}$ there exists an ℓ-path $\mathcal{Q} \subset \mathcal{H}$ with $V(\mathcal{Q}) = V(\mathcal{A}) \cup U$ such that $\mathcal{A}$ and $\mathcal{Q}$ have the same ends.*

PROOF. Let $\eta, \zeta > 0$ and let $k \geqslant 4$ and $1 \leqslant \ell < k/2$ be integers, and assume w.l.o.g. that $\zeta \leqslant 1$. Fix auxiliary constants

$$\widetilde{\eta} = \frac{\eta}{4k!} \quad \text{and} \quad q = 3k - 2\ell$$

and set

$$\varepsilon = \frac{\zeta \widetilde{\eta}^{10}}{56k}.$$

Let n be sufficiently large and let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph on n vertices that satisfies $\delta_{k-2}(\mathcal{H}) \geqslant \eta\binom{n}{2}$. First, we will show that for any $S \in \binom{V}{k-\ell}$ there exist many, i.e. $\Omega(n^q)$, 3-edge ℓ-paths that absorb S (see Claim 1 below). For that we will use the following consequence of the minimum degree condition. Let $A, B \subset V(\mathcal{H})$ be disjoint sets of vertices with $|A| \leqslant k - 2$ and $|B| \leqslant q + k$. Then,

$$\deg_{\mathcal{H}[V \setminus B]}(A) \geqslant \frac{(n - |A|) \cdot \dots \cdot (n - k + 3)}{(k - |A|)!} \cdot \eta \binom{n}{2} - |B| n^{k-|A|-1} \geqslant \widetilde{\eta} n^{k-|A|}. \quad (56)$$

Claim 1. *For every $S \in \binom{V}{k-\ell}$ there exist at least $\widetilde{\eta}^5 n^q$ many 3-edge ℓ-paths that absorb S.*

PROOF. Let $S_1 \cup S_2 = S$ be chosen in some way such that

$$|S_1| \geqslant |S_2| \geqslant |S_1| - 1 \quad \text{and} \quad \max\{0, 3\ell - k\} \leqslant |S_1 \cap S_2| < \ell \quad (57)$$

and set $s_1 = |S_1|$, $s_2 = |S_2|$, and $s_3 = |S_1 \cap S_2|$. Clearly, we have

$$s_1 + s_2 - s_3 = |S| = k - \ell. \quad (58)$$

It follows from the choices above that $s_1 + s_2 \geqslant 2\ell$. Indeed, since $s_3 \geqslant 3\ell - k$ we have $k - \ell = s_1 + s_2 - s_3 \leqslant s_1 + s_2 - 3\ell + k$ and, hence, $s_1 + s_2 \geqslant 2\ell$. Furthermore, $s_1 \geqslant s_2 \geqslant s_1 - 1$ yields

$$s_1 \geqslant s_2 \geqslant \ell. \quad (59)$$

We then select the following sets. See Figure 2 for a drawing of the chosen sets and edges containing them. In each step, we will only select sets that are disjoint from S and anything chosen in a previous step.

(*i*) Since $s_1 \leqslant k - \ell - 1 \leqslant k - 2$, by (56) there exist $\tilde{\eta} n^{k-s_1}$ choices for a $(k-s_1)$-set X such that $f_1 = X \cupdot S_1$ is an edge of $\mathcal{H}$. Since $|X| = k - s_1 \overset{(58)}{=} \ell + s_2 - s_3$ it follows from (59) that we may partition $X = L_1 \cupdot F \cupdot F_1$ such that $|L_1| = \ell$ and $|F| = \ell - s_3 \overset{(57)}{>} 0$.

(*ii*) Since $k \geqslant 4$ we have $k - \ell \geqslant 3$ and, consequently, $s_1 \geqslant \lceil (k-\ell)/2 \rceil \geqslant 2$. Thus, by (56) and $|S_2 \cup F| = s_2 + \ell - s_3 = k - s_1$, there exist $\tilde{\eta} n^{s_1}$ choices for a set Y of size s_1 such that $f_2 = S_2 \cupdot F \cupdot Y$ is an edge of $\mathcal{H}$. Again owing to (59) we may partition $Y = L_2 \cupdot F_2$ such that $|L_2| = \ell$.

(*iii*) Fix $L_1' \subset L_1$ and $L_2' \subset L_2$ subsets of size $\ell - 1$. Note that

$$|L_1' \cupdot L_2' \cupdot F \cupdot F_1 \cupdot F_2| = |X| + |Y| - 2 = k - 2.$$

Therefore, there exist at least $\tilde{\eta} n^2$ choices for a pair of vertices $\{x_1, x_2\}$ such that $e_2 = \{x_1, x_2\} \cupdot L_1' \cupdot L_2' \cupdot F \cupdot F_1 \cupdot F_2$ is an edge of $\mathcal{H}$.

(*iv*) Since $k \geqslant 4$ we have $\ell + 1 \leqslant k - 2$. Therefore, there exist $\tilde{\eta} n^{k-(\ell+1)}$ choices each for two disjoint edges e_1 and e_3 such that $\{x_1\} \cupdot L_1 \subset e_1$ and $\{x_2\} \cupdot L_2 \subset e_3$.

By construction we have

$$e_1 \cap e_2 = \{x_1\} \cupdot L_1' \quad \text{and} \quad e_2 \cap e_3 = \{x_2\} \cupdot L_2',$$

so the edges e_1, e_2, and e_3 form an ℓ-path $\mathcal{P}$ in $\mathcal{H}$. Moreover, since

$$e_1 \cap f_1 = L_1, \quad |f_1 \cap f_2| = |(S_1 \cap S_2) \cup F| \overset{(i)}{=} \ell, \quad \text{and} \quad f_2 \cap e_3 = L_2,$$

the edges e_1, f_1, f_2, and e_3 form an ℓ-path $\mathcal{P}'$. Since $k - \ell - 1 \geqslant \ell$, we may select for $\mathcal{P}$ and $\mathcal{P}'$ the same ends in e_1 and e_3. Moreover, $V(\mathcal{P}') = V(\mathcal{P}) \cupdot S$ and, therefore, the ℓ-path $\mathcal{P}$ absorbs S. From (*i*)–(*iv*) it is clear that there are at least $\tilde{\eta}^5 n^q$ choices for $\mathcal{P}$. $\square$

Following the scheme from [**48**], let $\mathcal{F} \subset V(\mathcal{H})^q$ be a family of ordered q-sets of vertices such that each of these sets are selected from $V(\mathcal{H})^q$ independently with

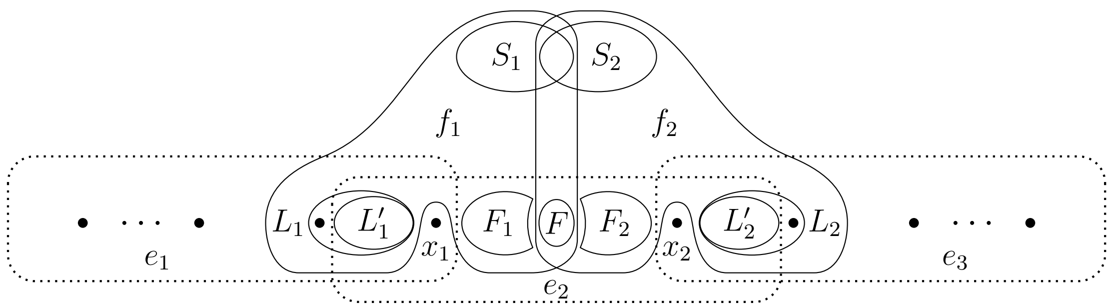

FIGURE 2. The path $\mathcal{P}$, consisting of e_1, e_2, and e_3, that absorbs S.

probability

$$p = \frac{4\varepsilon}{\widetilde{\eta}^5 n^{q-1}}.$$

An ℓ-path in $V(\mathcal{H})^q$ is an ordered set $(v_1, \ldots, v_q)$ of vertices such that

$$e_1 = \{v_1, \ldots, v_k\}, \quad e_2 = \{v_{k-\ell+1}, \ldots, v_{2k-\ell}\}, \quad \text{and} \quad e_3 = \{v_{2k-2\ell+1}, \ldots, v_{3k-2\ell}\}$$

are edges in $\mathcal{H}$. Using Chernoff's inequality, with high probability we have

$$|\mathcal{F}| \leqslant 2pn^q = \frac{8\varepsilon}{\widetilde{\eta}^5} n.$$

By Claim 1, for each set S of size $k - \ell$, at least $\widetilde{\eta}^5 n^q$ ℓ-paths in $V(\mathcal{H})^q$ absorb S. By Chernoff's inequality w.h.p. for all $S \in \binom{V}{k-\ell}$, there are at least $2\varepsilon n$ ℓ-paths in $\mathcal{F}$ that absorb S. The expected value of the number of intersecting pairs of q-sets in $\mathcal{F}$ is at most

$$nn^{2q-2}p^2 = n^{2q-1}\left(\frac{4\varepsilon}{\widetilde{\eta}^5 n^{q-1}}\right)^2 = \varepsilon n \frac{16\zeta}{56k} \leqslant \frac{1}{2}\varepsilon n$$

So by Markov's inequality the number of intersecting pairs of q-sets in $\mathcal{F}$ is at most εn with probability at least $1/2$.

Let $\mathcal{F}$ be a family that satisfies the above conditions. For each of the intersecting pairs in $\mathcal{F}$, delete one of the q-sets and let $\mathcal{F}' \subset \mathcal{F}$ be the remaining family. Using Lemma 56 with $R = V$ (so $r = |R|$ is sufficiently large), we can connect all ℓ-paths in $\mathcal{F}'$ to an ℓ-path $\mathcal{A}$ with

$$|V(\mathcal{A})| \leqslant |\mathcal{F}'| \cdot (4k + t) \leqslant 2pn^q \cdot 7k = \frac{56k}{\widetilde{\eta}^5}\varepsilon n \leqslant \zeta n$$

and this path absorbs all sets $U \subset V \backslash V(\mathcal{A})$ with $|U| \in (k - \ell)\mathbb{N}$ and $|U| \leqslant \varepsilon n$. $\square$

5.1.4. Path-Tiling. In this part we will find a path-tiling of ℓ-paths in $\mathcal{H}$ that covers all but a small fraction of the vertices of $\mathcal{H}$. For that purpose we use the so-called *weak regularity lemma* for hypergraphs, which is the straightforward extension of Szemerédi's regularity lemma for graphs [**57**] (see also Lemma 37). Roughly speaking, we will show that there exists a fractional $\mathcal{C}_\ell$-tiling, a so-called β-hom$(\mathcal{C}_\ell)$-tiling in the resulting reduced hypergraph $\mathcal{R}$ of $\mathcal{H}$, where $\mathcal{C}_\ell$ is the k-uniform "cherry" consisting of two hyperedges that share exactly 2ℓ vertices. The fractional $\mathcal{C}_\ell$-tiling of $\mathcal{R}$ will transfer to a path-tiling of $\mathcal{H}$.

First, we introduce the standard notation for the regularity lemma. Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph and let $V_1, \ldots, V_k$ be non-empty, mutually disjoint subsets of V. We denote the number of edges with one vertex in each V_i by $e_{\mathcal{H}}(V_1, \ldots, V_k)$ and define the density of $\mathcal{H}$ w.r.t. $(V_1, \ldots, V_k)$ by

$$d_{\mathcal{H}}(V_1, \ldots, V_k) = \frac{e_{\mathcal{H}}(V_1, \ldots, V_k)}{|V_1| \cdots |V_k|}.$$

For $\varepsilon > 0$ and $d > 0$, a k-tuple $(V_1, \ldots, V_k)$ of mutually disjoint subsets of vertices is called (ε, d)*-regular* if for all k-tuples $(A_1, \ldots, A_k)$ of subsets $A_i \subseteq V_i$ with $|A_i| \geqslant \varepsilon |V_i|$, we have

$$|d_{\mathcal{H}}(A_1, \ldots, A_k) - d| \leqslant \varepsilon.$$

Moreover, the tuple $(V_1, \ldots, V_k)$ is called ε*-regular* if it is (ε, d)-regular for some $d > 0$. Below we state the weak hypergraph regularity lemma (see, e.g. [**7**, **18**, **56**]).

Lemma 59 (Weak regularity lemma). *For all integers $k \geqslant 2$ and $t_0 \geqslant 1$ and for every $\varepsilon > 0$, there exists $T_0 = T_0(k, t_0, \varepsilon)$ such that for every sufficiently large k-uniform hypergraph $\mathcal{H} = (V, E)$ on n vertices, there exists a partition $V = V_0 \,\dot\cup\, V_1 \,\dot\cup\, \ldots \,\dot\cup\, V_t$ satisfying*

(*i*) $t_0 \leqslant t \leqslant T_0$,

(*ii*) $|V_1| = \cdots = |V_t|$ *and* $|V_0| \leqslant \varepsilon n$, *and*

(*iii*) *for all but at most $\varepsilon\binom{t}{k}$ many k-subsets $\{i_1, \ldots, i_k\} \subset [t]$, the k-tuple $(V_{i_1}, \ldots, V_{i_k})$ is ε-regular.*

A vertex partition of a hypergraph $\mathcal{H}$ satisfying (i)–(iii) of the conclusion of Lemma 59 will be referred to as an ε-regular partition. For $\varepsilon > 0$ and $d > 0$, we define the *reduced hypergraph* $\mathcal{R} = \mathcal{R}(\varepsilon, d)$ of $\mathcal{H}$ w.r.t. such a partition as the k-uniform hypergraph on the vertex set $[t]$ and

$$\{i_1, \ldots, i_k\} \in E(\mathcal{R}) \iff (V_{i_1}, \ldots, V_{i_k}) \text{ is } (\varepsilon, d')\text{-regular, for some } d' \geqslant d.$$

In typical applications of the regularity lemma, the reduced hypergraph inherits some key features of the given hypergraph $\mathcal{H}$. In fact, the following observation shows that the reduced hypergraph inherits approximately the minimum degree condition of the original hypergraph. A similar result can be found in [**27**, Proposition 16] and for completeness we include its proof below.

LEMMA 60. *Given $c, \varepsilon, d > 0$ and integers $k \geqslant 3$ and $t_0 \geqslant 2k/d$. Let $\mathcal{H}$ be a k-uniform hypergraph on $n \geqslant t \geqslant t_0$ vertices such that*

$$\delta_{k-2}(\mathcal{H}) \geqslant c\binom{n}{2}.$$

If $\mathcal{H}$ has an ε-regular partition $V_0 \cup V_1 \cup \ldots \cup V_t$ with reduced hypergraph $\mathcal{R} = \mathcal{R}(\varepsilon, d)$, then at most $\sqrt{\varepsilon}\binom{t}{k-2}$ many $(k-2)$-subsets K of $[t]$ violate

$$\deg_{\mathcal{R}}(K) \geqslant (c - 2d - \sqrt{\varepsilon})\binom{t}{2}.$$

PROOF. Let $\mathcal{D} = \mathcal{D}(d)$ and $\mathcal{N} = \mathcal{N}(\varepsilon)$ be the hypergraphs with vertex set $[t]$ and

- $E(\mathcal{D})$ consists of all sets $\{i_1, \ldots, i_k\}$ such that $d(V_{i_1}, \ldots, V_{i_k}) \geqslant d$,
- $E(\mathcal{N})$ consists of all sets $\{i_1, \ldots, i_k\}$ such that $(V_{i_1}, \ldots, V_{i_k})$ is not ε-regular.

Note that the reduced hypergraph $\mathcal{R}(\varepsilon, d)$ is the hypergraph with vertex set $[t]$ and edge set $E(\mathcal{D}) \setminus E(\mathcal{N})$. For an arbitrary $K = \{i_1, \ldots, i_{k-2}\} \in \binom{[t]}{k-2}$ we will show that

$$\deg_{\mathcal{D}}(K) \geqslant (c - 2d)\binom{t}{2}. \tag{60}$$

Let $n/t \geqslant |V_{i_j}| = m \geqslant (1-\varepsilon)n/t$ be the size of the partition classes and let x be the number of edges in $\mathcal{H}$ that intersect each V_{i_j} in exactly one vertex for each $j \in [k-2]$. By the condition on $\delta_{k-2}(\mathcal{H})$ and $t \geqslant t_0 \geqslant 2k/d$, we obtain

$$x \geqslant m^{k-2}\left(c\binom{n}{2} - (k-2)mn\right) \geqslant (c-d)\, m^{k-2}\binom{n}{2}.$$

If (60) did not hold, then we would find for x the upper bound

$$x < (c-2d)\binom{t}{2}m^k + \binom{t}{2}dm^k \leqslant (c-d)\, m^{k-2}\binom{n}{2}$$

contradicting the lower bound for x.

Next we observe that at most $\sqrt{\varepsilon}\binom{t}{k-2}$ many $(k-2)$-sets K satisfy $\deg_{\mathcal{N}}(K) \leqslant \sqrt{\varepsilon}\binom{t}{2}$ since the number of non-ε-regular k-tuples in $\mathcal{R}$ is at most $\varepsilon\binom{t}{k}$. Consequently, it follows from the degree conditions on $\mathcal{D}$ and $\mathcal{N}$ that all but at most $\sqrt{\varepsilon}\binom{t}{k-2}$ many $(k-2)$-sets K satisfy

$$\deg_{\mathcal{R}}(K) \geqslant \left(c - 2d - \sqrt{\varepsilon}\right)\binom{t}{2}.$$

□

We will find a suitable fractional $\mathcal{C}_\ell$-tiling in the reduced hypergraph $\mathcal{R}$, where the cherry $\mathcal{C}_\ell$ is the k-uniform hypergraph with vertex set $[2k-2\ell]$ and edges $\{1,\dots,k\}$ and $\{k-2\ell+1,\dots,2k-2\ell\}$.

Definition 61. *Let $\mathcal{C}$ and $\mathcal{R}$ be k-uniform hypergraphs, $\beta > 0$, and let Φ be a multiset of hypergraph homomorphisms from $\mathcal{C}$ to $\mathcal{R}$. A function $h\colon \Phi \to \{a\beta\colon a \in \mathbb{N}_{>0}\}$ is called a β-hom$(\mathcal{C})$-tiling if the weight $w_h(v)$ of a vertex v satisfies*

$$w_h(v) = \sum_{u \in V(\mathcal{C})} \sum_{\phi \in \Phi: v = \phi(u)} h(\phi) \leqslant 1$$

for all $v \in V(\mathcal{R})$. We call

$$w(h) = \sum_{v \in V(\mathcal{R})} w_h(v) = \sum_{\phi \in \Phi} h(\phi)|V(\mathcal{C})|$$

the weight of the tiling.

The following building block allows us to easily define a tiling on a single edge.

FACT 62. *Given an edge $e = \{v_1, \dots, v_k\}$, there exists a $\frac{1}{2(k-\ell-1)}$-hom$(\mathcal{C}_\ell)$-tiling h that is non-zero only on e, such that $w_h(v_i) = 1$ for $i \in [k-2]$ and $w_h(v_{k-1}) = w_h(v_k) = \frac{k-2}{2(k-\ell-1)}$. Note that we may scale the weight of h by any $q \in (0,1]$ and obtain a $\frac{q}{2(k-\ell-1)}$-hom$(\mathcal{C}_\ell)$-tiling with $w_h(v_i) = q$ for $i \in [k-2]$ and $w_h(v_{k-1}) = w_h(v_k) = \frac{q(k-2)}{2(k-\ell-1)}$. Similarly, for any $q \in (0,1]$ there exists a $\frac{q}{2(k-\ell)}$-hom$(\mathcal{C}_\ell)$-tiling with $w_h(v_i) = q$ for $i \in [k]$.*

PROOF. For this consider the homomorphism that maps $\mathcal{C}_\ell$ to e such that $v_1, \dots, v_{2\ell-2}, v_{k-1}$ and v_k are the image of the intersection of the two edges of $\mathcal{C}_\ell$. By cyclically shifting the image of the first $2\ell - 2$ vertices of the intersection and appropriate scaling, we obtain all homomorphisms for the required tiling. We obtain the even weight distribution for the last part of the fact by cyclically shifting the whole image k times. $\square$

The following lemma is the main part of the proof of the Path-Tiling Lemma. For this we introduce a fractional notion of extremality. We say that a k-uniform hypergraph $\mathcal{R}$ on t vertices is *β-fractionally (ℓ, ξ)-extremal* if there is a function $b \colon V(\mathcal{R}) \to \{0\} \cup [\beta, 1]$ with

$$\sum_{v \in V(\mathcal{R})} b(v) \geqslant \frac{2(k-\ell)-1}{2(k-\ell)} t \qquad \text{and} \qquad \sum_{e \in E(\mathcal{R})} \prod_{v \in e} b(v) \leqslant \xi \binom{t}{k}.$$

Note that the function b can be viewed as a set of weighted vertices, which plays the rôle of the vertex set B in the definition of extremality.

LEMMA 63. *For all integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$, there exist C and γ_0 such that for all $\alpha > 0$ and $\gamma \in (0, \gamma_0)$, there exist $\beta > 0$ and $\varepsilon > 0$ such that the following holds for sufficiently large t. Let $\mathcal{R}$ be a k-uniform hypergraph on t vertices that is not β-fractionally $(\ell, C\gamma)$-extremal and*

$$\deg(K) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - \gamma \right) \binom{t}{2} \tag{61}$$

holds for all but at most $\varepsilon\binom{t}{k-2}$ *sets* $K \in \binom{V(\mathcal{R})}{k-2}$. *Then there exists a* β-hom$(\mathcal{C}_\ell)$-*tiling* h *with weight at least* $(1-\alpha)t$.

PROOF. Clearly, it is sufficient to prove the lemma for small values of α. Consequently the quantification of the lemma allows us to fix the parameters and auxiliary constants C' and c to satisfy the following hierarchy of constants

$$\frac{1}{k}, \frac{1}{\ell} \gg \frac{1}{C'} \gg \frac{1}{C} \gg \gamma_0 \geqslant \gamma \gg \alpha \gg c, \varepsilon, \tag{62}$$

where "$\gg x$" denotes that x is chosen sufficiently small with regard to all constants to its left. Moreover, we fix β inductively such that

$$1 = \beta_0 \gg \beta_1 \gg \cdots \gg \beta_{\lfloor 1/c \rfloor} = \beta \quad \text{and} \quad 16 \cdot k! \text{ divides } \tfrac{\beta_i}{\beta_{i+1}},$$

and let t be sufficiently large such that c, ε, $\beta \gg 1/t$. Note that any β_i-hom$(\mathcal{C}_\ell)$-tiling is also a β-hom$(\mathcal{C}_\ell)$-tiling as β_i is a multiple of β. To prove the lemma, we show that given a β_i-hom$(\mathcal{C}_\ell)$-tiling h with weight $w(h) < (1-\alpha)t$, there exists a β_{i+1}-hom$(\mathcal{C}_\ell)$-tiling h' with weight $w(h') \geqslant w(h) + ct$. We can begin with the trivial 1-hom$(\mathcal{C}_\ell)$-tiling with weight zero and hence, after at most $1/c$ steps, we obtain a β-hom$(\mathcal{C}_\ell)$-tiling with weight at least $(1-\alpha)t$.

For the rest of the proof fix a β_i-hom$(\mathcal{C}_\ell)$-tiling h with weight $w(h) < (1-\alpha)t$ and assume for a contradiction that there is no β_{i+1}-hom$(\mathcal{C}_\ell)$-tiling with weight $w(h) + ct$. It follows from the upper bound on the weight that there are at least $\alpha t/2$ vertices $v \in V(\mathcal{R})$ with $w_h(v) < 1 - \alpha/2$ and we may fix a subset W of them of size $\alpha t/2$.

Although a tiling with bigger weight implies the existence of more edges, we will not use these edges for an improvement. So we may actually assume w.l.o.g. that

$$(1-2\alpha)t \leqslant w(h) < (1-\alpha)t,$$

as we can otherwise add edges on $V \smallsetminus W$ until this weight can be trivially achieved and remove these edges after the improvement step.

We view Φ, the set of homomorphisms from $\mathcal{C}_\ell$ to $\mathcal{R}$, as a multiset, where we include ϕ with multiplicity $\frac{h(\phi)}{\beta_i}$, so that we can assume $h\colon \Phi \to \{\beta_i\}$. With this notion the following bounds on the size of Φ follow from the above

$$(1-2\alpha)\frac{t}{\beta_i v(\mathcal{C}_\ell)} \leqslant |\Phi| < (1-\alpha)\frac{t}{\beta_i v(\mathcal{C}_\ell)}. \tag{63}$$

Also, we identify a homomorphism ϕ in Φ with the – not necessarily distinct – vertices $(v_1,\dots,v_{2k-2\ell})$ in its image, where $v_i = \phi(i)$ so that $\{v_1,\dots,v_k\}$ and $\{v_{k-2\ell+1},\dots,v_{2k-2\ell}\}$ form edges in $\mathcal{R}$. We refer to the elements of Φ as *cherries* $\mathcal{C} \in \Phi$.

Consider the $(k-2)$-sets in W that satisfy the degree condition (61) of the lemma. Since $\alpha \gg \varepsilon$, among those $(k-2)$-sets we find a collection $\mathcal{W}$ whose elements are pairwise disjoint and cover at least $|W|/2$ vertices. For later reference we note

$$|\mathcal{W}| \geqslant \frac{|W|}{2(k-2)} > \frac{\alpha t}{4k}. \tag{64}$$

For $K \in \mathcal{W}$ we consider the *link graph* L_K of K in $\mathcal{R}$, which is the (2-uniform) graph containing all edges e such that $K \cup e \in E(\mathcal{R})$. At most $\frac{t}{\beta_i}\binom{v(\mathcal{C}_\ell)}{2} \leqslant \gamma\binom{t}{2}$ edges have both ends in the same $\mathcal{C} \in \Phi$ and at most $\alpha t^2/2 \leqslant \gamma\binom{t}{2}$ edges contain a vertex from W, so let L'_K be the graph obtained from L_K by removing all these edges. Combined with the degree condition (61) we have

$$e(L'_K) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - 3\gamma\right)\binom{t}{2} \tag{65}$$

for every such $(k-2)$-set $K \in \mathcal{W}$.

We will find pairs $\mathcal{C}, \mathcal{C}' \in \Phi$ on which the link graph allows us to find a better tiling. For this we only want to consider edges in the *bipartite induced link graph* $L_K(\mathcal{C}, \mathcal{C}')$. Formally the vertex classes of $L_K(\mathcal{C}, \mathcal{C}')$ are given by two disjoint copies of $[2k-2\ell]$. In particular, $L_K(\mathcal{C}, \mathcal{C}')$ has $4k-4\ell$ vertices even when $\mathcal{C}$ and $\mathcal{C}'$ intersect or when $\mathcal{C}$ or $\mathcal{C}'$ are not given by injective homomorphisms from $\mathcal{C}_\ell$. Moreover, two vertices i and j from different classes are adjacent in $L_K(\mathcal{C}, \mathcal{C}')$ if $\{v_i, v'_j\}$ is an edge in the link graph L_K, where v_i is the image of $i \in V(\mathcal{C}_\ell)$ in $\mathcal{C}$

and v'_j is the image of j in $\mathcal{C}'$. However, similar as above we canonically identify the vertices of $L_K(\mathcal{C},\mathcal{C}')$ with the vertices of $\mathcal{C}$ and $\mathcal{C}'$.

We show in the following that for most $K \in \mathcal{W}$ the bipartite link graph between most $\mathcal{C}$ and $\mathcal{C}'$ has a very specific structure. We call $\mathcal{C},\mathcal{C}' \in \Phi$ an *extremal pair for* K if there exist *special vertices* $u \in \mathcal{C}$ and $u' \in \mathcal{C}'$ such that $L_K(\mathcal{C},\mathcal{C}')$ contains exactly all edges incident to these two vertices. In particular, in such a case $L_K(\mathcal{C},\mathcal{C}')$ has $4(k-\ell)-1$ edges.

Claim 2. *There exists a* β_{i+1}-$\hom(\mathcal{C}_\ell)$*-tiling* h' *with* $w(h') > w(h) + ct$*, or for every* $\mathcal{C} \in \Phi$ *there exists* $u_{\mathcal{C}} \in \mathcal{C}$ *such that the following holds. For all but at most* $\gamma|\mathcal{W}|$ *sets* $K \in \mathcal{W}$ *all but at most* $C'\gamma|\Phi|^2$ *pairs* $\mathcal{C},\mathcal{C}' \in \Phi^2$ *are extremal for* K *with special vertices* $u_{\mathcal{C}}$ *and* $u_{\mathcal{C}'}$*.*

Proof. The proof of the claim consists of three steps. First we show that if for a given $(k-2)$-tuple $K \in \mathcal{W}$ and some pair of cherries $\mathcal{C},\mathcal{C}' \in \Phi$ the induced bipartite link graph $L_K(\mathcal{C},\mathcal{C}')$ contains more than $4(k-\ell)-1$ edges, then there is a *local improvement* of the tiling by a weight of at least $\beta_i/4$. In a second step we shall bound the number of possible local improvements, as otherwise we could combine them to arrive at a desired tiling h' with a weight increased by ct, which would conclude the proof. In the last step we utilise this bound on the number of local improvements to show that "typically" $L_K(\mathcal{C},\mathcal{C}')$ contains only $4(k-\ell)-1$ edges and displays the structural conditions stated in the claim.

For the first step we consider two cases. Suppose that there is a matching with three edges in $L_K(\mathcal{C},\mathcal{C}')$ for some $K \in \mathcal{W}$ and $\mathcal{C}$, $\mathcal{C}' \in \Phi$. Recall that $L_K(\mathcal{C},\mathcal{C}')$ is a bipartite graph with partition classes of size $2(k-\ell)$. Then we assign weight $\frac{k-2}{6(k-\ell-1)}\beta_i$ to the vertices of the matching edges and weight β_i to the vertices of K. Setting in addition $h'(\mathcal{C}) = h'(\mathcal{C}') = (1 - \frac{k-2}{6(k-\ell-1)})\beta_i$, this defines a valid β_{i+1}-$\hom(\mathcal{C}_\ell)$-tiling h' by applying Fact 62 (with $q = \frac{1}{3}\beta_i$) to the three edges in $\mathcal{R}$ corresponding to the matching edges in the link graph. Note that the weights on the vertices of K remain unchanged, and by considering the vertices on which the weight has changed, it is easy to see that

$$w(h') = w(h) + \left(k - 2 - (4k - 4\ell - 6) \cdot \frac{k-2}{6(k-\ell-1)}\right)\beta_i \geqslant w(h) + \frac{1}{3}\beta_i\,,$$

which yields a local improvement in this case.

For the next case suppose that there are two vertices in $\mathcal{C}$ each incident to two edges such that all four neighbours in $\mathcal{C}'$ are distinct. On the vertices of these edges we put weights $\frac{k-2}{8(k-\ell-1)}\beta_i$ and β_i on the vertices of K. Set $h'(\mathcal{C}) = (1 - \frac{k-2}{4(k-\ell-1)})\beta_i$ and $h'(\mathcal{C}') = (1 - \frac{k-2}{8(k-\ell-1)})\beta_i$. Again, this defines a tiling h' with

$$\begin{aligned} w(h') &= w(h) + \left(k-2 - (2k - 2\ell - 2)\frac{k-2}{4(k-\ell-1)} - (2k - 2\ell - 4)\frac{k-2}{8(k-\ell-1)}\right)\beta_i \\ &\geqslant w(h) + \frac{k-2}{4}\beta_i \geqslant w(h) + \frac{1}{4}\beta_i\,. \end{aligned} \tag{66}$$

This establishes a local improvement for this case and concludes the discussion of the first step.

For the second step suppose that there is a subset $\mathcal{W}' \subset \mathcal{W}$ of size at least $\gamma|\mathcal{W}|/2$, such that for each $K \in \mathcal{W}'$ we can define a local improvement for $\gamma|\Phi|^2$ cherry pairs. We apply these local improvements greedily, only using each cherry $\mathcal{C} \in \Phi$ at most once (over all $K \in \mathcal{W}'$), to increase the weight of the tiling. This procedure may end, either when every $K \in \mathcal{W}'$ contains a saturated vertex, in which case we enlarge the total weight by at least $\frac{\alpha}{2} \cdot |\mathcal{W}'|$, or when for every $K \in \mathcal{W}'$ for each of the $\gamma|\Phi|^2$ pairs of cherries at least one cherry was used for some local improvement already. Since any collection of $\gamma|\Phi|^2$ pairs of cherries contains $\gamma|\Phi|/2$ such pairs none of which share a cherry, then the latter case would imply that we applied $\gamma|\Phi|/2$ local improvements before.

In summary we showed that if for at least $\gamma|\mathcal{W}|/2 \geqslant \gamma\frac{\alpha t}{8k}$ (see (64)) many $K \in \mathcal{W}$ we can define a local improvement for $\gamma|\Phi|^2$ cherry pairs then we can aggregate local improvements leading to a β_{i+1}-hom$(\mathcal{C}_\ell)$-tiling h'' with weight at least

$$w(h'') \geqslant w(h) + \min\left\{\frac{\alpha}{2} \cdot \gamma\frac{\alpha t}{8k}\,,\ \frac{\beta_i}{4} \cdot \frac{\gamma}{2}|\Phi|\right\} \overset{(62),(63)}{>} w(h) + ct,$$

which would conclude the proof of Claim 2.

Consequently, for the third step we need only consider those $K \in \mathcal{W}$ for which at most a γ-fraction of its cherry pairs $\mathcal{C}$, $\mathcal{C}'$ allow one of the two local improvements discussed in the first step. In particular, those pairs induce no matching of size three in $L_K(\mathcal{C}, \mathcal{C}')$ and by König's theorem [**36**] $L_K(\mathcal{C}, \mathcal{C}')$ spans at most $4(k-\ell)$ edges. On the other hand, in view of (63) the degree condition (65) of $K \in \mathcal{W}$ translates to an average number of edges of at least $4(k-\ell)-1-4(3\gamma+4\alpha)(k-\ell)^2$ in the link graphs. So, as C' was chosen big enough, all but $(C'-1)\gamma|\Phi|^2$ cherry pairs $\mathcal{C}$, $\mathcal{C}'$ induce exactly $4(k-\ell)-1$ edges in $L_K(\mathcal{C}, \mathcal{C}')$. Since in addition these pairs allow no local improvement as considered in (66), there must be a vertex on each side that has a complete neighbourhood on the other side, so most pairs are indeed extremal.

It remains to show that typically the special vertex $u \in \mathcal{C}$ in an extremal pair $L_K(\mathcal{C}, \mathcal{C}')$ is independent of K and $\mathcal{C}'$. So assume for a moment that there are two vertices u and v in $\mathcal{C} \in \Phi$ such that u is a special vertex for an extremal pair $L_K(\mathcal{C}, \mathcal{C}')$ and v is special for an extremal pair $L_{K'}(\mathcal{C}, \mathcal{C}'')$ for some (possibly non-distinct) K, $K' \in \mathcal{W}$ and $\mathcal{C}'$, $\mathcal{C}'' \in \Phi$. In this case we can define a local improvement by "splitting" the case with four edges above. Indeed choose four edges incident with u in $L_K(\mathcal{C}, \mathcal{C}')$ and four for v in $L_{K'}(\mathcal{C}, \mathcal{C}'')$. Assign weights $\frac{1}{2}\beta_i$ to the vertices of K and K', and $\frac{k-2}{16(k-\ell-1)}\beta_i$ to the vertices of the eight chosen edges. Set $h'(\mathcal{C}) = (1 - \frac{k-2}{4(k-\ell-1)})\beta_i$ and reduce the weights on $\mathcal{C}'$ and $\mathcal{C}''$ by $\frac{k-2}{16(k-\ell-1)}\beta_i$ (or by $\frac{k-2}{8(k-\ell-1)}\beta_i$ in case $\mathcal{C}' = \mathcal{C}''$). Similar calculations as in (66) lead to a local improvement of $\beta_i/4$ involving the three cherries $\mathcal{C}$, $\mathcal{C}'$, and $\mathcal{C}''$.

For each cherry $\mathcal{C}$ fix $u_{\mathcal{C}} \in \mathcal{C}$ as the vertex that occurs most often as a special vertex over all extremal pairs $L_K(\mathcal{C}, \mathcal{C}')$. If for at least $\gamma|\mathcal{W}|/2$ many $K \in \mathcal{W}$ we can define a local improvement for $\gamma|\Phi|^2$ extremal pairs, i.e. pairs $\mathcal{C}$, $\mathcal{C}'$ for which there exist K' and $\mathcal{C}''$ as above, we can aggregate them as in the second step. Otherwise the chosen $u_{\mathcal{C}}$ satisfy the statement of the claim. $\square$

We call $\mathcal{C} \in \Phi$ *good* if it is contained in at least $\frac{1}{2}|\Phi|$ extremal pairs for at least $\frac{1}{2}|\mathcal{W}|$ many $K \in \mathcal{W}$ and *bad* otherwise. As a β_{i+1}-hom$(\mathcal{C}_\ell)$-tiling h' with $w(h') > w(h) + ct$ would complete the proof of Lemma 63, it follows from Claim 2 that at most $5C'\gamma|\Phi|$ cherries are bad. Moreover, for every vertex $v \in V$ we denote by $\Phi_{\text{BAD}}(v)$ the set of bad cherries $\mathcal{C} \in \Phi$ that contain it.

To complete the proof of Lemma 63 we will show that we find a matching M in $\mathcal{R}$ such that every vertex $v \in e \in M$ is contained in "many" good cherries. For each good cherry $\mathcal{C} \in \Phi$ there are a lot of choices for $\mathcal{C}'$ and $K \in \mathcal{W}$ such that $\mathcal{C}$ and $\mathcal{C}'$ are an extremal pair for K. We will redistribute the weights to transfer weight from the non-special vertices of $\mathcal{C}$ (and $\mathcal{C}'$) to K, which will reduce the weight on v (since we will ensure that v is a non-special vertex). Repeating this for every $v \in e$ will allow us to obtain a local improvement for the tiling and repeating this for sufficiently many hyperedges $e \in M$ leads to the desired global improvement.

We define the function $a\colon V(\mathcal{R}) \to [0,1]$ by $v \mapsto \beta_i \cdot \sum_{\mathcal{C}\in\Phi} \mathbb{1}_{\{v\}}(u_{\mathcal{C}})$, which assigns to a vertex the sum of weights used by special vertices. As any cherry contains $2(k-\ell)$ vertices, it is clear that $\sum_{v\in V(\mathcal{R})} a(v) \leqslant \frac{t}{2(k-\ell)}$ and, therefore, we can utilise the β-fractional non-extremality of $\mathcal{R}$ for $b(\cdot) = 1 - a(\cdot)$ and obtain

$$\sum_{e\in E(\mathcal{R})} \prod_{v\in e} b(v) \geqslant C\gamma \binom{t}{k}.$$

Since there are at most $5C'\gamma|\Phi|$ bad cherries, they contribute at most

$$\beta_i \sum_{v\in V(\mathcal{R})} \left|\Phi_{\text{BAD}}(v)\right| \leqslant \beta_i v(\mathcal{C}_\ell) \cdot 5C'\gamma|\Phi| \overset{(63)}{\leqslant} 5C'\gamma t \tag{67}$$

to the overall weight of the β_i-hom$(\mathcal{C}_\ell)$-tiling h. We shall only use good cherries to redistribute weights for the desired β_{i+1}-hom$(\mathcal{C}_\ell)$-tiling, so we consider the function $b'\colon V(\mathcal{R}) \to [0,1]$ given by

$$b'(v) = \max\left\{0,\, b(v) - \beta_i \cdot |\Phi_{\text{BAD}}(v)|\right\}$$

and in view of (67) and $C' \ll C$ (cf. (62)) we have

$$\sum_{e\in E(\mathcal{R})}\prod_{v\in e} b'(v) \geqslant \frac{C}{2}\gamma\binom{t}{k}.$$

By a simple double counting argument there is a matching $M \subset E(\mathcal{R})$ with

$$\sum_{e\in M}\prod_{v\in e} b'(v) \geqslant \frac{C}{2}\gamma \cdot \frac{t}{k}$$

and since $b'(v) \in [0,1]$ we have

$$\sum_{e\in M} k\cdot \min_{v\in e}\{b'(v)\} \geqslant \sum_{e\in M} k\prod_{v\in e} b'(v) \geqslant \frac{C}{2}\gamma t. \tag{68}$$

In particular, we may assume that $\min_{v\in e}\{b'(v)\} > 0$ for every $e \in M$, since this has no effect on inequality (68). Moreover, from the definition of the function $b'(\cdot)$ it then follows that $\min_{v\in e}\{b'(v)\} \geqslant \beta_i$ for every $e \in M$.

For each vertex $v \in \bigcup M$, we consider good cherries that contain v as a non-special vertex. Assume that we have $K \in \mathcal{W}$ and an extremal pair $\mathcal{C}$, $\mathcal{C}'$ such that v is a non-special vertex in $\mathcal{C}$. Recall that this means that $L_K(\mathcal{C}, \mathcal{C}')$ contains all edges incident to the two special vertices. We define a *local weight shift* as follows. Assign weights $\frac{1}{2(k-\ell)-1}\cdot\frac{k-2}{4(k-\ell-1)}\beta_i$ to the vertices of all edges incident with exactly one of the special vertices, β_i to the vertices of K and set $h'(\mathcal{C}) = h'(\mathcal{C}') = (1 - \frac{k-2}{4(k-\ell-1)})\beta_i$. By similar calculations as before, this defines a valid β_{i+1}-hom$(\mathcal{C}_\ell)$-tiling h' with $w(h') = w(h)$. On the other hand, the weight of the vertex v and all other non-special vertices in $L_K(\mathcal{C}, \mathcal{C}')$ is reduced by $\frac{k-2}{4(k-\ell)-2}\beta_i$, i.e.

$$w_{h'}(v) = w_h(v) - \frac{k-2}{4(k-\ell)-2}\beta_i\,.$$

It follows from the definition of $b'(v)$ that we have at least $b'(v)/\beta_i$ many good cherries that contain v as non-special vertex and we shall apply at most $\min_{u\in e}\{b'(u)\}/\beta_i$ local weight shifts for a vertex $v \in e \in M$.

For every edge $e \in M$ we would like to apply these local weight shifts for every vertex $v \in e$, where we cycle through all k vertices and apply one shift at a time. In other words, we evenly reduce the weights on the vertices of e. Note that we can apply these local weight shifts using $K, \mathcal{C}$, and $\mathcal{C}'$ unless we have saturated the vertices in K or used one of the cherries before. The procedure stops as soon as we reach a vertex for which no local weight shift is possible.

We first discuss the ideal case that this procedure does not stop, i.e. for every $e \in M$ and every $v \in e$ we applied $\min_{u \in e}\{b'(u)\}/\beta_i$ local weight shifts. In this case, for every $e \in M$ we reduced the weight of all vertices $v \in e$ by at least

$$\frac{1}{\beta_i}\min_{u\in e}\{b'(u)\} \cdot \frac{k-2}{4(k-\ell)-2}\beta_i = \frac{k-2}{4(k-\ell)-2}\min_{u\in e}\{b'(u)\}.$$

Consequently, we may appeal to Fact 62 to increase the tiling on the edge e by the same amount. Repeating this for all $e \in M$, we obtain a β_{i+1}-hom$(\mathcal{C}_\ell)$-tiling h'' satisfying

$$\begin{aligned} w(h'') &\geqslant w(h) + \sum_{e \in M} k \cdot \frac{k-2}{4(k-\ell)-2}\min_{u\in e}\{b'(u)\} \\ &\overset{(68)}{\geqslant} w(h) + \frac{C\gamma t}{2} \cdot \frac{k-2}{4(k-\ell)-2} \\ &\overset{(62)}{\geqslant} w(h) + ct, \end{aligned}$$

which would conclude the proof of Lemma 63 in this case.

In the case that the procedure stops, there is some $v \in V(M)$ and a good cherry $\mathcal{C}$ for v such that $\mathcal{C}$ cannot be used for a local weight shift for v. This means, since $\mathcal{C}$ is a good cherry, that either $\frac{1}{2}|\mathcal{W}|$ many $K \in \mathcal{W}$ contain a saturated vertex or that at least $\frac{1}{2}|\Phi|$ cherries were used in local weight shifts before. In the case that $\frac{1}{2}|\mathcal{W}|$ many $K \in \mathcal{W}$ contain a saturated vertex, each of these vertices was used in at least $\frac{\alpha}{2\beta_i}$ local weight shifts, so in total we have applied

$$\frac{1}{2}|\mathcal{W}| \cdot \frac{\alpha}{2\beta_i} \overset{(64)}{\geqslant} \frac{\alpha t}{8k} \cdot \frac{\alpha}{2\beta_i}$$

local weight shifts. If on the other hand all $\frac{1}{2}|\Phi|$ possible cherries $\mathcal{C}'$ were used in local weight shifts before, then we have applied at least $\frac{1}{4}|\Phi|$ local weight shifts. As in the ideal case, using Fact 62, we conclude that we can increase the tiling on the edges in M and obtain a β_{i+1}-hom$(\mathcal{C}_\ell)$-tiling h'' with

$$w(h'') \geqslant w(h) + \Big(\min\Big\{\frac{\alpha^2 t}{16k\beta_i}, \frac{|\Phi|}{4}\Big\} - k\Big) \cdot \frac{(k-2)\beta_i}{4(k-\ell)-2} \overset{(62),(63)}{\geqslant} w(h) + ct,$$

which concludes the proof of Lemma 63. □

Next we want to transfer the β-hom$(\mathcal{C}_\ell)$-tiling of $\mathcal{R}$ into a path-tiling of $\mathcal{H}$. For that purpose we will use the following lemma from [**28**, Lemma 2.7].

Lemma 64. *Fix $k \geqslant 3$, $1 \leqslant \ell < k/2$ and ε, $d > 0$ such that $d > 2\varepsilon$. Let $m > \frac{k^2}{\varepsilon^2(d-\varepsilon)}$. Suppose $\mathcal{V} = (V_1, \dots, V_k)$ is an (ε, d)-regular k-tuple with*

$$|V_1| = \cdots = |V_{2\ell}| = m \text{ and } |V_{2\ell+1}| = \cdots = |V_k| = 2m.$$

Then there are at most $\frac{2k}{(d-\varepsilon)\varepsilon}$ vertex disjoint ℓ-paths that together cover all but at most $2k\varepsilon m$ vertices of $\mathcal{V}$. □

Finally, by using Lemma 64 on the edges of the β-hom$(\mathcal{C}_\ell)$-tiling of $\mathcal{R}$ given by Lemma 63, we obtain a path-tiling from $\mathcal{H}$ of the desired size.

Lemma 65 (Path-Tiling Lemma). *For all integers $k \geqslant 3$ and $1 \leqslant \ell < k/2$, there exist $C, \gamma_0 > 0$ such that for all $\alpha > 0$, $\gamma \leqslant \gamma_0$ there exists an integer s such that the following holds for all sufficiently large n. Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices and*

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - \gamma\right)\binom{n}{2}.$$

Then either there is a family of at most s disjoint ℓ-paths that cover all but at most αn vertices of $\mathcal{H}$ or $\mathcal{H}$ is $(\ell, C\gamma)$-extremal.

Proof. Let $k \geqslant 3$ and $1 \leqslant \ell < k/2$ be given. Let C' and γ_0' be the constants given by Lemma 63 for k and ℓ. Set $C = 6C'$ and $\gamma_0 = \frac{\gamma_0'}{4}$, and let $\alpha > 0$ and $\gamma \leqslant \gamma_0$. Following the quantification of Lemma 63 with $\frac{\alpha}{2}$ and γ we obtain β and ε' and a sufficiently large t_0. Let ε be sufficiently small. Then the weak regularity lemma (Lemma 59) for $\varepsilon_0 = \frac{\beta\varepsilon}{2} < \gamma^2$ and t_0 yields T_0. Let s be a sufficiently large constant. Let $\mathcal{H}$ be a k-uniform hypergraph on n vertices such that

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - \gamma\right)\binom{n}{2}$$

and n is sufficiently large. By the regularity lemma there exists an ε_0-regular partition $V_0 \mathbin{\dot\cup} \ldots \mathbin{\dot\cup} V_t$ of $\mathcal{H}$ with $|V_1| = \cdots = |V_t| = m$, $|V_0| \leqslant \varepsilon_0 n$ and $t_0 \leqslant t \leqslant T_0$ and the corresponding reduced hypergraph $\mathcal{R} = \mathcal{R}(\varepsilon_0, \gamma)$ on t vertices satisfies, by Lemma 60,

$$\deg_{\mathcal{R}}(K) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - 4\gamma\right)\binom{t}{2}$$

for all but at most $\sqrt{\varepsilon_0}\binom{t}{k-2} \leqslant \varepsilon'\binom{t}{k-2}$ many $(k-2)$-sets $K \in \binom{[t]}{k-2}$. We split the remainder of the proof in two cases, depending on whether $\mathcal{R}$ is β-fractionally $(\ell, 4C'\gamma)$-extremal.

Suppose that $\mathcal{R}$ is not β-fractionally $(\ell, 4C'\gamma)$-extremal. Then Lemma 63 implies that there exists a β-hom$(\mathcal{C}_\ell)$-tiling h of $\mathcal{R}$ with weight $(1-\frac{\alpha}{2})t$. Let Φ^+ be the set of homomorphisms ϕ from $\mathcal{C}_\ell$ to $\mathcal{R}$ with $h(\phi) > 0$, which implies in fact $h(\phi) \geqslant \beta$. We will use Lemma 64 to obtain ℓ-paths covering almost all vertices of $\mathcal{H}$ and for this we split the partition classes according to the tiling h. Let $\{R_1^\phi, \ldots, R_{2k-2\ell}^\phi\}_{\phi\in\Phi^+}$ be a family such that for all $\phi \neq \phi' \in \Phi^+$

- $R_i^\phi \subset V_{\phi(i)}$ for all $i \in [2k-2\ell]$,
- $R_i^\phi \cap R_j^{\phi'} = \varnothing$ for all $i, j \in [2k-2\ell]$,
- $|R_i^\phi| = 2\lfloor\frac{h(\phi)m}{2}\rfloor$ for all $i \in [2k-2\ell]$.

For each $\phi \in \Phi^+$ and all $i \in \{k-2\ell+1, \ldots, k\}$ let $S_i^\phi \cup U_i^\phi = R_i^\phi$ be a partition of R_i^ϕ into two classes of equal size. Note that

$$(R_1^\phi, \ldots, R_{k-2\ell}^\phi, S_{k-2\ell+1}^\phi, \ldots, S_k^\phi) \quad \text{and} \quad (U_{k-2\ell+1}^\phi, \ldots, U_k^\phi, R_{k+1}^\phi, \ldots, R_{2k-2\ell}^\phi)$$

are (ε, γ)-regular, where we used that $h(\phi) \geqslant \beta$ for all $\phi \in \Phi^+$. Then, with Lemma 64 we obtain at most $\frac{2k}{(\gamma-\varepsilon)\varepsilon}$ many ℓ-paths that cover all but $k\varepsilon|R_i^\phi|$ vertices. Applying this to each homomorphism $\phi \in \Phi^+$ we obtain at most s many ℓ-paths.

We claim that the number of vertices in $V(\mathcal{H})$ that are not covered by these paths is less then αn. For this note that the uncovered vertices are the vertices from the partition class V_0, the vertices that are not contained in any R_i^ϕ and those vertices in some R_i^ϕ that are not contained in any ℓ-path. At most $\frac{\alpha}{2}n$ vertices are not in any R_i^ϕ due to the weight of the β-hom(C_ℓ)-tiling h and we lose at most $\frac{2t}{\beta}$ vertices due to the rounding in the definition of R_i^ϕ. The ℓ-paths cover all but a $(k\varepsilon)$-fraction of vertices in $\bigcup_{i,\phi} R_i^\phi$. Consequently the total number of uncovered vertices is at most

$$\varepsilon_0 n + \frac{\alpha}{2}n + \frac{2t}{\beta} + k\varepsilon n < \alpha n.$$

Now suppose that $\mathcal{R}$ is β-fractionally $(\ell, 4C'\gamma)$-extremal. This means by definition that there is a function $b\colon V(\mathcal{R}) \to \{0\} \cup [\beta, 1]$ with

$$\sum_{v \in V(\mathcal{R})} b(v) \geqslant \frac{2(k-\ell)-1}{2(k-\ell)}t \qquad \text{and} \qquad \sum_{e \in E(\mathcal{R})} \prod_{v \in e} b(v) \leqslant 4C'\gamma \binom{t}{k}.$$

For each $i \in [t]$ we fix a subset $A_i \subseteq V_i$ with $|A_i| = \lfloor b(i)|V_i| \rfloor$ and define $B = \bigcup_{i \in [t]} A_i$. Thus, recalling the definition of the reduced hypergraph $\mathcal{R} = \mathcal{R}(\varepsilon_0, \gamma)$

$$\begin{aligned} e_{\mathcal{H}}(B) &\leqslant \sum_{e \in E(\mathcal{R})} \prod_{v \in e} \left(b(v)\frac{n}{t}\right) + \binom{t}{k}\gamma\left(\frac{n}{t}\right)^k + \varepsilon_0 \binom{t}{k}\left(\frac{n}{t}\right)^k + t\binom{n/t}{2}\binom{n}{k-2} \\ &\leqslant 4C'\gamma\binom{n}{k} + \gamma\binom{n}{k} + \varepsilon_0\binom{n}{k} + \frac{k(k-1)}{2t}\binom{n}{k} \\ &\leqslant 5C'\gamma\binom{n}{k}. \end{aligned}$$

Note that

$$|B| \geqslant \left(\frac{2(k-\ell)-1}{2(k-\ell)}t\right)(1-\varepsilon_0)\frac{n}{t} - t \geqslant \left(\frac{2(k-\ell)-1}{2(k-\ell)} - \varepsilon_0\right)n.$$

Therefore, by adding at most $\varepsilon_0 n$ vertices from $V \setminus B$ to B we obtain a set B' with $|B'| = \left\lfloor \frac{2(k-\ell)-1}{2(k-\ell)}n \right\rfloor$ such that

$$e_{\mathcal{H}}(B') \leqslant e_{\mathcal{H}}(B) + \varepsilon_0 n\binom{n}{k-1} \leqslant 6C'\gamma\binom{n}{k} = C\gamma\binom{n}{k},$$

from which we conclude that $\mathcal{H}$ is $(\ell, C\gamma)$-extremal. $\square$

§5.2. Proof of Theorem 13

Below we give the proof of the main technical result, which details the outline from Section 5.1.1 and is based on the lemmas from the last section.

Proof of Theorem 13. Let $0 < \xi < 1$ and let $k \geqslant 4$ and $1 \leqslant \ell < k/2$ be integers. Let C and γ_0 be given by the Path-Tiling Lemma (Lemma 65) for k and ℓ. Let $\gamma < \gamma_0$ be a sufficiently small constant, in particular we may assume $\gamma \ll \xi$. By the Absorbing Lemma (Lemma 58) for γ, $\zeta = \gamma$, k, and ℓ we obtain ε. Following the quantification of the Path-Tiling Lemma for $\alpha = \varepsilon/2$ and 5γ we obtain an integer s. We will use the Reservoir Lemma (Lemma 57) with $\eta = \frac{4(k-\ell)-1}{4(k-\ell)^2} - 3\gamma$, $\varepsilon' = \min\{\varepsilon/2, \gamma\}$, k, and $m = s+1$. Let $n \in (k-\ell)\mathbb{N}$ be sufficiently large and let $\mathcal{H}$ be a k-uniform hypergraph on n vertices.

Suppose $\mathcal{H}$ is not (ℓ, ξ)-extremal and

$$\delta_{k-2}(\mathcal{H}) \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - \gamma\right)\binom{n}{2}.$$

Let $\mathcal{A}$ be the absorbing path obtained with the Absorbing Lemma and let X_0 and Y_0 be the ends of $\mathcal{A}$. Then $|V(\mathcal{A})| \leqslant \gamma n$ and $\mathcal{A}$ has the following absorption property: for every subset $U \subset V \setminus V(\mathcal{A})$ with $|U| \leqslant \varepsilon n$ and $|U| \in (k-\ell)\mathbb{N}$ there exists an ℓ-path $\mathcal{Q} \subset \mathcal{H}$ such that $V(\mathcal{Q}) = V(\mathcal{A}) \cup U$ and $\mathcal{Q}$ has the ends X_0 and Y_0.

Let $V' = (V \setminus V(\mathcal{A})) \cup \{X_0, Y_0\}$ and let $\mathcal{H}' = \mathcal{H}[V']$ be the subhypergraph of $\mathcal{H}$ induced by V'. Note that

$$\delta_{k-2}(\mathcal{H}') \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - 3\gamma\right)\binom{n}{2}.$$

The Reservoir Lemma guarantees the existence of a set $R \subset V'$ with $|R| \leqslant \varepsilon' n \leqslant \gamma n$ such that for every $j \leqslant s+1$ and every family $(X_i, Y_i)_{i\in[j]}$ of mutually disjoint pairs of sets of ℓ vertices can be connected by paths that contain vertices of $\bigcup_{i\in[j]}(X_i \cup Y_i) \cup R$ only.

Let $V'' = V \setminus (V(\mathcal{A}) \cup R)$ and let $\mathcal{H}'' = \mathcal{H}[V'']$ be the subhypergraph of $\mathcal{H}$ induced by V''. Then

$$\delta_{k-2}(\mathcal{H}'') \geqslant \left(\frac{4(k-\ell)-1}{4(k-\ell)^2} - 5\gamma\right)\binom{n}{2}.$$

Now we apply the Path-Tiling Lemma to $\mathcal{H}''$ and obtain a family of at most s disjoint ℓ-paths that cover all but at most $\alpha|V''| \leqslant \alpha n$ vertices of $\mathcal{H}''$, or $\mathcal{H}''$ is $(\ell, C\gamma)$-extremal. Set $n'' = |V''|$ and suppose for a contradiction that $\mathcal{H}''$ is $(\ell, C\gamma)$-extremal. Then there exists a set $B \subset V''$ such that $|B| = \left\lfloor \frac{2(k-\ell)-1}{2(k-\ell)} n'' \right\rfloor$ and $e(B) \leqslant C\gamma (n'')^k$. By adding at most $n - n'' \leqslant 2\gamma n$ vertices from $V \setminus B$ to B, we obtain a vertex set $B' \subset V$ such that $|B'| = \left\lfloor \frac{2(k-\ell)-1}{2(k-\ell)} n \right\rfloor$ and

$$e(B') \leqslant C\gamma (n'')^k + 2\gamma n \binom{n-1}{k-1} \leqslant \xi n^k,$$

a contradiction to the fact that $\mathcal{H}$ is not (ℓ, ξ)-extremal. Therefore, we may assume that there exist disjoint ℓ-paths $\mathcal{P}_1, \ldots, \mathcal{P}_j$ with $j \leqslant s$ that cover all but at most $\alpha|V''| \leqslant \alpha n$ vertices of $\mathcal{H}''$.

For all $i \in [j]$, we denote the ends of $\mathcal{P}_i$ by X_i and Y_i. Let $Y_{j+1} = Y_0$. By using the Reservoir Lemma to connect the family $(X_i, Y_{i+1})_{0 \leqslant i \leqslant j}$, we connect the ℓ-paths $\mathcal{A}, \mathcal{P}_1, \ldots, \mathcal{P}_j$ to an ℓ-cycle $\mathcal{C} \subset \mathcal{H}$.

Let $U = V \setminus V(\mathcal{C})$ be the set of vertices not contained in $\mathcal{C}$, i.e. the vertices that were leftover in the reservoir R or uncovered by the path-tiling. We have $|U| \leqslant (\varepsilon' + \alpha)n \leqslant \varepsilon n$. Furthermore, since $\mathcal{C}$ is an ℓ-cycle and $n \in (k-\ell)\mathbb{N}$, we have $|U| \in (k-\ell)\mathbb{N}$. Therefore, we can utilise the absorbing property of $\mathcal{A}$ to replace $\mathcal{A}$ in $\mathcal{C}$ by a path $\mathcal{Q}$ with the same ends as $\mathcal{A}$, obtaining a Hamiltonian ℓ-cycle of $\mathcal{H}$. $\square$

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Appendix

Summary/Zusammenfassung

In this thesis we investigate two problems in extremal and probabilistic combinatorics. In the first part we analyse sharp thresholds for Ramsey-type properties of random discrete structures, which contributes to the common theme in recent years to transfer classical results to sparse random structures. For two graphs F and G let $G \to (F)^e_r$ denote that for every edge colouring of G with r colours there exists a monochromatic copy of F. In 1995 Rödl and Ruciński determined the threshold $p = p(F, r)$ for $G(n, p) \to (F)^e_r$ for the binomial random graph $G(n, p)$ and any F and r. Furthermore, in 2006 Friedgut et al. proved that in the case that $r = 2$ and F being a triangle the threshold is *sharp*. In the first part we generalise Friedgut's result to a larger class of graphs F including all cycles. Related to this question we also show that the property that a random subset of the integers contains in every 2-colouring a monochromatic Schur triple has a sharp threshold.

In the second part we present a result concerning Hamiltonian cycles in hypergraphs. In 1952 Dirac showed that every graph on $n \geqslant 3$ vertices with minimum degree at least $n/2$ contains a Hamiltonian cycle. Transferring Dirac's Theorem to hypergraphs leads to multiple open questions since there are several notions of cycles and of minimum degree in k-uniform hypergraphs for $k \geqslant 3$. Over the last 20 years various researchers proved such extensions to hypergraphs. In this thesis we continue this line of research and obtain an approximate version for so-called loose ℓ-cycles and a δ_{k-2}-degree condition in k-uniform hypergraphs.

In dieser Arbeit werden zwei Probleme der extremalen Kombinatorik untersucht. Eine typische Fragestellung der letzten Jahre in diesem Forschungsbereich beschäftigt sich mit der Übertragung klassischer Resultate auf dünne zufällige Strukturen. In dieses Themengebiet fällt auch der erste Teil dieser Arbeit, in der Ramsey-Eigenschaften von zufälligen Teilmengen diskreter Strukturen analysiert

werden. Für zwei Graphen F und G schreibe dabei $G \to (F)^e_r$, wenn für jede Kantenfärbung von G mit r Farben eine einfarbige Kopie von F existiert. Im Jahr 1995 haben Rödl und Ruciński für den binomialen Zufallsgraphen $G(n,p)$ sowie alle Graphen F und jede Anzahl an Farben r den Schwellenwert $p = p(F,r)$ der Eigenschaft $G(n,p) \to (F)^e_r$ bestimmt. Friedgut et al. erweiterten dies 2006 für den Fall eines Dreiecks F und $r = 2$, indem sie zeigten, dass der Schwellenwert dann *scharf* ist. In dieser Arbeit wird Friedgut's Ergebnis auf eine größere Klasse von Graphen inklusive aller Kreise F erweitert. Auf eine ähnliche Weise wird zudem gezeigt, dass die Eigenschaft, dass eine zufällige Teilmenge der ganzen Zahlen in jeder 2-Färbung ein einfarbiges Schur Trippel enthält, einen scharfen Schwellenwert hat.

Der zweite Teil der Arbeit beschäftigt sich mit Hamiltonkreisen in Hypergraphen. 1952 hat Dirac gezeigt, dass jeder Graph auf $n \geqslant 3$ Ecken mit Minimalgrad mindestens $n/2$ einen Hamiltonkreis enthält. Die Übertragung von Dirac's Theorem auf Hypergraphen führt zu verschiedenen Fragestellungen, da es für Kreise und Minimalgrad unterschiedliche Konzepte in Hypergraphen gibt. Über die letzten 20 Jahre haben unterschiedliche Forscher Ergebnisse zu diesem Themenkomplex beigetragen. In dieser Arbeit wird diese Forschung fortgesetzt und eine approximative Version des Falles von sogenannten dünnen ℓ-Kreisen und einer δ_{k-2}-Gradbedingung in k-uniformen Hypergraphen vorgestellt.

Publications Related to This Thesis

Articles

- M. Schacht, F. Schulenburg, *Sharp thresholds for Ramsey properties of strictly balanced nearly bipartite graphs*, accepted, to appear in Random Structures and Algorithms
- J. de O. Bastos, G. O. Mota, M. Schacht, J. Schnitzer, F. Schulenburg, *Loose Hamiltonian cycles forced by large* $(k-2)$*-degree - approximate version*, submitted to Electronic Notes in Discrete Mathematics

Extended abstracts

- J. de O. Bastos, G. O. Mota, M. Schacht, J. Schnitzer, F. Schulenburg, *Loose Hamiltonian cycles forced by large* $(k-2)$*-degree - approximate version*, Proceedings of the Tenth Jornadas de Matematica Discreta y Algoritmica (JMDA 2016), vol. 54 series Electron. Notes Discrete Math., 325-330

Declaration on My Contributions

Chapter 2 and Chapter 3 is based on the paper *Sharp thresholds for Ramsey properties of strictly balanced and nearly bipartite graphs* [**53**], which is joint work with my PhD supervisor Mathias Schacht, who introduced me to the topic.

He suggested that the result given in [**24**] may be improved by using a technique similar to the the one in [**21**]. I read the relevant articles and together we discussed the different key points in the proof strategy. I drafted the first version, which we improved together to the version in [**53**].

During that project the question arose to which regular systems of equations (that are not density regular) these methods might be applied to. Owing to the well known connection between Ramsey-type results for triangles and solutions of the Schur-equation I focused on that case. This research leads to Chapter 4. The necessary changes in the lemmas and their proofs I developed on my own, while writing this thesis.

Chapter 5 is based on the paper *Loose Hamiltonian cycles forced by large* $(k-2)$*-degree – approximate version* [**2**], joint work with Bastos, Mota, Schacht and Schnitzer. When Bastos and Mota where in Hamburg in 2015, we began to investigate questions of this type and studied the absorption method. Together we discussed the necessary steps in the proof. We split the first draft of the paper between us, where I drafted mainly the first version of Section 5.1.1, of Section 5.1.2, and of the texts that connect different parts of the proof. Together we improved to the version appearing in [**2**].

Declaration on Oath/Eidesstattliche Erklärung

I hereby declare, on oath, that I have written the present dissertation by my own and have not used other than the acknowledged resources and aids.

Hiermit erkläre ich an Eides statt, dass ich die vorliegende Dissertationsschrift selbst verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Hamburg, den 29. November 2016

Fabian Schulenburg